Functional Analysis

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Hyperlinked and revised edition

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Introduction

These notes are an expanded version of a set written for a course given to final-year undergraduates at the University of Oxford.

A thorough understanding of the Oxford third-year b4 analysis course (an introduction to Banach and Hilbert spaces) or its equivalent is a prerequisite for this material. We use [24] as a compendium of results from that series of lectures. (Numbers in square brackets refer to items in the bibliography.)

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Convention

Throughout these notes we follow the Dirac-formalism convention that inner products on complex vector spaces are conjugate linear in the first argument and linear in the second, in contrast to many Oxford courses.

Spaces

One — Normed Spaces

Throughout, the scalar field of a vector space will be denoted by $\mathbb{F}$ and will be either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$.

Basic Definitions

Definition 1.1. A *norm* on a vector space X is a function

$$\|\cdot\|\colon X \to \mathbb{R}^+ := [0,\infty);\ x \mapsto \|x\|$$

that satisfies, for all x, $y \in X$ and $\alpha \in \mathbb{F}$,

$$\begin{array}{lll} & \text{(i)} & \|x\| = 0 \text{ if and only if } x = 0 & (\textit{faithfulness}), \\ & \text{(ii)} & \|\alpha x\| = |\alpha|\,\|x\| & (\textit{homogeneity}) \\ \text{and} & \text{(iii)} & \|x+y\| \leqslant \|x\| + \|y\| & (\textit{subadditivity}). \end{array}$$

A *seminorm* on X is a function $p\colon X \to \mathbb{R}^+$ that satisfies (ii) and (iii) above.

Definition 1.2. A *normed vector space* is a vector space X with a norm $\|\cdot\|$; if necessary we will denote the norm on the space X by $\|\cdot\|_X$. We will sometimes use the term *normed space* as an abbreviation.

Definition 1.3. A *Banach space* is a normed vector space $(E, \|\cdot\|)$ that is complete, i.e., every Cauchy sequence in E is convergent, where E is equipped with the metric $d(x,y) := \|x-y\|$.

Definition 1.4. Let $(x_n)_{n\geqslant 1}$ be a sequence in the normed vector space X. The series $\sum_{n=1}^{\infty} x_n$ is *convergent* if there exists $x \in X$ such that $\left(\sum_{k=1}^{n} x_k\right)_{n\geqslant 1}$ is convergent to x, and the series is said to have *sum* x. The series is *absolutely convergent* if $\sum_{n=1}^{\infty} \|x_n\|$ is convergent.

Theorem 1.5. (Banach's Criterion) A normed vector space X is complete if and only if every absolutely convergent series in X is convergent.

Proof

This is a b4 result: see [24, Theorem 1.2.9]. $\square$

Subspaces and Quotient Spaces

Definition 1.6. A *subspace* of a vector space X is a subset $M \subseteq X$ that is closed under vector addition and scalar multiplication: $M + M \subseteq M$ and $\alpha M \subseteq M$ for all $\alpha \in \mathbb{F}$, where

$$A + B := \{a + b : a \in A,\ b \in B\} \text{ and } \alpha A := \{\alpha a : a \in A\} \qquad \forall\, A, B \subseteq X,\ \alpha \in \mathbb{F}.$$

Example 1.7. Let $(X, \mathcal{T})$ be a topological space and let $(E, \|\cdot\|_E)$ be a Banach space over $\mathbb{F}$. The set of continuous, E-valued functions on X forms an vector space, denoted by $C(X, E)$, where the algebraic operations are defined pointwise: if $f, g \in C(X, E)$ and $\alpha \in \mathbb{F}$ then

$$(f + g)(x) := f(x) + g(x) \quad \text{and} \quad (\alpha f)(x) := \alpha f(x) \qquad \forall\, x \in X.$$

The subspace of bounded functions

$$C_{\mathrm{b}}(X, E) := \big\{f \in C(X, E) \bigm| \|f\|_\infty < \infty\big\},$$

where

$$\|\cdot\|_\infty \colon C_{\mathrm{b}}(X, E) \to \mathbb{R}^+;\ f \mapsto \sup\big\{\|f(x)\|_E : x \in X\big\},$$

is a Banach space, with *supremum norm* $\|\cdot\|_\infty$ (see Theorem 1.36). If X is compact then every continuous, E-valued function is bounded, hence $C(X, E) = C_{\mathrm{b}}(X, E)$ in this case. If $E = \mathbb{C}$ (the most common case of interest) we use the abbreviations $C(X)$ and $C_{\mathrm{b}}(X)$.

Proposition 1.8. A subspace of a Banach space is closed if and only if it is complete.

Proof

See [24, Theorem 1.2.10]. □

Definition 1.9. Given a vector space X with a subspace M, the *quotient space* X/M is the set

$$X/M := \big\{[x] := x + M \bigm| x \in X\big\}, \qquad \text{where} \quad x + M := \{x + m : m \in M\},$$

equipped with the vector-space operations

$$[x] + [y] := [x + y] \quad \text{and} \quad \alpha[x] := [\alpha x] \qquad \forall\, x, y \in X,\ \alpha \in \mathbb{F}.$$

(It is a standard result of linear algebra that this defines a vector space; for a full discussion see [7, Appendix A.4].) The dimension of X/M is the *codimension* of M (in X).

Theorem 1.10. Let X be a normed vector space with a subspace M and let

$$\big\|[x]\big\|_{X/M} := \inf\big\{\|x - m\| : m \in M\big\} \qquad \forall\, [x] \in X/M.$$

This defines a seminorm on X/M, which is a norm if and only if M is closed, called the *quotient seminorm* (or *quotient norm*) on X/M. If E is a Banach space and M is a closed subspace of E then $\big(E/M, \|\cdot\|_{E/M}\big)$ is a Banach space.

Proof

Clearly $\big\|[x]\big\|_{X/M} = 0$ if and only if $d(x, M) = 0$, which holds if and only if $x \in \bar{M}$. Hence $\|\cdot\|_{X/M}$ is faithful if and only if M is closed.

If $\alpha \in \mathbb{F}$ and $x \in X$ then

$$\begin{aligned}\big\|\alpha[x]\big\|_{X/M} = \big\|[\alpha x]\big\|_{X/M} &= \inf\big\{\|\alpha x - m\| : m \in M\big\} \\ &= \inf\big\{|\alpha|\,\|x - n\| : n \in M\big\} = |\alpha|\,\big\|[x]\big\|_{X/M},\end{aligned}$$

using the fact that $\alpha^{-1}M = M$ if $\alpha \neq 0$ (because M is a subspace).

For subadditivity, let $x,\ y \in X$ and note that

$$\big\|[x] + [y]\big\|_{X/M} = \big\|[x + y]\big\|_{X/M} \leqslant \|x + y - (m + n)\| \leqslant \|x - m\| + \|y - n\|$$

for all $m,\ n \in M$. Taking the infimum over such m and n gives the result.

We prove the final claim in Proposition 2.15 as a consequence of the open-mapping theorem; see also Exercise 1.2. □

Example 1.11. Let I be a subinterval of $\mathbb{R}$ and let $p \in [1, \infty)$. The vector space of Lebesgue-measurable functions on I that are *p-integrable* is denoted by $\mathcal{L}^p(I)$:

$$\mathcal{L}^p(I) := \big\{f \colon I \to \mathbb{C} \;\big|\; f \text{ is measurable and } \|f\|_p < \infty\big\},$$

with vector-space operations defined pointwise and

$$\|f\|_p := \Big(\int_I |f(x)|^p \,\mathrm{d}x\Big)^{1/p}.$$

(Note that

$$|f + g|^p \leqslant \big(|f| + |g|\big)^p \leqslant \big(2\max\{|f|, |g|\}\big)^p = 2^p \max\{|f|^p, |g|^p\} \leqslant 2^p\big(|f|^p + |g|^p\big),$$

so $\mathcal{L}^p(I)$ is closed under addition; it is simple to verify that $\mathcal{L}^p(I)$ is a vector space.) The map $f \mapsto \|f\|_p$ is a seminorm, but not a norm; the subadditivity of $\|\cdot\|_p$ is known as Minkowski's inequality (see [17, Theorem 28.19] for its proof).

If $\mathcal{N} := \{f \in \mathcal{L}^p(I) : \|f\|_p = 0\}$ then $L^p(I) := \mathcal{L}^p(I)/\mathcal{N}$ is a Banach space, with norm $[f] \mapsto \big\|[f]\big\|_p := \|f\|_p$. (A function lies in $\mathcal{N}$ if and only if it is zero almost everywhere.) As is usual practise in functional analysis, we shall frequently blur the distinction between f and $[f]$. (Discussion of $L^p(\mathbb{R})$ may be found in [17, Chapter 28] and [26, Chapter 7]; the generalisation from $\mathbb{R}$ to a subinterval I is trivial.)

Example 1.12. Let I be a subinterval of $\mathbb{R}$ and let $\mathcal{L}^\infty(I)$ denote the vector space of Lebesgue-measurable functions on I that are *essentially bounded*:

$$\mathcal{L}^\infty(I) := \big\{f \colon I \to \mathbb{C} \;\big|\; f \text{ is measurable and } \|f\|_\infty < \infty\big\},$$

with vector-space operations as usual and

$$\|f\|_\infty := \inf\{M : |f(x)| \leqslant M \text{ almost everywhere}\}.$$

(It is not difficult to show that $\|f\|_\infty = \sup\{|f(x)| : x \in I \setminus N\}$ for some null set N which may, of course, depend on f.)

As in the previous example, $f \mapsto \|f\|_\infty$ is a seminorm, $\mathcal{N} = \{f \in \mathcal{L}^\infty(I) : \|f\|_\infty = 0\}$ consists of those functions that are zero almost everywhere and $L^\infty(I) := \mathcal{L}^\infty/\mathcal{N}$ is a Banach space with respect to the norm $[f] \mapsto \big\|[f]\big\|_\infty := \|f\|_\infty$.

Although it may seem that we have two different meanings for $\|f\|_\infty$, the above and that in Example 1.7, they coincide if f is continuous.

Example 1.13. Let Ω be an open subset of the complex plane $\mathbb{C}$ and let

$$H_\mathrm{b}(\Omega) := \{f\colon \Omega \to \mathbb{C} \mid f \text{ is bounded and holomorphic in } \Omega\}.$$

Equipped with the supremum norm on Ω, $H_\mathrm{b}(\Omega)$ is a Banach space. (Completeness is most easily established *via* Morera's theorem [16, Theorem 5.6].)

Completions

Recall that a map $f\colon X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is an *isometry* if $d_Y\big(f(x_1), f(x_2)\big) = d_X(x_1, x_2)$ for all $x_1,\ x_2 \in X$, and an *isometric isomorphism* between normed vector spaces is an invertible linear isometry (the inverse of which is automatically linear and isometric).

Theorem 1.14. If X is a normed vector space then there exists a Banach space $\tilde{X}$ and a linear isometry $i\colon X \to \tilde{X}$ such that $i(X)$ is dense in $\tilde{X}$. The pair $\big(\tilde{X}, i\big)$ is a *completion* of X, and is unique in the following sense: if (Y, i) and (Z, j) are completions of X then there exists an isometric isomorphism $k\colon Y \to Z$ such that $k \circ i = j$.

Proof

We defer this until we have developed more machinery; see Propositions 3.11 and 2.6. □

As we have uniqueness, we talk about *the* completion of a normed vector space.

The process of completing a given space may often be simplified by realising it as a dense subspace of some known Banach space. The following examples demonstrate this.

Example 1.15. If $(X, \mathcal{T})$ is a topological space and $(E, \|\cdot\|)$ a Banach space then $C_\mathrm{b}(X, E)$ contains two subspaces worthy of note:

(i) $C_0(X, E)$, the continuous, E-valued functions on X that *vanish at infinity* (i.e., those $f \in C(X, E)$ such that, for all $\varepsilon > 0$, the set $\{x \in X : \|f(x)\| \geqslant \varepsilon\}$ is compact);

(ii) $C_{00}(X, E)$, the continuous, E-valued functions on X with *compact support* (i.e., those $f \in C(X, E)$ such that $\operatorname{supp} f := \overline{\{x \in X : f(x) \neq 0\}}$ is compact).

The first is a closed subspace of $C_b(X, E)$ (the proof of this is Exercise 2.1) and if X is Hausdorff and *locally compact* (for all $x \in X$ there exists $U \in \mathfrak{T}$ such that $x \in U$ and $\bar{U}$ is compact) then $C_{00}(X, E)$ is dense in $C_0(X, E)$. Hence the latter space is the completion of the former; for the proof of this claim see Proposition 2.18.

Example 1.16. It is immediate that $C[0, 1]$ is a subspace of $\mathcal{L}^1[0, 1]$ (because of the inequality $\|f\|_1 \leqslant \|f\|_\infty$). Furthermore, since step functions can be approximated arbitrarily well by continuous functions (with respect to the $\|\cdot\|_1$ norm), $C[0, 1]$ is dense in $L^1[0, 1]$; more accurately, its image under the map $f \mapsto [f]$ is. Hence the completion of $C[0, 1]$ (with respect to $\|\cdot\|_1$) is $L^1[0, 1]$.

Direct Sums

Throughout this section $\big(E_a : a \in A\big)$ denotes a family of Banach spaces with common scalar field $\mathbb{F}$.

Definition 1.17. Let $\sum_{a \in A} E_a$ denote the *algebraic direct sum* of the spaces E_a, i.e.,

$$\textstyle\sum_{a \in A} E_a := \Big\{x := (x_a)_{a \in A} \;\Big|\; x_a = 0 \text{ for all but finitely many } a \in A\Big\} \subseteq \bigtimes_{a \in A} E_a.$$

This is a vector space, with the vector-space operations defined pointwise:

$$x + y := (x_a + y_a)_{a \in A} \quad \text{and} \quad \alpha x := (\alpha x_a)_{a \in A} \qquad \forall\, x, y \in \sum_{a \in A} E_a,\ \alpha \in \mathbb{F}.$$

Theorem 1.18. The set

$$\textstyle\prod_{a \in A} E_a := \big\{x = (x_a)_{a \in A} : \|x\|_\infty < \infty\big\} \subseteq \bigtimes_{a \in A} E_a,$$

equipped with vector-space operations defined pointwise and norm

$$\|\cdot\|_\infty \colon x \mapsto \sup\big\{\|x_a\| : a \in A\big\},$$

is a Banach space, the *direct product* of the Banach spaces E_a.

Proof

The only thing that is not immediate is the proof of completeness; this follows the pattern of the b4 proof that ℓ^∞ is complete [24, Example 1.3.4] so we omit it. ☐

For other types of direct sum we need the following definition.

Definition 1.19. (Uncountable Sums) If A is an arbitrary set let

$$\sum_{a \in A} x_a := \sup\Big\{\sum_{a \in A_0} x_a : A_0 \text{ is a finite subset of } A\Big\}$$

for any collection $(x_a)_{a\in A}$ of non-negative real numbers. It is easy to show that this agrees with the usual definition if A is countable. (We are looking at the net of finite sums $\big(\sum_{a\in A_0} x_a\big)_{A_0\in\mathcal{A}}$ in $\mathbb{R}^+$, where $\mathcal{A}$ is the aggregate of all finite subsets of A, ordered by inclusion; see Definition 1.39.)

Theorem 1.20. For $p\in[1,\infty)$ and $x\in\prod_{a\in A}E_a$ let

$$\|x\|_p := \Big(\sum_{a\in A}\|x_a\|^p\Big)^{1/p}.$$

Then

$$\textstyle\sum^{(p)}_{a\in A}E_a := \{x\in\prod_{a\in A}E_a : \|x\|_p<\infty\}$$

is a subspace of $\prod_{a\in A}E_a$ and $\big(\sum^{(p)}_{a\in A}E_a,\ \|\cdot\|_p\big)$ is a Banach space, the *p-norm direct sum* of the Banach spaces E_a.

Proof

Let A_0 be a finite subset of A; the inequality

$$\sum_{a\in A_0}\|x_a+y_a\|^p \leqslant \sum_{a\in A_0}2^p\big(\|x_a\|^p+\|y_a\|^p\big)\leqslant 2^p\big(\|x\|_p^p+\|y\|_p^p\big)\qquad \forall\, x,y\in \textstyle\sum^{(p)}_{a\in A}E_a,$$

which may be proved as in Example 1.11, shows that $\sum^{(p)}_{a\in A}E_a$ is a subspace of $\prod_{a\in A}E_a$.

Subadditivity of $\|\cdot\|_p$ on $\sum^{(p)}_{a\in A}E_a$ follows from Minkowski's inequality on $\mathbb{C}^n$, and this can be obtained by applying the integral version of Minkowski's inequality in Example 1.11 to suitable step functions.

To see that we have completeness, let $(x^{(n)})_{n\geqslant1}$ be a Cauchy sequence in $\sum^{(p)}_{a\in A}E_a$. Since $\|x^{(n)}_a-x^{(m)}_a\|\leqslant\|x^{(n)}-x^{(m)}\|_p$, we have that $x_a:=\lim_{n\to\infty}x^{(n)}_a$ exists for all $a\in A$. If $A_0\subseteq A$ is finite then

$$\sum_{a\in A_0}\|x_a\|^p=\lim_{n\to\infty}\sum_{a\in A_0}\|x^{(n)}_a\|^p\leqslant\lim_{n\to\infty}\|x^{(n)}\|_p^p;$$

this last limit exists because $\big(\|x^{(n)}\|\big)_{n\geqslant1}$ is Cauchy: $\big|\|x^{(n)}\|_p-\|x^{(m)}\|_p\big|\leqslant\|x^{(n)}-x^{(m)}\|_p$. This bound shows that $x\in\sum^{(p)}_{a\in A}E_a$; it remains to prove that $x^{(n)}\to x$. Note first that if $A_0\subseteq A$ is finite and $m,\,n\in\mathbb{N}$ then $\sum_{a\in A_0}\|x^{(n)}_a-x^{(m)}_a\|^p\leqslant\|x^{(n)}-x^{(m)}\|_p^p$. Let $\varepsilon>0$ and suppose that $n_0\in\mathbb{N}$ is such that $\|x^{(n)}-x^{(m)}\|_p<\varepsilon$ for all $m,\,n\geqslant n_0$. Then

$$\sum_{a\in A_0}\|x_a-x^{(m)}_a\|^p=\lim_{n\to\infty}\sum_{a\in A_0}\|x^{(n)}_a-x^{(m)}_a\|^p\leqslant\varepsilon^p\qquad\forall\, m\geqslant n_0,$$

so $\|x-x^{(m)}\|_p\leqslant\varepsilon$ for all $m\geqslant n_0$, and this gives the result. □

Proposition 1.21. For all $p\in[1,\infty)$ the algebraic direct sum $\sum_{a\in A}E_a$ is dense in the p-norm direct sum $\sum^{(p)}_{a\in A}E_a$.

Proof

This is a simple consequence of the fact that if $x \in \sum^{(p)}_{a\in A} E_a$ then $x_a \neq 0$ for only countably many $a \in A$: see Exercise 1.7. $\square$

Definition 1.22. The previous proposition motives the definition of $\sum^{(\infty)}_{a\in A} E_a$ as the completion of $\sum_{a\in A} E_a$ with respect to $\|\cdot\|_\infty$; this space is the *direct sum* of the Banach spaces E_a. Clearly $\sum^{(\infty)}_{a\in A} E_a \subseteq \prod_{a\in A} E_a$, but the inclusion may be strict.

Example 1.23. **(Sequence Spaces)** Let $A = \mathbb{N} := \{1, 2, 3, \ldots\}$ and take $E_a = \mathbb{F}$ for all $a \in A$. Then the algebraic direct sum

$$\textstyle\sum_{a\in A} E_a = c_{00} := \big\{x = (x_n)_{n\in\mathbb{N}} : \exists\, N \in \mathbb{N} \text{ such that } x_{N+1} = x_{N+2} = \cdots = 0\big\},$$

the p-norm direct sum

$$\textstyle\sum^{(p)}_{a\in A} E_a = \begin{cases} \ell^p & \text{if } p \in [1, \infty), \\ c_0 & \text{if } p = \infty, \end{cases}$$

and the direct product $\prod_{a\in A} E_a = \ell^\infty$.

In general, if A is any set and $E_a = \mathbb{F}$ for all $a \in A$ then we define $c_{00}(A)$, $c_0(A)$ and $\ell^p(A)$ in this manner.

Initial Topologies

Definition 1.24. Let X be a set and F be collection of functions on X, such that $f\colon X \to Y_f$, where $(Y_f, \mathcal{S}_f)$ is a topological space, for all $f \in F$. The *initial topology generated by* F, denoted by $\mathcal{T}_F$, is the coarsest topology such that each function $f \in F$ is continuous. (Older books call $\mathcal{T}_F$ the *weak topology generated by* F: the adjective 'weak' is tremendously overworked in functional analysis so we prefer the modern term.) It is clear that $\mathcal{T}_F$ is the intersection of all topologies on X that contain

$$\bigcup_{f\in F} f^{-1}(\mathcal{S}_f) = \{f^{-1}(U) : f \in F,\ U \in \mathcal{S}_f\}.$$

In fact, every element of $\mathcal{T}_F$ is the arbitrary union of sets of the form

$$\bigcap_{i=1}^{n} f_i^{-1}(U_i) \qquad (n \in \mathbb{N},\ f_1, \ldots, f_n \in F,\ U_1 \in \mathcal{S}_{f_1}, \ldots, U_n \in \mathcal{S}_{f_n}); \tag{1.1}$$

these sets are a basis for this topology. (To see this, note that every set of this form lies in $\mathcal{T}_F$, and that the collection of arbitrary unions of these sets is a topology (cf. Exercise 1.4).)

