Calculus I Lecture Notes

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Contents

Chapter 1

Algebra / Precalculus Review

1.1 Exponent and logarithm rules

Exponent rules

Here is a list of exponent rules you should be familiar with. In calculus, we use these exponent rules to rewrite a given expression in a way that makes it easier to perform calculus operations on the expression.

Theorem 1.1 (Exponent rules I) *Let x, a, b and n be numbers, where $x \neq 0$. Then:*

- $x^a x^b = x^{a+b}$
- $\frac{x^a}{x^b} = x^{a-b}$
- $x^0 = 1$
- $x^{-a} = \frac{1}{x^a}$
- $(x^a)^b = x^{ab}$
- $\sqrt[n]{x} = x^{1/n}$ *(in particular, $\sqrt{x} = x^{1/2}$)*
- $x^{m/n} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$ *(this last way of writing $x^{m/n}$ is most useful)*

EXAMPLE 1

Simplify each expression as much as possible and write the answer so that it has no radical signs (i.e. no $\sqrt{\ }$ or $\sqrt[3]{\ }$, etc.) or fractions with xs in the denominators:

1. $64^{2/3}$
   Solution: $64^{2/3} = \left(\sqrt[3]{64}\right)^2 = 4^2 = 16.$
2. 2^{-3}
   Solution: $2^{-3} = \frac{1}{2^3} = \frac{1}{8}.$
3. $4^{-5/2}$
   Solution: $4^{-5/2} = \frac{1}{4^{5/2}} = \frac{1}{(\sqrt{4})^5} = \frac{1}{2^5} = \frac{1}{32}.$
4. $3x^4x^{-2}(x^3)^3$
   Solution: $3x^4x^{-2}(x^3)^3 = 3x^4x^{-2}x^9 = 3x^{4-2+9} = 3x^{11}.$
5. $\frac{1}{x^7}$
   Solution: $\frac{1}{x^7} = x^{-7}$
6. $\frac{2x^2}{x^4}$
   Solution: $\frac{2x^2}{x^4} = 2x^{2-4} = 2x^{-2}$
7. $\sqrt{x}$
   Solution: $\sqrt{x} = x^{1/2}$
8. $\frac{4}{\sqrt{x^7}}$
   Solution: $\frac{4}{\sqrt{x^7}} = \frac{4}{x^{7/2}} = 4x^{-7/2}$

Remark on existence of square roots: $\sqrt{x}$ DNE if $x < 0$, and $\sqrt{x}$ means only the nonnegative square root of x, i.e. $\sqrt{25} = 5$, not ± 5. This is so that the process of taking a square root is a function (later).

Remark on simplifying square roots: For any positive number x,

$$\left(\sqrt{x}\right)^2 = x.$$

But, if you do the square root and the squaring in the other order, the operations don't cancel:

$$\sqrt{x^2} =$$

In general, if n is even then $\sqrt[n]{x^n} = |x|$, but if n is odd, then $\sqrt[n]{x^n} = x$.

Theorem 1.2 (Exponent rules II) *Let x, a, b and n be numbers, where $x \neq 0$. Then:*

- $(xy)^a = x^a y^a$
- $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$

EXAMPLE 2

Simplify each expression as much as possible, and write the answer so that it has no radical signs or fractions with xs in the denominators:

1. $\left(\frac{x}{3}\right)^{-3}$

   Solution: $\left(\frac{x}{3}\right)^{-3} = \frac{x^{-3}}{3^{-3}} = \frac{x^{-3}}{\frac{1}{27}} = 27x^{-3}$.

2. $x^2\sqrt{\frac{x}{2}}$

   Solution: $x^2\sqrt{\frac{x}{2}} = x^2\frac{\sqrt{x}}{\sqrt{2}} = x^2\frac{x^{1/2}}{\sqrt{2}} = \frac{x^{2+1/2}}{\sqrt{2}} = \frac{x^{5/2}}{\sqrt{2}}$.

3. $\frac{(2x)^3x^4}{(4x)^2}$

   Solution: $\frac{(2x)^3x^4}{(4x)^2} = \frac{2^3x^3x^4}{4^2x^2} = \frac{8x^7}{16x^2} = \frac{1}{2}x^5$.

4. $x^0\sqrt[3]{2(2x)^2}$

   Solution: $x^0\sqrt[3]{2(2x)^2} = 1\sqrt[3]{2(2^2)(x^2)} = \sqrt[3]{8x^2} = \sqrt[3]{8}\sqrt[3]{x^2} = 2x^{2/3}$.

WARNING: In general,

$$(x+y)^a \neq x^a + y^a \qquad \text{and} \qquad (x-y)^a \neq x^a - y^a$$

As a special case of this, when $a = -1$ we see that

$$\frac{1}{x+y} \neq \frac{1}{x} + \frac{1}{y} \qquad \text{and } \frac{A}{x+y} \neq \frac{A}{x} + \frac{A}{y}.$$

Logarithm rules

Logarithms are the "inverse" operation of exponentials:

Definition 1.3 *Let y and b be positive numbers. To say that x is the* **logarithm base b of y** *means that $b^y = x$, i.e.*

$$x = \log_b y \iff b^x = y.$$

The **common logarithm** *of y is the logarithm base 10 of y, i.e.*

$$x = \log y \iff 10^x = y.$$

Euler's constant *is an irrational number denoted by e. It is approximately $2.7182818...$ The* **natural logarithm** *of y is the logarithm base e of y, i.e.*

$$x = \ln y \iff e^x = y.$$

(Logarithms of non-positive numbers are not defined.)

The reason why we care about the number e and natural logarithms has to do with calculus: it turns out that calculus operations are "easier" when dealing with natural exponentials and logarithms rather than exponentials and logarithms with bases other than e.

Notation: e^x is also written $\exp(x)$, so $\exp(3x^2 + y)$ means e^{3x^2+y}, etc.

EXAMPLE 3

Evaluate the following expressions:

1. $\log 10000$

   Solution: $10^4 = 10000$, so $\log 10000 = \log_{10} 10000 = 4$.

2. $\log_3 \frac{1}{27}$

   Solution: $3^{-3} = \frac{1}{3^3} = \frac{1}{27}$, so $\log_3 \frac{1}{27} = -3$.

3. $\log_6 36$

   Solution: $6^2 = 36$ so $\log_6 36 = 2$.

4. $\log_4 32$

   Solution: We know $32 = 4 \cdot 4 \cdot 2 = 4 \cdot 4 \cdot 4^{1/2} = 4^{1+1+1/2} = 4^{5/2}$ so $\log_4 32 = \frac{5}{2}$.

5. $\ln e^9$

   Solution: $\ln e^9 = 9$.

As with exponent rules, it is often convenient to rewrite expressions with logarithms in them before doing calculus:

Theorem 1.4 (Logarithm Rules) *Let b, C and D be positive numbers, and let n be any number. Then:*

- $\log_b(CD) = \log_b C + \log_b D$
- $\log_b \left(\frac{C}{D}\right) = \log_b C - \log_b D$
- $\log_b 1 = 0$
- $\log_b \left(\frac{1}{D}\right) = -\log_b D$
- $\log_b(C^n) = n \log_b C$
- $b^{\log_b C} = C$
- $\log_b b^C = C$

EXAMPLE 4

Use properties of logarithms to expand each logarithmic expression as much as possible:

1. $\ln \frac{a^5}{b^2}$

   Solution: $\ln \frac{a^5}{b^2} = \ln a^5 - \ln b^2 = 5 \ln a - 2 \ln b$

2. $\log_3 \sqrt{5r}$

   Solution: $\log_3 \sqrt{5r} = \log_3(5r)^{1/2} = \frac{1}{2} \log_3(5r) = \frac{1}{2} \log_3 5 + \frac{1}{2} \log_3 r$

EXAMPLE 5

Suppose $x = \log_2 A$, $y = \log_2 B$ and $z = \log_2 C$. Find the following, in terms of x, y and z:

1. $\log_2 AB^2$

2. $\log_2 \frac{8C}{A\sqrt{B}}$

3. A^8

EXAMPLE 6

Write each of these expressions as the logarithm of a single quantity:

1. $\log x + \log y$

   Solution: $\log x + \log y = \log(xy)$

2. $3 \log x - \frac{1}{2} \log y$

   Solution: $3 \log x - \frac{1}{2} \log y = \log x^3 - \log y^{1/2} = \log \frac{x^3}{y^{1/2}}$

The world's most underrated exponent rule allows you to write an arbitrary exponent as an exponent with base e:

Theorem 1.5 (Change of base formula for exponents) *Let A and B be numbers with $A > 0$. Then:*

$$A^B = e^{B \ln A}.$$

EXAMPLE 7

Rewrite the following expressions as a single exponent, so that the base of the exponent is e:

1. $e^3 e^x e^{-2y}$

   Solution: $e^3 e^x e^{-2y} = e^{3+x-2y}$.

2. $(e^{2x})^4 e^{-3x}$

   Solution: $(e^{2x})^4 e^{-3x} = e^{2x \cdot 4} e^{-3x} = e^{8x} e^{-3x} = e^{8x-3x} = e^{5x}$.

3. 5^{3x}

   Solution: $5^{3x} = e^{3x \ln 5}$ by the preceding Theorem.

4. $e^{2x} 4^x$

   Solution: $e^{2x} 4^x = e^{2x} e^{x \ln 4} = e^{(2+\ln 4)x}$.

A similar rule allows you to write an arbitrary logarithm in terms of natural logarithms:

Theorem 1.6 (Change of base formula for logarithms) *Let B and C be positive numbers. Then:*

$$\log_B C = \frac{\ln C}{\ln B}.$$

EXAMPLE 8

1. Rewrite the following expression in terms of natural logarithms: $\log x$

2. Rewrite the following expression in terms of natural logarithms: $4\log_3(4x)$

3. Suppose $x = \ln A$ and $y = \ln B$. What is $\log_A B$ in terms of x and y?

   Solution: $\log_A B = \frac{\ln B}{\ln A} = \frac{y}{x}$.

EXAMPLE 9

Solve the following equations for x:

1. $e^{3x} - 4 = 0$

2. $\log_3(4x - 1) = 2$

1.2 Functions

Question: What is a function?

Definition 1.7 *Let A and B be sets. A* **function** *f from A to B is*

We denote such a function by writing "$f : A \to B$". The set A of inputs is called the **domain** *of f. The set of outputs of the function is called the* **range** *of f.*

In Math 220, we study functions where:

- the domain is $\mathbb{R}$, the set of real numbers (sometimes the domain is a subset of $\mathbb{R}$ like an interval), and
- the outputs are also real numbers.

Such a function f is often denoted by the symbols "$f : \mathbb{R} \to \mathbb{R}$".

Example: Let f be the function $\mathbb{R} \to \mathbb{R}$ which takes the input, squares it, and then adds 3 to produce the output.

To describe this function f, we could take some example inputs and see what the outputs are, arranging the results in a table:

INPUT	OUTPUT
-2	
-1	
0	
1	
2	

Rather than continuing to list inputs and outputs like this, it is easier to take a generic input (something we call x and figure out what the generic output is. This output is called $f(x)$. Writing down a formula for $f(x)$ in terms of x is sufficient to describe any function $f : \mathbb{R} \to \mathbb{R}$; such a formula is called a "rule" for the function.

In the example on the previous page, we can therefore describe the function by writing

$$f(x) = x^2 + 3.$$

Definition 1.8 *Let $f : A \to B$ and let $x \in A$. We write the output associated to input x as $f(x)$; this is pronounced "f* **of** *x". A formula for $f(x)$ in terms of x is called a* **rule** *for the function.*

Notation:

$$f(x)$$

Idea: Think of the x as a placeholder which represents where the input goes. Given a rule for f, you take whatever input you are given and replace all the xs in the rule with that input.

EXAMPLE 1

Let $f(x) = 2x^2 + x$. Compute and simplify the following expressions:

1. $f(2) = 2 \cdot 2^2 + 2 = 2 \cdot 4 + 2 = 10.$
2. $f(-1)$
3. $f(x) + f(3)$
4. $f(\text{trumpet})$
5. $f(\text{hamburger}) = 2(\text{hamburger})^2 + \text{hamburger}$
6. $f(2x)$
7. $f(x - 1)$
8. $f(x) - f(1)$

9. $f(x+h)$

10. $\frac{f(x+3)-f(x)}{3}$

<u>WARNING</u>: All your life you have been told that parenthesis means multiplication, i.e. $3(2) = 6$ or $a(b+c) = ab + ac$. **The parenthesis in the definition of $f(x)$ do not mean multiplication.** In particular, $f(x)$ does not mean f times x, and $f(a+b)$ is not the same thing as $f(a)+f(b)$ (in general). $f(x)$ means:

"the output of function f when x is the input".

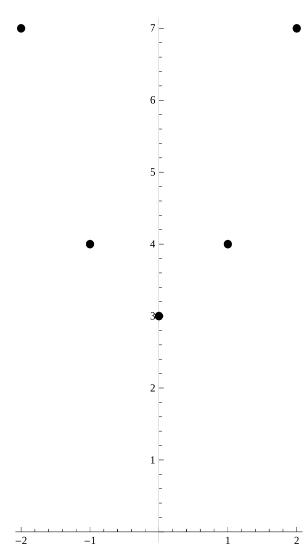
and is better denoted by the diagram

$$x \xrightarrow{f} f(x)$$

The graph of a function $f : \mathbb{R} \to \mathbb{R}$

Earlier, we saw the following table of values for the function whose rule is $f(x) = x^2 + 3$:

INPUT x	OUTPUT $f(x)$
-2	7
-1	4
0	3
1	4
2	7



Turning each of the inputs and outputs to the function into an ordered pair and plotting all these points produces a picture called the **graph** of the function. Note that since every input has at most one output, functions from $\mathbb{R}$ to $\mathbb{R}$ must pass the **Vertical Line Test** (i.e. every vertical line must hit the graph in at most one point).

Operations on functions

Definition 1.9 *Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$ and let c be a constant. Then, the functions $f+g$, $f-g$, fg, cf, $\frac{f}{g}$ and $f \circ g$ are defined by*

- $(f+g)(x) = f(x) + g(x)$
- $(f-g)(x) = f(x) - g(x)$
- $(fg)(x) = f(x)g(x)$
- $(cf)(x) = c\,f(x)$
- $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$
- $(f \circ g)(x) = f(g(x))$

$f \circ g$ is called the **composition** *of f and g.*

EXAMPLE 2

Suppose $f(x) = x + 2$ and $g(x) = x^2$. Then:

1. $(f+g)(x) = x + 2 + x^2$
2. $(fg)(x) = (x+2)x^2$
3. $(2g)(4) = 2g(4) = 2(4^2) = 2 \cdot 16 = 32$
4. $(f-g)(3) = (3+2) - (3^2) = -4$
5. $(f \circ g)(x) =$
6. $(g \circ f)(x) =$
7. $(f \circ f)(x) =$

EXAMPLE 3

Given each function F, write $F = f \circ g$ where f and g are "easy" functions:

1. $F(x) = (3x-2)^{12}$

   Solution: $f(x) = x^{12}$; $g(x) = 3x - 2$

2. $F(x) = \ln^7 x$

3. $F(x) = \ln x^7$

4. $F(x) = 5\cos(e^x + 2x - 1)$

5. $F(x) = e^{-x}$

Piecewise-defined functions

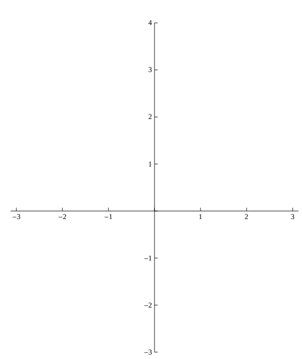
Consider the function

$$f(x) = \begin{cases} 1-x & x < -1 \\ x^2 & x \geq -1 \end{cases}.$$

This means that to evaluate f at a number x, you look at which inequality x satisfies, then apply the corresponding formula. So a table of values for this f looks like

x	-3	-2	-1.5	-1	$-.5$	0	1	2
$f(x)$								

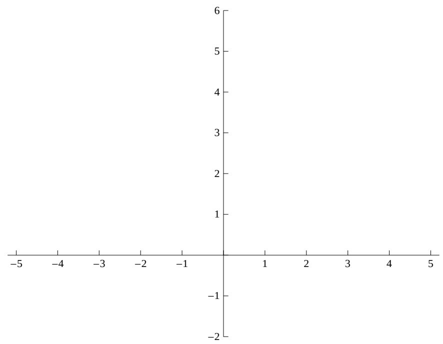
and the graph of f looks like



EXAMPLE 4

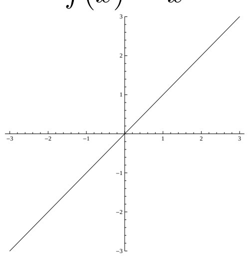
Graph this function:

$$f(x) = \begin{cases} x^2 & x \neq 2 \\ -1 & x = 2 \end{cases}$$

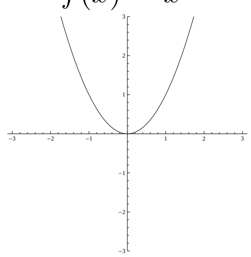


Common functions whose graphs you should know

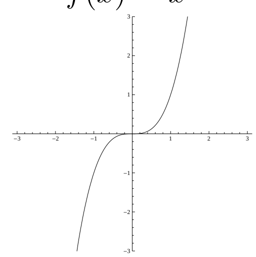
$f(x) = x$



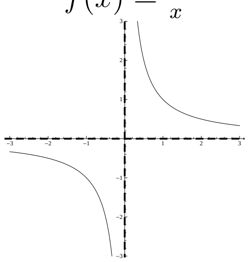
$f(x) = x^2$



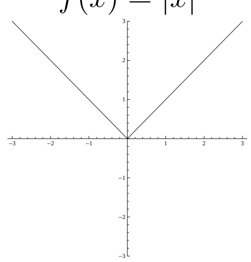
$f(x) = x^3$



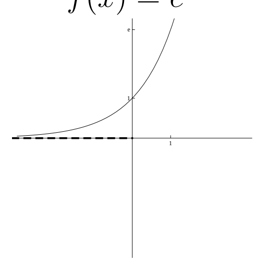
$f(x) = \frac{1}{x}$



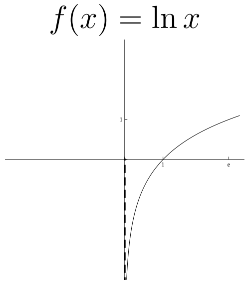
$f(x) = |x|$



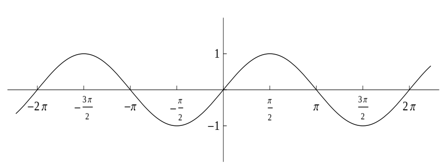
$f(x) = e^x$



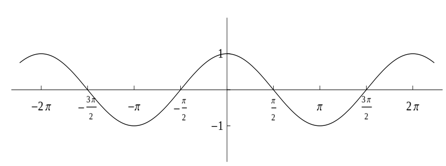
$f(x) = \ln x$



$f(x) = \sin x$



$f(x) = \cos x$



Transformations on functions

It is useful to know how the graph of a function changes if you alter the rule of the function a little bit. Suppose you know the graph of function f. Then:

Altered version of function f (all cs are positive numbers)	Corresponding transformation on the graph
$f(x) + c$	graph shifts up c units
$f(x) - c$	graph shifts down c units
$f(x + c)$	graph shifts left c units
$f(x - c)$	graph shifts right c units
$cf(x)$	graph stretched vertically by factor of c (taller if $c > 1$, shorter if $0 < c < 1$)
$f(-x)$	graph reflected through y-axis
$-f(x)$	graph reflected through x-axis

1.3 Lines

By far the most important class of functions are lines. Reasons:

1. Linear equations model a large class of real-world problems
2. Linear equations are relatively easy to work with.
3. You can often approximate the solution to hard problems (using calculus techniques) by considering something related to a linear equation.

Question: What "determines" a line? That is, what makes one line different from another one?

1.

2.

Definition 1.10 *The* **slope** *of a line is the ratio of the rise of the line to its run, i.e. for any two points on the line* (x_1, y_1) *and* (x_2, y_2)*, the slope of the line is given by*

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

If $m > 0$, then the line goes up from left to right. In this case, the greater m is, the steeper the line is.

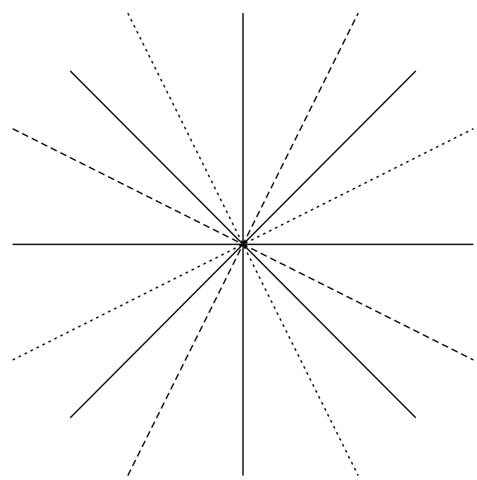
If $m = 1$, the line goes up at a $45°$ angle.

If $m = 0$, the line is horizontal.

If $m < 0$, then the line goes down from left to right. In this case, the more negative m is, the steeper the line is.

If $m = -1$, the line goes down at a $45°$ angle.

Vertical lines have undefined slope.



EXAMPLE 1

Find the slope of the line passing through the points $(2, -5)$ and $(4, 11)$.

Solution: $m = \frac{y_2-y_1}{x_2-x_1} = \frac{11-(-5)}{4-2} = \frac{16}{2} = 8.$

Given the slope m of a line, and a point (x_0, y_0) on the line, one can write the equation of the line as follows:

Definition 1.11 *The* **point-slope formula** *of a line with slope m passing through (x_0, y_0) is*

$$y = y_0 + m(x - x_0).$$

You may be familiar with the "slope-intercept" formula $y = mx + b$ for a line. The point-slope formula

$$y = y_0 + m(x - x_0)$$

is equivalent, because it can be rewritten as

It is extremely useful to know the point-slope formula, because it is easier than the $y = mx + b$ formula to apply in calculus.

EXAMPLE 2

Write the equation of the line passing through $(2, -5)$ and $(6, -7)$.

EXAMPLE 3

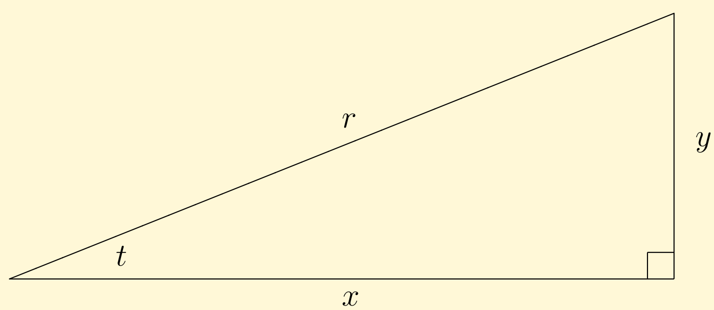
Write the equation of the line passing through $(-3, -2)$ with slope $\frac{2}{5}$.

NOTE: Vertical lines do not have a slope, so their equation cannot be written using the point-slope formula. The equation of a vertical line is $x = h$, where h is a constant. For example, the vertical line passing through $(6, -5)$ is $x = 6$.

1.4 Trigonometry

Trigonometry is the study of triangles. In particular, the key concept of trigonometry is that given a triangle, the ratio of the lengths of any two sides of that triangle depend only on the angles of the triangle (and not on the side lengths). To relate side lengths and ratios of side lengths to angle measurements, we invent six trigonometric functions, which can be arrived at two different ways:

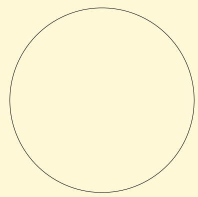
Definition 1.12 (Triangle definition of the trig functions) *Consider a right triangle with one angle measuring t, labelled as below:*



Then we define:

- *the* **sine** *of t, by* $\sin t = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r}$
- *the* **cosine** *of t, by* $\cos t = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r}$
- *the* **tangent** *of t, by* $\tan t = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}$
- *the* **cotangent** *of t, by* $\cot t = \frac{\text{adjacent}}{\text{opposite}} = \frac{x}{y}$
- *the* **secant** *of t, by* $\sec t = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{r}{x}$
- *the* **cosecant** *of t, by* $\csc t = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{r}{y}$

Definition 1.13 (Unit circle definition of the trig functions) *Let t be a real number. Imagine a string of length t, which is laid along the circle of radius 1 centered at the origin starting at the point $(1,0)$ (the string is laid counterclockwise if $t > 0$ and clockwise if $t < 0$):*

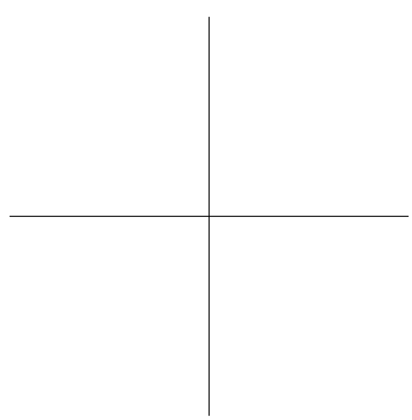


Wherever the string stops, call that point (x, y). Based on this, we define

- $\sin t = y$
- $\cos t = x$
- $\tan t = \frac{y}{x}$
- $\cot t = \frac{x}{y}$
- $\sec t = \frac{1}{x}$
- $\csc t = \frac{1}{y}$

The two definitions above are the same, **so long as the angle t is measured in radians**. (This is one of the many reasons why mathematicians prefer radians to degrees.) The advantage of the unit circle method is that it allows you to evaluate trig functions at angles measuring less than 0 or more than $90° = \frac{\pi}{2}$.

Notice that we can determine the signs of the six trig functions by looking at the signs of x and y, i.e. looking at the quadrant the angle t lies in:



Either way you choose to define the trig functions, it is straightforward to deduce the following relationships:

Theorem 1.14 (Trigonometric identites) *Given the trig functions as defined above, the following identities hold for all x:*

- Quotient identities:
  1. $\tan x = \frac{\sin x}{\cos x}$
  2. $\cot x = \frac{\cos x}{\sin x}$
- Reciprocal identities:
  1. $\cot x = \frac{1}{\tan x}$
  2. $\sec x = \frac{1}{\cos x}$
  3. $\csc x = \frac{1}{\sin x}$
- Pythagorean identities:
  1. $\sin^2 x + \cos^2 x = 1$
  2. $1 + \cot^2 x = \csc^2 x$
  3. $1 + \tan^2 x = \sec^2 x$
- Odd-even identities:
  1. $\sin(-x) = -\sin x$
  2. $\cos(-x) = \cos x$
  3. $\tan(-x) = -\tan x$

Using these identities and the "All Scholars Take Calculus" rules, you can find the values of all six trig functions if you are given the value of one trig function, and the sign of a second trig function:

EXAMPLE 1

Find the values of all six trig functions of θ, if $\sin\theta = \frac{7}{11}$ and $\tan\theta < 0$.

Computing values of the trig functions at special angles

You are responsible for computing any trig function at any multiple of $\pi/6$ or $\pi/4$ radians; virtually all problems in this course will use radians rather than degrees. You should especially know the following values of sine, cosine and tangent:

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
x in degrees	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\tan x$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	DNE	0	DNE

EXAMPLE 2

Compute each of these:

1. $\cos \frac{3\pi}{4}$
2. $\sin \frac{-7\pi}{6}$
3. $\cot \frac{5\pi}{6}$
4. $\sec \frac{3\pi}{2}$
5. $\tan \frac{-\pi}{3}$
6. $\cos \frac{5\pi}{3}$
7. $\sin \frac{7\pi}{2}$
8. $\tan \frac{5\pi}{4}$

Inverse trigonometric functions

Definition 1.15 *The* **arctangent** *(a.k.a.* **inverse tangent***) function is the function* $\arctan : \mathbb{R} \to \mathbb{R}$ *defined by*

$$\arctan x = \textit{ an angle (in radians) between } \frac{-\pi}{2} \textit{ and } \frac{\pi}{2}\textit{, whose tangent is } x.$$

The **arcsine** *(a.k.a.* **inverse sine***) function is the function* $\arcsin : [-1, 1] \to \mathbb{R}$ *defined by*

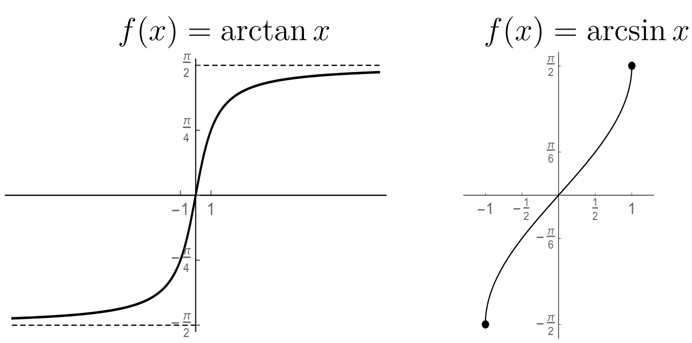
$$\arcsin x = \textit{ an angle (in radians) between } \frac{-\pi}{2} \textit{ and } \frac{\pi}{2}\textit{, whose sine is } x.$$

EXAMPLES

- $\arctan 1 = \frac{\pi}{4}$ because $\tan \frac{\pi}{4} = 1$.
- $\arcsin \frac{\sqrt{3}}{2} = \frac{\pi}{3}$ because $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

<u>Notation</u>: $\arctan x$ is sometimes written as $\tan^{-1} x$, and $\arcsin x$ is sometimes written as $\sin^{-1} x$.

Graphs of arctangent and arcsine:



Theorem 1.16 (Properties of arctangent and arcsin) *These hold for all real numbers* x, y*:*

- $\arctan(-x) = -\arctan x$ *and* $\arcsin(-x) = -\arcsin x$.
- $y = \arctan x \iff x = \tan y$
- $y = \arcsin x \iff x = \sin y$

1.5 Homework exercises

1. Evaluate each expression:

   a) 4^{-2} b) $144^{1/2}$ c) $27^{5/3}$ d) $\left(\frac{1}{4}\right)^{-3/2}$

2. Write each expression in the form $\square x^{\square}$, where each of the two squares represent constants:

   a) $\sqrt[5]{-32x^3}$ c) $(3x)^4x^2$ e) $\sqrt{7x}$
   b) $\frac{8}{\sqrt[3]{8x^4}}$ d) $\frac{(3x)^2}{18x^3}$ f) $\sqrt{x^4}$

3. Simplify each expression as much as possible:

   a) $\sqrt{25x^2}$ b) $\sqrt{2(2\sqrt{2(2x)^2})^2}$

4. Use properties of logarithms to expand the given logarithmic expression as much as possible:

   a) $\log_{17}\frac{2}{3}$ b) $\ln s^2\sqrt{r^3}$ c) $\log 3e^2x^4$

5. Suppose $A = \ln x$, $B = \ln y$ and $C = \ln z$. Rewrite the following in terms of A, B and C:

   a) $\ln\left(\frac{x}{y}\right)^2$ c) $\ln x^{-3}z^2e$ e) e^A
   b) $\ln xe^4\sqrt{y}$ d) $\log_x y$ f) e^{4C}

6. Write each of these as a logarithm of a single quantity:

   a) $\ln(x-2) - \ln(x+2)$
   b) $2\ln x + 3\ln y - \frac{1}{4}\ln z$
   c) $3\left[\ln x + \ln(x+1)\right]$

7. Simplify the following expressions:

   a) $e^{4\ln 2}$ c) $5^{\log_5 7}$ e) $2e^{2\ln 3}$ g) $\log 10^{99}$
   b) $\ln(e^{-2/3})$ d) $e^{\frac{1}{3}\ln 27}$ f) $e^{x\ln y}$ h) $2^{\log_8 32}$

8. Write each of the following expressions as a single exponential term, where the base of the exponent is e:

   a) $e^5 e^{-3}$
   b) $\frac{e^3}{\sqrt[3]{e}}$
   c) $\frac{e^{-2}}{e^{-5}}$
   d) $\frac{e^{2x} e e^{4y}}{e^3 e^{2y} e^{7x}}$
   e) 5^7
   f) $(3x)^{4x}$
   g) 23
   h) $e^{7x} 3^{2x}$
   i) $\frac{e^{-3x}}{2^x}$

9. Rewrite the following in terms of natural logarithms:

   a) $\log y$
   b) $3 \log_4 11$
   c) $\log_{1/2} \frac{2}{3}$

10. Solve the following equations for x:

    a) $e^x = 4$
    b) $\ln x = 5$
    c) $\ln(x-1) - 1 = 0$
    d) $e^{2 \ln x} = 3$
    e) $5e^{2x} = 17$
    f) $\ln x - 3 = 0$

11. Let $f(x) = x^2 - 3$ and let $g(x) = 3 - x$. Compute and simplify:

    a) $f(-4)$
    b) $g(-2)$
    c) $(f - g)(1)$
    d) $(fg)(4)$
    e) $(f \circ g)(2)$
    f) $(f \circ f)(0)$
    g) $g(\text{bulldog})$
    h) $g(x + 3)$
    i) $\sqrt{g(1)}$
    j) $g(\sqrt{x})$
    k) $(f + g)(x + 1)$
    l) $(fg)(2x)$

12. Let $h(x) = 2 + x^4$. Compute and simplify:

    a) $\sqrt{h(\sqrt{x})}$
    b) $h(x - 1)$
    c) $h(x) - h(1)$
    d) $h(x) - 1$
    e) $4h(2x)$
    f) $h(x^2 + 1)$

13. Let $f(x) = x^3$. Compute and simplify:

$$\frac{f(x+h) - f(x)}{h}$$

14. Let $f(x) = x^2 - x$. Compute and simplify:

$$\frac{f(1+h) - f(1)}{h}$$

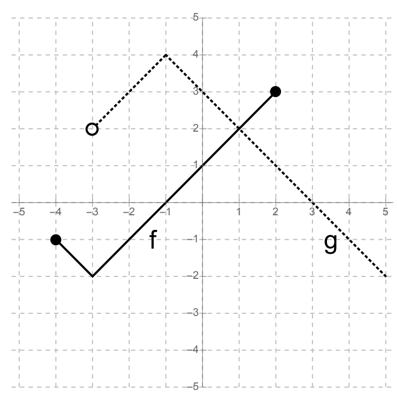
15. Let $f(x) = x + 2$. Compute and simplify:
$$\frac{f(x) - f(t)}{x - t}$$

16. Let
$$f(x) = \begin{cases} 2x + 1 & x < 1 \\ 2x + 2 & x \geq 1 \end{cases}$$
Evaluate $f(-1)$, $f(0)$, $f(1)$ and $f(2)$.

17. Sketch the graph of the function
$$f(x) = \begin{cases} 1 - x & x < 1 \\ x + 1 & x \geq 1 \end{cases}$$

18. Sketch the graph of the function
$$f(x) = \begin{cases} x & x \neq 1 \\ -1 & x = 1 \end{cases}$$

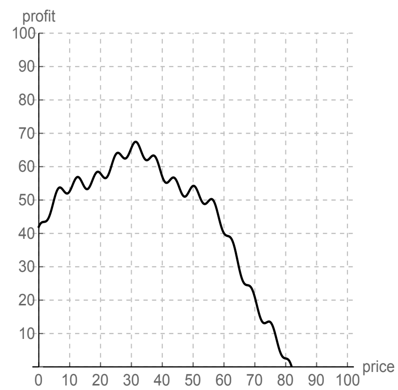
19. The graphs of unknown functions f and g are given below:



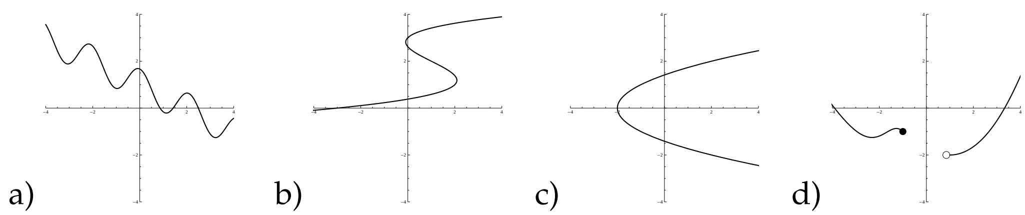
Use these graphs to estimate answers to the following questions:

a) Find $f(-2)$
b) Find $g(3)$.
c) Find $g(-3)$.
d) Find $(f + g)(-1)$.
e) Find $(f \circ g)(3)$.
f) Find $(fg)(1)$.
g) Find all value(s) x (if any) such that $f(x) = g(x)$.
h) Find all value(s) x (if any) for which $f(x) = -1$.
i) Find all value(s) x (if any) for which $g(x) = 0$.

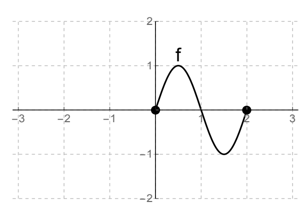
20. Suppose the graph below is the picture of a function where the output of the function is a company's profit (in millions of dollars), and the input is the price at which the company sells its product. At (roughly) what price should the company sell its product, if its goal is to make as much money as possible? How much profit will be made at this price?



21. Determine which one or ones of the following pictures (a)-(d) depict situations where y is a function of x.



22. Suppose $y = f(x)$ is a function whose graph is:



Sketch the graphs of the following functions:

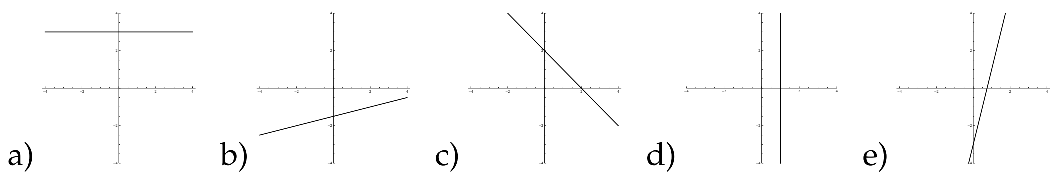
a) $y = f(x+5)$ c) $y = f(-x)$ e) $y = f(x-2)+1$

b) $y = f(x)-5$ d) $y = -f(x)+5$ f) $y = -f(-x)$

23. Sketch the graphs of the following functions:

a) $y = 2\sin x$
b) $y = (x-3)^2 + 1$
c) $y = -\ln x$
d) $y = e^{-x}$
e) $y = \cos(-x)$
f) $y = -(x+2)^3$
g) $y = -|x| + 2$
h) $y = -\frac{1}{x}$
i) $y = -(x+2)^2 - 4$

24. Estimate the slope of each of the following lines by looking at its graph (assume the scales on the x- and y-axes are the same):



25. Find the slope of the line passing through these pairs of points:

a) $(3, -4)$ and $(5, 2)$
b) $(\frac{-1}{2}, \frac{2}{3})$ and $(\frac{-3}{4}, \frac{1}{6})$
c) (a, b) and $(a+s, b+r)$
d) $(2, 7)$ and $(2, -1)$
e) $(x, f(x))$ and $(x+h, f(x+h))$
f) $(-1, 4)$ and $(5, -8)$

26. Find the equation of the line with each set of properties:

a) passes through $(0, 3)$ and has slope $\frac{3}{4}$
b) passes through the origin; $m = \frac{2}{3}$
c) passes through $(2, 1)$ and $(0, -3)$
d) passes through $(-3, -2)$; $m = 4$
e) passes through $(2, 6)$ and is vertical
f) passes through $(-4, 2)$ and is horizontal
g) passes through $(5, 1)$ and $(5, 8)$
h) passes through $(-7, 3)$ and $(2, -5)$

27. Suppose $\sin x = \frac{5}{13}$. Assuming the values of the other five trig functions of x are positive, find them.

28. Suppose $\cos x = \frac{7}{25}$. If $\tan x < 0$, find the values of the other five trig functions of x.

29. Suppose $\csc x = \frac{5}{2}$. What is $\sin x$?

30. Suppose $\tan x = 2$. If $\cos x < 0$, what is $\sin x$?

31. Compute each of the following (if they are not defined, say so). Try to do these without looking anything up (to simulate how you will have to do these things on quizzes and exams).

a) $\sin \frac{\pi}{3}$
b) $\cos \frac{\pi}{2}$
c) $\tan \frac{\pi}{4}$
d) $\cos 0$
e) $\cos \frac{\pi}{6}$
f) $\cos \frac{2\pi}{3}$
g) $\sin \frac{3\pi}{2}$
h) $\cot \frac{3\pi}{4}$
i) $\sin 0$
j) $\sec \pi$
k) $\tan \frac{\pi}{6}$
l) $\sin \frac{7\pi}{4}$
m) $\sec \frac{\pi}{3}$
n) $\tan \frac{3\pi}{2}$
o) $\sin \pi$
p) $\csc \frac{5\pi}{6}$
q) $\sec \frac{-\pi}{4}$
r) $\sin \frac{-5\pi}{6}$
s) $\tan -\pi$
t) $\tan \frac{-8\pi}{3}$

32. Evaluate each of the following:

a) $\arcsin \frac{1}{2}$
b) $\arcsin 0$
c) $\arctan 1$
d) $\arctan\left(-\sqrt{3}\right)$
e) $\arctan \frac{\sqrt{3}}{3}$
f) $\arcsin \frac{-\sqrt{3}}{2}$
g) $\arcsin -1$
h) $\arctan -1$
i) $\arcsin \frac{-\sqrt{2}}{2}$

Answers

DISCLAIMER: Throughout the lecture notes, the provided answers are answers only (not complete solutions) and may contain errors and/or typos.

1. a) $\frac{1}{16}$ b) 12 c) 243 d) 8

2. a) $-2x^{3/5}$ b) $2x^{-4/3}$ c) $81x^6$ d) $\frac{1}{2}x^{-1}$ e) $\sqrt{7}\,x^{1/2}$ f) x^2

3. a) $5|x|$ b) $8|x|$

4. a) $\log_{17} 2 - \log_{17} 3$
   b) $2\ln s + \frac{3}{2}\ln r$
   c) $\log 3 + 2\log e + 4\log x$

5. a) $2A - 2B$ b) $A + 4 + \frac{1}{2}B$ c) $-3A + 2C + 1$ d) $\frac{B}{A}$ e) x f) z^4

6. a) $\ln\left(\frac{x-2}{x+2}\right)$ b) $\ln\left(x^2 y^3 \sqrt[4]{z}\right)$ c) $\ln\left[x(x+1)\right]^3$

7. a) 16 b) $\frac{-2}{3}$ c) 7 d) 3 e) 18 f) y^x g) 99 h) $2^{5/3}$

8. a) e^2 b) $e^{8/3}$ c) e^3 d) $e^{-5x+2y-2}$ e) $e^{7\ln 5}$ f) $e^{4x\ln(3x)}$ g) $e^{\ln 23}$ h) $e^{7x+2x\ln 3}$ i) $e^{-3x-x\ln 2}$

9. a) $\frac{\ln y}{\ln 10}$ b) $\frac{3\ln 11}{\ln 4}$ c) $\frac{\ln(2/3)}{\ln(1/2)} = \frac{\ln 2 - \ln 3}{-\ln 2}$

10. a) $x = \ln 4$ b) $x = e^5$ c) $x = e + 1$ d) $x = \sqrt{3}$ e) $x = \frac{1}{2}\ln\frac{17}{5}$ f) $x = e^3$

11. a) 13 b) 5 c) -4 d) -13 e) -2 f) 6 g) $3-$ bulldog h) $-x$ i) $\sqrt{2}$ j) $3-\sqrt{x}$ k) $(x+1)^2 - x - 1$ l) $(4x^2-3)(3-2x)$

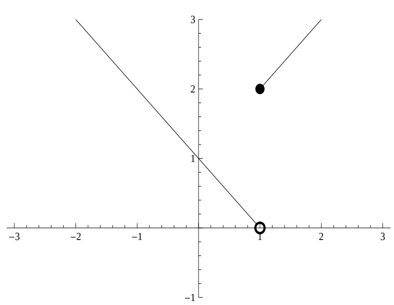
12. a) $\sqrt{2+x^2}$ b) $2+(x-1)^4$ c) $x^4 - 1$ d) $x^4 + 1$ e) $8 + 64x^4$ f) $2 + (x^2+1)^4$

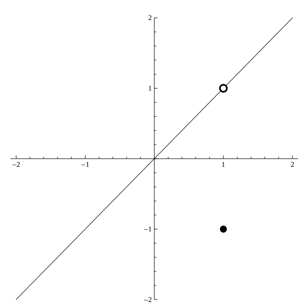
13. $3x^2 + 3xh + h^2$

14. $h + 1$

15. 1

16. $f(-1) = -1$; $f(0) = 1$; $f(1) = 4$; $f(2) = 6$.

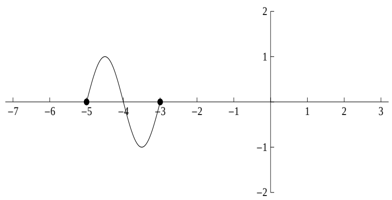
17. 

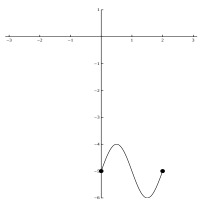
18. 

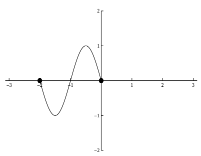
19. a) -1 b) 0 c) DNE d) 4 e) 1 f) 4 g) $x = 1$ h) $x = -4, x = -2$ i) $x = 3$

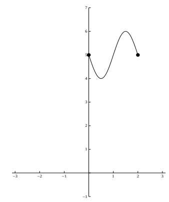
20. The price should be roughly $\$32$, and their profit will be about $\$67,000,000$.

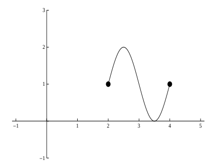
21. (a) and (d) are functions $y = f(x)$; (b) and (c) are not.

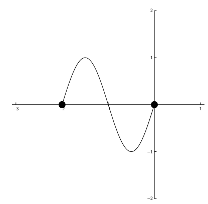
22. a) 

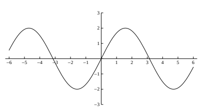
b) 

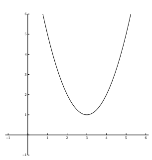
c) 

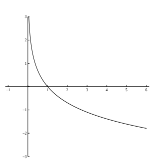
d) 

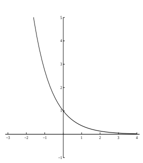
e) 

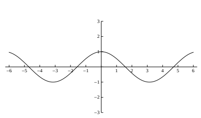
f) 

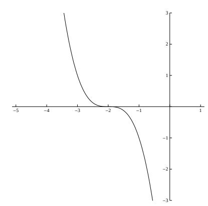
23. a) 

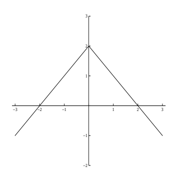
b) 

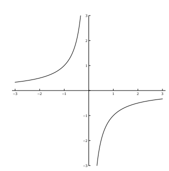
c) 

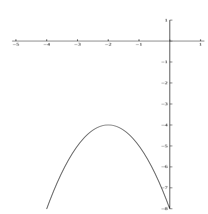
d) 

e) 

f) 

g) 

h) 

i) 

24. a) 0 b) $\approx \frac{1}{3}$ c) ≈ -1 d) DNE e) ≈ 4

25. a) 3 b) 2 c) $\frac{r}{s}$ d) DNE e) $\frac{f(x+h)-f(x)}{h}$ f) -2

26. a) $y = \frac{3}{4}x + 3$
b) $y = \frac{2}{3}x$
c) $y = 2x - 3$
d) $y = -2 + 4(x + 3)$
e) $x = 2$
f) $y = 2$
g) $x = 5$
h) $y = -5 + \frac{-8}{9}(x - 2)$

27. $\cos x = \frac{12}{13}$; $\tan x = \frac{5}{12}$; $\cot x = \frac{12}{5}$; $\sec x = \frac{13}{12}$; $\csc x = \frac{13}{5}$.

28. $\sin x = \frac{-24}{25}$; $\tan x = \frac{-24}{7}$; $\cot x = \frac{-7}{24}$; $\sec x = \frac{25}{7}$; $\csc x = \frac{-25}{24}$.

29. $\frac{2}{5}$

30. $\frac{-2}{\sqrt{5}}$

31. a) $\frac{\sqrt{3}}{2}$ b) 0 c) 1 d) 1 e) $\frac{\sqrt{3}}{2}$ f) $\frac{-1}{2}$ g) -1 h) -1 i) 0 j) -1 k) $\frac{\sqrt{3}}{3}$ l) $\frac{-\sqrt{2}}{2}$ m) 2 n) DNE o) 0 p) 2 q) $\sqrt{2}$ r) $\frac{-1}{2}$ s) 0 t) $\sqrt{3}$

32. a) $\frac{\pi}{6}$ b) 0 c) $\frac{\pi}{4}$ d) $\frac{-\pi}{3}$ e) $\frac{\pi}{6}$ f) $\frac{-\pi}{3}$ g) $\frac{-\pi}{2}$ h) $\frac{-\pi}{4}$ i) $\frac{-\pi}{4}$

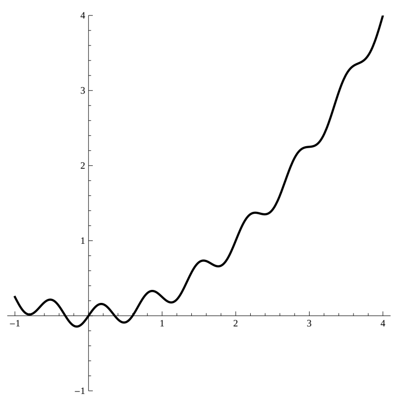
Chapter 2

Limits

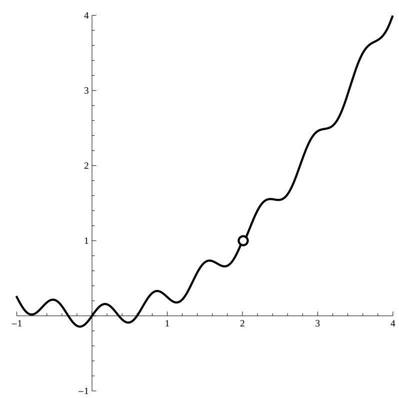
2.1 The idea of the limit

Warmup: Given the graphs of each of these functions, tell me the value of $f(2)$:

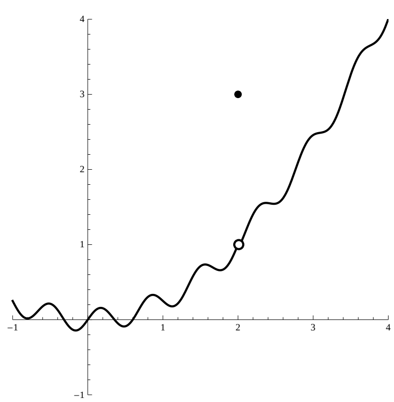
(1)



(3)



(2)



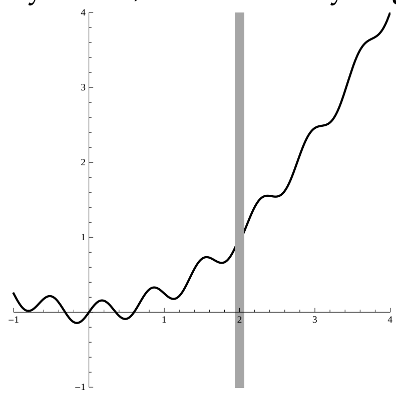
Answers:

(1)

(2)

(3)

Modified warmup: Here is the graph of some function f. The portion of the graph above $x = 2$ is "covered" (by a strip of tape, for example). Based only on what you see, what would you **guess** the value of $f(2)$ is?



First idea of the limit: graphical interpretation

Suppose you can see the entire graph of a function f except for the possible point on the graph sitting above (or below) $x = a$. If, based on the picture, you'd guess that $f(a) = L$, then you say

"the limit as x approaches a of $f(x)$ is L"

and you'd write

$$\lim_{x \to a} f(x) = L \quad \text{or} \quad \text{“}f(x) \to L \text{ as } x \to a\text{”}.$$

Examples: In the modified warmup above,

$$\lim_{x \to 2} f(x) =$$

In all three warmup examples,

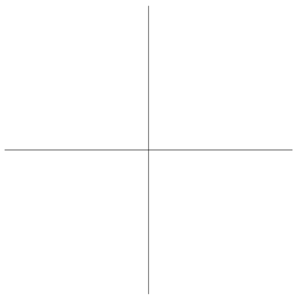
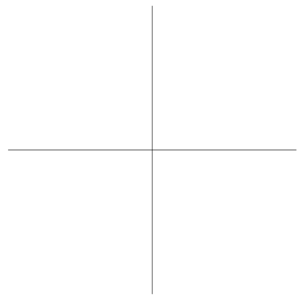
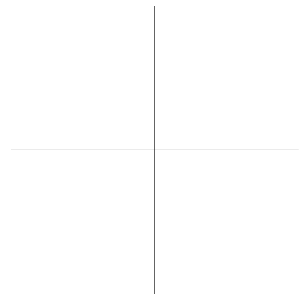
$$\lim_{x \to 2} f(x) =$$

Note: $f(2)$ is different in the three warmup examples. In one example, $f(2)$ doesn't even exist!

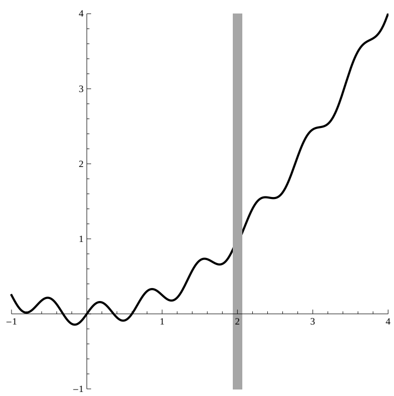
In general, if you have a function f which satisfies

$$\lim_{x\to a} f(x) = L,$$

then the graph of f should look like one of these three pictures:

 or  or 

Back to the modified warmup:



In this example, we said $\lim_{x\to 2} f(x) = 1$.
Why is 1 the most reasonable guess for the value of $f(2)$?

Second idea of the limit: approximation via tables

To say

$$\lim_{x\to a} f(x) = L$$

means that as x gets closer and closer to a (without ever reaching a), the corresponding values $f(x)$ of the function get closer and closer to L.

EXAMPLE 1

$$\lim_{x \to 0} \frac{\sin x - x}{x^3} = ?$$

Solution: The **first idea of the limit** requires a graph to apply, and we don't have a graph.

To implement the **second idea of the limit**, let's take $x-$values which get closer and closer to 0 and see if the corresponding $f(x)-$values approach a number:

x	.1	.05	.01	.001	.0000001
$\frac{\sin x - x}{x^3}$	$-.166583$	$-.166646$	$-.166666$	$-.166666$	$-.166666$

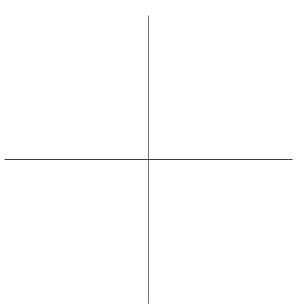
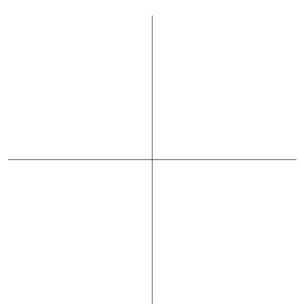
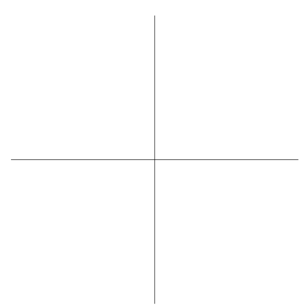
x	-.1	- .05	-.01	-.001	-.0000001
$\frac{\sin x - x}{x^3}$	$-.166583$	$-.166646$	$-.166666$	$-.166666$	$-.166666$

Based on this, we can conjecture that

$$\lim_{x \to 0} \frac{\sin x - x}{x^3} =$$

IMPORTANT:

This suggests that the graph of $f(x) = \frac{\sin x - x}{x^3}$ looks like

 or  or 

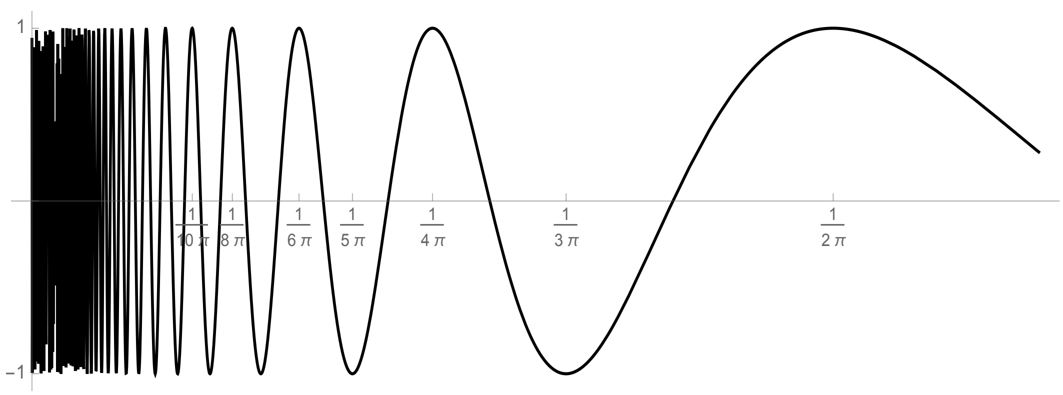
The method of the previous example sometimes works well, but it can lie:

EXAMPLE 2

Let $f(x) = \cos \frac{1}{x}$.

$$\lim_{x \to 0} f(x) = ?$$

Solution: Here's a graph of $f(x)$:



Let's try the method of Example 1:

x	$\frac{1}{2\pi}$	$\frac{1}{4\pi}$	$\frac{1}{6\pi}$	$\frac{1}{100\pi}$	$\frac{1}{1000\pi}$
$f(x)$					

x	$\frac{1}{3\pi}$	$\frac{1}{5\pi}$	$\frac{1}{7\pi}$	$\frac{1}{101\pi}$	$\frac{1}{1001\pi}$
$f(x)$					

Third idea of the limit: formal definition

Suppose $f(x)$ is defined for all x near a but possibly not at a. If $f(x)$ is as close to L as we like **for all** x sufficiently close to a (but not a itself), we say

$$\lim_{x\to a} f(x) = L.$$

In Example 2, there is no L such that $f(x) = \cos\frac{1}{x}$ stays close to L for <u>all</u> x near 0. Therefore

2.2 One-sided limits

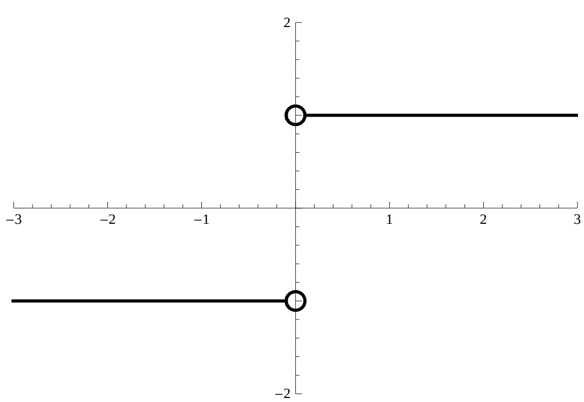
EXAMPLE 3

Let

$$f(x) = \frac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

$$\lim_{x\to 0} f(x) = ?$$

Solution: Here is a graph of f:



Definition 2.1 *Suppose $f(x)$ is defined for all x near a with $x > a$. If (whenever x gets closer and closer to a from the right, $f(x)$ approaches L), then we say the* **limit of $f(x)$ as x approaches a from the right** *is L and we write*

$$\lim_{x \to a^+} f(x) = L.$$

Suppose $f(x)$ is defined for all x near a with $x < a$. If (whenever x gets closer and closer to a from the left, $f(x)$ approaches L), then we say **the limit of $f(x)$ as x approaches a from the left is** *L and we write*

$$\lim_{x \to a^-} f(x) = L.$$

These are also called, respectively, **left-hand limits** and **right-hand limits**. Collectively, left- and right-hand limits are referred to as **one-sided limits**.

Example: In the previous example where $f(x) = \frac{|x|}{x}$,

$$\lim_{x \to 0^+} f(x) = \qquad\qquad\qquad \lim_{x \to 0^-} f(x) =$$

Theorem 2.2 $\lim\limits_{x \to a} f(x)$ *exists* ***only if*** $\lim\limits_{x \to a^+} f(x)$ *and* $\lim\limits_{x \to a^-} f(x)$ *both exist and are equal. In this situation,*

$$\lim_{x \to a} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x).$$

Example: For the function $f(x) = \frac{|x|}{x}$, since

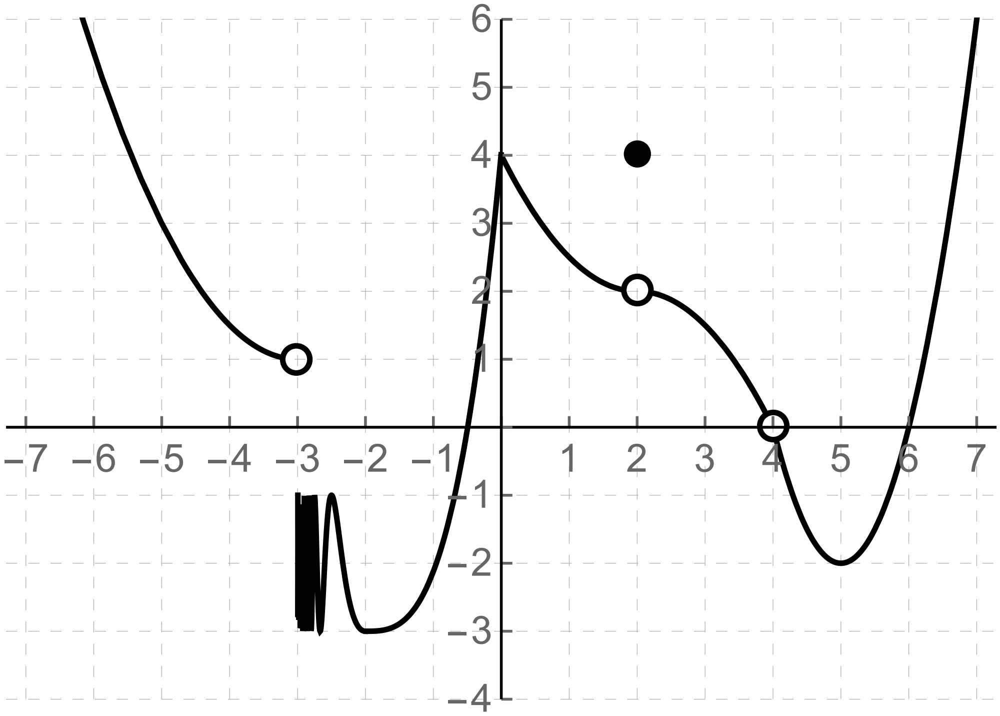
$$\lim_{x \to 0^+} f(x) = 1 \neq -1 = \lim_{x \to 0^-} f(x),$$

we see that

$$\lim_{x \to 0} f(x) \text{ DNE}.$$

EXAMPLE 4

Consider the following graph of some unknown function f:



Based on this graph, find the following:

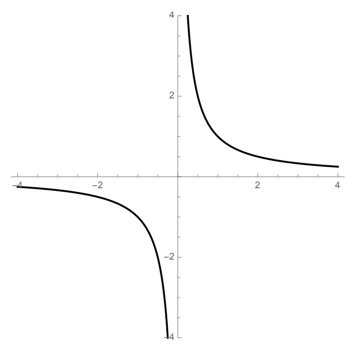
1. $\lim_{x \to 2} f(x)$
2. $\lim_{x \to 0} f(x)$
3. $f(2)$
4. $f(0)$
5. $\lim_{x \to 2^+} f(x)$
6. $\lim_{x \to -3^-} f(x)$
7. $\lim_{x \to -3^+} f(x)$
8. $\lim_{x \to -3} f(x)$
9. $f(4)$
10. $\lim_{x \to 4^+} f(x)$
11. $\lim_{x \to 4^-} f(x)$
12. $\lim_{x \to 4} f(x)$

2.3 Infinite limits and limits at infinity

Consider the function $f(x) = \frac{1}{x}$. What happens to $f(x)$ as $x \to 0$?

x	1	.5	.1	.001	.0000001
$f(x)$	1	2	10	1000	1000000

x	-1	- .5	-.1	-.001	-.0000001
$f(x)$	-1	-2	-10	-1000	-1000000

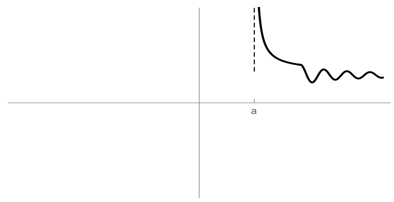


We invent new notation to describe this situation. We say

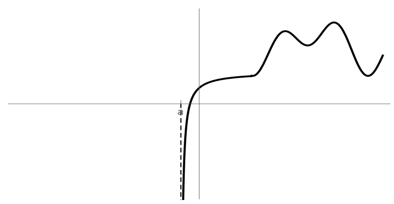
$$\lim_{x\to 0^+} \frac{1}{x} = \infty \qquad \text{and} \qquad \lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

Formally:

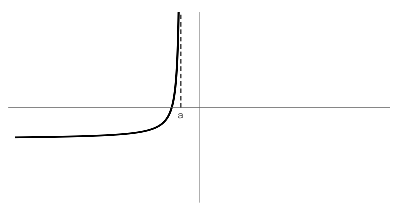
- to say $\lim_{x\to a^+} f(x) = \infty$ means that as x gets closer and closer to a from the right, the numbers $f(x)$ grow without bound.



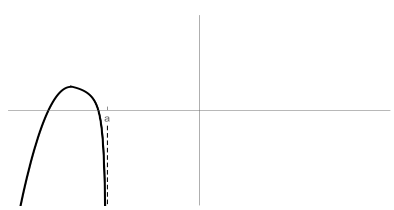
- to say $\lim_{x\to a^+} f(x) = -\infty$ means that as x gets closer and closer to a from the right, the numbers $f(x)$ become more and more negative without bound.



