Lecture Notes in Mathematics

An Introduction to Riemannian Geometry

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Preface

These lecture notes grew out of an M.Sc. course on differential geometry which I gave at the University of Leeds in Spring 1992. Their main purpose is to introduce the beautiful theory of Riemannian geometry, a still very active area of mathematical research.

This is a subject with no lack of interesting examples. They are indeed the key to a good understanding of it and will therefore play a major role throughout this work. Of special interest are the classical Lie groups allowing concrete calculations of many of the abstract notions on the menu.

The study of Riemannian geometry is rather meaningless without some basic knowledge on Gaussian geometry i.e. the geometry of curves and surfaces in 3-dimensional Euclidean space. For this we recommend the following text: M. P. do Carmo, *Differential geometry of curves and surfaces*, Prentice Hall (1976).

These lecture notes are written for students with a good understanding of linear algebra, real analysis of several variables, the classical theory of ordinary differential equations and some topology. The most important results stated in the text are also proven there. Others are left to the reader as exercises, which follow at the end of each chapter. This format is aimed at students willing to put **hard work** into the course. For further reading we recommend the excellent standard text: M. P. do Carmo, *Riemannian Geometry*, Birkhäuser (1992).

I am very grateful to my enthusiastic students and many other readers who have, throughout the years, contributed to the text by giving numerous valuable comments on the presentation.

Norra Nöbbelöv the 15th of April 2024

Sigmundur Gudmundsson

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CHAPTER 1

Introduction

On the 10th of June 1854 Georg Friedrich Bernhard Riemann (1826-1866) gave his famous "Habilitationsvortrag" in the Colloquium of the Philosophical Faculty at Göttingen. His talk "Über die Hypothesen, welche der Geometrie zu Grunde liegen" is often said to be the most important in the history of differential geometry. Johann Carl Friedrich Gauss (1777-1855) was in the audience, at the age of 77, and is said to have been very impressed by his former student.

Riemann's revolutionary ideas generalised the geometry of surfaces which had earlier been initiated by Gauss. Later this lead to an exact definition of the modern concept of an abstract Riemannian manifold.

The development of the 20th century has turned Riemannian geometry into one of the most important parts of modern mathematics. For an excellent survey on this vast field we recommend the following work written by one of the main actors: M. Berger, *A Panoramic View of Riemannian Geometry*, Springer (2003).

CHAPTER 2

Differentiable Manifolds

In this chapter we introduce the important concept of a differentiable manifold. This generalises the curves and surfaces in $\mathbb{R}^3$ studied in classical differential geometry. Our manifolds are modelled on the standard differentiable structure on the classical vector spaces $\mathbb{R}^m$ via compatible local charts. We give many explicit examples of differentiable manifolds, study their submanifolds and differentiable maps between them.

Let $\mathbb{R}^m$ be the m-dimensional real vector space equipped with its standard topology $\mathcal{T}_m$ induced by the Euclidean metric d on $\mathbb{R}^m$ given by

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_m - y_m)^2}.$$

For a natural number r and an open subset U of $\mathbb{R}^m$ we will by $C^r(U, \mathbb{R}^n)$ denote the r-times continuously **differentiable** maps from U to $\mathbb{R}^n$. By **smooth** maps $U \to \mathbb{R}^n$ we mean the elements of the set

$$C^\infty(U, \mathbb{R}^n) = \bigcap_{r=0}^{\infty} C^r(U, \mathbb{R}^n).$$

The set of **real analytic** maps from U to $\mathbb{R}^n$ will be denoted by $C^\omega(U, \mathbb{R}^n)$. For the theory of real analytic maps we recommend the important text: S. G. Krantz and H. R. Parks, *A Primer of Real Analytic Functions*, Birkhäuser (1992).

Definition 2.1. Let $(M, \mathcal{T})$ be a topological Hausdorff space with a countable basis. Then M is called a **topological manifold** if there exists a positive integer $m \in \mathbb{Z}^+$ such that for each point $p \in M$ we have an open neighbourhood U of p, an open subset V of $\mathbb{R}^m$ and a homeomorphism $x : U \to V$. The pair (U, x) is called a **local chart** (or **local coordinates**) on M. The natural number m is called the **dimension** of M. To denote that the dimension of M is m we write M^m.

According to Definition 2.1, an m-dimensional topological manifold $(M^m, \mathcal{T})$ is locally homeomorphic to the standard $\mathbb{R}^m$. We will now introduce a differentiable structure $\hat{\mathcal{A}}$ on M via its local charts and turn it into a differentiable manifold.

Definition 2.2. Let M be an m-dimensional topological manifold. Then a C^r-**atlas** on M is a collection

$$\mathcal{A} = \{(U_\alpha, x_\alpha) \mid \alpha \in \mathcal{I}\}$$

of local charts on M such that $\mathcal{A}$ covers the whole of M i.e.

$$M = \bigcup_\alpha U_\alpha$$

and for all $\alpha, \beta \in \mathcal{I}$ the corresponding **transition maps**

$$x_\beta \circ x_\alpha^{-1}|_{x_\alpha(U_\alpha \cap U_\beta)} : x_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^m \to \mathbb{R}^m$$

are r-times continuously differentiable i.e. of class C^r.

A local chart (U, x) on M is said to be **compatible** with a C^r-atlas $\mathcal{A}$ on M if the union $\mathcal{A} \cup \{(U, x)\}$ is a C^r-atlas on M. A C^r-atlas $\hat{\mathcal{A}}$ on M is said to be **maximal** if it contains all the local charts that are compatible with it.

A maximal atlas $\hat{\mathcal{A}}$ on M is also called a C^r-**structure** on M. The pair $(M, \hat{\mathcal{A}})$ is said to be a C^r-**manifold**, or a **differentiable manifold** of class C^r, if M is a topological manifold and $\hat{\mathcal{A}}$ is a C^r-structure on M. A differentiable manifold is said to be **smooth** if its transition maps are C^∞ and **real analytic** if they are C^ω.

Remark 2.3. It should be noted that a given C^r-atlas $\mathcal{A}$ on a topological manifold M determines a unique C^r-structure $\hat{\mathcal{A}}$ on M containing $\mathcal{A}$. It simply consists of all the local charts on M compatible with $\mathcal{A}$.

Example 2.4. For the standard topological space $(\mathbb{R}^m, \mathcal{T}_m)$ we have the trivial C^ω-atlas

$$\mathcal{A} = \{(\mathbb{R}^m, x) \mid x : p \mapsto p\}$$

inducing the standard C^ω-structure $\hat{\mathcal{A}}$ on $\mathbb{R}^m$.

Example 2.5. Let S^m denote the unit sphere in $\mathbb{R}^{m+1}$ i.e.

$$S^m = \{p \in \mathbb{R}^{m+1} \mid p_1^2 + \cdots + p_{m+1}^2 = 1\},$$

equipped with the subset topology $\mathcal{T}$ induced by the standard $\mathcal{T}_{m+1}$ on $\mathbb{R}^{m+1}$. Let N be the north pole $N = (1, 0) \in \mathbb{R} \times \mathbb{R}^m$ and S be the south pole $S = (-1, 0)$ on S^m, respectively. Put $U_N = S^m \setminus \{N\}$,

$U_S = S^m \setminus \{S\}$ and define the homeomorphisms $x_N : U_N \to \mathbb{R}^m$ and $x_S : U_S \to \mathbb{R}^m$ by

$$x_N : (p_1, \dots, p_{m+1}) \mapsto \frac{1}{1 - p_1}(p_2, \dots, p_{m+1}),$$

$$x_S : (p_1, \dots, p_{m+1}) \mapsto \frac{1}{1 + p_1}(p_2, \dots, p_{m+1}).$$

Then the C^ω transition maps

$$x_S \circ x_N^{-1}, x_N \circ x_S^{-1} : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^m \setminus \{0\}$$

are both given by

$$x \mapsto \frac{x}{|x|^2},$$

so $\mathcal{A} = \{(U_N, x_N), (U_S, x_S)\}$ is a C^ω-atlas on S^m. The corresponding C^ω-manifold $(S^m, \hat{\mathcal{A}})$ is called the m-dimensional **standard sphere**.

Another interesting example of a differentiable manifold is the m-dimensional real projective space $\mathbb{R}P^m$.

Example 2.6. On the set $\mathbb{R}^{m+1} \setminus \{0\}$ we define the equivalence relation $\equiv$ by

$$p \equiv q \text{ if and only if there exists a } \lambda \in \mathbb{R}^* \text{ such that } p = \lambda \cdot q.$$

Let $\mathbb{R}P^m$ be the quotient space $(\mathbb{R}^{m+1} \setminus \{0\})/\equiv$ and

$$\pi : \mathbb{R}^{m+1} \setminus \{0\} \to \mathbb{R}P^m$$

be the natural projection, mapping a point $p \in \mathbb{R}^{m+1} \setminus \{0\}$ onto the equivalence class $[p] \in \mathbb{R}P^m$ i.e. the punctured line

$$[p] = \{\lambda \cdot p \in \mathbb{R}^{m+1} \,|\, \lambda \in \mathbb{R}^*\},$$

generated by p.

We then equip the set $\mathbb{R}P^m$ with the quotient topology $\mathcal{T}$ induced by π and $\mathcal{T}_{m+1}$ on $\mathbb{R}^{m+1}$. This means that a subset U of $\mathbb{R}P^m$ is open if and only if its pre-image $\pi^{-1}(U)$ is open in $\mathbb{R}^{m+1} \setminus \{0\}$. For $k \in \{1, \dots, m+1\}$ we then define the open subset U_k of $\mathbb{R}P^m$ by

$$U_k = \{[p] \in \mathbb{R}P^m \,|\, p_k \neq 0\}$$

and the local charts $x_k : U_k \subset \mathbb{R}P^m \to \mathbb{R}^m$ by

$$x_k : [p] \mapsto (\frac{p_1}{p_k}, \dots, \frac{p_{k-1}}{p_k}, 1, \frac{p_{k+1}}{p_k}, \dots, \frac{p_{m+1}}{p_k}).$$

If $[p] = [q] \in U_k$ then $p = \lambda \cdot q$ for some $\lambda \in \mathbb{R}^*$ so $p_l/p_k = q_l/q_k$ for all l. This shows that the maps $x_k : U_k \subset \mathbb{R}P^m \to \mathbb{R}^m$ are all well defined.

A line $[p] \in \mathbb{R}P^m$ is represented by a non-zero point $p \in \mathbb{R}^{m+1}$ so at least one of its components is non-zero. This shows that

$$\mathbb{R}P^m = \bigcup_{k=1}^{m+1} U_k.$$

The corresponding transition maps

$$x_k \circ x_l^{-1}|_{x_l(U_l \cap U_k)} : x_l(U_l \cap U_k) \subset \mathbb{R}^m \to \mathbb{R}^m$$

are given by

$$(\frac{p_1}{p_l}, \dots, \frac{p_{l-1}}{p_l}, 1, \frac{p_{l+1}}{p_l}, \dots, \frac{p_{m+1}}{p_l}) \mapsto (\frac{p_1}{p_k}, \dots, \frac{p_{k-1}}{p_k}, 1, \frac{p_{k+1}}{p_k}, \dots, \frac{p_{m+1}}{p_k}),$$

so the collection

$$\mathcal{A} = \{(U_k, x_k) \,|\, k = 1, \dots, m+1\}$$

is a C^ω-atlas on $\mathbb{R}P^m$. The real-analytic manifold $(\mathbb{R}P^m, \hat{\mathcal{A}})$ is called the m-dimensional **real projective space**.

Remark 2.7. The above definition of the real projective space $\mathbb{R}P^m$ might seem very abstract. But later on we will embed $\mathbb{R}P^m$ into the vector space $\mathrm{Sym}(\mathbb{R}^{m+1})$ of symmetric $(m+1) \times (m+1)$ real matrices. For this see Example 3.26.

Example 2.8. Let $\hat{\mathbb{C}}$ be the extended complex plane given by

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

and put $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $U_0 = \mathbb{C}$ and $U_\infty = \hat{\mathbb{C}} \setminus \{0\}$. Then define the local charts $x_0 : U_0 \to \mathbb{C}$ and $x_\infty : U_\infty \to \mathbb{C}$ on $\hat{\mathbb{C}}$ by $x_0 : z \mapsto z$ and $x_\infty : w \mapsto 1/w$, respectively. Then the corresponding transition maps

$$x_\infty \circ x_0^{-1}, x_0 \circ x_\infty^{-1} : \mathbb{C}^* \to \mathbb{C}^*$$

are both given by $z \mapsto 1/z$ so $\mathcal{A} = \{(U_0, x_0), (U_\infty, x_\infty)\}$ is a C^ω-atlas on $\hat{\mathbb{C}}$. The real analytic manifold $(\hat{\mathbb{C}}, \hat{\mathcal{A}})$ is called the **Riemann sphere**.

For the product of two differentiable manifolds we have the following important result.

Proposition 2.9. *Let $(M_1, \hat{\mathcal{A}}_1)$ and $(M_2, \hat{\mathcal{A}}_2)$ be two differentiable manifolds of class C^r. Let $M = M_1 \times M_2$ be the product space with the product topology. Then there exists an atlas $\mathcal{A}$ on M turning $(M, \hat{\mathcal{A}})$ into a differentiable manifold of class C^r and the dimension of M satisfies*

$$\dim M = \dim M_1 + \dim M_2.$$

Proof. See Exercise 2.1. □

The concept of a submanifold of a given differentiable manifold will play an important role as we go along and we will be especially interesting in the connection between the geometry of a submanifold and that of its ambient space.

Definition 2.10. Let m, n be positive integers with $m \leq n$ and $(N^n, \hat{\mathcal{A}}_N)$ be a C^r-manifold. A subset M of N is said to be a **submanifold** of N if for each point $p \in M$ there exists a local chart $(U_p, x_p) \in \hat{\mathcal{A}}_N$ such that $p \in U_p$ and $x_p : U_p \subset N \to \mathbb{R}^m \times \mathbb{R}^{n-m}$ satisfies

$$x_p(U_p \cap M) = x_p(U_p) \cap (\mathbb{R}^m \times \{0\}).$$

The natural number $(n - m)$ is called the **codimension** of M in N.

Proposition 2.11. *Let m, n be positive integers with $m \leq n$ and $(N^n, \hat{\mathcal{A}}_N)$ be a C^r-manifold. Let M be a submanifold of N equipped with the subset topology and $\pi : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$ be the natural projection onto the first factor. Then*

$$\mathcal{A}_M = \{(U_p \cap M, (\pi \circ x_p)|_{U_p \cap M}) \mid p \in M\}$$

is a C^r-atlas for M. Hence the pair $(M, \hat{\mathcal{A}}_M)$ is an m-dimensional C^r-manifold. The differentiable structure $\hat{\mathcal{A}}_M$ is called the **induced differentiable structure** *on M by $\hat{\mathcal{A}}_N$ on N.*

Proof. See Exercise 2.2. □

Remark 2.12. Our next aim is to prove Theorem 2.16 which is a useful tool for the construction of submanifolds of $\mathbb{R}^m$. For this we use the classical inverse mapping theorem stated below. Note that if

$$F : U \to \mathbb{R}^n$$

is a differentiable C^r-map defined on an open subset U of $\mathbb{R}^m$ then its differential $dF_p : \mathbb{R}^m \to \mathbb{R}^n$ at the point $p \in U$ is a linear map given by the $n \times m$ matrix

$$dF_p = \begin{pmatrix} \partial F_1/\partial x_1(p) & \dots & \partial F_1/\partial x_m(p) \\ \vdots & & \vdots \\ \partial F_n/\partial x_1(p) & \dots & \partial F_n/\partial x_m(p) \end{pmatrix}.$$

If $\gamma : \mathbb{R} \to U$ is a curve in U such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v \in \mathbb{R}^m$, then the composition $F \circ \gamma : \mathbb{R} \to \mathbb{R}^n$ is a curve in $\mathbb{R}^n$ and according to the chain rule we have

$$dF_p \cdot v = \frac{d}{ds}(F \circ \gamma(s))|_{s=0}.$$

This is the tangent vector of the curve $F \circ \gamma$ at $F(p) \in \mathbb{R}^n$.

The above shows that the differential dF_p can be seen as a linear map that maps tangent vectors at $p \in U$ to tangent vectors at the image point $F(p) \in \mathbb{R}^n$. This will later be generalised to the manifold setting.

We now state the classical inverse mapping theorem well known from multivariable analysis.

Fact 2.13. *Let U be an open subset of $\mathbb{R}^m$ and $F : U \to \mathbb{R}^m$ be a C^r-map. If $p \in U$ and the differential*

$$dF_p : \mathbb{R}^m \to \mathbb{R}^m$$

of F at p is invertible then there exist open neighbourhoods U_p around p and U_q around $q = F(p)$ such that $\hat{F} = F|_{U_p} : U_p \to U_q$ is bijective and the inverse $(\hat{F})^{-1} : U_q \to U_p$ is a C^r-map. The differential $(d\hat{F}^{-1})_q$ of $\hat{F}^{-1}$ at q satisfies

$$(d\hat{F}^{-1})_q = (dF_p)^{-1}$$

i.e. it is the inverse of the linear differential dF_p of F at p.

Before stating the classical implicit mapping theorem we remind the reader of the following well known notions.

Definition 2.14. Let m, n be positive integers, U be an open subset of $\mathbb{R}^m$ and $F : U \to \mathbb{R}^n$ be a C^r-map. A point $p \in U$ is said to be **regular** for F, if the differential

$$dF_p : \mathbb{R}^m \to \mathbb{R}^n$$

is of full rank, but **critical** otherwise. A point $q \in F(U)$ is said to be a **regular value** of F if every point in the pre-image $F^{-1}(\{q\})$ of q is regular.

Remark 2.15. Note that if m, n are positive integers with $m \geq n$ then $p \in U$ is a regular point for

$$F = (F_1, \dots, F_n) : U \to \mathbb{R}^n$$

if and only if the gradients $\text{grad}F_1, \dots, \text{grad}F_n$ of the coordinate functions $F_1, \dots, F_n : U \to \mathbb{R}$ are linearly independent at p, or equivalently, the differential dF_p of F at p satisfies the following condition

$$\det(dF_p \cdot (dF_p)^t) \neq 0.$$

The next result is a useful tool for constructing submanifolds of the classical vector space $\mathbb{R}^m$.

Theorem 2.16 (The implicit function theorem)**.** *Let m, n be positive integers with $m > n$ and $F : U \to \mathbb{R}^n$ be a C^r-map from an open subset U of $\mathbb{R}^m$. If $q \in F(U)$ is a regular value of F then the pre-image $F^{-1}(\{q\})$ of q is an $(m-n)$-dimensional submanifold of $\mathbb{R}^m$ of class C^r.*

Proof. Let p be an element of $F^{-1}(\{q\})$ and K_p be the kernel of the differential dF_p i.e. the $(m-n)$-dimensional subspace of $\mathbb{R}^m$ given by $K_p = \{v \in \mathbb{R}^m \,|\, dF_p \cdot v = 0\}$. Let $\pi_p : \mathbb{R}^m \to \mathbb{R}^{m-n}$ be a linear map such that $\pi_p|_{K_p} : K_p \to \mathbb{R}^{m-n}$ is bijective, $\pi_p|_{K_p^\perp} = 0$ and define the map $G_p : U \to \mathbb{R}^n \times \mathbb{R}^{m-n}$ by

$$G_p : x \mapsto (F(x), \pi_p(x)).$$

Then the differential $(dG_p)_p : \mathbb{R}^m \to \mathbb{R}^m$ of G_p, with respect to the decompositions $\mathbb{R}^m = K_p^\perp \oplus K_p$ and $\mathbb{R}^m = \mathbb{R}^n \oplus \mathbb{R}^{m-n}$, is given by

$$(dG_p)_p = \begin{pmatrix} dF_p|_{K_p^\perp} & 0 \\ 0 & \pi_p \end{pmatrix},$$

hence bijective. It now follows from the inverse function theorem that there exist open neighbourhoods V_p around p and W_p around $G_p(p)$ such that $\hat{G}_p = G_p|_{V_p} : V_p \to W_p$ is bijective, the inverse $\hat{G}_p^{-1} : W_p \to V_p$ is C^r, $d(\hat{G}_p^{-1})_{G_p(p)} = (dG_p)_p^{-1}$ and $d(\hat{G}_p^{-1})_y$ is bijective for all $y \in W_p$. Now put $\tilde{U}_p = F^{-1}(\{q\}) \cap V_p$ then

$$\tilde{U}_p = \hat{G}_p^{-1}((\{q\} \times \mathbb{R}^{m-n}) \cap W_p)$$

so if $\pi : \mathbb{R}^n \times \mathbb{R}^{m-n} \to \mathbb{R}^{m-n}$ is the natural projection onto the second factor, then the map

$$\tilde{x}_p = \pi \circ G_p|_{\tilde{U}_p} : \tilde{U}_p \to (\{q\} \times \mathbb{R}^{m-n}) \cap W_p \to \mathbb{R}^{m-n}$$

is a local chart on the open neighbourhood $\tilde{U}_p$ of p. The point $q \in F(U)$ is a regular value so the set

$$\mathcal{A} = \{(\tilde{U}_p, \tilde{x}_p) \,|\, p \in F^{-1}(\{q\})\}$$

is a C^r-atlas for $F^{-1}(\{q\})$. $\square$

Applying the implicit function theorem, we obtain the following interesting examples of the m-dimensional sphere S^m and its tangent bundle TS^m as differentiable submanifolds of $\mathbb{R}^{m+1}$ and $\mathbb{R}^{2m+2}$, respectively.

Example 2.17. Let $F : \mathbb{R}^{m+1} \to \mathbb{R}$ be the C^ω-map given by

$$F : (p_1, \dots, p_{m+1}) \mapsto p_1^2 + \cdots + p_{m+1}^2.$$

Then the differential dF_p of F at the point $p \in \mathbb{R}^{m+1}$ is given by $dF_p = 2 \cdot p$, so

$$dF_p \cdot (dF_p)^t = 4|p|^2 \in \mathbb{R}.$$

This means that $1 \in \mathbb{R}$ is a regular value of F, so the fibre

$$S^m = \{p \in \mathbb{R}^{m+1} \,|\, |p|^2 = 1\} = F^{-1}(\{1\})$$

of F is an m-dimensional submanifold of $\mathbb{R}^{m+1}$. This is the m-dimensional **standard sphere** introduced in Example 2.5.

Example 2.18. Let $F : \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \to \mathbb{R}^2$ be the C^ω-map given by

$$F : (p, v) \mapsto ((|p|^2 - 1)/2, \langle p, v\rangle).$$

Then the differential $dF_{(p,v)}$ of F at (p, v) satisfies

$$dF_{(p,v)} = \begin{pmatrix} p & 0 \\ v & p \end{pmatrix} \in \mathbb{R}^{2\times(2m+2)}.$$

A simple calculation shows that

$$\det(dF \cdot (dF)^t) = \det \begin{pmatrix} |p|^2 & \langle p, v\rangle \\ \langle p, v\rangle & |v|^2 + |p|^2 \end{pmatrix} = 1 + |v|^2 > 0$$

on the fibre $F^{-1}(\{0\})$. This means that

$$F^{-1}(\{0\}) = \{(p, v) \in \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \,|\, |p|^2 = 1 \text{ and } \langle p, v\rangle = 0\}$$

is a $2m$-dimensional submanifold of $\mathbb{R}^{2m+2}$. We will later see that the set $TS^m = F^{-1}(\{0\})$ is what is called the **tangent bundle** of the m-dimensional sphere S^m.

We now employ the implicit function theorem to construct the important orthogonal group $\mathbf{O}(m)$ as a submanifold of the linear space $\mathbb{R}^{m\times m}$.

Example 2.19. Let $\mathbb{R}^{m\times m}$ be the m^2-dimensional vector space of real $m \times m$ matrices and $\mathrm{Sym}(\mathbb{R}^m)$ be its linear subspace consisting of the symmetric matrices given by

$$\mathrm{Sym}(\mathbb{R}^m) = \{x \in \mathbb{R}^{m\times m} \,|\, x^t = x\}.$$

A generic element $x \in \mathrm{Sym}(\mathbb{R}^m)$ is of the form

$$x = \begin{pmatrix} x_{11} & \cdots & x_{1m} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mm} \end{pmatrix},$$

where $x_{kl} = x_{lk}$ for all $k, l = 1, 2, \dots m$. With this at hand, it is easily seen that the dimension of the subspace $\mathrm{Sym}(\mathbb{R}^m)$ is $m(m + 1)/2$.

Let $F : \mathbb{R}^{m\times m} \to \mathrm{Sym}(\mathbb{R}^m)$ be the map defined by

$$F : x \mapsto x^t \cdot x.$$

Then the inverse image $\mathbf{O}(m) = F^{-1}(\{e\})$ of F clearly satisfies

$$F^{-1}(\{e\}) = \{x \in \mathbb{R}^{m\times m} \,|\, x^t x = e\}.$$

If $\gamma : I \to \mathbb{R}^{m\times m}$ is a curve in $\mathbb{R}^{m\times m}$ such that $\gamma(0) = x$ and $\dot\gamma(0) = X$, then

$$\begin{aligned} dF_x(X) &= \frac{d}{ds}(F \circ \gamma(s))|_{s=0} \\ &= \frac{d}{ds}(\gamma(s)^t \cdot \gamma(s))|_{s=0} \\ &= (\dot\gamma(s)^t \cdot \gamma(s) + \gamma(s)^t \cdot \dot\gamma(s))|_{s=0} \\ &= X^t \cdot x + x^t \cdot X. \end{aligned}$$

This means that for arbitrary elements $x \in \mathbf{O}(m)$ and $X \in \mathbb{R}^{m\times m}$ we have

$$\begin{aligned} dF_x(xX) &= (xX)^t \cdot x + x^t \cdot (xX) \\ &= X^t x^t \cdot x + x^t \cdot x\; X \\ &= X^t + X. \end{aligned}$$

It is a well know fact from linear algebra, that for the linear vector space $\mathbb{R}^{m\times m}$ of real $m \times m$ matrices we have the direct sum

$$\mathrm{Skew}(\mathbb{R}^m) \oplus \mathrm{Sym}(\mathbb{R}^m)$$

i.e. every matrix $X \in \mathbb{R}^{m\times m}$ has a unique decomposition $X = Y + Z$ where

$$Y = \tfrac{1}{2}(X - X^t) \in \mathrm{Skew}(\mathbb{R}^m) \quad \text{and} \quad Z = \tfrac{1}{2}(X + X^t) \in \mathrm{Sym}(\mathbb{R}^m).$$

This means that $dF_x(xY) = 0$, $dF_x(xZ/2) = Z$ and shows that the differential dF_x is surjective, so the identity matrix $e \in \mathrm{Sym}(\mathbb{R}^m)$ is a regular value for F.

It is now a direct consequence of the implicit function theorem that $\mathbf{O}(m)$ is a submanifold of $\mathbb{R}^{m\times m}$ of dimension $m(m-1)/2$. We will later see that the set $\mathbf{O}(m)$ can be equipped with a group structure and is then called the **orthogonal group**.

The concept of a differentiable map $U \to \mathbb{R}^n$, defined on an open subset of $\mathbb{R}^m$, can be generalised to mappings between manifolds. We will see that the most important properties of these objects, in the classical case, are also valid in the manifold setting.

Definition 2.20. Let $(M^m, \hat{\mathcal{A}}_M)$ and $(N^n, \hat{\mathcal{A}}_N)$ be C^r-manifolds. A map $\phi : M \to N$ is said to be **differentiable** of class C^r **at a point** $p \in M$ if there exist local charts $(U, x) \in \hat{\mathcal{A}}_M$ around p and $(V, y) \in \hat{\mathcal{A}}_N$ around $q = \phi(p)$ such that the **transition** map

$$y \circ \phi \circ x^{-1}|_{x(U \cap \phi^{-1}(V))} : x(U \cap \phi^{-1}(V)) \subset \mathbb{R}^m \to \mathbb{R}^n$$

is of class C^r. The map ϕ is said to be **differentiable** of class C^r if it is differentiable of class C^r at every point $p \in M$.

A differentiable map $\gamma : I \to M$, defined on an open interval I of $\mathbb{R}$, is called a **differentiable curve** in M. A real-valued differentiable map $f : M \to \mathbb{R}$ is called a **differentiable function** on M. The set of smooth functions defined on M is denoted by $C^\infty(M)$.

Remark 2.21. It should be noted that, in Definition 2.20, the differentiablility of $\phi : M_1 \to M_2$ at a point $p \in M$ is independent of the choice of the local charts (U, x) and (V, y).

It is an easy exercise, using Definition 2.20, to prove the following result concerning the composition of differentiable maps between manifolds.

Proposition 2.22. *Let $(M_1, \hat{\mathcal{A}}_1), (M_2, \hat{\mathcal{A}}_2), (M_3, \hat{\mathcal{A}}_3)$ be C^r-manifolds and $\phi : (M_1, \hat{\mathcal{A}}_1) \to (M_2, \hat{\mathcal{A}}_2)$, $\psi : (M_2, \hat{\mathcal{A}}_2) \to (M_3, \hat{\mathcal{A}}_3)$ be two differentiable maps of class C^r. Then the composition $\psi \circ \phi : (M_1, \hat{\mathcal{A}}_1) \to (M_3, \hat{\mathcal{A}}_3)$ is a differentiable map of class C^r.*

Proof. See Exercise 2.5. □

Definition 2.23. Two manifolds $(M, \hat{\mathcal{A}}_M)$ and $(N, \hat{\mathcal{A}}_N)$ of class C^r are said to be **diffeomorphic** if there exists a bijective C^r-map $\phi : M \to N$ such that the inverse $\phi^{-1} : N \to M$ is of class C^r. In that case the map ϕ is called a **diffeomorphism** between $(M, \hat{\mathcal{A}}_M)$ and $(N, \hat{\mathcal{A}}_N)$.

Proposition 2.24. *Let $(M, \hat{\mathcal{A}})$ be an m-dimensional C^r-manifold and (U, x) be a local chart on M. Then the bijective continuous map $x : U \to x(U) \subset \mathbb{R}^m$ is a diffeomorphism.*

Proof. See Exercise 2.6. □

It can be shown that the 2-dimensional unit sphere S^2, in the Euclidean $\mathbb{R}^3$, and the Riemann sphere $\hat{\mathbb{C}}$ are diffeomorphic, see Exercise 2.7.

Definition 2.25. For a differentiable manifold $(M, \hat{\mathcal{A}})$ we denote by $\mathcal{D}(M)$ the set of all its diffeomorphisms. If $\phi, \psi \in \mathcal{D}(M)$ then it is clear that the composition $\psi \circ \phi$ and the inverse ϕ^{-1} are also diffeomorphisms. The operation is clearly associative and the identity map is its neutral element. This means that the pair $(\mathcal{D}(M), \circ)$ forms a group, called the **diffeomorphism group** of $(M, \hat{\mathcal{A}})$.

Definition 2.26. Two C^r-structures $\hat{\mathcal{A}}_1$ and $\hat{\mathcal{A}}_2$ on the same topological manifold M are said to be **different** if the identity map $\mathrm{id}_M : (M, \hat{\mathcal{A}}_1) \to (M, \hat{\mathcal{A}}_2)$ is not a diffeomorphism.

It can be seen that even the real line $\mathbb{R}$ carries infinitely many different differentiable structures, see Exercise 2.8.

Deep Result 2.27. *Let $(M, \hat{\mathcal{A}}_M)$ and $(N, \hat{\mathcal{A}}_N)$ be differentiable manifolds of class C^r of the same dimension m. If M and N are homeomorphic as topological spaces and $m \leq 3$ then $(M, \hat{\mathcal{A}}_M)$ and $(N, \hat{\mathcal{A}}_N)$ are diffeomorphic.*

The following remarkable result was proven by M. A. Kervaire and J. M. Milnor in their celebrated paper *Groups of Homotopy Spheres: I*, Annals of Mathematics **77** (1963), 504-537.

Deep Result 2.28. *The 7-dimensional sphere S^7 has exactly 28 different smooth differentiable structures.*

The next useful statement generalises a classical result from the real analysis of several variables.

Proposition 2.29. *Let $(N_1, \hat{\mathcal{A}}_1)$ and $(N_2, \hat{\mathcal{A}}_2)$ be two differentiable manifolds of class C^r and M_1, M_2 be submanifolds of N_1 and N_2, respectively. If $\phi : N_1 \to N_2$ is a differentiable map of class C^r such that $\phi(M_1)$ is contained in M_2 then the restriction $\phi|_{M_1} : M_1 \to M_2$ is differentiable of class C^r.*

Proof. See Exercise 2.9. □

Example 2.30. The above Propositon 2.29 provides the following list of interesting examples of differentiable maps between the manifolds which we have introduced above.

(i) $\phi_1 : \mathbb{R}^1 \to S^1 \subset \mathbb{C}$, $\phi_1 : t \mapsto e^{it}$,
(ii) $\phi_2 : S^2 \subset \mathbb{R}^3 \to S^3 \subset \mathbb{R}^4$, $\phi_2 : (x, y, z) \mapsto (x, y, z, 0)$,
(iii) $\phi_3 : S^3 \subset \mathbb{C}^2 \to S^2 \subset \mathbb{C} \times \mathbb{R}$, $\phi_3 : (z_1, z_2) \mapsto (2z_1\bar{z}_2, |z_1|^2 - |z_2|^2)$,
(iv) $\phi_4 : \mathbb{R}^{m+1} \setminus \{0\} \to S^m \subset \mathbb{R}^{m+1}$, $\phi_4 : x \mapsto x/|x|$,
(v) $\phi_5 : S^m \to \mathbb{R}P^m$, $\phi_5 : x \mapsto [x]$.
(vi) $\phi_6 = \phi_5 \circ \phi_4 : \mathbb{R}^{m+1} \setminus \{0\} \to \mathbb{R}P^m$, $\phi_6 : x \mapsto [x/|x|]$,
(vii) $\phi_7 : \mathbf{U}(m) \to \mathbb{R} : x \mapsto x_{11}$.

In differential geometry, we are interested in manifolds carrying a group structure compatible with their differentiable structures. Such manifolds are named after the famous mathematician Sophus Lie (1842-1899) and will play an important role throughout this work.

Definition 2.31. A **Lie group** is a smooth manifold G with a group structure $\cdot$ such that the map $\rho : G \times G \to G$ with

$$\rho : (p, q) \mapsto p \cdot q^{-1}$$

is smooth.

Example 2.32. Let $(\mathbb{R}^m, +, \cdot)$ be the m-dimensional real vector space equipped with its standard differential structure. Then $(\mathbb{R}^m, +)$ with $\rho : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ given by

$$\rho : (p, q) \mapsto p - q$$

is a Lie group.

Definition 2.33. Let $(G, \cdot)$ be a Lie group and p be an element of G. Then we define the **left translation** $L_p : G \to G$ of G by p with

$$L_p : q \mapsto p \cdot q.$$

Proposition 2.34. *Let G be a Lie group and p be an element of G. Then the left translation $L_p : G \to G$ is a smooth diffeomorphism.*

Proof. See Exercise 2.11 $\square$

Proposition 2.35. *Let $(G, \cdot)$ be a Lie group and K be a submanifold of G which is a subgroup. Then $(K, \cdot)$ is a Lie group.*

Proof. The statement is a direct consequence of Definition 2.31 and Proposition 2.29. $\square$

Example 2.36. Let $(\mathbb{C}^*, \cdot)$ be the set of non-zero complex numbers equipped with its standard multiplication. Then $(\mathbb{C}^*, \cdot)$ is a Lie group. The unit circle $(S^1, \cdot)$ is an interesting compact Lie subgroup of $(\mathbb{C}^*, \cdot)$. Another subgroup is the set of the non-zero real numbers $(\mathbb{R}^*, \cdot)$ containing the positive real numbers $(\mathbb{R}^+, \cdot)$ as a subgroup.

Definition 2.37. Let $(G, \cdot)$ be a Lie group and V be a finite-dimensional real vector space of dimension n. Then an n-dimensional **linear representation** of G on V is a map

$$\rho : G \to \mathrm{Aut}(V)$$

into the space of automorphisms of V i.e. the invertible linear endomorphisms such that for all $g, h \in G$ we have

$$\rho(g \cdot h) = \rho(g) \circ \rho(h).$$

Here $\circ$ denotes the composition in $\mathrm{Aut}(V)$. The linear representation $\rho : G \to \mathrm{Aut}(V)$ is said to be **faithful** if it is injective.

Remark 2.38. It should be noted that for a given basis for the vector space V and an element $g \in G$ the automorphism $\rho(g) \in \mathrm{Aut}(V)$ can be represented by an invertible matrix with respect to this basis and then the operation $\circ$ is just the standard matrix multiplication.

Example 2.39. The Lie group of non-zero complex numbers $(\mathbb{C}^*, \cdot)$ has a well known linear representation $\rho : \mathbb{C}^* \to \mathrm{Aut}(\mathbb{R}^2)$ on $\mathbb{R}^2$ given by

$$\rho : a + ib \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

This is obviously injective and it respects the standard multiplicative structures of $\mathbb{C}^*$ and $\mathbb{R}^{2\times 2}$ since

$$\begin{aligned}
\rho((a+ib)\cdot(x+iy)) &= \rho((ax-by)+i(bx+ay)) \\
&= \begin{pmatrix} ax-by & -(bx+ay) \\ bx+ay & ax-by \end{pmatrix} \\
&= \begin{pmatrix} a & -b \\ b & a \end{pmatrix} * \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \\
&= \rho(a+ib) * \rho(x+iy).
\end{aligned}$$

As an introduction to Example 2.41 we now play the same game in the complex case.

Example 2.40. Let $\rho : \mathbb{C}^2 \to \mathbb{C}^{2\times 2}$ be the real linear map given by

$$\rho : (z, w) \mapsto \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix}.$$

Then an easy calculation shows that the following is true

$$\begin{aligned}
\rho(z_1, w_1) * \rho(z_2, w_2) &= \begin{pmatrix} z_1 & -\bar{w}_1 \\ w_1 & \bar{z}_1 \end{pmatrix} * \begin{pmatrix} z_2 & -\bar{w}_2 \\ w_2 & \bar{z}_2 \end{pmatrix} \\
&= \begin{pmatrix} z_1 z_2 - \bar{w}_1 w_2 & -(\bar{w}_1 \bar{z}_2 + z_1 \bar{w}_2) \\ w_1 z_2 + \bar{z}_1 w_2 & \bar{z}_1 \bar{z}_2 - w_1 \bar{w}_2 \end{pmatrix} \\
&= \rho(z_1 z_2 - \bar{w}_1 w_2, w_1 z_2 + \bar{z}_1 w_2).
\end{aligned}$$

We now introduce the quaternions $\mathbb{H}$ and the three dimensional sphere S^3 which carries a natural group structure.

Example 2.41. Let $\mathbb{H}$ be the set of **quaternions** given by

$$\mathbb{H} = \{(z, w) \in \mathbb{C}^2 \mid z, w \in \mathbb{C}\}.$$

We equip $\mathbb{H}$ with an addition, a multiplication and the conjugation satisfying

(i) $(z_1, w_1) + (z_2, w_2) = (z_1 + z_2, w_1 + w_2)$,
(ii) $(z_1, w_1) \cdot (z_2, w_2) = (z_1 z_2 - \bar{w}_1 w_2, w_1 z_2 + \bar{z}_1 w_2)$,
(iii) $\overline{(z, w)} = (\bar{z}, -w)$.

