An Introduction to Fourier Analysis

Fourier Series, Partial Differential Equations and Fourier Transforms

Solutions for MA3139 Problems

Arthur L. Schoenstadt
Department of Applied Mathematics
Naval Postgraduate School
Code MA/Zh
Monterey, California 93943

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PROBLEMS

1. For each of the following sequences, determine if the sequence converges or diverges. If the sequence converges, determine the limit

a. $a_n = \dfrac{2^{n+1}}{3^{n+2}}$ b. $a_n = \dfrac{(n+1)^2}{5n^2+2n+1}$ c. $a_n = \dfrac{\sin(n)}{n+1}$

d. $a_n = \cos(n)$ e. $a_n = \dfrac{2(n+1)^2 + e^{-n}}{3n^2+5n+10}$ f. $a_n = \dfrac{n\cos(\frac{n\pi}{2})}{n+1}$

g. $a_n = \dfrac{\cos(n\pi)}{n^2+1}$ h. $a_n = \dfrac{e^n}{n!}$ i. $a_n = \dfrac{n\sin(n\pi)}{n+1}$

2. Determine the order ("big Oh") of the following sequences

a. $a_n = \dfrac{n^3+2n^2+1000}{n^7+600n^6+n}$ b. $a_n = \dfrac{\cos(n\pi)}{n^2+1}$

c. $a_n = \left[\dfrac{n}{n^2-1} - \dfrac{n}{n^2+1}\right]\sin((n+\dfrac{1}{2})\pi)$

d. $a_n = \dfrac{10n^3e^{-n}+n^2}{(2n+1)^2}\cos(n^2\pi)$

3. Consider the infinite series

$$\sum_{n=0}^{\infty} \frac{(n+1)^2 2^n}{(2n)!}$$

a. Compute, explicitely, the partial sums S_3 and S_6

b. Write the equivalent series obtained by replacing n by $k-2$, i.e. by shifting the index.

4. Determine whether each of the following infinite series diverges or converges:

a. $\displaystyle\sum_{n=0}^{\infty} e^{-n}$ b. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2+1}{(n+1)^3}$ c. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2\cos(n\pi)}{(n^3+1)^2}$

d. $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n+3}$ e. $\displaystyle\sum_{n=0}^{\infty} \frac{e^n}{n!}\cos(n\pi)$ f. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

5. Determine an (approximate) upper bound to the error when each of the following infinite series is approximated by a twenty-term partial sum (S_{20}).

a. $\displaystyle\sum_{n=0}^{\infty} \frac{2n+1}{3n^4+n+1}$ b. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$ c. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n+1)^2}{n^4}$

6. Consider the series:

$$\sum_{n=0}^{\infty} x^n$$

a. plot the partial sums $S_1(x)$, $S_5(x)$, $S_{10}(x)$, and $S_{20}(x)$ for $-2 < x < 2$.

b. What can you conclude about the convergence of the partial sums in this interval?

c. What, if anything, different can you conclude about the convergence of these partial sums in the interval $-\frac{1}{2} < x < \frac{1}{2}$.

1. a.

$$a_n = \frac{2^{n+1}}{3^{n+2}} = \frac{2}{9}\left(\frac{2}{3}\right)^n \to 0 \text{ since } \frac{2}{3} < 1$$

1. b.

$$a_n = \frac{(n+1)^2}{5n^2+2n+1} = \frac{n^2+2n+1}{5n^2+2n+1} = \frac{1+\frac{2}{n}+\frac{1}{n^2}}{5+\frac{2}{n}+\frac{1}{n^2}} \to \frac{1+0+0}{5+0+0} = \frac{1}{5}$$

1. c.

$$a_n = \frac{\sin n}{n+1} \quad \text{Note } |\sin n| \leq 1$$

Thus $|a_n| \leq \dfrac{1}{n+1} \to 0$.
Thus $a_n \to 0$.

1. d.

$$a_n = \cos n \quad \text{does not approach a limit}$$

Thus the sequence diverges.

1. e.

$$a_n = \frac{2(n+1)^2 + e^{-n}}{3n^2+5n+10} \to \frac{2n^2}{3n^2} = \frac{2}{3} \quad \text{since } e^{-n} \to 0$$

1. f.

$$a_n = \frac{n}{n+1}\cos\left(\frac{n\pi}{2}\right) \to \cos\left(\frac{n\pi}{2}\right)$$

Since $\cos\left(\dfrac{n\pi}{2}\right) = 0, \pm 1,$ depending on n the sequence diverges.

1. g.

$$a_n = \frac{\cos(n\pi)}{n^2+1}$$

Thus $|a_n| \leq \dfrac{1}{n^2+1} \to 0$ and $a_n \to 0$.

1. h.

$$a_n = \frac{e^n}{n!} = \frac{e^n}{n(n-1)\cdots 2\cdot 1}$$

Note $n! = \underbrace{n(n-1)\cdots 2\cdot 1}_{n\ terms} \geq \underbrace{3\cdots 3}_{n-2\ terms} \cdot 2\cdot 1 = 3^{n-2}\cdot 2$

Thus $a_n = \dfrac{e^n}{n!} \leq \dfrac{e^n}{2\cdot 3^{n-2}} = \dfrac{e^2}{2}\left(\dfrac{e}{3}\right)^{n-2} \to 0$ since $e < 3$.

1. i.

$$a_n = \frac{n\sin(n\pi)}{n+1} = 0 \quad \text{since } \sin(n\pi) = 0$$

Thus $a_n \to 0$.

2. a.

$$a_n = \frac{n^3 + 2n^2 + 1000}{n^7 + 600n^6 + n}$$

For n "large" a_n behaves like $\frac{n^3}{n^7} = \frac{1}{n^4}$. Thus $a_n = O\left(\frac{1}{n^4}\right)$.

Note that for $n \geq 10$ the numerator is $\leq n^3 + 2n^3 + n^3 = 4n^3$ and the denominator $\geq n^7$. Thus for $n \geq 10$ we have $a_n \leq \frac{4n^3}{n^7} = 4\frac{1}{n^4}$.

2. b.

$$a_n = \frac{cos(n\pi)}{n^2+1}$$

Since $|\cos(n\pi)| = 1$, we have $|a_n| = \frac{1}{n^2+1} \leq \frac{1}{n^2}$. Thus $a_n = O\left(\frac{1}{n^2}\right)$.

2. c.

$$a_n = \left[\frac{n}{n^2-1} - \frac{n}{n^2+1}\right]\sin((n+1/2)\pi)$$

Note $\sin((n+1/2)\pi) = \underbrace{\sin(n\pi)}_{=0}\cos(\pi/2) + \cos(n\pi)\underbrace{\sin(\pi/2)}_{=1} = \cos(n\pi)$

The first factor in brackets is $\frac{n^3 + n - (n^3 - n)}{n^4 - 1} = \frac{2n}{n^4-1}$.

So $a_n = \frac{2n}{n^4-1}\cos(n\pi) = O\left(\frac{1}{n^3}\right)$ since $|\cos(n\pi)| = 1$.

2. d.

$$a_n = \frac{10n^3e^{-n} + n^2}{(2n+1)^2}\cos(n^2\pi)$$

Note $\cos(n^2\pi) = \pm 1$ depending on n even or odd. Also $x^3e^{-x} \to 0$ by L'Hospital rule, so the first term in numerator is negligible compared to the second term there.

So $|a_n| \to \frac{n^2}{(2n)^2} = \frac{1}{4}$. Therefore $a_n = O(1)$.

3.

$$\sum_{n=0}^{\infty} \frac{(n+1)^2 2^n}{(2n)!} = 1 + \frac{4\cdot 2}{2!} + \frac{9\cdot 4}{4!} + \frac{16\cdot 8}{6!} + \frac{25\cdot 16}{8!} + \frac{36\cdot 32}{10!} + \ldots$$

a. $$S_3 = \sum_{n=0}^{3} \frac{(n+1)^2 2^n}{(2n)!} = 1 + \frac{4\cdot 2}{2!} + \frac{9\cdot 4}{4!} + \frac{16\cdot 8}{6!} = 6.678$$

$$S_6 = \sum_{n=0}^{6} \frac{(n+1)^2 2^n}{(2n)!} = 1 + \frac{4\cdot 2}{2!} + \frac{9\cdot 4}{4!} + \frac{16\cdot 8}{6!} + \frac{25\cdot 16}{8!} + \frac{36\cdot 32}{10!} + \frac{49\cdot 64}{12!} = 6.688$$

b. Let $k = n+2$ (or $n = k-2$)

$$\sum_{n=0}^{\infty} \frac{(n+1)^2 2^n}{(2n)!} = \sum_{k-2=0}^{\infty} \frac{(k-1)^2 2^{k-2}}{(2k-4)!} = \frac{1}{4}\sum_{k=2}^{\infty} \frac{(k-1)^2 2^k}{(2k-4)!}$$

4. a. $\displaystyle\sum_{n=0}^{\infty} e^{-n} = \sum_{n=0}^{\infty} \left(e^{-1}\right)^n$ converges as a geometric series with $r = e^{-1} < 1$.

The sum is $\displaystyle\sum_{n=0}^{\infty} e^{-n} = \frac{1}{1-e^{-1}} = \frac{e}{e-1}$

b. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2+1}{(n+1)^3}$. Note that for large n a_n is of the order of $1/n$.

Now use the comparison test, comparing our series to $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n}$.

Since the latter diverges, our series diverges.

c. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2 \cos n}{(n^3+1)^2}$. Note that for large n, $|a_n|$ behaves like $\frac{n^2}{n^6}$.

Now use the comparison test, comparing our series to $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^4}$.

Since the latter converges (p test with $p=4$), our series converges.

d. $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n+3}$. Note that for large n, a_n behaves like $1 \neq 0$.

Since $\lim_{n\to\infty} a_n \neq 0$ the series diverges.

4. e. $\sum_{n=0}^{\infty} \frac{e^n}{n!}\cos(n\pi)$. See problem 1.h.

Let us estimate $n!$, by takeing $n! = 1 \cdot 2 \cdot 3 \cdots n > 1 \cdot 2 \cdot \underbrace{3 \cdots 3}_{n-2 times}$. Therefore

$$|a_n| = \frac{e^n}{n!} \leq \frac{e^2}{2}\left(\frac{e}{3}\right)^{n-2} = \frac{e^2}{2}r^{n-2}$$

where $|r| = \frac{e}{3} < 1$. Thus $\sum_{n=0}^{\infty} \frac{e^2}{2} r^{n-2} = \frac{e^2}{2r^2} \sum_{n=0}^{\infty} r^n$ converges as a geometric series.

Thus our series coverges by comparison test.

f. $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$. Note that $a_n > 0$. Taking $f(x) = \frac{1}{x\ln(x)}$ we can use the integral test.

$$\int_2^{\infty} \frac{dx}{x\ln(x)} = \int_{\ln(2)}^{\infty} \frac{du}{u}, \text{ where we used } u = \ln(x).$$

Thus the anti derivative is $\ln(\ln(x))$ which tend to infinity at the upper limit. Therefore the series diverges.

5. a. $\sum_{n=0}^{\infty} \frac{2n+1}{3n^4+n+1}$, $a_n = \frac{2n+1}{3n^4+n+1}$

For large n, $|a_n| \leq \frac{2n}{3n^4} \leq \frac{1}{n^3}$. Thus $a_n = O\left(\frac{1}{n^3}\right)$.

$$E_{20} = S - S_{20} = \sum_{n=21}^{\infty} \frac{2n+1}{3n^4+n+1}$$

$$|E_{20}| \leq \sum_{n=21}^{\infty} \left|\frac{2n+1}{3n^4+n+1}\right| \leq \sum_{n=21}^{\infty} \frac{1}{n^3} \leq \frac{1}{2 \cdot 20^2}$$

Note the above is $\frac{1}{(p-1)N^{p-1}}$ for $N = 20$, $p = 3$. Thus

$$|E_{20}| \leq \frac{1}{800} = .00125$$

b. $\sum_{n=1}^{\infty} \frac{1}{n^5}$, $a_n = \frac{1}{n^5} = O\left(\frac{1}{n^5}\right)$.

So

$$|E_N| = \sum_{n=N+1}^{\infty} \frac{1}{n^5} \leq \int_N^{\infty} \frac{dx}{x^5} = \frac{1}{4N^4}$$

$$|E_{20}| \leq \frac{1}{4 \cdot 20^4} = 1.56 \cdot 10^{-6}$$

c. $\displaystyle\sum_{n=1}^{\infty}\frac{(2n+1)^2}{n^4}$, $a_n = \dfrac{(2n+1)^2}{n^4} \sim \dfrac{4}{n^2}$. So

$$|E_N| \sim \sum_{n=N+1}^{\infty}\frac{4}{n^2} \leq \frac{4}{N}$$

$$|E_{20}| \leq \frac{4}{20} = .2$$

6. Consider

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

$$S_N = \sum_{n=0}^{N} x^n = 1 + x + x^2 + x^3 + \cdots + x^N = \begin{cases} \dfrac{1-x^{N+1}}{1-x}, & x \neq 1 \\ N+1, & x = 1 \end{cases}$$

a. $S_1 = 1 + x$

$$S_5 = 1 + x + x^2 + x^3 + x^4 + x^5 = \frac{1-x^6}{1-x}$$

$$S_{10} = 1 + x + x^2 + x^3 + \cdots + x^{10} = \frac{1-x^{11}}{1-x}$$

$$S_{20} = 1 + x + x^2 + x^3 + \cdots + x^{20} = \frac{1-x^{21}}{1-x}$$

In Table 1 we list the sum for various values of $-1 < x < 1$. The graphs of $f(x) = \frac{1}{1-x}$ along with the partial sums S_1, S_5, S_{10}, S_{20} is given in Figure 1.

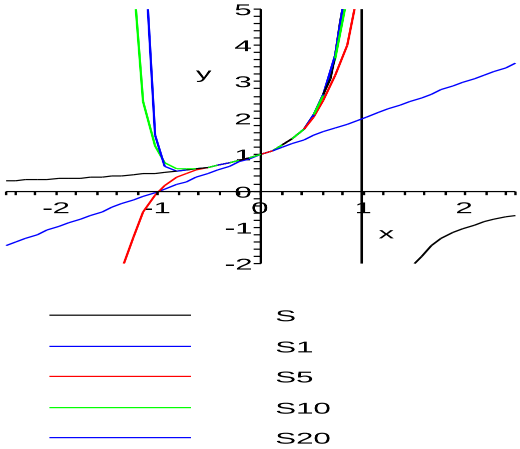


Figure 1: Plot of $f(x) = \frac{1}{1-x}$ along with the partial sums S_1, S_5, S_{10}, S_{20}

x	S	S_1	S_5	S_{10}	S_{20}
-1.00	0.50	0.00	0.00	1.00	1.00
-0.96	0.51	0.04	0.11	0.84	0.73
-0.92	0.52	0.08	0.21	0.73	0.61
-0.88	0.53	0.12	0.28	0.66	0.57
-0.84	0.54	0.16	0.35	0.62	0.56
-0.80	0.56	0.20	0.41	0.60	0.56
-0.76	0.57	0.24	0.46	0.60	0.57
-0.72	0.58	0.28	0.50	0.60	0.58
-0.68	0.60	0.32	0.54	0.60	0.60
-0.64	0.61	0.36	0.57	0.61	0.61
-0.60	0.63	0.40	0.60	0.63	0.63
-0.56	0.64	0.44	0.62	0.64	0.64
-0.52	0.66	0.48	0.64	0.66	0.66
-0.48	0.68	0.52	0.67	0.68	0.68
-0.44	0.69	0.56	0.69	0.69	0.69
-0.40	0.71	0.60	0.71	0.71	0.71
-0.36	0.74	0.64	0.73	0.74	0.74
-0.32	0.76	0.68	0.76	0.76	0.76
-0.28	0.78	0.72	0.78	0.78	0.78
-0.24	0.81	0.76	0.81	0.81	0.81
-0.20	0.83	0.80	0.83	0.83	0.83
-0.16	0.86	0.84	0.86	0.86	0.86
-0.12	0.89	0.88	0.89	0.89	0.89
-0.08	0.93	0.92	0.93	0.93	0.93
-0.04	0.96	0.96	0.96	0.96	0.96
0.00	1.00	1.00	1.00	1.00	1.00
0.04	1.04	1.04	1.04	1.04	1.04
0.08	1.09	1.08	1.09	1.09	1.09
0.12	1.14	1.12	1.14	1.14	1.14
0.16	1.19	1.16	1.19	1.19	1.19
0.20	1.25	1.20	1.25	1.25	1.25
0.24	1.32	1.24	1.32	1.32	1.32
0.28	1.39	1.28	1.39	1.39	1.39
0.32	1.47	1.32	1.47	1.47	1.47
0.36	1.56	1.36	1.56	1.56	1.56
0.40	1.67	1.40	1.66	1.67	1.67
0.44	1.79	1.44	1.77	1.79	1.79
0.48	1.92	1.48	1.90	1.92	1.92
0.52	2.08	1.52	2.04	2.08	2.08
0.56	2.27	1.56	2.20	2.27	2.27
0.60	2.50	1.60	2.38	2.49	2.50
0.64	2.78	1.64	2.59	2.76	2.78
0.68	3.13	1.68	2.82	3.08	3.12
0.72	3.57	1.72	3.07	3.48	3.57
0.76	4.17	1.76	3.36	3.96	4.15
0.80	5.00	1.80	3.69	4.57	4.95
0.84	6.25	1.84	4.05	5.33	6.09
0.88	8.33	1.88	4.46	6.29	7.76
0.92	12.50	1.92	4.92	7.50	10.33
0.96	25.00	1.96	5.43	9.04	14.39
1.00	∞	2.00	6.00	11.00	21.00

b. The series only converges for $-1 < x < 1$.

The series diverges for $-2 < x \leq -1$ or for $1 \leq x < 2$.

c. Series converges uniformly to a continuous function for $-\dfrac{1}{2} < x < \dfrac{1}{2}$.

2 Fourier Series

2.1 Introduction

2.2 Derivation of the Fourier Series Coefficients

PROBLEMS

1. Derive the formula for the Fourier sine coefficients, b_n

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad ,$$

using a method similar to that used to derive

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad .$$

2. For each of the following functions, find the Fourier coefficients, the Fourier series, and sketch the partial sums $S_2(x)$, $S_5(x)$, and $S_{10}(x)$:

a. $f(x) = \begin{cases} 0 & , \quad -1 < x < 0 \\ 1 & , \quad 0 \le x < 1 \end{cases}$

$f(x+2) = f(x)$

b. $f(x) = \begin{cases} 3+x & , \quad -3 < x < 0 \\ 3-x & , \quad 0 \le x < 3 \end{cases}$ $\qquad f(x+6) = f(x)$

c. $f(x) = \begin{cases} 0 & , \quad -2 < x < 0 \\ x & , \quad 0 \le x < 1 \\ 2-x & , \quad 1 \le x < 2 \end{cases}$

$f(x+4) = f(x)$

d. $f(x) = 1 - \cos(\pi x), \quad -1 \le x \le 1$

3. a. Show that the alternative Fourier Series representation

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left\{a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right\}$$

leads to the formulas

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \quad , \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad , \quad n > 0 \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad , \quad n > 0 \end{aligned}$$

where the formula for a_0 is no longer the $n = 0$ special case of the formula for a_n.

b. Show that the alternative Fourier Series representation

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{2n\pi t}{T_0}\right) + b_n \sin\left(\frac{2n\pi t}{T_0}\right) \right\}$$

(note the independent variable here is t), where L has been replaced by $T_0/2$, leads to

$$\begin{aligned} a_n &= \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} f(x) \cos\left(\frac{2n\pi t}{T_0}\right) dt \quad , \quad n \geq 0 \\ b_n &= \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} f(x) \sin\left(\frac{2n\pi t}{T_0}\right) dt \quad , \quad n > 0 \quad . \end{aligned}$$

Note that here the formula for a_0 is again the $n = 0$ special case of the formula for a_n.

4. In each of the following, find each point in $-L < x < L$ where $f(x)$ has a discontinuity. Find the left and right-hand limits of $f(x)$ and $f'(x)$ at each point of discontinuity and at the end points of the interval. Without computing the Fourier coefficients, indicate to what values the series should converge at these points.

a. $f(x) = \begin{cases} x^2 & , \quad 1 \leq x \leq 3 \\ 0 & , \quad -2 \leq x < 1 \\ 2x & , \quad -3 \leq x < -2 \end{cases}$

b. $f(x) = \begin{cases} 3 & , \quad \pi/2 \leq x \leq \pi \\ 2x - 2 & , \quad -\pi \leq x < \pi/2 \end{cases}$

c. $f(x) = \begin{cases} x^2 & , \quad -2 \leq x < 0 \\ 0 & , \quad 0 \leq x < 1 \\ 4(x-1) & , \quad 1 \leq x \leq 2 \end{cases}$

1.

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left\{a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)\right\}$$

Let k denote a fixed integer. Multiply both sides by $\sin\left(\dfrac{k\pi x}{L}\right)$ and integrate

$$\textstyle\int_{-L}^{L} f(x)\sin\left(\frac{k\pi x}{L}\right)dx = a_0\underbrace{\displaystyle\int_{-L}^{L}\sin\left(\frac{k\pi x}{L}\right)dx}_{=0}+\sum_{n=1}^{\infty}\left\{a_n\underbrace{\displaystyle\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{k\pi x}{L}\right)dx}_{=0\text{ for all }n,k}\right.$$

$$\left.+\,b_n\underbrace{\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{k\pi x}{L}\right)dx}_{=0\text{ for all }n\neq k,\ =L\text{ for }n=k}\right\}$$

or

$$\int_{-L}^{L} f(x)\sin\left(\frac{k\pi x}{L}\right)dx = Lb_k$$

$$b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{k\pi x}{L}\right)dx$$

Replacing k by n on both sides

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

2. a. $f(x)=\begin{cases}0 & , \quad -1<x<0\\ 1 & , \quad 0\le x<1\end{cases}$

$f(x+2)=f(x)$

$$L=1$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\{a_n\cos(n\pi x)+b_n\sin(n\pi x)\}$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos(n\pi x)dx$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin(n\pi x)dx$$

$$a_0 = \int_{-1}^{1} f(x)dx = \int_0^1 dx = x\Big|_0^1 = 1 \Rightarrow \frac{a_0}{2}=\frac{1}{2}$$

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)dx = \int_0^1\cos(n\pi x)dx = \frac{1}{n\pi}\sin(n\pi x)\Big|_0^1 = 0$$

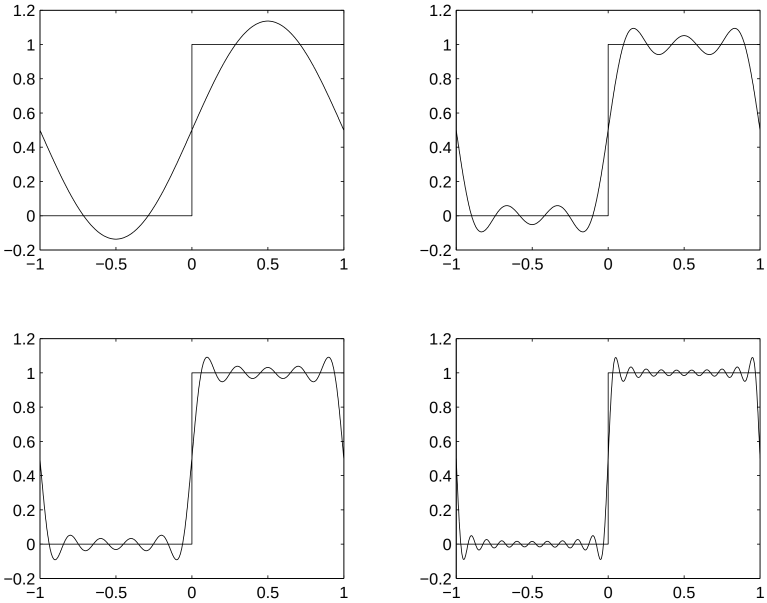


Figure 2: Graph of $f(x)$ and the N^{th} partial sums for $N = 2, 5, 10, 20$

$$b_n = \int_{-1}^{1} f(x)\sin(n\pi x)dx = \int_0^1 \sin(n\pi x)dx = -\frac{1}{n\pi}\cos(n\pi x)\Big|_0^1 = \frac{1 - cos(n\pi)}{n\pi}$$

Thus

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1 - cos(n\pi)}{n\pi}\sin(n\pi x)$$

or

$$f(x) = \frac{1}{2} + \frac{2)}{\pi}\sin(\pi x) + \frac{2)}{3\pi}\sin(3\pi x) + \frac{2)}{5\pi}\sin(5\pi x) + \cdots$$

2. b. $f(x) = \begin{cases} 3 + x \quad , & -3 < x < 0 \\ 3 - x \quad , & 0 \le x < 3 \end{cases}$

$f(x+6) = f(x)$

$$L = 3$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n \cos\left(\frac{n\pi x}{3}\right) + b_n \sin\left(\frac{n\pi x}{3}\right)\right\}$$

where

$$a_n = \frac{1}{3}\int_{-3}^{3} f(x)\cos(\frac{n\pi x}{3})dx$$

$$b_n = \frac{1}{3}\int_{-3}^{3} f(x)\sin(\frac{n\pi x}{3})dx$$

$$a_0 = \frac{1}{3}\int_{-3}^{3} f(x)dx = \frac{1}{3}\left[\int_{-3}^{0}(3+x)dx + \int_0^3 (3-x)dx\right]$$

$$a_0 = \frac{1}{3}\left[\frac{(3+x)^2}{2}\Big|_{-3}^{0} - \frac{(3-x)^2}{2}\Big|_{0}^{3}\right] = \frac{1}{3}\left[\frac{9}{2} - \left(-\frac{9}{2}\right)\right] = 3$$

$$\Rightarrow \frac{a_0}{2} = \frac{3}{2}.$$

$$a_n = \frac{1}{3}\left[\int_{-3}^{0}(3+x)\cos(\frac{n\pi x}{3})dx + \int_{0}^{3}(3-x)\cos(\frac{n\pi x}{3})dx\right]$$

$$a_n = \frac{1}{3}\left[\frac{3}{n\pi}(3+x)\underbrace{\sin(\frac{n\pi x}{3})}_{=0}\Big|_{-3}^{0} + \left(\frac{3}{n\pi}\right)^2\cos(\frac{n\pi x}{3})\Big|_{-3}^{0}\right.$$

$$\left.\frac{3}{n\pi}(3-x)\underbrace{\sin(\frac{n\pi x}{3})}_{=0}\Big|_{0}^{3} - \left(\frac{3}{n\pi}\right)^2\cos(\frac{n\pi x}{3})\Big|_{0}^{3}\right]$$

$$a_n = \frac{1}{3}\left[\frac{3}{n\pi}\right]^2(1-\cos(-n\pi)-\cos(n\pi)+1) = \frac{6}{n^2\pi^2}(1-\cos(n\pi))$$

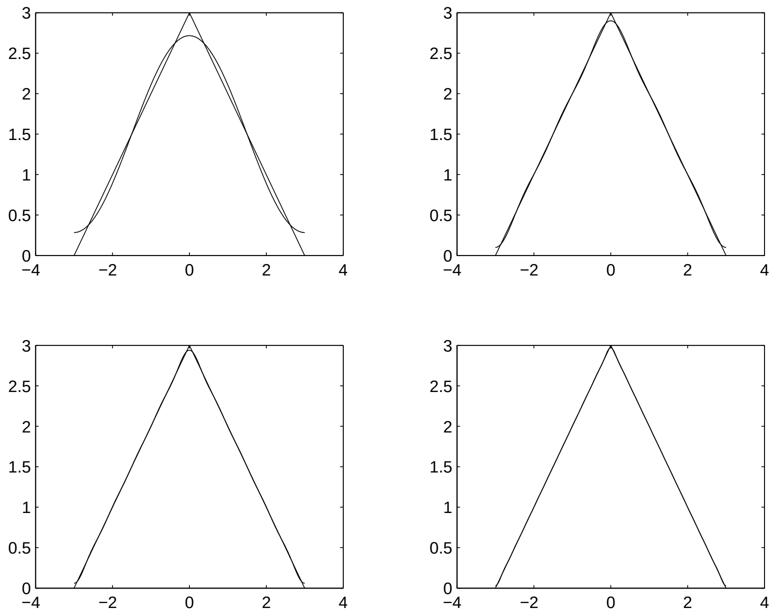


Figure 3: Graph of $f(x)$ and the N^{th} partial sums for $N = 2, 5, 10, 20$

$$b_n = \frac{1}{3}\left[\int_{-3}^{0}(3+x)\sin(\frac{n\pi x}{3})dx + \int_{0}^{3}(3-x)\sin(\frac{n\pi x}{3})dx\right]$$

$$b_n = \frac{1}{3}\left[-\frac{3}{n\pi}(3+x)\cos(\frac{n\pi x}{3})\Big|_{-3}^{0} + \left(\frac{3}{n\pi}\right)^2 \underbrace{\sin(\frac{n\pi x}{3})}_{=0}\Big|_{-3}^{0} \right.$$

$$\left. -\frac{3}{n\pi}(3-x)\cos(\frac{n\pi x}{3})\Big|_{0}^{3} - \left(\frac{3}{n\pi}\right)^2 \underbrace{\cos(\frac{n\pi x}{3})}_{=0}\Big|_{0}^{3} \right]$$

$$b_n = \frac{1}{3}\left[-\frac{3}{n\pi}\cdot 3\cdot\cos(0) - \left(-\frac{3}{n\pi}\cdot 3\cdot\cos(0)\right)\right] = 0$$

Thus

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{6}{n^2\pi^2}\left(1-\cos(n\pi)\right)\cos\left(\frac{n\pi x}{3}\right)$$

$$f(x) = \frac{3}{2} + \frac{12}{\pi^2}\cos\left(\frac{\pi x}{3}\right) + \frac{12}{9\pi^2}\cos\left(\frac{3\pi x}{3}\right) + \frac{12}{25\pi^2}\cos\left(\frac{5\pi x}{3}\right) + \cdots$$

c. $f(x) = \begin{cases} 0 & , \quad -2 < x < 0 \\ x & , \quad 0 \le x < 1 \\ 2-x & , \quad 1 \le x < 2 \end{cases}$

$f(x+4) = f(x)$

$$L = 2$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{ a_n \cos\left(\frac{n\pi x}{2}\right) + b_n \sin\left(\frac{n\pi x}{2}\right)\right\}$$

where

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos(\frac{n\pi x}{2})dx$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin(\frac{n\pi x}{2})dx$$

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)dx = \frac{1}{2}\left[\int_0^1 x dx + \int_1^2 (2-x)dx\right]$$

$$a_0 = \frac{1}{2}\left[\frac{x^2}{2}\Big|_0^1 - \frac{(2-x)^2}{2}\Big|_1^2\right] = \frac{1}{2}\left[\frac{1}{2}+\frac{1}{2}\right] = \frac{1}{2}$$

$$\Rightarrow \frac{a_0}{2} = \frac{1}{4}$$

$$a_n = \frac{1}{2}\left[\int_0^1 x\cos(\frac{n\pi x}{2})dx + \int_1^2 (2-x)\cos(\frac{n\pi x}{2})dx\right]$$

$$a_n = \frac{1}{2}\left[\left(\frac{2}{n\pi}x\sin(\frac{n\pi x}{2}) + \left(\frac{2}{n\pi}\right)^2\cos(\frac{n\pi x}{2})\right)\Big|_0^1\right.$$

$$\left. + \left(\frac{2}{n\pi}(2-x)\sin(\frac{n\pi x}{2}) - \left(\frac{2}{n\pi}\right)^2\cos(\frac{n\pi x}{2})\right)\Big|_1^2\right]$$

$$a_n = \frac{1}{2}\left[\frac{2}{n\pi}\sin(\frac{n\pi}{2}) + \left(\frac{2}{n\pi}\right)^2\left(\cos(\frac{n\pi}{2}) - 1\right)\right.$$

$$\left. -\frac{2}{n\pi}\sin(\frac{n\pi}{2}) - \left(\frac{2}{n\pi}\right)^2\left(\cos(n\pi) - \cos(\frac{n\pi}{2}\right)\right]$$

Note that the terms with $\sin(\frac{n\pi}{2})$ cancel out.

$$a_n = \frac{2}{(n\pi)^2}\left[2\cos(\frac{n\pi}{2}) - \cos(n\pi) - 1\right]$$

and

$$b_n = \frac{1}{2}\left[\int_0^1 x\sin(\frac{n\pi x}{2})dx + \int_1^2 (2-x)\sin(\frac{n\pi x}{2})dx\right]$$

$$b_n = \frac{1}{2}\left[\left(-\frac{2}{n\pi}x\cos(\frac{n\pi x}{2}) + \left(\frac{2}{n\pi}\right)^2\sin(\frac{n\pi x}{2})\right)\Big|_0^1\right.$$

$$\left.\left(-\frac{2}{n\pi}(2-x)\cos(\frac{n\pi x}{2}) - \left(\frac{2}{n\pi}\right)^2\sin(\frac{n\pi x}{2})\right)\Big|_1^2\right]$$

$$b_n = \frac{1}{2}\left[-\frac{2}{n\pi}\cos(\frac{n\pi}{2}) + \left(\frac{2}{n\pi}\right)^2\sin(\frac{n\pi}{2}) + \frac{2}{n\pi}\cos(\frac{n\pi}{2}) + \left(\frac{2}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}\right)\right]$$

$$b_n = \left(\frac{2}{n\pi}\right)^2\sin(\frac{n\pi}{2})$$

Thus

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\left\{\left(\frac{2}{n\pi}\right)^2\frac{2\cos(\frac{n\pi}{2}) - \cos(n\pi) - 1}{2}\cos\left(\frac{n\pi x}{2}\right) + \left(\frac{2}{n\pi}\right)^2\sin(\frac{n\pi}{2})\sin\left(\frac{n\pi x}{2}\right)\right\}$$

$$f(x) = \frac{1}{4} + \underbrace{\left(\frac{2}{\pi}\right)^2\sin\left(\frac{\pi x}{2}\right)}_{n=1} - \underbrace{2\left(\frac{2}{2\pi}\right)^2\cos\left(\frac{2\pi x}{2}\right)}_{n=2} - \underbrace{\left(\frac{2}{3\pi}\right)^2\sin\left(\frac{3\pi x}{2}\right)}_{n=3} - \underbrace{2\left(\frac{2}{6\pi}\right)^2\cos\left(\frac{6\pi x}{2}\right)}_{n=6} + \cdots$$

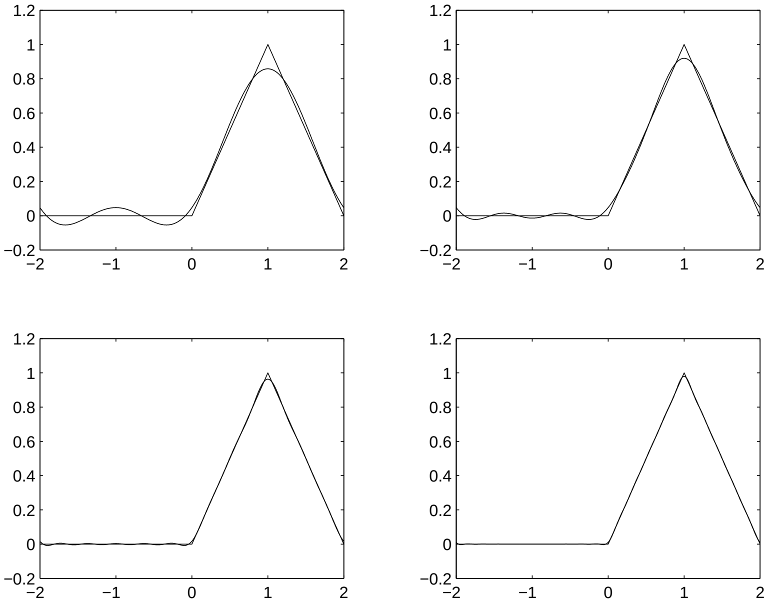


Figure 4: Graph of $f(x)$ and the N^{th} partial sums for $N = 2, 5, 10, 20$

d. $f(x) = 1 - \cos(\pi x), \quad -1 \le x \le 1$

$$L = 1$$

so

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos(n\pi x) + b_n \sin(n\pi x)\}$$

But $f(x)$ is already in that form, i.e.

$$a_0 = 2, \ a_1 = -1, \ a_n = 0, \quad \text{for } n \ge 2$$

$$b_n \ = \ 0 \ \text{ for all } n$$

If you don't see this, it will emerge by "brute force" (recalling the orthogonality integrals.)