- to say $\lim_{x\to a^-} f(x) = \infty$ means that as x gets closer and closer to a from the left, the numbers $f(x)$ grow without bound.



- to say $\lim_{x\to a^-} f(x) = -\infty$ means that as x gets closer and closer to a from the left, the numbers $f(x)$ become more and more negative without bound.



All these situations are called **infinite limits**. The graphical description of an infinite limit is as follows:

Definition 2.3 *If* $\lim_{x \to a^+} f(x) = \pm\infty$ *or* $\lim_{x \to a^-} f(x) = \pm\infty$*, we say the vertical line* $x = a$ *is a* **vertical asymptote (VA)** *for* $f(x)$*.*

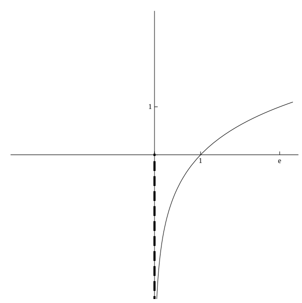
Example: $x = 0$ is a VA for $f(x) = \frac{1}{x}$.

NOTE: ∞ is **not a number**. It is only a symbol. However, in the context of limits, ∞ can be manipulated in some ways as if it was a number (we'll see how in Chapter 3). For now you should remember these facts:

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \qquad \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

<u>**One infinite limit to memorize**</u>**:**

$$\lim_{x \to 0^+} \ln x = -\infty$$



Other infinite limits are computed using techniques we will study later, using some rules of arithmetic with ∞.

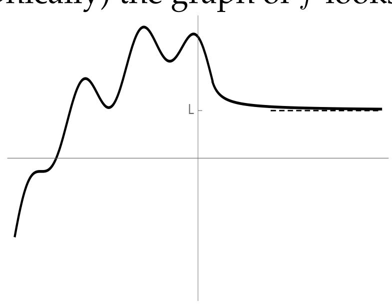
Limits at infinity

We want to consider the values of $f(x)$ when x gets larger and larger without bound. For example, suppose $f(x) = \frac{1}{x}$:

x	1	10	10000	10^{100}	10^{10000}
$f(x)$	1	$\frac{1}{10}$	$\frac{1}{10000}$	$\frac{1}{10^{100}}$	$\frac{1}{10^{10000}}$

We say $\lim\limits_{x\to\infty} f(x) = L$ if

- (heuristically) when x grows without bound, $f(x)$ approaches L.
- (graphically) the graph of f looks like



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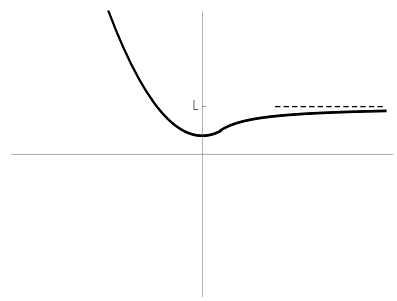
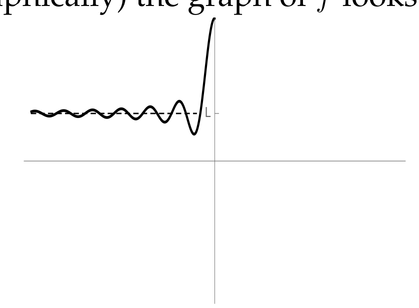
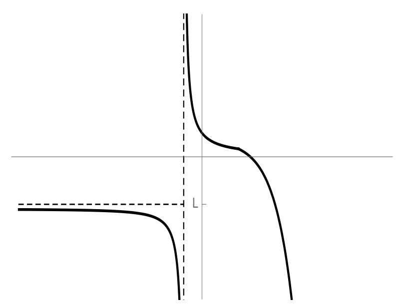


We say $\lim\limits_{x\to-\infty} f(x) = L$ if

- (heuristically) when x becomes more and more negative without bound, $f(x)$ approaches L.
- (graphically) the graph of f looks like



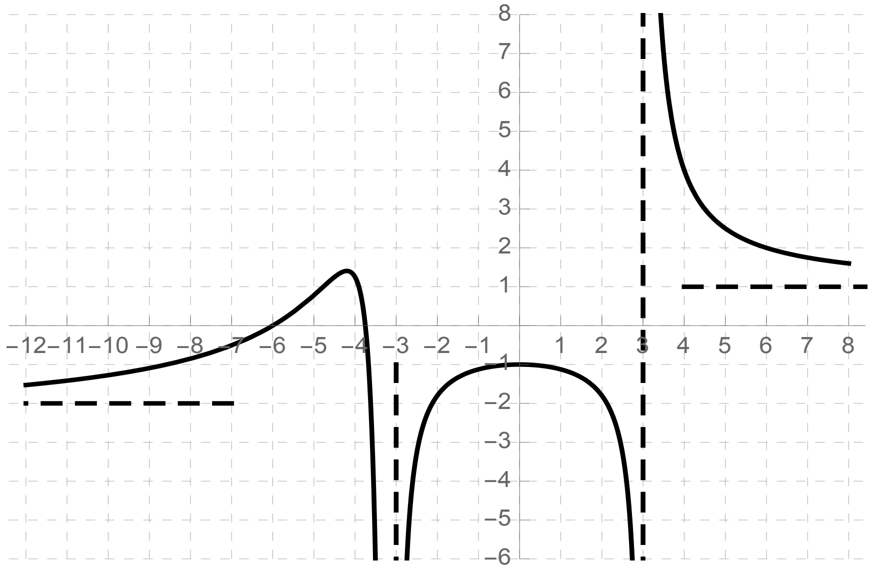
or



Definition 2.4 *If* $\lim\limits_{x\to\infty} f(x) = L$ *or* $\lim\limits_{x\to-\infty} f(x) = L$*, we say the horizontal line* $y = L$ *is a* **horizontal asymptote (HA)** *for* $f(x)$*.*

EXAMPLE

Consider the following graph of some unknown function f:



Based on this graph, find the following:

1. $\lim_{x\to\infty} f(x)$
2. $\lim_{x\to-\infty} f(x)$
3. $\lim_{x\to-3^+} f(x)$
4. $\lim_{x\to-3^-} f(x)$
5. $\lim_{x\to-3} f(x)$
6. $\lim_{x\to3^+} f(x)$
7. $\lim_{x\to3^-} f(x)$
8. $\lim_{x\to3} f(x)$
9. the equation(s) of any vertical asymptote(s) of f
10. the equation(s) of any horizontal asymptote(s) of f

2.4 Homework exercises

In Problems 1-2 below, use a calculator or computer to complete the tables and use the results to estimate the value of the limit:

1. $$\lim_{x \to 3} \frac{x-3}{x^2-7x+12}$$

x	2.9	2.99	2.999
$f(x)$			

x	3.1	3.01	3.001
$f(x)$			

2. $$\lim_{x \to -2} \frac{\sqrt{2-x}-2}{x+2}$$

x	-2.1	-2.01	-2.001
$f(x)$			

x	-1.9	-1.99	-1.999
$f(x)$			

3. Find the value of the following limit using tables similar to Problems 1 and 2. (This time, you have to pick your own x values.)
$$\lim_{x \to 0} \frac{\ln(x+1)-x}{x^2}$$

4. Find the value of the following limit using tables similar to Problems 1 and 2. (Again, you have to pick your own x values.)
$$\lim_{x \to 1} \frac{1-\frac{2}{x+1}}{x-1}$$

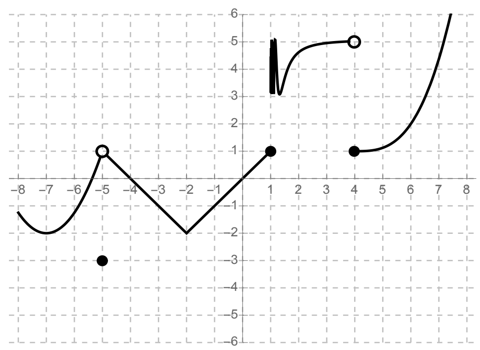
5. Complete the following charts for the function $f(x) = \frac{|x-5|}{x-5}$:

x	5.1	5.01	5.001
$f(x)$			

x	4.9	4.99	4.999
$f(x)$			

What do these charts suggest to you about $\lim_{x \to 5} \frac{|x-5|}{x-5}$?

6. Given the graph of f below, evaluate the given expressions. If the quantity does not exist, say so.



a) $\lim\limits_{x\to -5} f(x)$

b) $f(-5)$

c) $\lim\limits_{x\to -1} f(x)$

d) $\lim\limits_{x\to 1^-} f(x)$

e) $\lim\limits_{x\to 1^+} f(x)$

f) $\lim\limits_{x\to 1} f(x)$

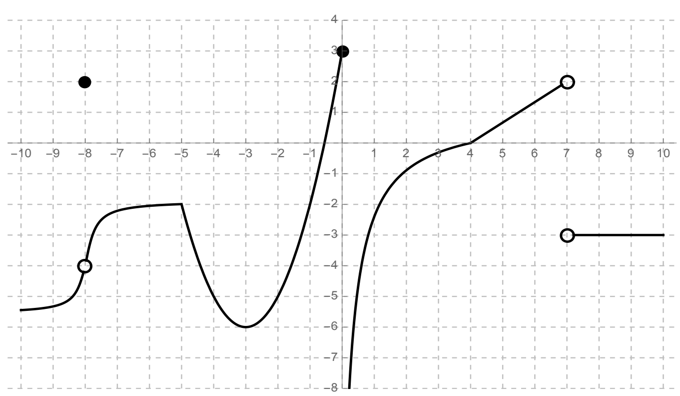
g) $\lim\limits_{x\to 4^-} f(x)$

h) $\lim\limits_{x\to 4^+} f(x)$

i) $\lim\limits_{x\to 4} f(x)$

j) $f(4)$

7. Given the graph of g below, evaluate the given expressions. If the quantity does not exist, say so.



a) $\lim\limits_{x\to -8} g(x)$

b) $f(-8)$

c) $\lim\limits_{x\to -5} g(x)$

d) $\lim\limits_{x\to 0^-} g(x)$

e) $\lim\limits_{x\to 1} g(x)$

f) $\lim\limits_{x\to 7^-} g(x)$

g) $\lim\limits_{x\to 7^+} g(x)$

h) $\lim\limits_{x\to 7} g(x)$

i) $g(7)$

j) $\lim\limits_{x\to -2^-} g(x)$

8. Sketch a graph of a function f which has all of the following four properties (there are many possible correct answers):
   - $f(0)$ is not defined;
   - $\lim_{x\to 0} f(x) = 4$;
   - $f(2) = 6$;
   - $\lim_{x\to 2} f(x) = 3$.

9. Sketch a graph of a function f which has all of the following five properties (there are many possible correct answers):
   - $\lim_{x\to -1^+} f(x) = 3$;
   - $\lim_{x\to -1^-} f(x) = -2$;
   - $\lim_{x\to 2^-} f(x)$ DNE;
   - $f(2) = 0$;
   - $\lim_{x\to 2^+} f(x) = 3$.

10. Sketch a graph of a function f which has all of the following four properties (there are many possible correct answers):
    - $\lim_{x\to 3} f(x) = -1$;
    - $f(3) = 2$;
    - $\lim_{x\to -4^-} f(x) = -5$;
    - $\lim_{x\to -4^+} f(x) = -1$.

In Problems 11-12 below, complete the tables (use a calculator or computer if necessary) and use the results to estimate the value of the limit:

11.

$$\lim_{x\to\infty} \frac{4x+3}{2x-1}$$

x	10	100	1000	10^6	10^{10}
$f(x)$					

12.

$$\lim_{x\to\infty} \frac{-6x}{\sqrt{4x^2+5}}$$

x	10	100	1000	10^6	10^{10}
$f(x)$					

In Problems 13-18 below, graph the function inside the limit using *Mathematica* (or a calculator) and use the graph of the function to estimate $\lim_{x\to\infty} f(x)$:

13. $f(x) = \frac{|x|}{x+1}$

    Hint: Mathematica code to plot this function (where x ranges from -10 to 10) is

    ```
    Plot[ Abs[x] / (x+1), {x, -10, 10}]
    ```

14. $f(x) = \frac{\ln x}{\sqrt{x}}$

    Hint: Mathematica code to plot this function is

    ```
    Plot[ Log[x] / Sqrt[x], {x, -10, 10}]
    ```

15. $f(x) = \frac{\sin x}{x}$

16. $f(x) = x \arctan\left(\frac{1}{x}\right)$

17. $f(x) = x - \sqrt{x(x-1)}$

18. $f(x) = \frac{x+1}{x\sqrt{x}}$

In Problems 19-20 below, complete the tables (use a calculator or computer if necessary) and use the results to estimate the value of the limit:

19.
$$\lim_{x\to 1^+} \frac{2+x}{1-x}$$

x	2	1.1	1.01	1.0001	1.000001
$f(x)$					

20.
$$\lim_{x\to 3^-} \frac{x^2+7}{x-3}$$

x	2	2.9	2.99	2.9999	2.999999
$f(x)$					

In Problems 21-26 below, graph the function inside the limit using *Mathematica* (or a calculator) and use the graph of the function to estimate the given limit:

21. $\lim_{x\to\pi^-} \sec(\frac{x}{2})$

    Hint: Mathematica code to plot this function (where x ranges from -10 to 10) is

    ```
    Plot[ Sec[x/2], {x, -10, 10}]
    ```

22. $\lim\limits_{x\to\pi^+} \sec(\frac{x}{2})$

23. $\lim\limits_{x\to 0} \frac{(x-1)^2}{x^2}$

24. $\lim\limits_{x\to\pi^-} (\cot x - \sec x)$

25. $\lim\limits_{x\to 4^-} \frac{3x^2-6x+5}{x^2-5x+4}$

26. $\lim\limits_{x\to 1^-} \frac{x-4}{e^x-e}$

27. By using *Mathematica* to graph the function, find the equation of any horizontal and/or vertical asymptotes of the function

$$f(x) = \frac{x^2+3}{x^3-5x^2+4x}.$$

Hint: Mathematica code to plot this function (where x ranges from -10 to 10) is

```
Plot[ (x^2 + 3)/(x^3 - 5x^2 + 4x), {x, -10, 10}]
```

Make sure to use parentheses to surround the numerator and denominator when using *Mathematica*.

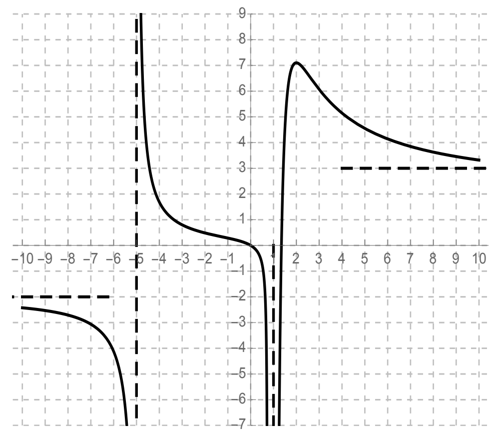
28. By using *Mathematica* to graph the function, find the equation of any horizontal and/or vertical asymptotes of the function

$$f(x) = \frac{2x^2-8x-42}{x^2-25}.$$

29. By using *Mathematica* to graph the function, find the equation of any horizontal and/or vertical asymptotes of the function

$$f(x) = \frac{(x-3)(x+4)(x-7)}{x(x-3)(x+1)}.$$

30. Given the graph of f below, evaluate the given limits.



a) $\lim\limits_{x\to\infty} f(x)$

b) $\lim\limits_{x\to-\infty} f(x)$

c) $\lim\limits_{x\to-5^-} f(x)$

d) $\lim\limits_{x\to-5^+} f(x)$

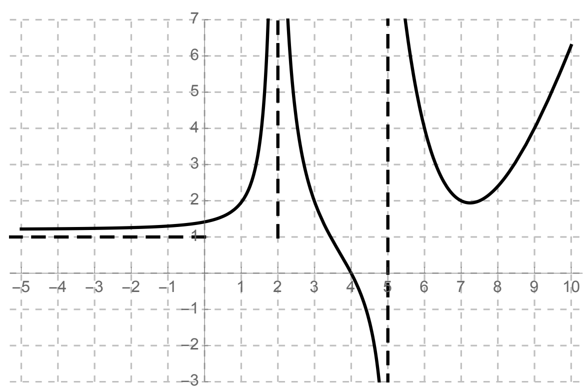
e) $\lim\limits_{x\to-5} f(x)$

f) $\lim\limits_{x\to1^+} f(x)$

g) $\lim\limits_{x\to1^-} f(x)$

h) $\lim\limits_{x\to1} f(x)$

31. Given the graph of g below, evaluate the given limits.



a) $\lim\limits_{x\to\infty} g(x)$

b) $\lim\limits_{x\to-\infty} g(x)$

c) $\lim\limits_{x\to2^-} g(x)$

d) $\lim\limits_{x\to2^+} g(x)$

e) $\lim\limits_{x\to2} g(x)$

f) $\lim\limits_{x\to5^+} g(x)$

g) $\lim\limits_{x\to5^-} g(x)$

h) $\lim\limits_{x\to5} g(x)$

32. Sketch a graph of a function f which has all of the following three properties (there are many possible correct answers):
    - $\lim_{x\to 1^+} f(x) = \infty$;
    - $\lim_{x\to 1^-} f(x) = -\infty$;
    - $\lim_{x\to\infty} f(x) = 3$.

33. Sketch a graph of a function f which has all of the following four properties (there are many possible correct answers):
    - $\lim_{x\to -4} f(x) = \infty$;
    - $\lim_{x\to -\infty} f(x) = 5$;
    - $\lim_{x\to\infty} f(x) = -2$;
    - $f(0) = -3$.

34. Sketch a graph of a function f which has all of the following four properties (there are many possible correct answers):
    - $\lim_{x\to 2^-} f(x) = 4$;
    - $\lim_{x\to 2^+} f(x) = \infty$;
    - $f(2) = -1$;
    - $\lim_{x\to -\infty} f(x) = \lim_{x\to\infty} f(x) = 2$.

Answers

1. $\lim_{x\to 3} \frac{x-3}{x^2-7x+12} = -1$
2. $\lim_{x\to -2} \frac{\sqrt{2-x}-2}{x+2} = \frac{-1}{4}$
3. $\frac{-1}{2}$
4. $\frac{1}{2}$
5. $\lim_{x\to 5} \frac{|x-5|}{x-5}$ DNE (left- and right-hand limits are unequal)
6. a) 1
   b) -3
   c) -1
   d) 1
   e) DNE
   f) DNE
   g) 5
   h) 1
   i) DNE
   j) 1
7. a) -4
   b) 2
   c) -2
   d) 3
   e) about -2.5
   f) about 1.5
   g) -3
   h) DNE

   i) DNE
   j) -5

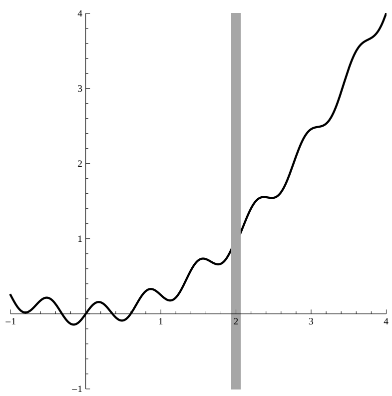
8. Many answers are possible.
9. Many answers are possible.
10. Many answers are possible.
11. $\lim_{x\to\infty} \frac{4x+3}{2x-1} = 2$
12. $\lim_{x\to\infty} \frac{-6x}{\sqrt{4x^2+5}} = -3$
13. 1
14. 0
15. 0
16. 1
17. $\frac{1}{2}$
18. 0
19. $\lim_{x\to 1^+} \frac{2+x}{1-x} = -\infty$
20. $\lim_{x\to 3^-} \frac{x^2+7}{x-3} = -\infty$
21. ∞
22. $-\infty$
23. ∞
24. $-\infty$
25. $-\infty$
26. ∞
27. HA: $y = 0$
    VA: $x = 0, x = 1, x = 4$.
28. HA: $y = 2$
    VA: $x = 5, x = -5$.
29. HA: $y = 1$
    VA: $x = 0, x = -1$.
30. a) 3
    b) -2
    c) $-\infty$
    d) ∞
    e) DNE
    f) $-\infty$
    g) $-\infty$
    h) $-\infty$
31. a) ∞
    b) 1
    c) ∞
    d) ∞
    e) ∞
    f) ∞
    g) $-\infty$
    h) DNE
32. Many answers are possible.
33. Many answers are possible.
34. Many answers are possible.

Chapter 3

Computing Limits

3.1 Continuity

Recall the modified warmup example from an earlier lecture:

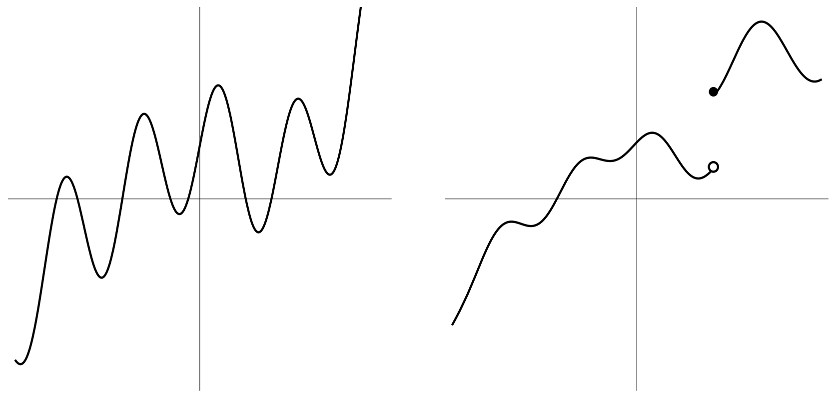


What is $f(2)$?

We don't know the answer, but we said that the most "reasonable" guess was 1.

Why was this the most "reasonable" guess?

Mathematically, this is described by the notion of "continuity". For example:



Functions whose graphs have no breaks are called "continuous".

More precisely:

Definition 3.1 *A function f is called* **continuous at the point** $x = a$ *if*

1. $f(a)$ *exists;*
2. $\lim_{x \to a} f(x)$ *exists (i.e.* $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$*); and*
3. $\lim_{x \to a} f(x) = f(a)$ *(i.e.* $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a)$*).*

Otherwise we say f is **discontinuous at** $x = a$*.*

The word continuous is abbreviated "cts".

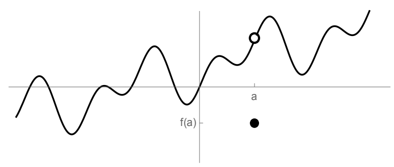
Definition 3.2 *A function f is called* **continuous on an interval** *if it is continuous at every point in that interval. A function f is called* **continuous** *if it is continuous at every point in its domain.*

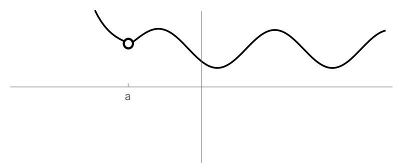
Classification of discontinuities

When looking at the graph of a function, its easier to tell where the function is *discontinuous* than where it is continuous, because the discontinuities in a function usually stand out.

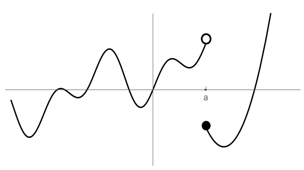
It turns out that there are four types of discontinuities (it's not critical that you know this vocabulary):

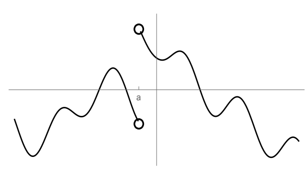
1. **removable discontinuity** (a.k.a. **hole discontinuity**): $\lim\limits_{x\to a} f(x)$ exists but either $f(a)$ DNE or $f(a) \neq \lim\limits_{x\to a} f(x)$:

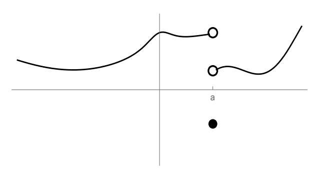




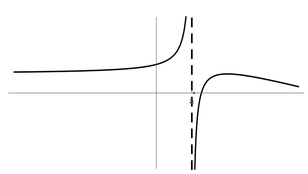
2. **jump discontinuity**: $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ both exist but are not equal:

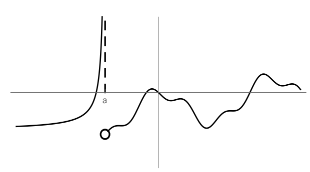




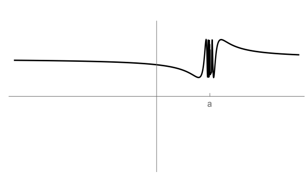


3. **infinite discontinuity**: $\lim\limits_{x\to a^+} f(x)$ or $\lim\limits_{x\to a^-} f(x) = \pm\infty$:





4. **oscillating discontinuity**: $\lim\limits_{x\to a^+} f(x)$ or $\lim\limits_{x\to a^-} f(x)$ DNE because of too many wiggles:



Dictionary of continuous functions

What is important is that you have a working knowledge of functions which are continuous.

In particular, the following functions are continuous, because there are no breaks in their graphs:

Theorem 3.3 *Suppose f and g are continuous functions. Then:*

1. *$f+g$, $f-g$, fg, and $f \circ g$ are continuous; and*
2. *$\frac{f}{g}$ is continuous at all x where $g(x) \neq 0$.*

Theorem 3.4 *Any function made up of powers of x, sines and cosines, arcsines, arctangents, exponentials and/or logarithms using addition, subtraction, and/or multiplication is continuous (at every point of its domain).*

Theorem 3.5 *Any function which is the quotient of functions made up of powers of x, sines, cosines, arcsines, arctangents, exponentials and/or logarithms is continuous everywhere* ***except where the denominator is zero****.*

EXAMPLES

$$f(x) = 3\arcsin(x^2+4)\cos^5\left(\frac{3x}{x^2+4}\right) - 5e^{\sin(3x^8-5)}\ln(x^4+3)$$

is continuous everywhere on its domain.

$$g(x) = \frac{x^3 + 3\cos(2x^2-5) - 6^{x-4\sin\sqrt[3]{x}}}{x-3}$$

is continuous everywhere except $x = 3$.

3.2 Evaluation of limits

First concept: limits behave "nicely" with respect to arithmetic

Theorem 3.6 (Main Limit Theorem) *Suppose* $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ *both exist and are finite, where* a *is either* $\pm\infty$ *or a finite number. Then:*

1. $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$;
2. $\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$;
3. $\lim_{x\to a} [k\ f(x)] = k \cdot \lim_{x\to a} f(x)$ *for any constant* k;
4. $\lim_{x\to a} [f(x)g(x)] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right]$;
5. $\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *provided the denominator is nonzero.*
6. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ *provided both sides exist.*
7. $\lim_{x\to a} e^{f(x)} = \exp\left(\lim_{x\to a} f(x)\right)$.
8. $\lim_{x\to a} \ln f(x) = \ln\left(\lim_{x\to a} f(x)\right)$.

Second concept: evaluate limits of cts functions by plugging in

If f is continuous at a, then $\lim_{x\to a} f(x) = f(a)$.

Third concept: ignore $f(a)$ in general

$f(a)$ has nothing to do with the value of $\lim_{x\to a} f(x)$.

The second and third concepts seem contradictory, but aren't.

Fourth concept: manipulate expressions with ∞ using rules

Although ∞ is not a number, it can be mainpulated in some ways as if it is a number.

See the next page for details.

<u>Some arithmetic rules with ∞</u>: (useful to compute some limits)

Let $c \in \mathbb{R}$. Then:

$$\begin{aligned}
\infty \pm c &= \infty \\
c \cdot \infty &= \infty \quad \text{if } c > 0 \\
\infty \cdot \infty &= \infty \\
\infty^{\infty} &= \infty \\
c \cdot \infty &= -\infty \quad \text{if } c < 0 \\
\frac{\infty}{c} &= \infty \quad \text{if } c > 0 \\
\frac{\infty}{c} &= -\infty \quad \text{if } c < 0 \\
\frac{c}{\infty} &= 0 \\
\sqrt{\infty} &= \infty \\
\ln \infty &= \infty
\end{aligned}$$

$$\begin{aligned}
e^{\infty} &= \infty \\
e^{-\infty} &= 0 \\
\frac{\infty}{0} &= \pm\infty \\
&\text{(need careful analysis to determine the sign)} \\
\frac{c}{0} &= \pm\infty \text{ so long as } c \neq 0 \\
&\text{(need careful analysis to determine the sign)} \\
\infty^{c} &= \begin{cases} \infty & \text{if } c > 0 \\ 0 & \text{if } c < 0 \end{cases}
\end{aligned}$$

Warning:

$$\begin{aligned}
\frac{0}{0} &\text{ is indeterminate} \\
\infty - \infty &\text{ is indeterminate} \\
0 \cdot \infty &\text{ is indeterminate} \\
\frac{\infty}{\infty} &\text{ is indeterminate} \\
\infty^{0} &\text{ is indeterminate} \\
0^{0} &\text{ is indeterminate}
\end{aligned}$$

"Indeterminate" means that these expressions can work out to be different things depending on the context (we will see how to compute some of these in Section 8.2).

Evaluating limits at infinity

The most important examples of limits to understand how to evaluate are those for which $x \to \infty$ (i.e. **limits at infinity**):

EXAMPLE 1

$$\lim_{x \to \infty} \frac{4 + 3x^2}{8x^2 + 3x + 2}$$

Remark: this example could have been phrased differently: suppose you were asked to find the horizontal asymptotes of $f(x) = \frac{4+3x^2}{8x^2+3x+2}$. In this case, you'd compute the limit as above, and identify the HA as

Rephrasing this as a story problem: Suppose the population of an endangered species in a national park at time x, in thousands, is given by $f(x) = \frac{4+3x^2}{8x^2+3x+2}$ (the function from Example 1). What is the long-term population of this species in this park projected to be?

EXAMPLE 2

$$\lim_{x \to \infty} \frac{-3 - 5x^2}{2x^4 - x + \frac{5}{2}}$$

EXAMPLE 3

$$\lim_{x\to\infty} \frac{2x^2+7x-2}{x-1}$$

EXAMPLE 4

$$\lim_{x\to\infty} \frac{x}{\sqrt{2x^2+1}}$$

General principle behind examples 1-4: Suppose f is a rational function, i.e. has form

$$f(x) = \frac{a_m x^m + a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \dots + a_2x^2 + a_1x + a_0}{b_n x^n + b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \dots + b_2x^2 + b_1x + b_0}.$$

Then:

1. If $m < n$ (i.e. largest power in numerator $<$ largest power in denominator), then
$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0.$$
2. If $m > n$ (i.e. largest power in numerator $>$ largest power in denominator), then
$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \pm\infty.$$
3. If $m = n$ (i.e. largest powers in numerator and denominator are equal), then
$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \frac{a_m}{b_n}.$$

Other limits at infinity to memorize:

$$\lim_{x\to\infty} e^{-x} = 0 \qquad \lim_{x\to\infty} e^{x} = \infty$$

$$\lim_{x\to\infty} \ln x = \infty$$

$$\lim_{x\to\infty} \arctan x = \frac{\pi}{2} \qquad \lim_{x\to-\infty} \arctan x = \frac{-\pi}{2}$$

EXAMPLE 5

$$\lim_{x\to\infty} \left(e^{-3x} + \arctan 2x\right)$$

EXAMPLE 6

$$\lim_{x\to\infty} \sin x$$

Evaluation of limits of continuous functions

Key fact: If f is continuous at a, then $\lim_{x \to a} f(x) = f(a)$.

EXAMPLE 7

$$\lim_{x \to 3} \frac{x^2 + 3}{x - 1}$$

EXAMPLE 8

$$\lim_{x \to \frac{\pi}{2}} 3 \cos 2x$$

EXAMPLE 9

$$\lim_{x \to 0} e^{2x}$$

Evaluation of limits of functions which are not known to be continuous

Given limit $\lim_{x \to a} f(x)$, start by plugging in a to the function f.

1. if you get a number when you plug in, almost always this is the answer (and the function is actually continuous at a);
2. if you get $\frac{\text{nonzero}}{0}$, the limit is infinite; carefully analyze the sign of $f(x)$ to determine whether the answer is ∞ or $-\infty$;
3. if you get $\frac{0}{0}$, use an algebraic technique to rewrite f:
   a) if f can be factored, factor and cancel terms;
   b) if f contains square roots which are added or subtracted, multiply through by the conjugate (then factor and cancel);
   c) if f contains "fractions within fractions", clear the denominators of the interior fractions (then factor and cancel).

EXAMPLE 10

$$\lim_{x \to 3^+} \left(\frac{2}{(x-3)^2} + 2x^2 \right)$$

EXAMPLE 11

$$\lim_{x \to 5} \frac{x^2 - 3x - 10}{x^2 + x - 20}$$

EXAMPLE 11

$$\lim_{x \to 2^+} \frac{4}{4 - x^2}$$

EXAMPLE 12

$$\lim_{x \to 2^-} \frac{4}{4 - x^2}$$

EXAMPLE 13

$$\lim_{x \to 2} \frac{4}{4 - x^2}$$

Solution: From Examples 12 and 13, we see that

$$\lim_{x \to 2^+} \frac{4}{4 - x^2} \neq \lim_{x \to 2^-} \frac{4}{4 - x^2}.$$

Therefore the two-sided limit

$$\lim_{x \to 2} \frac{4}{4 - x^2} \text{ DNE.}$$

EXAMPLE 14

$$\lim_{x \to 3} \frac{x^2 + 2x - 15}{x^2 - 7x + 12}$$

EXAMPLE 15

$$\lim_{x \to -2} \frac{x^2 - 3x - 15}{x^3 + 5x^2 + 6x}$$

More complicated examples

Key idea: if you get $\frac{0}{0}$ when you plug in, eventually you have to factor and cancel.

But in complicated situations, you first have to do some preliminary algebra to rewrite the function. Here are some worked-out examples which illustrate some techniques:

EXAMPLE 16

$$\lim_{x \to -1} \frac{\frac{1}{x} + 1}{\frac{1}{x+2} - 1}$$

Solution: When I look at this, I see "fractions inside fractions". In such a problem, here is the procedure:

Multiply through the top and bottom of the "big fraction" by the "small denominators".

In this example, the "small denominators" are x and $x+2$, and the "big fraction" is the entire function $\frac{\frac{1}{x}+1}{\frac{1}{x+2}-1}$. So you get

$$\lim_{x \to -1} \frac{\frac{1}{x} + 1}{\frac{1}{x+2} - 1} = \lim_{x \to -1} \frac{\left(\frac{1}{x} + 1\right)(x)(x+2)}{\left(\frac{1}{x+2} - 1\right)(x)(x+2)} \quad \text{Now distribute:}$$

$$= \lim_{x \to -1} \frac{\frac{1}{x}(x)(x+2) + 1(x)(x+2)}{\frac{1}{x+2}(x)(x+2) - 1(x)(x+2)}$$

$$= \lim_{x \to -1} \frac{x + 2 + x(x+2)}{x - x(x+2)}$$

$$= \lim_{x \to -1} \frac{x + 2 + x^2 + 2x}{x - x^2 - 2x}$$

$$= \lim_{x \to -1} \frac{x^2 + 3x + 2}{-x^2 - x} \quad \text{Now factor and cancel:}$$

$$= \lim_{x \to -1} \frac{(x+2)(x+1)}{-x(x+1)}$$

$$= \lim_{x \to -1} \frac{x+2}{-x} = \frac{-1+2}{-(-1)} = 1.$$

EXAMPLE 17

$$\lim_{t \to 4} \frac{\sqrt{t} - 2}{t - 4}$$

Solution: I see a square root term plus/minus another term in the numerator of the fraction. In such a situation, here is the procedure:

Multiply through the top and bottom by the "conjugate" of the square root term.

In this problem, the "conjugate" of $\sqrt{t} - 2$ is $\sqrt{t} + 2$ (see below for more on conjugates). So you get

$$\lim_{t \to 4} \frac{\sqrt{t} - 2}{t - 4} = \lim_{t \to 4} \frac{\left(\sqrt{t} - 2\right)\left(\sqrt{t} + 2\right)}{(t - 4)\left(\sqrt{t} + 2\right)}$$

Now notice the numerator is of the form $(A - B)(A + B)$, which becomes $A^2 - B^2$.
Don't multiply out the bottom.

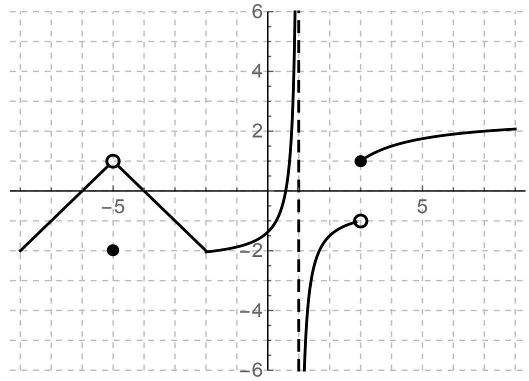
$$= \lim_{t \to 4} \frac{t - 4}{(t - 4)\left(\sqrt{t} + 2\right)}$$
$$= \lim_{t \to 4} \frac{1}{\sqrt{t} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{4}.$$

How to find conjugates:

Expression	Conjugate	Example	Conjugate of example
$\square + \sqrt{\triangle}$	$\square - \sqrt{\triangle}$	$3 + \sqrt{x - 1}$	$3 - \sqrt{x - 1}$
$\square - \sqrt{\triangle}$	$\square + \sqrt{\triangle}$	$5 - \sqrt{2x}$	$5 + \sqrt{2x}$
$\sqrt{\square} + \sqrt{\triangle}$	$\sqrt{\square} - \sqrt{\triangle}$	$\sqrt{t + 3} + \sqrt{x - 1}$	$\sqrt{t + 3} - \sqrt{x + 1}$
$\sqrt{\square} - \sqrt{\triangle}$	$\sqrt{\square} + \sqrt{\triangle}$	$\sqrt{u} - \sqrt{3x}$	$\sqrt{u} + \sqrt{3x}$

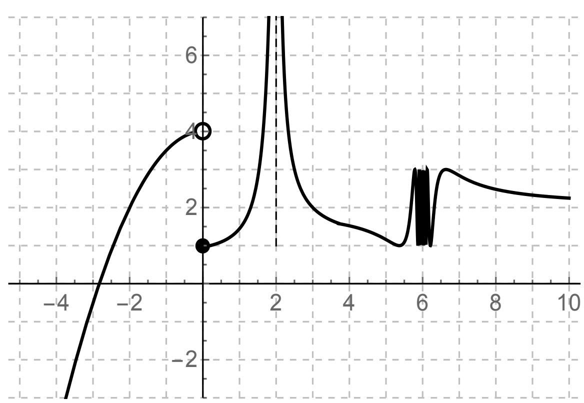
3.3 Homework exercises

1. Consider the function f whose graph is given below:



   a) At what value(s) of x, if any, is f **not** continuous?
   b) At what value(s) of x, if any, does f have a removable discontinuity?
   c) At what value(s) of x, if any, does f have a jump discontinuity?
   d) At what value(s) of x, if any, does f have an infinite discontinuity?
   e) At what value(s) of x, if any, does f have an oscillating discontinuity?

2. Consider the function g whose graph is given below:



   a) At what value(s) of x, if any, does g have a removable discontinuity?
   b) At what value(s) of x, if any, does g have a jump discontinuity?
   c) At what value(s) of x, if any, does g have an infinite discontinuity?
   d) At what value(s) of x, if any, does g have an oscillating discontinuity?

In Problems 3-14, evaluate the given limit (algebraically, by hand). If the limit does not exist, say so.

3. $\lim_{x\to\infty} \frac{x^2+3}{x^3-2}$
4. $\lim_{x\to\infty} \frac{x^2+3}{x^2-2}$
5. $\lim_{x\to\infty} \frac{x^2+3}{x-2}$
6. $\lim_{x\to\infty} \frac{3-2x^2+x}{4x(x-1)}$
7. $\lim_{x\to\infty} \frac{\sqrt{x}}{x+1}$
8. $\lim_{x\to\infty} \frac{x+1}{\sqrt{4x^2-x}}$
9. $\lim_{x\to-\infty} \frac{7x^3}{x^3+1}$
10. $\lim_{x\to\infty} \ln(4x+1)$
11. $\lim_{x\to\infty} 8\arctan x^2$
12. $\lim_{x\to\infty} \frac{4}{e^x+x}$
13. $\lim_{x\to\infty} e^{4x-5}$
14. $\lim_{x\to\infty} e^{-x^2}$
15. Suppose that the population of emperor penguins (in thousands of penguins) in Antarctica at time t (in years) is given by the function $p(t) = \frac{350}{1+\frac{3}{4}e^{-t/35}}$. Estimate the long-term population of emperor penguins in Antarctica.
16. After taking a certain antibiotic, the concentration C (in ppm) of a drug in a patient's bloodstream is given by $C(t) = \frac{t}{40t^2-80}$ where t (in hours) is the time after taking the antibiotic. What is the long-term concentration of the drug in the patient's bloodstream? (Write your answer with correct units.)
17. If you are r km from the center of a black hole, general relativity theory suggests that the velocity of a light wave at your position is given by $v(r) = \frac{300000r-7800000}{r}$ km/sec. If you are very, very far away from the black hole, what is the velocity of a light wave at your position? (Write your answer with correct units.)

In Problems 18-47, evaluate the given limit (algebraically, by hand). If the limit does not exist, say so.

18. $\lim_{x\to 2^-} \frac{x-3}{x-2}$
19. $\lim_{x\to 5^+} \frac{x^2}{x^2-25}$
20. $\lim_{x\to -2^+} \frac{x+3}{x^2+x-2}$
21. $\lim_{x\to 4} \frac{x+2}{(x-4)^2}$
22. $\lim_{x\to 0^-} \left(x^2 - \frac{1}{x}\right)$
23. $\lim_{x\to 0^+} \frac{x+1}{e^x-1}$
24. $\lim_{x\to 0^+} \frac{3}{\sin x}$
25. $\lim_{x\to 0^-} \frac{3}{\sin x}$
26. $\lim_{x\to 4} \frac{x+2}{x-4}$
27. $\lim_{x\to 3^+} \ln(x-3)$
28. $\lim_{x\to -2} (x^2-4x)$

29. $\lim_{x \to 3} \frac{x+5}{x^2-1}$
30. $\lim_{x \to 0} e^{-x}$
31. $\lim_{x \to 5} \sqrt[3]{x+3}$
32. $\lim_{x \to \pi} \tan\left(\frac{x}{3}\right)$
33. $\lim_{x \to -3} \sin \pi x$
34. $\lim_{x \to e^2} \ln x^2$
35. $\lim_{x \to 5^+} \frac{x}{x^2-5}$
36. $\lim_{x \to -1} \arctan x$
37. $\lim_{x \to 3} \frac{\frac{1}{x}-\frac{1}{3}}{x+3}$

38. $\lim_{x \to 2} f(x), \text{ where } f(x) = \begin{cases} x+2 & x < 2 \\ x^2 & x \geq 2 \end{cases}$

Hint: Consider the left- and right-hand limits at $x = 2$ separately.

39. $\lim_{x \to 2} f(x), \text{ where } f(x) = \begin{cases} 2x+1 & x < 2 \\ 8 & x = 2 \\ x^2-1 & x > 2 \end{cases}$

Hint: Consider the left- and right-hand limits at $x = 2$ separately.

40. $\lim_{x \to -2} \frac{x^2-3x-10}{x^2+5x+6}$
41. $\lim_{x \to 4} \frac{x^2-16}{x^2+x-20}$
42. $\lim_{x \to 1} \frac{x^3-3x^2+2x}{x-1}$
43. $\lim_{x \to 0} \frac{x^3+2x^2+x}{x^2-x}$
44. $\lim_{x \to 3} \frac{x^2-x-6}{2x^2-7x+3}$
45. $\lim_{x \to 2} \frac{\frac{1}{x}-\frac{1}{2}}{x-2}$
   Hint: Use the method of Example 16.
46. $\lim_{x \to 0} \frac{\sqrt{x+7}-\sqrt{7}}{x}$
   Hint: Use the method of Example 17.
47. $\lim_{x \to 1} \frac{1-x}{\sqrt{x+3}-2}$
   Hint: Use the method of Example 17.

In Problems 48-51, find the equations of all horizontal and vertical asymptotes of the given function.

Hint: for the VA, you need to find values of c for which $\lim_{x \to c^{\pm}} f(x) = \pm\infty$. This means that when you evaluate the limit, you need to get $\frac{\text{nonzero}}{0}$.

48. $f(x) = \frac{3-x}{x+2}$
49. $f(x) = \frac{x^2-4}{x+1}$
50. $f(x) = \frac{x+10}{x^2-8x+15}$
51. $f(x) = \frac{2x^2-8x+10}{x^2-11x+30}$

Answers

1. a) $x = -5, x = 1, x = 3$
   b) $x = -5$
   c) $x = 3$
   d) $x = 1$
   e) no such x
2. a) no such x
   b) $x = 0$
   c) $x = 2$
   d) $x = 6$
3. 0
4. 1
5. ∞
6. $\frac{-1}{2}$
7. 0
8. $\frac{1}{2}$
9. 7
10. ∞
11. 4π
12. 0
13. ∞
14. 0
15. 200000 penguins
16. 0 ppm
17. 300000 km/sec
18. ∞
19. ∞
20. $-\infty$
21. ∞
22. ∞
23. ∞
24. ∞
25. $-\infty$
26. DNE
27. $-\infty$
28. 12

29. 1
30. 1
31. 2
32. $\sqrt{3}$
33. 0
34. 4
35. $\frac{1}{4}$
36. $\frac{-\pi}{4}$
37. 0
38. 4
39. DNE
40. -7
41. $\frac{8}{9}$
42. -1
43. -1
44. 1
45. $\frac{-1}{4}$
46. $\frac{1}{2\sqrt{7}}$
47. -4
48. HA: $y = -1$; VA: $x = -2$
49. HA: none; VA: $x = -1$
50. HA: $y = 0$; VA: $x = 3, x = 5$
51. HA: $y = 2$; VA: $x = 6$ (not $x = 5$)

Chapter 4

Introduction to Derivatives

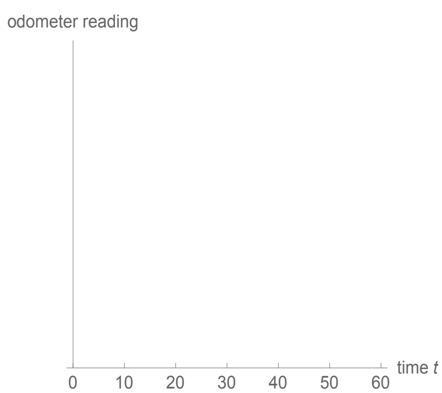
4.1 Odometers and speedometers

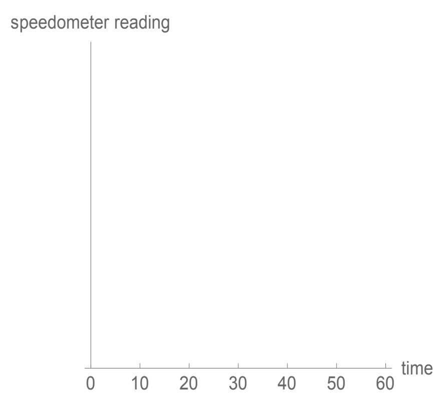
Suppose you get in your car and drive to Grand Rapids. There are two ways to record your motion as a function of elapsed time t:

1.

2.

As an example, here are two graphs representing the same trip:





Essentially, the content of Math 220 centers on the conversion from one of these pictures to the other. In particular, we want to know:

1.

2.

In Chapters 4-8, we focus on the first question above and its other applications. We will turn to the second question in Chapter 9.

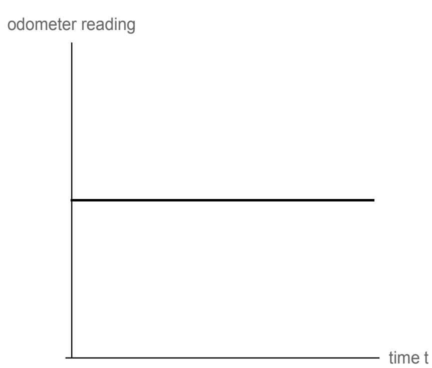
First major problem of calculus

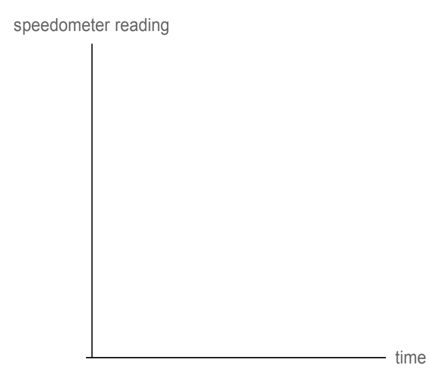
Given a function $f = f(t)$ which represents the position of an object at time t, compute the object's instantaneous velocity at time t.

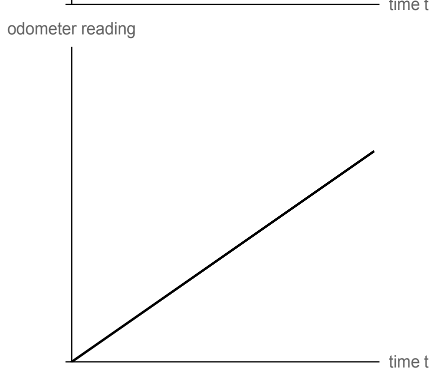
Motivation: Given the graph of a position function (i.e. a function which represents an odometer), what attribute(s) of that graph are relevant to understanding the velocity (i.e. speedometer)?

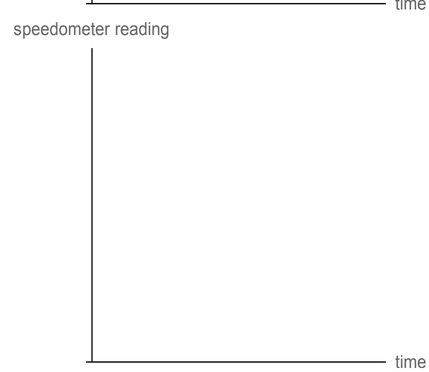
EXAMPLES

Here, you are given a series of pictures which represent odometers (that is, the $x-$axis represents time and the $y-$axis represents an odometer reading). On the blank graph to the right, sketch the graph of the corresponding **speedometer** (that is, the graph of a function where x represents elapsed time and y represents the velocity at time x).

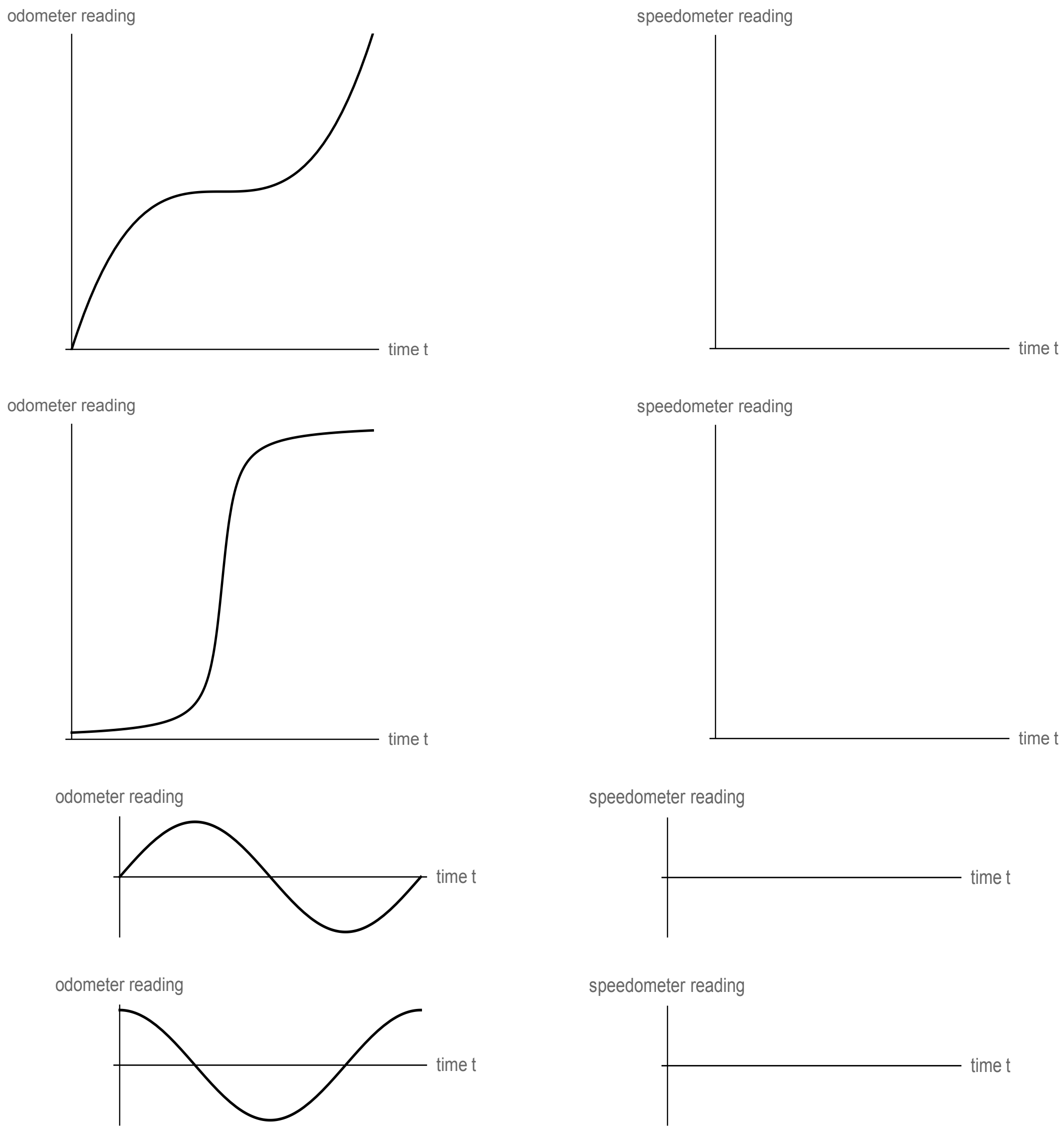








(four more graphs on the next page)



Punchline: Given a function f which measures distance traveled at time t, the corresponding velocity at time t is the slope or steepness of the graph of f at time t.

But what is meant by "slope"? We know how to find the slope of a line (from high-school algebra), but what is meant by the "slope" of a curve?

The Big Idea:

Tangent lines and differentiability

Definition 4.1 *Given a function f and a number x in the domain of f, the* **tangent line to f at x** *(if it exists) is the line which most closely approximates the graph of f at points very near x.*

Question 1: What is meant by "most closely" approximating the graph of f? What makes one line a "better" approximation than another?

Question 1 (a): Is it possible for a function f to have more than one tangent line at x?

Question 2: What does it mean (conceptually) for the tangent line to f to exist at x? Why might a tangent line not exist at x?

Definition 4.2 *A function is called* **differentiable** *at x if it has a tangent line at x.*

Theorem 4.3 (Differentiability implies continuity) *Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x. Then f must be continuous at x.*

Theorem 4.4 *A function f fails to be differentiable at x if:*

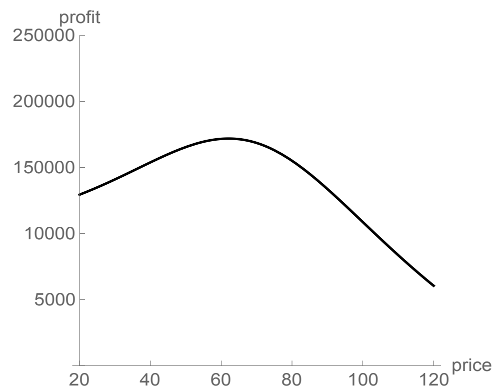
1. *f is not continuous at x; or*
2. *the tangent line to f at x is vertical; or*
3. *the graph of f has a corner or cusp at x.*

Second major problem of calculus

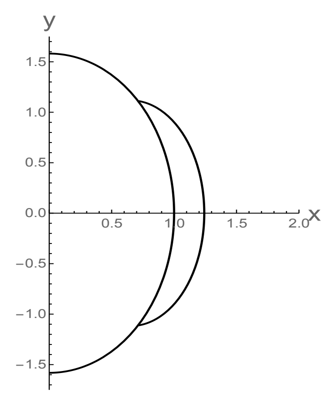
Given a function f and a particular number x
(sometimes I'll use a for the value of x),
find (if possible) the slope of the line tangent to f at x.

Why else might we care about finding the slope of a tangent line to a graph?

Business / economics:



Optometry:



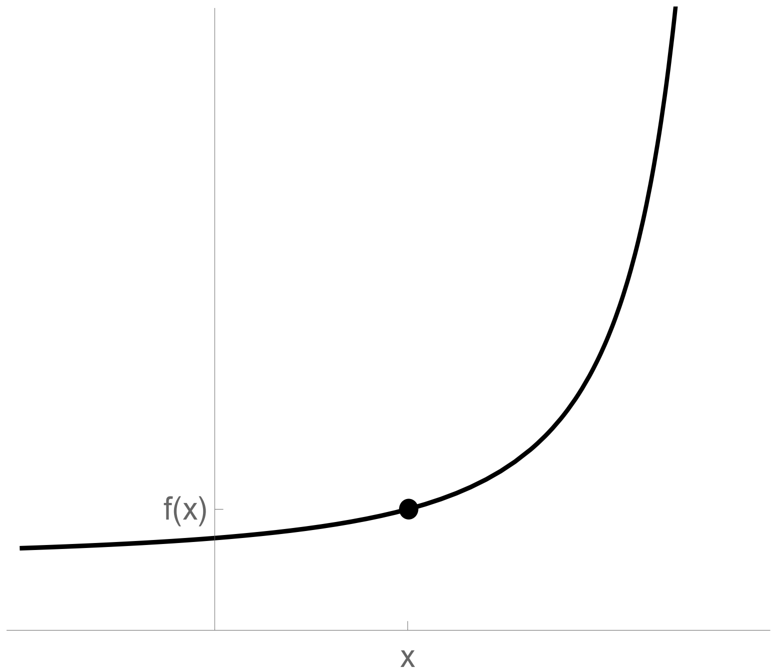
(There will be other reasons coming later.)

Can we find the slope of a tangent line to a graph using just algebra?

4.2 Definition of the derivative

Recall: the second major problem of calculus is to find the slope of the line tangent to f at x.

Let's try to solve this problem theoretically, thinking of the following picture:



Back to the first major problem
(find instantaneous velocity given position function)

Note: An object's *average* velocity over some interval of time is given by

$$v_{avg} = \frac{\text{change in object's position}}{\text{elapsed time}}.$$

Therefore if the object's position at time t is given by $f(t)$, then the object's average velocity between times t_1 and t_2 is

$$v_{[t_1,t_2]} =$$

So the object's velocity over the time interval $[x, x+h]$ is

$$v_{[x,x+h]} =$$

and its instantaneous velocity at time x should be

Notice that the formula

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

solves both of the two major problems of calculus posed earlier in this chapter. This motivates the following definition:

Definition 4.5 (Limit definition of the derivative) *Let $f:\mathbb{R}\to\mathbb{R}$ be a function and let x be in the domain of f. If the limit*

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

exists and is finite, say that f is **differentiable at** *x. In this case, we call the value of this limit* **the derivative of** *f and denote it by $f'(x)$ or $\frac{df}{dx}$ or $\frac{dy}{dx}$.*

The word "differentiable" is abbreviated "diffble".

Some algebraic manipulation of the derivative formula:

Theorem 4.6 (Alternate limit definition of the derivative) *Let $f:\mathbb{R}\to\mathbb{R}$ be a function and let f be differentiable at x. Then*

$$f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x}.$$

Notation and verbiage

- "derivative" is a noun. The verb form of this noun is "differentiate", i.e. to "differentiate" a function means to compute the derivative of that function.
- Given a function f and a particular value of x (say 4), the derivative of f at $x = 4$ is denoted

  These denote a **number**, which is the slope of the line tangent to f at $x = 4$.
- The fractional notation with "d"s above is called Leibniz notation.

The derivative as an operator

We can also think of the derivative as a **function**. But it is a different kind of function than the ones you are used to. You are used to functions like $f(x) = x^2$, where

The derivative is a new kind of function. Its inputs and outputs aren't numbers; they are *functions*. This makes differentiation into something called an *operator*:

> **Definition 4.7** *An* **operator** *is a function whose inputs and outputs are themselves functions.*

When thought of as an operator, the operation of differentiation is usually denoted $\frac{d}{dx}$ or D or just $'$. In particular,

$$\begin{aligned} \frac{d}{dx}(\text{blah}) &= \text{derivative of (blah)} \\ (\text{blah})' &= \text{derivative of (blah)} \end{aligned}$$

The output of the differentiation operator is itself a function, which we denote by

The function f' takes input x (a number) and produces as its output $f'(x)$ the slope of the line tangent to f at x.

At this point, we know that the derivative is used to compute the following quantities:

1. $f'(x)$ gives the slope of the tangent line to f at the value x;
2. $f'(x)$ gives the slope of the curve f at the value x;
3. $f'(t)$ gives the instantaneous velocity of an object at time t, given that the object's position at time t is $f(t)$;
4. $f'(x)$ gives the instantaneous rate of change of $y = f(x)$ with respect to x.

Units: If $y = f(x)$ is measured in some unit U_y and x is measured in some unit U_x, then the units of $f'(x)$ are U_y/U_x. For example, if y is measured in lbs and x is measured in ft, then $f'(x)$ will be measured in lbs/ft.

Question: What is the equation of the line tangent to differentiable function f at the point where $x = a$ (a is a constant)?

We will return to this formula many times, so it is good to remember it:

Theorem 4.8 (Tangent line equation) *Suppose f is differentiable at a. Then the equation of the line tangent to f at $x = a$ is*

$$y = f(a) + f'(a)(x - a).$$

EXAMPLE 1

Use the definition of derivative to compute the slope of the line tangent to $f(x) = \sqrt{x}$ at the point $(9, 3)$.

EXAMPLE 2

Use the definition of derivative to compute the instantaneous velocity of an object at time 4, given that the object's position (in m) at time t (in sec) is given by $f(t) = t^2 - t$.

EXAMPLE 3

Let $f(x) = |x|$. Find $f'(0)$.

Conceptual solution: Sketch the graph of f:

Justification of this: Again, use the definition:

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{|h| - 0}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

4.3 Estimating derivatives using tables or graphs

EXAMPLE 1

A straight piece of wire is placed over a heat source, so that at various points on the wire, the temperature of the wire is different. Here is a table which gives some temperature measurements at various points on the wire:

t (cm from left end of wire)	$T(t)$ (degrees Fahrenheit)
0	76
6	94
10	110
12	102
16	85

1. Use the information in this table to estimate $T(8)$. Show the computations that lead to your answer, and write your answer with correct units.

2. What does your answer to Question 1 mean, in the context of this problem?

3. Use the information in this table to estimate $T'(8)$. Show the computations that lead to your answer, and write your answer with correct units.

4. What does your answer to Question 3 mean, in the context of this problem?

EXAMPLE 2

During a flight, an airplane crew takes periodic measurements of the distance they have travelled and the amount of fuel left in their fuel tank. Their results are described in the following chart:

t (minutes after takeoff)	0	30	60	120	200	240
x (miles travelled)	0	170	405	945	1595	1775
f (thousands of gallons)	18	14	12	7	5	1.5

1. Use the information in this table to estimate $\left.\frac{dx}{dt}\right|_{t=90}$. Show the computations that lead to your answer, and write your answer with correct units.

2. What does your answer to Question 1 mean, in the context of this problem?

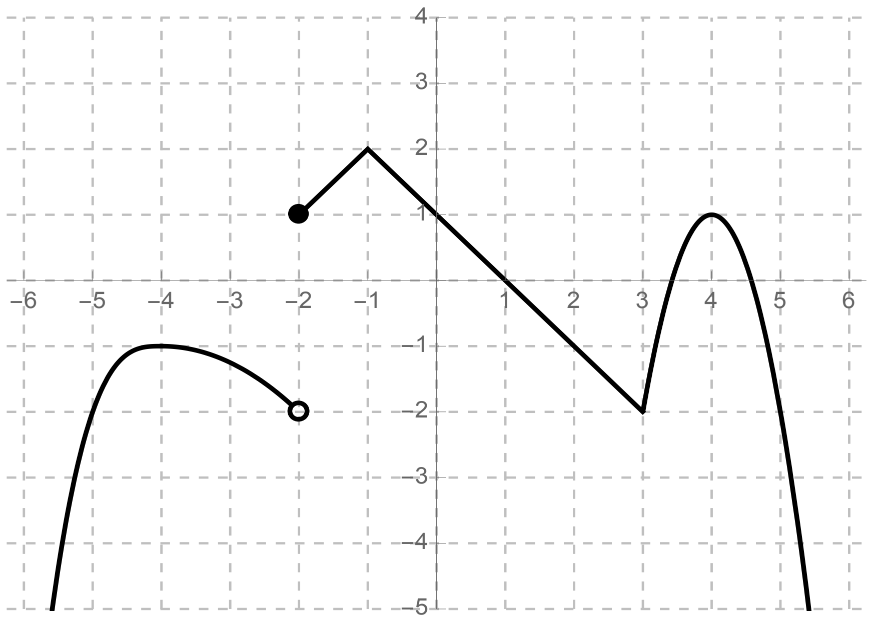
3. Use the information in this table to estimate $\left.\frac{df}{dt}\right|_{t=220}$. Show the computations that lead to your answer, and write your answer with correct units.

4. What does your answer to Question 1 mean, in the context of this problem?

5. What is the rate of fuel consumption of this aircraft per mile travelled, when the aircraft is at cruising speed? Show the computations that lead to your answer, and write your answer with correct units.