These extend the standard operations on $\mathbb{C}$ as a subset of $\mathbb{H}$. It is easily seen that the non-zero quaternions $(\mathbb{H}^*, \cdot)$ form a Lie group. Then the map $\rho : \mathbb{H}^* \to \mathrm{Aut}(\mathbb{C}^2)$ with

$$\rho : (z, w) \mapsto \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix}$$

is a linear representation of $\mathbb{H}^*$ on $\mathbb{C}^2$. On $\mathbb{H}$ we define the quaternionic scalar product

$$\mathbb{H} \times \mathbb{H} \to \mathbb{H}, \qquad (p, q) \mapsto p \cdot \bar{q}$$

and a real-valued norm given by $|p|^2 = p \cdot \bar{p}$. Then the 3-dimensional unit sphere

$$S^3 = \{p \in \mathbb{H} \mid |p| = 1\}$$

in $\mathbb{H} \cong \mathbb{C}^2 \cong \mathbb{R}^4$, with the restricted multiplication, forms a compact Lie subgroup $(S^3, \cdot)$ of $(\mathbb{H}^*, \cdot)$. They are both non-abelian.

We will now introduce some of the classical real and complex matrix Lie groups. As a reference on this topic we recommend the wonderful book: A. W. Knapp, *Lie Groups Beyond an Introduction*, Birkhäuser (2002).

Example 2.42. Let Nil be the subset of $\mathbb{R}^{3\times3}$ given by

$$\mathrm{Nil} = \{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{3\times3} \mid x, y, z \in \mathbb{R}\}.$$

Then Nil has a natural differentiable structure determined by the global coordinates $\phi : \mathrm{Nil} \to \mathbb{R}^3$ with

$$\phi : \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto (x, y, z).$$

It is easily seen that if $*$ is the standard matrix multiplication, then $(\mathrm{Nil}, *)$ is a Lie group.

Example 2.43. Let Sol be the subset of $\mathbb{R}^{3\times3}$ given by

$$\mathrm{Sol} = \{ \begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{3\times3} \mid x, y, z \in \mathbb{R}\}.$$

Then Sol has a natural differentiable structure determined by the global coordinates $\phi : \mathrm{Sol} \to \mathbb{R}^3$ with

$$\phi : \begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto (x, y, z).$$

It is easily seen that if $*$ is the standard matrix multiplication, then $(\mathrm{Sol}, *)$ is a Lie group.

Example 2.44. The set of invertible real $m \times m$ matrices

$$\mathbf{GL}_m(\mathbb{R}) = \{x \in \mathbb{R}^{m\times m} \mid \det x \neq 0\},$$

equipped with the standard matrix multiplication, has the structure of a Lie group. It is called the **real general linear group** and its neutral element e is the identity matrix. The subset $\mathbf{GL}_m(\mathbb{R})$ of $\mathbb{R}^{m\times m}$ is open so $\dim \mathbf{GL}_m(\mathbb{R}) = m^2$.

As a subgroup of $\mathbf{GL}_m(\mathbb{R})$ we have the **real special linear group** $\mathbf{SL}_m(\mathbb{R})$ given by

$$\mathbf{SL}_m(\mathbb{R}) = \{x \in \mathbb{R}^{m\times m} \mid \det x = 1\}.$$

We will show in Example 3.11 that the dimension of the submanifold $\mathbf{SL}_m(\mathbb{R})$ of $\mathbb{R}^{m\times m}$ is $m^2 - 1$.

Another subgroup of $\mathbf{GL}_m(\mathbb{R})$ is the **orthogonal group**

$$\mathbf{O}(m) = \{x \in \mathbb{R}^{m\times m} \mid x^t x = e\}.$$

As we have already seen in Example 2.19 this is a submanifold of $\mathbb{R}^{m\times m}$ of dimension of $m(m-1)/2$.

As a subgroup of $\mathbf{O}(m)$ and even $\mathbf{SL}_m(\mathbb{R})$ we have the **special orthogonal group** $\mathbf{SO}(m)$ which is defined as

$$\begin{aligned} \mathbf{SO}(m) &= \mathbf{O}(m) \cap \mathbf{SL}_m(\mathbb{R}) \\ &= \{x \in \mathbb{R}^{m\times m} \mid x^t x = e \text{ and } \det x = 1\}. \end{aligned}$$

It can be shown that $\mathbf{O}(m)$ is diffeomorphic to $\mathbf{SO}(m) \times \mathbf{O}(1)$, see Exercise 2.10. Note that $\mathbf{O}(1) = \{\pm 1\}$ so $\mathbf{O}(m)$ can be seen as double cover of $\mathbf{SO}(m)$. This means that

$$\dim \mathbf{SO}(m) = \dim \mathbf{O}(m) = m(m-1)/2.$$

To the above mentioned real Lie groups we have their following complex close relatives.

Example 2.45. The set of invertible complex $m \times m$ matrices

$$\mathbf{GL}_m(\mathbb{C}) = \{z \in \mathbb{C}^{m\times m} \mid \det z \neq 0\},$$

equipped with the standard matrix multiplication, has the structure of a Lie group. It is called the **complex general linear group** and its neutral element e is the identity matrix. The subset $\mathbf{GL}_m(\mathbb{C})$ of $\mathbb{C}^{m\times m}$ is open so $\dim \mathbf{GL}_m(\mathbb{C}) = 2m^2$.

As a subgroup of $\mathbf{GL}_m(\mathbb{C})$ we have the **complex special linear group $\mathbf{SL}_m(\mathbb{C})$** given by

$$\mathbf{SL}_m(\mathbb{C}) = \{z \in \mathbb{C}^{m\times m} \mid \det x = 1\}.$$

The dimension of the submanifold $\mathbf{SL}_m(\mathbb{C})$ of $\mathbb{C}^{m\times m}$ is $2(m^2 - 1)$.

Another subgroup of $\mathbf{GL}_m(\mathbb{C})$ is the **unitary group $\mathbf{U}(m)$** given by

$$\mathbf{U}(m) = \{z \in \mathbb{C}^{m\times m} \mid \bar{z}^t z = e\}.$$

Calculations similar to those for the orthogonal group show that the dimension of $\mathbf{U}(m)$ is m^2.

As a subgroup of $\mathbf{U}(m)$ and $\mathbf{SL}_m(\mathbb{C})$ we have the **special unitary group $\mathbf{SU}(m)$** which is defined as

$$\begin{aligned} \mathbf{SU}(m) &= \mathbf{U}(m) \cap \mathbf{SL}_m(\mathbb{C}) \\ &= \{z \in \mathbb{C}^{m\times m} \mid \bar{z}^t z = e \text{ and } \det z = 1\}. \end{aligned}$$

It can be shown that $\mathbf{U}(1)$ is diffeomorphic to the circle S^1 and that $\mathbf{U}(m)$ is diffeomorphic to $\mathbf{SU}(m) \times \mathbf{U}(1)$, see Exercise 2.10. This means that $\dim \mathbf{SU}(m) = m^2 - 1$.

For the rest of this work we will assume, when not stating otherwise, that all our manifolds and maps are smooth i.e. in the C^∞-category.

Exercises

Exercise 2.1. Find a proof of Proposition 2.9.

Exercise 2.2. Find a proof of Proposition 2.11.

Exercise 2.3. Let S^1 be the unit circle in the complex plane $\mathbb{C}$ given by $S^1 = \{z \in \mathbb{C} \,|\, |z|^2 = 1\}$. Use the maps $x : \mathbb{C} \setminus \{i\} \to \mathbb{C}$ and $y : \mathbb{C} \setminus \{-i\} \to \mathbb{C}$ with

$$x : z \mapsto \frac{i+z}{1+iz}, \quad y : z \mapsto \frac{1+iz}{i+z}$$

to show that S^1 is a 1-dimensional submanifold of $\mathbb{C} \cong \mathbb{R}^2$.

Exercise 2.4. Use the implicit function theorem to show that the m-dimensional **torus**

$$\begin{aligned} T^m &= \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^m \,|\, x_1^2 + y_1^2 = \cdots = x_m^2 + y_m^2 = 1\} \\ &\cong \{z \in \mathbb{C}^m \,|\, |z_1|^2 = \cdots = |z_m|^2 = 1\} \end{aligned}$$

is a differentiable submanifold of $\mathbb{R}^{2m} \cong \mathbb{C}^m$.

Exercise 2.5. Find a proof of Proposition 2.22.

Exercise 2.6. Find a proof of Proposition 2.24.

Exercise 2.7. Prove that the 2-dimensional sphere S^2 as a differentiable submanifold of the standard $\mathbb{R}^3$ and the Riemann sphere $\hat{\mathbb{C}}$ are diffeomorphic.

Exercise 2.8. Equip the real line $\mathbb{R}$ with the standard topology and for each odd integer $k \in \mathbb{Z}^+$ let $\hat{\mathcal{A}}_k$ be the C^ω-structure defined on $\mathbb{R}$ by the atlas

$$\mathcal{A}_k = \{(\mathbb{R}, x_k) \,|\, x_k : p \mapsto p^k\}.$$

Show that the differentiable structures $\hat{\mathcal{A}}_k$ are all different but that the differentiable manifolds $(\mathbb{R}, \hat{\mathcal{A}}_k)$ are all diffeomorphic.

Exercise 2.9. Find a proof of Proposition 2.29.

Exercise 2.10. Let the spheres S^1, S^3 and the Lie groups $\mathbf{SO}(n)$, $\mathbf{O}(n)$, $\mathbf{SU}(n)$, $\mathbf{U}(n)$ be equipped with their standard differentiable structures. Use Proposition 2.29 to prove the following diffeomorphisms

$$S^1 \cong \mathbf{SO}(2), \quad S^3 \cong \mathbf{SU}(2),$$

$$\mathbf{SO}(n) \times \mathbf{O}(1) \cong \mathbf{O}(n), \quad \mathbf{SU}(n) \times \mathbf{U}(1) \cong \mathbf{U}(n).$$

Exercise 2.11. Find a proof of Proposition 2.34.

Exercise 2.12. Let $(G, *)$ and $(H, \cdot)$ be two Lie groups. Prove that the product manifold $G \times H$ has the structure of a Lie group.

CHAPTER 3

The Tangent Space

In this chapter we introduce the notion of the tangent space T_pM of a differentiable manifold M at a point p in M. This is a vector space of the same dimension as M. We first study the standard $\mathbb{R}^m$ and show how a tangent vector v at a point $p \in \mathbb{R}^m$ can be interpreted as a first order linear differential operator, annihilating constants, when acting on real-valued functions locally defined around p. Then we generalise to the manifold setting. To explain the notion of the tangent space we give several explicit examples. Here the classical Lie groups play an important role. We then conclude this chapter by introducing the notions of an immersion, an embedding and a submersion.

Let $\mathbb{R}^m$ be the m-dimensional real vector space with its standard differentiable structure. If p is a point in $\mathbb{R}^m$ and $\gamma : I \to \mathbb{R}^m$ is a C^1-curve such that $\gamma(0) = p$, then the **tangent vector**

$$\dot\gamma(0) = \lim_{t\to 0} \frac{\gamma(t) - \gamma(0)}{t}$$

of γ at p is an element of $\mathbb{R}^m$. Conversely, for an arbitrary element v of $\mathbb{R}^m$ we can easily find a curve $\gamma : I \to \mathbb{R}^m$ such that $\gamma(0) = p$ and $\dot\gamma(0) = v$. One example is given by

$$\gamma : t \mapsto p + t \cdot v.$$

This shows that the **tangent space** i.e. the set of tangent vectors at the point $p \in \mathbb{R}^m$ can be identified with $\mathbb{R}^m$.

We will now describe how the first order linear differential operators, annihilating constants, can be interpreted as tangent vectors. For a point $p \in \mathbb{R}^m$ we denote by $\varepsilon(p)$ the set of differentiable real-valued functions defined locally around p. Then it is well known from multivariable analysis that if $v \in \mathbb{R}^m$ and $f \in \varepsilon(p)$ then the **directional derivative** $\partial_v f$ of f at the point p in the direction of v satisfies

$$\partial_v f = \lim_{t\to 0} \frac{f(p + tv) - f(p)}{t} = \langle \mathrm{grad}(f), v\rangle.$$

Furthermore, the operator ∂ has the following properties

$$\partial_v(\lambda \cdot f + \mu \cdot g) \;=\; \lambda \cdot \partial_v f + \mu \cdot \partial_v g,$$

$$\begin{aligned} \partial_v(f \cdot g) &= \partial_v f \cdot g(p) + f(p) \cdot \partial_v g, \\ \partial_{(\lambda \cdot v + \mu \cdot w)} f &= \lambda \cdot \partial_v f + \mu \cdot \partial_w f, \end{aligned}$$

for all $\lambda, \mu \in \mathbb{R}$, $v, w \in \mathbb{R}^m$ and $f, g \in \varepsilon(p)$.

Motivated by the above well-known classical results, we now present the following.

Definition 3.1. For a point $p \in \mathbb{R}^m$, let $T_p\mathbb{R}^m$ be the set of **first order linear differential operators at p annihilating constants** i.e. the set of mappings $\alpha : \varepsilon(p) \to \mathbb{R}$ such that

(i) $\alpha(\lambda \cdot f + \mu \cdot g) = \lambda \cdot \alpha(f) + \mu \cdot \alpha(g)$,
(ii) $\alpha(f \cdot g) = \alpha(f) \cdot g(p) + f(p) \cdot \alpha(g)$,

for all $\lambda, \mu \in \mathbb{R}$ and $f, g \in \varepsilon(p)$.

The set of first order linear differential operators, annihilating constants, carries a natural structure of a real vector space. This is simply given by the addition $+$ and the multiplication $\cdot$ by real numbers satisfying

$$\begin{aligned} (\alpha + \beta)(f) &= \alpha(f) + \beta(f), \\ (\lambda \cdot \alpha)(f) &= \lambda \cdot \alpha(f), \end{aligned}$$

for all $\alpha, \beta \in T_p\mathbb{R}^m$, $f \in \varepsilon(p)$ and $\lambda \in \mathbb{R}$.

The following result provides an important identification between $\mathbb{R}^m$ and the tangent space $T_p\mathbb{R}^m$ as defined above.

Theorem 3.2. *For a point $p \in \mathbb{R}^m$ the map $\Phi : \mathbb{R}^m \to T_p\mathbb{R}^m$ defined by $\Phi : v \mapsto \partial_v$ is a linear vector space isomorphism.*

PROOF. The linearity of the map $\Phi : \mathbb{R}^m \to T_p\mathbb{R}^m$ follows directly from the fact that for all $\lambda, \mu \in \mathbb{R}$, $v, w \in \mathbb{R}^m$ and $f \in \varepsilon(p)$ we have

$$\partial_{(\lambda \cdot v + \mu \cdot w)} f = \lambda \cdot \partial_v f + \mu \cdot \partial_w f.$$

Let $v, w \in \mathbb{R}^m$ be such that $v \neq w$. Choose an element $u \in \mathbb{R}^m$ such that $\langle u, v \rangle \neq \langle u, w \rangle$ and define $f : \mathbb{R}^m \to \mathbb{R}$ by $f(x) = \langle u, x \rangle$. Then

$$\partial_v f = \langle u, v \rangle \neq \langle u, w \rangle = \partial_w f$$

so $\partial_v \neq \partial_w$. This proves that the linear map Φ is injective.

Let α be an arbitrary element of $T_p\mathbb{R}^m$. For $k = 1, \dots, m$ let the real-valued function $\hat{x}_k : \mathbb{R}^m \to \mathbb{R}$ be the natural projection onto the k-th component given by

$$\hat{x}_k : (x_1, \dots, x_m) \mapsto x_k$$

and put $v_k = \alpha(\hat{x}_k)$. For the constant function $1 : (x_1, \dots, x_m) \mapsto 1$ we have

$$\alpha(1) = \alpha(1 \cdot 1) = \alpha(1) \cdot 1 + 1 \cdot \alpha(1) = 2 \cdot \alpha(1),$$

so $\alpha(1) = 0$. By the linearity of α it then follows that $\alpha(c) = 0$ for any constant $c \in \mathbb{R}$. Let $f \in \varepsilon(p)$ and following Lemma 3.3, locally write

$$f(x) = f(p) + \sum_{k=1}^{m} (\hat{x}_k(x) - p_k) \cdot \psi_k(x),$$

where $\psi_k \in \varepsilon(p)$ with

$$\psi_k(p) = \frac{\partial f}{\partial x_k}(p).$$

We can now apply the differential operator $\alpha \in T_p\mathbb{R}^m$ on the function f and yield

$$\begin{aligned}
\alpha(f) &= \alpha(f(p) + \sum_{k=1}^{m} (\hat{x}_k - p_k) \cdot \psi_k) \\
&= \alpha(f(p)) + \sum_{k=1}^{m} \alpha(\hat{x}_k - p_k) \cdot \psi_k(p) + \sum_{k=1}^{m} (\hat{x}_k(p) - p_k) \cdot \alpha(\psi_k) \\
&= \sum_{k=1}^{m} v_k \cdot \frac{\partial f}{\partial x_k}(p) \\
&= \langle v, (\mathrm{grad} f)(p) \rangle \\
&= \partial_v f,
\end{aligned}$$

where $v = (v_1, \dots, v_m) \in \mathbb{R}^m$. This means that $\Phi(v) = \partial_v = \alpha$ so the linear map $\Phi : \mathbb{R}^m \to T_p\mathbb{R}^m$ is surjective and hence a vector space isomorphism. $\square$

Lemma 3.3. *Let p be a point in $\mathbb{R}^m$ and $f : U \to \mathbb{R}$ be a differentiable function defined on an open ball around p. Then for each $k = 1, 2, \dots, m$, there exist functions $\psi_k : U \to \mathbb{R}$ such that for all $x \in U$*

$$f(x) = f(p) + \sum_{k=1}^{m} (x_k - p_k) \cdot \psi_k(x) \quad \text{and} \quad \psi_k(p) = \frac{\partial f}{\partial x_k}(p).$$

Proof. It follows from the fundamental theorem of calculus that

$$\begin{aligned}
f(x) - f(p) &= \int_0^1 \frac{\partial}{\partial t}(f(p + t(x - p)))dt \\
&= \sum_{k=1}^{m} (x_k - p_k) \cdot \int_0^1 \frac{\partial f}{\partial x_k}(p + t(x - p))dt.
\end{aligned}$$

The statement then immediately follows by setting

$$\psi_k(x) = \int_0^1 \frac{\partial f}{\partial x_k}(p + t(x-p))dt.$$

$\square$

As a direct consequence of Theorem 3.2 we now have the following important result.

Corollary 3.4. *Let p be a point in $\mathbb{R}^m$ and $\{e_k \,|\, k = 1, \dots, m\}$ be a basis for $\mathbb{R}^m$. Then the set $\{\partial_{e_k} \,|\, k = 1, \dots, m\}$ is a basis for the tangent space $T_p\mathbb{R}^m$ at p.*

Remark 3.5. Let p be a point in $\mathbb{R}^m$, $v \in T_p\mathbb{R}^m$ be a tangent vector at p and $f : U \to \mathbb{R}$ be a C^1-function defined on an open subset U of $\mathbb{R}^m$ containing p. Let $\gamma : I \to U$ be a curve such that $\gamma(0) = p$ and $\dot\gamma(0) = v$. Then the identification given by Theorem 3.2 tells us that v acts on f by

$$v(f) = \partial_v(f) = \langle v, \mathrm{grad} f_p\rangle = df_p(\dot\gamma(0)) = \frac{d}{dt}(f \circ \gamma(t))|_{t=0}.$$

This implies that the real number $v(f)$ is independent of the choice of the curve γ as long as $\gamma(0) = p$ and $\dot\gamma(0) = v$.

We will now employ the ideas presented above to generalise to the manifold setting. Let M be a differentiable manifold and for a point $p \in M$ let $\varepsilon(p)$ denote the set of differentiable real-valued functions defined on an open neighborhood of p.

Definition 3.6. Let M be a differentiable manifold and p be a point in M. A **tangent vector** X_p at p is a map $X_p : \varepsilon(p) \to \mathbb{R}$ such that

(i) $X_p(\lambda \cdot f + \mu \cdot g) = \lambda \cdot X_p(f) + \mu \cdot X_p(g)$,
(ii) $X_p(f \cdot g) = X_p(f) \cdot g(p) + f(p) \cdot X_p(g)$,

for all $\lambda, \mu \in \mathbb{R}$ and $f, g \in \varepsilon(p)$. The set of tangent vectors at p is called the **tangent space** at p and denoted by T_pM.

The tangent space T_pM of M at p has a natural structure of a real vector space. The addition $+$ and the multiplication $\cdot$ by real numbers are simply given by

$$\begin{aligned}(X_p + Y_p)(f) &= X_p(f) + Y_p(f),\\ (\lambda \cdot X_p)(f) &= \lambda \cdot X_p(f),\end{aligned}$$

for all $X_p, Y_p \in T_pM$, $f \in \varepsilon(p)$ and $\lambda \in \mathbb{R}$.

We have not yet defined the differential of a map between manifolds, see Definition 3.14, but still think that the following remark is appropriate at this point. This will make it possible for us to explicitly determine the tangent spaces of some of the manifolds introduced earlier.

Remark 3.7. Let M be an m-dimensional manifold and (U, x) be a local chart around $p \in M$. Then the differential

$$dx_p : T_pM \to T_{x(p)}\mathbb{R}^m$$

is a bijective linear map such that for a given element $X_p \in T_pM$ there exists a tangent vector v in $T_{x(p)}\mathbb{R}^m \cong \mathbb{R}^m$ such that $dx_p(X_p) = v$. The image $x(U)$ is an open subset of $\mathbb{R}^m$ containing $x(p)$ so we can easily find a curve $c : I \to x(U)$ with $c(0) = x(p)$ and $\dot{c}(0) = v$. Then the composition $\gamma = x^{-1} \circ c : I \to U$ is a curve in M through p since $\gamma(0) = p$. The element $d(x^{-1})_{x(p)}(v)$ of the tangent space T_pM denoted by $\dot{\gamma}(0)$ is called the **tangent** to the curve γ at p. It follows from the relation

$$\dot{\gamma}(0) = d(x^{-1})_{x(p)}(v) = X_p$$

that the tangent space T_pM can be thought of as the set of all tangents to curves through the point p.

If $f : U \to \mathbb{R}$ is a C^1-function defined locally on U then it follows from Definition 3.14 that

$$\begin{aligned} X_p(f) &= (dx_p(X_p))(f \circ x^{-1}) \\ &= \frac{d}{dt}(f \circ x^{-1} \circ c(t))|_{t=0} \\ &= \frac{d}{dt}(f \circ \gamma(t))|_{t=0} \end{aligned}$$

It should be noted that the real number $X_p(f)$ is independent of the choice of the local chart (U, x) around p and the curve $c : I \to x(U)$ as long as $\gamma(0) = p$ and $\dot{\gamma}(0) = X_p$.

We are now ready to determine the tangent spaces of some of the differentiable manifolds that were introduced in Chapter 2. We start with the m-dimensional unit sphere S^m in $\mathbb{R}^{m+1}$. This should be seen as an introduction to our Example 3.10.

Example 3.8. Let $\gamma : I \to S^m$ be a differentiable curve into the m-dimensional unit sphere in $\mathbb{R}^{m+1}$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = X$. Then the curve satisfies

$$\langle \gamma(t), \gamma(t) \rangle = 1$$

and differentiation yields

$$\langle \dot\gamma(t), \gamma(t)\rangle + \langle \gamma(t), \dot\gamma(t)\rangle = 0.$$

This means that $\langle p, X\rangle = 0$, so every tangent vector $X \in T_pS^m$ must be orthogonal to p. On the other hand if $X \neq 0$ satisfies $\langle p, X\rangle = 0$ then $\gamma : \mathbb{R} \to S^m$ with

$$\gamma : t \mapsto \cos(t|X|)\cdot p + \sin(t|X|)\cdot X/|X|$$

is a differentible curve into S^m with $\gamma(0) = p$ and $\dot\gamma(0) = X$. This shows that the tangent space T_pS^m is actually given by

$$T_pS^m = \{X \in \mathbb{R}^{m+1} \mid \langle p, X\rangle = 0\}.$$

For the following we need the next well known result from matrix theory.

Proposition 3.9. *Let $\mathbb{C}^{m\times m}$ be the set of complex $m \times m$ matrices. Then the exponential map* $\mathrm{Exp} : \mathbb{C}^{m\times m} \to \mathbb{C}^{m\times m}$ *is defined by the convergent power series*

$$\mathrm{Exp} : Z \mapsto \sum_{k=0}^{\infty} \frac{Z^k}{k!}.$$

If Z, W are elements of $\mathbb{C}^{m\times m}$, then the following statements hold

(i) $\mathrm{Exp}(Z^t) = \mathrm{Exp}(Z)^t$,
(ii) $\mathrm{Exp}(\bar Z) = \overline{\mathrm{Exp}(Z)}$,
(iii) $\det(\mathrm{Exp}(Z)) = \exp(\mathrm{trace}\, Z)$,
(iv) if $ZW = WZ$ then $\mathrm{Exp}(Z + W) = \mathrm{Exp}(Z)\cdot \mathrm{Exp}(W)$.

Proof. See Exercise 3.2. □

We are now equipped with the necessary tools for determining the tangent space $T_e\mathbf{O}(m)$ of the orthogonal group $\mathbf{O}(m)$ at the neutral element $e \in \mathbf{O}(m)$.

Example 3.10. Let $\gamma : I \to \mathbf{O}(m)$ be a differentiable curve into the orthogonal group $\mathbf{O}(m)$ such that $\gamma(0) = e$ and $\dot\gamma(0) = X$. Then $\gamma(s)^t \cdot \gamma(s) = e$ for all $s \in I$ and differentiation gives

$$\begin{aligned} 0 &= (\dot\gamma(s)^t \cdot \gamma(s) + \gamma(s)^t \cdot \dot\gamma(s))|_{s=0} \\ &= X^t \cdot e + e^t \cdot X \\ &= X^t + X. \end{aligned}$$

This implies that each tangent vector $X \in T_e\mathbf{O}(m)$ of the orthogonal group $\mathbf{O}(m)$ at the neutral e is a skew-symmetric matrix.

On the other hand, for an arbitrary skew-symmetric matrix $X \in \mathbb{R}^{m\times m}$ we define the curve $A : \mathbb{R} \to \mathbb{R}^{m\times m}$ by $A : s \mapsto \mathrm{Exp}(sX)$. Then

$$\begin{aligned} A(s)^t \cdot A(s) &= \mathrm{Exp}(sX)^t \cdot \mathrm{Exp}(sX) \\ &= \mathrm{Exp}(sX^t) \cdot \mathrm{Exp}(sX) \\ &= \mathrm{Exp}(s(X^t + X)) \\ &= \mathrm{Exp}(0) \\ &= e. \end{aligned}$$

This shows that A is a curve in the orthogonal group, $A(0) = e$ and $\dot{A}(0) = X$, so X is an element of the tangent space $T_e\mathbf{O}(m)$. Hence

$$T_e\mathbf{O}(m) = \{X \in \mathbb{R}^{m\times m} \,|\, X^t + X = 0\}.$$

It now immediately follows that the dimension of the tangent space $T_e\mathbf{O}(m)$ is $m(m-1)/2$. We have seen in Example 2.19 that this is exactly the dimension of the orthogonal group $\mathbf{O}(m)$.

According to Exercise 2.10, the orthogonal group $\mathbf{O}(m)$ is diffeomorphic to $\{\pm 1\} \times \mathbf{SO}(m)$ so $\dim \mathbf{SO}(m) = \dim \mathbf{O}(m)$. Hence

$$T_e\mathbf{SO}(m) = T_e\mathbf{O}(m) = \{X \in \mathbb{R}^{m\times m} \,|\, X^t + X = 0\}.$$

The real general linear group $\mathbf{GL}_m(\mathbb{R})$ is an open subset of $\mathbb{R}^{m\times m}$ so its tangent space $T_p\mathbf{GL}_m(\mathbb{R})$ is simply $\mathbb{R}^{m\times m}$ at any point $p \in \mathbf{GL}_m(\mathbb{R})$.

The tangent space $T_e\mathbf{SL}_m(\mathbb{R})$ of the special linear group $\mathbf{SL}_m(\mathbb{R})$ at the neutral element $e \in \mathbf{SL}_m(\mathbb{R})$ can be determined as follows.

Example 3.11. If X is a matrix in $\mathbb{R}^{m\times m}$ with trace $X = 0$ then we define the differentiable curve $A : \mathbb{R} \to \mathbb{R}^{m\times m}$ by

$$A : s \mapsto \mathrm{Exp}(sX).$$

Then $A(0) = e$, $\dot{A}(0) = X$ and

$$\det(A(s)) = \det(\mathrm{Exp}(sX)) = \exp(\mathrm{trace}(sX)) = \exp(0) = 1.$$

This shows that A is a curve in the special linear group $\mathbf{SL}_m(\mathbb{R})$ and that X is an element of the tangent space $T_e\mathbf{SL}_m(\mathbb{R})$ of $\mathbf{SL}_m(\mathbb{R})$ at the neutral element e. Hence the (m^2-1)-dimensional linear space

$$\{X \in \mathbb{R}^{m\times m} \,|\, \mathrm{trace} X = 0\}$$

is contained in the tangent space $T_e\mathbf{SL}_m(\mathbb{R})$ of $\mathbf{SL}_m(\mathbb{R})$ at the neutral element e.

On the other hand, the curve $B : I \to \mathbf{GL}_m(\mathbb{R})$ given by

$$B : s \mapsto \mathrm{Exp}(s \cdot e) = \exp(s) \cdot e$$

is not contained in $\mathbf{SL}_m(\mathbb{R})$ so the dimension of $T_e\mathbf{SL}_m(\mathbb{R})$ is at most $m^2-1=\dim\mathbf{GL}_m(\mathbb{R})-1$. This shows that

$$T_e\mathbf{SL}_m(\mathbb{R})=\{X\in\mathbb{R}^{m\times m}\,|\,\mathrm{trace}X=0\}.$$

With the above arguments we have proven the following result.

Theorem 3.12. *Let e be the neutral element of the classical real Lie groups $\mathbf{GL}_m(\mathbb{R})$, $\mathbf{SL}_m(\mathbb{R})$, $\mathbf{O}(m)$, $\mathbf{SO}(m)$. Then their tangent spaces at e are given by*

$$\begin{aligned}T_e\mathbf{GL}_m(\mathbb{R}) &= \mathbb{R}^{m\times m},\\ T_e\mathbf{SL}_m(\mathbb{R}) &= \{X\in\mathbb{R}^{m\times m}\,|\,\mathrm{trace}X=0\},\\ T_e\mathbf{O}(m) &= \{X\in\mathbb{R}^{m\times m}\,|\,X^t+X=0\},\\ T_e\mathbf{SO}(m) &= T_e\mathbf{O}(m)=T_e\mathbf{O}(m)\cap T_e\mathbf{SL}_m(\mathbb{R}).\end{aligned}$$

For the classical complex Lie groups, similar methods can be used to prove the following result.

Theorem 3.13. *Let e be the neutral element of the classical complex Lie groups $\mathbf{GL}_m(\mathbb{C})$, $\mathbf{SL}_m(\mathbb{C})$, $\mathbf{U}(m)$, $\mathbf{SU}(m)$. Then their tangent spaces at e are given by*

$$\begin{aligned}T_e\mathbf{GL}_m(\mathbb{C}) &= \mathbb{C}^{m\times m},\\ T_e\mathbf{SL}_m(\mathbb{C}) &= \{Z\in\mathbb{C}^{m\times m}\,|\,\mathrm{trace}\,Z=0\},\\ T_e\mathbf{U}(m) &= \{Z\in\mathbb{C}^{m\times m}\,|\,\bar{Z}^t+Z=0\},\\ T_e\mathbf{SU}(m) &= T_e\mathbf{U}(m)\cap T_e\mathbf{SL}_m(\mathbb{C}).\end{aligned}$$

Proof. See Exercise 3.4 □

We now introduce the notion of the differential of a map between manifolds. This will play an important role in what follows.

Definition 3.14. Let $\phi:M\to N$ be a differentiable map between differentiable manifolds. Then the **differential** $d\phi_p$ of ϕ at a point p in M is the map $d\phi_p:T_pM\to T_{\phi(p)}N$ such that for all $X_p\in T_pM$ and $f\in\varepsilon(\phi(p))$ we have

$$(d\phi_p(X_p))(f)=X_p(f\circ\phi).$$

Remark 3.15. Let M and N be differentiable manifolds, $p\in M$ and $\phi:M\to N$ be a differentiable map. Further let $\gamma:I\to M$ be a curve in M such that $\gamma(0)=p$ and $\dot\gamma(0)=X_p$. Let $c:I\to N$ be the curve $c=\phi\circ\gamma$ in N with $c(0)=\phi(p)$ and put $Y_{\phi(p)}=\dot c(0)$. Then it is an immediate consequence of Definition 3.14 that for each function $f \in \varepsilon(\phi(p))$, defined locally around $\phi(p)$, we have

$$\begin{aligned}(d\phi_p(X_p))(f) &= X_p(f \circ \phi) \\ &= \frac{d}{dt}(f \circ \phi \circ \gamma(t))|_{t=0} \\ &= \frac{d}{dt}(f \circ c(t))|_{t=0} \\ &= Y_{\phi(p)}(f).\end{aligned}$$

Hence $d\phi_p(X_p) = Y_{\phi(p)}$, or equivalently, $d\phi_p(\dot\gamma(0)) = \dot c(0)$. This statement should be compared with Remark 2.12.

The following result describes the most important properties of the differential, in particular, the so called chain rule.

Proposition 3.16. *Let $\phi : M_1 \to M_2$ and $\psi : M_2 \to M_3$ be differentiable maps between differentiable manifolds. Then for each point $p \in M_1$ we have*

(i) the map $d\phi_p : T_pM_1 \to T_{\phi(p)}M_2$ is linear,
(ii) if $id_{M_1} : M_1 \to M_1$ is the identity map, then $d(id_{M_1})_p = id_{T_pM_1}$,
(iii) $d(\psi \circ \phi)_p = d\psi_{\phi(p)} \circ d\phi_p$.

Proof. The statement (i) follows immediately from the fact that for $\lambda, \mu \in \mathbb{R}$, $X_p, Y_p \in T_pM$ and $f \in \varepsilon(\phi(p))$ we have

$$\begin{aligned}d\phi_p(\lambda \cdot X_p + \mu \cdot Y_p)(f) &= (\lambda \cdot X_p + \mu \cdot Y_p)(f \circ \phi) \\ &= \lambda \cdot X_p(f \circ \phi) + \mu \cdot Y_p(f \circ \phi) \\ &= \lambda \cdot d\phi_p(X_p)(f) + \mu \cdot d\phi_p(Y_p)(f).\end{aligned}$$

The statement (ii) is obvious. The statement (iii) is called the **chain rule**. If $X_p \in T_pM_1$ and $f \in \varepsilon(\psi \circ \phi(p))$, then

$$\begin{aligned}(d\psi_{\phi(p)} \circ d\phi_p)(X_p)(f) &= (d\psi_{\phi(p)}(d\phi_p(X_p)))(f) \\ &= (d\phi_p(X_p))(f \circ \psi) \\ &= X_p(f \circ \psi \circ \phi) \\ &= (d(\psi \circ \phi)_p(X_p))(f).\end{aligned}$$

This proves the last statement. □

As an immediate consequence of Proposition 3.16 we have the following interesting result generalising the corresponding statement in multivariable analysis.

Corollary 3.17. *Let $\phi : M \to N$ be a diffeomorphism with the inverse $\psi = \phi^{-1} : N \to M$. If p is a point in M then the differential $d\phi_p : T_pM \to T_{\phi(p)}N$ of ϕ at p is bijective and satisfies $(d\phi_p)^{-1} = d\psi_{\phi(p)}$.*

PROOF. The statement is a direct consequence of the following relations

$$d\psi_{\phi(p)} \circ d\phi_p = d(\psi \circ \phi)_p = d(\mathrm{id}_M)_p = \mathrm{id}_{T_pM},$$
$$d\phi_p \circ d\psi_{\phi(p)} = d(\phi \circ \psi)_{\phi(p)} = d(\mathrm{id}_N)_{\phi(p)} = \mathrm{id}_{T_{\phi(p)}N}.$$

□

We are now ready to prove the following important result. This is of course a direct generalisation of the corresponding statement in Gaussian geometry i.e. the classical theory of surfaces in $\mathbb{R}^3$.

Theorem 3.18. *Let M^m be an m-dimensional differentable manifold and p be a point in M. Then the tangent space T_pM of M at p is an m-dimensional real vector space.*

PROOF. Let (U, x) be a local chart on M. Then Proposition 2.24 tells us that the map $x : U \to x(U)$ is a diffeomorphism. This implies that the linear differential $dx_p : T_pM \to T_{x(p)}\mathbb{R}^m$ is a vector space isomorphism. The statement now follows directly from Theorem 3.2 and Corollary 3.17. □

Proposition 3.19. *Let M^m be a differentiable manifold, (U, x) be a local chart on M and $\{e_k \,|\, k = 1, \dots, m\}$ be the canonical basis for $\mathbb{R}^m$. For an arbitrary point p in U we define the differential operator $(\frac{\partial}{\partial x_k})_p$ in T_pM by*

$$\Big(\frac{\partial}{\partial x_k}\Big)_p : f \mapsto \frac{\partial f}{\partial x_k}(p) = \partial_{e_k}(f \circ x^{-1})(x(p)).$$

Then the set

$$\{(\frac{\partial}{\partial x_k})_p \,|\, k = 1, 2, \dots, m\}$$

is a basis for the tangent space T_pM of M at p.

PROOF. The local chart $x : U \to x(U)$ is a diffeomorphism and the differential $(dx^{-1})_{x(p)} : T_{x(p)}\mathbb{R}^m \to T_pM$ of the inverse $x^{-1} : x(U) \to U$ satisfies

$$\begin{aligned} (dx^{-1})_{x(p)}(\partial_{e_k})(f) &= \partial_{e_k}(f \circ x^{-1})(x(p)) \\ &= \Big(\frac{\partial}{\partial x_k}\Big)_p(f) \end{aligned}$$

for all $f \in \varepsilon(p)$. The statement is then a direct consequence of Corollary 3.4. □

The rest of this chapter is devoted to the introduction of special types of differentiable maps. They are the immersions, the embeddings and the submersions.

Definition 3.20. For positive integers $m, n \in \mathbb{Z}^+$ with $m \leq n$, a differentiable map $\phi : M^m \to N^n$ between manifolds is said to be an **immersion** if for each $p \in M$ the differential $d\phi_p : T_pM \to T_{\phi(p)}N$ is injective. An **embedding** is an immersion $\phi : M \to N$ which is a homeomorphism onto its image $\phi(M)$.

Example 3.21. For positive integers m, n with $m < n$ we have the inclusion map $\phi : \mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ given by

$$\phi : (x_1, \dots, x_{m+1}) \mapsto (x_1, \dots, x_{m+1}, 0, \dots, 0).$$

The differential $d\phi_x$ at x is injective since $d\phi_x(v) = (v, 0)$. The map ϕ is obviously a homeomorphism onto its image $\phi(\mathbb{R}^{m+1})$ hence an embedding. It is easily seen that even the restriction $\phi|_{S^m} : S^m \to S^n$ of ϕ to the m-dimensional unit sphere S^m in $\mathbb{R}^{m+1}$ is an embedding.

Definition 3.22. Let M be an m-dimensional differentiable manifold and U be an open subset of $\mathbb{R}^m$. An immersion $\phi : U \to M$ is called a **local parametrisation** of M. If the immersion ϕ is surjective then it is said to be a **global** parametrisation.

Remark 3.23. If M is a differentiable manifold and (U, x) is a local chart on M, then the inverse $x^{-1} : x(U) \to U$ of x is a global parametrisation of the open subset U of M.

Example 3.24. Let S^1 be the unit circle in the complex plane $\mathbb{C}$. For a non-zero integer $k \in \mathbb{Z}$ define $\phi_k : S^1 \to \mathbb{C}$ by $\phi_k : z \mapsto z^k$. For a point $w \in S^1$ let $\gamma_w : \mathbb{R} \to S^1$ be the curve with $\gamma_w : t \mapsto we^{it}$. Then $\gamma_w(0) = w$ and $\dot{\gamma}_w(0) = iw$. For the differential of ϕ_k we have

$$(d\phi_k)_w(\dot{\gamma}_w(0)) = \frac{d}{dt}(\phi_k \circ \gamma_w(t))|_{t=0} = \frac{d}{dt}(w^k e^{ikt})|_{t=0} = kiw^k \neq 0.$$

This shows that the differential $(d\phi_k)_w : T_wS^1 \cong \mathbb{R} \to T_{w^k}\mathbb{C} \cong \mathbb{R}^2$ is injective, so the map ϕ_k is an immersion. It is easily seen that ϕ_k is an embedding if and only if $k = \pm 1$.