Proposition 1.25. Let F be a collection of functions as in Definition 1.24 and let $(Z, \mathcal{U})$ be a topological space. A function $g\colon (Z, \mathcal{U}) \to (X, \mathcal{T}_F)$ is continuous if and only if $f \circ g\colon (Z, \mathcal{U}) \to (Y_f, \mathcal{S}_f)$ is continuous for all $f \in F$.

Proof

As the composition of continuous functions is continuous, one implication is immediate. For the converse, suppose that $f \circ g$ is continuous for all $f \in F$ and let $U \in \mathcal{T}_F$. We may assume that $U = \bigcap_{i=1}^n f_i^{-1}(U_i)$ for $f_1, \ldots, f_n \in F$ and $U_1 \in \mathcal{S}_{f_1}, \ldots, U_n \in \mathcal{S}_{f_n}$, and then

$$g^{-1}(U) = \bigcap_{i=1}^n g^{-1}\big(f_i^{-1}(U_i)\big) = \bigcap_{i=1}^n (f_i \circ g)^{-1}(U_i) \in \mathcal{U},$$

as required. □

The above may remind the reader of a result concerning the product topology, which is an initial topology (that generated by the coordinate projections). We shall see other examples of initial topologies later.

Proposition 1.26. Let F be a collection of functions as in Definition 1.24, such that $(Y_f, \mathcal{S}_f)$ is Hausdorff for all $f \in F$. The initial topology $\mathcal{T}_F$ is Hausdorff if F *separates points*: for all $x, y \in Y$ such that $x \neq y$ there exists $f \in F$ such that $f(x) \neq f(y)$.

Proof

Let $x, y \in X$ be distinct and suppose that $f \in F$ is such that $f(x) \neq f(y)$. Since $\mathcal{S}_f$ is Hausdorff there exist disjoint sets $U, V \in \mathcal{S}_f$ such that $f(x) \in U$ and $f(y) \in V$. Then $f^{-1}(U), f^{-1}(V) \in \mathcal{T}_F$ are such that

$$x \in f^{-1}(U), \quad y \in f^{-1}(V) \quad \text{and} \quad f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = \emptyset.$$

Hence $\mathcal{T}_F$ is Hausdorff, as claimed. □

Nets

In a metric space (X, d) it is readily proven that, given a set $M \subseteq X$, the element $x \in \bar{M}$ if and only if there exists a sequence $(x_n)_{n \geqslant 1} \subseteq M$ such that $x_n \to x$. Hence closed sets (those such that $M = \bar{M}$) may be characterised by means of sequences, and therefore so can the topology generated by the metric d.

If $X = \mathbb{R}$ is equipped with the cocountable topology (which consists of the empty set and the complement of each countable subset of $\mathbb{R}$) then $M = \mathbb{R} \setminus \{0\}$ has closure $\bar{M} = \mathbb{R}$ but there is no sequence $(x_n)_{n \geqslant 1} \subseteq M$ such that $x_n \to 0$. This shows that the result of the previous paragraph does not hold for general topological spaces (and that the cocountable topology on $\mathbb{R}$ is non-metrizable).

However, arguments with sequences are often very natural and easy to follow, whereas arguments involving open sets can sometimes appear rather opaque. Is it possible to replace the notion of sequence with some generalisation which allows a version of the result above? The answer is, of course, yes.

Definition 1.27. Let A be a set. A *preorder* $\leqslant$ on A is a binary relation that satisfies, for all $a, b, c \in A$,

(i) $a \leqslant a$ (*reflexivity*)

and (ii) $a \leqslant b$ and $b \leqslant c$ imply that $a \leqslant c$ (*transitivity*);

we say that A is ordered by $\leqslant$. For convenience we write $b \geqslant a$ if and only if $a \leqslant b$. A *directed set* $(A, \leqslant)$ is a set A and a preorder $\leqslant$ on A with the following property: for all a, $b \in A$ there exists $c \in A$ such that $a \leqslant c$ and $b \leqslant c$; the element c is called an *upper bound* or *majorant* for a and b. A *net* in a set X is a directed set $(A, \leqslant)$ and a function $x\colon A \to X$; this is denoted by $(x_a)_{a\in A}$ (the preorder being understood).

Example 1.28. If $\mathbb{N}$ is equipped with the usual order $\leqslant$ then it is a directed set, and the net $(x_n)_{n\in\mathbb{N}}$ is the same thing as the sequence $(x_n)_{n\geqslant 1}$.

Example 1.29. Let f be a continuous function on the interval $[0, 1]$, and let A be the collection of (real-valued) step functions ϕ on $[0, 1]$ that are bounded above by f: a step function $\phi \in A$ if $\phi(x) \leqslant f(x)$ for all $x \in [0, 1]$ (where $\leqslant$ is the usual order on $\mathbb{R}$). Order A by saying that $\phi \leqslant \psi$ if and only if $\phi(x) \leqslant \psi(x)$ for all $x \in [0, 1]$; since $\phi \vee \psi\colon x \mapsto \max\{\phi(x), \psi(x)\}$ is a step function if ϕ and ψ are, the pair $(A, \leqslant)$ forms a directed set.

Definition 1.30. Let $(X, \mathcal{T})$ be a topological space. A net $(x_a)_{a\in A}$ in X is *convergent* if there exists $x \in X$ such that, for all $U \in \mathcal{T}$ with $x \in X$, there exists $a_0 \in A$ such that $x_a \in U$ for all $a \geqslant a_0$; the element x is the *limit* of this net, and we write $x_a \to x$ or $\lim_{a\in A} x_a = x$. (This latter notation is a slight abuse as, in general, limits need not be unique.)

Proposition 1.31. A net in a Hausdorff topological space has at most one limit.

Proof

This may be proved in the same manner as the corresponding result for sequences [22, Proposition 4.2.2]. □

Example 1.32. A net $(x_n)_{n\in\mathbb{N}}$ in a topological space $(X, \mathcal{T})$ converges to x if and only if the sequence $(x_n)_{n\geqslant 1}$ converges to x; the proof of this is immediate. Hence nets are generalisations of sequences.

Example 1.33. If $(A, \leqslant)$ is the directed set of Example 1.29 and $\int \phi$ denotes the integral of the step function ϕ then $(\int \phi)_{\phi\in A}$ is a net in $\mathbb{R}$ that converges to $\int f$. (To see this, note first that $\int f = \sup\{\int \phi : \phi \in A\}$: see [17, §9.6].)

We now give some theorems that illustrate how nets can be used to answer topological questions.

Theorem 1.34. Let $(X, \mathcal{T})$ be a topological space and suppose that $M \subseteq X$. Then M is closed if and only if $\lim_{a\in A} x_a \in M$ for all convergent nets $(x_a)_{a\in A} \subseteq M$.

Proof

Suppose M is closed and $(x_a)_{a\in A}$ is a net in M, such that $x_a \to x$ for some $x \in X$. If $x \notin M$ then $X \setminus M$ is an open set containing x, so it contains some elements of $(x_a)_{a\in A}$, contrary to hypothesis. Hence $\lim_{a\in A} x_a \in M$.

Conversely, let $x \in \bar{M}$ and let $\mathcal{A} = \{U \in \mathcal{T} : x \in U\}$ denote the collection of open sets containing x; this forms a directed set if ordered by *reverse inclusion*, i.e., $A \leqslant B$

if and only if $A \supseteq B$. If $A \in \mathcal{A}$ then $A \cap M \neq \emptyset$, for otherwise $X \setminus A$ is a closed set containing M and so must contain x. Hence we may choose $x_A \in A \cap M$ for all $A \in \mathcal{A}$, and the net $(x_A)_{A \in \mathcal{A}}$ converges to x: note that if $U \in \mathcal{T}$ contains x then $x_A \in U$ for all $A \geqslant U$. This gives the result. $\square$

The following proposition shows why nets and initial topologies work so well together.

Proposition 1.35. Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces. A function $f\colon X \to Y$ is continuous if and only if $f(x_a) \to f(x)$ for every net $(x_a)_{a \in A}$ such that $x_a \to x$.

Proof

Let f be continuous, suppose that $(x_a)_{a \in A}$ is convergent to x and let $U \in \mathcal{S}$ contain $f(x)$. Then $V = f^{-1}(U) \in \mathcal{T}$ contains x, so there exists $a_0 \in A$ such that $x_a \in V$ for all $a \geqslant a_0$. Hence $f(x_a) \in U$ for all $a \geqslant a_0$, as required.

Conversely, let $U \in \mathcal{S}$ and suppose for contradiction that $V = f^{-1}(U)$ is not open. Then there exists $x \in V \setminus V^\circ$, and no open set containing x is contained in V: for all $A \in \mathcal{A} := \{W \in \mathcal{T} : x \in W\}$ there exists $x_A \in A \cap (X \setminus V)$. If $\mathcal{A}$ is ordered by reverse inclusion then $(x_A)_{A \in \mathcal{A}}$ is a net converging to x, so $f(x_A) \to f(x)$. In particular there exists $A_0 \in \mathcal{A}$ such that $f(x_{A_0}) \in U$, and $x_{A_0} \in V$, the desired contradiction. $\square$

This proposition allows us to give a simple proof of the completeness of $C_{\mathrm{b}}(X, E)$.

Theorem 1.36. If X is a topological space and E is a Banach space then $C_{\mathrm{b}}(X, E)$ is a Banach space with respect to the supremum norm $\|\cdot\|_\infty$.

Proof

We prove only completeness; everything else is trivial. Let $(f_n)_{n \geqslant 1}$ be a Cauchy sequence in $C_{\mathrm{b}}(X, E)$ and note that $\|f_n(x) - f_m(x)\| \leqslant \|f_n - f_m\|_\infty$ for all $x \in X$, so

$$f\colon X \to E;\ x \mapsto \lim_{n \to \infty} f_n(x)$$

is well defined. If $\varepsilon > 0$ then there exists $n_0 \in \mathbb{N}$ such that $\|f_n - f_m\|_\infty < \varepsilon$ if $m, n \geqslant n_0$, so

$$\|f(x) - f_m(x)\| = \lim_{n \to \infty} \|f_n(x) - f_m(x)\| \leqslant \varepsilon \qquad \forall\, m \geqslant n_0,\ x \in X$$

whence $\|f - f_m\|_\infty \to 0$ as $m \to \infty$ and also $\|f\|_\infty \leqslant \|f - f_{n_0}\|_\infty + \|f_{n_0}\|_\infty \leqslant \varepsilon + \|f_{n_0}\|_\infty$, i.e., f is bounded. Finally, if $x_a \to x$ then choose $a_0 \in A$ such that $\|f_{n_0}(x_a) - f_{n_0}(x)\| < \varepsilon$ for all $a \geqslant a_0$ and note that

$$\|f(x_a) - f(x)\| \leqslant \|f(x_a) - f_{n_0}(x_a)\| + \|f_{n_0}(x_a) - f_{n_0}(x)\| + \|f_{n_0}(x) - f(x)\| < 3\varepsilon$$

for all $a \geqslant a_0$. This shows that $f(x_a) \to f(x)$, so f is continuous. $\square$

Proposition 1.37. Let F be a collection of functions as in Definition 1.24, and suppose that $(x_a)_{a \in A}$ is a net in X. Then $x_a \to x$ in $(X, \mathcal{T}_F)$ if and only if $f(x_a) \to f(x)$ in $(Y_f, \mathcal{S}_f)$ for all $f \in F$.

Proof

As $\mathcal{T}_F$ makes each $f \in F$ continuous, half of the proof follows from Proposition 1.35. For the other half, suppose that $(x_a)_{a \in A} \subseteq X$ and $x \in X$ are such that $f(x_a) \to f(x)$

for all $f \in F$. Let $U \in \mathcal{T}_F$ be such that $x \in U$; it is sufficient to consider U of the form (1.1) in Definition 1.24. Note that $x \in f_i^{-1}(U_i)$ for $i = 1, \ldots, n$, so $f_i(x) \in U_i$ and there exists $a_i \in A$ such that $f_i(x_a) \in U_i$ for all $a \geqslant a_i$. If $a_0 \in A$ is such that $a_i \leqslant a_0$ for each i then $x_a \in \bigcap_{i=1}^n f_i^{-1}(U_i) = U$ for all $a \geqslant a_0$, as required. $\square$

Example 1.38. Let $\Omega \subseteq \mathbb{C}$ be open and let

$$\iota_K \colon C(\Omega) \to C(K);\ f \mapsto f|_K$$

be the restriction map to K, where $K \subseteq \Omega$. If

$$F = \{\iota_K : K \text{ is a compact subset of } \Omega\}$$

and each $C(K)$ has the supremum norm then $\mathcal{T}_F$ is the topology of *locally uniform convergence*:

$$f_n \to f \iff f_n|_K \to f|_K \text{ uniformly on } K \text{ for all compact } K \subseteq \Omega.$$

Nets allow us to give a proper treatment of summability, which coincides with the *ad hoc* method used in Definition 1.19.

Definition 1.39. Let X be a normed vector space. A family $(x_a)_{a\in A} \subseteq X$ is *summable* if the net of partial sums $\left(\sum_{a\in A_0} x_a\right)_{A_0\in\mathcal{A}}$ is convergent, where $\mathcal{A}$ is the collection of finite subsets of A, ordered by inclusion. If $(x_a)_{a\in A}$ is summable then $\sum_{a\in A} x_a$ denotes the limit of the net of partial sums.

Example 1.40. If $(x_a)_{a\in A}$ is a family of non-negative real numbers then this definition agrees with Definition 1.19 (Exercise 1.6). If $(z_n)_{n\in\mathbb{N}}$ is a family of complex numbers then it is summable if and only if $\sum_{n=1}^{\infty} z_n$ is absolutely convergent (see Exercises 1.7 and 1.8). Note also that if $(x_a)_{a\in A}$ is a family of vectors in a Banach space then

$$\sum_{a\in A} \|x_a\| < \infty \Longrightarrow \sum_{a\in A} x_a \text{ is convergent,}$$

i.e., absolute summability implies summability (Exercise 1.7).

Nets were introduced by Moore and Smith in [13] – the theory is also called Moore-Smith convergence, especially in older references – and they were applied to general topological spaces by Garrett Birkhoff [2]. As to the choice of nomenclature, the reader may care to reflect upon the following, taken from [12, Third footnote on p.3]:

> *[J. L.] Kelley writes me that ["net"] was suggested by Norman Steenrod in a conversation between Kelley, Steenrod and Paul Halmos. Kelley's own inclination was to the name "way"; the analogue of a subsequence would then be a "subway"!*

An aged but excellent introduction to nets is [9, Chapter 2]; McShane's article [12] is a very pleasant introductory exposition. Pedersen [14] refers to the viewpoints of topology in terms of nets and of open sets as dynamic and static, respectively.

Exercises 1

Exercise 1.1. Let X be a normed vector space and let M be a closed subspace of X. Prove that

$$\pi\{y \in X : \|y - x\| < \varepsilon\} = \big\{[y] \in X/M : \big\|[y] - [x]\big\| < \varepsilon\big\} \qquad \forall\, x \in X,\ \varepsilon > 0,$$

where $\pi\colon X \to X/M;\ x \mapsto [x]$ is the natural map from X onto X/M (the *quotient map*). Deduce that the quotient norm yields the *quotient topology* on X/M given by

$$\mathcal{Q} := \{U \subseteq X/M : \pi^{-1}(U) \in \mathcal{T}\},$$

where $\mathcal{T}$ denotes the norm topology on X, and that the quotient map is open (i.e., sends open sets to open sets). Prove also that the quotient map is linear and continuous.

Exercise 1.2. Prove directly that if E is a Banach space and M is a closed subspace of E then the quotient space $\big(E/M, \|\cdot\|_{E/M}\big)$ is complete. [Use Banach's criterion.]

Exercise 1.3. Let M and N be subspaces of the normed space X. Prove that if M is finite dimensional and N is closed then $M + N$ is closed. [Recall that finite-dimensional subspaces of normed spaces are closed [24, Corollary 1.2.18] and use the quotient map.]

Exercise 1.4. Prove that if $\{A^i_j : i \in I,\ j \in J\}$ and $\{B^k_l : k \in K,\ l \in L\}$ are families of sets, where the index sets I, J, K and L are arbitrary, then

$$\Big(\bigcup_{i\in I}\bigcap_{j\in J} A^i_j\Big) \cap \Big(\bigcup_{k\in K}\bigcap_{l\in L} B^k_l\Big) = \bigcup_{(i,k)\in I\times K}\ \bigcap_{(j,l)\in J\times L} A^i_j \cap B^k_l.$$

What does this have to do with initial topologies?

Exercise 1.5. Prove that if $\mathcal{T}_F$ is the initial topology on X generated by a collection of functions F and $Y \subseteq X$ then $\mathcal{T}_F|_Y$, the relative initial topology on Y, is the initial topology generated by $F|_Y = \{f|_Y : f \in F\}$, the restrictions of the functions in F to Y.

Exercise 1.6. Let $(x_a)_{a\in A}$ be a family of non-negative real numbers and let $\mathcal{A}$ denote the collection of finite subsets of A. Prove that $(x_a)_{a\in A}$ is summable (with sum α) if and only if $\beta = \sup\big\{\sum_{a\in A_0} x_a : A_0 \in \mathcal{A}\big\} < \infty$ and in this case $\alpha = \beta$.

Exercise 1.7. Let E be a Banach space and let $(x_a)_{a\in A}$ a family of vectors in E. Prove that if $\sum_{a\in A}\|x_a\|$ is convergent then $S := \{a \in A : x_a \neq 0\}$ is countable. [Consider the sets $S_n := \{a \in A : \|x_a\| > 1/n\}$ for $n \in \mathbb{N}$.] Deduce that $(x_a)_{a\in A}$ is summable with sum

$$\sum_{a\in A} x_a = \begin{cases} \sum_{a\in S} x_a & \text{if } S \text{ is finite,} \\ \sum_{j=1}^{\infty} x_{a_j} & \text{if } S \text{ is infinite,} \end{cases}$$

where (if S is infinite) $j \mapsto a_j$ is a bijection between $\mathbb{N}$ and S.

Exercise 1.8. Prove that a family of complex numbers $(z_a)_{a\in A}$ is summable if and only if $\big(|z_a|\big)_{a\in A}$ is summable. [Consider real and imaginary parts to reduce to the real case and then consider positive and negative parts.]

Exercise 1.9. Find a Hilbert space H and a countable family of vectors $(x_n)_{n\in\mathbb{N}}$ in H that is summable but not absolutely summable (i.e., $\big(\|x_n\|\big)_{n\in\mathbb{N}}$ is not summable).

Exercise 1.10. Prove the converse to Proposition 1.31, that in a space with a non-Hausdorff topology there exists a net that converges to two distinct points. [Take two points that cannot be separated by open sets and define a net that converges to both of them.]

Exercise 1.11. A sequence in a normed vector space that is convergent is necessarily bounded. Is the same true for nets?

Two Linear Operators

Preliminaries

Let X be a normed vector space; for all $r \in \mathbb{R}^+$ let X_r denote the closed ball in X with radius r and centre the origin:

$$X_r := \big\{x \in X : \|x\| \leqslant r\big\}.$$

Definition 2.1. A *bounded (linear) operator* from X to Y is a linear transformation $T\colon X \to Y$ such that the *operator norm* $\|T\|$ is finite, where

$$\begin{aligned} \|T\| &:= \inf\{M \in \mathbb{R}^+ : \|Tx\| \leqslant M\|x\| \text{ for all } x \in X\} \\ &= \sup\{\|Tx\| : x \in X_1\} \\ &= \sup\big\{\|Tx\| : x \in X,\ \|x\| = 1\big\}. \end{aligned}$$

Proposition 2.2. Let $T\colon X \to Y$ be a linear transformation. The following statements are equivalent:

(i) T is a bounded linear operator;
(ii) T is uniformly continuous;
(iii) T is continuous;
(iv) T is continuous at 0;
(v) $T(X_1)$ is bounded: $T(X_1) \subseteq Y_r$ for some $r \in \mathbb{R}^+$.

Proof
The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i) are immediate. □

We denote the collection of bounded linear operators from X to Y by $\mathcal{B}(X,Y)$ (or $\mathcal{B}(X)$ if $X = Y$). Note that $\mathcal{B}(X,Y)$ is a normed vector space, where

$$(T+S)x := Tx + Sx \quad \text{and} \quad (\alpha T)x := \alpha Tx \qquad \forall\, S,\, T \in \mathcal{B}(X,Y),\ x \in X,\ \alpha \in \mathbb{F}$$

and the norm on $\mathcal{B}(X,Y)$ is the operator norm.

Theorem 2.3. If $T\colon X \to Y$ be a linear transformation then

(i) the *kernel* $\ker T := \{x \in X : Tx = 0\}$ is a subspace of X;

(ii) the *image* $\operatorname{im} T := \{Tx : x \in X\}$ is a subspace of Y;

(iii) $X/\ker T \cong \operatorname{im} T$ *via* the linear transformation

$$\tilde{T}\colon X/\ker T \to \operatorname{im} T;\ [x] \mapsto Tx.$$

If X and Y are normed spaces and $T \in \mathcal{B}(X, Y)$ then $\ker T$ is closed, $X/\ker T$ is a normed space and $\tilde{T}$ is continuous, with $\|\tilde{T}\| = \|T\|$.