EXAMPLE 3

Given below is the graph of some unknown function f:



Use this graph to answer the questions below:

1. Give the values of x at which f is not continuous.

2. Give the values of x at which f is not differentiable.

3. Estimate $f(1)$.

4. Estimate $f'(1)$.

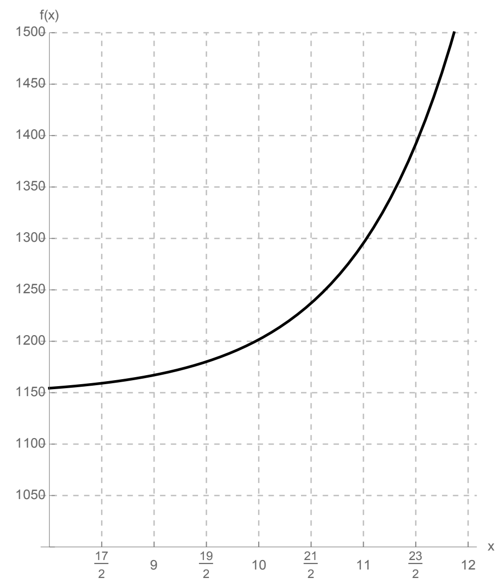
5. Estimate $f'(-5)$.

6. Find two values of x for which $f'(x) = 0$.

7. Estimate $\left.\frac{df}{dx}\right|_{x=5}$

EXAMPLE 4 (TRICKIER)

The graph of some unknown function f is given below.

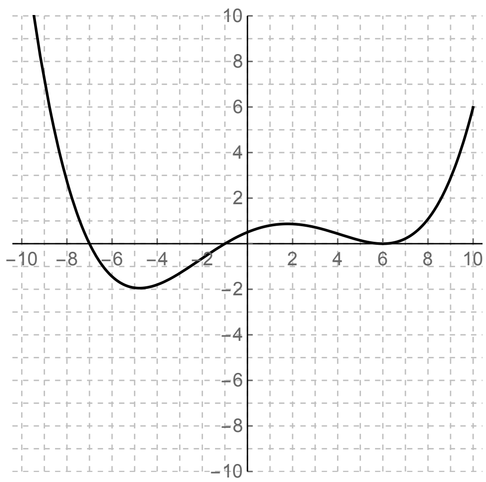


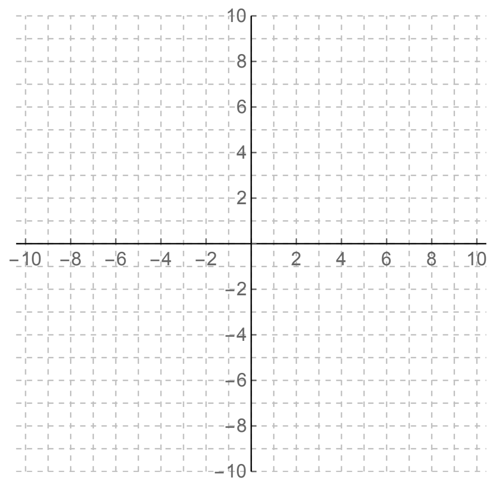
1. Use this graph to estimate $f'(10)$.

2. Use your estimate from Question 1 to write the equation of the line tangent to f at $x = 10$.

EXAMPLE 5

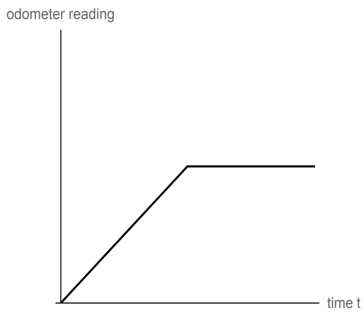
The graph of some unknown function f is given below at left. On the right-hand axes, sketch the graph of f'.

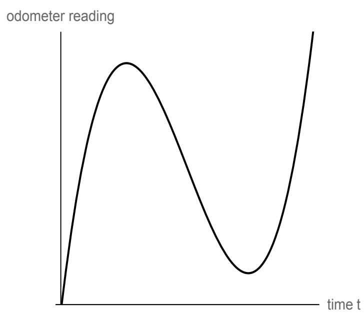


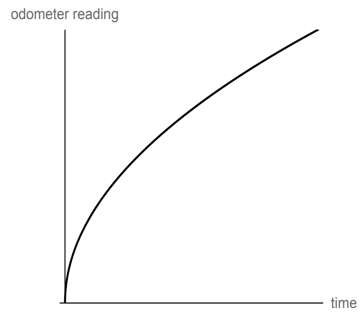


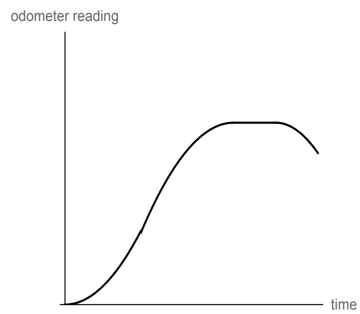
4.4 Homework exercises

In Problems 1-4, you are given the graph of an odometer. Sketch the graph of the corresponding speedometer.

1.


2.


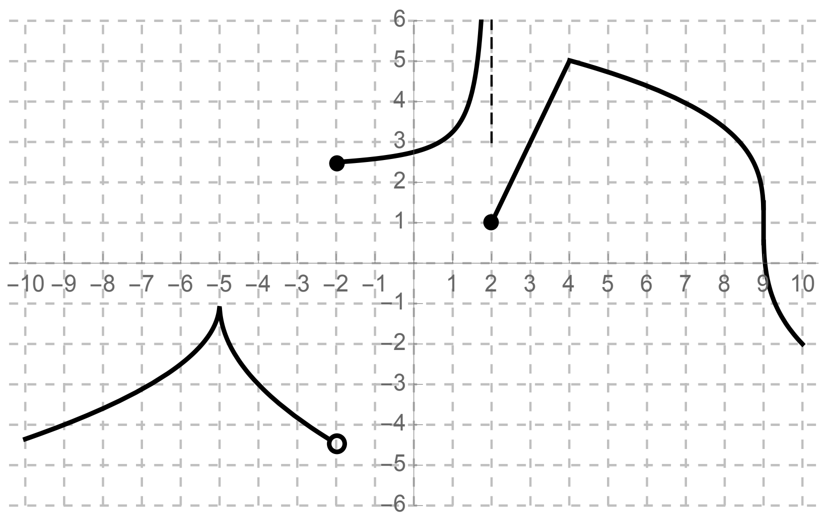
3.


4.


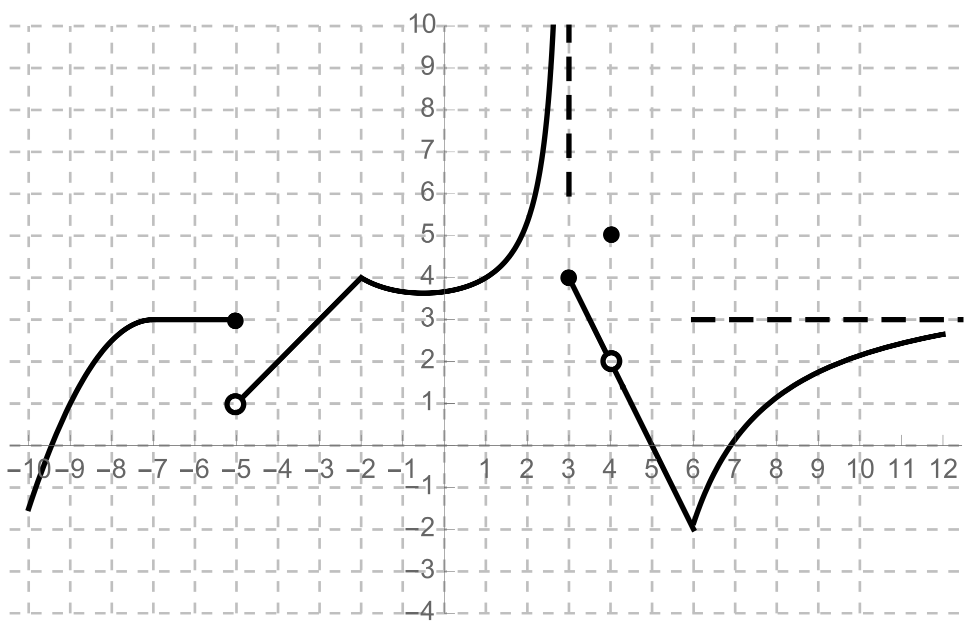
In Problems 5-10, you must compute all derivatives using the definition of derivative (do not use any "rules" you may know if you have already taken calculus).

5. Let $f(x) = 4 - \frac{2}{3}x$. Find $f'(x)$.
6. Find the derivative of $f(x) = \frac{1}{x+3}$.
7. Compute $\frac{dy}{dx}$ if $y = \sqrt{3x-2}$.
8. Find the equation of the line tangent to the function $f(x) = x^3+1$ when $x = 1$.
9. Suppose that the power supplied to a machine (in kilowatts) at time t (in hours) is $P = \sqrt{t}$. Find the instantaneous rate of change in the power supplied to the machine at time 4; write your answer with correct units.
10. Find the instantaneous velocity of an object at time 6, given that the object's position (in miles) at time t (in hours) is $f(t) = 2t^2+3t-1$; write your answer with correct units.
11. Use *Mathematica* to sketch a graph of the function $f(x) = |3x^2 - 15x + 12|$; use this graph to determine the values of x at which f is not differentiable.

12. Given the following graph of function f, give all the values x at which f is not differentiable:



13. Use the graph of the function f given below to answer the following questions:



a) Estimate $f(-6)$.

b) Estimate $f'(-6)$.

c) Estimate a value of x between -3 and 5 for which $f'(x) = 0$.

d) Find all values of x at which f is not continuous.

e) Find all values of x at which f is not differentiable.

f) Estimate $\left.\frac{df}{dx}\right|_{x=-3}$.

g) Is $f'(2)$ positive, negative or zero? Explain.

h) Estimate $\left.\frac{dy}{dx}\right|_{x=5}$.

i) Estimate $\lim_{x\to\infty} f(x)$.

j) Estimate $\lim_{x\to\infty} f'(x)$.

k) Find the slope of the function f when $x = -3$.

l) Find the equation of the line tangent to f when $x = -3$.

m) Find the equation of the line tangent to f when $x = 8$.

n) On the graph above, sketch the graph of the tangent line to x when $x = 2$.

14. A botanist measures the height, in inches, of a plant each day after it sprouts. Her data is gathered in the following table:

t (days)	0	1	3	4	8	10
h (height in inches)	0	2	8	9	10	10

a) Use the given data to estimate $h'(6)$. Show the computations that lead to your answer, and write your answer with correct units.

b) What does your answer to part (a) mean, in the context of this problem?

c) Use the given data to estimate $h'(1)$. Show the computations that lead to your answer, and write your answer with correct units.

d) What does your answer to part (c) mean, in the context of this problem?

15. As time passes, a scientist records the temperature and pressure of a gas inside a chamber as the chamber is heated. His data is summarized in the following table:

time t (minutes after start of experiment)	0	1	2	4	5	6	8
pressure P (pressure in kPa)	696	764	818	891	916	935	963
temperature T ($^\circ$ C)	20	48	71	102	112	120	132

a) Use the given data to estimate $\left.\frac{dP}{dt}\right|_{t=1}$. Show the computations that lead to your answer, and write your answer with correct units.

b) What does your answer to part (a) mean, in the context of this problem?

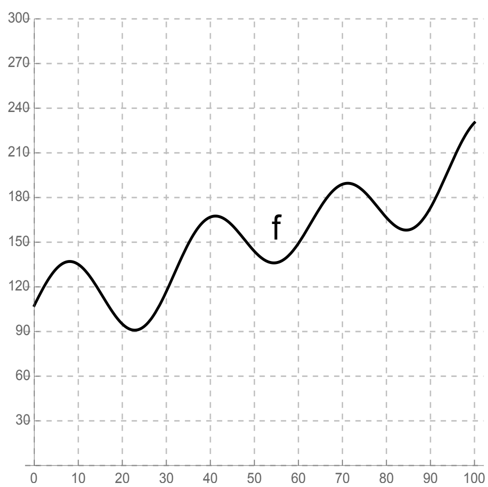
c) Use the given data to estimate $\left.\frac{dT}{dt}\right|_{t=2}$. Show the computations that lead to your answer, and write your answer with correct units.

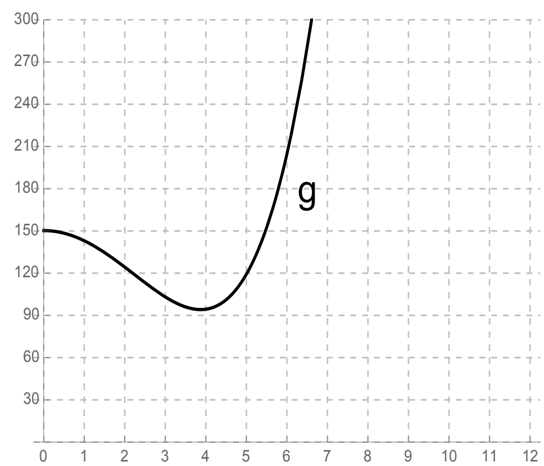
d) What does your answer to part (c) mean, in the context of this problem?

e) Use the given data to estimate the rate of change in temperature with respect to time when $t = 3$. Show the computations that lead to your answer, and write your answer with correct units.

f) Use the given data to estimate the rate of change in temperature with respect to the change in pressure when $t = 5$. Show the computations that lead to your answer, and write your answer with correct units.

16. Given the graph of f below at left, estimate $f'(30)$ and $f'(80)$.





17. Given the graph of g above at right:

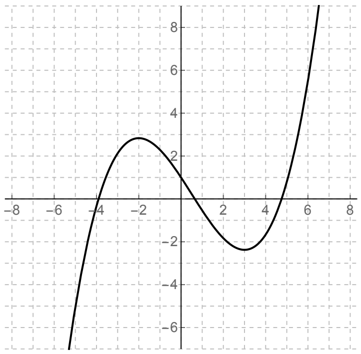
a) Estimate $g'(5)$.

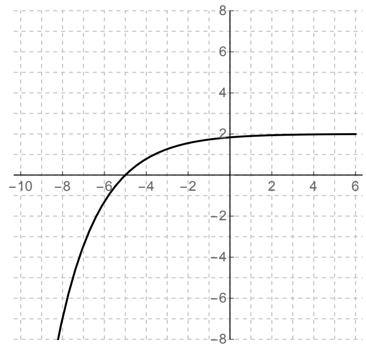
b) Write the equation of the line tangent to g when $x = 5$.

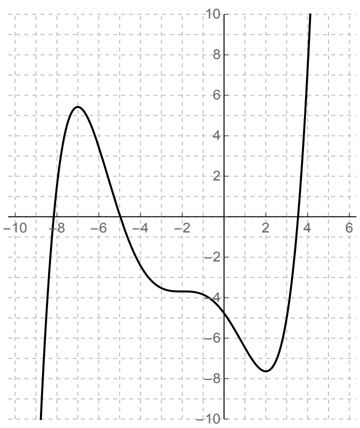
c) Estimate $\left.\frac{dg}{dx}\right|_{x=6}$.

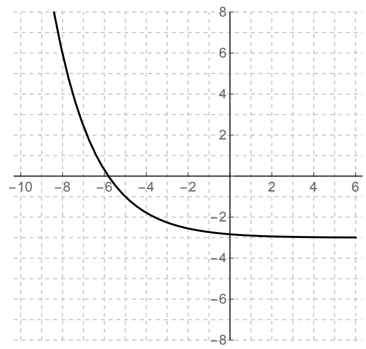
d) Sketch the graph of $g'(x)$.

In Problems 18-21, you are given the graph of an unknown function f. Sketch the graph of the function f'.

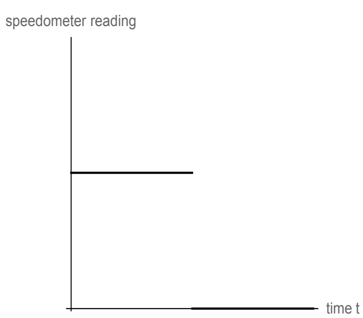
18. 

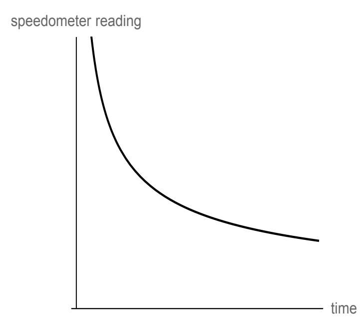
20. 

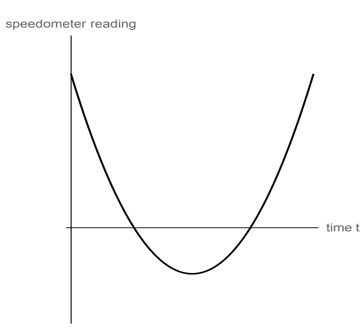
19. 

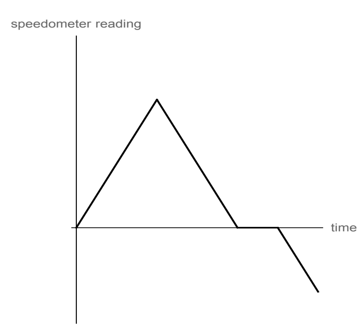
21. 

Answers

1. 

3. 

2. 

4. 

5. $\frac{-2}{3}$

6. $\frac{-1}{x^2+6x+9}$

7. $\frac{3}{2\sqrt{3x-2}}$

8. $y = 2 + 3(x-1)$

9. $P'(4) = \frac{1}{4}$ kw/hr

10. 27 mi/hr

11. $x = 1$ and $x = 4$

12. $x = -5$ (cusp), $x = -2$ (discontinuous), $x = 2$ (discontinuous), $x = 4$ (corner), $x = 9$ (vertical tangency)

13. a) 3
    b) 0
    c) $x \approx -1/2$
    d) $x = -5, x = 3, x = 4$
    e) $x = -5, x = -2, x = 3, x = 4, x = 6$
    f) 1
    g) Positive, since the graph goes up from left to right at $x = 2$.
    h) -2
    i) 3
    j) 0
    k) 1
    l) $y = 1(x+3) + 3$ (a.k.a. $y = x + 6$)
    m) $y \approx (2/3)(x-8) + 1.5$
    n) The line should go through $(2, f(2))$ and have positive slope, lying tangent to the graph at $(2, f(2))$.

14. a) $h'(6) \approx \frac{h(8)-h(4)}{8-4} = \frac{10-9}{8-4} = \frac{1}{4}$ in/day (answer can vary somewhat)
    b) On day 6, the plant is growing at a rate of $\frac{1}{4}$ inches per day.
    c) $h'(1) \approx \frac{h(1)-h(0)}{1-0} = \frac{2-0}{1-0} = 2$ and $h'(1) \approx \frac{h(3)-h(1)}{3-1} = \frac{8-2}{3-1} = 3$; averaging these gives $h'(1) \approx 2.5$ in/day (this answer can vary somewhat)
    d) On day 1, the plant is growing at a rate of 2.5 inches per day.

15. a) $\left.\frac{dP}{dt}\right|_{t=1} \approx \frac{P(1)-P(0)}{1-0} = \frac{764-696}{1} = 68$ and $\left.\frac{dP}{dt}\right|_{t=1} \approx \frac{P(2)-P(1)}{2-1} = \frac{818-764}{1} = 54$; averaging these gives $\left.\frac{dP}{dt}\right|_{t=1} \approx 61$ kPa/min (answer can vary somewhat)

b) 1 minute after the start of the experiment, the pressure in the chamber is increasing at a rate of 61 kPa/min.

c) $\left.\frac{dT}{dt}\right|_{t=2} \approx \frac{T(2)-T(1)}{2-1} = \frac{71-48}{1} = 23$ and $\left.\frac{dT}{dt}\right|_{t=2} \approx \frac{T(4)-T(2)}{4-2} = \frac{102-71}{2} = 15.5$; averaging these gives $\left.\frac{dT}{dt}\right|_{t=2} \approx 19.25$ kPa/min (answer can vary somewhat)

d) $\left.\frac{dT}{dt}\right|_{t=3} \approx \frac{T(4)-T(2)}{4-2} = \frac{102-71}{2} = 15.5$ kPa/min (answer can vary somewhat)

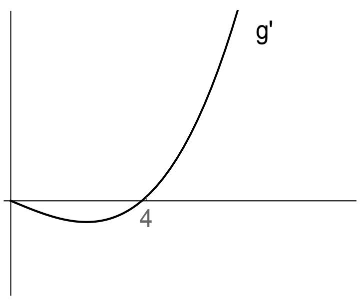
e) $\left.\frac{dT}{dP}\right|_{t=5} \approx \frac{T(5)-T(4)}{P(5)-P(4)} = \frac{112-102}{916-891} = \frac{10}{25} = .4$ and $\left.\frac{dT}{dP}\right|_{t=5} \approx \frac{T(6)-T(5)}{P(6)-P(5)} = \frac{120-112}{935-916} = \frac{8}{19} \approx .421$; averaging these gives $\left.\frac{dT}{dP}\right|_{t=5} \approx .411$ kPa/min (answer can vary somewhat)

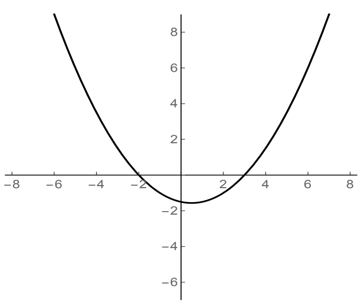
16. $f'(30) \approx 6$; $f'(80) \approx -3$

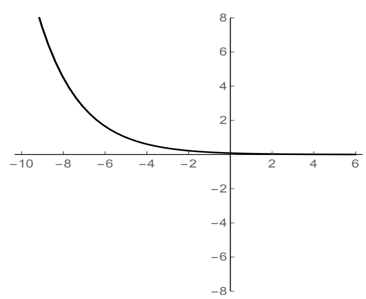
17. a) ≈ 45

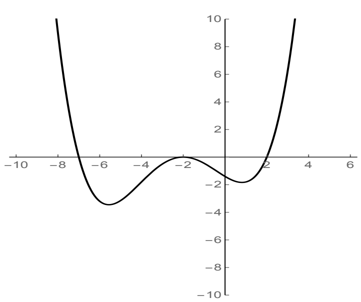
 b) $y = 120 + 45(x-5)$

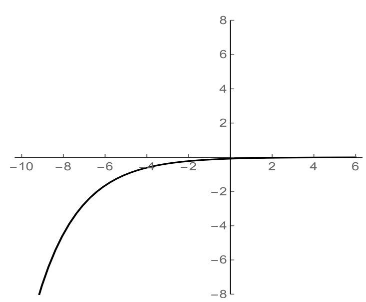
 c) ≈ 120

 d) 

18. 

20. 

19. 

21. 

4.5 Review problems for Exam 1

WARNING!

Most, **but not all** exam questions look like things you've seen before (in homework, examples from class, review sheets, quizzes old exams, etc.)

I reserve the right to occasionally ask original questions, that may be different from anything you've seen before, that apply course concepts.

WARNING!

At any time, you may be asked to compute something which doesn't exist or is equal to $\pm\infty$. You are always responsible for recognizing these situations and answering appropriately.

Mathematica questions

1. For each problem, you are given a problem that a student was trying to solve on *Mathematica*, and what the student typed in. What they typed in is WRONG. Explain why what they typed in is wrong, and write what the command should have been:

   a) The student wants to find the sine of $\pi/6$, but types in `Sin(Pi/6)`

   b) The student wants to find $\log 7$, but types in `Log[7]`

   c) The student wants to solve the equation $x^2 + 3x = 7$, but types in `Solve[x^2 + 3x = 7, x]`

   d) The student wants to define function $f(x) = x^2$, but types in `f[x] = x^2`

   e) The student wants to evaluate $\frac{32+9}{63-17}$, but types in `[32+9]/[63-17]`

   f) The student wants to define function $f(x) = \frac{x-1}{x+1}$, but types in `f[x_] = x-1/x+1`

2. In each problem, you are given some code in *Mathematica* (the code works). Determine what output *Mathematica* will give you.

   a) `f[x_] = x^2 + x;  f[3]`

   b) `Cos[2 Pi/3]`

   c) `g[x_] = 1/x-1; g[x+1]`

   d) `Solve[x+3 ==5, x]`

e) `Factor[x^2 - 4, x]`

3. Suppose you typed in the following command into *Mathematica*:

```
Plot[x^3 Log[x^2 + 1], {x, -3, 5}, PlotRange -> {0,4}]
```

   a) What function is being plotted? (Write the function in hand-written notation, not *Mathematica* syntax.)
   b) What $x-$value will be at the left edge of the graph?
   c) What $y-$value will be at the top of the graph?

Questions related mostly to Chapter 1

4. Write each[expression in the form $\square x^{\square}$, where each of the two squares represent constants:

   a) $\frac{2}{x^7}$ b) $\frac{\sqrt[5]{x^2}}{8}$ c) $2(3x^4)^2x^5$ d) $\frac{3\sqrt{x}}{9x^2}$

5. Let $f(x) = x^2$.

   a) Find $f(3)$.
   b) Find all x such that $f(x) = 4$.
   c) Simplify $3f(2x) - f(x) + 2x$.
   d) Simplify $\frac{f(x+h)-f(x)}{h}$.

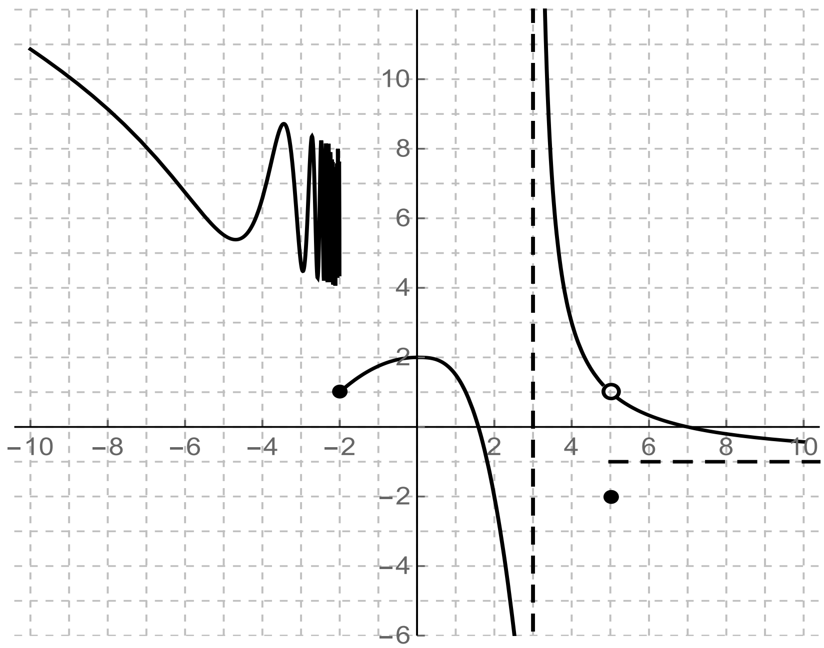
6. Write the equation of the line that has slope -2 and passes through the point $(-2, 3)$.

Questions related mostly to Chapter 2

7. Sketch the graph of some function g with all six of the following properties:

   - $g(-3)$ and $g(4)$ DNE;
   - $\lim_{x\to -3} g(x) = \infty$;
   - $\lim_{x\to 1} g(x) = 2$
   - $g(1) = 0$;
   - $\lim_{x\to 4^+} g(x) = -3$;
   - $\lim_{x\to 4^-} g(x) = 2$.

8. The graph of some unknown function f is given.



Use that graph to estimate the following quantities;

a) $\lim\limits_{x\to -2^-} f(x)$

b) $\lim\limits_{x\to -2^+} f(x)$

c) $\lim\limits_{x\to 0} f(x)$

d) $\lim\limits_{x\to 3^-} f(x)$

e) $f(3)$

f) $\lim\limits_{x\to 3^+} f(x)$

g) $\lim\limits_{x\to 5} f(x)$

h) $f(5)$

i) $\lim\limits_{x\to \infty} f(x)$

j) the equation of any horizontal asymptote of f

k) the equation of any vertical asymptote of f

Questions related mostly to Chapter 3

In Problems 9-19, evaluate the following limits:

9. $\lim\limits_{x\to -3} \frac{2x+6}{4x^2-36}$

10. $\lim\limits_{x\to 3^+} \frac{(x+3)^2}{\sqrt{x-3}}$

11. $\lim\limits_{x\to -1} \frac{2-\sqrt{x+5}}{x^2-1}$

12. $\lim\limits_{x\to 10} \sqrt{\frac{10x}{2x+5}}$

13. $\lim\limits_{x\to \pi} \cos\left(\frac{4}{3}x\right)$

14. $\lim\limits_{x\to 0^+} \csc x$

15. $\lim_{x \to \infty} \frac{x+3}{\sqrt{x^2-1}}$

16. $\lim_{x \to \infty} (3 + 2e^{-4x})$

17. $\lim_{x \to \infty} \frac{2x-5}{8x+3}$

18. $\lim_{x \to 2} \frac{x^3-3x^2+2x}{x^2+5x-14}$

19. $\lim_{x \to 2} h(x)$, where

$$h(x) = \begin{cases} x^2 & x \geq 2 \\ 3x & x < 2 \end{cases}$$

20. Suppose that at time t, some quantity q is given by the function $q(t) = \frac{x(x+2)}{2x^2-3x+5}$. What will the value of q be in the long run?

21. Find all horizontal and vertical asymptotes of the function $f(x) = \frac{x+3}{x^2-16x+15}$.

Questions related mostly to Chapter 4

22. Compute the derivative of $f(x) = \frac{2}{x}$ using the definition of derivative.

23. Write the equation of the line tangent to $g(x) = 4x^2$ when $x = -1$. (You must use the definition of derivative to compute any derivatives used in this problem.)

24. Sketch the graph of a function f which has all four of the following properties:
    - f is continuous at all real numbers other than -3 and 3;
    - f is differentiable at all real numbers other than $-3, -1, 1$ and 3;
    - $f'(2) = 0$;
    - $f'(-4) = 1$.

25. Sketch the graph of a function g which has all six of the following properties:
    - g is continuous at all real numbers other than -4 and 1;
    - $\lim_{x \to \infty} g(x) = 0$;
    - $\lim_{x \to -4^+} g(x) = 2$;
    - $\lim_{x \to -4^-} g(x) = -3$;
    - $g'(5) = 2$;
    - $\lim_{x \to 1} g(x) = \infty$.

26. A nurse measures the heart rate of a patient at various times t, where t is the number of hours that have passed since $12:00$ PM. Her data is recorded in the following chart:

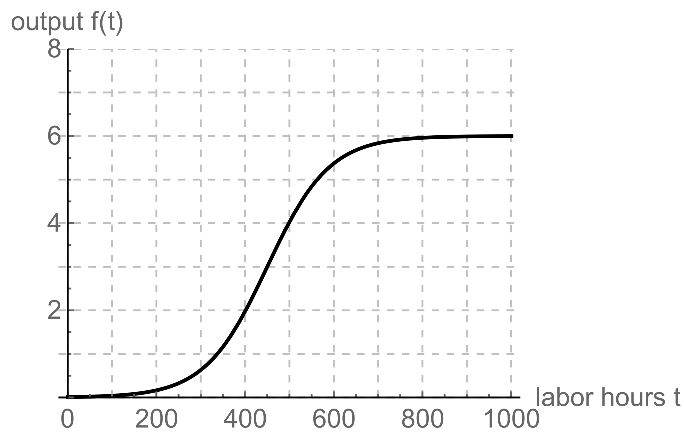
t (hours after $12:00$ PM)	0	$\frac{1}{2}$	1	2	4	6
$r(t)$ (beats per minute)	92	80	74	70	66	64

a) Use the given data to estimate $r'(3)$. Show the computations leading to your answer, and write your answer with correct units.

b) What does your answer to part (a) mean, in the context of this problem?

c) Estimate $r'(\frac{1}{2})$ using the given data, showing the computations leading to your answer and writing your answer with correct units.

27. In business, the rate of change of the output with respect to the number of employee labor hours is called the **productivity** of the labor force. Suppose that the function f, whose graph is given below, represents the output (in thousands of units produced) from t labor hours of work.



Use this graph to estimate the answers to these questions (answer with correct units):

a) What is the output from 350 labor hours of work?

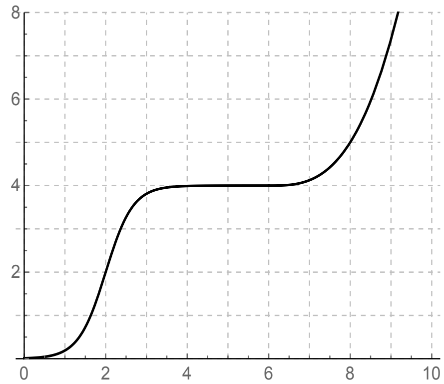
b) What is the productivity at 650 labor hours of work?

c) For what number of labor hours is the productivity greatest?

d) Sketch a graph of the productivity, as a function of the number of labor hours.

28. The graph of some unknown function h is pictured below. Use this graph to

sketch a graph of h'.



Answers to these review problems

1. a) Used parentheses instead of brackets: command should have been `Sin[Pi/6]`
   b) Log computes natural logarithm, not logarithm base 10: command should have been `Log10[7]` or `Log[10,7]`.
   c) Equation inside solve command needs two equal signs, not one: should have been `Solve[x^2 + 3x == 7, x]`
   d) Missing underscore after the x: command should have been `f[x_] = x^2`
   e) Used brackets instead of parentheses: should have been `(32+9)/(63-17)`
   f) Forgot parentheses: should have been `f[x_] = (x-1)/(x+1)`

2. a) 12
   b) $-1/2$
   c) $\frac{1}{x+1} - 1$
   d) 2
   e) $(x-2)(x+2)$ (the order doesn't matter)

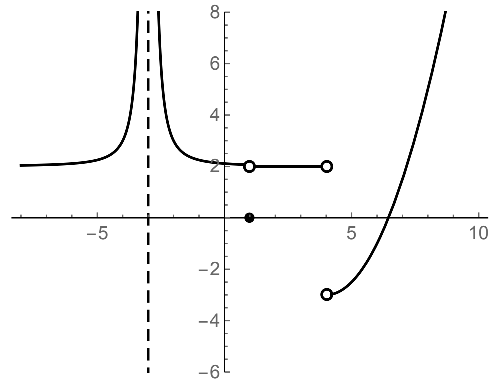
3. a) $x^3 \ln(x^2+1)$
   b) -3
   c) 4

4. a) $2x^{-7}$
   b) $\frac{1}{8}x^{2/5}$
   c) $18x^{13}$
   d) $\frac{1}{3}x^{-3/2}$

5. a) 9
   b) $x = 2, -2$
   c) $11x^2 + 2x$
   d) $2x + h$

6. $y = 3 - 2(x+2)$

7. Answers may vary; one possibility is given below:



8. a) DNE
   b) 1
   c) 2
   d) $-\infty$
   e) DNE
   f) ∞
   g) 1
   h) -2
   i) -1
   j) $y = -1$ (I would accept $y = 6$ also although that was not my intent)
   k) $x = 3$

9. $-1/12$

10. ∞

11. $1/8$

12. 2

13. $-1/2$

14. ∞

15. 1

16. 5

17. $\frac{1}{4}$

18. $\frac{2}{9}$

19. DNE (left- and right- hand limits unequal)

20. $\frac{1}{2}$

21. HA: $y = 0$; VA: $x = 1, x = 15$

22. You must use the limit definition here:

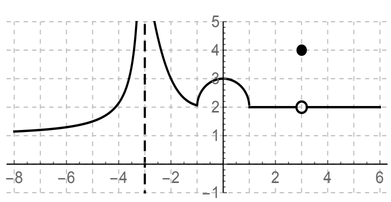
$$\begin{aligned} f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{2}{x+h} - \frac{2}{x}}{h} &= \lim_{h\to 0} \frac{\left(\frac{2}{x+h} - \frac{2}{x}\right)(x+h)x}{h(x+h)x} \\ &= \lim_{h\to 0} \frac{2x - 2(x+h)}{h(x+h)x} \\ &= \lim_{h\to 0} \frac{2x - 2x - 2h}{h(x+h)x} \\ &= \lim_{h\to 0} \frac{-2h}{h(x+h)x} \\ &= \lim_{h\to 0} \frac{-2}{(x+h)x} = \frac{-2}{x^2}. \end{aligned}$$

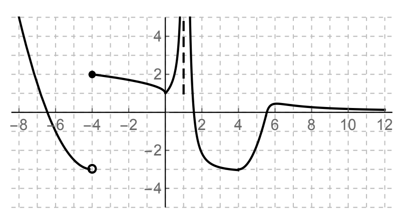
23. The tangent line passes through $(-1, g(-1)) = (-1, 4)$. The slope of this line is $g'(-1)$ which we compute using the limit definition:

$$\begin{aligned} g'(-1) = \lim_{t\to -1} \frac{g(t) - g(-1))}{t - (-1)} = \lim_{h\to 0} \frac{4t^2 - 4}{t+1} &= \lim_{t\to -1} \frac{4(t+1)(t-1)}{t+1} \\ &= \lim_{t\to -1} 4(t-1) = 4(-2) = 8. \end{aligned}$$

The tangent line therefore has equation $y = 4 + 8(x+1)$.

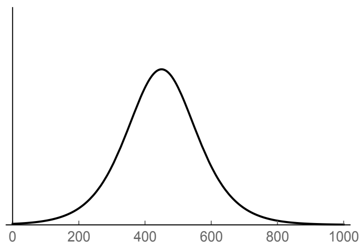
24. Answers may vary; one graph is shown below at left.

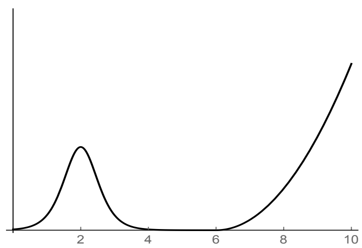




25. Answers may vary; one graph is shown above at right.

26.
  a) $r'(3) \approx \frac{r(4)-r(2)}{4-2} = \frac{66-70}{2} = -2$.
  b) At $3:00$ PM, the patient's heart rate is decreasing at a rate of 2 (beats per minute) per hour.
  c) $r'(\frac{1}{2}) \approx \frac{r(\frac{1}{2})-r(0)}{\frac{1}{2}-0} = \frac{80-92}{\frac{1}{2}-0} = -24$; also, $r'(\frac{1}{2}) \approx \frac{r(1)-r(\frac{1}{2})}{1-\frac{1}{2}} = \frac{74-80}{\frac{1}{2}} = -6$. Averaging these, the best estimate is $r'(\frac{1}{2}) = -15$ (beats per minute) per hour.

27. a) $f(350) \approx 750$ units.
    b) $f'(650) \approx 5$ units per hour.
    c) f' is greatest at around $t = 450$.
    d) 

28. 

Chapter 5

Elementary Differentiation Rules

MOTIVATING EXAMPLE

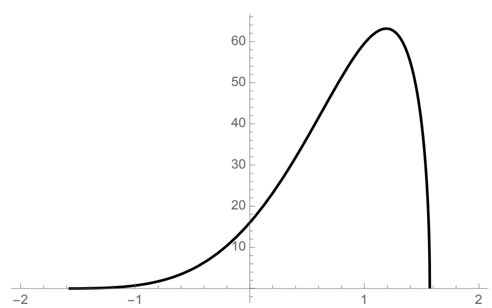
Compute the derivative $f'(0)$, given that

$$f(x) = (x+2)^4\sqrt{\cos x}.$$

Practical approach to the answer: Graph f using the *Mathematica* code

```
Plot[(x+2)^4 Sqrt[Cos[x]], {x, -2, 2}]
```

to obtain this graph of f, then estimate the value of $f'(0)$:



Problem with this practical approach:

Analytic solution: based on what we know so far, the exact answer is

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{(h+2)^4\sqrt{\cos h}-2^4}{h} =$$

Goal: we want to figure out how to compute derivatives without using the limit definition (and without having to resort to estimates coming from graphs and/or tables).

General procedure for computing derivatives

1. Memorize the derivatives of a few basic functions
   (power, exponential, trigonometric, logarithmic, etc.)
2. Learn some rules which tell you how to compute the derivatives of more complicated functions in terms of the derivatives you have memorized.

Over the next two chapters we will develop these rules, which allow us to compute derivatives without having to resort to the limit definition. Eventually we will come to a list of rules which are given on page 162 in Section 6.6.

5.1 Constant function and power rules

EXAMPLE 1

Find the derivative of $f(x) = c$, where c is a constant.

First, what should this be? The graph of $f(x) = c$ is a ____________________,

whose slope is ____. So $f'(x)$ should equal ____.

Justification of this intuition:

Theorem 5.1 (Constant Function Rule) *Let c be a constant. Then $\frac{d}{dx}(c) = 0$.*

EXAMPLE 2

Find the derivative of $f(x) = mx + b$, where m and b are constants.

First, what <u>*should*</u> *this be?* The graph of $f(x) = mx + b$ is a (straight) line, whose slope is ____. So $f'(x)$ should equal ____.

Justification of this intuition:

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{[m(x+h)+b] - [mx+b]}{h} \\
&=
\end{aligned}
$$

Theorem 5.2 (Linear Function Rule) *If $f(x) = mx + b$, then $f'(x) = m$.*

EXAMPLE 3

Find the derivative of $f(x) = x^2$.

Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[(x+h)^2] - [x^2]}{h}$$

$$= \lim_{h \to 0} \frac{[x^2 + 2xh + h^2] - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$=$$

EXAMPLE 4

Find the derivative of $f(x) = x^n$, where n is a nonnegative integer.

Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[(x+h)^n] - [x^n]}{h}$$

$$= \lim_{h \to 0} \frac{[x^n + nx^{n-1}h + ... + h^n] - x^n}{h}$$

(continued on next page)

From the previous page,

$$f'(x) = \lim_{h \to 0} \frac{[x^n + nx^{n-1}h + \ldots + h^n] - x^n}{h}$$

$$= \lim_{h \to 0} \frac{nx^{n-1}h + \ldots + h^n}{h}$$

$$= \lim_{h \to 0} \left(nx^{n-1} + \ldots + h^{n-1} \right)$$

$$= nx^{n-1}.$$

EXAMPLE 5

Find the derivative of $f(x) = \sqrt{x}$.

Solution: Just to show you that you can use either definition of derivative, we'll do this example with the alternate definition:

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

$$= \lim_{t \to x} \frac{\sqrt{t} - \sqrt{x}}{t - x}$$

$$= \lim_{t \to x} \frac{\left(\sqrt{t} - \sqrt{x}\right)}{(t - x)} \cdot \frac{\left(\sqrt{t} + \sqrt{x}\right)}{\left(\sqrt{t} + \sqrt{x}\right)}$$

$$= \lim_{t \to x} \frac{t - x}{(t - x)\left(\sqrt{t} + \sqrt{x}\right)}$$

$$= \lim_{t \to x} \frac{1}{\sqrt{t} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

EXAMPLE 6

Find the derivative of $f(x) = \frac{1}{x}$.

Solution:

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

$$= \lim_{t \to x} \frac{\frac{1}{t} - \frac{1}{x}}{t - x}$$

$$=$$

Examples 1-6 illustrate the following general principle:

Theorem 5.3 (Power Rule) *Let $f(x) = x^n$, where $n \neq 0$. Then $f'(x) = nx^{n-1}$.*

The Power Rule can also be written this way: $\frac{d}{dx}(x^n) = nx^{n-1}$ whenever $n \neq 0$.

Theorem 5.4 (Special cases of the Power Rule)

- $\frac{d}{dx}(x) = 1$
- $\frac{d}{dx}(mx + b) = m$
- $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$
- $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$
- $\frac{d}{dx}(x^2) = 2x$

EXAMPLE 7

An object's position (in meters) at time t (measured in seconds) is given by $y = t^4$. Find the object's velocity at time 3.

OLD SOLUTION:

$$v(3) = f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{(3+h)^4 - 3^4}{h} = \cdots$$

NEW SOLUTION:

5.2 Linearity rules

Question: Do derivatives add/subtract/multiply/divide as expected?

In particular, if f and g are differentiable functions,

- does $(f+g)' = f' + g'$?
- does $(f-g)' = f' - g'$?
- does $(cf)' = c \cdot f'$ when c is a constant?
- does $(fg)' = f' \cdot g'$?
- does $\left(\frac{f}{g}\right)' = \frac{f'}{g'}$?

Theorem 5.5 (Sum Rule) *If f and g are differentiable at x, then $f+g$ is differentiable at x and $(f+g)'(x) = f'(x) + g'(x)$.*

Proof of the Sum Rule: By definition, $(f+g)(x)$ means $f(x)+g(x)$. Now using the definition of the derivative,

$$\begin{aligned}
(f+g)'(x) &= \lim_{h\to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h} \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(x) + g'(x).
\end{aligned}$$

Theorem 5.6 (Difference Rule) *If f and g are differentiable at x, then $f-g$ is differentiable at x and $(f-g)'(x) = f'(x) - g'(x)$.*

Proof of the Difference Rule: similar to the proof of the Sum Rule.

Theorem 5.7 (Constant Multiple Rule) *If f is differentiable at x, then cf is differentiable at x for any constant c and $(cf)'(x) = c \cdot f'(x)$.*

Proof of the Constant Multiple Rule:

$$\begin{aligned}
(cf)'(x) &= \lim_{h\to 0} \frac{(cf)(x+h) - (cf)(x)}{h} \\
&= \lim_{h\to 0} \frac{c\,f(x+h) - c\,f(x)}{h} \\
&= \lim_{h\to 0} \frac{c[f(x+h) - f(x)]}{h} \\
&= c \lim_{h\to 0} \frac{[f(x+h) - f(x)]}{h} \\
&= c\,f'(x).
\end{aligned}$$

Together, the Sum Rule, Difference Rule and Constant Multiple Rule are called the **linearity rules** for differentiation (for reasons that you learn in linear algebra (Math 322)).

EXAMPLE 1

Compute the derivative of $y = 3x^2 + 2\sqrt{x} - 1$.

EXAMPLE 2

Suppose the cost of producing x units of a drug is given by $c(x) = 10x^{15} - 8x + 7$. Find the instantaneous rate of change in the cost when $x = 1$.

EXAMPLE 3

Let $y = 3\sqrt[3]{x} - \frac{2}{3x^3} + (3x-2)^2$. Find $\frac{dy}{dx}$.

WARNING: Products do not behave nicely under differentiation. Here is an example to show why $(fg)' \neq f' \cdot g'$:

Suppose $f(x) = x^2$ and $g(x) = x^3$.

Then $f'(x) = 2x$ and $g'(x) = 3x^2$.
Therefore the product of the derivatives is $f'(x)g'(x) = (2x)(3x^2) = 6x^3$.

BUT $(fg)(x) = f(x)g(x) = x^2x^3 = x^5$.

Therefore the derivative of the product is $(fg)'(x) =$

5.3 Derivatives of sine, cosine and tangent

Question:

$$\frac{d}{dx}(\sin x) = ?$$
$$\frac{d}{dx}(\cos x) = ?$$
$$\frac{d}{dx}(\tan x) = ?$$

To address these questions, we will use the limit definition of derivative for each function. In computing these derivatives, we will need some trigonometric identities, which are listed here for convenience:

$$\sin(x+h) = \sin x \cos h + \cos x \sin h \tag{5.1}$$
$$\cos x = 1 - 2\sin^2\left(\frac{x}{2}\right) \tag{5.2}$$
$$\cos(x+h) = \cos x \cos h - \sin x \sin h \tag{5.3}$$
$$\tan(x+h) = \frac{\tan x + \tan h}{1 - \tan x \tan h} \tag{5.4}$$
$$1 + \tan^2 x = \sec^2 x \tag{5.5}$$

EXAMPLE

Find the derivative of $f(x) = \sin x$.

What should this derivative be? **Hint:** look back at page 78.

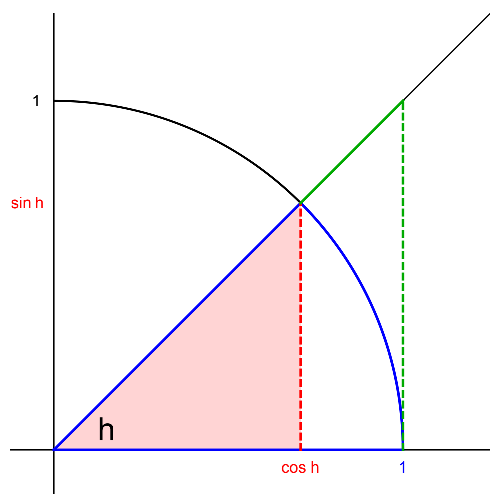
Justification of this intuition:

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x\cos h + \cos x \sin h - \sin x}{h} \quad \text{(by trig identity (5.1) above)} \\
&= \lim_{h\to 0}(\cos x)\frac{\sin h}{h} + \lim_{h\to 0}(\sin x)\frac{\cos h - 1}{h} \\
&= \cos x\left(\lim_{h\to 0}\frac{\sin h}{h}\right) + \sin x\left(\lim_{h\to 0}\frac{\cos h - 1}{h}\right).
\end{aligned}$$

Side question 1:

$$\lim_{h\to 0}\frac{\sin h}{h} = ?$$

Solution to side question 1:



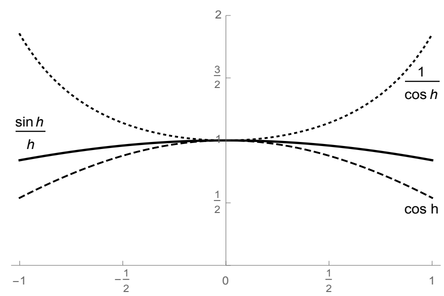
From this picture, it is clear that

area of pink triangle	$\leq$	area of blue pizza wedge	$\leq$	area of green triangle
$\frac{1}{2}$(base)(height)	$\leq$	$\frac{\text{angle}}{2\pi}(\pi \text{ radius}^2)$	$\leq$	$\frac{1}{2}$(base)(height)

From the previous page, we have

$$\begin{array}{ccccc}
\frac{1}{2}(\cos h)(\sin h) & \leq & \frac{h}{2} & \leq & \frac{1}{2}(1)(\tan h) \\[1em]
\frac{1}{2}\cos h \sin h & \leq & \frac{h}{2} & \leq & \frac{1}{2}\frac{\sin h}{\cos h} \\[1em]
\cos h \sin h & \leq & h & \leq & \frac{\sin h}{\cos h} \\[1em]
\cos h & \leq & \frac{h}{\sin h} & \leq & \frac{1}{\cos h} \\[1em]
\frac{1}{\cos h} & \geq & \frac{\sin h}{h} & \geq & \cos h
\end{array}$$

These inequalities "prove" the relationships between the graphs of $\cos h$, $\frac{\sin h}{h}$ and $\frac{1}{\cos h}$ seen below:



We can conclude that since

$$\lim_{h\to 0} \cos h = \cos 0 = 1 \qquad \text{and} \qquad \lim_{h\to 0} \frac{1}{\cos h} = \frac{1}{1} = 1,$$

that

Side question 2:

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = ?$$

Solution to side question 2:

$$\begin{aligned}
\lim_{h\to 0}\frac{\cos h - 1}{h} &= \lim_{h\to 0}\frac{1-2\sin^2\left(\frac{h}{2}\right)-1}{h} && \text{(by trig identity (5.2) on page 118)}\\
&= \lim_{h\to 0}\frac{-2\sin^2\left(\frac{h}{2}\right)}{h}\\
&= \lim_{u\to 0}\frac{-2\sin^2 u}{2u} && \text{(by setting } u = \tfrac{h}{2} \text{ so that } h = 2u\text{)}\\
&= \lim_{u\to 0} -1\cdot\sin u\cdot\frac{\sin u}{u}\\
&=
\end{aligned}$$

Returning to the computation of the derivative of $\sin x$:

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \cos x\left(\lim_{h\to 0}\frac{\sin h}{h}\right) + \sin x\left(\lim_{h\to 0}\frac{\cos h - 1}{h}\right) && \text{(from page 119)}\\
&= \cos x\cdot 1 + \sin x\cdot 0\\
&= \cos x.
\end{aligned}$$

EXAMPLE

Find the derivative of $f(x) = \cos x$.

What should this derivative be? **Hint:** look back at page 78.

Justification of this intuition:

$$\begin{aligned}
\frac{d}{dx}(\cos x) &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}\\
&= \lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h} && \text{(by trig identity (5.3) on page 118)}\\
&= \lim_{h\to 0}(\cos x)\frac{\cos h - 1}{h} - \lim_{h\to 0}(\sin x)\frac{\sin h}{h}\\
&= \cos x\cdot 0 - \sin x\cdot 1 && \text{(by the two side questions)}\\
&= -\sin x.
\end{aligned}$$

EXAMPLE

Find the derivative of $f(x) = \tan x$.

Solution:

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \lim_{h\to 0} \frac{\tan(x+h) - \tan x}{h} \\
&= \lim_{h\to 0} \frac{\frac{\tan x + \tan h}{1 - \tan x \tan h} - \tan x}{h} && \text{(by trig identity (5.4) on page 118)} \\
&= \lim_{h\to 0} \frac{\tan x + \tan h - \tan x(1 - \tan x \tan h)}{h(1 - \tan x \tan h)} \\
&= \lim_{h\to 0} \frac{\tan x + \tan h - \tan x + \tan^2 x \tan h}{h(1 - \tan x \tan h)} \\
&= \lim_{h\to 0} \frac{\tan h + \tan h \tan^2 x}{h(1 - \tan x \tan h)} \\
&= \lim_{h\to 0} \frac{\tan h(1 + \tan^2 x)}{h(1 - \tan x \tan h)} \\
&= \lim_{h\to 0} \frac{\tan h \sec^2 x}{h(1 - \tan x \tan h)} && \text{(by trig identity (5.5) on page 118)} \\
&= \sec^2 x \cdot \lim_{h\to 0} \frac{\sin h}{\cos h \cdot h \cdot (1 - \tan x \tan h)} \\
&= \sec^2 x \cdot \lim_{h\to 0} \left(\frac{1}{\cos h}\right)\left(\frac{\sin h}{h}\right) \frac{1}{1 - \tan x \tan h} \\
&= \sec^2 x \cdot \frac{1}{1} \cdot (1) \cdot \frac{1}{1 - \tan x \cdot 0} && \text{(by the first side question)} \\
&= \sec^2 x.
\end{aligned}$$

Theorem 5.8 (Derivatives of Sine, Cosine and Tangent)

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\tan x) = \sec^2 x$$

The derivatives of $\cot x$, $\sec x$ and $\csc x$, as well as the derivatives of $\arctan x$ and $\arcsin x$ will be derived in the next chapter.

EXAMPLE 1

Find $f'(x)$ if $f(x) = 2\sin x - 3\cos x + 4$.

EXAMPLE 2

Find $\frac{dy}{dx}$ if $y = \sqrt{x^3} + 5x - 2\tan x$.

EXAMPLE 3

Let $y = 4\cos x - 2$. Find the slope of the line tangent to this curve when $x = \frac{\pi}{2}$.

5.4 Derivatives of exponential and logarithmic functions

Question:

$$\frac{d}{dx}(e^x) = ?$$
$$\frac{d}{dx}(\ln x) = ?$$

EXAMPLE

Find the derivative of $f(x) = e^x$.

Solution: Go back to the limit definition of derivative:

$$\begin{aligned}\frac{d}{dx}(e^x) &= \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}\\ &= \lim_{h\to 0}\frac{e^x e^h - e^x}{h}\\ &= \lim_{h\to 0}\frac{e^x(e^h-1)}{h}\\ &= e^x\cdot\left[\lim_{h\to 0}\frac{e^h-1}{h}\right]\end{aligned}$$

Side question 3:

$$\lim_{h\to 0}\frac{e^h-1}{h} = ?$$

Solution to side question 3: We will estimate this limit by means of charts, as in Chapter 2:

h	.1	.05	.01	.001	.0000001
$\frac{e^h-1}{h}$	1.052	1.025	1.005	1.0005	1.

h	-.1	- .05	-.01	-.001	-.0000001
$\frac{e^h-1}{h}$	.9516	.9754	.995	.9995	1.

Based on these charts, it seems reasonable to conclude that

$$\lim_{h\to 0}\frac{e^h-1}{h} =$$

and therefore, returning to the example,

$$\frac{d}{dx}\left(e^x\right) = e^x \left[\lim_{h\to 0} \frac{e^h - 1}{h}\right] = e^x \cdot 1 = e^x.$$

Theorem 5.9 (Derivative of the natural exponential function)) $\frac{d}{dx}\left(e^x\right) = e^x$.

In Section 6.5, we will derive the following rule for the natural logarithm function:

Theorem 5.10 (Derivative of the natural logarithm function) $\frac{d}{dx}\left(\ln x\right) = \frac{1}{x}$.

EXAMPLE 1

Find the slope of the line tangent to the function $f(x) = 3\ln x + \sqrt{x}$ at $x = 4$.

EXAMPLE 2

Find the derivative of $y = 2e^x - 4\sin x + \cos x - 2x^6 - 1$.

EXAMPLE 3

Find the derivative of $\lambda(z) = \log_{10} z$.

5.5 Higher-order derivatives

We will see that many problems can be studied not just by differentiating a function once, but by repeatedly differentiating it many times. First, we establish notation to describe this procedure:

Definition 5.11 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function.*

- *The* **zeroth derivative** *of f, sometimes denoted $f^{(0)}$, is just the function f itself.*
- *The* **first derivative** *of f, sometimes denoted $f^{(1)}$ or $\frac{dy}{dx}$, is just f'.*
- *The* **second derivative** *of f, denoted f'' or $f^{(2)}$ or $\frac{d^2y}{dx^2}$, is the derivative of f'; in other words, $f'' = (f')'$. The* **third derivative** *of f, denoted f''' or $f^{(3)}$ or $\frac{d^3y}{dx^3}$, is the derivative of f''; in other words, $f''' = ((f')')'$.*
- *More generally, the n^{th}* **derivative** *of f, denoted $f^{(n)}$ or $\frac{d^ny}{dx^n}$,is the derivative of $f^{(n-1)}$; in other words $f^{(n)} = ((((f')') \cdots ')'$.*

EXAMPLE 1

Let $f(x) = 2x^6$. Find $f'''(x)$.

EXAMPLE 2

If $y = \cos x + \sin x$, find $\left.\frac{d^2y}{dx^2}\right|_{x=\pi/4}$.

Physical interpretation of the second derivative

Suppose an object's position on a number line after t seconds of elapsed time is given by $f(t)$. Then

$$f'(t) = \text{ rate of change of position } = \text{ velocity}$$

$$f''(t) = (f')'(t) = \text{ rate of change of velocity } =$$

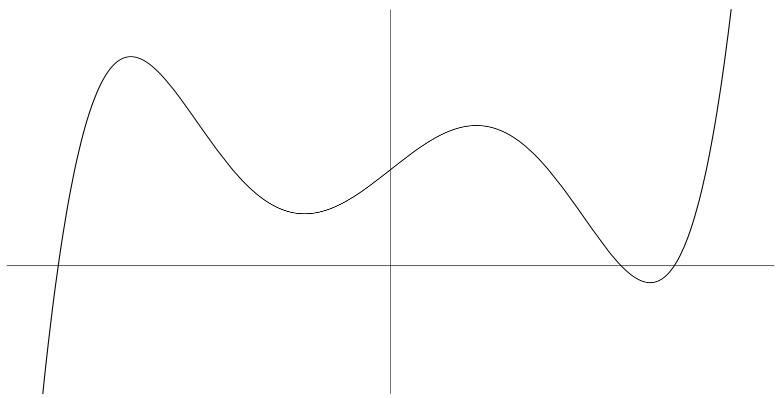
EXAMPLE 3

A bee is flying back and forth along a number line, so that its position after t units of time is $f(t) = \frac{-1}{3}t^3 + 3t^2$. What is the velocity of the object at the instant where its acceleration is zero?

Graphical interpretation of the second derivative

Let f be a twice- differentiable function. Then

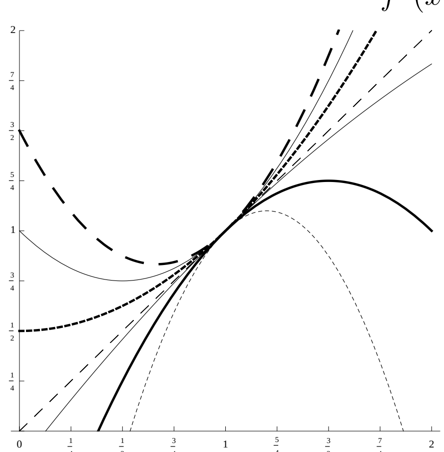
$$f'(x) = \text{ slope of graph of } f \text{ at } x$$
$$f''(x) = (f')'(x) = \text{ rate of change of slope at } x$$



EXAMPLE 4

Let k be a constant and define $f(x) = \frac{1}{2}kx^2 + (1-k)x + \frac{1}{2}k$. Examine the behavior of $f(x)$ at $x = 1$ for various k:

$$f(1) = \frac{1}{2}k + 1 - k + \frac{1}{2}k = 1$$
$$f'(x) = kx + 1 - k \Rightarrow f'(1) = k + 1 - k = 1$$
$$f''(x) = k \Rightarrow f''(1) = k$$



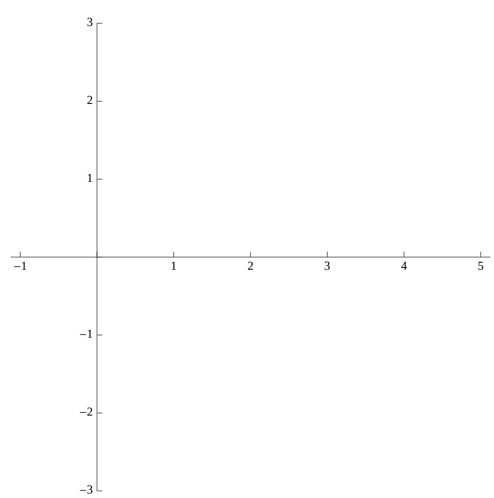
Compiling information from the first and second derivative, we can determine the general shape of a graph near a value x as follows:

	$f'(x) > 0$	$f'(x) < 0$	$f'(x) = 0$
$f''(x) > 0$			
$f''(x) < 0$			
$f''(x) = 0$			

Before the days of *Mathematica* and graphics calculators, this is how people learned to sketch the graphs of functions.

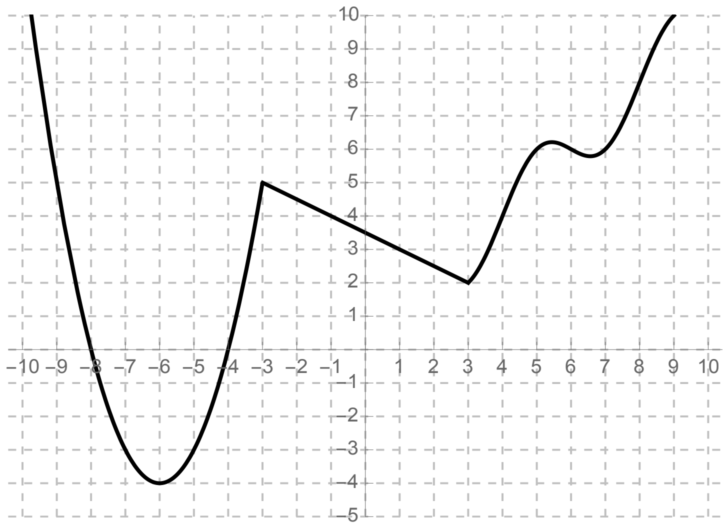
EXAMPLE 5

Suppose f is some unknown function such that $f(3) = -2$, $f'(3) = 1$ and $f''(3) = 2$. Sketch a picture of what the graph of f looks like near $x = 3$:



EXAMPLE 6

Suppose f is some function whose graph is given below:



1. Estimate $f(-6)$.

2. Estimate $f'(-6)$.

3. Estimate $f'(1)$.

4. Estimate $f''(1)$.

5. Estimate $f''(-3)$.

6. Estimate a value of x for which $f'(x) = 0$ but $f''(x) < 0$.

7. Estimate a value of x for which $f'(x) < 0$ but $f''(x) > 0$.

8. Is $f''(9)$ positive, negative, or zero? Explain.

9. Is $f''(-7)$ positive, negative, or zero? Explain.

EXAMPLE 7

Suppose that you look at your Fitbit periodically to measure the number of steps you have walked and record what you see in the following table:

time t (minutes after noon)	0	2	5	7	11	15
steps taken $f(t)$	0	35	115	147	163	191

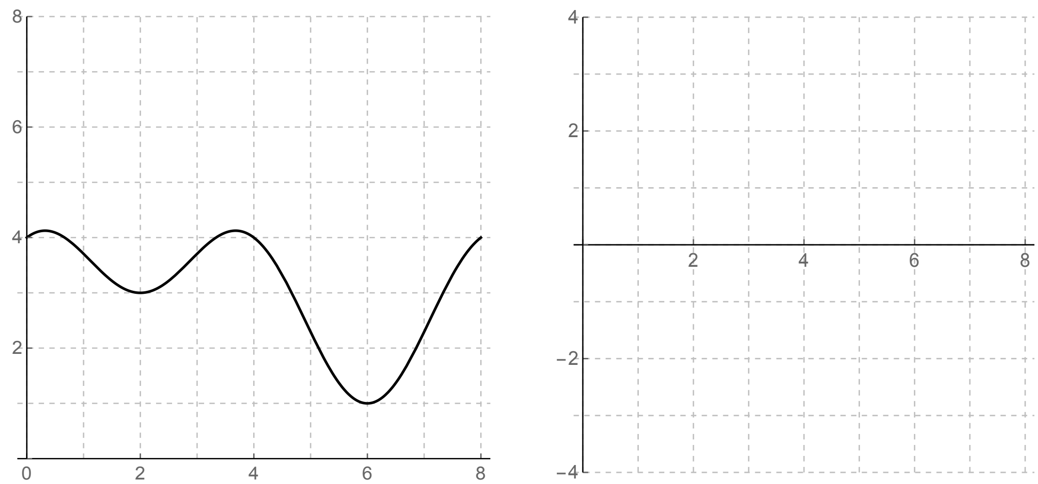
Use the table above to estimate the answers to these questions. Show your work; use correct mathematical language and use appropriate units.