Example 3.25. Let $q \in S^3$ be a quaternion of unit length and $\phi_q : S^1 \to S^3$ be the map defined by $\phi_q : z \mapsto qz$. For $w \in S^1$ let $\gamma_w : \mathbb{R} \to S^1$ be the curve given by $\gamma_w(t) = we^{it}$. Then $\gamma_w(0) = w$, $\dot{\gamma}_w(0) = iw$ and $\phi_q(\gamma_w(t)) = qwe^{it}$. By differentiating we yield

$$d\phi_q(\dot{\gamma}_w(0)) = \frac{d}{dt}(\phi_q(\gamma_w(t)))|_{t=0} = \frac{d}{dt}(qwe^{it})|_{t=0} = qiw.$$

Then $|d\phi_q(\dot{\gamma}_w(0))| = |qwi| = |q||w| = 1 \neq 0$ implies that the differential $d\phi_q$ is injective. It is easily checked that the immersion ϕ_q is an embedding.

We have introduced the real projective space $\mathbb{R}P^m$ as an abstract manifold in Example 2.6. In the next example we construct an interesting embedding of $\mathbb{R}P^m$ into the real vector space $\mathrm{Sym}(\mathbb{R}^{m+1})$ of symmetric real $(m+1) \times (m+1)$ matrices.

Example 3.26. Let S^m be the m-dimensional unit sphere in $\mathbb{R}^{m+1}$. For a point $p \in S^m$, let

$$\ell_p = \{\lambda \cdot p \in \mathbb{R}^{m+1} \mid \lambda \in \mathbb{R}\}$$

be the line, through the origin, generated by p. Further let

$$R_p : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$$

be the reflection about the line ℓ_p. Then R_p is an element of $\mathrm{End}(\mathbb{R}^{m+1})$ i.e. the set of linear endomorphisms of $\mathbb{R}^{m+1}$ which can be identified with the set $\mathbb{R}^{(m+1)\times(m+1)}$ of real $(m+1)\times(m+1)$ matrices. It is easily checked that the reflection R_p about the line ℓ_p is given by

$$R_p : q \mapsto 2\langle p, q\rangle p - q.$$

It then immediately follows from the relation

$$R_p(q) = 2\langle p, q\rangle p - q = 2p\langle p, q\rangle - q = (2p \cdot p^t - e) \cdot q$$

that the symmetric matrix in $\mathbb{R}^{(m+1)\times(m+1)}$ corresponding to R_p is just

$$(2p \cdot p^t - e).$$

We will now show that the map $\phi : S^m \to \mathrm{Sym}(\mathbb{R}^{m+1})$ given by

$$\phi : p \mapsto R_p$$

is an immersion. Let p be an arbitrary point on S^m and $\alpha, \beta : I \to S^m$ be two curves meeting at p i.e. $\alpha(0) = p = \beta(0)$, with $X_p = \dot\alpha(0)$ and $Y_p = \dot\beta(0)$. For $\gamma \in \{\alpha, \beta\}$ we have

$$\phi \circ \gamma : t \mapsto (q \mapsto 2\langle q, \gamma(t)\rangle\gamma(t) - q)$$

so

$$\begin{aligned}(d\phi)_p(\dot\gamma(0)) &= \frac{d}{dt}(\phi \circ \gamma(t))|_{t=0} \\ &= (q \mapsto 2\langle q, \dot\gamma(0)\rangle\gamma(0) + 2\langle q, \gamma(0)\rangle\dot\gamma(0)).\end{aligned}$$

This means that

$$d\phi_p(X_p) = (q \mapsto 2\langle q, X_p\rangle p + 2\langle q, p\rangle X_p)$$

and

$$d\phi_p(Y_p) = (q \mapsto 2\langle q, Y_p\rangle p + 2\langle q, p\rangle Y_p).$$

Let us now assume that the tangent vectors $X_p, Y_p \in T_pS^m$ are linearly independent. Then the symmetric linear operators

$$d\phi_p(X_p), d\phi_p(Y_p) : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$$

satisfy

$$d\phi_p(X_p)(p) = 2X_p \neq 2Y_p = d\phi_p(Y_p)(p).$$

This implies that the linear differential $d\phi_p$ of ϕ at p is injective and hence the map $\phi : S^m \to \text{Sym}(\mathbb{R}^{m+1})$ is an immersion.

If two points $p, q \in S^m$ are linearly independent, then the corresponding lines ℓ_p and ℓ_q are different. But these are exactly the eigenspaces of R_p and R_q with the eigenvalue $+1$, respectively. This shows that the linear endomorphisms R_p and R_q of $\mathbb{R}^{m+1}$ are different in this case.

On the other hand, if p and q are parallel i.e. $p = \pm q$ then $R_p = R_q$. This means that the image $\phi(S^m)$ can be identified with the quotient space $S^m/\equiv$ where $\equiv$ is the equivalence relation defined by

$$x \equiv y \ \text{ if and only if } \ x = \pm y.$$

The quotient space is of course the real projective space $\mathbb{R}P^m$, introduced in Example 2.6. This implies that the map ϕ induces an embedding $\Phi : \mathbb{R}P^m \to \text{Sym}(\mathbb{R}^{m+1})$ satisfying $\Phi : [p] \mapsto R_p$.

For each point $p \in S^m$ the reflection $R_p : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ about the line ℓ_p satisfies

$$R_p \cdot R_p^t = R_p \cdot R_p = e.$$

This shows that the image $\Phi(\mathbb{R}P^m) = \phi(S^m)$ is not only contained in the linear space $\text{Sym}(\mathbb{R}^{m+1})$ but also in the orthogonal group $\mathbf{O}(m+1)$, which we know from Example 2.19 is a submanifold of $\mathbb{R}^{(m+1)\times(m+1)}$.

The next result was proven by Hassler Whitney (1907-1989) in his celebrated paper, *Differentiable Manifolds*, Ann. of Math. **37** (1936), 645-680.

Deep Result 3.27. *For $1 \leq r \leq \infty$ let M be an m-dimensional C^r-manifold. Then there exists a C^r-embedding $\phi : M \to \mathbb{R}^{2m+1}$ of M into the $(2m+1)$-dimensional real vector space $\mathbb{R}^{2m+1}$.*

The following is interesting in view of Witney's result.

Example 3.28. According to Example 3.26, the m-dimensional real projective space $\mathbb{R}P^m$ can be embedded into the linear space

$\mathrm{Sym}(\mathbb{R}^{m+1})$. The embedding $\Phi : \mathbb{R}P^m \to \mathrm{Sym}(\mathbb{R}^{m+1})$ is given by

$$\Phi : [p] \mapsto R_p = \begin{pmatrix} 2p_1^2 - 1 & 2p_1p_2 & \cdots & 2p_1p_{m+1} \\ 2p_2p_1 & 2p_2^2 - 1 & \cdots & 2p_2p_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ 2p_{m+1}p_1 & 2p_{m+1}p_2 & \cdots & 2p_{m+1}^2 - 1 \end{pmatrix}.$$

In the special case of the two dimensional real projective plane $\mathbb{R}P^2$ we have the embedding $\Phi : \mathbb{R}P^2 \to \mathrm{Sym}(\mathbb{R}^3)$ into the 6-dimensional linear space $\mathrm{Sym}(\mathbb{R}^3)$ of symmetric real 3×3 matrices. This is given by

$$\Phi : [(x, y, z)] \mapsto \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{12} & r_{22} & r_{23} \\ r_{13} & r_{23} & r_{33} \end{pmatrix} = \begin{pmatrix} 2x^2 - 1 & 2xy & 2xz \\ 2yx & 2y^2 - 1 & 2yz \\ 2zx & 2zy & 2z^2 - 1 \end{pmatrix}.$$

The image $\Phi(\mathbb{R}P^2)$ is clearly contained in the 5-dimensional hyperplane of $\mathbb{R}^6$ defined by

$$r_{11} + r_{22} + r_{33} = -1.$$

With the following, we now show that the classical inverse function theorem generalises to the manifold setting. The reader should compare this with Fact 2.13.

Theorem 3.29 (The Inverse Mapping Theorem)**.** *Let $\phi : M \to N$ be a differentiable map between manifolds with $\dim M = \dim N$. If p is a point in M such that the differential $d\phi_p : T_pM \to T_{\phi(p)}N$ at p is bijective then there exist open neighborhoods U_p around p and U_q around $q = \phi(p)$ such that $\psi = \phi|_{U_p} : U_p \to U_q$ is bijective and the inverse $\psi^{-1} : U_q \to U_p$ is differentiable.*

Proof. See Exercise 3.8 □

We will now generalise the classical implicit mapping theorem to manifolds. For this we need the following definition. Compare this with Definition 2.14.

Definition 3.30. Let m, n be positive integers and $\phi : M^m \to N^n$ be a differentiable map between manifolds. A point $p \in M$ is said to be **regular** for ϕ if the differential

$$d\phi_p : T_pM \to T_{\phi(p)}N$$

is of full rank, but **critical** otherwise. A point $q \in \phi(M)$ is said to be a **regular value** of ϕ if every point in the pre-image $\phi^{-1}(\{q\})$ of $\{q\}$ is regular.

The reader should compare the following result with Theorem 2.16.

Theorem 3.31 (The Implicit Mapping Theorem)**.** *Let $\phi : M^m \to N^n$ be a differentiable map between manifolds such that $m > n$. If $q \in \phi(M)$ is a regular value, then the pre-image $\phi^{-1}(\{q\})$ of q is a submanifold of M^m of dimension an $(m - n)$. The tangent space $T_p\phi^{-1}(\{q\})$ of $\phi^{-1}(\{q\})$ at p is the kernel of the differential $d\phi_p$ i.e.*

$$T_p\phi^{-1}(\{q\}) = \{X \in T_pM \mid d\phi_p(X) = 0\}.$$

Proof. Let (V, y) be a local chart on N with $q \in V$ and $y(q) = 0$. For a point $p \in \phi^{-1}(\{q\})$ we choose a local chart (U, x) on M such that $p \in U$, $x(p) = 0$ and $\phi(U) \subset V$. Then the differential of the map

$$\psi = y \circ \phi \circ x^{-1}|_{x(U)} : x(U) \to \mathbb{R}^n$$

at the point 0 is given by

$$d\psi_0 = (dy)_q \circ d\phi_p \circ (dx^{-1})_0 : T_0\mathbb{R}^m \to T_0\mathbb{R}^n.$$

The pairs (U, x) and (V, y) are local charts so the differentials $(dy)_q$ and $(dx^{-1})_0$ are bijective. This means that $d\psi_0$ is surjective since $d\phi_p$ is. It then follows from Theorem 2.16 that $x(\phi^{-1}(\{q\}) \cap U)$ is an $(m - n)$-dimensional submanifold of $x(U)$. Hence $\phi^{-1}(\{q\}) \cap U$ is an $(m - n)$-dimensional submanifold of U. This is true for each point $p \in \phi^{-1}(\{q\})$ so we have proven that $\phi^{-1}(\{q\})$ is a submanifold of M^m of dimension $(m - n)$.

Let $\gamma : I \to \phi^{-1}(\{q\})$ be a curve such that $\gamma(0) = p$. Then

$$(d\phi)_p(\dot\gamma(0)) = \frac{d}{dt}(\phi \circ \gamma(t))|_{t=0} = \frac{dq}{dt}|_{t=0} = 0.$$

This implies that $T_p\phi^{-1}(\{q\})$ is contained in and has the same dimension as the kernel of $d\phi_p$, so $T_p\phi^{-1}(\{q\}) = \text{Ker } d\phi_p$. $\square$

We conclude this chapter with a discussion on the important submersions between differentiable manifolds.

Definition 3.32. For positive integers $m, n \in \mathbb{Z}^+$ with $m \geq n$ a differentiable map $\phi : M^m \to N^n$ between two manifolds is said to be a **submersion** if for each $p \in M$ the differential $d\phi_p : T_pM \to T_{\phi(p)}N$ is surjective.

The reader should compare Definition 3.32 with Definition 3.20.

Example 3.33. If $m, n \in \mathbb{Z}^+$ such that $m \geq n$ then we have the projection map $\pi : \mathbb{R}^m \to \mathbb{R}^n$ given by $\pi : (x_1, \dots, x_m) \mapsto (x_1, \dots, x_n)$. Its differential $d\pi_x$ at a point x is surjective since

$$d\pi_x(v_1, \dots, v_m) = (v_1, \dots, v_n).$$

This means that the projection is a submersion.

The next item of the menu is the famous Hopf map. This is named after the distingued differential topologist Heinz Hopf (1894-1971). His construction has been very important both in topology and differential geometry and has later been generalised in several different directions. It provides us with an important submersion between spheres.

Example 3.34. Let S^2 and S^3 be the unit spheres in $\mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3$ and $\mathbb{C}^2 \cong \mathbb{R}^4$, respectively. Then the **Hopf map** $\phi : S^3 \to S^2$ is given by

$$\phi : (z, w) \mapsto (2z\bar{w}, |z|^2 - |w|^2).$$

For a point $p = (z, w)$ in S^3 the **Hopf circle** C_p through p is defined by

$$C_p = \{e^{i\theta}(z, w) \mid \theta \in \mathbb{R}\}.$$

The following shows that the Hopf map is constant along each Hopf circle

$$\begin{aligned} \phi(e^{i\theta}(z, w)) &= (2e^{i\theta} z e^{-i\theta} \bar{w}, |e^{i\theta} z|^2 - |e^{i\theta} w|^2) \\ &= (2z\bar{w}, |z|^2 - |w|^2) \\ &= \phi((z, w)). \end{aligned}$$

Now define the vectors $v_1 = (i, 0), v_2 = (0, 1), v_3 = (0, i) \in \mathbb{C}^2$ and, for $k = 1, 2, 3$, the curves $\gamma_k : \mathbb{R} \to S^3$ by

$$\gamma_k : t \mapsto \cos t \cdot (1, 0) + \sin t \cdot v_k.$$

Then $\gamma_k(0) = e$ and $\dot{\gamma}_k(0) = v_k$, so v_1, v_2, v_3 are elements of the tangent space $T_e S^3$ of S^3 at the neutral element e. They are linearly independent and hence form a basis for the 3-dimensional $T_e S^3$.

It can be shown that the Hopf map $\phi : S^3 \to S^2$ is surjective and that the same applies to its differential $d\phi_p : T_p S^3 \to T_{\phi(p)} S^2$ for each $p \in S^3$. This means that ϕ is a submersion, so each point $q \in S^2$ is a regular value of ϕ and the fibres $\phi^{-1}(\{q\})$ of ϕ are 1-dimensional submanifolds of S^3. They are actually the Hopf circles given by

$$\phi^{-1}(\{(2z\bar{w}, |z|^2 - |w|^2)\}) = \{e^{i\theta}(z, w) \mid \theta \in \mathbb{R}\}.$$

This means that the 3-dimensional sphere S^3 is a disjoint union of great circles

$$S^3 = \bigcup_{q \in S^2} \phi^{-1}(\{q\}).$$

Exercises

Exercise 3.1. Let p be an arbitrary point of the unit sphere S^{2n+1} in $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$. Determine the tangent space $T_p S^{2n+1}$ and show that this contains an n-dimensional complex vector subspace of $\mathbb{C}^{n+1}$.

Exercise 3.2. Use your local library to find a proof of Proposition 3.9.

Exercise 3.3. Prove that the matrices

$$X_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

form a basis for the tangent space $T_e\mathbf{SL}_2(\mathbb{R})$ of the real special linear group $\mathbf{SL}_2(\mathbb{R})$ at the neutral element e. For each $k = 1, 2, 3$ find an explicit formula for the curve $\gamma_k : \mathbb{R} \to \mathbf{SL}_2(\mathbb{R})$ given by

$$\gamma_k : s \mapsto \mathrm{Exp}(sX_k).$$

Exercise 3.4. Find a proof of Theorem 3.13.

Exercise 3.5. Prove that the matrices

$$Z_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad Z_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

form a basis for the tangent space $T_e\mathbf{SU}(2)$ of the special unitary group $\mathbf{SU}(2)$ at the neutral element e. For each $k = 1, 2, 3$ find an explicit formula for the curve $\gamma_k : \mathbb{R} \to \mathbf{SU}(2)$ given by

$$\gamma_k : s \mapsto \mathrm{Exp}(sZ_k).$$

Exercise 3.6. For each non-negative integer k define $\phi_k : \mathbb{C} \to \mathbb{C}$ and $\psi_k : \mathbb{C}^* \to \mathbb{C}$ by $\phi_k, \psi_k : z \mapsto z^k$. For which such k are ϕ_k, ψ_k immersions, embeddings or submersions ?

Exercise 3.7. Prove that the differentiable map $\phi : \mathbb{R}^m \to T^m$ given by

$$\phi : (x_1, \dots, x_m) \mapsto (e^{ix_1}, \dots, e^{ix_m})$$

is a parametrisation of the m-dimensional torus T^m in $\mathbb{C}^m$.

Exercise 3.8. Find a proof of Theorem 3.29.

Exercise 3.9. Prove that the differential $d\phi_p : T_p S^3 \to T_{\phi(p)} S^2$ of the Hopf-map $\phi : S^3 \to S^2$, with

$$\phi : (z, w) \mapsto (2\, z\bar{w}, |z|^2 - |w|^2),$$

is surjective at the point $p = (1, 0) \in S^3$.

CHAPTER 4

The Tangent Bundle

In this chapter we introduce the tangent bundle TM of a differentiable manifold M. Intuitively, this is the object that we obtain by glueing at each point p in M the corresponding tangent space T_pM. The differentiable structure on M induces a natural differentiable structure on the tangent bundle TM turning it into a differentiable manifold of twice the dimension of M. To explain the notion of the tangent bundle we investigate several concrete examples. The classical Lie groups will here play a particular important role.

We have already seen that for a point $p \in \mathbb{R}^m$ the **tangent space** $T_p\mathbb{R}^m$ can be identified with the m-dimensional vector space $\mathbb{R}^m$. This means that if we at each point $p \in \mathbb{R}^m$ glue the tangent space $T_p\mathbb{R}^m$ to $\mathbb{R}^m$ we obtain the so called **tangent bundle** of $\mathbb{R}^m$

$$T\mathbb{R}^m = \{(p, v) \,|\, p \in \mathbb{R}^m \text{ and } v \in T_p\mathbb{R}^m\}.$$

For this we have the **natural projection** $\pi : T\mathbb{R}^m \to \mathbb{R}^m$ defined by

$$\pi : (p, v) \mapsto p$$

and for each point p in M the fibre $\pi^{-1}(\{p\})$ over p is precisely the tangent space $T_p\mathbb{R}^m$ at p.

Remark 4.1. Classically, a **vector field** X on $\mathbb{R}^m$ is a differentiable map $X : \mathbb{R}^m \to \mathbb{R}^m$ but we would like to view it as a map $X : \mathbb{R}^m \to T\mathbb{R}^m$ into the tangent bundle and write

$$X : p \mapsto (p, X_p).$$

Following Proposition 3.19, two vector fields $X, Y : \mathbb{R}^m \to T\mathbb{R}^m$ can be written as

$$X = \sum_{k=1}^{m} a_k \cdot \frac{\partial}{\partial x_k} \quad \text{and} \quad Y = \sum_{k=1}^{m} b_k \cdot \frac{\partial}{\partial x_k},$$

where $a_k, b_k : \mathbb{R}^m \to \mathbb{R}$ are differentiable functions defined on $\mathbb{R}^m$. If $f : \mathbb{R}^m \to \mathbb{R}$ is another such function the **commutator** $[X, Y]$ acts on f as follows

$$[X, Y](f) \quad = \quad X(Y(f)) - Y(X(f))$$

$$
\begin{aligned}
&= \sum_{k,l=1}^{m} \big(a_k \frac{\partial}{\partial x_k}(b_l \frac{\partial}{\partial x_l}) - b_k \frac{\partial}{\partial x_k}(a_l \frac{\partial}{\partial x_l})\big)(f) \\
&= \sum_{k,l=1}^{m} \big(a_k \frac{\partial b_l}{\partial x_k}\frac{\partial}{\partial x_l} + a_k b_l \frac{\partial^2}{\partial x_k \partial x_l} \\
&\qquad\qquad - b_k \frac{\partial a_l}{\partial x_k}\frac{\partial}{\partial x_l} - b_k a_l \frac{\partial^2}{\partial x_k \partial x_l}\big)(f) \\
&= \sum_{l=1}^{m} \Big\{ \sum_{k=1}^{m} \big(a_k \frac{\partial b_l}{\partial x_k} - b_k \frac{\partial a_l}{\partial x_k}\big) \Big\} \frac{\partial}{\partial x_l}(f).
\end{aligned}
$$

This shows that the commutator $[X, Y]$ is actually a differentiable vector field on $\mathbb{R}^m$.

In this chapter we will generalise the above important ideas to the manifold setting. We first introduce the following general notion of a topological vector bundle.

Definition 4.2. Let E and M be topological manifolds and $\pi : E \to M$ be a continuous surjective map. The triple (E, M, π) is said to be an n-dimensional **topological vector bundle** over M if

(i) for each point p in M, the **fibre** $E_p = \pi^{-1}(\{p\})$ is an n-dimensional vector space,

(ii) for each point p in M, there exists a **local bundle chart** $(\pi^{-1}(U), \psi)$ consisting of the pre-image $\pi^{-1}(U)$ of an open neighbourhood U in M containing the point p and a homeomorphism $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ such that for each $q \in U$ the map $\psi_q = \psi|_{E_q} : E_q \to \{q\} \times \mathbb{R}^n$ is a vector space isomorphism.

A **bundle atlas** for the topological vector bundle (E, M, π) is a collection

$$\mathcal{B} = \{(\pi^{-1}(U_\alpha), \psi_\alpha) \,|\, \alpha \in \mathcal{I}\}$$

of local bundle charts such that

$$M = \bigcup_{\alpha \in \mathcal{I}} U_\alpha$$

and for all $\alpha, \beta \in \mathcal{I}$ there exists a map $A_{\alpha,\beta} : U_\alpha \cap U_\beta \to \mathbf{GL}_n(\mathbb{R})$ such that the corresponding continuous map

$$\psi_\beta \circ \psi_\alpha^{-1}|_{(U_\alpha \cap U_\beta) \times \mathbb{R}^n} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \to (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

is given by

$$(p, v) \mapsto (p, (A_{\alpha,\beta}(p))(v)).$$

The elements of $\{A_{\alpha,\beta} \,|\, \alpha, \beta \in \mathcal{I}\}$ are called the **transition maps** of the bundle atlas $\mathcal{B}$.

Definition 4.3. Let (E, M, π) be an n-dimensional topological vector bundle over M. A continuous map $v : M \to E$ is called a **section** of the bundle (E, M, π) if $\pi \circ v(p) = p$ for each $p \in M$.

Definition 4.4. A topological vector bundle (E, M, π) over M of dimension n is said to be **trivial** if there exists a global bundle chart $\psi : E \to M \times \mathbb{R}^n$.

We now give two examples of trivial topological vector bundles.

Example 4.5. Let M be the one dimensional unit circle S^1 in $\mathbb{R}^2$, E be the two dimensional cylinder $E = S^1 \times \mathbb{R}^1$ and $\pi : E \to M$ be the projection map given by $\pi : (p, t) \mapsto p$. Then (E, M, π) is a **trivial line bundle** i.e. a trivial 1-dimensional vector bundle over the circle. This because the identity map $\psi : S^1 \times \mathbb{R}^1 \to S^1 \times \mathbb{R}^1$ with $\psi : (p, t) \to (p, t)$ is a global bundle chart.

Example 4.6. For a positive integer n and a topological manifold M we have the n-dimensional **trivial vector bundle** $(M \times \mathbb{R}^n, M, \pi)$ over M, where $\pi : M \times \mathbb{R}^n \to M$ is the projection map with $\pi : (p, v) \mapsto p$. The bundle is trivial since the identity map $\psi : M \times \mathbb{R}^n \to M \times \mathbb{R}^n$ is a global bundle chart.

The famous Möbius band is an interesting example of a non-trivial topological vector bundle.

Example 4.7. Let M be the unit circle S^1 in $\mathbb{R}^4$ parametrised by $\gamma : \mathbb{R} \to \mathbb{R}^4$ with

$$\gamma : s \mapsto (\cos s, \sin s, 0, 0).$$

Further let E be the well known **Möbius band** in $\mathbb{R}^4$ parametrised by $\phi : \mathbb{R}^2 \to \mathbb{R}^4$ with

$$\phi : (s, t) \mapsto (\cos s, \sin s, 0, 0) + t \cdot (0, 0, \sin(s/2), \cos(s/2)).$$

Then E is a regular surface and the natural projection $\pi : E \to M$ given by $\pi : (x, y, z, w) \mapsto (x, y, 0, 0)$ is continuous and surjective. The triple (E, M, π) is a line bundle over the circle S^1. The Möbius band is not orientable and hence not homeomorphic to the product $S^1 \times \mathbb{R}$. This shows that the bundle (E, M, π) is not trivial.

We now introduce the notion of a smooth vector bundle. As we will see in Example 4.11 the tangent bundle (TM, M, π) of a smooth manifold M belongs to the C^∞-category.

Definition 4.8. Let E and M be differentiable manifolds and $\pi : E \to M$ be a differentiable map such that (E, M, π) is an n-dimensional topological vector bundle. A bundle atlas $\mathcal{B}$ for (E, M, π)

is said to be **differentiable** if the corresponding transition maps are differentiable. A **differentiable vector bundle** is a topological vector bundle together with a maximal differentiable bundle atlas. By $C^\infty(E)$ we denote the set of all smooth sections of (E, M, π).

From now on we will assume, when not stating otherwise, that all our vector bundles are smooth i.e. of the C^∞-category.

Definition 4.9. Let (E, M, π) be a smooth vector bundle over a manifold M. Then we define the operations $+$ and $\cdot$ on the set $C^\infty(E)$ of smooth sections of (E, M, π) by

(i) $(v + w)_p = v_p + w_p$,
(ii) $(f \cdot v)_p = f(p) \cdot v_p$

for all $p \in M$, $v, w \in C^\infty(E)$ and $f \in C^\infty(M)$.

If U is an open subset of M then a set $\{v_1, \dots, v_n\}$ of smooth sections $v_1, \dots, v_n : U \to E$ on U is called a **local frame** for E if for each $p \in U$ the set $\{(v_1)_p, \dots, (v_n)_p\}$ is a basis for the vector space E_p i.e. the fibre $\pi^{-1}(\{p\})$ over p.

Remark 4.10. According to Definition 2.20, the set of smooth real-valued functions on M is denoted by $C^\infty(M)$. This satisfies the algebraic axioms of a **ring** but not those of a **field**. With the above defined operations on $C^\infty(E)$ it becomes a **module** over the ring $C^\infty(M)$ and in particular a **vector space** over the field of real numbers, seen as the constant functions in $C^\infty(M)$.

The following example is the central part of this chapter. Here we construct the differentiable tangent bundle of a differentiable manifold.

Example 4.11. Let M^m be a differentiable manifold with maximal atlas $\hat{\mathcal{A}}$. Then define the **set** TM by

$$TM = \{(p, v) \,|\, p \in M \text{ and } v \in T_pM\}$$

and let $\pi : TM \to M$ be the **projection map** satisfying

$$\pi : (p, v) \mapsto p.$$

For each point $p \in M$, the **fibre** $\pi^{-1}(\{p\})$ is the **tangent space** T_pM isomorphic to $\mathbb{R}^m$. The triple (TM, M, π) is called the **tangent bundle** of M. We will now equip this with the structure of a differentiable vector bundle.

For every local coordinate $x : U \to \mathbb{R}^m$ on the manifold M, we define a local chart

$$x^* : \pi^{-1}(U) \to \mathbb{R}^m \times \mathbb{R}^m$$

on the tangent bundle TM of M by the formula

$$x^* : (p, \sum_{k=1}^{m} v_k(p) \cdot \big(\frac{\partial}{\partial x_k}\big)_p) \mapsto (x(p), (v_1(p), \dots, v_m(p))).$$

Proposition 3.19 shows that the map x^* is well defined. The collection

$$\{(x^*)^{-1}(W) \subset TM \,|\, (U, x) \in \hat{\mathcal{A}} \text{ and } W \subset x(U) \times \mathbb{R}^m \text{ open}\}$$

is a basis for a topology $\mathcal{T}_{TM}$ on TM and $(\pi^{-1}(U), x^*)$ is a local chart on the **topological manifold** $(TM, \mathcal{T}_{TM})$ of dimension $2m$. Note that $\mathcal{T}_{TM}$ is the weakest topology on TM such that the bundle charts are continuous.

If $(U, x), (V, y) \in \hat{\mathcal{A}}$ are two local charts on the differentiable manifold M such that $p \in U \cap V$ then it follows from Exercise 4.1 that the transition map

$$(y^*) \circ (x^*)^{-1} : x^*(\pi^{-1}(U \cap V)) \to \mathbb{R}^m \times \mathbb{R}^m$$

is given by

$$(a, b) \mapsto (y \circ x^{-1}(a), \Big(\sum_{k=1}^{m} \frac{\partial y_1}{\partial x_k}(x^{-1}(a)) \cdot b_k, \dots, \sum_{k=1}^{m} \frac{\partial y_m}{\partial x_k}(x^{-1}(a)) \cdot b_k\Big)).$$

Since we are assuming that $y \circ x^{-1}$ is differentiable it follows that the map $(y^*) \circ (x^*)^{-1}$ is also differentiable. Accordingly, the collection

$$\mathcal{A}^* = \{(\pi^{-1}(U), x^*) \,|\, (U, x) \in \hat{\mathcal{A}}\}$$

is a differentiable atlas on the tangent bundle TM so $(TM, \widehat{\mathcal{A}^*})$ is a **differentiable manifold**. Furthermore, it is clear that the surjective projection map $\pi : TM \to M$ is differentiable.

For each point $p \in M$ the fibre $\pi^{-1}(\{p\})$ is the linear tangent space T_pM isomorphic to $\mathbb{R}^m$. For a local coordinate $x : U \to \mathbb{R}^m$ on M we define the map $\bar{x} : \pi^{-1}(U) \to U \times \mathbb{R}^m$ by

$$\bar{x} : \big(p, \sum_{k=1}^{m} v_k(p) \cdot \big(\frac{\partial}{\partial x_k}\big)_p) \mapsto (p, (v_1(p), \dots, v_m(p))).$$

The restriction $\bar{x}_p = \bar{x}|_{T_pM} : T_pM \to \{p\} \times \mathbb{R}^m$ of $\bar{x}$ to the tangent space T_pM is given by

$$\bar{x}_p : \sum_{k=1}^{m} v_k(p) \cdot \big(\frac{\partial}{\partial x_k}\big)_p \mapsto (v_1(p), \dots, v_m(p)),$$

so it is clearly a vector space isomorphism. This implies that the map

$$\bar{x} : \pi^{-1}(U) \to U \times \mathbb{R}^m$$

is a local bundle chart. If $(U,x),(V,y)\in\hat{\mathcal{A}}$ are two local charts on M such that $p\in U\cap V$ then the transition map

$$(\bar{y})\circ(\bar{x})^{-1}:(U\cap V)\times\mathbb{R}^m\to(U\cap V)\times\mathbb{R}^m$$

is given by

$$(p,b)\mapsto(p,\Big(\sum_{k=1}^m\frac{\partial y_1}{\partial x_k}(p)\cdot b_k,\dots,\sum_{k=1}^m\frac{\partial y_m}{\partial x_k}(p)\cdot b_k\Big)).$$

It is clear that the matrix

$$\begin{pmatrix} \partial y_1/\partial x_1(p) & \dots & \partial y_1/\partial x_m(p) \\ \vdots & \ddots & \vdots \\ \partial y_m/\partial x_1(p) & \dots & \partial y_m/\partial x_m(p) \end{pmatrix}$$

is of full rank so the corresponding linear map $A(p):\mathbb{R}^m\to\mathbb{R}^m$ is a vector space isomorphism for all $p\in U\cap V$. This shows that

$$\bar{\mathcal{A}}=\{(\pi^{-1}(U),\bar{x})\,|\,(U,x)\in\hat{\mathcal{A}}\}$$

is a bundle atlas turning (TM,M,π) into a **topological vector bundle** of dimension m. It follows from the above that (TM,M,π) together with the maximal bundle atlas induced by $\bar{\mathcal{A}}$ is a **differentiable vector bundle**.

We now introduce the fundamental notion of a vector field on a differentiable manifold.

Definition 4.12. Let M be a differentiable manifold, then a section $X:M\to TM$ of the tangent bundle is called a **vector field**. The set of smooth vector fields $X:M\to TM$ is denoted by $C^\infty(TM)$.

Example 4.13. We have earlier seen that the 3-dimensional unit sphere S^3 in $\mathbb{H}\cong\mathbb{C}^2\cong\mathbb{R}^4$ carries a group structure $\cdot$ given by

$$(z_1,w_1)\cdot(z_2,w_2)=(z_1z_2-\bar{w}_1w_2,w_1z_2+\bar{z}_1w_2).$$

This turns $(S^3,\cdot)$ into a Lie group with neutral element $e=(1,0)$. Put $v_1=(0,1)$, $v_2=(i,0)$ and $v_3=(0,i)$ and for $k=1,2,3$ define the curves $\gamma_k:\mathbb{R}\to S^3$ by

$$\gamma_k:t\mapsto\cos t\cdot(1,0)+\sin t\cdot v_k.$$

Then $\gamma_k(0)=e$ and $\dot{\gamma}_k(0)=v_k$ so v_1, v_2, v_3 are elements of the tangent space T_eS^3 of S^3 at the neutral element e. They are linearly independent and hence form a basis for T_eS^3.

The group structure on S^3 can be used to extend vectors in T_eS^3 to vector fields on S^3 as follows. For a point $p\in S^3$, let the map $L_p:S^3\to S^3$ be the left-translation on the Lie group S^3 by p satisfying

$L_p : q \mapsto p \cdot q$. Then define the vector fields $X_1, X_2, X_3 \in C^\infty(TS^3)$ on S^3 by

$$\begin{aligned}(X_k)_p &= (dL_p)_e(v_k)\\ &= \frac{d}{dt}(L_p(\gamma_k(t)))|_{t=0}\\ &= \frac{d}{dt}(p \cdot \gamma_k(t)))|_{t=0}\\ &= p \cdot v_k.\end{aligned}$$

It is left as an exercise for the reader to show that at an arbitrary point $p = (z, w) \in S^3$ the values of X_k at p are given by

$$\begin{aligned}(X_1)_p &= (z, w) \cdot (0, 1) = (-\bar{w}, \bar{z}),\\ (X_2)_p &= (z, w) \cdot (i, 0) = (iz, iw),\\ (X_3)_p &= (z, w) \cdot (0, i) = (-i\bar{w}, i\bar{z}).\end{aligned}$$

Our next task is to introduce the important Lie brackets on the set of smooth vector fields $C^\infty(TM)$ on the manifold M.

Definition 4.14. Let M be a differentiable manifold. For two vector fields $X, Y \in C^\infty(TM)$ we define the **Lie bracket** $[X, Y]_p : C^\infty(M) \to \mathbb{R}$ of X and Y at $p \in M$ by

$$[X, Y]_p(f) = X_p(Y(f)) - Y_p(X(f)).$$

Remark 4.15. The reader should note that if M is a smooth manifold, $X \in C^\infty(TM)$ and $f \in C^\infty(M)$ then the derivative $X(f)$ is the smooth real-valued function on M given by $X(f) : p \mapsto X_p(f)$ for all $p \in M$.

The next result shows that the Lie bracket $[X, Y]_p$ is actually an element of the tangent space T_pM of M at p. The reader should compare this with Definition 3.6 and Remark 4.1.

Proposition 4.16. *Let M be a smooth manifold, $X, Y \in C^\infty(TM)$ be vector fields on M, $f, g \in C^\infty(M)$ and $\lambda, \mu \in \mathbb{R}$. Then*

(i) $[X, Y]_p(\lambda \cdot f + \mu \cdot g) = \lambda \cdot [X, Y]_p(f) + \mu \cdot [X, Y]_p(g)$,

(ii) $[X, Y]_p(f \cdot g) = [X, Y]_p(f) \cdot g(p) + f(p) \cdot [X, Y]_p(g)$.

Proof. The result is a direct consequence of the following calculations.

$$\begin{aligned}&[X, Y]_p(\lambda f + \mu g)\\ =\ & X_p(Y(\lambda f + \mu g)) - Y_p(X(\lambda f + \mu g))\\ =\ & \lambda X_p(Y(f)) + \mu X_p(Y(g)) - \lambda Y_p(X(f)) - \mu Y_p(X(g))\end{aligned}$$

$$= \lambda[X,Y]_p(f) + \mu[X,Y]_p(g).$$

$$\begin{aligned}
& [X,Y]_p(f\cdot g) \\
= \; & X_p(Y(f\cdot g)) - Y_p(X(f\cdot g)) \\
= \; & X_p(f\cdot Y(g) + g\cdot Y(f)) - Y_p(f\cdot X(g) + g\cdot X(f)) \\
= \; & X_p(f)Y_p(g) + f(p)X_p(Y(g)) + X_p(g)Y_p(f) + g(p)X_p(Y(f)) \\
& -Y_p(f)X_p(g) - f(p)Y_p(X(g)) - Y_p(g)X_p(f) - g(p)Y_p(X(f)) \\
= \; & f(p)\{X_p(Y(g)) - Y_p(X(g))\} + g(p)\{X_p(Y(f)) - Y_p(X(f))\} \\
= \; & f(p)[X,Y]_p(g) + g(p)[X,Y]_p(f).
\end{aligned}$$

□

Proposition 4.16 implies that if X, Y are smooth vector fields on M then the map $[X,Y] : M \to TM$ given by $[X,Y] : p \mapsto [X,Y]_p$ is a section of the tangent bundle. In Proposition 4.18 we will prove that this section is smooth. For this we need the following technical result.

Lemma 4.17. *Let M^m be a smooth manifold and $X : M \to TM$ be a section of TM. Then the following conditions are equivalent*

(i) the section X is smooth,

(ii) if (U,x) is a local chart on M then the functions $a_1, \dots, a_m : U \to \mathbb{R}$ given by

$$X|_U = \sum_{k=1}^m a_k \frac{\partial}{\partial x_k},$$

are smooth,

(iii) if $f : V \to \mathbb{R}$ defined on an open subset V of M is smooth, then the function $X(f) : V \to \mathbb{R}$ with $X(f)(p) = X_p(f)$ is smooth.

Proof. This proof is divided into three parts. First we show that (i) implies (ii): The functions

$$a_k = \pi_{m+k} \circ x^* \circ X|_U : U \to \pi^{-1}(U) \to x(U) \times \mathbb{R}^m \to \mathbb{R}$$

are compositions of smooth maps so therefore smooth.

Secondly, we now show that (ii) gives (iii): Let (U,x) be a local chart on M such that U is contained in V. By assumption the map

$$X(f|_U) = \sum_{i=1}^m a_i \frac{\partial f}{\partial x_i}$$

is smooth. This is true for each such local chart (U,x) so the function $X(f)$ is smooth on V.