Proof

The algebraic facts are immediately verified, as is the fact that $\ker T = T^{-1}\{0\}$ is closed if T is continuous. If

$$\pi\colon X \to X/\ker T;\ x \mapsto [x]$$

is the quotient map, the inequality $\|\pi(x)\| = \|[x]\| \leqslant \|x\|$ implies that $\|\pi\| \leqslant 1$ and so $\|T\| \leqslant \|\tilde{T}\|\,\|\pi\| \leqslant \|\tilde{T}\|$ (because $\tilde{T} \circ \pi = T$). Conversely, let $x \in X$, $\varepsilon > 0$ and choose $y \in \ker T$ such that $\|x - y\| < \|[x]\| + \varepsilon$. Then

$$\|\tilde{T}[x]\| = \|Tx\| = \|T(x - y)\| \leqslant \|T\|\,\|x - y\| \leqslant \|T\|\big(\|[x]\| + \varepsilon\big),$$

and since this holds for all $\varepsilon > 0$ and $x \in X$ we have the result. □

Completeness of $\mathcal{B}(X, Y)$

Proposition 2.4. If $X \neq \{0\}$ then the normed vector space $\mathcal{B}(X, Y)$ is a Banach space if and only if Y is a Banach space.

Proof

The fact that the completeness of Y entails the completeness of $\mathcal{B}(X, Y)$ is a result from b4 [24, Exercise 1.5.17(ii)]; a leisurely proof may be found in [10, Theorem 2.10-2] and a more concise version in [14, Theorem 2.1.4]. We prove the converse as an application of the Hahn-Banach theorem: see Proposition 3.12. □

Extension of Linear Operators

Often it is easiest to define a linear operator on some dense subspace of a Banach space, and extend it to the whole space "by continuity": the following theorem explains the meaning of this.

Theorem 2.5. (BLT) Let X_0 be a dense subspace of a normed vector space X and let $T_0 \in \mathcal{B}(X_0, Y)$, where Y is a Banach space. There exists a unique $T \in \mathcal{B}(X, Y)$ such that $T|_{X_0} = T_0$, and such satisfies $\|T\| = \|T_0\|$.

Proof

Existence is a b4 result [24, Exercise 1.5.17(iii)]; for a proof see [10, Theorem 2.7-11] or [14, Theorem 2.1.11]. For uniqueness, note that if $S|_{X_0} = T_0 = T|_{X_0}$ for $S, T \in \mathcal{B}(X, Y)$ then $(S - T)|_{X_0} = 0$ and so $S - T = 0$ by continuity: if $x \in X$ let $(x_n)_{n \geqslant 1} \subseteq X_0$ be such that $x_n \to x$ and note that $(S - T)x = \lim_{n\to\infty}(S - T)x_n = 0$. □

Uniqueness of Completions

We are now in the position to prove that the completion of a normed space is unique.

Proposition 2.6. Let $(Y,\, i)$ and $(Z,\, j)$ be completions of the normed space X. There exists an isometric isomorphism $k\colon Y \to Z$ such that the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, hook, "i"] \arrow[dr, hook, "j"'] & Y \arrow[d, hook, two heads, dotted, "k"] \\
 & Z
\end{tikzcd}$$

Proof

Note that

$$k_0\colon i(X) \to Z;\ i(x) \mapsto j(x)$$

is a well-defined linear isometry from a dense subspace of Y onto a dense subspace of Z, *viz* $j(X)$. Hence, by Theorem 2.5, k_0 extends to $k \in \mathcal{B}(Y, Z)$ and k is an isometry: if $y \in Y$ then there exists $(x_n)_{n\geqslant 1} \subseteq X$ such that $i(x_n) \to y$, but k and $x \mapsto \|x\|$ are continuous, so

$$\|k(y)\| = \lim_{n\to\infty} \big\|k\big(i(x_n)\big)\big\| = \lim_{n\to\infty} \big\|k_0\big(i(x_n)\big)\big\| = \lim_{n\to\infty} \|i(x_n)\| = \|y\|.$$

Furthermore, $k(Y)$ is closed in Z: to see this, let $\big(k(y_n)\big)_{n\geqslant 1}$ be a convergent sequence (with $y_n \in Y$) and note that it is Cauchy, so $(y_n)_{n\geqslant 1}$ is Cauchy (k being an isometry) and hence convergent, to $y \in Y$; the continuity of k gives $k(y_n) \to k(y) \in k(Y)$, as required. As $k(Y)$ contains $j(X)$, it contains its closure, Z, and therefore k is surjective. □

The Baire Category Theorem

Definition 2.7. A subset of a metric space is said to be *meagre* or *of the first category* if it is a countable union of closed sets with empty interior; a set is *non-meagre* or *of the second category* otherwise. (Recall that a set is *nowhere dense* if its closure has empty interior.)

We introduce some notation (which we learned from Dr J. M. Lindsay) that will be used in the proof of the next theorem. If (X, d) is a metric space then let

$$B_r^X(x) := \{y \in X : d(x, y) < r\} \quad \text{(the } \textit{open ball} \text{ of } \textit{radius } r \text{ and } \textit{centre } x\text{)}$$

and

$$B_r^X[x] := \{y \in X : d(x, y) \leqslant r\} \quad \text{(the } \textit{closed ball} \text{ of } \textit{radius } r \text{ and } \textit{centre } x\text{)}$$

for all $x \in X$ and $r \in (0, \infty)$; if the space is clear from the context then we abbreviate these to $B_\varepsilon(x)$ and $B_\varepsilon[x]$ respectively. If X is a normed space then

$$B_r^X(x) = x + rB_1^X(0) \qquad \text{and} \qquad B_r^X[x] = x + X_r = x + rX_1;$$

furthermore, $\overline{B_r^X(x)} = B_r^X[x]$.

Recall that G is said to be *dense* in the topological space $(X, \mathcal{T})$ if and only if $\bar{G} = X$. This is equivalent to the condition that $G \cap U \neq \emptyset$ for all $U \in \mathcal{T} \setminus \{\emptyset\}$ (because $\bar{G} = X$ if and only if $(X \setminus G)^\circ = X \setminus \bar{G} = \emptyset$). If $\mathcal{T}$ is given by a metric d then this condition is equivalent to requiring that $G \cap B_\varepsilon(x) \neq \emptyset$ for all $x \in X$ and $\varepsilon > 0$.

Theorem 2.8. (Baire) If $(G_n)_{n\geqslant 1}$ is a sequence of open, dense subsets of the complete metric space (X, d) then the intersection $\bigcap_{n\geqslant 1} G_n$ is dense in X.

Proof

Let $x_0 \in X$ and $r_0 > 0$; it suffices to prove that $B_{r_0}(x_0) \cap \bigcap_{n\geqslant 1} G_n \neq \emptyset$. To do this we construct sequences $(r_n)_{n\geqslant 1} \subseteq (0, \infty)$ and $(x_n)_{n\geqslant 1} \subseteq X$ such that

$$r_0 > r_1 > r_2 > \cdots \to 0 \tag{2.1}$$

as $n \to \infty$ and

$$B_{r_n}[x_n] \subseteq B_{r_{n-1}}(x_{n-1}) \cap G_n \qquad \forall\, n \geqslant 1. \tag{2.2}$$

Given (2.2), if $n > m \geqslant 1$ then $B_{r_n}[x_n] \subseteq B_{r_{n-1}}(x_{n-1}) \subseteq B_{r_{n-1}}[x_{n-1}] \subseteq \cdots \subseteq B_{r_m}(x_m)$; in particular,

$$d(x_n, x_m) < \max\{r_n, r_m\} \quad \forall\, m, n \geqslant 1,$$

which, if (2.1) holds, shows that $(x_n)_{n\geqslant 1}$ is Cauchy, and so convergent. As $x_n \in B_{r_m}[x_m]$ for all $n \geqslant m \geqslant 1$,

$$\lim_{n\to\infty} x_n \in \bigcap_{n\geqslant 1} B_{r_n}[x_n] \subseteq B_{r_0}(x_0) \cap \bigcap_{n\geqslant 1} G_n$$

and we are done. (This last step shows why we must consider both closed and open balls.)

To see why we may find sequences satisfying (2.1) and (2.2), consider first the case $n = 1$. Note that $B_{r_0}(x_0) \cap G_1$ is open (being the intersection of two open sets) and non-empty (as G_1 is dense in X). Hence there exists $x_1 \in X$ and $s \in (0, r_0)$ such that $B_s(x_1) \subseteq B_{r_0}(x_0) \cap G_1$; taking $r_1 \in \big(0, \min\{s, 1\}\big)$ shows that (2.2) holds in the case $n = 1$. This argument may be repeated for all n (each time ensuring that $r_n < 1/n$) which completes the proof. $\square$

An alternative formulation of Baire's category theorem as follows, which states that a complete metric space is non-meagre.

Theorem 2.9. Let $(F_n)_{n\geqslant 1}$ be a sequence of closed subsets of the complete metric space (X, d). If $X = \bigcup_{n\geqslant 1} F_n$ then there exists $n_0 \in \mathbb{N}$ such that F_{n_0} has non-empty interior.

Proof

Suppose for contradiction that $F_n^\circ = \emptyset$ for all $n \geqslant 1$, and let $G_n = X \setminus F_n$. Then

$$\bar{G}_n = \overline{X \setminus F_n} = X \setminus F_n^\circ = X$$

and so G_n is an open, dense subset of X, whence $\bigcap_{n\geqslant 1} G_n$ is dense, by Theorem 2.8, but $\bigcap_{n\geqslant 1} G_n = \bigcap_{n\geqslant 1} X \setminus F_n = X \setminus \bigcup_{n\geqslant 1} F_n = \emptyset$. $\square$

The Open-Mapping Theorem

Recall that a function between topological spaces is *open* if the image of every open set is open.

Proposition 2.10. Let X and Y be normed spaces; a linear transformation $T\colon X \to Y$ is open if and only if $T(X_1)$ contains Y_ε for some $\varepsilon > 0$.

Proof

If T is open then $T\big(B_1^X(0)\big)$ is an open subset of Y that contains $T0 = 0$, and so contains $B_{2\varepsilon}^Y(0)$ for some $\varepsilon > 0$. Hence $Y_\varepsilon \subseteq B_{2\varepsilon}^Y(0) \subseteq T\big(B_1^X(0)\big) \subseteq T(X_1)$.

Conversely, suppose there exists $\varepsilon > 0$ such that $Y_\varepsilon \subseteq T(X_1)$. Let U be an open subset of X; if $u \in U$ then $B_{2\delta}^X(u) \subseteq U$ for some $\delta > 0$. Hence

$$T(U) \supseteq T\big(u + \delta B_2^X(0)\big) \supseteq Tu + \delta T(X_1) \supseteq Tu + \delta Y_\varepsilon \supseteq B_{\delta\varepsilon}^Y(Tu),$$

which shows that $T(U)$ is open, as required. $\square$

The Open-Mapping Lemma

Definition 2.11. Let $(X,\, d)$ be a metric space and suppose that $A,\, B \subseteq X$; we say that A is *k-dense* in B if for all $b \in B$ there exists $a \in A$ such that $d(b, a) \leqslant k$. Equivalently, $B \subseteq \bigcup_{a\in A} B_k[a]$. (If A is dense in B then A is k-dense in B for all $k > 0$; if A is dense in B and B is k-dense in C then A is k'-dense in C for any $k' > k$.)

Lemma 2.12. (Open-Mapping Lemma) Let E be a Banach space, Y a normed space and $T \in \mathcal{B}(E, Y)$. If there exist $r > 0$ and $k \in (0, 1)$ such that $T(E_r)$ is k-dense in Y_1 then

(i) for all $y \in Y$ there exists $x \in E$ such that $\|x\| \leqslant \dfrac{r}{1-k}\|y\|$ and $Tx = y$, so T is surjective,

(ii) T is an open mapping

and (iii) Y is complete.

Proof

To prove (i) let $y \in Y$; without loss of generality we may take $y \in Y_1$ (the case $y = 0$ is trivial and otherwise we may replace y by $y/\|y\|$) so there exists $x_0 \in E_r$ such that

$$\|y - Tx_0\| \leqslant k.$$

As $k^{-1}(y - Tx_0) \in Y_1$ there exists $x_1 \in E_r$ such that

$$\|k^{-1}(y - Tx_0) - Tx_1\| \leqslant k \qquad \Longleftrightarrow \qquad \|y - T(x_0 + kx_1)\| \leqslant k^2.$$

Continuing in this way we see that there exists a sequence $(x_l)_{l\geqslant 0} \subseteq E_r$ such that

$$\Big\|y - T\sum_{l=0}^{n} k^l x_l\Big\| \leqslant k^{n+1} \qquad \forall\, n \geqslant 0. \tag{2.3}$$

Let $x := \sum_{l=0}^{\infty} k^l x_l$; note that this series is absolutely convergent, so convergent, and (2.3) shows that $Tx = y$. The inequality $\|x\| \leqslant r/(1-k)$ follows from the definition of x and the fact that the series $\sum_{l=0}^{\infty} k^l$ has sum $1/(1-k)$.

For (ii), note that (i) implies that $T(X_1) \supseteq Y_{(1-k)/r}$: if $y \in Y$ satisfies $\|y\| \leqslant (1-k)/r$ then there exists $x \in X$ such that $Tx = y$ and $\|x\| \leqslant r\|y\|/(1-k) \leqslant 1$. Hence T is open, by Proposition 2.10.

Finally, let $\tilde{Y}$ denote the completion of Y and regard Y as a dense subspace of $\tilde{Y}$, so that T is a bounded operator from E into $\tilde{Y}$. The hypotheses of the theorem hold with Y replaced by $\tilde{Y}$, i.e., $T(E_r)$ is k'-dense in $\tilde{Y}_1$ (where $k' \in (k,1)$, say $k' = (k+1)/2$: to prove this, use the density of Y_1 in $\tilde{Y}_1$) and so by (i) we must have that $T(E) = \tilde{Y}$. Since $T(E) \subseteq Y$ this gives the equality $Y = \tilde{Y}$, showing that Y is complete. □

Theorem 2.13. (Open-Mapping Theorem) If $T \in \mathcal{B}(E,F)$ is surjective, where E and F are Banach spaces, then T is an open map.

Proof

Since T is surjective, $F = \bigcup_{n\geqslant 1} \overline{T(E_n)}$ and so, by Theorem 2.9, there exists $n_0 \in \mathbb{N}$ such that the closure of $T(E_{n_0})$ has non-empty interior: let $y_0 \in F$ and $\varepsilon > 0$ be such that $B_\varepsilon^F(y_0) \subseteq \overline{T(E_{n_0})}$. If $y \in F_1$ then

$$y = \frac{1}{2\varepsilon}\big((y_0+\varepsilon y)-(y_0-\varepsilon y)\big) \in \frac{1}{2\varepsilon}\big(B_\varepsilon^F[y_0]-B_\varepsilon^F[y_0]\big) \subseteq \frac{1}{2\varepsilon}\Big(\overline{T(E_{n_0})}-\overline{T(E_{n_0})}\Big) \subseteq \overline{T(E_{n_0/\varepsilon})},$$

where $A - B := \{a - b : a \in A,\ b \in B\}$ for all $A, B \subseteq E$. (For the last inclusion, let $(x_n)_{n\geqslant 1}$, $(y_n)_{n\geqslant 1} \subseteq E_{n_0}$ be such that $(Tx_n)_{n\geqslant 1}$ and $(Ty_n)_{n\geqslant 1}$ are convergent, and note that

$$\frac{1}{2\varepsilon}\big(\lim_{n\to\infty} Tx_n - \lim_{n\to\infty} Ty_n\big) = \lim_{n\to\infty} T\Big(\frac{1}{2\varepsilon}(x_n - y_n)\Big) \in \overline{T(E_{n_0/\varepsilon})}$$

since $\big\|(x_n - y_n)/(2\varepsilon)\big\| \leqslant (n_0 + n_0)/(2\varepsilon) = n_0/\varepsilon$.) This shows that $T(E_{n_0/\varepsilon})$ is dense in F_1, and the result follows by Lemma 2.12. □

Corollary 2.14. If $T \in \mathcal{B}(E,F)$ is a bijection between Banach spaces E and F then the inverse $T^{-1} \in \mathcal{B}(F,E)$.

Proof

This is immediate. □

Exercises 2

Exercise 2.1. Let X be a topological space and E a Banach space; recall that $C_b(X,E)$, the space of E-valued, bounded, continuous functions on X, is complete with respect to the norm

$$f \mapsto \|f\|_\infty := \sup\{\|f(x)\|_E : x \in X\}.$$

Prove that $C_0(X,E)$, the continuous, E-valued functions on X that *vanish at infinity* (i.e., those $f \in C(X,E)$ such that $\{x \in X : \|f(x)\|_E \geqslant \varepsilon\}$ is compact for all $\varepsilon > 0$) is a closed subspace of $C_b(X,E)$.

Exercise 2.2. Let $(X, \mathcal{T})$ be a Hausdorff, locally compact space and let ∞ denote a point not in X. Show that

$$\dot{\mathcal{T}} := \mathcal{T} \cup \{U \subseteq \dot{X} : \infty \in U,\ X \setminus U \text{ is compact}\}$$

is a Hausdorff, compact topology on $\dot{X} := X \cup \{\infty\}$. [$(\dot{X}, \dot{\mathcal{T}})$ is the *one-point compactification* of $(X, \mathcal{T})$.] Prove that there is a natural correspondence between $C_0(X, E)$ and $\{f \in C(\dot{X}, E) : f(\infty) = 0\}$.

Exercise 2.3. Let X be a separable normed space. Prove that X_1 is separable (in the norm topology). Prove that any separable Banach space E is isometrically isomorphic to a quotient space of ℓ^1. [Consider the map $x \mapsto \sum_{n=1}^{\infty} x_n e_n$ for suitable $(e_n)_{n\geqslant 1} \subseteq E_1$.]

Exercise 2.4. Prove that no infinite-dimensional Banach space E has a countable Hamel basis (where a *Hamel basis* is a linearly independent set S such that every vector in E is a finite linear combination of elements of S).

Exercise 2.5. Let $T\colon X \to Y$ be a linear transformation from the normed space X onto the finite-dimensional normed space Y. Prove that T is continuous if and only if $\ker T$ is closed and that if T is continuous then T is open. [Recall that all norms on a finite-dimensional space are equivalent.]

Exercise 2.6. Let $X = C([0,1], \mathbb{R})$ denote the Banach space of continuous, real-valued functions on the unit interval and for all $k \in \mathbb{N}$ let

$$D_k := \{f \in X : \text{ there exists } t \in [0,1] \text{ such that } |f(s) - f(t)| \leqslant k|s-t| \text{ for all } s \in [0,1]\}.$$

Prove that D_k is closed. [You may find the Bolzano-Weierstrass theorem useful.] Prove further that D_k is nowhere dense. [Consider suitable piecewise-linear functions.] Deduce that there exist continuous functions on $[0,1]$ that are differentiable at no point in $(0,1)$.

Exercise 2.7. Let H be an infinite-dimensional, separable Hilbert space. Prove that $\mathcal{B}(H)_1$ is not separable in the norm topology. [Take an orthonormal basis $\{e_1, e_2, \ldots\}$ of H, define suitable projection operators on its linear span and extend these by continuity. Recall the b4 proof that ℓ^∞ is closed; you may assume that $2^{\mathbb{N}}$, the set of all subsets of $\mathbb{N}$, is uncountable.]

Completeness of Quotient Spaces

The open-mapping theorem provides a quick proof of the following proposition.

Proposition 2.15. Let M be a closed subspace of the Banach space E. The quotient space $\big(E/M, \|\cdot\|_{E/M}\big)$ is complete, i.e., a Banach space.

Proof

If $[x] \in (E/M)_1$ then, by definition of the quotient norm, there exists $m \in M$ such that $x - m \in E_2$ (any number greater than 1 will do) and so $(E/M)_1 \subseteq \pi(E_2)$, where $\pi\colon x \mapsto [x]$ is the quotient map. Since π is a bounded linear operator, the result follows by Lemma 2.12. $\square$

Urysohn's Lemma

To prove certain facts about $C(X)$, the continuous functions on a compact, Hausdorff space (which is the most important of all commutative Banach algebras) we need a result from analytic topology.

Definition 2.16. A topological space X is *normal* if every pair of disjoint, closed sets can be separated by open sets: if C, $D \subseteq X$ are closed and disjoint there there exist disjoint, open sets U, $V \subseteq X$ such that $C \subseteq U$ and $D \subseteq V$. Equivalently, X is normal if for every open set W and closed set C such that $C \subseteq W \subseteq X$ there exists an open set U such that $C \subseteq U \subseteq \bar{U} \subseteq W$. [To see the equivalence, let $D = X \setminus W$.]

Some authors require the additional condition that all singleton sets to be closed for a topology to be normal; we follow [3], [9] and [22], but the definition in [21] includes this and [14] insists on the seemingly stronger requirement that normal spaces be Hausdorff; in fact, in a normal space the conditions that singletons are closed and the Hausdorff property are equivalent.