1. How fast are you walking at 12:06 PM?

2. What is your acceleration at 12:07 PM? Use appropriate units.

EXAMPLE 8

The graph of some unknown function f is given below at left. Sketch the graph of f'' on the right-hand axes:



5.6 Homework exercises

In these problems (and in all future problems), you may (and should) use differentiation rules to compute any necessary derivatives (i.e. you do not have to use the limit definition).

1. Find $\frac{dy}{dx}$ if $y = 3$.
2. Find $f'(x)$ if $f(x) = x^6 + 2\sqrt{x}$.
3. Find $\frac{d}{dx}\left(3x - \frac{4}{5x} + 1\right)$.
4. A business estimates that if it employs x thousands of people, then its profit, in millions of dollars, is given by the function $f(x) = 2x^3 + 2 - x^{-1}$. Find the rate of change of the business' profit relative to the change in x, when $x = 2$.
5. Find the derivative of $h(x) = (x-2)(x^2+4)$.
6. Find the slope of the line tangent to $y = 2x^{5/2} - x^{3/2}$ when $x = 4$.
7. Find the equation of the line tangent to $f(x) = 3 - x^4 + \frac{1}{x}$ when $x = 1$.
8. Suppose an object is traveling along a number line so that its position at time t is $f(t) = 2t - 4t^{-2}$. Find the object's velocity when $t = 2$.
9. Suppose an object's position at time t is given by $f(t) = 4t^2 - 5t + 2$. Find all times t where the velocity of the object is -1.
10. Find $f'(x)$ if $f(x) = 3 + 4x - \sqrt[3]{x}$.
11. Find $\frac{d}{dx}\sqrt{2x}$.
12. Find the derivative of $f(x) = \frac{x^2-1}{\sqrt{x}}$.
13. Find $f'(x)$ if $f(x) = \frac{2}{3}\sin x + \frac{3}{4}\cos x - x^2$.
14. Find the derivative of $y = 2 - x - 4\tan x$.
15. Let $f(x) = \cos x - 3$. Find $\left.\frac{df}{dx}\right|_{x=\pi/4}$.
16. Find the slope of the line tangent to $y = 3\tan x - \cos x$ when $x = \frac{\pi}{6}$.
17. Find the instantaneous velocity of an object at time t, if the object's position at time t is $f(t) = 3t + \sin t$.
18. Let $g(x) = 4e^x - 5x + \sin x$. Find $g'(x)$.
19. Find the derivative of $f(x) = 4 - \frac{3}{x} + 2\ln x$.

20. Find the derivative of $y = 3e^x$ when $x = 0$.

21. Find the derivative of $f(x) = \ln x + 4\sqrt[3]{x}$.

22. Find the second derivative of $f(x) = x^3 - \frac{1}{x} + 4\sin x$.

23. Find $\sigma''(x)$ if $\sigma(x) = \frac{2}{3}x^6 - \frac{2}{x} + 4$.
    Note: σ is the Greek letter sigma.

24. Let $y = 2\sin\theta$. Find $\frac{d^2y}{d\theta^2}$.

25. Find $\left.\frac{d^2f}{dx^2}\right|_{x=1}$ if $f(x) = \left(\frac{2}{x} + \sqrt{x}\right)$.

26. If $f(x) = 4e^x - 5x^4 + 3x$, find $f''(x)$.

27. Find the third derivative of $f(x) = \ln x$ when $x = 2$.

28. Find the 33rd derivative of $f(x) = e^x$.

29. Let $f(x) = \sin x$. Find $f^{(801)}(x)$.

30. Find the acceleration of an object at time 3, if the object's position at time t is $f(t) = 2t^3 - t^2 + 4t$.

31. Find the acceleration of an object at time $\frac{2\pi}{3}$, if the object's velocity at time t is $v(t) = 3\sin t + 2$.

32. An object moves in such a fashion that its position after t units of time is $f(t) = e^t - 2t$. As time passes, is the object speeding up or slowing down?

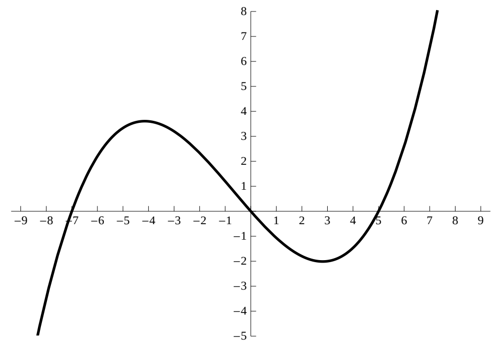
33. An object moves in such a fashion that its position at time t is $f(t) = t^3 - 9t^2$. Find all times t where the acceleration of the object is zero.

34. Suppose f is some unknown function such that $f(4) = 0$, $f'(4) = -1$ and $f''(4) = 5$. Sketch a picture of what the graph of f looks like near $x = 4$.

35. Suppose g is some unknown function such that $g(-1) = 3$, $g'(-1) = 0$ and $g''(-1) = \frac{-2}{5}$. Sketch a picture of what the graph of g looks like near $x = -1$.

36. Suppose f is some unknown function such that $f(4) = 1$, $f'(4) = \frac{1}{7}$ and $f''(4) = \frac{-2}{3}$. Sketch a picture of what the graph of f looks like near $x = 4$.

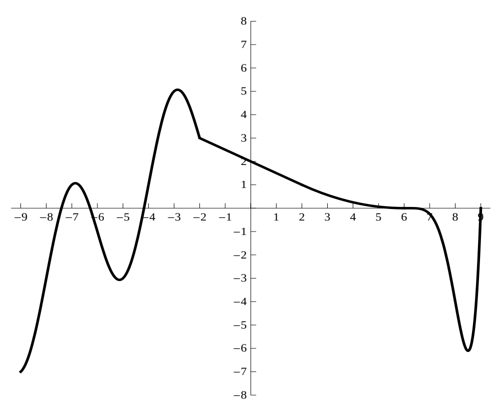
37. Pictured below is the graph of some unknown function f.



Use the graph to determine, with justification, whether each of the following quantities are positive, negative, or zero:

a) $f(5)$	d) $f(-6)$	g) $f(-1)$	j) $f(3)$
b) $f'(5)$	e) $f'(-6)$	h) $f'(-1)$	k) $f'(3)$
c) $f''(5)$	f) $f''(-6)$	i) $f''(-1)$	l) $f''(3)$

38. Pictured below is the graph of some unknown function g.



Use the graph to answer the following questions:

a) Estimate $g''(1)$.
b) Estimate $g''(5)$.
c) Estimate $g''(-7)$.
d) Find a value of x such that $g'(x) = 0$ but $g''(x) > 0$.
e) Find a value of x such that $g'(x) = 0$ but $g''(x) < 0$.
f) Find a value of x for which $g''(x)$ DNE.

39. The position of a bug which is crawling back and forth along the x-axis at various times t are given in the following chart:

time t (seconds)	0	1	4	8	12
position $x(t)$ (inches)	14	7	-5	40	220

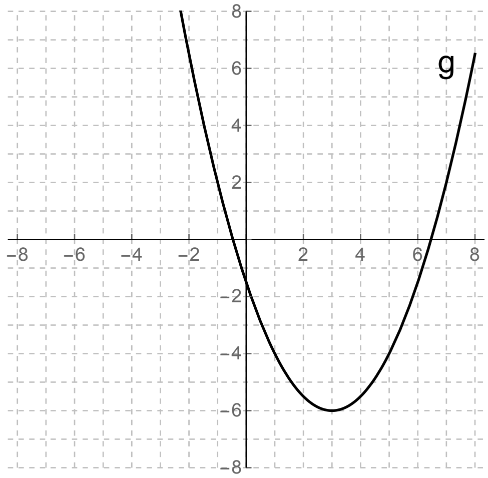
   a) Use the information in the chart to estimate $x'(3)$. Show the computations that lead to your answer, and write your answer with appropriate units.
   b) In the context of this problem, what does your answer to part (a) mean?
   c) In the context of this problem, what is the significance of the sign of your answer to part (a)?
   d) Use the information in the chart to estimate $x''(6)$. Show the computations that lead to your answer, and write your answer with appropriate units.
   e) In the context of this problem, what does your answer to part (d) mean?
   f) In the context of this problem, what is the significance of the sign of your answer to part (d)?

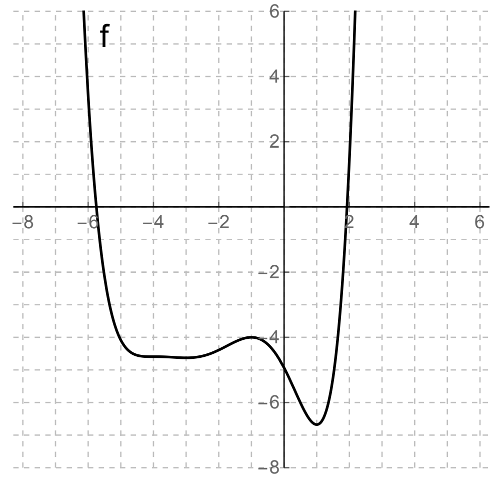
40. During a snowstorm, you periodically measure the depth of snow that has fallen outside your house. Your observations are recorded in the following table:

time t (hours)	0	1	3	4	5	7	8
depth of snow $f(t)$ (inches)	0	6	15	18	20	21	24

   a) Use the information in the chart to estimate $f'(4)$. Show the computations that lead to your answer, and write your answer with appropriate units.
   b) In the context of this problem, what does your answer to part (a) mean?
   c) Use the information in the chart to estimate $f''(6)$. Show the computations that lead to your answer, and write your answer with appropriate units.
   d) In the context of this problem, what does your answer to part (c) mean?

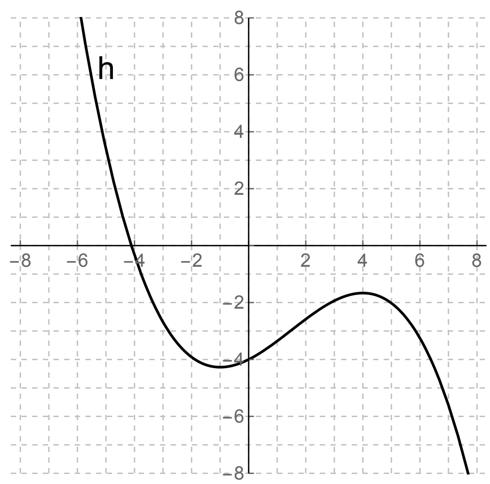
41. The graph of some unknown function g is shown below at left. Use this graph to sketch graphs of the functions g' and g''.





42. The graph of some unknown function f is shown above at right. Use this graph to sketch graphs of the functions f' and f''.

43. The graph of some unknown function h is shown below. Use this graph to sketch graphs of the functions h', h'' and h'''.



44. Sketch the graph of any differentiable function f which has all of the following properties:

- $f'(3) > 0$;
- $f''(3) < 0$;
- $f'(-1) > 0$;
- $f''(-1) > 0$.

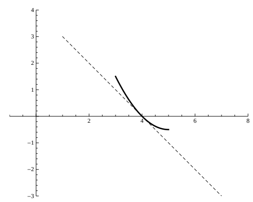
45. Sketch the graph of any differentiable function g which has all of the following properties:
    - $g'(5) < 0$;
    - $g''(5) < 0$;
    - $g'(0) = 0$;
    - $g''(0) < 0$.

46. Sketch the graph of any differentiable function h which has all of the following properties:
    - $h'(2) = 0$;
    - $h'(2) > 0$;
    - $h'(-4) > 0$;
    - $h''(-4) = 0$.

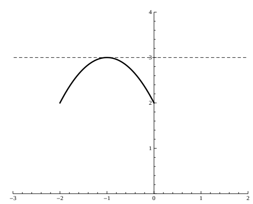
Answers

1. 0
2. $6x^5 + \frac{1}{\sqrt{x}}$
3. $3 + \frac{4}{5}x^{-2}$
4. $\frac{97}{4}$
5. $3x^2 - 4x + 4$
6. 37
7. $y = 3 - 5(x - 1)$
8. 3
9. $t = \frac{1}{2}$
10. $4 - \frac{1}{3}x^{-2/3}$
11. $\frac{\sqrt{2}}{2\sqrt{x}}$
12. $\frac{3}{2}x^{1/2} - x^{-1/2}$
13. $\frac{2}{3}\cos x - \frac{3}{4}\sin x - 2x$
14. $-1 - 4\sec^2 x$
15. $\frac{-\sqrt{2}}{2}$
16. $\frac{9}{2}$
17. $3 + \cos t$
18. $4e^x - 5 + \cos x$
19. $3x^{-2} + \frac{2}{x}$
20. 3
21. $\frac{1}{x} + \frac{4}{3}x^{-2/3}$
22. $6x - 2x^{-3} - 4\sin x$
23. $20x^4 - 4x^{-3}$
24. $-2\sin\theta$
25. $\frac{15}{4}$
26. $4e^x - 60x^2$
27. $\frac{1}{4}$
28. e^x.
29. $f(x) = \cos x$
30. 34
31. $\frac{-3}{2}$
32. Speeding up (since acceleration is positive)
33. $t = 3$

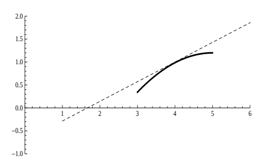
34. Passes through $(4,0)$, slope of tangent line is -1 and lies above the tangent line at 4:



35. Passes through $(-1,3)$, curved downward such that the "peak" of the graph is at $(-1,3)$:



36. Passes through $(4,1)$, slope of tangent line is $\frac{1}{7}$, and graph lies below the tangent line:



37. a) $f(5) = 0$ (graph at $x-$axis at $x = 5$)
    b) $f'(5) > 0$ (graph going up from left to right)
    c) $f''(5) > 0$ (graph lies above tangent line)
    d) $f(-6) > 0$ (graph above $x-$axis at $x = -6$)
    e) $f'(-6) > 0$ (graph going up from left to right)
    f) $f''(-6) < 0$ (graph lies below tangent line)
    g) $f(-1) > 0$ (graph above $x-$axis at $x = -1$)
    h) $f'(-1) < 0$ (graph going down from left to right)
    i) $f''(-1) = 0$ (graph is straight at $x = -1$)
    j) $f(3) < 0$ (graph below $x-$axis at $x = 3$)
    k) $f'(3) = 0$ (tangent line horizontal)
    l) $f''(3) > 0$ (graph lies above tangent line)

38. a) 0

b) $\approx \frac{1}{4}$ (a small positive number)

c) ≈ -5 (a negative number)

d) $x \approx -5.25$, $x \approx 8.1$

e) $x \approx -6.8$, $x \approx -2.8$

f) $x = -2$

39. a) $x'(3) \approx \frac{x(4)-x(1)}{4-1} = \frac{-5-7}{4-1} = -4$ in/sec.

b) The bug's velocity at time 3 is -4 in/sec.

c) Since the velocity is negative, the bug is moving from right to left at time 3.

d) $x'(6) \approx \frac{x(8)-x(4)}{8-4} = \frac{40-(-5)}{8-4} = 11.25$ in/sec;
$x'(10) \approx \frac{x(12)-x(8)}{12-8} = \frac{220-40}{12-8} = 45$ in/sec;
$x''(6) \approx \frac{x'(10)-x'(6)}{10-6} = \frac{45-11.25}{10-6} \approx 8$ in/sec^2.

e) The bug's acceleration at time 6 is 8 in/sec^2.

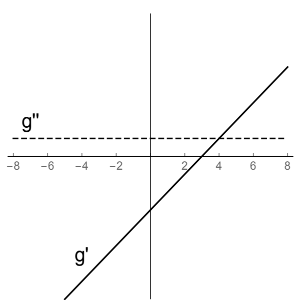
f) Since the acceleration is positive, the bug is speeding up at time 6.

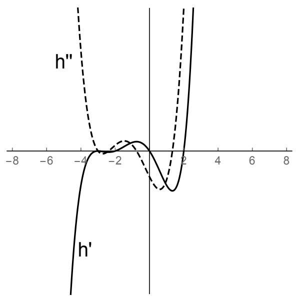
40. a) $f'(4) \approx \frac{f(5)-f(4)}{5-4} = 2$ and $f'(4) \approx \frac{x(4)-x(3)}{4-3} = 3$; averaging these we estimate $f'(4) \approx 2.5$ in/hr.

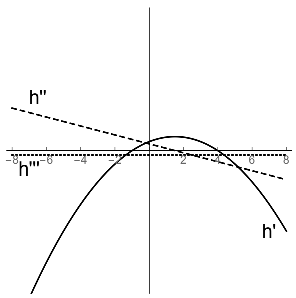
b) At time 4, the snow is falling at a rate of 2.5 inches per hour.

c) $f'(5) \approx \frac{f(5)-f(4)}{5-4} = 2$ and $f'(7) \approx \frac{f(7)-f(5)}{7-5} = \frac{1}{2}$. Then, $f''(6) \approx \frac{f'(7)-f'(5)}{7-5} = \frac{\frac{1}{2}-2}{2} = \frac{-3}{4}$ in/hr^2.

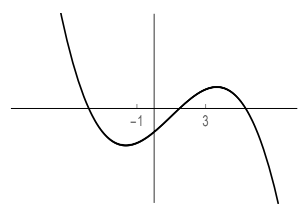
d) At time 6, since $f''(6) < 0$, the rate at which the snow is falling is decreasing (i.e. the snowstorm is "letting up").

41. 

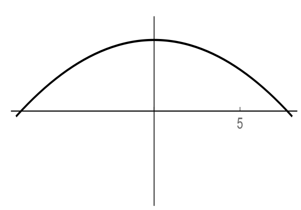
42. 

43. 

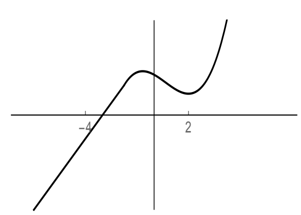
44. Answers may vary; one possible answer is



45. Answers may vary; one possible answer is



46. Answers may vary; one possible answer is



Chapter 6

Intermediate Differentiation Rules

6.1 Product rule

Question: What is $\frac{d}{dx}(fg)$ (a.k.a. $(fg)'$) in terms of f, g, f' and g'?

*First, what is $(fg)'$ **not** equal to?*

Some intiution involving units: Suppose x is time (measured in sec) and $f(x)$ and $g(x)$ are both distances (measured in meters). Then

$f'(x)$ is ________________, which is measured in ____________.

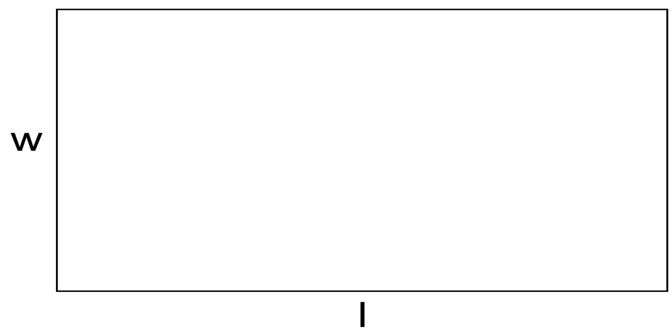
$g'(x)$ is ________________, which is measured in ____________.

So $f'(x)g'(x)$ would be measured in ____________.

But $(fg)(x) = f(x)g(x)$ is ____________, which is measured in ____________,

which means $(fg)'(x)$ would be measured in ____________.

More intuition: Suppose you have a rectangle whose length is l and whose width is w. This makes the area lw. Suppose you increase l and w by a small amount. How much does the area change?



Justification of this intuition:

$$\begin{aligned}
(fg)'(x) &= \lim_{h\to 0} \frac{(fg)(x+h) - (fg)(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x+h) + f(x)\,g(x+h) - f(x)\,g(x)}{h} \\
&= \lim_{h\to 0} \left[\frac{f(x+h)\,g(x+h) - f(x)\,g(x+h)}{h} + \frac{f(x)\,g(x+h) - f(x)\,g(x)}{h}\right] \\
&= \lim_{h\to 0} \left[g(x+h)\frac{f(x+h) - f(x)}{h}\right] + \lim_{h\to 0} \left[f(x)\frac{g(x+h) - g(x)}{h}\right] \\
&= g(x)\,f'(x) + f(x)\,g'(x).
\end{aligned}$$

This work proves the following theorem:

Theorem 6.1 (Product Rule) *Let f and g be differentiable at x. Then fg is differentiable at x and*

$$(fg)'(x) = f'(x)\,g(x) + g'(x)\,f(x).$$

The Product Rule says, in English, the following:

the derivative of a product is "*the derivative of the first times the second plus the derivative of the second times the first*".

EXAMPLE 1

Find y' if $y = 3x^2 \sin x$.

EXAMPLE 2

Find the slope of the line tangent to $f(x) = (2x^3 + 4x - 1)\tan x$, at $x = 0$.

EXAMPLE 3

Find $\frac{d^2y}{dx^2}$ if $y = x^4 e^x$.

Solution: First, by the Product Rule,

$$\frac{dy}{dx} = 4x^3 e^x + e^x x^4.$$

EXAMPLE 4

Find $f'(x)$ if $f(x) = \cos^2 x$.

6.2 Quotient rule

Theorem 6.2 (Quotient Rule) *Let f and g be differentiable at x, where $g(x) \neq 0$. Then $\frac{f}{g}$ is differentiable at x and*

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)\,g(x) - g'(x)\,f(x)}{[g(x)]^2}.$$

The proof of this is similar to the proof of the Product Rule and is omitted.

The Quotient Rule says, in English, the following:

the derivative of a quotient is "*the derivative of the top times the bottom minus the derivative of the bottom times the top, all over the bottom squared*".

EXAMPLE 1

Find $\theta'(x)$ if $\theta(x) = \frac{2\sqrt{x}-3x+1}{5\ln x}$.

EXAMPLE 2

Let $f(x) = \frac{x^2+1}{x^2-1}$. Find the slope of the line tangent to f when $x = 0$.

EXAMPLE 3

Suppose that at time t (measured in seconds), the energy in a nuclear reaction is $\frac{3e^t}{t}$ Joules. Find the rate of change of the energy with respect to time.

EXAMPLE 4

Find $f'(x)$ if

$$f(x) = \frac{3\tan x + 6x^2 - 5x + 2}{-4\cos x - 3x^{-2/3} + 2}.$$

6.3 Derivatives of secant, cosecant and cotangent

The quotient rule can be used to compute the derivatives of $\sec x$, $\csc x$ and $\cot x$. You can either memorize the answers that are derived below, or remember how to "re-compute" them using the quotient rule, as necessary.

EXAMPLE 1

Find the derivative of $f(x) = \sec x$.

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right)$$

$$=$$

EXAMPLE 2

Find the derivative of $f(x) = \csc x$.

Solution:

$$\begin{aligned}\frac{d}{dx}(\csc x) &= \frac{d}{dx}\left(\frac{1}{\sin x}\right)\\ &= \frac{(1)' \cdot \sin x - (\sin x)' \cdot 1}{(\sin x)^2}\\ &= \frac{0 \cdot \sin x - \cos x \cdot 1}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} = -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x.\end{aligned}$$

EXAMPLE 3

Find the derivative of $f(x) = \cot x$.

Solution:

$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$$

$$=$$

Theorem 6.3 (Derivatives of secant, cosecant and cotangent)

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

EXAMPLE 4

Find the instantaneous rate of change of the function $f(x) = 2x \sec x + 1$ when $x = 0$.

EXAMPLE 5

Let $y = \frac{\sec x + 3\cot x}{x - \sin x}$. Find $\frac{dy}{dx}$.

EXAMPLE 6

Suppose an object's position, measured in feet, at time t, measured in seconds, is given by $f(t) = e^t \sec t$. Find the object's velocity and acceleration at time 0.

EXAMPLE 7

Find $\frac{d}{dt}\left(2\sqrt{t}\csc t\right)$.

EXAMPLE 8

Find $g'\left(\frac{\pi}{3}\right)$ if $g(t) = t^2 \sin t$.

Solution: First, by the Product Rule, $g'(t) = 2t\sin t + (\cos t)t^2$.

$$\begin{aligned}\Rightarrow g'\left(\frac{\pi}{3}\right) &= 2\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right)\cdot\left(\frac{\pi}{3}\right)^2 \\ &= 2\left(\frac{\pi}{3}\right)\frac{\sqrt{3}}{2} + \frac{1}{2}\left(\frac{\pi^2}{9}\right) \\ &= \frac{\pi\sqrt{3}}{3} + \frac{\pi^2}{18}.\end{aligned}$$

6.4 Chain rule

Question: Given differentiable functions f and g, find the derivative of $f \circ g$ in terms of f, f', g and g'.

Motivating example:

Suppose Mrs. Young is moving 5 times as fast as Mrs. Underwood.

Suppose also that Mrs. Underwood is moving 3 times as fast as Mrs. Xavier.

What is the relationship between Mrs. Young's speed and Mrs. Xavier's speed?

Answer:

In the language of derivatives, the motivating example becomes the following question:

$$\text{“If } \frac{dy}{du} = 5 \text{ and } \frac{du}{dx} = 3\text{, what is } \frac{dy}{dx}\text{?”}$$

The answer is found as follows:

The general idea described here is what is called the Chain Rule:

Theorem 6.4 (Chain Rule, Leibniz notation)

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Example 1

Find $\frac{dy}{dx}$ if $y = \sqrt{\sin x}$.

Continuing with this example, let $F(x) = \sqrt{\sin x}$. Then,

Theorem 6.5 (Chain Rule, prime notation) *If f and g are differentiable functions, then $f \circ g$ is differentiable and*

$$(f \circ g)'(x) = f'(g(x))\, g'(x).$$

I like to think of a composition as having an "outside" part (the f) and an "inside" part (the g). The Chain Rule says, in English, the following:

the derivative of a composition is "*the derivative of a composition is the derivative of the outside, with the inside plugged in, times the derivative of the inside*".

EXAMPLE 2

Find $\frac{d}{dx}\left[\left(\frac{1}{x} - \sin x\right)^4\right]$.

EXAMPLE 3

Find y', if $y = \sqrt{3x+4}$.

EXAMPLE 4

Find the equation of the line tangent to $F(x) = (3x^2 - 3x - 1)^9$ when $x = 1$.

EXAMPLE 5

If an object's position, in feet, at time x (measured in minutes) is given by $f(x) = e^{-t}$, find the object's velocity and acceleration at time x.

EXAMPLE 6

Compute the second derivative of $y = \cos(x^2)$.

When to use the Product Rule, as opposed to the Chain Rule

EXAMPLE

$$\frac{d}{dx}\left(x^2 \sin x\right) \qquad \text{vs.} \qquad \frac{d}{dx}\left(\sin x^2\right)$$

Use of the Chain Rule in conjunction with other rules

EXAMPLES

Find the derivative of each of these functions:

1. $y = 2x \ln(4x^2 + 1)$

2. $f(x) = x^2 \cos^3 x - 4x \tan^2 x$

3. $y = \frac{(e^x+x^2-2)^3}{(x^{-3}-1)^{3/2}}$

4. $f(x) = \sin\left(\frac{\ln x-2}{\cos x+x}\right)$

   Solution: Start with the Chain Rule, because "$\sin$" doesn't mean anything by itself:

$$\begin{aligned} f'(x) &= \text{outside}'(\text{inside}) \cdot (\text{inside})' \\ &= \cos\left(\frac{\ln x - 2}{\cos x + x}\right) \cdot \left(\frac{\ln x - 2}{\cos x + x}\right)' \end{aligned}$$

   Now use the Quotient Rule to compute the inside$'$:

$$f'(x) = \cos\left(\frac{\ln x - 2}{\cos x + x}\right) \cdot \frac{\frac{1}{x}(\cos x + x) - (-\sin x + 1)(\ln x - 2)}{(\cos x + x)^2}$$

5. $g(x) = \cos(\sqrt{\sec x})$

6.5 Implicit differentiation

Another application of the Chain Rule

Suppose $z = \sin y$ and $y = f(x)$, where you don't know what the function f is.

$$\frac{dz}{dx} = ?$$

Answer: By the Chain Rule,

EXAMPLE 1

Suppose that y is some unknown function of x. Find $\frac{d}{dx}\left(y^2 + 6y - 2\right)$.

Note: If y is a constant, rather than a function of x, then $\frac{d}{dx}\left(y^2 + 6y - 2\right) =$

EXAMPLE 2

$\frac{d}{dx}\left(x^4 - \sin y + 5\right) = ?$

EXAMPLE 3

$\frac{d}{dx}\left(y^3 \sin x\right) = ?$

Implicit differentiation of equations

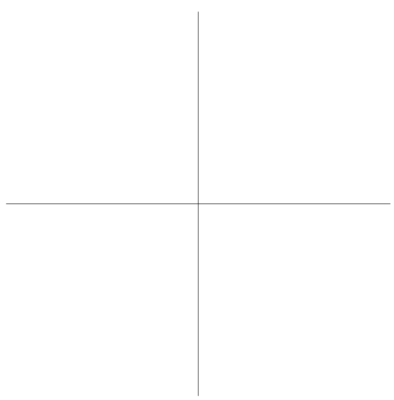
MOTIVATING EXAMPLE

Consider the equation $x^2 + y^2 = 25$.

This equation is not a function, for two reasons:

1.

2.



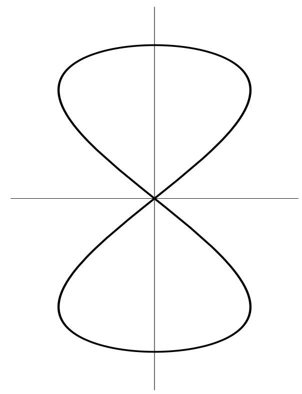
Suppose you wanted to write the equation of the tangent line to $x^2 + y^2 = 25$ at some point. You would need to compute $\frac{dy}{dx}$ at that point to get the slope. But which equation do you differentiate:

$$y = \sqrt{25 - x^2} \quad \text{or} \quad y = -\sqrt{25 - x^2}$$

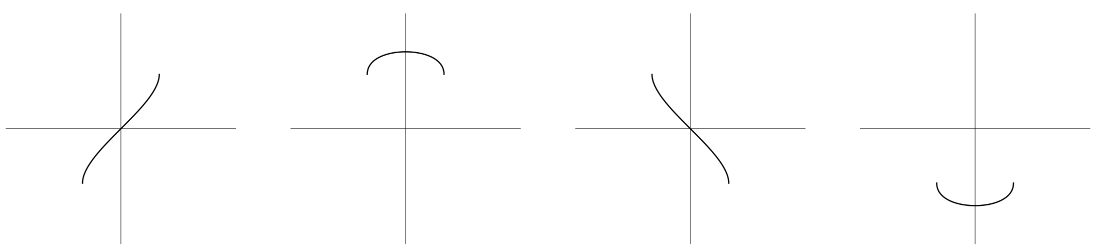
In this example, the choice is obvious:

But for a more interesting equation, there is no way to tell which equation to use. Consider the equation

$$4(y^2 - x^2) = y^4.$$



If you solve for y, you will get four different solutions:



There's no (easy) way to tell which solution goes with which graph.

Question: Is there a way to compute $\frac{dy}{dx}$ for some equation without solving for y in terms of x?

Answer: Yes. The method is called **implicit differentiation**. To implement it, start with the equation and differentiate both sides with respect to x (i.e. "take $\frac{d}{dx}$ of both sides").

General procedure to implement implicit differentiation:

1. Take $\frac{d}{dx}$ of both sides (as with the examples earlier).
2. If you are given x and/or y values, plug them in.
3. Solve for $\frac{dy}{dx}$.

EXAMPLE 4

Find the slope of the line tangent to the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

Follow up question # 1: What is the equation of the line tangent to the circle $x^2 + y^2 = 25$ at $(3, -4)$?

Follow up question # 2: In the preceding example, how would you determine the value of the second derivative at $(3, -4)$ (i.e. how would you measure the concavity of the circle)?

EXAMPLE 5

Find $\left.\frac{dy}{dx}\right|_{x=3,y=3}$ for the equation $x^3 + y^3 - 6xy = 0$.

EXAMPLE 6

Find $\frac{dy}{dx}$ for the equation $x + e^{2xy} = 10$.

Theoretical applications of implicit differentiation

Implicit differentiation can be used to justify derivative rules for complicated functions:

EXAMPLE 7

Use implicit differentiation to verify that the derivative of $\ln x$ is $\frac{1}{x}$:

$$y = \ln x \quad \Leftrightarrow \quad e^y = x$$
$$\frac{d}{dx}(e^y) = \frac{d}{dx}(x)$$

EXAMPLE 8

Find the derivative of $f(x) = \arctan x$.

Solution: As in the previous example, rewrite the function and use implicit differentiation:

$$\begin{aligned} y = \arctan x \Leftrightarrow x &= \tan y \\ 1 &= \sec^2 y \,\frac{dy}{dx} \\ 1 &= (1 + \tan^2 y)\,\frac{dy}{dx} \\ 1 &= (1 + x^2)\,\frac{dy}{dx} \\ \frac{1}{1 + x^2} &= \frac{dy}{dx} \end{aligned}$$

EXAMPLE 9

Find the derivative of $f(x) = \arcsin x$.

6.6 Summary of differentiation rules

Derivatives of functions that you should memorize:

Constant Functions	$\frac{d}{dx}(c) = 0$
Power Rule	$\frac{d}{dx}(x^n) = nx^{n-1}$ (so long as $n \neq 0$)
Special cases of the Power Rule:	$\frac{d}{dx}(mx+b) = m$
	$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$
	$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$
	$\frac{d}{dx}(x^2) = 2x$
Trigonometric Functions	$\frac{d}{dx}(\sin x) = \cos x$
	$\frac{d}{dx}(\cos x) = -\sin x$
	$\frac{d}{dx}(\tan x) = \sec^2 x$
	$\frac{d}{dx}(\cot x) = -\csc^2 x$
	$\frac{d}{dx}(\sec x) = \sec x \tan x$
	$\frac{d}{dx}(\csc x) = -\csc x \cot x$
Exponential Function	$\frac{d}{dx}(e^x) = e^x$
Natural Log Function	$\frac{d}{dx}(\ln x) = \frac{1}{x}$
Inverse Trig Functions	$\frac{d}{dx}(\arctan x) = \frac{1}{x^2+1}$
	$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$

Rules that tell you how to differentiate more complicated functions:

Sum Rule	$(f+g)'(x) = f'(x) + g'(x)$
Difference Rule	$(f-g)'(x) = f'(x) - g'(x)$
Constant Multiple Rule	$(kf)'(x) = k \cdot f'(x)$ for any constant k
Product Rule	$(fg)'(x) = f'(x)\, g(x) + g'(x)\, f(x)$
Quotient Rule	$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)\, g(x) - g'(x)\, f(x)}{[g(x)]^2}$
Chain Rule	$(f \circ g)'(x) = f'(g(x))\, g(x)$

6.7 Homework exercises

1. Let $g(x) = (x^2+1)(x^2-3x+4)$. Find $g'(x)$.
2. Let $f(x) = 4x^2 \ln x$. Find $\frac{df}{dx}$.
3. Find the derivative of $f(x) = \frac{x}{x^2-x+1}$.
4. Find $g'(1)$ if $g(x) = \frac{x}{\sqrt{x}+1}$.
5. Differentiate $f(x) = \frac{\sin x}{x^2}$.
6. Find the derivative of $f(x) = \sqrt{x}\sin x$.
7. Find $\frac{dy}{dx}$ if $y = (2x^3 - x^{-2/3})e^x$.
8. Find the instantaneous velocity of an object at time $t = \frac{\pi}{3}$, if the position of the object is given by $f(t) = t^2 \sin t$.
9. Find $\frac{d}{dx}\left[(\frac{1}{4}x^2-1)\ln x\right]$.
10. Find the second derivative of $f(x) = x\ln x$.
11.
    a) Find $f'(2)$ if $f(x) = 2\sin x\sqrt[5]{x}$.
    b) Explain in your own words what your answer to part (a) means.
12. Differentiate $f(x) = \frac{x^2+1}{x^3-1}$.
13. Find y' if $y = \frac{\cos x}{\sqrt{x}}$.
14. Find the slope of the line tangent to the graph of $f(x) = 4\cos x \sin x$ when $x = \frac{\pi}{4}$.
15. Find the acceleration of a particle at time t, given that the particle's position at time t is $\frac{3t^2-4}{t^2+1}$.
16. Let $y = \frac{8x^9 - \sin x}{\ln x + 5}$. Find $\frac{dy}{dx}$.
17. Find $f'(\pi)$ if $f(x) = x^2 \sin x$.
18. Find the equation of the line tangent to the graph of $y = \frac{\cos x}{x}$ when $x = \pi/2$.
19. Find the equation of the line tangent to $f(x) = (x-1)(x^2-2)$ at the point $(0,2)$.
20. Suppose $f'''(x) = 2x\cos x$. Find $f^{(4)}(x)$.
21. Suppose f and g are functions such that $f(3) = 2$, $f'(3) = -1$, $g(3) = 4$ and $g'(3) = 2$. Find $(fg)'(3)$ and $\left(\frac{f}{g}\right)'(3)$.

22. Here is a table which lists of values of functions f, g, f' and g':

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	2	1	-2	-5	3	0	1	2	3
$f'(x)$	3	-1	4	2	-1	3	2	2	5
$g(x)$	2	-5	0	3	1	-4	2	0	-2
$g'(x)$	3	-2	-1	-2	4	1	0	3	7

Use this information to compute the following quantities:

a) $(fg)'(2)$

b) $(fg)'(0)$

c) $\left(\frac{f}{g}\right)'(4)$

d) $(f+3g)'(-1)$

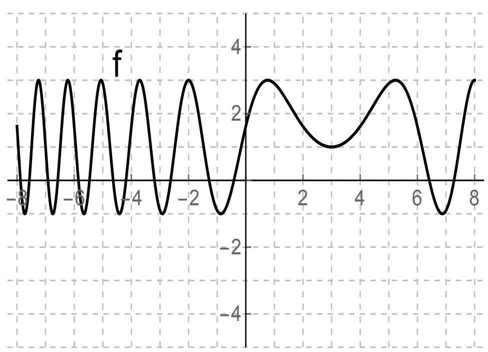
e) $\left(\frac{f}{f+g}\right)'(2)$

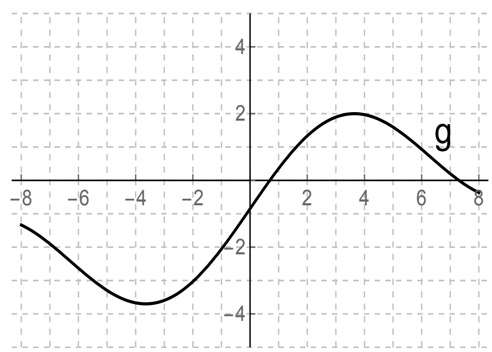
f) $h'(3)$, if $h(x) = x^2 f(x)$

g) $k'(-2)$, if $k(x) = 4x^3 g(x)$

h) $\frac{d}{dx}\left(\frac{x}{g(x)}\right)\Big|_{x=-1}$

23. The graphs of two functions f and g are shown below:





Use the graphs to estimate these quantities:

a) $(fg)'(0)$

b) $(fg)'(3)$

c) $\left(\frac{f}{g}\right)'(2)$

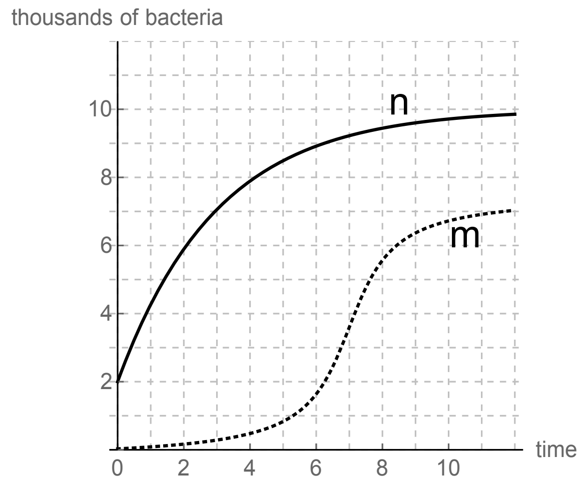
d) $\left(\frac{f}{g}\right)'(-5)$

e) $b'(6)$, if $b(x) = 5xg(x)$

f) $\frac{d}{dx}\left(\frac{g(x)}{x}\right)\Big|_{x=-6}$

24. A team of biologists studies the behavior of a bacteria colony under the effect of exposure to radiation as time passes. They produce graphs of functions n and m, where $n(t)$ is the bacteria population (measured in thousands of bacteria) at time t (measured in hours), and $m(t)$ is the number of mutated bacteria (measured in thousands of bacteria) at time t (measured in hours).

Graphs of these functions are shown below:

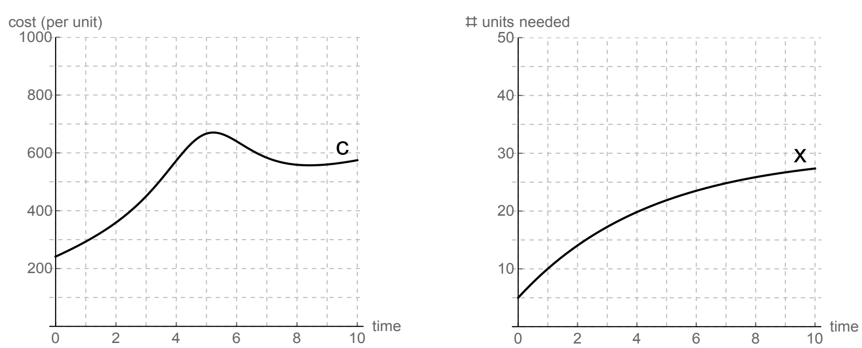


a) Let p be the proportion of bacteria that have mutated at time t. Write p as a function of m and n.

b) Estimate $p'(7)$ from the given graphs. Write your answer with appropriate units.

c) In terms of the context of this problem, what does your answer to part (b) mean?

25. Suppose that at time t (measured in months), a raw material needed by a business costs $c(t)$ dollars per unit. Suppose also that at time t (in months), the business estimates that it needs $x(t)$ units of the material. If the graphs of c and x are as given below, what is the instantaneous rate of change of the company's total raw material costs relative to time, when $t = 7$?



In Problems 26-33, find the derivative of the given function.

26. $f(x) = 2\cot x$
27. $y = 3x^4 \csc x$
28. $f(x) = \frac{-1}{x^2} + \sec x - 4\sin x$
29. $f(x) = \frac{\sec x}{x}$
30. $y = x\sin x - \frac{2x}{\cot x}$
31. $f(x) = \frac{1}{4}\sqrt{x} + 3 - 5\csc x$
32. $y = \sqrt[4]{x} + 6\tan x - 3\cot x$
33. $f(x) = \ln x \sin x$
34. a) Find the derivative of $f(x) = (x-3)^{-3}$ using the Chain Rule.
    b) Find the derivative of $f(x) = (x-3)^{-3}$ by rewriting the function (to get rid of the negative exponent) and using the Quotient Rule.
    c) Verify that the answers you got in (a) and (b) are the same.
35. Find the derivative of $y = (2x-3)^8$.
36. Find $f'(2)$ if $f(x) = \sqrt{8-x}$.
37. Find $\frac{dy}{dx}$ if $y = \sqrt[3]{4x^2+5}$.
38. Differentiate $\pi(x) = \csc^2 x$.

    Note: In this problem, π is not the number π; it is just the name of the function.
39. Find the derivative of $f(x) = 4\ln(\cos x)$.
40. Suppose an object's position at time t is $\cos\left(\frac{3\pi t}{2}\right)$. Find the velocity of the object at the instant $t = 1$.
41. Find $\frac{d^2y}{dx^2}$ if $y = (5x-1)^{-3}$.
42. Find the derivative of $f(x) = e^{5x}$.
43. Find the derivative of $f(x) = \sin(\frac{x}{2})$.
44. Find the slope of the line tangent to $f(x) = 3\cos(x^2)$ when $x = 0$.
45. Find y' if $y = \sec\frac{1}{x} - x^2$.
46. Let $f(x) = \frac{1}{4}\sin^4(2x)$. Find $f'(x)$.
47. Find the equation of the line tangent to $f(x) = \sqrt{x^2+2x+8}$ when $x = 2$.
48. Suppose an object's position at time t is given by $f(t) = (t^2+3)e^{2t}$. Find the velocity of the object when $t = 0$.
49. Suppose f and g are functions such that $f(1) = 4$, $f'(1) = -3$, $f(3) = 2$, $f'(3) = 5$, $g(3) = 1$ and $g'(3) = 2$. Find $(fg)'(3)$ and $(f \circ g)'(3)$.

50. Suppose $\frac{dy}{du} = 3$ and $\frac{du}{dx} = 6$. What is $\frac{dy}{dx}$?

51. Suppose $\frac{dy}{dx} = 8$ and $\frac{du}{dx} = 4$. What is $\frac{dy}{du}$?

52. Suppose $\frac{dy}{dv} = 5$ and $\frac{dx}{dv} = 3$. What is $\frac{dy}{dx}$?

53. Use the table of values given in Problem 22 above to compute the following quantities:

a) $(f \circ g)'(2)$

b) $(g \circ f)'(-3)$

c) $(f \circ f)'(0)$

d) $(g \circ f)'(4)$

e) $r'(1)$, if $r(x) = g(2x)$

f) $h'(2)$, if $h(x) = (f(x))^2$

g) $H'(2)$, if $H(x) = f(x^2)$

h) $k'(0)$, if $k(x) = f(g(x) \cos x)$

i) $z'(-2)$, if $z(t) = x^2 f(g(t))$

j) $w'(1)$, if $w(x) = g(f(x)g(x))$

54. Use the graphs given in Problem 23 above to estimate these quantities:

a) $(f \circ g)'(0)$

b) $(g \circ f)'(0)$

c) $(f \circ g)'(6)$

d) $(f \circ f)'(-5)$

e) $r'(-1)$, if $r(x) = f(2x)$

f) $h'(-2)$, if $h(x) = (g(x))^2$

In Problems 55-74, find the derivative of the given function.

55. $f(x) = x^2(x-2)^4$

56. $f(x) = x\sqrt{4-x^2}$

57. $f(x) = \left(\frac{1-2x}{x+1}\right)^5$

58. $f(x) = \sqrt{x \cos x}$

59. $f(x) = \frac{x^3-2}{\sqrt{x^6+1}}$

60. $f(x) = \sin\left(\frac{x+1}{x-1}\right)$

61. $f(x) = \cos(\tan x)$

62. $f(x) = \cos x \tan x$

63. $f(x) = \cos(x \tan x)$

64. $y = \cot^4(5x+1)$

65. $y = \sqrt{\frac{x}{x-1}}$

66. $f(x) = e^{\sin x}$

67. $f(x) = e^{2x-5}$

68. $g(x) = \ln(x^2 + 8x + 5)$

69. $f(x) = \sec^2(4x)$

70. $f(x) = \frac{3}{x} - \sqrt{x} + x^2 \sin x$

71. $f(x) = 2 + \ln x - x^7 e^{4x}$

72. $f(x) = \frac{3+x^2 \cot x}{4\sqrt{x} - \sin(e^x)}$

73. $f(x) = 5^x$
Hint: Use the log rule $A^B = e^{B \ln A}$ to rewrite f; then take the derivative using the Chain Rule.

74. $f(x) = x^{2x}$

75. Compute $\frac{d}{dx}\left(3y^2+5y\right)$.
76. Compute $\frac{d}{dx}\left(4y^5-3x^3\right)$.
77. Compute $\frac{d}{dx}\left(y^2e^{3x}\right)$.
78. Compute $\frac{d}{dx}\left(4x^3y^2\right)$.

In problems 79-84, find the derivative $\frac{dy}{dx}$.

79. $x^2+y^2=49$
80. $x^3-xy+y^2=4$
81. $\sin x+2\cos 2y=1$
82. $x=\cos(xy)$
83. $e^x=\frac{x}{e^y}$
84. $\ln y=\cos x$
85. Find the slope of the line tangent to $x^2y-y^3=-8$ at the point $(0,2)$.
86. Find the equation of the line tangent to $(x^2+y^2)^2=4x^2y$ at the point $(1,1)$.
87. Find the equation of the line tangent to the elllipse $\frac{x^2}{2}+\frac{y^2}{8}=1$ at $(1,2)$.
88. Find the slope of the line tangent to the hyperbola $\frac{y^2}{6}-\frac{x^2}{8}=1$ at the point $(-2,-3)$.
89. Find $\frac{d^2y}{dx^2}$ if $x^2+y^2=4$.

    Hint: First find $\frac{dy}{dx}$, then take the derivative of that expression implicitly.
90. Find $\frac{d^2y}{dx^2}$ if $y^2=\sin x$.

In Problems 91-96, find the derivative of the given function.

91. $f(x)=\arctan 2x$
92. $f(x)=x^3\arctan x$
93. $f(x)=\arcsin x-\frac{1}{x}+\ln x-2$
94. $f(x)=4\arcsin 3x$
95. $f(x)=x\arctan 2x$
96. $f(x)=e^{\arctan x}$

Answers

1. $2x(x^2 - 3x + 4) + (2x - 3)(x^2 + 1)$
2. $8x \ln x + 4x$
3. $\frac{x^2-x+1-x(2x-1)}{(x^2-x+1)^2}$.
4. $\frac{3}{8}$
5. $\frac{x^2 \cos x - 2x \sin x}{x^4}$
6. $\frac{1}{2\sqrt{x}} \sin x + \sqrt{x} \cos x$
7. $(6x^2 + \frac{2}{3}x^{-5/3})e^x + (2x^3 - x^{-2/3})e^x$
8. $\frac{\pi\sqrt{3}}{3} + \frac{\pi^2}{18}$
9. $\frac{1}{2}x \ln x + (\frac{1}{4}x^2 - 1)\frac{1}{x}$
10. $\frac{1}{x}$
11. a) $2\sqrt[5]{2} \cos 2 + \frac{2}{5}2^{-4/5} \sin 2$
    b) The answer in part (a) is some number which gives the slope of the line tangent to f at $x = 2$.
12. $\frac{2x(x^3-1)-3x^2(x^2+1)}{(x^3-1)^2}$
13. $\frac{-\sqrt{x}\sin x - \frac{1}{2\sqrt{x}}\cos x}{x}$.
14. 0
15. $\frac{14(1-3x^2)}{(1+x^2)^3}$
16. $\frac{(72x^8 - \cos x)(\ln x + 5) - \frac{1}{x}(8x^9 - \sin x)}{(\ln x + 5)^2}$
17. $-\pi^2$.
18. $y = -\frac{2}{\pi}(x - \frac{\pi}{2})$
19. $y = 2 - 2x$
20. $2 \cos x - 2x \sin x$
21. $(fg)'(3) = 0$; $\left(\frac{f}{g}\right)'(3) = \frac{-1}{2}$.
22. a) 4
    b) 11
    c) $\frac{-31}{4}$
    d) -4
    e) $\frac{4}{9}$
    f) 30
    g) 32
    h) $\frac{-7}{9}$
23. Answers can vary a bit here:
    a) 0
    b) $\frac{1}{3}$
    c) -2
    d) $\frac{3}{25}$
    e) -15
    f) $\frac{1}{4}$
24. a) $p(t) = \frac{m(t)}{n(t)}$
    b) $p'(7) \approx \frac{1}{4}$ hr^{-1}
    c) At time 7, the proportion of mutated bacteria is increasing at a rate of $1/4$ per hour.
25. -675 dollars per month
26. $-2 \csc^2 x$
27. $12x^3 \csc x - 3x^4 \csc x \cot x$
28. $2x^{-3} + \sec x \tan x - 4 \cos x$
29. $\frac{x \sec x \tan x - \sec x}{x^2}$
30. $\sin x + x \cos x - \frac{2 \cot x + 2x \csc^2 x}{\cot^2 x}$
31. $\frac{1}{8\sqrt{x}} + 5 \csc x \cot x$
32. $\frac{1}{4}x^{-3/4} + 6 \sec^2 x + 3 \csc^2 x$

33. $\frac{1}{x}\sin x + \ln x \cos x$
34. $-3(x-3)^{-4}$
35. $16(2x-3)^7$
36. $\frac{-1}{2\sqrt{6}}$
37. $\frac{8x}{3}(4x^2+5)^{-2/3}$
38. $-2\csc^2 x \cot x$
39. $-4\tan x$
40. $\frac{3\pi}{2}$
41. $300(5x-1)^{-5}$
42. $5e^{5x}$
43. $\frac{1}{2}\cos\frac{x}{2}$
44. 0
45. $\frac{-1}{x^2}\sec\frac{1}{x}\tan\frac{1}{x} - 2x$
46. $2\sin^3(2x)\cos(2x)$
47. $y = 4 + \frac{3}{4}(x-2)$
48. 6
49. $(fg)'(3) = 9$; $(f \circ g)'(3) = -6$.
50. 18
51. 2
52. $\frac{5}{3}$
53. a) 0
    b) -1
    c) -2
    d) 15
    e) -4
    f) 4
    g) 20
    h) 12
    i) -8
    j) -48
54. a) 0
    b) $\frac{4}{3}$
    c) $\frac{1}{4}$
    d) 0
    e) 2
    f) -3
55. $2x(x-2)^4 + 4(x-2)^3x^2$
56. $\sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}}$
57. $5\left(\frac{1-2x}{x+1}\right)^4 \cdot \frac{-2(x+1)-(1-2x)}{(x+1)^2}$
58. $\frac{1}{2\sqrt{x\cos x}} \cdot (\cos x - x\sin x)$
59. $\frac{3x^2\sqrt{x^6+1} - \frac{6x^5}{2\sqrt{x^6+1}}(x^3-2)}{x^6+1}$
60. $\cos\left(\frac{x+1}{x-1}\right) \cdot \frac{-2}{(x-1)^2}$
61. $-\sin(\tan x)\sec^2 x$
62. $-\sin x \tan x + \cos x \sec^2 x$
63. $-\sin(x\tan x)\cdot(\tan x + x\sec^2 x)$
64. $-20\cot^3(5x+1)\csc^2(5x+1)$
65. $\frac{1}{2\sqrt{\frac{x}{x-1}}} \cdot \frac{-1}{(x-1)^2}$
66. $e^{\sin x}\cos x$
67. $2e^{2x-5}$
68. $\frac{2x+8}{x^2+8x+5}$
69. $8\sec^2(4x)\tan(4x)$
70. $\frac{-3}{x^2} - \frac{1}{2\sqrt{x}} + 2x\sin x + x^2\cos x$
71. $\frac{1}{x} - 7x^6e^{4x} - 4x^7e^{4x}$

72. $\frac{(2x\cot x - x^2\csc^2 x)(4\sqrt{x}-\sin(e^x)) - (\frac{2}{\sqrt{x}} - e^x\cos(e^x))(3+x^2\cot x)}{(4\sqrt{x}-\sin(e^x))^2}$

73. $5^x \ln 5$

74. $x^{2x}(2\ln x + 2)$

75. $6y\frac{dy}{dx} + 5\frac{dy}{dx}$

76. $20y^4\frac{dy}{dx} - 9x^2$

77. $2y\frac{dy}{dx}e^{3x} + 3e^{3x}y^2$

78. $12x^2y^2 + 8x^3y\frac{dy}{dx}$

79. $\frac{-x}{y}$

80. $\frac{y-3x^2}{2y-x}$

81. $\frac{\cos x}{4\sin 2y}$

82. $\frac{-\csc xy - y}{x}$

83. $\frac{e^{x+2y}-e^y}{-xe^y}$

84. $-y\sin x$

85. 0

86. $y = 1$

87. $y = 2 - 2(x-1)$

88. $\frac{1}{2}$

89. $\frac{-4}{y^3}$.

90. $\frac{-2y^2\sin x - \cos^2 x}{4y^3}$

91. $\frac{2}{1+(2x)^2}$

92. $3x^2\arctan x + \frac{x^3}{x^2+1}$

93. $\frac{1}{\sqrt{1-x^2}} + \frac{1}{x^2} + \frac{1}{x}$

94. $\frac{12}{\sqrt{1-9x^2}}$

95. $\arctan 2x + \frac{2x}{1+4x^2}$

96. $e^{\arctan x}\cdot\frac{1}{1+x^2}$

6.8 Review problems for Exam 2

Mathematica questions

1. Write *Mathematica* commands which will compute the derivative of the function $f(x) = 3\sin(2x^4 - 8)\tan(3\ln x)$ when $x = 4$.

2. Write *Mathematica* commands which will compute the eighth derivative of the function $f(x) = \frac{2}{x} - \csc x$.

3. Write the output you will get (either in *Mathematica* syntax or hand-written notation) when you execute the following commands in *Mathematica*:

```
g[x_] = Log[x] +3
g''[2]
```

4. Write the output you will get (either in *Mathematica* syntax or hand-written notation) when you execute the following commands in *Mathematica*:

```
h[x_] = Cos[x] + 3x^(20)
D[h[x], {x, 42}]
```

Questions from Chapters 5 and 6

5. Explain in your own words, without referring to the Product Rule, why $(fg)'(x)$ is generally not equal to $f'(x)g'(x)$.

6. a) Find $f'(1)$ if $f(x) = (\sqrt{x} + 3)(2x^3 - 4x + 1)$.
   b) Explain, in your own words, what the answer to part (a) means.

7. Find the equation of the tangent line to the graph of $y = 3x^2 - \sqrt{x}$ at $x = 4$.

8. Suppose that a company's profit (in millions of dollars), if it produces x units of a product, is given by $P(x) = \frac{3}{\sqrt[3]{x}} + 1$. Find the instantaneous rate of change of the company's product with respect to the number of units produced, when 27 units are produced.

9. Suppose the position, in inches, of an object moving along an axis at time t, in seconds, is given by
$$f(t) = t^2 + \cos t + 2.$$
   a) Find the velocity of the object at time π.
   b) Find the acceleration of the object at time 0.

10. Differentiate the function $f(x) = 4\sqrt{x} + 3\cos x - 2e^x + 4$.

11. Find $g'(x)$, if $g(x) = \frac{1+\sin x}{x \ln x}$.

12. Find y' if $y = (1-3x)(\cot x - \csc x)$.

13. Find the slope of the line tangent to the curve $12x^5 - 2xy^3 + y^4 = 12$ at the point $(1, 2)$.

14. Find the derivative of $f(x) = (9x^3+1)^2 \sin 5x$.

15. Find the second derivative of $y = \arctan(6x+1)$.

16. Find y' if $y = \frac{x^2 - 3\cos x}{2x + \sec 4x}$.

17. Find the derivative of $h(x) = \sqrt[3]{x \cos x}$.

18. Find $\frac{d}{dx}\left(\frac{3}{2x^2} - 2x^{-1/5} + \frac{3}{8x^{3/2}}\right)$.

19. Suppose $y = (x^5 - 3x^2 + \sin x)$. Find $\frac{d}{dx}(y^3 - 2y + 1)$ (in terms of x).

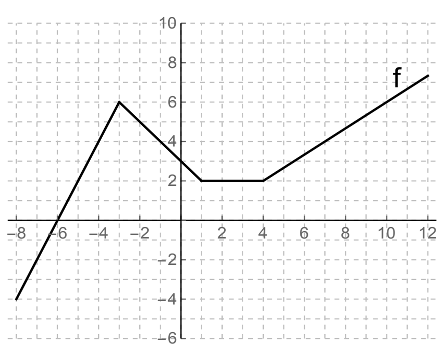
20. a) Suppose $f(x) = \cos 4x + 2x^3 - 1$. Find $f'''(x)$.
    b) Suppose $f(x) = \sin 2x$. What is $f^{(400)}(x)$?

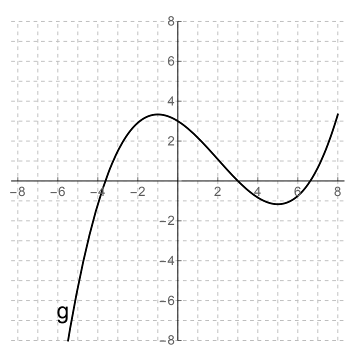
21. Find the zeroth derivative of the function $f(x) = e^{\sin x}$.

22. Suppose f is some function which satisfies $f(5) = 2$, $f'(5) = -3$, $f''(5) = -1$. Sketch a picture which illustrates what the graph of f looks like near $x = 5$.

23. Sketch a graph of a function g which has all six of the given properties:
    - $f'(-3) = 2$;
    - $f''(-3) = -1$;
    - $f'(2) = \frac{1}{4}$;
    - $f''(2) = 2$;
    - $f'(7) = 0$;
    - $f''(7) = \frac{2}{5}$.

24. Given below are the graphs of two unknown functions f and g.





Use these graphs to estimate the following quantities:

a) $f''(5)$

b) $(f \circ g)'(1)$

c) $(g \circ f)'(-1)$

d) $(fg)'(-2)$

e) $(fg)'(-3)$

f) $\left(\frac{g}{f}\right)'(0)$

g) $h'(7)$, where $h(x) = x^2 f(x)$

h) $\frac{d}{dx}(g(2x))\Big|_{x=3}$

25. Given the graph of g in Problem 24, sketch rough graphs of the functions g' and g''.

26. Given the graphs of f and g in Problem 24, determine whether the following quantities are positive, negative, zero, or do not exist:

a) $g''(-2))$

b) $g'(2)$

c) $g'(5)$

d) $g''(5)$

e) $f''(0)$

f) $g''(0)$

27. The national debt of a country, in millions of dollars, is given in the following table:

time t (years after 1980)	0	6	10	15	20	28	40
debt $f(t)$ (millions of dollars)	30	40	48	60	80	118	210

a) The **national deficit** is the rate of change of the country's debt. Use the data in the table to estimate the country's deficit in 1998. Show the work that leads to your answer, and write your answer with correct units.

b) What is the calculus notation (with f and/or t) that expresses what you estimated in part (a)?

c) Use the data in the table to estimate the rate of change of the country's deficit in 2008. Show the work that leads to your answer, and write your answer with correct units.

d) What is the calculus notation (with f and/or t) that expresses what you estimated in part (c)?

Answers

1. This takes two lines as shown here:

```
f[x_] = 3 Sin[2x^4 - 8] Tan[3 Log[x]]
f'[4]
```

2. This could be done in one line:

```
D[2/x - Csc[x], {x, 8}]
```

   A different (but less good) way to do this is in two lines:

```
f[x_] = 2/x - Csc[x]
f''''''''[x]
```

3. $\frac{-1}{4}$

4. $-\cos x$

5. There are several different answers, but one involves units. The units of $(fg)'(x)$ are

$$\frac{(\text{units of } f)(\text{units of } g)}{(\text{units of } x)}$$

   but the units of $f'(x)g'(x)$ are

$$\frac{(\text{units of } f)}{(\text{units of } x)} \cdot \frac{(\text{units of } g)}{(\text{units of } x)}$$

   which aren't the same.

6. a) Use the Product Rule to find $f'(x)$; then $f'(1) = \frac{15}{2}$.
   b) The slope of the line tangent to f at $x = 1$ is $\frac{15}{2}$; alternatively, the instantaneous rate of change of $f(x)$ with respect to x at $x = 1$ is $\frac{15}{2}$.

7. $y = 46 + \frac{95}{4}(x - 4)$.

8. $\frac{-1}{81}$ millions of dollars per unit.

9. a) $v(\pi) = 2\pi$
   b) $a(0) = 1$.

10. $f(x) = \frac{2}{\sqrt{x}} + -3 \sin x - 2e^x$.

11. Use the Quotient Rule to get

$$g'(x) = \frac{x \cos x \ln x - (\ln x + 1)(1 + \sin x)}{(x \ln x)^2}.$$

12. Use the Product Rule to get
$$y' = -3(\cot x - \csc x) + (-\csc^2 x + \csc x \cot x)(1 - 3x).$$

13. Use implicit differentiation to get the slope; then the tangent line has equation
$$y = 2 + (x - 1).$$

14. Use the Product Rule, then the Chain Rule to get
$$f'(x) = 2(9x^3 + 1)(27x^2) \sin 5x + (5 \cos 5x)(9x^3 + 1)^2.$$

15. Use the Chain Rule for the first derivative, then the Quotient and Chain rules for the second derivative to get $y'' = \frac{-72(1+6x)}{(1+(1+6x)^2)^2}$.

16. Use the Quotient Rule to get
$$y' = \frac{(2x + 3\sin x)(2x + \sec 4x) - (2 + \sec 4x \tan 4x \cdot 4)(x^2 - 3\cos x)}{(2x + \sec 4x)^2}.$$

17. Use the Chain Rule, then the Product Rule for the IN' part to get
$$h'(x) = \frac{1}{3} (x \cos x)^{-2/3} \cdot [\cos x - x \sin x] .$$

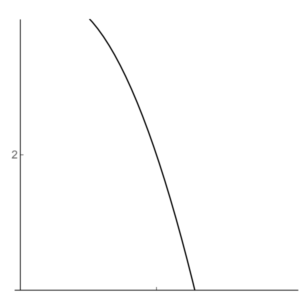
18. $-3x^{-3} + \frac{2}{5}x^{-6/5} - \frac{9}{16}x^{-5/2}$

19. $(3(x^5 - 3x^2 + \sin x)^2 - 2)(5x^4 - 6x + \cos x)$

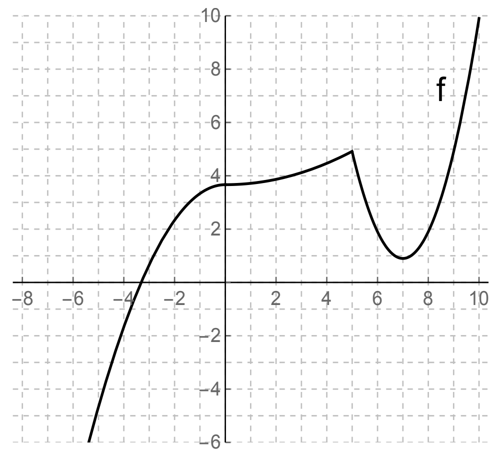
20. a) $f'''(x) = -64 \sin 3x$
    b) $f^{(400)}(x) = 2^{400} \sin 2x$

21. $e^{\sin x}$.