3. a. Show that the alternative Fourier Series representation

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right\} \tag{2.2.1}$$

leads to the formulas

$$\begin{aligned} a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x)dx \quad , \\ a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \quad , \quad n > 0 \\ b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad , \quad n > 0 \end{aligned}$$

where the formula for a_0 is no longer the $n = 0$ special case of the formula for a_n.

Since the only change is in the first term before the summation, we will show how a_0 is computed. Integrate both sides of (2.2.1)

$$\int_{-L}^{L} f(x)dx = \int_{-L}^{L} a_0 dx + \sum_{n=1}^{\infty} \left\{ a_n \underbrace{\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) dx}_{=0} + b_n \underbrace{\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) dx}_{=0} \right\}$$

so

$$\int_{-L}^{L} f(x)dx = 2La_0$$

and therefore

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)dx$$

b. Show that the alternative Fourier Series representation

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{2n\pi t}{T_0}\right) + b_n \sin\left(\frac{2n\pi t}{T_0}\right) \right\} \tag{2.2.2}$$

(note the independent variable here is t), where L has been replaced by $T_0/2$, leads to

$$\begin{aligned} a_n &= \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} f(x) \cos\left(\frac{2n\pi t}{T_0}\right) dt \quad , \quad n \geq 0 \\ b_n &= \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} f(x) \sin\left(\frac{2n\pi t}{T_0}\right) dt \quad , \quad n > 0 \quad . \end{aligned}$$

Note that (2.2.2) is identical to the Fourier series in the text, except that x is replaced here by t and L is here $T_0/2$.

Thus

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(t)\cos(\frac{n\pi t}{L})dt \;=\; \frac{1}{T_0/2}\int_{-T_0/2}^{T_0/2} f(t)\cos(\frac{n\pi t}{T_0/2})dt \\ &= \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\cos(\frac{2n\pi t}{T_0})dt \end{aligned}$$

and

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(t)\sin(\frac{n\pi t}{L})dx \;=\; \frac{1}{T_0/2}\int_{-T_0/2}^{T_0/2} f(t)\sin(\frac{n\pi t}{T_0/2})dt \\ &= \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\sin(\frac{2n\pi t}{T_0})dt \end{aligned}$$

4. In each of the following, find each point in $-L < x < L$ where $f(x)$ has a discontinuity. Find the left and right-hand limits of $f(x)$ and $f'(x)$ at each point of discontinuity and at the end points of the interval. Without computing the Fourier coefficients, indicate to what values the series should converge at these points.

a. $f(x) = \begin{cases} x^2 & , \quad 1 \le x \le 3 \\ 0 & , \quad -2 \le x < 1 \\ 2x & , \quad -3 \le x < -2 \end{cases}$

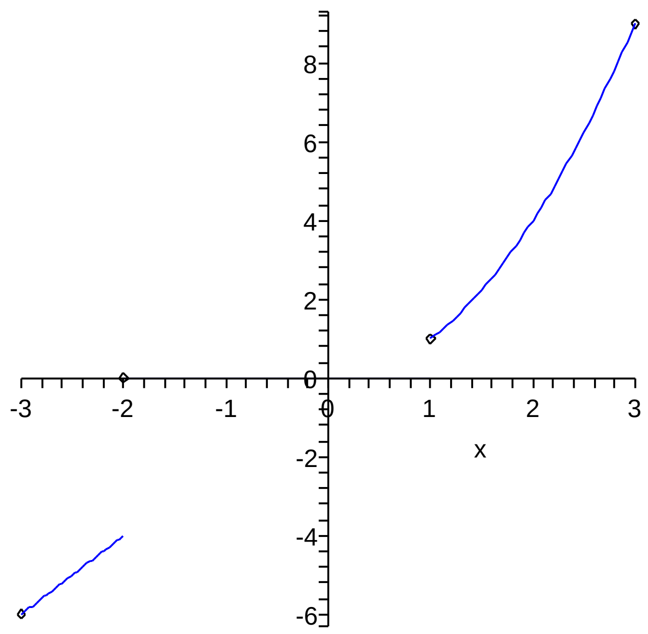


Figure 5: Graph of $f(x)$ for problem 4a

Clearly $f(x)$ is discontinuous at $x = -2$ and $x = 1$, and the periodic extension will also be discontinuos at $x = \pm 3$.

Since $f(-2_-) = -4$ and $f(-2_+) = 0$, the series converges to $(-4+0)/2 = -2$ at $x = -2$.
Since $f(1_-) = 0$ and $f(1_+) = 1$, the series converges to $\frac{1}{2}$ at $x = 1$.
Since $f(3_-) = 9$ and $f(3_+) = -6$, the series converges to $\frac{3}{2}$ at $x = \pm 3$.

b. $f(x) = \begin{cases} 3 & , \quad \pi/2 \leq x \leq \pi \\ 2x - 2 & , \quad -\pi \leq x < \pi/2 \end{cases}$

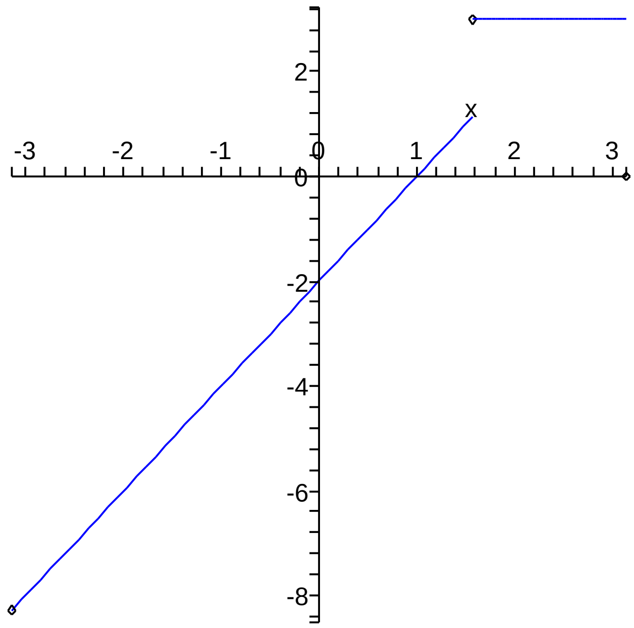


Figure 6: Graph of $f(x)$ for problem 4b

Clearly $f(x)$ is discontinuous at $x = \pi/2$ and the periodic extension is also discontinuous at $x = \pm\pi$.

Since $f\left(\dfrac{\pi}{2}_-\right) = \pi - 2$ and $f\left(\dfrac{\pi}{2}_+\right) = 3$, the series converges to $\dfrac{\pi + 1}{2}$ at $x = \pi/2$.
Since $f(\pi_-) = 3$ and $f(\pi_+) = -2\pi - 2$, the series converges to $\dfrac{-2\pi + 1}{2}$ at $x = \pm\pi$.

c. $f(x) = \begin{cases} x^2 & , \quad -2 \le x < 0 \\ 0 & , \quad 0 \le x < 1 \\ 4(x-1) & , \quad 1 \le x \le 2 \end{cases}$

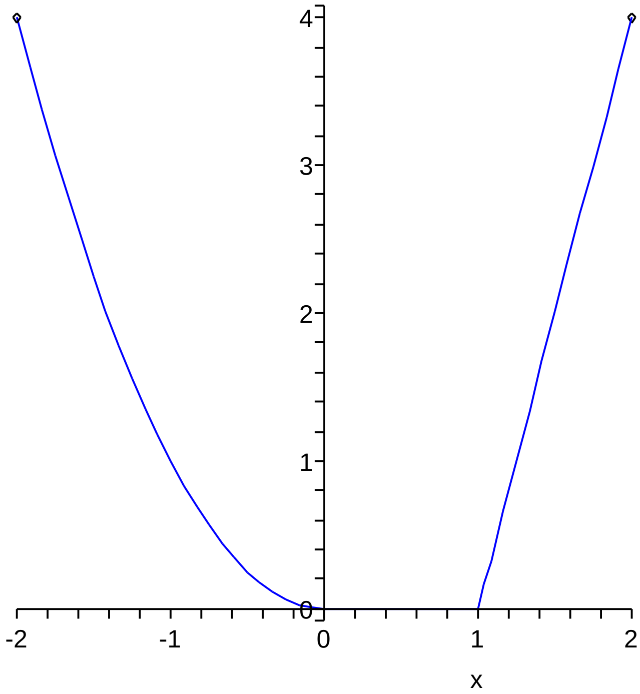


Figure 7: Graph of $f(x)$ for problem 4c

Clearly $f(x)$ is continuous and so is its periodic extension. So the series converges to $f(x)$ **everywhere**.

2.3 Odd and Even Functions

PROBLEMS

1. Find the Fourier series for the following functions

a. $f(x) = \begin{cases} -1 & , \quad -2 \leq x \leq 0 \\ \;\;1 & , \quad \;\;0 < x < 2 \end{cases}$

$f(x+4) = f(x)$

$f(x)$ is clearly odd, and $L = 2$. Thus

$$f(x) \;=\; \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

where

$$b_n \;=\; \frac{2}{2}\int_0^2 1 \cdot \sin\left(\frac{n\pi x}{2}\right) dx \;=\; -\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Bigg|_0^2 \;=\; \frac{2}{n\pi}\left[1 - \cos(n\pi)\right]$$

or

$$f(x) \;=\; \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1 - \cos(n\pi)\right] \sin\left(\frac{n\pi x}{2}\right)$$

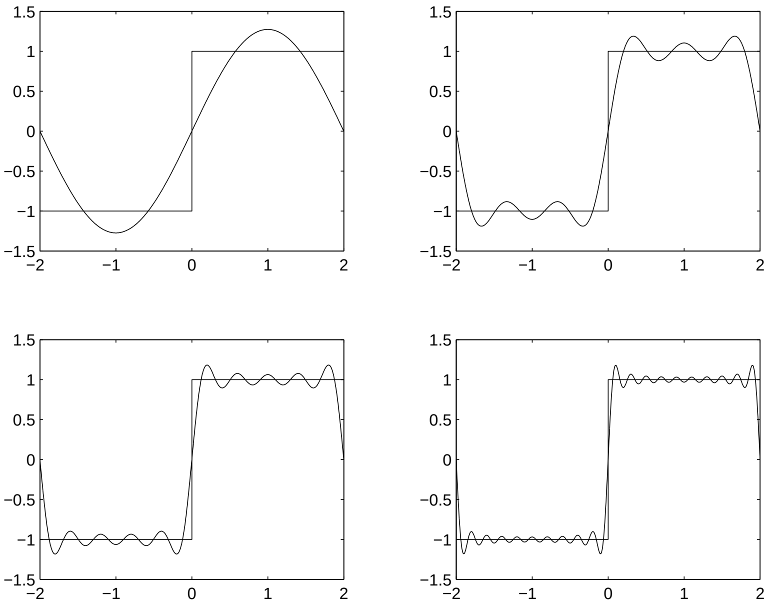


Figure 8: Graph of $f(x)$ for problem 1a of 2.3

b. $f(x) = |x| \;\; , \qquad f(x+2) = f(x)$
$f(x)$ is clearly even, $L = 1$. Thus

$$f(x) \; = \; \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos{(n\pi x)}$$

where (note that $|x| = x$ for $x \geq 0$)

$$a_n \; = \; \frac{2}{1}\int_0^1 x \cdot \cos{(n\pi x)}\, dx \; = \; 2\left[\frac{x}{n\pi}\sin{(n\pi x)} + \frac{1}{n^2\pi^2}\cos(n\pi x)\right]\Bigg|_0^1 \quad \text{for } n \neq 0$$

$$a_n = \; \frac{2}{n^2\pi^2}\left[\cos(n\pi) - 1\right] \quad \text{for } n \neq 0$$

and

$$a_0 \; = \; 2\int_0^1 x dx \; = \; x^2\Big|_0^1 \; = \; 1$$

$$\Rightarrow \frac{a_0}{2} = \frac{1}{2}$$

or

$$f(x) \; = \; \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\left[\cos(n\pi) - 1\right]\cos{(n\pi x)}$$

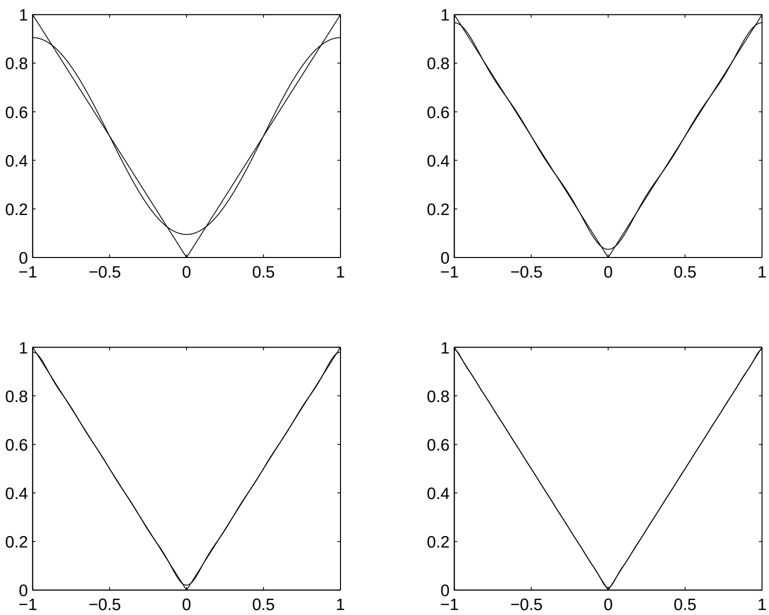


Figure 9: Graph of $f(x)$ for problem 1b of 2.3

c. $f(x) = |\sin(x)|$
Pick $L = \pi/2$. Since $f(x)$ is even

$$f(x) \;=\; \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

where

$$a_n \;=\; \frac{2}{\pi/2}\int_0^{\pi/2} |\sin(x)| \cos(2nx)\,dx \;=\; \frac{4}{\pi}\int_0^{\pi/2} \sin(x)\cos(2nx)\,dx$$

or

$$a_n \;=\; \frac{4}{\pi}\left[-\frac{\cos(1-2n)x}{2(1-2n)} - \frac{\cos(1+2n)x}{2(1+2n)}\right]\Bigg|_0^{\pi/2} \quad \text{for } n \neq 0$$

$$a_n = \frac{4}{\pi}\left[-\frac{\cos(1-2n)\pi/2}{2(1-2n)} - \frac{\cos(1+2n)\pi/2}{2(1+2n)} + \frac{1}{2(1-2n)} + \frac{1}{2(1+2n)}\right] \quad \text{for } n \neq 0$$

Note that $\cos(1-2n)\pi/2 \;=\; \underbrace{\cos(\pi/2)}_{=0}\cos(n\pi) + \sin(\pi/2)\underbrace{\sin(n\pi)}_{=0}$, also $\cos(1+2n)\pi/2 \;=\; 0.$

Thus

$$a_n = \frac{4}{2\pi}\left[\frac{1}{1-2n} + \frac{1}{1+2n}\right] \;=\; \frac{4}{\pi(1-4n^2)} \quad \text{for } n \neq 0$$

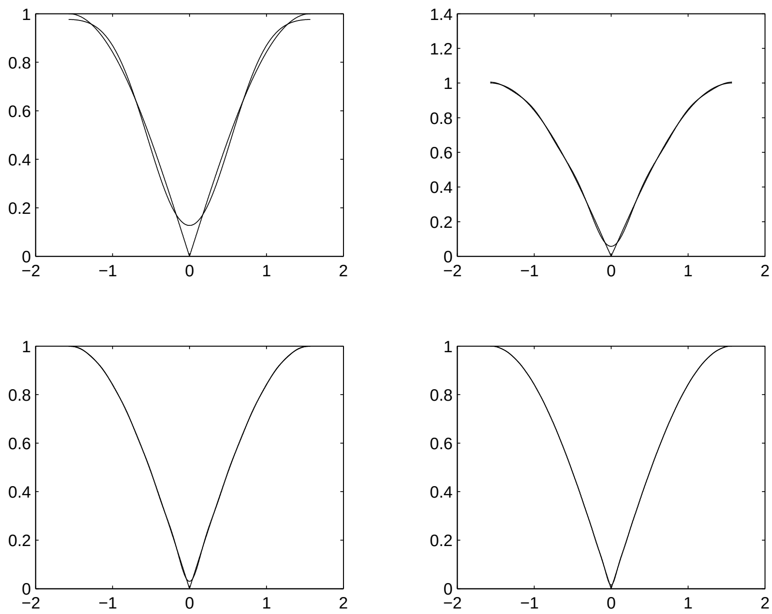


Figure 10: Graph of $f(x)$ for problem 1c of 2.3

and

$$a_0 \;=\; \frac{4}{\pi}\int_0^{\pi/2} \sin x\,dx \;=\; -\frac{4}{\pi}\cos x\Big|_0^{\pi/2} \;=\; \frac{4}{\pi}$$

$$\Rightarrow \frac{a_0}{2} = \frac{2}{\pi}$$

or

$$f(x) \;=\; \frac{1}{2} + \sum_{n=1}^{\infty} \frac{4}{\pi(1-4n^2)} \cos(2nx)$$

d. $f(x) = \begin{cases} -2-x & , \quad -2 \leq x \leq -1 \\ x & , \quad -1 \leq x \leq \;\; 1 \\ 2-x & , \quad \;\; 1 \leq x \leq \;\; 2 \end{cases}$

$f(x+4) = f(x)$

$$L = 2.$$

Since $f(x)$ is odd,

$$f(x) \;=\; \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{2}x\right)$$

where

$$\begin{aligned}
b_n &= \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{n\pi}{2}x\right) \\
&= \int_0^1 x\sin\left(\frac{n\pi}{2}x\right) + \int_1^2 (2-x)\sin\left(\frac{n\pi}{2}x\right) \\
&= \left\{-\frac{2}{n\pi}x\cos\left(\frac{n\pi}{2}x\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}x\right)\right\}\Big|_0^1 \\
&+ \left\{-\frac{2}{n\pi}(2-x)\cos\left(\frac{n\pi}{2}x\right) - \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}x\right)\right\}\Big|_1^2 \\
&= -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) + \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) \\
&= 2\left(\tfrac{2}{n\pi}\right)^2 \sin\left(\tfrac{n\pi}{2}\right)
\end{aligned}$$

Thus

$$f(x) \;=\; \sum_{n=1}^{\infty} 2\left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{2}x\right)$$

or

$$f(x) \;=\; \frac{8}{\pi^2}\sin\left(\frac{\pi}{2}x\right) - \frac{8}{9\pi^2}\sin\left(\frac{3\pi}{2}x\right) + \frac{8}{25\pi^2}\sin\left(\frac{5\pi}{2}x\right) \pm \cdots$$

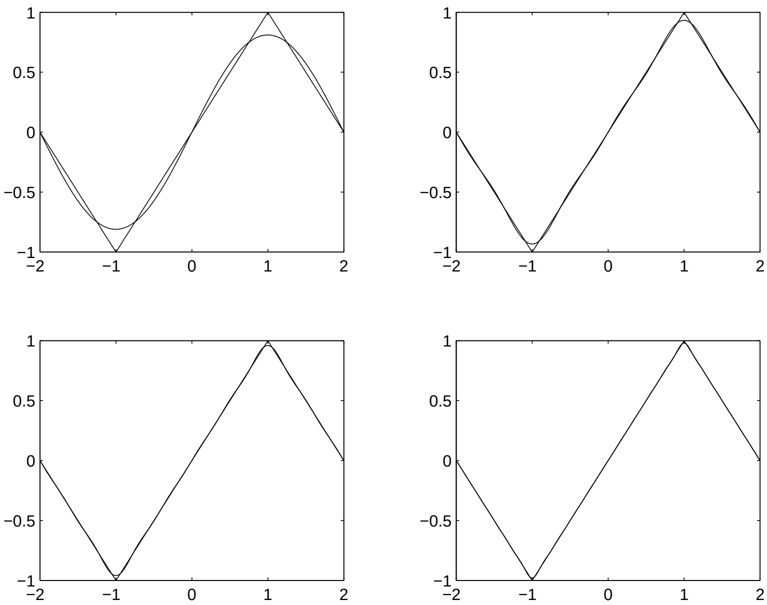


Figure 11: Graph of $f(x)$ for problem 1d of 2.3

2.4 Convergence Properties of Fourier Series

PROBLEMS

1. For each of the following Fourier series, determine whether the series will converge uniformly, converge only in mean square, or diverge:

a. $\displaystyle \frac{1}{3} + \sum_{n=1}^{\infty} \frac{1}{n^3\pi^3} \sin\left(\frac{n\pi x}{3}\right)$

b. $\displaystyle 1 + \sum_{n=1}^{\infty} \frac{1}{n\pi} \cos(n\pi) \cos\left(\frac{n\pi x}{2}\right)$

c. $\displaystyle -2 + \sum_{n=1}^{\infty} \left\{ \frac{n\cos(n\pi)}{n^2+1} \cos(nx) - \frac{1}{n^2\pi^2} \sin(nx) \right\}$

d. $\displaystyle \sum_{n=1}^{\infty} \frac{n}{(n+1)\pi} \sin(n\pi x)$

2. For each convergent Fourier series in problem 1 above, determine the highest derivative of the periodic extension of $f(x)$ that should be continuous.

3. Consider the Fourier series for

$$f(x) = |x|, \qquad -1 \leq x \leq 1$$

(found earlier). Differentiate this series once term by term, and compare your answer to the actual series of $f'(x)$.

1. a. $a_n = 0,\ \ b_n \ = \ \dfrac{1}{n^3\pi^2} \ = O\left(\dfrac{1}{n^3}\right),\ \Rightarrow p = 3 > 1$
The series converges uniformly.

1. b. $\ a_n = \dfrac{1}{n\pi}\cos(n\pi) \ = \ O\left(\dfrac{1}{n}\right),\ \Rightarrow p = 1\ \ b_n \ = \ 0$
The series converges in the mean but not uniformly.

1. c. $\ a_n = \dfrac{n\cos(n\pi)}{n^2+1} \ = O\left(\dfrac{1}{n}\right),\ \Rightarrow p = 1\ \ b_n \ = \ -\dfrac{1}{n^2\pi^2} \ = O\left(\dfrac{1}{n^2}\right)$
The series converges in the mean but not uniformly (because the dominant term is $O\left(\frac{1}{n}\right)$).

1. d. $\ a_n = 0,\ \ b_n \ = \ -\dfrac{n}{n+1} \ = O\left(1\right)$
The series does not converge.
2. a. $a_n = 0,\ \ b_n \ = \ \dfrac{1}{n^3\pi^2} \ = O\left(\dfrac{1}{n^3}\right),\ \Rightarrow p = 3 > 1$
Since $p = 3$, both $f(x)$ and $f'(x)$ should be continuous but not $f''(x)$

2. b. $\ a_n = \dfrac{1}{n\pi}\cos(n\pi) \ = \ O\left(\dfrac{1}{n}\right),\ \Rightarrow p = 1\ \ b_n \ = \ 0$
$f(x)$ should not be continuous.

2. c. $\ a_n = \dfrac{n\cos(n\pi)}{n^2+1} \ = O\left(\dfrac{1}{n}\right),\ \Rightarrow p = 1\ \ b_n \ = \ -\dfrac{1}{n^2\pi^2} \ = O\left(\dfrac{1}{n^2}\right)$
$f(x)$ should not be continuous.

2. d. $\ a_n = 0,\ \ b_n \ = \ -\dfrac{n}{n+1} \ = O\left(1\right)$
The series does not converge and so we can't talk about the limit function.

3. The Fourier series

$$f(x) \ = \ |x| \ = \ \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2(\cos(n\pi) - 1)}{n^2\pi^2}\cos(n\pi x),\ -1 \le x \le 1$$

Differentiate this series once term by term,

$$f'(x) \ = \ \sum_{n=1}^{\infty} \frac{2(\cos(n\pi) - 1)}{n^2\pi^2}(-n\pi\sin(n\pi x)),\ = \ \sum_{n=1}^{\infty} \frac{2(1 - \cos(n\pi))}{n\pi}\sin(n\pi x)$$

which is exactly the Fourier series of the function

$$g(x) \ = \ \begin{cases} 1, & 0 \le x \le 1 \\ -1, & -1 \le x < 0 \end{cases}$$

Note that $g(x)$ is the derivative of $|x|$ and also that it is odd and so we expect a Fourier sine series.

2.5 Interpretation of the Fourier Coefficients

PROBLEMS

1. Plot the amplitude and phase as a function of frequency for each of the Fourier series found for the problems in the first section of this chapter.

2. Prove Parseval's Theorem

$$\frac{1}{2L}\int_{-L}^{L}[f(x)]^2\,dx = A_0^2 + \sum_{n=1}^{\infty}\frac{1}{2}A_n^2 \equiv \frac{a_0^2}{4} + \sum_{n=1}^{\infty}\frac{\{a_n^2+b_n^2\}}{2}$$

(Hint: first show

$$\frac{1}{2L}\int_{-L}^{L}[f(x)]^2\,dx = \frac{a_0^2}{4} + \sum_{n=1}^{\infty}\frac{\{a_n^2+b_n^2\}}{2}$$

then use the definition of the A_n.)

1. a.

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1 - cos(n\pi)}{n\pi} \sin(n\pi x)$$

$$f(x) \;=\; A_0 + \sum_{n=1}^{\infty} A_n \cos\left(n\pi x - \phi_n\right)$$

where

$$A_n = \sqrt{a_n^2 + b_n^2}, \;\; \phi_n \;=\; \tan^{-1}\left(\frac{b_n}{a_n}\right)$$

Note that $\phi_n = 0$ by definition if $A_n = 0$. Now

$$a_n = 0, \; b_n = \frac{1 - cos(n\pi)}{n\pi}, \;\Rightarrow A_n \;=\; \frac{|1 - \cos(n\pi)|}{n\pi}, \; A_0 = \frac{1}{2}$$

$$\phi_n \;=\; \begin{cases} 0, & n = 0, 2, 4, \ldots \\ \pi/2, & n = 1, 3, 5, \ldots \end{cases}$$

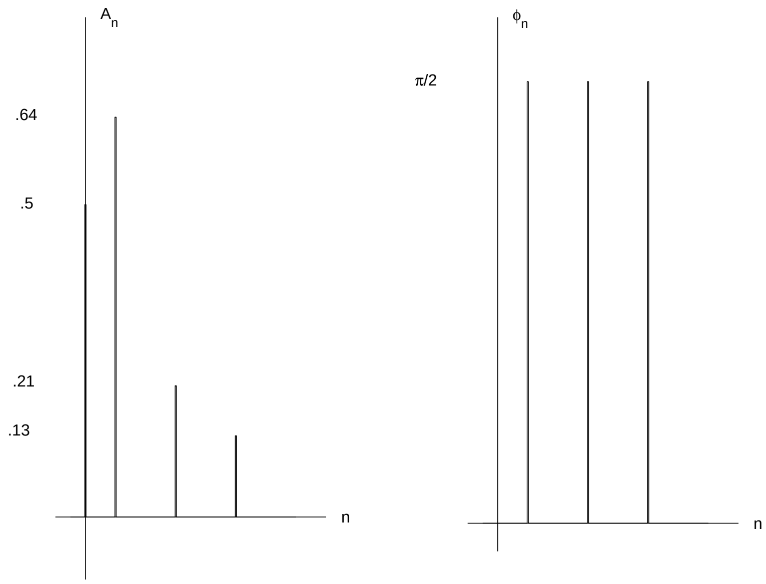


Figure 12: Graph of spectrum for problem 1a of 2.5

1. b.

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{6}{n^2\pi^2}\left(1-\cos(n\pi)\right)\cos\left(\frac{n\pi x}{3}\right)$$

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(n\pi x - \phi_n\right)$$

where

$$A_n = \sqrt{a_n^2 + b_n^2}, \;\; \phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$$

Now

$$\frac{a_0}{2} = \frac{3}{2}, \;\; a_n = \frac{6}{n^2\pi^2}(1-\cos(n\pi)), \;\; b_n = 0,$$

$$\Rightarrow A_0 = \frac{3}{2}, \;\; A_n = \begin{cases} \dfrac{12}{n^2\pi^2}, & n = 1,3,5,\ldots \\ 0, & n = 2,4,6,\ldots \end{cases}$$

$$\phi_n = 0$$

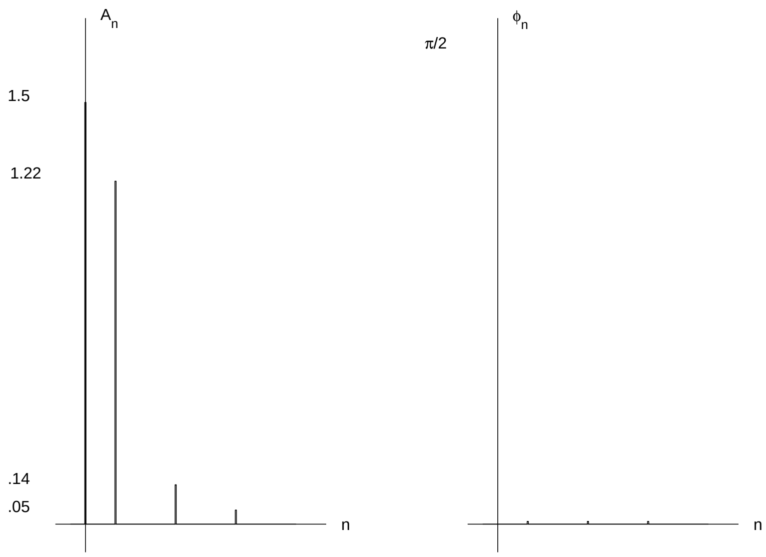


Figure 13: Graph of spectrum for problem 1b of 2.5

1. c.

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \left\{ \left(\frac{2}{n\pi}\right)^2 \frac{2\cos(\frac{n\pi}{2}) - \cos(n\pi) - 1}{2} \cos\left(\frac{n\pi x}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin(\frac{n\pi}{2}) \sin\left(\frac{n\pi x}{2}\right) \right\}$$

Now

$$\frac{a_0}{2} = \frac{1}{4},\ a_n = \left(\frac{2}{n\pi}\right)^2 \frac{2\cos(\frac{n\pi}{2}) - \cos(n\pi) - 1}{2},\ b_n = \left(\frac{2}{n\pi}\right)^2 \sin(\frac{n\pi}{2}),$$

$$\Rightarrow A_0 = \frac{1}{4},\ A_n = \left(\frac{2}{n\pi}\right)^2 \sqrt{\frac{(2\cos(\frac{n\pi}{2}) - \cos(n\pi) - 1)^2}{4} + \sin^2(\frac{n\pi}{2})}$$

$$\phi_n = \tan^{-1}\left(\frac{2\sin(\frac{n\pi}{2})}{2\cos(\frac{n\pi}{2}) - \cos(n\pi) - 1}\right)$$

$$\phi_1 = \frac{\pi}{2},\ \phi_2 = \pm\pi,\ \phi_3 = -\frac{\pi}{2},\ \phi_4 = 0$$

For $n \geq 5$ these values repeat.

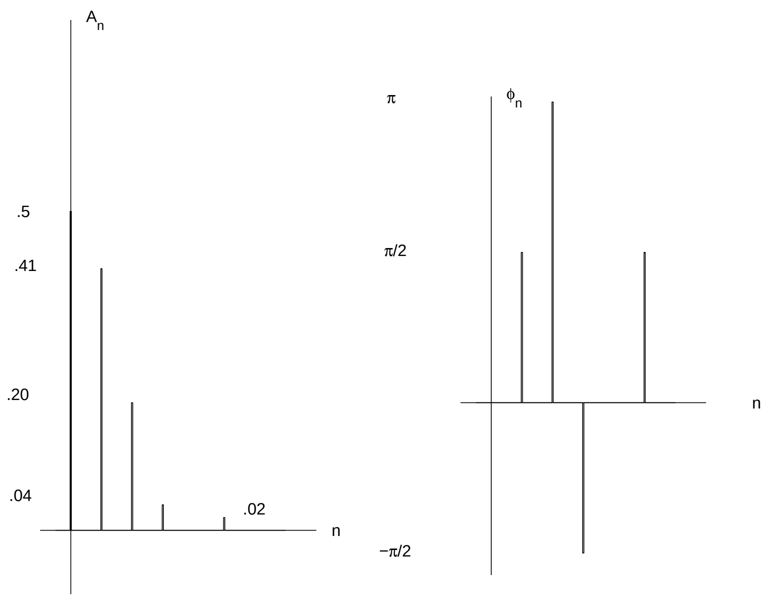


Figure 14: Graph of spectrum for problem 1c of 2.5

1. d.

$$f(x) = 1 - \cos(\pi x)$$

$$a_0 = 1,\ a_1 = -1, \quad \text{all others are zero}$$

$$A_0 = 1,\ A_1 = 1,\ A_n = 0,\ n \geq 2$$

$$\phi_1 = \pi$$

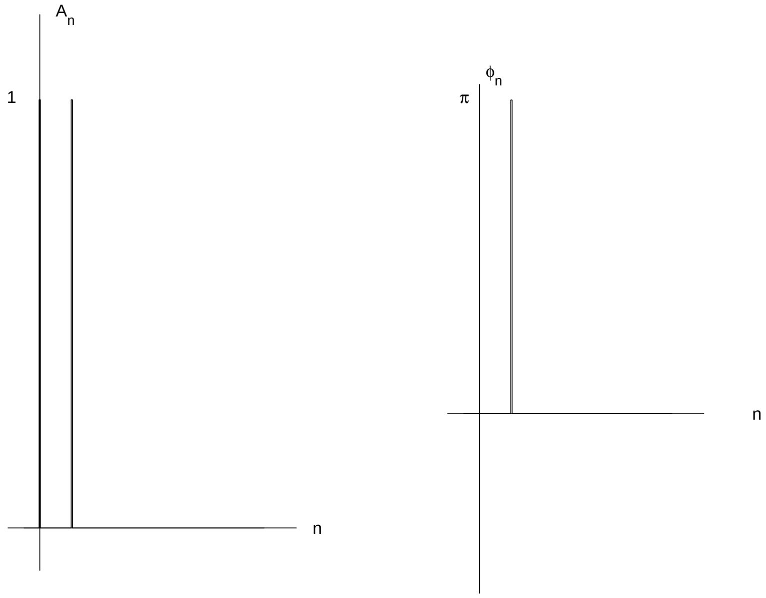


Figure 15: Graph of spectrum for problem 1d of 2.5

2.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right)$$

So

$$\frac{1}{2L}\int_{-L}^{L} [f(x)]^2\,dx = \frac{1}{2L}\int_{-L}^{L} \left[\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right) \right]^2 dx$$

$$= \frac{1}{2L}\int_{-L}^{L} \left[\frac{a_0^2}{4} + 2\frac{a_0}{2} \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right) \right.$$

$$\left. + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right) \left(a_m \cos\left(\frac{m\pi}{L}x\right) + b_m \sin\left(\frac{m\pi}{L}x\right) \right) \right] dx$$

$$= \frac{1}{2L}\int_{-L}^{L} \frac{a_0^2}{4} dx + \frac{1}{2L} a_0 \sum_{n=1}^{\infty} \left[a_n \underbrace{\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right) dx}_{=0} + b_n \underbrace{\int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right) dx}_{=0} \right]$$

$$+ \frac{1}{2L} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \left(a_n a_m \underbrace{\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right) \cos\left(\frac{m\pi}{L}x\right) dx}_{=0,\ m\neq n} + a_n b_m \underbrace{\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx}_{=0,\ \text{ for all } m,n} \right.$$

$$\left. + a_m b_n \underbrace{\int_{-L}^{L} \cos\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx}_{=0,\ \text{ for all } m,n} + b_n b_m \underbrace{\int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx}_{=0,\ m\neq n} \right)$$

$$= \frac{1}{2L} 2L \frac{a_0^2}{4} + \frac{1}{2L} \sum_{n=1}^{\infty} \left(a_n^2 \underbrace{\int_{-L}^{L} \cos^2\left(\frac{n\pi}{L}x\right) dx}_{=L} + b_n^2 \underbrace{\int_{-L}^{L} \sin^2\left(\frac{n\pi}{L}x\right) dx}_{=L} \right)$$

$$= \frac{a_0^2}{4} + \frac{1}{2L} \sum_{n=1}^{\infty} \left(L a_n^2 + L b_n^2 \right) = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2 \right)$$

$$= A_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} A_n^2$$

2.6 The Complex Form of the Fourier Series

PROBLEMS

1. Find the complex Fourier series for each of the following functions:

   a. $f(x) = x, -3 \le x \le 3 \quad , \qquad f(x+6) = f(x)$

   b. $f(x) = \begin{cases} 0 & , \quad -1 < x < 0 \\ 1 & , \quad 0 < x < 1 \end{cases}$
   $f(x+2) = f(x)$

2. Plot the complex amplitude spectrum for each of the series found in problem 1 above.

3. Show that if we use T_0 for the period of a signal, rather than $2L$, the formula for the complex Fourier series coefficients reduces to

$$c_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f(x) e^{-2in\pi x/T_0} dx$$

4. Using the complex form of the Fourier series, prove the following form of Parseval's theorem

$$\frac{1}{2L} \int_{-L}^{L} |f(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} |c_n|^2$$

(Hint: Show

$$\begin{aligned} |f(x)|^2 = f(x) f(x)^* &= \left[\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \right] \left[\sum_{m=-\infty}^{\infty} c_m^* e^{-im\pi x/L} \right] \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left[c_n c_m^* e^{i(n-m)\pi x/L} \right] \quad , \end{aligned}$$

then integrate.)