It is an easy exercise [22, Exercise 5.10.17] to prove that a compact, Hausdorff space is normal (for a proof see [3, Lemma 6.1], [14, Theorem 1.6.6] or [21, Theorem 27.A]). The following lemma yields the fact that compact, Hausdorff spaces (indeed, Hausdorff spaces that are normal) have sufficient continuous functions to separate points.

Lemma 2.17. (Urysohn) Let X be a normal space and let C, $D \subseteq X$ be disjoint and closed. There exists a continuous function $f\colon X \to [0,1]$ such that $f|_C = 0$ and $f|_D = 1$.

Proof

Let $U_1 = X \setminus D$; by normality there exists an open set $U_{1/2}$ such that $C \subseteq U_{1/2}$ and $\bar{U}_{1/2} \subseteq U_1$, and then open sets $U_{1/4}$ and $U_{3/4}$ such that

$$C \subseteq U_{1/4} \subseteq \bar{U}_{1/4} \subseteq U_{1/2} \qquad \text{and} \qquad \bar{U}_{1/2} \subseteq U_{3/4} \subseteq \bar{U}_{3/4} \subseteq U_1.$$

Continuing in this manner we find a family of open sets $\{U_{m2^{-n}} : 1 \leqslant m \leqslant 2^n,\ n \geqslant 1\}$ such that $C \subseteq U_r \subseteq \bar{U}_r \subseteq U_s$ if $r < s$. (Throughout this proof the letters r, s and t refer to dyadic rationals in $(0,1]$, i.e., numbers of the form $m2^{-n}$, where n, $m \in \mathbb{N}$ and $1 \leqslant m \leqslant 2^n$; these are dense in $[0,1]$.) We set

$$f\colon X \to [0,1];\ x \mapsto \begin{cases} 1 & \text{if } x \in D, \\ \inf\{r : x \in U_r\} & \text{if } x \notin D. \end{cases}$$

It is immediate that $f|_D = 1$, and $f|_C = 0$ since $C \subseteq U_r$ for all r; it remains to prove that f is continuous.

If $a \in (0,1]$ then $f(x) < a$ if and only if $\inf\{r : x \in U_r\} < a$, which holds exactly when $x \in U_r$ for some $r < a$, and hence $f^{-1}[0,a) = \bigcup_{r<a} U_r$ is open. If $b \in [0,1)$ then $\inf\{r : x \in U_r\} \leqslant b$ if and only if for all $r > b$ there exists $s < r$ such that $x \in U_s$, which gives that

$$f^{-1}[0,b] = \bigcap_{r>b} \bigcup_{s<r} U_s \subseteq \bigcap_{t>b} \bar{U}_t.$$

This inclusion is actually an equality; let $x \in \bar{U}_t$ for all $t > b$ and suppose that $r > b$. There exist s, t such that $r > s > t > b$ and so $x \in \bar{U}_t \subseteq U_s$, whence $x \in \bigcap_{r>b} \bigcup_{s<r} U_s$, as claimed. This shows that $f^{-1}[0, b]$ is closed, and so $f^{-1}(b, 1]$ is open, for all $b \in [0, 1)$. As $\{(0, a],\ (b, 1] : a \in (0, 1],\ b \in [0, 1)\}$ is a subbase for the usual topology on $[0, 1]$ we have the result. $\square$

Proposition 2.18. Let X be a locally compact, Hausdorff space and let E be a Banach space. The space $C_{00}(X, E)$ of compactly supported, continuous, E-valued functions on X is dense in $C_0(X, E)$, the space of continuous, E-valued functions on X that vanish at infinity.

Proof

Let $\varepsilon > 0$ and $f \in C_0(X, E)$; by definition, $K = \{x \in X : \|f(x)\| \geqslant \varepsilon\}$ is compact. For all $x \in K$ let U_x be an open set containing x and with compact closure; these exist by local compactness. As K is compact, $K \subseteq \bigcup_{i=1}^n U_{x_i}$ for $x_1, \ldots, x_n \in K$, and $L = \bigcup_{i=1}^n \bar{U}_{x_i}$ is a compact set such that $K \subseteq L^\circ$.

By Urysohn's lemma there exists a continuous function $g\colon L \to [0, 1]$ such that $g|_K = 1$ and $g|_{L \setminus L^\circ} = 0$; extend g to X by defining

$$h\colon X \to [0, 1];\ x \mapsto \begin{cases} g(x) & \text{if } x \in L, \\ 0 & \text{if } x \in X \setminus L. \end{cases}$$

Then h has compact support and is continuous: if $C \subseteq [0, 1]$ is closed then $g^{-1}(C)$ is closed in L, so in X, and $h^{-1}(C)$ equals $g^{-1}(C)$ (if $0 \notin C$) or $g^{-1}(C) \cup (X \setminus L^\circ)$ (if $0 \in C$). Hence $fh \in C_{00}(X, E)$, and $\|fh - f\|_\infty < \varepsilon$, as required: if $x \in K$ then $f(x)h(x) = f(x)$, and if $x \in X \setminus K$ then $\|f(x)\| < \varepsilon$, so $\|f(x)h(x) - f(x)\| = (1 - h(x))\|f(x)\| < \varepsilon$. $\square$

The following theorem can be deduced from Urysohn's lemma directly, or with an application of the open-mapping lemma. It is a theorem of Hahn-Banach type, but applies to continuous functions on normal spaces.

Theorem 2.19. (Tietze) Let X be a normal space and let Y be a closed subset of X. If f is a continuous, bounded, real-valued function on Y then there exists a continuous, bounded, real-valued function F on X such that $F|_Y = f$ and $\|F\|_\infty = \|f\|_\infty$.

Proof

Let

$$T\colon C_{\mathrm{b}}(X, \mathbb{R}) \to C_{\mathrm{b}}(Y, \mathbb{R});\ f \mapsto f|_Y$$

be the restriction map and note that T is continuous. Let $f \in C_{\mathrm{b}}(Y, \mathbb{R})$ be such that $\|f\|_\infty \leqslant 1$, and let $C = f^{-1}[-1, -1/3]$ and $D = f^{-1}[1/3, 1]$. These are closed subsets of Y, so of X, and by Urysohn's lemma there exists $g \in C_{\mathrm{b}}(X, \mathbb{R})$ such that $\|g\|_\infty \leqslant 1/3$, $g|_C = -1/3$ and $g|_D = 1/3$. Hence $\|Tg - f\|_\infty \leqslant 2/3$, and so T satisfies the conditions of the open-mapping lemma: $T(C_{\mathrm{b}}(X, \mathbb{R})_{1/3})$ is 2/3-dense in $C_{\mathrm{b}}(Y, \mathbb{R})_1$. In particular, T is surjective, so there exists $F \in C_{\mathrm{b}}(X, \mathbb{R})$ such that $T(F) = f$ and

$$\|f\|_\infty \leqslant \|F\|_\infty \leqslant \frac{1/3}{1 - 2/3}\|f\|_\infty = \|f\|_\infty,$$

as required. □

It is an exercise to extend the above to unbounded real-valued functions, and to complex-valued functions; see Exercise 3.3.

The Closed-Graph Theorem

Recall that if $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ are topological spaces then the *product topology* on $X \times Y$ has basis

$$\{U \times V : U \in \mathcal{T},\ V \in \mathcal{S}\}.$$

If X and Y are normed spaces then this topology is given by the *product norm*,

$$\|\cdot\|_{X\times Y} \colon X \times Y \to \mathbb{R}^+;\ (x, y) \mapsto \|x\|_X + \|y\|_Y.$$

Theorem 2.20. (Closed-Graph Theorem) Let E, F be Banach spaces. A linear transformation $T\colon E \to F$ is bounded if and only if the *graph* of T,

$$\mathcal{G}(T) := \{(x, Tx) : x \in E\} \subseteq E \times F,$$

is closed (with respect to the product topology on $E \times F$).

Proof

It is a standard (and simple) result from point-set topology that any continuous function with values in a Hausdorff space has closed graph; this is often set as an exercise [21, Exercise 26.6], [22, Exercise 4.3.3] and a proof may be found in [19, Proposition 2.14].

Now suppose that $\mathcal{G}(T)$ is closed and note that $\mathcal{G}(T)$ is a subspace of $E \times F$, so a Banach space with respect to the product norm. Let

$$\pi_1 \colon \mathcal{G}(T) \to E;\ (x, Tx) \mapsto x;$$

this linear transformation is norm-decreasing (so continuous) and bijective, so by the open-mapping theorem π_1^{-1} is bounded. Furthermore

$$\pi_2 \colon \mathcal{G}(T) \to F;\ (x, Tx) \mapsto Tx$$

is continuous, hence $T = \pi_2 \circ \pi_1^{-1}$ is bounded, as required. □

The closed-graph theorem is often used in the following manner. *A priori*, to show that a linear transformation between Banach spaces is continuous we must show that if $x_n \to x$ then $Tx_n \to Tx$, for any sequence $(x_n)_{n\geqslant 1}$. The closed-graph theorem means that we need only show that the graph of T contains its limit points, i.e., if $(x_n)_{n\geqslant 1}$ is such that $x_n \to x$ and $Tx_n \to y$ then $Tx = y$. In this case we have control over $(x_n)_{n\geqslant 1}$ and also over $(Tx_n)_{n\geqslant 1}$, which is a considerable improvement. An application of this idea occurs in the solution to Exercise 4.8.

The Principle of Uniform Boundedness

We employ the closed-graph theorem to give a proof of the principle of uniform boundedness, also known as the Banach-Steinhaus theorem.

Theorem 2.21. (Banach-Steinhaus) Let E be a Banach space, Y a normed space and suppose that $\{T_a : a \in A\} \subseteq \mathcal{B}(E, Y)$. If $\{T_a x : a \in A\} \subseteq Y$ is bounded, for all $x \in E$, then $\{\|T_a\| : a \in A\}$ is bounded.

Proof

Note first that we may replace Y by its completion, so without loss of generality we assume that Y is a Banach space. Let $Y_a := Y$ for all $a \in A$ and let $Z := \prod_{a\in A} Y_a$ be their direct product. Define

$$T\colon E \to Z;\ x \mapsto (T_a x)_{a\in A}$$

and note that the pointwise boundedness of the T_a ensures that T is well defined. For all $b \in A$ let

$$\pi_b\colon Z \to Y_b;\ (y_a)_{a\in A} \mapsto y_b$$

and observe that this map is linear and norm-decreasing. Let $(x_n, Tx_n)_{n\geqslant 1} \subseteq \mathcal{G}(T)$ be convergent, say $x_n \to x$ and $Tx_n \to y$. For all $a \in A$ we have that

$$\pi_a y = \lim_{n\to\infty} \pi_a T x_n = \lim_{n\to\infty} T_a x_n = T_a x = \pi_a T x,$$

by the continuity of π_a and T_a, which shows that $y = Tx$. Hence T has closed graph, so is bounded, and

$$\|T_a x\| = \|\pi_a T x\| \leqslant \|Tx\| \leqslant \|T\|\,\|x\| \qquad \forall\, x \in E,$$

which shows that $\|T_a\| \leqslant \|T\|$ for all $a \in A$. □

The Strong Operator Topology

Let H be a Hilbert space with orthonormal basis $\{e_1, e_2, \ldots\}$ and define orthogonal projections $P_n \in \mathcal{B}(H)$ by setting

$$P_n\colon H \to H;\ x \mapsto \sum_{k=1}^{n} \langle e_k, x\rangle e_k.$$

It is easy to see that $P_n x \to x$ as $n \to \infty$ for all $x \in H$, by the Parseval equality, but as $\|P_m - P_n\| \geqslant 1$ for all $m \neq m$ it cannot be the case that $\|P_n - I\| \to 0$ as $n \to \infty$. This example highlights the utility of a weaker sense of convergence for operators.

Definition 2.22. Let X and Y be normed spaces; the initial topology on $\mathcal{B}(X, Y)$ generated by the family of maps $\{T \mapsto Tx : x \in X\}$ (where Y is equipped with its norm topology) is called the *strong operator topology*.

Since

$$\|Tx\| \leqslant \|T\|\,\|x\| \qquad \forall\, x \in X,\ T \in \mathcal{B}(X,Y),$$

we see that norm convergence implies strong operator convergence: a net $(T_a)_{a\in A}$ in $\mathcal{B}(X,Y)$ is convergent to T in the norm topology if and only if for all $\varepsilon > 0$ there exists $a_0 \in A$ such that $\|T_a - T\| < \varepsilon$ for all $a \geqslant a_0$, and similarly for strong operator convergence. Hence sets that are strong operator closed are also norm closed, and so the strong operator topology is coarser that the norm topology on $\mathcal{B}(X,Y)$.

Exercises 3

Exercise 3.1. Let H be a separable Hilbert space with orthonormal basis $\{e_1, e_2, \ldots\}$. For $n \geqslant 1$ let P_n denote the orthogonal projection onto $\mathbb{F}e_1 + \cdots + \mathbb{F}e_n$; prove that $P_nTP_nx \to Tx$ as $n \to \infty$ for all $T \in \mathcal{B}(H)$ and $x \in H$. Deduce that $\mathcal{B}(H)$ is separable in the strong operator topology.

Exercise 3.2. Prove that if E is a Banach space with respect to two different norms then they are either equivalent or non-comparable (i.e., neither is coarser than the other).

Exercise 3.3. Prove the following extension of Tietze's theorem to complex-valued functions: if X is a normal space, Y a closed subset of X and $f \in C_{\mathrm{b}}(Y)$ then there exists $F \in C_{\mathrm{b}}(X)$ such that $F|_Y = f$ and $\|F\|_\infty = \|f\|_\infty$. Prove also that Tietze's theorem applies to unbounded, real-valued functions: if X and Y are as above and $f\colon Y \to \mathbb{R}$ is continuous then there exists $F\colon X \to \mathbb{R}$ such that $F|_Y = f$.

Exercise 3.4. Let E be a Banach space, Y a normed vector space and suppose that $(T_n)_{n\geqslant 1} \subseteq \mathcal{B}(E,Y)$ is such that $\lim_{n\to\infty} T_nx$ exists for all $x \in E$. Prove that there exists $T \in \mathcal{B}(E,Y)$ such that $T_n \to T$ in the strong operator topology. What can be said about the norm of T?

Exercise 3.5. Let $x = (x_n)_{n\geqslant 1}$ be a sequence of complex numbers such that the series $\sum_{n=1}^\infty x_ny_n$ is convergent for all $y \in c_0$. Prove that $x \in \ell^1$. [Consider the mappings $f_n\colon y \mapsto \sum_{j=1}^n x_jy_j$.]

Exercise 3.6. Let E be a Banach space with closed subspaces F and G such that $E = F \oplus G$ (i.e., every element of E can be expressed uniquely as the sum of an element of F and an element of G). Define P_F and P_G by setting

$$P_F\colon E \to E;\ f + g \mapsto f \quad \text{and} \quad P_G\colon E \to E;\ f + g \mapsto g \qquad \forall\, f \in F,\ g \in G.$$

Prove that P_F and P_G are bounded linear operators such that $P_F^2 = P_F$, $P_G^2 = P_G$ and $P_FP_G = P_GP_F = 0$.

Exercise 3.7. Find a Banach space E with closed subspaces F and G such that $E = F \oplus G$ and

$$P : E \to E;\ f + g \mapsto f \qquad \forall\, f \in F,\ g \in G$$

has norm strictly greater than one. [Let $E = \mathbb{R}^3$ with the norm $\|(x_1,x_2,x_3)\| = \max\{|x_1|,|x_2|,|x_3|\}$.]

Exercise 3.8. Let E be a Banach space with closed subspaces F and G such that $F \cap G = \{0\}$. Prove that $F \oplus G$ is closed if and only if there exists $C > 0$ such that

$$\|f\| \leqslant C\|f + g\| \qquad \forall\, f \in F,\ g \in G.$$

Deduce that $F \oplus G$ is closed if and only if

$$c := \inf\{\|f - g\| : f \in F,\ g \in G,\ \|f\| = \|g\| = 1\} > 0.$$

Three Dual Spaces

Initial Definitions

Definition 3.1. Let X be a vector space over the field $\mathbb{F}$. A *linear functional* on X is a linear map $\phi\colon X \to \mathbb{F}$. The set of all linear functionals on X is a vector space denoted by X', the *algebraic dual space* of X, where the vector-space structure is defined pointwise:

$$(\phi + \psi)(x) := \phi(x) + \psi(x) \quad \text{and} \quad (\alpha\phi)(x) := \alpha\phi(x) \qquad \forall\, \phi, \psi \in X',\ \alpha \in \mathbb{F}.$$

If X is a normed space then X' contains $X^* := \mathcal{B}(X, \mathbb{F})$, the *topological dual space* of X. An element of X^* is said to be a *bounded linear functional* on X.

Our notation for the algebraic and topological dual spaces is the opposite of that adopted in [10].

If X is infinite dimensional then the inclusion of X^* in X' is proper; if X is finite dimensional then the spaces coincide. Being interested primarily in analysis, henceforth the term *dual space* will refer to the topological dual.

Example 3.2. Recall that

$$(c_0)^* \cong \ell^1 \quad \text{and} \quad (\ell^p)^* \cong \ell^q$$

if $p \in [1, \infty)$ and $1/p + 1/q = 1$, where $\cong$ denotes isometric isomorphism. More generally, if I is a subinterval of $\mathbb{R}$ (or a σ-finite measure space)

$$L^p(I)^* \cong L^q(I)$$

for the same pairs p and q; the isomorphism is analogous to the ℓ^p case: $g \in L^q(I)$ yields an element of $L^p(I)^*$ *via* $f \mapsto \int_I \overline{g(t)} f(t)\, \mathrm{d}t$. Proof that every element of $L^p(I)^*$ arises this way requires the Radon-Nikodým theorem [18, Theorem 6.16].

Example 3.3. The Riesz-Fréchet theorem implies that $H^* \cong H$ for any Hilbert space H.

The Weak Topology

Definition 3.4. Any normed space X gains a natural topology from its dual space, its *weak topology*. This is the initial topology generated by X^*, i.e., the coarsest topology to make each map $\phi \in X^*$ continuous. The weak topology on X is denoted by $\sigma(X, X^*)$. (The letter σ is used here because the German word for weak is *schwach*.)

It is by no means clear that an infinite-dimensional space has any continuous functionals, but the Hahn-Banach theorem guarantees a plentiful supply (enough to ensure that the weak topology is Hausdorff). In order to prove the Hahn-Banach theorem in full generality, we need a version of the Axiom of Choice.

Zorn's Lemma

Definition 3.5. A *partial order* on a set A is a preorder $\leqslant$ (see Definition 1.27) that is *antisymmetric*: for all a, $b \in A$,

$$a \leqslant b \text{ and } b \leqslant a \text{ imply that } a = b.$$

Let A be a set with a partial order $\leqslant$. A *chain* C in A is a subset of A such that, for all a, $b \in C$, either $a \leqslant b$ or $b \leqslant a$. An *upper bound* or *majorant* for $B \subseteq A$ is an element $a \in A$ such that $b \leqslant a$ for all $b \in B$. An element $a \in A$ is *maximal* if $a \leqslant b$ implies that $a = b$ for all $b \in X$.

Lemma 3.6. (Zorn) Let A be a non-empty set with a partial order $\leqslant$. If every chain in A has an upper bound then A has a maximal element.

Proof

We take the lemma as axiomatic; it is equivalent to the Axiom of Choice. (For a proof of this, see [14, Theorem 1.1.6].) □

The Hahn-Banach Theorem

Definition 3.7. Let X be a real vector space. A *sublinear functional* on X is a function $p\colon X \to \mathbb{R}$ such that, for all x, $y \in X$ and $\alpha \in \mathbb{R}^+$,

$$\begin{array}{llll} & \text{(i)} & p(\alpha x) = \alpha p(x) & (\textit{positive homogeneity}) \\ \text{and} & \text{(ii)} & p(x+y) \leqslant p(x) + p(y) & (\textit{subadditivity}). \end{array}$$

Theorem 3.8. (Hahn-Banach) Let p be a sublinear functional on the real vector space X and suppose that M is a subspace of X. If $f \in M'$ is a linear functional that satisfies $f(m) \leqslant p(m)$ for all $m \in M$ (p is a majorant for f) then there exists a linear functional $F \in X'$ such that $F|_M = f$ and $F(x) \leqslant p(x)$ for all $x \in X$.

Proof

The proof of the Hahn-Banach theorem falls naturally into two parts. The first involves showing that f has a "one-dimensional extension", i.e., f extends to N, where N has codimension one in M: this should be familiar from the b4 course. The second part is an application of Zorn's lemma (which is necessary only if X is non-separable; otherwise simple induction will suffice). Throughout we assume (as we may) that M is a proper subspace of X.

Choose a vector $x_0 \in X \setminus M$, let $N := M + \mathbb{R}x_0$ and for all $\gamma \in \mathbb{R}$ define

$$f_\gamma \colon N \to \mathbb{R};\ m + \alpha x_0 \mapsto f(m) + \alpha\gamma.$$

This is a sound definition because N is the direct sum of M and $\mathbb{R}x_0$, and furthermore $f_\gamma|_M = f$. It remains to show that we may find $\gamma \in \mathbb{R}$ such that $f_\gamma(x) \leqslant p(x)$ for all $x \in N$, i.e.,

$$f(m) + \alpha\gamma \leqslant p(m + \alpha x_0) \qquad \forall\, m \in M,\ \alpha \in \mathbb{R}.$$

By positive homogeneity of p and linearity of f, the above inequality will be satisfied if

$$f(m) + \gamma \leqslant p(m + x_0) \quad \forall\, m \in M \qquad \text{and} \qquad f(m) - \gamma \leqslant p(m - x_0) \quad \forall\, m \in M.$$

Hence we wish to find $\gamma \in \mathbb{R}$ such that

$$f(m) - p(m - x_0) \leqslant \gamma \leqslant -f(n) + p(n + x_0) \quad \forall\, m, n \in M$$

but, by the subadditivity of p and linearity of f,

$$-f(n) + p(n + x_0) - f(m) + p(m - x_0) \geqslant -f(n + m) + p(n + m) \geqslant 0,$$

so such γ exists.