Finally we show that (iii) leads to (i): Note that the smoothness of the section X is equivalent to $x^* \circ X|_U : U \to \mathbb{R}^{2m}$ being smooth for all local charts (U, x) on M. On the other hand, this is equivalent to

$$x_k^* = \pi_k \circ x^* \circ X|_U : U \to \mathbb{R}$$

being smooth for all $k = 1, 2, \dots, 2m$ and all local charts (U, x) on M. It is trivial that the coordinate functions $x_k^* = x_k$ for $k = 1, \dots, m$ are smooth. But $x_{m+k}^* = a_k = X(x_k)$ for $k = 1, \dots, m$ hence also smooth by assumption. $\square$

Proposition 4.18. *Let M be a manifold and $X, Y \in C^\infty(TM)$ be vector fields on M. Then the section $[X, Y] : M \to TM$ of the tangent bundle given by $[X, Y] : p \mapsto [X, Y]_p$ is smooth.*

PROOF. Let $f : M \to \mathbb{R}$ be an arbitrary smooth function on M then $[X, Y](f) = X(Y(f)) - Y(X(f))$ is smooth so it follows from Lemma 4.17 that the section $[X, Y]$ is smooth. $\square$

For later use we prove the following important result.

Lemma 4.19. *Let M be a smooth manifold and*

$$[\cdot, \cdot] : C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM)$$

be the Lie bracket on the tangent bundle TM of M. Then

(i) $[X, f \cdot Y] = X(f) \cdot Y + f \cdot [X, Y]$,
(ii) $[f \cdot X, Y] = f \cdot [X, Y] - Y(f) \cdot X$,

for all $X, Y \in C^\infty(TM)$ and $f \in C^\infty(M)$.

PROOF. If $g \in C^\infty(M)$, then

$$\begin{aligned} [X, f \cdot Y](g) &= X(f \cdot Y(g)) - f \cdot Y(X(g)) \\ &= X(f) \cdot Y(g) + f \cdot X(Y(g)) - f \cdot Y(X(g)) \\ &= (X(f) \cdot Y + f \cdot [X, Y])(g). \end{aligned}$$

This proves the first statement and the second follows from the skew-symmetry of the Lie bracket. $\square$

We now define the general notion of a Lie algebra. This is a fundamental concept in differential geometry.

Definition 4.20. A real vector space $(V, +, \cdot)$ equipped with an operation $[\cdot, \cdot] : V \times V \to V$ is said to be a real **Lie algebra** if the following relations hold

(i) $[\lambda X + \mu Y, Z] = \lambda [X, Z] + \mu [Y, Z]$,
(ii) $[X, Y] = -[Y, X]$,
(iii) $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$,

for all $X, Y, Z \in V$ and $\lambda, \mu \in \mathbb{R}$. The important equation (iii) is called the **Jacobi identity**.

Example 4.21. Let $\mathbb{R}^3$ be the 3-dimensional real vector space generated by $X = (1,0,0)$, $Y = (0,1,0)$ and $Z = (0,0,1)$. Let $\times$ be the standard cross product on $\mathbb{R}^3$ and define the skew-symmetric bilinear operation $[\cdot,\cdot] : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ by

$$[X, Y] = X \times Y = Z,$$

$$[Z, X] = Z \times X = Y,$$

$$[Y, Z] = Y \times Z = X.$$

This turns $\mathbb{R}^3$ into a Lie algebra. Compare this with Exercise 4.7.

Theorem 4.22. *Let M be a smooth manifold. The vector space $C^\infty(TM)$ of smooth vector fields on M equipped with the Lie bracket $[\cdot,\cdot] : C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM)$ is a Lie algebra.*

Proof. See Exercise 4.4. □

Definition 4.23. If $\phi : M \to N$ is a **surjective** map between differentiable manifolds, then two vector fields $X \in C^\infty(TM)$ and $\bar{X} \in C^\infty(TN)$ are said to be ϕ-**related** if $d\phi_p(X_p) = \bar{X}_{\phi(p)}$ for all $p \in M$. In that case we write $d\phi(X) = \bar{X}$.

Example 4.24. Let S^1 be the unit circle in the complex plane and $\phi : S^1 \to S^1$ be the map given by $\phi(z) = z^2$. Note that this is surjective but not bijective. Further let X be the vector field on S^1 satisfying $X(z) = iz$. Then

$$d\phi_z(X_z) = \frac{d}{d\theta}(\phi(ze^{i\theta}))|_{\theta=0} = \frac{d}{d\theta}((ze^{i\theta})^2)|_{\theta=0} = 2iz^2 = 2X_{\phi(z)}.$$

This shows that the vector field X is ϕ-related to $\bar{X} = 2X$.

Example 4.25. Let $f : \mathbb{R} \to \mathbb{R}$ be a surjective C^1-function and $x, y \in \mathbb{R}$ such that $x \neq y$, $f(x) = f(y)$ and $f'(x) \neq f'(y)$. Further let $\gamma : \mathbb{R} \to \mathbb{R}$ be the curve with $\gamma(t) = t$ and define the vector field $X \in C^1(T\mathbb{R})$ by $X_t = \dot{\gamma}(t)$. Then for each $t \in \mathbb{R}$ we have

$$df_t(X_t) = (f \circ \gamma(t))' = f'(t).$$

If $\bar{X} \in C^1(T\mathbb{R})$ is a vector field which is f-related to X then

$$\bar{X}_{f(x)} = df_x(X_x) = f'(x) \neq f'(y) = df_y(X_y) = \bar{X}_{f(y)}.$$

This contradicts the existence of such a vector field $\bar{X}$.

The next item is hopefully helpful for understanding the proof of Proposition 4.27.

Remark 4.26. Let $\phi : M \to N$ be a differentiable map between differentiable manifolds. For this situation we have in Definition 3.14 introduced the linear differential $d\phi_p : T_pM \to T_{\phi(p)}N$ of ϕ at a point $p \in M$ such that for all $X_p \in T_pM$ and $f \in \varepsilon(\phi(p))$ we have

$$(d\phi_p(X_p))(f) = X_p(f \circ \phi),$$

or equivalently,

$$d\phi(X)(f)(\phi(p)) = X(f \circ \phi)(p).$$

This equation is a comparison of two real numbers. But since it is true for all points $p \in M$ it induces the following relation at the level of functions

$$d\phi(X)(f) \circ \phi = X(f \circ \phi).$$

The next result turns out to be important and will be employed several times in what follows.

Proposition 4.27. *Let $\phi : M \to N$ be a surjective map between differentiable manifolds, $X, Y \in C^\infty(TM)$ and $\bar{X}, \bar{Y} \in C^\infty(TN)$ such that $d\phi(X) = \bar{X}$ and $d\phi(Y) = \bar{Y}$. Then the Lie brackets $[X, Y] \in C^\infty(TM)$ and $[\bar{X}, \bar{Y}] \in C^\infty(TN)$ are ϕ-related i.e.*

$$d\phi([X, Y]) = [\bar{X}, \bar{Y}].$$

Proof. Let $p \in M$ and $f : N \to \mathbb{R}$ be a smooth function, then

$$\begin{aligned}
d\phi_p([X, Y]_p)(f) &= [X, Y]_p(f \circ \phi) \\
&= X_p(Y(f \circ \phi)) - Y_p(X(f \circ \phi)) \\
&= X_p(d\phi(Y)(f) \circ \phi) - Y_p(d\phi(X)(f) \circ \phi) \\
&= d\phi(X)_{\phi(p)}(d\phi(Y)(f)) - d\phi(Y)_{\phi(p)}(d\phi(X)(f)) \\
&= [\bar{X}, \bar{Y}]_{\phi(p)}(f).
\end{aligned}$$

□

For the important special case of a diffeomorphism we have the following natural consequence of Proposition 4.27.

Proposition 4.28. *Let M and N be differentiable manifolds and $\phi : M \to N$ be a diffeomorphism. If $X, Y \in C^\infty(TM)$ are vector fields on M, then $d\phi(X)$ and $d\phi(Y)$ are vector fields on N and the tangent map $d\phi : C^\infty(TM) \to C^\infty(TN)$ is a Lie algebra homomorphism i.e.*

$$d\phi([X, Y]) = [d\phi(X), d\phi(Y)].$$

Proof. The fact that ϕ is bijective implies that $d\phi(X)$ is a section of the tangent bundle TN. That $d\phi(X)$ is smooth follows directly from the fact that

$$d\phi(X)(f)(\phi(p)) = X(f \circ \phi)(p),$$

for all $p \in M$ and $f \in \varepsilon(\phi(p))$. The rest is an immediate consequence of Proposition 4.27. $\square$

Definition 4.29. Let M be a differentiable manifold. Two vector fields $X, Y \in C^\infty(TM)$ are said to **commute** if their Lie bracket vanishes i.e. $[X, Y] = 0$.

The fact that a local chart on a differentiable manifold is a diffeomorphism has the following important consequence.

Proposition 4.30. *Let M be a differentiable manifold, (U, x) be a local chart on M and*

$$\{\frac{\partial}{\partial x_k} \,|\, k = 1, 2, \dots, m\}$$

be the induced local frame for the tangent bundle TM. Then the local frame fields commute i.e.

$$[\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}] = 0 \quad \text{for all} \quad k, l = 1, \dots, m.$$

Proof. The map $x : U \to x(U)$ is a diffeomorphism. The vector field $\partial/\partial x_k \in C^\infty(TU)$ is x-related to the coordinate vector field $\partial_{e_k} \in C^\infty(Tx(U))$. Then Proposition 4.28 implies that

$$dx([\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}]) = [\partial_{e_k}, \partial_{e_l}] = 0.$$

The last equation is an immediate consequence of the following well known fact

$$[\partial_{e_k}, \partial_{e_l}](f) = \partial_{e_k}(\partial_{e_l}(f)) - \partial_{e_l}(\partial_{e_k}(f)) = 0$$

for all $f \in C^2(x(U))$. The result now follows from the fact that the linear map $dx_p : T_pM \to T_{x(p)}\mathbb{R}^m$ is bijective for all $p \in U$. $\square$

We now introduce the notion of a left-invariant vector field on a Lie group. This will play an important role later on and should be compared with Example 4.13.

Definition 4.31. Let G be a Lie group. Then a vector field $X \in C^\infty(TG)$ on G is said to be **left-invariant** if it is L_p-related to itself for all $p \in G$ i.e.

$$(dL_p)_q(X_q) = X_{pq} \quad \text{for all} \quad p, q \in G.$$

The set of left-invariant vector fields on G is called the **Lie algebra** of G and denoted by $\mathfrak{g}$.

Remark 4.32. It should be noted that if e is the neutral element of the Lie group G and $X \in \mathfrak{g}$ is a left-invariant vector field on G, then

$$X_p = (dL_p)_e(X_e).$$

This shows that the value X_p of the left-invariant vector field $X \in \mathfrak{g}$ at p is completely determined by its value X_e at e. Hence the linear map $\Phi : T_eG \to \mathfrak{g}$ given by

$$\Phi : X_e \mapsto (X : p \mapsto (dL_p)_e(X_e))$$

is a vector space isomorphism. As a direct consequence of this fact we see that the Lie algebra $\mathfrak{g}$ is a finite dimensional subspace of $C^\infty(TG)$ of the same dimension as that of the Lie group G.

Proposition 4.33. *Let G be a Lie group. Then its Lie algebra $\mathfrak{g}$ is a* **Lie subalgebra** *of $C^\infty(TG)$ i.e. if $X, Y \in \mathfrak{g}$ are left-invariant then $[X, Y] \in \mathfrak{g}$.*

Proof. If $p \in G$ then the left translation $L_p : G \to G$ is a diffeomorphism so it follows from Proposition 4.28 that

$$dL_p([X, Y]) = [dL_p(X), dL_p(Y)] = [X, Y]$$

for all $X, Y \in \mathfrak{g}$. This proves that the Lie bracket $[X, Y]$ of two left-invariant vector fields $X, Y \in \mathfrak{g}$ is also left-invariant. $\square$

The following shows that the Lie algebra of a Lie group can be identified with the tangent space at its neutral element. This identification turns out to be very useful.

Remark 4.34. The reader should note that the linear isomorphism $\Phi : T_eG \to \mathfrak{g}$ given by

$$\Phi : X_e \mapsto (X : p \mapsto (dL_p)_e(X_e))$$

induces a natural Lie bracket $[\cdot, \cdot] : T_eG \times T_eG \to T_eG$ on the tangent space T_eG of G at e via

$$[X_e, Y_e] = [X, Y]_e.$$

This shows that we can simply identify the Lie algebra $\mathfrak{g}$ of G with its tangent space T_eG at the neutral element $e \in G$.

Notation 4.35. For the classical matrix Lie groups, introduced in Chapter 3, we denote their Lie algebras by $\mathfrak{gl}_m(\mathbb{R})$, $\mathfrak{sl}_m(\mathbb{R})$, $\mathfrak{o}(m)$, $\mathfrak{so}(m)$, $\mathfrak{gl}_m(\mathbb{C})$, $\mathfrak{sl}_m(\mathbb{C})$, $\mathfrak{u}(m)$ and $\mathfrak{su}(m)$, respectively.

The next result is a useful tool for handling the Lie brackets of the classical matrix Lie groups. They can simply be calculated by means of the standard matrix multiplication.

Proposition 4.36. *Let G be one of the classical matrix Lie groups and T_eG be the tangent space of G at the neutral element e. Then the Lie bracket $[\cdot,\cdot] : T_eG \times T_eG \to T_eG$ on T_eG is given by*

$$[X_e, Y_e] = X_e \cdot Y_e - Y_e \cdot X_e,$$

where $\cdot$ is the standard matrix multiplication.

Proof. We prove the result for the general linear group $\mathbf{GL}_m(\mathbb{R})$. For the other real groups the result follows from the fact that they are all subgroups of $\mathbf{GL}_m(\mathbb{R})$. The same proof can be used in the complex cases.

Let $X, Y \in \mathfrak{gl}_m(\mathbb{R})$ be left-invariant vector fields, $f : U \to \mathbb{R}$ be a function defined locally around the identity element e and p be an arbitrary point of U. Then the first order derivative $X_p(f)$ of f at p is given by

$$X_p(f) = \frac{d}{dt}(f(p \cdot \mathrm{Exp}(tX_e)))|_{t=0} = df_p(p \cdot X_e) = df_p(X_p).$$

The general linear group $\mathbf{GL}_m(\mathbb{R})$ is an open subset of $\mathbb{R}^{m\times m}$ so we can apply standard arguments from multivariable analysis. The second order derivative $Y_e(X(f))$ satisfies

$$\begin{aligned} Y_e(X(f)) &= \frac{d}{dt}(X_{\mathrm{Exp}(tY_e)}(f))|_{t=0} \\ &= \frac{d}{dt}(df_{\mathrm{Exp}(tY_e)}(\mathrm{Exp}(tY_e) \cdot X_e))|_{t=0} \\ &= d^2 f_e(Y_e, X_e) + df_e(Y_e \cdot X_e). \end{aligned}$$

Here $d^2 f_e$ is the symmetric Hessian of the function f. As an immediate consequence we obtain

$$\begin{aligned} [X, Y]_e(f) &= X_e(Y(f)) - Y_e(X(f)) \\ &= d^2 f_e(X_e, Y_e) + df_e(X_e \cdot Y_e) \\ &\qquad - d^2 f_e(Y_e, X_e) - df_e(Y_e \cdot X_e) \\ &= df_e(X_e \cdot Y_e - Y_e \cdot X_e). \end{aligned}$$

This last calculation implies the statement. □

Corollary 4.37. *Let G be one of the classical matrix Lie groups and T_eG be the tangent space of G at the neutral element e. If $p \in G$*

and $X, Y \in \mathfrak{g}$ are left-invariant vector fields on G then their Lie bracket $[X, Y] \in \mathfrak{g}$ satisfies

$$[X, Y]_p = p \cdot (X_e \cdot Y_e - Y_e \cdot X_e),$$

where $\cdot$ is the standard matrix multiplication.

PROOF. The statement is an immediate consequence of Proposition 4.36. $\square$

The next remarkable result shows that the tangent bundle of any Lie group is trivial.

Theorem 4.38. *Let G be a Lie group. Then its tangent bundle TG is trivial.*

PROOF. Let $\{(X_1)_e, \dots, (X_m)_e\}$ be a basis for the tangent space T_eG of G at the neutral element e. Then extend each tangent vector $(X_k)_e \in T_eG$ to the corresponding left-invariant vector field $X_k \in \mathfrak{g}$ satisfying

$$(X_k)_p = (dL_p)_e((X_k)_e).$$

For a point $p \in G$, the left translation $L_p : G \to G$ is a diffeomorphism so the set $\{(X_1)_p, \dots, (X_m)_p\}$ is a basis for the tangent space T_pG of G at p. This means that the map $\psi : TG \to G \times \mathbb{R}^m$ given by

$$\psi : (p, \sum_{k=1}^{m} v_k \cdot (X_k)_p) \mapsto (p, (v_1, \dots, v_m))$$

is globally well-defined. This is a global bundle chart for TG, which therefore is trivial. $\square$

Exercises

Exercise 4.1. Let $(M, \hat{\mathcal{A}})$ be a smooth manifold, (U, x), (V, y) be local charts such that $U \cap V$ is non-empty and

$$f = y \circ x^{-1} : x(U \cap V) \to \mathbb{R}^m$$

be the corresponding transition map. Show that the local frames

$$\{\frac{\partial}{\partial x_i} \,|\, i = 1, \dots, m\} \quad \text{and} \quad \{\frac{\partial}{\partial y_j} \,|\, j = 1, \dots, m\}$$

for TM on $U \cap V$ are related as follows

$$\frac{\partial}{\partial x_i} = \sum_{j=1}^{m} \frac{\partial (f_j \circ x)}{\partial x_i} \cdot \frac{\partial}{\partial y_j}.$$

Exercise 4.2. Let $\mathbf{SO}(m)$ be the special orthogonal group.

(i) Find a basis for the tangent space $T_e\mathbf{SO}(m)$,
(ii) construct a non-vanishing vector field $Z \in C^\infty(T\mathbf{SO}(m))$,
(iii) determine all smooth vector fields on $\mathbf{SO}(2)$.

The Hairy Ball Theorem. There does not exist a continuous non-vanishing vector field $X \in C^0(TS^{2m})$ on the even dimensional sphere S^{2m}.

Exercise 4.3. Employ the Hairy Ball Theorem to show that the tangent bundle TS^{2m} is not trivial. Then construct a non-vanishing vector field $X \in C^\infty(TS^{2m+1})$ on the odd-dimensional sphere S^{2m+1}.

Exercise 4.4. Find a proof of Theorem 4.22.

Exercise 4.5. The Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ of the special linear group $\mathbf{SL}_2(\mathbb{R})$ is generated by

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Show that the Lie brackets of $\mathfrak{sl}_2(\mathbb{R})$ satisfy

$$[X, Y] = 2Z, \quad [Z, X] = 2Y, \quad [Y, Z] = -2X.$$

Exercise 4.6. The Lie algebra $\mathfrak{su}(2)$ of the special unitary group $\mathbf{SU}(2)$ is generated by

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Show that the corresponding Lie bracket relations are given by

$$[X, Y] = 2Z, \quad [Z, X] = 2Y, \quad [Y, Z] = 2X.$$

Exercise 4.7. The Lie algebra $\mathfrak{so}(3)$ of the special orthogonal group $\mathbf{SO}(3)$ is generated by

$$X = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Show that the corresponding Lie bracket relations are given by

$$[X, Y] = Z, \quad [Z, X] = Y, \quad [Y, Z] = X.$$

Compare this result with Example 4.21.

Exercise 4.8. Prove that the differential $d\phi_p : T_pS^3 \to T_{\phi(p)}S^2$ of the Hopf-map $\phi : S^3 \to S^2$, with

$$\phi : (z, w) \mapsto (2\, z\bar{w}, |z|^2 - |w|^2),$$

is surjective at each point $p \in S^3$.

CHAPTER 5

Riemannian Manifolds

In this chapter we introduce the notion of a Riemannian manifold. The Riemannian metric provides us with a scalar product on each tangent space and can be used to measure angles and the lengths of curves on the manifold. This defines a distance function and turns the manifold into a metric space in a natural way. The Riemannian metric is the most important example of what is called a tensor field.

Let M be a smooth manifold, $C^\infty(M)$ denote the commutative ring of smooth functions on M and $C^\infty(TM)$ be the set of smooth vector fields on M forming a **module** over $C^\infty(M)$. Put

$$C_0^\infty(TM) = C^\infty(M)$$

and for each positive integer $r \in \mathbb{Z}^+$ let

$$C_r^\infty(TM) = C^\infty(TM) \otimes \cdots \otimes C^\infty(TM)$$

be the r-fold tensor product of $C^\infty(TM)$ over the commutative ring $C^\infty(M)$.

Definition 5.1. Let M be a differentiable manifold. A smooth **tensor field** A on M of type (s, r) is a map

$$A : C_r^\infty(TM) \to C_s^\infty(TM)$$

which is multilinear over the commutative ring $C^\infty(M)$ i.e. satisfying

$$\begin{aligned} & A(X_1 \otimes \cdots \otimes X_{k-1} \otimes (f \cdot Y + g \cdot Z) \otimes X_{k+1} \otimes \cdots \otimes X_r) \\ = \quad & f \cdot A(X_1 \otimes \cdots \otimes X_{k-1} \otimes Y \otimes X_{k+1} \otimes \cdots \otimes X_r) \\ & +g \cdot A(X_1 \otimes \cdots \otimes X_{k-1} \otimes Z \otimes X_{k+1} \otimes \cdots \otimes X_r), \end{aligned}$$

for all $X_1, \dots, X_r, Y, Z \in C^\infty(TM)$, $f, g \in C^\infty(M)$ and $k = 1, \dots, r$.

Notation 5.2. For the rest of this work we will for $A(X_1 \otimes \cdots \otimes X_r)$v use the notation $A(X_1, \dots, X_r)$.

The next fundamental result provides us with the most important property of a tensor field. It shows that the value $A(X_1, \dots, X_r)(p)$ of $A(X_1, \dots, X_r)$ at a point $p \in M$ only depends on the values

$$(X_1)_p, \dots, (X_r)_p$$

of the vector fields $X_1, \dots, X_r$ at p and is independent of their values away from p.

Proposition 5.3. *Let $A : C_r^\infty(TM) \to C_s^\infty(TM)$ be a tensor field of type (s, r) and $p \in M$. Let $X_1, \dots, X_r$ and $Y_1, \dots, Y_r$ be smooth vector fields on M such that $(X_k)_p = (Y_k)_p$ for each $k = 1, \dots, r$. Then*

$$A(X_1, \dots, X_r)(p) = A(Y_1, \dots, Y_r)(p).$$

Proof. We will prove the statement for $r = 1$, the rest follows by induction. Put $X = X_1$ and $Y = Y_1$ and let (U, x) be a local chart on M. Choose a function $f \in C^\infty(M)$ such that $f(p) = 1$,

$$\text{support}(f) = \overline{\{p \in M \mid f(p) \neq 0\}}$$

is contained in U and define the vector fields $v_1, \dots, v_m \in C^\infty(TM)$ on M by

$$(v_k)_q = \begin{cases} f(q) \cdot (\frac{\partial}{\partial x_k})_q & \text{if } q \in U, \\ 0 & \text{if } q \notin U. \end{cases}$$

Then there exist functions $\rho_k, \sigma_k \in C^\infty(M)$ such that

$$f \cdot X = \sum_{k=1}^{m} \rho_k \cdot v_k \quad \text{and} \quad f \cdot Y = \sum_{k=1}^{m} \sigma_k \cdot v_k.$$

This implies that

$$\begin{aligned} A(X)(p) &= f(p)\, A(X)(p) \\ &= (f \cdot A(X))(p) \\ &= A(f \cdot X)(p) \\ &= A(\sum_{k=1}^{m} \rho_k \cdot v_k)(p) \\ &= \sum_{k=1}^{m} (\rho_k \cdot A(v_k))(p) \\ &= \sum_{k=1}^{m} \rho_k(p) A(v_k)(p), \end{aligned}$$

and similarly,

$$A(Y)(p) = \sum_{k=1}^{m} \sigma_k(p) A(v_k)(p).$$

The fact that $X_p = Y_p$ shows that $\rho_k(p) = \sigma_k(p)$ for all k. As a direct consequence we see that

$$A(X)(p) = A(Y)(p).$$

$\square$

The result of Proposition 5.3 shows that the following multilinear operator A_p is well-defined.

Notation 5.4. For a tensor field $A : C_r^\infty(TM) \to C_s^\infty(TM)$ of type (s, r) we will by A_p denote the real multilinear restriction of A to the r-fold tensor product $T_pM \otimes \cdots \otimes T_pM$ of the real vector space T_pM given by

$$A_p : ((X_1)_p, \dots, (X_r)_p) \mapsto A(X_1, \dots, X_r)(p).$$

Next we introduce the notion of a Riemannian metric. This is the most important example of a tensor field in Riemannian geometry.

Definition 5.5. Let M be a smooth manifold. A **Riemannian metric** g on M is a tensor field $g : C_2^\infty(TM) \to C_0^\infty(TM)$ such that for each $p \in M$ the restriction g_p of g to the tensor product $T_pM \otimes T_pM$ with

$$g_p : (X_p, Y_p) \mapsto g(X, Y)(p)$$

is a real scalar product on the tangent space T_pM. The pair (M, g) is called a **Riemannian manifold**. The study of Riemannian manifolds is called **Riemannian geometry**. The geometric properties of (M, g) which only depend on the metric g are said to be **intrinsic** or **metric** properties.

The classical Euclidean spaces are Riemannian manifolds defined as follows.

Example 5.6. The m-dimensional **Euclidean space** $\mathbb{E}^m$ is the standard real vector space $\mathbb{R}^m$ equipped with its natural scalar product given by

$$\langle X, Y \rangle = X^t \cdot Y = \sum_{k=1}^{m} X_k Y_k.$$

On Riemannian manifolds we have the notion of lengths of curves, in a natural way.

Definition 5.7. Let (M, g) be a Riemannian manifold and $\gamma : I \to M$ be a C^1-curve in M. Then the **length** $L(\gamma)$ of γ is defined by

$$L(\gamma) = \int_I \sqrt{g(\dot\gamma(t), \dot\gamma(t))}dt.$$

The standard punctured round sphere Σ^m has the following description as a Riemannian manifold.

Example 5.8. Equip the vector space $\mathbb{R}^m$ with the Riemannian metric g given by

$$g_p(X, Y) = \frac{4}{(1 + |p|^2)^2}\langle X, Y\rangle.$$

The Riemannian manifold $\Sigma^m = (\mathbb{R}^m, g)$ is called the m-dimensional **punctured round sphere**. Let $\gamma : \mathbb{R}^+ \to \Sigma^m$ be the curve with

$$\gamma : t \mapsto (t, 0, \dots, 0).$$

Then the length $L(\gamma)$ of γ can be determined as follows.

$$L(\gamma) = 2\int_0^\infty \frac{\sqrt{\langle\dot\gamma, \dot\gamma\rangle}}{1 + |\gamma|^2} dt = 2\int_0^\infty \frac{dt}{1 + t^2} = 2[\arctan(t)]_0^\infty = \pi.$$

The important real hyperbolic space H^m can be modelled in different ways. In the following Example 5.9 we present it as the open unit ball. For the upper half space model see Exercise 8.8.

Example 5.9. Let $B_1^m(0)$ be the open unit ball in $\mathbb{R}^m$ given by

$$B_1^m(0) = \{p \in \mathbb{R}^m \mid |p|^2 < 1\}.$$

By the m-dimensional **real hyperbolic space** we mean $B_1^m(0)$ equipped with the Riemannian metric

$$g_p(X, Y) = \frac{4}{(1 - |p|^2)^2}\langle X, Y\rangle.$$

Let $\gamma : (0, 1) \to B_1^m(0)$ be the curve given by

$$\gamma : t \mapsto (t, 0, \dots, 0).$$

Then the length $L(\gamma)$ of γ can be determined as follows.

$$L(\gamma) = 2\int_0^1 \frac{\sqrt{\langle\dot\gamma, \dot\gamma\rangle}}{1 - |\gamma|^2} dt = 2\int_0^1 \frac{dt}{1 - t^2} = [\log(\frac{1+t}{1-t})]_0^1 = \infty\ .$$

The following result tells us that a path-connected Riemannian manifold (M, g) has the structure of a metric space (M, d) in a natural way.

Proposition 5.10. *Let (M, g) be a path-connected Riemannian manifold. For two points $p, q \in M$ let C_{pq} denote the set of C^1-curves $\gamma : [0, 1] \to M$ such that $\gamma(0) = p$ and $\gamma(1) = q$ and define the function $d : M \times M \to \mathbb{R}_0^+$ by*

$$d(p, q) = \inf\{L(\gamma) \mid \gamma \in C_{pq}\}.$$

Then (M, d) is a **metric space** *i.e. for all $p, q, r \in M$ we have*

(i) $d(p, q) \geq 0$,

(ii) $d(p, q) = 0$ *if and only if* $p = q$,
(iii) $d(p, q) = d(q, p)$,
(iv) $d(p, q) \leq d(p, r) + d(r, q)$.

The topology on M *induced by the metric* d *is identical to the one* M *carries as a topological manifold* $(M, \mathcal{T})$*, see Definition 2.1.*

Proof. See for example: P. Petersen, *Riemannian Geometry*, Graduate Texts in Mathematics **171**, Springer (1998). □

A Riemannian metric on a differentiable manifold induces a Riemannian metric on its submanifolds as follows.

Definition 5.11. Let (N, h) be a Riemannian manifold and M be a submanifold. Then the smooth tensor field $g : C_2^\infty(TM) \to C_0^\infty(M)$ given by

$$g(X, Y) : p \mapsto h_p(X_p, Y_p)$$

is a Riemannian metric on M. It is called the **induced metric** on M in (N, h).

We can now easily equip some of the manifolds introduced in Chapter 2 with a Riemannian metric.

Example 5.12. The standard Euclidean metric $\langle , \rangle$ on $\mathbb{R}^n$ induces Riemannian metrics on the following submanifolds.

(i) the unit sphere S^m in $\mathbb{R}^n$, with $n = m + 1$,
(ii) the tangent bundle TS^m in $\mathbb{R}^n$, where $n = 2(m + 1)$,
(iii) the torus T^m in $\mathbb{R}^n$, with $n = 2m$,

Example 5.13. The vector space $\mathbb{C}^{m \times m}$ of complex $m \times m$ matrices carries the standard Riemannian metric h given by

$$h(Z, W) = \mathrm{Re}(\mathrm{trace}(\bar{Z}^t \cdot W))$$

for all $Z, W \in \mathbb{C}^{m \times m}$. This induces natural metrics on the submanifolds of $\mathbb{C}^{m \times m}$ such as $\mathbb{R}^{m \times m}$ and the classical Lie groups $\mathbf{GL}_m(\mathbb{R})$, $\mathbf{SL}_m(\mathbb{R})$, $\mathbf{O}(m)$, $\mathbf{SO}(m)$, $\mathbf{GL}_m(\mathbb{C})$, $\mathbf{SL}_m(\mathbb{C})$, $\mathbf{U}(m)$, $\mathbf{SU}(m)$.

Our next aim is to prove that every differentiable manifold M can be equipped with a Riemannian metric g. For this we need Fact 5.15.

Definition 5.14. Let $(M, \hat{\mathcal{A}})$ be a differentiable manifold. Then a **partition of unity** on M consists of a family $\{f_\alpha : M \to \mathbb{R} \mid \alpha \in \mathcal{I}\}$ of differentiable real-valued functions such that

(i) $0 \leq f_\alpha \leq 1$ for all $\alpha \in \mathcal{I}$,
(ii) every point $p \in M$ has a neighbourhood which intersects only finitely many of the sets

$$\mathrm{support}(f_\alpha) = \overline{\{p \in M \mid f_\alpha(p) \neq 0\}},$$

(iii)

$$\sum_{\alpha \in \mathcal{I}} f_\alpha = 1.$$

Note that the sum in (iii) is finite at each point $p \in M$.

For the proof of the following interesting result, it is important that M is a Hausdorff space with a countable basis, see Defintion 2.1.

Fact 5.15. *Let M be a differentiable manifold and $(U_\alpha)_{\alpha \in \mathcal{I}}$ be an open covering of M such that for each $\alpha \in \mathcal{I}$ the pair (U_α, ϕ_α) is a local chart on M. Then there exist*

(i) a locally finite open cover $(W_\beta)_{\beta \in \mathcal{J}}$ such that each W_β is contained in U_α for some $\alpha \in \mathcal{I}$. Furthermore, for each $\beta \in \mathcal{J}$, W_β is an open neighbourhood for a local chart (W_β, x_β), and

(ii) a partition of unity $(f_\beta)_{\beta \in \mathcal{J}}$ such that the support(f_β) is contained in the open subset W_β.

Proof. See for example J. R. Munkres, *Topology*, Prentice Hall (2000). $\square$

We are now ready to prove the following important statement.

Theorem 5.16. *Let $(M, \hat{\mathcal{A}})$ be a differentiable manifold. Then there exists a Riemannian metric g on M.*

Proof. For each point $p \in M$, let $(U_p, \phi_p) \in \hat{\mathcal{A}}$ be a local chart such that $p \in U_p$. Then $(U_p)_{p \in M}$ is an open covering and let (W_β, x^β) be local charts on M as in Fact 5.15. Let $(f_\beta)_{\beta \in \mathcal{J}}$ be a partition of unity such that the support(f_β) is contained in W_β. Further, let $\langle , \rangle_{\mathbb{R}^m}$ be the standard Euclidean metric on $\mathbb{R}^m$. Then for each $\beta \in \mathcal{J}$ we define

$$g_\beta : C_2^\infty(TM) \to C_0^\infty(TM)$$

by

$$g_\beta(\frac{\partial}{\partial x_k^\beta}, \frac{\partial}{\partial x_l^\beta})(p) = \begin{cases} f_\beta(p) \cdot \langle e_k, e_l \rangle_{\mathbb{R}^m} & \text{if } p \in W_\beta \\ 0 & \text{if } p \notin W_\beta \end{cases}$$

Note that at each point only finitely many of g_β are non-zero. This means that the well defined tensor $g : C_2^\infty(TM) \to C_0^\infty(TM)$ given by

$$g = \sum_{\beta \in \mathcal{J}} g_\beta$$

is a Riemannian metric on M. $\square$

We will now introduce the notion of isometries of a given Riemannian manifold. These play a central role in differential geometry.

Definition 5.17. A map $\phi : (M, g) \to (N, h)$ between Riemannian manifolds is said to be **conformal** if there exists a function $\lambda : M \to \mathbb{R}$ such that

$$e^{\lambda(p)} \cdot g_p(X_p, Y_p) = h_{\phi(p)}(d\phi_p(X_p), d\phi_p(Y_p)),$$

for all $X, Y \in C^\infty(TM)$ and $p \in M$. The positive real-valued function e^λ is called the **conformal factor** of ϕ. A conformal map with $\lambda \equiv 0$ i.e. $e^\lambda \equiv 1$ is said to be **isometric**. An isometric diffeomorphism is called an **isometry**.

It is interesting that the isometries of a given Riemannian manifold actually form a group.

Definition 5.18. For a Riemannian manifold (M, g) we denote by $\text{Iso}(M)$ the set of its isometries. If $\phi, \psi \in \text{Iso}(M)$ then it is clear that the composition $\psi \circ \phi$ and the inverse ϕ^{-1} are also isometries. The operation is clearly associative and the identity map is its neutral element. The pair $(\text{Iso}(M), \circ)$ is called the **isometry group** of (M, g).

Remark 5.19. It can be shown that the isometry group $(\text{Iso}(M), \circ)$ of a Riemannian manifold (M, g) has the structure of a Lie group. For this see: R. S. Palais, *On the differentiability of isometries*, Proc. Amer. Math. Soc. **8** (1957), 805-807.

We next introduce the notion of a Riemannian homogeneous space. The classical reference for this important class of manifolds is: S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, Vol. II, John Wiley & Sons (1969).

Definition 5.20. The isometry group $\text{Iso}(M)$ of a Riemannian manifold (M, g) is said to be **transitive** if for all $p, q \in M$ there exists an isometry $\phi_{pq} : M \to M$ such that $\phi_{pq}(p) = q$. In that case (M, g) is called a **Riemannian homogeneous space**.

An important subclass of Riemannian homogeneous spaces is that of symmetric spaces introduced in Definition 7.31.

Example 5.21. Let S^m be the unit sphere in the $(m+1)$-dimensional Euclidean space $\mathbb{E}^{m+1}$. Then we have a natural action $\alpha : \mathbf{SO}(m+1) \times S^m \to S^m$ of the special orthogonal group $\mathbf{SO}(m+1)$ on S^m given by

$$\alpha : (x, p) \mapsto x \cdot p,$$

where $\cdot$ is the standard matrix multiplication. The following shows that this action is isometric

$$\langle x \cdot X, x \cdot Y \rangle = X^t x^t x Y = X^t Y = \langle X, Y \rangle.$$

This means that the special orthogonal group $\mathbf{SO}(m+1)$ is a subgroup of the isometry group $\mathrm{Iso}(S^m)$. The full isometry group $\mathrm{Iso}(S^m)$ of the unit sphere is the orthogonal group $\mathbf{O}(m+1)$. It is easily seen that $\mathbf{SO}(m+1)$ acts transitively on the sphere S^m so this is a Riemannian homogeneous space.

Example 5.22. The standard Euclidean scalar product on the real vector space $\mathbb{R}^{m\times m}$ induces a Riemannian metric on the special orthogonal group $\mathbf{SO}(m)$ given by

$$g(X,Y) = \mathrm{trace}(X^t \cdot Y).$$

Applying the left translation $L_p : \mathbf{SO}(m) \to \mathbf{SO}(m)$, with $L_p : q \mapsto pq$, we see that the tangent space $T_p\mathbf{SO}(m)$ of $\mathbf{SO}(m)$ at p is simply

$$T_p\mathbf{SO}(m) = \{p \cdot X \mid X^t + X = 0\}.$$

The differential $(dL_p)_q : T_q\mathbf{SO}(m) \to T_{pq}\mathbf{SO}(m)$ of L_p at $q \in \mathbf{SO}(m)$ satisfies

$$(dL_p)_q : qX \mapsto pqX.$$

We then have

$$\begin{aligned}
g_{pq}((dL_p)_q(qX),(dL_p)_q(qY)) &= \mathrm{trace}((pqX)^t pqY) \\
&= \mathrm{trace}(X^t q^t p^t pqY) \\
&= \mathrm{trace}(qX)^t(qY). \\
&= g_q(qX,qY).
\end{aligned}$$

This shows that the left translation $L_p : \mathbf{SO}(m) \to \mathbf{SO}(m)$ is an isometry for all $p \in \mathbf{SO}(m)$.

We next introduce the important notion of a left-invariant metric on a Lie group.

Definition 5.23. A Riemannian metric g on a Lie group G is said to be **left-invariant** if for each $p \in G$ the left translation $L_p : G \to G$ is an isometry. A Lie group (G,g) equipped with a left-invariant metric is called a **Riemannian Lie group**.

Remark 5.24. It should be noted that if (G,g) is a Riemannian Lie group and $X,Y \in \mathfrak{g}$ are left-invariant vector fields on G then

$$g_p(X_p,Y_p) = g_p((dL_p)_e(X_e),(dL_p)_e(Y_e)) = g_e(X_e,Y_e).$$

This tells us that a left-invariant metric g on G is completely determined by the scalar product $g_e : T_eG \times T_eG \to \mathbb{R}$ on the tangent space at the neutral element $e \in G$.

Theorem 5.25. *A Riemannian Lie group (G,g) is a Riemannian homogeneous space.*

Proof. For arbitrary elements $p, q \in G$ the left-translation $\phi_{pq} = L_{qp^{-1}}$ by $qp^{-1} \in G$ is an isometry satisfying $\phi_{pq}(p) = q$. This shows that the isometry group $\mathrm{Iso}(G)$ is transitive. $\square$

In Example 2.6 we have introduced the real projective space $\mathbb{R}P^m$ as an abstract differentiable manifold. We will now equip this with a natural Riemannian metric.

