Introduction to fractional calculus (Based on lectures by R. Gorenflo, F. Mainardi and I. Podlubny)

R. Vilela Mendes

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Contents

Fractional Calculus was born in 1695

$$\frac{d^n f}{dt^n}$$

What if the order will be $n = 1/2$?

It will lead to a paradox, from which one day useful consequences will be drawn.

G.F.A. de L'Hôpital (1661–1704)

G.W. Leibniz (1646–1716)

G. W. Leibniz (1695–1697)

In the letters to J. Wallis and J. Bernulli (in 1697) Leibniz mentioned the possible approach to fractional-order differentiation in that sense, that for non-integer values of n the definition could be the following:

$$\frac{d^n e^{mx}}{dx^n} = m^n e^{mx},$$

L. Euler (1730)

$$\frac{d^n x^m}{dx^n} = m(m-1)\ldots(m-n+1)x^{m-n}$$

$$\Gamma(m+1) = m(m-1)\ldots(m-n+1)\,\Gamma(m-n+1)$$

$$\frac{d^n x^m}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}.$$

Euler suggested to use this relationship also for negative or non-integer (rational) values of n. Taking $m = 1$ and $n = \frac{1}{2}$, Euler obtained:

$$\frac{d^{1/2} x}{dx^{1/2}} = \sqrt{\frac{4x}{\pi}} \qquad \left(= \frac{2}{\sqrt{\pi}} x^{1/2} \right)$$

S. F. Lacroix adopted Euler's derivation for his successful textbook (*Traité du Calcul Différentiel et du Calcul Intégral*, Courcier, Paris, t. 3, 1819; pp. 409–410).

TRAITÉ ÉLÉMENTAIRE

DE

CALCUL DIFFÉRENTIEL

ET DE

CALCUL INTÉGRAL,

PAR S.-F. LACROIX.

SEPTIÈME ÉDITION,

REVUE ET AUGMENTÉE DE NOTES

Par MM. HERMITE et J.-A. SERRET,

MEMBRES DE L'INSTITUT.

J. B. J. Fourier (1820–1822)

The first step to generalization of the notion of differentiation for **arbitrary functions** was done by J. B. J. Fourier (*Théorie Analytique de la Chaleur*, Didot, Paris, 1822; pp. 499–508).

After introducing his famous formula

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(z)dz \int_{-\infty}^{\infty} \cos(px - pz)dp,$$

Fourier made a remark that

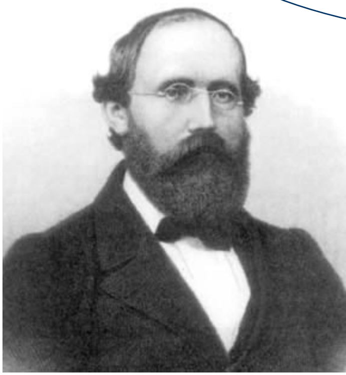
$$\frac{d^n f(x)}{dx^n} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(z)dz \int_{-\infty}^{\infty} \cos(px - pz + n\frac{\pi}{2})dp,$$

and this relationship could serve as a definition of the n-th order derivative for non-integer n.

Riemann–Liouville definition

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_a^t \frac{f(\tau)\, d\tau}{(t-\tau)^{\alpha-n+1}} $$

$$ (n-1 \le \alpha < n) $$



G.F.B. Riemann (1826–1866)

J. Liouville (1809–1882)

Fractional integral according to Riemann-Liouville

- According to Riemann-Liouville the notion of fractional integral of order α ($\alpha > 0$) for a function $f(t)$, is a natural consequence of the well known formula (Cauchy-Dirichlet ?), that reduces the calculation of the $n-$fold primitive of a function $f(t)$ to a single integral of convolution type

$$J^n_{a+} f(t) := \frac{1}{(n-1)!} \int_a^t (t-\tau)^{n-1} f(\tau)\, d\tau\,, \qquad n \in \mathbb{N} \qquad (1)$$

vanishes at $t = a$ with its derivatives of order $1, 2, \ldots, n-1$. Require $f(t)$ and $J^n_{a+} f(t)$ to be *causal* functions, that is, vanishing for $t < 0$.

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- Extend to any positive real value by using the Gamma function, $(n-1)! = \Gamma(n)$

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$$J_{a+}^{n} f(t) := \frac{1}{(n-1)!} \int_a^t (t-\tau)^{n-1} f(\tau)\, d\tau\,, \qquad n \in \mathbb{N} \qquad (1)$$

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- Extend to any positive real value by using the Gamma function, $(n-1)! = \Gamma(n)$
- **Fractional Integral of order α>0** (right-sided)

$$J_{a+}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau\,, \qquad \alpha \in \mathbb{R} \qquad (2)$$

Define $J_{a+}^{0} := I$, $J_{a+}^{0} f(t) = f(t)$

Fractional integral according to Riemann-Liouville

- Alternatively (left-sided integral)

$$J_{b-}^{\alpha}\, f(t) := \frac{1}{\Gamma(\alpha)} \int_t^b (\tau - t)^{\alpha-1}\, f(\tau)\, d\tau\,, \qquad \alpha \in \mathbb{R}$$

$(a = 0, b = +\infty)$ Riemann $\qquad (a = -\infty, b = +\infty)$ Liouville

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- Let

$$J^{\alpha} := J_{0+}^{\alpha}$$

Semigroup properties $\quad J^{\alpha} J^{\beta} = J^{\alpha+\beta}\,, \qquad \alpha\,, \beta \geq 0$

Commutative property $\quad J^{\beta} J^{\alpha} = J^{\alpha} J^{\beta}$

Effect on power functions

$J^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)}\, t^{\gamma+\alpha}\,, \qquad \alpha > 0\,, \gamma > -1\,, t > 0$

(Natural generalization of the positive integer properties).

Fractional integral according to Riemann-Liouville

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(Natural generalization of the positive integer properties).

- Introduce the following causal function (vanishing for $t < 0$)

$$\Phi_{\alpha}(t) := \frac{t_{+}^{\alpha-1}}{\Gamma(\alpha)}\,, \qquad \alpha > 0$$

Fractional integral according to Riemann-Liouville

- $$\Phi_\alpha(t) * \Phi_\beta(t) = \Phi_{\alpha+\beta}(t)\,, \qquad \alpha\,,\beta > 0$$
  $$J^\alpha f(t) = \Phi_\alpha(t) * f(t)\,, \qquad \alpha > 0$$

Fractional integral according to Riemann-Liouville

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- Laplace transform

$$\mathcal{L}\{f(t)\} := \int_0^\infty e^{-st} f(t)\, dt = \widetilde{f}(s)\,, \qquad s \in \mathbb{C}$$

Fractional integral according to Riemann-Liouville

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- Laplace transform
  $$\mathcal{L}\left\{f(t)\right\} := \int_0^\infty e^{-st} f(t)\,dt = \widetilde{f}(s)\,, \qquad s \in \mathbb{C}$$
- Defining the Laplace transform pairs by $f(t) \div \widetilde{f}(s)$
  $$J^\alpha f(t) \div \frac{\widetilde{f}(s)}{s^\alpha}\,, \qquad \alpha > 0$$

Fractional derivative according to Riemann-Liouville

- Denote by D^n with $n \in \mathbb{N}$, the derivative of order n. Note that

$$D^n J^n = I \,, \qquad J^n D^n \neq I \,, \qquad n \in \mathbb{N}$$

D^n is a left-inverse (not a right-inverse) to J^n. In fact

$$J^n D^n f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!} \,, \qquad t > 0$$

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- Then, define D^α as a left-inverse to J^α. With a positive integer m, $m - 1 < \alpha \le m$, define:
  Fractional Derivative of order α : $\qquad D^\alpha\, f(t) := D^m\, J^{m-\alpha}\, f(t)$

$$D^\alpha\, f(t) := \begin{cases} \frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}}\,d\tau\right], & m-1 < \alpha < m \\ \frac{d^m}{dt^m} f\,(t)\,, & \alpha = m \end{cases}$$

Fractional derivative according to Riemann-Liouville

- Define $D^0 = J^0 = I$.
  Then $D^\alpha \, J^\alpha = I \,, \qquad \alpha \geq 0$

$$D^\alpha \, t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} \, t^{\gamma-\alpha} \,, \qquad \alpha > 0 \,, \gamma > -1 \,, t > 0$$