Let

$$\mathcal{S} = \big\{(g, N) : N \text{ is a subspace of } X,\ M \subseteq N,\ g \in N',\ g|_M = f,\ g(x) \leqslant p(x)\ \forall\, x \in N\big\}$$

denote all suitable extensions of f; the previous part shows that this set is non-empty. Define $\leqslant$ on $\mathcal{S}$ by saying that

$$(g, N) \leqslant (h, P) \iff N \subseteq P \ \text{ and } \ h|_N = g$$

and note that this is a partial order. Let $\{(g_a, N_a) : a \in A\}$ be a chain in $(\mathcal{S}, \leqslant)$, let $N := \bigcup_{a \in A} N_a$ and define

$$g \colon N \to \mathbb{R};\ x \mapsto g_a(x) \text{ if } x \in N_a.$$

It is a simple exercise to see that $(g, N) \in \mathcal{S}$ and $(g_a, N_a) \leqslant (g, N)$ for all $a \in A$. By Zorn's lemma we conclude that there exists $(h, P) \in \mathcal{S}$ that is maximal for $\leqslant$; if we can show that $P = X$ then we are done. Suppose otherwise; then P is a proper subspace of X and there exists a proper extension of h, by the first part of this proof. This contradicts the maximality of (h, P). □

Theorem 3.9. (Bohnenblust-Sobczyk) Let p be a seminorm on the vector space X and suppose that M is a subspace of X. If $\phi \in M'$ is a linear functional such that $|\phi(m)| \leqslant p(m)$ for all $m \in M$ (ϕ is *dominated* by p) then there exists a linear functional $\Phi \in X'$ that extends ϕ (i.e., $\Phi|_M = \phi$) and is dominated by p (i.e., $|\Phi(x)| \leqslant p(x)$ for all $x \in X$).

Proof

Suppose first that X is a real vector space. Note that a seminorm is a sublinear functional, so we may apply the Hahn-Banach theorem to obtain $\Phi \in X'$ such that $\Phi|_M = \phi$ and $\Phi(x) \leqslant p(x)$ for all $x \in X$, but also

$$-\Phi(x) \leqslant p(-x) = p(x) \quad \forall\, x \in X$$

by the homogeneity of the seminorm p. Hence $|\Phi(x)| \leqslant p(x)$ for all $x \in X$, as required.

Now suppose that X is a complex vector space. We may regard it as a real vector space and apply the first part of this proof to obtain a real-linear functional F on X that extends $\operatorname{Re}\phi$ and is dominated by p. Define Φ by

$$\Phi\colon X \to \mathbb{C};\ x \mapsto F(x) - iF(ix).$$

It is clear that Φ is additive, and if $a, b \in \mathbb{R}$ then

$$\begin{aligned}\Phi((a+ib)x) &= F((a+ib)x) - iF(i(a+ib)x)\\ &= F(ax) + F(ibx) - iF(iax) - iF(-bx)\\ &= (a+ib)F(x) - i(a+ib)F(ix) \;=\; (a+ib)\Phi(x) \quad \forall\, x \in X,\end{aligned}$$

so $\Phi \in X'$. Note also that

$$\begin{aligned}\Phi(m) = F(m) - iF(im) &= \operatorname{Re}\phi(m) - i\operatorname{Re} i\phi(m)\\ &= \operatorname{Re}\phi(m) + i\operatorname{Im}\phi(m) = \phi(m) \quad \forall\, m \in M,\end{aligned}$$

so $\Phi|_M = \phi$. Finally, let $x \in X$ and choose $\alpha \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ such that $\alpha\Phi(x) \in \mathbb{R}^+$. Then

$$|\Phi(x)| = |\alpha\Phi(x)| = \operatorname{Re}\alpha\Phi(x) = \operatorname{Re}\Phi(\alpha x) = F(\alpha x) \leqslant p(\alpha x) = p(x)$$

and so Φ is dominated by p, as required. □

The Dual Space Separates Points

Theorem 3.10. Let X be a normed space. For all $x \in X \setminus \{0\}$ there exists $\phi \in X^*$ such that $\phi(x) = \|x\|$ and $\|\phi\| = 1$.

Proof

Let $M = \mathbb{F}x := \{\alpha x : \alpha \in \mathbb{F}\}$ and define $f\colon M \to \mathbb{F};\ \alpha x \mapsto \alpha\|x\|$. Note that $|f(\alpha x)| = |\alpha|\,\|x\| = \|\alpha x\|$ and so f is dominated on M by $\|\cdot\|$. By Theorem 3.9 there exists a linear functional $\phi\colon X \to \mathbb{F}$ such that $\phi|_M = f$ (in particular, $\phi(x) = \|x\|$) and $|\phi(y)| \leqslant \|y\|$ for all $y \in X$ (so that $\|\phi\| \leqslant 1$). Combining these observations gives the result. □

This proves that the weak topology on X is Hausdorff: if $x, y \in X$ are such that $x \neq y$ then there exists $\phi \in X^*$ such that $\phi(x-y) = \|x-y\| \neq 0$; the claim follows by Proposition 1.26.

Existence of Completions

Recall that $X^{**} = (X^*)^*$ is the *bidual* or *second dual* of the normed space X. For all $x \in X$ define a linear functional $\hat{x}$ on X^* by setting $\hat{x}(\phi) = \phi(x)$ and note that

$$|\hat{x}(\phi)| = |\phi(x)| \leqslant \|\phi\| \, \|x\|$$

so that $\hat{x} \in X^{**}$ with $\|\hat{x}\| \leqslant \|x\|$. Theorem 3.10 shows that the map

$$\Gamma \colon X \to X^{**}; \; x \mapsto \hat{x}$$

is an isometry, called the *canonical embedding* of X into its bidual. If the canonical embedding is surjective then X is said to be *reflexive*.

Proposition 3.11. If X is a normed vector space then it has a completion: there exists a Banach space $\tilde{X}$ and a linear isometry $i \colon X \to \tilde{X}$ such that $i(X)$ is dense in $\tilde{X}$.

Proof

Let $\tilde{X} := \overline{\Gamma(X)}$ be the closure in the bidual X^{**} of the image of X under the canonical embedding. As X^{**} is complete (being the dual of a normed space) and closed subspaces of Banach spaces are complete (Proposition 1.8), $\big(\tilde{X}, \| \cdot \|_{X^{**}}|_{\tilde{X}}\big)$ is a Banach space containing X as a dense subspace. Let $i \colon X \to \tilde{X}$; $x \mapsto \hat{x}$; it is immediate from the previous remarks that i is a linear isometry, as required. □

There is another, more pedestrian way of finding the completion of a normed space (or any metric space) which mimics the way that the real numbers may be constructed as a collection of equivalence classes of sequences of rational numbers: see [3, pp. 34–35] or [22, Theorem 11.2.2]; it is not difficult to check that the completion inherits the structure of a normed space [3, Theorem 2.7].

Y is Complete if $\mathcal{B}(X, Y)$ is Complete

Proposition 3.12. If X and Y are normed vector spaces with $X \neq \{0\}$ and $\mathcal{B}(X, Y)$ complete then Y is complete.

Proof

Let $(y_n)_{n \geqslant 1} \subseteq Y$ be a Cauchy sequence and let $x_0 \in X$ be a unit vector. By Theorem 3.10 there exists a linear functional $\phi \in X^*$ such that $\|\phi\| = 1 = \phi(x_0)$. For $n \geqslant 1$ define $T_n \in \mathcal{B}(X, Y)$ by setting $T_n x = \phi(x) y_n$ and note that $T_n x_0 = y_n$. Then

$$\|(T_n - T_m)x\| = |\phi(x)| \, \|y_n - y_m\| \leqslant \|y_n - y_m\| \, \|x\| \qquad \forall \, x \in X$$

and so $\|T_n - T_m\| \leqslant \|y_n - y_m\|$, which shows that $(T_n)_{n \geqslant 1}$ is a Cauchy sequence in $\mathcal{B}(X, Y)$. Let T be the limit of this sequence and conclude by noting that

$$\lim_{n \to \infty} y_n = \lim_{n \to \infty} T_n x_0 = T x_0.$$

□

Vector-valued Holomorphic Functions

Definition 3.13. Let X be a complex normed space and let U be an open subset of $\mathbb{C}$. A function $f\colon U \to X$ is *weakly holomorphic* if $\phi \circ f\colon U \to \mathbb{C}$ is holomorphic for all $\phi \in X^*$, and is *strongly holomorphic* if

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} \quad \text{exists} \quad \forall\, z \in U,$$

where this limit is taken with respect to the norm topology on X. Note that every strongly holomorphic function is weakly holomorphic.

Theorem 3.14. (Liouville) Let $f\colon \mathbb{C} \to X$ be a weakly holomorphic function into the complex normed space X. If f is bounded, i.e., there exists $r \in \mathbb{R}^+$ such that $\|f(z)\| \leqslant r$ for all $z \in \mathbb{C}$ (equivalently, $f(\mathbb{C}) \subseteq X_r$) then f is constant.

Proof

For all $\phi \in X^*$ we have that $\phi \circ f\colon \mathbb{C} \to \mathbb{C}$ is bounded and holomorphic everywhere, and so constant, by the classical Liouville's theorem [16, § 5.2]. Hence

$$\phi(f(z)) = \phi(f(0)) \quad \forall z \in \mathbb{C},\ \phi \in X^*.$$

As X^* separates points in X we must have that $f(z) = f(0)$ for all $z \in \mathbb{C}$, i.e., f is constant. □

The proof holds if the holomorphic function f is only required to be weakly bounded, i.e., for all $\phi \in X^*$ there exists $r_\phi \in \mathbb{R}^+$ with $|\phi(f(z))| \leqslant r_\phi$ for all $z \in \mathbb{C}$. This is not a generalisation, however, as a subset of a normed space is weakly bounded if and only if it is norm bounded (Exercise 4.6).

The Weak Operator Topology

Definition 3.15. Let X, Y be normed spaces; the intial topology on $\mathcal{B}(X,Y)$ generated by the collection of maps $\{T \mapsto \phi(Tx) : x \in X,\ \phi \in Y^*\}$ (where $\mathbb{F}$ is equipped with its usual topology) is called the *weak operator topology*.

In the same manner as we compared strong operator and norm convergence, the fact that

$$|\phi(Tx)| \leqslant \|\phi\|\,\|Tx\| \qquad \forall\, x \in X,\ \phi \in Y^*,\ T \in \mathcal{B}(X,Y)$$

shows that strong operator convergence implies weak operator convergence; thus sets that are weak operator closed are also strong operator closed, and so the weak operator topology is coarser than the strong operator topology.

Adjoint Operators

Theorem 3.16. Let X, Y be normed vector spaces and let $T \in \mathcal{B}(X,Y)$. There exists $T^* \in \mathcal{B}(Y^*,X^*)$ such that

$$\phi(Tx) = (T^*\phi)(x) \qquad \forall\, x \in X,\ \phi \in Y^*.$$

Furthermore $\|T^*\| = \|T\|$.

Proof

This is a b4 result [24, Theorem 2.2.13]. $\square$

Definition 3.17. If M is a subspace of the normed space X and N is a subspace of X^* then

$$M^\perp := \{\phi \in X^* : \phi(x) = 0 \text{ for all } x \in M\}$$

is the *annihilator* of M and

$${}^\perp N := \{x \in X : \phi(x) = 0 \text{ for all } \phi \in N\}$$

is the *pre-annihilator* of N.

Theorem 3.18. Let X and Y be normed spaces and let $T \in \mathcal{B}(X, Y)$. Then

$$\ker T = {}^\perp(\operatorname{im} T^*) \qquad \text{and} \qquad \ker T^* = (\operatorname{im} T)^\perp.$$

Proof

Note that, since Y^* separates points in Y,

$$\begin{aligned} \ker T = \{x \in X : Tx = 0\} &= \{x \in X : \phi(Tx) = 0 \text{ for all } \phi \in Y^*\} \\ &= \{x \in X : (T^*\phi)(x) = 0 \text{ for all } \phi \in Y^*\} = {}^\perp(\operatorname{im} T^*). \end{aligned}$$

The other identity can be established in the same manner. $\square$

The Weak* Topology

Definition 3.19. Let X be a normed vector space. The *weak* topology* on X^* is the initial topology generated by the maps

$$\hat{x} \colon X^* \to \mathbb{F};\ \phi \mapsto \phi(x) \qquad (x \in X),$$

i.e., the coarsest topology to make these maps continuous. The weak* topology on X^* is denoted by $\sigma(X^*, X)$.

Note that the weak* topology is Hausdorff, by Proposition 1.26: if $\phi, \psi \in X^*$ are distinct then there exists $x \in X$ such that $\phi(x) \neq \psi(x)$, and so the map $\hat{x}$ separates these points. Note also that $\phi \mapsto \phi(x) \in X^{**}$ for all $x \in X$, so the weak* topology $\sigma(X^*, X)$ is even coarser than the weak topology $\sigma(X^*, X^{**})$: there are fewer functions required to be continuous.

Exercises 4

Exercise 4.1. A closed subspace M of the normed space X is *complemented in* X if there exists a closed subspace N such that $M \oplus N = X$, i.e., $M + N = X$ and $M \cap N = \{0\}$. Prove that M is complemented in X if M is finite dimensional. [Start by considering a basis of M^*.] Prove also that M is complemented in X if M has finite codimension, i.e., $\dim X/M < \infty$.

Exercise 4.2. Let M be a finite-dimensional subspace of the normed space X and let N be a closed subspace of X such that $X = M \oplus N$. Prove that if ϕ_0 is a linear functional on M then

$$\phi\colon M \oplus N \to \mathbb{F};\ m + n \mapsto \phi_0(m) \qquad \forall\, m \in M,\ n \in N$$

is an element of the dual space X^*.

Exercise 4.3. Prove that a normed vector space X is separable if its dual X^* is. [You may assume that if M is a non-empty subspace of X and $x_0 \in X \setminus \bar{M}$ then there exists $\phi \in X^*$ such that $\phi|_M = 0$ and $\phi(x_0) = 1$.] Find a separable Banach space E such that E^* is not separable. [Proof of (non-)separability is not required.] Prove that a reflexive Banach space E is separable if and only if E^* is.

Exercise 4.4. Prove that a Banach space E is reflexive if and only its dual E^* is reflexive.

Exercise 4.5. Prove that any infinite-dimensional normed space has a discontinuous linear functional defined on it.

Exercise 4.6. Let A be a subset of the normed vector space X. Prove that A is *norm bounded* (there exists $r \in \mathbb{R}^+$ such that $\|a\| \leqslant r$ for all $a \in A$) if and only if it is *weakly bounded* (for all $\phi \in X^*$ there exists $r_\phi \in \mathbb{R}^+$ such that $|\phi(a)| \leqslant r_\phi$ for all $a \in A$). [Use the principle of uniform boundedness and the canonical embedding $\Gamma\colon x \mapsto \hat{x}$.] Deduce that a weakly holomorphic function is (strongly) continuous. [Cauchy's integral formula may be useful.]

Exercise 4.7. Let H be a Hilbert space. Prove that the adjoint $T \mapsto T^*$ is continuous with respect to the weak operator topology on $\mathcal{B}(H)$, but not necessarily with respect to the strong operator topology. [For the latter claim, consider the operators $T_n \in \mathcal{B}(\ell^2)$ such that $T_n x = \langle e_1, x \rangle e_n$ for all $x \in \ell^2$, where $\{e_k : k \geqslant 1\}$ is the standard orthonormal basis of ℓ^2.]

Exercise 4.8. Let E and F be Banach spaces. Show that if $T\colon E \to F$ and $S\colon F^* \to E^*$ are linear transformations that satisfy

$$\phi(Tx) = (S\phi)(x) \qquad \forall\, x \in E,\ \phi \in F^*$$

then S and T are bounded, with $S = T^*$. [Use the closed-graph theorem.]

Exercise 4.9. Let E and F be Banach spaces and suppose that $T \in \mathcal{B}(E, F)$ has closed range, i.e., $\operatorname{im} T$ is closed in F. Prove that $\operatorname{im} T^* = (\ker T)^\perp$ (where

$$M^\perp := \{\phi \in E^* : \phi(x) = 0 \text{ for all } x \in M\}$$

is the annihilator of the subspace $M \subseteq E$).

Exercise 4.10. Let $E = c_0$, so that $E^* = \ell^1$ and $E^{**} = \ell^\infty$. Prove that $x \mapsto \sum_{n=1}^\infty x_n$ is weakly continuous on ℓ^1 but is not weak* continuous.

Exercise 4.11. Prove that a compact metric space is separable. Prove that if X is a separable normed space then X_1^*, the closed unit ball of the dual space X^*, is metrizable when equipped with the weak* topology. [Let $(x_n)_{n\geqslant 1} \subseteq X_1$ be dense in X_1 and consider $d(\phi, \psi) := \sum_{n=1}^{\infty} 2^{-n}|\phi(x_n) - \psi(x_n)|$.] Deduce that X^* is separable in the weak* topology.

Exercise 4.12. Let X and Y be normed spaces and for all $x \in X$ and $y \in Y$ let

$$x \otimes y \colon \mathcal{B}(X, Y^*) \to \mathbb{F};\ T \mapsto (Tx)(y).$$

Prove that $x \otimes y \in \mathcal{B}(X, Y^*)^*$, with $\|x \otimes y\| = \|x\|\,\|y\|$, and that the mapping

$$X \times Y \to \mathcal{B}(X, Y^*); (x, y) \mapsto x \otimes y$$

is bilinear. If Z is the closed linear span of $\{x \otimes y : x \in X,\ y \in Y\}$ in $\mathcal{B}(X, Y^*)^*$, prove that

$$j \colon \mathcal{B}(X, Y^*) \to Z^*;\ j(T)z = z(T)$$

is an isometric isomorphism. [For surjectivity, let $\phi \in Z^*$ and consider $\phi_x \colon y \mapsto \phi(x \otimes y)$.]

Tychonov's Theorem

Definition 3.20. Let $\big((X_a, \mathcal{T}_a) : a \in A\big)$ be a collection of topological spaces. Their *topological product* is $(X, \mathcal{T})$, where

$$X = \bigtimes_{a\in A} X_a := \{(x_a)_{a\in A} : x_a \in X_a\ \forall\, a \in A\}$$

is the Cartesian product of the sets X_a and $\mathcal{T} = \prod_{a\in A} \mathcal{T}_a$ is the *product topology*, i.e., the initial topology generated by the projection maps

$$\pi_b \colon X \to X_b;\ (x_a)_{a\in A} \mapsto x_b \qquad (b \in A).$$

The fundamental fact about the product topology is that is preserves compactness; this is Tychonov's theorem. The proof of Tychonov's theorem is not particularly simple; to avoid clutter, we remind ourselves of some minor points from basic topology.

Let $(X, \mathcal{T})$ be a topological space.

(i) The space $(X, \mathcal{T})$ is compact if and only if every collection of closed subsets of X with the finite-intersection property has non-empty intersection; a collection $\mathcal{F}$ of subsets of X has the *finite-intersection property* if $F_1 \cap \ldots \cap F_n \neq \emptyset$ for all $F_1, \ldots, F_n \in \mathcal{F}$. [For a proof of this, see [21, Theorem 21.D] or [3, p.116].]

(ii) If $A \subseteq X$ then $x \in \bar{A}$ if and only if every open set containing x meets A. [Otherwise $x \in (X \setminus A)^\circ = X \setminus \bar{A}$.]

(iii) If X is the product of $\{(X_a, \mathcal{T}_a) : a \in A\}$ then every set in $\mathcal{T}$ is the union of sets of the form

$$\bigcap_{i=1}^{n} \pi_{a_i}^{-1}(U_i) \qquad (n \in \mathbb{N},\ a_1, \ldots, a_n \in A,\ U_1 \in \mathcal{T}_{a_1}, \ldots, U_n \in \mathcal{T}_{a_n});$$

these sets form a base for $\mathcal{T}$.

Theorem 3.21. (Tychonov) Let $\{(X_a,\,\mathcal{T}_a) : a \in A\}$ be a collection of compact topological spaces. The product space $(X,\,\mathcal{T})$ is compact.

Proof

Let $\mathcal{F}$ be a family of closed subsets of X with the finite-intersection property. By Zorn's lemma we may find a maximal family $\mathcal{H}$ of (not necessarily closed) subsets of X such that $\mathcal{H}$ contains $\mathcal{F}$ and has the finite-intersection property. Note that $\mathcal{H}$ is closed under finite intersections: if $A_1,\,\ldots,\,A_n \in \mathcal{H}$ then $A = A_1 \cap \ldots \cap A_n \in \mathcal{H}$, as otherwise $\mathcal{H} \cup \{A\}$ strictly contains $\mathcal{H}$, contains $\mathcal{F}$ and has the finite-intersection property, which contradicts the maximality of $\mathcal{H}$. Furthermore, if $A \subseteq X$ is such that $A \cap H \neq \emptyset$ for all $H \in \mathcal{H}$ then $A \in \mathcal{H}$; otherwise considering $\mathcal{H} \cup \{A\}$ leads to the same contradiction.