Example 5.26. Let S^m be the unit sphere in $\mathbb{E}^{m+1}$ and $\mathrm{Sym}(\mathbb{R}^{m+1})$ be the vector space of real symmetric $(m+1) \times (m+1)$ matrices equipped with the Riemannian metric g given by

$$g(X, Y) = \frac{1}{8} \cdot \mathrm{trace}(X^t \cdot Y).$$

As in Example 3.26, we define the immersion $\phi : S^m \to \mathrm{Sym}(\mathbb{R}^{m+1})$ by

$$\phi : p \mapsto (R_p : q \mapsto 2\langle q, p\rangle p - q).$$

This maps a point $p \in S^m$ to the reflection $R_p : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ about the real line ℓ_p generated by p. This is clearly a symmetric bijective linear map.

Let $\alpha, \beta : \mathbb{R} \to S^m$ be two curves meeting at a point $p \in S^m$ i.e. $\alpha(0) = p = \beta(0)$ and put $X = \dot\alpha(0)$, $Y = \dot\beta(0)$. Then for a curve $\gamma \in \{\alpha, \beta\}$ we have

$$d\phi_p(\dot\gamma(0)) = (q \mapsto 2\langle q, \dot\gamma(0)\rangle p + 2\langle q, p\rangle \dot\gamma(0)).$$

If $\mathcal{B}$ is an orthonormal basis for $\mathbb{R}^{m+1}$, then

$$\begin{aligned}
g(d\phi_p(X), d\phi_p(Y)) &= \frac{1}{8} \cdot \mathrm{trace}(d\phi_p(X)^t \cdot d\phi_p(Y)) \\
&= \frac{1}{8} \sum_{q \in \mathcal{B}} \langle q, d\phi_p(X)^t \cdot d\phi_p(Y) q\rangle \\
&= \frac{1}{8} \sum_{q \in \mathcal{B}} \langle d\phi_p(X) q, d\phi_p(Y) q\rangle \\
&= \frac{1}{2} \sum_{q \in \mathcal{B}} \langle \langle q, X\rangle p + \langle q, p\rangle X, \langle q, Y\rangle p + \langle q, p\rangle Y\rangle \\
&= \frac{1}{2} \sum_{q \in \mathcal{B}} \{\langle p, p\rangle \langle X, q\rangle \langle q, Y\rangle + \langle X, Y\rangle \langle p, q\rangle \langle p, q\rangle\} \\
&= \frac{1}{2} \{\langle X, Y\rangle + \langle X, Y\rangle\} \\
&= \langle X, Y\rangle.
\end{aligned}$$

This proves that the immersion $\phi : S^m \to \mathrm{Sym}(\mathbb{R}^{m+1})$ is isometric. In Example 3.26 we have seen that the image $\phi(S^m)$ can be identified with the real projective space $\mathbb{R}P^m$. This inherits the induced metric from $\mathrm{Sym}(\mathbb{R}^{m+1})$. The map $\phi : S^m \to \mathbb{R}P^m$ is what is called an isometric **double cover** of $\mathbb{R}P^m$.

Proposition 5.27. *Let $\mathbb{R}P^2$ be the two dimensional real projective plane equipped with the Riemannian metric introduced in Example 5.26. Then the surface area of $\mathbb{R}P^2$ is 2π.*

PROOF. Example 5.26 shows that if m is a positive integer then the map $\phi : S^m \to \mathbb{R}P^m$ is an isometric double cover. Hence this is locally volume preserving. This implies that the m-dimensional volume satisfies

$$\mathrm{vol}(S^m) = 2 \cdot \mathrm{vol}(\mathbb{R}P^m).$$

In particular,

$$\mathrm{area}(\mathbb{R}P^2) = \tfrac{1}{2} \cdot \mathrm{area}(S^2) = 2\pi.$$

□

Long before John Nash became famous in Hollywood he proved the next remarkable result in his paper: J. Nash, *The imbedding problem for Riemannian manifolds*, Ann. Math. **63** (1956), 20-63. It implies that every Riemannian manifold can be realised as a submanifold of a Euclidean space. The original proof of Nash has later been simplified, see for example: M. Günther, *On the perturbation problem associated to isometric embeddings of Riemannian manifolds*, Ann. Global Anal. Geom. **7** (1989), 69-77.

Deep Result 5.28. *For $3 \leq r \leq \infty$, let (M, g) be a Riemannian C^r-manifold. Then there exists an isometric C^r-embedding of (M, g) into a Euclidean space $\mathbb{R}^n$. If the manifold (M, g) is compact then $n \leq m(m+1)$ but $n \leq (3m+11)/2$ otherwise.*

Remark 5.29. Note that in Example 5.26 we have embedded the compact Riemannian manifold $\mathbb{R}P^m$ isometrically into the Euclidean space $\mathrm{Sym}(\mathbb{R}^{m+1})$ of dimension $(m+2)(m+1)/2$.

Remark 5.30. We will now see that local parametrisations are very useful tools for studying the intrinsic geometry of a Riemannian manifold (M, g). Let p be a point on M and $\hat{\psi} : U \to M$ be a local parametrisation of M with $q \in U$ and $\hat{\psi}(q) = p$. The differential $d\hat{\psi}_q : T_q\mathbb{R}^m \to T_pM$ is bijective so, following the inverse mapping theorem, there exist neighbourhoods U_q of q and U_p of p such that the restriction $\psi = \hat{\psi}|_{U_q} : U_q \to U_p$ is a diffeomorphism. On U_q we have the canonical frame $\{e_1, \ldots, e_m\}$ for TU_q so $\{d\psi(e_1), \ldots, d\psi(e_m)\}$ is a local

frame for TM over U_p. We then define the pull-back metric $\tilde{g} = \psi^* g$ on U_q by
$$\tilde{g}(e_k, e_l) = g(d\psi(e_k), d\psi(e_l)).$$
Then $\psi : (U_q, \tilde{g}) \to (U_p, g)$ is an isometry so the intrinsic geometry of $(U_q, \tilde{g})$ and that of (U_p, g) are exactly the same.

Example 5.31. Let G be a classical Lie group and e be the neutral element of G. Let $\{X_1, \dots, X_m\}$ be a basis for the Lie algebra $\mathfrak{g}$ of G. For $p \in G$ define $\psi_p : \mathbb{R}^m \to G$ by
$$\psi_p : (t_1, \dots, t_m) \mapsto L_p(\prod_{k=1}^{m} \mathrm{Exp}(t_k X_k(e)))$$
where $L_p : G \to G$ is the left translation given by $L_p(q) = pq$. Then
$$(d\psi_p)_0(e_k) = X_k(p)$$
for all k. This means that the differential $(d\psi_p)_0 : T_0\mathbb{R}^m \to T_pG$ is an isomorphism so there exist open neighbourhoods U_0 of 0 and U_p of p such that the restriction of ψ to U_0 is bijective onto its image U_p and hence a local parametrisation of G around p.

The following idea will later turn out to be very useful. It provides us with the existence of a local orthonormal frame of the tangent bundle of a Riemannian manifold.

Example 5.32. Let (M, g) be a Riemannian manifold and (U, x) be a local chart on M. Then it follows from Proposition 3.19 that the set
$$\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_m}\}$$
of local vector fields is a frame for the tangent bundle TM on the open subset U of M. Then the Gram-Schmidt process produces a local orthonormal frame
$$\{E_1, E_2, \dots, E_m\}$$
of TM on U.

We will now study the normal bundle of a submanifold of a given Riemannian manifold. This is an important example of the notion of a vector bundle over a manifold, see Definition 4.2.

Definition 5.33. Let (N, h) be a Riemannian manifold and (M, g) be a smooth submanifold equipped with the induced metric. For a point $p \in M$ we define the **normal space** N_pM of M at p by
$$N_pM = \{X \in T_pN \mid h_p(X, Y) = 0 \ \text{ for all } \ Y \in T_pM\}.$$

For all $p \in M$ we have the orthogonal decomposition

$$T_pN = T_pM \oplus N_pM.$$

The **normal bundle** of M in N is defined by

$$NM = \{(p, X) \,|\, p \in M \text{ and } X \in N_pM\}.$$

Theorem 5.34. *Let (N, h) be a Riemannian manifold and (M, g) be a smooth submanifold equipped with the induced metric. Then the normal bundle (NM, M, π) is a smooth vector bundle over M of dimension $(n - m)$.*

Proof. See Exercise 5.6. □

Example 5.35. Let S^m be the unit sphere in $\mathbb{R}^{m+1}$ equipped with its standard Euclidean metric $\langle , \rangle$. If $p \in S^m$ then the tangent space T_pS^m of S^m at p is

$$T_pS^m = \{X \in \mathbb{R}^{m+1} \,|\, \langle p, X\rangle = 0\},$$

so the normal space N_pS^m of S^m at p satisfies

$$N_pS^m = \{\lambda \cdot p \in \mathbb{R}^{m+1} \,|\, \lambda \in \mathbb{R}\}.$$

This shows that the normal bundle NS^m of S^m in $\mathbb{R}^{m+1}$ is given by

$$NS^m = \{(p, \lambda \cdot p) \in \mathbb{R}^{2m+2} \,|\, p \in S^m \text{ and } \lambda \in \mathbb{R}\}.$$

We will now determine the normal bundle $N\mathbf{SO}(m)$ of the special orthogonal group $\mathbf{SO}(m)$ as a submanifold of $\mathbb{R}^{m\times m}$.

Example 5.36. Let the linear space $\mathbb{R}^{m\times m}$ of real $m \times m$ matrices be equipped with its standard Euclidean scalar product satisfying

$$g(X, Y) = \operatorname{trace}(X^tY).$$

Then we have a natural action $\alpha : \mathbf{SO}(m) \times \mathbb{R}^{m\times m} \to \mathbb{R}^{m\times m}$ of the orthogonal group $\mathbf{SO}(m)$ on $\mathbb{R}^{m\times m}$ given by

$$\alpha : (p, x) \mapsto L_p(x) = p \cdot x.$$

Then for any point $p \in \mathbf{SO}(m)$ and tangent vectors $X, Y \in \mathbb{R}^{m\times m}$ it follows that

$$\begin{aligned} g(pX, pY) &= \operatorname{trace}((pX)^t(pY)) \\ &= \operatorname{trace}(X^tp^tpY) \\ &= \operatorname{trace}(X^tY) \\ &= g(X, Y). \end{aligned}$$

This tells us that this action of $\mathbf{SO}(m)$ on $\mathbb{R}^{m\times m}$ is isometric.

As we have already seen in Example 3.10 the tangent space $T_e\mathbf{SO}(m)$ of $\mathbf{SO}(m)$ at the neutral element e satisfies

$$T_e\mathbf{SO}(m) = \{X \in \mathbb{R}^{m\times m} \,|\, X^t + X = 0\}.$$

This means that the tangent bundle $T\mathbf{SO}(m)$ of $\mathbf{SO}(m)$ is given by

$$T\mathbf{SO}(m) = \{(p, pX) \,|\, p \in \mathbf{SO}(m) \text{ and } X \in T_e\mathbf{SO}(m)\}.$$

The real vector space $\mathbb{R}^{m\times m}$ has a natural linear decomposition

$$\mathbb{R}^{m\times m} = \mathrm{Sym}(\mathbb{R}^m) \oplus T_e\mathbf{SO}(m),$$

where every element $X \in \mathbb{R}^{m\times m}$ can be decomposed $X = X^\top + X^\perp$ into its skew-symmetric and symmetric parts given by

$$X^\top = \frac{1}{2}(X - X^t) \quad \text{and} \quad X^\perp = \frac{1}{2}(X + X^t).$$

If $X \in T_e\mathbf{SO}(m)$ and $Y \in \mathrm{Sym}(\mathbb{R}^m)$ then

$$\begin{aligned} g(X,Y) &= \mathrm{trace}(X^tY) \\ &= \mathrm{trace}(Y^tX) \\ &= \mathrm{trace}(XY^t) \\ &= \mathrm{trace}(-X^tY) \\ &= -g(X,Y). \end{aligned}$$

This shows that $g(X,Y) = 0$ so the normal space $N_e\mathbf{SO}(m)$, of $\mathbf{SO}(m)$ in $\mathbb{R}^{m\times m}$ at the neutral element e, satisfies

$$N_e\mathbf{SO}(m) = \mathrm{Sym}(\mathbb{R}^m).$$

This means that in this situation the normal bundle $N\mathbf{SO}(m)$ of $\mathbf{SO}(m)$ is given by

$$N\mathbf{SO}(m) = \{(p, pY) \,|\, p \in \mathbf{SO}(m) \text{ and } Y \in \mathrm{Sym}(\mathbb{R}^m)\}.$$

A Riemannian metric g on a differentiable manifold M can be used to construct families of natural metrics on the tangent bundle TM of M. The best known such examples are the Sasaki and Cheeger-Gromoll metrics. For a detailed survey on the geometry of tangent bundles equipped with these metrics we recommend the paper: S. Gudmundsson, E. Kappos, *On the geometry of tangent bundles*, Expo. Math. **20** (2002), 1-41.

Exercises

Exercise 5.1. Let $\mathbb{R}^m$ and $\mathbb{C}^m$ be equipped with their standard Euclidean metrics given by

$$g(z, w) = \mathrm{Re}\sum_{k=1}^{m} z_k \bar{w}_k$$

and let

$$T^m = \{z \in \mathbb{C}^m \,|\, |z_1| = ... = |z_m| = 1\}$$

be the m-dimensional torus in $\mathbb{C}^m$ with the induced metric. Let $\phi : \mathbb{R}^m \to T^m$ be the standard parametrisation of the m-dimensional torus in $\mathbb{C}^m$ satisfying $\phi : (x_1, \dots, x_m) \mapsto (e^{ix_1}, \dots, e^{ix_m})$. Show that ϕ is isometric.

Exercise 5.2. The **stereographic projection** from the north pole of the m-dimensional sphere

$$\phi : (S^m \setminus \{(1, 0, \dots, 0)\}, \langle, \rangle) \to (\mathbb{R}^m, \frac{4}{(1 + |x|^2)^2} \cdot \langle, \rangle)$$

is given by

$$\phi : (x_0, \dots, x_m) \mapsto \frac{1}{1 - x_0} \cdot (x_1, \dots, x_m).$$

Show that ϕ is an isometry.

Exercise 5.3. Let $B_1^2(0)$ be the open unit disk in the complex plane equipped with the hyperbolic metric

$$g(X, Y) = \frac{4}{(1 - |z|^2)^2} \cdot \langle X, Y\rangle.$$

Equip the upper half plane $\{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$ with the Riemannian metric

$$g(X, Y) = \frac{1}{\mathrm{Im}(z)^2} \cdot \langle X, Y\rangle.$$

Prove that the holomorphic function $f : B_1^2(0) \to \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$ given by

$$f : z \mapsto \frac{i + z}{1 + iz}$$

is an isometry.

Exercise 5.4. Equip the unitary group $\mathbf{U}(m)$ with the Riemannian metric g given by

$$g(Z, W) = \mathrm{Re}(\mathrm{trace}(\bar{Z}^t \cdot W)).$$

Show that for each $p \in \mathbf{U}(m)$ the left translation $L_p : \mathbf{U}(m) \to \mathbf{U}(m)$ is an isometry.

Exercise 5.5. For the general linear group $\mathbf{GL}_m(\mathbb{R})$ we have two Riemannian metrics g and h satisfying

$$g_p(pZ, pW) = \mathrm{trace}((pZ)^t \cdot pW), \quad h_p(pZ, pW) = \mathrm{trace}(Z^t \cdot W).$$

Further let $\hat{g}, \hat{h}$ be their induced metrics on the special linear group $\mathbf{SL}_m(\mathbb{R})$ as a subset of $\mathbf{GL}_m(\mathbb{R})$.

(i) Which of the metrics $g, h, \hat{g}, \hat{h}$ are left-invariant ?
(ii) Determine the normal space $N_e\mathbf{SL}_m(\mathbb{R})$ of $\mathbf{SL}_m(\mathbb{R})$ in $\mathbf{GL}_m(\mathbb{R})$ with respect to g
(iii) Determine the normal bundle $N\mathbf{SL}_m(\mathbb{R})$ of $\mathbf{SL}_m(\mathbb{R})$ in $\mathbf{GL}_m(\mathbb{R})$ with respect to h.

Exercise 5.6. Find a proof of Theorem 5.34. (Hint: Use Example 5.32).

Exercise 5.7. Equip the tangent space $T_e\mathbf{SL}_2(\mathbb{R})$, of the special linear group $\mathbf{SL}_2(\mathbb{R})$ at the neutral element e, with the scalar product

$$\langle A, B \rangle = \frac{1}{2}\,\mathrm{trace}\,(A^t \cdot B).$$

Show that $\{X, Y, Z\}$ is an orthonornal basis for $T_e\mathbf{SL}_2(\mathbb{R})$, where

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Exercise 5.8. Equip the tangent space $T_e\mathbf{SU}(2)$, of the special unitary group $\mathbf{SU}(2)$ at the neutral element e, with the scalar product

$$\langle A, B \rangle = \frac{1}{2}\,\mathrm{Re}\,\mathrm{trace}\,(\bar{A}^t \cdot B).$$

Show that $\{X, Y, Z\}$ is an orthonornal basis for $T_e\mathbf{SU}(2)$, where

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

CHAPTER 6

The Levi-Civita Connection

In this chapter we introduce the Levi-Civita connection on the tangent bundle of a Riemannian manifold. This is the most important example of the general notion of a connection on a smooth vector bundle.

We deduce the explicit Koszul formula for the Levi-Civita connection and show how this simplifies in the important cases of Riemannian Lie groups. We also give an example of a metric connection on the normal bundle of a submanifold of a Riemannian manifold and study its properties.

On the m-dimensional Euclidean vector space $\mathbb{R}^m$ we have the well known differential operator

$$\partial : C^\infty(T\mathbb{R}^m) \times C^\infty(T\mathbb{R}^m) \to C^\infty(T\mathbb{R}^m)$$

on the tangent bundle $T\mathbb{R}^m$. This maps a pair of vector fields X, Y on $\mathbb{R}^m$ to the classical **directional derivative** $\partial_X Y$ of Y in the direction of X given by

$$(\partial_X Y)(x) = \lim_{t \to 0} \frac{Y(x + t \cdot X(x)) - Y(x)}{t}.$$

The best known fundamental properties of the operator ∂ are expressed by the following: If $\lambda, \mu \in \mathbb{R}$, $f, g \in C^\infty(\mathbb{R}^m)$ and $X, Y, Z \in C^\infty(T\mathbb{R}^m)$ then

(i) $\partial_X(\lambda \cdot Y + \mu \cdot Z) = \lambda \cdot \partial_X Y + \mu \cdot \partial_X Z$,
(ii) $\partial_X(f \cdot Y) = X(f) \cdot Y + f \cdot \partial_X Y$,
(iii) $\partial_{(f \cdot X + g \cdot Y)} Z = f \cdot \partial_X Z + g \cdot \partial_Y Z$.

The next result shows that the classical differential operator ∂ is compatible with both the standard differentiable structure on $\mathbb{R}^m$ and its Euclidean metric.

Proposition 6.1. *Let the real vector space $\mathbb{R}^m$ be equipped with the standard Euclidean metric $\langle , \rangle$ and $X, Y, Z \in C^\infty(T\mathbb{R}^m)$ be smooth vector fields on $\mathbb{R}^m$. Then*

(iv) $\partial_X Y - \partial_Y X = [X, Y]$,

(v) $X(\langle Y, Z\rangle) = \langle \partial_X Y, Z\rangle + \langle Y, \partial_X Z\rangle$.

Our principal aim is now to generalise the differential operator ∂, on the classical Euclidean space $E^m = (\mathbb{R}^m, <,>)$, to the so called Levi-Civita connection ∇ on a general Riemannian manifold (M, g).

In this important process, we first introduce the general concept of a connection on a smooth vector bundle, see Definition 4.8.

Definition 6.2. Let M be a smooth manifold and (E, M, π) be a smooth vector bundle over M. Then a **connection** $\hat{\nabla}$ on (E, M, π) is an operator

$$\hat{\nabla} : C^\infty(TM) \times C^\infty(E) \to C^\infty(E),$$

such that for all $\lambda, \mu \in \mathbb{R}$, $f, g \in C^\infty(M)$, $X, Y \in C^\infty(TM)$ and smooth sections $v, w \in C^\infty(E)$, we have

(i) $\hat{\nabla}_X(\lambda \cdot v + \mu \cdot w) = \lambda \cdot \hat{\nabla}_X v + \mu \cdot \hat{\nabla}_X w$,
(ii) $\hat{\nabla}_X(f \cdot v) = X(f) \cdot v + f \cdot \hat{\nabla}_X v$,
(iii) $\hat{\nabla}_{(f \cdot X + g \cdot Y)} v = f \cdot \hat{\nabla}_X v + g \cdot \hat{\nabla}_Y v$.

A smooth section $v \in C^\infty(E)$ of the vector bundle (E, M, π) is said to be **parallel** with respect to the connection $\hat{\nabla}$ if and only if, for all vector fields $X \in C^\infty(TM)$, we have

$$\hat{\nabla}_X v = 0.$$

In the special important case when the vector bundle, over a differentiable manifold, is the tangent bundle we have the following notion of torsion. It should be noted that here we are not assuming that the manifold is equipped with a Riemannian metric.

Definition 6.3. Let M be a smooth manifold and $\hat{\nabla}$ be a connection on the tangent bundle (TM, M, π). Then we define its **torsion**

$$T : C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM)$$

by

$$T(X, Y) = \hat{\nabla}_X Y - \hat{\nabla}_Y X - [X, Y],$$

where $[,]$ is the Lie bracket on $C^\infty(TM)$. The connection $\hat{\nabla}$ is said to be **torsion-free** if its torsion T vanishes i.e. if for all $X, Y \in C^\infty(TM)$, we have

$$[X, Y] = \hat{\nabla}_X Y - \hat{\nabla}_Y X.$$

For the tangent bundle of a Riemannian manifold we have the following natural notion.

Definition 6.4. Let (M, g) be a Riemannian manifold. Then a connection $\hat{\nabla}$ on the tangent bundle (TM, M, π) is said to be **metric**, or compatible with the Riemannian metric g, if for all $X, Y, Z \in C^\infty(TM)$

$$X(g(Y,Z)) = g(\hat{\nabla}_X Y, Z) + g(Y, \hat{\nabla}_X Z).$$

The following turns out to be very important for what follows.

Observation 6.5. Let (M, g) be a Riemannian manifold and ∇ be a metric and torsion-free connection on its tangent bundle (TM, M, π). Then it is easily seen that the following equations hold

$$g(\nabla_X Y, Z) = X(g(Y,Z)) - g(Y, \nabla_X Z),$$

$$\begin{aligned} g(\nabla_X Y, Z) &= g([X,Y], Z) + g(\nabla_Y X, Z) \\ &= g([X,Y], Z) + Y(g(X,Z)) - g(X, \nabla_Y Z), \end{aligned}$$

$$\begin{aligned} 0 &= -Z(g(X,Y)) + g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \\ &= -Z(g(X,Y)) + g(\nabla_X Z + [Z,X], Y) + g(X, \nabla_Y Z - [Y,Z]). \end{aligned}$$

When adding these relations we yield the following so called **Koszul formula** for the operator ∇

$$\begin{aligned} 2 \cdot g(\nabla_X Y, Z) = \{&X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ &+ g(Z, [X,Y]) + g([Z,X], Y) + g([Z,Y], X)\}. \end{aligned}$$

If $\{E_1, \dots, E_m\}$ is a local orthonormal frame for the tangent bundle, see Example 5.32, then

$$\nabla_X Y = \sum_{i=1}^m g(\nabla_X Y, E_i) E_i.$$

It follows from the Koszul formula that the coefficients in this sum are uniquely determined by the Lie bracket $[,]$ and the Riemannian metric g. This sum is also independent of the chosen local orthonormal frame. As a direct consequence we see that there exists **at most** one torsion-free and metric connection on the tangent bundle of (M, g).

This leads us to the following natural definition of the all important Levi-Civita connection.

Definition 6.6. Let (M, g) be a Riemannian manifold then the operator

$$\nabla : C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM)$$

given by

$$g(\nabla_X Y, Z) = \tfrac{1}{2}\{X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y))$$

$$+g(Z,[X,Y])+g([Z,X],Y)+g([Z,Y],X)\}$$

is called the **Levi-Civita connection** on M.

Remark 6.7. It is very important to note that the Levi-Civita connection is an intrinsic object on (M,g) i.e. only depending on the differentiable structure of the manifold and its Riemannian metric.

Proposition 6.8. *Let (M,g) be a Riemannian manifold. Then the Levi-Civita connection ∇ is a connection on the tangent bundle (TM,M,π).*

Proof. It follows from Definition 3.6, Theorem 4.22 and the fact that g is a tensor field that

$$g(\nabla_X(\lambda\cdot Y_1+\mu\cdot Y_2),Z)=\lambda\cdot g(\nabla_X Y_1,Z)+\mu\cdot g(\nabla_X Y_2,Z)$$

and that

$$g(\nabla_{Y_1+Y_2}X,Z)=g(\nabla_{Y_1}X,Z)+g(\nabla_{Y_2}X,Z)$$

for all $\lambda,\mu\in\mathbb{R}$ and $X,Y_1,Y_2,Z\in C^\infty(TM)$. Furthermore, for all $f\in C^\infty(M)$, we have

$$\begin{aligned}
&2\cdot g(\nabla_X fY,Z)\\
=\ &\{X(f\cdot g(Y,Z))+f\cdot Y(g(X,Z))-Z(f\cdot g(X,Y))\\
&+f\cdot g([Z,X],Y)+g([Z,f\cdot Y],X)+g(Z,[X,f\cdot Y])\}\\
=\ &\{X(f)\cdot g(Y,Z)+f\cdot X(g(Y,Z))+f\cdot Y(g(X,Z))\\
&-Z(f)\cdot g(X,Y)-f\cdot Z(g(X,Y))+f\cdot g([Z,X],Y)\\
&+g(Z(f)\cdot Y+f\cdot[Z,Y],X)+g(Z,X(f)\cdot Y+f\cdot[X,Y])\}\\
=\ &2\cdot\{X(f)\cdot g(Y,Z)+f\cdot g(\nabla_X Y,Z)\}\\
=\ &2\cdot g(X(f)\cdot Y+f\cdot\nabla_X Y,Z)
\end{aligned}$$

and

$$\begin{aligned}
&2\cdot g(\nabla_{f\cdot X}Y,Z)\\
=\ &\{f\cdot X(g(Y,Z))+Y(f\cdot g(X,Z))-Z(f\cdot g(X,Y))\\
&+g([Z,f\cdot X],Y)+f\cdot g([Z,Y],X)+g(Z,[f\cdot X,Y])\}\\
=\ &\{f\cdot X(g(Y,Z))+Y(f)\cdot g(X,Z)+f\cdot Y(g(X,Z))\\
&-Z(f)\cdot g(X,Y)-f\cdot Z(g(X,Y))\\
&+g(Z(f)\cdot X,Y)+f\cdot g([Z,X],Y)\\
&+f\cdot g([Z,Y],X)+f\cdot g(Z,[X,Y])-g(Z,Y(f)\cdot X)\}\\
=\ &2\cdot f\cdot g(\nabla_X Y,Z).
\end{aligned}$$

This proves that ∇ is a connection on the tangent bundle (TM, M, π). $\square$

The next result is generally called the **Fundamental Theorem of Riemannian Geometry**.

Theorem 6.9. *Let (M, g) be a Riemannian manifold. Then the Levi-Civita connection is the unique metric and torsion-free connection on the tangent bundle (TM, M, π).*

PROOF. The difference $g(\nabla_X Y, Z) - g(\nabla_Y X, Z)$ equals twice the skew-symmetric part (w.r.t the pair (X, Y)) of the right hand side of the equation in Definition 6.6. This implies that

$$\begin{aligned} g(\nabla_X Y, Z) - g(\nabla_Y X, Z) &= \tfrac{1}{2}\{g(Z, [X, Y]) - g(Z, [Y, X])\} \\ &= g([X, Y], Z). \end{aligned}$$

This proves that the Levi-Civita connection is torsion-free.

The sum $g(\nabla_X Y, Z) + g(\nabla_X Z, Y)$ equals twice the symmetric part (w.r.t the pair (Y, Z)) on the right hand side of Definition 6.6. This yields

$$\begin{aligned} g(\nabla_X Y, Z) + g(Y, \nabla_X Z) &= \tfrac{1}{2}\{X(g(Y, Z)) + X(g(Z, Y))\} \\ &= X(g(Y, Z)). \end{aligned}$$

This shows that the Levi-Civita connection is compatible with the Riemannian metric g on M. The stated result follows now immediately from Proposition 6.8. $\square$

From Lie theory, we have the following important notion of the adjoint representation of a Lie algebra.

Definition 6.10. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then the **adjoint representation** of $\mathfrak{g}$ is the linear operator $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ mapping an element $Z \in \mathfrak{g}$ onto the linear endomorphism $\mathrm{ad}_Z : \mathfrak{g} \to \mathfrak{g}$ with

$$\mathrm{ad}_Z : X \mapsto [Z, X].$$

For the compact classical Lie groups, introduced in Chapter 2, we have the following interesting result.

Proposition 6.11. *Let G be one of the classical compact Lie groups $\mathbf{O}(m)$, $\mathbf{SO}(m)$, $\mathbf{U}(m)$ or $\mathbf{SU}(m)$, equipped with its left-invariant Riemannian metric given by*

$$g(X, Y) = \mathrm{Re}(\mathrm{trace}(\bar{X}^t \cdot Y)).$$

If $Z \in \mathfrak{g}$ is a left-invariant vector field on G then the linear endomorphism $\mathrm{ad}_Z : \mathfrak{g} \to \mathfrak{g}$ is skew-symmetric i.e. for all $X, Y \in \mathfrak{g}$ we have

$$g(\mathrm{ad}_Z(X), Y) + g(X, \mathrm{ad}_Z(Y)) = 0.$$

PROOF. See Exercise 6.2. $\square$

The following result shows that the Koszul formula simplifies considerably in the important case when the manifold is a Riemannian Lie group.

Proposition 6.12. *Let (G, g) be a Lie group equipped with a left-invariant metric and $X, Y, Z \in \mathfrak{g}$ be left-invariant vector fields on G. Then its Levi-Civita connection ∇ satisfies*

$$g(\nabla_X Y, Z) = \tfrac{1}{2}\{g(Z, [X, Y]) + g(\mathrm{ad}_Z(X), Y) + g(X, \mathrm{ad}_Z(Y))\}.$$

In particular, if for all $Z \in \mathfrak{g}$ the linear endomorphism $\mathrm{ad}_Z : \mathfrak{g} \to \mathfrak{g}$ is skew-symmetric with respect to the Riemannian metric g, then

$$\nabla_X Y = \tfrac{1}{2}[X, Y].$$

PROOF. See Exercise 6.3. $\square$

The next example shows how the Levi-Civita connection can be presented by means of local coordinates. Hopefully, this will convince the reader that those should be avoided whenever possible.

Example 6.13. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. Further let (U, x) be a local chart on M and put $X_i = \partial/\partial x_i \in C^\infty(TU)$, so $\{X_1, \dots, X_m\}$ is a local frame for TM on U. Then we define the **Christoffel symbols** $\Gamma_{ij}^k : U \to \mathbb{R}$ of the connection ∇ with respect to (U, x) by

$$\nabla_{X_i} X_j = \sum_{k=1}^m \Gamma_{ij}^k \cdot X_k.$$

On the open subset $x(U)$ of $\mathbb{R}^m$ we define the Riemannian metric $\tilde{g}$ by

$$\tilde{g}(e_i, e_j) = g_{ij} = g(X_i, X_j).$$

This turns the diffeomorphism $x : U \to x(U)$ into an isometry, so that the local geometry of U with g and that of $x(U)$ with $\tilde{g}$ are precisely the same. The differential dx is bijective so Proposition 4.28 implies that

$$dx([X_i, X_j]) = [dx(X_i), dx(X_j)] = [\partial_{e_i}, \partial_{e_j}] = 0$$

and hence $[X_i, X_j] = 0$. It now follows from the definition of the Christoffel symbols and the Koszul formula that for each $l = 1, 2, \dots, m$ we have

$$\begin{aligned}
\sum_{k=1}^{m} g_{kl} \cdot \Gamma_{ij}^k &= \sum_{k=1}^{m} g(X_k, X_l) \cdot \Gamma_{ij}^k \\
&= g(\sum_{k=1}^{m} \Gamma_{ij}^k \cdot X_k, X_l) \\
&= g(\nabla_{X_i} X_j, X_l) \\
&= \tfrac{1}{2} \{X_i(g(X_j, X_l)) + X_j(g(X_l, X_i)) - X_l(g(X_i, X_j))\} \\
&= \tfrac{1}{2} \{\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{li}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l}\}.
\end{aligned}$$

This means that for each pair (i, j) we have a system of m linear equations in the m variables Γ_{ij}^k where $k = 1, 2, \dots, m$. Because the metric g is positive definite we can solve this as follows: Let $g^{kl} = (g^{-1})_{kl}$ be the components of the inverse g^{-1} of g then the Christoffel symbols Γ_{ij}^k satisfy

$$\Gamma_{ij}^k = \tfrac{1}{2} \sum_{l=1}^{m} g^{kl} \{\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{li}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l}\}.$$

We are now interested in the relation between the Levi-Civita connection of a Riemannian manifold and that of its submanifolds, see Theorem 6.20. For this we need the following natural notion of an extension.

Definition 6.14. Let (N, h) be a Riemannian manifold and (M, g) be a submanifold of N equipped with the induced metric. Further let $\tilde{X} \in C^\infty(TM)$ be a vector field on M and $\tilde{Y} \in C^\infty(NM)$ be a section of its normal bundle. Let U be an open subset of N such that $U \cap M \neq \emptyset$. Two vector fields $X, Y \in C^\infty(TU)$ are said to be **local extensions** of $\tilde{X}$ and $\tilde{Y}$ to U if $\tilde{X}_p = X_p$ and $\tilde{Y}_p = Y_p$ for all $p \in U \cap M$. If $U = N$ then X, Y are said to be **global extension** of $\tilde{X}$ and $\tilde{Y}$, respectively.

Fact 6.15. *Let (N, h) be a Riemannian manifold and (M, g) be a submanifold equipped of N with the induced metric, $\tilde{X} \in C^\infty(TM)$, $\tilde{Y} \in C^\infty(NM)$ and $p \in M$. Then there exists an open neighbourhood U of N containing p and $X, Y \in C^\infty(TU)$ extending $\tilde{X}$ and $\tilde{Y}$ on U, respectively.*

Remark 6.16. Let (N, h) be a Riemannian manifold and (M, g) be a submanifold of N equipped with the induced metric. Let $Z \in$

$C^\infty(TN)$ be a vector field on N and $\tilde{Z} = Z|_M : M \to TN$ be the restriction of Z to M. Note that $\tilde{Z}$ is not necessarily an element of $C^\infty(TM)$ i.e. a vector field on the submanifold M. For each $p \in M$ the tangent vector $\tilde{Z}_p \in T_pN$ has a unique orthogonal decomposition

$$\tilde{Z}_p = \tilde{Z}_p^\top + \tilde{Z}_p^\perp,$$

into its tangential part $\tilde{Z}_p^\top \in T_pM$ and its normal part $\tilde{Z}_p^\perp \in N_pM$. For this we write $\tilde{Z} = \tilde{Z}^\top + \tilde{Z}^\perp$.

Proposition 6.17. *Let (N, h) be a Riemannian manifold and (M, g) be a submanifold of N equipped with the induced metric. If $Z \in C^\infty(TN)$ is a vector field on N then the sections $\tilde{Z}^\top$ of the tangent bundle TM and $\tilde{Z}^\perp$ of the normal bundle NM are smooth.*

PROOF. See Exercise 6.8. □

The following important remark depends on a later observation. For pedagogical reasons we have chosen to first present the argument needed in Remark 7.3.

Remark 6.18. Let (N, h) be a Riemannian manifold and (M, g) be a submanifold of N equipped with the induced metric. Further let $\tilde{X}, \tilde{Y} \in C^\infty(TM)$ be vector fields on M and $X, Y \in C^\infty(TU)$ extend $\tilde{X}, \tilde{Y}$ on an open neighbourhood U of p in N. It will be shown in Remark 7.3 that $(\nabla_X Y)_p$ only depends on the value $X_p = \tilde{X}_p$ and the value of Y along **some** curve $\gamma : (-\epsilon, \epsilon) \to N$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = X_p = \tilde{X}_p$.

Since $X_p \in T_pM$ we may choose the curve γ such that the image $\gamma((-\epsilon, \epsilon))$ is contained in M. Then $\tilde{Y}_{\gamma(t)} = Y_{\gamma(t)}$ for $t \in (-\epsilon, \epsilon)$. This means that $(\nabla_X Y)_p$ only depends on $\tilde{X}_p$ and the value of $\tilde{Y} \in C^\infty(TM)$ along γ, hence independent of how the vector fields $\tilde{X}$ and $\tilde{Y}$ are extended.

Remark 6.18 shows that the following important operators $\tilde{\nabla}$ and B are well defined.

Definition 6.19. Let (N, h) be a Riemannian manifold and (M, g) be a submanifold of N with the induced metric. Then we define the operators

$$\tilde{\nabla} : C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM)$$

and

$$B : C^\infty(TM) \times C^\infty(TM) \to C^\infty(NM)$$

by

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = (\nabla_X Y)^\top \quad \text{and} \quad B(\tilde{X},\tilde{Y}) = (\nabla_X Y)^\perp.$$

Here X and Y are some local extensions of $\tilde{X},\tilde{Y} \in C^\infty(TM)$. The operator B is called the **second fundamental form** of M in (N,h).

The next result provides us with the important relationship between the Levi-Civita connection of a Riemannian manifold and that of its submanifolds.

Theorem 6.20. *Let (N,h) be a Riemannian manifold and (M,g) be a submanifold of N with the induced metric. Then the operator*

$$\tilde{\nabla} : C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM),$$

given by

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = (\nabla_X Y)^\top,$$

is the Levi-Civita connection of the submanifold (M,g).

Proof. See Exercise 6.9. $\square$

The important second fundamental form of a submanifold of a Riemannian manifold has the following important properties.

Proposition 6.21. *Let (N,h) be a Riemannian manifold and (M,g) be a submanifold of N with the induced metric. Then the second fundamental form B of M in N is symmetric and tensorial in both its arguments.*

Proof. See Exercise 6.10. $\square$

We now introduce the notion of a minimal submanifold of a Riemannian manifold.

Definition 6.22. Let (N,h) be a Riemannian manifold and (M,g) be a submanifold of N with the induced metric. Then M is said to be **minimal** in N if its second fundamental form

$$B : C^\infty(TM) \otimes C^\infty(TM) \to C^\infty(NM)$$

is **traceless** i.e.

$$\text{trace } B = \sum_{k=1}^{m} B(\tilde{E}_k,\tilde{E}_k) = 0.$$

Here $\{\tilde{E}_1,\tilde{E}_2,\dots,\tilde{E}_m\}$ is any local orthonormal frame for the tangent bundle TM.

In the next Example 6.23, we show how the second fundamental form of a surface in the Euclidean 3-space corresponds to the classical shape operator.

Example 6.23. Let us now consider the classical Gaussian situation of a regular surface Σ^2 as a submanifold of the three dimensional Euclidean space $\mathbb{R}^3$.

Let U be an open subset of Σ and $\{\tilde{Z}, \tilde{W}\}$ be a local orthonormal frame for the tangent bundle TU of U around a point $p \in U$ and $\tilde{N}$ be the local Gauss map with $\tilde{N} = \tilde{Z} \times \tilde{W}$. Further let Z, W, N be local extensions of $\tilde{Z}, \tilde{W}, \tilde{N}$, forming a local orthonormal frame for $T\mathbb{R}^3$.