1. a.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i(n\pi x/L)}$$

where $L = 3$ and

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i(n\pi x/L)}dx = \frac{1}{6}\int_{-3}^{3} xe^{-i(n\pi x/3)}$$

$$c_n = \frac{1}{6}\left[-\frac{3}{in\pi}xe^{-i(n\pi x/3)} - \left(\frac{3}{in\pi}\right)^2 e^{-i(n\pi x/3)}\right]\Big|_{-3}^{3}$$

$$c_n = \frac{1}{6}\left[-\frac{9}{in\pi}e^{-in\pi} - \left(\frac{3}{in\pi}\right)^2 e^{-in\pi} - \frac{9}{in\pi}e^{in\pi} + \left(\frac{3}{in\pi}\right)^2 e^{in\pi}\right],\ n \neq 0$$

Note that $e^{ix} = \cos x + i\sin x \ \Rightarrow e^{\pm in\pi} = \cos(n\pi) \pm i\underbrace{\sin(n\pi)}_{=0} = \cos(n\pi)$. Also

$$c_0 = \frac{1}{6}\int_{-3}^{3} xdx = 0$$

So

$$c_n = \frac{1}{6}\left[-\frac{18}{in\pi}\cos(n\pi)\right] = \frac{3i}{n\pi}\cos(n\pi), \quad \text{note } -\frac{1}{i} = i$$

Thus

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{3i}{n\pi}\cos(n\pi)e^{i(n\pi x/3)}$$

$$|c_n| = |\frac{3i}{n\pi}\cos(n\pi)| = |\frac{3}{n\pi}|,\ n \neq 0$$

$$|c_0| = 0$$

To compute the argument of c_n, we note that

$$c_n = -\frac{3i}{n\pi}, \quad \text{for } n = \pm 1, 3, 5, \ldots$$

and

$$c_n = \frac{3i}{n\pi}, \quad \text{for } n = \pm 2, 4, 6, \ldots$$

Notice that since n changes sign, we have

$$arg(c_n) = \begin{cases} -\frac{\pi}{2}, & n = 1, 3, 5, \ldots \\ \frac{\pi}{2}, & n = -1, -3, -5, \ldots \end{cases}$$

and

$$arg(c_n) = \begin{cases} \frac{\pi}{2}, & n = 2, 4, 6, \ldots \\ -\frac{\pi}{2}, & n = -2, -4, -6, \ldots \end{cases}$$

1. b.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i(n\pi x/L)}$$

where $L = 1$ and

$$c_n = \frac{1}{2}\int_{-1}^{1} f(x)e^{-i(n\pi x)}dx = \frac{1}{2}\int_{0}^{1} e^{-i(n\pi x)}$$

$$c_n = -\frac{1}{2in\pi}e^{-i(n\pi x)}\Big|_0^1 = -\frac{1}{2in\pi}\left[e^{-in\pi} - 1\right], \; n \neq 0$$

Note that $e^{\pm in\pi} = \cos(n\pi) \pm i\underbrace{\sin(n\pi)}_{=0} = \cos(n\pi)$. Also

$$c_0 = \frac{1}{2}\int_0^1 dx = \frac{1}{2}$$

So

$$c_n = \frac{i}{2n\pi}\left(\cos(n\pi) - 1\right)$$

Thus

$$f(x) = \frac{1}{2} + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{i}{2n\pi}\left(\cos(n\pi) - 1\right)e^{i(n\pi x)}$$

$$|c_n| = |\frac{i}{2n\pi}\left(\cos(n\pi) - 1\right)| = \begin{cases} |\frac{1}{n\pi}|, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

$$|c_0| = \frac{1}{2}$$

Argument of c_n is $\mp\frac{\pi}{2}$ where $n = \pm 1, 3, 5, \ldots$ and is 0 for $n = \pm 2, 4, 6, \ldots$

2. a.

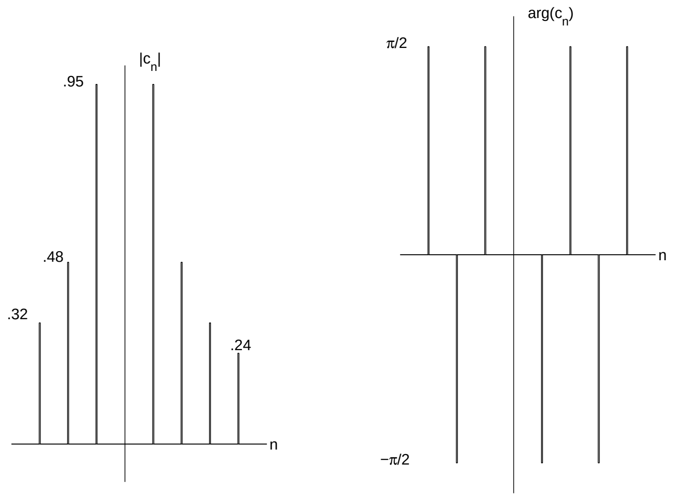


Figure 16: Graph of spectrum for problem 1a of 2.6

2. b.

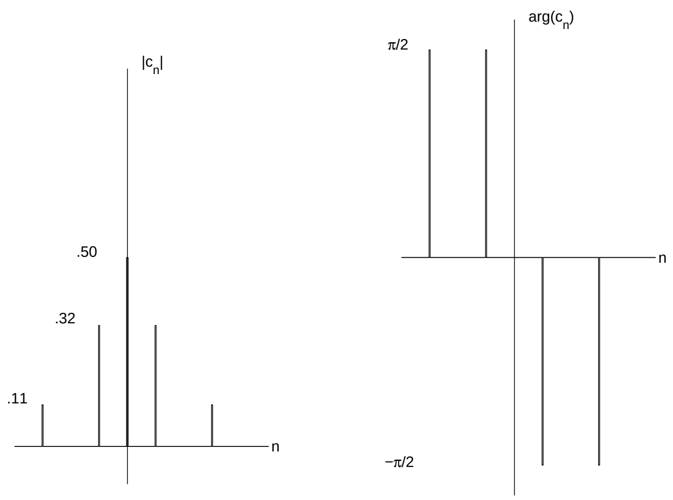


Figure 17: Graph of spectrum for problem 1b of 2.6

3.

If $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i(n\pi x/L)}$ then

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i(n\pi x/L)}dx$$

Now let $T_0 = 2L, \Rightarrow L = \frac{T_0}{2}$. Thus

$$c_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(x)e^{-in\pi x/(T_0/2)}dx$$

$$c_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(x)e^{-2in\pi x/T_0}dx$$

4.

$$\frac{1}{2L}\int_{-L}^{L} |f(x)|^2 \, dx = \frac{1}{2L}\int_{-L}^{L} f(x)f(x)^* dx = \frac{1}{2L}\int_{-L}^{L} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_n c_m^* e^{i(n-m)\pi x/L}dx$$

$$= \frac{1}{2L}\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_n c_m^* \underbrace{\int_{-L}^{L} e^{i(n-m)\pi x/L}dx}_{=0 \text{ for } n\neq m} = \frac{1}{2L}\sum_{n=-\infty}^{\infty} c_n c_n^* \underbrace{\int_{-L}^{L} dx}_{=2L}$$

Thus

$$\frac{1}{2L}\int_{-L}^{L} |f(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} |c_n|^2$$

2.7 Fourier Series and Ordinary Differential Equations

PROBLEM

1. Use Fourier series to construct a non-homogeneous solution to the ordinary differential equations:

$$y'' + 2y' + y = f(x),$$

where:

$$f(x) = \begin{cases} x & , \quad 0 \leq x < 1 \\ 2 - x & , \quad 1 \leq x \leq 2 \end{cases}$$
$$f(x+2) = f(x)$$

1. A plot of the periodic extension of $f(x)$ shows that $f(x) = |x|$, $-1 \le x \le 1$. Therefore

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2(\cos(n\pi) - 1)}{n^2\pi^2} \cos(n\pi x)$$

So, if

$$y'' + 2y' + y = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2(\cos(n\pi) - 1)}{n^2\pi^2} \cos(n\pi x)$$

then $y(x) = y_0(x) + \sum\limits_{n=1}^{\infty} y_n(x)$ where

$$y_0'' + 2y_0' + y_0 = \frac{1}{2} \quad \Rightarrow y_0 = \frac{1}{2}$$

$$y_n'' + 2y_n' + y_n = \frac{2(\cos(n\pi) - 1)}{n^2\pi^2} \cos(n\pi x)$$

Thus

$$y_n(x) = \alpha_n \cos(n\pi x) + \beta_n \sin(n\pi x)$$

and substitution yields (after comparing like terms)

$$\left(-n^2\pi^2 + 1\right) \alpha_n + 2n\pi\beta_n = \frac{2(\cos(n\pi) - 1)}{n^2\pi^2}$$

$$-2n\pi\alpha_n + \left(-n^2\pi^2 + 1\right) \beta_n = 0$$

Solving the system of two equations for the two unknowns, we have

$$\alpha_n = \frac{2(\cos(n\pi) - 1)}{n^2\pi^2} \frac{(1 - n^2\pi^2)}{(1 - n^2\pi^2)^2 + 4n^2\pi^2} = O\left(\frac{1}{n^4}\right)$$

$$\beta_n = \frac{2(\cos(n\pi) - 1)}{n^2\pi^2} \frac{2n\pi}{(1 - n^2\pi^2)^2 + 4n^2\pi^2} = O\left(\frac{1}{n^5}\right)$$

So

$$y(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2(\cos(n\pi) - 1)}{n^2\pi^2 \left[(1 - n^2\pi^2)^2 + 4n^2\pi^2\right]} \left\{\left(1 - n^2\pi^2\right) \cos(n\pi x) + 2n\pi \sin(n\pi x)\right\}$$

Plot of 10 terms of the series is given in Figure 18.

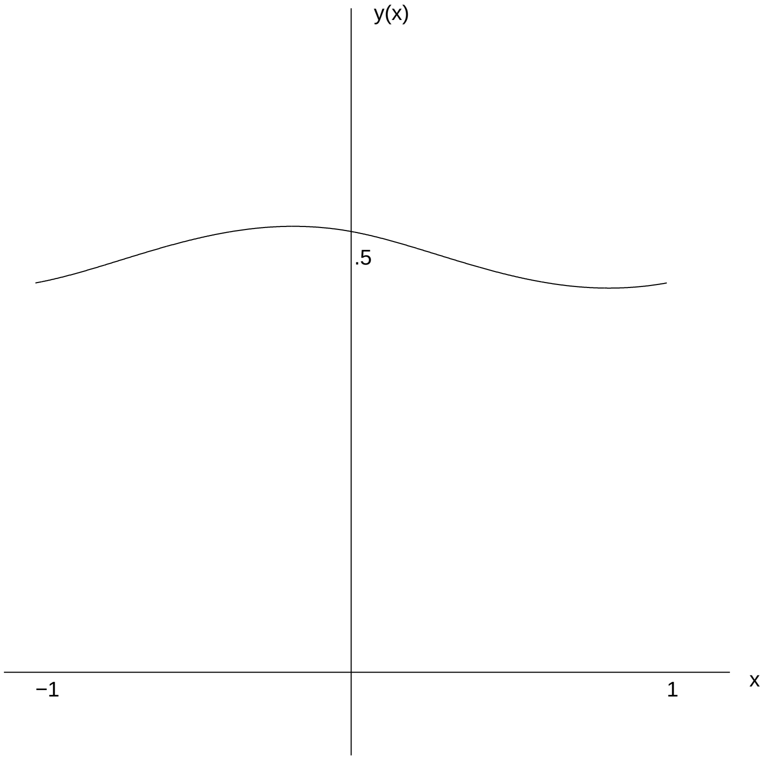


Figure 18: Graph of S_{10} for problem 1 of 2.7

2.8 Fourier Series and Digital Data Transmission

3 The One-Dimensional Wave Equation

3.1 Introduction

3.2 The One-Dimensional Wave Equation

PROBLEMS

1. Show that if a uniform, thin, tightly stretched elastic string is acted upon by no forces other than internal tension and an external *air resistance* proportional to the vertical velocity, then Newton's second law leads to a partial differential equation of the form:

$$\frac{\partial^2 u}{\partial t^2} + \kappa_d \frac{\partial u}{\partial t} = \frac{\tau}{\rho} \frac{\partial^2 u}{\partial x^2} \quad ,$$

where κ_d is some positive constant of proportionality.

2. Show that if a uniform, thin, tightly stretched elastic string is acted upon by no forces other than internal tension and an external *spring-like restoring force* proportional to the vertical displacement, then Newton's second law leads to a partial differential equation of the form:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\tau}{\rho} \frac{\partial^2 u}{\partial x^2} - \kappa_s u \quad ,$$

where κ_s is some positive constant of proportionality.

1. If we add air resistance proportional to the vertical velocity then the net vertical force acting on the small segment of the string becomes

$$\tau\left\{\frac{\partial u}{\partial x}(x+\Delta x,t)-\frac{\partial u}{\partial x}(x,t)\right\}+\rho g(x+\frac{1}{2}\Delta x,t)\Delta x-K\frac{\partial u}{\partial t}\Delta x$$

Therefore

$$\rho\frac{\partial^2 u}{\partial t^2}(x,t)=\tau\frac{\partial^2 u}{\partial x^2}(x,t)+\rho g(x,t)-K_d\frac{\partial u}{\partial t}$$

2. If instead of air resistance we have external spring-like restoring force then the first equation in problem 1 will have $-Ku\Delta x$ instead of $K\frac{\partial u}{\partial t}\Delta x$ and thus the final equation becomes

$$\rho\frac{\partial^2 u}{\partial t^2}(x,t)=\tau\frac{\partial^2 u}{\partial x^2}(x,t)+\rho g(x,t)-K_s u(x,t)$$

3.3 Boundary Conditions

PROBLEMS

1. Having physically correct algebraic signs in boundary conditions can be critical. Show, both mathematically and physically, that the following boundary conditions:
a. $u(L,t) - u_x(L,t) = 0$
b. $u(0,t) + u_x(0,t) = 0$
are **not** physically realistic.

2. Show that if a uniform, thin, tightly stretched elastic string is attached at its right-hand boundary to a slip-ring on a pole that is **not** frictionless, but in which the friction is proportional to the vertical velocity along the pole, then the boundary condition at that point becomes

$$\tau \frac{\partial u}{\partial x} + \kappa_d \frac{\partial u}{\partial t} = 0 \quad ,$$

where κ_d is some positive constant of proportionality.

1. a.-b. The balance of forces, as discussed in the notes, leads to

$$\kappa_s u(L,t) + \tau \frac{\partial u}{\partial x}(L,t) = 0$$

$$\kappa_s u(0,t) - \tau \frac{\partial u}{\partial x}(0,t) = 0$$

Physically, the spring constant κ_s and the string tension τ cannot be negative.

2. The vertical force acting on a weightless slip ring due to string tension is, as discussed in the notes, $-\tau \frac{\partial u}{\partial x}(x,t)$. The vertical force acting on the ring due to pole friction is opposite to the direction of its motion: $-\kappa_d \frac{\partial u}{\partial t}(x,t)$. The sum of forces acting on the ring must be zero, therefore

$$-\tau \frac{\partial u}{\partial x}(x,t) - \kappa_d \frac{\partial u}{\partial t}(x,t) = 0$$

3.4 Initial Conditions

3.5 Introduction to the Solution of the Wave Equation

PROBLEMS

1. Briefly describe, in a few sentences, a physical model for each of the following boundary value problems:

a.

$$\begin{aligned}
u_{tt} &= 4u_{xx} \\
u(0,t) &= u_x(3,t) = 0 \\
u(x,0) &= \begin{cases} 2x & ,0 < x < 1 \\ 0 & ,1 \le x < 3 \end{cases} \\
u_t(x,0) &= 0
\end{aligned}$$

b.

$$\begin{aligned}
&u_{tt} = u_{xx} \\
&u(0,t) = u(1,t) = 0 \\
&u(x,0) = 0 \\
&u_t(x,0) = 1
\end{aligned}$$

c.

$$\begin{aligned}
&u_{tt} = 9u_{xx} \\
&u_x(0,t) = u_x(2,t) = 0 \\
&u(x,0) = x \\
&u_t(x,0) = 0
\end{aligned}$$

d.

$$\begin{aligned}
&u_{tt} = u_{xx} \\
&u(0,t) = 0 \\
&u(3,t) + 2u_x(3,t) = 0 \\
&u(x,0) = \begin{cases} 2x & ,0 < x < 1 \\ 0 & ,1 \le x < 3 \end{cases} \\
&u_t(x,0) = 10
\end{aligned}$$

1. a. A vibrating string of length $L = 3$ and $c = 2$. The initial displacement is linear from 0 to 1 and at rest from 1 to 3. The initial velocity is zero. The left end is fixed and the right end is free.

1. b. A vibrating string of length $L = 1$ and $c = 1$. The initial displacement is zero and the initial velocity is 1. Both ends are fixed.

1. c. A vibrating string of length $L = 2$ and $c = 3$. The initial displacement is linear and the initial velocity is zero. Both ends are free.

1. d. A vibrating string of length $L = 3$ and $c = 1$. The initial displacement is as in problem 1a and the initial velocity is 10. The left end is fixed and the right end is attached to a spring.

3.6 The Fixed End Condition String

PROBLEMS

1. Solve:

$$\begin{aligned}
&u_{tt} = u_{xx} \\
&u(0,t) = u(3,t) = 0 \\
&u(x,0) = \begin{cases} 2x & ,0 < x < 1/2 \\ 2-2x & ,1/2 \leq x < 1 \\ 0 & ,1 \leq x < 3 \end{cases} \\
&u_t(x,0) = 0
\end{aligned}$$

Sketch the ten-term partial sum of your computed solution at

$$t = 0, 1, 2, 4 \ .$$

2. Solve:

$$\begin{aligned}
&u_{tt} = u_{xx} \\
&u(0,t) = u(\pi,t) = 0 \\
&u(x,0) = \begin{cases} 0 & , \quad 0 < x < \pi/4 \\ (4x-\pi)/\pi & , \quad \pi/4 \leq x \leq \pi/2 \\ (3\pi-4x)/\pi & , \quad \pi/2 \leq x \leq 3\pi/4 \\ 0 & , \quad 3\pi/4 < x < \pi \end{cases} \\
&u_t(x,0) = 0
\end{aligned}$$

3. Solve:

$$\begin{aligned}
&u_{tt} = u_{xx} \\
&u(0,t) = u(\pi,t) = 0 \\
&u(x,0) = x(\pi - x), 0 < x < \pi \\
&u_t(x,0) = 0
\end{aligned}$$

4. Solve:

$$\begin{aligned}
&u_{tt} = u_{xx} \\
&u(0,t) = u(3,t) = 0 \\
&u(x,0) = 0 \\
&u_t(x,0) = x
\end{aligned}$$

5. Solve:

$$\begin{aligned}
&u_{tt} = 9u_{xx} \\
&u(0,t) = u(\pi,t) = 0 \\
&u(x,0) = sin(x) \\
&u_t(x,0) = 1
\end{aligned}$$

6. Solve:

$$\begin{aligned}
&u_{tt} = 4u_{xx} \\
&u(0,t) = u(5,t) = 0 \\
&u_t(x,0) = \begin{cases} x & , \quad 0 < x < 5/2 \\ 5 - x & , \quad 5/2 \le x < 5 \end{cases} \\
&u(x,0) = 0
\end{aligned}$$

7. The dissipation of heat in a "very large" solid slab of thickness L whose faces are held at a fixed reference temperature of 0^o is described by the partial differential equation:

$$\begin{aligned}
&u_t = k u_{xx} \\
&u(0,t) = u(L,t) = 0 \\
&u(x,0) = f(x)
\end{aligned}$$

where $u(x,t)$ denotes the temperature at location x and time t.

a. Why is only one initial condition required in this problem?

b. Show that the method of Separation of Variables also "works" in this problem, and leads formally to the general solution:

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-k(n\pi/L)^2 t} \sin\left(\frac{n\pi x}{L}\right) \quad ,$$

where:

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad .$$

1.

$$\begin{aligned}
&u_{tt} = u_{xx}\\
&u(0,t) = u(3,t) = 0\\
&u(x,0) = \begin{cases} 2x & ,0 < x < 1/2 \\ 2-2x & ,1/2 \le x < 1 \\ 0 & ,1 \le x < 3 \end{cases}\\
&u_t(x,0) = 0
\end{aligned}$$

Separation of variables leads to

$$\ddot{T}(t) + \lambda T(t) = 0 \qquad\qquad \begin{aligned} X''(x) + \lambda X(x) &= 0 \\ X(0) &= 0 \\ X(3) &= 0 \end{aligned}$$

Eigenvalues are $\lambda_n = \xi_n^2 = \dfrac{n^2\pi^2}{9}$ and eigenfunctions are $X_n(x) = \sin\left(\dfrac{n\pi}{3}x\right)$. The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ A_n \cos\left(\frac{n\pi}{3}t\right) + B_n \sin\left(\frac{n\pi}{3}t\right) \right\} \sin\left(\frac{n\pi}{3}x\right)$$

Initial conditions

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{3}x\right) = f(x) \Rightarrow A_n = \frac{2}{3}\int_0^3 f(x)\sin\left(\frac{n\pi}{3}x\right)dx$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \left(\frac{n\pi}{3}\right) B_n \sin\left(\frac{n\pi}{3}x\right) = 0 \Rightarrow B_n = 0$$

Computing A_n, we have

$$A_n = \frac{2}{3}\left\{ \int_0^{1/2} 2x \sin\left(\frac{n\pi}{3}x\right)dx + \int_{1/2}^{1} (2-2x)\sin\left(\frac{n\pi}{3}x\right)dx \right\}$$

$$\begin{aligned}
A_n = &\frac{2}{3}\left\{ \left[-2x\left(\frac{3}{n\pi}\right)\cos\left(\frac{n\pi}{3}x\right) + 2\left(\frac{3}{n\pi}\right)^2 \sin\left(\frac{n\pi}{3}x\right)\right]\Bigg|_0^{1/2} \right.\\
&\left. + \left[-(2-2x)\left(\frac{3}{n\pi}\right)\cos\left(\frac{n\pi}{3}x\right) - 2\left(\frac{3}{n\pi}\right)^2 \sin\left(\frac{n\pi}{3}x\right)\right]\Bigg|_{1/2}^{1} \right\}
\end{aligned}$$

$$\begin{aligned}
&= \frac{2}{3}\left\{ -\left(\frac{3}{n\pi}\right)\cos\left(\frac{n\pi}{6}\right) + 2\left(\frac{3}{n\pi}\right)^2 \sin\left(\frac{n\pi}{6}\right) - 2\left(\frac{3}{n\pi}\right)^2 \sin\left(\frac{n\pi}{3}\right) \right.\\
&\left. + \left(\frac{3}{n\pi}\right)\cos\left(\frac{n\pi}{6}\right) + 2\left(\frac{3}{n\pi}\right)^2 \sin\left(\frac{n\pi}{6}\right) \right\}
\end{aligned}$$

$$\begin{aligned}
&= \frac{2}{3}\cdot 2 \cdot \left(\frac{3}{n\pi}\right)^2 \left\{ 2\sin\left(\frac{n\pi}{6}\right) - \sin\left(\frac{n\pi}{3}\right) \right\}\\
&= \frac{12}{(n\pi)^2}\left\{ 2\sin\left(\frac{n\pi}{6}\right) - \sin\left(\frac{n\pi}{3}\right) \right\} \qquad\qquad \Rightarrow O\left(\frac{1}{n^2}\right)
\end{aligned}$$

Thus

$$u(x,t)=\sum_{n=1}^{\infty}\frac{12}{(n\pi)^2}\left\{2\sin\left(\frac{n\pi}{6}\right)-\sin\left(\frac{n\pi}{3}\right)\right\}\cos\left(\frac{n\pi}{3}t\right)\sin\left(\frac{n\pi}{3}x\right)$$

The plot of $u(x,t)$ for various values of t is given in Figure 19

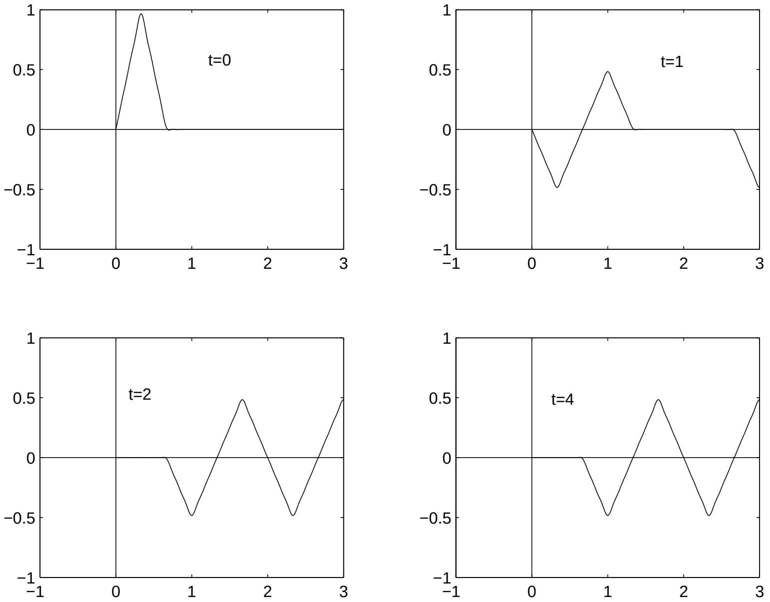


Figure 19: Graph of $u(x,t)$ for problem 1 of 3.6 for $t=1,2,3,4$

2.

$$\begin{aligned}
&u_{tt}=u_{xx}\\
&u(0,t)=u(\pi,t)=0\\
&u(x,0)=\begin{cases}0 & , \ 0<x<\pi/4\\ (4x-\pi)/\pi & , \ \pi/4\le x\le \pi/2\\ (3\pi-4x)/\pi & , \ \pi/2\le x\le 3\pi/4\\ 0 & , \ 3\pi/4<x<\pi\end{cases}\\
&u_t(x,0)=0
\end{aligned}$$

Separation of variables yields:

$$u(x,t)\;=\;\sum_{n=1}^{\infty}\{A_n\cos(nt)+B_n\sin(nt)\}\sin(nx)$$

Initial conditions

$$u(x,0)\;=\;\sum_{n=1}^{\infty}A_n\sin(nx)\;=\;f(x)\;\Rightarrow A_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin(nx)\,dx$$

$$u_t(x,0)\;=\;\sum_{n=1}^{\infty}nB_n\sin(nx)\;=\;0\;\Rightarrow B_n=0$$

Thus

$$A_n = \frac{2}{\pi}\left\{\int_{\pi/4}^{\pi/2} \frac{4}{\pi}(x-\frac{\pi}{4})\sin(nx)\,dx + \int_{\pi/2}^{3\pi/4} \frac{4}{\pi}(\frac{3\pi}{4}-x)\sin(nx)\,dx\right\}$$

$$A_n = \frac{8}{\pi}\left\{\left[-\frac{x-\frac{\pi}{4}}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right]\Big|_{\pi/4}^{\pi/2}\right.$$
$$\left. + \left[-\frac{\frac{3\pi}{4}-x}{n}\cos(nx) - \frac{1}{n^2}\sin(nx)\right]\Big|_{\pi/2}^{3\pi/4}\right\}$$

$$= \frac{8}{\pi^2}\left\{-\frac{\pi}{4n}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right) - \frac{1}{n^2}\sin\left(\frac{n\pi}{4}\right)\right.$$
$$\left. + \frac{\pi}{4n}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n^2}\sin\left(\frac{3n\pi}{4}\right) + \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\right\}$$

$$= \frac{8}{\pi^2}\left\{\frac{2}{n^2}\sin\left(\frac{n\pi}{2}\right) - \frac{1}{n^2}\sin\left(\frac{n\pi}{4}\right) - \frac{1}{n^2}\sin\left(\frac{3n\pi}{4}\right)\right\}$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{8}{\pi^2}\left\{\frac{2}{n^2}\sin\left(\frac{n\pi}{2}\right) - \frac{1}{n^2}\sin\left(\frac{n\pi}{4}\right) - \frac{1}{n^2}\sin\left(\frac{3n\pi}{4}\right)\right\}\cos(nt)\sin(nx)$$

3.

$$\begin{aligned}
&u_{tt} = u_{xx}\\
&u(0,t) = u(\pi,t) = 0\\
&u(x,0) = x(\pi - x), 0 < x < \pi\\
&u_t(x,0) = 0
\end{aligned}$$

Separation of variables yields:

$$u(x,t) \;=\; \sum_{n=1}^{\infty} \{A_n \cos{(nt)} + B_n \sin{(nt)}\} \sin{(nx)}$$

Initial conditions

$$u(x,0) \;=\; \sum_{n=1}^{\infty} A_n \sin{(nx)} \;=\; (\pi x - x^2) \;\Rightarrow A_n = \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2) \sin{(nx)}\, dx$$

$$u_t(x,0) \;=\; \sum_{n=1}^{\infty} nB_n \sin{(nx)} \;=\; 0 \;\Rightarrow B_n = 0$$

Thus

$$A_n \;=\; \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2) \sin{(nx)}\, dx$$

$$A_n = \frac{2}{\pi}\left\{-\frac{\pi x - x^2}{n}\cos{(nx)} + \frac{\pi - 2x}{n^2}\sin{(nx)} - \frac{2}{n^3}\cos{(nx)}\right\}\Bigg|_0^{\pi}$$

$$= \frac{2}{\pi}\left\{0 + 0 - \frac{2}{n^3}\left(\cos{(n\pi)} - 1\right)\right\}$$

$$= \frac{4}{\pi n^3}\left(1 - \cos{(n\pi)}\right)$$

$$u(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1 - \cos{(n\pi)}}{n^3} \cos{(nt)} \sin{(nx)}$$

4.

$$\begin{aligned}&u_{tt} = u_{xx}\\&u(0,t) = u(3,t) = 0\\&u(x,0) = 0\\&u_t(x,0) = x\end{aligned}$$

Separation of variables leads to

$$\ddot{T}(t) + \lambda T(t) = 0 \qquad\qquad \begin{aligned} X''(x) + \lambda X(x) &= 0\\ X(0) &= 0\\ X(3) &= 0\end{aligned}$$

Eigenvalues are $\lambda_n = \left(\frac{n\pi}{3}\right)^2$ and eigenfunctions are $X_n(x) = \sin\left(\frac{n\pi}{3}x\right)$. The general solution is

$$u(x,t) = \sum_{n=1}^{\infty}\left\{A_n\cos\left(\frac{n\pi}{3}t\right) + B_n\sin\left(\frac{n\pi}{3}t\right)\right\}\sin\left(\frac{n\pi}{3}x\right)$$

Initial conditions

$$u(x,0) = \sum_{n=1}^{\infty} A_n\sin\left(\frac{n\pi}{3}x\right) = 0 \Rightarrow A_n = 0$$

$$u_t(x,0) = \sum_{n=1}^{\infty}\left(\frac{n\pi}{3}\right)B_n\sin\left(\frac{n\pi}{3}x\right) = x \Rightarrow B_n = \frac{3}{n\pi}\cdot\frac{2}{3}\int_0^3 x\sin\left(\frac{n\pi}{3}x\right)dx$$

Computing B_n, we have

$$B_n = \frac{2}{n\pi}\left\{-\frac{9}{n\pi}\cos(n\pi)\right\} = -\frac{18}{n^2\pi^2}\cos(n\pi)$$

Thus

$$u(x,t) = -\sum_{n=1}^{\infty}\frac{18\cos(n\pi)}{(n\pi)^2}\sin\left(\frac{n\pi}{3}t\right)\sin\left(\frac{n\pi}{3}x\right)$$

5.

$$\begin{aligned}
&u_{tt} = 9u_{xx}\\
&u(0,t) = u(\pi,t) = 0\\
&u(x,0) = sin(x)\\
&u_t(x,0) = 1
\end{aligned}$$

Separation yields

$$u(x,t) \; = \; \sum_{n=1}^{\infty} \{A_n \cos(3nt) + B_n \sin(3nt)\} \sin(nx)$$

Initial conditions

$$u(x,0) \; = \; \sum_{n=1}^{\infty} A_n \sin(nx) \; = \; \sin x \;\; \Rightarrow A_1 = 1, \;\; A_n = 0, \, n > 1$$

$$u_t(x,0) \; = \; \sum_{n=1}^{\infty} 3nB_n \sin(nx) \; = \; 1 \;\; \Rightarrow B_n = \frac{1}{3n} \cdot \frac{2}{\pi} \int_0^{\pi} 1 \cdot \sin(nx)\, dx$$

Computing B_n, we have

$$B_n \; = \; \frac{2(1-\cos(n\pi))}{3n^2\pi}$$

Thus

$$u(x,t) = \cos(3t)\sin(x) + \frac{2}{3\pi} \sum_{n=1}^{\infty} \frac{1-\cos(n\pi)}{n^2} \sin(3nt)\sin(nx)$$

6.

$$
\begin{aligned}
&u_{tt} = 4u_{xx} \\
&u(0,t) = u(5,t) = 0 \\
&u_t(x,0) = \begin{cases} x & , \quad 0 < x < 5/2 \\ 5-x & , \quad 5/2 \le x < 5 \end{cases} \\
&u(x,0) = 0
\end{aligned}
$$

Separation yields

$$u(x,t) \;=\; \sum_{n=1}^{\infty} \left\{ A_n \cos\left(\frac{2n\pi}{5}t\right) + B_n \sin\left(\frac{2n\pi}{5}t\right) \right\} \sin\left(\frac{n\pi}{5}x\right)$$

Initial conditions

$$u(x,0) \;=\; \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{5}x\right) \;=\; 0 \;\Rightarrow A_n = 0$$

$$u_t(x,0) \;=\; \sum_{n=1}^{\infty} \left(\frac{2n\pi}{5}\right) B_n \sin\left(\frac{n\pi}{5}x\right) \;=\; \begin{cases} x, & 0 < x < 5/2 \\ 5-x, & 5/2 < x < 5 \end{cases}$$

So

$$B_n = \frac{5}{2n\pi} \cdot \frac{2}{5} \left\{ \int_0^{5/2} x \sin\left(\frac{n\pi}{5}x\right) dx + \int_{5/2}^{5} (5-x) \sin\left(\frac{n\pi}{5}x\right) dx \right\}$$

Computing B_n, we have

$$B_n \;=\; \frac{1}{n\pi} \left\{ 2 \cdot \left(\frac{5}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) \right\}$$

Thus

$$u(x,t) = \sum_{n=1}^{\infty} \frac{50}{(n\pi)^3} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{2n\pi}{5}t\right) \sin\left(\frac{n\pi}{5}x\right)$$

7.

$$\begin{aligned} u_t &= k u_{xx} \\ u(0,t) &= u(L,t) = 0 \\ u(x,0) &= f(x) \end{aligned}$$

Separation of variables, as in the wave equation case, yields:

$$\frac{\dot{T}(t)}{kT(t)} = \frac{\ddot{X}(x)}{X(x)} = -\lambda$$

and

$$u(0,t) = X(0)T(t) = 0, \Rightarrow X(0) = 0$$

$$u(L,t) = X(L)T(t) = 0, \Rightarrow X(L) = 0$$

Thus

$$\dot{T}(t) + k\lambda T(t) = 0 \qquad \begin{aligned} X''(x) + \lambda X(x) &= 0 \\ X(0) &= 0 \\ X(L) &= 0 \end{aligned}$$

The problem for $X(x)$ was already solved and the eigenvalues are: $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n = 1, 2, \ldots$ and the eigenfunctions are: $X_n(x) = \sin\left(\frac{n\pi}{L}x\right)$, $n = 1, 2, \ldots$ So now

$$\dot{T}(t) + k\left(\frac{n\pi}{L}\right)^2 T(t) = 0 \Rightarrow T(t) = e^{-k(n\pi/L)^2 t}$$

So the linearly independent solutions for $u(x,t)$ are

$$u_n(x,t) = e^{-k(n\pi/L)^2 t} \sin\left(\frac{n\pi}{L}x\right), \; n = 1, 2, \ldots$$

So to form the general solution, take the linear combination

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-k(n\pi/L)^2 t} \sin\left(\frac{n\pi}{L}x\right)$$

The initial condition is now

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$$

and so

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right) dx$$

3.7 The Free End Conditions Problem

PROBLEMS

1. Solve:

$$u_{tt} = 25u_{xx}$$
$$u_x(0,t) = u_x(1,t) = 0$$
$$u(x,0) = \begin{cases} 0 & , \quad 0 < x < 1/4 \\ x - 1/4 & , 1/4 < x < 3/4 \\ 1/2 & , \quad 3/4 < x < 1 \end{cases}$$
$$u_t(x,0) = 0$$

Interpret the solution physically.