22. The graph passes through $(5, 2)$, has tangent line with slope -3 (i.e. goes down steeply from left to right) and bends below the tangent line (i.e. frowns) since $f''(5) < 0$. An example of such a graph is this one:

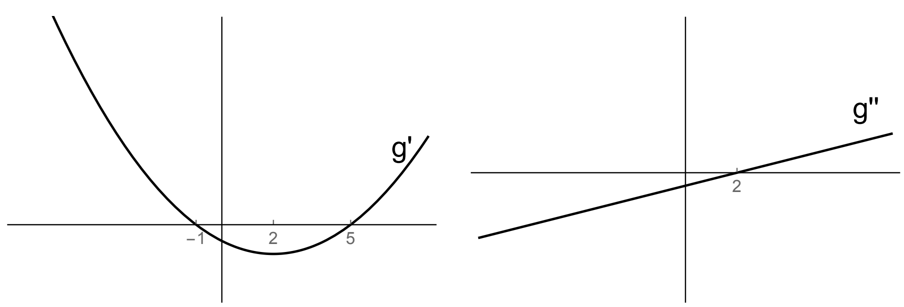


23. Answers may vary; here is one possible graph:



24. a) $f''(5) = 0$
b) $(f \circ g)'(1) = f'(g(1))g'(1) = 0$
c) $(g \circ f)'(-1) = g'(f(-1))f'(-1) = \frac{1}{2}$
d) $(fg)'(-2) = f'(-2)g(-2) + f(-2)g'(-2) = -3 + 5 = 2$
e) $(fg)'(-3) = f'(-3)g(-3) + f(-3)g'(-3)$ which DNE
f) $\left(\frac{g}{f}\right)'(0) = \frac{g'(0)f(0) - f'(0)g(0)}{[g(0)]^2} = \frac{1}{6}$
g) $h'(7) = 56 + 49(2/3)$
h) $\frac{d}{dx}(g(2x))\Big|_{x=3} = g'(6) \cdot 2 = 1$

25. Here are the graphs of g' and g'', with the relevant x-values marked:



26. a) $g''(-2))$ is negative
b) $g'(2)$ is positive
c) $g'(5)$ is zero
d) $g''(5)$ is positive
e) $f''(0)$ is zero
f) $g''(0)$ is negative

27. a) This is roughly the rate of change of the debt when $t = 18$, which is approximately

$$\frac{f(20) - f(15)}{20 - 15} = \frac{20}{5} = 4\text{million dollars / year}.$$

b) $f'(28) \approx 4$.

c) From the table, the deficit at about $t = 24$ is roughly

$$\frac{f(28) - f(20)}{28 - 20} = \frac{118 - 80}{8} = \frac{38}{8} = \frac{19}{4}$$

and the deficit at about time $t = 34$ is roughly

$$\frac{f(40) - f(28)}{40 - 28} = \frac{210 - 118}{40 - 28} = \frac{92}{12} = \frac{23}{3}.$$

That means the rate of change of the deficit when $t = 28$ is roughly

$$\frac{\frac{23}{3} - \frac{19}{4}}{34 - 24} = \frac{92}{12} - \frac{\frac{57}{12}}{10} = \frac{35}{120} = \frac{7}{24}\text{million dollars per year per year.}$$

d) $f''(28) \approx \frac{7}{24}$.

Chapter 7

Optimization Analysis

7.1 What is an optimization problem?

There are many situations in the real world where you need to determine how to make some quantity as large or as small as possible. Here are some examples:

EXAMPLE 1

If an archer shoots an arrow into the air at angle θ from the ground, it will travel a horizontal distance of $\frac{v \sin 2\theta}{g}$, where v and g are constants. At what angle should the archer shoot the arrow to make it travel as far as possible? (Equivalently, what is the maximum range of the archer?)

EXAMPLE 2

An epidemic spreads through a population in such a way that the number of infected people, I, is a function of the number of susceptible people, x, by the formula

$$I(x) = 4 \ln \left(\frac{x}{30} \right) - x + 30.$$

What is the maximum number of people who will become infected?

EXAMPLE 3

A patient's temperature change T, when given dose d of some medicine, is given by

$$T = \left(1 - \frac{d}{3} \right) d^2$$

What dosage maximizes this temperature change?

EXAMPLE 4

A farmer has 50 feet of fence with which to build a rectangular pen. What dimensions of the pen make its area as big as possible?

EXAMPLE 5

A box with a square base and no top is to be constructed from plywood. If there is 48 square feet of plywood available, and if the length, width and height of the box must be at least 1 foot, what is the largest volume of a box that can be made?

Common characteristics of Examples 1-5

1. In each example, there is some quantity you are allowed to "choose"; this quantity is the **variable**.
2. In each example, there is a second quantity which depends on the variable. This quantity is called the **utility**; the goal of the problem is to maximize or minimize the utility.

Any problem which asks you to maximize or minimize a utility function depending on one (or more) variables is called an **optimization problem**.

Here is the variable and utility for each of the first three examples on the previous page:

	Variable	Utility
Example 1		
Example 2		
Example 3		

Constrained optimization problems

Examples 4 and 5 are a little different, because there are two variables present in the problem.

In Math 220, we can only solve an optimization problem with two or more variables if there is some extra information which relates the variables. This extra information is called a **constraint** on the variables. (Take Math 320 - Calculus III - if you want to learn how to solve general optimization problems with more than one variable.)

	Variables	Utility	Constraint
Example 4			
Example 5			

We call problems like Examples 1 to 3 *free optimization problems* and problems like Examples 4 and 5 *constrained optimization problems*.

- **Free optimization problem:**

- **Constrained optimization problem:**

Converting a constrained optimization problem to a free optimization problem

The techniques of Math 220 are best suited to solving free optimization problems. So if you are given a constrained optimization problem, you first have to convert it to a free optimization problem by

1.

2.

Let's see how this works in Examples 4 and 5:

EXAMPLE 4

(variables x and y) (utility $A = xy$) (constraint $2x + 2y = 50$)

EXAMPLE 5

(variables x and y) (utility $V = x^2y$) (constraint $x^2 + 4xy = 20$)

Henceforth we will focus on solving free optimization problems. Keep in mind that whenever you are given a constrained optimization problem, the first step is to convert it to a free optimization as above.

7.2 Theory of optimization

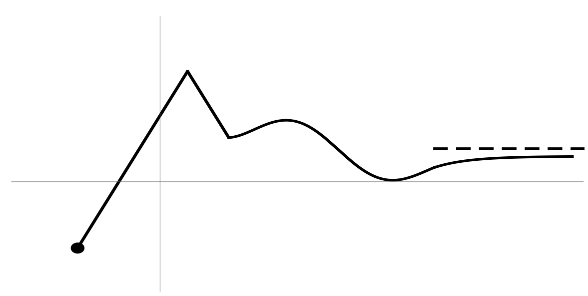
Our goal is to determine the maximum and minimum of some utility function $f(x)$. To understand how this is done, we first need a lot of vocabulary:

Definition 7.1 *Given a function f and a specified domain D of that function:*

1. *We say f has an* **absolute maximum** *(a.k.a.* **global maximum***) at $x = c$ if $f(x) \leq f(c)$ for all $x \in D$. In this case $f(c)$ is called the* **absolute (global) maximum value** *of f on D.*
2. *We say f has an* **absolute minimum** *(a.k.a.* **global minimum***) at $x = c$ if $f(x) \geq f(c)$ for all $x \in D$. In this case $f(c)$ is called the* **absolute (global) minimum value** *of f on D.*
3. *We say f has a* **local maximum** *(a.k.a.* **relative maximum***) at $x = c$ if $f(x) \leq f(c)$ for all $x \in D$ sufficiently close to c. In this case $f(c)$ is called a* **local (relative) maximum value** *of f on D.*
4. *We say f has a* **local minimum** *(a.k.a.* **relative minimum***) at $x = c$ if $f(x) \geq f(c)$ for all $x \in D$ sufficiently close to c. In this case $f(c)$ is called a* **local (relative) minimum value** *of f on D.*
5. *Collectively, all maxima and minima are called* **extrema***.*

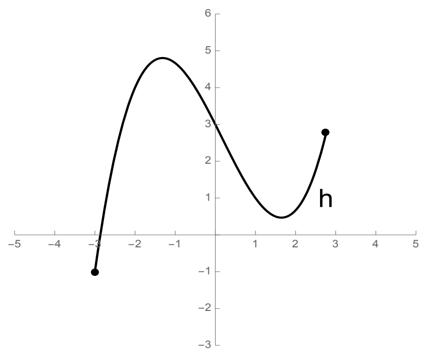
Note: If one says "f has a local maximum of 5 at 3", then one means that 5 is the $y-$value and 3 is the $x-$value, i.e. that the maximum is at the point $(3, 5)$.

Note: A function can have lots of local maxs/local mins, but has at most one global max and at most one global min. A list of all the local maxs (local mins) of a function always includes the global max (global min).



EXAMPLES

For each of the following graphs, identify all global extrema and all local minima. At all local extrema which are **not** endpoints, find the derivative of the function at the extrema.

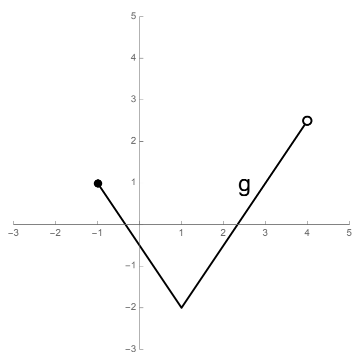


GLOBAL MAX:

GLOBAL MIN:

LOCAL MAX:

LOCAL MIN:

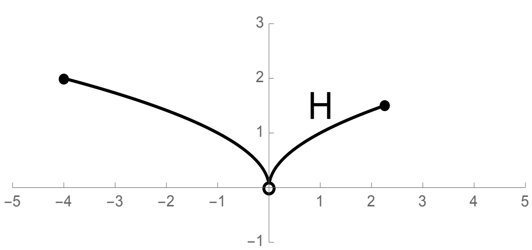


GLOBAL MAX:

GLOBAL MIN:

LOCAL MAX:

LOCAL MIN:

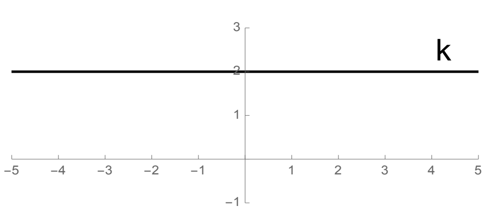


GLOBAL MAX: 2, at $x = -4$

GLOBAL MIN: none (there is no point on the graph at $(0, 0)$)

LOCAL MAX: $\begin{cases} 2 \text{ at } x = -4 \\ 1.5 \text{ at } x = 2 \end{cases}$

LOCAL MIN: none

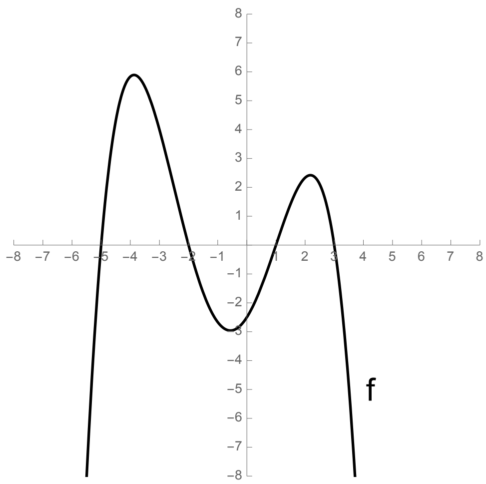


GLOBAL MAX: 3, at all x

GLOBAL MIN: 3, at all x

LOCAL MAX: 3, at all x

LOCAL MIN: 3, at all x



GLOBAL MAX: 6 at $x = -4$

GLOBAL MIN: DNE

LOCAL MAX: $\begin{cases} 2.5 \text{ at } x = 2 & f'(2) = \\ 6 \text{ at } x = -4 & f'(-4) = \end{cases}$

LOCAL MIN: -3 at $x = -1$ $\quad f'(-1) =$

Definition 7.2 *An* **optimization problem** *is a problem in which you are asked to find the absolute maximum and/or absolute minimum value of a function on some domain.*

Question 1: Does a function necessarily have an absolute maximum and/or absolute minimum? (In other words, does a generic optimization problem necessarily have a solution?)

Theorem 7.3 (Max-Min Existence Theorem) *If f is continuous on a closed and bounded interval $[a, b]$, then f has a global maximum value and a global minimum value on that interval.*

Note: The preceding theorem may fail if f is not cts, or if the interval is not closed, or if it is not bounded.

Question 2: How do you find the absolute maximum value and/or absolute minimum value of some function on some domain? (In other words, how do you solve an optimization problem?)

Definition 7.4 *A* **critical point** *(a.k.a.* **CP***) of a function f is a number c such that $f'(c) = 0$ or $f'(c)$ does not exist.*

Note: Critical points are numbers, not points. (They are the x-coordinates of points).

Theorem 7.5 (Critical Point Theorem) *All local extrema of a function (and therefore all global extrema) on an interval must occur at*

1. *endpoints of the interval, and/or*
2. *critical points of f lying in the interval.*

Note: Not all critical points are local extrema.

The Critical Point Theorem suggests a method of finding the global extrema of a function on an interval:

To optimize function f on interval $[a, b]$:

1. Find the critical points of f by
   (a) setting $f'(x) = 0$ and solving for x, and
   (b) finding all x for which $f'(x)$ DNE.
2. Discard any critical points which are not inside the interval $[a, b]$.
3. Plug each of the remaining critical points, as well as the two endpoints a and b, into the function f.

   The largest number you get is the absolute maximum, and the smallest number you get is the absolute minimum.

EXAMPLE A

Find the absolute extrema of the function $f(x) = 8 - x^2$ on the interval $[-4, 2]$.

EXAMPLE B

Find the absolute extrema of the function $f(x) = 2x^3 - 6x^2 + 1$ on the interval $[1, 3]$.

Solution: First, find CPs:

$f'(x) = 6x^2 - 12x$

$\underline{f'(x) = 0}$: $6x^2 - 12x = 0$
$6x(x-2) = 0$
$x = 0, x = 2$

$\underline{f'(x) \text{ DNE}}$: no such points

$\Rightarrow$ CPs: $x = 0, x = 2$

Second, test CPs and endpoints:

EXAMPLE C

Find the absolute extrema of the function $f(x) = 9\sqrt[3]{x}$ on the interval $[-1, 8]$.

Solution: First, find CPs:

$f'(x) = 9\left(\frac{1}{3}\right)x^{-2/3} = \frac{3}{x^{2/3}}$

$\underline{f'(x) = 0}$: $\frac{3}{x^{2/3}} = 0$
$3 = 0$
no such points

$\underline{f'(x) \text{ DNE}}$: $\frac{3}{x^{2/3}}$ DNE

Second, test CPs and endpoints:

Solving optimization word problems

General procedure to solve optimization word problems

1. Read the problem carefully, and draw a picture if necessary.
2. Identify any variable(s) and the utility (the quantity that needs to be maximized and/or minimized).
3. If there is more than one variable, find a constraint and convert the problem to a free optimization problem using the procedure outlined on p. 182.
4. Optimize the utility function using the procedure on p. 186 (find critical points, plug in critical points and endpoints to the utility, and choose the maximum and/or minimum value).
5. Make sure you answer the question that is asked.

EXAMPLE 1 (FROM PAGE 180)

If an archer shoots an arrow into the air at angle θ from the ground, it will travel a horizontal distance of $1000 \sin 2\theta$ ft. What is the maximum range of the archer?

EXAMPLE 4 (FROM PAGE 181)

A farmer has 50 feet of fence with which to build a rectangular pen. What dimensions of the pen make its area as big as possible?

EXAMPLE 5 (FROM PAGE 181)

A box with a square base and no top is to be constructed from plywood. If there is 48 square feet of plywood available, and if the length, width and height of the box must be at least 1 foot, what is the largest volume of a box that can be made?

EXAMPLE 2 (FROM PAGE 180)

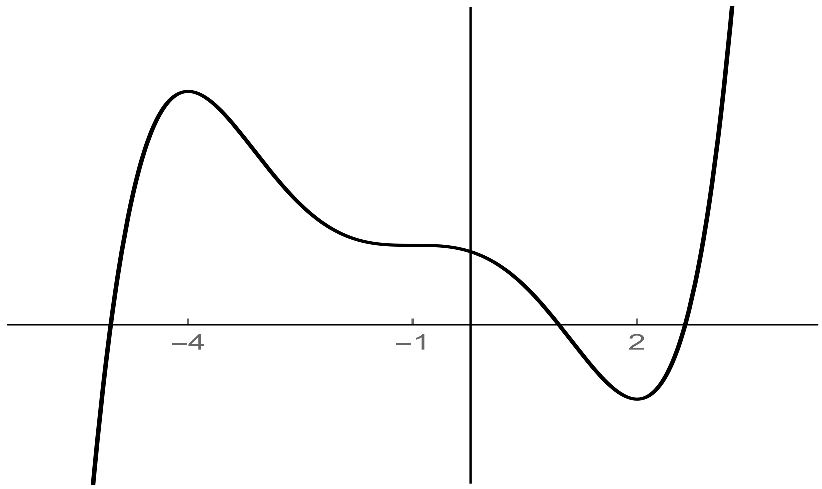
An epidemic spreads through a population in such a way that the number of infected people, I (measured in thousands), is a function of the number of susceptible people, x (measured in thousands), by the formula

$$I(x) = 4\ln\left(\frac{x}{30}\right) - x + 30.$$

What is the maximum number of people who will become infected?

7.3 Graphical analysis using derivatives

Tone



Definition 7.6

1. *A function f is called* **increasing** *on an open interval if for any x_1 and x_2 in that interval,*
$$x_1 \leq x_2 \textit{ implies } f(x_1) \leq f(x_2).$$
2. *A function f is called* **decreasing** *on an open interval if for any x_1 and x_2 in that interval,*
$$x_1 \leq x_2 \textit{ implies } f(x_1) \geq f(x_2).$$
3. *A function f is called* **monotone** *on an open interval if it is either increasing or decreasing on that interval.*

Note: Constant functions are both increasing and decreasing.

Note: Functions are always said to increase or decrease *on an open interval*, not at a point.

Theorem 7.7 (Monotonicity Test) *If f is differentiable on (a, b), then*

1. $f'(x) > 0$ *on* $(a, b) \Rightarrow f$ *is increasing;*
2. $f'(x) < 0$ *on* $(a, b) \Rightarrow f$ *is decreasing.*

EXAMPLE

Determine whether or not the function $f(x) = \frac{\ln x}{x}$ is increasing or decreasing on the interval $(0, 1)$. Determine whether or not f is increasing or decreasing on the interval $(4, 5)$.

Solution: Whether or not the function is increasing or decreasing depends on whether the derivative $f'(x)$ is positive or negative. By the Quotient Rule,

$$f'(x) = \frac{\frac{1}{x} \cdot x - 1 \cdot \ln x}{x^2} = \frac{1 - \ln x}{x^2}.$$

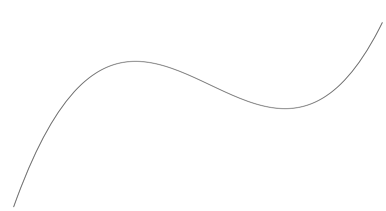
When $x \in (0, 1)$,

When $x \in (4, 5)$,

Concavity

Definition 7.8 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function.*

1. *f is called* **concave up** *on an open interval if f' is increasing on that interval.*
2. *f is called* **concave down** *on an open interval if f' is decreasing on that interval.*
3. *A number c is called an* **inflection point** *of f if the concavity of f changes at c.*



Theorem 7.9 (Concavity Test) *Let f be a function so that f'' exists on (a, b). Then:*

1. *if $f''(x) > 0$ for all $x \in (a, b)$, then f is concave up on (a, b);*
2. *if $f''(x) < 0$ for all $x \in (a, b)$, then f is concave down on (a, b).*
3. *c is an inflection point of f if and only if the sign of f'' changes at c.*

Remark: Based on the discussion from Chapter 5, if a function is concave up at/near x, then it will lie above the tangent line at x. If a function is concave down at/near x, then it will lie below the tangent line at x. If the function crosses its tangent line at x, then x is an inflection point of f.

EXAMPLE

Determine whether the function $f(x) = x^2 e^{-x}$ is concave up or concave down on the interval $(1, 2)$.

Solution: By the Product and Chain Rules,

$$f'(x) = 2xe^{-x} - x^2 e^{-x}$$

and

$$\begin{aligned} f''(x) = [2e^{-x} - 2xe^{-x}] - [2xe^{-x} - x^2 e^{-x}] &= 2e^{-x} - 4xe^{-x} + x^2 e^{-x} \\ &= e^{-x}(2 - 4x + x^2) \end{aligned}$$

When $x \in (1, 2)$,

EXAMPLE

Find the inflection points of the function $f(x) = x^3 + 3x^2 - 2x + 1$.

Solution: Compute the second derivative of f:

$$f'(x) = 3x^2 + 6x - 2$$

$$f''(x) = 6x + 6$$

The second derivative can also be used to classify critical points as local maxima or local minima using the following test:

Theorem 7.10 (Second Derivative Test) *Suppose $f'(c) = 0$ and that f'' is continuous on an open interval containing c. Then:*

1. *if $f''(c) > 0$, then f has a local minimum at c;*
2. *if $f''(c) < 0$, then f has a local maximum at c;*
3. *if $f''(c) = 0$, then this test is inconclusive.*

More sophisticated ideas along the lines of the Second Derivative Test were developed in your lab assignment on applications of derivatives. These ideas are summarized in this theorem:

Theorem 7.11 (n^{th} Derivative Test) *Suppose f is continuous on an open interval containing c and $f'(c) = f''(c) = f'''(c) = \ldots f^{(n-1)}(c) = 0$ but $f^{(n)}(c) \neq 0$. Then:*

1. *if n is even and $f^{(n)}(c) > 0$, then f has a local minimum at c;*
2. *if n is even and $f^{(n)}(c) < 0$, then f has a local maximum at c;*
3. *if n is odd, then f has no local extremum at c.*

Before the days of *Mathematica* and graphics calculators, this is how people learned to sketch the graphs of functions. In 2019, it is more useful to use these ideas to study applied optimization problems.

EXAMPLE 2 (FROM PAGES 180 AND 191)

An epidemic spreads through a population in such a way that the number of infected people, I (measured in thousands), is a function of the number of susceptible people, x (measured in thousands of people), by the formula

$$I(x) = 4\ln\left(\frac{x}{30}\right) - x + 30.$$

What is the maximum number of people who will become infected?

Recall from the work on page 191: we said our goal was to maximize $I(x)$ on $(0, \infty)$. We found earlier that

$$I'(x) = \frac{4}{x} - 1$$

and therefore the critical points of I are

$$x = 0 \text{ (where } I'(x) \text{ DNE);}$$

$$x = 4 \text{ (where } I'(x) = 0).$$

To determine whether or not $x = 4$ is the location of an extremum, we can use some analysis from this section:

$$I''(x) = \frac{-4}{x^2} \quad \Rightarrow \quad I''(4) = \frac{-4}{4^2} = \frac{-1}{4} < 0$$

7.4 More examples of optimization problems

EXAMPLE 6

A farmer grows zucchini. He has 10 acres available to plant; if he plants x acres his profit/loss will be $2x^3 - 33x^2 + 108x$ dollars. How many acres should the farmer plant (assuming he wants to make as much money as possible)?

EXAMPLE 7

In the human body, arteries must branch repeatedly to deliver blood to the entire body. Suppose a small artery branches off from a large artery at angle $\theta \in [0, \frac{\pi}{2}]$; the energy lost due to friction in this setting is approximately

$$E = \csc\theta + \frac{1-\cot\theta}{16}.$$

Find the value of θ that minimizes the energy loss.

Solution: First, write E as $E = \csc\theta + \frac{1}{16}(1-\cot\theta)$ and differentiate to get

$$E'(\theta) = -\csc\theta\cot\theta + \frac{1}{16}\csc^2\theta.$$

Next, find critical points: let $E'(\theta) = 0$ and solve for θ to get

$$\begin{aligned} 0 &= -\csc\theta\cot\theta + \frac{1}{16}\csc^2\theta \\ 0 &= \csc\theta\left(-\cot\theta + \frac{1}{16}\csc\theta\right) \end{aligned}$$

$\csc\theta = \frac{1}{\sin\theta}$ is never zero, so the only critical point is where $-\cot\theta + \frac{1}{16}\csc\theta = 0$. Rewriting with trig identities, we get

$$\frac{-\cos\theta}{\sin\theta} + \frac{1}{16\sin\theta} = 0 \;\Rightarrow\; -\cos\theta + \frac{1}{16} = 0 \;\Rightarrow\; \cos\theta = \frac{1}{16} \;\Rightarrow\; \theta = \arccos\frac{1}{16}.$$

Plug the endpoints $\theta = 0$ and $\theta = \frac{\pi}{2}$ and the critical point $\arccos\frac{1}{16}$ into the utility E:

$$\theta = 0: \qquad E = \csc 0 + \tfrac{1-\cot 0}{16} = 1 + \tfrac{1-\text{DNE}}{16} \text{ which DNE.}$$

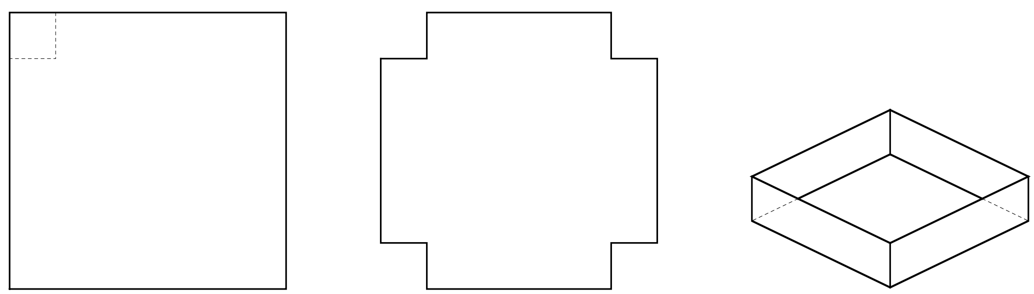
$$\begin{aligned} \theta = \arccos\tfrac{1}{16}: \quad E &= \csc(\arccos\tfrac{1}{16}) + \tfrac{1}{16}\left(1 - \cot(\arccos\tfrac{1}{16})\right) \\ &= \tfrac{16}{\sqrt{255}} + \tfrac{1}{16}\left(1 - \tfrac{1}{\sqrt{255}}\right) \\ &= \tfrac{1}{16}\left(1+\sqrt{255}\right). \end{aligned}$$

$$\theta = \tfrac{\pi}{2}: \qquad E = \csc\tfrac{\pi}{2} + \tfrac{1-\cot\frac{\pi}{2}}{16} = 1 + \tfrac{1-0}{16} = \tfrac{17}{16}.$$

Notice $\frac{1}{16}\left(1+\sqrt{255}\right) < \frac{1}{16}\left(1+\sqrt{256}\right) = \frac{1}{16}(1+16) = \frac{17}{16}$, so the absolute minimum is at $\theta = \arccos\frac{1}{16}$.

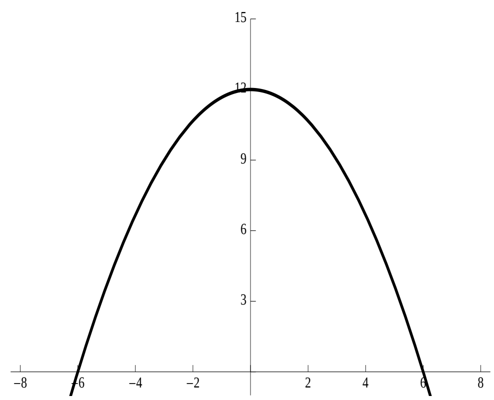
EXAMPLE 8

A $12''$ by $12''$ square sheet of cardboard is made into an open box (i.e. no top) by cutting squares of equal size out of each corner and folding up the sides along the dotted lines (see the pictures below). Find the dimensions of the box with the largest volume.



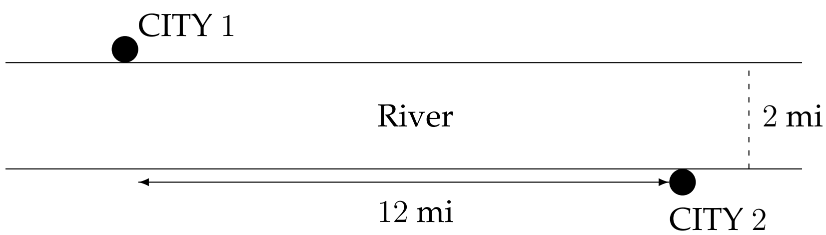
EXAMPLE 9

Find the maximum area of a rectangle if one side of the rectangle is on the $x-$axis and two corners of the rectangle are to be on the graph of $y = 12 - \frac{1}{3}x^2$ (this graph is shown below):

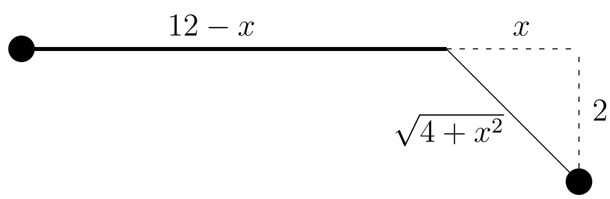


EXAMPLE 10

Michigan wants to build a new stretch of highway to link two sites on either side of a river (see the picture below) which is 2 miles wide. The second site is 12 miles downriver from the first site. It costs the state $\$13$ million per mile to build over water and $\$5$ million per mile to build over land. How should the state build its road to minimize costs?



Solution: First, it only makes sense to build a bridge in a straight line over the river, then to build along the riverbank to the other city. So the road goes along the solid lines shown below:



Letting x be as indicated in the picture, that means the cost of the road is

$$\begin{aligned} C(x) &= \text{cost of road along shore} + \text{cost of bridge} \\ &= 5(12 - x) + 13\sqrt{4 + x^2}. \end{aligned}$$

Our goal is to maximize this utility on the interval $[0, 12]$. First, differentiate (use the Chain Rule on the second term):

$$C'(x) = -5 + \frac{13}{2\sqrt{4 + x^2}} \cdot (2x) = -5 + \frac{13x}{\sqrt{4 + x^2}}.$$

Set this equal to zero and solve for x (details omitted, ask me if you don't follow this):

$$0 = -5 + \frac{13x}{\sqrt{4 + x^2}} \Rightarrow 5 = \frac{13x}{\sqrt{4 + x^2}} \Rightarrow x = \pm\frac{5}{6}$$

Plug the endpoints $x = 0$ and $x = 12$ and the critical point $x = \frac{5}{6}$ into the utility C; you will find that the minimum value of C is when $x = \frac{5}{6}$. Therefore the state should angle the bridge so that it goes $5/6$ mile downastream as it crosses the river.

7.5 Homework exercises

In Problems 1-10, you are given a word problem. Identify the utility and write the utility as a function of one variable. (You do not need to actually solve the problem.)

1. Find the maximum product of two numbers whose sum is 12.

2. On a given day, the rate of traffic flow on a conjested roadway is given by $F(v) = \frac{v}{24+.01v^2}$, where v is the velocity of the traffic. Find the velocity which maximizes the rate of traffic flow.

3. A farmer will build a rectangular pen, where one side of the pen is against a river (and does not need to be fenced). If he wants the pen to enclose an area of 3 acres, what is the minimum amount of fence that he can use?

4. Find the maximum sum of two numbers, where the second number is three times the reciprocal of the first.

5. A box has a square base. Find the maximum volume of the box, if the surface area of the box is 300 square cm (assume that the box has a top and a bottom).

6. The potential energy of a particle moving along an axis (say the $x-$axis) is $E = b\left(\frac{a^2}{x^2} - \frac{a}{x}\right)$ where a, b are positive constants and $x > 0$. What value of x minimizes this potential energy?

7. A box with four sides and a bottom, but no top, has a square base. Find the minimum surface area of the box, if its volume is to be 80 cubic cm.

8. A rectangular box (with a top and bottom) has its length equal to twice its width. Find the maximum volume of the box, if the surface area of the box is 120 square inches.

9. Suppose the perimeter of a rectangle is P units, where P is a constant. Find the maximum area of such a rectangle.

10. A 6-foot tall wall runs parallel to the side of a building, 4 feet away from the building. Find the minimum length of a ladder that can lean up against the building and touch the ground, while just touching the top of the wall.

    Hint: Write the utility as a function of the angle the ladder makes with the ground.

In Problems 11-16, you are given a graph of some unknown function f. In each picture, you should assume the graph continues to the left and right (i.e. that the extreme left and right ends of the graph have arrows on them). For each function:

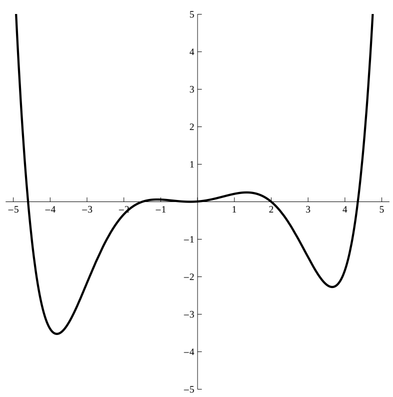
(a) Give the location of any local minima of f;

(b) Find the global minimum value of f on the interval $[-1, 4]$;

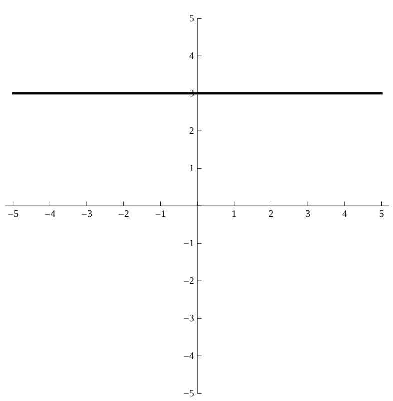
(c) Give the location of any local maxima of f;

(d) Find the global maximum value of f on the interval $[-1, 4]$.

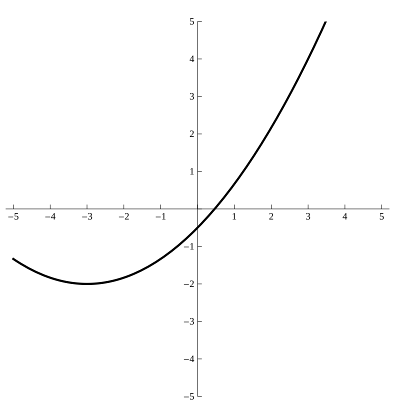
11.



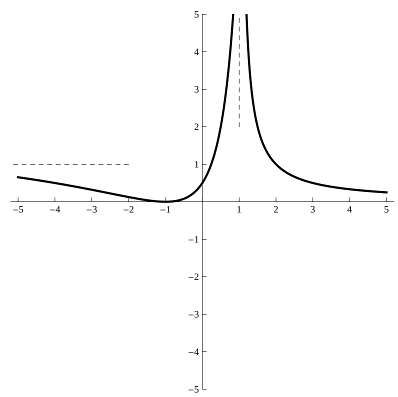
12.



13.



14.



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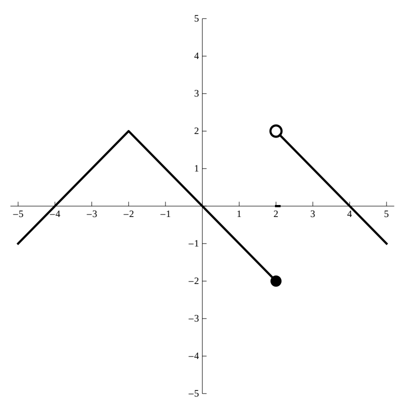
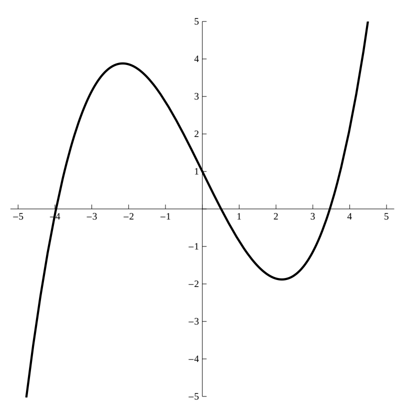


16.



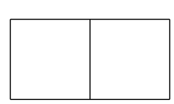
In Problems 17-24, find all critical points of the given function.

17. $f(x) = x^2(x^2 - 4)$
18. $f(x) = 3x^{1/5} + 2$
19. $f(x) = x^3 - 3x + 4$
20. $f(x) = |x|$
    Hint: consider the graph of f.
21. $f(x) = 4e^{-x}$
22. $f(x) = x^{7/3} - 28x^{1/3}$
23. $f(x) = \frac{3x}{x^2-1}$
24. $f(x) = \sin x + \cos x$
    Hint: in #24, find only the critical points of f between 0 and 2π.
25. Show that the functions $f(x)$ and $e^{f(x)}$ have the same set of critical points.
    Hint: Let $g(x) = e^{f(x)}$. Explain why solving $g'(x) = 0$ and $f'(x) = 0$ gives the same solutions.

In Problems 26-35, find the absolute extrema of the given function on the indicated interval.

26. $f(x) = \sin x + \cos x$ on $[0, 2\pi]$
27. $f(x) = x^{2/3}$ on $[-1, 27]$
28. $f(x) = x^3 - 12x + 4$ on $[-3, 5]$
29. $f(x) = x^3 - 12x + 4$ on $[-3, 0]$
30. $f(x) = \frac{1}{2}e^{-x^2}$ on $[-4, 4]$
31. $f(x) = 5$ on $[-3, 4]$
32. $f(x) = \frac{x}{x-2}$ on $[3, 5]$
33. $f(x) = \frac{3x}{x^2-1}$ on $[0, 2]$
34. $f(x) = 5 - x$ on $[1, 4]$
35. $f(x) = \arctan(x^2)$ on $[0, 1]$

36. If a person eats n sausages, then they will get heartburn in the amount of $h(n) = -n^3 + 12n$. If a person has the most amount of heartburn possible from eating sausages, how many sausages do they eat?
37. A farmer has 96 feet of fence with which to build a rectangular pen divided into two pieces as follows:



What dimensions should the farmer use to build her pen, if she wants the enclosed area to be as big as possible?

38. In an endurance contest, athletes start 2 miles out to sea need to reach a location which is 2 miles inland and three miles east of their initial location (assume the seashore runs east-west). If an athlete can run 10 miles per hour and swim 5 miles per hour, what is the minimum amount of time she will need to reach the finish? (Use *Mathematica* to compute the derivative of your utility function, then use the `NSolve` command in *Mathematica* to solve for the critical point.)

39. Suppose that if a company spends x hundred dollars on advertising, then their profit will be $P(x) = -3x^3 + 225x^2 - 3600x + 18000$. How much should the company spend on advertising if they want to maximize their profit, assuming that they only have enough capital to spend $\$3000$ on advertising?

40. A box is made with a square base and no top. If the surface area of the box is 80 square units, what is the largest possible volume of the box?

In Problems 41-44, you are given a function f and an interval (a, b). Determine, with justification, the sign of f' on (a, b). Use the sign of f' to draw a conclusion about the behavior of f on (a, b).

41. $f(x) = x^2 + \frac{1}{x^2}$ on $(0, 1)$
42. $f(x) = e^x - e^{-x}$ on $(-1, 1)$
43. $f(x) = -2x^3 + 3x^2 - 5$ on $(2, 3)$
44. $f(x) = \ln(x + \frac{1}{x})$ on $(0, 1)$

In Problems 45-49, find all the local extrema of the given function, and classify them as local maxima or local minima.

45. $y = x^4 + 4x^3 + 4x^2 - 3$
46. $f(x) = x \ln x$
47. $f(x) = x^2 - \frac{16}{x}$
48. $f(x) = e^{1/x^2}$

    Hint: The result of Problem 25 may be useful.
49. $f(x) = x + \frac{1}{x}$

In Problems 50-53, you are given a function f and an interval (a, b). Determine, with justification, the sign of f'' on (a, b). Use the sign of f'' to draw a conclusion about the behavior of f on (a, b).

50. $f(x) = e^x + e^{-x}$ on $(-\infty, \infty)$
51. $f(x) = x^4 - 16x^3 + 5$ on $(6, 7)$
52. $f(x) = -5 \sin x$ on $(\frac{\pi}{2}, \pi)$
53. $f(x) = \ln(x + \frac{1}{x})$ on $(0, 1)$

In Problems 54-58, find all inflection points of the function.

54. $f(x) = x^3 - 3x^2 + 4x - 1$

55. $f(x) = x + \frac{1}{x}$

56. $f(x) = xe^{-2x}$

57. $f(x) = x^2 + 2x + 3$

58. $f(x) = \sin x - \cos x$

(**Hint:** in #58, only find the inflection points between 0 and 2π)

59. Suppose f is some function such that $f'(2) = f''(2) = f'''(2) = 0$ and $f^{(4)}(2) = -3$. Is $x = 2$ the location of a local maximum, local minimum, or neither?

60. Suppose f is some function such that $f'(4) = f''(4) = f'''(4) = ... = f^{(14)}(4) = 0$ and $f^{(15)}(4) = 2$. Is $x = 4$ the location of a local maximum, local minimum, or neither?

61. Suppose f is some function such that $f'(-1) = f''(-1) = f'''(-1) = ... = f^{(11)}(-1) = 0$ and $f^{(12)}(-1) = -5$. Is $x = -1$ the location of a local maximum, local minimum, or neither?

62. Suppose f is some function such that $f'(0) = f''(0) = f'''(0) = ... = f^{(100)}(0) = 0$ and $f^{(101)}(0) = -17$. Is $x = 0$ the location of a local maximum, local minimum, or neither?

63. Suppose f is some function such that $f'(4) = f''(4) = f'''(4) = ... = f^{(99)}(4) = 0$ and $f^{(100)}(4) = \frac{2}{3}$. Is $x = 4$ the location of a local maximum, local minimum, or neither?

64. Let $f(x) = \frac{x^2}{x^2+1}$ for $x > 0$. Determine where the graph of f is steepest (i.e. where the slope of the graph is a maximum).

65. The Gompertz growth curve, whose formula is

$$W(t) = ae^{-be^{-t}},$$

is useful in several fields (including biology and economics). Assuming a and b are positive constants, find the value of t at which the rate of change of $W(t)$ with respect to t is largest.

66. Suppose that a worker can make $Q(t) = -t^3 + 12t^2 + 60t$ items in t hours.

    a) Explain why the efficiency of the worker at time t can be measured by $Q'(t)$.

    b) Find the time at which the worker is most efficient.

67. A rectangular poster is to be made which consists of a printed region and an unprinted margin, which is 3 inches on the top and bottom but 2 inches on the left and right side. If the total area of the poster is to be 120 square inches, what dimensions of the poster maximize the area of the printed region?

68. Suppose a wire of length 4 ft is cut into two pieces. Each piece is bent to form a square; find the largest possible combined area from the two pieces.

69. The velocity of air moving through a person's windpipe is $V(r) = Cr^2(A-r)$ for constants C and A, where r is the radius of the windpipe.

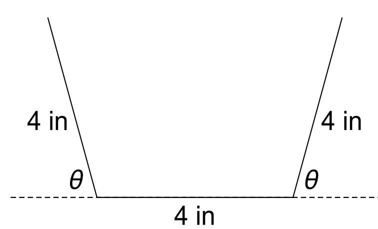
    a) Find the radius which maximizes this velocity.

    b) Suppose that normally, a person's windpipe has radius A. When a person coughs, the windpipe changes radius so that air moves through the windpipe as quickly as possible. Based on your answer to (a), does a person's windpipe get wider or narrower when a person coughs?

70. Find the point on the curve $y = \sqrt{x}$ which is closest to the point $(2, 0)$. *Hint:* Don't minimize the distance to the point; minimize the square of the distance to the point.

71. To transmit data (like a music file) electronically, the file has to be translated into a sequence of 0s and 1s so that it can be read by a computer or phone. An important computation related to the coding of files by 0s and 1s is the computation of a quantity called *entropy*, which is given by the following formula:
$$h(x) = x \ln x + (1-x)\ln(1-x)$$
Find the value of $x \in (0, 1)$ which maximizes the entropy h.

72. A $12-$inch wide piece of sheet metal is bent to form a rain gutter. A cross-section of the gutter is shown in the picture below. What value of θ maximizes the volume of water that can be held by the gutter?



Answers

1. The utility is the product, denoted by $U(x) = x(12 - x)$.

2. The utility is the rate of traffic flow, denoted by $F(v) = \frac{v}{24+.01v^2}$.

3. The utility is the amount of fence used, denoted by $A(x) = x + \frac{6}{x}$ or $A(x) = 2x + \frac{3}{x}$, depending on your setup.
4. The utility is the sum of the numbers, denoted by $U(x) = x + \frac{3}{x}$.
5. The utility is the volume of the box, denoted by $V(x) = x^2 \cdot \frac{300-2x^2}{4x}$.
6. The utility is the energy, denoted by $E(x) = b\left(\frac{a^2}{x^2} - \frac{a}{x}\right)$.
7. The utility is the surface area of the box, denoted by $S(x) = x^2 + \frac{320}{x}$.
8. The utility is the volume of the box, denoted by $V(w) = \frac{2}{3}w(60 - 2w^2)$.
9. The utility is the area of the rectangle, denoted by $A(x) = x(\frac{P-2x}{2})$.
10. The utility is the length of the ladder, denoted by $L(\theta) = 6\csc\theta + 4\sec\theta$.
11. a) $x = -4, x = 0, x = 3.5$
    b) -2.2
    c) $x = -1.2, x = 1.4$
    d) .4
12. a) all x are local mins
    b) 3
    c) all x are local maxs
    d) 3
13. a) $x = -3$
    b) -1.2
    c) no local max
    d) 5
14. a) $x = -1$
    b) 0
    c) no local max
    d) no absolute max
15. a) $x = 2$
    b) -2
    c) $x = -2$
    d) no absolute max
16. a) $x = 2$
    b) -2
    c) $x = -2$
    d) 3
17. $x = 0, x = \sqrt{2}, x = -\sqrt{2}$
18. $x = 0$
19. $x = 1, x = -1$
20. $x = 0$
21. None
22. $x = 0, x = 2, x = -2$
23. $x = 1, x = -1$
24. $x = \frac{\pi}{4}, \frac{5\pi}{4}$
25. Let $g(x) = e^{f(x)}$. By the Chain Rule, the derivative of g is $g'(x) = e^{f(x)}f'(x)$. Since $e^{f(x)}$ always exists and is never zero, $g'(x) = 0$ only if $f'(x) = 0$ and $g'(x)$ DNE only if $f'(x)$ DNE. Thus $g(x)$ and $f(x)$ have the same critical points.

26. Max is $\sqrt{2}$ at $\pi/4$; min is $-\sqrt{2}$ at $5\pi/4$
27. Max is 9 at $x = 27$; min is 0 at $x = 0$
28. Max is 69 at $x = 5$; min is -12 at $x = 2$
29. Max is 20 at $x = -2$; min is 4 at $x = 0$
30. Max is $\frac{1}{2}$ at $x = 0$; min is $\frac{1}{2}e^{-16}$ at $x = \pm 4$
31. Max and min are 5 occuring at all x
32. Max is 3 at $x = 3$; min is $\frac{5}{3}$ at $x = 5$
33. No max or min because of the asymptote at $x = 1$
34. Max is 4 at $x = 1$; min is 1 at $x = 4$
35. Max is $\frac{\pi}{4}$ at $x = 1$; min is 0 at $x = 0$
36. 2 sausages
37. Relative to the picture in the homework assignment, the height should be 16 feet and the width (all the way across) should be 24 feet.
38. $.728134$ hours
39. $\$3000$
40. $160\sqrt{\frac{5}{3}}$ cubic units.
41. $f'(x) = 2x - 2x^{-3} = 2x^{-3}(x^4 - 1) = (+)(-) < 0$ on $(0, 1)$, so f is decreasing on $(0, 1)$.
42. $f'(x) = e^x + e^{-x} > 0$ on $(-1, 1)$, so f is increasing on $(-1, 1)$.
43. $f'(x) = -6x^2 + 6x = -6(x)(x+1) = (-)(+)(+) < 0$ on $(2, 3)$, so f is decreasing on $(2, 3)$.
44. $f'(x) = \frac{1}{x+\frac{1}{x}} \cdot \left(1 - \frac{1}{x^2}\right) = \frac{1}{+}(-) < 0$ on $(0, 1)$, so f is decreasing on $(0, 1)$.
45. $x = 0$ local min; $x = -1$ local max; $x = -2$ local min
46. $x = \frac{1}{e}$ local min
47. $x = -2$ local min
48. No local extrema
49. $x = -1$ local max; $x = 1$ local min

50. $f''(x) = e^x + e^(-x) > 0$ on $(-\infty, \infty)$, so f is concave up on $(-\infty, \infty)$.
51. $f''(x) = 12x^2 - 96x = 12x(x-8) = 12(+)(-) < 0$ on $(6,7)$, so f is concave down on $(6,7)$.
52. $f''(x) = 5\sin x > 0$ on $(\frac{\pi}{2}, \pi)$, so f is concave up on $(\frac{\pi}{2}, \pi)$.
53. $f''(x) = \frac{4x}{(x^2+1)^2} = \frac{+}{+} > 0$ on $(0,1)$, so f is concave up on $(0,1)$.
54. $x = 1$
55. None
56. $x = 1$
57. None
58. $x = \frac{\pi}{4}$; $x = \frac{5\pi}{4}$
59. local maximum
60. neither
61. local maximum
62. neither
63. local minimum
64. The graph is steepest at $x = \frac{1}{\sqrt{3}}$.
65. At $t = \ln b$.
66. a) The efficiency of the worker is the rate at which the worker makes items; this rate is given by the derivative $Q'(t)$.
    b) At $t = 4$.
67. The width should be $4\sqrt{5}$ inches and the height should be $6\sqrt{5}$ inches.
68. 1 sq ft. (Cut the wire into a piece of length 4 ft and a piece of length 0 ft.)
69. a) $r = 2A/3$.
    b) It gets narrower, since $2A/3$ is less than A.
70. $\left(\frac{3}{2}, \sqrt{\frac{3}{2}}\right)$
71. $x = \frac{1}{2}$
72. $\theta = \frac{\pi}{4}$

Chapter 8

Other Applications of Differentiation

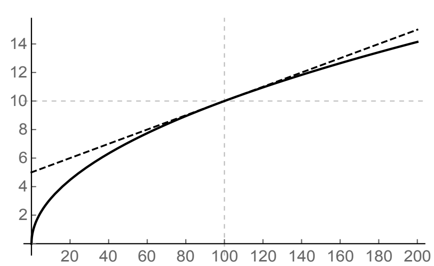
8.1 Tangent line and quadratic approximation

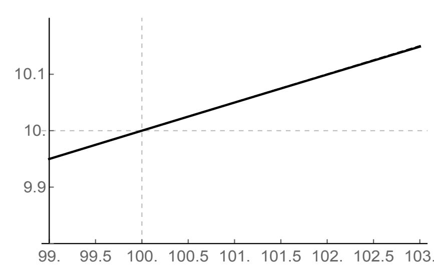
Motivation: Suppose you wanted to estimate $\sqrt{102}$ without the use of a calculator. (Put another way, how does your calculator produce an approximation of $\sqrt{102}$?)

A way of rephrasing this is as follows: let $f(x) = \sqrt{x}$. What is the approximate value of $f(102)$?

What we know is that $f(100) = \sqrt{100} = 10$, and since 102 is a little bit bigger than 100, $\sqrt{102}$ should be a bit bigger than 10. But how much bigger?

To address this issue, we use the ideas of calculus. Recall from Chapter 4 that the tangent line to a function at $x = 100$ is the line which most closely approximates the function at values near 100. Let's give a name to the tangent line at 100 and call it L.





Now from a calculation we did on page 85 of these notes, we found that the tangent line to a function f at a is

$$L(x) = f(a) + f'(a)(x - a).$$

In our setting, $f(x) = \sqrt{x}$ so $f'(x) = \frac{1}{2\sqrt{x}}$ and $a = 100$. So we have

$$\begin{aligned} L(x) &= f(a) + f'(a)(x-a) \\ &= f(100) + f'(100)(x-100) \\ &= \sqrt{100} + \frac{1}{2\sqrt{100}}(x-100) \\ &= 10 + \frac{1}{20}(x-100). \end{aligned}$$

The whole point of this is that the tangent line **closely approximates the original function**, so

$$\sqrt{102} = f(102) \approx L(102) = 10 + \frac{1}{20}(102-100) = 10 + \frac{2}{20} = 10.1.$$

Note: the actual value of $\sqrt{102}$ is $10.0995...$ so our approximation of 10.1 is correct to four decimal places.

Definition 8.1 (Linear approximation) *Given a differentiable function f and a number a at which you can easily compute $f(a)$ and $f'(a)$, the values $f(x)$ for x close to a can be estimated by the formula*

$$f(x) \approx L(x) = f(a) + f'(a)(x-a).$$

This procedure is called **tangent line approximation** *or* **linear approximation**.

The function $L(x)$ described above (which depends on f and a) has lots of names. It is also denoted $P_1(x)$ and is called:

1. the **tangent line to f at** a;
2. the **linearization of f at** a;
3. the **standard linear approximation to f at** a; and
4. the **first Taylor polynomial of f centered at** a.

EXAMPLE 1

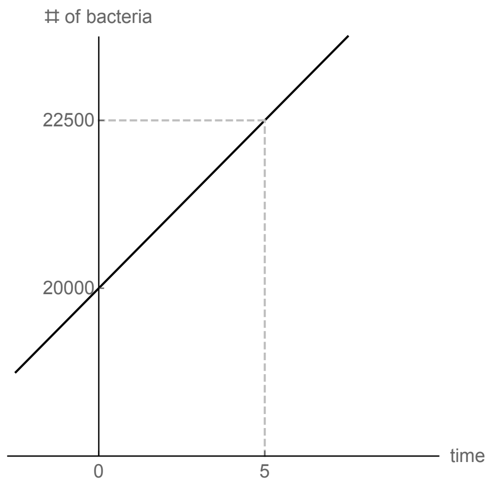
Estimate $\sqrt[4]{17}$ using tangent line approximation.

Note: You do not need to know the formula for f to perform a tangent line approximation. All you need to know are the values of $f(a)$ and $f'(a)$ (these two numbers can often be determined experimentally if f is some unknown function dealing with some experiment).

EXAMPLE 2

A biologist is growing bacteria in a petri dish. At 2:00 PM, she estimates that there are 20000 living bacteria in the dish, and that the number of bacteria is growing at a rate of 500 bacteria per minute. Use tangent line approximation to estimate the number of bacteria in the dish at 2:05 PM.

A more interesting calculus problem: In the example above will the answer overestimate, or underestimate the number of bacteria that are actually in the dish?



To get a better approximation which accounts for this kind of error, we approximate f not by a line but by a parabola which has the same slope and concavity as f at a.

Question: What would the equation of this parabola be?

Let's call this parabola $Q(x)$. Since $Q(x)$ is a parabola, we could write

$$Q(x) =$$

but it is actually easier to write the equation of this parabola "centered at a", i.e.

$$Q(x) = c_0 + c_1(x - a) + c_2(x - a)^2.$$

To find c_0, c_1 and c_2, use the concept that Q has to have the same value, slope and concavity as f at a.

The value of Q at a is

This should be the same as the value of f at a, which is

Conclusion:

The slope of Q at a is

This should be the same as the slope of f at a, which is

Conclusion:

The concavity of Q at a is $Q''(a) = 2c_2$.

This should be the same as the concavity of f at a, which is

Conclusion:

From all this, we know that

$$Q(x) = c_0 + c_1(x-a) + c_2(x-a)^2.$$

where

$$c_0 = f(a) \qquad c_1 = f'(a) \qquad c_2 = \frac{1}{2}f''(a).$$

To summarize:

Definition 8.2 (Quadratic approximation) *Given a twice-differentiable function f and a number a at which you can easily compute $f(a)$, $f'(a)$ and $f''(a)$, the values $f(x)$ for x close to a can be approximated by the formula*

$$f(x) \approx Q(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2.$$

This procedure is called **quadratic approximation**.

In general, quadratic approximation of a function is more accurate than linear approximation. In Math 230, you will learn how to approximate functions f by polynomials of larger degree which can produce highly accurate estimates to problems.

The function $Q(x)$ described above (which depends on f and a) also has lots of names. It is also denoted $P_2(x)$ and is called:

1. **parabolic approximation to** f **at** a;
2. the **standard quadratic approximation to** f **at** a; and
3. the **second Taylor polynomial of** f **at** a.

EXAMPLE 3

Approximate $\sqrt{102}$ using quadratic approximation.

EXAMPLE 4

Suppose the biologist in Example 2 assumes (in addition to what she knew in Example 2) that the number of bacteria in her petri dish at time t is given by a function whose second derivative at 2:00 PM is 10. Estimate the number of bacteria in her dish at 2:15 PM using quadratic approximation.

EXAMPLE 5

A pharmacy researcher measures a patient's blood pressure periodically after receiving a dose of an experimental medicine. His data is collected in the following table:

t (minutes after dosage)	0	1	2	3	4
$P(t)$ (blood pressure in mmHg)	230	190	162	142	128

Use quadratic approximation at $t = 3$ to estimate the patient's blood pressure at time 6.

Differentials

We will now establish some additional notation which will be used later in the course. Given a function $y = f(x)$, we create a new function with 2 inputs and one output. The two inputs are:

$$\begin{aligned} x &= \text{an "initial" value of } x \\ dx &= \text{a change in the value of } x \end{aligned}$$

Thus, we think of x as changing from x to $x + dx$. Given these inputs, we define dy to be the estimated change in y that we would compute using tangent line approximation at x:

$$\begin{aligned} dy &= L(x + dx) - L(x) \\ &= [f(x) + f'(x)(x + dx - x)] - [f(x) + f'(x)(x - x)] \\ &= [f(x) + f'(x)\,dx] - [f(x)] \\ &= f'(x)\,dx. \end{aligned}$$

The quantities dy and dx are called *differentials*. They represent small changes in y and x, respectively and are related by the formula

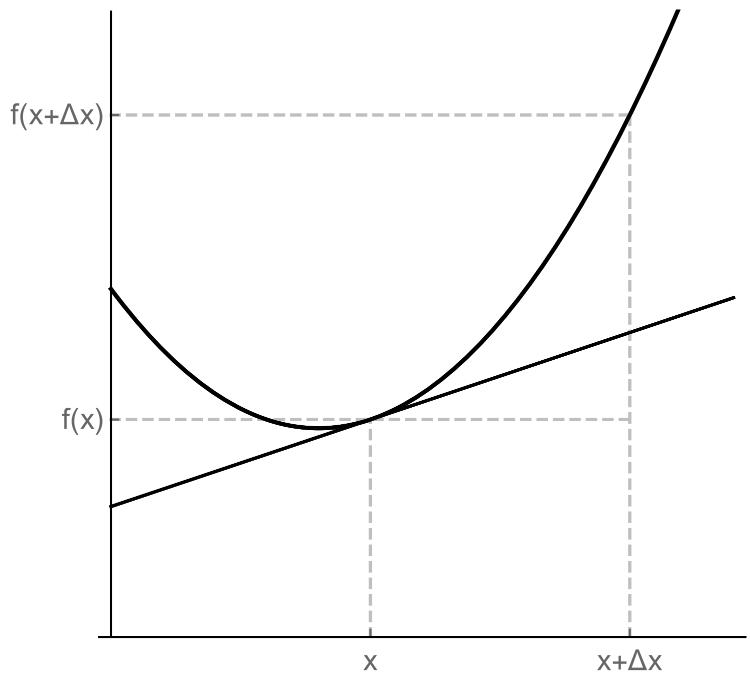
$$dy = f'(x)\,dx$$

.

EXAMPLE 6

Suppose $y = 2x^6 + \sin x - 3$. Compute dy (in terms of x and dx).

A picture associated to differentials:



In principle $dy \approx$ the actual change in y, since $L(x) \approx f(x)$.

8.2 L'Hôpital's rule

Recall that most limits are evaluated by "plugging in", i.e.

$$\lim_{x \to 5} \frac{2x+1}{x-3} = \frac{2(5)+1}{5-3} = \frac{11}{2}.$$

Other limits are not so easy:

$$\lim_{x \to 2} \frac{x^2-4}{x-2}$$

$$\lim_{x \to 0} \frac{\sin x}{x}$$

The $\frac{0}{0}$ obtained by plugging in 2 to the expression $\frac{x^2-4}{x-2}$ or by plugging in 0 to $\frac{\sin x}{x}$ is called an "indeterminate form". Note that both examples above are of the form $\frac{0}{0}$, but evaluate to different answers. More generally:

Definition 8.3 *An* **indeterminate form** *is an expression which can work out to one of many different answers, depending on the context.*

Examples of indeterminate forms:

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 0\cdot\infty \qquad \infty-\infty \qquad 1^{\infty} \qquad \infty^{0} \qquad 0^{0}$$

Forms which are <u>not</u> indeterminate:

$$\frac{0}{\text{nonzero constant}} = 0 \qquad \frac{\text{nonzero constant}}{0} = \pm\infty$$

$$\frac{\infty}{0} = \pm\infty \qquad \frac{0}{\infty} = 0 \qquad 0\cdot 0 = 0 \qquad 0^1 = 0 \qquad 1^0 = 1$$

$$(\text{nonzero constant})^0 = 1 \qquad \frac{\infty}{\text{nonzero constant}} = \pm\infty \qquad \frac{\text{nonzero constant}}{\infty} = 0$$

In Chapter 3, we learned to evaluate limits that have indeterminate forms in them by factoring and cancelling, or performing other algebraic manipulations (like conjugating square roots and clearing fractions within fractions).

One additional, and very useful, method to evaluate indeterminate forms in limits is called L'Hôpital's Rule:

Theorem 8.4 (L'Hôpital's Rule) *Suppose f and g are differentiable functions. Suppose also that either*

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$$

or

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty.$$

Then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} \stackrel{L}{=} \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Application: Expressions like $\frac{0}{0}$ or $\frac{\infty}{\infty}$ can often be evaluated by taking derivatives of the top and bottom independently, then plugging in a.

<u>WARNING</u>: We are **not** differentiating $\frac{f}{g}$ here. To do this, use the quotient rule (but that has nothing to do with the evaluation of the limit).

<u>WARNING</u>: Be sure that the limit you are calculating is a common (i.e. easy) indeterminate form before using L'Hôpital's Rule.

Notation: The symbol $\stackrel{L}{=}$ is used to denote usage of L'Hôpital's Rule. It is just an equals sign, and the L tells the reader that you are using L'Hôpital's Rule.

EXAMPLE 1

$$\lim_{x\to 2} \frac{x^2-4}{x-2}$$

EXAMPLE 2

$$\lim_{x\to 0} \frac{\sin x}{x}$$

EXAMPLE 3

$$\lim_{x \to 0} \frac{3}{x^2}$$

EXAMPLE 4

$$\lim_{x \to 3} \frac{x-3}{2x+1}$$

EXAMPLE 5

$$\lim_{x \to \infty} \frac{7+2x^2}{x^2-3x+1}$$

EXAMPLE 6

$$\lim_{x \to 0} \frac{\cos x - 1}{x^2}$$

Solution:

$$\lim_{x \to 0} \frac{\cos x - 1}{x^2} = \frac{''0''}{0} \stackrel{L}{=} \lim_{x \to 0} \frac{-\sin x}{2x} = \frac{''0''}{0} \stackrel{L}{=} \lim_{x \to 0} \frac{-\cos x}{2} = \frac{0}{2} = 0.$$

EXAMPLE 7

$$\lim_{x\to\infty} \frac{x^2}{x+1}$$

EXAMPLE 8

Evaluate this limit, where n is a positive integer:

$$\lim_{x\to\infty} \frac{x^n}{e^x}$$

Harder indeterminate forms

You can also evaluate other indeterminate forms (like $0 \cdot \infty$, $\infty - \infty$, 1^∞, ∞^0, 0^0) by first doing some algebra, then using L'Hôpital's Rule:

EXAMPLE 9

$$\lim_{x\to\infty} x \sin\left(\frac{1}{x}\right)$$

EXAMPLE 10

$$\lim_{x\to 0^+}(\csc x - \cot x)$$

Solution:

$$\lim_{x\to 0^+}(\csc x - \cot x) = '' \infty - \infty '' \text{ which is indeterminate}$$

Rewrite as

$$\lim_{x\to 0^+}(\csc x - \cot x) = \lim_{x\to 0^+}\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right) = \lim_{x\to 0^+}\frac{1-\cos x}{\sin x} = \frac{''0''}{0}$$

$$\stackrel{L}{=} \lim_{x\to 0^+}\frac{\sin x}{\cos x} = \frac{0}{1} = 0.$$

EXAMPLE 11

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

Note: The answer to this problem should be memorized.

<u>WARNING</u>: L'Hôpital's Rule is a dangerous thing to rely on too much for two reasons:

(1)

EXAMPLE 12

$$\lim_{x \to 0} \frac{\cos x - 1 + \frac{1}{2}x^2}{e^x - 1 - x - \frac{1}{2}x^2 - \frac{1}{6}x^3}$$

(2)

EXAMPLE 13

$$\lim_{x \to \infty} \frac{\sqrt{x^2 + 1}}{x}$$

8.3 Newton's method

Goal: use calculus to quickly and accurately approximate solutions to equations.