Let $a \in A$ and note that $\{\pi_a(H) : H \in \mathcal{H}\}$ has the finite-intersection property (because $f(\bigcap_{b\in B} S_b) \subseteq \bigcap_{b\in B} f(S_b)$ for any function f and collection of sets $\{S_b\}$), so there exists $x_a \in \bigcap_{H\in\mathcal{H}} \overline{\pi_a(H)}$ by the compactness of X_a. We complete the proof by showing that $x := (x_a)_{a\in A} \in \bigcap_{H\in\mathcal{H}} \overline{H}$, which suffices to show that $\bigcap_{F\in\mathcal{F}} F \neq \emptyset$.

Any open set containing x contains a set of the form $U = \bigcap_{a\in A_0} \pi_a^{-1}(U_a)$, where $U_a \in \mathcal{T}_a$ contains x_a and A_0 is a finite subset of A. If $a \in A_0$ then $x_a \in U_a \cap \overline{\pi_a(H)}$, so $\pi_a^{-1}(U_a) \cap H \neq \emptyset$, for all $H \in \mathcal{H}$. Hence $\pi_a^{-1}(U_a) \in \mathcal{H}$ for all $a \in A_0$, and so $U \in \mathcal{H}$, as $\mathcal{H}$ is closed under finite intersections. As H has the finite-intersection property, $U \bigcap H \neq \emptyset$ for all $H \in \mathcal{H}$, and therefore $x \in \bigcap_{H\in\mathcal{H}} \bar{H}$, as required. □

The Banach-Alaoglu Theorem

Theorem 3.22. (Banach-Alaoglu) If X is a normed space then X_1^*, the closed unit ball of X^*, is compact in the weak* topology.

Proof

By Tychonov's theorem, the space $K = \bigtimes_{x\in X} \mathbb{F}_{\|x\|}$ is compact when equipped with the product topology. The map

$$F\colon X_1^* \to K;\ \phi \mapsto \big(\phi(x)\big)_{x\in X}$$

is well defined (since $|\phi(x)| \leqslant \|\phi\|\,\|x\| \leqslant \|x\|$) and injective, so F^{-1} is well defined on $F(X_1^*)$. This map is weak* continuous, by Proposition 1.25, since $\hat{x}|_{X_1^*} \circ F^{-1} = \pi_x|_{F(X_1^*)}$ for all $x \in X$, and $X_1^* = F^{-1}(F(X_1^*))$ is weak* compact if $F(X_1^*)$ is closed in K. To prove this, by Theorem 1.34 it suffices to take a net $(\phi_a)_{a\in A}$ in X_1^* such that $\big(F(\phi_a)\big)_{a\in A}$ has limit $f = (f_x)_{x\in X} \in K$ and show that $\phi\colon x \mapsto f_x \in X_1^*$. Note that $F(\phi_a) \to f$ if and only if $\phi_a(x) \to f_x$, by Proposition 1.37, and so

$$f_x + \alpha f_y = \lim_{a\in A} \phi_a(x) + \alpha \lim_{a\in A} \phi_a(y) = \lim_{a\in A} \phi_a(x+\alpha y) = f_{x+\alpha y} \qquad \forall\, x, y \in X,\ \ \alpha \in \mathbb{F}.$$

Hence $\phi\colon x \mapsto f_x \in X'$, and $f_x \in \mathbb{F}_{\|x\|}$ for all $x \in X$ implies that $|\phi(x)| = |f_x| \leqslant \|x\|$, so $\|\phi\| \leqslant 1$, as required. (The linearity result used here follows from continuity of addition and multiplication in $\mathbb{F}$.) □

Characterisation of Normed Vector Spaces

The following theorem reduces the study of normed vector spaces to the study of subspaces of a particular type of Banach space, the collection of continuous functions on a compact, Hausdorff space. In particular, all Banach spaces are isomorphic to closed subspaces of $C(K)$ for some compact, Hausdorff space K.

Theorem 3.23. Let $(X, \|\cdot\|)$ be a normed space. There exists a compact, Hausdorff space K and a linear isometry $i\colon X \to C(K)$ such that X is isometrically isomorphic to $i(X)$, a subspace of $C(K)$, which is closed if and only if X is complete.

Proof

Let $K = X_1^*$ be the closed unit ball of X^*, equipped with the (restriction of the) weak* topology. (This is Hausdorff, being a subspace of the Hausdorff space $\big(X^*, \sigma(X^*, X)\big)$.) Define $i\colon X \to C(K)$ by setting $i(x) = \hat{x}|_K$, i.e.,

$$i(x)\colon K \to \mathbb{F};\ \phi \mapsto \phi(x);$$

this map is continuous by the definition of the weak* topology, and i is clearly linear. Furthermore,

$$\|i(x)\|_\infty = \sup\{|i(x)(\phi)| : \phi \in K\} = \sup\{|\phi(x)| : \phi \in K\} = \|x\|,$$

by Theorem 3.10. Hence i is an isometry, so $i(X)$ is closed if and only if X is complete: a subspace of a Banach space is closed if and only if it is complete (Proposition 1.8) and isometries preserve Cauchy sequences. $\square$

The previous theorem is the starting point which motivates the theory of operator spaces: every Banach space is isometrically isomorphic to a closed subspace of some $C(K)$, which is the paradigm example of a commutative C^* algebra. An *operator space* is a closed subspace of some $\mathcal{B}(H)$, where H is a Hilbert space; $\mathcal{B}(H)$ is the natural non-commutative generalisation of $C(K)$. This is a very active area of current research (see [5] or [15]).

Topological Vector Spaces

Definition 3.24. Let X be a vector space. A set of linear functionals $A \subseteq X'$ is *separating* if for all $x \in X \setminus \{0\}$ there exists $\phi \in A$ such that $\phi(x) \neq 0$. If $M \subseteq X'$ is a separating subspace then $\sigma(X, M) := \mathcal{T}_M$ is the initial topology on X generated by M; this topology is Hausdorff by Proposition 1.26.

The weak and weak* topologies are defined in this fashion.

Proposition 3.25. Let X be a vector space with separating subspace $M \subseteq X'$. A linear functional $\phi \in X'$ is $\sigma(X, M)$-continuous if and only if $\phi \in M$.

Proof

See Exercise 5.1. □

Proposition 3.26. Let X be a vector space over $\mathbb{F}$ and let $M \subseteq X'$ be a separating subspace. The functions

$$\mathbb{F} \times X \to X;\ (\alpha, x) \mapsto \alpha x \qquad \text{and} \qquad X \times X \to X;\ (x, y) \mapsto x + y$$

are continuous (where X is equipped with the topology $\sigma(X, M)$ and $\mathbb{F}$ has its usual topology).

Proof

The function $(\alpha, x) \mapsto \alpha x$ is continuous if $(\alpha, x) \mapsto \phi(\alpha x)$ is continuous for all $\phi \in M$, by Proposition 1.25, and this function is continuous if $\phi(\alpha_a x_a) \to \phi(\alpha x)$ for any net $(\alpha_a, x_a)_{a \in A} \subseteq \mathbb{F} \times X$ such that $(\alpha_a, x_a) \to (\alpha, x)$, by Proposition 1.35. If $(\alpha_a, x_a) \to (\alpha, x)$ in $\mathbb{F} \times X$ then (by the definition of the product topology and Proposition 1.37) $\alpha_a \to \alpha$ and $x_a \to x$, i.e., $\phi(x_a) \to \phi(x)$ for all $\phi \in M$. Continuity of multiplication in $\mathbb{F}$ yields

$$\phi(\alpha_a x_a) = \alpha_a \phi(x_a) \to \alpha \phi(x) = \phi(\alpha x) \qquad \forall \phi \in M,$$

as required. The proof for the other function is similar (and depends upon the continuity of addition in $\mathbb{F}$). □

Definition 3.27. A *topological vector space* is a vector space X equipped with a Hausdorff topology such that the maps

$$X \times X \to X;\ (x, y) \mapsto x + y$$

and

$$\mathbb{F} \times X \to X;\ (\alpha, x) \mapsto \alpha x$$

are continuous.

Any normed space, or vector space equipped with the topology given by a separating subspace of linear functionals, is a topological vector space.

Lemma 3.28. Let X be a topological vector space. A linear functional $\phi \in X'$ is continuous if and only if $|\phi|^{-1}[0, 1)$ contains an open set containing 0.

Proof

One implication is immediate from the definitions. For the converse, suppose that $U \subseteq X$ is an open set containing 0 and such that $|\phi(u)| < 1$ for all $u \in U$. Let $A \subseteq \mathbb{F}$ be open and let $x \in \phi^{-1}(A)$; there exists $\varepsilon_x > 0$ such that $B_x := B^{\mathbb{F}}_{\varepsilon_x}\big(\phi(x)\big) \subseteq A$. As

$$|\phi(y) - \phi(x)| = |\phi(y - x)| < \varepsilon_x \qquad \forall\, y \in x + \varepsilon_x U,$$

we see that

$$\phi^{-1}(A) \supseteq \phi^{-1}(B_x) \supseteq x + \varepsilon_x U$$

and so $\phi^{-1}(A) = \bigcup_{x \in \phi^{-1}(A)} (x + \varepsilon_x U)$ is open, as required. (The fact that $x + \varepsilon_x U$ is open follows as the maps $y \mapsto y + x$ and $y \mapsto \varepsilon_x y$ are homeomorphisms of X to itself.) □

Separation

Definition 3.29. A subset C of a vector space is *convex* if $tC + (1-t)C \subseteq C$ for all $t \in (0,1)$, i.e., $tx + (1-t)y \in C$ for all x, $y \in C$ and $t \in (0,1)$. [Geometrically, this condition states that every line segment with endpoints in C lies in C. It is immediate that linear transformations preserve convexity.]

Lemma 3.30. Let X be a real topological vector space. If C is a convex, open set in X that contains the origin then the map

$$\mu_C \colon X \to \mathbb{R}^+;\ x \mapsto \inf\{t \in \mathbb{R}^+ : x \in tC\}$$

(called the *gauge* or *Minkowski functional* of C) is a sublinear functional on X such that $C = \mu_C^{-1}[0,1) = \{x \in X : \mu_C(x) < 1\}$.

Proof

If $x \in X$ then $m_x \colon \mathbb{R} \to X;\ t \mapsto tx$ is continuous, so $m_x^{-1}(C)$ is open and contains 0. Hence there exists $\delta > 0$ such that $tx \in C$ if $|t| < \delta$, i.e., $x \in sC$ if $|s| > \delta^{-1}$. This shows that μ_C is well defined.

Let x, $y \in X$ and suppose that $\varepsilon > 0$; we may find s, $t > 0$ such that $s < \mu_C(x) + \varepsilon$, $t < \mu_C(y) + \varepsilon$ and $x \in sC$, $y \in tC$. Then

$$x + y \in sC + tC = (s+t)\Big(\frac{s}{s+t}C + \frac{t}{s+t}C\Big) \subseteq (s+t)C,$$

by the convexity of C, and $\mu_C(x+y) \leqslant s + t < \mu_C(x) + \mu_C(y) + 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary we have the subadditivity of μ_C.

Positive homogeneity is immediate: if $s > 0$ then $x \in tC$ if and only if $sx \in stC$, hence $\mu_C(sx) = s\mu_C(x)$ for all $x \in X$ and $s \in \mathbb{R}^+$.

Finally, if $x \in C$ then $(1+\varepsilon)x \in C$ for some $\varepsilon > 0$ (because $m_x^{-1}(C)$ is open and contains 1) and so $x \in (1+\varepsilon)^{-1}C$, which yields $\mu_C(x) < 1$. Conversely, if $\mu_C(x) < 1$ then $x \in tC$ for some $t < 1$, whence $t^{-1}x \in C$ and so $x = (1-t)0 + t(t^{-1}x) \in C$ because $0 \in C$ and C is convex. □

Definition 3.31. A topological vector space is *locally convex* if every open set containing the origin contains a convex open set containing the origin. A normed space, or a topological vector space with topology given by a separating subspace of linear functionals, is locally convex (Exercise 5.8).

Theorem 3.32. Let X be a topological vector space and let A, B be non-empty, disjoint, convex subsets of X.

(i) If A is open then there exists a continuous linear functional $\phi \in X'$ and $s \in \mathbb{R}$ such that

$$\operatorname{Re}\phi(x) < s \leqslant \operatorname{Re}\phi(y) \qquad \forall\, x \in A,\ y \in B.$$

(ii) If X is locally convex, $x_0 \in X \setminus B$ and B is closed then there exists a continuous linear functional $\phi \in X'$ such that

$$\operatorname{Re}\phi(x_0) < \inf\{\operatorname{Re}\phi(y) : y \in B\}.$$

Proof

First, note that we may assume that the scalar field $\mathbb{F} = \mathbb{R}$, working as in the last part of the proof of Theorem 3.9: if ϕ is a continuous, real-linear functional on X satisfying (i) or (ii) then $\Phi\colon x \mapsto \phi(x) - i\phi(ix)$ is an continuous element of X' with the same property (since $\operatorname{Re}\Phi = \phi$).

For (i), let $a_0 \in A$, $b_0 \in B$ and consider $C = A - B + (b_0 - a_0)$; it contains 0, is open (being the union of translates of the open set A) and is convex (this is immediate upon checking the definition). Hence the gauge μ_C is sublinear on X.

Since A and B are disjoint, $x_0 := b_0 - a_0 \notin C$, and therefore $\mu_C(x_0) \geqslant 1$. Let $N = \mathbb{R}x_0$ and define ϕ_0 on N by setting $\phi_0(tx_0) = t$ for all $t \in \mathbb{R}$. Then $\phi_0 \leqslant \mu_C$ on N, and so, by Theorem 3.8, there exists a (real-linear) functional ϕ such that $\phi(x_0) = \phi_0(x_0) = 1$ and $\phi \leqslant \mu_C$. If $x \in C$ then $\phi(x) \leqslant \mu_C(x) < 1$ and if $x \in -C$ then $\phi(x) = -\phi(-x) > -1$. Hence $|\phi(x)| < 1$ on $C \cap (-C)$; since $C \cap (-C)$ is an open set containing the origin, ϕ is continuous (by Lemma 3.28).

If $a \in A$, $b \in B$ then $a - b + x_0 \in C$ and so

$$\phi(a) - \phi(b) + 1 = \phi(a - b + x_0) \leqslant \mu_C(a - b + x_0) < 1,$$

hence $\phi(a) < \phi(b)$. Then $\phi(A)$ and $\phi(B)$ are disjoint, convex subsets of $\mathbb{R}$ and $\phi(A)$ is open (see Exercise 5.3), so taking $s = \sup \phi(A)$ gives the result.

For (ii), note that $(X \setminus B) - x_0$ is open and contains 0, so by local convexity there exists a convex, open set U such that $0 \in U \subseteq (X \setminus B) - x_0$. Then $A := x_0 + U$ is convex, open and contained in $X \setminus B$; the result follows by (i). $\square$

Corollary 3.33. If X is a locally convex topological vector space and $x \in X \setminus \{0\}$ then there exists a continuous linear functional $\phi \in X'$ such that $\phi(x) \neq 0$. In other words, the topological dual of X separates points.

Proof

Let $B = \{0\}$, $x_0 = x$ and apply Theorem 3.32(ii). $\square$

Example 3.34. Let $0 < p < 1$,

$$\mathcal{L}^p[0,1] := \bigl\{ f\colon [0,1] \to \mathbb{C} \bigm| f \text{ is measurable and } d(f,0) < \infty \bigr\},$$

where

$$d(f,g) := \int_0^1 |f(t) - g(t)|^p \, \mathrm{d}t,$$

and let $L^p[0,1] = \mathcal{L}^p[0,1]/\mathcal{N}$, where $\mathcal{N} := \{ f \in \mathcal{L}^p[0,1] : d(f,0) = 0\}$ is the subspace of functions zero almost everywhere. The map $([f],[g]) \mapsto d(f,g)$ is a metric on $L^p[0,1]$ which makes it a topological vector space (with algebraic operations defined pointwise). However, this topology is not locally convex and the only continuous linear functional on $L^p[0,1]$ is the zero functional (Exercise 5.6), A consequence of Proposition 3.25 is the fact that the collections of norm-continuous and weakly continuous linear functionals on a normed space coincide; combined with the separation theorem this yields the following.

Corollary 3.35. A convex subset of a normed space is norm closed if and only if it is weakly closed.

Proof

Since the weak topology is coarser than the norm topology, we need only consider a non-empty, norm-closed, convex subset C of the normed space X. If $x \notin C$ then there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \cap C = \emptyset$; applying Theorem 3.32(i) with $A = B_\varepsilon(x)$ and $B = C$ yields $\phi \in X^*$ and $s \in \mathbb{R}$ such that

$$\{y \in X : \operatorname{Re}\phi(y) < s\} \cap C = \emptyset.$$

As $(\operatorname{Re}\phi)^{-1}(-\infty, s)$ is weakly open and contains x, it follows that C is weakly closed and we have the result. □

In fact, the previous proposition holds for all locally convex topological vector spaces: the proper generalisation of the weak topology is the initial topology generated by all the continuous linear functionals (Exercise 5.9).

The Krein-Milman Theorem

Definition 3.36. Let X be a vector space. A *face* of a convex set C is a non-empty, convex subset $F \subseteq C$ such that if $t \in (0,1)$ and x, $y \in C$ satisfy $tx + (1-t)y \in F$ then x and $y \in F$. An *extreme point* of a convex set C is a one-point face, i.e., an element of C that cannot be expressed as a non-trivial convex combination of elements of C. [We blur the distinction between an extreme point and the singleton set containing it.] The *extremal boundary* of C is the set of its extreme points, denoted by $\partial_e C$

In geometrical terms, a face F is a subset of the convex set C such that if $f \in F$ and ℓ is a line through f then $\ell \cap C \subseteq F$. An extreme point is a point of C that is not contained in the interior of any line segment in C.

[Some pictures would go well here.]

Definition 3.37. If X is a vector space and $A \subseteq X$ then

$$\operatorname{cnv} A := \Big\{ \sum_{i=1}^n \alpha_i x_i : n \in \mathbb{N},\ \alpha_1, \ldots, \alpha_n \in \mathbb{R}^+,\ \sum_{i=1}^n \alpha_i = 1,\ x_1, \ldots, x_n \in A \Big\}$$

is the *convex hull* of A. If X is a topological vector space then the *closed convex hull* of A, denoted by $\overline{\operatorname{cnv}}\, A$, is the closure of $\operatorname{cnv} A$.

Proposition 3.38. If X is a vector space and $A \subseteq X$ then $\operatorname{cnv} A$ is the smallest convex set containing A. If X is a topological vector space then $\overline{\operatorname{cnv}}\, A$ is the smallest closed, convex set containing A.

Proof

The convexity of $\operatorname{cnv} A$ is readily verified. Let B be any convex set containing A; we claim that $\operatorname{cnv} A \subseteq B$, i.e., for all $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} \alpha_i x_i \in B \qquad \text{if } x_1, \ldots, x_n \in A \text{ and } \alpha_1, \ldots, \alpha_n \in \mathbb{R}^+ \text{ with } \sum_{i=1}^{n} \alpha_i = 1. \tag{3.1}$$

To see this, we proceed by induction: the cases $n = 1$ and 2 are immediate, so suppose that $n > 2$ and that (3.1) holds for sums containing $n-1$ terms. Without loss of generality $\alpha_n \neq 1$ and

$$\sum_{i=1}^{n} \alpha_i x_i = (1-\alpha_n)\Big(\sum_{i=1}^{n-1} \frac{\alpha_i}{1-\alpha_n} x_i\Big) + \alpha_n x_n \in B$$

by the inductive hypothesis and convexity. This proves the first statement.

If B is a closed, convex set containing A then $\operatorname{cnv} A \subseteq B$ (since B is convex) and hence $\overline{\operatorname{cnv}}\, A \subseteq \bar{B} = B$ (taking closures). It remains to prove that $\overline{\operatorname{cnv}}\, A$ is convex; let x, $y \in \overline{\operatorname{cnv}}\, A$ and choose a net $(x_p, y_p)_{p \in P} \subseteq \operatorname{cnv} A \times \operatorname{cnv} A$ such that $(x_p, y_p) \to (x, y)$ (recall that the closure of a product is the product of the closures). By continuity of scalar multiplication and vector addition,

$$tx + (1-t)y = t \lim_{p \in P} x_p + (1-t) \lim_{p \in P} y_p = \lim_{p \in P} t x_p + (1-t) y_p \in \overline{\operatorname{cnv} A} =: \overline{\operatorname{cnv}}\, A$$

for all $t \in (0,1)$, as required. □

Lemma 3.39. Let X be a topological vector space and let C be a non-empty, compact, convex subset of X. If $\phi \in X'$ is continuous then

$$F := \big\{ x \in C : \operatorname{Re} \phi(x) = \min\{\operatorname{Re} \phi(y) : y \in C\} \big\}$$

is a closed face of C.