2. Solve:

$$u_{tt} = u_{xx}$$
$$u_x(0,t) = u_x(2,t) = 0$$
$$u(x,0) = \begin{cases} 2x & , \quad 0 < x < 1/2 \\ 2 - 2x & , \quad 1/2 \leq x \leq 3/2 \\ 2x - 4 & , \quad 3/2 < x < 2 \end{cases}$$
$$u_t(x,0) = 1$$

Interpret the solution physically.

1.

$$
\begin{aligned}
&u_{tt} = 25u_{xx} \\
&u_x(0,t) = u_x(1,t) = 0 \\
&u(x,0) = \begin{cases} 0 & , \quad 0 < x < 1/4 \\ x - 1/4 & , 1/4 < x < 3/4 \\ 1/2 & , \quad 3/4 < x < 1 \end{cases} \\
&u_t(x,0) = 0
\end{aligned}
$$

Separation of variables leads to

$$
\begin{aligned}
\ddot{T}(t) + 25\lambda T(t) &= 0 \qquad & X''(x) + \lambda X(x) &= 0 \\
\dot{T}(0) &= 0 & X'(0) &= 0 \\
& & X'(1) &= 0
\end{aligned}
$$

Eigenvalues are $\lambda_n = (n\pi)^2$ and eigenfunctions are $X_n(x) = \cos(n\pi x)$, $n = 0, 1, 2, \ldots$ The solution of the equation for $T(t)$ is

$$
T_0(t) = 1, \qquad T_n(t) = \cos(5n\pi t), \; n > 0
$$

The general solution is

$$
u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(5n\pi t)\cos(n\pi x)
$$

Initial conditions

$$
u(x,0) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x) = \begin{cases} 0 & , \quad 0 < x < 1/4 \\ x - 1/4 & , 1/4 < x < 3/4 \\ 1/2 & , \quad 3/4 < x < 1 \end{cases}
$$

Computing A_0, we have

$$
A_0 = 2\int_0^1 u(x,0)dx = \frac{1}{2}
$$

and

$$
A_n = 2\int_0^1 u(x,0)\cos(n\pi x)\,dx = \frac{2(\cos\left(\frac{3n\pi}{4}\right) - \cos\left(\frac{n\pi}{4}\right))}{n^2\pi^2}
$$

Thus

$$
u(x,t) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{2(\cos\left(\frac{3n\pi}{4}\right) - \cos\left(\frac{n\pi}{4}\right))}{n^2\pi^2}\cos(5n\pi t)\cos(n\pi x)
$$

2.

$$
\begin{aligned}
&u_{tt} = u_{xx}\\
&u_x(0,t) = u_x(2,t) = 0\\
&u(x,0) = \begin{cases} 2x & , \quad 0 < x < 1/2 \\ 2-2x & , \quad 1/2 \le x \le 3/2 \\ 2x-4 & , \quad 3/2 < x < 2 \end{cases}\\
&u_t(x,0) = 1
\end{aligned}
$$

Separation of variables leads to

$$
\ddot{T}(t) + \lambda T(t) \;=\; 0 \qquad\qquad \begin{aligned} X''(x) + \lambda X(x) &= 0 \\ X'(0) &= 0 \\ X'(2) &= 0 \end{aligned}
$$

Eigenvalues are $\lambda_n = \left(\frac{n\pi}{2}\right)^2$ and eigenfunctions are $X_n(x) = \cos\left(\frac{n\pi}{2}x\right)$, $n = 0, 1, 2, \ldots$ The solution of the equation for $T(t)$ is

$$
T_0(t) = A_0 + B_0 t, \qquad\qquad T_n(t) = A_n \cos\left(\frac{n\pi}{2}t\right) + B_n \sin\left(\frac{n\pi}{2}t\right),\ n > 0
$$

The general solution is

$$
u(x,t) \;=\; \frac{1}{2}\left(A_0 + B_0 t\right) + \sum_{n=1}^{\infty} \left\{ A_n \cos\left(\frac{n\pi}{2}t\right) + B_n \sin\left(\frac{n\pi}{2}t\right) \right\} \cos\left(\frac{n\pi}{2}x\right)
$$

Initial conditions

$$
u(x,0) \;=\; \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{2}x\right) \;=\; \begin{cases} 2x & , \quad 0 < x < 1/2 \\ 2-2x & , \quad 1/2 \le x \le 3/2 \\ 2x-4 & , \quad 3/2 < x < 2 \end{cases}
$$

$$
u_t(x,0) \;=\; \frac{1}{2}B_0 + \sum_{n=1}^{\infty} \frac{n\pi}{2} B_n \cos\left(\frac{n\pi}{2}x\right) \;=\; 1
$$

Computing A_0, B_0, we have

$$
A_0 \;=\; 0, \quad B_0 = 2
$$

and

$$
A_n \;=\; \frac{-8 + 16\cos\left(\frac{n\pi}{4}\right) - 16\cos\left(\frac{3n\pi}{4}\right) + 8\cos\left(n\pi\right)}{n^2\pi^2}
$$

and

$$
B_n \;=\; \frac{2}{n\pi} \cdot \frac{2}{2}\int_0^2 \cos\left(\frac{n\pi}{2}x\right) dx \;=\; 0
$$

Thus

$$
u(x,t) = t + \sum_{n=1}^{\infty} \frac{-8 + 16\cos\left(\frac{n\pi}{4}\right) - 16\cos\left(\frac{3n\pi}{4}\right) + 8\cos\left(n\pi\right)}{n^2\pi^2} \cos\left(\frac{n\pi}{2}t\right) \cos\left(\frac{n\pi}{2}x\right)
$$

3.8 The Mixed End Conditions Problem

PROBLEMS

1. Solve:

$$\begin{array}{l} u_{tt} = u_{xx} \\ u(0,t) = 0 \\ u_x(2,t) = 0 \\ u(x,0) = \left\{ \begin{array}{ccc} x & , & 0 < x \le 1 \\ 1 & , & 1 < x < 2 \end{array} \right. \\ u_t(x,0) = 0 \end{array}$$

2. Solve:

$$\begin{array}{l} u_{tt} = 4u_{xx} \\ u_x(0,t) = 0 \\ u(1,t) = 0 \\ u(x,0) = \left\{ \begin{array}{cc} 1 & ,0 < x \le 1/2 \\ 2-2x & ,1/2 < x < 1 \end{array} \right. \\ u_t(x,0) = 0 \end{array}$$

3. Solve:

$$\begin{array}{l} u_{tt} = 9u_{xx} \\ u_x(0,t) = u(2,t) = 0 \\ u(x,0) = 0 \\ u_t(x,0) = (2-x), 0 < x < 2 \end{array}$$

4. Show that the "normal" Fourier series reduces to

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_{2n-1} \sin\left(\tfrac{(2n-1)\pi x}{2L}\right) \\ &= b_1 \sin\left(\tfrac{\pi x}{2L}\right) + b_3 \sin\left(\tfrac{3\pi x}{2L}\right) + b_5 \sin\left(\tfrac{5\pi x}{2L}\right) + \cdots \end{aligned}$$

provided:

a. $f(x)$ is odd,
b. $f(x)$ is periodic of period $4L$, and
c. $f(x+L) = f(L-x)$, $0 < x < L$

1.

$$\begin{aligned}
&u_{tt} = u_{xx} \\
&u(0,t) = 0 \\
&u_x(2,t) = 0 \\
&u(x,0) = \begin{cases} x & , \quad 0 < x \le 1 \\ 1 & , \quad 1 < x < 2 \end{cases} \\
&u_t(x,0) = 0
\end{aligned}$$

Separation of variables leads to

$$\begin{aligned} \ddot{T}(t) + \lambda T(t) &= 0 \\ \dot{T}(0) &= 0 \end{aligned} \qquad\qquad \begin{aligned} X''(x) + \lambda X(x) &= 0 \\ X(0) &= 0 \\ X'(2) &= 0 \end{aligned}$$

Eigenvalues are $\lambda_n = \left[\left(n - \frac{1}{2}\right)\frac{\pi}{2}\right]^2$ and eigenfunctions are

$$X_n(x) = \sin\left[\left(n - \frac{1}{2}\right)\frac{\pi}{2}x\right], \; n = 1, 2, \ldots$$

The solution of the equation for $T(t)$ is

$$T_n(t) = \cos\left[\left(n - \frac{1}{2}\right)\frac{\pi}{2}t\right], n = 1, 2, \ldots$$

The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos\left[\left(n - \frac{1}{2}\right)\frac{\pi}{2}t\right] \sin\left[\left(n - \frac{1}{2}\right)\frac{\pi}{2}x\right]$$

Initial conditions

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left[\left(n - \frac{1}{2}\right)\frac{\pi}{2}x\right] = \begin{cases} x & , \quad 0 < x \le 1 \\ 1 & , \quad 1 < x < 2 \end{cases}$$

Computing A_n, we have

$$A_n = \frac{2}{2}\int_0^1 x \sin\left[\left(n - \frac{1}{2}\right)\frac{\pi}{2}x\right] dx + \frac{2}{2}\int_1^2 \sin\left[\left(n - \frac{1}{2}\right)\frac{\pi}{2}x\right] dx$$

$$A_n = \frac{16 \sin\left[\frac{2n-1}{4}\pi\right]}{(2n-1)^2\pi^2}$$

Thus

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ \frac{16 \sin\left[\frac{2n-1}{4}\pi\right]}{(2n-1)^2\pi^2} \right\} \cos\left[\left(n - \frac{1}{2}\right)\frac{\pi}{2}t\right] \sin\left[\left(n - \frac{1}{2}\right)\frac{\pi}{2}x\right]$$

2.

$$
\begin{aligned}
&u_{tt} = 4u_{xx}\\
&u_x(0,t) = 0\\
&u(1,t) = 0\\
&u(x,0) = \begin{cases} 1 & ,0 < x \le 1/2 \\ 2-2x & ,1/2 < x < 1 \end{cases}\\
&u_t(x,0) = 0
\end{aligned}
$$

Separation of variables leads to

$$
\begin{aligned}
\ddot{T}(t) + 4\lambda T(t) &= 0 \qquad & X''(x) + \lambda X(x) &= 0\\
\dot{T}(0) &= 0 & X'(0) &= 0\\
& & X(1) &= 0
\end{aligned}
$$

Eigenvalues are $\lambda_n = \left[\left(n - \frac{1}{2}\right)\pi\right]^2$ and eigenfunctions are

$$
X_n(x) = \cos\left[\left(n - \frac{1}{2}\right)\pi x\right],\ n = 1, 2, \ldots
$$

The general solution is

$$
u(x,t) = \sum_{n=1}^{\infty} \left\{A_n \cos\left[(2n-1)\pi t\right] + B_n \sin\left[(2n-1)\pi t\right]\right\} \cos\left[\left(n - \frac{1}{2}\right)\pi x\right]
$$

Initial conditions

$$
u(x,0) = \sum_{n=1}^{\infty} A_n \cos\left[\left(n - \frac{1}{2}\right)\pi x\right] = \begin{cases} 1 & ,0 < x \le 1/2 \\ 2-2x & ,1/2 < x < 1 \end{cases}
$$

$$
u_t(x,0) = \sum_{n=1}^{\infty} \left(n - \frac{1}{2}\right)\pi B_n \cos\left[\left(n - \frac{1}{2}\right)\pi x\right] = 0 \ \Rightarrow B_n = 0
$$

By Sturm-Liouville ($p(x) = w(x) = 1, q(x) = 0$), we have

$$
A_n = \frac{\int_0^1 f(x) X_n(x)dx}{\int_0^1 X_n^2(x)dx} = 2\int_0^1 f(x)\cos\left[\left(n - \frac{1}{2}\right)\pi x\right]dx
$$

$$
A_n = 2\left\{\int_0^{1/2} \cos\left[\left(n - \frac{1}{2}\right)\pi x\right]dx + \int_{1/2}^1 (2-2x)\cos\left[\left(n - \frac{1}{2}\right)\pi x\right]dx\right\}
$$

$$
\begin{aligned}
A_n &= 2\left\{\frac{2}{(2n-1)\pi}\sin\left[\left(n - \frac{1}{2}\right)\pi x\right]\Big|_0^{1/2}\right.\\
&+ \left.\left\{(2-2x)\frac{2}{(2n-1)\pi}\sin\left[\left(n - \frac{1}{2}\right)\pi x\right] - 2\left(\frac{2}{(2n-1)\pi}\right)^2 \cos\left[\left(n - \frac{1}{2}\right)\pi x\right]\right\}\Big|_{1/2}^1\right\}
\end{aligned}
$$

$$
A_n = 4\left(\frac{2}{(2n-1)\pi}\right)^2 \cos\left[\left(n - \frac{1}{2}\right)\frac{\pi}{2}\right]
$$

Thus

$$u(x,t) = \sum_{n=1}^{\infty} 2 \left(\frac{2}{(2n-1)\pi} \right)^2 \cos \left[\left(n - \frac{1}{2} \right) \frac{\pi}{2} \right] \cos \left[(2n-1) \pi t \right] \cos \left[\left(n - \frac{1}{2} \right) \pi x \right]$$

3.

$$\begin{aligned}&u_{tt} = 9u_{xx}\\&u_x(0,t) = u(2,t) = 0\\&u(x,0) = 0\\&u_t(x,0) = (2-x), 0 < x < 2\end{aligned}$$

Separation of variables leads to

$$\ddot{T}(t) + 9\lambda T(t) \;=\; 0 \qquad\qquad \begin{aligned} X''(x) + \lambda X(x) &= 0\\ X'(0) &= 0\\ X(2) &= 0\end{aligned}$$

Eigenvalues are $\lambda_n = \left[\left(n - \frac{1}{2}\right)\frac{\pi}{2}\right]^2$ and eigenfunctions are

$$X_n(x) = \cos\left[\left(n - \frac{1}{2}\right)\frac{\pi}{2}x\right],\; n = 1, 2, \ldots$$

The general solution is

$$u(x,t) \;=\; \sum_{n=1}^{\infty}\left\{A_n \cos\left[3\left(n-\frac{1}{2}\right)\frac{\pi}{2}t\right] + B_n \sin\left[3\left(n-\frac{1}{2}\right)\frac{\pi}{2}t\right]\right\}\cos\left[\left(n-\frac{1}{2}\right)\frac{\pi}{2}x\right]$$

Initial conditions

$$u(x,0) \;=\; \sum_{n=1}^{\infty} A_n \cos\left[\left(n-\frac{1}{2}\right)\frac{\pi}{2}x\right] \;=\; 0 \;\;\Rightarrow A_n \;=\; 0$$

$$u_t(x,0) \;=\; \sum_{n=1}^{\infty}\left\{3\left(n-\frac{1}{2}\right)\frac{\pi}{2}B_n \cos\left[\left(n-\frac{1}{2}\right)\frac{\pi}{2}x\right]\right\} \;=\; (2-x)$$

Thus

$$B_n \;=\; \left[\frac{4}{3\,(2n-1)\,\pi}\right]\cdot\frac{2}{2}\int_0^2 (2-x)\cos\left[\left(n-\frac{1}{2}\right)\frac{\pi}{2}x\right]dx$$

$$B_n \;=\; \frac{4}{3\,(2n-1)\,\pi}\left\{\frac{4}{(2n-1)\,\pi}(2-x)\sin\left[\frac{(2n-1)\,\pi}{4}x\right] - \left(\frac{4}{(2n-1)\,\pi}\right)^2\cos\left[\frac{(2n-1)\,\pi}{4}x\right]\right\}\Big|_0^2$$

$$B_n \;=\; \frac{1}{3}\left(\frac{4}{(2n-1)\,\pi}\right)^3$$

Thus

$$u(x,t) = \sum_{n=1}^{\infty}\frac{1}{3}\left(\frac{4}{(2n-1)\,\pi}\right)^3 \sin\left[3\left(n-\frac{1}{2}\right)\frac{\pi}{2}t\right]\cos\left[\left(n-\frac{1}{2}\right)\frac{\pi}{2}x\right]$$

4. Given an odd function $f(x)$ with a period of $4L$, we know that the Fourier series should have only sine functions, i.e.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2L}\right)$$

Now we would like to use the fact that $f(x+L) = f(L-x)$, $0 < x < L$, i.e. symmetry about $x = L$. This requires that the sine functions satisty the same thing.

$$\sin\left(\frac{n\pi}{2L}(x+L)\right) = \sin\left(\frac{n\pi}{2L}(L-x)\right)$$

$$\sin\left(\frac{n\pi}{2L}x\right)\cos\left(\frac{n\pi}{2}\right) + \underbrace{\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{2L}x\right)}_{=0} = \underbrace{\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{2L}x\right)}_{=0} - \sin\left(\frac{n\pi}{2L}x\right)\cos\left(\frac{n\pi}{2}\right)$$

This can only happen if

$$\cos\left(\frac{n\pi}{2}\right) = -\cos\left(\frac{n\pi}{2}\right), \quad \Rightarrow n \text{ must be odd}$$

Therefore $n = 2m - 1$ and the series is then

$$f(x) = \sum_{n=1}^{\infty} b_{2n-1} \sin\left(\frac{(2n-1)\pi x}{2L}\right)$$

3.9 Generalizations on the Method of Separation of Variables

3.10 Sturm-Liouville Theory

PROBLEMS

1. For each of the following problems, determine if the given equation is in Sturm-Liouville form, and, if so, identify the values of the appropriate function $p(x)$, $q(x)$, $w(x)$, and the values of α_i and β_i:

a.
$$\begin{aligned} &[(x+1)y']' + \lambda(x+1)y - y = 0 \\ &\quad y(1) = 0 \\ &\quad y(2) = 0 \end{aligned}$$

b.
$$\begin{aligned} &[(x^2-1)u']' + 3\lambda u = 0 \\ &\quad u(0) = 0 \\ &\quad u(1/2) = 0 \end{aligned}$$

c.
$$\begin{aligned} &y'' + \lambda xy = 0 \\ &\quad y(0) = 0 \\ &\quad y(3) + 2y'(3) = 0 \end{aligned}$$

d.
$$\begin{aligned} &y'' + xy' + \lambda y = 0 \\ &\quad y(0) = 0 \\ &\quad y(1) = 0 \end{aligned}$$

2. Following similar steps to those used in class, show that the eigenfunctions of the singular Sturm-Liouville problem:

$$\begin{aligned} [p(x)y']' \quad &+\lambda w(x)y + q(x)y = 0 \quad , \quad a < x < b \\ &y(a), y'(a) \text{ finite} \qquad\qquad\qquad\qquad , \\ &\alpha_2 y(b) + \beta_2 y'(b) = 0 \end{aligned}$$

where $p'(x)$, $w(x)$, and $q(x)$ are continuous, and

$$\begin{aligned} &p(x) > 0 \quad , \quad a < x \leq b \quad , \\ &p(a) = 0 \\ &w(x) > 0 \quad , \quad a < x < b \\ &q(x) \geq 0 \quad , \quad a \leq x \leq b \end{aligned}$$

corresponding to different eigenvalues are orthogonal with respect to the weighting function $w(x)$.

1. In all of these, the key is to identify values (if any) such that the given differential equation is a special case of the one in the notes.

a.

$$\begin{aligned} ((x+1)y')' + \lambda(x+1)y - y &= 0 \\ y(1) &= 0 \\ y(2) &= 0 \end{aligned}$$

This is in Sturm-Liouville form, since $p(x) = x+1$, $w(x) = x+1$, $q(x) = -1$ and the interval is $a = 1$, $b = 2$. The coefficients in the boundary conditions are: $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = 0$.

b.

$$\begin{aligned} \left[(x^2-1)\, u'\right]' + 3\lambda u &= 0 \\ u(0) &= 0 \\ u(1/2) &= 0 \end{aligned}$$

This is almost in Sturm-Liouville form, since $p(x) = (x^2 - 1)$, $w(x) = 3$, $q(x) = 0$ but $p(x) < 0$ for the interval from $a = 0$ to $b = 1/2$. This violates the condition on $p(x)$. If we multiply the equation by -1 and incorporating the sign with λ (i.e., let $\sigma = -\lambda$) then it is S-L. The coefficients in the boundary conditions are: $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = 0$.

c.

$$\begin{aligned} y'' + \lambda x y &= 0 \\ y(0) &= 0 \\ y(3) + 2y'(3) &= 0 \end{aligned}$$

This is in Sturm-Liouville form, since $p(x) = 1$, $w(x) = x$, $q(x) = 0$. The interval is from $a = 0$ to $b = 3$. The coefficients in the boundary conditions are: $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 0$, $\beta_2 = 2$.

d.

$$\begin{aligned} y'' + xy' + \lambda y &= 0 \\ y(0) &= 0 \\ y(1) &= 0 \end{aligned}$$

This is **not** in Sturm-Liouville form, since the equation has a separate y' term which does **not** occur in the S-L form.

2. Suppose $y_1(x)$ and $y_2(x)$ are solutions of

$$\begin{array}{ll} [p(x)y']' & +\lambda w(x)y + q(x)y = 0 \quad , \quad a < x < b \\ & y(a), y'(a) \text{ finite} \\ & \alpha_2 y(b) + \beta_2 y'(b) = 0 \end{array} ,$$

where $p'(x)$, $w(x)$, and $q(x)$ are continuous, and

$$\begin{array}{l} p(x) > 0 \quad , \quad a < x \leq b \quad , \\ p(a) = 0 \\ w(x) > 0 \quad , \quad a < x < b \\ q(x) \geq 0 \quad , \quad a \leq x \leq b \end{array}$$

corresponding to different eigenvalues, i.e.

$$[p(x)y_1']' + \lambda_1 w(x)y_1 + q(x)y_1 = 0$$

$$[p(x)y_2']' + \lambda_2 w(x)y_2 + q(x)y_2 = 0$$

Multiply the first equation by $y_1(x)$ and the second equation by $y_1(x)$ and subtract. Then

$$y_1[py_1']' - y_1[py_2']' + (\lambda_1 - \lambda_2)wy_1y_2 = 0$$

or

$$\frac{d}{dx}\left[y_2 p(x)y_1' - y_1 p(x)y_2\right] + (\lambda_1 - \lambda_2)wy_1y_2 = 0$$

Integrate

$$\left[y_2 p(x)y_1' - y_1 p(x)y_2\right]\Big|_a^b + (\lambda_1 - \lambda_2)\int_a^b wy_1y_2dx = 0$$

or

$$p(b)\left[y_1'(b)y_2(b) - y_2'(b)y_1(b)\right] - \underbrace{p(a)}_{=0}\left[y_1'(a)y_2(a) - y_2'(a)y_1(a)\right] + (\lambda_1 - \lambda_2)\int_a^b wy_1y_2dx = 0$$

Thus

$$p(b)\left[y_1'(b)y_2(b) - y_2'(b)y_1(b)\right] + (\lambda_1 - \lambda_2)\int_a^b wy_1y_2dx = 0$$

But now if both $y_1(x)$ and $y_2(x)$ satisfy the B.C. at $x = b$

$$\alpha_2 y_2(b) + \beta_2 y_2'(b) = 0$$

$$\alpha_2 y_1(b) + \beta_2 y_1'(b) = 0$$

and α_2 and β_2 not both zero implies that the homogeneous system can have a solution only if the determinant is zero, i.e. $y_1'(b)y_2(b) - y_2'(b)y_1(b) = 0$. Thus $p(b)\left[y_1'(b)y_2(b) - y_2'(b)y_1(b)\right] = 0$ and so

$$(\lambda_1 - \lambda_2)\int_a^b wy_1y_2dx = 0$$

Since $\lambda_1 \neq \lambda_2$ we must have

$$\int_a^b wy_1y_2dx = 0$$

3.11 The Frequency Domain Interpretation of the Wave Equation

PROBLEM

1. Find the three lowest natural frequencies, and sketch the associated modes, for the equation:

$$
\begin{aligned}
&u_{tt} = c^2 u_{xx} \\
&u(0,t) = u_x(L,t) = 0
\end{aligned}
$$

Plot, in the frequency domain, the natural frequencies of this "string."

1. Separation of variables yields

$$\ddot{T}(t) + \lambda c^2 T(t) = 0 \qquad \begin{aligned} X''(x) + \lambda X(x) &= 0 \\ X(0) &= 0 \\ X'(L) &= 0 \end{aligned}$$

For $\lambda > 0$ the solution is $X(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)$. Using the left boundary condition, $X(0) = 0$ implies $C_1 = 0$. The other boundary condition implies

$$C_2\sqrt{\lambda}\cos(\sqrt{\lambda}L) = 0, \Rightarrow \cos(\sqrt{\lambda}L) = 0$$

Therefore

$$\lambda_n = \left(\frac{\left(n+\frac{1}{2}\right)\pi}{L}\right)^2, \; n = 0, 1, 2, \ldots$$

and the eigenfunctions

$$X_n(x) = \sin\left(\frac{\left(n+\frac{1}{2}\right)\pi}{L}x\right), \; n = 0, 1, 2, \ldots$$

Notice that n starts at zero.
Thus

$$\ddot{T}_n + \lambda_n c^2 T_n =$$

$$T_n(t) = \alpha_n \cos\left(\frac{\left(n+\frac{1}{2}\right)\pi}{L}ct\right) + \beta_n \sin\left(\frac{\left(n+\frac{1}{2}\right)\pi}{L}ct\right)$$

or

$$T_n(t) = A_n \cos\left(\frac{\left(n+\frac{1}{2}\right)\pi}{L}ct - \phi_n\right)$$

where

$$A_n = \sqrt{\alpha_n^2 + \beta_n^2}$$

Thus the frequencies are

$$f_n = \frac{\sqrt{\lambda_n}c}{2\pi}, \; n = 0, 1, 2, \ldots$$

Thus

$$f_0 = \frac{\pi}{2L}\frac{c}{2\pi} = \frac{c}{4L}$$

$$f_1 = \frac{3\pi}{2L}\frac{c}{2\pi} = \frac{3c}{4L} = 3f_0$$

$$f_2 = \frac{5\pi}{2L}\frac{c}{2\pi} = \frac{5c}{4L} = 5f_0$$

The modes are

$$X_0(x) = \sin\left(\frac{\pi}{2L}x\right)$$

$$X_1(x) = \sin\left(\frac{3\pi}{2L}x\right)$$
$$X_2(x) = \sin\left(\frac{5\pi}{2L}x\right)$$

The modes are given in Figure 20 and the frequencies in Figure 21

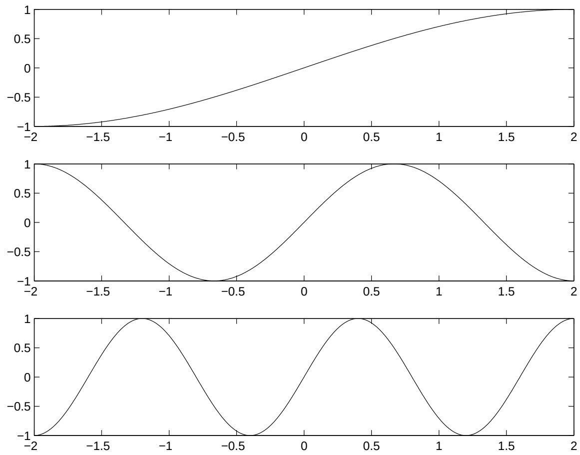


Figure 20: The first three modes for problem 1 of 3.11

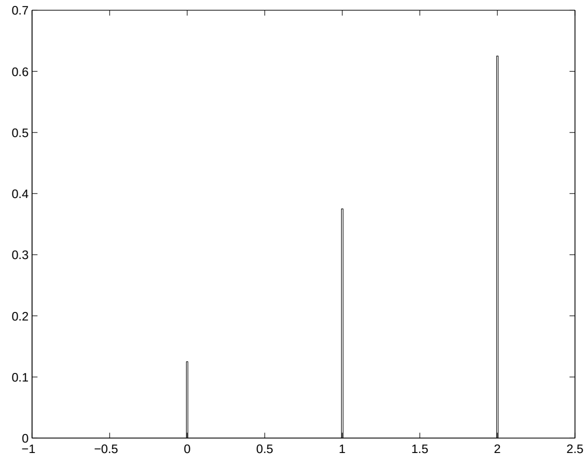


Figure 21: The first three frequencies for problem 1 of 3.11

3.12 The D'Alembert Solution of the Wave Equation

3.13 The Effect of Boundary Conditions

PROBLEM

1. Sketch the D'Alembert solutions at $t = 0, 1, 2.5$ and 4 to:

a.
$$\begin{aligned}
&u_{tt} = u_{xx} \\
&u(0,t) = u(3,t) = 0 \\
&u(x,0) = \begin{cases} 2x &, \ 0 < x < 1/2 \\ 1 &, \ 1/2 < x < 3/2 \\ 4 - 2x &, \ 3/2 < x < 2 \\ 0 &, \ 2 < x < 3 \end{cases} \\
&u_t(x,0) = 0
\end{aligned}$$

b.
$$\begin{aligned}
&u_{tt} = u_{xx} \\
&u(0,t) = u_x(3,t) = 0 \\
&u(x,0) = \begin{cases} 2x &, \ 0 < x < 1/2 \\ 1 &, \ 1/2 < x < 3/2 \\ 4 - 2x &, \ 3/2 < x < 2 \\ 0 &, \ 2 < x < 3 \end{cases} \\
&u_t(x,0) = 0
\end{aligned}$$

c.
$$\begin{aligned}
&u_{tt} = 4u_{xx} \\
&u_x(0,t) = u(1,t) = 0 \\
&u(x,0) = \begin{cases} 1 &, \ 0 < x < 1/2 \\ 2 - 2x &, \ 1/2 < x < 1 \end{cases} \\
&u_t(x,0) = 0
\end{aligned}$$

1. a. The initial displacement is given by:

$$u(x,0) = \begin{cases} 2x & , \quad 0 < x < 1/2 \\ 1 & , \quad 1/2 < x < 3/2 \\ 4-2x & , \quad 3/2 < x < 2 \\ 0 & , \quad 2 < x < 3 \end{cases}$$

The solution is

$$u(x,t) = F(x+t) + F(x-t)$$

where $F(x)$ is odd and periodic with period $2L = 6$. The solution at various time is given in the following 4 figures:

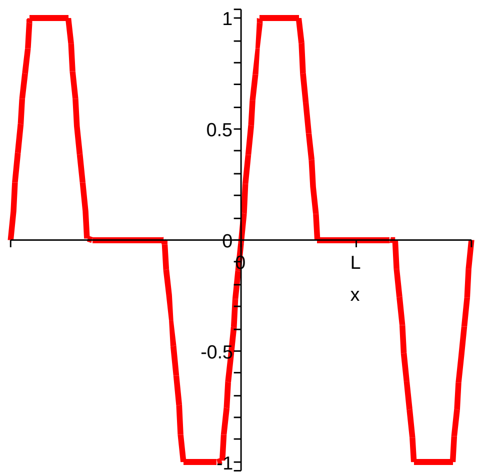


Figure 22: Graph of $u(x,0)$ for problem 1a of 13.3

1. b. The initial displacement is given by

$$u(x,0) = \begin{cases} 2x & , \quad 0 < x < 1/2 \\ 1 & , \quad 1/2 < x < 3/2 \\ 4-2x & , \quad 3/2 < x < 2 \\ 0 & , \quad 2 < x < 3 \end{cases}$$

Since the initial velocity is zero, the solution is:

$$u(x,t) = F(x+t) + F(x-t)$$

where the function $F(x)$ is odd around 0 and even around L of period $4L = 12$. The solution at various time is given in the following 4 figures:

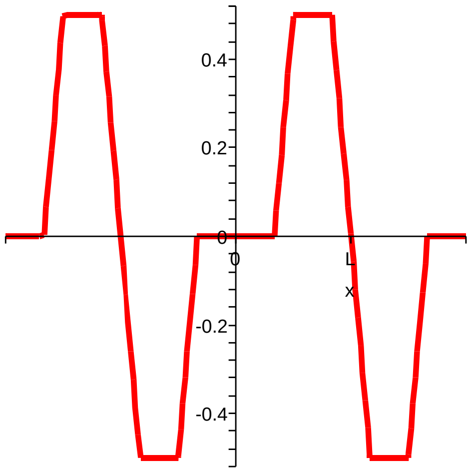


Figure 23: Graph of $u(x,1)$ for problem 1a of 13.3

1. c. The initial displacement is given by

$$u(x,0) = \begin{cases} 1 & , \quad 0 < x < 1/2 \\ 2-2x & , \quad 1/2 < x < 1 \end{cases}$$

Since the initial velocity is zero, the solution is:

$$u(x,t) = F(x+2t) + F(x-2t), \quad c = 2$$

where the function $F(x)$ is even around $x = 0$ and odd around $x = 1$ of period $4L = 4$. The solution at various time is given in the following 4 figures:

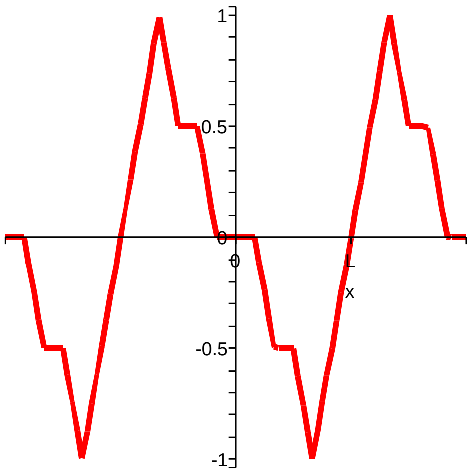


Figure 24: Graph of $u(x, 2.5)$ for problem 1a of 13.3

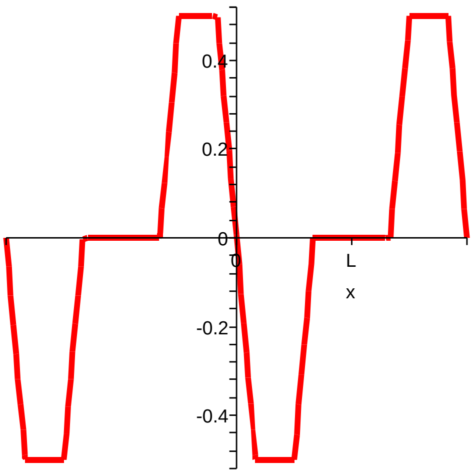


Figure 25: Graph of $u(x, 4)$ for problem 1a of 13.3

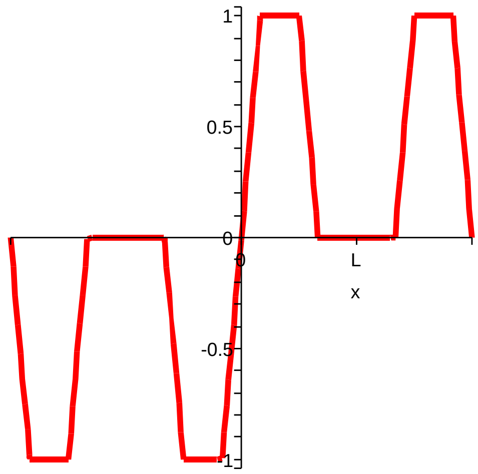


Figure 26: Graph of $u(x, 0)$ for problem 1b of 13.3

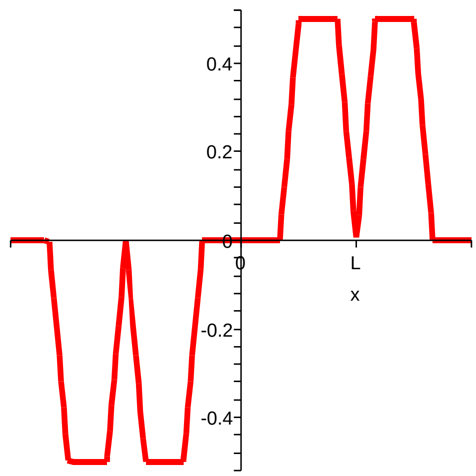


Figure 27: Graph of $u(x, 1)$ for problem 1b of 13.3

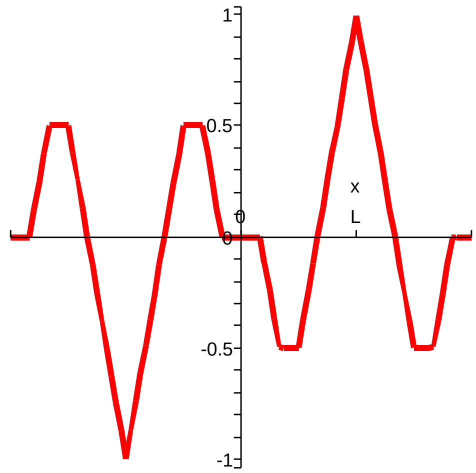


Figure 28: Graph of $u(x, 2.5)$ for problem 1b of 13.3

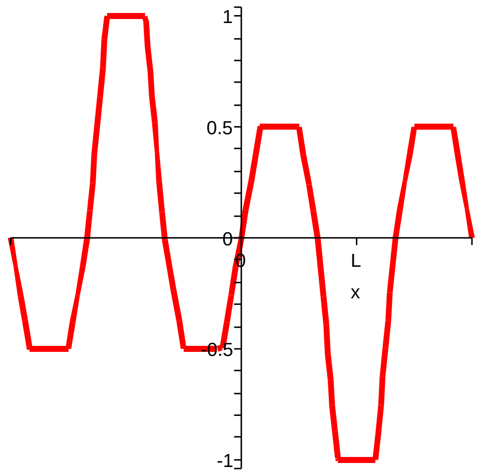


Figure 29: Graph of $u(x, 4)$ for problem 1b of 13.3

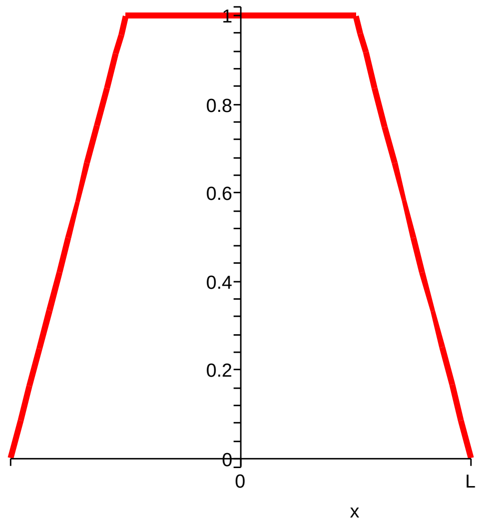


Figure 30: Graph of $u(x, 0)$ for problem 1c of 13.3

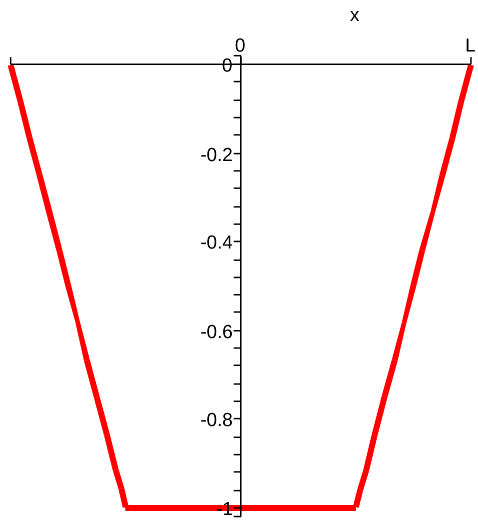


Figure 31: Graph of $u(x, 1)$ for problem 1c of 13.3

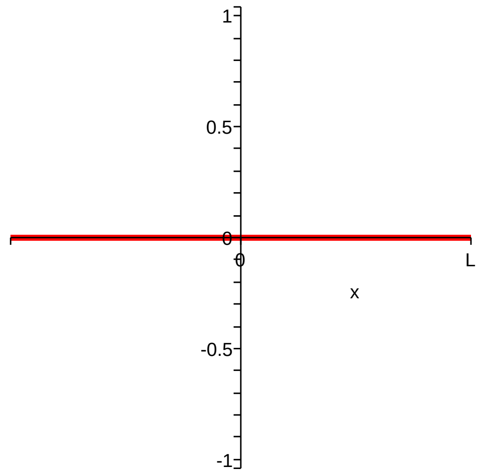


Figure 32: Graph of $u(x, 2.5)$ for problem 1c of 13.3

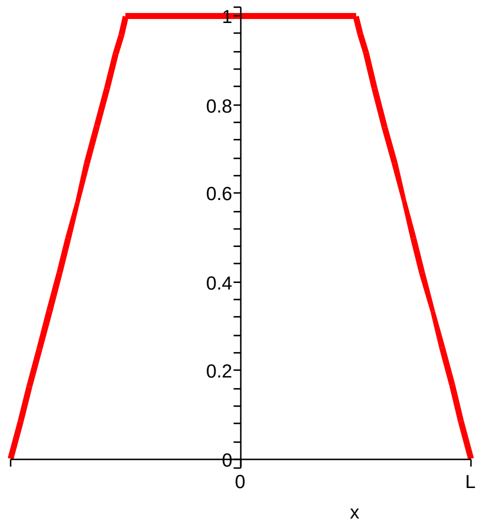


Figure 33: Graph of $u(x, 4)$ for problem 1c of 13.3

4 The Two-Dimensional Wave Equation

4.1 Introduction

4.2 The Rigid Edge Problem

4.3 Frequency Domain Analysis

4.4 Time Domain Analysis

4.5 The Wave Equation in Circular Regions

4.6 Symmetric Vibrations of the Circular Drum

4.7 Frequency Domain Analysis of the Circular Drum

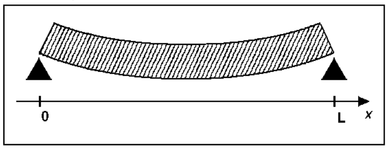
4.8 Time Domain Analysis of the Circular Membrane

PROBLEMS

1. It can be shown that the small free vertical vibrations of a uniform beam (e.g. a bridge girder) are governed by the fourth order partial differential equation:

$$\frac{\partial^2 u}{\partial t^2} + c^2 \frac{\partial^4 u}{\partial x^4} = 0$$

where c^2 is a constant involving the elasticity, moment of inertia, density and cross sectional area of the beam. If the beam if freely supported at both ends, e.g. sitting on a piling, the boundary conditions for this problem become:



$$\begin{aligned} u(0,t) &= u(L,t) = 0 \\ u_{xx}(0,t) &= u_{xx}(L,t) = 0 \end{aligned}$$

Show that separation of variables "works" in this problem, and, in case the beam is initially at rest, i.e.

$$u_t(x,0) = 0$$

produces a general solution of the form:

$$\sum_{n=1}^{\infty} A_n \cos\left(\frac{n^2\pi^2 ct}{L^2}\right) \sin\left(\frac{n\pi x}{L}\right)$$

2. Solve the two-dimensional rectangular wave equation:

$$\begin{aligned} u_{tt} &= u_{xx} + u_{yy} \\ u(0,y,t) &= u(1,y,t) = 0 \\ u(x,0,t) &= u(x,1,t) = 0 \\ u(x,y,0) &= .01xy(1-x)(1-y) \\ u_t(x,y,0) &= 0 \end{aligned}$$

3. Solve the two-dimensional rectangular wave equation:

$$\begin{aligned} u_{tt} &= 16\left(u_{xx} + u_{yy}\right) \\ u(0,y,t) &= u(3,y,t) = 0 \\ u(x,0,t) &= u(x,2,t) = 0 \\ u(x,y,0) &= y(2-y)\sin\left(\tfrac{2\pi x}{3}\right) \\ u_t(x,y,0) &= 0 \end{aligned}$$

4. Find the eigenvalues and the form of the eigenfunctions for:

$$
\begin{aligned}
u_{tt} &= 9\,(u_{xx} + u_{yy}) \\
u(0, y, t) &= u(4, y, t) = 0 \\
u_y(x, 0, t) &= u(x, 1, t) = 0
\end{aligned}
$$

Calculate the actual values of the four lowest natural frequencies.

5. One of the "quirks" of the two-dimensional wave equation in rectangular coordinates is that, unlike the one-dimensional problem, two different values of n and m may yield the same natural frequency, and therefore this single natural frequency may have two (or more) independent modes ("shapes") associated with it. For example, if $L = 2$ and $W = 1$, the eigenvalues and eigenfunctions are,

$$\lambda_{nm} = \left[\left(\tfrac{n}{2}\right)^2 + m^2\right]\pi^2$$

and

$$u_{nm} = \sin(m\pi y)\sin\left(\tfrac{n\pi x}{2}\right) \quad .$$

Show that the following eigenvalues are in fact equal:

$$\lambda_{41} = \lambda_{22} \ ; \quad \lambda_{61} = \lambda_{23} \ ; \quad \lambda_{62} = \lambda_{43} \ ; \quad \lambda_{72} = \lambda_{14}$$

6. Show that in the square membrane, certain natural frequencies may have four independent modes ("shapes") associated with them.

1.
$$\frac{\partial^2 u}{\partial t^2} + c^2 \frac{\partial^4 u}{\partial x^4} = 0$$

Since the PDE and BC are linear and homogeneous, assume

$$u(x,t) = F(x)G(t)$$

Substitution of this into the PDE and BC should lead to

$$\begin{array}{ll} \ddot{G}(t) - kc^2 G(t) = 0 & \qquad \begin{aligned} F''''(x) + kF(x) &= 0 \\ F(0) = F(L) &= 0 \\ F''(0) = F''(L) &= 0 \end{aligned} \\ \dot{G}(0) = 0 & \end{array}$$

where k is some constant. The condition $F''(0) = 0$ arises from

$$u_{xx} = (F(x)G(t))_{xx} = F''(x)G(t)$$

Thus $u_{xx}(0,t) = F''(0)G(t) = 0, \ \Rightarrow F''(0) = 0$. Similarly for the other conditions.
There is no a-priori reason to assume a particular value for k, so consider all three cases. If $k > 0$, we write $k = 4\eta^4$ (just to avoid radicals) then try $F(x) = e^{rx}$ to get

$$r^4 + 4\eta^4 = 0$$

The roots are

$$r = \eta(1+i),\ \eta(1-i),\ \eta(-1+i),\ \eta(-1-i)$$

The solutions are

$$e^{\eta(1+i)x},\ e^{\eta(1-i)x},\ e^{\eta(-1+i)x},\ e^{\eta(-1-i)x}$$

Converting to the equivalent real valued form gives

$$F(x) = C_1 \cosh \eta x \sin \eta x + C_2 \cosh \eta x \cos \eta x + C_3 \sinh \eta x \sin \eta x + C_4 \sinh \eta x \cos \eta x$$

Direct calculations yield

$$F''(x) = 2\eta^2 \left\{ C_1 \sinh \eta x \cos \eta x - C_2 \sinh \eta x \sin \eta x + C_3 \cosh \eta x \cos \eta x - C_4 \cosh \eta x \sin \eta x \right\}$$

Applying the BC at $x = 0$ leads to

$$F(0) = C_2 = 0$$

$$F''(0) = C_3 = 0$$

Thus at $x = L$

$$F(L) = C_1 \cosh \eta L \sin \eta L + C_4 \sinh \eta L \cos \eta L = 0$$

$$F''(L) = 2\eta^2 \left\{ C_1 \sinh \eta L \cos \eta L - C_4 \cosh \eta L \sin \eta L \right\} = 0$$

The determinant of the coefficients is equal

$$2\eta^2 \left\{ -\cosh^2 \eta L \sin^2 \eta L - \sinh^2 \eta L \cos^2 \eta L \right\} \neq 0$$

Thus by Cramer's rule, the only solution is $C_1 = C_4 = 0$ which is the trivial solution.

Now try $k = 0$. Thus $F'''' = 0$ and $F(x) = C_1 + C_2 x + C_3 x^2 + C_4 x^3$

$$F(0) = C_1 = 0$$

$$F''(0) = C_3 = 0$$

$$F(L) = C_2 L + C_4 L^3 = 0$$

$$F''(L) = 6C_4 L = 0, \Rightarrow C_4 = 0$$

Substituting in the previous equation $F(L) = C_2 L + C_4 L^3 = 0$ we have $C_2 = 0$ and again this is the trivial solution.

So $k < 0$, so we take $k = -\beta^4$ and we find that the roots are now

$$r = \pm\beta, \pm\beta i$$

Using real valued solutions

$$F(x) = C_1 \sinh \beta x + C_2 \cosh \beta x + C_3 \sin \beta x + C_4 \cos \beta x$$

$$F''(x) = \beta^2 \{C_1 \sinh \beta x + C_2 \cosh \beta x - C_3 \sin \beta x - C_4 \cos \beta x\}$$

So, the BC at $x = 0$ become

$$F(0) = C_2 + C_4 = 0$$

$$F''(0) = \beta^2 \{C_2 - C_4\} = 0$$

and we have $C_2 = C_4 = 0$. The BC at $x = L$ imply

$$F(L) = C_1 \sinh \beta L + C_3 \sin \beta L = 0$$

$$F''(L) = \beta^2 \{C_1 \sinh \beta L - C_3 \sin \beta L\} = 0$$

Adding the equations (after dividing the second by β^2 gives

$$C_1 \sinh \beta L = 0, \Rightarrow C_1 = 0$$

Therefore

$$C_3 \sin \beta L = 0, \ \Rightarrow \sin \beta L = 0$$

Therefore

$$\beta_n \ = \ \frac{n\pi}{L}$$

and

$$F_n(x) = \sin\left(\frac{n\pi}{L} x\right)$$

Now recall that $k = -\beta^4$, so

$$\ddot{G}(t) - kc^2 G = 0, \ \Rightarrow \ddot{G}(t) + \left(\frac{n\pi}{L}\right)^4 c^2 G = 0$$

Thus

$$G(t) = c_1 \cos\left(\frac{n^2\pi^2}{L^2}ct\right) + c_2 \sin\left(\frac{n^2\pi^2}{L^2}ct\right)$$

$$\dot{G}(0) = 0, \Rightarrow \left(\frac{n^2\pi^2}{L^2}c\right) c_2 = 0, \Rightarrow c_2 = 0$$

$$G_n(t) = \cos\left(\frac{n^2\pi^2}{L^2}ct\right)$$

So the solution becomes

$$u_n(x,t) = \cos\left(\frac{n^2\pi^2}{L^2}ct\right) \sin\left(\frac{n\pi}{L}x\right)$$

and

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n^2\pi^2}{L^2}ct\right) \sin\left(\frac{n\pi}{L}x\right)$$

2.

$$
\begin{aligned}
u_{tt} &= u_{xx} + u_{yy} \\
u(0,y,t) &= u(1,y,t) = 0 \\
u(x,0,t) &= u(x,1,t) = 0 \\
u(x,y,0) &= .01xy(1-x)(1-y) \\
u_t(x,y,0) &= 0
\end{aligned}
$$

Let $u(x,y,t) = X(x)Y(y)T(t)$, then

$$
\ddot{T}(t) + \lambda T(t) = 0 \qquad
\begin{aligned} X'' + \mu X &= 0 \\ X(0) &= 0 \\ X(1) &= 0 \end{aligned} \qquad
\begin{aligned} Y'' + (\lambda - \mu)Y &= 0 \\ Y(0) &= 0 \\ Y(1) &= 0 \end{aligned}
$$

The solution for X and Y are

$$
X_n(x) = \sin(n\pi x), \;\; Y_n(y) = \sin(m\pi y), \;\; n = 1, 2, \ldots, \; m = 1, 2, \ldots
$$

So

$$
u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left\{ A_{mn} \cos(\sqrt{\lambda_{mn}}t) + B_{mn} \sin(\sqrt{\lambda_{mn}}t) \right\} \sin(n\pi x) \sin(m\pi y)
$$

where $\sqrt{\lambda_{mn}} = \sqrt{m^2 + n^2}\pi$, so

$$
u(x,y,0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sin(n\pi x) \sin(m\pi y) = .01xy(1-x)(1-y)
$$

$$
u_t(x,y,0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sqrt{\lambda_{mn}} B_{mn} \sin(n\pi x) \sin(m\pi y) = 0, \; \Rightarrow B_{mn} = 0
$$

Thus

$$
A_{mn} = \frac{4}{1 \cdot 1} \int_0^1 \int_0^1 (.01xy(1-x)(1-y) \sin(n\pi x) \sin(m\pi y) dy dx
$$

$$
A_{mn} = .04 \int_0^1 x(1-x) \sin(n\pi x) dx \int_0^1 y(1-y) \sin(m\pi y) dy
$$

But

$$
\int_0^1 x(1-x)\sin(n\pi x)dx = \left[-\frac{x(1-x)}{n\pi}\cos(n\pi x) + \frac{1-2x}{(n\pi)^2}\sin(n\pi x) - \frac{2}{(n\pi)^3}\cos(n\pi x) \right] \Big|_0^1
$$

which is $2\dfrac{1-\cos(n\pi)}{(n\pi)^3}$.

Similarly

$$
\int_0^1 y(1-y)\sin(m\pi y)dy = 2\frac{1-\cos(m\pi)}{(m\pi)^3}
$$

Therefore

$$
A_{mn} = .16 \frac{1-\cos(m\pi)}{(m\pi)^3} \frac{1-\cos(n\pi)}{(n\pi)^3}
$$

and

$$u(x,y,t) \;=\; .16\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1-\cos(m\pi)}{(m\pi)^3}\frac{1-\cos(n\pi)}{(n\pi)^3}\cos(\sqrt{\lambda_{mn}}t)\sin(n\pi x)\sin(m\pi y)$$

where

$$\sqrt{\lambda_{mn}} \;=\; \sqrt{m^2+n^2}\pi$$

3.

$$
\begin{aligned}
u_{tt} &= 16\left(u_{xx}+u_{yy}\right)\\
u(0,y,t) &= u(3,y,t)=0\\
u(x,0,t) &= u(x,2,t)=0\\
u(x,y,0) &= y(2-y)\sin\left(\tfrac{2\pi x}{3}\right)\\
u_t(x,y,0) &= 0
\end{aligned}
$$

Let $u(x,y,t)=X(x)Y(y)T(t)$, then

$$
\begin{array}{rclcrclcrcl}
\ddot{T}(t)+16\lambda T(t) &=& 0 & \quad & X''+\mu X &=& 0 & \quad & Y''+(\lambda-\mu)Y &=& 0\\
 & & & & X(0) &=& 0 & & Y(0) &=& 0\\
 & & & & X(3) &=& 0 & & Y(2) &=& 0
\end{array}
$$

The solution for X and Y are

$$
X_n(x) \;=\; \sin\left(\frac{n\pi}{3}x\right),\;\; Y_m(y) \;=\; \sin\left(\frac{m\pi}{2}y\right),\;\; n=1,2,\ldots,\; m=1,2,\ldots
$$

with

$$
\mu_n \;=\; \left(\frac{n\pi}{3}\right)^2,\;\; (\lambda-\mu)_m \;=\; \left(\frac{m\pi}{2}\right)^2,\; \Rightarrow \lambda_{mn} \;=\; \left(\frac{n\pi}{3}\right)^2+\left(\frac{m\pi}{2}\right)^2
$$

Let $\nu_{mn} \;=\; \sqrt{\lambda_{mn}}$, then

$$
\nu_{mn} \;=\; \sqrt{\left(\frac{n}{3}\right)^2+\left(\frac{m}{2}\right)^2}\,\pi
$$

So

$$
u(x,y,t) \;=\; \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left\{A_{mn}\cos(4\nu_{mn}t)+B_{mn}\sin(4\nu_{mn}t)\right\}\sin\left(\frac{n\pi}{3}x\right)\sin\left(\frac{m\pi}{2}y\right)
$$

so

$$
\begin{aligned}
u(x,y,0) \;&=\; \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_{mn}\sin\left(\frac{n\pi}{3}x\right)\sin\left(\frac{m\pi}{2}y\right)=y(2-y)\sin\left(\frac{2\pi}{3}x\right)\\
u_t(x,y,0) \;&=\; \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\lambda_{mn}B_{mn}\sin\left(\frac{n\pi}{3}x\right)\sin\left(\frac{m\pi}{2}y\right)=0,\;\Rightarrow B_{mn}=0
\end{aligned}
$$

Thus

$$
A_{mn}=\frac{4}{3\cdot 2}\int_0^3\int_0^2 y(2-y)\sin\left(\frac{2\pi}{3}x\right)\sin\left(\frac{n\pi}{3}x\right)\sin\left(\frac{m\pi}{2}y\right)dxdy
$$

$$
A_{mn}=\frac{4}{6}\int_0^3\left\{\underbrace{\int_0^2 y(2-y)\sin\left(\frac{m\pi}{2}y\right)dy}_{2\left(\frac{2}{m\pi}\right)^3(1-\cos(m\pi))}\right\}\sin\left(\frac{2\pi}{3}x\right)\sin\left(\frac{n\pi}{3}x\right)dx
$$

$$A_{mn} = \frac{8}{6}\left(\frac{2}{m\pi}\right)^3 (1-\cos(m\pi)) \underbrace{\int_0^3 \sin\left(\frac{2\pi}{3}x\right)\sin\left(\frac{n\pi}{3}x\right)dx}_{=\frac{3}{2},\ n=2,\ \text{otherwise } =0}$$

Thus

$$A_{m\,2} \;=\; 2\left(\frac{2}{m\pi}\right)^3 (1-\cos(m\pi)),\ A_{m\,n} \;=\; 0,\ n\neq 2$$

So

$$u(x,y,t) \;=\; \sum_{m=1}^{\infty} 2\left(\frac{2}{m\pi}\right)^3 (1-\cos(m\pi))\cos(4\nu_{m\,2}t)\sin\left(\frac{2\pi}{3}x\right)\sin\left(\frac{m\pi}{2}y\right)$$

where

$$\nu_{m\,2} \;=\; \sqrt{\left(\frac{2}{3}\right)^2+\left(\frac{m}{2}\right)^2}\,\pi$$

4.
$$
\begin{aligned}
u_{tt} &= 9\left(u_{xx}+u_{yy}\right)\\
u(0,y,t) &= u(4,y,t)=0\\
u_y(x,0,t) &= u(x,1,t)=0
\end{aligned}
$$

Let $u(x,y,t)=X(x)Y(y)T(t)$, then

$$
\ddot{T}(t)+9\lambda T(t) = 0 \qquad
\begin{aligned} X''+\mu X &= 0\\ X(0) &= 0\\ X(4) &= 0\end{aligned} \qquad
\begin{aligned} Y''+(\lambda-\mu)Y &= 0\\ Y'(0) &= 0\\ Y(1) &= 0\end{aligned}
$$

The solution for X and Y are

$$
X_n(x) = \sin\left(\frac{n\pi}{4}x\right),\; Y_n(y) = \cos\left(\frac{(2m+1)\pi}{2}y\right),\; n=1,2,\ldots,\, m=0,1,2,\ldots
$$

with

$$
\mu_n = \left(\frac{n\pi}{4}\right)^2,\; (\lambda-\mu)_m = \left(\frac{(2m+1)\pi}{2}\right)^2,\; \Rightarrow \lambda_{mn} = \left(\frac{n\pi}{4}\right)^2+\left(\frac{(2m+1)\pi}{2}\right)^2
$$

Let $\nu_{mn} = \sqrt{\lambda_{mn}}$, then

$$
\nu_{mn} = \sqrt{\left(\frac{n}{4}\right)^2+\left(\frac{2m+1}{2}\right)^2}\,\pi
$$

So

$$
T_{mn} = A_{mn}\cos(3\nu_{mn}t)+B_{mn}\sin(3\nu_{mn}t)
$$

Thus the frequencies are given by

$$
f_{mn} = \frac{3}{2\pi}\nu_{mn} = \frac{3}{2}\sqrt{\left(\frac{n}{4}\right)^2+\left(\frac{2m+1}{2}\right)^2}
$$

$m=$	0	1	2
$n=1$	0.839	2.281	3.769
$n=2$	1.061	2.372	5.303
$n=3$	1.352	2.516	
$n=4$	1.677		

The four lowest frequencies are: 0.839, 1.061, 1.352, 1.677. Notice that in this case they all come from $m=0$.

5. If $L = 2$ and $W = 1$, the eigenvalues are,

$$\sqrt{\lambda_{nm}} = \sqrt{\left(\frac{n}{L}\right)^2 + \left(\frac{m}{W}\right)^2}\,\pi \;=\; \sqrt{\left(\frac{n}{2}\right)^2 + m^2}\,\pi \;=\; \frac{\pi}{2}\sqrt{n^2 + (2m)^2}$$

The eigenfunctions are

$$\sin(m\pi y)\sin\left(\frac{n\pi x}{2}\right)$$

Thus two different eigenfunctions correspond to the same eigenvalue if $n_1^2 + (2m_1)^2 = n_2^2 + (2m_2)^2$. For example, $m = 1$, $n = 4$ and $m = 2$, $n = 2$ give the same eigenvalue $\frac{\pi}{2}\sqrt{20}$. Also, by trial and error, we find: $m = 1$, $n = 6$ and $m = 3$, $n = 2$ give the same eigenvalue $\frac{\pi}{2}\sqrt{40}$, $m = 2$, $n = 6$ and $m = 3$, $n = 4$ give the same eigenvalue $\frac{\pi}{2}\sqrt{52}$, and $m = 2$, $n = 7$ and $m = 4$, $n = 1$ give the same eigenvalue $\frac{\pi}{2}\sqrt{65}$.

6. For a square membrane $L = W$ and the eigenvalues are $\sqrt{\lambda_{mn}} = \sqrt{\left(\frac{n}{L}\right)^2 + \left(\frac{m}{L}\right)^2}\,\pi = \frac{L}{\pi}\sqrt{n^2 + m^2}$. The eigenfunctions are $\sin\left(\frac{m\pi y}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$. Thus different eigenfunctions have the same eigenvalues only when different m, n have the same $m^2 + n^2$. The following 4 eigenfunctions have the same eigenvalue:

$$m = 1,\ n = 8;\ \ m = 4,\ n = 7;\ \ m = 8,\ n = 1;\ \ m = 7,\ n = 4;\ \Rightarrow m^2 + n^2 = 65$$

$$m = 2,\ n = 9;\ \ m = 6,\ n = 7;\ \ m = 9,\ n = 2;\ \ m = 7,\ n = 6;\ \Rightarrow m^2 + n^2 = 85$$

$$m = 2,\ n = 11;\ \ m = 5,\ n = 10;\ \ m = 11,\ n = 2;\ \ m = 10,\ n = 5;\ \Rightarrow m^2 + n^2 = 125$$

$$m = 3,\ n = 11;\ \ m = 7,\ n = 9;\ \ m = 11,\ n = 3;\ \ m = 9,\ n = 7;\ \Rightarrow m^2 + n^2 = 130$$

PROBLEMS

1. Show that separation of variables ($u(r,\theta,t) = R(r)\Theta(\theta)T(t)$), applied to the wave equation in a circular region of radius A,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left\{ \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right\}$$

$$u(A,\theta,t) = 0$$

$$u(0,\theta,t) \ , \quad \frac{\partial u}{\partial r}(0,\theta,t) \quad \text{finite}$$

$$u(r,\theta,t) = u(r,\theta+2\pi,t)$$

leads to

$$T'' + \lambda c^2 T = 0 \qquad \begin{array}{l} r\left[rR'\right]' + (\lambda r^2 - \mu)R = 0 \\ R(0) \ , \quad R'(0) \quad \text{finite} \\ R(A) = 0 \end{array} \qquad \begin{array}{l} \Theta'' + \mu\Theta = 0 \\ \Theta(\theta) = \Theta(\theta + 2\pi) \end{array}$$

2. Explain the mathematical and physical significance of the condition

$$u(r,\theta,t) = u(r,\theta+2\pi,t).$$

3. Find the three lowest natural frequencies of

$$u_{tt} = \frac{6}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r}\right]$$

$$u(4,t) = 0$$
$$u(0,t) \ , \quad \frac{\partial u}{\partial r}(0,t) \quad \text{finite}$$

$$u(r,0) = f(r)$$
$$u_t(r,0) = 0$$

4. Solve the following problems. (*Non-zero* coefficients may be left in terms of definite integrals of known functions.)

a.

$$u_{tt} = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r}\right]$$

$$u(2,t) = 0$$
$$u(0,t) \ , \quad \frac{\partial u}{\partial r}(0,t) \quad \text{finite}$$

$$u(r,0) = \sin(\pi r)$$
$$u_t(r,0) = 0$$

b.

$$
\begin{aligned}
u_{tt} &= \frac{4}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r}\right] \\
& u(1,t) = 0 \\
& u(0,t)\ ,\quad \frac{\partial u}{\partial r}(0,t) \quad \text{finite} \\
& u(r,0) = 1 - r^2 \\
& u_t(r,0) = 0
\end{aligned}
$$

c.

$$
\begin{aligned}
u_{tt} &= c^2\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r}\right] \\
& u(2,t) = 0 \\
& u(0,t)\ ,\quad \frac{\partial u}{\partial r}(0,t) \quad \text{finite} \\
& u(r,0) = 0 \\
& u_t(r,0) = 1
\end{aligned}
$$

5. Solve the following problem. (*Non-zero* coefficients may be left in terms of definite integrals of known functions.) Physically interpret the boundary conditions, and relate this to the properties of the solution:

$$
\begin{aligned}
u_{tt} &= c^2\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r}\right] \\
& u_r(L,t) = 0 \\
& u(0,t)\ ,\quad \frac{\partial u}{\partial r}(0,t) \quad \text{finite} \\
& u(r,0) = f(r) \\
& u_t(r,0) = 0
\end{aligned}
$$

1.
$$\frac{\partial^2 u}{\partial t^2} = c^2 \quad \left\{ \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r} \right] + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right\}$$

$$u(A, \theta, t) = 0$$

$$u(0, \theta, t)\ , \quad \frac{\partial u}{\partial r}(0, \theta, t) \quad \text{finite}$$

$$u(r, \theta, t) = u(r, \theta + 2\pi, t)$$

Let $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$, thus

$$R\Theta\ddot{T} = c^2 \left[\frac{1}{r}\Theta T (rR')' + \frac{1}{r^2} RT\Theta'' \right]$$

Divide by $c^2 R\Theta T$,

$$\underbrace{\frac{\ddot{T}}{c^2 T}}_{\text{function of } t \text{ only}} = \underbrace{\frac{1}{rR}(rR')' + \frac{1}{r^2}\frac{\Theta''}{\Theta}}_{\text{function of } r,\theta} = -\lambda$$

Thus

$$\ddot{T} + \lambda c^2 T = 0$$

and

$$\frac{1}{rR}(rR')' + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\lambda$$

or, multiply by r^2 and separate the variables:

$$\underbrace{\frac{r}{R}(rR')' + \lambda r^2}_{\text{function of } r \text{ only}} = \underbrace{-\frac{\Theta''}{\Theta}}_{\text{function of } \theta \text{ only}} = \mu$$

Thus

$$\Theta'' + \mu\Theta = 0, \quad r(rR')' + (\lambda r^2 - \mu)R = 0$$

The condition $u(A, \theta, t) = 0$ implies $R(A) = 0$ and $u(0, \theta, t)$ finite, implies $R(0)$ finite, similarly $\frac{\partial u}{\partial r}(0, \theta, t)$ finite implies $R'(0)$ is finite. The periodicity $u(r, \theta, t) = u(r, \theta + 2\pi, t)$ implies

$$\Theta(0) = \Theta(2\pi)$$

Thus

$$\ddot{T} + \lambda c^2 T = 0 \qquad \begin{array}{c} r(rR')' + (\lambda r^2 - \mu)R = 0 \\ R(0), R'(0) \text{ finite} \\ R(A) = 0 \end{array} \qquad \begin{array}{c} \Theta'' + \mu\Theta = 0 \\ \Theta(0) = \Theta(2\pi) \end{array}$$

2. Mathematically, the PDE is defined only for $0 \leq \theta \leq 2\pi$ (due to polar coordinates system.
Physically, (r, θ) and $(r, \theta + 2\pi)$ are the same point and must have the same displacement. Thus $u(r, \theta, t) = u(r, \theta + 2\pi, t)$.

3.

$$u_{tt} = \frac{6}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r}\right]$$

$$u(4,t) = 0$$
$$u(0,t)\ ,\quad \frac{\partial u}{\partial r}(0,t) \quad \text{finite}$$

$$u(r,0) = f(r)$$
$$u_t(r,0) = 0$$

Separation yields

$$\ddot{T} + 6\lambda c^2 T = 0 \qquad\qquad r(rR')' + (\lambda r^2 - \mu)R = 0$$
$$R(0), R'(0) \text{ finite}$$
$$R(4) = 0$$

Let $\lambda = \xi^2$ then $r(rR')' + (\xi^2 r^2 - \mu)R = 0$ is Bessel's equation of order zero. The general solution is

$$R(r) = C_1 J_0(\xi r) + C_2 Y_0(\xi r)$$

The BC $R(0)$ finite implies $C_2 = 0$. The BC $R(4) = 0$ implies

$$C_1 J_0(4\xi) = 0$$

Thus

$$J_0(4\xi) = 0, \ \Rightarrow 4\xi_n = \alpha_n, \ n = 1, 2, \ldots$$

where $J_0(\alpha_n) = 0$.
Thus

$$\begin{array}{rclrcl} 4\xi_1 &=& 2.4048 & \xi_1 &=& .6012 \\ 4\xi_2 &=& 5.5201 & \xi_2 &=& .1.3800 \\ 4\xi_3 &=& 8.6537 & \xi_3 &=& 2.1634 \\ 4\xi_4 &=& 11.7915 & \xi_4 &=& 2.9479 \end{array}$$

Now $\ddot{T} + 6\lambda c^2 T = 0$ implies $\ddot{T} + 6\xi_n^2 c^2 T = 0$ and the solution

$$T_n(t) = A_n \cos(\sqrt{6}\xi_n t) + B_n \sin(\sqrt{6}\xi_n t)$$

Thus

$$f_n = \frac{\sqrt{6}\xi_n}{2\pi}$$

$$f_1 = \frac{\sqrt{6}(.6012)}{2\pi} = .2344$$

$$f_2 = .5380$$

$$f_3 = .8434$$

$$f_4 = 1.1492$$

4. a.

$$u_{tt} = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r}\right]$$

$$u(2,t) = 0$$
$$u(0,t)\ ,\quad \frac{\partial u}{\partial r}(0,t) \quad \text{finite}$$

$$u(r,0) = \sin(\pi r)$$
$$u_t(r,0) = 0$$

Separation yields

$$\ddot{T} + \lambda T = 0 \qquad\qquad \begin{array}{c}(rR')' + \lambda r R = 0\\ R(0), R'(0) \text{ finite}\\ R(2) = 0\end{array}$$

Let $x = \xi r$ then $(rR')' + \lambda r R = 0$ becomes

$$x^2\frac{d^2R}{dx^2} + x\frac{dR}{dx} + x^2 R = 0$$

which is Bessel's equation of order zero. The general solution is

$$R(r) = C_1 J_0(\xi r) + C_2 Y_0(\xi r)$$

The BC $R(0)$ finite implies $C_2 = 0$. The BC $R(2) = 0$ implies

$$C_1 J_0(2\xi) = 0$$

Thus

$$J_0(2\xi) = 0,\ \Rightarrow 2\xi_n = \alpha_n,\ n = 1, 2, \dots$$

where $J_0(\alpha_n) = 0$.

$$\xi_1 = \frac{2.4048}{2},\ \ \xi_2 = \frac{5.5201}{2}, \dots$$

and

$$T_n(t) = A_n \cos(\xi_n t) + B_n \sin(\xi_n t)$$

The general solution is then

$$u(r,t) = \sum_{n=1}^{\infty}\left\{A_n \cos(\xi_n t) + B_n \sin(\xi_n t)\right\} J_0(\xi_n r)$$

Use the initial conditions

$$u(r,0) = \sum_{n=1}^{\infty} A_n J_0(\xi_n r) = \sin(\pi r)$$

$$u_t(r,0) = \sum_{n=1}^{\infty} \xi_n B_n J_0(\xi_n r) = 0, \ \Rightarrow B_n = 0$$

$$A_n = \frac{\int_0^2 r \sin(\pi r) J_0(\xi_n r) dr}{\int_0^2 r \left[J_0(\xi_n r)\right]^2 dr}$$

So the solution is

$$u(r,t) = \sum_{n=1}^{\infty} A_n \cos(\xi_n t) J_0(\xi_n r)$$

with A_n given above.