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- The fractional derivative $D^\alpha \, f$ is not zero for the constant function $f(t) \equiv 1$ if $\alpha \notin \mathbb{N}$

  $$D^\alpha 1 = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \,, \qquad \alpha \geq 0 \,, t > 0$$

  Is $\equiv 0$ for $\alpha \in \mathbb{N}$, due to the poles of the gamma function

Caputo fractional derivative

- $D_*^\alpha f(t) := J^{m-\alpha} D^m f(t)$ with $m-1<\alpha\leq m$, namely

$$D_*^\alpha f(t) := \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}d\tau, & m-1<\alpha<m \\ \frac{d^m}{dt^m}f(t), & \alpha=m \end{cases}$$

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- A definition more restrictive than the one before. It requires the absolute integrability of the derivative of order m. In general

$$D^\alpha f(t) := D^m J^{m-\alpha} f(t) \neq J^{m-\alpha} D^m f(t) := D_*^\alpha f(t)$$

unless the function $f(t)$ along with its first $m-1$ derivatives vanishes at $t = 0^+$. In fact, for $m-1 < \alpha < m$ and $t > 0$,

$$D^\alpha f(t) = D_*^\alpha f(t) + \sum_{k=0}^{m-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(0^+)$$

and therefore, recalling the fractional derivative of the power functions

$$D^\alpha \left(f(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} f^{(k)}(0^+) \right) = D_*^\alpha f(t), \qquad D_*^\alpha 1 \equiv 0, \ \alpha > 0$$

Riemann versus Caputo

- $$D^\alpha t^{\alpha-1} \equiv 0\,, \qquad \alpha > 0\,, t > 0$$

  D^α is not a right-inverse to J^α

  $$J^\alpha D^\alpha t^{\alpha-1} \equiv 0, \qquad \text{but} \qquad D^\alpha J^\alpha t^{\alpha-1} = t^{\alpha-1}\,, \qquad \alpha > 0\,, t > 0$$

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- Functions which for $t > 0$ have the same fractional derivative of order α, with $m-1 < \alpha \leq m$. (the c_j's are arbitrary constants)

  $$D^\alpha f(t) = D^\alpha g(t) \iff f(t) = g(t) + \sum_{j=1}^{m} c_j\, t^{\alpha-j}$$

  $$D_*^\alpha f(t) = D_*^\alpha g(t) \iff f(t) = g(t) + \sum_{j=1}^{m} c_j\, t^{m-j}$$

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- *Formal* limit as $\alpha \to (m-1)^+$

  $$\alpha \to (m-1)^+ \implies D^\alpha f(t) \to D^m J f(t) = D^{m-1} f(t)$$

  $$\alpha \to (m-1)^+ \implies D_*^\alpha f(t) \to J D^m f(t) = D^{m-1} f(t) - f^{(m-1)}(0^+)$$

Riemann versus Caputo

- The *Laplace transform*

$$D^{\alpha} f(t) \div s^{\alpha} \widetilde{f}(s) - \sum_{k=0}^{m-1} D^k J^{(m-\alpha)} f(0^+) s^{m-1-k}, \qquad m-1 < \alpha \leq m$$

Requires the knowledge of the (bounded) initial values of the fractional integral $J^{m-\alpha}$ and of its integer derivatives of order $k = 1, 2, m-1$

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  Requires the knowledge of the (bounded) initial values of the fractional integral $J^{m-\alpha}$ and of its integer derivatives of order $k = 1, 2, m-1$

- For the *Caputo fractional derivative*

$$D_*^\alpha f(t) \div s^\alpha \widetilde{f}(s) - \sum_{k=0}^{m-1} f^{(k)}(0^+) s^{\alpha-1-k}, \qquad m-1 < \alpha \leq m$$

  Requires the knowledge of the (bounded) initial values of the function and of its integer derivatives of order $k = 1, 2, m-1$ in analogy with the case when $\alpha = m$

Riesz - Feller fractional derivative

- For functions with Fourier transform

$$\begin{aligned} \mathcal{F}\left\{\phi(x)\right\} &= \overset{\wedge}{\phi}(k) = \int_{-\infty}^{\infty} e^{ikx}\phi(x)\,dx \\ \mathcal{F}^{-1}\left\{\overset{\wedge}{\phi}(k)\right\} &= \phi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\overset{\wedge}{\phi}(k)\,dx \end{aligned}$$

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- *Symbol* of an operator

$$\hat{A}(k)\,\overset{\wedge}{\phi}(k) = \int_{-\infty}^{\infty} e^{ikx}A\phi(x)\,dx$$

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- For the Liouville integral

$$J_{\infty+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-\xi)^{\alpha-1} f(\xi)\,d\xi$$

$$J_{\infty-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}(\xi-x)^{\alpha-1} f(\xi)\,d\xi\,, \qquad \alpha\in\mathbb{R}$$

Riesz - Feller fractional derivative

- Liouville derivatives $(m-1<\alpha<m)$

$$D^{\alpha}_{\infty\pm}=\begin{cases}\pm\left(D^{m}J^{m-\alpha}_{\infty\pm}\right)f\left(x\right), & m \text{ odd}\\ \left(D^{m}J^{m-\alpha}_{\infty\pm}\right)f\left(x\right), & m \text{ even}\end{cases}$$

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- Operator symbols

$$\begin{aligned} \overset{\wedge}{J^{\alpha}_{\infty\pm}} &= |k|^{-\alpha} e^{\pm i(\mathrm{sign} k)\alpha\pi/2} = (\mp ik)^{-\alpha} \\ \overset{\wedge}{D^{\alpha}_{\infty\pm}} &= |k|^{+\alpha} e^{\mp i(\mathrm{sign} k)\alpha\pi/2} = (\mp ik)^{+\alpha} \end{aligned}$$

$$\overset{\wedge}{J^{\alpha}_{\infty+}} + \overset{\wedge}{J^{\alpha}_{\infty-}} = \frac{2\cos(\alpha\pi/2)}{|k|^{\alpha}}$$

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$$\overset{\wedge}{J^{\alpha}_{\infty+}} + \overset{\wedge}{J^{\alpha}_{\infty-}} = \frac{2\cos(\alpha\pi/2)}{|k|^{\alpha}}$$

- Define a symmetrized version

$$I_0^{\alpha} f(x) = \frac{J^{\alpha}_{\infty+} f + J^{\alpha}_{\infty-} f}{2\cos(\alpha\pi/2)} = \frac{1}{2\Gamma(\alpha)\cos(\alpha\pi/2)} \int_{-\infty}^{\infty} |x-\xi|^{\alpha-1}\, f(\xi)\, d\xi$$

(wth exclusion of odd integers). The operator symbol is $\overset{\wedge}{I_0^{\alpha}} = |k|^{-\alpha}$

Riesz-Feller fractional derivative

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Riesz-Feller fractional derivative

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- Define the *Riesz fractional derivative* by analytical continuation

$$\mathcal{F}\left\{D_0^{\alpha} f\right\}\left(k\right) := \mathcal{F}\left\{-I_0^{-\alpha} f\right\}\left(k\right) = -\left|k\right|^{\alpha} \hat{f}\left(k\right)$$

generalized by Feller

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  generalized by Feller
- D_θ^α =Riesz-Feller fractional derivative of order α and skewness θ

$$\mathcal{F}\left\{D_0^\alpha f\right\}(k) := -\psi_\alpha^\theta(k)\hat{f}(k)$$

  with

$$\psi_\alpha^\theta(k) = |k|^\alpha e^{i(\operatorname{sign} k)\theta\pi/2}, \qquad 0 < \alpha \leq 2, |\theta| \leq \min\{\alpha, 2-\alpha\}$$

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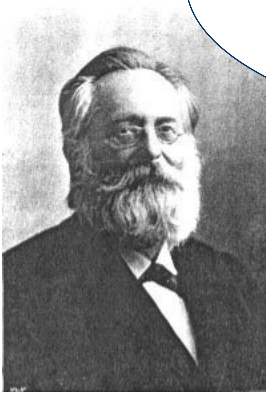
$$\psi_\alpha^\theta(k) = \left|k\right|^\alpha e^{i(\mathrm{sign} k)\theta\pi/2}, \qquad 0 < \alpha \leq 2, \left|\theta\right| \leq \min\left\{\alpha, 2-\alpha\right\}$$