First, to solve any equation in one variable, it is sufficient to solve equations where one side is equal to zero (i.e. to find **roots** a.k.a. **x-intercepts** of functions). This is because if you are given an equation of the form

$$g(x) = h(x)$$

you can rewrite it as

$$g(x) - h(x) = 0 \text{ or } h(x) - g(x) = 0.$$

So our goal is: given function f, find (or at least approximate) r such that $f(r) = 0$. The procedure we will use is called **Newton's method** and works as follows:

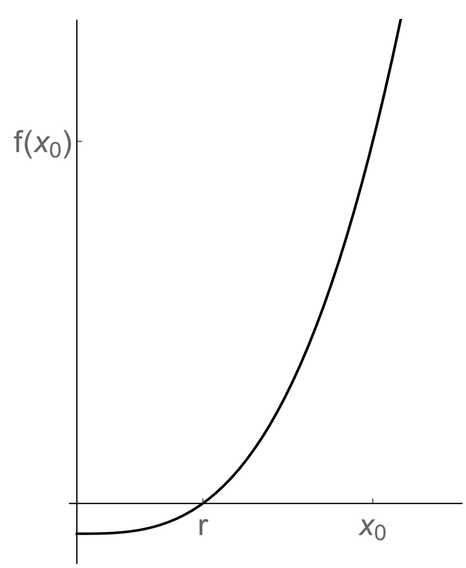
Newton's method

1. Guess the value of r. Call your guess x_0 (x_0 is called the "initial guess" or "seed").
2. Draw the tangent line to f at x_0.
3. Find the $x-$intercept of the tangent line from step (2). Call this $x-$int x_1.

   (Ideally, x_1 is closer to r than x_0 is.)
4. Draw the tangent line to f at x_1.
5. Find the $x-$intercept of the tangent line from step (2). Call this $x-$int x_2.

   (Ideally, x_2 is closer to r than x_1 is.)
6. Repeat the procedure over and over: given x_n, sketch the tangent line to f at x_n; call this $x-$int of this tangent line x_{n+1}.
7. You get a sequence of points $x_0, x_1, x_2, x_3, ...$

   The numbers x_n should (hopefully) get closer and closer to r, so they approximate r better and better as n gets larger.



Let's implement this procedure for an arbitrary function f and initial guess x_0:

The tangent line to f at x_n has equation

and x_{n+1}, the $x-$intercept of this line is found as follows:

EXAMPLE 1

Approximate a solution to $x^3 - x = 2$ by using Newton's method with initial guess 2 and two steps.

EXAMPLE 2

Approximate a solution to $x^3 - x = 2$ by using Newton's method and getting an approximation correct to 4 decimal places.

Newton's method on *Mathematica*

Newton's method is easy to implement on *Mathematica*. You need three lines of code, all in the same cell. For example, to implement Newton's method for the function $f(x) = x^2 - 2$ where $x_0 = 3$ and you want to perform 6 iterations (to find x_6), just type

```
f[x_] = x^2 - 2;
Newton[x_] = N[x - f[x]/f'[x]];
NestList[Newton, 3, 6]
```

and execute (all three lines at once). The first line defines the function f, the second line gives a name to the formula you iterate in Newton's method, and the last line iterates the formula and spits out the results.

The resulting output for the code listed above is:

```
{3, 1.83333, 1.46212, 1.415, 1.41421, 1.41421, 1.41421}
```

These numbers are $x_0, x_1, x_2, ..., x_6$ so for example, $x_2 = 1.46212$ and $x_4 = 1.41421...$ and $x_6 = 1.41421$ (the same as x_4 to 5 decimal places).

To implement Newton's method for a different function, different initial guess and different number of iterations, simply change the formula for f, change the 3 to the appropriate value of x_0 and the 6 to the number of times you want to iterate Newton's method.

EXAMPLE 3

Use Newton's method to approximate the solution to

$$\cos 2x + 3x = \sin x.$$

Obtain an approximation which is accurate to four decimal places.

***Mathematica* code:**

```
f[x_] = Cos[2x] + 3x - Sin[x];
Newton[x_] = N[x - f[x]/f'[x]];
NestList[Newton,-1/2,10]
```

output:

```
{-1/2, -0.373791, -0.367115, -0.367093, -0.367093, -0.367093,
-0.367093, -0.367093, -0.367093, -0.367093, -0.367093 }
```

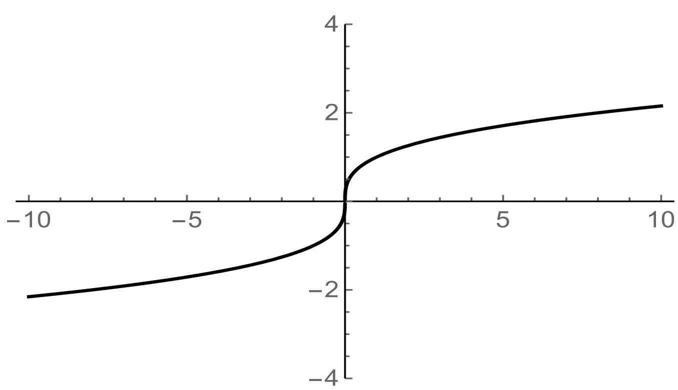
Potential problems with Newton's method

EXAMPLE 4

Use Newton's method to find a solution of

$$x^{1/3} = 0$$

using initial guess $x = 1$.

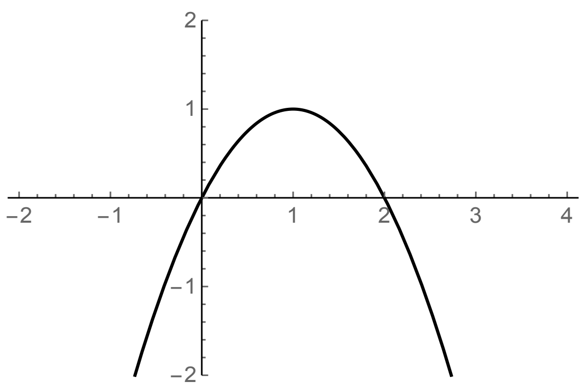


Example 5

Use Newton's method to find a solution of

$$2x - x^2 = 0$$

using initial guess $x = 1$.

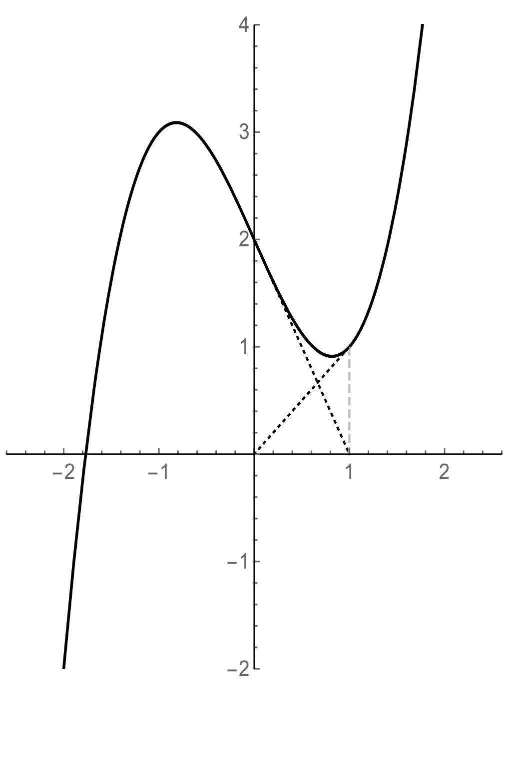


Example 6

Use Newton's method to find a solution of

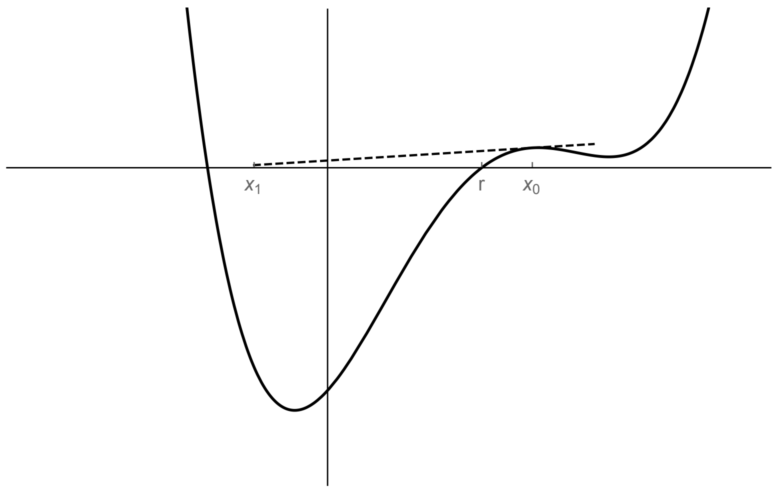
$$x^3 - 2x + 2 = 0$$

using initial guess $x = 0$.



EXAMPLE 7

Here is a graph of $f(x) = \frac{1}{2}(x+2)(x-3)(x-4)(x-5)+4$:



This function has two roots, a negative one which is about -2 and a positive one which is about 3. If you use initial guess $x_0 = 3.4$, you get the following:

$$x_1 = -8.9451; x_2 = -6.28878; x_3 = -4.38652; \dots\ x_n \to -1.96114$$

which is the negative root. In other words, you get a root, but not the root you wanted.

Major reasons why Newton's method fails

1. "overshooting" (as in Example 4) - caused by vertical tangency at the root
2. $f'(x_n)$ being equal to zero for some n (as in Example 5) - caused by horizontal tangency at x_n
3. periodicity in the sequence x_n (as in Example 6) - caused by "poor" or "unlucky" initial guess
4. getting an unexpected root (as in Example 7) - caused by having a point where $f'(x)$ is small too close to the root you want

Reasons for the failure of Newton's method can always be explained graphically.

8.4 Related rates

EXAMPLE 1

The volume of a spherical bubble is given by $V = \frac{4}{3}\pi r^3$ (where r is in mm and V is in mm^3). If the radius of the bubble is growing at a rate of 3 mm/sec, how fast is the volume growing when the radius is 2 mm?

This is an example of a **related rates** problem. Related rates problems are described as follows:

-

  (In the example, the radius and the volume are changing)

-

  (In the example, you are given the rate of change of the radius)

-

  (In the example, you are asked to compute the rate of change of the volume)

Conceptually, the key thing about solving related rates problems is to understand that there is a hidden variable in the problem:

Given information in the example:

To solve this problem, start with the equation relating V and r, and differentiate it with respect to t (i.e. take $\frac{d}{dt}$ of both sides):

General procedure to solve related rates problems:

1. Make sure the problem is a related rates problem (there should be 2 or more quantities changing that are related by an equation; you should be given the rate of change of one quantity and be asked for the rate of change of the other quantity).
2. Identify the quantities that are changing in the problem, and give them variable names. If necessary, draw a picture to help you do this. (Don't give variable names to quantities which don't change.)
3. Write the equation that relates the quantities from Step 2.
4. Take $\frac{d}{dt}$ of both sides of the equation from Step 3.
5. Plug in the values of the variables and solve for the unknown.

EXAMPLE 2

A 13-foot long ladder leans up against a wall. Someone comes along and pushes the bottom of the ladder toward the wall at a rate of 2 feet per second. How fast is the top of the ladder moving up the wall when the bottom of the ladder is 5 feet away from the wall?

EXAMPLE 3

A conical cup with height 12 in and radius 4 in is being filled with water at the rate of 6π in^3/sec. How fast is the water depth in the cone changing when the water is 6 in deep?

EXAMPLE 4

A wide receiver lines up 20 yards to the left of the center. He runs at a constant speed of 8 yards per second. The quarterback takes the snap, drops back 4 yards, and stops. The receiver runs downfield for 1 sec, then turns 90° left and runs toward the sideline. Let θ be the angle between the line from the center to the quarterback and the line from the quarterback to the receiver; find how fast θ is changing 1.5 seconds after the snap.

8.5 Homework exercises

In Problems 1-4 below, compute the linear approximation $L(x)$ to f at the given value of a:

1. $f(x) = \sqrt[5]{x}$, $a = 1$
2. $f(x) = \cot x$, $a = \pi/4$
3. $f(x) = \sin 3x$, $a = \pi$
4. $f(x) = xe^x$, $a = 0$

In Problems 5-8 below, compute the quadratic approximation $Q(x)$ to f at the given value of a:

5. $f(x) = x^{2/3}$, $a = 27$
6. $f(x) = 4\cos x$, $a = \pi$
7. $f(x) = \ln(x+1)$, $a = 0$
8. $f(x) = 3\sec x$, $a = 0$

In Problems 9-16 below, estimate the following quantities using tangent line approximation:

9. $\sqrt{50}$
10. $(8.1)^3$
11. $\ln(1.3)$
12. $e^{.2}$
13. $\sqrt[3]{66}$
14. $\sin(.2)$
15. $\arctan(1/3)$
16. $\cos(\frac{\pi}{2} + 1/8)$

17. Is the estimate you made in problem 9 an overestimate or an underestimate? Explain (without obtaining a decimal approximation to $\sqrt{50}$ using a computer or calculator).
18. Is the estimate you made in problem 10 an overestimate or an underestimate? Explain (without obtaining a decimal approximation to $(8.1)^3$ using a computer or calculator).

In Problems 19-22 below, estimate the following quantities using quadratic approximation:

19. $17^{3/2}$
20. $\cos \frac{1}{2}$
21. $e^{1/3}$
22. $\sqrt{150}$

23. After turning his gas grill on, a cook looks at the grill's internal temperature regularly, writing what he sees in the following table:

t (minutes after grill is lit)	0	1	2	4	5
$T(t)$ (temperature in $^\circ$F)	70	240	320	440	475

   a) Use linear approximation to estimate what the temperature of the grill will be 7 minutes after it is turned on.
   b) Use quadratic approximation to estimate what the temperature of the grill will be 7 minutes after it is turned on.
   c) Use the same quadratic approximation you computed in part (b) to estimate what the temperature of the grill will be 13 minutes after it is turned on.
   d) Does your answer to part (c) make sense? Explain.
   e) What about the procedure of quadratic approximation made our answer to part (c) so far off?

In Problems 24-27, compute the differential dy.

24. $y = 3x^2 - 4$
25. $y = x\sqrt{1 - x^2}$
26. $y = \arcsin x$
27. $y = e^{3x}$

28. a) Compute dy if $y = \frac{1}{2}x^3$, when $x = 2$ and $dx = .1$.
    b) Sketch a picture representing the computation done in part (a) of this problem, labelling x, dx and dy appropriately.

29. a) Compute dy if $y = 1 - x^4$, when $x = 1$ and $dx = .1$.
    b) Sketch a picture representing the computation done in part (a) of this problem, labelling x, dx and dy appropriately.

In problems 30-43, compute the indicated limit (indicating if the limit is $\pm\infty$ or does not exist):

30. $\lim\limits_{x \to 3} \frac{2x-6}{x^2-9}$
31. $\lim\limits_{x \to 0} \frac{\sqrt{4-x^2}-2}{x}$
32. $\lim\limits_{x \to 0} \frac{2e^x-2x-2}{x^2}$
33. $\lim\limits_{x \to 3} \frac{x^2-2x-3}{x-3}$
34. $\lim\limits_{x \to 2} \frac{x^2+10}{x+2}$
35. $\lim\limits_{x \to 0} \frac{\sin 4x}{\sin 9x}$

36. $\lim_{x\to\infty} \frac{3x^2-2x+1}{2x^2+3}$

37. $\lim_{x\to\infty} \frac{x^4}{e^{x/3}}$

38. $\lim_{x\to\infty} \frac{\ln x}{x^2}$

39. $\lim_{x\to\infty} \frac{e^x}{x^9}$

40. $\lim_{x\to\infty} \frac{2000x^{2014}}{e^x}$

41. $\lim_{x\to 2^+} \left(\frac{8}{x^2-4} - \frac{x}{x-2}\right)$

    Hint: Add the fractions by finding a common denominator.

42. $\lim_{x\to\infty} x^{1/x}$

    Hint: Follow the procedure of Example 11 on page 224.

43. $\lim_{x\to 0} x^x$

44. Approximate (by hand) a solution to $2x^3 + x^2 - x = -1$ by using Newton's method with initial guess $x = -1$ and two steps.

45. Approximate (by hand) a solution to $x^5 = 4$ by using Newton's method with initial guess $x = 1$ and two steps.

In Problems 46-49, use *Mathematica* to estimate a solution to the following equations using Newton's method; solutions should be correct to 4 decimal places:

46. $x^3 = 3$

47. $x^5 + x = 1$

48. $3\sqrt{x-1} = x$

49. $2x^3 = \cos x$

In Problems 50-52, use *Mathematica* to estimate all solutions to the following equations using Newton's method; solutions should be correct to 4 decimal places.

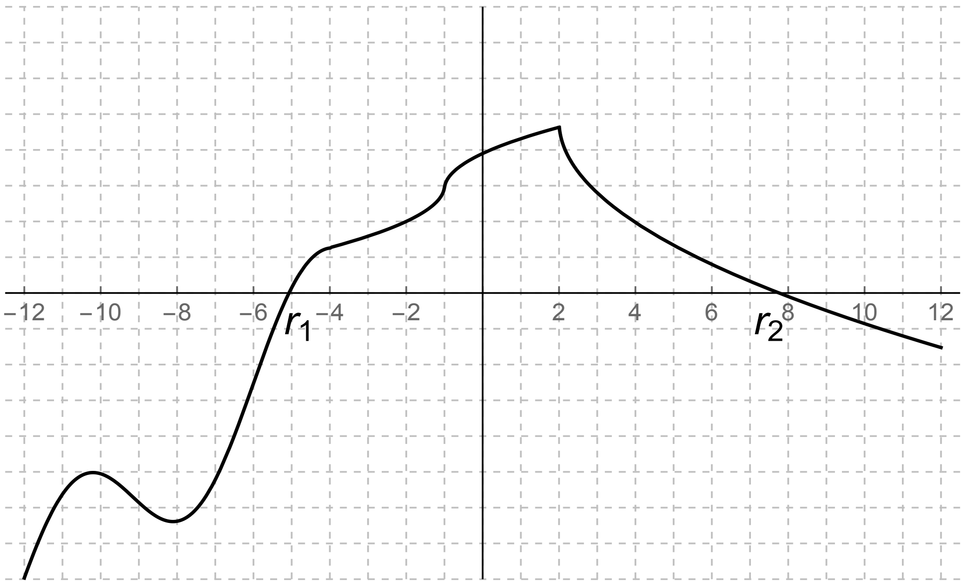
Hint: First, have *Mathematica* plot both functions on the same $xy-$plane; use the plot to determine the number of solutions to the equation. For each solution, run Newton's method with an initial guess close to the $x-$value of the appropriate solution.

50. $e^{x-5} = \ln x$

51. $\arctan 2x = x^2 - 1$

52. $6 \sin \frac{x}{6} = 8x - x^3$

In Problems 53-60, use the graph of some unknown function f shown here. As you can see, the equation $f(x) = 0$ has two solutions, r_1 (the negative one, near -5) and r_2 (the positive one, near 8).



53. Suppose you were to execute Newton's method for this function with initial guess $x = 4$. Will this produce an approximation to r_1 or r_2 (or will it not work at all)? Explain.

54. Suppose you were to execute Newton's method for this function with initial guess $x = -6$. Will this produce an approximation to r_1 or r_2 (or will it not work at all)? Explain.

55. Suppose you were to execute Newton's method for this function with initial guess $x = -10$ (assume that -10 is the $x-$coordinate of the "peak" of the function). Will this produce an approximation to r_1 or r_2 (or will it not work at all)? Explain.

56. Suppose you were to execute Newton's method for this function with initial guess $x = 3$. Will this produce an approximation to r_1 or r_2 (or will it not work at all)? Explain.

57. Suppose you were to execute Newton's method for this function with initial guess $x = -1$. Will this produce an approximation to r_1 or r_2 (or will it not work at all)? Explain.

58. Suppose you were to execute Newton's method for this function with initial guess $x = 2$. Will this produce an approximation to r_1 or r_2 (or will it not work at all)? Explain.

59. Suppose you were to execute Newton's method for this function with initial guess $x = -7.5$. Will this produce an approximation to r_1 or r_2 (assume that the slope at -7.5 is a very small positive number)? Explain.

60. Suppose you were to execute Newton's method for this function with initial guess $x = 10$. Will this produce an approximation to r_1 or r_2 (or will it not work at all)? Explain.

61. Attempt Newton's method on the function $f(x) = 4x^3 - 12x^2 + 12x - 3$ with initial guess $x = \frac{3}{2}$. Try lots of iterations.

    a) What happens?

    b) Sketch the graph of the function f using *Mathematica* and explain, via the graph, the phenomenon you observe in part (a).

62. If every edge of a cube is expanding at a rate of 2 cm/sec, how fast is the volume of the cube changing when each side is 12 cm long?

63. At a sand gravel plant, sand is falling off a conveyor onto a conical pile at a rate of 10 cubic feet per minute. The diameter of the base of the pile is roughly three times the altitude of the pile. At what rate is the height of the pile changing when the pile is 15 feet high?

64. The combined electrical resistance R of R_1 and R_2 connected in parallel is given by
$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$
where R, R_1 and R_2 are measured in ohms. If R_1 is increasing at the rate of 1 ohm per minute and R_2 is increasing at the rate of 1.5 ohms per minute, how fast is R changing when R_1 is 50 ohms and R_2 is 75 ohms?

65. An oil tanker has an accident and oil pours out of the tanker at a rate of 20 ft^3 per minute. If the oil spill is in the shape of a cylinder with thickness $\frac{1}{120}$ feet, determine the rate at which the radius of the oil spill is increasing when the radius is 500 feet.

66. A Ferris wheel is 50 ft in diameter and its center is located 30 feet above the ground. If the wheel rotates once every 2 minutes, how fast is a passenger rising when she is 42.5 feet above the ground?

    Hint: convert the rotational speed to radians per minute.

67. A rocket blasts off directly upward from the ground. If the height of the rocket at time t (measured in the number of seconds after lift-off) is $50t^2$ feet, find the rate of change of the angle of elevation to an observer who is 2000 feet

away from the launch pad, 10 seconds after lift-off. (The angle of elevation is measured between the ground and a line from the observer to the rocket.)

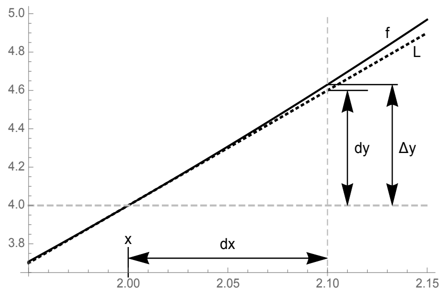
68. In the same situation as Problem 67, find the rate of change of the distance from the rocket to the observer (at the same instant as in Problem 67).

69. As a space shuttle moves into space, an astronaut's weight decreases. An astronaut weighing 650 N at sea level has a weight of $w = 650\left(\frac{6400}{6400+h}\right)$ at h kilometers above sea level. If the shuttle is moving away from Earth at 6 km/sec, at what weight is the astronaut's weight changing when $h = 1.2$ km?

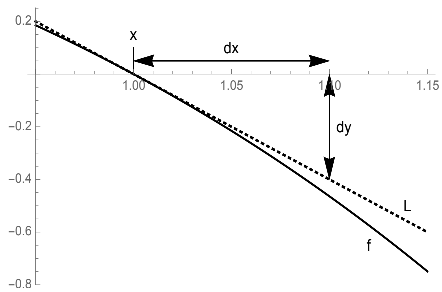
70. Two cars approach an intersection, one heading east toward the intersection at 30 km per hour and one heading south toward the intersection at 40 km per hour. At what rate are the cars approaching one another when the first car is $.1$ km from the intersection and the other is $.075$ m from the intersection?

Answers

1. $L(x) = 1 + \frac{1}{5}(x - 1)$
2. $L(x) = 1 - 2(x - \frac{\pi}{4})$
3. $L(x) = -3(x - \pi)$
4. $L(x) = x$
5. $Q(x) = 9 + \frac{2}{9}(x - 27) - \frac{1}{729}(x - 27)^2$
6. $Q(x) = -4 + 2(x - \pi)^2$
7. $Q(x) = x - \frac{x^2}{2}$
8. $Q(x) = 3 + \frac{3}{2}x^2$
9. $\frac{99}{14}$
10. 531.2
11. $.3$
12. 1.2
13. $\frac{97}{24}$
14. $.2$
15. $\frac{1}{3}$
16. $-1/8$
17. Overestimate, because $f''(49) < 0$.
18. Underestimate, because $f''(8) > 0$.
19. $\frac{2243}{32}$
20. $\frac{7}{8}$
21. $\frac{25}{18}$
22. $\frac{4703}{384}$
23. a) $545°$ F (answers may vary)
    b) $520°$ F (answers may vary)
    c) $355°$ F (answers may vary)
    d) No, because the grill should be hotter at time 13 than it was at time 5.
    e) Since we approximated using a parabola Q that opens downward, eventually Q starts to decrease. But the temperature T probably continues to increase; it is just that it increases at a slower rate.
24. $6x\,dx$
25. $\frac{\sqrt{1-x^2} - \frac{-2x^2}{2\sqrt{1-x^2}}}{1-x^2}\,dx = \frac{1}{(1-x^2)^{3/2}}\,dx$
26. $\frac{1}{\sqrt{1-x^2}}\,dx$
27. $3e^{3x}\,dx$
28. a) $.6$

b) 

29. a) $-.4$

b) 

30. $\frac{1}{3}$

31. 0

32. 1

33. 4

34. $\frac{7}{2}$ (this is not indeterminate)

35. $\frac{4}{9}$

36. $\frac{3}{2}$

37. 0

38. 0

39. ∞

40. 0

41. $\frac{-3}{2}$

42. 1

43. 1

44. $\frac{-235}{189}$

45. $\frac{35893}{25600}$

46. 1.44225

47. $.754878$

48. 7.8541

49. $.721406$

50. 1.01884 and 5.53738

51. $-.482303$ and 1.49966

52. -2.65184, 0 and 2.65184

53. r_2

54. r_1

55. won't work (since tangent line at $x = -10$ never hits $x-$axis)

56. r_2

57. won't work (tangent line at $x = -1$ is vertical)

58. won't work (function not differentiable at $x = 2$

59. r_2 (tangent line at -7.5 hits $x-$axis close to r_2)

60. r_2

61. a) Starting with the second iteration, you get infinity.
    b) If you sketch the picture associated to Newton's method, after the first iteration the tangent line is horizontal.

62. 864 cm^3/sec

63. $\frac{8}{405\pi}$ feet per minute

64. $\frac{3}{5}$ ohm per minute

65. $\frac{12}{5\pi}$ feet per minute

66. $\frac{25\sqrt{3}\pi}{2} \approx 68.018$ ft / min.

67. $\frac{2}{29}$ radians per sec

68. $\frac{5000}{\sqrt{29}}$ feet per sec

69. $-.432$ N/sec

70. 48 km/hour

8.6 Review problems for Exam 3

Note: There are no questions on this exam involving *Mathematica* syntax.

Questions from Chapter 6

1. Find all the critical points of the following functions:

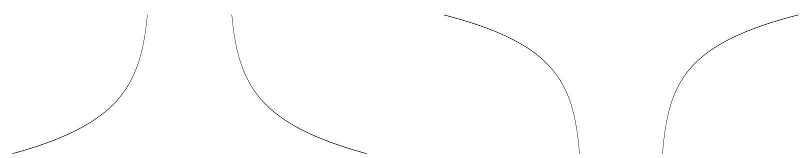
$$f(x) = \frac{2}{3}x^3 + 4x^2 - 24x + 2 \qquad\qquad f(x) = x^{8/3} - x^{2/3}$$

2. Suppose $f(x) = x + \frac{1}{x^2}$.

   a) Determine, with justification, whether f is increasing or decreasing on the interval $(0, 1)$.

   b) Determine, with justification, whether f is concave up or concave down on the interval $(0, 1)$.

   c) Based on your answers to (a) and (b), select the shape below which best represents the graph of f on the interval $(0, 1)$.



3. Find the absolute maximum value and absolute minimum value of the function $f(x) = \frac{x-1}{x^2+8}$ on the interval $[-5, 5]$.

4. Find the absolute maximum value and absolute minimum value of the function $f(x) = 8 - 12x - 3x^2$ on the interval $[0, 2]$.

5. Suppose that a company's profit if they make x units of their product is $P(x) = 20000 + 4000x - x^2$ dollars.

   a) Assuming they have the capability to make up to 1000 units, what is the maximum profit they can make? How many units should they make to ensure this profit?

   b) Assuming they have the capability to make 3000 units, what is the maximum profit they can make? How many units should they make to ensure this profit?

6. A box with a square base must have a surface area of 150 square units. Find the dimensions of the box, if the box is to be as large as possible.

7. Find the point on the curve $y = x^2$ which is closest to $(0, 3)$.

8. A sheet of paper of size 10 inches by 10 inches is made into a box by cutting squares out from each of the four corners of the paper and folding the tabs up to make a box. If the volume of the box is to be as big as possible, what size squares should be cut out from the box?

Questions from Chapter 8

9. Compute the linear approximation $L(x)$, and the quadratic approximation $Q(x)$, to $f(x) = e^{2x}$ when $a = 0$.

10. Approximate the following quantities using tangent line approximation:

$$(1.25)^{10} \qquad \sqrt{124}$$

Determine, with justification, whether each of these approximations are over-estimates or underestimates.

11. Approximate $\cos(1/5)$ using quadratic approximation.

12. Evaluate each of the following limits:

$$\lim_{x\to 3} \frac{x^3 - 27}{x^2 - 9} \qquad \lim_{x\to 0} \frac{\ln(x+1)}{x} \qquad \lim_{x\to\infty} \frac{\sqrt{9x^4 - 1}}{x^2 + 3}$$

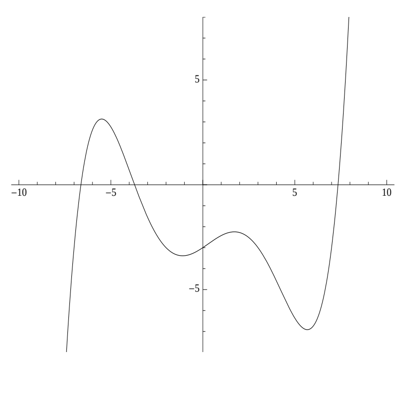
13. Evaluate each of the following limits:

$$\lim_{x\to\infty} xe^{-x} \qquad \lim_{x\to 0^+} (x^2+1)^{\ln x}$$

14. Use Newton's method, with 2 steps and initial guess 1, to approximate a solution of the equation $x^4 = 2x + 1$.

15. Suppose you are trying to find a solution of the equation $\ln x + x = 0$ and that after three steps of Newton's method, you have obtained $x_3 = 1$. Compute x_4.

16. Suppose the roots of the following function are labelled, from left to right, as r_1, r_2 and r_3.



a) If you implement Newton's method with initial guess 5, which of the three roots will you get? Explain.

b) Suppose you implement Newton's method with initial guess 0. Estimate the value of x_1 from looking at the graph.

Answers (with some comments)

1. For the first function, $x = -6$ and $x = 2$.

   For the second function, $x = \frac{1}{2}$, $x = \frac{-1}{2}$, $x = 0$ (0 comes from setting the denominator of the derivative equal to zero).

2. a) $f'(x) = 1 - \frac{2}{x^3} < 0$ on $(0, 1)$, so f is decreasing on $(0, 1)$.

   b) $f''(x) = \frac{6}{x^4} > 0$, so f is concave up on $(0, 1)$.

   c) The graph which is decreasing and concave up is the second one from the left.

3. The derivative is $f'(x) = \frac{1(x^2+8)-2x(x-1)}{(x^2+8)^2} = \frac{-x^2+2x+8}{(x^2+8)^2}$; the denominator of this derivative is never zero but the derivative is zero when $x = -2, x = 4$. Testing these critical points and the endpoints, we see the absolute maximum is $1/8$ at $x = 4$ and the absolute minimum is $-1/4$ at $x = -2$.

4. The only critical point is where $f'(x) = 0$ which is at $x = -2$, which you discard since it isn't in $[0, 2]$. The absolute maximum is 8 at $x = 0$ and the absolute minimum is -28 at $x = 2$.

5. a) Their largest profit is 3020000, if they make 1000 units. (Throw out the critical point 2000 since it is not in $[0, 1000]$.)

   b) Their largest profit is 4020000, if they make 2000 units.

6. The surface area is $2x^2 + 4xh = 150$; solve this constraint for h to get $h = \frac{150-2x^2}{4x}$. We want to maximize $V = x^2h = x^2\frac{150-2x^2}{4x} = \frac{1}{4}(150x - 2x^3)$. Set $V'(x) = 0$ to obtain $x = 5$. Solve for h to get $h = 5$ as well; the dimensions are therefore $5 \times 5 \times 5$.

7. Minimize the square of the distance from the point (x, x^2) to $(0, 3)$; the square of this distance is $f(x) = (x-0)^2 + (x^2-3)^2 = x^4 - 5x^2 + 9$. Take the derivative to get $f'(x) = 4x^3 - 10x$; set this equal to zero and solve for x to get $x = \sqrt{\frac{5}{2}}$. The corresponding $y-$value is $\frac{5}{2}$, so the point is $\left(\sqrt{\frac{5}{2}}, \frac{5}{2}\right)$.

8. Once you fold the box, the dimensions of the box are length $10 - 2x$, width $10 - 2x$ and height x so the volume is $V(x) = x(10 - 2x)^2 = 4x^3 - 40x^2 + 100x$. We need to maximize this on $[0, 5]$. Now $V'(x) = 12x^2 - 80x + 100 = (12x -$

$20)(x-5)$. So the critical points are $x = 5$ and $x = \frac{5}{3}$; the maximum is at $x = \frac{5}{3}$, so the squares you need to cut out are $\frac{5}{3} \times \frac{5}{3}$.

9. $L(x) = 1 + 2x$; $Q(x) = 1 + 2x + 2x^2$

10. $(1.25)^{10} \approx 3.5$ (use $f(x) = x^{10}$ and $a = 1$); $\sqrt{124} \approx 11 + \frac{3}{22}$ (use $f(x) = \sqrt{x}$ and $a = 121$). The first answer is an underestimate since $f''(1) = 90 > 0$ and the second answer is an overestimate since $f''(121) < 0$.

11. $\cos(1/5) \approx 1 - \frac{1}{2}\left(\frac{1}{5}\right)^2 = \frac{49}{50}$.

12. $$\lim_{x\to 3} \frac{x^3 - 27}{x^2 - 9} = \frac{9}{2}; \qquad \lim_{x\to 0} \frac{\ln(x+1)}{x} = 1; \qquad \lim_{x\to\infty} \frac{\sqrt{9x^4 - 1}}{x^2 + 3} = 3$$

13. For the first limit, rewrite it as $\lim\limits_{x\to\infty} \frac{x}{e^x}$ then use L'Hôpital's Rule. The answer works out to be 0.

    For the second limit, rewrite it as $\lim\limits_{x\to 0^+} e^{\ln x \ln(x^2+1)}$ and compute the limit of the term inside the exponent by rewriting it as $\frac{\ln(x^2+1)}{\frac{1}{\ln x}}$ and using L'Hôpital's Rule. Eventually, the limit of the term inside the exponent is 0 so the original limit works out to $e^0 = 1$.

14. $\frac{49}{30}$ (I get $x_1 = 2$ along the way.)

15. Apply Newton's method once to get $x_4 = \frac{1}{2}$.

16. a) r_2, since the tangent line at $x = 5$ intercepts the $x-$axis near r_2.

    b) It looks like the tangent line at $x = 0$ intercepts the $x-$axis near 7, so $x_1 \approx 7$.

Chapter 9

Theory of the Definite Integral

9.1 Motivating problems: area and displacement

Recall that to define the derivative of a function, we started with a real-world problem we wanted to solve:

Then, we approximated the solution to that problem (by finding the slope of some secant line):

Next, we observed how the approximation got better:

This told us how to define the answer to the problem (using a limit):

(In principle, we don't use this definition to compute derivatives; we use rules like the Power Rule, Product Rule, Chain Rule, etc.)

For the rest of the semester, we will consider two new classes (actually only one class) of real-world problems.

We need to define a new mathematical object which will solve these problems.

To create this new object, we will:
1. Approximate the answer to the problem.
2. Observe how the approximation gets better.
3. Define the answer to the problem using a limit.

What are the two new classes of real-world problems?

1.

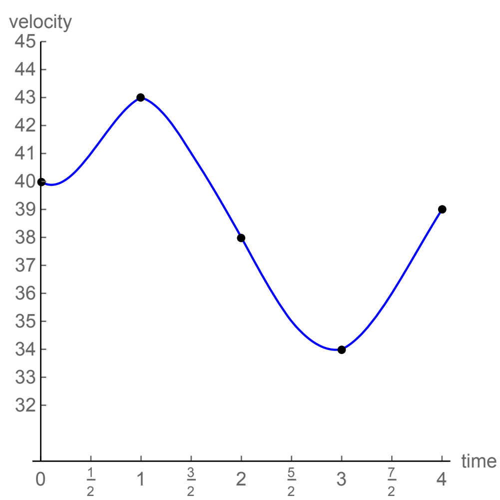
2.

First remark: Problems (1) and (2) above are really the same problem in disguise. Why?

Suppose you are in a car and you look at the speedometer once an hour:

(hr)	t	0	1	2	3
(mph)	$v(t)$	40	43	38	34

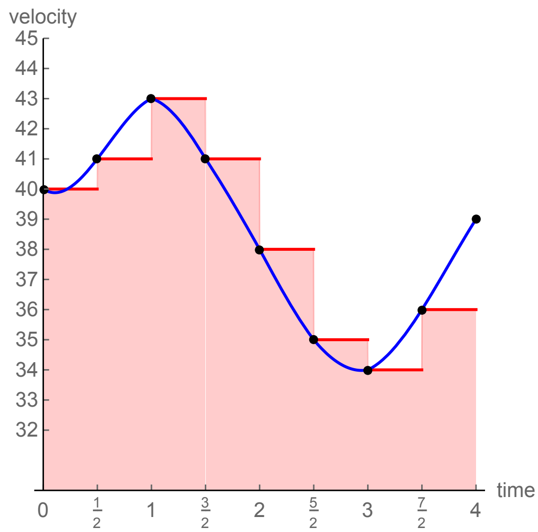
How far do you travel from $t = 0$ to $t = 4$ (i.e. what is your **displacement** from time 0 to time 4)?



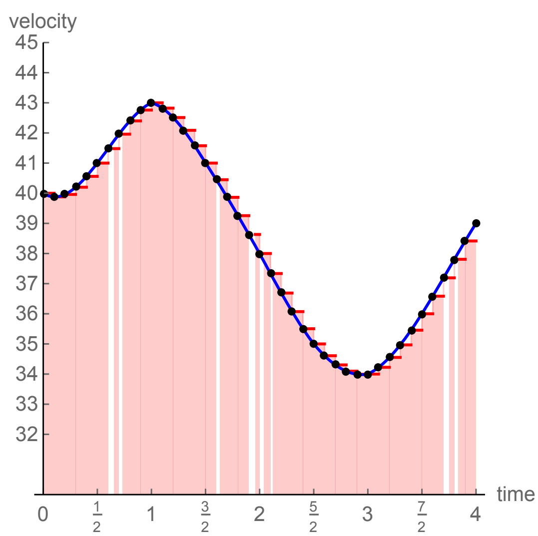
This is just an approximation. How might the approximation improve?

Suppose you look at the speedometer every 30 minutes:

(hr)	t	0	1/2	1	$\cdots$
(mph)	$v(t)$	40	41	43	$\cdots$



Take more and more measurements:



This suggests:

$$\begin{array}{c}\text{displacement of an object}\\ \text{from } t=0 \text{ to } t=4,\\ \text{given velocity function } v(t)\end{array} \quad = \quad \begin{array}{c}\text{area under the graph of } v\\ \text{from } t=0 \text{ to } t=4\end{array}$$

More generally: Suppose an object's position at time t is given by function $f(t)$. Then its displacement from time $t = a$ to time $t = b$ is $f(b) - f(a)$.

At the same time, its velocity at time t is given by $f'(t)$, and the displacement from time a to time b is equal to the area under the graph of f' from $t = a$ to $t = b$. Putting this together, we have the following important idea:

area under the graph of f' from $t = a$ to $t = b$	$=$	$f(b) - f(a)$

This means: the problems of finding the area between the graph of a function and the $x-$axis, and the problem of finding displacement given velocity, are really the same problem. The process that solves these problems is probably something like "differentiation in reverse".

EXAMPLE 1

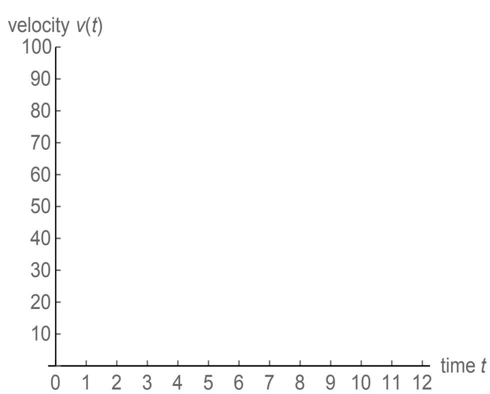
Suppose that the velocity (in m/sec) of a bird at time t (in seconds) is given by $v(t) = \frac{2}{3}t + \frac{4}{3}$. Find the distance travelled by the bird between time 0 and time 6.

EXAMPLE 2

In each situation A through D described below:

1. Based on the description given, sketch a graph of the velocity, plotted against time.
2. Determine how far you travel between time $t = 0$ and $t = 3$ (throughout this assignment, t is in hours).
3. Determine how far you travel between times $t = 5$ and $t = 9$.
4. Without being given any other information, do you know what your odometer reading is at time $t = 4$? If so, what is it?
5. Without being given any other information, do you know what your odometer reading is at time $t = 8$? If so, what is it?
6. Suppose your odometer reading at time $t = 0$ is 0. Now, do you know the odometer readings at time 8? If so, what is it?
7. Suppose your odometer reading at time $t = 0$ is 10000. Now, do you know the odometer readings at time 8? If so, what is it?
8. Suppose your odometer reading at time $t = 0$ is C, where C is an arbitrary constant. What is the odometer reading at time 4? What is the odometer reading at time 8?

Situation A: Assume that the velocity at all times is 60 miles per hour.



Displacement from $t = 0$ to $t = 3$:

Displacement from $t = 5$ to $t = 9$:

Odometer reading at $t = 4$:

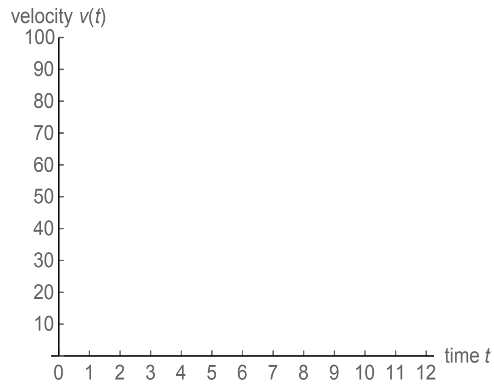
Odometer reading at $t = 8$:

If initially 0, odometer reading at $t = 8$:

If initially 10000, odometer reading at $t = 8$:

If initially C, odometer reading at $t = 8$:

Situation B: Assume that the velocity is 50 miles per hour for the first six hours, then 80 miles per hour at all times after the first six hours.



Displacement from $t = 0$ to $t = 3$:

Displacement from $t = 5$ to $t = 9$:

Odometer reading at $t = 4$:

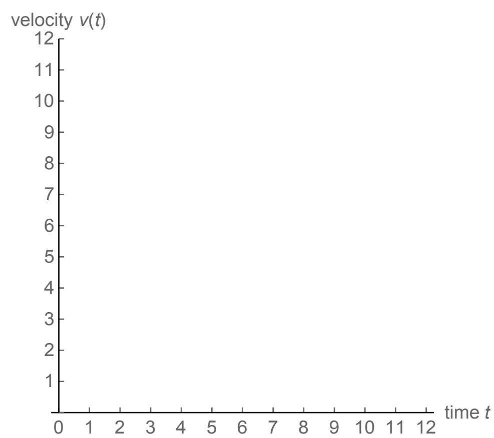
Odometer reading at $t = 8$:

If initially 0, odometer reading at $t = 8$:

If initially 10000, odometer reading at $t = 8$:

If initially C, odometer reading at $t = 8$:

Situation C: Assume that the velocity at time x is equal to x.



Displacement from $t = 0$ to $t = 3$:

Displacement from $t = 5$ to $t = 9$:

Odometer reading at $t = 4$:

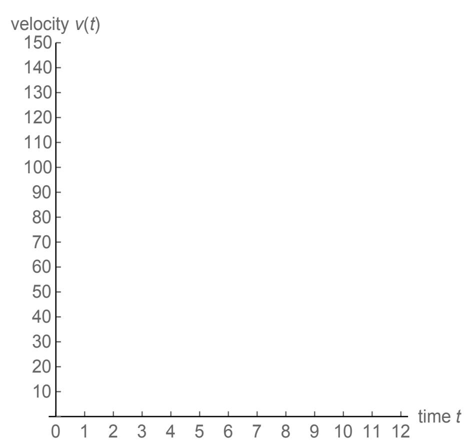
Odometer reading at $t = 8$:

If initially 0, odometer reading at $t = 8$:

If initially 10000, odometer reading at $t = 8$:

If initially C, odometer reading at $t = 8$:

Situation D: Assume that the velocity at time x is equal to x^2.



Displacement from $t = 0$ to $t = 3$:

Displacement from $t = 5$ to $t = 9$:

Odometer reading at $t = 4$:

Odometer reading at $t = 8$:

If initially 0, odometer reading at $t = 8$:

If initially 10000, odometer reading at $t = 8$:

If initially C, odometer reading at $t = 8$:

Concepts illustrated in the preceding example

- At the beginning of the semester, we discussed the "big picture" problem of converting from a function which represents an odometer to a function which represents a speedometer. The operation we eventually cooked up to do this is **differentiation**. In other words:

$$\begin{array}{rcl} \text{ODOMETER} & \xrightarrow{\text{DERIVATIVE}} & \text{SPEEDOMETER} \\ \text{POSITION} & \xrightarrow{\text{DERIVATIVE}} & \text{VELOCITY} \\ f(x) & \xrightarrow{\text{DERIVATIVE}} & f'(x) \end{array}$$

- Now, we are looking at the same problem in the other direction. That is, we want to assume we are given a speedometer (i.e. a function that represents velocity), and we want to determine the function that was the odometer:

$$\begin{array}{rcl} \text{ODOMETER} & \longleftarrow & \text{SPEEDOMETER} \\ \text{POSITION} & \longleftarrow & \text{VELOCITY} \\ ? & \xleftarrow{?} & f(x) \end{array}$$

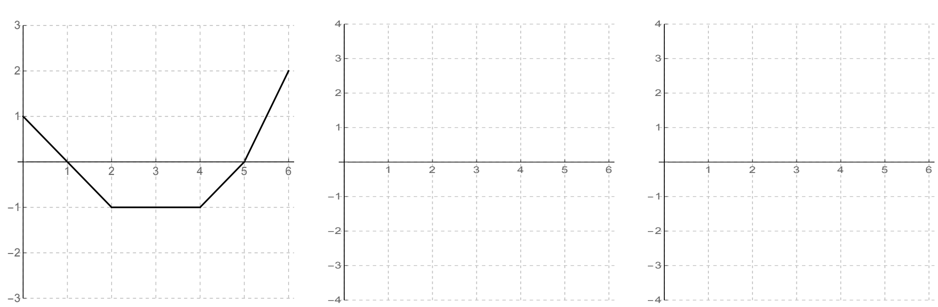
- If you are given a function f which represents your velocity, then you **cannot** use f by itself to determine your odometer reading at a certain time (because you didn't know what the original odometer reading was).

- But, if you are given a function f which represents your velocity **and you are given an initial odometer reading** (a.k.a. the odometer reading at time a), then you can determine your odometer reading at any time t by the formula

$$\begin{array}{c} \text{odometer reading} \\ \text{at time } t \end{array} = \begin{array}{c} \text{original odometer reading} \\ + \\ \text{area under velocity function} \\ \text{from time } a \text{ to time } t \end{array}$$

EXAMPLE 3

The graph of some function f' is given below at left. If $f(0) = 2$, sketch the graph of f on the middle axes. On the right-hand axes, sketch all possible graphs of f (if you don't know $f(0)$).



9.2 Riemann sums

Summation notation

Suppose a_k is some expression which can be computed in terms of k. (a_k is like $a(k)$.) For example, if $a_k = k^2 + k$, then

$$a_1 = 1^2 + 1 = 2 \qquad a_2 = 2^2 + 2 = 6 \qquad a_3 = 3^2 + 3 = 12 \qquad \text{etc.}$$

Frequently in mathematics we want to **add** together values of a_k where k ranges over some set. For example, we might want to add up

$$a_2 + a_3 + a_4 + a_5 + ... + a_{20}.$$

We use the following notation to represent this kind of addition:

Definition 9.1 *Given numbers a_1, a_2, ... the* **sum from $k = 1$ to n of a_k** *is*

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 ... + a_n.$$

(More generally, $\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + ... + a_n$.)

EXAMPLE 1

Write the expression $\frac{3^2}{2} + \frac{4^2}{2} + \frac{5^2}{2} + ... + \frac{17^2}{2}$ in Σ-notation.

EXAMPLE 2

Compute $\sum_{k=0}^{3} \frac{2}{k+1}$.

Approximating the area under a function

Idea: Approximate the area under a function by finding the total area of some rectangles.

> **Definition 9.2** *Given an interval $[a, b]$, a* **partition** $\mathcal{P}$ *is a (finite) list of numbers* $\{x_0, x_1, x_2, ..., x_n\}$ *such that* $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$*. Such a partition divides* $[a, b]$ *into* n **subintervals***; the* k^{th} **subinterval** *is* $[x_{k-1}, x_k]$*. For each* k*, set* $\Delta x_k = x_k - x_{k-1}$*;* Δx_k *is called the* **width** *of the* k^{th} *subinterval. Call the largest* Δx_k *the* **norm** *of the partition; denote the norm by* $||\mathcal{P}||$*.*

EXAMPLE 3

$a = 0$; $b = 1$; $\mathcal{P} = \{0, \frac{1}{4}, \frac{3}{4}, \frac{7}{8}, 1\}$.

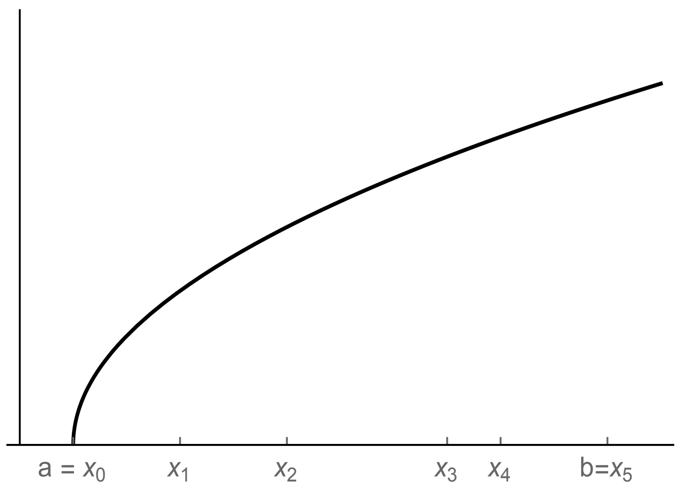
EXAMPLE 4

$\mathcal{P} =$ partition of $[a, b]$ into n equal-length subintervals.

Definition 9.3 *Given function $f : [a, b] \to \mathbb{R}$ and given partition $\mathcal{P} = \{x_0, ..., x_n\}$ of $[a, b]$, a* **Riemann sum** *associated to $\mathcal{P}$ for f is any expression of the form*

$$\sum_{k=1}^{n} f(c_k)\Delta x_k$$

where for all k, c_k belongs to the k^{th} subinterval of $\mathcal{P}$. The points $c_1, c_2, ..., c_n$ are called test points *for the Riemann sum.*



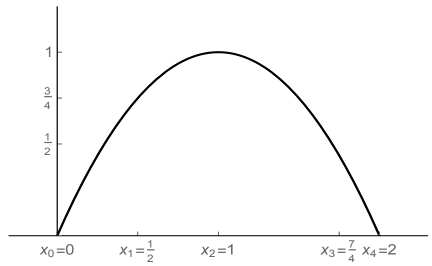
A Riemann sum approximates the area under $f(x)$ from $x = a$ to $x = b$ by adding up areas of rectangles as above. Different choices of $\mathcal{P}$ and different choices of c_k (even for the same $\mathcal{P}$) give different Riemann sums.

In the following examples, $f(x) = 2x - x^2$, $[a, b] = [0, 2]$, and $\mathcal{P} = \{0, \frac{1}{2}, 1, \frac{7}{4}, 2\}$. Therefore:

$$\begin{cases} \Delta x_1 = x_1 - x_0 = \frac{1}{2} - 0 = \frac{1}{2} \\ \Delta x_2 = x_2 - x_1 = 1 - \frac{1}{2} = \frac{1}{2} \\ \Delta x_3 = x_3 - x_2 = \frac{3}{4} \\ \Delta x_4 = \frac{1}{4}. \end{cases}$$

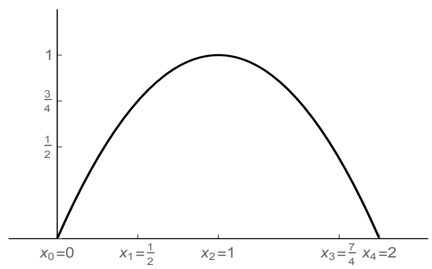
By choosing different test points, we get different Riemann sums for this partition. On the next page, we describe some specific kinds of Riemann sums one should know how to compute:

1. **Left sum:** choose $c_k = x_{k-1} =$ left endpoint of k^{th} subinterval.



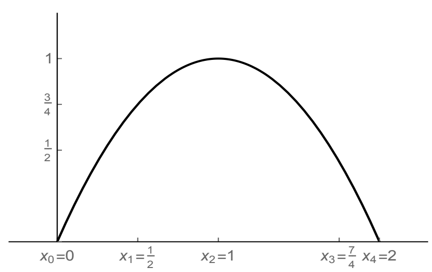
$$\sum_{k=1}^{4} f(c_k)\Delta x_k =$$

2. **Right sum:** choose $c_k = x_k =$ right endpoint of k^{th} subinterval.



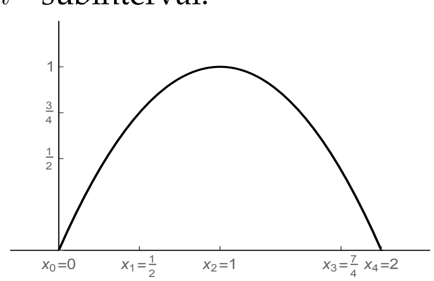
$$\sum_{k=1}^{4} f(c_k)\Delta x_k =$$

3. **Upper sum:** choose c_k to be the $x-$value corresponding to abs max of f on k^{th} subinterval.



$$\sum_{k=1}^{4} f(c_k)\Delta x_k =$$

4. **Lower sum:** choose c_k to be the $x-$value corresponding to abs min of f on k^{th} subinterval.



$$\sum_{k=1}^{4} f(c_k)\Delta x_k =$$

Note: The actual area under a function f is always greater than any lower sum, and less than any upper sum (no matter the choice of $\mathcal{P}$).

Note: If f is increasing, then

Note: If f is decreasing, then

EXAMPLE

Estimate the area under $f(x) = x^3$ from $x = 0$ to $x = 1$ by using a lower sum for a partition into 6 subintervals of equal length.

This sum works out to be

$$0\left(\frac{1}{6}\right) + \frac{1}{216}\left(\frac{1}{6}\right) + \frac{1}{27}\left(\frac{1}{6}\right) + \frac{1}{8}\left(\frac{1}{6}\right) + \frac{8}{27}\left(\frac{1}{6}\right) + \frac{125}{216}\left(\frac{1}{6}\right) = \frac{25}{144}.$$

9.3 Definition of the definite integral

In the last section: we approximated the area under f from a to b by the Riemann sum

$$\sum_{k=1}^{n} f(c_k)\Delta x_k$$

Next task:

Key observation: As $||\mathcal{P}|| \to 0$, the rectangles under the graph of f get skinnier and skinnier, so the corresponding Riemann sum estimates become more and more exact. So

$$\lim_{||\mathcal{P}|| \to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k$$

should give the exact area under the graph. This motivates the following definition:

Definition 9.4 (Limit definition of the integral) *Given function $f : [a, b] \to \mathbb{R}$, the* **definite integral** *of f from a to b is*

$$\int_a^b f(x)\, dx = \lim_{||\mathcal{P}|| \to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k$$

if this limit exists (in Math 220 and Math 230, it always will). If the limit exists, we say f is **integrable** *on $[a, b]$.)*

Notation:

Two ways to think about the integral:

1. The definite integral is "continuous addition of areas of rectangles of infinitely small width".
2. The definite integral is "accumulation" of values of f from $x = a$ to $x = b$.

Some integrals can be computed without doing any sophisticated calculus:

EXAMPLE 1

Evaluate the following definite integrals:

1. $\displaystyle\int_4^7 5\,dx$

2. $\displaystyle\int_4^8 \frac{1}{4}x\,dx$

3. $\displaystyle\int_{-3}^3 \sqrt{9-x^2}\,dx$

4. $\displaystyle\int_{-2}^1 (3-|x|)\;dx$

Evaluating an integral using the definition of integral is much harder:

EXAMPLE 2

Compute

$$\int_0^1 x\,dx$$

using the limit definition of the definite integral.

Aside: The answer is

Computation using the limit definition:

$$\int_0^1 x\,dx = \lim_{||\mathcal{P}||\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k$$

We need to choose partitions whose norm $\to 0$,
and we need to choose a type of Riemann sum.
I will choose a partition into n equal-length subintervals,
and compute a left-hand Riemann sum, for simplicity.

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(0 + k\left[\frac{1-0}{n}\right]\right) \cdot \left(\frac{1-0}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \cdot \frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n} \cdot \frac{1}{n}$$

$$= \lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n} k$$

$$= \lim_{n\to\infty} \frac{1}{n^2}(1 + 2 + 3 + \ldots + n).$$

Now the question is, what is $1 + 2 + 3 + \ldots + n$?

From the previous page,

$$\int_0^1 x\,dx = \lim_{n\to\infty} \frac{1}{n^2}(1+2+3+...+n)$$

$$= \lim_{n\to\infty} \frac{1}{n^2}\cdot\frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty} \frac{n^2+n}{2n^2}$$

$$= \lim_{n\to\infty} \frac{n+1}{2n}$$

$$\stackrel{L}{=} \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}.$$

This example was very hard (even though the integrand was very simple). This suggests that computing integrals like

$$\int_0^\pi \sin x\,dx \quad \text{or} \quad \int_1^4 x^5\,dx$$

using the definition of definite integral is impossible. We need another method, which we will discuss in Section 9.5.

9.4 Elementary properties of Riemann integrals

Theorem 9.5 *All continuous functions are integrable.*

Definition 9.6 *Let $a < b$ and let $f : [a, b] \to \mathbb{R}$ be integrable. Then*

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Definition 9.7 *Let $f : [a, b] \to \mathbb{R}$. Then $\int_a^a f(x)\,dx = 0$.*

Theorem 9.8 (Linearity properties of integrals) *Let f and g be integrable; let k be any constant. Then:*

1. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$;
2. $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$;
3. $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$.

WARNING: integrals are **not** multiplicative nor divisive:

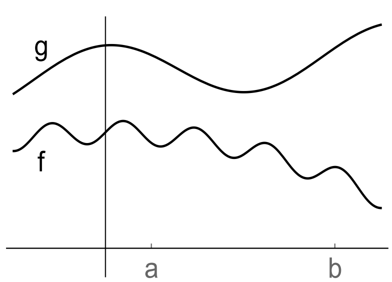
$$\int_a^b [f(x)g(x)]\,dx \neq \left[\int_a^b f(x)\,dx\right]\left[\int_a^b g(x)\,dx\right]$$

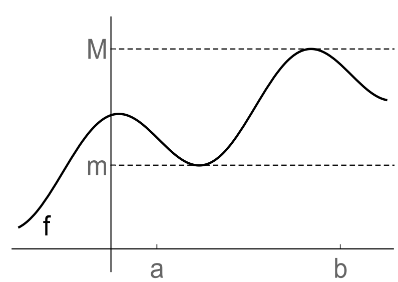
$$\int_a^b \frac{f(x)}{g(x)}\,dx \neq \frac{\int_a^b f(x)\,dx}{\int_a^b g(x)\,dx}$$

Theorem 9.9 (Inequality properties of integrals) *Let f and g be integrable. Then:*

1. **(Positivity Law)** *If $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\,dx \geq 0$.*
2. **(Monotonicity Law)** *If $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.*
3. **(Max-Min Inequality)** *Let m and M be the absolute min value and absolute max value of f on $[a, b]$, respectively. Then*

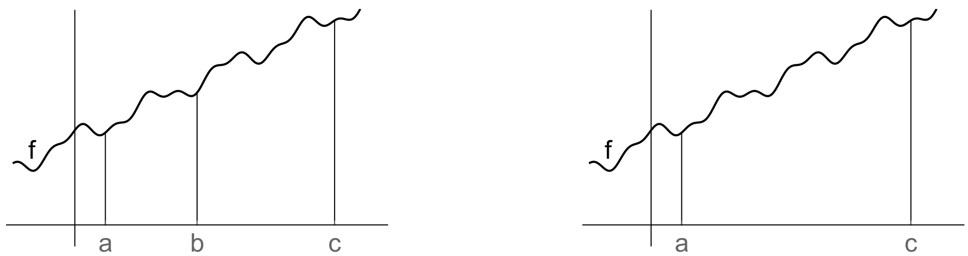
$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$



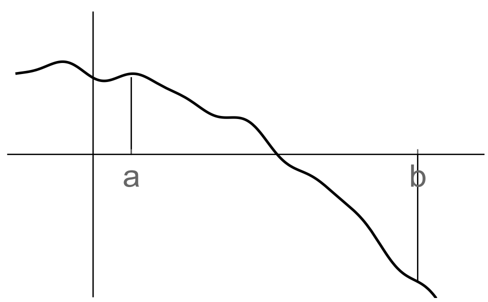


Theorem 9.10 (Additivity property of integrals) *Suppose f is integrable. Then for any numbers a, b and c,*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



Note: It is possible for integrals to be negative (so integrals actually compute something called "signed area"):



EXAMPLE 1

Suppose f and g are functions such that

$$\int_3^7 f(x)\,dx = 6 \quad \int_7^8 f(x)\,dx = 4 \text{ and } \int_3^7 g(x)\,dx = 2.$$

1. Find $\int_3^8 f(x)\,dx$.

2. Find $\int_7^3 f(x)\,dx$.

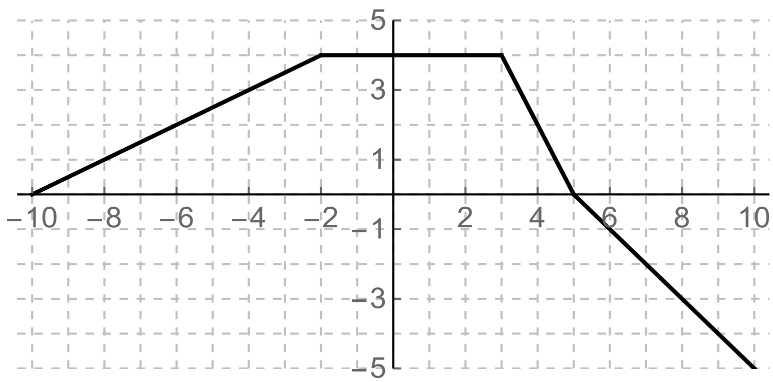
3. Find $\int_4^4 f(x)\,dx$.

4. Find $\int_3^7 [4f(x) + 5g(x)]\,dx$.

5. Find $\int_3^7 [f(x) + 2x]\,dx$.

EXAMPLE 2

Here is the graph of some unknown function f:



Use the graph to estimate the answers to the following integrals:

1. $\int_{-1}^{2} f(x)\,dx$

2. $\int_{-6}^{-4} 10f(x)\,dx$

3. $\int_{5}^{3} f(x)\,dx$

4. $\int_{5}^{7} f(x)\,dx$

5. $\int_{-2}^{-2} f(x)\,dx$

9.5 Fundamental Theorem of Calculus

Recall from Section 9.3 that it is virtually impossible to compute integrals using the limit definition. So, in order to compute integrals, we need some new ideas. The theory that follows is motivated by the idea from page 252, which suggests that

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

Based on this idea, to evaluate an integral like

$$\int_1^4 x^5\,dx,$$

we should think of x^5 as $f'(x)$ and try to find $f(x)$. Just by "guessing" (for now), we see that $f(x) = \frac{1}{6}x^6$ works. So if we let $f(x) = \frac{1}{6}x^6$, we have

$$\int_1^4 x^5\,dx = \int_1^4 f'(x)\,dx = f(4) - f(1) = \frac{1}{6}4^6 - \frac{1}{6}1^6 = \frac{1365}{2}.$$

In this section we justify that this idea works in general. To do this, we need some new terminology:

Definition 9.11 *Given function f, an* **antiderivative** *of f is a function F such that $F' = f$.*

EXAMPLES

$F(x) = x^2 - 3$ is an antiderivative of $f(x) = 2x$.

$F(x) = x^2$ is an antiderivative of $f(x) = 2x$.

$F(x) = x^2 + C$ is an antiderivative of $f(x) = 2x$ for any constant C.

Question: Are there any other antiderivatives of $f(x) = 2x$?

Theorem 9.12 (Antiderivative Theorem) *Suppose F and G are both antiderivatives of the same function f. Then, for all x, $F(x) = G(x) + C$.*

Proof: Let $H(x) = F(x) - G(x)$. Then

$$H'(x) = F'(x) - G'(x) = f(x) - f(x) = 0.$$

so H is a function whose derivative is everywhere zero. That means H have slope zero, so it be a horizontal line, i.e. must be a constant function (this seems obvious, but is actually very deep - take Math 430 (Advanced Calculus) to see how to prove this).