Proof

The set F is non-empty (since C is compact and $x \mapsto \operatorname{Re} \phi(x)$ is continuous), closed (since it is the pre-image of a point under the continuous function $\operatorname{Re} \phi$) and convex (since $\operatorname{Re} \phi$ is real-linear). Furthermore, if $t \in (0,1)$ and x, $y \in C$ are such that $tx + (1-t)y \in F$ then

$$\min_C \operatorname{Re} \phi = \operatorname{Re} \phi(tx + (1-t)y) = t \operatorname{Re} \phi(x) + (1-t) \operatorname{Re} \phi(y) \geqslant \min_C \operatorname{Re} \phi. \tag{3.2}$$

(The notation $\min\limits_C$ means that the minimum is taken over the set C.) If x is not in F then $\operatorname{Re} \phi(x) > \min_C \operatorname{Re} \phi$, which gives a strict inequality in (3.2), a contradiction, and similarly for y. Hence x, $y \in F$, as required. □

Theorem 3.40. (Krein-Milman) Let X be a locally convex topological vector space and let C be a non-empty, compact, convex subset of X. Then $C = \overline{\operatorname{cnv}}\, \partial_{\mathrm{e}} C$, i.e., C is the closed convex hull of its extreme points. (In particular, $\partial_{\mathrm{e}} C$ is non-empty.)

Proof

Let $\mathcal{F}$ denote the collection of closed faces of C; it is an exercise to verify that $(\mathcal{F}, \supseteq)$ is a non-empty, partially ordered set, such that every chain in $\mathcal{F}$ has an upper bound. Applying Zorn's lemma we obtain a maximal element of $\mathcal{F}$, i.e., a closed face F_0 such that if $F_1 \in \mathcal{F}$ satisfies $F_0 \supseteq F_1$ then $F_1 = F_0$. We claim that $F_0 \in \partial_e C$; to see this, suppose otherwise, so that there exist distinct x, $y \in F_0$. By Corollary 3.33 there exists a continuous $\phi \in X'$ such that $\phi(x) \neq \phi(y)$, and without loss of generality $\operatorname{Re}\phi(x) < \operatorname{Re}\phi(y)$ (else we replace ϕ by one of $-\phi$, $i\phi$ or $-i\phi$). Let

$$F_1 = \big\{z \in F_0 : \operatorname{Re}\phi(z) = \min\{\operatorname{Re}\phi(w) : w \in F_0\}\big\};$$

this is a proper subset of F_0 (since $y \notin F_1$) and a closed face of F_0, by Lemma 3.39, and so a closed face of C: a face of a face of C is itself a face of C. This contradicts the minimality of F_0 and so $\partial_e C \neq \emptyset$.

It is immediate that $C \supseteq \overline{\operatorname{cnv}}\,\partial_e C$, so to complete the proof suppose for contradiction that $x \in C \setminus \overline{\operatorname{cnv}}\,\partial_e C$. By Theorem 3.32(ii) (applied to x and $\overline{\operatorname{cnv}}\,\partial_e C$) we may find a continuous $\psi \in X'$ such that $\operatorname{Re}\psi(x) < \min\{\operatorname{Re}\psi(y) : y \in \overline{\operatorname{cnv}}\,\partial_e C\}$. Let

$$F = \big\{z \in C : \operatorname{Re}\psi(z) = \min\{\operatorname{Re}\psi(w) : w \in C\}\big\};$$

this is a closed face of C, by Lemma 3.39, and applying the first part of this proof, with F in place of C, yields $z \in \partial_e F \subseteq \partial_e C$. Hence

$$\min_C \operatorname{Re}\psi = \operatorname{Re}\psi(z) > \operatorname{Re}\psi(x) \geqslant \min_C \operatorname{Re}\psi,$$

the desired contradiction. □

Exercises 5

Exercise 5.1. Prove that if X is a vector space with separating subspace $M \subseteq X'$ and $\phi \in X'$ is a linear functional that is $\sigma(X, M)$-continuous then there exist $\phi_1, \ldots, \phi_n \in M$ such that

$$|\phi(x)| \leqslant \max_{1 \leqslant i \leqslant n} |\phi_i(x)| \qquad \forall\, x \in X.$$

Deduce that $\bigcap_{i=1}^n \ker \phi_i \subseteq \ker \phi$ and that there exists $f \in (\mathbb{F}^n)^*$ such that

$$f\big(\phi_1(x), \ldots, \phi_n(x)\big) = \phi(x) \qquad \forall\, x \in X.$$

Conclude that $\phi \in M$.

Exercise 5.2. Let X be an infinite-dimensional normed space and let $V \subseteq X$ be a weakly open set containing the origin. Show that V contains a closed subspace of finite codimension in X. Deduce that the weak topology on X is strictly coarser than the norm topology.

Exercise 5.3. Let X be a topological vector space. Prove that every $\phi \in X' \setminus \{0\}$ is open. [Note that $m_x \colon \alpha \mapsto \alpha x$ is continuous for all $x \in X$ and that there exists $x_0 \in X$ such that $\phi(x_0) = 1$.]

Exercise 5.4. Suppose that X is a vector space equipped with a topology that makes vector addition and scalar multiplication, i.e., the maps

$$X \times X \to X;\ (x,y) \mapsto x+y \qquad \text{and} \qquad \mathbb{F} \times X \to X;\ (\alpha, x) \mapsto \alpha x,$$

continuous. Show that if this topology is such that singleton sets are closed (i.e., $\{x\}$ is closed for all $x \in X$) then the topology is Hausdorff (so X is a topological vector space).

Exercise 5.5. Let X be a topological vector space. Prove that every open set containing the origin contains a non-empty open set which is *balanced*: a set B is balanced if $\lambda b \in B$ for all $b \in B$ and $\lambda \in \mathbb{F}_1$. [Balanced sets are in some ways analogous to open balls about the origin in normed spaces.] Deduce that if $C \subseteq X$ is compact and does not contain the origin then there exist disjoint open sets A, $B \subseteq X$ such that $C \subseteq A$ and B is a balanced set containing 0. Show that a balanced set is connected and give an example to show that a balanced set need not be convex.

Exercise 5.6. Let $p \in (0,1)$,

$$\mathcal{L}^p[0,1] := \bigl\{ f\colon [0,1] \to \mathbb{C} \bigm| f \text{ is measurable and } \Delta(f) < \infty \bigr\},$$

where $\Delta(f) := \int_0^1 |f(x)|^p \,\mathrm{d}x$, and let $L^p[0,1] := \mathcal{L}^p[0,1]/\mathcal{N}$, where

$$\mathcal{N} := \bigl\{ f\colon [0,1] \to \mathbb{C} \bigm| f \text{ is measurable and zero almost everywhere} \bigr\}.$$

Prove that $d([f],[g]) := \Delta(f-g)$ is a metric on $L^p[0,1]$ and that $L^p[0,1]$ is a topological vector space (when equipped with this topology). Prove further that $L^p[0,1]$ has no convex, open sets other than $\emptyset$ and $L^p[0,1]$. Deduce that the only continuous linear functional on $L^p[0,1]$ is the zero functional.

Exercise 5.7. Let X be a topological vector space over $\mathbb{F}$ and let M be a finite-dimensional subspace of X. Prove that M is linearly homeomorphic to $\mathbb{F}^n$, where n is the dimension of M. Prove also that M is closed in X.

Exercise 5.8. Prove that a topological vector space with topology given by a separating family of linear functionals is locally convex.

Exercise 5.9. Suppose that X is a locally convex topological vector space and M is the collection of continuous linear functionals on X. Prove that a convex subset of X is closed (with respect to the original topology) if and only if it is closed with respect to $\sigma(X,M)$, the initial topology generated by M. Need this hold if X is not locally convex?

Exercise 5.10. Let X be a locally convex topological vector space. Show that if N is a non-empty subspace of X and $x_0 \in X \setminus \bar{N}$ then there exists a continuous linear functional $\phi \in X'$ such that $\phi|_N = 0$ and $\phi(x_0) = 1$. [Use the separation theorem.]

Exercise 5.11. Let X be a topological vector space and suppose V is an open set containing 0. Prove there exists an open set U containing 0 such that $U + U \subseteq V$. Deduce or prove otherwise that if $A \subseteq B \subseteq X$, where A is compact and B is open, then there exists an open set $U \subseteq X$ such that $0 \in U$ and $A + U \subseteq B$.

Exercise 5.12. Suppose that X is a topological vector space such that the continuous elements of X' separate points. Prove that given disjoint, non-empty, compact, convex A, $B \subseteq X$ there exists a continuous $\phi \in X'$ such that

$$\sup_{x \in A} \operatorname{Re} \phi(x) < \inf_{x \in B} \operatorname{Re} \phi(x).$$

[Consider X equipped with the topology $\sigma(X, M)$, where M is the collection of continuous linear functionals on X.] Deduce that Theorem 3.40 is true for topological spaces with continuous dual spaces that separate points.

Exercise 5.13. Let X be a topological vector space and suppose that C is a non-empty, compact, convex subset of X. Prove that $(\mathcal{F}, \supseteq)$, the collection of closed faces of C ordered by reverse inclusion, is a non-empty, partially ordered set such that every chain in $\mathcal{F}$ has an upper bound.

Exercise 5.14. Prove that the closed unit ball of c_0 has no extreme points.

Exercise 5.15. Let H be a Hilbert space. Prove that every unit vector in H is an extreme point of the closed unit ball H_1. [Note that 1 is an extreme point of $\mathbb{F}_1$.] Deduce that every isometry in $\mathcal{B}(H)$ is an extreme point of the closed unit ball $\mathcal{B}(H)_1$.

Exercise 5.16. Let C be a convex subset of a topological vector space X. Prove that if $x \in C$ and $y \in C^\circ$, the interior of C, then $tx + (1-t)y \in C^\circ$ for all $t \in [0, 1)$. Prove also that the interior C° and the extremal boundary $\partial_e C$ are disjoint (as long as $X \neq \{0\}$).

Exercise 5.17. Let $C \subseteq \mathbb{R}^n$ be compact and convex. Prove that every element of C can be written as a convex combination of at most $(n+1)$ elements of $\partial_e C$. [Use induction.]

Algebras

Four Normed Algebras

We start with some purely algebraic definitions. All the algebras that we consider will have scalar field $\mathbb{R}$ or $\mathbb{C}$.

Definition 4.1. An *algebra* (more correctly, an *associative algebra*) is a vector space A equipped with a bilinear map (called *multiplication*)

$$\cdot : A \times A \to A;\ (a, b) \mapsto ab$$

that is associative:

$$(ab)c = a(bc) \qquad \forall\, a, b, c \in A.$$

An algebra is *commutative* if its multiplication is, i.e., $ab = ba$ for all a, $b \in A$. An algebra is *unital* if there exists $1 \in A$ such that $1a = a1 = a$ for all $a \in A$; such an element is unique if it exists. A *subalgebra* of the algebra A is a subspace B that is closed under multiplication: $ab \in A$ for all $a, b \in B$ (more briefly, $B^2 \subseteq B$).

Example 4.2. A field is a unital algebra over itself. (If the underlying scalar field $\mathbb{F}$ of an algebra needs to be mentioned explicitly we refer to an *algebra over* $\mathbb{F}$.) More generally, the collection of $n \times n$ matrices over a field $\mathbb{F}$ forms a unital algebra, denoted by $M_n(\mathbb{F})$, when equipped with the usual multiplication:

$$\big(a^i_j\big)^n_{i,j=1}\big(b^j_k\big)^n_{j,k=1} = \big(\textstyle\sum^n_{j=1} a^i_j b^j_k\big)^n_{i,k=1}.$$

The examples above are finite-dimensional (the dimension of an algebra is its dimension as a vector space) and every finite-dimensional algebra A over $\mathbb{F}$ is isomorphic to a subalgebra of $M_n(\mathbb{F})$ (with $n = \dim A$ if A is unital, or $n = 1 + \dim A$ otherwise), where homomorphism and isomorphism are defined in the usual manner: see Definition 4.7

Example 4.3.
If X is a topological space then the set of complex-valued, continuous functions on X is an algebra over $\mathbb{C}$, denoted by $C(X)$, with the algebraic operations defined pointwise: if f, $g \in C(X)$ and $\alpha \in \mathbb{C}$ then

$$(f + \alpha g)(x) := f(x) + \alpha g(x) \quad \text{and} \quad (fg)(x) := f(x)g(x) \qquad \forall\, x \in X.$$

This algebra has three important subalgebras:

(i) $C_{00}(X)$, the continuous functions on X with compact support;

(ii) $C_0(X)$, the continuous functions on X that vanish at infinity;

(iii) $C_{\mathrm{b}}(X)$, the bounded, continuous functions on X.

We have
$$C_{00}(X) \subseteq C_0(X) \subseteq C_{\mathrm{b}}(X) \subseteq C(X),$$
with equality if and only if X is compact. Note that $C_{\mathrm{b}}(X)$ and $C(X)$ are unital, but $C_{00}(X)$ and $C_0(X)$ are unital only if X is compact.

Definition 4.4. A *normed algebra* is a normed space that is also an associative algebra, such that the norm is *submultiplicative*: $\|ab\| \leqslant \|a\|\,\|b\|$ for all a, $b \in A$. A *Banach algebra* is a complete normed algebra, i.e., a normed algebra that is also a Banach space (with respect to its norm). A normed algebra is *unital* if it is a unital algebra and $\|1\| = 1$.

Note that the submultiplicativity of the norm means that multiplication in normed algebras is jointly continuous: if $a_n \to a$ and $b_n \to b$ then $(a_n)_{n\geqslant 1}$ is bounded and
$$\begin{aligned}\|a_n b_n - ab\| = \|a_n(b_n - b) + (a_n - a)b\| &\leqslant \|a_n\|\,\|b_n - b\| + \|a_n - a\|\,\|b\| \\ &\leqslant \sup\bigl\{\|a_n\|\bigr\}\,\|b_n - b\| + \|b\|\,\|a_n - a\| \to 0\end{aligned}$$
as $n \to \infty$.

Example 4.5. If X is a Banach space then $\bigl(B(X), \|\cdot\|\bigr)$ is a unital Banach algebra, where $\|\cdot\|$ is the operator norm; this example generalises Example 4.2, as if X is finite-dimensional then $B(X)$ is isomorphic to $M_n(\mathbb{F})$ (where $n = \dim X$).

Example 4.6. Let X be a locally compact, Hausdorff space. When equipped with the supremum norm
$$\|\cdot\|_\infty \colon f \mapsto \|f\|_\infty := \sup\{|f(x)| : x \in X\},$$
$C_{\mathrm{b}}(X)$ is a unital Banach algebra, $C_0(X)$ is a closed subalgebra of $C_{\mathrm{b}}(X)$ (so a Banach algebra in its own right) and $C_{00}(X)$ is a dense subalgebra of $C_0(X)$. [This follows from Proposition 2.18.]

Quotient Algebras

Definition 4.7. Let A and B be algebras over the field $\mathbb{F}$. An *algebra homomorphism* is an $\mathbb{F}$-linear map $\phi\colon A \to B$ such that $\phi(ab) = \phi(a)\phi(b)$ for all a, $b \in A$. An *algebra isomorphism* is a bijective algebra homomorphism (which implies that ϕ^{-1} is an algebra homomorphism from B to A). The *kernel* of an algebra homomorphism $\phi\colon A \to B$ is $\ker\phi := \{a \in A : \phi(a) = 0\}$ and the *image* is $\operatorname{im}\phi := \{\phi(a) : a \in A\}$.

The obvious fundamental theorem about homomorphisms is true; in order to state it we need to define the concept of a *quotient algebra.*

Definition 4.8. Let A be an algebra. An *ideal* of the algebra A is a subspace I such that ab, $ba \in I$ for all $a \in A$ and $b \in I$ (i.e., AI, $IA \subseteq I$). If I is an ideal of A then the quotient space A/I is an algebra, called the *quotient algebra* of A by I, when equipped with the multiplication

$$(a+I)(b+I) = ab + I \qquad \forall\, a, b \in A,$$

and A/I is unital if A is unital.

Theorem 4.9. Let A be an algebra and $\phi\colon A \to B$ an algebra homomorphism. Then $\ker\phi$ is an ideal of A, $\operatorname{im}\phi$ is a subalgebra of B and $A/\ker\phi \cong \operatorname{im}\phi$, *via*

$$\tilde{\phi}\colon A \to \operatorname{im}\phi;\ [a] := a + \ker\phi \mapsto \phi(a).$$

Proof

This is trivial. □

Proposition 4.10. If A is a normed algebra and I is a closed, proper ideal then A/I is a normed algebra when equipped with the quotient norm, unital if A is unital; if A is a Banach algebra then so is A/I.

Proof

To see that the quotient norm is submultiplicative, note that if a, $b \in A$ then

$$\begin{aligned}
&\big\|[a]\big\|_{A/I}\big\|[b]\big\|_{A/I} \\
&\qquad = \inf\big\{\|a-x\| : x \in I\big\}\inf\big\{\|b-y\| : y \in I\big\} \\
&\qquad \geqslant \inf\big\{\|ab-ay-xb+xy\| : x, y \in I\big\} \geqslant \inf\big\{\|ab-x\| : x \in I\big\} = \big\|[ab]\big\|_{A/I}.
\end{aligned}$$

In particular, if A is unital then

$$\|[1]\|_{A/I} = \|[1]^2\|_{A/I} \leqslant \|[1]\|_{A/I}^2$$

so $\|[1]\|_{A/I} \geqslant 1$ (since I is proper, $1 \notin I$ and so $[1] \neq [0]$, whence $\|[1]\| > 0$). We have also that $\|[1]\|_{A/I} \leqslant \|1\|_A = 1$ and therefore $\|[1]\|_{A/I} = 1$. Everything else follows from Theorem 1.10. □

Unitization

In many cases an algebra has a unit; there are some situation, however, when none exists but it would be useful to act as though one did.

Definition 4.11. If A is an algebra over the field $\mathbb{F}$ then A^{u} is the *unitization* of A, defined by setting $A^{\mathrm{u}} = A \oplus \mathbb{F}$ and

$$(a,\alpha)(b,\beta) = (ab + \alpha b + \beta a, \alpha\beta) \qquad \forall\, a, b \in A,\ \alpha, \beta \in \mathbb{F}.$$

The algebra A is an ideal of A^{u}. The unitization A^{u} is commutative if and only if A is commutative.

If A is a normed algebra over $\mathbb{R}$ or $\mathbb{C}$ then A^{u} is a normed algebra, where

$$\|\cdot\|_{A^{\mathrm{u}}} \colon A^{\mathrm{u}} \to \mathbb{R}^+;\ (a, \alpha) \mapsto \|a\|_A + |\alpha|.$$

The unitization A^{u} is complete if and only if A is complete, and A is a closed ideal of A^{u}.

Example 4.12. Let $L^1(\mathbb{R})$ denote the space of (equivalence classes of) complex-valued, Lebesgue-integrable functions on the real line, with norm

$$\|\cdot\|_1 \colon L^1(\mathbb{R}) \to \mathbb{R}^+;\ f \mapsto \int_{\mathbb{R}} |f|.$$

This is a commutative Banach algebra when equipped with the *convolution* product:

$$f \star g \colon \mathbb{R} \to \mathbb{R};\ t \mapsto \int_{\mathbb{R}} f(t-s)g(s)\,\mathrm{d}s.$$

(By the theorems of Fubini and Tonelli, if f and g are integrable then this integral exists almost everywhere and defines an element of $L^1(\mathbb{R})$.) This algebra lacks a unit; it is easy to see that $L^1(\mathbb{R})^{\mathrm{u}}$ is isomorphic to the algebra given by adjoining the Dirac measure δ_0: by definition

$$(f \star \delta_0)(t) = \text{“}\int_{\mathbb{R}} f(t-s)\delta_0(s)\,\mathrm{d}s\text{”} = f(t) \qquad \forall\, t \in \mathbb{R}$$

and $\delta_0 \star \delta_0 = \delta_0$.

Approximate Identities

Definition 4.13. Let A be a Banach algebra. An *approximate identity* for A is a net $(e_\lambda)_{\lambda\in\Lambda} \subseteq A_1$ such that

$$\lim_\lambda e_\lambda a = a = a \lim_\lambda e_\lambda \qquad \forall\, a \in A.$$

Example 4.14. The sequence $(e_n)_{n\geqslant 1}$ is an approximate identity for $L^1(\mathbb{R})$, where

$$e_n(x) = \frac{n}{\sqrt{2\pi}}\, e^{-\frac{1}{2}n^2x^2} = n e_1(nx) \qquad \forall\, x \in \mathbb{R}.$$

For a proof of this, see [17, Section 33.13 and Theorem 33.14] (but note that $k_\lambda(x)$ should equal $\sqrt{\frac{\lambda}{2\pi}}e^{-\frac{1}{2}\lambda x^2}$). Another possibility is $(h_n)_{n\geqslant 1}$, where

$$h_n(x) = \frac{1}{\pi}\,\frac{n}{1+n^2x^2} = n h_1(nx) \qquad \forall\, x \in \mathbb{R}:$$

see [18, Section 9.7 and Theorem 9.10].