Further, let $\tilde{X}, \tilde{Y}$ be local vector fields on $M \cap U$ around a point $p \in \Sigma$ and X, Y be some local extension of $\tilde{X}, \tilde{Y}$ to U. Then the second fundamental form B of Σ in $\mathbb{R}^3$ satisfies

$$\begin{aligned}
B(\tilde{X}, \tilde{Y}) &= (\partial_X Y)^\perp \\
&= < \partial_X Y, N > N \\
&= - < Y, \partial_X N > N \\
&= - < Y, dN(X) > N \\
&= < \tilde{Y}, S_p(\tilde{X}) > \tilde{N},
\end{aligned}$$

where $S_p : T_p\Sigma \to T_p\Sigma$ is the classical **shape operator** at p.

Then the trace of B satisfies

$$\begin{aligned}
\text{trace } B &= (< S_p(\tilde{Z}), \tilde{Z} > + < S_p(\tilde{W}), \tilde{W} >)\, \tilde{N} \\
&= (\text{trace } S_p)\, \tilde{N} \\
&= (k_1 + k_2)\, \tilde{N}.
\end{aligned}$$

Here k_1 and k_2 are the eigenvalues of the symmetric shape operator S_p i.e. the **principal curvatures** at p. This shows that the surface Σ is a minimal submanifold of $\mathbb{R}^3$ if and only if the classical **mean curvature** vanishes i.e.

$$H = \frac{1}{2}(k_1 + k_2) = 0.$$

We conclude this chapter by observing that the Levi-Civita connection of a Riemannian manifold induces a metric connection on the normal bundle of its submanifolds, in a natural way.

Proposition 6.24. *Let (N, h) be a Riemannian manifold and (M, g) be a submanifold of N with the induced metric. Then the operator*

$$\bar{\nabla} : C^\infty(TM) \times C^\infty(NM) \to C^\infty(NM)$$

given by

$$\bar{\nabla}_{\tilde{X}} \tilde{Y} = (\nabla_X Y)^\perp$$

is a well defined connection on the normal bundle NM. Here X and Y are some local extensions of $\tilde{X} \in C^\infty(TM)$ and $\tilde{Y} \in C^\infty(NM)$, respectively. Furthermore, the connection $\bar{\nabla}$ is **metric** *i.e. it satisfies*

$$\tilde{X}(h(\tilde{Y}, \tilde{Z})) = h(\bar{\nabla}_{\tilde{X}}\tilde{Y}, \tilde{Z}) + h(\tilde{Y}, \bar{\nabla}_{\tilde{X}}\tilde{Z}),$$

for all $\tilde{X} \in C^\infty(TM)$ and $\tilde{Y}, \tilde{Z} \in C^\infty(NM)$.

PROOF. See Exercise 6.11. $\square$

Exercises

Exercise 6.1. Let M be a smooth manifold and $\hat{\nabla}$ be a connection on the tangent bundle (TM, M, π). Prove that the torsion of $\hat{\nabla}$

$$T : C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM),$$

given by

$$T(X, Y) = \hat{\nabla}_X Y - \hat{\nabla}_Y X - [X, Y],$$

is a tensor field of type $(1, 2)$.

Exercise 6.2. Find a proof of Proposition 6.11.

Exercise 6.3. Find a proof of Proposition 6.12.

Exercise 6.4. Let Sol be the 3-dimensional subgroup of $\mathbf{SL}_3(\mathbb{R})$ given by

$$\mathrm{Sol} = \{ \begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix} \mid p = (x, y, z) \in \mathbb{R}^3 \}.$$

Let $X, Y, Z \in \mathfrak{g}$ be left-invariant vector fields on Sol such that

$$X_e = \frac{\partial}{\partial x}|_{p=0}, \quad Y_e = \frac{\partial}{\partial y}|_{p=0} \quad \text{and} \quad Z_e = \frac{\partial}{\partial z}|_{p=0}.$$

Show that

$$[X, Y] = 0, \quad [Z, X] = X \quad \text{and} \quad [Z, Y] = -Y.$$

Let g be the left-invariant Riemannian metric on G such that $\{X, Y, Z\}$ is an orthonormal basis for the Lie algebra $\mathfrak{g}$. Calculate the following vector fields:

$$\nabla_X Y, \quad \nabla_Y X, \quad \nabla_X Z, \quad \nabla_Z X, \quad \nabla_Y Z \quad \text{and} \quad \nabla_Z Y.$$

Exercise 6.5. Let the special orthogonal group $\mathbf{SO}(m)$ be equipped with the Riemannian metric

$$g(X, Y) = \tfrac{1}{2} \cdot \mathrm{trace}(X^t \cdot Y).$$

Prove that g is left-invariant and that for vector fields $X, Y \in \mathfrak{so}(m)$ we have

$$\nabla_X Y = \tfrac{1}{2} \cdot [X, Y].$$

Let $A, B, C \in \mathfrak{so}(3)$ be the left-invariant vector fields on $\mathbf{SO}(3)$ such that

$$A_e = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ B_e = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ C_e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Show that $\{A, B, C\}$ is an orthonormal basis for $\mathfrak{so}(3)$ and determine the value of the left-invariant vector fields $\nabla_A B, \nabla_B C, \nabla_C A \in \mathfrak{so}(3)$ at the neutral element $e \in \mathbf{SO}(3)$.

Exercise 6.6. Let the special linear group $\mathbf{SL}_2(\mathbb{R})$ be equipped with the left-invariant metric

$$g_p(pA, pB) = \tfrac{1}{2} \cdot \operatorname{trace}(A^t \cdot B).$$

Let $X, Y, Z \in \mathfrak{sl}_2(\mathbb{R})$ be the left-invariant vector fields on $\mathbf{SL}_2(\mathbb{R})$ with

$$X_e = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Y_e = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Z_e = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Determine the value of the left-invariant vector fields $\nabla_X Y, \nabla_Y Z, \nabla_Z X \in \mathfrak{sl}_2(\mathbb{R})$ at the neutral element $e \in \mathbf{SL}_2(\mathbb{R})$.

Exercise 6.7. Let the special unitary group $\mathbf{SU}(2)$ be equipped with the left-invariant metric

$$g_p(pA, pB) = \tfrac{1}{2} \operatorname{Re} \operatorname{trace}(\bar{A}^t \cdot B).$$

Let $X, Y, Z \in \mathfrak{su}(2)$ be the left-invariant vector fields on $\mathbf{SU}(2)$ with

$$X_e = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Y_e = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad Z_e = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Determine the value of the left-invariant vector fields $\nabla_X Y, \nabla_Y Z, \nabla_Z X \in \mathfrak{su}(2)$ at the neutral element $e \in \mathbf{SU}(2)$.

Exercise 6.8. Find a proof of Proposition 6.17.

Exercise 6.9. Find a proof of Theorem 6.20.

Exercise 6.10. Find a proof of Proposition 6.21.

Exercise 6.11. Find a proof of Proposition 6.24.

CHAPTER 7

Geodesics

The main purpose of this chapter is the introduction of the important notion of geodesics on Riemannian manifolds. Geodesics are solutions to a second order system, of non-linear ordinary differential equations, heavily depending on the geometry of the manifolds involved.

In this process we develop the idea of parallel vector fields along curves in Riemannian manifolds. We show that geodesics are solutions to two different variational problems. They are both critical points of the so called energy functional and locally the shortest paths between their endpoints. We then study the important notion of totally geodesic submanifolds.

Definition 7.1. Let (TM, M, π) be the tangent bundle of a smooth manifold M. A **vector field** X **along a curve** $\gamma : I \to M$ is a smooth map $X : I \to TM$ such that $\pi \circ X = \gamma$. By $C^\infty_\gamma(TM)$ we denote the set of all smooth vector fields along γ. For $X, Y \in C^\infty_\gamma(TM)$ and $f \in C^\infty(I)$ we define the addition $+$ and the multiplication $\cdot$ by

(i) $(X + Y)(t) = X(t) + Y(t)$,

(ii) $(f \cdot X)(t) = f(t) \cdot X(t)$.

This turns $(C^\infty_\gamma(TM), +, \cdot)$ into a module over $C^\infty(I)$ and a real vector space over the constant functions, in particular. For a given smooth curve $\gamma : I \to M$ in M the smooth vector field $X : I \to TM$ with $X : t \mapsto (\gamma(t), \dot\gamma(t))$ is called the **tangent field** along γ.

The next result provides us with a differential operator for vector fields along a given curve and shows how this is closely related to the Levi-Civita connection.

Proposition 7.2. *Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇ and $\gamma : I \to M$ be a C^1-curve in M. Then there exists a unique operator*

$$\frac{D}{dt} : C^\infty_\gamma(TM) \to C^\infty_\gamma(TM),$$

such that for all $\lambda, \mu \in \mathbb{R}$, $f \in C^\infty(I)$ and $X, Y \in C^\infty_\gamma(TM)$, we have *(i)* $D(\lambda \cdot X + \mu \cdot Y)/dt = \lambda \cdot (DX/dt) + \mu \cdot (DY/dt)$,

(ii) $D(f \cdot X)/dt = df/dt \cdot X + f \cdot (DX/dt)$, *and*

(iii) *for each* $t_0 \in I$, *there exists an open subinterval* J *of* I *such that* $t_0 \in J$ *and if* $X \in C^\infty(TM)$ *is a vector field with* $X_{\gamma(t)} = Y(t)$ *for all* $t \in J$, *we have*

$$\Big(\frac{DY}{dt}\Big)(t_0) = (\nabla_{\dot\gamma} X)_{\gamma(t_0)}.$$

Proof. Here we start by proving the uniqueness part of the statement, hence we assume that such an operator exists. For a point $t_0 \in I$, choose a local chart (U, x) on M and an open subinterval $J \subset I$ such that $t_0 \in J$, $\gamma(J) \subset U$ and for $i = 1, 2, \dots, m$ we put $X_i = \partial/\partial x_i \in C^\infty(TU)$. Then any vector field Y along the restriction of γ to J can be written in the form

$$Y(t) = \sum_{j=1}^m \alpha_j(t) \cdot \big(X_j\big)_{\gamma(t)},$$

for some functions $\alpha_j \in C^\infty(J)$. The conditions (i) and (ii) imply that

$$\Big(\frac{DY}{dt}\Big)(t) = \sum_{k=1}^m \dot\alpha_k(t) \cdot \big(X_k\big)_{\gamma(t)} + \sum_{j=1}^m \alpha_j(t) \cdot \Big(\frac{DX_j}{dt}\Big)_{\gamma(t)}. \tag{7.1}$$

For the local chart (U, x), the composition

$$x \circ \gamma(t) = (\gamma_1(t), \dots, \gamma_m(t)) = \sum_{i=1}^m \gamma_i(t) \cdot e_i$$

parametrises a curve in $\mathbb{R}^m$ contained in $x(U)$. Hence the tangent map dx satisfies

$$dx_{\gamma(t)}(\dot\gamma(t)) = \frac{d}{dt}(x \circ \gamma(t)) = (\dot\gamma_1(t), \dots, \dot\gamma_m(t)).$$

Because the local coordinate $x : U \to x(U)$ is a diffeomorphism, its linear differential $dx : TU \to T\mathbb{R}^m$ is bijective, satisfying

$$dx\Big(\frac{\partial}{\partial x_i}\Big) = e_i,$$

for $i = 1, 2, \dots, m$. This immediately implies that

$$\dot\gamma(t) = \sum_{i=1}^m \dot\gamma_i(t) \cdot \big(X_i\big)_{\gamma(t)}$$

and the condition (iii) shows that

$$\Big(\frac{DX_j}{dt}\Big)_{\gamma(t)} \quad = \quad (\nabla_{\dot\gamma} X_j)_{\gamma(t)}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} \dot\gamma_i(t) \cdot (\nabla_{X_i} X_j)_{\gamma(t)} \\
&= \sum_{i,k=1}^{m} \dot\gamma_i(t) \cdot \Gamma_{ij}^k(\gamma(t)) \cdot (X_k)_{\gamma(t)}.
\end{aligned}
$$

By substituting this into relation into equation (7.1) we yield

$$
(7.2) \quad \big(\frac{DY}{dt}\big)(t) = \sum_{k=1}^{m} \big\{ \dot\alpha_k(t) + \sum_{i,j=1}^{m} \alpha_j(t) \cdot \dot\gamma_i(t) \cdot \Gamma_{ij}^k(\gamma(t)) \big\} \cdot \big(X_k\big)_{\gamma(t)}.
$$

This shows that there exists **at most** one such differential operator.

It is easily seen that if we use equation (7.2) for defining an operator D/dt then this satisfies the necessary conditions of Proposition 7.2. That proves the existence part of the stated result. □

The calculations of the last proof have the following important consequence.

Remark 7.3. Let us assume the set up of Proposition 7.2. It then follows from the fact that the Levi-Civita connection is tensorial in its first argument and the following equation

$$
(\nabla_{\dot\gamma} X)_{\gamma(t_0)} = \sum_{k=1}^{m} \big\{ \dot\alpha_k(t_0) + \sum_{i,j=1}^{m} \alpha_j(t_0) \cdot \dot\gamma_i(t_0) \cdot \Gamma_{ij}^k(\gamma(t_0)) \big\} \cdot \big(X_k\big)_{\gamma(t_0)}
$$

that the value $(\nabla_Z X)_p$ of $\nabla_Z X$ at p only depends on the value Z_p of Z at p and the values of X along some curve γ satisfying $\gamma(0) = p$ and $\dot\gamma(0) = Z_p$. This allows us to use the notation $\nabla_{\dot\gamma} Y$ for DY/dt.

The Levi-Civita connection can now be used to define the notions of parallel vector fields and geodesics on a Riemannian manifold. We will show that they are solutions to ordinary differential equations.

Definition 7.4. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇ and $\gamma : I \to M$ be a C^1-curve. A vector field X along γ is said to be **parallel** if

$$
\nabla_{\dot\gamma} X = 0.
$$

A C^2-curve $\gamma : I \to M$ is said to be a **geodesic** if its tangent field $\dot\gamma$ is parallel along γ i.e.

$$
\nabla_{\dot\gamma} \dot\gamma = 0.
$$

The next result shows that for a given initial value at a point we yield a parallel vector field globally defined along any curve through that point.

Theorem 7.5. *Let (M, g) be a Riemannian manifold and $I = (a, b)$ be an open interval on the real line $\mathbb{R}$. Further let $\gamma : [a, b] \to M$ be a continuous curve which is C^1 on I, $t_0 \in I$ and $v \in T_{\gamma(t_0)}M$. Then there exists a unique parallel vector field Y along γ such that $Y(t_0) = v$.*

PROOF. Let (U, x) be a local chart on M such that $\gamma(t_0) \in U$ and for $i = 1, 2, \ldots, m$ define $X_i = \partial/\partial x_i \in C^\infty(TU)$. Let J be an open subinterval of I such that the image $\gamma(J)$ is contained in U. Then the tangent of the restriction of γ to J can be written as

$$\dot\gamma(t) = \sum_{i=1}^{m} \dot\gamma_i(t) \cdot \big(X_i\big)_{\gamma(t)}.$$

Similarly, let Y be a vector field along γ presented by

$$Y(t) = \sum_{j=1}^{m} \alpha_j(t) \cdot \big(X_j\big)_{\gamma(t)}.$$

Then

$$\begin{aligned}\big(\nabla_{\dot\gamma} Y\big)(t) &= \sum_{j=1}^{m} \big\{\dot\alpha_j(t) \cdot \big(X_j\big)_{\gamma(t)} + \alpha_j(t) \cdot \big(\nabla_{\dot\gamma} X_j\big)_{\gamma(t)}\big\} \\ &= \sum_{k=1}^{m} \big\{\dot\alpha_k(t) + \sum_{i,j=1}^{m} \alpha_j(t) \cdot \dot\gamma_i(t) \cdot \Gamma_{ij}^k(\gamma(t))\big\} \big(X_k\big)_{\gamma(t)}.\end{aligned}$$

This implies that the vector field Y is parallel i.e. $\nabla_{\dot\gamma} Y = 0$ if and only if the following first order **linear** system of ordinary differential equations is satisfied

$$\dot\alpha_k(t) + \sum_{i,j=1}^{m} \alpha_j(t) \cdot \dot\gamma_i(t) \cdot \Gamma_{ij}^k(\gamma(t)) = 0,$$

for all $k = 1, \ldots, m$. It follows from Fact 7.6 that to each initial value $\alpha(t_0) = (v_1, \ldots, v_m) \in \mathbb{R}^m$, with

$$Y_0 = \sum_{k=1}^{m} v_k \cdot \big(X_k\big)_{\gamma(t_0)},$$

there exists a unique solution $\alpha = (\alpha_1, \dots, \alpha_m)$ to the above system. This gives us the unique parallel vector field Y

$$Y(t) = \sum_{k=1}^{m} \alpha_k(t) \cdot \big(X_k\big)_{\gamma(t)}$$

along J. Since the Christoffel symbols are bounded along the compact set $[a, b]$ it is clear that the parallel vector field can be extended to the whole of $I = (a, b)$. □

The following result is the well-known theorem of Picard-Lindelöf.

Fact 7.6. *Let $f : U \to \mathbb{R}^n$ be a continuous map defined on an open subset U of $\mathbb{R} \times \mathbb{R}^n$ and $L \in \mathbb{R}^+$ such that*

$$|f(t, y_1) - f(t, y_2)| \leq L \cdot |y_1 - y_2|$$

for all $(t, y_1), (t, y_2) \in U$. If $(t_0, x_0) \in U$ then there exists a unique local solution $x : I \to \mathbb{R}^n$ to the following initial value problem

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0.$$

For parallel vector fields we have the following important result.

Lemma 7.7. *Let (M, g) be a Riemannian manifold, $\gamma : I \to M$ be a C^1-curve and X, Y be parallel vector fields along γ. Then the function $g(X, Y) : I \to \mathbb{R}$, given by*

$$g(X, Y) : t \mapsto g_{\gamma(t)}(X_{\gamma(t)}, Y_{\gamma(t)}),$$

is constant. In particular, if γ is a geodesic then $g(\dot{\gamma}, \dot{\gamma})$ is constant along γ.

Proof. Using the fact that the Levi-Civita connection is metric we obtain

$$\frac{d}{dt}(g(X, Y)) = g(\nabla_{\dot{\gamma}} X, Y) + g(X, \nabla_{\dot{\gamma}} Y) = 0.$$

This proves that the function $g(X, Y)$ is constant along γ. □

The following result turns out to be a very useful tool. We will employ this in Chapter 9.

Proposition 7.8. *Let (M, g) be a Riemannian manifold, $p \in M$ and $\{v_1, \dots, v_m\}$ be an orthonormal basis for the tangent space T_pM. Let $\gamma : I \to M$ be a C^1-curve such that $\gamma(0) = p$ and $X_1, \dots, X_m$ be the parallel vector fields along γ such that $X_k(0) = v_k$ for $k = 1, 2, \dots, m$. Then the set $\{X_1(t), \dots, X_m(t)\}$ is an orthonormal basis for the tangent space $T_{\gamma(t)}M$ for all $t \in I$.*

Proof. This is a direct consequence of Lemma 7.7. □

Geodesics play a very important role in Riemannian geometry. For these we have the following fundamental existence and uniqueness result.

Theorem 7.9. *Let (M, g) be a Riemannian manifold. If $p \in M$ and $v \in T_pM$ then there exists an open interval $I = (-\epsilon, \epsilon)$ and a unique geodesic $\gamma : I \to M$ such that $\gamma(0) = p$ and $\dot\gamma(0) = v$.*

Proof. Let $\gamma : I \to M$ be a C^2-curve in M such that $\gamma(0) = p$ and $\dot\gamma(0) = v$. Further let (U, x) be a local chart on M such that $p \in U$ and for $i = 1, 2, \dots, m$ put $X_i = \partial/\partial x_i \in C^\infty(TU)$. Let J be an open subinterval of I such that the image $\gamma(J)$ is contained in U. Then the tangent of the restriction of γ to J can be written as

$$\dot\gamma(t) = \sum_{i=1}^m \dot\gamma_i(t) \cdot \big(X_i\big)_{\gamma(t)}.$$

By differentiation we then obtain

$$\begin{aligned}
\nabla_{\dot\gamma}\dot\gamma &= \sum_{j=1}^m \nabla_{\dot\gamma}\big(\dot\gamma_j(t) \cdot \big(X_j\big)_{\gamma(t)}\big) \\
&= \sum_{j=1}^m \big\{\ddot\gamma_j(t) \cdot \big(X_j\big)_{\gamma(t)} + \sum_{i=1}^m \dot\gamma_i(t) \cdot \dot\gamma_j(t) \cdot \big(\nabla_{X_i} X_j\big)_{\gamma(t)}\big\} \\
&= \sum_{k=1}^m \big\{\ddot\gamma_k(t) + \sum_{i,j=1}^m \dot\gamma_i(t) \cdot \dot\gamma_j(t) \cdot \Gamma_{ij}^k(\gamma(t))\big\} \cdot \big(X_k\big)_{\gamma(t)}.
\end{aligned}$$

Hence the curve γ is a geodesic if and only if

$$\ddot\gamma_k(t) + \sum_{i,j=1}^m \dot\gamma_i(t) \cdot \dot\gamma_j(t) \cdot \Gamma_{ij}^k(\gamma(t)) = 0$$

for all $k = 1, 2, \dots, m$. It follows from Fact 7.10 that for initial values $q = x(p)$ and $w = (dx)_p(v)$ there exists an open interval $(-\epsilon, \epsilon)$ and a unique solution $(\gamma_1, \dots, \gamma_m)$ satisfying the initial conditions

$$(\gamma_1(0), \dots, \gamma_m(0)) = q \quad \text{and} \quad (\dot\gamma_1(0), \dots, \dot\gamma_m(0)) = w.$$

$\square$

The following result is a second order consequence of the well-known theorem of Picard-Lindelöf.

Fact 7.10. *Let $f : U \to \mathbb{R}^n$ be a continuous map defined on an open subset U of $\mathbb{R} \times \mathbb{R}^{2n}$ and $L \in \mathbb{R}^+$ such that*

$$|f(t, y_1) - f(t, y_2)| \le L \cdot |y_1 - y_2|$$

for all $(t, y_1), (t, y_2) \in U$. *If* $(t_0, (x_0, x_1)) \in U$ *and* $x_0, x_1 \in \mathbb{R}^n$ *then there exists a unique local solution* $x : I \to \mathbb{R}^n$ *to the following initial value problem*

$$x''(t) = f(t, x(t), x'(t)), \quad x(t_0) = x_0, \quad x'(t_0) = x_1.$$

Remark 7.11. The Levi-Civita connection ∇ on a given Riemannian manifold (M, g) is an inner object i.e. completely determined by the differentiable structure on M and the Riemannian metric g, see Remark 6.7. Hence the same applies for the condition

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0$$

for any given curve $\gamma : I \to M$. This means that the image of a geodesic under a local isometry is again a geodesic.

We can now determine the geodesics in the Euclidean spaces.

Example 7.12. Let $E^m = (\mathbb{R}^m, \langle , \rangle)$ be the standard Euclidean space of dimension m. For the global chart $\mathrm{id}_{\mathbb{R}^m} : \mathbb{R}^m \to \mathbb{R}^m$ the metric on E^m is given by $g_{ij} = \delta_{ij}$. As a direct consequence of Example 6.13 we see that the corresponding Christoffel symbols satisfy

$$\Gamma_{ij}^k = 0 \ \text{ for all } \ i, j, k = 1, \dots, m.$$

Hence a C^2-curve $\gamma : I \to \mathbb{R}^m$ is a geodesic if and only if $\ddot{\gamma}(t) = 0$. For any $p \in \mathbb{R}^m$ and any $v \in T_p\mathbb{R}^m \cong \mathbb{R}^m$ define the curve

$$\gamma_{(p,v)} : \mathbb{R} \to \mathbb{R}^m \ \text{ by } \ \gamma_{(p,v)}(t) = p + t \cdot v.$$

Then $\gamma_{(p,v)}(0) = p$, $\dot{\gamma}_{(p,v)}(0) = v$ and $\ddot{\gamma}_{(p,v)} = 0$. It now follows from the uniqueness part of Theorem 7.9 that the geodesics in E^m are the straight lines.

For the classical situation of a surface in the three dimensional Euclidean space we have the following well known result.

Example 7.13. Let Σ be a regular surface as a submanifold of the three dimensional Euclidean space E^3. If $\gamma : I \to \Sigma$ is a C^2-curve, then Theorem 6.20 tells us that

$$\nabla_{\dot{\gamma}} \dot{\gamma} = (\partial_{\dot{\gamma}} \dot{\gamma})^\top = \ddot{\gamma}^\top.$$

This means that γ is a geodesic if and only if the tangential part $\ddot{\gamma}^\top$ of its second derivative $\ddot{\gamma}$ vanishes.

Definition 7.14. A geodesic $\gamma : J \to (M, g)$ in a Riemannian manifold is said to be **maximal** if it can not be extended to a geodesic defined on an interval I strictly containing J. The manifold (M, g)

is said to be **complete** if for each point $(p, v) \in TM$ there exists a geodesic $\gamma : \mathbb{R} \to M$, defined on the whole of $\mathbb{R}$, such that $\gamma(0) = p$ and $\dot\gamma(0) = v$.

The next statement generalises the classical result of Example 7.13.

Proposition 7.15. *Let (N, h) be a Riemannian manifold with Levi-Civita connection ∇ and M be a submanifold of N equipped with the induced metric g. A C^2-curve $\gamma : I \to M$ is a geodesic in M if and only if*

$$(\nabla_{\dot\gamma}\dot\gamma)^\top = 0.$$

PROOF. The result is an immediate consequence of Theorem 6.20 stating that the Levi-Civita connection $\tilde\nabla$ of (M, g) satisfies

$$\tilde\nabla_{\dot\gamma}\dot\gamma = (\nabla_{\dot\gamma}\dot\gamma)^\top.$$

□

With this at hand, we can now determine the geodesics on the standard unit spheres.

Example 7.16. Let $E^{m+1} = (\mathbb{R}^{m+1}, \langle , \rangle)$ be the standard $(m+1)$-dimensional Euclidean space and S^m be the unit sphere in E^{m+1} with the induced metric. At a point $p \in S^m$ the normal space N_pS^m of S^m in E^{m+1} is simply the line generated by p. If $\gamma : I \to S^m$ is a C^2-curve on the sphere, then

$$\tilde\nabla_{\dot\gamma}\dot\gamma = (\nabla_{\dot\gamma}\dot\gamma)^\top = (\partial_{\dot\gamma}\dot\gamma)^\top = \ddot\gamma^\top = \ddot\gamma - \ddot\gamma^\perp = \ddot\gamma - \langle\ddot\gamma, \gamma\rangle\gamma.$$

This shows that γ is a geodesic on the sphere S^m if and only if

$$\ddot\gamma = \langle\ddot\gamma, \gamma\rangle\gamma. \tag{7.3}$$

For a point $(p, X) \in TS^m$ define the curve $\gamma = \gamma_{(p,X)} : \mathbb{R} \to S^m$ by

$$\gamma : t \mapsto \begin{cases} p & \text{if } X = 0 \\ \cos(|X|t) \cdot p + \sin(|X|t) \cdot X/|X| & \text{if } X \neq 0. \end{cases}$$

Then one easily checks that $\gamma(0) = p$, $\dot\gamma(0) = X$ and that γ satisfies the geodesic equation (7.3). This shows that the non-constant geodesics on S^m are precisely the great circles and that the sphere is complete.

Having determined the geodesics on the standard spheres, we can now easily find the geodesics on the real projective spaces.

Example 7.17. Let $\mathrm{Sym}(\mathbb{R}^{m+1})$ be equipped with the metric

$$g(A, B) = \frac{1}{8}\,\mathrm{trace}(A^t \cdot B).$$

Then we know from Example 5.26 that the map $\phi : S^m \to \mathrm{Sym}(\mathbb{R}^{m+1})$ with

$$\phi : p \mapsto (2pp^t - e)$$

is an isometric immersion and that the image $\phi(S^m)$ is isometric to the m-dimensional real projective space $\mathbb{R}P^m$. This means that the geodesics on $\mathbb{R}P^m$ are exactly the images of geodesics on S^m. This shows that the real projective space is complete.

We will now show that the geodesics are critical points of the so called energy functional. For this we need the following two definitions.

Definition 7.18. Let (M, g) be a Riemannian manifold and $\gamma : I \to M$ be a C^r-curve on M. A **variation** of γ is a C^r-map

$$\Phi : (-\epsilon, \epsilon) \times I \to M$$

such that for all $s \in I$, $\Phi_0(s) = \Phi(0, s) = \gamma(s)$. If the interval is compact i.e. of the form $I = [a, b]$, then the variation Φ is said to be **proper** if for all $t \in (-\epsilon, \epsilon)$ we have $\Phi_t(a) = \gamma(a)$ and $\Phi_t(b) = \gamma(b)$.

Definition 7.19. Let (M, g) be a Riemannian manifold and $\gamma : I \to M$ be a C^2-curve on M. For every compact interval $[a, b] \subset I$ we define the **energy functional** $E_{[a,b]}$ by

$$E_{[a,b]}(\gamma) = \frac{1}{2}\int_a^b g(\dot\gamma(t), \dot\gamma(t))dt.$$

A C^2-curve $\gamma : I \to M$ is called **a critical point** for the energy functional if every proper variation Φ of $\gamma|_{[a,b]}$ satisfies

$$\frac{d}{dt}(E_{[a,b]}(\Phi_t))|_{t=0} = 0.$$

We will now prove that geodesics can be characterised as the critical points of the energy functional.

Theorem 7.20. *A C^2-curve $\gamma : I = [a, b] \to M$ is a critical point for the energy functional if and only if it is a geodesic.*

Proof. For a C^2-map $\Phi : (-\epsilon, \epsilon) \times I \to M$, $\Phi : (t, s) \mapsto \Phi(t, s)$ we define the vector fields $X = d\Phi(\partial/\partial s)$ and $Y = d\Phi(\partial/\partial t)$ along Φ. The following shows that the vector fields X and Y commute.

$$\begin{aligned}\nabla_X Y - \nabla_Y X &= [X, Y]\\ &= [d\Phi(\partial/\partial s), d\Phi(\partial/\partial t)]\\ &= d\Phi([\partial/\partial s, \partial/\partial t])\\ &= 0,\end{aligned}$$

since $[\partial/\partial s, \partial/\partial t] = 0$. We now assume that Φ is a proper variation of γ. Then

$$\begin{aligned}
\frac{d}{dt}(E_{[a,b]}(\Phi_t)) &= \frac{1}{2}\frac{d}{dt}(\int_a^b g(X,X)ds) \\
&= \frac{1}{2}\int_a^b \frac{d}{dt}(g(X,X))ds \\
&= \int_a^b g(\nabla_Y X, X)ds \\
&= \int_a^b g(\nabla_X Y, X)ds \\
&= \int_a^b (\frac{d}{ds}(g(Y,X)) - g(Y, \nabla_X X))ds \\
&= [g(Y,X)]_a^b - \int_a^b g(Y, \nabla_X X)ds.
\end{aligned}$$

The variation is proper, so $Y(t,a) = Y(t,b) = 0$. Furthermore

$$X(0,s) = \partial\Phi/\partial s(0,s) = \dot{\gamma}(s),$$

so

$$\frac{d}{dt}(E_{[a,b]}(\Phi_t))|_{t=0} = -\int_a^b g(Y(0,s), (\nabla_{\dot{\gamma}}\dot{\gamma})(s))ds.$$

The last integral vanishes for every proper variation Φ of γ if and only if $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$. $\square$

A geodesic $\gamma : I \to (M,g)$ is a special case of what is called a **harmonic map** $\phi : (M,g) \to (N,h)$ between Riemannian manifolds. Other examples are the conformal immersions $\psi : (M^2,g) \to (N,h)$ which parametrise the minimal surfaces in (N,h). The study of harmonic maps between Riemannian manifolds was initiated by the seminal paper: J. Eells, J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86**, (1964), 109-160. For a modern reference on harmonic maps see H. Urakawa, *Calculus of Variations and Harmonic Maps*, Translations of Mathematical Monographs **132**, AMS (1993).

Our next goal is to prove the important result of Theorem 7.23. For this we introduce the exponential map, which is a fundamental tool in Riemannian geometry.

Definition 7.21. Let (M^m, g) be an m-dimensional Riemannian manifold, $p \in M$ and

$$S_p^{m-1} = \{v \in T_pM \,|\, g_p(v, v) = 1\}$$

be the unit sphere in the tangent space T_pM at p. Then every non-zero element $w \in T_pM$ can be written as $w = r_w \cdot v_w$, where $r_w = |w|$ and $v_w = w/|w| \in S_p^{m-1}$. For $v \in S_p^{m-1}$ let $\gamma_v : (-\alpha_v, \beta_v) \to M$ be the maximal geodesic such that $\alpha_v, \beta_v \in \mathbb{R}^+ \cup \{\infty\}$, $\gamma_v(0) = p$ and $\dot\gamma_v(0) = v$. The unit sphere is compact and for this reason it can be shown that the real number

$$\epsilon_p = \inf\{\alpha_v, \beta_v \,|\, v \in S_p^{m-1}\}$$

is positive so the open ball

$$B_{\epsilon_p}^m(0) = \{v \in T_pM \,|\, g_p(v, v) < \epsilon_p^2\}$$

is non-empty. The **exponential map** $\exp_p : B_{\epsilon_p}^m(0) \to M$ at p is defined by

$$\exp_p : w \mapsto \begin{cases} p & \text{if } w = 0 \\ \gamma_{v_w}(r_w) & \text{if } w \neq 0. \end{cases}$$

Note that for $v \in S_p^{m-1}$ the line segment $\lambda_v : (-\epsilon_p, \epsilon_p) \to T_pM$ with $\lambda_v : t \mapsto t \cdot v$ is mapped onto the geodesic γ_v i.e. locally we have $\gamma_v = \exp_p \circ \lambda_v$. One can prove that the map $\exp_p$ is differentiable and it follows from its definition that the differential

$$d(\exp_p)_0 : T_pM \to T_pM$$

is the identity map for the tangent space T_pM. Then the Inverse Mapping Theorem 3.29 tells us that there exists an $r_p \in \mathbb{R}^+$ such that if $U_p = B_{r_p}^m(0)$ and $V_p = \exp_p(U_p)$ then $\exp_p|_{U_p} : U_p \to V_p$ is a diffeomorphism parametrising the open subset V_p of M.

Example 7.22. Let S^m be the unit sphere in the standard Euclidean $\mathbb{R}^{m+1}$ and $\exp_p : T_pS^m \to S^m$ be the exponential map of S^m at the north pole $p = (0, 1) \in \mathbb{R}^m \times \mathbb{R}$. Then we clearly have $\exp_p(0) = p$. If $Y \in T_pS^m$ is a unit vector i.e. $|Y| = 1$, then the line through the origin, generated by Y, is parametrised by $\lambda_Y : \mathbb{R} \to T_pS^m$ satisfying $\lambda_Y(s) = s \cdot Y$ with

$$\lambda_Y(0) = 0 \quad \text{and} \quad \dot\lambda_Y(0) = Y.$$

Furthermore, there exists a unique geodesic $\gamma_Y : \mathbb{R} \to S^m$ such that

$$\gamma_Y(0) = p \quad \text{and} \quad \dot\gamma_Y(0) = Y.$$

According to Example 7.16, this satisfies

$$\gamma_Y(s) = \cos s \cdot p + \sin s \cdot (Y, 0).$$

From this we see that the exponential map $\exp_p : T_pS^m \to S^m$ satisfies

$$\exp_p : s \cdot Y \mapsto (\cos s \cdot p + \sin s \cdot (Y, 0)).$$

This maps the line λ_Y onto the geodesic γ_Y and is clearly injective on the open ball

$$B^m_\pi(0) = \{X \in T_pS^m \,|\, |X| < \pi\}$$

of radius π. We will see in Theorem 7.23 that the geodesic

$$\gamma_Y : s \mapsto \exp_p(s \cdot Y)$$

is the shortest path between p and $\gamma_Y(r)$ as long as $r < \pi$. Note that each point on the $(m-1)$-dimensional sphere

$$T^\pi_p S^m = \{Z \in T_pS^m \,|\, |Z| = \pi\}$$

is mapped to the south pole $-p = (0, -1)$, so the globally defined exponential map $\exp_p : T_pS^m \to S^m$ is not injective.

The exponential map $\exp_p$ takes the origin $0 \in T_pS^m$ to the point $p \in S^m$. This means that its tangent map $d(\exp_p)_0$ at 0 is defined on the tangent space $T_0T_pS^m$ of T_pS^m at $0 \in T_pS^m$, which we identify with T_pS^m. Since the two tangents $\dot\lambda_Y(0)$ and $\dot\gamma_Y(0)$ satisfy $\dot\lambda_Y(0) = \dot\gamma_Y(0)$ we see that the tangent map

$$d(\exp_p)_0 : T_pS^m \to T_pS^m$$

is simply the identity map of the tangent space T_pS^m.

The next result shows that on a Riemannian manifold the geodesics are locally the shortest paths between their endpoints.

Theorem 7.23. *Let (M, g) be a Riemannian manifold. Then the geodesics are locally the shortest paths between their endpoints.*

Proof. Let $p \in M$, $U = B^m_r(0)$ in T_pM and $V = \exp_p(U)$ be such that the restriction

$$\phi = \exp_p|_U : U \to V$$

of the exponential map at p is a diffeomorphism. We define a metric $\tilde g$ on U such that for each $X, Y \in C^\infty(TU)$ we have

$$\tilde g(X, Y) = g(d\phi(X), d\phi(Y)).$$

This turns $\phi : (U, \tilde g) \to (V, g)$ into an isometry. It then follows from the construction of the exponential map, that the geodesics in $(U, \tilde g)$ through the point $0 = \phi^{-1}(p)$ are exactly the lines $\lambda_v : t \mapsto t \cdot v$ where $v \in T_pM$.

Now let q be an arbitrary non-zero element of $B^m_r(0)$ and $\lambda_q : [0, 1] \to B^m_r(0)$ be the geodesic $\lambda_q : t \mapsto t \cdot q$. Further let $\sigma : [0, 1] \to U$ be any C^1-curve such that $\sigma(0) = 0$ and $\sigma(1) = q$. Along the curve

σ we define the vector field X with $X : t \mapsto \sigma(t)$ and the tangent field $\dot\sigma : t \to \dot\sigma(t)$ to σ. Then the radial component $\dot\sigma_{\text{rad}}$ of $\dot\sigma$ is the orthogonal projection of $\dot\sigma$ onto the line generated by X i.e.

$$\dot\sigma_{\text{rad}} : t \mapsto \frac{\tilde g(\dot\sigma(t), X(t))}{\tilde g(X(t), X(t))} X(t).$$

Then it is easily checked that

$$|\dot\sigma_{\text{rad}}(t)| = \frac{|\tilde g(\dot\sigma(t), X(t))|}{|X(t)|}$$

and

$$\frac{d}{dt}|X(t)| = \frac{d}{dt}\sqrt{\tilde g(X(t), X(t))} = \frac{\tilde g(\dot\sigma(t), X(t))}{|X(t)|}.$$

Combining these two relations we yield

$$|\dot\sigma_{\text{rad}}(t)| \geq \frac{d}{dt}|X(t)|.$$

This means that

$$\begin{aligned} L(\sigma) &= \int_0^1 |\dot\sigma(t)| dt \\ &\geq \int_0^1 |\dot\sigma_{\text{rad}}(t)| dt \\ &\geq \int_0^1 \frac{d}{dt}|X(t)| dt \\ &= |X(1)| - |X(0)| \\ &= |q| \\ &= L(\lambda_q). \end{aligned}$$

This proves that in fact that λ_q is the shortest path connecting p and q. $\square$

We now introduce the important notion of totally geodesic submanifolds of a Riemannian manifold.

Definition 7.24. Let (N, h) be a Riemannian manifold and M be a submanifold of N with the induced metric. Then M is said to be **totally geodesic** in N if its second fundamental form vanishes identically i.e. $B \equiv 0$.

For the totally geodesic submanifolds we have the following important characterisation.