Thus $H(x) = F(x) - G(x) = C$ so $F(x) = G(x) + C$ where C is constant. □

Remark: The point of the Antiderivative Theorem is that any two antiderivatives of the same function must differ by at most a constant.
(So there are no other antiderivatives of $f(x) = 2x$ other than $F(x) = x^2 + C$.)

Essentially this means that if you have found one antiderivative of a function, you have found them all (by adding an arbitrary constant).

Theorem 9.13 (Fundamental Theorem of Calculus I) ***(Differentiation of Integrals)*** *Let f be continuous on $[a, b]$. Consider a new function*

$$F(x) = \int_a^x f(t)\,dt.$$

Then:

1. *F is cts and diffble on $[a, b]$; and*
2. *$F'(x) = f(x)$.*

Picture:

Physical interpretation:

Mathematical significance of this part of the FTC:

1. The FTC reveals that differentiation and integration are inverse operations (because it says that if you start with a function f, take its integral (to get F) and then take the derivative of that, you get back to the function f that you started with).

2. The FTC guarantees that every continuous function has an antiderivative: given function $f(x)$, the function $F(x) = \int_a^x f(t)\,dt$ is an antiderivative of f for any choice of a.

Proof of FTC Part I: By the definition of derivative,

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right) \\ . &= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt. \end{aligned}$$

Now by the Max-Min Inequality, by letting m and M be the minimum and maximum values of f on $[x, x+h]$, we have

$$\begin{aligned} m(x+h-x) &\le \int_x^{x+h} f(t)\,dt \le M(x+h-x) \\ \Rightarrow mh &\le \int_x^{x+h} f(t)\,dt \le Mh \\ \Rightarrow m &\le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le M \end{aligned}$$

As $h \to 0$, m and M both go to $f(x)$, so the inside quantity must go to $f(x)$ as well, i.e.

$$F'(x) = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x). \ \square$$

Now for our last big theorem. Remember that the goal is to develop a method to evaluate integrals that doesn't use the limit definition. We are now able to achieve this goal:

Theorem 9.14 (Fundamental Theorem of Calculus Part II) ***(Evaluation of Integrals)*** *Let f be continuous on $[a, b]$. Suppose F is **any** antiderivative of f. Then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Notation: The expression $F(b) - F(a)$ is written $[F(x)]_a^b$ or $F(x)|_a^b$.

Proof: Let $G(x) = \int_a^x f(t)\,dt$. $G'(x) = f(x)$ by the first part of the Fundamental Theorem of Calculus. By the Antiderivative Theorem, if F is any antiderivative of f, $F(x) = G(x) + C$. Then

$$\begin{aligned} F(b) - F(a) &= G(b) + C - (G(a) + C) \\ &= G(b) - G(a) \\ &= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt \\ &= \int_a^b f(t)\,dt - 0 \\ &= \int_a^b f(t)\,dt \\ &= \int_a^b f(x)\,dx \quad \text{(since the } t \text{ and } x \text{ are dummy variables).} \end{aligned}$$

Physical interpretation of this part of the FTC: Suppose $F(x)$ gives the position of an object at time x. Then the object's velocity is $F'(x)$. This part of the FTC says that the displacement of the object from time a to time b equals the area under the velocity function F' from a to b, as suggested earlier in this chapter.

More general interpretation: Suppose $F(x)$ is any quantity. Then the rate of change of F with respect to x is $F'(x)$. This part of the FTC says that the integral of the rate of change, i.e. the <u>accumulation of the rate of change</u>, is equal to the <u>net change</u> in F from $x = a$ to $x = b$.

Mathematical significance of this part of the FTC: This result provides a mechanism to evaluate definite integrals without having to compute limits of Riemann sums. In particular, if you can find any one antiderivative of f that is easy to work with (say F), then you can evaluate integrals of f by subtracting values of F.

You are responsible for being able to state both parts of the FTC and explain their physical interpretation and mathematical significance.

EXAMPLE 1

Evaluate the integral:

$$\int_3^4 x\,dx$$

EXAMPLE 2

Suppose an object is moving along a line so that its velocity at time t is $3\sec^2 t$. Find the distance traveled by the object between times $t = 0$ and $t = \pi/4$.

EXAMPLE 3

In an electrical circuit, the **current** is the instantaneous rate of change of the charge. If the current in a circuit at time t (in seconds) is $2 + \frac{1}{4}\sin t$ amperes, find the net change in the charge from time $\frac{\pi}{2}$ to time π. (P.S. An ampere times a second is a coulomb, a unit of charge.)

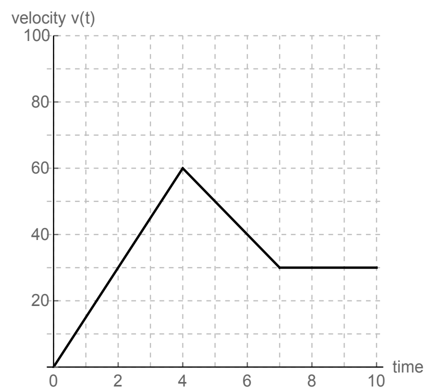
EXAMPLE 4

A tank is being filled with fluid at a non-constant rate: at time t (in seconds), the rate at which the tank is being filled is $2t(4-t)$ L/sec. Find the amount of fluid that is poured in the tank during the first 3 seconds.

The Fundamental Theorem of Calculus reduces the problem of computing integrals to the problem of finding antiderivatives. Thus it is important to be able to find antiderivatives of functions, and we address this task in the next chapter.

9.6 Homework exercises

1. During a car trip, let $v(t)$ represent the car's speedometer reading (in miles per hour) at time t (measured in hours after the beginning of the car trip). Suppose that the graph of $v(t)$ for $0 \leq t \leq 10$ is as given below:



Use this graph to estimate the answers to the following questions (answer with appropriate units):

a) What is the speedometer reading 2 hours after the trip starts?

b) What is the acceleration of the car at time 6?

c) Is the car speeding up, or slowing down at time 3? Explain.

d) Is the car moving forward or backward at time 6? Explain.

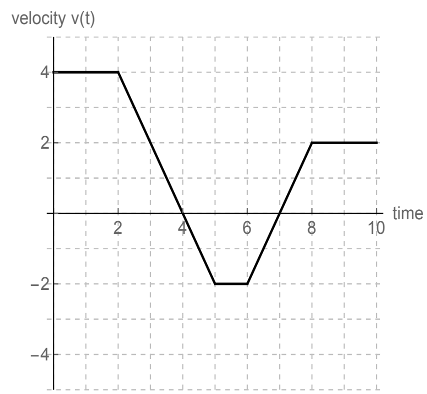
e) Find the distance the car travels during the first 3 hours of the trip.

f) Find the distance the car travels between times 4 and 9.

g) If the odometer reading of the car at the beginning of the trip is 1000, find the odometer reading six hours later.

h) If the odometer reading of the car at time 5 is 2000, what was the odometer reading three hours earlier?

2. On Mars, a rover is moving back and forth along a dirt track so that at time t (measured in seconds), its velocity (measured in cm/sec) is given by the function v whose graph is given below for $0 \leq t \leq 10$:



Use this graph to estimate the answers to the following questions (answer with appropriate units):

a) What is the velocity of the rover at time 8?

b) At what time(s) is the velocity of the rover equal to 1 cm/sec?

c) What is the acceleration of the rover at time 7?

d) Is the rover moving forward or backward at time 6? Explain.

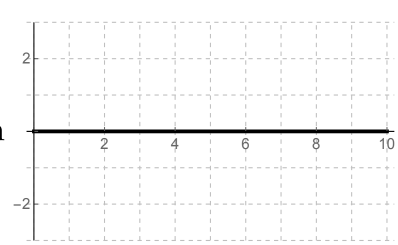
e) Find the displacement of the rover from time 0 to time 3.

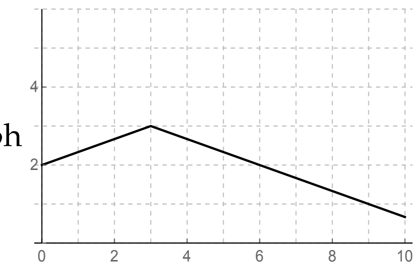
f) Find the displacement of the rover from time 6 to time 10.

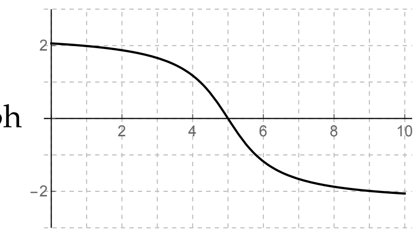
g) Suppose the initial position of the rover is 0. Find all times when the position of the rover is 8.

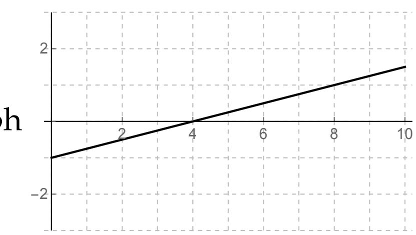
h) Suppose the initial position of the rover is 4. Sketch a crude graph of the position of the rover, as a function of t.

3. In each part of this problem, you are given the graph of the derivative f' of some function f for $0 \leq x \leq 10$, and the value of f at one value of x. Use this information to sketch the graph of f for $0 \leq x \leq 10$.

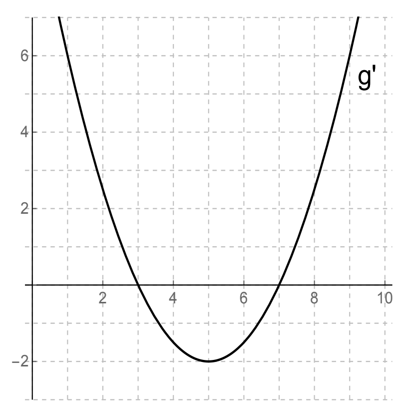
a) $f(3) = 2$; f' has graph 

b) $f(0) = 4$; f' has graph 

c) $f(0) = -3$; f' has graph 

d) $f(0) = 5$; f' has graph 

4. Suppose the graph of some derivative g' is as given below. On a single set of axes, sketch all possible graphs of g:



In Problems 5-8, write the following sums in $\Sigma-$notation:

5. $\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + ... + \frac{1}{19}$
6. $\frac{5}{7^2} + \frac{5}{8^2} + \frac{5}{9^2} + ... + \frac{5}{26^2}$
7. $\frac{2^4}{4} + \frac{2^5}{5} + \frac{2^6}{6} + ... + \frac{2^{14}}{14}$
8. $\frac{3}{8^2}\sqrt{2} + \frac{3}{8^3}\sqrt{3} + \frac{3}{8^4}\sqrt{4} + ... + \frac{3}{8^{25}}\sqrt{25}$

In Problems 9-11, evaluate the given sum by hand (simplify your answer):

9. $\sum_{n=1}^{7} 2n$

10. $\sum_{n=0}^{4} \cos \pi n$

11. $\sum_{n=3}^{6} n^2$

In Problems 12-14, evaluate each of the following sums using *Mathematica*. *Note:* to evaluate a sum of the form $\sum_{n=M}^{N} a_n$ in *Mathematica*, use the following syntax:

$$\texttt{Sum[}a_n\texttt{, \{n, M, N\}]}$$

For example, to evaluate $\sum_{n=2}^{9} n^2$, execute `Sum[n^2, {n, 2, 9}]`. (You can also get a Σ on the Basic Math Assistant Pallette.)

12. $\sum_{n=2}^{13} \frac{1}{n}$

13. $\sum_{n=1}^{35} \frac{12n+4n^2+n^3}{6400}$

14. $\sum_{n=1}^{17} \cos\left(\frac{\pi}{2}n\right) 3n^2$

15. Consider the partition $\mathcal{P} = \{2, 3, 8, 10, 13\}$.
    a) Sketch a picture of this partition.
    b) What interval is this a partition of?
    c) How many subintervals comprise this partition?
    d) What is x_3 for this partition?
    e) What is the second subinterval of the partition?
    f) What is Δx_1?
    g) What is $||\mathcal{P}||$?

16. Consider the partition $\mathcal{P}$ of $[5, 12]$ into 70 equal-length subintervals.
    a) What is x_{20} for this partition?
    b) What is the twelfth subinterval of the partition?
    c) What is Δx_{32}?
    d) What is $||\mathcal{P}||$?

17. Let $f(x) = 1 + 2x - x^2$. Consider the partition $\mathcal{P} = \{0, \frac{1}{4}, \frac{1}{2}, 1\}$ of the interval $[0, 1]$.
    a) Calculate the value of the Riemann sum associated to $\mathcal{P}$ where the test points c_j are chosen to be the midpoints of their respective subintervals.
    b) Sketch a picture which reflects the area being calculated in the Riemann sum you computed in part (a).
    c) What is the smallest possible value of any Riemann sum associated to the partition $\mathcal{P}$? Explain your answer.
    d) What is the largest possible value of any Riemann sum associated to the partition $\mathcal{P}$? Explain your answer.
    e) What do your answers to parts (c) and (d) of this question tell you about the possible value of the area under f from $x = 0$ to $x = 1$?

18. Let $f(x) = 4\sin x$ and let $\mathcal{P} = \{0, \pi/3, \pi/2, 5\pi/6, \pi\}$.

    a) Calculate the right-hand Riemann sum associated to this partition.
    b) Sketch a picture which reflects the area being calculated in part (a).
    c) Calculate the lower Riemann sum associated to this partition.
    d) Sketch a picture which reflects the area being calculated in part (c).

19. Let $f(x) = 2x^2 + 1$.

    a) Compute the left-hand Riemann sum associated to the partition of $[1, 4]$ into three equal-length subintervals.
    b) Sketch a picture which reflects the area being calculated in part (a).
    c) Compute the upper sum associated to the partition of $[1, 4]$ into six equal-length subintervals.
    d) Sketch a picture which reflects the area being calculated in part (c).

20. Let f be an unknown function with the following table of values:

x	-3	-1	1	4	10	11
$f(x)$	2	1	2	0	3	5

    a) Use a left-hand Riemann sum to estimate the area under the graph of f from $x = -1$ to $x = 4$.
    b) Use a right-hand Riemann sum to estimate the area under the graph of f from $x = 1$ to $x = 11$.
    c) Can you compute an upper Riemann sum for f associated to the partition $\mathcal{P} = \{-3, -1, 1, 4\}$? If so, explain why and compute it. If not, explain why you do not have enough information to compute this Riemann sum.

21. Suppose that the velocity of a rocket t seconds after it is launched is given by function v, some of whose values are given in the following table:

t (seconds after launch)	0	1	2	4	8	9	10	12
$v(t)$ (m/sec)	0	2	5	13	30	75	110	240

    Suppose also that the acceleration of the rocket is positive at all times between $t = 0$ and $t = 12$.

    a) Use a left-hand Riemann sum to estimate the distance the rocket travels in the first 8 seconds after it is launched.

b) Use a right-hand Riemann sum to estimate the distance the rocket travels between times $t = 4$ and $t = 10$.

c) Can you compute an upper Riemann sum for f associated to the partition $\mathcal{P} = \{0, 2, 4, 8, 12\}$? If so, explain why and compute it. If not, explain why you do not have enough information to compute this Riemann sum.

In Problems 22-24, write a definite integral which computes the desired area. (You do not actually need to compute the integral.)

22. The area between the graph of the function $f(x) = \sin x$ and the $x-$axis from $x = 0$ to $x = \pi/2$.

23. The area between the graph of the function $f(x) = x^6$ and the $x-$axis from $x = -3$ to $x = 4$

24. The area between the graph of the function $f(x) = \arctan x$ and the $x-$axis from $x = 0$ to $x = 1$.

In Problems 25-32, evaluate each definite integral:

25. $\int_2^6 3\,dx$
26. $\int_8^{11} 0\,dx$
27. $\int_0^7 5x\,dx$
28. $\int_0^2 (2x+3)\,dx$
29. $\int_5^8 (20-2x)\,dx$
30. $\int_{-4}^4 \sqrt{16-x^2}\,dx$
31. $\int_0^4 |x-3|\,dx$
32. $\int_0^2 \sqrt{4-x^2}\,dx$

33. Assuming the following two statements,

$$\int_0^5 f(x)\,dx = 10 \quad \text{and} \quad \int_5^9 f(x)\,dx = 2.$$

compute each of the following:

(a) $\displaystyle\int_0^9 f(x)\,dx$ (b) $\displaystyle\int_0^5 2f(x)\,dx$ (c) $\displaystyle\int_5^0 f(x)\,dx$ (d) $\displaystyle\int_3^3 f(x)\,dx$

34. Assuming the following two statements,

$$\int_0^4 f(x)\,dx = 7 \quad \text{and} \quad \int_2^4 f(x)\,dx = 6.$$

compute each of the following:

(a) $\displaystyle\int_0^2 f(x)\,dx$ (b) $\displaystyle\int_0^4 7f(x)\,dx$ (c) $\displaystyle\int_0^4 [f(x)+9]\,dx$

35. Assuming the following two statements,

$$\int_5^8 f(x)\,dx = 4 \quad \text{and} \quad \int_5^8 g(x)\,dx = 7.$$

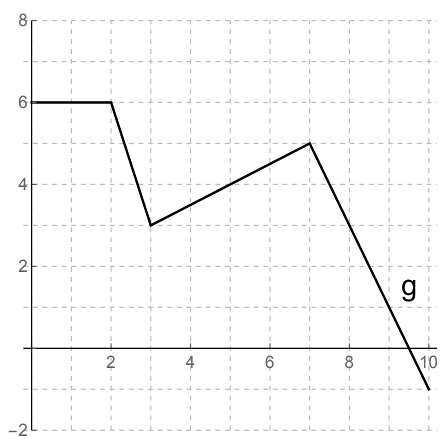
compute each of the following:

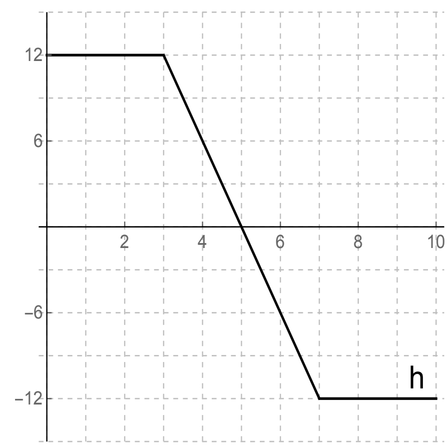
(a) $\int_5^8 [2f(x)+3g(x)]\,dx$ (b) $\int_5^8 [g(x)-f(x)]\,dx$ (c) $\int_3^7 f(x)\,dx+\int_7^3 f(x)\,dx$

36. Assume that f is an unknown function with the following properties:

$$\int_0^3 f(x)\,dx = 7 \qquad \int_3^5 f(x)\,dx = -3 \qquad \int_5^8 f(x)\,dx = 2$$

Also, assume g and h are unknown functions whose graphs are given below:





Use this information to compute the folllowing quantities:

a) $\int_0^3 [f(x)+g(x)]\ dx$

b) $\int_3^7 3g(x)\,dx$

c) $\int_8^5 [f(x)-h(x)]\ dx$

d) $\int_3^8 [2g(x)+4f(x)]\ dx$

e) $\int_0^8 [f(x)+g(x)+h(x)]\ dx$

f) $\int_5^3 [h(x)-2f(x)]\ dx$

g) $\int_0^5 (h(x)-2)\ dx$

h) $\int_0^2 (g(x)+3x)\ dx$

In Problems 37-44, classify each statement as TRUE or FALSE:

37. $F(x) = \sin(x^2)$ is an antiderivative of $f(x) = \cos(x^2)$.

38. $F(x) = 3x^2$ is an antiderivative of $f(x) = 6x$.

39. $F(x) = 3x^2$ is the only antiderivative of $f(x) = 6x$.

40. If F is an antiderivative of f, then for any constant C, $F(x) - C$ is an antiderivative of f as well.

41. If F is an antiderivative of f, then for any constant C, $C\,F(x)$ is an antiderivative of f as well.

42. All antiderivatives of $f(x) = \sec^2 x$ are of the form $\tan x + C$.

43. If F is an antiderivative of some continuous function f, then $\int f(x)\,dx = F(x)$.

44. If F is an antiderivative of some continuous function f, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

In Problems 45-49, compute the indicated definite integral by using the Fundamental Theorem of Calculus:

45. $\int_3^7 4x^3\,dx$

46. $\int_{\pi/3}^{\pi/2} \cos x\,dx$

47. $\int_0^3 (12t^2 - 6t)\,dt$

48. $\int_0^{\ln 6} \frac{1}{2}e^x\,dx$

49. $\int_7^{10} \frac{1}{x}\,dx$

50. A syringe is being emptied at a non-constant rate: at time t (in seconds), the rate at which the syringe is being emptied is $4\sin t + 2\cos t$ mL/sec. Find the amount of liquid drained from the syringe in the first $\frac{\pi}{4}$ seconds.

51. If the current in an electrical current at time t (in seconds) is $t - \frac{2}{t}$ amperes, find the net change in the charge in the circuit from time 1 to time 3.

52. A truck's velocity at time t (in hours) is $v(t) = 40t(t+1)$ miles per hour. How far does the truck travel in the first 30 minutes of its journey?

Answers

1. a) 30 mi/hr
   b) -10 mi/hr^2
   c) The car is speeding up, because the acceleration (i.e. the slope of v) is positive at $t = 3$.
   d) The car is moving forward, because the velocity (i.e. the height of the graph of v) is positive at $t = 6$.
   e) $\frac{135}{2} = 67.5$ miles
   f) 195 miles

g) 1220

h) 1855

2. a) $v(8) = 2$ cm/sec

   b) $t = 3.5$ sec, $t = 7.5$ sec

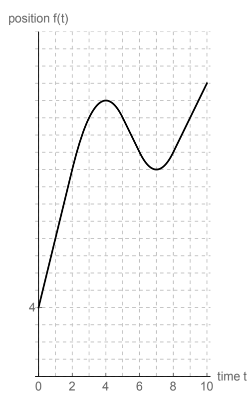
   c) 2 cm/sec^2

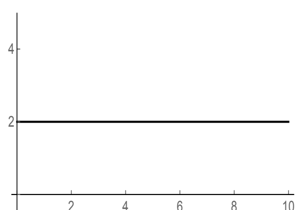
   d) The rover is moving backward, because the velocity is negative at $t = 6$.

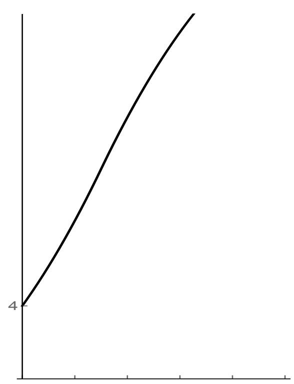
   e) 11 cm

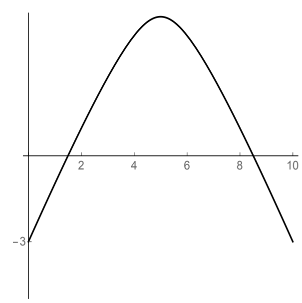
   f) 4 cm

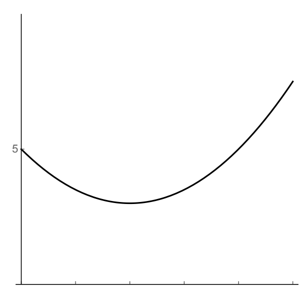
   g) $t = 2, t = 7$

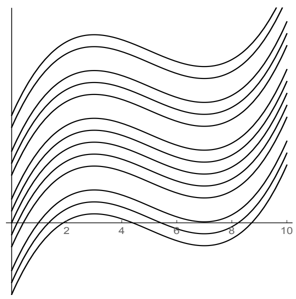
   h) 

3. a) 

   b) 

   c) 

   d) 

4. 

5. $\sum_{n=3}^{19} \frac{1}{n}$

6. $\sum_{n=7}^{26} \frac{5}{n^2}$

7. $\sum_{n=4}^{14} \frac{2^n}{n}$

8. $\sum_{n=2}^{25} \frac{3}{8^n}\sqrt{n}$

9. 56

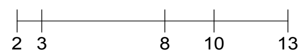
10. 1

11. 86

12. $\frac{785633}{360360}$

13. $\frac{4641}{64}$

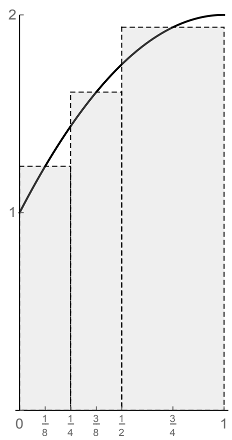
14. 432

15. a) 

    b) $[2, 13]$
    c) 4
    d) 10
    e) $[3, 8]$
    f) 1
    g) 5

16. a) $[6.9, 7]$
    b) $[6.1, 6.2]$
    c) $.1$
    d) $.1$

17. a) $\frac{215}{128}$

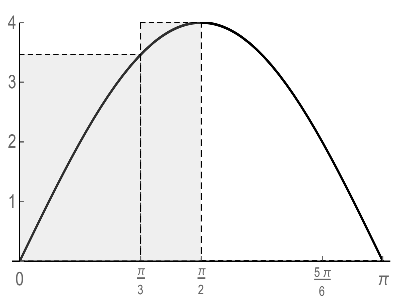
b) 

c) The smallest possible value is the lower Riemann sum associated to $\mathcal{P}$, which is $\frac{95}{64}$.

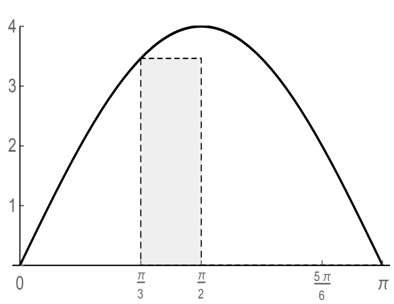
d) The largest possible value is the upper Riemann sum associated to $\mathcal{P}$, which is $\frac{115}{64}$.

e) The actual area under the function f must be greater than the lower sum (i.e. greater than $\frac{95}{64}$ and less than the upper sum (i.e. less than $\frac{115}{64}$).

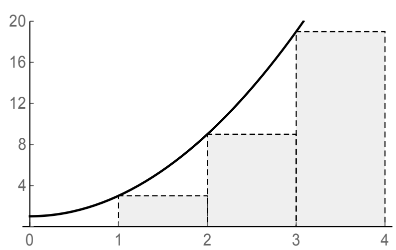
18. a) $\frac{(4+2\sqrt{3})\pi}{3}$

b) 

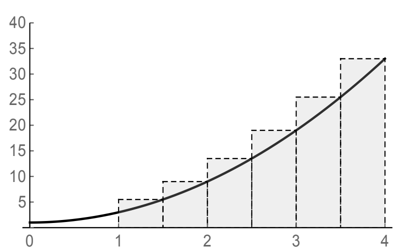
c) $\frac{\pi\sqrt{3}}{3}$

d) 

19. a) 31

b) 

c) $\frac{211}{4} = 52.75$

d) 

20. a) 2

b) 23

c) You do not have enough information, because you do not know the maximum value f achieves on each subinterval of $\mathcal{P}$.

21. a) 64 m
    b) 305 m
    c) Since the acceleration is positive, v is increasing. This means that the upper sum coincides with the right-hand sum, which is 1116 m.
22. $\int_0^{\pi/2} \sin x\, dx$
23. $\int_{-3}^4 x^6\, dx$
24. $\int_0^1 \arctan x\, dx$
25. 12
26. 0
27. $\frac{245}{2}$
28. 10
29. 21
30. 8π
31. 5
32. π
33. a) 12
    b) 20
    c) -10
    d) 0
34. a) 1
    b) 49
    c) 43
35. a) 29
    b) 3
    c) 0
36. a) 23.5
    b) 48
    c) -26
    d) 36
    e) 66.5
    f) -18
    g) 38
    h) 18
37. FALSE (the derivative of $\sin(x^2)$ is $\cos(x^2) \cdot 2x$)
38. TRUE (the derivative of $3x^2$ is $6x$)
39. FALSE ($3x^2 + 1$ is also an antiderivative)
40. TRUE (the derivative of $F(x) - c$ is also $f(x)$)
41. FALSE (the derivative of $2F(x)$ is $2f(x)$, not $f(x)$)
42. TRUE (by the Antiderivative Theorem)
43. FALSE ($\int f(x)\, dx = F(x) + C$)
44. TRUE (this is the Fund. Thm. of Calculus Part 2)
45. $7^4 - 3^4$
46. $1 - \frac{\sqrt{3}}{2}$
47. 81
48. $\frac{5}{2}$
49. $\ln 10 - \ln 7$
50. $4 - \sqrt{2}$ mL
51. $4 - \ln 9$ coulombs
52. $\frac{20}{3}$ miles

Chapter 10

Integration Rules

10.1 General integration concepts

In Chapter 9, we formally defined the definite integral as a limit of Riemann sums:

$$\int_a^b f(x)\,dx = \lim_{||\mathcal{P}||\to 0} \sum_{j=1}^{n} f(c_j)\,\Delta x_j$$

This expression gives the area under function f from $x = a$ to $x = b$.

We saw that in practice, it is mostly impossible to compute integrals by evaluating these limits. To actually compute a definite integral, one uses the following theorem:

Theorem 10.1 (Fundamental Theorem of Calculus Part II) ***(Evaluation of Integrals)*** *Let f be continuous on $[a, b]$. Suppose F is* <u>*any*</u> *antiderivative of f. Then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

This suggests that it is important to find antiderivatives of functions. Recall:

Definition 10.2 *Given function f, an* **antiderivative** *of f is a function F such that $F' = f$.*

Definition 10.3 *Given function f, the* **indefinite integral** *of f, denoted $\int f(x)\,dx$, is the set of all antiderivatives of f.*

At this point, we have two objects which look the same but are very different:

Definite Integral	Indefinite Integral
$\int_a^b f(x)\,dx$	$\int f(x)\,dx$

EXAMPLE 1

If $f(x) = 2x$, then

$$\int f(x)\,dx =$$

$$\int_{-1}^{4} f(x)\,dx =$$

EXAMPLE 2

Suppose $\int f(x)\,dx = \cos x + C$. Compute

$$\int_{\pi/3}^{\pi/2} f(x)\,dx$$

and find $f(x)$.

General principle illustrated by the previous example:

This means that each of the derivatives we learned earlier in the semester turns into an integral that we know now:

$$\begin{aligned}
\frac{d}{dx}(C) &= 0 && \Rightarrow \\
\frac{d}{dx}(x^n) &= nx^{n-1}(\text{if } n \neq 0) && \Rightarrow
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}(e^x) &= e^x && \Rightarrow && \int e^x \, dx = e^x + C \\
\frac{d}{dx}(\ln x) &= \frac{1}{x} && \Rightarrow && \int \frac{1}{x} \, dx = \ln x + C \\
\frac{d}{dx}(\arctan x) &= \frac{1}{x^2+1} && \Rightarrow && \int \frac{1}{x^2+1} \, dx = \arctan x + C \\
\frac{d}{dx}(\arcsin x) &= \frac{1}{\sqrt{1-x^2}} && \Rightarrow && \int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C \\
\frac{d}{dx}(\sin x) &= \cos x && \Rightarrow && \int \cos x \, dx = \sin x + C \\
\frac{d}{dx}(\cos x) &= -\sin x && \Rightarrow && \int (-\sin x) \, dx = \cos x + C
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \sec^2 x && \Rightarrow && \int \sec^2 x \, dx = \tan x + C \\
\frac{d}{dx}(\cot x) &= -\csc^2 x && \Rightarrow && \int \csc^2 x \, dx = -\cot x + C \\
\frac{d}{dx}(\sec x) &= \sec x \tan x && \Rightarrow && \int \sec x \tan x \, dx = \sec x = C \\
\frac{d}{dx}(\csc x) &= -\csc x \cot x && \Rightarrow && \int \csc x \cot x \, dx = -\csc x + C
\end{aligned}$$

Furthermore, since differentiation is linear, so is integration. We have:

Theorem 10.4 (Linearity of Definite Integration) *Suppose f and g are integrable functions. Then:*

1. $\int_a^b [f(x) + g(x)] \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx$;
2. $\int_a^b [f(x) - g(x)] \, dx = \int_a^b f(x) \, dx - \int_a^b g(x) \, dx$;
3. $\int_a^b [k \cdot f(x)] \, dx = k \int_a^b f(x) \, dx$ *for any constant k.*

Theorem 10.5 (Linearity of Indefinite Integration) *Suppose f and g are integrable functions. Then:*

1. $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$;
2. $\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$;
3. $\int [k \cdot f(x)]\,dx = k \int f(x)\,dx$ *for any constant* k.

NOTE: Integration is not multiplicative nor divisive:

$$\int f(x)g(x)\,dx \neq \left(\int f(x)\,dx\right) \cdot \left(\int g(x)\,dx\right)$$

$$\int \left(\frac{f(x)}{g(x)}\right)\,dx \neq \frac{\int f(x)\,dx}{\int g(x)\,dx}$$

EXAMPLE 3

Compute the integral:

$$\int -\frac{1}{4}\csc^2 x\,dx$$

EXAMPLE 4

Suppose a bicyclist is driving down a road so that her velocity at time t is $3-2t+t^9$.

(a) Find the displacement of the bicyclist from time 1 to time 2.

(b) If the position of the bicyclist at time 0 is 4, find the position at time 1.

EXAMPLE 5

Compute the integral:

$$\int \left(\frac{2}{\sqrt[3]{x}} + \frac{5}{x}\right)\,dx$$

EXAMPLE 6

Compute the integral:

$$\int (4\cos x - 3x^5 + 2e^x)\, dx$$

EXAMPLE 7

Compute the integral:

$$\int \left(\frac{\sin x}{7} + \frac{4}{1+x^2} - 2 \right)\, dx$$

EXAMPLE 8

A Math 230 student is asked to compute this integral:

$$\int x \sec^2 x \, dx$$

After some substantial work, the student obtains the answer

$$\ln(\cos x) + x \tan x + C.$$

Is the student's answer correct? Why or why not?

Here is a list of integration rules which, together with linearity, allows you to do most easy integrals:

Theorem 10.6 (Integration Rules to Memorize)

Constant Rules:

$$\int 0\,dx = C$$
$$\int M\,dx = Mx + C$$

Power Rules:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \text{ (so long as } n \neq -1\text{)}$$
$$\int \frac{1}{x}\,dx = \ln|x| + C$$

Trig Functions:

$$\int \sin x\,dx = -\cos x + C$$
$$\int \cos x\,dx = \sin x + C$$
$$\int \sec^2 x\,dx = \tan x + C$$
$$\int \csc^2 x\,dx = -\cot x + C$$
$$\int \sec x \tan x\,dx = \sec x + C$$
$$\int \csc x \cot x\,dx = -\csc x + C$$

Exponential Functions:

$$\int e^x\,dx = e^x + C$$

Inverse Trig Functions :

$$\int \frac{1}{x^2+1}\,dx = \arctan x + C$$
$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

Note: There are some integrals which we don't know yet. They include:

$$\int \tan x\,dx \qquad \int \cot x\,dx \qquad \int \sec x\,dx \qquad \int \sin(x^2)\,dx \qquad \int \ln x\,dx, \text{ etc.}$$

Some (most) of these integrals will be discussed in Math 230 (alternatively, some of them can be computed using *Mathematica*, but some integrals are known to be impossible to compute, even with an infinitely powerful computer!).

10.2 Rewriting the integrand

Sometimes it is useful to use algebra, or a trigonometric identity, or a logarithm rule, to rewrite the integrand before computing an integral.

EXAMPLE 1

Find the area under the graph of $f(x) = \frac{(x^2-1)^2}{x}$ between $x = 1$ and $x = 2$.

EXAMPLE 2

Compute the integral

$$\int \tan^2 x \, dx$$

EXAMPLE 3

Compute the integral

$$\int \ln\left(2^x\right) \, dx$$

10.3 Elementary u-substitutions

MOTIVATING EXAMPLE

Let $f(x) = \sin(x^3)$.

Goal: Recognize integrands which arise as the result of the Chain Rule.

Idea: Identify the presence of a function and its derivative in the integrand.

Generalization of the Motivating Example

Consider the function $F(g(x))$, where $F' = f$. Then

$$\frac{d}{dx}\left[F(g(x))\right] = F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x)$$

Theorem 10.7 (Integration by $u-$substitution - Indefinite Integrals)

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(u)\, du \text{ by setting } u = g(x).$$

Procedure for *indefinite* integration by $u-$substitution:

1. Make sure you can't just "write the answer" to the integral without a substitution.
2. Check the integrand to make sure a $u-$substitution is appropriate:
   - The integral should not be one you have memorized.
   - The integrand should have two terms multiplied together.
   - One of the terms being multiplied should be the derivative of part of the other term (i.e. the terms should be related).
3. Let $u =$ the term whose derivative stands by itself.
4. Write the derivative of u in Leibniz notation, then multiply through by an appropriate constant to match what is in the integral.
5. Substitute in the integral so that all xs are replaced with us as appropriate.
6. Integrate with respect to u.
7. Back-substitute to get an answer in terms of x.

A picture to explain the logic:

EXAMPLE 1

Compute the integral:

$$\int \left(6x^2 + 3\right)^4 x\,dx$$

EXAMPLE 2

Compute the integral:

$$\int 27(z^3+1)^9 z^2\,dz$$

EXAMPLE 3

Compute the integral:

$$\int \tan^3(3x+1)\sec^2(3x+1)\,dx$$

Solution:

$$\begin{aligned} \text{Let } u &= \tan(3x+1). \\ \Rightarrow \frac{du}{dx} &= \sec^2(3x+1)\cdot 3 \\ \Rightarrow du &= 3\sec^2(3x+1)\,dx \\ \Rightarrow \frac{1}{3}du &= \sec^2(3x+1)\,dx \end{aligned}$$

EXAMPLE 4

Compute the integral:

$$\int \sin(3x+2)\,dx$$

EXAMPLE 5

Compute the integral:

$$\int (5x-2)^{12}\,dx$$

Solution:

$$\begin{aligned} \text{Let } u &= 5x-2 \\ \Rightarrow \frac{du}{dx} &= 5 \\ \Rightarrow du &= 5\,dx \\ \Rightarrow \frac{1}{5}du &= dx \end{aligned}$$

Now the integral becomes

$$\int u^{12}\frac{1}{5}\,du = \frac{1}{5}\cdot\frac{u^{13}}{13} + C = \frac{1}{65}u^{13} + C = \frac{1}{65}(5x-2)^{13} + C.$$

The general principle illustrated by the preceding examples is called the Linear Replacement Principle:

Theorem 10.8 (Linear Replacement Principle) *Suppose* $\int f(x)\,dx = F(x) + C$. *Then for any constants* m *and* b,

$$\int f(mx+b)\,dx = \frac{1}{m}F(mx+b) + C.$$

EXAMPLE 6

Compute these integrals without actually doing the u-substitution:

$$\int 2e^{5x}\,dx \qquad \int \frac{4}{2x-3}\,dx$$

Some integrals require rewriting before performing a substitution:

EXAMPLE 7

Compute the integral:

$$\int \tan x\,dx$$

EXAMPLE 8

Find all functions g whose derivative is $g'(x) = xe^{-x/2}$.

Substitution in definite integrals

EXAMPLE 9

Compute the integral:

$$\int_1^4 \frac{1}{\sqrt{x}(\sqrt{x}+1)^3}\,dx$$

Theorem 10.9 (Integration by $u-$substitution - Definite Integrals) *By way of the u-substitution $u = g(x)$,*

$$\int_a^b f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Procedure for *definite* integration by $u-$substitution:

1. Make sure you can't just "write the answer" to the integral without a substitution.
2. Check the integrand to make sure a $u-$substitution is appropriate:
   - The integral should not be one you have memorized.
   - The integrand should have two terms multiplied together.
   - One of the terms being multiplied should be the derivative of part of the other term (i.e. the terms should be related).
3. Let $u =$ the term whose derivative stands by itself.
4. Write the derivative of u in Leibniz notation, then multiply through by an appropriate constant to match what is in the integral.
5. Substitute in the integral so that all xs are replaced with us as appropriate.
6. Change the limits of integration to values of u using the formula from Step 3.
7. Integrate with respect to u (don't back-substitute for x).

EXAMPLE 10

Find the distance travelled by an object between times 0 and $\pi/2$, if its velocity at time t is $e^{\cos t} \sin t$.

EXAMPLE 11

Compute the integral:

$$\int_0^{\ln 3} \frac{e^x}{e^x + 1}\, dx$$

10.4 Homework exercises

In Problems 1-6, an advanced student was asked to compute the given integral. Determine, in part by taking an appropriate derivative, whether or not the student's answer was correct:

1. Problem: $\int \cos x\, dx$; proposed answer: $\sin x$
2. Problem: $\int \ln x\, dx$; proposed answer: $\frac{1}{x} + C$
3. Problem: $\int \frac{1}{x+3}\, dx$; proposed answer: $\ln(x+3) + C$
4. Problem: $\int \frac{1}{x^2+25}\, dx$; proposed answer: $\ln(x^2+25) + C$
5. Problem: $\int \frac{1}{x^2+25}\, dx$; proposed answer: $\frac{1}{5} \arctan \frac{x}{5} + C$
6. Problem: $\int \frac{4}{x^2-1}\, dx$; proposed answer: $2\ln(1-x) + 2\ln(1+x) + C$

In Problems 7-12, an advanced student was asked to compute the given integral, and got an answer which is close, but wrong. After taking a derivative of the student's answer, use the derivative you get to "fix" the student's answer, making it correct.

7. Problem: $\int \cos 2x\, dx$; wrong answer: $\sin 2x + C$
8. Problem: $\int \frac{1}{3x-4}\, dx$; wrong answer: $\ln(3x-4) + C$
9. Problem: $\int \csc^2 x\, dx$; wrong answer: $\cot x + C$
10. Problem: $\int e^{3x}\, dx$; wrong answer: $e^{3x} + C$
11. Problem: $\int 2e^{-x/4}\, dx$; wrong answer: $2e^{-x/4} + C$
12. Problem: $\int \sin^3 2x \cos 2x\, dx$; wrong answer: $\sin^4 2x + C$
13. Compute $\int 0\, dx$.
14. Compute $\int_4^6 5\, dx$.
15. Evaluate $\int_3^5 x\, dx$.
16. Find $\int 4x\, dx$.
17. Find all antiderivatives of $f(x) = x + 3$.
18. Compute $\int_0^1 \sqrt[3]{x^2}\, dx$
19. Suppose that the rate at which a tank is being filled with water at time t is $5e^t$ gal/min. Find the amount of water put in the tank between times 0 and 4.

20. Find the area under the graph of $f(x) = \frac{4}{x}$ between $x = 2$ and $x = 9$.

21. Compute $\int (2x^3 - x)\, dx$.

22. Compute the integral $\int (\sec^2 x - 7 \sin x)\, dx$.

23. Compute the indefinite integral of $\frac{3}{x^2} + \csc^2 x$ with respect to x.

24. Suppose that the rate at which energy is used by a machine at time t is given by $2 \sec t \tan t$ J/sec. Find the energy consumption between times $\frac{\pi}{4}$ and $\frac{\pi}{3}$.

25. Find the area under the graph of $y = 1 + \frac{4}{x^2}$ from $x = 1$ to $x = 2$.

26. Compute $\int \frac{e^x}{4}\, dx$.

27. Evaluate $\int \left(\frac{6}{\sqrt{x}} + \frac{1}{x}\right)\, dx$.

28. Find all antiderivatives of $f(x) = x^{3/2} + 4x + 2$.

29. Compute $\int_{-1}^{1} (x^3 - x^2)\, dx$.

30. Find $\int dx$.

31. Compute $\int (2 - \csc x \cot x)\, dx$.

32. Find the indefinite integral of $f(x) = x^3 + 4 \cos x$ with respect to x.

33. Suppose an object is moving back and forth along a line so that its acceleration at time t is $a(t) = -5t$ in/sec^2. If the object's velocity at time 2 is 3 in/sec, what is its velocity at time 5?

34. Suppose a bee is moving along a number line so that its velocity at time t is $v(t) = t^2 + 3$ cm/sec. If at time 1 the bee is at position -4, what is its position at time 4?

35. Suppose a bug is crawling along a number line so that its acceleration at time t is $a(t) = \frac{1}{10} \cos t$ meters per hour squared.

    a) If its velocity at time 0 is $\frac{1}{5}$ meters per hour and its position at time 0 is 1, what is its position at time π?

    b) If its velocity at time 0 is 1 meter per hour and its position at time 0 is 0, what is its position at time $\frac{\pi}{3}$?

36. Suppose f is a function such that the slope of the line tangent to f at x is $4x - 1$. If f passes through the point $(4, 0)$, what is $f(-2)$?

In Problems 37-42, use *Mathematica* to compute the indicated integrals (write the answers as you would write them by hand).

Note: *Mathematica* computes integrals using the `Integrate` command. For example, to compute the definite integral $\int_2^4 x^2\,dx$ using *Mathematica*, execute

```
Integrate[x^2, {x, 2, 4}]
```

and to compute the indefinite integral $\int x^2\,dx$ using *Mathematica*, execute

```
Integrate[x^2, x]
```

(The `x` in the command is necessary and corresponds to the dx in the integral.) You can also get an integral sign on the Basic Math Assistant.

37. $\int \sec x\,dx$

    Note: When you compute an indefinite integral using *Mathematica*, something important is missing from its answer.

38. $\int \ln x\,dx$

39. $\int_0^{\pi/4} \tan x\,dx$

40. $\int_0^1 \arctan x\,dx$

41. $\int x^3 e^{-x}\,dx$

42. $\int \frac{3}{x^2-x}\,dx$

In Problems 43-47, you are given a definite integral and a u-substitution. Perform the u-substitution to rewrite the integral as a simpler integral (be sure to change the limits from x-values to u-values). You do **not** need to evaluate the integral.

43. $\int_{-2}^3 \frac{2x}{x^2+5}\,dx$; $u = x^2+5$

44. $\int_3^7 e^{8x}\,dx$; $u = 8x$

45. $\int_0^1 20(x^7+3)x^6\,dx$; $u = x^7+3$

46. $\int_0^{\pi/4} \sin^3 x \cos x\,dx$; $u = \sin x$

47. $\int_0^{\ln 4} \frac{e^x}{e^x+1}\,dx$; $u = e^x+1$

In Problems 48-66, compute the indicated integral:

48. $\int_1^8 \sqrt{\frac{2}{x}}\, dx$
49. $\int (2-x)\sqrt{x}\, dx$
50. $\int \frac{x^2+2-3x^3+1}{x^4}\, dx$
51. $\int_0^2 (x+1)(3x-2)\, dx$
52. $\int \frac{5-e^x}{e^{2x}}\, dx$
53. $\int \frac{(\ln x)^2}{x}\, dx$
54. $\int \sqrt{3-x^2}(-2x)\, dx$
55. $\int x^3(x^4-1)^5\, dx$
56. $\int \frac{3}{2+7x}\, dx$
57. $\int 5x\sqrt[3]{1-x^2}\, dx$
58. $\int \frac{6x^2}{1+x^3}\, dx$
59. $\int_2^3 \frac{6x^2}{(1+x^3)^3}\, dx$
60. $\int \sin \pi x\, dx$
61. $\int \cos 2x\, dx$
62. $\int \frac{3}{4} \cos \frac{x}{2}\, dx$
63. $\int \tan^4 x \sec^2 x\, dx$
64. $\int_{\pi/6}^{\pi/2} \cot x\, dx$
65. $\int 3e^{2x}\, dx$
66. $\int_1^9 \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\, dx$
67. Suppose that the rate of fuel consumption of a motor at time t is te^{-t^2} L/min. Compute the amount of fuel consumed by the motor in the first minute of operation.
68. Find the area under the graph of $y = x(x^2+1)^3$ from $x = 0$ to $x = 1$.
69. Suppose that an object is moving back and forth along a number line so that its velocity at time t is $v(t) = 4t^2\sqrt{t^3+1}$ ft/sec. What is the object's displacement from time 1 sec to time 2 sec?

Answers

1. Wrong (missing the $+C$)
2. Wrong
3. Correct
4. Wrong
5. Correct
6. Correct
7. $\frac{1}{2}\sin 2x + C$
8. $\frac{1}{3}\ln(3x-4) + C$
9. $-\cot x + C$
10. $\frac{1}{3}e^{3x} + C$
11. $-8e^{-x/4} + C$
12. $\frac{1}{8}\sin^4 2x + C$
13. C
14. 10
15. 8
16. $2x^2 + C$
17. $\frac{1}{2}x^2 + 3x + C$
18. $\frac{3}{5}$
19. $5e^4 - 5$ gal
20. $4\ln 9 - 4\ln 2$
21. $\frac{1}{2}x^4 - \frac{1}{2}x^2 + C$
22. $\tan x + 7\cos x + C$
23. $-\frac{3}{x} - \cot x + C$
24. $4 - 2\sqrt{2}$ J
25. 3
26. $\frac{e^x}{4} + C$
27. $12\sqrt{x} + \ln|x| + C$
28. $\frac{2}{5}x^{5/2} + 2x^2 + 2x + C$
29. $-\frac{2}{3}$
30. $x + C$
31. $2x + \csc x + C$
32. $\frac{1}{4}x^4 + 4\sin x + C$
33. $\frac{-99}{2}$ in/sec
34. 26
35. a) $\frac{\pi+6}{5}$
    b) $\frac{1}{20} + \frac{\pi}{3}$
36. -18
37. $-\ln[\cos(x/2) - \sin(x/2)] + \ln[\cos(x/2) + \sin(x/2)] + C$
38. $-x + x\ln x + C$
39. $\frac{\ln 2}{2}$
40. $\frac{1}{4}(\pi - \ln 4)$
41. $e^{-x}(-6 - 6x - 3x^2 - x^3) + C$
42. $3(\ln(1-x) - \ln x) + C$
43. $\int_9^{14} \frac{1}{u}\,du$
44. $\int_{24}^{56} \frac{1}{8}e^u\,du$
45. $\int_3^4 \frac{10}{3}u\,du$
46. $\int_0^{\sqrt{2}/2} u^3\,du$
47. $\int_2^5 \frac{1}{u}\,du$
48. $8 - \sqrt{8}$

49. $\frac{4}{3}x^{3/2} - \frac{2}{5}x^{5/2} + C$
50. $-x^{-3} - x^{-1} - 3\ln x + C$
51. 6
52. $\frac{-5}{2}e^{-2x} + e^{-x} + C$
53. $\frac{1}{3}(\ln x)^3 + C$
54. $\frac{2}{3}(3 - x^2)^{3/2} + C$
55. $\frac{1}{24}(x^4 - 1)^6 + C$
56. $\frac{3}{7}\ln(2 + 7x) + C$
57. $\frac{-15}{8}(1 - x^2)^{4/3} + C$
58. $2\ln(x^3 + 1) + C$
59. $\frac{-1}{28^2} + \frac{1}{81}$
60. $\frac{-1}{\pi}\cos \pi x + C$
61. $\frac{1}{2}\sin 2x + C$
62. $\frac{3}{2}\sin \frac{x}{2} + C$
63. $\frac{1}{5}\tan^5 x + C$
64. $\ln 2$
65. $\frac{3}{2}e^{2x} + C$
66. $\frac{1}{2}$
67. $\frac{e-1}{2e}$ L
68. $\frac{49}{40}$
69. $24 - \frac{16}{9}\sqrt{2}$ ft

Chapter 11

Introduction to Differential Equations

11.1 Terminology and examples

Mathematics in a nutshell

1. Start with some real-world problem.
2. Build a mathematical model or mathematical object that represents the problem (this often involves writing down variables and equations).
3. Find the solution of the model from Step 2.
4. Generalize the solution and develop the corresponding theory (so you can adapt the solution technique to as many other similar problems as possible).

EXAMPLES 1-4

Example 1 A rectangle is 3 units longer than it is wide. If its area is 54, what is its side length?	$\to$	Let $x =$ width. $x + 3 =$ length. $A = 54$ $x(x + 3) = 54.$	$\to$	$x = 6$ (6×9) (throw out $x = -9$)

Example 2 (physics)
An object falls to earth, starting at height 100m. What is its height after t seconds?

Example 3 (biology)
If you eat a raw hamburger, how much *E. coli* is in your system t minutes later?

Example 4 (pharmacokinetics)
Give a patient dose D of a drug every 24 hours. What is the concentration of the drug in the patient's bloodstream after t hours?

The model of Example 1 above is a "numerical equation":

- both sides of the equation are **numbers**;
- the object is to solve for **numbers** x such that the equation holds.

The models of Examples 2, 3 and 4 above are examples of **(ordinary) differential equations** (ODEs):

- both sides of the equation are thought of as **functions** of an independent variable (t in these examples);
- there is another variable (y in Example 2, x in Example 3, C in Example 4) which depends on that independent variable;
- the equation contains derivatives of the dependent variable with respect to the independent variable;
- the object is to solve for the explicit relationship between the variables.

More examples of ODEs:

EXAMPLES 5-8

5. $y' + xy = 3x^2$

Assumption:

This equation can also be written as:

6. $x^2y''' + (x^2 - 3xy)y'' + 2y'y = 3xy' - 4$

7. $y' \cdot y''' = y$

Assumption:

8. $x'' + c_1x' + c_0x = 0$ (describes motion of damped oscillator)

Assumption:

EXAMPLE 9

$$\frac{dy}{dt} = 3y$$

Assumption: $y = y(t)$

In Example 9,

- $y = e^{3t}$ is a solution because
- $y = 2e^{3t}$ is a solution because
- $y = Ke^{3t}$ is a solution for any constant K because
- $y = t^2$ is not a solution because

Question: Are there any solutions to Example 9 other than $y = Ke^{3t}$?

Some ODEs are particularly easy to solve. Suppose $\frac{dy}{dx} = g(x)$ (i.e. there's only an x on the right-hand side). Then

$$y(x) = \int g(x)\,dx$$

by the Fundamental Theorem of Calculus (in particular, notice that there will be infinitely many solutions, parameterized by a single constant C).

EXAMPLE 10

$$\frac{dy}{dt} = 6t^2$$

EXAMPLE 11

$$y'' = 2$$

These constants (the K in Example 9, the C in Example 10, and the C and D in Example 11) are typical of solutions to ODEs, because solving an ODE is akin to indefinite integration. For a general ODE (not just one of the form $\frac{dy}{dx} = g(x)$), we expect arbitrary constants in the description of the solution. **The number of constants in the solution is equal to the highest order of derivative occuring in the equation**, i.e.

$$y''' - 3y = x^2 + 3 \Rightarrow$$

$$y + x\frac{d^2y}{dx^2} = x^3 - \frac{dy}{dx} \Rightarrow$$

$$y^{(8)} + 3 = x \Rightarrow$$

Sometimes you know additional information which allows you to solve for the constant(s):

EXAMPLE 12

Suppose a bug travels along a line with velocity at time t given by $v(t) = 2t - 4$. If at time 0, the bug is at position 7, what is the bug's position at time t?

11.2 Vector fields and qualitative analysis of first-order ODEs

Definition 11.1 *An ODE is called* **first-order** *if it can be rewritten $y' = g(x, y)$ for some function g of x and y.*

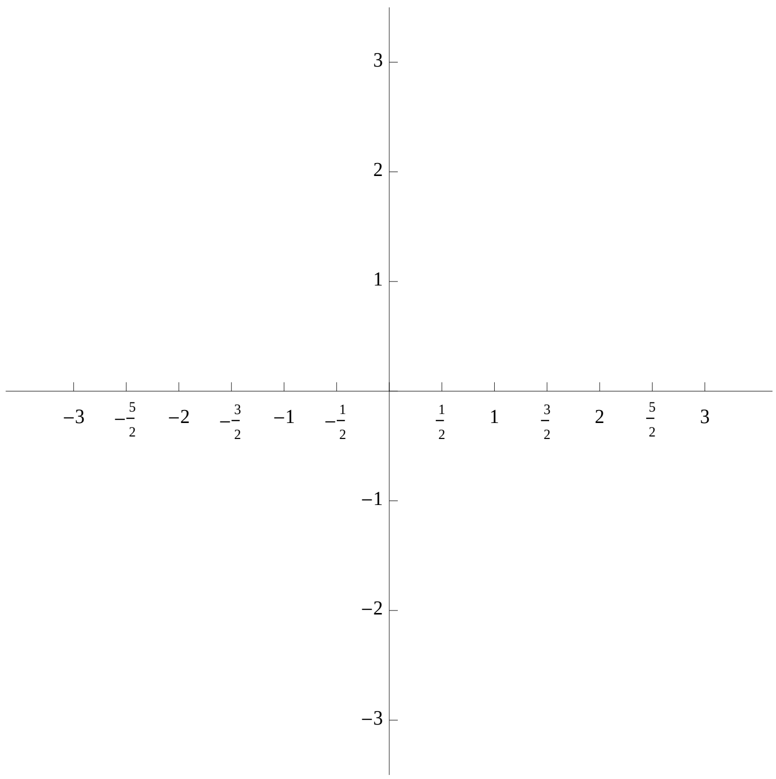
EXAMPLE 1

$$3xy' + 2xy = \cos y \cdot y' - e^x$$

Suppose we have a first-order ODE $y' = g(x, y)$. Associated to this ODE is a **vector field** (a.k.a. **slope field**) in the following sense: at every point (x, y) in the plane, draw a short line segment or arrow of slope $g(x, y)$ passing through (x, y). The collection of all these line segments/arrows is the "vector field" of the ODE.

EXAMPLE 2

$$y' = 2x$$



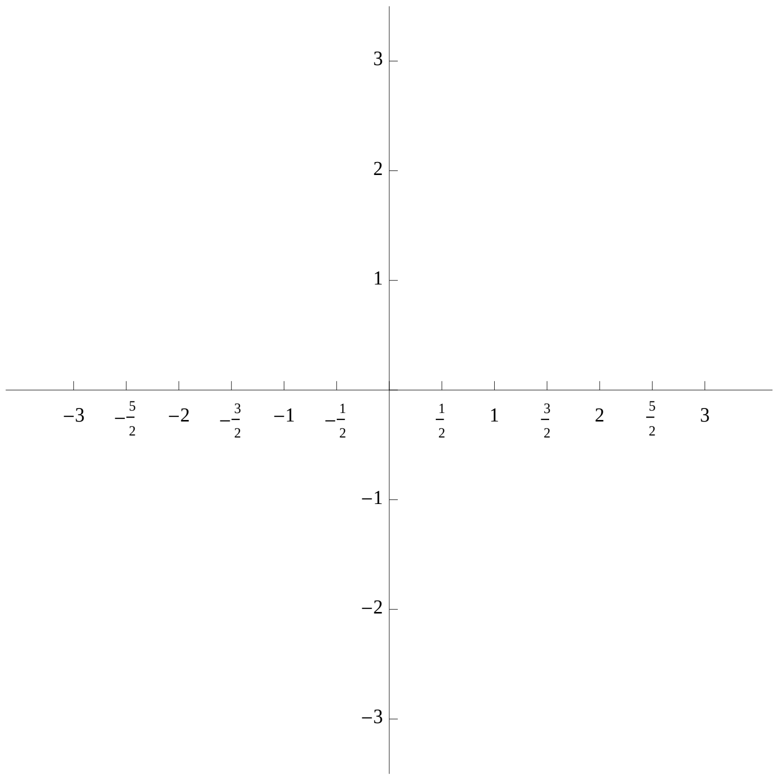
The vector field suggests what the solutions of the first-order ODE $y' = g(x, y)$ look like. If $y(x)$ is a solution, then the graph of $y(x)$ must be tangent to all of the line segments associated to all the (x, y) on the graph. Put another way,

*"the solutions have to have graphs which **flow with** the line segments of the vector field"*

In the example $y' = 2x$, it appears that the solutions are parabolas. That is the case, because

EXAMPLE 3

$$y' = xy$$



Notice first that

- there appear to be infinitely many different solutions,
- but given any one point (x_0, y_0), there is only one possible solution which passes through that point.

What are the solutions to the ODE of this example ($y' = xy$)?

Mathematica is an extremely useful tool for drawing vector fields associated to ODEs. To sketch the vector field associated to ODE $y' = g(x, y)$, use the following code (which is explained below). The code is to be typed in one *Mathematica* cell and executed all together.

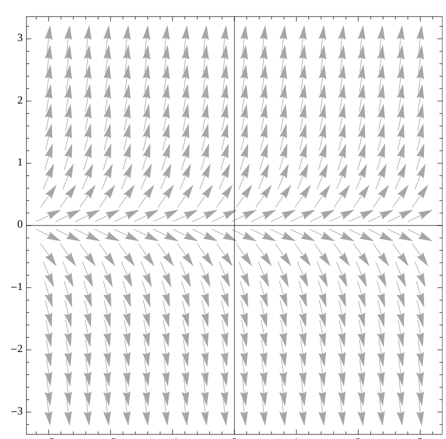
```
g[x_,y_] := 3y;
VectorPlot[{1,g[x,y]}, {x, -3, 3}, {y, -3, 3},
 VectorPoints -> 20, Axes -> True,
 VectorScale -> {Automatic, Automatic, None},
 VectorStyle -> Orange]
```

An explanation of the code (and things you can change):

1. The first line defines $g(x, y)$. For example, this code will produce the vector field for $y' = 3y$.
2. The second line controls the range of the picture; for example, this will sketch the vector field for x ranging from -3 to 3 and y ranging from -3 to 3.
3. The third line tells *Mathematica* how many arrows to draw in each direction, and to include the $x-$ and $y-$axes in the picture.
4. The fourth line controls the size of the arrows and is optional, but I think these choices make for a nice picture.
5. The last line tells *Mathematica* what color to draw the arrows.

Note: In principle, you don't type this code over and over. You can get this code from the file "vectorfields.nb" (available on my webpage) and you simply copy and paste the cells into your *Mathematica* notebooks, editing the formula for $g(x, y)$ and the plot range as necessary.

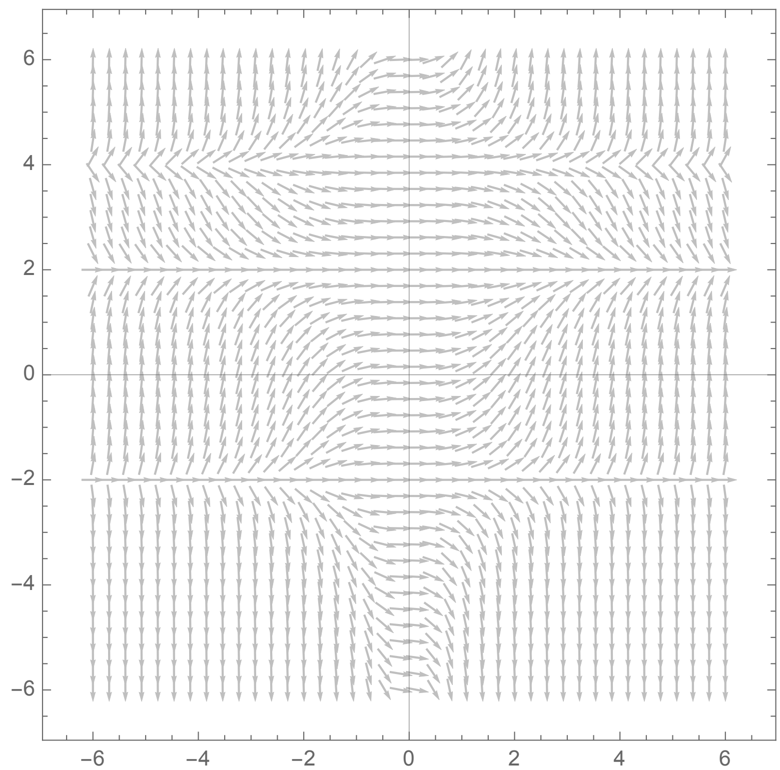
The above code produces this picture:



As mentioned earlier, solutions to an ODE must "flow with" the vector field of the ODE. This allows you to qualitatively study solutions to ODEs by examining the picture of the vector field associated to the ODE.

EXAMPLE 4

Below is a picture of the vector field associated to some first-order ODE $y' = g(x,y)$:



1. Write the equation of two explicit solutions to this ODE:

2. On the above picture, sketch the graph of the solution satisfying $y(-2) = 1$.

3. Suppose $y(-2) = 0$. Estimate $y(2)$.

4. Suppose $y(0) = 3$. Find $\lim_{x \to \infty} y(x)$.

5. Suppose $y(2) = 0$. Find $\lim_{x \to -\infty} y(x)$.

In many of the questions of the previous example, you are given a point (x_0, y_0) that the solution of the ODE is to pass through (for example, in Question 2 you are given $x_0 = -2, y_0 = 1$).

Definition 11.2 *A point (x_0, y_0) through which a solution to an ODE must pass is called an* **initial value**. *An ODE, together with an initial value, is called an* **initial value problem**. *A solution of an initial value problem is called a* **particular solution** *of the ODE; the set of all particular solutions of the ODE is called the* **general solution** *of the ODE.*

EXAMPLE

$y' = 2x$ is a first-order ODE, whose general solution is $y = x^2 + C$.

$\begin{cases} y' = 2x \\ y(0) = 1 \end{cases}$ is an initial value problem, whose particular solution is $y = x^2 + 1$.

Our observation that solutions to ODEs have to "flow with" the vector field leads to the following important theorem:

Theorem 11.3 (Existence / Uniqueness Theorem for First-Order ODEs) *Given a "reasonable" first-order ODE and an initial value (x_0, y_0), there is one and only one solution of that ODE satisfying that initial value.*

Logic behind this theorem:

11.3 Solving separable ODEs

In the previous section we learned how to study a first-order ODE qualitatively (by looking at a picture of the associated vector field).

Question: Can you explicitly solve a first-order ODE $y' = g(x, y)$ for the solutions $y(x)$?