- The symbol $-\psi_\alpha^\theta(k)$ is the logarithm of the characteristic function of a Lévy stable probability distribution with index of stability α and asymmetry parameter θ

Grünwald–Letnikov definition

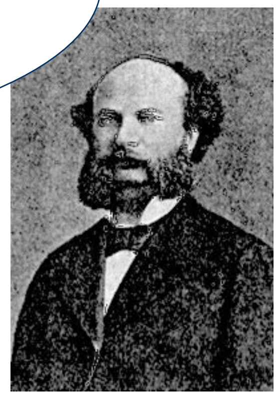
$$ {}_aD_t^{\alpha} f(t) = \lim_{h \to 0} h^{-\alpha} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j \binom{\alpha}{j} f(t - jh) $$

$[x]$ – integer part of x



A.K. Grünwald

A.V. Letnikov



Grünwal - Letnikov

- From

$$
\begin{aligned}
D\phi(x) &= \lim_{h\to 0} \frac{\phi(x)-\phi(x-h)}{h} \\
&\cdots \\
D^n &= \lim_{h\to 0} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \phi(x-kh)
\end{aligned}
$$

Grünwal - Letnikov

- From

$$D\phi(x) = \lim_{h\to 0} \frac{\phi(x) - \phi(x-h)}{h}$$

$$\cdots$$

$$D^n = \lim_{h\to 0} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \phi(x - kh)$$

- the Grünwal-Letnikov fractional derivatives are

$$_{GL}D^{\alpha}_{a+} = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{k=0}^{[(x-a)/h]} (-1)^k \binom{\alpha}{k} \phi(x - kh)$$

$$_{GL}D^{\alpha}_{b-} = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{k=0}^{[(b-x)/h]} (-1)^k \binom{\alpha}{k} \phi(x + kh)$$

$[\bullet]$ denotes the integer part

Integral equations

- **Abel's equation (1st kind)**

$$\frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(\tau)}{(t-\tau)^{1-\alpha}} \, d\tau = f(t), \qquad 0 < \alpha < 1$$

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- The mechanical problem of the *tautochrone*, that is, determining a curve in the vertical plane, such that the time required for a particle to slide down the curve to its lowest point is independent of its initial placement on the curve.

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- Found many applications in diverse fields:
  - Evaluation of spectroscopic measurements of cylindrical gas discharges
  - Study of the solar or a planetary atmosphere
  - Star densities in a globular cluster
  - Inversion of travel times of seismic waves for determination of terrestrial sub-surface structure
  - Inverse boundary value problems in partial differential equations

Abel's equation

- - Heating (or cooling) of a semi-infinite rod by influx (or efflux) of heat across the boundary into (or from) its interior

$$u_t - u_{xx} = 0\,, \qquad u = u(x,t)$$

in the semi-infinite intervals $0 < x < \infty$ and $0 < t < \infty$. Assume initial temperature, $u(x,0) = 0$ for $0 < x < \infty$ and given influx across the boundary $x = 0$ from $x < 0$ to $x > 0$,

$$-u_x(0,t) = p(t)$$

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- Then,

  $$u(x,t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{p(\tau)}{\sqrt{t-\tau}} e^{-x^2/[4(t-\tau)]} d\tau\,, \qquad x > 0\,,\ t > 0$$

Abel's equation (1st kind)

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- Is
$$J^\alpha\, u(t) = f(t)$$
and consequently is solved by
$$u(t) = D^\alpha\, f(t)$$
using $D^\alpha\, J^\alpha = I$. Let us now solve using the Laplace transform
$$\frac{\widetilde{u}(s)}{s^\alpha} = \widetilde{f}(s) \implies \widetilde{u}(s) = s^\alpha\, \widetilde{f}(s)$$

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$$\frac{\widetilde{u}(s)}{s^\alpha} = \widetilde{f}(s) \implies \widetilde{u}(s) = s^\alpha\, \widetilde{f}(s)$$
- The solution is obtained by the inverse Laplace transform: Two possibilities :

Abel's equation (1st kind)

- 1)

$$\widetilde{u}(s) = s\left(\frac{\widetilde{f}(s)}{s^{1-\alpha}}\right)$$

$$u(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha}}\,d\tau$$

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- 2)

$$\tilde{u}(s) = \frac{1}{s^{1-\alpha}}\left[s\tilde{f}(s) - f(0^+)\right] + \frac{f(0^+)}{s^{1-\alpha}}$$

$$u(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha}\,d\tau + f(0^+)\,\frac{t^{-\alpha}}{\Gamma(1-a)}$$

Solutions expressed in terms of the fractional derivatives D^α and D_*^α, respectively

Abel's equation (2nd kind)

- $$u(t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t \frac{u(\tau)}{(t-\tau)^{1-\alpha}}\, d\tau = f(t)\,, \qquad \alpha > 0\,, \lambda \in \mathbb{C}$$

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- In terms of the fractional integral operator

  $$(1 + \lambda\, J^\alpha)\, u(t) = f(t)$$

  solved as

  $$u(t) = (1 + \lambda J^\alpha)^{-1}\; f(t) = \left(1 + \sum_{n=1}^{\infty} (-\lambda)^n\, J^{\alpha n}\right)\, f(t)$$

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- Noting that
$$J^{\alpha n}\, f(t) = \Phi_{\alpha n}(t) \,*\, f(t) = \frac{t_+^{\alpha n - 1}}{\Gamma(\alpha n)} * f(t)$$
$$u(t) = f(t) + \left(\sum_{n=1}^{\infty} (-\lambda)^n \frac{t_+^{\alpha n - 1}}{\Gamma(\alpha n)}\right) * f(t)$$

Abel's equation (2nd kind)

- Relation to the Mittag-Leffler functions

$$e_\alpha(t;\lambda) := E_\alpha(-\lambda\, t^\alpha) = \sum_{n=0}^{\infty} \frac{(-\lambda\, t^\alpha)^n}{\Gamma(\alpha n + 1)}\,, \qquad t > 0\,, \alpha > 0\,, \lambda \in \mathbb{C}$$

$$\sum_{n=1}^{\infty} (-\lambda)^n \frac{t_+^{\alpha n - 1}}{\Gamma(\alpha n)} = \frac{d}{dt} E_\alpha(-\lambda t^\alpha) = e'_\alpha(t;\lambda)\,, \qquad t > 0$$

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- Finally,

$$u(t) = f(t) + e'_\alpha(t;\lambda)$$

- **Relaxation and oscillation equations. Integer order**

$$u'(t) = -u(t) + q(t)$$

the solution, under the initial condition $u(0^+) = c_0$, is

$$u(t) = c_0 e^{-t} + \int_0^t q(t-\tau) e^{-\tau} d\tau$$

Fractional differential equations

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- For the *oscillation* differential equation

$$u''(t) = -u(t) + q(t)$$

the solution, under the initial conditions $u(0^+) = c_0$ and $u'(0^+) = c_1$, is

$$u(t) = c_0 \cos t + c_1 \sin t + \int_0^t q(t-\tau) \sin \tau \, d\tau$$

Fractional version

$$D_*^\alpha \, u(t) = D^\alpha \left((u(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} \, u^{(k)}(0^+) \right) = -u(t) + q(t) \,, \qquad t > 0$$

$m - 1 < \alpha \le m$, initial values $u^{(k)}(0^+) = c_k \,, \; k = 0, \ldots, m-1$ .When α is the integer m

$$u(t) = \sum_{k=0}^{m-1} c_k \, u_k(t) + \int_0^t q(t-\tau) \, u_\delta(\tau) \, d\tau$$

$u_k(t) = J^k \, u_0(t) \,, \; u_k^{(h)}(0^+) = \delta_{k\,h} \,, h, k = 0, \cdots, m-1, \; u_\delta(t) = - \, u_0'(t)$

The $u_k(t)$'s are the *fundamental solutions,* linearly independent solutions of the *homogeneous* equation satisfying the initial conditions. The function $u_\delta(t)$, which is convoluted with $q(t)$, is the *impulse-response solution* of the *inhomogeneous* equation with $c_k \equiv 0 \,, \; k = 0, \ldots, m-1$, $q(t) = \delta(t)$ .For ordinary relaxation and oscillation, $u_0(t) = e^{-t} = u_\delta(t)$ and $u_0(t) = \cos\, t \,, \; u_1(t) = J \, u_0(t) = \sin\, t = \cos(t - \pi/2) = u_\delta(t)$.