4. b.

$$u_{tt} = \frac{4}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r}\right]$$

$$u(1,t) = 0$$
$$u(0,t)\ , \quad \frac{\partial u}{\partial r}(0,t) \quad \text{finite}$$

$$u(r,0) = 1 - r^2$$
$$u_t(r,0) = 0$$

Proceeding as in the previous case, the general solution is

$$u(r,t) = \sum_{n=1}^{\infty} \left\{A_n \cos(2\xi_n t) + B_n \sin(2\xi_n t)\right\} J_0(\xi_n r)$$

where $J_0(\xi_n) = 0$ as before. Use the initial conditions

$$u(r,0) = \sum_{n=1}^{\infty} A_n J_0(\xi_n r) = 1 - r^2$$

$$u_t(r,0) = \sum_{n=1}^{\infty} \xi_n B_n J_0(\xi_n r) = 0, \ \Rightarrow B_n = 0$$

$$A_n = \frac{\int_0^1 r\,(1-r^2)\, J_0(\xi_n r) dr}{\int_0^1 r \left[J_0(\xi_n r)\right]^2 dr}$$

So the solution is

$$u(r,t) = \sum_{n=1}^{\infty} A_n \cos(2\xi_n t) J_0(\xi_n r)$$

with A_n given above.

4. c.

$$
\begin{aligned}
u_{tt} &= c^2 \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r}\right] \\
&u(2,t) = 0 \\
&u(0,t)\ ,\quad \frac{\partial u}{\partial r}(0,t) \quad \text{finite} \\
&u(r,0) = 0 \\
&u_t(r,0) = 1
\end{aligned}
$$

Proceeding as in the previous case, 4a, the general solution is

$$u(r,t) \;=\; \sum_{n=1}^{\infty} \{A_n \cos(\xi_n ct) + B_n \sin(\xi_n ct)\}\, J_0(\xi_n r)$$

where $J_0(2\xi_n) = 0$ as before. Use the initial conditions

$$u(r,0) \;=\; \sum_{n=1}^{\infty} A_n J_0(\xi_n r) = 0, \;\Rightarrow A_n = 0$$

$$u_t(r,0) \;=\; \sum_{n=1}^{\infty} \xi_n c B_n J_0(\xi_n r) = 1$$

$$B_n = \frac{1}{\xi_n c}\frac{\int_0^2 r J_0(\xi_n r) dr}{\int_0^2 r\left[J_0(\xi_n r)\right]^2 dr}$$

So the solution is

$$u(r,t) \;=\; \sum_{n=1}^{\infty} B_n \sin(\xi_n ct) J_0(\xi_n r)$$

with B_n given above.

5.
$$u_{tt} = c^2 \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r}\right]$$

$$\begin{aligned}
&u_r(L,t) = 0\\
&u(0,t)\ ,\quad \frac{\partial u}{\partial r}(0,t) \quad \text{finite}\\
&u(r,0) = f(r)\\
&u_t(r,0) = 0
\end{aligned}$$

Separation of variables yields

$$\ddot{T} + \lambda c^2 T = 0 \qquad\qquad \begin{array}{c} (rR')' + \lambda r R = 0 \\ R(0), R'(0) \text{ finite} \\ R'(L) = 0 \end{array}$$

The case $\lambda = 0$ gives a solution $R_0(r) = 1$, for $\lambda > 0$ the solution is

$$R(r) = C_1 J_0(\xi r) + C_2 Y_0(\xi r)$$

The condition $R(0), R'(0)$ finite implies $C_2 = 0$ and the condition $R'(L) = 0$ gives

$$J_0'(\xi L) = 0$$

which is the same as

$$-J_1(\xi L) = 0$$

So

$$\xi_1 = \frac{3.8317}{L},\ \xi_2 = \frac{7.0156}{L}, \dots$$

The solution is then

$$u(r,t) \ = \ (A_0 + B_0 t) + \sum_{n=1}^{\infty} \left\{A_n \cos(\xi_n c t) + B_n \sin(\xi_n c t)\right\} J_0(\xi_n r)$$

Note that the eigenfunctions are still Bessel functions of order zero, due to the ODE. Only the eigenvalues are different. The initial conditions

$$u(r,0) \ = \ A_0 + \sum_{n=1}^{\infty} A_n J_0(\xi_n r) = f(r)$$

$$u_t(r,t) \ = \ B_0 + \sum_{n=1}^{\infty} \xi_n c B_n J_0(\xi_n r) = 0, \ \Rightarrow B_n = 0$$

and

$$A_n = \frac{\int_0^L r f(r) J_0(\xi_n r) dr}{\int_0^L r \left[J_0(\xi_n r)\right]^2 dr}, \qquad A_0 = \frac{\int_0^L r f(r) dr}{\int_0^L r dr}$$

5 Introduction to the Fourier Transform

5.10 Further Extensions of the Fourier Transform

PROBLEMS

1. Compute, from the definition, and using the properties of even and odd functions where appropriate, the Fourier transform of each of the following functions. In each case, plot $h(t)$ and the Amplitude spectrum and phase angle graphs.

a. $h(t) = e^{-\alpha|t|} \;, \quad -\infty < t < \infty \;, \quad \alpha > 0.$

(Plot for $\alpha = 1$ and $\alpha = .05$)

b. $h(t) = \begin{cases} 1 & , \quad 0 \leq t \leq 1 \\ 0 & , \quad \text{otherwise} \end{cases}$

c. $h(t) = \begin{cases} te^{-t} & , \quad 0 < t < \infty \\ 0 & , \quad \text{otherwise} \end{cases}$

d. $h(t) = \begin{cases} (1 - t^2) & , \quad -1 < t < 1 \\ 0 & , \quad \text{otherwise} \end{cases}$

e. $h(t) = \begin{cases} (1 - t)^2 & , \quad -1 < t < 1 \\ 0 & , \quad \text{otherwise} \end{cases}$

f. $h(t) = Ae^{-\alpha|t|} \cos(2\pi t) \;, \quad -\infty < t < \infty \;, \quad \alpha > 0 \;.$

(Plot for $\alpha = 1$ and $\alpha = .05$)

g. $h(t) = \begin{cases} (1 + t) & , \quad -1 < t < 0 \\ 1 & , \quad 0 \leq t \leq 1 \\ (2 - t) & , \quad 1 < t < 2 \\ 0 & , \quad \text{otherwise} \end{cases}$

h. $h(t) = Ate^{-\alpha|t|} \;, \quad -\infty < t < \infty \;, \quad \alpha > 0$

i. $h(t) = \begin{cases} t & , \quad -1 < t < 1 \\ 0 & , \quad \text{otherwise} \end{cases}$

2. Find, directly from the definition, the inverse of the following Fourier transforms, and plot $h(t)$ and the amplitude and phase graphs:

a. $H(f) = \begin{cases} (1 - f^2)^2 & , \quad -1 < f < 1 \\ 0 & , \quad \text{otherwise} \end{cases}$

b. $H(f) = |f| \, e^{-2|f|} \;, \quad -\infty < f < \infty \;.$

1. a. $h(t) = e^{-\alpha|t|}$, $\quad -\infty < t < \infty$, $\quad \alpha > 0$. Note that $h(t)$ is even (see Figure 34).

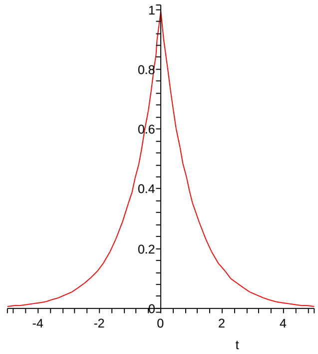


Figure 34: Graph of $h(t)$ for problem 1a of first set of Chapter 5 for $\alpha = 1$

Thus

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-2\pi jft}dt = 2\int_{0}^{\infty} e^{-\alpha t}\cos(2\pi ft)dt$$

$$H(f) = 2e^{-\alpha t}\left\{\frac{-\alpha\cos(2\pi ft) + 2\pi f\sin(2\pi ft)}{\alpha^2 + (2\pi f)^2}\right\}\Big|_{0}^{\infty}$$

$$H(f) = \frac{2\alpha}{\alpha^2 + (2\pi f)^2}$$

For $\alpha = 1$, $H(f) = \dfrac{2}{1 + (2\pi f)^2}$, real and positive, $\Theta(f) = 0$. The plot of $H(f)$ is given in Figure 35 on the left. If we decrease α to 0.05, then the plot of $H(f)$ is now on the right of the same figure. Note the vertical scale, does this suggest something?

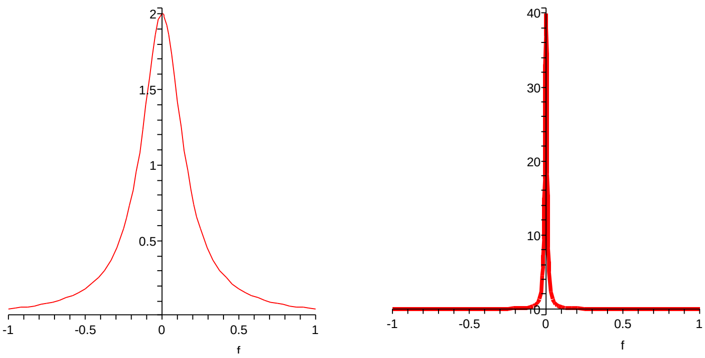


Figure 35: Graph of $H(f)$ for problem 1a of first set of Chapter 5. The left plot for $\alpha = 1$ and the right for $\alpha = 0.05$

1. b. $h(t) = \begin{cases} 1 & , \quad 0 \le t \le 1 \\ 0 & , \quad \text{otherwise} \end{cases}$ Note that $h(t)$ is neither even nor odd, see Figure 36. Thus

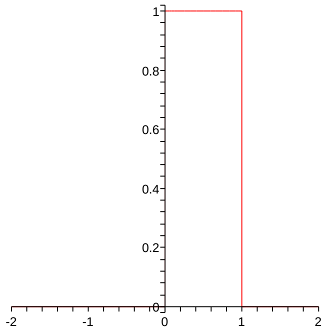


Figure 36: Graph of $h(t)$ for problem 1b of first set of Chapter 5

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-2\pi jft}dt = \int_0^1 1 \cdot e^{-2\pi jft}dt$$

$$H(f) = -\frac{e^{-2\pi jft}}{2\pi jf}\bigg|_0^1 = \frac{1 - e^{-2\pi jf}}{2\pi jf}$$

Or, in terms of real and imaginary parts

$$H(f) = \frac{1 - \cos(-2\pi f) + j\sin(2\pi f)}{2\pi jf} = \frac{\sin(2\pi f)}{2\pi f} - j\frac{1 - cos(2\pi f)}{2\pi f}$$

Note that $H(f) = O\left(\frac{1}{f}\right)$

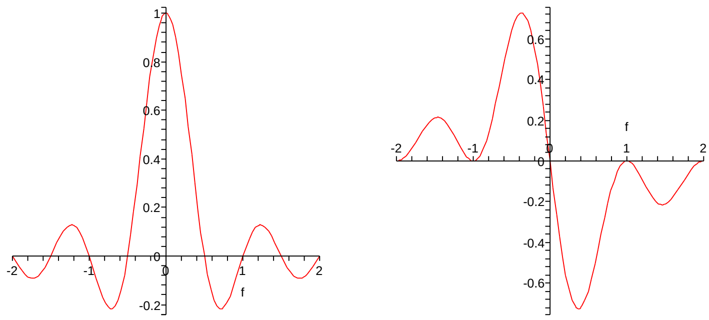


Figure 37: Graph of $\Re(H(f))$ (on the left) and $\Im(H(f))$ (on the right) for problem 1b of first set of Chapter 5

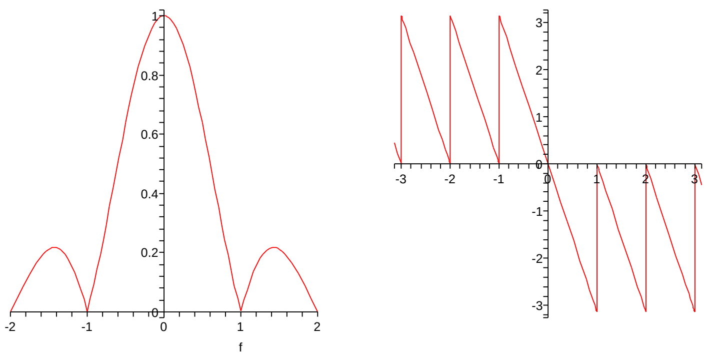


Figure 38: Graph of $|H(f)|$ (on the left) and $\Theta(f)$ (on the right) for problem 1b of first set of Chapter 5

1. c. $h(t) = \begin{cases} te^{-t} & , \quad 0 < t < \infty \\ 0 & , \quad \text{otherwise} \end{cases}$

Note that $h(t)$ is neither even nor odd, see Figure 36. Thus

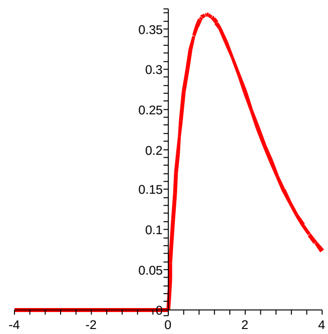


Figure 39: Graph of $h(t)$ for problem 1c of first set of Chapter 5

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-2\pi jft}dt = \int_{0}^{\infty} te^{-t}e^{-2\pi jft}dt$$

$$H(f) = \left\{ -\frac{te^{-(1+2\pi jf)t}}{1+2\pi jf} - \frac{e^{-(1+2\pi jf)t}}{(1+2\pi jf)^2} \right\} \Big|_0^\infty$$

$$H(f) = \frac{1}{(1+2\pi jf)^2}$$

Or, in terms of real and imaginary parts

$$H(f) = \frac{1-4\pi^2f^2}{(1+4\pi^2f^2)^2} - j\frac{4\pi f}{(1+4\pi^2f^2)^2}$$

Thus

$$\Re(H(f)) = \frac{1-4\pi^2f^2}{(1+4\pi^2f^2)^2}$$

and

$$\Im(H(f)) = -\frac{4\pi f}{(1+4\pi^2f^2)^2}$$

and

$$|H(f)| = \frac{1}{1+4\pi^2f^2}$$

Note that $|H(f)| = O\left(\frac{1}{f^2}\right)$

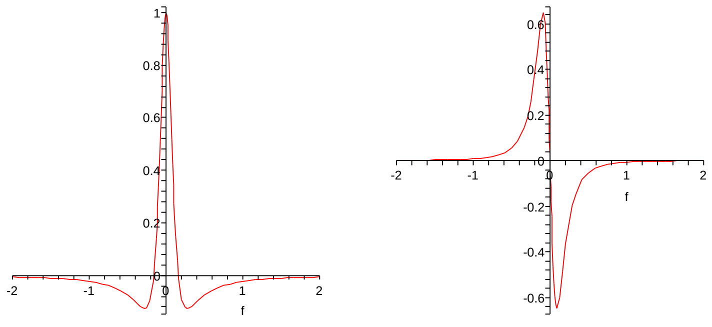


Figure 40: Graph of $\Re(H(f))$ (on the left) and $\Im(H(f))$ (on the right) for problem 1c of first set of Chapter 5

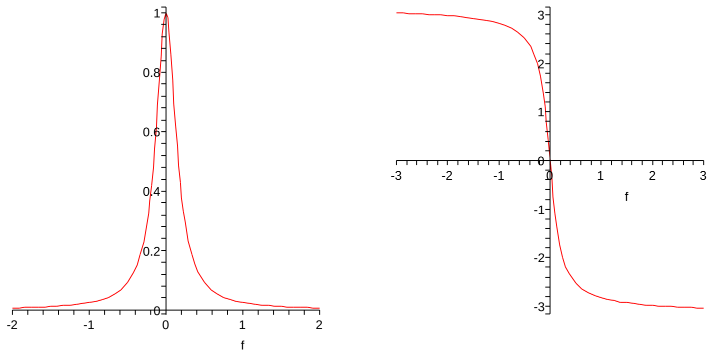


Figure 41: Graph of $|H(f)|$ (on the left) and $\Theta(f)$ (on the right) for problem 1c of first set of Chapter 5

d. $h(t) = \begin{cases} (1-t^2) & , \quad -1 < t < 1 \\ 0 & , \quad \text{otherwise} \end{cases}$

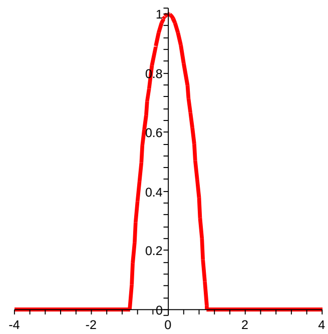


Figure 42: Graph of $h(t)$ for problem 1d of first set of Chapter 5

$$H(f) \;=\; \int_{-\infty}^{\infty} h(t)e^{-2\pi jft}dt \;=\; 2\int_0^1 (1-t^2)\cos(2\pi ft)dt, \qquad \text{since } h(t) \text{ is even}$$

$$H(f) = 2\left\{\frac{(1-t^2)\sin(2\pi ft)}{2\pi f} - \frac{2t\cos(2\pi ft)}{(2\pi f)^2} + \frac{2\sin(2\pi ft)}{(2\pi f)^3}\right\}\Big|_0^1$$

$$H(f) \;=\; \left\{-\frac{4\cos(2\pi f)}{(2\pi f)^2} + \frac{4\sin(2\pi f)}{(2\pi f)^3}\right\}$$

Or

$$H(f) \;=\; \frac{1}{2\pi^3 f^3}\left\{\sin(2\pi f) - 2\pi f\cos(2\pi f)\right\}$$

Note that $H(f)$ is real and even!! Thus

$$|H(f)| \;=\; \frac{1}{2\pi^3|f|^3}\left\{|\sin(2\pi f) - 2\pi f\cos(2\pi f)|\right\}$$

Note that $|H(f)| = O\left(\frac{1}{f^2}\right)$

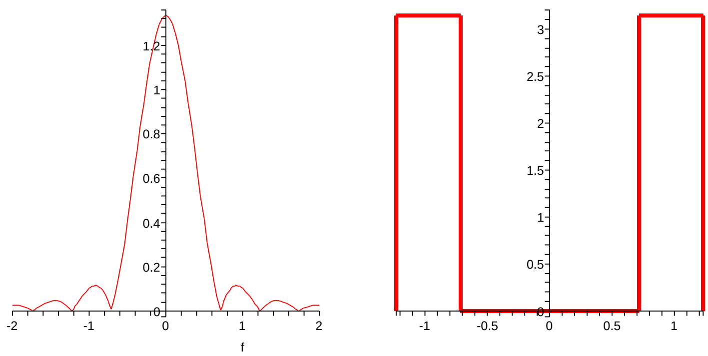


Figure 43: Graph of $|H(f)|$ (on the left) and $\Theta(f)$ (on the right) for problem 1d of first set of Chapter 5

e. $h(t) = \begin{cases} (1-t)^2 & , \quad -1 < t < 1 \\ 0 & , \quad \text{otherwise} \end{cases}$

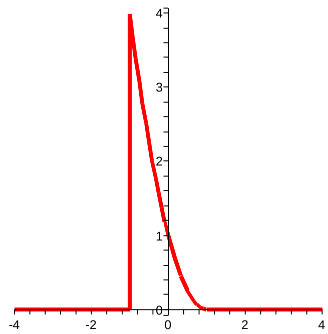


Figure 44: Graph of $h(t)$ for problem 1e of first set of Chapter 5

$$H(f) \;=\; \int_{-\infty}^{\infty} h(t)e^{-2\pi jft}dt \;=\; \int_{-1}^{1}(1-t)^2 e^{-2\pi jft}dt$$

$$H(f) = \left\{-\frac{(1-t)^2 e^{-2\pi jft}}{2\pi jf} + \frac{2(1-t)e^{-2\pi jft}}{(2\pi jf)^2} - \frac{2e^{-2\pi jft}}{(2\pi jf)^3}\right\}\Bigg|_{-1}^{1}$$

$$H(f) \;=\; \left\{\frac{4e^{2\pi jf}}{2\pi jf} - \frac{4e^{2\pi jf}}{(2\pi jf)^2} + \frac{2}{(2\pi f)^3}\left[e^{2\pi jf} - e^{-2\pi jf}\right]\right\}$$

Or

$$H(f) \;=\; \frac{4(2\pi jf-1)e^{2\pi jf}}{(2\pi jf)^2} + \frac{4j}{(2\pi jf)^3}\sin(2\pi f)$$

$$H(f) \;=\; \frac{(1-2\pi jf)}{(\pi f)^2}\left[\cos(2\pi f) + j\sin(2\pi f)\right] - \frac{1}{2(\pi f)^3}\sin(2\pi f)$$

$$H(f) \;=\; \frac{2\pi f\cos(2\pi f) + (4\pi^2 f^2 - 1)\sin(2\pi f)}{2(\pi f)^3} + j\frac{\sin(2\pi f) - 2\pi f\cos(2\pi f)}{(\pi f)^2}$$

Note that $H(f) \;=\; O\left(\frac{1}{f}\right)$

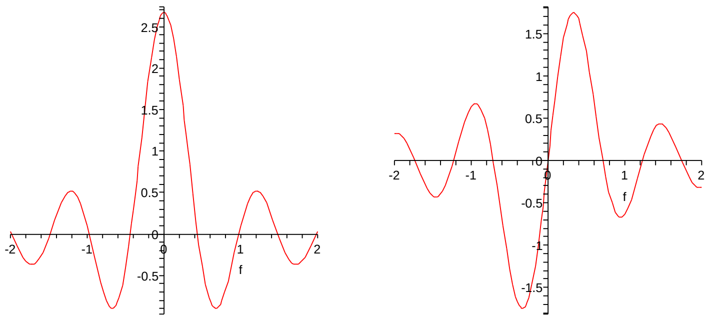


Figure 45: Graph of $\Re(H(f))$ (on the left) and $\Im(H(f))$ (on the right) for problem 1e of first set of Chapter 5

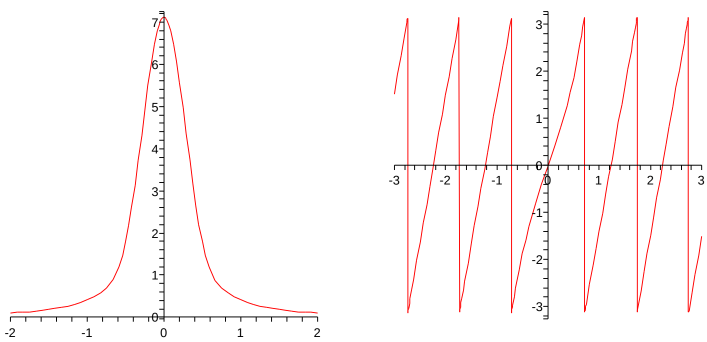


Figure 46: Graph of $|H(f)|$ (on the left) and $\Theta(f)$ (on the right) for problem 1e of first set of Chapter 5

f. $h(t) = Ae^{-\alpha|t|}\cos(2\pi t)\ , \quad -\infty < t < \infty\ , \quad \alpha > 0\ .$

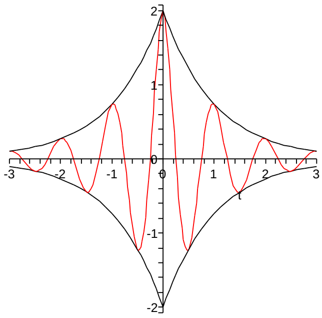


Figure 47: Graph of $h(t)$ for problem 1f of first set of Chapter 5, using $A = 2$, $\alpha = 1$

Note that $h(t)$ is even.

$$H(f) \;=\; \int_{-\infty}^{\infty} h(t)e^{-2\pi jft}dt \;=\; 2A\int_{0}^{\infty} e^{-\alpha t}\cos(2\pi t)\cos(2\pi ft)dt$$

$$\begin{aligned} H(f) \;&=\; 2Ae^{-\alpha t}\left\{\frac{(2\pi f - 2\pi)\sin((2\pi f - 2\pi)t) - \alpha\cos((2\pi f - 2\pi)t)}{2\left[\alpha^2 + (2\pi f - 2\pi)^2\right]}\right\}\Bigg|_0^{\infty} \\ &+\; 2Ae^{-\alpha t}\left\{\frac{(2\pi f + 2\pi)\sin((2\pi f + 2\pi)t) - \alpha\cos((2\pi f + 2\pi)t)}{2\left[\alpha^2 + (2\pi f + 2\pi)^2\right]}\right\}\Bigg|_0^{\infty} \end{aligned}$$

$$H(f) \;=\; \frac{\alpha A}{\alpha^2 + (2\pi(f-1))^2} + \frac{\alpha A}{\alpha^2 + (2\pi(f+1))^2}$$

Note that $H(f)$ is real and positive. Note the change of scale as we change the value of α from $\alpha = 1$ in Figure 48 to a value of $\alpha = 0.05$ in Figure 49.

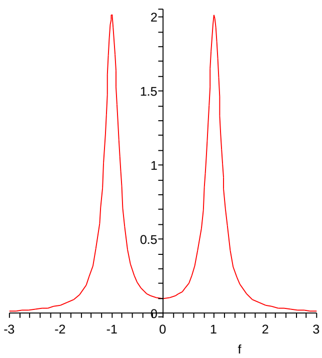


Figure 48: Graph of $H(f)$ for problem 1e of first set of Chapter 5, using $A = 2$, $\alpha = 1$

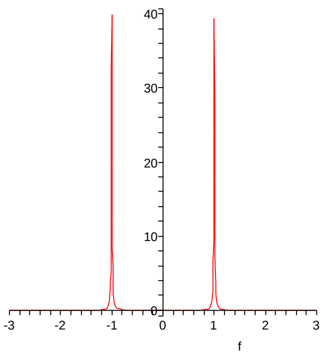


Figure 49: Graph of $H(f)$ for problem 1e of first set of Chapter 5, using $A = 2$, $\alpha = .05$

g. $h(t) = \begin{cases} (1+t) & , \quad -1 < t < 0 \\ 1 & , \quad 0 \le t \le 1 \\ (2-t) & , \quad 1 < t < 2 \\ 0 & , \quad \text{otherwise} \end{cases}$ Since $h(t)$ is neither even nor odd

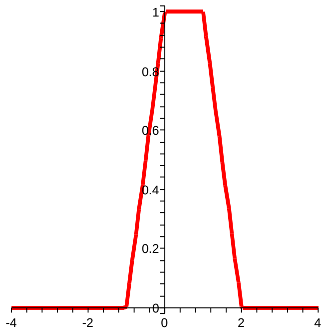


Figure 50: Graph of $h(t)$ for problem 1g of first set of Chapter 5

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-2\pi jft}dt = \int_{-1}^{0}(1+t)e^{-2\pi jft}dt + \int_{0}^{1} e^{-2\pi jft}dt + \int_{1}^{2}(2-t)e^{-2\pi jft}dt$$

$$\begin{aligned}
H(f) &= \left\{-\frac{(1+t)e^{-2\pi jft}}{2\pi jf} - \frac{e^{-2\pi jft}}{(2\pi jf)^2}\right\}\Big|_{-1}^{0} \\
&+ \left\{-\frac{e^{-2\pi jft}}{2\pi jf}\right\}\Big|_{0}^{1} + \left\{-\frac{(2-t)e^{-2\pi jft}}{2\pi jf} + \frac{e^{-2\pi jft}}{(2\pi jf)^2}\right\}\Big|_{1}^{2} \\
&= -\frac{1}{2\pi jf} - \frac{1}{(2\pi jf)^2} + \frac{e^{2\pi jf}}{(2\pi jf)^2} - \frac{e^{-2\pi jf}}{2\pi jf} + \frac{1}{2\pi jf} + \frac{e^{-2\pi jf}}{2\pi jf} + \frac{e^{-4\pi jf}}{(2\pi jf)^2} - \frac{e^{-2\pi jf}}{(2\pi jf)^2} \\
&= \frac{-1 + e^{2\pi jf} + e^{-4\pi jf} - e^{-2\pi jf}}{(2\pi jf)^2} \\
&= \frac{-1 + \cos(2\pi f) + j\sin(2\pi f) + \cos(4\pi f) - j\sin(4\pi f) - \cos(2\pi f) + j\sin(2\pi f)}{-(2\pi f)^2}
\end{aligned}$$

Or

$$H(f) = \frac{[1 - \cos(4\pi f)] + j\,[\sin(4\pi f) - 2\sin(2\pi f)]}{(2\pi f)^2}$$

Note that $H(f) = O\left(\frac{1}{f^2}\right)$. So

$$|H(f)| = \frac{\sqrt{[1-\cos(4\pi f)]^2 + [\sin(4\pi f) - 2\sin(2\pi f)]^2}}{(2\pi f)^2}$$

and

$$\Theta(f) = \arctan\left\{\frac{\underbrace{\sin(4\pi f)}_{2\sin(2\pi f)\cos(2\pi f)} \quad -2\sin(2\pi f)}{\underbrace{1-\cos(4\pi f)}_{2\sin^2(2\pi f)}}\right\} = \arctan\left\{\frac{\cos(2\pi f)-1}{\sin(2\pi f)}\right\}$$

Or

$$\Theta(f) = \arctan\left\{-\frac{\sin(\pi f)}{\cos(\pi f)}\right\}$$

In the following figures we plot the real and imaginary parts of $H(f)$, the absolute value of $H(f)$ and $\Theta(f)$.

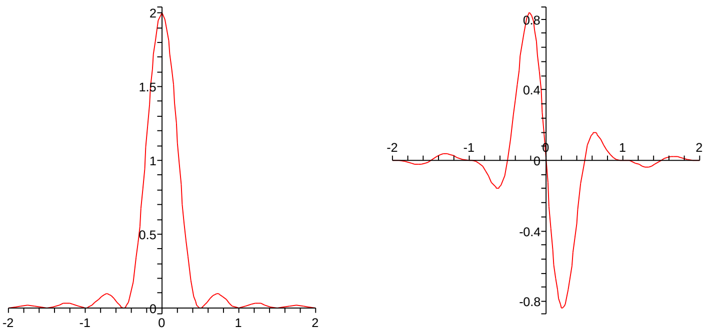


Figure 51: Graph of $\Re(H(f))$ (on the left) and $\Im(H(f))$ (on the right) for problem 1g of first set of Chapter 5

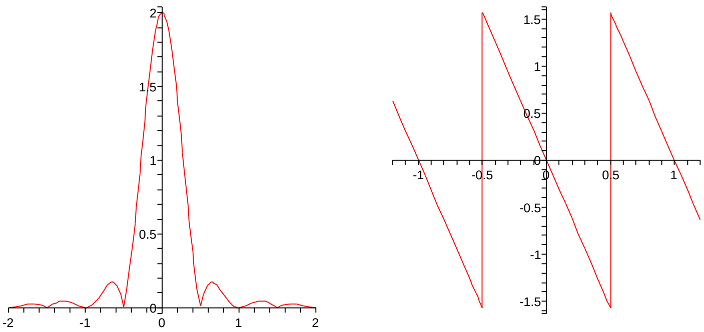


Figure 52: Graph of $|H(f)|$ (on the left) and $\Theta(f)$ (on the right) for problem 1g of first set of Chapter 5

h. $h(t) = Ate^{-\alpha|t|}$, $\quad -\infty < t < \infty$, $\quad \alpha > 0$
Note that $h(t)$ is odd, see Figure 53. Thus

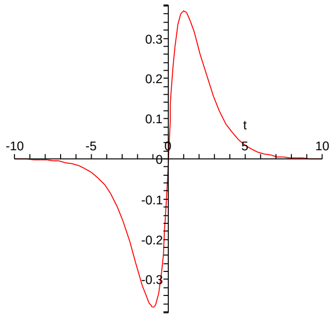


Figure 53: Graph of $h(t)$ with $A = \alpha = 1$ for problem 1h of first set of Chapter 5

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-2\pi jft}dt = -2jA\int_0^{\infty} te^{-t}\sin(2\pi ft)dt$$

$$\begin{aligned} H(f) &= -2jA\left\{\frac{te^{-\alpha t}\left[-\alpha\sin(2\pi ft) - 2\pi f\cos(2\pi ft)\right]}{\alpha^2 + (2\pi f)^2}\Bigg|_0^{\infty}\right. \\ &- \left.\frac{e^{-\alpha t}\left[(\alpha^2 - (2\pi f)^2)\sin(2\pi ft) + 2\alpha(2\pi f)\cos(2\pi ft)\right]}{\left[\alpha^2 + (2\pi f)^2\right]^2}\right\}\Bigg|_0^{\infty} \\ H(f) &= -2jA\left[\frac{2\alpha(2\pi f)}{\left[\alpha^2 + (2\pi f)^2\right]^2}\right] = -\frac{8\pi A\alpha f}{\left[\alpha^2 + (2\pi f)^2\right]^2}j = O\left(\frac{1}{f^3}\right) \end{aligned}$$

Note that $H(f)$ is purely imaginary. The $|H(f)|$ is given by

$$|H(f)| = \frac{8\pi A\alpha|f|}{\left[\alpha^2 + (2\pi f)^2\right]^2}$$

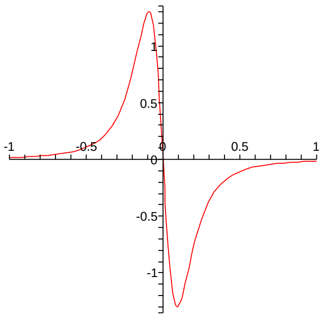
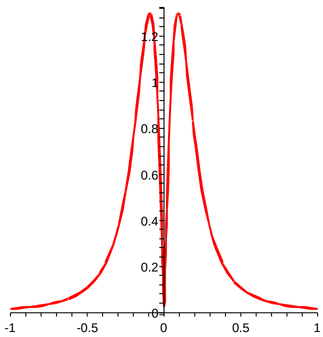


Figure 54: Graph of $\Im(H(f))$ with $A = \alpha = 1$ for problem 1h of first set of Chapter 5



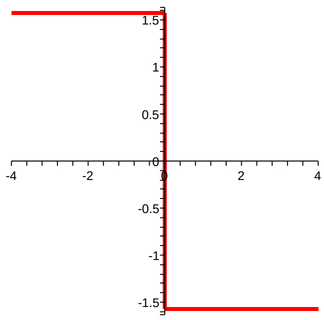


Figure 55: Graph of $|H(f)|$ (on the left) and $\Theta(f)$ (on the right) with $A = \alpha = 1$ for problem 1h of first set of Chapter 5

i. $h(t) = \begin{cases} t & , \quad -1 < t < 1 \\ 0 & , \quad \text{otherwise} \end{cases}$
Note that $h(t)$ is odd, see Figure 56. Thus

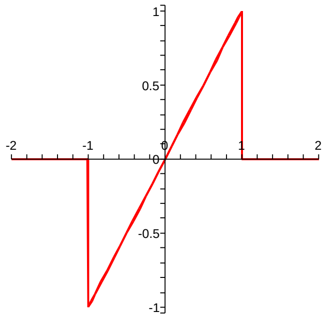


Figure 56: Graph of $h(t)$ for problem 1i of first set of Chapter 5

$$
\begin{aligned}
H(f) &= \int_{-\infty}^{\infty} h(t) e^{-2\pi j f t} dt = -2j \int_0^1 t \sin(2\pi f t) dt \\
H(f) &= -2j \left\{ \frac{t \sin(2\pi f t)}{(2\pi f)^2} - \frac{t \cos(2\pi f t)}{2\pi f} \right\} \Big|_0^1 \\
&= -2j \left\{ \frac{\sin(2\pi f)}{(2\pi f)^2} - \frac{\cos(2\pi f)}{2\pi f} \right\} \\
&= -2j \frac{\sin(2\pi f) - 2\pi f \cos(2\pi f)}{(2\pi f)^2} = O\left(\frac{1}{f}\right)
\end{aligned}
$$

Note that $H(f)$ is purely imaginary. The $|H(f)|$ is given by

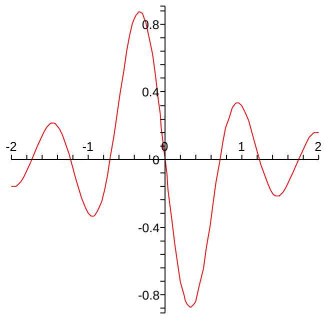


Figure 57: Graph of $\Im(H(f))$ for problem 1i of first set of Chapter 5

$$
|H(f)| = \frac{|\sin(2\pi f) - 2\pi f \cos(2\pi f)|}{2(\pi f)^2}
$$

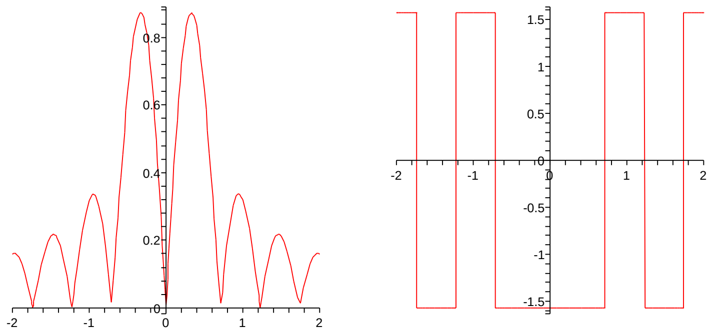


Figure 58: Graph of $|H(f)|$ (on the left) and $\Theta(f)$ (on the right) for problem 1i of first set of Chapter 5

2. a. $H(f) = \begin{cases} (1-f^2)^2 & , \quad -1 < f < 1 \\ 0 & , \quad \text{otherwise} \end{cases}$ $H(f)$ is even, see Figure 59. Thus

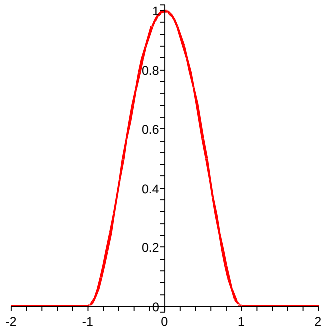
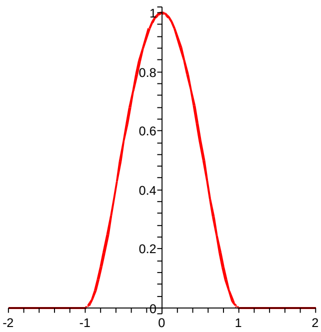


Figure 59: Graph of $H(f)$ for problem 2a of first set of Chapter 5

$$h(t) = \int_{-\infty}^{\infty} H(f) e^{2\pi jft} df = \int_{-1}^{1} (1-f^2)^2 e^{2\pi jft} df$$

$$\begin{aligned}
h(t) &= 2\int_0^1 (1 - 2f^2 + f^4)\cos(2\pi ft)df, \qquad \text{Since } H(f) \text{ is even} \\
&= 2\left\{\frac{1-2f^2+f^4}{2\pi t}\sin(2\pi ft) + \frac{-4f+4f^3}{(2\pi t)^2}\cos(2\pi ft) - \frac{-4+12f^2}{(2\pi t)^3}\sin(2\pi ft)\right. \\
&- \left. -\frac{24f}{(2\pi f)^4}\cos(2\pi ft) + \frac{24}{(2\pi t)^5}\sin(2\pi ft)\right\}\Bigg|_{f=0}^{f=1} \\
&= 2\left[-\frac{8}{(2\pi t)^3}\sin(2\pi t) - \frac{24}{(2\pi t)^4}\cos(2\pi t) + \frac{24}{(2\pi t)^5}\sin(2\pi t)\right] \\
&= \frac{16}{(2\pi t)^5}\left\{\left[-(2\pi t)^3 + 3\right]\sin(2\pi t) - 3(2\pi t)\cos(2\pi t)\right\}
\end{aligned}$$

Note that $|H(f)| = H(f)$.