Answer:

Definition 11.4 *A first-order ODE is called* **separable** *if it can be rewritten in the form $f(y)y' = h(x)$ for functions f of y and h of x.*

In other words, an ODE is separable if one can **separate the variables**, i.e. put all the y on one side and all the x on the other side.

Theoretical solution of separable, first-order ODEs

Suppose you have a separable, first-order ODE. Then, by replacing the y' with $\frac{dy}{dx}$, it can be rewritten as

$$f(y)\,\frac{dy}{dx} = h(x).$$

Integrate both sides with respect to x to get

$$\int f(y)\,\frac{dy}{dx}\,dx = \int h(x)\,dx$$

On the left-hand side, perform the $u-$substitution $u = y(x)$, $du = \frac{dy}{dx}\,dx$ to get

$$\int f(u)\,du = \int h(x)\,dx$$

Assuming F and H are antiderivatives of f and h, respectively, we get

$$F(u) = H(x) + C$$

which, since $u = y = y(x)$, is equivalent to the solution

$$F(y) = H(x) + C.$$

Note: We only need a constant on one side of the equation, because the constants on the two sides can be combined into one.

A shortcut:

The following method involves the writing of things that are technically incorrect, but always works. Again start with a separable, first-order ODE. Again, replace the y' with $\frac{dy}{dx}$ to get

$$f(y)\,\frac{dy}{dx} = h(x).$$

Now, pretend that the $\frac{dy}{dx}$ is a fraction and "multiply" through by dx to get

$$f(y)\,dy = h(x)\,dx.$$

Now, integrate both sides:

$$\int f(y)\,dy = \int h(x)\,dx$$

This gives the same solution as before:

$$F(y) = H(x) + C.$$

The shortcut above suggests the following method to solve separable ODEs:

Procedure to solve separable ODEs:

1. Replace y' with $\frac{dy}{dx}$ in the equation.
2. Separate the variables, i.e. put all the y (with the dy) on one side of the equation, and all the x (with the dx) on the other side of the equation.
3. Integrate both sides, putting the arbitrary constant on one side.
4. If the problem contains an initial value, plug it in for x and y and solve for the constant.
5. If the problem asks for a solution of the form $y(x)$, solve the solution for y.

EXAMPLE 4

Find the general solution of the ODE $y' = xy$.
(Recall that this example was studied qualitatively on page 316.)

EXAMPLE 5

Solve the following initial value problem:

$$\begin{cases} y' = 2x - 2xy \\ y(1) = -2 \end{cases}$$

EXAMPLE 6

Solve the ODE $y' = \frac{-x}{y}$. Without sketching a vector field, describe geometrically what the graphs of the solutions look like.

11.4 Exponential and logistic models

EXAMPLE 1

Experiments have shown that the rate at which a radioactive element decays is directly proportional to the amount present. (Radioactive elements are chemically unstable elements that decay, or decompose, into stable elements as time passes.) Suppose that if you start with 20 grams of a radioactive substance, in 2 years you will have 15 grams of radioactive substance left.

1. How much radioactive substance will you have after 7 years?
2. How long will it take for you to only have 5 grams of radioactive substance left?

Exponential growth and decay

The preceding example can be generalized as follows:

Whenever the rate of change of quantity y is proportional to y itself, then $y(t)$ satisfies the ODE

$$y' = ky,$$

where k is called the **proportionality constant**. This ODE has solution

$$y(t) = Ce^{kt}$$

where $C = y(0)$ is the initial amount of quantity y present in the system.

When $k < 0$, this model is called **exponential decay**. We see the following:

When $k > 0$, this model is called **exponential growth**. We see the following:

EXAMPLE 2

The amount of money in an account increases via an exponential growth model. If there is $\$200$ in the account initally and $\$350$ in the account after 3 years, how much will be in the account after 10 years?

Logistic growth and decay

Population biology models seek to determine or estimate the population y of a species in terms of the time t. Suppose the species reproduces at rate K. This means that the rate of change in y should be something like K times y. This makes sense if the population is small. But if the population gets too big (say greater than some number L), there is not enough food in the ecosystem to support all of the organisms, so the population won't grow despite reproduction, because the organisms starve. A differential equation representing this type of situation is called a **logistic equation** and has the following form:

$$y' = K\, y\, (L - y)$$

In a logistic equation, K is a constant called the **rate of reproduction** and L is a constant called the **carrying capacity** or **limiting capacity** of the system.

Question: What is the general solution of a logistic equation?

Solution: First, a preliminary algebra calculation (related to something you learn in Math 230 called "partial fraction decompositions"):

$$\left[\frac{\frac{1}{L}}{y} + \frac{\frac{1}{L}}{L-y}\right] = \frac{\frac{1}{L}(L-y)}{y(L-y)} + \frac{\frac{1}{L}(y)}{y(L-y)} = \frac{\frac{1}{L}(L-y+y)}{y(L-y)} = \frac{\frac{1}{L}(L)}{y(L-y)} = \frac{1}{y(L-y)}.$$

Now for the differential equation:

$$\begin{aligned}
\frac{dy}{dx} &= K\, y\, (L-y) \\
\frac{1}{y(L-y)}\, dy &= K\, dx \\
\left[\frac{\frac{1}{L}}{y} + \frac{\frac{1}{L}}{L-y}\right] dy &= K\, dx \qquad \text{(by the algebra calculation above)} \\
\int \left[\frac{\frac{1}{L}}{y} + \frac{\frac{1}{L}}{L-y}\right] dy &= \int K\, dx \\
\frac{1}{L}\ln y - \frac{1}{L}\ln(L-y) &= Kx + C \\
\ln y - \ln(L-y) &= KLx + C
\end{aligned}$$

Because the method of solution on the previous page is so long and complicated, the answer we get should be memorized:

Theorem 11.5 (Solution of the logistic equation) *The general solution of the logistic equation* $y' = Ky(L - y)$ *is*

$$y = \frac{L}{1 + Ce^{-KLt}}.$$

EXAMPLE 3

Suppose you release 1000 elk into a game refuge. Suppose that 5 years later there are 104 elk. If the carrying capacity of the game refuge is 4000 elk, and the elk population follows a logistic model, how many elk will be in the game refuge 15 years after the initial release?

EXAMPLE 4

Suppose y measures the concentration of a certain medicine in a patient's blood (so that the maximum possible value of y is 1. If the drug is removed from the patient's system at rate $.002$, and the patient receives a dose of the medicine that makes her initial concentration of the medicine be $.045$, and if the concentration of the drug follows a logistic model,

1. Write the initial value problem represented by this situation.

2. Write the formula which gives the concentration of the drug at time t.

11.5 Homework exercises

1. Verify that the equation $y = 3e^{4x}$ is a solution of the differential equation $y' = 4y$.

2. Verify that for any constant C, $y = Ce^{4x}$ is a solution of the differential equation $y' = 4y$.

3. Verify that for any constant C, $y = Ce^{x^2}$ is a solution of the differential equation $\frac{dy}{dx} = 2xy$.

4. Verify that the equation $y = \sin x \cos x - \cos^2 x$ is a solution of the differential equation $2y + y' = 2\sin(2x) - 1$ satisfying $y(\pi/4) = 0$.

5. Solve for y if $\frac{dy}{dx} = 3x^2$.

6. Solve for y if $\frac{dy}{dx} = \frac{x-2}{x}$.

7. Solve for y if $\frac{d^2y}{dx^2} = x^{-2}$.

In Problems 8-11, use *Mathematica* to sketch the vector field corresponding to the given first-order ODE (the appropriate *Mathematica* code can be found on page 317 or in the file `vectorfields.nb` on my web page). Your viewing window should be $[-6, 6] \times [-6, 6]$.

8. $\frac{dy}{dx} = \frac{1}{10}y(4 - y)$

9. $\frac{dy}{dx} = \frac{x}{x+y}$

10. $\frac{dy}{dx} = 2xy - y^2$

11. $\frac{dy}{dx} = e^{-x}\sin(\pi y)$

12. Consider the first-order ODE $\frac{dy}{dx} = \frac{1}{5}x\cos(\pi y/6)$.

    a) Use *Mathematica* to sketch the vector field associated to this ODE.

    b) Based on the picture you see, give one explicit solution of the ODE.

    c) Let $y = f(x)$ be the solution to this ODE passing through the origin. Find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.

    d) Let $y = f(x)$ be as in part (c). Estimate $f(4)$ and $f(-4)$.

    e) Let $y = h(x)$ be the solution to this ODE passing through $(0, -2)$. Estimate the smallest positive value of x such that $h(x) = 0$.

13. Consider the first-order ODE $\frac{dy}{dx} = \frac{1}{4}(y^3 - 16y)$.

    a) Use *Mathematica* to sketch the vector field associated to this ODE.
    b) Based on the picture you see, give three explicit solutions of the ODE.
    c) Let $y = f(x)$ be the solution to this ODE passing through $(0, 1)$. Find $\lim_{x\to\infty} f(x)$.
    d) Let $y = h(x)$ be the solution to this ODE passing through $(2, -2)$. Find $\lim_{x\to\infty} h(x)$.

14. Consider the first-order ODE $\frac{dy}{dx} = 2x + y$.

    a) Use *Mathematica* to sketch the vector field associated to this ODE.
    b) Let $y = f(x)$ be the solution to this ODE passing through $(-1, 1)$. Estimate $f(1)$.

In Problems 15-17, solve the indicated differential equation.

15. $\frac{dy}{dx} = \frac{x}{y}$
16. $(2 + x)y' = 3y$
17. $yy' = \sin x$

In Problems 18-22, solve the indicated initial value problem.

18. $\begin{cases} \sqrt{x} + \sqrt{y}y' = 0 \\ y(1) = 4 \end{cases}$
19. $\begin{cases} y(1 + x^2)y' - x(1 + y^2) = 0 \\ y(0) = \sqrt{3} \end{cases}$
20. $\begin{cases} yy' - e^x = 0 \\ y(0) = 6 \end{cases}$
21. $\begin{cases} \frac{dy}{dx} = e^{x-2y} \\ y(0) = 0 \end{cases}$
22. $\begin{cases} \frac{dy}{dx} = \frac{y}{4+x} \\ y(0) = 1 \end{cases}$

23. Suppose you have 300 mg of a radioactive substance initially, and that after 8 years you have 280 mg of the substance remaining.

    a) How much will you have left after 35 years?
    b) How long will it take you to have 200 mg of the substance left?

24. Radioactive radium has a half-life of 1599 years (this means that if you start with a certain amount of it, after 1599 years you will have half as much as you started with). What percent of an initial sample of radium will be present after 100 years?

25. Assume that an initial investment of $\$750$ is placed in an account. If the amount of money in that account increases exponentially so that it takes 7.75 years to double, what is the annual rate of growth?

26. Assume that an initial investment of $\$50$ is placed in an account. If the amount of money in that account increases exponentially and there is $\$63$ in the account after 3 years,

    a) How long will it take the account to be worth $\$100$?

    b) How much money will be in the account after 10 years?

27. Suppose that the size of a bacteria culture follows a logistic equation. At time $t = 0$ the bacteria culture weighs 1 gram, and two hours later the culture weighs 2 grams. Suppose further that the maximum possible mass of the culture is 10 grams.

    a) Write the initial value problem which models the mass of the bacteria culture.

    b) Find the mass of the culture after 5 hours.

    c) At what time will the mass of the culture be 8 grams?

28. A conservation organization releases 25 Florida panthers into a game preserve. After 2 years, there are 39 panthers in the preserve. Assume that the number of panthers in the preserve follows a logistic model with carrying capacity 200.

    a) Write the initial value problem which models the panther population.

    b) How many panthers will be alive after 5 years?

    c) When will the population reach 100?

29. Suppose y measures the concentration of dope in a professional athlete's system (scaled so that the maximum possible value of y is 1). Suppose the amount of dope in the athlete's system is $.3$ immediately after he takes a dose. If the concentration of dope in the athlete's system needs to be less than $.1$ for him to pass a drug test, how long does he need to wait after his dose in order to pass the test (assume the concentration in his system follows a logistic model with $k = -.05$)?

Answers

1. $y' = 12e^{4x} = 4y$.

2. $y' = 4Ce^{4x} = 4y$.

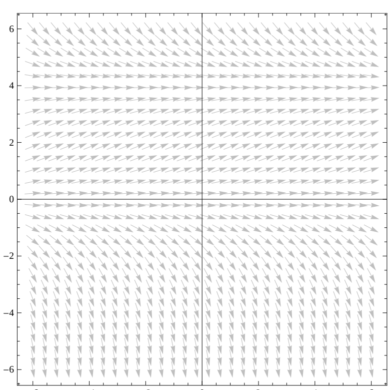
3. $y' = Ce^{x^2}2x = 2xy$.

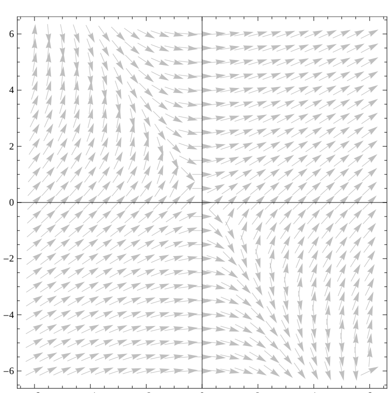
4. Differentiate to get $y' = \cos^2 x - \sin^2 x + 2\sin x \cos x$. Now $2y + y' = 2\sin x \cos x - 2\cos^2 x + \cos^2 x - \sin^2 x + 2\sin x \cos x = 4\sin x \cos x - 1 = 2\sin 2x - 1$ as desired. Last, $y(\pi/4) = 0$ by direct calculation.

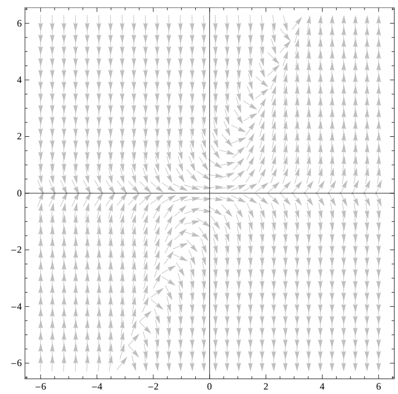
5. $y = x^3 + C$

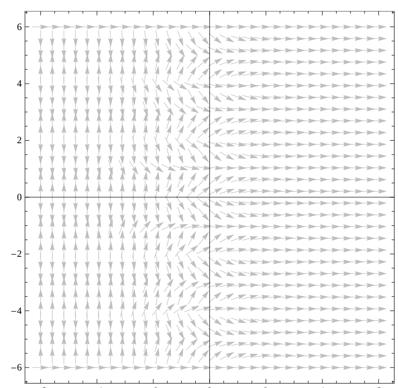
6. $y = x - 2\ln x + C$

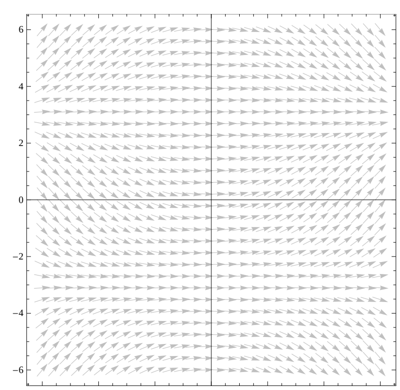
7. $y = -\ln x + Cx + D$

8. 

9. 

10. 

11. 

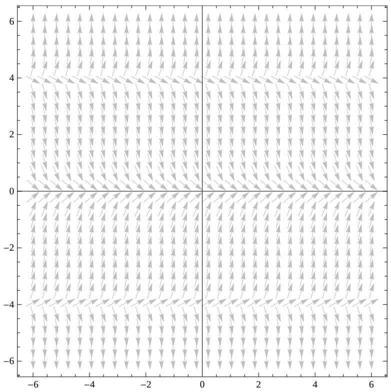
12. a) 

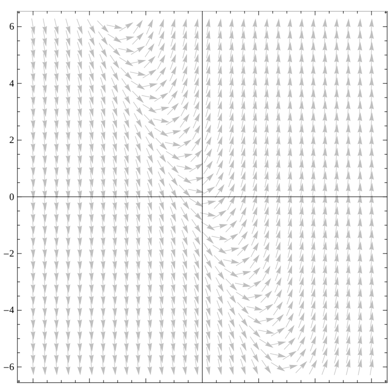
    b) $y = 3$ (also $y = -3, y = 9, y = -9$)

    c) Both answers are 3.

    d) Both answers are about 1.7.

    e) Roughly $x = 5$.

13. a) 
    b) $y = 4, y = 0, y = -4$
    c) 0
    d) 0

14. a) 
    b) 1

15. $\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$

16. $\ln y = 3\ln(x+2) + C$ (this simplifies to $y = K(x+2)^3$

17. $\frac{1}{2}y^2 = -\cos x + C$

18. $y = (9 - x^{3/2})^{2/3}$

19. $\frac{1+y^2}{1+x^2} = 4$

20. $\frac{1}{2}y^2 = e^x + 17$

21. $\frac{1}{2}e^{2y} = e^x - \frac{1}{2}$

22. $\ln y = \ln(x+4) - \ln 4$ (this simplifies to $y = \frac{1}{4}(x+4)$)

23. a) 221.836 mg
    b) 47 years

24. 95.7%

25. 8.943%

26. a) 8.9976 years
    b) $\$108.03$

27. a) $\begin{cases} y' = \left(\frac{1}{20}\ln\frac{9}{4}\right) y(10-y) \\ y(0) = 1 \end{cases}$
    b) $\frac{270}{59}$
    c) $t \approx 8.838$

28. a) $\begin{cases} y' = \left(\frac{1}{20}\ln\frac{9}{4}\right) y(10-y) \\ y(0) = 25 \end{cases}$
    b) 70
    c) After 7.37 years

29. 27 days

11.6 Review problems for Exam 4

Problems from Chapter 9

1. Precisely state both parts of the Fundamental Theorem of Calculus.

2. What is the difference between the concepts of *definite integral* and *indefinite integral*?

   Note: It is not correct to say that one of them has numbers on the integral sign and one doesn't; this only tells you the difference between the notation, not the difference between the underlying concepts.

3. Why is it important to write the dx with the integral sign? There is a theoretical answer and a practical answer; you should give both.

4. Suppose f and g are functions such that

   $$\int_2^4 f(x)\,dx = 5; \quad \int_2^7 f(x)\,dx = 3; \quad \int_2^7 g(x)\,dx = 2.$$

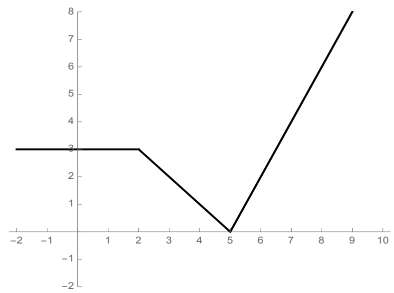
   Compute the following:

   a) $\int_4^7 f(x)\,dx$
   b) $\int_4^2 f(x)\,dx$
   c) $\int_2^7 (3f(x) - 2g(x))\,dx$
   d) $\int_2^4 (f(x) - x)\,dx$

5. Let $f(x) = x^2$, and let $\mathcal{P}$ be the partition $\{-6, -3, -2, 0\}$.

   a) What is $||\mathcal{P}||$?
   b) Using the usual indexing of numbers in a partition, what is Δx_2?
   c) Using the usual indexing of numbers in a partition, what number is x_1?
   d) Estimate the area under $f(x)$ from $x = -6$ to $x = 0$ by finding the left Riemann sum associated to the partition $\mathcal{P}$.
   e) Is your answer to part (d) an overestimate, or an underestimate, of the actual area under f from $x = -5$ to $x = 0$? Explain your answer.

6. Use the graph of the function f given below to estimate the given values:



a) $\int_{-1}^{2} 2f(x)\,dx$

b) $\int_{3}^{6} f(x)\,dx$

c) $\int_{2}^{0} f(x)\,dx$

d) $\int_{1}^{1} 4f(x)\,dx$

Problems from Chapter 10

In Problems 7-16, evaluate the indicated integrals:

7. $\int_{-2}^{3}(x^2-2x)\,dx$

8. $\int\left(\frac{2}{x^5}+3-\sqrt{x}\right)\,dx$

9. $\int_{\pi/4}^{\pi}\frac{\sin x}{2}\,dx$

10. $\int 2\csc 3x\cot 3x\,dx$

11. $\int\frac{(x-2)^2}{2x^3}\,dx$

12. $\int\sin^8 x\cos x\,dx$

13. $\int_{2}^{\sqrt{19}}3x\sqrt{x^2-3}\,dx$

14. $\int 6(3x^4-7)^{12}x^3\,dx$

15. $\int\frac{5}{2x-1}\,dx$

16. $\int(e^x+1)^2\,dx$

17. Suppose an object is moving so that its acceleration at time t is $a(t)=24t^2+120t$ m/sec^2. If the velocity of the object at time 0 is 20 m/sec and the position of the object at time 0 is -30, find the position and velocity of the object at time 2.

18. Suppose that the rate at which electricity is used by a house at time t (measured in days) is given by $f(t)=(\frac{1}{4}t^3+4)^3t^2$ kW/day. Find the total amount of electricity used by the house from $t=2$ to $t=4$.

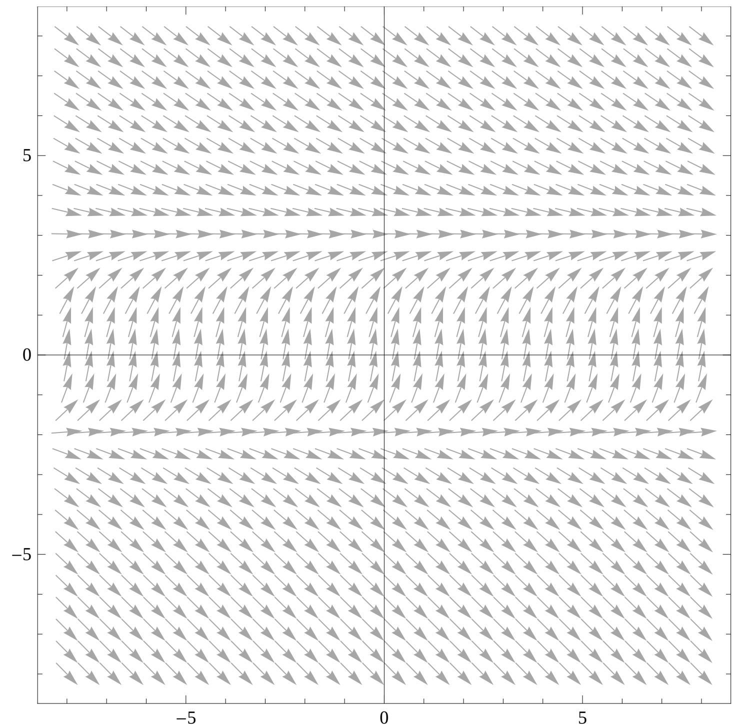
Problems from Chapter 11

19. Find the general solution of the ODE $4y^2y'=\sin x$.

20. Find the solution to this initial value problem:

$$\begin{cases}\frac{dy}{dx}=y^2x^2+y^2\\ y(2)=3\end{cases}$$

21. Use the picture below of a vector field associated to some ODE to answer the following questions:



a) Give one explicit solution to the differential equation.
b) On the vector field above, sketch the graph of the solution to this ODE passing through the point $(5, -4)$.
c) Let $y = f(x)$ be the solution to this ODE passing through the point $(2, 0)$.
   i. Estimate $f(3)$.
   ii. Estimate $f'(3)$.
   iii. Estimate $\lim_{x\to\infty} f(x)$.
   iv. Estimate $\lim_{x\to-\infty} f(x)$.

22. Suppose a bacterial culture has 300 cells in it initially. If after 1 hour, the culture has increased to 800 cells, what will the population be after 12 hours? Assume the number of cells in the culture grows exponentially.

23. Suppose the population of fish in a lake is given by a logistic equation where the carrying capacity is 800 fish, and the rate of reproduction is $.025$ (per year). If there are currently 8 fish in the lake, how many fish will be in the lake 20 years from now?

Answers

1.
   - **One part (differentiation of integrals):** Let f be continuous on $[a, b]$ and let $F(x) = \int_a^x f(t)\,dt$. Then F is differentiable on $[a, b]$ and $F'(x) = f(x)$.
   - **Other part (evaluation of integrals):** Let f be continuous on $[a, b]$ and let F be any antiderivative of f. Then $\int_a^b f(x)\,dx = F(b) - F(a)$.

2. A definite integral is a number which represents the area under the graph of f on some interval of $x-$values; an indefinite integral is the set of antiderivatives of a function, i.e. is a set of functions.

3. From a theoretical perspective, the dx represents the widths of really thin rectangles whose areas are being added up to give the area under a function, so without the dx, these widths are incorrectly being set to 1. From a practical perspective, the dx tells you the type of integral you are computing, tells you the variable of the integral, and helps with u-substitutions.

4.
   a) $\int_4^7 f(x)\,dx = \int_2^7 f(x)\,dx - \int_2^4 f(x)\,dx = 3 - 5 = -2$.

   b) $\int_4^2 f(x)\,dx = -\int_2^4 f(x)\,dx = -5$.

   c) $\int_2^7 (3f(x) - 2g(x))\,dx = 3\int_2^7 f(x)\,dx - 2\int_2^7 g(x)\,dx = 3(3) - 2(2) = 5$.

   d) $\int_2^4 (f(x) - x)\,dx = \int_2^4 f(x)\,dx - \int_2^4 x\,dx = 5 - [\frac{1}{2}x^2]_2^4 = 5 - [8 - 2] = -1$.

5.
   a) $||\mathcal{P}|| = 3$, the width of the widest subinterval.

   b) $\Delta x_2 = -2 - (-3) = 1$.

   c) $x_1 = -3$. (The first number is x_0.)

   d) The Riemann sum is $f(-6)\cdot 3 + f(-3)\cdot 1 + f(-2)\cdot 2 = 36\cdot 3 + 9\cdot 1 + 4\cdot 2 = 108 + 9 + 8 = 125$.

   e) Since $f'(x) = 2x$, $f'(x) < 0$ for $x \in [-6, 0]$ so f is decreasing on the interval. Thus the left Riemann sum is also the upper sum for this partition, hence it **overestimates** the actual area under f from $x = -6$ to $x = 0$.

6.
   a) $\int_{-1}^2 2f(x)\,dx = 2\int_{-1}^2 f(x)\,dx = 2(3)(3) = 18$ (the integral is $3(3)$ since it is the area of a rectangle with height 3 and width 3).

   b) $\int_3^6 f(x)\,dx$ is the area of two triangles; the left-hand triangle has vertices $(3, 2)$, $(3, 0)$ and $(5, 0)$ so it has area $\frac{1}{2}(2)(2) = 2$ and the right-hand triangle has vertices $(5, 0)$, $(6, 0)$ and $(6, 1)$ so it has area $\frac{1}{2}(1)(1) = \frac{1}{2}$. So the total area is $2 + \frac{1}{2} = \frac{5}{2}$.

   c) $\int_2^0 f(x)\,dx = -\int_0^2 f(x)\,dx = -(2)(3) = -6$.

   d) $\int_1^1 4f(x)\,dx = 0$.

7. $\int_{-2}^3 (x^2 - 2x)\,dx = \left[\frac{x^3}{3} - x^2\right]_{-2}^3 = \left[\frac{3^3}{3} - 3^2\right] - \left[\frac{(-2)^3}{3} - (-2)^2\right] = [9 - 9] - [\frac{-8}{3} - 4] = \frac{20}{3}$.

8. $\int \left(\frac{2}{x^5} + 3 - \sqrt{x}\right) dx = 2\int x^{-5}\, dx + \int 3\, dx - \int x^{1/2}\, dx = 2\frac{x^{-4}}{4} + 3x - \frac{x^{3/2}}{3/2} + C = \frac{-1}{2}x^{-4} + 3x - \frac{2}{3}x^{3/2} + C.$

9. $\int_{\pi/4}^{\pi} \frac{\sin x}{2}\, dx = \frac{-1}{2}\cos x|_{\pi/4}^{\pi} = \frac{-1}{2}\cos\pi - \frac{-1}{2}\cos\frac{\pi}{4} = \frac{1}{2} + \frac{1}{2\sqrt{2}} = \frac{1}{2} + \frac{\sqrt{2}}{4}.$

10. Let $u = 3x$ so that $du = 3dx$ and $\frac{2}{3}du = 2dx$. Then after substituting, the integral becomes $\int \frac{2}{3}\csc u \cot u\, du = -\frac{2}{3}\csc u + C = -\frac{2}{3}\csc 3x + C$.

11. Multiply out the numerator and split the integral:

$$\begin{aligned}\int \frac{(x-2)^2}{2x^3}\, dx &= \int \frac{x^2 - 4x + 4}{2x^3}\, dx \\ &= \int \left(\frac{1}{2}x^{-1} - 2x^{-2} + 2x^{-3}\right) dx \\ &= \frac{1}{2}\int x^{-1}\, dx - 2\int x^{-2}\, dx + 2\int x^{-3}\, dx. \\ &= \frac{1}{2}\ln x + 2x^{-1} - x^{-2} + C.\end{aligned}$$

12. Let $u = \sin x$ so that $du = \cos x\, dx$. After substituting, this becomes $\int u^8\, du = \frac{1}{9}u^9 + C = \frac{1}{9}\sin^9 x + C$.

13. Let $u = x^2 - 3$ so that $du = 2x\, dx$ and $\frac{3}{2}du = 3x\, dx$. Now, when $x = 2$, $u = 2^2 - 3 = 1$ and when $x = \sqrt{19}$, $u = (\sqrt{19})^2 - 3 = 16$. So substituting into the integral, we get $\int_1^{16} \frac{3}{2}\sqrt{u}\, du = \frac{3}{2}\frac{u^{3/2}}{3/2}|_1^{16} = u^{3/2}|_1^{16} = 16^{3/2} - 1^{3/2} = 64 - 1 = 63$.

14. Let $u = 3x^4 - 7$ so that $du = 12x^3\, dx$ and $\frac{1}{2}du = 6x^3\, dx$. Substituting into the integral, we obtain $\int \frac{1}{2}u^{12}\, du = \frac{1}{2}\frac{u^{13}}{13} + C = \frac{1}{26}(3x^4 - 7)^{13} + C$.

15. Let $u = 2x - 1$ so that $du = 2dx$ and $\frac{1}{2}du = dx$. Then the integral becomes $\frac{5}{2}\frac{1}{u}\, du = \frac{5}{2}\ln u + C = \frac{5}{2}\ln(2x-1) + C$.

16. Multiply out, then integrate: $\int (e^{2x} + 2e^x + 1)\, dx = \frac{1}{2}e^{2x} + 2e^x + x + C$.

17. First, $v(t) = \int a(t)\, dt = \int (24t^2 + 120t)\, dt = 8t^3 + 60t^2 + C$. To find C, note that $v(0) = 20$ so $20 = 8(0)^3 + 60(0)^2 + C$ so $C = 20$. Then $v(t) = 8t^3 + 60t^2 + 20$.

    Now, $p(t) = \int v(t)\, dt = \int (8t^3 + 60t^2 + 20)\, dt = 2t^4 + 20t^3 + 20t + D$. To find D, note that $p(0) = -30$ so $-30 = 2(0)^4 + 20(0)^3 + 20(0) + D$ so $D = -30$. Therefore the position is $p(t) = 2t^4 + 20t^3 + 20t - 30$ m.

    Finally, the position at time 2 is $p(2) = 2(2)^4 + 20(2)^3 + 20(2) - 30 = 32 + 160 + 40 - 30 = 202$ and the velocity at time 2 is $v(2) = 8(2)^3 + 60(2)^2 + 20 = 64 + 240 + 20 = 324$ m/sec.

18. We need to compute $\int_2^4 (\frac{1}{4}t^3 + 4)^3 t^2\, dt$. Use the $u-$sub $u = \frac{1}{4}t^3 + 4$, $du = \frac{3}{4}t^2$ to get $\int_6^{20} \frac{4}{3}u^3\, du = \frac{1}{3}u^4|_6^{20} = \frac{1}{3}(20^4 - 6^4)$ kW.

19. Separate variables to get $4y^2\,dy = \sin x\,dx$; then integrate both sides to get $\frac{4}{3}y^3 = -\cos x + C$.

20. Factor the right-hand side to get $\frac{dy}{dx} = y^2(x^2+1)$; then divide both sides by y^2 and write in differential form (i.e. "multiply by dx") to get $\frac{1}{y^2}dy = (x^2+1)dx$. Integrate both sides to get $\int y^{-2}\,dy = \int(x^2+1)\,dx$, i.e. $-y^{-1} = \frac{1}{3}x^3 + x + C$. Plugging in 2 for x and 3 for y and solving for C, we get $C = -5$. So the solution is $-y^{-1} = \frac{1}{3}x^3 + x - 5$.

21. a) $y = -2$ and $y = 3$ are both acceptable answers.
    b) This curve approaches -2 as $x \to -\infty$, decreases, passes through $(5,4)$, then heads southeast.
    c) i. $f(3) \approx 2$.
       ii. $f'(3) \approx 1$, the slope of the curve at $x = 3$.
       iii. $\lim\limits_{x\to\infty} f(x) = 3$.
       iv. $\lim\limits_{x\to-\infty} f(x) = -2$.

22. The exponential growth equation is $y = Ce^{kt}$. Substituting in $t = 0, y = 300$, we get $C = 300$ so $y = 300e^{kt}$. Now substituting in $t = 1, y = 800$ we get $800 = 300e^{k\cdot 1}$, i.e. $k = \ln\frac{8}{3}$. Now when $t = 12$, $y = 300e^{\ln(8/3)\cdot 12} = 300\cdot(8/3)^{12}$.

23. The logistic equation is $y = \frac{L}{1+Ce^{-kLt}} = \frac{800}{1+Ce^{-.025\cdot 800t}} = \frac{800}{1+Ce^{-20t}}$. Substituting in $t = 0$ and $y = 8$, we get $8 = \frac{800}{1+C}$, i.e. $C = 99$. This gives the equation $y = \frac{800}{1+99e^{-20t}}$; plugging in $t = 20$ we get $\frac{800}{1+99e^{-400}}$ fish in the lake after 20 years.

Appendix A

Mathematica information

A.1 General *Mathematica* principles

Mathematica is an extremely useful and powerful software package / programming language invented by a mathematician named Stephen Wolfram. Early versions of *Mathematica* came out in the late 1980s and early 1990s; the most recent version (which is loaded onto machines at FSU as of 2017) is *Mathematica* 11.

Mathematica does symbolic manipulation of mathematical expressions; it solves all kinds of equations; it has a library of important functions from mathematics which it recognizes while doing computations; it does 2- and 3-dimensional graphics; it has a built-in word processor tool; it works well with Java and C++; etc. One thing it doesn't do is prove theorems, so it is less useful for a theoretical mathematician than it is for an engineer or college student.

A bit about how *Mathematica* works: When you use the *Mathematica* program, you are actually running *two* programs. The "front end" of *Mathematica* is the part that you type on and the part you see. This part actually resides on the machine at which you are seated. The "kernel" is the part of *Mathematica* that actually does the calculations. If you type in `2 + 2` and hit [SHIFT]+[ENTER], the front end "sends" that information to the kernel which actually does the computation. The kernel then "sends" the result back to the front end, which displays the output `4` on the screen. Essentially, the way one uses *Mathematica* is by typing some "stuff" in, hitting [SHIFT]+[ENTER] to execute that stuff, and getting some output back from the program.

About *Mathematica* notebooks and cells: The actual files that *Mathematica* produces that you can edit and save are called *notebooks* and carry the file designation `*.nb`; they take up little space and can easily be saved to Google docs or on a flash drive, or emailed to yourself if you want them somewhere you can retrieve them. **Suggestion:** when saving any file, include the date in the file name (so it is easier to remember which file you are supposed to be open).

A *Mathematica* notebook is broken into *cells*. A cell can contain text, input, or output. A cell is indicated by a dark blue, right bracket (a "]") on the right-hand side of the notebook. To select a cell, click that bracket. This highlights the "]" in blue. Once selected, you can cut/copy/paste/delete cells as you would highlighted blocks of text in a Word document.

To change the formatting of a cell, select the cell, then click"Format / Style" and select the style you want. You may want to play around with this to see what the various styles look like. There are three particularly important styles:

- **input:** this is the default style for new cells you type
- **output:** this is the default style for cells the kernel produces from your commands
- **text:** changing a cell to text style allows you to make comments in between the calculations

Executing mathematical commands: To execute an input cell, put the cursor anywhere in the cell and hit [ENTER]. Well, not any [ENTER]; you have to use the [ENTER] on the numeric keypad at the far-right edge of the keyboard. The [ENTER] next to the apostrophe key (a.k.a. [RETURN]) gives you only a carriage return. You can also hold down the [SHIFT] key and hit either [ENTER] or [RETURN] to execute a command.

Important general concepts re: *Mathematica* syntax

1. **Multiplication:** use a star or a space: `2 * 3` or `2 3` will multiply numbers; `a x` means a times x; `ax` means the variable ax (in *Mathematica*, variables do not have to be named after one letter; they can be named by words or other strings of characters as well).

2. **Parentheses:** used for grouping and multiplication only. Parentheses mean "times" in *Mathematica*, and always mean that you intend to **multiply** what is in front of the parenthesis by what is inside the parenthesis.

3. **Brackets:** must be used to surround the input of any function or built-in *Mathematica* command. For example, to evaluate a function $f(x)$, you would type `f[x]`, not `f(x)`. Essentially, square brackets mean "**of**" in *Mathematica*.

4. **Capitalization:** All *Mathematica* commands and built-in functions begin with capital letters. For example, to find the sine of π, typing `sin(pi)` or `sin[pi]` does you no good (the first version would be the variable "sin" times the variable "pi", for instance). The correct syntax is `Sin[Pi]`. Similarly, e is `E` and i is `I` in *Mathematica*.

5. **Spaces:** *Mathematica* commands do not have spaces in them; for example, the inverse function of sine is `ArcSin`, not `Arc Sin` or `Arcsin`.

6. **Pallettes:** Lots of useful commands are available on the Basic Math Assistant Pallette, which can be brought up by clicking "Pallettes / Basic Math Assistant" on the toolbar. If you click on a button in the pallette, what you see appears in the cell. The tab halfway down this palette marked $d \int \Sigma$ has calculus commands, and the tab to the right of the $d \int \Sigma$ has matrix commands.

7. **Logarithms:** *Mathematica* does not know what `Ln` is. For natural logarithms (base e), type "Log[ ]". For common logarithms (base 10), type "Log10[ ]".

8. $\%$ refers to the last output (like "Ans" on a TI-calculator).

9. **Help:** To get help on a command, type "?" followed by the command you don't understand. If necessary, click the $\gg$ you get at the end of the help blurb to open a help browser. You can also find out how to do lots of stuff in *Mathematica* by using Google: search for what you want help on.

10. *Mathematica* gives exact answers (i.e. not decimals) for everything if possible. If you need a decimal approximation, use the command `N[ ]`. For example, `N[Pi]` spits out 3.14159...

11. If *Mathematica* freezes up in the middle of a calculation, click "Evaluation / Abort Evaluation" on the toolbar.

A.2 *Mathematica* quick reference guides

Basic operations

	Expression	*Mathematica* syntax
SPECIAL SYMBOLS	e	`E`
	π	`Pi`
	i (i.e. $\sqrt{-1}$)	`I`
	∞	`Infinity` (or use Basic Math Assistant palette)
ARITHMETIC	$3+4x$	`3 + 4x`
	$5-7$	`5 - 7`
	$8z$	`8z` or `8 z` or `8 * z`
	xy	`x y` (don't forget the space)
	$\frac{7}{3}$	`7/3`
	$\frac{x-7+2y}{a-7b}$	To get the fraction bar, type [CONTROL]+/ then use [TAB] to move between the top and bottom
	$\sqrt{32}$	`Sqrt[32]` (or type [CONTROL]+2 to get a $\sqrt{}$ sign) (or use Basic Math Assistant palette)
	$\sqrt[4]{40}$	`40^(1/4)` (or use Basic Math Assistant palette)
	$\lvert x-3\rvert$	`Abs[x-3]`
	$30!$ (factorial)	`30!`
EXPS AND LOGS	$\ln 3$	`Log[3]`
	$\log_6 63$	`Log[6,63]`
	$\log 18$	`Log10[18]` or `Log[10,18]`
	2^{7y}	`2^(7y)` (or type `2`, then [CONTROL]+6, then `7y`) (or use Basic Math Assistant palette)
	e^{x-5+x^2}	`E^(x-5+x^2)` or `Exp[x-5+x^2]` (or use Basic Math Assistant palette)
TRIG	$\sin\pi$	`Sin[Pi]`
	$\cos(x(y+1))$	`Cos[x(y+1)]`
	$\cot\left(\frac{2\pi}{3}+\frac{3\pi}{4}\right)$	`Cot[2 Pi/3 + 3 Pi/4]`
	$\arctan 1$	`ArcTan[1]`

Objective	*Mathematica* syntax
To call the preceding output	`%`
To get a decimal approximation to the preceding output	`N[%]` (or click **numerical value**)

Solving equations (see also section A.4)

Objective	*Mathematica* syntax
Find exact solution(s) to equation of form $lhs = rhs$	`Solve[`lhs `==` rhs`, x]` (two equals signs) (works only with polynomials or other relatively "easy" equations)
Find decimal approximation(s) to solution(s) of equation $lhs = rhs$	`NSolve[`lhs `==` rhs`, x]` (two equals signs) (works only with "easy" equations)
Find decimal approximation(s) to to solution(s) of equation $lhs = rhs$	`FindRoot[`lhs `==` rhs`, {x,`$guess$`}]` (two equals signs)

Defining functions

Objective	*Mathematica* syntax
Define a function $f(x) = formula$	`f[x_] = `$formula$ (one equals sign, underscore after `x`)

Tables and graphs (see also Section A.3)

Objective	*Mathematica* syntax
Generate table of values for f	`Table[{x,f[x]}, {x,`$xmin$`,`$xmax$`,`$step$`}]` (put `//TableForm` at end of command to arrange output in a table)
Plot the graph of $f(x) = formula$	`Plot[`$formula$`, {x,`$xmin$`,`$xmax$`}]`
Plot multiple graphs at once	`Plot[{`$formula$`,`$formula$`, ..., `$formula$`},` `{x,`$xmin$`,`$xmax$`}]`
Plot the graph of $f(x) = formula$ with range of $y-$values specified	`Plot[`$formula$`, {x,`$xmin$`,`$xmax$`},` `PlotRange -> {`$ymin$`,`$ymax$`}]`

Plot the graph of $f(x) = formula$ with x- and y-axes on same scale	`Plot[`$formula$`, {x,`$xmin$`,`$xmax$`},`
	`PlotRange -> {`$ymin$`,`$ymax$`},`
	`AspectRatio -> Automatic]`

Function operations and calculus

Expression	*Mathematica* syntax
$f(x+3)$ (if f is a function)	`f[x+3]`
$xf(2x) - x^2 f(x)$	`x f[2x] - x^2 f[x]` (spaces important)
$(f \circ g)(x)$	`f[g[x]]`
$\lim_{x \to 4} f(x)$	`Limit[f[x], x -> 4]`
$f'(3)$	`f'[3]`
$g'''(x)$	`g'''[x]` or `D[g[x],{x,3}]`
$\int x^2\,dx$	`Integrate[x^2,x]` (or use $\int$ sign on Basic Math Assistant palette)
$\int_2^5 \cos x\,dx$	`Integrate[Cos[x], {x, 2, 5}]` (or use $\int_{\square}^{\square}$ sign on Basic Math Assistant palette) (for a decimal approximation, use `NIntegrate`)
$\sum_{k=1}^{12} f(k)$	`Sum[f[k], {k, 1, 12}]` (or use Basic Math Assistant palette)
$\sum_{k=1}^{\infty} blah$	`Sum[`$blah$`, {k, 1, Infinity}]` (or use Basic Math Assistant palette)

Other

Objective	*Mathematica* syntax
Factor a polynomial	`Factor[ ]`
Multiply an expression out (i.e. "FOIL" an expression) (i.e. "undo" factoring)	`Expand[ ]`
Simplify an expression	`Simplify[ ]`

A.3 Graphing functions with *Mathematica*

Defining a function in *Mathematica*

To graph a function $y = f(x)$ on *Mathematica*, you usually start by defining the function. For example, to define a function like $f(x) = 3\cos 4x - x$, execute

```
f[x_] = 3 Cos[4x] - x
```

You could just as well use a different letter for the independent variable. For example, typing

```
f[t_] = 3 Cos[4t] - t
```

would accomplish the same thing as above. However, don't mix and match! Typing

```
f[x_] = 3 Cos[4t] - t
```

doesn't accomplish anything, because there is a `x` on the left-hand side, and a `t` on the right-hand side.

The general syntax for defining a function is

$$function\,name\texttt{[}variable\texttt{\_] = }formula$$

it is important to include the underscore after the variable to tell *Mathematica* you are defining a function.

The basic Plot command

Immediately after defining a function as above, you will get (underneath your output) a list of suggested follow-up commands. One of these is plot. If you click the word plot, you will get a graph of the function you just defined. Here, *Mathematica* picks a range of x- and y-values it thinks will work well for the function you defined. It is useful to remember the syntax of this Plot command:

$$\texttt{Plot[}formula\texttt{, \{}variable\texttt{, }xmin\texttt{, }xmax\texttt{\}]}$$

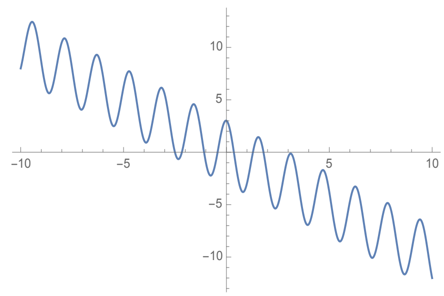
In this command:

- $formula$ is the function you want the graph of. It could be an expression like `f[x]` or `f[t]`, or a typed-out formula like `3 Cos[4x] - x`.
- $variable$ is the name of the independent variable (usually `x` or `t`); this must match the variable in the formula.

- $xmin$ and $xmax$ are numbers which represent, respectively, the left-most and right-most values of the independent variable shown on the graph. For example, if your Plot command has `{x,-3,5}` in it, then the graph will go from $x = -3$ to $x = 5$.

Here is an example, which plots $f(x) = 3\cos 4x - x$ from $x = -10$ to $x = 10$:

```
Plot[3 Cos[4x] - x, {x, -10,10}]
```



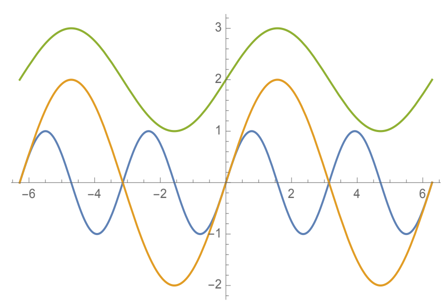
Plotting multiple functions at once

Suppose you want to plot more than one function on the same set of axes. To do this, you tweak the earlier Plot command by replacing the formula with a list of formulas inside squiggly braces, separated by commas. Thus the command you execute looks something like this:

Plot[{$formula1$, $formula2$, ...}, {$variable$, $xmin$, $xmax$}]

For example, the following command plots $\sin 2x$, $2\sin x$ and $\sin x + 2$ on the same set of axes:

```
Plot[{Sin[2x], 2 Sin[x], Sin[x] + 2}, {x, -2 Pi, 2 Pi}]
```



In *Mathematica* 10, the first graph you type will be blue; the second graph you type will be orange; the third graph you type is green; other graphs are in other colors. To change the way the graphs look, consult the end of this section.

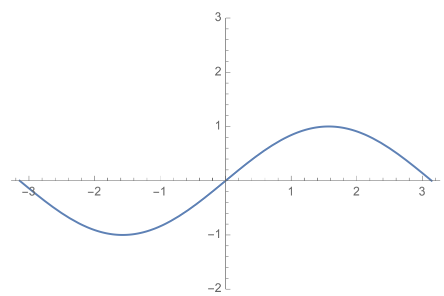
Specifying a range of y-values

By default, *Mathematica* just chooses a range of y-values it thinks will make the graph look good. If you want to force *Mathematica* to use a particular range of y-values, then you have to insert a phrase in the Plot command called `PlotRange`. This goes after the {x, *xmin*, *xmax*} and after another comma, but before the closing square bracket. The general command is

Plot[{*formulas*}, {*var*, *xmin*, *xmax*}, PlotRange -> {*ymin*, *ymax*}]

and an example of the code, which plots $\sin x$ on the viewing window $[-\pi, \pi] \times [-2, 3]$ is

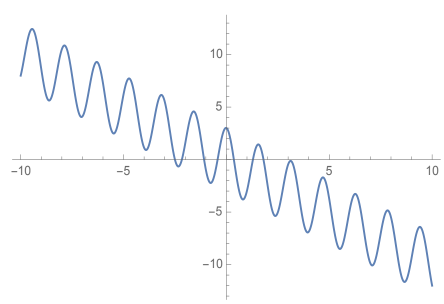
```
Plot[Sin[x], {x, -Pi, Pi}, PlotRange -> {-2,3}]
```



Making the x- and y-axes have the same scale on the screen

Here is the graph of $f(x) = 3\cos 4x - x$ that *Mathematica* produces with the command

```
Plot[3 Cos[4x] - x, {x, -10,10}]
```



If you look at this graph, the distance from the origin to $(5, 0)$ looks a lot longer than the distance from the origin to $(0, 5)$. But in actuality, both these distances are 5 units. The graph is distorted so that it fits nicely on your screen. To fix the distortion (you might want to do this if you needed to estimate the slope of a graph accurately), insert the command `AspectRatio -> Automatic` into the Plot command (similar to how you would insert a `PlotRange` command). This forces the number of pixels on your screen representing one unit in the x direction to be equal to the number of pixels on your screen representing one unit in the y direction. Here is the general syntax:

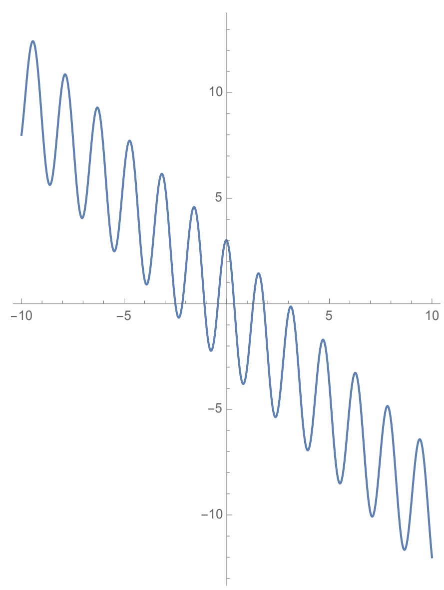
```
Plot[{formulas}, {var,xmin,xmax}, AspectRatio -> Automatic]
```

This command can also be used with the `PlotRange` command:

```
Plot[{formulas}, {var,xmin,xmax}, PlotRange -> {ymin,ymax},
 AspectRatio -> Automatic]
```

Here is an example command:

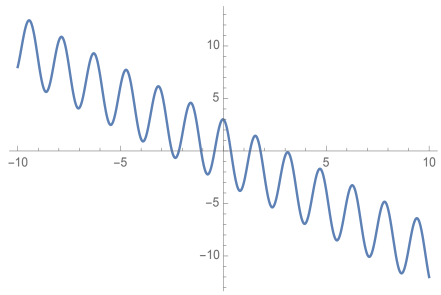
```
Plot[3 Cos[4x] - x, {x, -10,10}, AspectRatio -> Automatic]
```



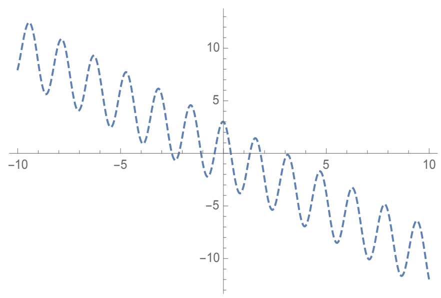
Changing the appearance of the curves

As mentioned earlier, by default *Mathematica* graphs all the functions with solid lines, using different colors for different formulas on the same picture. To change this, insert various directives into the Plot command using `PlotStyle`. Here are some examples:

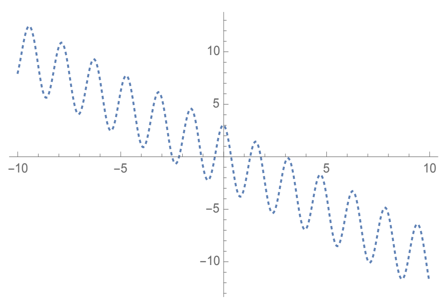
```
Plot[3 Cos[4x] - x, {x, -10,10}, PlotStyle -> Thick]
```




```
Plot[3 Cos[4x] - x, {x, -10,10}, PlotStyle -> Dashed]
```




```
Plot[3 Cos[4x] - x, {x, -10,10}, PlotStyle -> Dotted]
```



If you are plotting more than one function at once, then after the `PlotStyle ->`, you can type a list of graphics directives, separated by commas, enclosed by a set of squiggly braces. The directives will be applied to each function you are graphing, in the same order as they are typed after the `PlotStyle ->`. For example, this command plots $x, 2x$ and $3x$, where x is thick and black, $2x$ is red and dotted, and $3x$ is blue and dashed:

```
Plot[{x,2x,3x}, {x, -3,3},
 PlotStyle -> {{Thick, Black}, {Dotted, Red}, {Blue, Dashed}}]
```

A.4 Solving equations with *Mathematica*

There are three methods to solve an equation using *Mathematica*. They have something in common: to solve an equation, the equation **must be typed with two equals signs** where the $=$ is. (A single equal sign is used in *Mathematica* to assign values to variables, which doesn't apply in the context of solving equations.)

The Solve command

To solve an equation of the form $lhs = rhs$, execute

`Solve[`lhs `==` rhs`,` $variable$`]`

where $variable$ is the name of the variable you want to solve for. For example, to solve $x^2 - 2x - 7 = 0$ for x, execute `Solve[x^2 - 2x - 7 == 0, x]`.

You can solve an equation for one variable in terms of others: for example, `Solve[a x + b == c, x]` solves for x in terms of a, b and c.

WARNING: The advantage of the `Solve` command is that it gives exact answers (no decimals); this can be a pro or con (as sometimes the exact answers are horrible to write down). The disadvantage is that it only works on polynomial, rational and other "easy" equations. It won't work on equations that mix-and-match trigonometry and powers of x like $x^2 = \cos x$.

The NSolve command

`NSolve` works exactly like `Solve`, except that it gives decimal approximations to the solutions. It has the same drawback as `Solve` in that it only works on reasonably "easy" equations. The syntax is

`NSolve[`lhs `==` rhs`,` $variable$`]`

The FindRoot command

To find decimal approximations to equations that are too hard for the `Solve` and `NSolve` commands, use `FindRoot`. This executes a numerical algorithm to estimate a solution to an equation. The good news is that this command always works; the bad news is that it requires an initial "guess" as to what the solution is (usually you determine the initial guess by graphing both sides of the equation and seeing roughly where the graphs cross). For example, to find a solution to $x^2 = \cos x$ near $x = 1$, execute

```
FindRoot[x^2 == Cos[x], {x, 1}]
```

and to find a solution to the same equation near $x = -1$, execute

```
FindRoot[x^2 == Cos[x], {x, -1}]
```

(these probably won't give the same solution). The general syntax for this command is

FindRoot[*lhs*==*rhs*, {*variable*,*guess*}]

A.5 Code for Newton's method

You need three lines of code, all in the same cell. For example, to implement Newton's method for the function $f(x) = x^2 - 2$ where $x_0 = 3$ and you want to perform 6 iterations (to find x_6), just type

```
f[x_] = x^2 - 2;
Newton[x_] = N[x - f[x]/f'[x]];
NestList[Newton, 3, 6]
```

and execute (all three lines at once). The first line defines the function f, the second line gives a name to the formula you iterate in Newton's Method, and the last line iterates the formula and spits out the results.

The resulting output for the code listed above is:

```
{3, 1.83333, 1.46212, 1.415, 1.41421, 1.41421, 1.41421}
```

These numbers are $x_0, x_1, x_2, ..., x_6$ so for example, $x_2 = 1.46212$ and $x_4 = 1.41421...$ and $x_6 = 1.41421$ (the same as x_4 to 5 decimal places).

To implement Newton's method for a different function, different initial guess and different number of iterations, simply change the formula for f, change the 3 to the appropriate value of x_0 and the 6 to the number of times you want to iterate Newton's method.

A.6 Code for Riemann sums

In this section we discuss how to compute left- and right- Riemann sums using *Mathematica*. Ultimately, to do a Riemann sum you need to execute three commands found later in this section; for now we explain where these commands come from.

1. Defining the function f

First, recall that to define a function you use an underscore. For example, the following command defines f to be the function $f(x) = x^2$:

```
f[x_] = x^2
```

2. Defining the partition $\mathcal{P}$

Defining a partition in *Mathematica* is easy. Just use braces, and list the numbers from smallest to largest. For example, to define the partition $\mathcal{P} = \{0, 1, \frac{5}{2}, 4, 7\}$, just execute

```
P = {0, 1, 5/2, 4, 7}
```

We often use partitions which divide $[a, b]$ into n equal-length subintervals. To create such a partition in *Mathematica*, use the `Range` command. For example, to define a partition of $[0, 2]$ into 10 equal-length subintervals, execute the following:

```
P = Range[0, 2, (2-0)/10]
```

The `0` is a, the `2` is b, and the last number `2-0/10` is $\frac{b-a}{n}$, the width of each subinterval. In general, to split $[a, b]$ into n equal-length subintervals, execute

```
P = Range[a,b,(b-a)/n]
```

3. How to get to the individual numbers in a partition $\mathcal{P}$

Suppose you have defined a partition $\mathcal{P} = \{x_0, x_1, ..., x_n\}$ in *Mathematica*. To call one of the elements of $\mathcal{P}$, use double brackets as shown below. **There is a catch:** in handwritten math notation, we write our partitions starting with index 0. But *Mathematica* starts its partitions with index 1. So if $\mathcal{P} = \{0, 1, 5/2, 4, 7\}$ has been defined in *Mathematica*, executing

```
P[[3]]
```

generates the output $\frac{5}{2}$, which we think of as x_2, not x_3.

In general, once you have typed in a partition $\mathcal{P}$,

- execute `P[[j]]` to get the $(j-1)^{th}$ term x_{j-1}, and
- execute `P[[j+1]]` to get the j^{th} term x_j.

4. How to do sums (not necessarily Riemann sums) in *Mathematica*

Suppose you want to compute some sum which is written in $\Sigma-$notation. To do this, open the Basic Math Assistant pallette and click the $[d \int \sum]$ button (located under the phrase "Basic Commands"). In the first column of buttons, you will see a $\sum$ which you can click on to put a $\sum$ in your cell. You will get boxes to type all the pieces of the sum in.

5. An explanation of how to generate a Riemann sum for a function

First, remember that in any Riemann sum, $\Delta x_j = x_j - x_{j-1}$. From the remarks earlier in this section, we know that in *Mathematica* this expression is `P[[j+1]] - P[[j]]`.

Next, suppose we are doing a left-hand sum. Then the test points c_j satisfy

$$\begin{aligned} c_j &= \text{ left endpoint of the } j^{th} \text{ subinterval} \\ &= \text{ left endpoint of } [x_{j-1}, x_j] \\ &= x_{j-1}. \end{aligned}$$

Therefore, $c_j = x_{j-1}$ should be `P[[j]]` in *Mathematica* code, and $f(c_j)$ is `f[ P[[j]] ]`.

Putting this together, the right *Mathematica* code for a left-hand Riemann sum (assuming you have defined your function `f` and your partition `P`) is

```
  n
  ∑  f[ P[[j]] ] (P[[j + 1]] - P[[j]])
j = 1
```

6. The final commands for left- and right-hand Riemann sums

From above, we came up with the following sequence of commands for computing a left-hand Riemann sum:

Syntax to compute a left-hand Riemann sum

To evaluate a left-hand Riemann sum, execute the following commands:

```
f[x_] = x^2
```
(or whatever your function is)

```
P = {0, 1/2, 3/4, 1}
```
(or whatever your partition is)

$\sum_{\texttt{j = 1}}^{n}$ `f[ P[[j]] ] (P[[j+1]] - P[[j]])`
(n is the number of subintervals)

To evaluate a right-hand sum, the only thing that changes is the test point c_j, which goes from the left endpoint x_{j-1} (i.e. `P[[j]]`) to the right endpoint x_j (i.e. `P[[j+1]]`). Thus the commands for computing a right-hand Riemann sum are similar:

Syntax to compute a right-hand Riemann sum

To evaluate a right-hand Riemann sum, execute the following commands:

```
f[x_] = x^2
```
(or whatever your function is)

```
P = {0, 1/2, 3/4, 1}
```
(or whatever your partition is)

$\sum_{\texttt{j = 1}}^{\texttt{n}}$ `f[ P[[j+1]] ] (P[[j+1]] - P[[j]])`
(n is the number of subintervals)

A.7 Code for vector fields and differential equations

Code to plot a vector field

Here is an example of the syntax which produces the vector field associated to a first-order ODE (type this all in one cell):

```
g[x_,y_] : = 3y;
VectorPlot[{1,g[x,y]},{x,-3,-3},{y,-3,3}, VectorPoints -> 20,
Axes -> True, VectorScale -> {Automatic, Automatic, None},
VectorStyle -> Orange]
```

What this command does: First, the function $g(x, y)$ is defined in the first line. If you wanted to sketch a vector field for a different equation, you can change the `3y` here to whatever formula is given by $g(x, y)$. Second, the `VectorPlot` command tells *Mathematica* to sketch the vector field. The relevant ingredients here are as follows:

- The `{x, -3, 3}` and `{y, -3, 3}` tell *Mathematica* to produce a picture that runs from $x = -3$ to $x = 3$ and $y = -3$ to $y = 3$, i.e. it specifies the range of the picture.
- The `VectorPoints -> 20` tells *Mathematica* how many vectors to draw. If you change 20 to a larger number, you get more densely drawn arrows. This produces a more accurate picture of the vector field, but the program will take longer to run. On the other hand, if you change 20 to a smaller (positive) number, less arrows will be drawn, resulting in a less dense picture.
- The `Axes -> True` tells *Mathematica* to draw the x- and y- axes on the picture.
- The `VectorScale -> {Automatic, Automatic, None}` tells *Mathematica* to make sure all the vectors are scaled so that you can see them (without this command many of the vectors drawn would be too small to see).
- The `VectorStyle -> Orange` tells *Mathematica* to make the vectors orange. An interesting thing to do is to replace this command with the following phrase: `VectorColorFunction -> Hue` to get a 'tie-dye' looking picture.

Code to produce a stream plot

Here is a command which plots both the vector field and some solution curves to the differential equation $y' = g(x, y)$. It comes from taking the command above and adding some additional programming inside the `VectorPlot` command:

```
g[x_,y_] : = 3y;
VectorPlot[{1,g[x,y]},{x,-3,-3},{y,-3,3}, VectorPoints -> 20,
```

```
Axes -> True, VectorScale -> {Automatic, Automatic, None},
VectorStyle -> Orange, StreamPoints -> 35, StreamScale -> Full,
StreamStyle -> {Blue, Thick}]
```

As before, the first line defines $g(x, y)$; the `{x, -3, 3}` and `{y, -3, 3}` determine the range of the picture; the `VectorPoints -> 20` tell *Mathematica* how many arrows to draw; `Axes -> True` tells *Mathematica* to draw the x- and y- axes; and the `VectorScale` command ensures the vectors are big enough to see.

What' s new here are the commands regarding "Streams". The `StreamPoints -> 35` asks *Mathematica* to draw 35 stream lines (a stream line is a graph that ' follows' the vector field as described in lecture). If you lower the number 35, less stream lines are drawn and if you increase the number, more stream lines are drawn.

The `StreamScale -> Full` command ensures that the stream lines are connected when they should be (so that you get a much better picture). The last bit tells *Mathematica* what color to make the stream lines and to make them thick.

Code to produce a steam plot through a particular point

Suppose you wanted to draw one stream line that went through one specified point. In this case, you modify the above command by changing the `StreamPoints` directive from `35` to `{{a,b}}` where you want to draw the stream line through the point (a, b). For example, in the above picture, if you wanted the stream line passing through the point $(1, 2)$, your command would be :

```
g[x_,y_] : = 3y;
VectorPlot[{1,g[x,y]},{x,-3,-3},{y,-3,3}, VectorPoints -> 20,
Axes -> True, VectorScale -> {Automatic, Automatic, None},
VectorStyle -> Orange, StreamPoints -> {{1,2}}, StreamScale -> Full,
StreamStyle -> {Blue, Thick}]
```

Euler's method

All this code is available in the file `eulermethod.nb`, available on my web page.

To implement Euler's method using *Mathematica*, first run this block of code **once** each time you start *Mathematica* to define a program called `euler`:

```
euler[f_, {t_, t0_, tn_}, {y_, y0_}, steps_] :=
 Block[{told = t0, yold = y0, thelist = {{t0, y0}}, t, y, h},
  h = N[(tn - t0)/steps];
  Do[tnew = told + h;
```

```
 ynew = yold + h *(f /.{t -> told, y -> yold});
 thelist = Append[thelist, {tnew, ynew}];
 told = tnew;
 yold = ynew, {steps}];
Return[thelist];]
```

Once the above command is executed, you can then implement Euler's method with:

euler[*formula*, {t,t_0, t_n}, {y,y_0}, n]

Here,
- (t_0, y_0) is the initial value;
- t_n is the value of t where you want to estimate y (i.e. the ending value of t);
- n is the number of steps.

To get only the last point in the list (which is usually what you are most interested in), tweak this command as follows:

```
euler[3y, {t,1,3}, {y,-1}, 400][[401]]
```

The number in the double brackets should always be one more than the number of steps.

Plotting the points coming from Euler's method

Surround the `euler` command with `ListPlot[ ]`:

```
ListPlot[euler[3y, {t, 1,3}, {y, -1}, 400]]
```

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