- **Solution of the fractional equation by Laplace transform**

  Applying the operator J^α to the fractional equation

$$u(t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} - J^\alpha\, u(t) + J^\alpha\, q(t)$$

  Laplace transforming yields

$$\widetilde{u}(s) = \sum_{k=0}^{m-1} \frac{1}{s^{k+1}} - \frac{1}{s^\alpha}\widetilde{u}(s) + \frac{1}{s^\alpha}\widetilde{q}(s)$$

  hence

$$\widetilde{u}(s) = \sum_{k=0}^{m-1} \frac{s^{\alpha-k-1}}{s^\alpha + 1} + \widetilde{q}(s)$$

Relaxation and oscillation equations

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- Introducing the Mittag-Leffler type functions

$$e_\alpha(t) \equiv e_\alpha(t; 1) := E_\alpha(-t^\alpha) \div \frac{s^{\alpha-1}}{s^\alpha + 1}$$

Relaxation and oscillation equations

- $$u_k(t) := J^k e_\alpha(t) \div \frac{s^{\alpha-k-1}}{s^\alpha+1}\,, \qquad k = 0, 1, \ldots, m-1$$

  we find

$$u(t) = \sum_{k=0}^{m-1} u_k(t) - \int_0^t q(t-\tau)\, u_0'(\tau)\, d\tau$$

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- When α is not integer, $m-1$ represents the integer part of α ($[\alpha]$) and m the number of initial conditions necessary and sufficient to ensure the uniqueness of the solution $u(t)$. The m functions $u_k(t) = J^k e_\alpha(t)$ with $k = 0,1,\dots,m-1$ represent those particular solutions of the *homogeneous* equation which satisfy the initial conditions
  $$u_k^{(h)}(0^+) = \delta_{k\,h}\,,\, h,k = 0,1,\dots,m-1$$
  and therefore they represent the *fundamental solutions* of the fractional equation Furthermore, the function $u_\delta(t) = -e_\alpha'(t)$ represents the *impulse-response solution*.

Fractional diffusion equation

- Fractional diffusion equation, obtained from the standard diffusion equation by replacing the second-order space derivative with a Riesz-Feller derivative of order $\alpha \in (0, 2]$ and skewness θ and the first-order time derivative with a Caputo derivative of order $\beta \in (0, 2]$

$$
{}_x D_\theta^\alpha \, u(x,t) \;=\; {}_t D_*^\beta \, u(x,t) \,, \qquad x \in \mathbb{R} \,, t \in \mathbb{R}^+
$$

$$
0 < \alpha \le 2 \,, |\theta| \le \min\{\alpha, 2-\alpha\} \,, \quad 0 < \beta \le 2
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- Space-fractional diffusion $\{0 < \alpha \leq 2\,,\, \beta = 1\}$
  Time-fractional diffusion $\{\alpha = 2\,,\, 0 < \beta \leq 2\}$
  Neutral-fractional diffusion $\{0 < \alpha = \beta \leq 2\}$

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  Time-fractional diffusion $\{\alpha = 2\,,\, 0 < \beta \le 2\}$
  Neutral-fractional diffusion $\{0 < \alpha = \beta \le 2\}$
- *Riesz-Feller space-fractional derivative*

$$ \mathcal{F}\left\{ {}_x D_\theta^\alpha \, f(x); \kappa \right\} = -\psi_\alpha^\theta(\kappa)\, \widehat{f}(\kappa) $$

$$ \psi_\alpha^\theta(\kappa) = |\kappa|^\alpha \, e^{i(sign k)\theta\pi/2}\,, \qquad 0 < \alpha \le 2\,, |\theta| \le \min\{\alpha, 2-\alpha\} $$

Fractional diffusion equation

- *Caputo time-fractional derivative*

$$D_*^\alpha f(t) := \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, & m-1 < \alpha < m \\ \frac{d^m}{dt^m} f(t), & \alpha = m \end{cases}$$

Fractional diffusion equation

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- Cauchy problem

$$u(x,0) = \varphi(x), \qquad x \in \mathbb{R}, u(\pm\infty, t) = 0, t > 0$$

$$u_{\alpha,\beta}^\theta(x,t) = \int_{-\infty}^{+\infty} G_{\alpha,\beta}^\theta(\xi, t)\, \varphi(x-\xi)\, d\xi$$

$$G_{\alpha,\beta}^\theta(ax, bt) = b^{-\gamma} G_{\alpha,\beta}^\theta(ax/b^\gamma, t), \qquad \gamma = \beta/\alpha$$

Fractional diffusion equation

- *Caputo time-fractional derivative*

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- *Similarity variable* x/t^{γ}

$$G_{\alpha,\beta}^{\theta}(x,t) = t^{-\gamma} K_{\alpha,\beta}^{\theta}(x/t^{\gamma})\,, \qquad \gamma = \beta/\alpha$$

Fractional diffusion equation

- Solution by Fourier transform for the space variable and the Laplace transform for the time variable

$$-\psi_{\alpha}^{\theta}(\kappa)\,\widehat{\widetilde{G_{\alpha,\beta}^{\theta}}} = s^{\beta}\,\widehat{\widetilde{G_{\alpha,\beta}^{\theta}}} - s^{\beta-1}$$

$$\widehat{\widetilde{G_{\alpha,\beta}^{\theta}}} = \frac{s^{\beta-1}}{s^{\beta} + \psi_{\alpha}^{\theta}(\kappa)}$$

Fractional diffusion equation

- Solution by Fourier transform for the space variable and the Laplace transform for the time variable

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$$\widetilde{\widehat{G^\theta_{\alpha,\beta}}} = \frac{s^{\beta-1}}{s^\beta + \psi_\alpha^\theta(\kappa)}$$

- Inverse Laplace transform

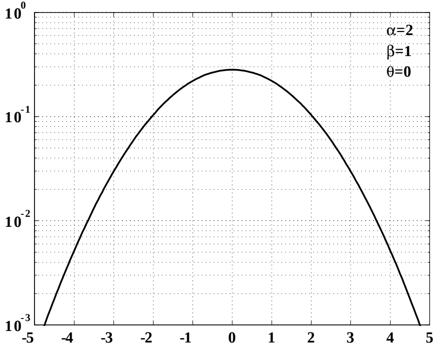
$$\widehat{G^\theta_{\alpha,\beta}}\,(k,t) = E_\beta\left[-\psi_\alpha^\theta(\kappa)\,t^\beta\right]\,, \qquad E_\beta(z) := \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\beta\,n+1)}$$

$$G^\theta_{\alpha,\beta}(x\,,\;t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ikx}\,E_\beta\left[-\psi_\alpha^\theta(\kappa)\,t^\beta\right]\,dk$$

Particular cases

$\{\alpha = 2\,,\ \beta = 1\}$ (*Standard diffusion*)

$$G^0_{2,1}(x,t) = t^{-1/2}\,\frac{1}{2\sqrt{\pi}}\,\exp[-x^2/(4t)]$$



Fractional diffusion equation

$\{0 < \alpha \le 2\,,\ \beta = 1\}$ (*Space fractional diffusion*)

The Mittag-Leffler function reduces to the exponential function and we obtain a characteristic function of the class $\{L_\alpha^\theta(x)\}$ of Lévy strictly stable densities

$$\widehat{L_\alpha^\theta}(\kappa) = e^{-\psi_\alpha^\theta(\kappa)}\,, \qquad \widehat{G_{\alpha,1}^\theta}(\kappa, t) = e^{-t\psi_\alpha^\theta(\kappa)}$$

The Green function of the space-fractional diffusion equation can be interpreted as a Lévy strictly stable *pdf*, evolving in time

$$G_{\alpha,1}^\theta(x,t) = t^{-1/\alpha}\, L_\alpha^\theta(x/t^{1/\alpha})\,, \qquad -\infty < x < +\infty\,, t \ge 0$$