Proposition 7.25. *Let (N, h) be a Riemannian manifold with its Levi-Civita connection ∇ and M be a submanifold of N equipped with the induced metric. Then the following conditions are equivalent*

(i) M is totally geodesic in N

(ii) a curve $\gamma : I \to M$ is a geodesic in M if and only it is geodesic in N.

PROOF. The result is a direct consequence of the decomposition formula

$$\nabla_{\dot{\gamma}}\dot{\gamma} = (\nabla_{\dot{\gamma}}\dot{\gamma})^{\top} + (\nabla_{\dot{\gamma}}\dot{\gamma})^{\perp} = \tilde{\nabla}_{\dot{\gamma}}\dot{\gamma} + B(\dot{\gamma}, \dot{\gamma})$$

and the polar identity for the symmetric second fundamental form

$$4 \cdot B(X, Y) = B(X + Y, X + Y) - B(X - Y, X - Y).$$

□

Corollary 7.26. *Let (N, h) be a Riemannian manifold, $p \in N$ and V be an m-dimensional linear subspace of the tangent space T_pN of N at p. Then there exists (locally) at most one totally geodesic submanifold M of N such that $T_pM = V$.*

PROOF. See Exercise 7.5. □

Proposition 7.27. *Let (N, h) be a Riemannian manifold and M be a submanifold of N with the induced metric. For a point (p, v) of the tangent bundle TM, let $\gamma_{(p,v)} : I \to N$ be the maximal geodesic in N with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. Then M is totally geodesic in (N, h) if $\gamma_{(p,v)}(I)$ is contained in M for all $(p, v) \in TM$. The converse is true if M is complete.*

PROOF. See Exercise 7.6. □

Proposition 7.28. *Let (N, h) be a Riemannian manifold and M be a submanifold of N which is the fixpoint set of an isometry $\phi : N \to N$. Then M is totally geodesic in N.*

PROOF. Let $p \in M$, $v \in T_pM$ and $c : J \to M$ be a curve in M such that $c(0) = p$ and $\dot{c}(0) = v$. Since M is the fix point set of ϕ we know that $\phi(c(t)) = c(t)$ for all $t \in J$ and hence that $\phi(p) = p$ and $d\phi_p(v) = v$. Further let $\gamma : I \to N$ be the maximal geodesic in N with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. The map $\phi : N \to N$ is an isometry so the curve $\phi \circ \gamma : I \to N$ is also a geodesic. The uniqueness result of Theorem 7.9, $\phi(\gamma(0)) = \gamma(0)$ and $d\phi(\dot{\gamma}(0)) = \dot{\gamma}(0)$ then imply that $\phi(\gamma) = \gamma$. Hence the image of the geodesic $\gamma : I \to N$ is contained in M, so following Proposition 7.27 the submanifold M is totally geodesic in N. □

Corollary 7.29. *Let $m < n$ be positive integers. Then the m-dimensional sphere*

$$S^m = \{(x,0) \in \mathbb{R}^{m+1} \times \mathbb{R}^{n-m} \mid |x|^2 = 1\}$$

is a totally geodesic submanifold of

$$S^n = \{(x,y) \in \mathbb{R}^{m+1} \times \mathbb{R}^{n-m} \mid |x|^2 + |y|^2 = 1\}.$$

Proof. The statement is a direct consequence of the fact that S^m is the fixpoint set of the isometry $\phi : S^n \to S^n$ of S^n with $(x,y) \mapsto (x,-y)$. □

Corollary 7.30. *Let $m < n$ be positive integers. Let H^n be the n-dimensional hyperbolic space modelled on the upper half space $\mathbb{R}^+ \times \mathbb{R}^{n-1}$ equipped with the Riemannian metric*

$$g(X,Y) = \frac{1}{x_1^2} \cdot \langle X, Y \rangle,$$

where $x = (x_1, \dots, x_n) \in H^n$. Then the m-dimensional hyperbolic space

$$H^m = \{(x,0) \in H^n \mid x \in \mathbb{R}^m\}$$

is totally geodesic in H^n.

Proof. See Exercise 7.8. □

We conclude this chapter by introducing the important notion of a Riemannian symmetric space.

Definition 7.31. A **symmetric space** is a connected Riemannian manifold (M,g) such that for each point $p \in M$ there exists a global isometry $\phi : M \to M$ which is a geodesic symmetry fixing p. By this we mean that $\phi(p) = p$ and the tangent map $d\phi_p : T_pM \to T_pM$ satisfies $d\phi_p(X) = -X$ for all $X \in T_pM$.

Example 7.32. Let p be an arbitrary point on the unit sphere S^m in the standard Euclidean $\mathbb{R}^{n+1}$. Then the reflection $\rho_p : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ about the line generated by p is given by

$$\rho_p : q \mapsto 2\langle q, p \rangle p - q.$$

This is a linear map hence identical to is differential $\rho_p : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$. The restriction $\phi = \rho_p|_{S^m} : S^m \to S^m$ is an isometry that fixes p. Its tangent map $d\phi_p : T_pS^m \to T_pS^m$ satisfies $d\phi_p(X) = -X$ for all $X \in T_pS^m$. This shows that the homogeneous space S^m is symmetric.

Proposition 7.33. *Every Riemannian symmetric space is complete.*

Proof. See Exercise 7.10. □

The following important result is a direct consequence of the famous Hopf-Rinow theorem.

Theorem 7.34. *Let (M, g) be a complete Riemannian manifold which is path-connected. If $p, q \in M$ then there exists a geodesic $\gamma : \mathbb{R} \to M$ such that $\gamma(0) = p$ and $\gamma(1) = q$.*

Proof. See Exercise 7.11. □

The following shows that every Riemannian symmetric space is homogeneous, see Definition 5.20.

Theorem 7.35. *Every Riemannian symmetric space is homogeneous.*

Proof. See Exercise 7.12. □

The Riemannian symmetric spaces were classified by Élie Cartan in his seminal study from 1926. They constitute 20 countably infinite families and 24 single exceptional cases and are quotents of Riemannian Lie groups. They come in dual **pairs** $(U/K, G/K)$, where U/K is **compact** and G/K is **non-compact**.

The best known simply connected examples are the dual spheres S^m and the hyperbolic spaces H^m of constant sectional curvature

$$S^m = \mathbf{SO}(m+1)/\mathbf{SO}(m), \quad H^m = \mathbf{SO}_o(m,1)/\mathbf{SO}(m).$$

We also have their complex counterparts i.e. the complex projective and hyperbolic spaces

$$\mathbb{C}P^m = \mathbf{SU}(m+1)/\mathbf{S}(\mathbf{U}(m) \times \mathbf{U}(1)),$$
$$\mathbb{C}H^m = \mathbf{SU}(m,1)/\mathbf{S}(\mathbf{U}(m) \times \mathbf{U}(1)).$$

The standard reference to the theory of Riemannian symmetric spaces is: Sigurdur Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Graduate Studies in Mathematics **34**, AMS (2001).

Exercises

Exercise 7.1. The result of Exercise 5.3 shows that the two dimensional hyperbolic disc H^2 introduced in Example 5.9 is isometric to the upper half plane $M = \{(x, y) \in \mathbb{R}^2 \,|\, y \in \mathbb{R}^+\}$ equipped with the Riemannian metric

$$g(X, Y) = \frac{1}{y^2} \cdot \langle X, Y \rangle.$$

Use your local library to find all geodesics in (M, g).

Exercise 7.2. Let the special orthogonal group $\mathbf{SO}(m)$ be equipped with its standard left-invariant Riemannian metric

$$g(A, B) = \mathrm{trace}(A^t \cdot B).$$

Prove that a C^2-curve $\gamma : (-\epsilon, \epsilon) \to \mathbf{SO}(m)$ is a geodesic if and only if

$$\gamma^t \cdot \ddot{\gamma} = \ddot{\gamma}^t \cdot \gamma.$$

Exercise 7.3. Let the special orthogonal group $\mathbf{SO}(m)$ be equipped with its standard left-invariant Riemannian metric

$$g(A, B) = \mathrm{trace}(A^t \cdot B).$$

Use the result of Exercise 7.2 to show that every geodesic $\gamma : \mathbb{R} \to \mathbf{SO}(m)$, satisfying $\gamma(0) = p$ and $\dot{\gamma}(0) = p \cdot X$, is of the form

$$\gamma(s) = p \cdot \mathrm{Exp}(s\, X),$$

where $p \in \mathbf{SO}(m)$ and $X \in T_e\mathbf{SO}(m)$.

Exercise 7.4. For the real parameter $\theta \in (0, \pi/2)$ define the 2-dimensional torus T^2_θ by

$$T^2_\theta = \{(\cos\theta \cdot e^{i\alpha}, \sin\theta \cdot e^{i\beta}) \in S^3 \,|\, \alpha, \beta \in \mathbb{R}\}.$$

Determine for which $\theta \in (0, \pi/2)$ the torus T^2_θ is a minimal submanifold of the 3-dimensional sphere

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\}.$$

Exercise 7.5. Find a proof of Corollary 7.26.

Exercise 7.6. Find a proof of Proposition 7.27.

Exercise 7.7. Determine the totally geodesic submanifolds of the m-dimensional real projective space $\mathbb{R}P^m$. (Hint: Use the result of Example 5.26).

Exercise 8.8. Find a proof of Corollary 7.30.

Exercise 7.9. Let the special orthogonal group $\mathbf{SO}(m)$ be equipped with the left-invariant metric

$$g(A, B) = \text{trace}(A^t \cdot B)$$

and let K be a Lie subgroup of $\mathbf{SO}(m)$. Prove that K is totally geodesic in $\mathbf{SO}(m)$.

Exercise 7.10. Find a proof of Proposition 7.33.

Exercise 7.11. Use your local library to find a proof of Theorem 7.34.

Exercise 7.12. Find a proof of Theorem 7.35.

CHAPTER 8

The Riemann Curvature Tensor

In this chapter we introduce the Riemann curvature tensor and the sectional curvature of a Riemannian manifold. These notions generalise the Gaussian curvature playing a central role in Gaussian geometry i.e. the classical differential geometry of curves and surfaces. We derive the important Gauss equation comparing the sectional curvatures of a submanifold and that of its ambient space. We prove that the Euclidean spaces, the standard spheres and the hyperbolic spaces all have constant sectional curvature. We then determine the Riemannian curvature tensor for manifolds of constant sectional curvature and also for an important class of Lie groups.

Definition 8.1. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. Then for a vector field $X \in C^\infty(TM)$ we have the **first order covariant derivative**

$$\nabla_X : C^\infty(TM) \to C^\infty(TM)$$

of vector fields in the direction of the given X satisfying

$$\nabla_X : Z \mapsto \nabla_X Z.$$

We will now generalise this idea and introduce the important covariant derivatives of tensor fields of types $(0, r)$ and $(1, r)$. Before we do this, in a formal way, we now provide the following motivation.

Motivation 8.2. Let (M, g) be a Riemannian manifold with its Levi-Civita connection ∇. Let $A : C_2^\infty(TM) \to C_1^\infty(TM)$ be a tensor field on M of type $(1, 2)$. If we differentiate the vector field $A(Y, Z)$ in the direction of X applying the following "naive" product rule

$$\nabla_X(A(Y, Z)) = (\nabla_X A)(Y, Z) + A(\nabla_X Y, Z) + A(Y, \nabla_X Z)$$

we obtain

$$(\nabla_X A)(Y, Z) = \nabla_X(A(Y, Z)) - A(\nabla_X Y, Z) - A(Y, \nabla_X Z).$$

Here $\nabla_X A$ is called the "covariant derivative" of the tensor field A in the direction of X.

The above idea turns out to be very useful and leads to the following formal Definitions 8.3 and 8.6.

Definition 8.3. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. For a tensor field $A : C^\infty_r(TM) \to C^\infty_0(TM)$ of type $(0, r)$ we define its **covariant derivative**

$$\nabla A : C^\infty_{r+1}(TM) \to C^\infty_0(TM)$$

by

$$\nabla A : (X, X_1, \dots, X_r) \mapsto (\nabla_X A)(X_1, \dots, X_r) =$$

$$X(A(X_1, \dots, X_r)) - \sum_{k=1}^{r} A(X_1, \dots, X_{k-1}, \nabla_X X_k, X_{k+1}, \dots, X_r).$$

A tensor field A of type $(0, r)$ is said to be **parallel** if $\nabla A \equiv 0$.

The following result can be seen as, yet another, compatibility of the Levi-Civita connection ∇ of (M, g) with the Riemannian metric g.

Proposition 8.4. *Let (M, g) be a Riemannian manifold with its Levi-Civita connection ∇. Then the Riemannian metric g is a parallel tensor field of type $(0, 2)$.*

Proof. See Exercise 8.1. □

Example 8.5. Let (M, g) be a Riemannian manifold. Then we already know that its Levi-Civita connection ∇ is tensorial in its first argument i.e. if $X, Y \in C^\infty(TM)$ and $f, g \in C^\infty(M)$ then we have

$$\nabla_{(f \cdot X + g \cdot Y)} Z = f \cdot \nabla_X Z + g \cdot \nabla_Y Z.$$

This means that a vector field $Z \in C^\infty(TM)$ on M induces the **natural tensor field** $\mathcal{Z} : C^\infty_1(TM) \to C^\infty_1(TM)$ of type $(1, 1)$ given by

$$\mathcal{Z} : X \mapsto \nabla_X Z,$$

satisfying

$$\mathcal{Z}(f \cdot X + g \cdot Y) = f \cdot \mathcal{Z}(X) + g \cdot \mathcal{Z}(Y).$$

For a tensor field of type $(1, r)$ we now have the following definition of its covariant derivative, much in the spirit of the above mentioned Motivation 8.2.

Definition 8.6. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. For a tensor field $A : C^\infty_r(TM) \to C^\infty_1(TM)$ of type $(1, r)$ we define its **covariant derivative**

$$\nabla A : C^\infty_{r+1}(TM) \to C^\infty_1(TM)$$

by

$$\nabla A : (X, X_1, \dots, X_r) \mapsto (\nabla_X A)(X_1, \dots, X_r) =$$

$$\nabla_X (A(X_1, \dots, X_r)) - \sum_{k=1}^{r} A(X_1, \dots, X_{k-1}, \nabla_X X_k, X_{k+1}, \dots, X_r).$$

A tensor field A of type $(1, r)$ is said to be **parallel** if $\nabla A \equiv 0$.

Definition 8.7. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇ and $X, Y \in C^\infty(TM)$ be two vector fields on M. Then the **second order covariant derivative**

$$\nabla^2_{X,Y} : C^\infty(TM) \to C^\infty(TM)$$

is defined by

$$\nabla^2_{X,Y} : Z \mapsto (\nabla_X \mathcal{Z})(Y),$$

where $\mathcal{Z}$ is the natural tensor field of type $(1,1)$ induced by $Z \in C^\infty(TM)$, see Example 8.5.

As a direct consequence of Definitions 8.6 and 8.7 we see that if $X, Y, Z \in C^\infty(TM)$ are vector fields on M, then the second order covariant derivative $\nabla^2_{X,Y}$ satisfies

$$\nabla^2_{X,Y} Z = \nabla_X(\mathcal{Z}(Y)) - \mathcal{Z}(\nabla_X Y) = \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z.$$

This leads us to the following important definition.

Definition 8.8. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. Then its **Riemann curvature operator**

$$R : C^\infty(TM) \times C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM)$$

is defined as twice the skew-symmetric part of the second covariant derivative ∇^2 i.e.

$$R(X, Y)Z = \nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z.$$

The next remarkable result shows that the curvature operator is actually a tensor field.

Theorem 8.9. *Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. Then the Riemann curvature operator*

$$R : C^\infty_3(TM) \to C^\infty_1(TM)$$

satisfying

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

is a tensor field on M of type $(1, 3)$.

Proof. See Exercise 8.2. □

The reader should note that the Riemann curvature tensor is an intrinsic object since it only depends on the intrinsic Levi-Civita connection. The following result shows that the curvature tensor has many beautiful symmetries.

Proposition 8.10. *Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. Then its Riemann curvature tensor R satisfies the following symmetry conditions.*

(i) $R(X,Y)Z = -R(Y,X)Z$,
(ii) $g(R(X,Y)Z,W) = -g(R(X,Y)W,Z)$,
(iii) $R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0$,
(iv) $g(R(X,Y)Z,W) = g(R(Z,W)X,Y)$,
(v)
$$\begin{aligned} 6 \cdot R(X,Y)Z = {} & R(X,Y+Z)(Y+Z) - R(X,Y-Z)(Y-Z) \\ & + R(X+Z,Y)(X+Z) - R(X-Z,Y)(X-Z). \end{aligned}$$

Here $X, Y, Z, W \in C^\infty(TM)$ are vector fields on M.

Proof. See Exercise 8.3. □

Part (iii) of Proposition 8.10 is the so called first **Bianchi identity**. The second Bianchi identity is a similar result concerning the covariant derivative ∇R of the curvature tensor. This will not be treated here.

Our next task is to obtain a better understanding of the Riemann curvature tensor and compare it with the Gaussian curvature, so important in the Gaussian geometry of surfaces in the three dimensional Euclidean space. For this see Example 8.19.

Definition 8.11. Let (M, g) be a Riemannian manifold and $p \in M$. Then a **section** V at p is a 2-dimensional subspace of the tangent space T_pM. The set

$$G_2(T_pM) = \{V \mid V \text{ is a section of } T_pM\}$$

of sections is called the **Grassmannian** of 2-planes at p.

Remark 8.12. In Gaussian geometry the tangent space $T_p\Sigma$ of a surface Σ in the Euclidean $\mathbb{R}^3$ is two dimensional. This means that in this case there is only one section at $p \in \Sigma$, namely the full two dimensional tangent plane $T_p\Sigma$.

Before introducing the notion of the sectional curvature we need the following useful technical lemma.

Lemma 8.13. *Let (M, g) be a Riemannian manifold, $p \in M$ and $X, Y, Z, W \in T_pM$ be tangent vectors at p such that the two sections* *span*$_{\mathbb{R}}\{X, Y\}$ *and span*$_{\mathbb{R}}\{Z, W\}$ *are identical. Then*

$$\frac{g(R(X,Y)Y,X)}{|X|^2|Y|^2 - g(X,Y)^2} = \frac{g(R(Z,W)W,Z)}{|Z|^2|W|^2 - g(Z,W)^2}.$$

Proof. See Exercise 8.4. □

We now introduce the notion of sectional curvature at a point. The result of Lemma 8.13 shows that this is well defined.

Definition 8.14. Let (M, g) be a Riemannian manifold and $p \in M$. Then the function $K_p : G_2(T_pM) \to \mathbb{R}$ given by

$$K_p : \mathrm{span}_{\mathbb{R}}\{X, Y\} \mapsto \frac{g(R(X,Y)Y,X)}{|X|^2|Y|^2 - g(X,Y)^2}$$

is called the **sectional curvature** of the section $V = \mathrm{span}_{\mathbb{R}}\{X, Y\}$ at the point $p \in M$. In this case we usually write $K(X, Y)$ for $K(V)$.

It can be shown that, for a fixed $p \in M$, the Grassmannian $G_2(T_pM)$ is diffeomorphic to the compact quotient manifold $\mathbf{SO}(m)/\mathbf{SO}(2) \times \mathbf{SO}(m-2)$. Hence the continuous real-valued function $K_p : G_2(T_pM) \to \mathbb{R}$ both has a minimum and a maximum at $p \in M$.

Definition 8.15. Let (M, g) be a Riemannian manifold and $K_p : G_2(T_pM) \to \mathbb{R}$ be the sectional curvature function at an arbitrary point $p \in M$. Then we define the functions $\delta, \Delta : M \to \mathbb{R}$ by

$$\delta : p \mapsto \min_{V \in G_2(T_pM)} K_p(V) \quad \text{and} \quad \Delta : p \mapsto \max_{V \in G_2(T_pM)} K_p(V).$$

The Riemannian manifold (M, g) is said to be

(i) of **non-negative curvature** if $\delta(p) \geq 0$ for all p,
(ii) of **positive curvature** if $\delta(p) > 0$ for all p,
(iii) of **non-positive curvature** if $\Delta(p) \leq 0$ for all p,
(iv) of **negative curvature** if $\Delta(p) < 0$ for all p,
(v) of **constant curvature** if $\delta = \Delta$ is constant,
(vi) **flat** if $\delta \equiv \Delta \equiv 0$.

The next example shows how the Riemann curvature tensor can be presented by means of local coordinates. Hopefully this will convince the reader that those should be avoided whenever possible.

Example 8.16. Let (M, g) be a Riemannian manifold and (U, x) be a local chart on M. For $i, j, k, l = 1, \ldots, m$ put

$$X_i = \frac{\partial}{\partial x_i}, \quad g_{ij} = g(X_i, X_j) \quad \text{and} \quad R_{ijk}^l = g(R(X_i, X_j)X_k, X_l).$$

Then

$$R^l_{ijk} = \sum_{s=1}^m g_{sl}\Big(\frac{\partial \Gamma^s_{jk}}{\partial x_i} - \frac{\partial \Gamma^s_{ik}}{\partial x_j} + \sum_{r=1}^m \{\Gamma^r_{jk}\cdot\Gamma^s_{ir} - \Gamma^r_{ik}\cdot\Gamma^s_{jr}\}\Big),$$

where the functions Γ^k_{ij} are the Christoffel symbols of the Levi-Civita connection ∇ of (M,g) with respect to (U,x), see Example 6.13.

Proof. Using the fact that $[X_i, X_j] = 0$, see Proposition 4.30, we then obtain

$$\begin{aligned}
&R(X_i,X_j)X_k = \nabla_{X_i}\nabla_{X_j}X_k - \nabla_{X_j}\nabla_{X_i}X_k \\
&= \sum_{s=1}^m \{\nabla_{X_i}(\Gamma^s_{jk}\cdot X_s) - \nabla_{X_j}(\Gamma^s_{ik}\cdot X_s)\} \\
&= \sum_{s=1}^m \Big(\frac{\partial \Gamma^s_{jk}}{\partial x_i}\cdot X_s + \sum_{r=1}^m \Gamma^s_{jk}\Gamma^r_{is}X_r - \frac{\partial \Gamma^s_{ik}}{\partial x_j}\cdot X_s - \sum_{r=1}^m \Gamma^s_{ik}\Gamma^r_{js}X_r\Big) \\
&= \sum_{s=1}^m \Big(\frac{\partial \Gamma^s_{jk}}{\partial x_i} - \frac{\partial \Gamma^s_{ik}}{\partial x_j} + \sum_{r=1}^m \{\Gamma^r_{jk}\Gamma^s_{ir} - \Gamma^r_{ik}\Gamma^s_{jr}\}\Big)X_s.
\end{aligned}$$

□

Example 8.17. Let $E^m = (\mathbb{R}^m, \langle,\rangle)$ be the standard m-dimensional Euclidean space. Then the set

$$\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\dots, \frac{\partial}{\partial x_m}\}$$

is a global frame for the tangent bundle $T\mathbb{R}^m$. In this situation we have $g_{ij} = \delta_{ij}$, so $\Gamma^k_{ij} \equiv 0$ by Example 6.13. This implies that $R \equiv 0$ so E^m is flat.

We will now present the famous **Gauss equation** comparing the curvature tensor of a submanifold and that of its ambient space in terms of the second fundamental form of the submanifold. This is a fundamental result in Riemannian geometry.

Theorem 8.18. *Let (N,h) be a Riemannian manifold with Levi-Civita connection ∇. Further let (M,g) be a submanifold of N equipped with the induced metric and Levi-Civita connection $\tilde\nabla$. Let $X,Y,Z,W \in C^\infty(TN)$ be vector fields on N extending $\tilde X, \tilde Y, \tilde Z, \tilde W \in C^\infty(TM)$ on M. Then we have*

$$\begin{aligned}
&g(\tilde R(\tilde X,\tilde Y)\tilde Z,\tilde W) - h(R(X,Y)Z,W) \\
= \ &h(B(\tilde Y,\tilde Z),B(\tilde X,\tilde W)) - h(B(\tilde X,\tilde Z),B(\tilde Y,\tilde W)).
\end{aligned}$$

Here $\tilde{R}$ and R are the Riemann curvature tensors of (M, g) and (N, h), respectively, and B the second fundamental of M as a submanifold of N.

Proof. Employing the definitions of the curvature tensors $\tilde{R}$, R, the Levi-Civita connection $\tilde{\nabla}$ and the second fundamental form B of M as a submanifold of N we obtain the following:

$$\begin{aligned}
& g(\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z}, \tilde{W}) \\
&= g(\tilde{\nabla}_{\tilde{X}}\tilde{\nabla}_{\tilde{Y}}\tilde{Z} - \tilde{\nabla}_{\tilde{Y}}\tilde{\nabla}_{\tilde{X}}\tilde{Z} - \tilde{\nabla}_{[\tilde{X}, \tilde{Y}]}\tilde{Z}, \tilde{W}) \\
&= h((\nabla_X(\nabla_Y Z)^\top)^\top - (\nabla_Y(\nabla_X Z)^\top)^\top - (\nabla_{[X, Y]} Z)^\top, W) \\
&= h((\nabla_X(\nabla_Y Z - (\nabla_Y Z)^\perp))^\top - (\nabla_Y(\nabla_X Z - (\nabla_X Z)^\perp))^\top, W) \\
&\quad - h((\nabla_{[X, Y]} Z - (\nabla_{[X, Y]} Z)^\perp)^\top, W) \\
&= h(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, W) \\
&\quad - h((\nabla_X(\nabla_Y Z)^\perp, W) + h(\nabla_Y(\nabla_X Z)^\perp, W) \\
&= h(R(X, Y)Z, W) \\
&\quad + h((\nabla_Y Z)^\perp, (\nabla_X W)^\perp) - h((\nabla_X Z)^\perp, (\nabla_Y W)^\perp) \\
&= h(R(X, Y)Z, W) \\
&\quad + h(B(\tilde{Y}, \tilde{Z}), B(\tilde{X}, \tilde{W})) - h(B(\tilde{X}, \tilde{Z}), B(\tilde{Y}, \tilde{W})).
\end{aligned}$$

$\square$

We will now employ the Gauss equation to the classical situation of a surface in the three dimensional Euclidean space.

Example 8.19. Let Σ be a regular surface in the Euclidean 3-dimensional $\mathbb{E} = (\mathbb{R}^3, \langle, \rangle)$. Let $\{\tilde{X}, \tilde{Y}\}$ be a local orthonormal frame for the tangent bundle $T\Sigma$ of Σ around a point $p \in \Sigma$ and $\tilde{N}$ be the local Gauss map with $\tilde{N} = \tilde{X} \times \tilde{Y}$. Further let X, Y, N be local extensions of $\tilde{X}, \tilde{Y}, \tilde{N}$, such that $\{X, Y, N\}$ is a local orthonormal frame for $T\mathbb{R}^3$. Then the second fundamental form B of Σ in $\mathbb{R}^3$ satisfies

$$\begin{aligned}
B(\tilde{X}, \tilde{Y}) &= (\partial_X Y)^\perp \\
&= <\partial_X Y, N> N \\
&= -<Y, \partial_X N> N \\
&= -<Y, dN(X)> N \\
&= <\tilde{Y}, S_p(\tilde{X})> \tilde{N},
\end{aligned}$$

where $S_p : T_p\Sigma \to T_p\Sigma$ is the shape operator of Σ at p. If we now apply the fact that $\mathbb{R}^3$ is flat, then the Gauss equation tells us that the sectional curvature $K(\tilde{X}, \tilde{Y})$ of Σ at p satisfies

$$\begin{aligned} K(\tilde{X}, \tilde{Y}) &= < \tilde{R}(\tilde{X}, \tilde{Y})\tilde{Y}, \tilde{X} > \\ &= < B(\tilde{Y}, \tilde{Y}), B(\tilde{X}, \tilde{X}) > - < B(\tilde{X}, \tilde{Y}), B(\tilde{Y}, \tilde{X}) > \\ &= \det S_p. \end{aligned}$$

In other words, the sectional curvature $K(\tilde{X}, \tilde{Y})$ is the determinant of the shape operator S_p i.e. the classical **Gaussian curvature**.

An interesting consequence of the Gauss equation is the following useful result. For important applications see Exercises 8.7 and 8.8.

Corollary 8.20. *Let (N, h) be a Riemannian manifold and M be a totally geodesic submanifold of N equipped with the induced metric g. Let $X, Y, Z, W \in C^\infty(TN)$ be vector fields extending $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W} \in C^\infty(TM)$. Then we have*

$$g(\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z}, \tilde{W}) = h(R(X, Y)Z, W).$$

Proof. This follows directly from the fact that the second fundamental for B of M in N vanishes identically. □

Corollary 8.21. *Let (N, h) be a Riemannian manifold and M be a totally geodesic submanifold of N equipped with the induced metric g. Let $X, Y \in C^\infty(TN)$ be orthogonal unit vector fields extending $\tilde{X}, \tilde{Y} \in C^\infty(TM)$. Then at a point $p \in M$ we have*

$$\tilde{K}_p(\tilde{X}, \tilde{Y}) = K_p(X, Y).$$

Here $\tilde{K}$ and K are the sectional curvatures on (M, g) and (N, h), respectively.

Proof. The statement is a direct consequence of Corollary 8.20. □

Example 8.22. The unit sphere S^m in the standard Euclidean $\mathbb{E}^{m+1}$ has constant sectional curvature $+1$ (see Exercises 8.6 and 8.7) and the real hyperbolic space H^m has constant sectional curvature -1 (see Exercise 8.8).

The next example provides an interesting geometric connection between the classical Gaussian curvature of a surface and the sectional curvature operator of its general Riemannian ambient manifold.

Example 8.23. Let (M, g) be a Riemannian manifold, $p \in M$ and V be a section at p i.e. a 2-dimensional subspace of the tangent space T_pM. Further, let U_p be an open neighbourhood of T_pM, containing the origin $0 \in T_pM$, such that the exponential map $\exp_p : U_p \to M$ is a local diffeomorphism onto the open image $\exp_p(U_p)$ in M. Then

$$\Sigma_p(V) = \exp_p(U_p \cap V)$$

is a Gaussian surface in M i.e. a 2-dimensional submanifold with the induced metric. Further let ∇ and $\tilde{\nabla}$ be the Levi-Civita connections on M and $\Sigma_p(V)$, respectively, and B be the second fundamental form of $\Sigma_p(V)$ in M.

If $X \in V$ is a tangent vector then the curve $\gamma : I \to \Sigma_p(V)$ with $\gamma(s) = \exp_p(s \cdot X)$ is a geodesic in M such that $\gamma(0) = p$ and $\dot{\gamma}(0) = X$. This implies that, at the point $p \in \Sigma$, we have

$$0 = \nabla_X X = (\nabla_X X)^{\top} + (\nabla_X X)^{\perp} = \tilde{\nabla}_X X + B(X, X).$$

In particular, $B(X, X) = 0$ for all $X \in V$. If $\{X, Y\}$ is an orthonormal basis for V, then the polar identity gives

$$4 \cdot B(X, Y) = B(X + Y, X + Y) - B(X - Y, X - Y) = 0.$$

This shows that the second fundamental form B vanishes at the point p. It then follows by the Gauss equation in Theorem 8.18 that the sectional curvature $K_p(V)$ and the Gaussian curvature of $\Sigma_p(V)$ are equal at p.

Our next aim is to show that the curvature tensor, of a manifold of constant sectional curvature, has a rather simple form. This we present as Theorem 8.28. But first we need some preparations.

Lemma 8.24. *Let (M, g) be a Riemannian manifold, $p \in M$ and $Y \in T_pM$. Then the linear map $\bar{Y} : T_pM \to T_pM$ given by*

$$\bar{Y} : X \mapsto R(X, Y)Y$$

is a symmetric endomorphism of the tangent space T_pM.

PROOF. If $X, Y, Z \in T_pM$ then it follows from Proposition 8.10 that

$$\begin{aligned} g(\bar{Y}(X), Z) &= g(R(X, Y)Y, Z) \\ &= g(R(Y, Z)X, Y) \\ &= g(R(Z, Y)Y, X) \\ &= g(X, \bar{Y}(Z)). \end{aligned}$$

$\square$

Remark 8.25. For a Riemannian manifold (M, g) and $p \in M$ let $Y \in T_pM$ be a tangent vector at p with $|Y| = 1$. Further let $\mathcal{N}(Y)$ be the orthogonal complement of the line generated by Y in T_pM i.e.

$$\mathcal{N}(Y) = \{X \in T_pM \mid g(X, Y) = 0\}.$$

The fact that $\bar{Y}(Y) = 0$ and Lemma 8.24 ensure the existence of an orthonormal basis of eigenvectors $X_1, \dots, X_{m-1}$ of the restriction of the symmetric endomorphism $\bar{Y}$ to $\mathcal{N}(Y)$. Without loss of generality, we can assume that the corresponding eigenvalues satisfy

$$\lambda_1(p) \leq \cdots \leq \lambda_{m-1}(p).$$

If $X \in \mathcal{N}(Y)$, $|X| = 1$ and $\bar{Y}(X) = \lambda \cdot X$ then

$$K_p(X, Y) = g(R(X, Y)Y, X) = g(\bar{Y}(X), X) = \lambda.$$

This means that the eigenvalues must satisfy the following inequalities

$$\delta(p) \leq \lambda_1(p) \leq \cdots \leq \lambda_{m-1}(p) \leq \Delta(p).$$

In order to prove the interesting result of Theorem 8.28 we introduce the following tensor field.

Definition 8.26. For a Riemannian manifold (M, g) let the tensor field $R_1 : C_3^\infty(TM) \to C_1^\infty(TM)$, of type $(1, 3)$, be defined by

$$R_1(X, Y)Z = g(Y, Z)X - g(X, Z)Y.$$

We now have the following useful technical lemma. The proof is based on standard arguments from linear algebra.

Lemma 8.27. *Let (M, g) be a Riemannian manifold and $X, Y, Z \in C^\infty(TM)$ be vector fields on M. Then*

(i) $|R(X, Y)Y - \frac{\delta+\Delta}{2} R_1(X, Y)Y| \leq \frac{1}{2}(\Delta - \delta)|X||Y|^2$

(ii) $|R(X, Y)Z - \frac{\delta+\Delta}{2} R_1(X, Y)Z| \leq \frac{2}{3}(\Delta - \delta)|X||Y||Z|$

Proof. Because of linearity we can, without loss of generality, assume that $|X| = |Y| = |Z| = 1$. If $X = X^\perp + X^\top$ with $X^\perp \perp Y$ and $X^\top$ is a multiple of Y then $R(X, Y)Z = R(X^\perp, Y)Z$ and $|X^\perp| \leq |X|$ so we can also assume that $X \perp Y$. Then

$$R_1(X, Y)Y = g(Y, Y)X - g(X, Y)Y = X.$$

The first statement (i) follows from the fact that the symmetric endomorphism of T_pM with

$$X \mapsto \big(R(X, Y)Y - \frac{\Delta + \delta}{2} \cdot X\big)$$

restricted to $\mathcal{N}(Y)$ has eigenvalues in the closed interval $[\frac{\delta-\Delta}{2}, \frac{\Delta-\delta}{2}]$.

It is easily checked that the operator R_1 satisfies the symmetry conditions of Proposition 8.10 and hence $D = R - \frac{\Delta+\delta}{2} \cdot R_1$ does so as well. This implies that

$$\begin{aligned} 6 \cdot D(X,Y)Z &= D(X,Y+Z)(Y+Z) - D(X,Y-Z)(Y-Z) \\ &+ D(X+Z,Y)(X+Z) - D(X-Z,Y)(X-Z). \end{aligned}$$

The second statement (ii) then follows from (i) and

$$\begin{aligned} 6|D(X,Y)Z| &\leq \frac{1}{2}(\Delta-\delta)\{|X|(|Y+Z|^2+|Y-Z|^2) \\ &\qquad +|Y|(|X+Z|^2+|X-Z|^2)\} \\ &= \frac{1}{2}(\Delta-\delta)\{2|X|(|Y|^2+|Z|^2)+2|Y|(|X|^2+|Z|^2)\} \\ &= 4(\Delta-\delta). \end{aligned}$$

□

The following result is an immediate consequence of Lemma 8.27.

Theorem 8.28. *Let (M,g) be a Riemannian manifold of constant sectional curvature κ. Then its curvature tensor R satisfies*

$$R(X,Y)Z = \kappa \cdot (g(Y,Z)X - g(X,Z)Y).$$

Proof. The result is an immediate consequence of Lemma 8.27 and the fact that $\kappa = \delta = \Delta$. □

The following result shows that the curvature tensor takes a rather simple form for the important class of Lie groups treated in Proposition 6.12.

Proposition 8.29. *Let (G,g) be a Lie group equipped with a left-invariant metric, such that for all $X \in \mathfrak{g}$ the endomorphism*

$$\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$$

is skew-symmetric, with respect to g. Then, for left-invariant vector fields $X,Y,Z \in \mathfrak{g}$, the curvature tensor R satisfies

$$R(X,Y)Z = -\frac{1}{4} \cdot [[X,Y],Z].$$

Proof. See Exercise 8.9. □

Corollary 8.30. *Let (G,g) be a Lie group equipped with a left-invariant metric, such that for all $Z \in \mathfrak{g}$ the endomorphism*

$$\mathrm{ad}_Z : \mathfrak{g} \to \mathfrak{g}$$

is skew-symmetric, with respect to g. Let $X, Y \in \mathfrak{g}$ be left-invariant vector fields such that $|X| = |Y| = 1$ and $g(X, Y) = 0$. Then the sectional curvature $K(X, Y)$ satisfies

$$K(X,Y) = \frac{1}{4} \cdot |[X,Y]|^2 \geq 0.$$

PROOF. See Exercise 8.10. $\square$

We conclude this chapter by defining the Ricci and scalar curvatures of a Riemannian manifold. These are obtained by taking traces over the curvature tensor and play an important role in Riemannian geometry.

Definition 8.31. Let (M, g) be a Riemannian manifold, then we define

(i) the **Ricci operator** $\mathrm{ric} : C_1^\infty(TM) \to C_1^\infty(M)$ by

$$\mathrm{ric}(X) = \sum_{i=1}^m R(X, e_i)e_i,$$

(ii) the **Ricci curvature** $\mathrm{Ric} : C_2^\infty(TM) \to C_0^\infty(TM)$ by

$$\mathrm{Ric}(X,Y) = \sum_{i=1}^m g(R(X, e_i)e_i, Y),$$

(iii) the **scalar curvature** $\mathrm{Scal} \in C^\infty(M)$ by

$$\mathrm{Scal} = \sum_{j=1}^m \mathrm{Ric}(e_j, e_j) = \sum_{j=1}^m \sum_{i=1}^m g(R(e_i, e_j)e_j, e_i).$$

Here $\{e_1, \dots, e_m\}$ is any local orthonormal frame for the tangent bundle.

In the case of constant sectional curvature we have the following result.

Corollary 8.32. *Let (M^m, g) be a Riemannian manifold of constant sectional curvature κ. Then its scalar curvature satisfies the following*

$$\mathrm{Scal} = m \cdot (m-1) \cdot \kappa.$$

PROOF. Let $\{e_1, \dots, e_m\}$ be any local orthonormal frame. Then Theorem 8.28 implies that

$$\begin{aligned} \mathrm{Ric}(e_j, e_j) &= \sum_{i=1}^m g(R(e_j, e_i)e_i, e_j) \\ &= \sum_{i=1}^m g(\kappa(g(e_i, e_i)e_j - g(e_j, e_i)e_i), e_j) \end{aligned}$$

$$
\begin{aligned}
&= \kappa\left(\sum_{i=1}^{m} g(e_i, e_i)g(e_j, e_j) - \sum_{i=1}^{m} g(e_i, e_j)g(e_i, e_j)\right) \\
&= \kappa\left(\sum_{i=1}^{m} 1 - \sum_{i=1}^{m} \delta_{ij}\right) = (m-1)\cdot\kappa.
\end{aligned}
$$

To obtain the formula for the scalar curvature Scal we only need to multiply the constant Ricci curvature $\mathrm{Ric}(e_j, e_j)$ by m. $\square$

As a reference on further notions of curvature we recommend the interesting book, W. Kühnel, Differential Geometry: Curves - Surfaces - Manifolds, Student Mathematical Library **77**, AMS (2015).