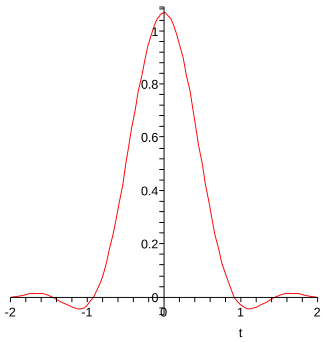


Figure 60: Graph of $|H(f)|$ (on the left) and $h(t)$ (on the right) for problem 2a of first set of Chapter 5

2. b. $H(f) = |f|\, e^{-2|f|}$, $\quad -\infty < f < \infty$.
$H(f)$ is real and even, see Figure 62. Thus

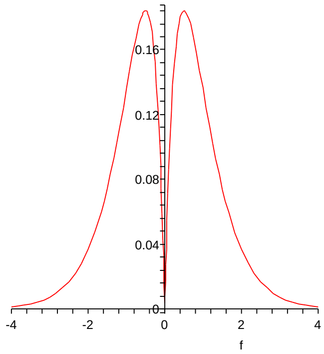


Figure 61: Graph of $H(f)$ for problem 2b of first set of Chapter 5

$$h(t) \;=\; \int_{-\infty}^{\infty} H(f)e^{2\pi jft}df \;=\; 2\int_0^{\infty} H(f)\cos(2\pi ft)df$$

$$\begin{aligned}
h(t) &= 2\int_0^{\infty} fe^{-2f}\cos(2\pi ft)df \\
&= 2\mathcal{L}\left[f\cos(\omega f)\right], \qquad \text{where } \omega = 2\pi t,\ s = 2 \\
&= 2\frac{s^2-\omega^2}{(s^2+\omega^2)^2}, \qquad \text{Using Laplace Transform tables} \\
&= 2\frac{4-(2\pi t)^2}{\left(4+(2\pi t)^2\right)^2}
\end{aligned}$$

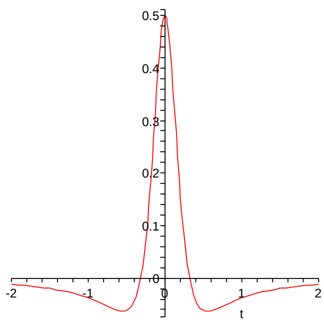


Figure 62: Graph of $h(t)$ for problem 2b of first set of Chapter 5

PROBLEMS

1. Compute the Fourier transform of each of the following functions, using tables, shifting and scaling, etc. when appropriate. In each case, plot $h(t)$ and the Amplitude spectrum and phase angle graphs.

a. $h(t) = \begin{cases} 2 & , & -1 < t < 5 \\ 0 & , & \text{otherwise} \end{cases}$

b. $h(t) = \begin{cases} t & , & 0 \le t \le 2 \\ 4-t & , & 2 < t \le 4 \\ 0 & , & \text{otherwise} \end{cases}$

c. $h(t) = \sin\left(\dfrac{t}{3}\right)$

d. $h(t) = \begin{cases} 2 & , & 3 < t < \infty \\ 0 & , & \text{otherwise} \end{cases}$

e. $h(t) = \dfrac{1}{4+t^2}$

f. $h(t) = \dfrac{\sin^2(3t)}{6t^2}$

g. $h(t) = \begin{cases} e^{-t} & , & 0 < t < 2 \\ 0 & , & \text{otherwise} \end{cases}$

2. Find, using tables, shifting and/or scaling, etc., the inverse of each of the following Fourier transforms, and plot $h(t)$ and the amplitude and phase graphs:

a. $H(f) = \dfrac{1}{1+f^2} \quad , \quad -\infty < f < \infty$

b. $H(f) = e^{-3j\pi f} e^{-2|f|} \quad , \quad -\infty < f < \infty$

c. $H(f) = \begin{cases} 2 & , & -3 < f < 3 \\ 0 & , & \text{otherwise} \end{cases}$

1. a. $h(t) = \begin{cases} 2 & , \quad -1 < t < 5 \\ 0 & , \quad \text{otherwise} \end{cases}$

The function $h(t)$ is given in Figure 63. The closest function for which we have a Fourier transform is the function

$$g(t) = \begin{cases} 1 & , \quad -T_0/2 < t < T_0/2 \\ 0 & , \quad \text{otherwise} \end{cases}$$

given in Figure 64.

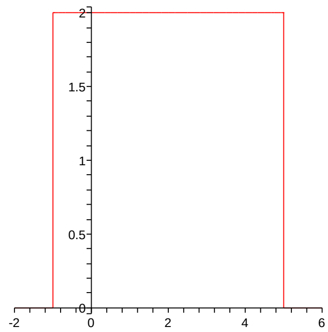


Figure 63: Graph of $h(t)$ for problem 1a of second set of Chapter 5

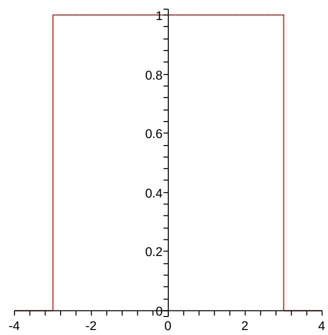


Figure 64: Graph of $g(t)$ for problem 1a of second set of Chapter 5

We need to write $h(t)$ as a shifted and scaled version of $g(t)$, specifically

$$h(t) = 2g(t-2), \qquad \text{when } T_0 = 6$$

So

$$H(f) = \mathcal{F}\left[2g(t-2)\right] = 2\mathcal{F}\left[g(t-2)\right], \qquad \text{by linearity}$$

$$H(f) = 2\left\{e^{-2\pi j(2)f}G(f)\right\}, \qquad \text{by shifting}$$

$$H(f) = 2e^{-4\pi jf}\frac{\sin(6\pi f)}{\pi f}$$

The real and imaginary parts of $H(f)$ are given in Figure 65, the $|H(f)|$ and $\Theta(f)$ are given in Figure 66.

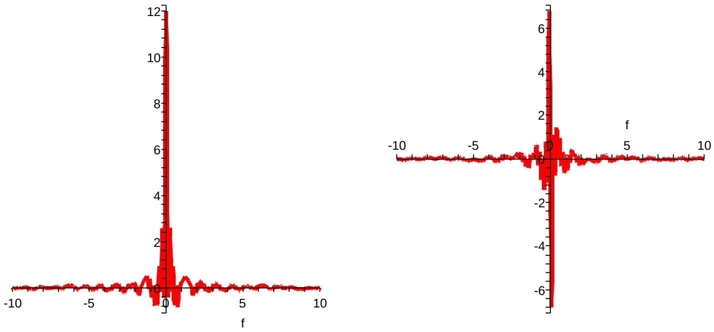


Figure 65: Graph of $Re(H(f))$ (on the left) and $Im(H(f))$ (on the right) for problem 1a of second set of Chapter 5

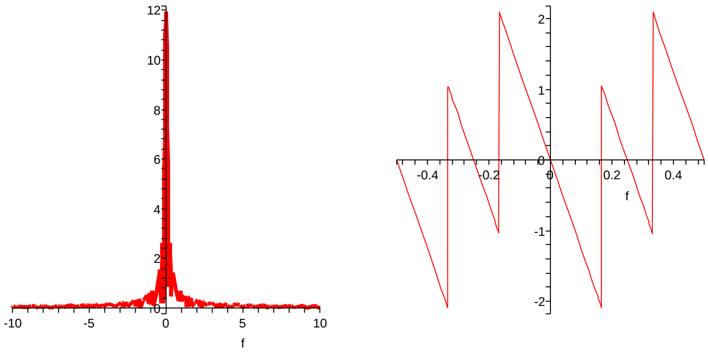


Figure 66: Graph of $|H(f)|$ (on the left) and $\Theta(f)$ (on the right) for problem 1a of second set of Chapter 5

1. b. $h(t) = \begin{cases} t & , \quad 0 \leq t \leq 2 \\ 4 - t & , \quad 2 < t \leq 4 \\ 0 & , \quad \text{otherwise} \end{cases}$

The function $h(t)$ is given in Figure 67. The closest function for which we have a Fourier transform is the function

$$g(t) = \begin{cases} 1 + t/T_0 & , \quad -T_0 < t \leq 0 \\ 1 - t/T_0 & , \quad 0 \leq t < T_0 \\ 0 & , \quad \text{otherwise} \end{cases}$$

given in Figure 68.

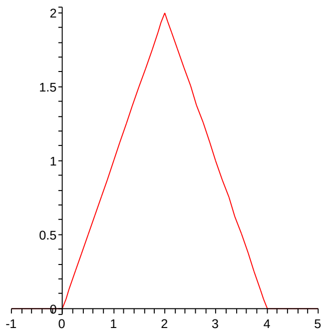


Figure 67: Graph of $h(t)$ for problem 1b of second set of Chapter 5

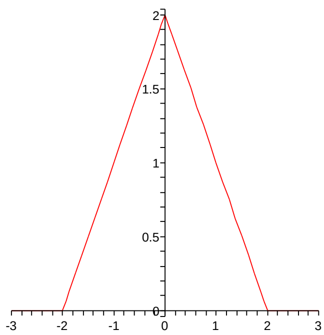


Figure 68: Graph of $g(t)$ for problem 1b of second set of Chapter 5

We need to write $h(t)$ as a shifted and scaled version of $g(t)$, i.e.

$$h(t) = 2g(t-2), \qquad \text{for } T_0 = 2$$

So

$$H(f) = \mathcal{F}\left[2g(t-2)\right] = 2\mathcal{F}\left[g(t-2)\right], \qquad \text{by linearity}$$

$$H(f) = 2\left\{e^{-2\pi j(2)f} G(f)\right\}, \qquad \text{by shifting}$$

$$H(f) = 2e^{-4\pi j f}\frac{1}{2}\left(\frac{\sin(2\pi f)}{\pi f}\right)^2$$

or

$$H(f) = e^{-4\pi j f}\left(\frac{\sin(2\pi f)}{\pi f}\right)^2$$

The real and imaginary parts of $H(f)$ are given in Figure 69, the $|H(f)|$ and $\Theta(f)$ are given in Figure 70.

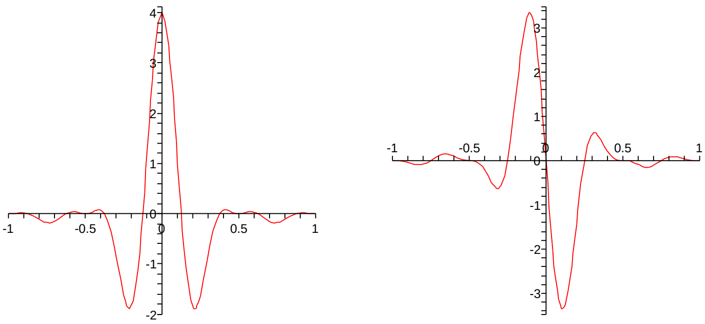


Figure 69: Graph of $Re(H(f))$ (on the left) and $Im(H(f))$ (on the right) for problem 1b of second set of Chapter 5

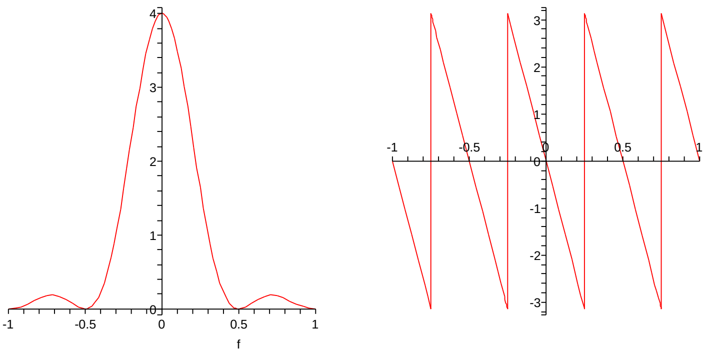


Figure 70: Graph of $|H(f)|$ (on the left) and $\Theta(f)$ (on the right) for problem 1b of second set of Chapter 5

1. c. $h(t) = \sin\left(\frac{t}{3}\right)$

The function $h(t)$ is given in Figure 71. The closest function for which we have a Fourier transform is the function $g(t) = A\sin(2\pi f_0 t)$ given in Figure 72.

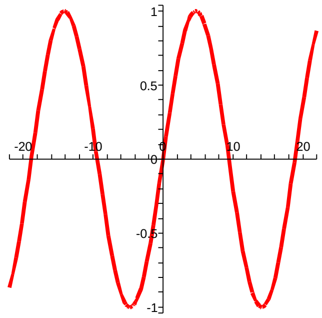


Figure 71: Graph of $h(t)$ for problem 1c of second set of Chapter 5

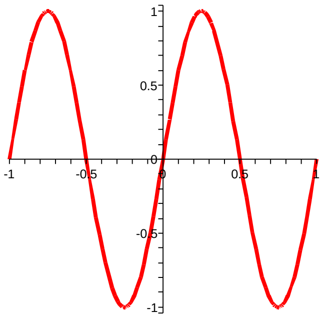


Figure 72: Graph of $g(t)$ for problem 1c of second set of Chapter 5

In this case

$$h(t) = g(t), \qquad \text{for } A = 1, \text{ and } 2\pi f_0 = \frac{1}{3} \Rightarrow f_0 = \frac{1}{6\pi}$$

So

$$H(f) = \mathcal{F}[g(t)] = j\frac{1}{2}\left\{\delta\left(f + \frac{1}{6\pi}\right) - \delta\left(f - \frac{1}{6\pi}\right)\right\}$$

Note that $H(f)$ is purely imaginary, so we plot $Im(H(f))$ and $|H(f)|$ in Figure 73. The graph of $\Theta(f)$ is given in Figure 74.

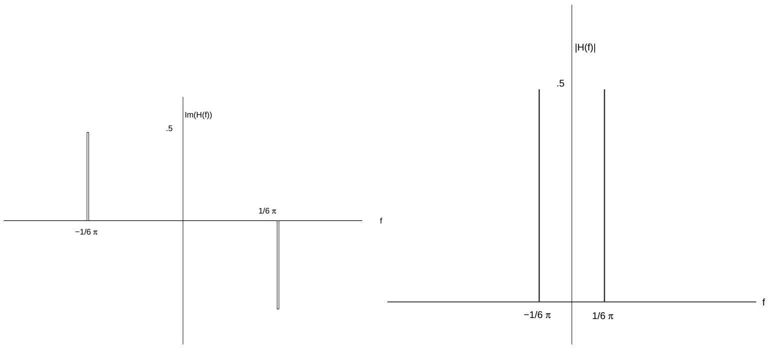


Figure 73: Graph of $Im(H(f))$ (on the left) and $|H(f)|$ (on the right) for problem 1c of second set of Chapter 5

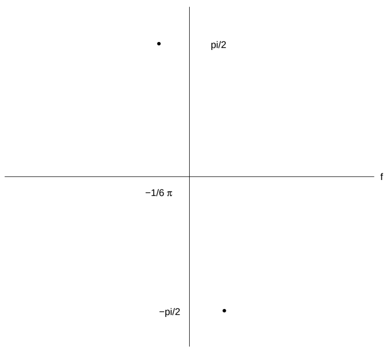


Figure 74: Graph of $\Theta(f)$ for problem 1c of second set of Chapter 5

1. d. $h(t) = \begin{cases} 2 & , \quad 3 < t < \infty \\ 0 & , \quad \text{otherwise} \end{cases}$

The function $h(t)$ is given in Figure 75. The closest function for which we have a Fourier transform is the function $g(t) = sgn(t)$ given in Figure 76.

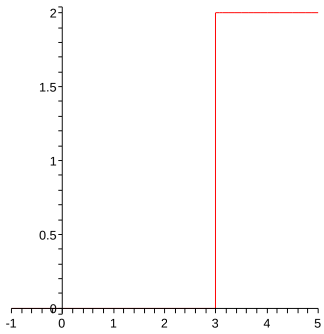


Figure 75: Graph of $h(t)$ for problem 1d of second set of Chapter 5

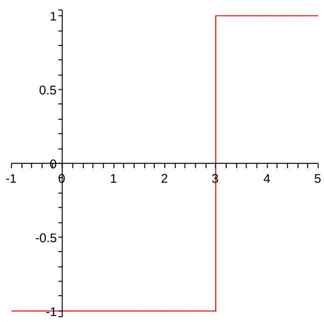


Figure 76: Graph of $g(t-3)$ for problem 1d of second set of Chapter 5

In this case $h(t) = 1 + g(t-3)$, so

$$H(f) = \mathcal{F}\left[1 + g(t-3)\right] \;=\; \mathcal{F}[1] + \mathcal{F}[g(t-3)], \qquad \text{by linearity}$$

$$H(f) \;=\; \delta(f) + e^{-2\pi j(3)f} G(f) \;=\; \delta(f) - \frac{j}{\pi f} e^{-6\pi j f}$$

or

$$H(f) \;=\; \delta(f) - j\frac{e^{-6\pi j f}}{\pi f}$$

The real and imaginary parts of $H(f)$ are given in Figure 77, the $|H(f)|$ and $\Theta(f)$ are given in Figure 78.

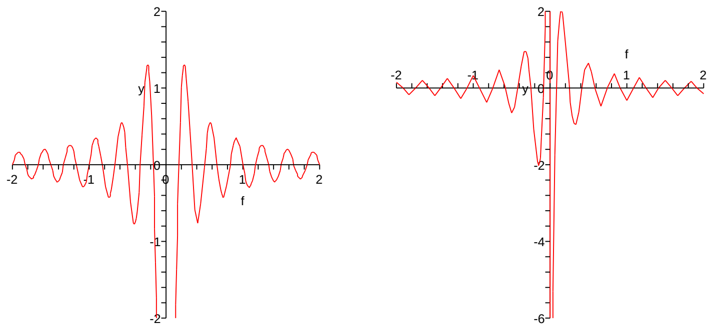


Figure 77: Graph of $Re(H(f))$ (on the left) and $Im(H(f))$ (on the right) for problem 1d of second set of Chapter 5

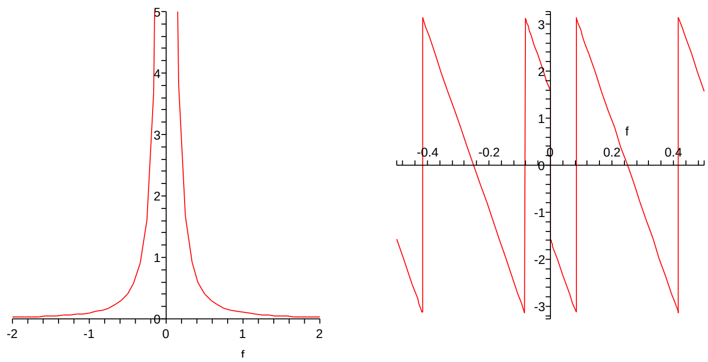


Figure 78: Graph of $|H(f)|$ (on the left) and $\Theta(f)$ (on the right) for problem 1d of second set of Chapter 5

1. e. $h(t) = \dfrac{1}{4+t^2}$

The closest transform pair to $h(t)$, given in Figure 79, is

$$G(f) \;=\; \frac{2\alpha}{\alpha^2 + 4\pi^2 f^2}$$

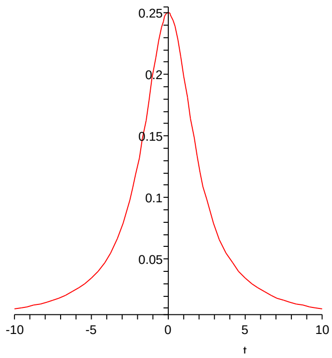


Figure 79: Graph of $h(t)$ for problem 1e of second set of Chapter 5

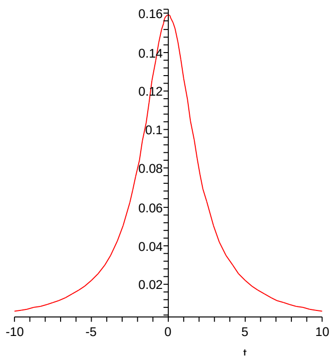


Figure 80: Graph of $g(t)$ for problem 1e of second set of Chapter 5

Thus we have to use the fact that if $X(f)$ is the Fourier transform of $x(t)$ then $x(-f)$ is the Fourier transform of $X(t)$. So for $g(t) \;=\; \dfrac{2\alpha}{\alpha^2 + 4\pi^2 t^2}$ (see Figure 80) we have

$$G(f) = e^{-\alpha|-f|} = e^{-\alpha|f|}.$$

But

$$h(t) \;=\; \frac{\pi}{2}\frac{2(4\pi)}{(4\pi)^2 + 4\pi^2 t^2} \;=\; \frac{\pi}{2} g(t), \qquad \text{with } \alpha = 4\pi$$

Thus

$$H(f) \;=\; \mathcal{F}\left[\frac{\pi}{2} g(t)\right] \;=\; \frac{\pi}{2} G(f)$$

So

$$H(f) = \frac{\pi}{2} e^{-4\pi|f|}$$

Note that the transform is a real function, so we have plotted $H(f)$ which is the same as $Re(H(f))$ and $|H(f)|$ in Figure 81.

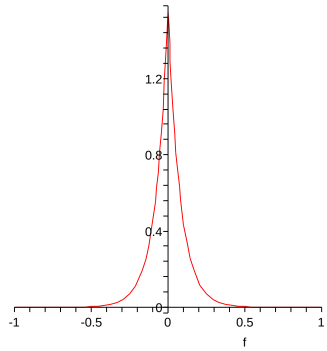


Figure 81: Graph of $Re(H(f))$ for problem 1e of second set of Chapter 5

1. f. $h(t) = \dfrac{\sin^2(3t)}{6t^2}$

The closest transform pair to $h(t)$, given in Figure 82, is

$$g(t) = \frac{\sin^2(\pi f_0 t)}{f_0 \pi^2 t^2}$$

whose transform is given in Figure 68 and can be described as

$$G(f) = \left(1 - \frac{|f|}{f_0}\right)(1 - u_{f_0}(|f|))$$

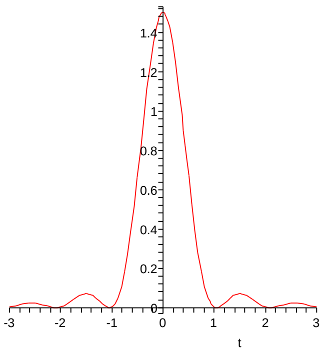


Figure 82: Graph of $h(t)$ for problem 1f of second set of Chapter 5

But now

$$h(t) = \frac{\sin^2(3t)}{6t^2} = \frac{\pi}{2}\frac{\sin^2(\pi f_0 t)}{\pi^2 t^2 f_0}, \qquad \text{with } f_0 = \frac{3}{\pi}$$

So

$$H(f) = \frac{\pi}{2}\left(1 - \frac{\pi|f|}{3}\right)\left(1 - u_{3/\pi}(|f|)\right) = \begin{cases} \frac{\pi}{2}\left(1 - \frac{\pi|f|}{3}\right), & |f| < \frac{3}{\pi} \\ 0, & \text{otherwise} \end{cases}$$

Note that $H(f)$ is real, so the plot of $Re(H(f))$ and $|H(f)|$ are identical and given in Figure 83.

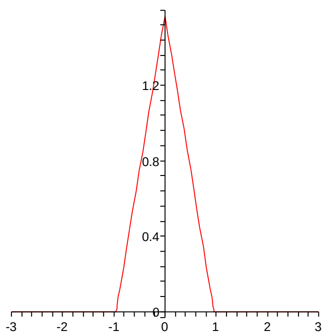


Figure 83: Graph of $Re(H(f))$ for problem 1f of second set of Chapter 5

1. g. $h(t) = \begin{cases} e^{-t} & , \quad 0 < t < 2 \\ 0 & , \quad \text{otherwise} \end{cases}$

The plot of $h(t)$ is given in Figure 84.

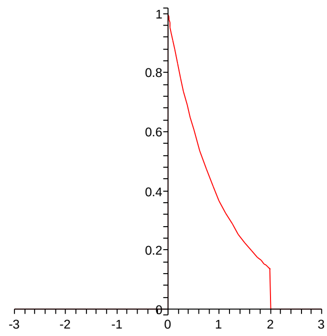


Figure 84: Graph of $h(t)$ for problem 1g of second set of Chapter 5

It is just as easy to get the transform directly.

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-2\pi jft}dt = \int_0^2 e^{-t}e^{-2\pi jft}dt$$

$$H(f) = -\frac{e^{-(1+2\pi jf)t}}{1+2\pi jf}\bigg|_0^2 = \frac{1-e^{-2(1+2\pi jf)}}{1+2\pi jf}$$

The plots of $Re(H(f))$ and $Im(H(f))$ are given in Figure 85. The plots of $|H(f)|$ and $\Theta(H(f))$ are given in Figure 86.

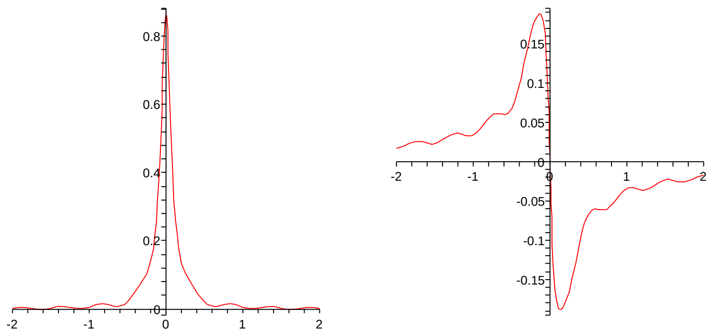


Figure 85: Graph of $Re(H(f))$ (on the left) and $Im(H(f))$ (on the right) for problem 1g of second set of Chapter 5

Another way to solve the problem is to view $h(t)$ as the product of $h_1(t)$ and $h_2(t)$ shown in Figure 87. The functions are

$$h_1(t) = e^{-t}, \;\; t > 0$$

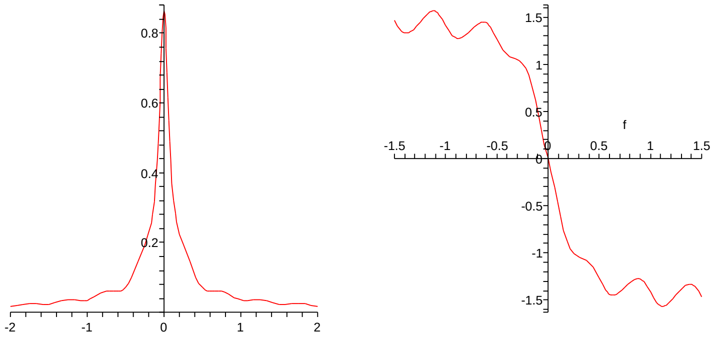


Figure 86: Graph of $|H(f)|$ (on the left) and $\Theta(f)$ (on the right) for problem 1g of second set of Chapter 5

and

$$h_2(t) = \begin{cases} 1, & 0 < t < 2 \\ 0, & \text{otherwise} \end{cases}$$

The transforms of these functions are

$$H_1(f) = \frac{1}{1 + 2\pi j f}, \qquad H_2(f) = e^{-2\pi j f}\frac{\sin(2\pi f)}{\pi f}$$

So

$$H(f) = \mathcal{F}[h_1(t)h_2(t)] = H_1(f) * H_2(f) = \int_{-\infty}^{\infty} H_1(u)H_2(f-u)du$$

or

$$H(f) = \int_{-\infty}^{\infty} \frac{1}{1 + 2\pi j u} e^{-2\pi j(f-u)} \frac{\sin(2\pi(f-u))}{\pi(f-u)} du$$

Now you see that the previous method is easier!!!

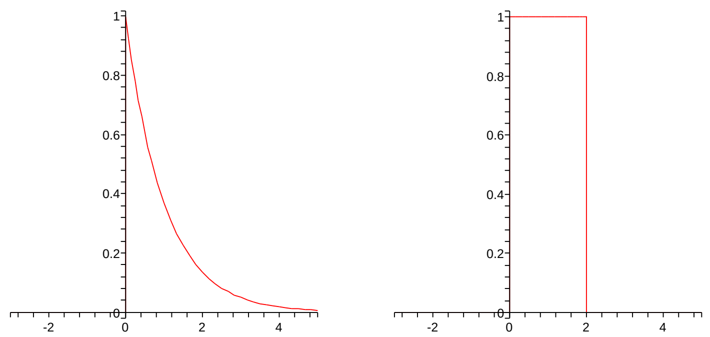


Figure 87: Graph of $h_1(t)$ (on the left) and $h_2(t)$ (on the right) for problem 1g of second set of Chapter 5

2. a. $H(f) = \dfrac{1}{1+f^2} \quad , \quad -\infty < f < \infty$

The graph of $H(f)$ is given in Figure 88. The closest pair is

$$G(f) \;=\; \frac{2\alpha}{\alpha^2 + 4\pi^2 f^2}, \qquad\qquad g(t) \;=\; e^{-\alpha|t|}.$$

With $\alpha = 2\pi$ we have

$$H(f) \;=\; \pi G(f)$$

See Figure 89 for the graphs of $G(f)$ and $g(t)$ and Figure 90 for $h(t)$.

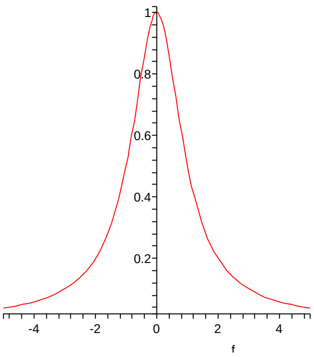


Figure 88: Graph of $H(f)$ for problem 2a of second set of Chapter 5

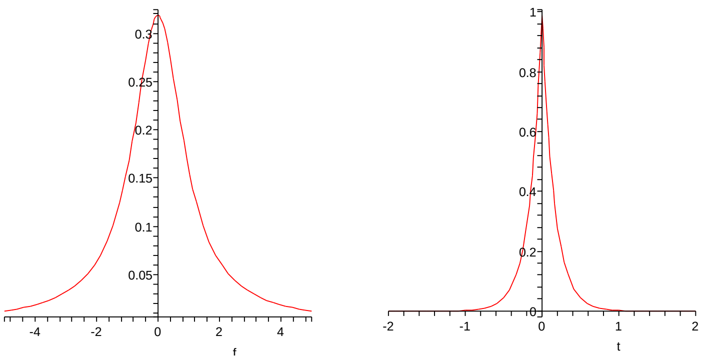


Figure 89: Graph of $G(f)$ (on the left) and $g(t)$ (on the right) for problem 2a of second set of Chapter 5

Thus

$$h(t) \;=\; \mathcal{F}^{-1}[H(f)] \;=\; \pi\mathcal{F}^{-1}[G(f)] \;=\; \pi e^{-2\pi|t|}$$

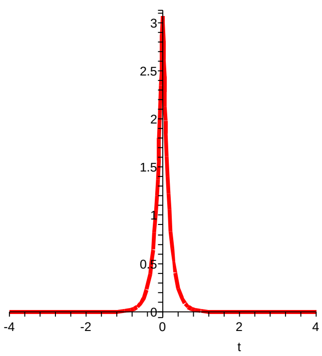


Figure 90: Graph of $h(t)$ for problem 2a of second set of Chapter 5

2. b. $H(f) = e^{-3j\pi f} e^{-2|f|} \quad , \quad -\infty < f < \infty$

$$H(f) \;=\; e^{-3\pi jf} e^{-2|f|} \;=\; e^{-2\pi jf\cdot(3/2)} e^{-2|f|} \;=\; \mathcal{F}[g(t-3/2)]$$

where

$$G(f) \;=\; e^{-2|f|}, \quad \text{even function of } f$$

so

$$g(t) \;=\; \frac{4}{4+4\pi^2 t^2}, \qquad \text{using tables with a=2}$$

Therefore

$$h(t) \;=\; g(t-3/2) \;=\; \frac{1}{1+\pi^2(t-3/2)^2}$$

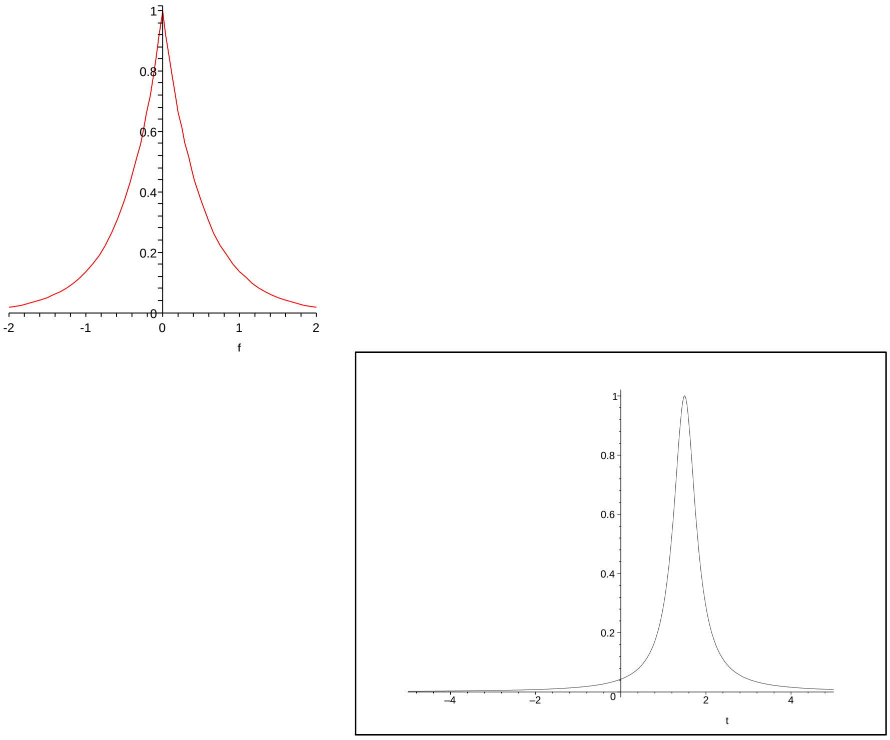


Figure 91: Graph of $G(f)$ (on the left) and $h(t)$ (on the right) for problem 2b of second set of Chapter 5

2. c. $H(f) = \begin{cases} 2 & , \quad -3 < f < 3 \\ 0 & , \quad \text{otherwise} \end{cases}$

Note that $H(f)$, given in Figure 92, is real and even and we can find $h(t)$ directly or from the tables.