Particular cases:

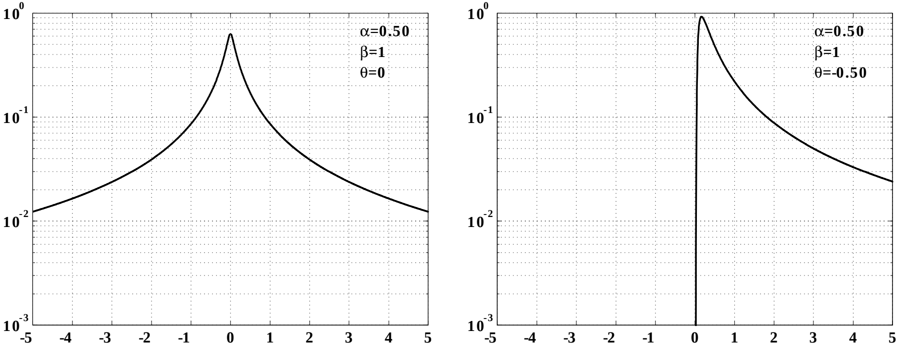
$\alpha = 1/2\,, \theta = -1/2\,,$ Lévy-Smirnov

$$e^{-s^{1/2}} \overset{\mathcal{L}}{\leftrightarrow} L_{1/2}^{-1/2}(x) = \frac{x^{-3/2}}{2\sqrt{\pi}}\, e^{-1/(4x)}\,, \qquad x \ge 0$$

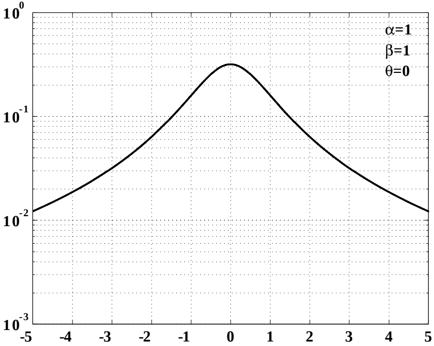
$\alpha = 1\,, \theta = 0\,,$ Cauchy

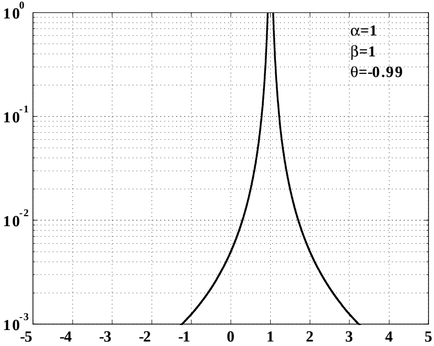
$$e^{-|\kappa|} \overset{\mathcal{F}}{\leftrightarrow} L_1^0(x) = \frac{1}{\pi}\frac{1}{x^2+1}\,, \qquad -\infty < x < +\infty$$

Fractional diffusion equation

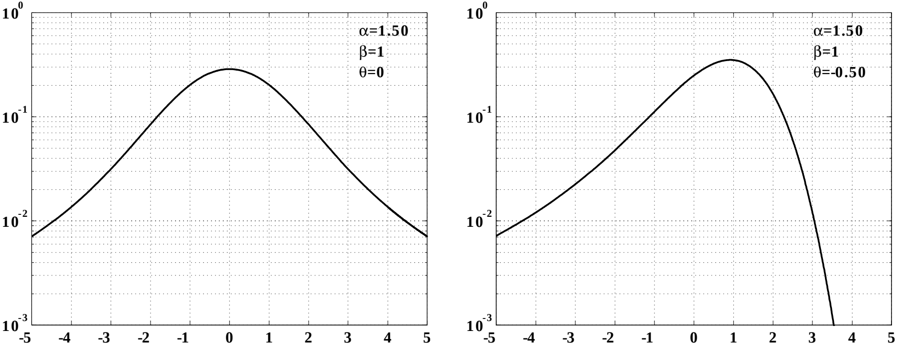


Fractional diffusion equation





Fractional diffusion equation



Fractional diffusion equation

$\{\alpha = 2\,,\, 0 < \beta < 2\}$ (*Time-fractional diffusion*)

$$\widehat{G^0_{2,\beta}}(\kappa, t) = E_\beta\left(-\kappa^2\, t^\beta\right)\,, \qquad \kappa \in \mathbb{R}\,,\, t \geq 0$$

or with the equivalent Laplace transform

$$\widetilde{G^0_{2,\beta}}(x, s) = \frac{1}{2}\, s^{\beta/2-1}\, e^{-|x|s^{\beta/2}}\,, \qquad -\infty < x < +\infty\,, \Re(s) > 0$$

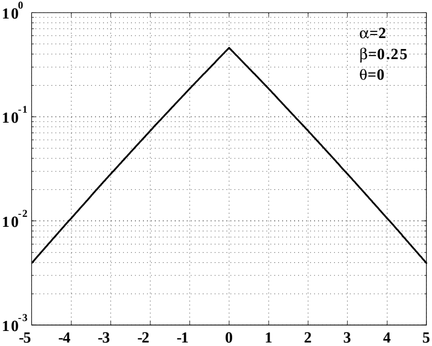
with solution

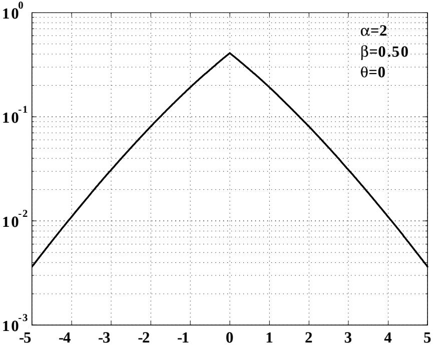
$$G^0_{2,\beta}(x, t) = \frac{1}{2} t^{-\beta/2}\, M_{\beta/2}\left(|x|/\, t^{\beta/2}\right), \qquad -\infty < x < +\infty\,,\, t \geq 0$$

$M_{\beta/2}$ is a function of Wright type of order $\beta/2$ defined for any order $\nu \in (0, 1)$ by

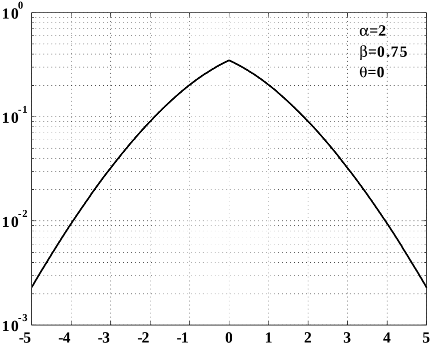
$$M_\nu(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\,\Gamma[-\nu n + (1-\nu)]} = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{(n-1)!}\, \Gamma(\nu n)\, \sin(\pi \nu n)$$