Exercises

Exercise 8.1. Let (M, g) be a Riemannian manifold. Prove that the tensor field g of type $(0, 2)$ is parallel with respect to the Levi-Civita connection.

Exercise 8.2. Let (M, g) be a Riemannian manifold. Prove that the Riemann curvature operator R is a tensor field of type $(1, 3)$.

Exercise 8.3. Find a proof for Proposition 8.10.

Exercise 8.4. Find a proof for Lemma 8.13.

Exercise 8.5. Let $\mathbb{R}^m$ and $\mathbb{C}^m$ be equipped with their standard Euclidean metric g given by

$$g(z, w) = \text{Re} \sum_{k=1}^{m} z_k \bar{w}_k$$

and let $T^m = \{z \in \mathbb{C}^m \,|\, |z_1| = ... = |z_m| = 1\}$ be the m-dimensional torus in $\mathbb{C}^m$ with the induced metric. Find an isometric immersion $\phi : \mathbb{R}^m \to T^m$, determine all geodesics on T^m and prove that the torus is flat.

Exercise 8.6. Let the Lie group $S^3 \cong \mathbf{SU}(2)$ be equipped with the Riemannian metric

$$g(Z, W) = \tfrac{1}{2} \cdot \text{Re}(\text{trace}(\bar{Z}^t W)).$$

(i) Find an orthonormal basis for $T_e\mathbf{SU}(2)$.
(ii) Prove that $(\mathbf{SU}(2), g)$ has constant sectional curvature $+1$.

Exercise 8.7. Let S^m be the unit sphere in $\mathbb{R}^{m+1}$ equipped with the standard Euclidean metric $\langle , \rangle$. Use the results of Corollaries 7.29, 8.20 and Exercise 8.6 to prove that $(S^m, \langle , \rangle)$ has constant sectional curvature $+1$.

Exercise 8.8. Let H^m be the m-dimensional hyperbolic space modelled on the upper half space $\mathbb{R}^+ \times \mathbb{R}^{m-1}$ equipped with the Riemannian metric

$$g(X, Y) = \frac{1}{x_1^2} \cdot \langle X, Y \rangle,$$

where $x = (x_1, \ldots, x_m) \in H^m$. For $k = 1, \ldots, m$ let the vector fields $X_k \in C^\infty(TH^m)$ be given by

$$(X_k)_x = x_1 \cdot \frac{\partial}{\partial x_k}$$

and define the operation $*$ on H^m by

$$(\alpha, x) * (\beta, y) = (\alpha \cdot \beta, \alpha \cdot y + x).$$

Prove that

(i) $(H^m, *)$ is a Lie group,
(ii) the vector fields $X_1, \dots, X_m$ are left-invariant,
(iii) $[X_k, X_l] = 0$ and $[X_1, X_k] = X_k$ for $k, l = 2, ..., m$,
(iv) the metric g is left-invariant,
(v) (H^m, g) has constant curvature -1.

Compare with Exercises 6.4 and 7.1.

Exercise 8.9. Find a proof for Proposition 8.29.

Exercise 8.10. Find a proof for Corollary 8.30.

CHAPTER 9

Curvature and Local Geometry

This chapter is devoted to the study of the local geometry of a Riemannian manifold and how this is controlled by its curvature tensor. For this we introduce the notion of a Jacobi field which is a standard tool in differential geometry. With this at hand we obtain a fundamental comparison result describing the curvature dependence of local distances.

Definition 9.1. Let (M, g) be a Riemannian manifold. By a 1-parameter **family of geodesics** we mean a C^3-map

$$\Phi : (-\epsilon, \epsilon) \times I \to M$$

such that the curve $\gamma_t : I \to M$ given by $\gamma_t : s \mapsto \Phi(t, s)$ is a geodesic for all $t \in (-\epsilon, \epsilon)$. The variable $t \in (-\epsilon, \epsilon)$ is called the **family parameter** of Φ.

The following result suggests that the Riemann curvature tensor is closely related to the local behaviour of geodesics.

Proposition 9.2. *Let (M, g) be a Riemannian manifold and $\Phi : (-\epsilon, \epsilon) \times I \to M$ be a 1-parameter family of geodesics. Then for each $t \in (-\epsilon, \epsilon)$ the vector field $J_t : I \to TM$ along γ_t, given by*

$$J_t(s) = \frac{\partial \Phi}{\partial t}(t, s),$$

satisfies the second order linear ordinary differential equation

$$\nabla_{\dot{\gamma}_t} \nabla_{\dot{\gamma}_t} J_t + R(J_t, \dot{\gamma}_t)\dot{\gamma}_t = 0.$$

Proof. Along Φ we define the vector fields $X(t, s) = \partial\Phi/\partial s$ and $J(t, s) = \partial\Phi/\partial t$. The fact that $[\partial/\partial t, \partial/\partial s] = 0$ implies that

$$[J, X] = [d\Phi(\partial/\partial t), d\Phi(\partial/\partial s)] = d\Phi([\partial/\partial t, \partial/\partial s]) = 0.$$

Since Φ is a family of geodesics we have $\nabla_X X = 0$ and the definition of the curvature tensor then implies that

$$\begin{aligned} R(J, X)X &= \nabla_J \nabla_X X - \nabla_X \nabla_J X - \nabla_{[J, X]} X \\ &= -\nabla_X \nabla_J X \end{aligned}$$

$$= \quad -\nabla_X \nabla_X J.$$

Hence for each $t \in (-\epsilon, \epsilon)$ we have

$$\nabla_{\dot{\gamma}_t} \nabla_{\dot{\gamma}_t} J_t + R(J_t, \dot{\gamma}_t)\dot{\gamma}_t = 0.$$

□

The result of Proposition 9.2 leads to the following natural notion.

Definition 9.3. Let (M, g) be a Riemannian manifold, $\gamma : I \to M$ be a geodesic and $X = \dot{\gamma}$ be the tangent vector field along γ. A C^2 vector field J along γ is called a **Jacobi field** if and only if

$$\nabla_X \nabla_X J + R(J, X)X = 0 \tag{9.1}$$

along γ. We denote the space of all Jacobi fields along γ by $\mathcal{J}_\gamma(TM)$.

We now give an example of a 1-parameter family of geodesics in the Euclidean space E^{m+1}.

Example 9.4. Let $c, n : \mathbb{R} \to E^{m+1}$ be smooth curves such that the image $n(\mathbb{R})$ of n is contained in the unit sphere S^m. If we define a map $\Phi : \mathbb{R} \times \mathbb{R} \to E^{m+1}$ by

$$\Phi : (t, s) \mapsto c(t) + s \cdot n(t)$$

then for each $t \in \mathbb{R}$ the curve $\gamma_t : s \mapsto \Phi(t, s)$ is a straight line and hence a geodesic in E^{m+1}. By differentiating this with respect to the family parameter t we yield the Jacobi field $J \in \mathcal{J}_{\gamma_0}(TE^{m+1})$ along γ_0 satisfying

$$J(s) = \frac{d}{dt}\Phi(t, s)|_{t=0} = \dot{c}(0) + s \cdot \dot{n}(0).$$

The Jacobi equation (9.1) is linear in J. This means that the space of Jacobi fields $\mathcal{J}_\gamma(TM)$, along the geodesic γ, is a vector space. We are now interested in determining its dimension.

Proposition 9.5. *Let (M^m, g) be a Riemannian manifold, $p \in M$, $\gamma : I \to M$ be a geodesic with $\gamma(0) = p$ and $X = \dot{\gamma}$ be the tangent vector field along γ. If $v, w \in T_pM$ are two tangent vectors at p then there exists a unique Jacobi field J along γ such that*

$$J_p = v \quad \text{and} \quad (\nabla_X J)_p = w.$$

Proof. In the spirit of Proposition 7.8 let $\{X_1, \dots, X_m\}$ be an orthonormal frame of parallel vector fields along γ. If J is a vector field along γ then

$$J = \sum_{i=1}^m a_i X_i,$$

where $a_i = g(J, X_i)$ are C^2-functions on the real interval I. The vector fields $X_1, \dots, X_m$ are parallel so

$$\nabla_X J = \sum_{i=1}^m \dot{a}_i X_i \quad \text{and} \quad \nabla_X \nabla_X J = \sum_{i=1}^m \ddot{a}_i X_i.$$

For the curvature tensor we have

$$R(X_i, X)X = \sum_{k=1}^m b_i^k X_k,$$

where $b_i^k = g(R(X_i, X)X, X_k)$ are smooth functions on the real interval I, heavily depending on the geometry of (M, g). This means that $R(J, X)X$ is given by

$$R(J, X)X = \sum_{i,k=1}^m a_i b_i^k X_k$$

and that J is a Jacobi field if and only if

$$\sum_{i=1}^m (\ddot{a}_i + \sum_{k=1}^m a_k b_k^i) X_i = 0.$$

This is clearly equivalent to the following second order system of linear ordinary differential equations in $a = (a_1, \dots, a_m) : I \to \mathbb{R}^m$

$$\ddot{a}_i + \sum_{k=1}^m a_k b_k^i = 0 \quad \text{for all } i = 1, 2, \dots, m.$$

A global solution will always exist and is uniquely determined by the initial values $a(0)$ and $\dot{a}(0)$. This implies that the Jacobi field J exists globally and is uniquely determined by the initial conditions

$$J(0) = v \quad \text{and} \quad (\nabla_X J)(0) = w.$$

□

As an immediate consequence of Proposition 9.5 we have the following interesting result.

Corollary 9.6. *Let (M^m, g) be a Riemannian manifold and $\gamma : I \to M$ be a geodesic in M. Then the vector space $\mathcal{J}_\gamma(TM)$, of Jacobi fields along γ, has the dimension $2m$.*

The following Lemma 9.7 shows that when proving results about Jacobi fields along a geodesic γ we can always assume, without loss of generality, that that they are parametrised by arclength i.e. $|\dot{\gamma}| = 1$.

Lemma 9.7. *Let (M, g) be a Riemannian manifold, $\gamma : I \to M$ be a geodesic and J be a Jacobi field along γ. If λ is a non-zero real number and $\sigma : \lambda I \to I$ is given by $\sigma : t \mapsto t/\lambda$, then $\gamma \circ \sigma : \lambda I \to M$ is a geodesic and $J \circ \sigma$ is a Jacobi field along $\gamma \circ \sigma$.*

Proof. See Exercise 9.1. □

The next result shows that both the tangential and the normal parts of a Jacobi field are again Jacobi fields. Furthermore we completely determine the tangential Jacobi fields.

Proposition 9.8. *Let (M, g) be a Riemannian manifold, $\gamma : I \to M$ be a geodesic with $|\dot\gamma| = 1$ and J be a Jacobi field along γ. Let $J^\top$ be the tangential part of J given by*

$$J^\top = g(J, \dot\gamma)\dot\gamma \quad \textit{and} \quad J^\perp = J - J^\top$$

be its normal part. Then $J^\top$ and $J^\perp$ are Jacobi fields along γ and there exist $a, b \in \mathbb{R}$ such that $J^\top(s) = (as + b)\dot\gamma(s)$ for all $s \in I$.

Proof. In this situation we have

$$\begin{aligned}
\nabla_{\dot\gamma}\nabla_{\dot\gamma}J^\top + R(J^\top, \dot\gamma)\dot\gamma &= \nabla_{\dot\gamma}\nabla_{\dot\gamma}(g(J, \dot\gamma)\dot\gamma) + R(g(J, \dot\gamma)\dot\gamma, \dot\gamma)\dot\gamma \\
&= g(\nabla_{\dot\gamma}\nabla_{\dot\gamma}J, \dot\gamma)\dot\gamma \\
&= -g(R(J, \dot\gamma)\dot\gamma, \dot\gamma)\dot\gamma \\
&= 0.
\end{aligned}$$

This shows that the tangential part $J^\top$ of J is a Jacobi field. The fact that $\mathcal{J}_\gamma(TM)$ is a vector space implies that the normal part $J^\perp = J - J^\top$ of J also is a Jacobi field.

By differentiating $g(J, \dot\gamma)$ twice along γ we obtain

$$\frac{d^2}{ds^2}(g(J, \dot\gamma)) = g(\nabla_{\dot\gamma}\nabla_{\dot\gamma}J, \dot\gamma) = -g(R(J, \dot\gamma)\dot\gamma, \dot\gamma) = 0$$

so $g(J, \dot\gamma(s)) = (as + b)$ for some $a, b \in \mathbb{R}$. □

Corollary 9.9. *Let (M, g) be a Riemannian manifold, $\gamma : I \to M$ be a geodesic and J be a Jacobi field along γ. If*

$$g(J(t_0), \dot\gamma(t_0)) = 0 \quad \textit{and} \quad g((\nabla_{\dot\gamma}J)(t_0), \dot\gamma(t_0)) = 0$$

for some $t_0 \in I$, then $g(J(t), \dot\gamma(t)) = 0$ for all $t \in I$.

Proof. This is a direct consequence of the fact that the function $g(J, \dot\gamma)$ satisfies the second order ordinary differential equation $\ddot f = 0$ and the initial conditions $f(t_0) = 0$ and $\dot f(t_0) = 0$. □

Our next aim is to show that if the Riemannian manifold (M, g) has constant sectional curvature then we can completely solve the Jacobi equation

$$\nabla_X \nabla_X J + R(J, X)X = 0$$

along any given geodesic $\gamma : I \to M$. For this we introduce the following useful notation. For a real number $\kappa \in \mathbb{R}$ we define the functions $c_\kappa, s_\kappa : \mathbb{R} \to \mathbb{R}$ by

$$c_\kappa(s) = \begin{cases} \cosh(\sqrt{|\kappa|}s) & \text{if } \kappa < 0, \\ 1 & \text{if } \kappa = 0, \\ \cos(\sqrt{\kappa}s) & \text{if } \kappa > 0. \end{cases}$$

and

$$s_\kappa(s) = \begin{cases} \sinh(\sqrt{|\kappa|}s)/\sqrt{|\kappa|} & \text{if } \kappa < 0, \\ s & \text{if } \kappa = 0, \\ \sin(\sqrt{\kappa}s)/\sqrt{\kappa} & \text{if } \kappa > 0. \end{cases}$$

It is a well known fact that the unique solution to the initial value problem

$$\ddot{f} + \kappa \cdot f = 0, \quad f(0) = a \text{ and } \dot{f}(0) = b$$

is the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(s) = a \cdot c_\kappa(s) + b \cdot s_\kappa(s)$.

We now give examples of Jacobi fields in the three model geometries of dimension two, the Euclidean plane, the sphere and hyperbolic plane, all of constant sectional curvature.

Example 9.10. Let $\mathbb{C}$ be the complex plane equipped with the standard Euclidean metric $\langle , \rangle$ of constant sectional curvature $\kappa = 0$. The rotations about the origin produce a 1-parameter family of geodesics $\Phi_t : s \mapsto s \cdot e^{it}$. Along the geodesic $\gamma_0 : s \mapsto s$ we yield the Jacobi field

$$J_0(s) = \frac{\partial \Phi_t}{\partial t}(0, s) = is$$

with $|J_0(s)|^2 = s^2 = |s_\kappa(s)|^2$.

Example 9.11. Let S^2 be the unit sphere in the standard three dimensional Euclidean space $\mathbb{C} \times \mathbb{R}$ equipped with the induced metric of constant sectional curvature $\kappa = +1$. Rotations about the $\mathbb{R}$-axis produce a 1-parameter family of geodesics $\Phi_t : s \mapsto (\sin(s) \cdot e^{it}, \cos(s))$. Along the geodesic $\gamma_0 : s \mapsto (\sin(s), \cos(s))$ we have the Jacobi field

$$J_0(s) = \frac{\partial \Phi_t}{\partial t}(0, s) = (i \sin(s), 0)$$

with $|J_0(s)|^2 = \sin^2(s) = |s_\kappa(s)|^2$.

Example 9.12. Let $B_1^2(0)$ be the open unit disk in the complex plane equipped with the hyperbolic metric

$$g(X,Y) = \frac{4}{(1-|z|^2)^2}\langle X, Y\rangle$$

of constant sectional curvature $\kappa = -1$. Rotations about the origin produce a 1-parameter family of geodesics $\Phi_t : s \mapsto \tanh(s/2) \cdot e^{it}$. Along the geodesic $\gamma_0 : s \mapsto \tanh(s/2)$ we obtain the Jacobi field

$$J_0(s) = \frac{\partial \Phi_t}{\partial t}(0,s) = i \cdot \tanh(s/2)$$

with

$$|J_0(s)|^2 = \frac{4 \cdot \tanh^2(s/2)}{(1-\tanh^2(s/2))^2} = \sinh^2(s) = |s_\kappa(s)|^2.$$

We are now ready to show that, in the case of constant sectional curvature, we can completely solve the Jacobi equation along any geodesic.

Example 9.13. Let (M,g) be a Riemannian manifold of constant sectional curvature κ and $\gamma : I \to M$ be a geodesic with $|X| = 1$ where $X = \dot\gamma$ is the tangent vector field along γ. Following Proposition 7.8 let $P_1, P_2, \dots, P_{m-1}$ be parallel vector fields along γ such that

$$g(P_i, P_j) = \delta_{ij} \quad \text{and} \quad g(P_i, X) = 0.$$

Then any vector field J along γ may be written as

$$J(s) = \sum_{i=1}^{m-1} f_i(s) P_i(s) + f_m(s) X(s).$$

Since the vector fields $P_1, P_2, \dots, P_{m-1}, X$ are parallel along the curve γ, this means that J is a Jacobi field if and only if

$$\begin{aligned}
\sum_{i=1}^{m-1} \ddot f_i(s) P_i(s) + \ddot f_m(s) X(s) &= \nabla_X \nabla_X J \\
&= -R(J,X)X \\
&= -R(J^\perp, X)X \\
&= -\kappa(g(X,X)J^\perp - g(J^\perp, X)X) \\
&= -\kappa J^\perp \\
&= -\kappa \sum_{i=1}^{m-1} f_i(s) P_i(s).
\end{aligned}$$

This is equivalent to the following system of ordinary differential equations

$$(9.2) \quad \ddot{f}_m(s) = 0 \ \text{ and } \ \ddot{f}_i(s) + \kappa f_i(s) = 0 \ \text{ for all } i = 1, 2, \dots, m-1.$$

It is clear that for the initial values

$$\begin{aligned} J(s_0) &= \sum_{i=1}^{m-1} v_i P_i(s_0) + v_m X(s_0), \\ (\nabla_X J)(s_0) &= \sum_{i=1}^{m-1} w_i P_i(s_0) + w_m X(s_0) \end{aligned}$$

or equivalently

$$f_i(s_0) = v_i \ \text{ and } \ \dot{f}_i(s_0) = w_i \ \text{ for all } \ i = 1, 2, \dots, m$$

we have a unique explicit solution to the system (9.2) on the whole of the interval I. It is given by

$$f_m(s) = v_m + s w_m \ \text{ and } \ f_i(s) = v_i c_\kappa(s) + w_i s_\kappa(s)$$

for all $i = 1, 2, \dots, m-1$. It should be noted that if $g(J, X) = 0$ and $J(0) = 0$ then

$$(9.3) \qquad |J(s)| = |(\nabla_X J)(0)| \cdot |s_\kappa(s)|.$$

In the next example we give a complete description of the Jacobi fields along a geodesic on the 2-dimensional sphere.

Example 9.14. Let S^2 be the unit sphere in the three dimensional Euclidean space $\mathbb{C} \times \mathbb{R}$ equipped with the induced metric of constant sectional curvature $\kappa = +1$. Further let $\gamma : \mathbb{R} \to S^2$ be the geodesic given by $\gamma : s \mapsto (e^{is}, 0)$. Then the tangent vector field along γ satisfies

$$\dot{\gamma}(s) = (ie^{is}, 0).$$

It then follows from Proposition 9.8 that all the Jacobi fields tangent to γ are given by

$$J^T_{(a,b)}(s) = (as + b)(ie^{is}, 0),$$

where $a, b \in \mathbb{R}$. The unit vector field $P : \mathbb{R} \to TS^2$ given by

$$s \mapsto ((e^{is}, 0), (0, 1))$$

is clearly normal along γ. In S^2 the tangent vector field $\dot{\gamma}$ is parallel along γ so P must be parallel. This implies that all the Jacobi fields orthogonal to $\dot{\gamma}$ are given by

$$J^N_{(a,b)}(s) = (0, a\cos s + b \sin s),$$

where $a, b \in \mathbb{R}$.

In the general situation, when we do not assume constant sectional curvature, the exponential map can be used to produce Jacobi fields as follows.

Example 9.15. Let (M, g) be a complete Riemannian manifold, $p \in M$ and $v, w \in T_pM$. Then $s \mapsto s(v + tw)$ defines a 1-parameter family of lines in the tangent space T_pM which all pass through the origin $0 \in T_pM$. Remember that the exponential map

$$\exp_p|_{B^m_{\varepsilon_p}(0)} : B^m_{\varepsilon_p}(0) \to \exp_p(B^m_{\varepsilon_p}(0))$$

maps lines in T_pM through the origin onto geodesics on M. Hence the map

$$\Phi_t : s \mapsto \exp_p(s(v + tw))$$

is a 1-parameter family of geodesics through $p \in M$, as long as $s(v+tw)$ is an element of $B^m_{\varepsilon_p}(0)$. This means that

$$J(s) = \frac{\partial \Phi_t}{\partial t}(t, s)|_{t=0} = d(\exp_p)_{s(v+tw)}(sw)|_{t=0} = d(\exp_p)_{sv}(sw)$$

is a Jacobi field along the geodesic $\gamma : s \mapsto \Phi_0(s)$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. Here

$$d(\exp_p)_{s(v+tw)} : T_{s(v+tw)}T_pM \to T_{\exp_p(s(v+tw))}M$$

is the linear tangent map of the exponential map $\exp_p$ at $s(v + tw)$. Now differentiating with respect to the parameter s gives

$$(\nabla_X J)(0) = \frac{d}{ds}(d(\exp_p)_{sv}(sw))|_{s=0} = d(\exp_p)_0(w) = w.$$

The above calculations show that

$$J(0) = 0 \quad \text{and} \quad (\nabla_X J)(0) = w. \tag{9.4}$$

For the proof of our main result, stated in Theorem 9.17, we need the following technical lemma.

Lemma 9.16. *Let (M, g) be a Riemannian manifold with sectional curvature uniformly bounded above by Δ and $\gamma : [0, \alpha] \to M$ be a geodesic on M with $|X| = 1$ where $X = \dot{\gamma}$. Further let $J : [0, \alpha] \to TM$ be a Jacobi field along γ such that $g(J, X) = 0$ and $|J| \neq 0$ on $(0, \alpha)$. Then*

(i) $\frac{d^2}{ds^2}|J| + \Delta \cdot |J| \geq 0$,
(ii) if $f : [0, \alpha] \to \mathbb{R}$ *is a* C^2*-function such that*
  (a) $\ddot{f} + \Delta \cdot f = 0$ *and* $f > 0$ *on* $(0, \alpha)$,
  (b) $f(0) = |J|(0)$, *and*
  (c) $\dot{f}(0) = \frac{d}{ds}|J|(0)$,

then $f(s) \leq |J(s)|$ *on* $(0, \alpha)$,

(iii) if $J(0) = 0$, *then* $|\nabla_X J(0)| \cdot s_\Delta(s) \leq |J(s)|$ *for all* $s \in (0, \alpha)$.

Proof. (i) Using the facts that $|X| = 1$ and $\langle X, J\rangle = 0$ we obtain

$$\begin{aligned}
\frac{d^2}{ds^2}|J| &= \frac{d^2}{ds^2}\sqrt{g(J,J)} = \frac{d}{ds}\Big(\frac{g(\nabla_X J, J)}{|J|}\Big) \\
&= \frac{g(\nabla_X \nabla_X J, J)}{|J|} + \frac{|\nabla_X J|^2 |J|^2 - g(\nabla_X J, J)^2}{|J|^3} \\
&\geq \frac{g(\nabla_X \nabla_X J, J)}{|J|} \\
&= -\frac{g(R(J,X)X, J)}{|J|} \\
&= -K(X, J) \cdot |J| \\
&\geq -\Delta \cdot |J|.
\end{aligned}$$

(ii) Define the function $h : [0, \alpha) \to \mathbb{R}$ by

$$h(s) = \begin{cases} \frac{|J(s)|}{f(s)} & \text{if } s \in (0, \alpha), \\ \lim_{s \to 0} \frac{|J(s)|}{f(s)} = 1 & \text{if } s = 0. \end{cases}$$

Then

$$\begin{aligned}
\dot{h}(s) &= \frac{1}{f^2(s)}\Big\{\frac{d}{ds}|J(s)| \cdot f(s) - |J(s)| \cdot \dot{f}(s)\Big\} \\
&= \frac{1}{f^2(s)}\int_0^s \frac{d}{dt}\Big\{\frac{d}{dt}|J(t)| \cdot f(t) - |J(t)| \cdot \dot{f}(t)\Big\}dt \\
&= \frac{1}{f^2(s)}\int_0^s \Big\{\frac{d^2}{dt^2}|J(t)| \cdot f(t) - |J(t)| \cdot \ddot{f}(t)\Big\}dt \\
&= \frac{1}{f^2(s)}\int_0^s f(t) \cdot \Big\{\frac{d^2}{dt^2}|J(t)| + \Delta \cdot |J(t)|\Big\}dt \\
&\geq 0.
\end{aligned}$$

This implies that $\dot{h}(s) \geq 0$ so $f(s) \leq |J(s)|$ for all $s \in (0, \alpha)$.

(iii) The function $f(s) = |\nabla_X J(0)| \cdot s_\Delta(s)$ satisfies the differential equation

$$\ddot{f}(s) + \Delta f(s) = 0$$

and the initial conditions $f(0) = |J(0)| = 0$, $\dot{f}(0) = |\nabla_X J(0)|$ so it follows from (ii) that $|\nabla_X J(0)| \cdot s_\Delta(s) = f(s) \leq |J(s)|$ for all $s \in (0, \alpha)$. □

Let (M, g) be a Riemannian manifold of sectional curvature which is uniformly bounded above, i.e. there exists a $\Delta \in \mathbb{R}$ such that $K_p(V) \leq \Delta$ for all $V \in G_2(T_pM)$ and $p \in M$. Let (M_Δ, g_Δ) be another Riemannian manifold which is complete and of constant sectional curvature $K \equiv \Delta$. Let $p \in M$, $p_\Delta \in M_\Delta$ and identify $T_pM \cong \mathbb{R}^m \cong T_{p_\Delta}M_\Delta$.

Let U be an open neighbourhood of $\mathbb{R}^m$ around 0 such that the exponential maps $(\exp)_p$ and $(\exp)_{p_\Delta}$ are diffeomorphisms from U onto their images $(\exp)_p(\mathrm{U})$ and $(\exp)_{p_\Delta}(U)$, respectively. Let (r, p, q) be a geodesic triangle i.e. a triangle with sides which are shortest paths between their endpoints. Furthermore let $c : [a, b] \to M$ be the geodesic connecting r and q and $v : [a, b] \to T_pM$ be the curve defined by $c(t) = (\exp)_p(v(t))$. Put $c_\Delta(t) = (\exp)_{p_\Delta}(v(t))$ for $t \in [a, b]$ and then it directly follows that $c(a) = r$ and $c(b) = q$. Finally put $r_\Delta = c_\Delta(a)$ and $q_\Delta = c_\Delta(b)$.

Theorem 9.17. *For the above situation the following inequality for the distance function d is satisfied*

$$d(q_\Delta, r_\Delta) \leq d(q, r).$$

Proof. Define a 1-parameter family $s \mapsto s \cdot v(t)$ of straight lines in T_pM through 0. Then

$$\Phi_t : s \mapsto (\exp)_p(s \cdot v(t)) \quad \text{and} \quad \Phi_t^\Delta : s \mapsto (\exp)_{p_\Delta}(s \cdot v(t))$$

are 1-parameter families of geodesics through $p \in M$, and $p_\Delta \in M_\Delta$, respectively. Hence

$$J_t = \partial\Phi_t/\partial t \quad \text{and} \quad J_t^\Delta = \partial\Phi_t^\Delta/\partial t$$

are Jacobi fields satisfying the initial conditions

$$J_t(0) = 0 = J_t^\Delta(0) \quad \text{and} \quad (\nabla_X J_t)(0) = \dot{v}(t) = (\nabla_X J_t^\Delta)(0).$$

Employing Equation (9.3), Lemma 9.16 and the fact that M_Δ has constant sectional curvature Δ we now yield

$$\begin{aligned}
|\dot{c}_\Delta(t)| &= |J_t^\Delta(1)| \\
&= |(\nabla_X J_t^\Delta)(0)| \cdot s_\Delta(1) \\
&= |(\nabla_X J_t)(0)| \cdot s_\Delta(1) \\
&\leq |J_t(1)| \\
&= |\dot{c}(t)|
\end{aligned}$$

The curve c is the shortest path between r and q so we have

$$d(r_\Delta, q_\Delta) \leq L(c_\Delta) \leq L(c) = d(r, q).$$

□

We now add the assumption that the sectional curvature of the manifold (M, g) is uniformly bounded below i.e. there exists a $\delta \in \mathbb{R}$ such that $\delta \leq K_p(V)$ for all $V \in G_2(T_pM)$ and $p \in M$. Let (M_δ, g_δ) be a complete Riemannian manifold of constant sectional curvature δ. Let $p \in M$ and $p_\delta \in M_\delta$ and identify $T_pM \cong \mathbb{R}^m \cong T_{p_\delta}M_\delta$. Then a similar construction as above gives two pairs of points $q, r \in M$ and $q_\delta, r_\delta \in M_\delta$ and shows that

$$d(q, r) \leq d(q_\delta, r_\delta).$$

Combining these two results we obtain **locally**

$$d(q_\Delta, r_\Delta) \leq d(q, r) \leq d(q_\delta, r_\delta).$$

Exercises

Exercise 9.1. Find a proof of Lemma 9.7.

Exercise 9.2. Let (M, g) be a Riemannian manifold and $\gamma : I \to M$ be a geodesic such that $X = \dot{\gamma} \neq 0$. Further let J be a non-vanishing Jacobi field along γ with $g(X, J) = 0$. Prove that if $g(J, J)$ is constant along γ then (M, g) does not have strictly negative curvature.

APPENDIX A

A Note on Classical Lie Algebras

Let $\mathbb{R}^{m\times m}$ be the vector space of real $m\times m$ matrices. For this we have the direct sum

$$\mathbb{R}^{m\times m} = \mathrm{Skew}(\mathbb{R}^m) \oplus \mathrm{Sym}(\mathbb{R}^m)$$

of its linear subspaces

$$\mathrm{Skew}(\mathbb{R}^m) = \{X \in \mathbb{R}^{m\times m} \,|\, X^t + X = 0\}$$

and

$$\mathrm{Sym}(\mathbb{R}^m) = \{Y \in \mathbb{R}^{m\times m} \,|\, Y^t - Y = 0\}$$

of skew-symmetric and symmetric matrices, respectively. This means that every matrix $A \in \mathbb{R}^{m\times m}$ has a unique decomposition $A = X + Y$, where

$$X = \tfrac{1}{2}(A - A^t) \in \mathrm{Skew}(\mathbb{R}^m) \quad \text{and} \quad Y = \tfrac{1}{2}(A + A^t) \in \mathrm{Sym}(\mathbb{R}^m).$$

We now equip $\mathbb{R}^{m\times m}$ with its standard Euclidean scalar product given by

$$\langle E, F\rangle = \mathrm{trace}\,(E^t F).$$

Then it is easily seen that the two subspaces $\mathrm{Skew}(\mathbb{R}^m)$ and $\mathrm{Sym}(\mathbb{R}^m)$ are orthogonal i.e. if $X^t = -X$ and $Y^t = Y$ then $\langle X, Y\rangle = 0$.

The real special linear group $\mathbf{SL}_m(\mathbb{R}) = \{x \in \mathbb{R}^{m\times m} \,|\, \det x = 1\}$ has Lie algebra $\mathfrak{sl}_m(\mathbb{R})$ consisting of the real traceless matrices i.e.

$$\mathfrak{sl}_m(\mathbb{R}) = \{A \in \mathbb{R}^{m\times m} \,|\, \mathrm{trace}\, A = 0\}.$$

For this we have an orthogonal decomposition $\mathfrak{sl}_m(\mathbb{R}) = \mathfrak{so}(m) \oplus \mathfrak{p}$, where $\mathfrak{so}(m) = \mathrm{Skew}(\mathbb{R}^m)$ is the Lie algebra of the special orthogonal subgroup $\mathbf{SO}(m)$ of $\mathbf{SL}_m(\mathbb{R})$ and $\mathfrak{p}$ consists of the symmetric traceless elements of $\mathbb{R}^{m\times m}$ i.e.

$$\mathfrak{p} = \{Y \in \mathbb{R}^{m\times m} \,|\, Y = Y^t \text{ and } \mathrm{trace}\, Y = 0\}.$$

For the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ we have the basis $\{X, Y_1, Y_2\}$ with

$$X = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad Y_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Here the Lie subalgebra $\mathfrak{so}(2)$ is generated by X and its orthogonal complement $\mathfrak{p}$ by Y_1 and Y_2. If we now employ the exponential map $\mathrm{Exp} : \mathbb{R}^{m\times m} \to \mathbb{R}^{m\times m}$ for real matrices we get

$$s \mapsto \mathrm{Exp}(s\,X) = \begin{bmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{bmatrix}, \ s \mapsto \mathrm{Exp}(s\,Y_1) = \begin{bmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{bmatrix},$$

$$s \mapsto \mathrm{Exp}(s\,Y_2) = \begin{bmatrix} e^s & 0 \\ 0 & e^{-s} \end{bmatrix}.$$

These are all curves into the special linear group

$$\mathbf{SL}_2(\mathbb{R}) = \{x \in \mathbb{R}^{2\times 2} \mid \det x = 1\}.$$

Two of them show that this is unbounded in $\mathbb{R}^{m\times m}$ and hence non-compact.

$$- - - - - - - - * - - - - - - - -$$

The vector space $\mathbb{C}^{m\times m}$ of complex $m \times m$ matrices is the complexification of the real vector space $\mathbb{R}^{m\times m}$ i.e. the direct sum

$$\mathbb{C}^{m\times m} = \mathbb{R}^{m\times m} \oplus i\,\mathbb{R}^{m\times m}.$$

For this we have the decomposition

$$\mathbb{C}^{m\times m} = \mathrm{sHerm}(\mathbb{C}^m) \oplus \mathrm{Herm}(\mathbb{C}^m)$$

into its linear subspaces

$$\mathrm{sHerm}(\mathbb{C}^m) = \{Z \in \mathbb{C}^{m\times m} \mid \bar{Z}^t + Z = 0\}$$

and

$$\mathrm{Herm}(\mathbb{C}^m) = \{W \in \mathbb{C}^{m\times m} \mid \bar{W}^t - W = 0\}$$

of skew-Hermitian and Hermitian matrices, respectively. This means that every matrix $B \in \mathbb{C}^{m\times m}$ has a unique decomposition $B = Z + W$, where

$$Z = \tfrac{1}{2}(B - \bar{B}^t) \in \mathrm{sHerm}(\mathbb{C}^m) \ \text{ and } \ W = \tfrac{1}{2}(B + \bar{B}^t) \in \mathrm{Herm}(\mathbb{C}^m).$$

We can now extend the Euclidean scalar product on $\mathbb{R}^{m\times m}$ to the standard Hermitian scalar product on $\mathbb{C}^{m\times m}$, given by

$$\langle E, F\rangle = \mathrm{Re}\,\mathrm{trace}\,(\bar{E}^t F).$$

Then it is easily seen that the two subspaces $\mathrm{sHerm}(\mathbb{C}^m)$ and $\mathrm{Herm}(\mathbb{C}^m)$ are orthogonal i.e. if $\bar{Z}^t = -Z$ and $\bar{W}^t = W$ then $\langle Z, W\rangle = 0$.

The complex special linear group $\mathbf{SL}_m(\mathbb{C}) = \{z \in \mathbb{C}^{m\times m} \mid \det z = 1\}$ has Lie algebra $\mathfrak{sl}_m(\mathbb{C})$ consisting of the complex traceless matrices i.e.

$$\mathfrak{sl}_m(\mathbb{C}) = \{B \in \mathbb{C}^{m\times m} \mid \mathrm{trace}\, B = 0\}.$$

This is clearly $\mathfrak{sl}_m(\mathbb{R}) \oplus i\,\mathfrak{sl}_m(\mathbb{R})$ i.e. the complexification of the Lie algebra $\mathfrak{sl}_m(\mathbb{R})$ of $\mathbf{SL}_m(\mathbb{R})$. For $\mathfrak{sl}_m(\mathbb{C})$ we have an orthogonal decomposition $\mathfrak{sl}_m(\mathbb{C}) = \mathfrak{su}(m) \oplus \mathfrak{m}$, where

$$\mathfrak{su}(m) = \{Z \in \mathbb{C}^{m\times m} \,|\, \bar{Z} + Z = 0,\ \operatorname{trace} Z = 0\}$$

is the Lie algebra of the special unitary subgroup $\mathbf{SU}(m)$ of $\mathbf{SL}_m(\mathbb{C})$ and $\mathfrak{m}$ consists of the Hermitian traceless elements of $\mathbb{C}^{m\times m}$ i.e.

$$\mathfrak{m} = \{W \in \mathbb{C}^{m\times m} \,|\, \bar{W}^t - W = 0,\ \operatorname{trace} W = 0\}.$$

It should be noted that the Lie algebra $\mathfrak{su}(m)$ satisfies

$$\mathfrak{su}(m) = \mathfrak{so}(m) \oplus i\,\mathfrak{p},$$

where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{so}(m)$ in $\mathfrak{sl}_m(\mathbb{R}) = \mathfrak{so}(m) \oplus \mathfrak{p}$ discussed above. This shows that $\mathfrak{so}(m)$ is the intersection

$$\mathfrak{so}(m) = \mathfrak{su}(m) \cap \mathfrak{sl}_m(\mathbb{R})$$

and at the group level we have $\mathbf{SO}(m) = \mathbf{SU}(m) \cap \mathbf{SL}_m(\mathbb{R})$.

For the Lie algebra $\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{R}) \oplus i\,\mathfrak{sl}_2(\mathbb{R})$ we have orthogonal basis

$$\mathcal{B} = \{X, Y_1, Y_2, i\,X, i\,Y_1, i\,Y_2\}.$$

Note that here the Lie algebra $\mathfrak{su}(2)$ of $\mathbf{SU}(2)$ satisfies

$$\mathfrak{su}(2) = \mathfrak{so}(2) \oplus i\,\mathfrak{p}$$

and is generated by $X, i\,Y_1, i\,Y_2$, where

$$X = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad i\,Y_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad i\,Y_2 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

If we now employ the exponential $\operatorname{Exp} : \mathbb{C}^{m\times m} \to \mathbb{C}^{m\times m}$ for complex matrices we get

$$s \mapsto \operatorname{Exp}(s\,X) = \begin{bmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{bmatrix}, \ s \mapsto \operatorname{Exp}(s\,i\,Y_1) = \begin{bmatrix} \cos s & i\sin s \\ i\sin s & \cos s \end{bmatrix},$$

$$s \mapsto \operatorname{Exp}(s\,i\,Y_2) = \begin{bmatrix} e^{is} & 0 \\ 0 & e^{-is} \end{bmatrix}.$$

These are all curves into the special unitary group

$$\mathbf{SU}(2) = \{z \in \mathbb{C}^{2\times 2} \,|\, \bar{z}^t z = e,\ \det z = 1\}$$

and they are bounded in $\mathbb{C}^{m\times m}$ since $\mathbf{SU}(2)$ is compact.