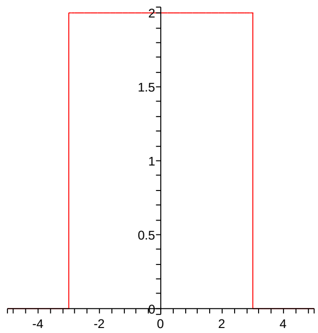


Figure 92: Graph of $H(f)$ for problem 2c of second set of Chapter 5

$$h(t) \;=\; \int_{-\infty}^{\infty} H(f)e^{2\pi jft}df \;=\; \int_{-3}^{3} 2e^{2\pi jft}df \;=\; 2\int_{0}^{3} 2\cos(2\pi ft)df$$

So

$$h(t) \;=\; 4\frac{\sin(2\pi ft)}{2\pi f}\Big|_{f=0}^{f=3} \;=\; 2\frac{\sin(6\pi t)}{\pi t}$$

The other way is to note that $H(f) = 12G(f)$ where $G(f) \;=\; \begin{cases} 1/(2f_0), & -f_0 < f < f_0 \\ 0, & \text{otherwise} \end{cases}$ is given in Figure 93 with $f_0 = 3$. Thus

$$h(t) \;=\; \mathcal{F}^{-1}[H(f)] \;=\; \mathcal{F}^{-1}[12G(f)] \;=\; 12g(t)$$

So

$$h(t) = 12\frac{\sin(2\pi(3)t)}{2\pi(3)t} \;=\; 2\frac{\sin(6\pi t)}{\pi t}$$

as before.

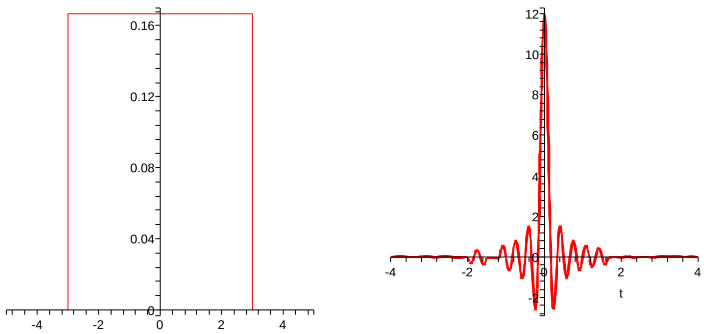


Figure 93: Graph of $G(f)$ (on the left) and $h(t)$ (on the right) for problem 2c of second set of Chapter 5

6 Applications of the Fourier Transform

6.1 Introduction

6.2 Convolution and Fourier Transforms

PROBLEMS

1. Compute, using the definition, the convolution ($h(t) * g(t)$) in the following cases. Then, in each case, compute the Fourier transform of the convolution and verify the result agrees with the convolution theorem:

a. $h(t) = \begin{cases} 2 & , \quad 0 < t < 2 \\ 0 & , \quad \text{otherwise} \end{cases}$

$g(t) = \begin{cases} e^{-t} & , \quad 0 < t \\ 0 & , \quad \text{otherwise} \end{cases}$

b. $h(t) = g(t) = \begin{cases} 2 & , \quad -2 < t < 2 \\ 0 & , \quad \text{otherwise} \end{cases}$

c. $h(t) = e^{-|t|} \quad , \quad -\infty < t < \infty$

$g(t) = \cos(2\pi t) \quad , \quad -\infty < t < \infty$

1. a. $h(t) = \begin{cases} 2 & , \quad 0 < t < 2 \\ 0 & , \quad \text{otherwise} \end{cases}$

$g(t) = \begin{cases} e^{-t} & , \quad 0 < t \\ 0 & , \quad \text{otherwise} \end{cases}$

The graphs of $h(t)$ and $g(t)$ are given in Figure 94 and the graph of $h(t-u)$ is given in Figure 95.

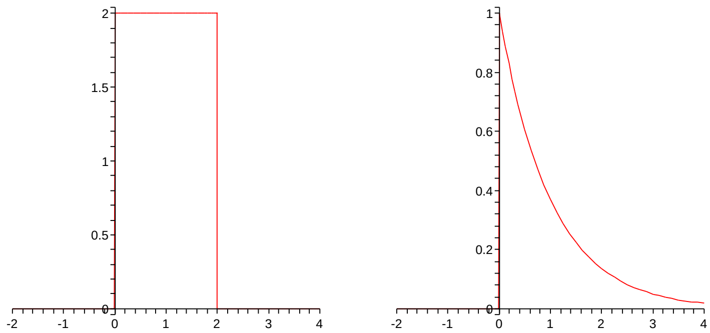


Figure 94: Graph of $h(t)$ (on the left) and $g(t)$ (on the right) for problem 1a of Chapter 6.2

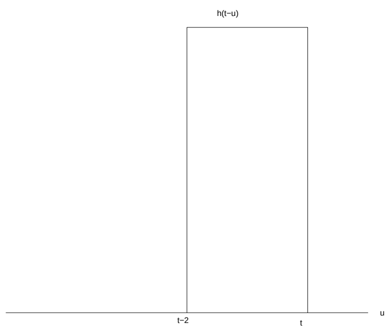


Figure 95: Graph of $h(t-u)$ for problem 1a of Chapter 6.2

Three possible cases are described below. Case I (Figure 96) when $t < 0$ and there is no overlap, Case II (Figure 97) when $0 < t < 2$ and there is an overlap for $0 < u < t$, Case III (Figure 98) when $2 < t$ and there is an overlap for $t - 2 < u < t$. The convolution

$$g * h = \int_{-\infty}^{\infty} g(u)h(t-u)du = \begin{cases} 0 & \text{case I} \\ \displaystyle\int_0^t 2e^{-u}du = 2\left(1 - e^{-t}\right) & \text{case II} \\ \displaystyle\int_{t-2}^t 2e^{-u}du = 2\left(e^{-(t-2)} - e^{-t}\right) & \text{case III} \end{cases}$$

This convolution is plotted in Figure 99.

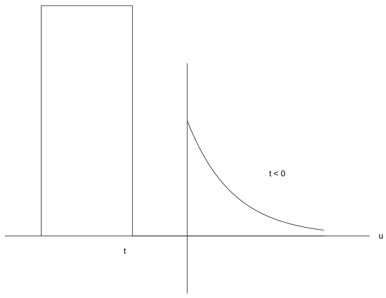


Figure 96: Graph of g and h for problem 1a of Chapter 6.2 when $t < 0$

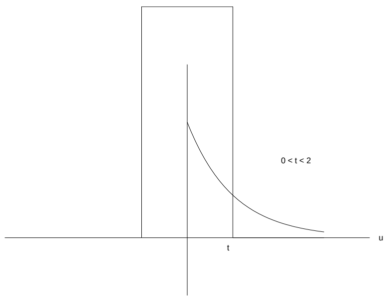


Figure 97: Graph of g and h for problem 1a of Chapter 6.2 when $0 < t < 2$

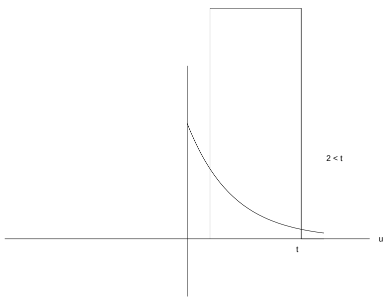


Figure 98: Graph of g and h for problem 1a of Chapter 6.2 when $2 < t$

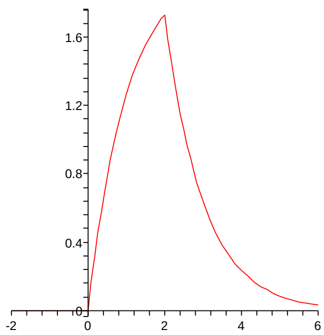


Figure 99: Graph of the convolution for problem 1a of Chapter 6.2

By definition

$$\mathcal{F}[g*h] \;=\; \int_{-\infty}^{\infty} g*he^{-2\pi jft}dt$$

$$\mathcal{F}[g*h] \;=\; 2\int_0^2 \left(1-e^{-t}\right)e^{-2\pi jft}dt + 2\int_2^{\infty}\left(e^{-(t-2)}-e^{-t}\right)e^{-2\pi jft}dt$$

$$\mathcal{F}[g*h] \;=\; 2\left\{-\frac{e^{-2\pi jft}}{2\pi jf}\bigg|_0^2 + \frac{e^{-(1+2\pi jf)t}}{1+2\pi jf}\bigg|_0^2 - 2e^2\frac{e^{-(1+2\pi jf)t}}{1+2\pi jf}\bigg|_2^{\infty} + 2\frac{e^{-(1+2\pi jf)t}}{1+2\pi jf}\bigg|_2^{\infty}\right\}$$

$$\mathcal{F}[g*h] \;=\; 2\left\{\frac{1}{2\pi jf} - \frac{e^{-4\pi jf}}{2\pi jf} + \frac{e^{-2(1+2\pi jf)}}{1+2\pi jf} - \frac{1}{1+2\pi jf} + e^2\frac{e^{-2(1+2\pi jf)}}{1+2\pi jf} - \frac{e^{-2(1+2\pi jf)}}{1+2\pi jf}\right\}$$

$$\mathcal{F}[g*h] \;=\; 2\left\{\frac{1}{2\pi jf} - \frac{e^{-4\pi jf}}{2\pi jf} - \frac{1}{1+2\pi jf} + \frac{e^{-4\pi jf)}}{1+2\pi jf}\right\}$$

$$\mathcal{F}[g*h] \;=\; 2\left(1-e^{-4\pi jf}\right)\left\{\frac{1}{2\pi jf} - \frac{1}{1+2\pi jf}\right\}$$

$$\mathcal{F}[g*h] \;=\; 2\frac{1-e^{-4\pi jf}}{2\pi jf(1+2\pi jf)}$$

By convolution theorem

$$H(f) = \int_0^2 2e^{-2\pi jft}dt \;=\; 2\frac{1-e^{-4\pi jf}}{2\pi jf}$$

$$G(f) \;=\; \frac{1}{1+2\pi jf}$$

therefore

$$\mathcal{F}[g*h] \;=\; G(f)H(f) \;=\; 2\frac{1-e^{-4\pi jf}}{2\pi jf(1+2\pi jf)}$$

1. b. $h(t) = g(t) = \begin{cases} 2 & , \quad -2 < t < 2 \\ 0 & , \quad \text{otherwise} \end{cases}$

The graphs of $h(t)$ and $g(t)$ are the same and given in Figure 100 and the graph of $h(t-u)$ is given in Figure 101.

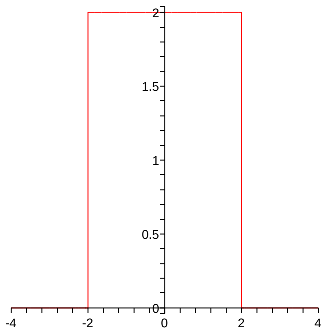


Figure 100: Graph of $h(t)$ for problem 1b of Chapter 6.2

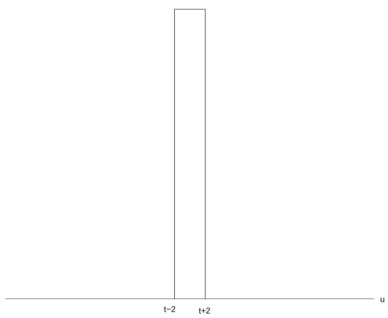


Figure 101: Graph of $h(t-u)$ for problem 1b of Chapter 6.2

Four possible cases are described below. Case I (Figure 102) when $t < -4$ and there is no overlap, Case II (Figure 103) when $-4 \leq t \leq 0$ and there is an overlap for $-2 < u < t+2$, Case III (Figure 104) when $0 < t < 4$ and there is an overlap for $t-2 < u < 2$, case IV (Figure 105) when $4 < t$ and there is no overlap again. The convolution

$$g * h = \int_{-\infty}^{\infty} g(u)h(t-u)du = \begin{cases} 0 & \text{case I} \\ \displaystyle\int_{-2}^{t+2} 4du = 4(t+4) & \text{case II} \\ \displaystyle\int_{t-2}^{2} 4du = 4(2-(t-2)) = 4(4-t)) & \text{case III} \\ 0 & \text{case IV} \end{cases}$$

This convolution is plotted in Figure 106.

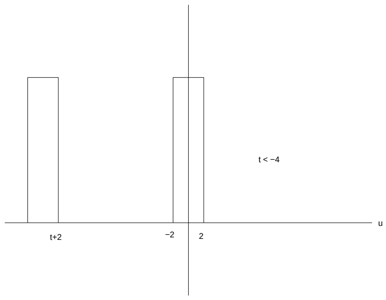


Figure 102: Graph of h and h for problem 1b of Chapter 6.2 when $t < -4$

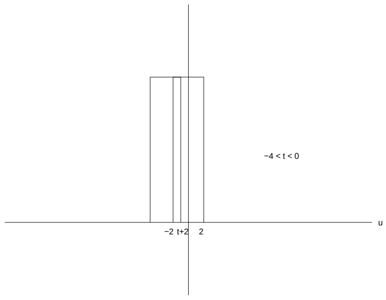


Figure 103: Graph of h and h for problem 1b of Chapter 6.2 when $-4 \leq t \leq 0$

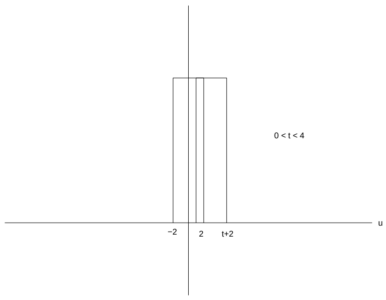


Figure 104: Graph of g and h for problem 1b of Chapter 6.2 when $0 < t < 4$

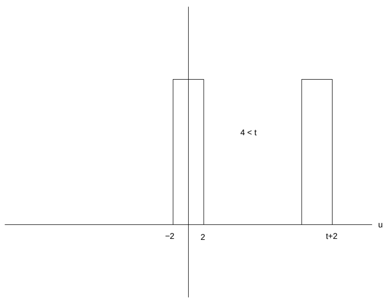


Figure 105: Graph of g and h for problem 1b of Chapter 6.2 when $4 < t$

By definition

$$\mathcal{F}[g*h] \;=\; \int_{-\infty}^{\infty} g*he^{-2\pi jft}dt$$

$$\mathcal{F}[g*h] \;=\; \int_{-4}^{0} 4(4+t)e^{-2\pi jft}dt + \int_{0}^{4} 4(4-t)e^{-2\pi jft}dt$$

$$\mathcal{F}[g*h] \;=\; 4\frac{\sin^2(4\pi f)}{(\pi f)^2}$$

By convolution theorem

$$H(f) = G(f) = 2\frac{\sin(4\pi f)}{\pi f}$$

therefore

$$\mathcal{F}[g*h] \;=\; G(f)H(f) \;=\; 4\frac{\sin^2(4\pi f)}{(\pi f)^2}$$

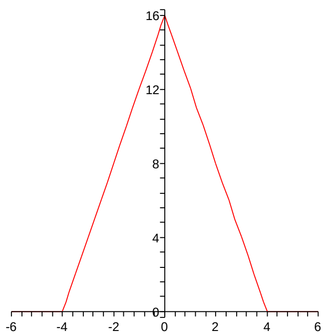


Figure 106: Graph of the convolution for problem 1b of Chapter 6.2

1. c. $h(t) = e^{-|t|} \quad , \quad -\infty < t < \infty$

$g(t) = \cos(2\pi t) \quad , \quad -\infty < t < \infty$

The graphs of $h(t)$ and $g(t)$ are given in Figure 107 and the graph of $h(t-u)$ is given in Figure 108.

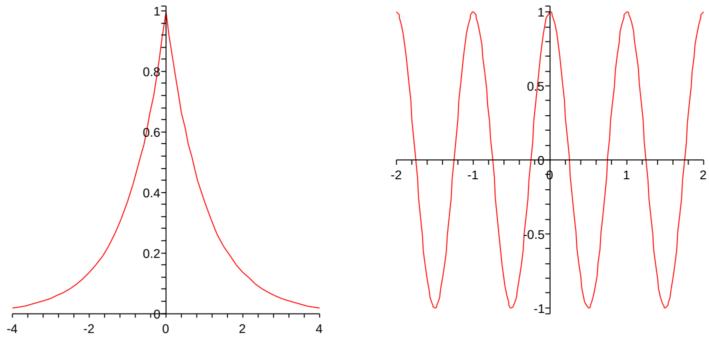


Figure 107: Graph of $h(t)$ (on the left) and $g(t)$ (on the right) for problem 1c of Chapter 6.2

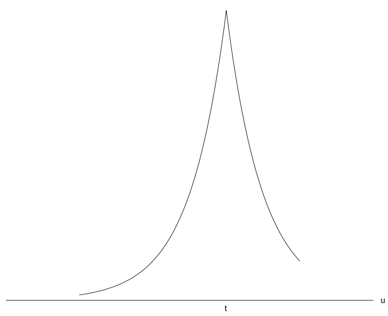


Figure 108: Graph of $h(t-u)$ for problem 1c of Chapter 6.2

By definition

$$g * h \;=\; \int_{-\infty}^{\infty} g(u)h(t-u)du$$

or the equivalent form which is easier to integrate

$$g * h \;=\; \int_{-\infty}^{\infty} h(u)g(t-u)du$$

$$g * h \;=\; \int_{-\infty}^{\infty} e^{-|u|}\cos(2\pi(t-u))du$$

$$g * h \;=\; \int_{-\infty}^{\infty} e^{-|u|}\left[\cos(2\pi t)\cos(2\pi u) + \sin(2\pi t)\sin(2\pi u)\right]du$$

$$g * h \;=\; \cos(2\pi t)\int_{-\infty}^{\infty} e^{-|u|}\cos(2\pi u)du + \sin(2\pi t)\underbrace{\int_{-\infty}^{\infty} e^{-|u|}\sin(2\pi u)du}_{\text{odd function on a symmetric interval, integral } = 0}$$

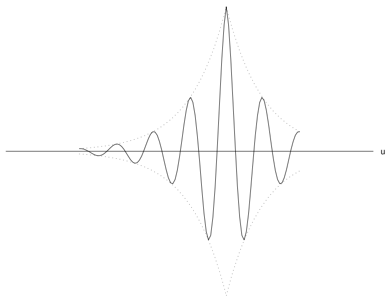


Figure 109: Graph of $g(u)h(t-u)$ for problem 1c of Chapter 6.2

$$g*h \;=\; 2\cos(2\pi t)\int_0^{\infty} e^{-|u|}\cos(2\pi u)du$$
$$g*h \;=\; \frac{2}{1+4\pi^2}\cos(2\pi t)$$

The graph of the convolution is given in Figure 110.

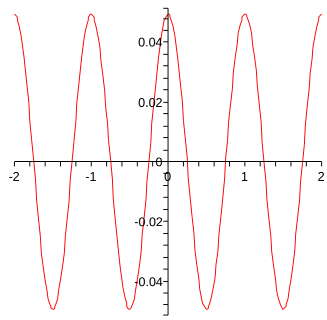


Figure 110: Graph of the convolution for problem 1c of Chapter 6.2

By definition

$$\mathcal{F}[g*h] \;=\; \int_{-\infty}^{\infty} g*he^{-2\pi jft}dt$$
$$\mathcal{F}[g*h] \;=\; \mathcal{F}\left[\frac{2}{1+4\pi^2}\cos(2\pi t)\right]$$
$$\mathcal{F}[g*h] \;=\; \frac{1}{1+4\pi^2}\left[\delta(f-1)+\delta(f+1)\right]$$

By convolution theorem

$$H(f) = 2\frac{1}{1+4\pi^2 f^2}$$
$$G(f) = \frac{1}{2}\left[\delta(f-1)+\delta(f+1)\right]$$

therefore

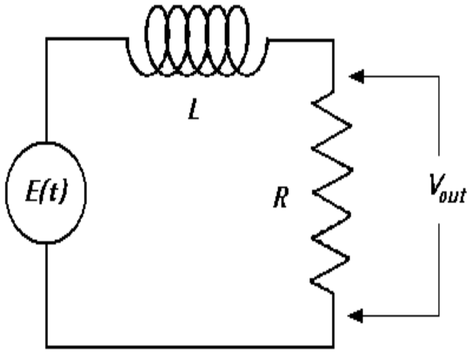
$$\mathcal{F}[g*h] \;=\; G(f)H(f) \;=\; \frac{1}{1+4\pi^2}\left[\delta(f-1)+\delta(f+1)\right]$$

Note that this last step uses the fact that $\phi(x)\delta(x - x_0) \equiv \phi(x_0)\delta(x - x_0)$.

6.10 Summary

PROBLEMS

1. Consider the linear, shift-invariant system represented by the following circuit



$$
\begin{aligned}
L\frac{dI}{dt} + RI &= E(t) \\
V_{out} &= RI
\end{aligned}
$$

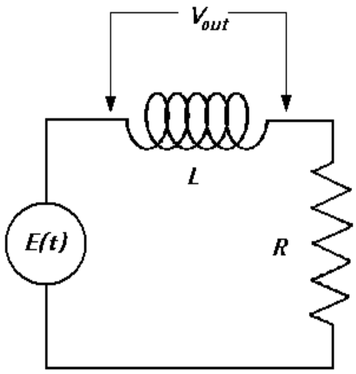
a. Directly determine, e.g. using the Laplace transform, the impulse response of this system. Sketch this response.

b. (1) Find the Transfer function of this system by computing the Fourier transform of the impulse response determined in part a. above.

b. (2) Show that the alternative method of finding the Transfer function, i.e. as the response of a system to the forcing function $e^{2\pi j f_0 t}$, produces the same result as in part (1),

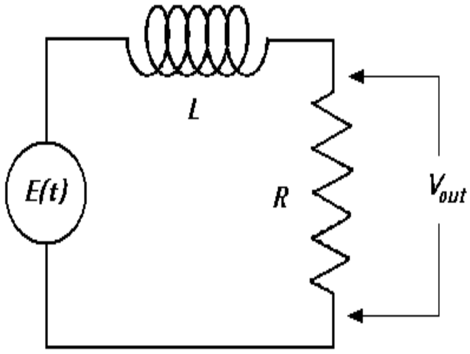
c. Sketch the amplitude and phase spectra of the Transfer function computed in part b.

2. Repeat problem 1 for the same circuit, except with the output taken as the voltage across the inductor, i.e.



$$
\begin{aligned}
L\frac{dI}{dt} + RI &= E(t) \\
V_{out} &= L\tfrac{dI}{dt}
\end{aligned}
$$

1. a.



$$
\begin{aligned}
L\frac{dI}{dt} + RI &= E(t) \\
V_{out} &= RI
\end{aligned}
$$

Laplace transform: Let us denote $\mathcal{L}[I(t)] = \tilde{I}(s)$, then

$$sL\tilde{I}(s) + R\tilde{I}(s) = 1$$

$$\tilde{V}_{out}(s) = R\tilde{I}(s)$$

Thus

$$\tilde{I}(s) \; = \; \frac{1}{sL + R} \; \Rightarrow \tilde{V}_{out}(s) = \frac{R}{sL+R} = \frac{R}{L}\frac{1}{s + (R/L)}$$

So

$$V_{out}(t) \; = \; \begin{cases} \frac{R}{L}e^{-Rt/L}, & t > 0 \\ 0, & t < 0 \end{cases}$$

This V_{out} is plotted in Figure 111.

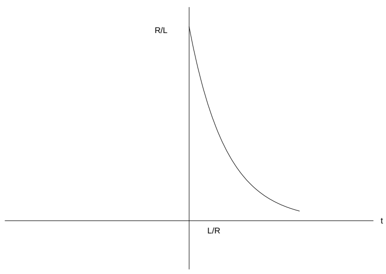


Figure 111: Graph of $h(t)$ for problem 1a of Chapter 6.10

1. b. (1)
Transfer function

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-2\pi jft}dt = \frac{R}{L}\int_{0}^{\infty} e^{-Rt/L}e^{-2\pi jft}dt = \frac{R}{L}\frac{1}{R/L + 2\pi jf}$$

$$H(f) = \frac{R}{R + 2\pi jfL}$$

1. b. (2) We can find the transfer function by computing the response to $e^{2\pi jf_0t}$:

$$L\frac{dI}{dt} + RI = e^{2\pi jf_0t}$$

Using undetermined coefficients, we substitute $I(t) = A(f_0)e^{2\pi jf_0t}$ in the equation and we have

$$(2\pi jf_0LA + RA)e^{2\pi jf_0t} = e^{2\pi jf_0t}$$

or

$$A(f_0) = \frac{1}{R + 2\pi jf_0L}$$

Therefore

$$I(t) = \frac{1}{R + 2\pi jf_0L}e^{2\pi jf_0t}$$

and

$$V_{out} = RI = \frac{R}{R + 2\pi jf_0L}e^{2\pi jf_0t} = H(f_0)e^{2\pi jf_0t}$$

Therefore

$$H(f_0) = \frac{R}{R + 2\pi jf_0L}$$

or

$$H(f) = \frac{R}{R + 2\pi jfL}$$

1. c.
We rewrite $H(f)$ as

$$H(f) = \frac{R^2}{R^2 + (2\pi fL)^2} - j\frac{2\pi fRL}{R^2 + (2\pi fL)^2}$$

Therefore

$$|H(f)| = \frac{R}{\sqrt{R^2 + (2\pi fL)^2}}$$

The plots of $|H(f)|$ and $\Theta(f)$ are given in Figures 112 and 113.

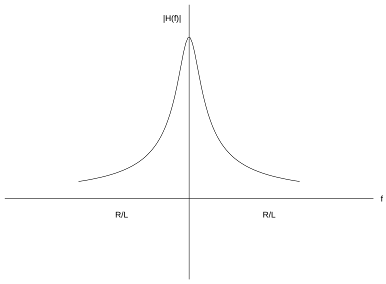


Figure 112: Graph of $|H(f)|$ for problem 1c of Chapter 6.10

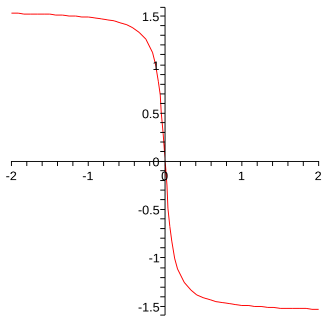
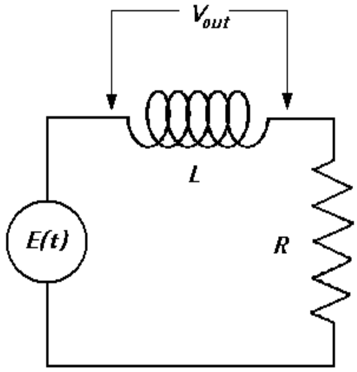


Figure 113: Graph of $\Theta(f)$ for problem 1c of Chapter 6.10

2. a.



$$\begin{aligned} L\frac{dI}{dt} + RI &= E(t) \\ V_{out} &= L\tfrac{dI}{dt} \end{aligned}$$

Impulse response:

$$\begin{aligned} L\frac{dI}{dt} + RI &= \delta(t) \\ V_{out} &= L\tfrac{dI}{dt} \end{aligned}$$

Laplace transform

$$\begin{aligned} sL\tilde{I}(s) + R\tilde{I}(s) &= 1 \\ \tilde{V}_{out}(s) &= sL\tilde{I}(s) \end{aligned}$$

or

$$\tilde{I}(s) \;=\; \frac{1}{sL+R} \;\Rightarrow \tilde{V}_{out}(s) \;=\; \frac{sL}{sL+R}$$

Now we can rewrite

$$\tilde{V}_{out}(s) = 1 - \frac{R}{sL+R} = 1 - \frac{R}{L}\frac{1}{s+(L/R)}$$

so

$$V_{out}(t) \;=\; \delta(t) - \frac{R}{L}e^{-Rt/L}$$

This function is given in Figure 114.

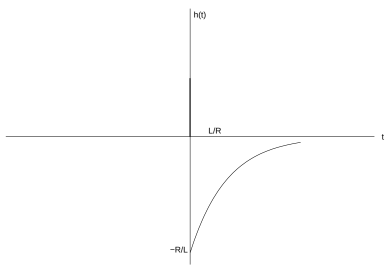


Figure 114: Graph of $V_{out}(t)$ for problem 2a of Chapter 6.10

2. b. (1)

$$H(f) = \mathcal{F}[h(t)] = \mathcal{F}\left[\delta(t) - \frac{R}{L}e^{-Rt/L}\right]$$
$$H(f) = 1 - \frac{R}{R + 2\pi jfL} = \frac{2\pi jfL}{R + 2\pi jfL}$$

2. b. (2)

$$L\frac{dI}{dt} + RI = e^{2\pi jf_0t}$$

As before

$$I(t) = \frac{1}{R + 2\pi jf_0L}e^{2\pi jf_0t}$$
$$V_{out} = L\frac{dI}{dt} = \underbrace{\frac{2\pi jf_0L}{R + 2\pi jf_0L}}_{H(f_0)}e^{2\pi jf_0t}$$

So

$$H(f) = \frac{2\pi jfL}{R + 2\pi jfL}$$

same as before.

2. c.
We rewrite $H(f)$ as

$$H(f) = \frac{(2\pi fL)^2}{R^2 + (2\pi fL)^2} + j\frac{2\pi fLR}{R^2 + (2\pi fL)^2}$$

Therefore

$$|H(f)| = \frac{2\pi L|f|}{\sqrt{R^2 + (2\pi fL)^2}}$$

and

$$\tan\Theta(f) = \frac{R}{2\pi fL}$$

The $|H(f)|$ and $\Theta(f)$ are given in Figures 115 and 116, respectively.

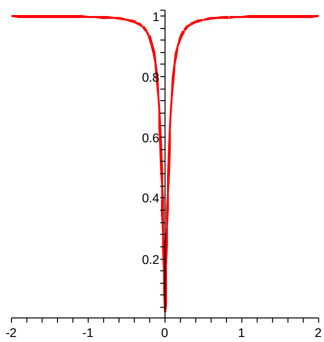


Figure 115: Graph of $|H(f)|$ for problem 2c of Chapter 6.10

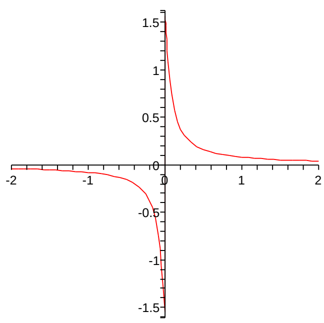


Figure 116: Graph of $\Theta(f)$ for problem 2c of Chapter 6.10

7 Appendix A - Bessel's Equation

7.1 Bessel's Equation

7.2 Properties of Bessel Functions

PROBLEMS

1. Using the recurrence formulas, and a table of values for $J_0(x)$ and $J_1(x)$, find
   a. $J_1'(x)$ in terms of $J_0(x)$ and $J_1(x)$
   b. $J_2(2.0)$
   c. $J_3'(1.0)$

2. Write, in terms of $J_n(x)$ and $Y_n(x)$, the general solution to
   a. $x^2y'' + xy' + 4x^2y = 0$
   b. $x^2y'' + xy' + (9x^2 - 4)y = 0$
   c. $4x^2y'' + 4xy' + (x^2 - 1)y = 0$

1. a. Using the recurrence relation $NJ_N(x) + xJ'_N(x) = xJ_{N-1}(x)$ with $N = 1$, we have

$$J_1(x) + xJ'_1(x) = xJ_0(x)$$

Therefore

$$J'_1(x) = \frac{xJ_0(x) - J_1(x)}{x}$$

1. b. Using the recurrence relation $J_{N+1} = \dfrac{2N}{x} J_N(x) - J_{N-1}(x)$ with $N = 1$, we have

$$J_2 = \frac{2}{x} J_1(x) - J_0(x)$$

From CRC tables (or Maple, Matlab), $J_0(2.0) = .2239$, $J_1(2.0) = .5767$ therefore

$$J_2(2.0) = \frac{2}{2}.0(.5767) - .2239 = .3528$$

1. c. Using the recurrence relation $NJ_N(x) + xJ'_N(x) = xJ_{N-1}(x)$ with $N = 3$ we have

$$3J_3(x) + xJ'_3(x) = xJ_2(x)$$

But

$$J_{N+1} = \frac{2N}{x} J_N(x) - J_{N-1}(x)$$

with $N = 2$ yields

$$J_3(x) = \frac{4}{x} J_2(x) - J_1(x)$$

Combining these two, we get

$$3\left(\frac{4}{x} J_2(x) - J_1(x)\right) + xJ'_3(x) = xJ_2(x)$$

So

$$J'_3(x) = \frac{1}{x}\left[\left(x - \frac{12}{x}\right) J_2(x) + 3J_1(x)\right]$$

$$J'_3(x) = \frac{1}{x}\left[\left(x - \frac{12}{x}\right)\left(\frac{2}{x} J_1(x) - J_0(x)\right) + 3J_1(x)\right]$$

Thus

$$J'_3(1.0) = \left[-11\left(2J_1(1.0) - J_0(1.0)\right) + 3J_1(1.0)\right]$$

$$J'_3(1.0) = -19J_1(1.0) + 11J_0(1.0) = -19(.4401) + 11(.7652) = .0553$$

2. a. The general solution of $x^2y'' + xy' + (\xi^2 x^2 - n^2)\, y = 0$ is

$$C_1 J_n(\xi x) + C_2 Y_n(\xi x)$$

In our case $n = 0$ and $\xi = 2$, therefore the solution is

$$C_1 J_0(2x) + C_2 Y_0(2x)$$

2. b. The equation $x^2y'' + xy' + (9x^2 - 4)y = 0$ matches the general case with $n = 2$ and $\xi = 3$, therefore the solution is

$$C_1 J_2(3x) + C_2 Y_2(3x)$$

2. c. The equation $4x^2y'' + 4xy' + (x^2 - 1)y = 0$ matches the general case with $n = \dfrac{1}{2}$ and $\xi = \dfrac{1}{2}$, therefore the solution is

$$C_1 J_{1/2}\left(\frac{1}{2}x\right) + C_2 Y_{1/2}\left(\frac{1}{2}x\right)$$

7.3 Variants of Bessel's Equation

PROBLEMS

Use Bessel Functions (the big ugly equation) to find the *general solution* to each ODE below.

1. $$x^2y'' + 3\,xy' + \left(-3 + 4\,x^4\right)y = 0$$

2. $$x^2y'' + \left(x + 2\,x^2\right)y' + \left(-4 + 9\,x^{-2} + x + x^2\right)y = 0$$

3. $$x^2y'' - 5\,xy' + \left(9 + 4\,x^2\right)y = 0$$

4. $$x^2y'' + \left(x - 2\,x^3\right)y' + \left(-1/4 + x^{-2} - 2\,x^2 + x^4\right)y = 0$$

Answers

1.
$$y(x) = A\,J_1(x^2)/x + B\,Y_1(x^2)/x$$

2.
$$y(x) = e^{-x}\;(A\,J_2(3/x) + B\,Y_2(3/x))$$

3.
$$y(x) = x^3\,(A\,J_0(2x) + B\,Y_0(2x))$$

4.
$$y(x) = e^{x^2/2}\;\Big(A\,J_{1/2}(1/x) + B\,Y_{1/2}(1/x)\Big)$$