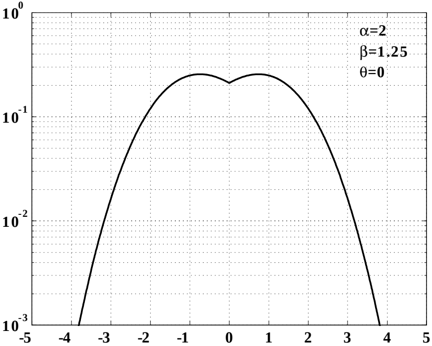
Fractional diffusion equation



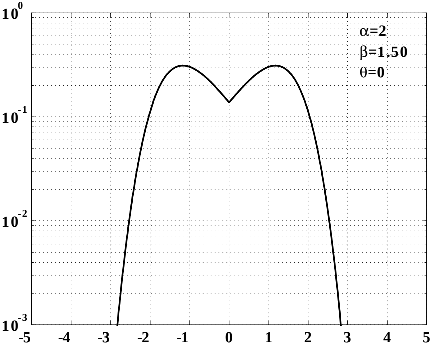


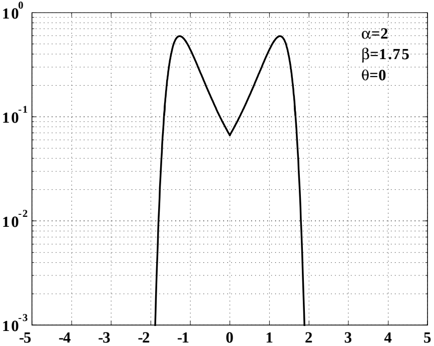
Fractional diffusion equation



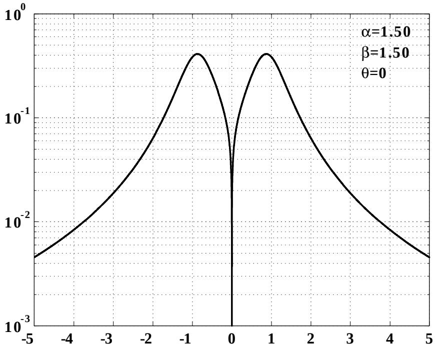


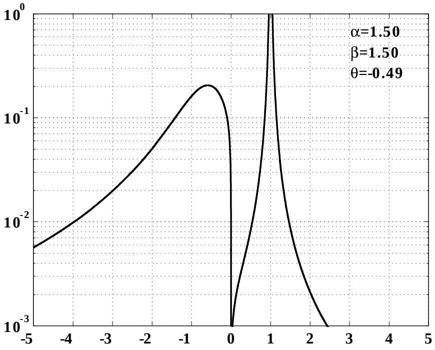
Fractional diffusion equation



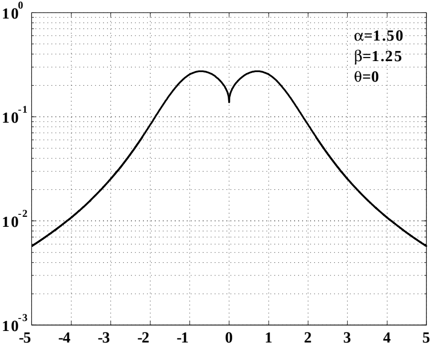


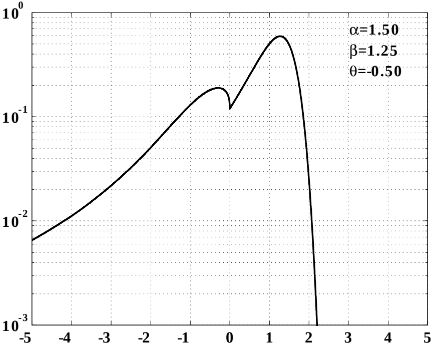
Space-time fractional diffusion equation. Some examples





Fractional diffusion equation





A fractional nonlinear equation. Stochastic solution

A fractional version of the KPP equation, studied by McKean

- $$\boxed{{}_tD_*^{\alpha} u(t,x) = \frac{1}{2}\, {}_xD_{\theta}^{\beta} u(t,x) + u^2(t,x) - u(t,x)}$$

  ${}_tD_*^{\alpha}$ is a Caputo derivative of order α

  $$ {}_tD_*^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(m-\beta)} \int_0^t \frac{f^{(m)}(\tau)\,d\tau}{(t-\tau)^{\alpha+1-m}} & m-1<\alpha<m \\ \frac{d^m}{dt^m} f(t) & \alpha = m \end{cases} $$

  ${}_xD_{\theta}^{\beta}$ is a Riesz-Feller derivative defined through its Fourier symbol

  $$\mathcal{F}\left\{ {}_xD_{\theta}^{\beta} f(x) \right\}(k) = -\psi_{\beta}^{\theta}(k)\, \mathcal{F}\{f(x)\}(k)$$

  with $\psi_{\beta}^{\theta}(k) = |k|^{\beta} e^{i(\operatorname{sign} k)\theta\pi/2}$.

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- Physically it describes a nonlinear diffusion with growing mass and in our fractional generalization it would represent the same phenomenon taking into account memory effects in time and long range correlations in space.

A fractional nonlinear equation

The first step towards a probabilistic formulation is the rewriting as an integral equation. Take the Fourier transform $(\mathcal{F})$ in space and the Laplace transform $(\mathcal{L})$ in time

$$s^{\alpha}\tilde{\hat{u}}(s,k) = s^{\alpha-1}\hat{u}\left(0^{+},k\right) - \frac{1}{2}\psi_{\beta}^{\theta}(k)\,\tilde{\hat{u}}(s,k) - \tilde{\hat{u}}(s,k) + \int_{0}^{\infty} dt e^{-st}\mathcal{F}\left(u^{2}\right)$$

where

$$\hat{u}(t,k) = \mathcal{F}(u(t,x)) = \int_{-\infty}^{\infty} e^{ikx}u(t,x)$$

$$\tilde{u}(s,x) = \mathcal{L}(u(t,x)) = \int_{0}^{\infty} e^{-st}u(t,x)$$

This equation holds for $0<\alpha\leq 1$ or for $0<\alpha\leq 2$ with $\frac{\partial}{dt}u\left(0^{+},x\right)=0$.

Solving for $\tilde{\hat{u}}(s,k)$ one obtains an integral equation

$$\tilde{\hat{u}}(s,k) = \frac{s^{\alpha-1}}{s^{\alpha}+\frac{1}{2}\psi_{\beta}^{\theta}(k)}\hat{u}\left(0^{+},k\right) + \int_{0}^{\infty} dt\frac{e^{-st}}{s^{\alpha}+\frac{1}{2}\psi_{\beta}^{\theta}(k)}\mathcal{F}\left(u^{2}(t,x)\right)$$

A fractional nonlinear equation

Taking the inverse Fourier and Laplace transforms

$$
\begin{aligned}
& u(t,x) \\
= \ & \boxed{E_{\alpha,1}\left(-t^{\alpha}\right)} \int_{-\infty}^{\infty} dy \mathcal{F}^{-1}\left(\frac{E_{\alpha,1}\left(-\left(1+\frac{1}{2}\psi_{\beta}^{\theta}(k)\right)t^{\alpha}\right)}{E_{\alpha,1}\left(-t^{\alpha}\right)}\right)(x-y)\, u(0,y) \\
& + \int_{0}^{t} d\tau \boxed{(t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-(t-\tau)^{\alpha}\right)} \\
& \int_{-\infty}^{\infty} dy \mathcal{F}^{-1}\left(\frac{E_{\alpha,\alpha}\left(-\left(1+\frac{1}{2}\psi_{\beta}^{\theta}(k)\right)(t-\tau)^{\alpha}\right)}{E_{\alpha,\alpha}\left(-(t-\tau)^{\alpha}\right)}\right)(x-y)\, u^{2}(\tau,y)
\end{aligned}
$$

$E_{\alpha,\rho}$ is the generalized Mittag-Leffler function $E_{\alpha,\rho}(z)=\sum_{j=0}^{\infty}\frac{z^{j}}{\Gamma(\alpha j+\rho)}$

$$E_{\alpha,1}\left(-t^{\alpha}\right)+\int_{0}^{t} d\tau\,(t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-(t-\tau)^{\alpha}\right)=1$$

A fractional nonlinear equation

We define the following propagation kernel

$$G^{\beta}_{\alpha,\rho}\left(t,x\right)=\mathcal{F}^{-1}\left(\frac{E_{\alpha,\rho}\left(-\left(1+\frac{1}{2}\psi^{\theta}_{\beta}\left(k\right)\right)t^{\alpha}\right)}{E_{\alpha,\rho}\left(-t^{\alpha}\right)}\right)\left(x\right)$$

$$\begin{aligned} & u\left(t,x\right) \\ = \quad & \boxed{E_{\alpha,1}\left(-t^{\alpha}\right)}\int_{-\infty}^{\infty}dy\boxed{G^{\beta}_{\alpha,1}\left(t,x-y\right)}u\left(0^{+},y\right) \\ & +\int_{0}^{t}d\tau\boxed{\left(t-\tau\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\left(t-\tau\right)^{\alpha}\right)} \\ & \int_{-\infty}^{\infty}dy\boxed{G^{\beta}_{\alpha,\alpha}\left(t-\tau,x-y\right)}u^{\boxed{2}}\left(\tau,y\right) \end{aligned}$$

$E_{\alpha,1}\left(-t^{\alpha}\right)$ and $\left(t-\tau\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\left(t-\tau\right)^{\alpha}\right)$ = survival probability up to time t and the probability density for the branching at time τ (branching process B_{α})

A fractional nonlinear equation

The propagation kernels satisfy the conditions to be the Green's functions of stochastic processes in $\mathbb{R}$:

$$u(t,x) = \mathbb{E}_x \left(u(0^+, x+\xi_1) u(0^+, x+\xi_2) \cdots u(0^+, x+\xi_n) \right)$$

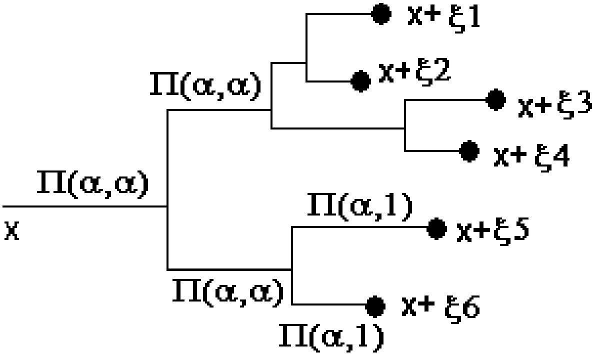
Denote the processes associated to $G^{\beta}_{\alpha,1}(t,x)$ and $G^{\beta}_{\alpha,\alpha}(t,x)$, respectively by $\Pi^{\beta}_{\alpha,1}$ and $\Pi^{\beta}_{\alpha,\alpha}$

Theorem: *The nonlinear fractional partial differential equation, with* $0 < \alpha \leq 1$, *has a stochastic solution, the coordinates* $x+\xi_i$ *in the arguments of the initial condition obtained from the exit values of a propagation and branching process, the branching being ruled by the process* B_α *and the propagation by* $\Pi^{\beta}_{\alpha,1}$ *for the first particle and by* $\Pi^{\beta}_{\alpha,\alpha}$ *for all the remaining ones.*

A sufficient condition for the existence of the solution is

$$\left| u(0^+, x) \right| \leq 1$$

A fractional nonlinear equation



Geometric interpretation of fractional integration: shadows on the walls

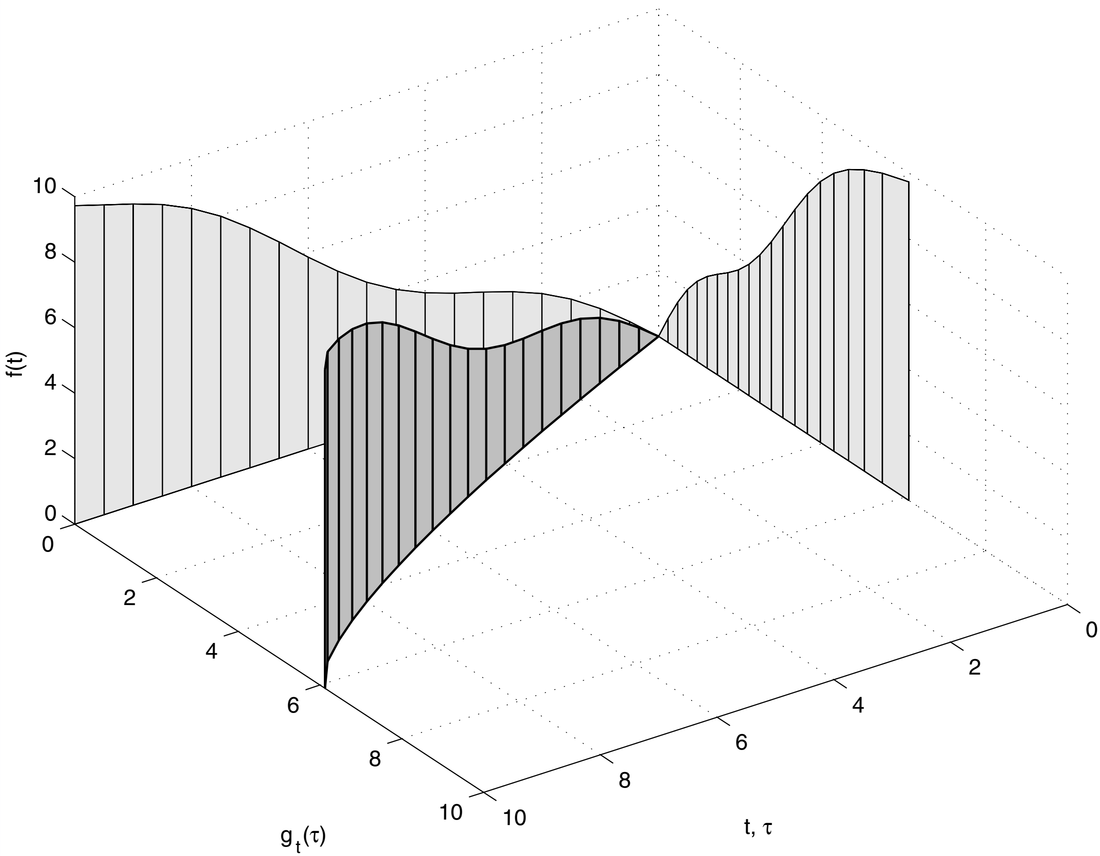
$$ {}_0I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t f(\tau)(t-\tau)^{\alpha-1}\, d\tau, \quad t \geq 0, $$

$$ {}_0I_t^{\alpha} f(t) = \int\limits_0^t f(\tau)\, dg_t(\tau), $$

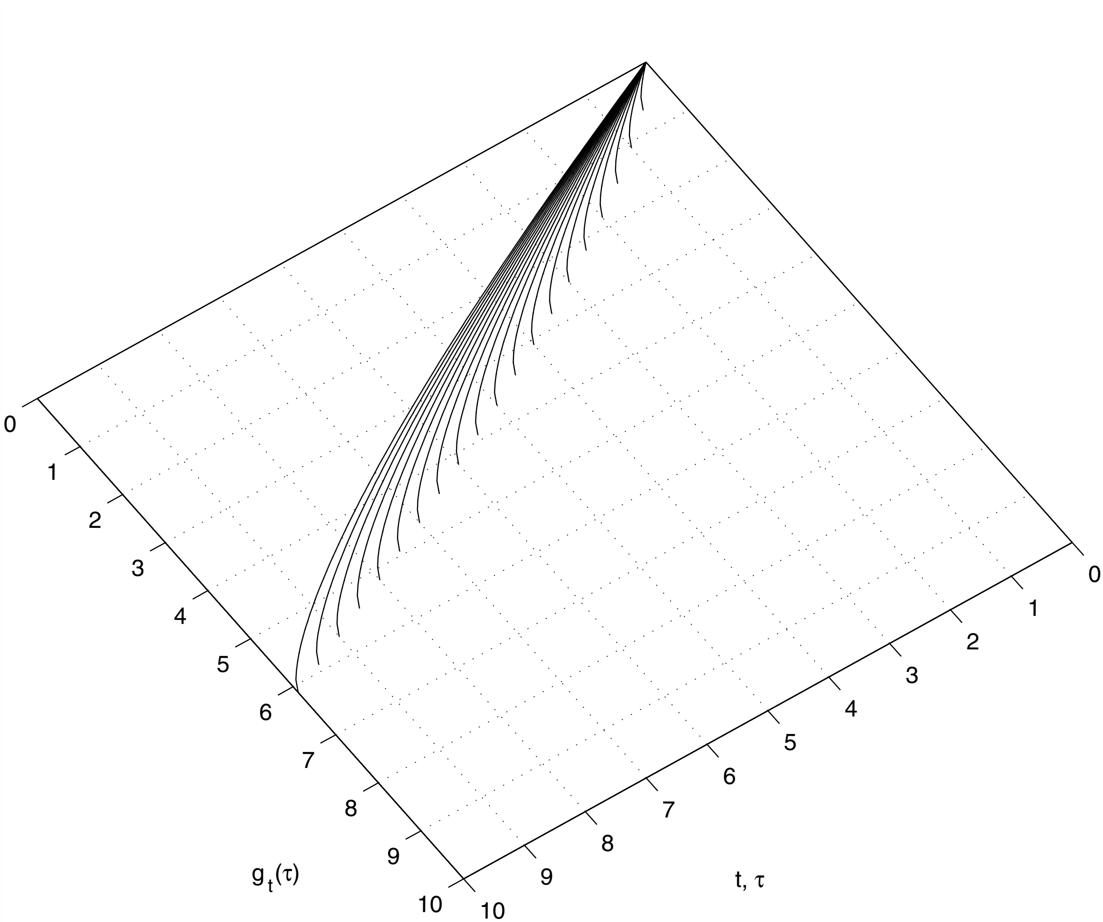
$$ g_t(\tau) = \frac{1}{\Gamma(\alpha+1)} \{t^{\alpha} - (t-\tau)^{\alpha}\}. $$

For $t_1 = kt$, $\tau_1 = k\tau$ $(k > 0)$ we have:

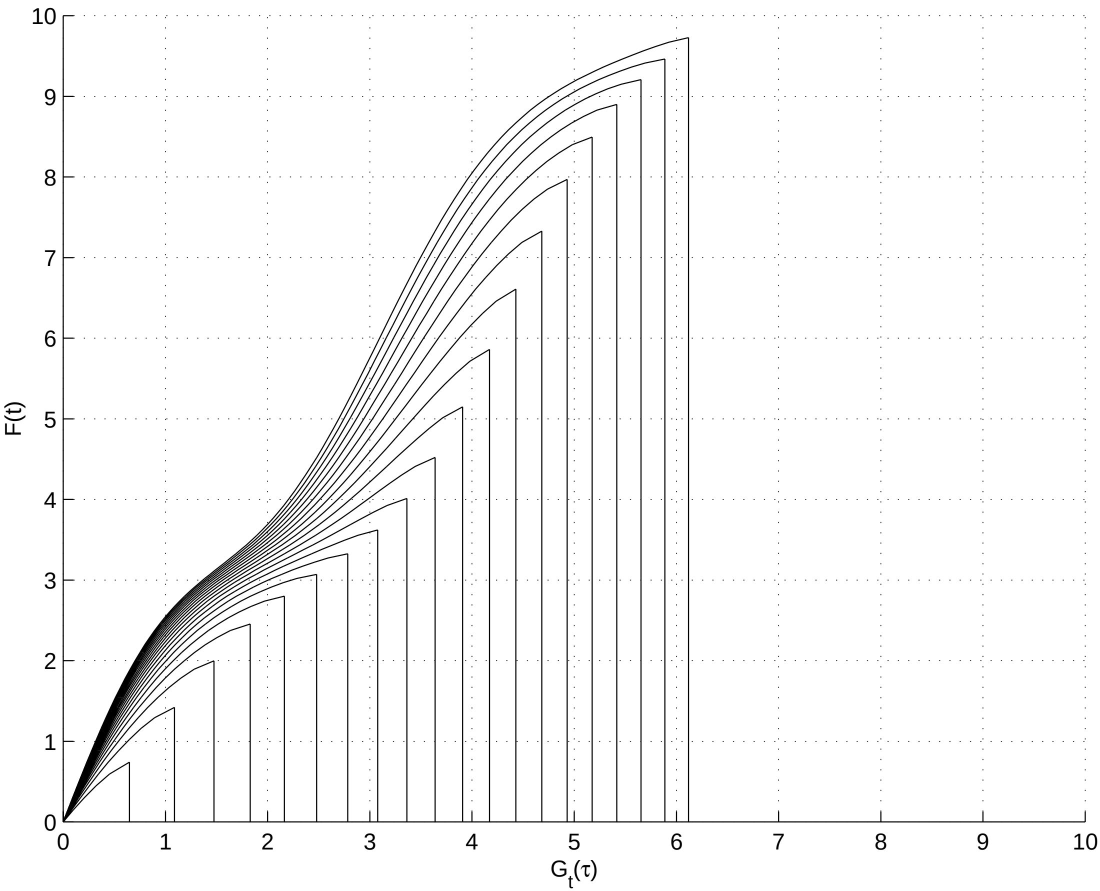
$$ g_{t_1}(\tau_1) = g_{kt}(k\tau) = k^{\alpha} g_t(\tau). $$



"Live fence" and its shadows: ${}_0I_t^1 f(t)$ a ${}_0I_t^\alpha f(t)$,
for $\alpha = 0.75$, $f(t) = t + 0.5\sin(t)$, $0 \le t \le 10$.



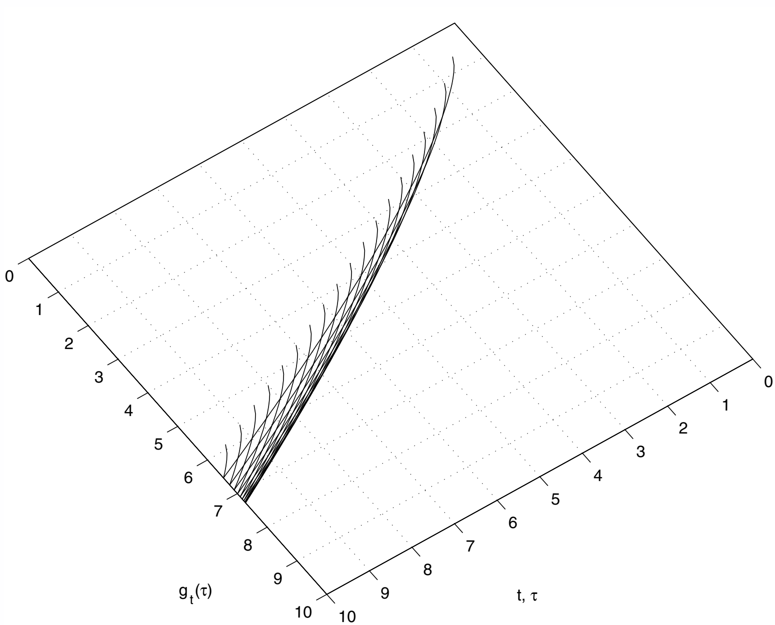
"Live fence": basis shape is changing for ${}_0I_t^\alpha f(t)$, $\alpha = 0.75$, $0 \le t \le 10$.



Snapshots of the changing "shadow" of changing "fence" for ${}_0I_t^\alpha f(t)$, $\alpha = 0.75$, $f(t) = t + 0.5\sin(t)$, with the time interval $\Delta t = 0.5$ between the snapshops.

Right-sided R-L integral

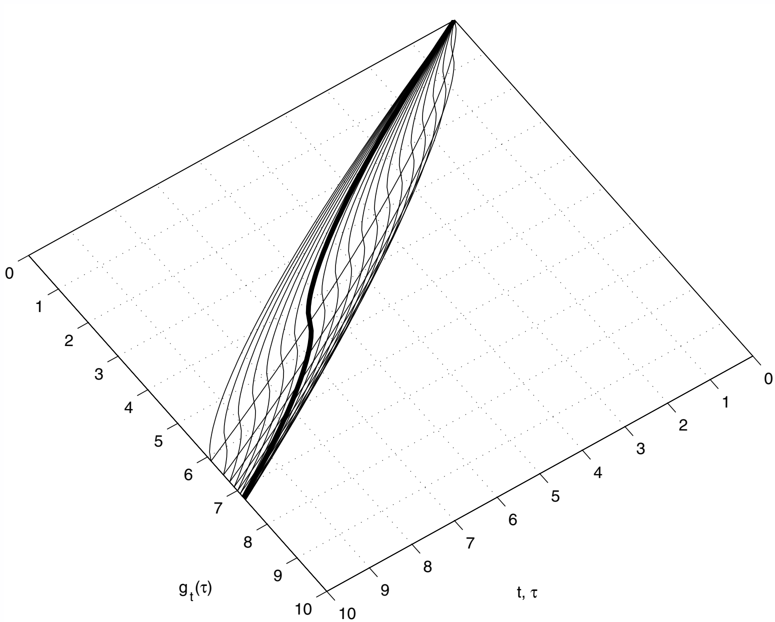
$$
{}_tI_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b f(\tau)(\tau - t)^{\alpha-1}\, d\tau, \quad t \le b,
$$



${}_tI_{10}^\alpha f(t)$, $\alpha = 0.75$, $0 \le t \le 10$

Riesz potential

$$ {}_0R_b^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^b f(\tau)|\tau - t|^{\alpha-1}\, d\tau, \quad 0 \le t \le b, $$



${}_0R_{10}^{\alpha} f(t)$, $\alpha = 0.75$, $0 \le t \le 10$

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