Example 4.15. Let $A = L^1[-\pi, \pi]$, which we can identify with the closed subalgebra of $L^1(\mathbb{R})$ consisting of 2π-periodic functions. This has approximate identity $(P_r)_{r\in[0,1)}$, where $[0,1)$ is directed in the usual manner and

$$P_r(t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} r^{|n|}e^{int} = \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos t + r^2} = \frac{1}{2\pi}\operatorname{Re}\frac{1+re^{it}}{1-re^{it}}.$$

[The function P is the *Poisson kernel*; it occurs in the theory of harmonic functions on the unit disc.] To prove that this is an approximate identity, note that if $f\colon t \mapsto e^{ikt}$ (where $k \in \mathbb{Z}$) then

$$(P_r \star f)(t) = \int_{-\pi}^{\pi} \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} r^{|n|}e^{in(t-s)}e^{iks}\,\mathrm{d}s = \sum_{n=-\infty}^{\infty}\frac{r^{|n|}e^{int}}{2\pi}\int_{-\pi}^{\pi} e^{i(k-n)s}\,\mathrm{d}s = r^{|k|}f(t),$$

so $\|(P_r \star f) - f\|_1 = (1 - r^{|k|})\|f\|_1 \to 0$ as $r \to 1-$. The linear span of $\{e^{ikt} : k \in \mathbb{Z}\}$ (the *trigonometric polynomials*) is dense in $L^1[-\pi,\pi]$ (an easy consequence of Fejér's theorem [16, Theorem 30.4]) so a simple $\varepsilon/2$ argument gives this result for general f. In fact, Fejér's theorem implies that if f is continuous and $\varepsilon > 0$ then there exists a trigonometric polynomial $p\colon t \mapsto \sum_{n=-N}^{N} a_n e^{int}$ such that $\|f - p\|_\infty < \varepsilon/2$, so that

$$|(P_r\star f)(t) - f(t)| \leqslant |(P_r\star f)(t) - p(t)| + |p(t) - f(t)| \leqslant \left(\sum_{n=-N}^{N} 1 - r^{|n|}\right)\|f\|_\infty + \|p - f\|_\infty < \varepsilon$$

if r is near enough to 1. Hence $P_r \star f \to f$ uniformly in this case. [Compare this proof to that given in [16, Section 10.36(2)].]

Completion

We have the notion of completion as for a normed space; an isometric isomorphism is now required to be multiplicative, of course.

Theorem 4.16. If A is a normed algebra then there exists a Banach algebra $\bar{A}$ and an isometric homomorphism $i\colon A \to \bar{A}$ such that $i(A)$ is a dense subalgebra of $\bar{A}$, with $\bar{A}$ unital if A is; $(\bar{A}, i)$ is called a *completion* of A. Furthermore, if (B, j) and (C, k) are completions of A then there exists an isometric isomorphism $l\colon B \to C$ such that $l \circ j = k$.

Proof

Let $(\bar{A}, i)$ be the completion of A considered as a normed space (see Theorem 1.14). Define a product on $\bar{A}$ by setting

$$ab := \lim_{n\to\infty} i(a_n b_n) \qquad \forall\, a, b \in \bar{A}$$

if $(a_n)_{n\geqslant 1}$, $(b_n)_{n=1}^{\infty} \subseteq A$ are such that $i(a_n) \to a$ and $i(b_n) \to b$ as $n \to \infty$; this limit exists (as $(i(a_n b_n))_{n=1}^{\infty}$ is Cauchy), is independent of the choice of sequences $(a_n)_{n=1}^{\infty}$ and

$(b_n)_{n=1}^\infty$ and the new product agrees with the old on $i(A)$, i.e., $i(a)i(b) = i(ab)$ for all $a, b \in A$. (To see the first claim, note that

$$\begin{aligned}\|i(a_nb_n) - i(a_mb_m)\| &= \|i(a_nb_n - a_mb_m)\| \\ &= \|a_nb_n - a_mb_m\| \\ &\leqslant \|a_n\|\,\|b_n - b_m\| + \|a_n - a_m\|\,\|b_m\| \\ &= \|i(a_n)\|\,\|i(b_n) - i(b_m)\| + \|i(a_n) - i(a_m)\|\,\|i(b_m)\| \\ &\to 0\end{aligned}$$

as $m, n \to \infty$; proof of uniqueness is similar.) It is easy to verify that this product makes $\bar{A}$ an associative algebra (which is unital if A is) and the norm on $\bar{A}$ is submultiplicative because

$$\|i(a_n)i(b_n)\| = \|i(a_nb_n)\| = \|a_nb_n\| \leqslant \|a_n\|\,\|b_n\| = \|i(a_n)\|\,\|i(b_n)\|.$$

Hence $\bar{A}$ is a Banach algebra with dense subalgebra $i(A)$.

If (B, j) and (C, k) are completions of A then there exists an isometric linear bijection $l\colon B \to C$ such that $l \circ j = k$, by Proposition 2.6. Furthermore, if $a, b \in A$ then

$$l\big(j(a)j(b)\big) = l\big(j(ab)\big) = k(ab) = k(a)k(b) = l\big(j(a)\big)l\big(j(b)\big),$$

which shows that l is multiplicative on $j(A)$. Since $j(A)$ is dense in B and multiplication is jointly continuous, l is multiplicative on B and so is an isometric isomorphism. □

Five Invertibility

From now on, A denotes a unital Banach algebra over $\mathbb{C}$.

Definition 5.1. An element $a \in A$ is *invertible* if there exists $b \in A$ such that $ab = ba = 1$. [If such an inverse exists, it is unique.] The collection of invertible elements in A is denoted $G(A)$; this is a group. [This last claim is obvious once we recall that $(ab^{-1})^{-1} = ba^{-1}$ if a and b are invertible.]

Proposition 5.2. Let $a \in A$ be such that $\|a\| < 1$. Then $1 - a \in G(A)$ and

$$(1-a)^{-1} = \sum_{n=0}^{\infty} a^n, \tag{5.1}$$

where this series converges in the norm topology. Hence $G(A)$ is an open subset of A; furthermore $a \mapsto a^{-1}$ is a homeomorphism of $G(A)$.

Proof

Since $\|a\| < 1$, the Neumann series (5.1) is absolutely convergent, so convergent. Furthermore,

$$\lim_{n\to\infty} \Big(\sum_{j=0}^{n} a^j\Big)(1-a) = (1-a)\lim_{n\to\infty} \sum_{j=0}^{n} a^j = \lim_{n\to\infty} 1 - a^{n+1} = 1,$$

so the sum of this series is an inverse for $(1-a)$, as claimed.

If $a \in G(A)$ let $b \in A$ and note that

$$b = a - (a-b) = a\big(1 - a^{-1}(a-b)\big) \in G(A)$$

if $\|a^{-1}(a-b)\| < 1$, e.g., if $b \in B^A_{\|a^{-1}\|^{-1}}(a)$; this shows that $G(A)$ is open.

Finally, if $h \in A$ is such that $\|h\| < \frac{1}{2}\|a^{-1}\|^{-1}$ then $\|a^{-1}h\| < \frac{1}{2}$, which implies that $a + h = a(1 + a^{-1}h) \in G(A)$ and

$$(a+h)^{-1} - a^{-1} = a^{-1}\big((1 + a^{-1}h)^{-1} - 1\big) = a^{-1}\sum_{n=1}^{\infty}(-a^{-1}h)^n$$

has norm at most

$$\|a^{-1}\|\,\|a^{-1}h\|/(1 - \|a^{-1}h\|) \leqslant 2\|a^{-1}\|^2\|h\|$$

(using the fact that $\|\sum_{n=1}^{\infty} x^n\| \leqslant \|x\|/(1-\|x\|)$ if $\|x\| < 1$). Hence $(a+h)^{-1} \to a^{-1}$ as $h \to 0$, as required. $\square$

The Spectrum and Resolvent

Definition 5.3. For all $a \in A$ the *spectrum* of a is

$$\sigma(a) := \{\lambda \in \mathbb{C} : \lambda 1 - a \notin G(A)\}.$$

The *resolvent set* of a is $\rho(a) := \mathbb{C} \setminus \sigma(a)$, the complement of the spectrum of a. The *resolvent* of a is the function defined on $\rho(a)$ by $\lambda \mapsto r_\lambda(a) := (\lambda 1 - a)^{-1}$.

Theorem 5.4. The resolvent satisfies the *resolvent equation*,

$$r_\lambda(a) - r_\mu(a) = -(\lambda - \mu) r_\lambda(a) r_\mu(a) \qquad \forall\, \lambda, \mu \in \rho(a),$$

and is strongly holomorphic on $\rho(a)$, which is an open set. The spectrum is a compact, non-empty subset of $\mathbb{C}_{\|a\|}$.

Proof

For the first claim, note that if λ, $\mu \in \rho(a)$ then

$$\begin{aligned}(\lambda 1 - a)^{-1} - (\mu 1 - a)^{-1} &= (\lambda 1 - a)^{-1}(1 - (\lambda 1 - a)(\mu 1 - a)^{-1}) \\ &= (\lambda 1 - a)^{-1}\big((\mu 1 - a) - (\lambda 1 - a)\big)(\mu 1 - a)^{-1} \\ &= -(\lambda - \mu)(\lambda 1 - a)^{-1}(\mu 1 - a)^{-1}.\end{aligned}$$

From this equation and Proposition 5.2 it follows that

$$\frac{r_\lambda(a) - r_\mu(a)}{\lambda - \mu} = -r_\lambda(a) r_\mu(a) \to -r_\mu(a)^2$$

as $\lambda \to \mu$: the resolvent is strongly holomorphic on $\rho(a)$ (recall Definition 3.13). To see that this set is open, let $f \colon \mathbb{C} \to A;\ \lambda \mapsto \lambda 1 - a$ and note that f is continuous, so $f^{-1}(G(A)) = \rho(a)$ is open.

If $a \in A$ and $|\lambda| > \|a\|$ then $\|\lambda^{-1} a\| < 1$, so by Proposition 5.2 we have that $\lambda 1 - a = \lambda(1 - \lambda^{-1} a)$ is invertible. Hence $\sigma(a) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leqslant \|a\|\}$. By the Heine-Borel theorem and the fact that $\sigma(a) = \mathbb{C} \setminus \rho(a)$ is closed, $\sigma(a)$ is compact.

Suppose for contradiction that $\sigma(a)$ is empty, so that the resolvent is a strongly holomorphic function defined on the whole of $\mathbb{C}$. If $|\lambda| > \|a\|$ then

$$r_\lambda(a) = \lambda^{-1}(1 - \lambda^{-1} a)^{-1} = \sum_{n=0}^{\infty} \lambda^{-(n+1)} a^n \tag{5.2}$$

and so

$$\|r_\lambda(a)\| \leqslant |\lambda|^{-1} / \big(1 - \|\lambda^{-1} a\|\big) = \big(|\lambda| - \|a\|\big)^{-1};$$

this shows that $r_\lambda(a) \to 0$ as $\lambda \to \infty$. In particular the resolvent is bounded, so constant (by Theorem 3.14) and therefore equal to 0; this is the desired contradiction. □

The Gelfand-Mazur Theorem

It is rather surprising that a theorem with as short a proof as the following has such important consequences. Note that commutativity forms no part of the hypotheses but is part of the conclusion.

Theorem 5.5. (Gelfand-Mazur) If A is a unital Banach algebra over $\mathbb{C}$ in which every non-zero element is invertible then A is isometrically isomorphic to $\mathbb{C}$.

Proof

Let $a \in A$; since $\sigma(a)$ is non-empty and

$$\lambda \in \sigma(a) \quad \Leftrightarrow \quad \lambda 1 - a \notin G(A) \quad \Leftrightarrow \quad \lambda 1 - a = 0 \quad \Leftrightarrow \quad a = \lambda 1,$$

we see that for all $a \in A$ there exists $\lambda_a \in \mathbb{C}$ such that $a = \lambda_a 1$. The map $a \mapsto \lambda_a$ is the desired isomorphism; note that $\|a\| = \|\lambda_a 1\| = |\lambda_a|$. □

The Spectral-Radius Formula

Definition 5.6. The *spectral radius* of $a \in A$ is the radius of the smallest disc about the origin that contains the spectrum of a:

$$\nu(a) := \inf\{r \geqslant 0 : \sigma(a) \subseteq \mathbb{C}_r\} = \sup\{|\lambda| : \lambda \in \sigma(a)\}.$$

We recall the spectral mapping theorem for polynomials; it is stated in b4 for bounded operators on a Banach space but its proof holds in any Banach algebra

Theorem 5.7. Let $a \in A$ and suppose that $p(z) \in \mathbb{C}[z]$ is a complex polynomial. Then $\sigma\big(p(a)\big) = p\big(\sigma(a)\big)$, i.e.,

$$\sigma\big(p(a)\big) := \{\lambda \in \mathbb{C} : \lambda 1 - p(a) \notin G(A)\} = \{p(\lambda) : \lambda 1 - a \notin G(A)\}.$$

Proof

This follows the same pattern as the proof which may be found in Dr Vincent-Smith's b4 notes ([25, Theorem 5.2.11]). □

The following theorem gives the Beurling-Gelfand spectral-radius formula. (According to [11, p.525], Beurling led Sweden's effort to crack the Enigma code during the second world war.)

Theorem 5.8. If $a \in A$ then the sequence $\big(\|a^n\|^{1/n}\big)_{n \geqslant 1}$ is convergent and

$$\nu(a) = \inf_{n \geqslant 1} \|a^n\|^{1/n} = \lim_{n \to \infty} \|a^n\|^{1/n}.$$

Proof

Firstly, the spectral mapping theorem for polynomials implies that

$$\nu(a)^n = \nu(a^n) \leqslant \|a^n\| \qquad \forall\, n \geqslant 1. \tag{5.3}$$

Next, $f\colon \lambda \mapsto (\lambda 1 - a)^{-1}$ is continuous on $\rho(a)$, so $M_r := \sup\big\{\|(\lambda 1 - a)^{-1}\| : |\lambda| = r\big\} < \infty$ for all $r > \nu(a)$. Let $\phi \in A^*$ and note that f is strongly holomorphic, so $\phi \circ f$ is holomorphic, on $S = \{\lambda \in \mathbb{C} : |\lambda| > \nu(a)\}$, with

$$(\phi \circ f)(\lambda) = \sum_{n=0}^{\infty} \phi(a^n)\lambda^{-(n+1)} \qquad (|\lambda| > \|a\|)$$

by (5.2), so for all $\lambda \in S$, by the uniqueness of Laurent series. Furthermore, by the integral formula for Laurent coefficients,

$$|\phi(a^n)| = \left| \frac{1}{2\pi i} \oint_{\{\lambda : |\lambda| = r\}} (\phi \circ f)(\lambda)\lambda^n \, \mathrm{d}\lambda \right| \leqslant r^{n+1}\|\phi\| M_r$$

and so $\|a^n\| \leqslant r^{n+1} M_r$ (by Theorem 3.10) for all $r > \nu(a)$. This and (5.3) yield

$$\nu(a) \leqslant \|a^n\|^{1/n} \leqslant \nu(r M_r)^{1/n} \to r$$

as $n \to \infty$; the result follows. (If $\alpha \leqslant a_n$ for all n and $a_n \to \alpha$ then $\alpha = \inf a_n$; that $\|a^n\|^{1/n} \to \nu(a)$ is immediate.) $\square$

The following theorem allows us to restrict our attention to commutative algebras when considered certain questions about the spectrum. Zorn's lemma allows us to prove that any commutative set in A is contained in a maximal commutative subalgebra.

Theorem 5.9. Let B be a maximal commutative subalgebra of A. If $a \in B$ then

$$\sigma_B(a) := \{\lambda \in \mathbb{C} : \lambda 1 - a \notin G(B)\} = \sigma_A(a) \ \big(= \{\lambda \in \mathbb{C} : \lambda 1 - a \notin G(A)\}\big).$$

Proof

Clearly $G(B) \subseteq G(A)$, so we need to prove that if $\lambda 1 - a$ is invertible in A then it is invertible in B. Suppose that $\lambda 1 - a \in G(A)$, let $b = (\lambda 1 - a)^{-1}$ and let $c \in B$; note that

$$bc = bc(\lambda 1 - a)b = b(\lambda 1 - a)cb = cb$$

and so $b \in B$ (else B is not maximal) as required. $\square$

Example 5.10. If S is a subset of the unital Banach algebra A then

$$S^{\mathrm{c}} := \{a \in A : sa = as \text{ for all } s \in S\}$$

is the *commutant* (or *centraliser*) of S. It is readily verified that S^{c} is a unital, closed subalgebra of A, that if $S \subseteq T \subseteq A$ then $T^{\mathrm{c}} \subseteq S^{\mathrm{c}}$ and that S is commutative if and only if $S \subseteq S^{\mathrm{c}}$. Furthermore, if S is commutative then the *double commutant* (or *bicommutant*) $S^{\mathrm{cc}} = (S^{\mathrm{c}})^{\mathrm{c}}$ is a commutative subalgebra of A containing S. It is an exercise to prove that a commutative subalgebra B is maximal if and only if $B = B^{\mathrm{c}}$, and that the proof of Theorem 5.9 goes through with the weaker hypothesis that B is a commutative subalgebra of A containing $\{a\}^{\mathrm{cc}}$.

Exercises 6

Exercise 6.1. Prove that the vector space $L^1(\mathbb{R})$ is a commutative Banach algebra when equipped with the convolution product and the norm $\|f\|_1 := \int |f|$. [To show associativity, use the convolution theorem and the inversion theorem for the Fourier transform.] Prove further that this algebra is not unital. [You may assume that the functions $f_n \colon x \mapsto (n/\sqrt{2\pi}) \exp(-\frac{1}{2}n^2x^2)$ are such that $\|f_n \star g - g\|_1 \to 0$ as $n \to \infty$ for all $g \in L^1(\mathbb{R})$.]

Exercise 6.2. Let X be a Hausdorff, locally compact space. Prove that $C_0(X)^{\mathrm{u}}$, the unitization of the algebra of continuous functions on X that vanish at infinity, is topologically isomorphic to $C(\dot{X})$, the algebra of continuous functions on $\dot{X}$, the one-point compactification of X.

Exercise 6.3. Let $A = \mathbb{C}[z]$ denote the unital algebra of complex polynomials and let $\|p\| := \sup\{|p(\alpha)| : |\alpha| \leqslant 1\}$ for all $p \in A$. Show that $(A, \|\cdot\|)$ is a unital, normed algebra which is not complete. [For the latter statement, consider invertibility and the polynomials $p_n(z) = 1 + z/n$.]

Exercise 6.4. Let A be a (non-unital) Banach algebra such that every element is nilpotent (i.e., for all $a \in A$ there exists $n \in \mathbb{N}$ such that $a^n = 0$). Prove that A is uniformly nilpotent: there exists $N \in \mathbb{N}$ such that $a^N = 0$ for all $a \in A$. [Consider the decomposition $A = \bigcup_{n \in \mathbb{N}} \{a \in A : a^n = 0\}$.]

Exercise 6.5. Let A be a unital Banach algebra over $\mathbb{C}$ and let $e^a := \sum_{n=0}^{\infty} a^n/n!$ for all $a \in A$. Prove that $e^{a+b} = e^a e^b$ if a and b commute. Deduce that e^a is invertible. Prove further that $f \colon \lambda \mapsto e^{\lambda a}$ is holomorphic everywhere, with $f'(\lambda) = af(\lambda) = f(\lambda)a$, for all $a \in A$.

Exercise 6.6. Let A be a unital Banach algebra over $\mathbb{C}$ and let $a, b \in A$. Use the identity $(ab)^n = a(ba)^{n-1}b$ to prove that ab and ba have the same spectral radius.

Exercise 6.7. Let A be a unital Banach algebra over $\mathbb{C}$. Suppose that there exists $K > 0$ such that $\|a\| \leqslant K\nu(a)$ for all $a \in A$, where $\nu(a)$ denotes the spectral radius of a. Prove that A is commutative. [Let $a, b \in A$ and consider the function $g \colon \lambda \mapsto e^{\lambda a} b e^{-\lambda a}$.]

Exercise 6.8. Let A be a Banach algebra and suppose that $(x_p)_{p \in P}$, $(y_q)_{q \in Q} \subseteq A$ are absolutely summable. Prove that

$$\sum_{p \in P} \sum_{q \in Q} x_p y_q = \sum_{(p,q) \in P \times Q} x_p y_q = \sum_{q \in Q} \sum_{p \in P} x_p y_q.$$

Six Characters and Maximal Ideals

Proposition 6.1. Let A be a unital algebra. If $\phi\colon A \to \mathbb{C}$ is a non-zero algebra homomorphism then $\phi(1) = 1$ and $\phi(a) \neq 0$ for all $a \in G(A)$. Furthermore, if A is a Banach algebra then $\phi \in A^*$ with $\|\phi\| = 1$.

Proof

Note that $\phi(a) = \phi(1a) = \phi(1)\phi(a)$ for all $a \in A$; it cannot be the case that $\phi(a) = 0$ for all $a \in A$ (as $\phi \neq 0$) and so $\phi(1) = 1$. Thus $1 = \phi(1) = \phi(aa^{-1}) = \phi(a)\phi(a^{-1})$ for all $a \in G(A)$, whence $\phi(a) \neq 0$ if a is invertible.

To see the statement about the norm of ϕ, note that $\phi(1) = 1$ (so $\|\phi\| \geqslant 1$) and suppose for contradiction that there exists $a \in A$ such that $\|a\| \leqslant 1$ and $|\phi(a)| > 1$. Let $b = \phi(a)^{-1}a$, so that $\|b\| < 1$ and $1 - b \in G(A)$, but $\phi(1 - b) = 1 - \phi(a)^{-1}\phi(a) = 0$, the desired contradiction. $\square$

From now on, A is a **commutative** unital Banach algebra over $\mathbb{C}$, unless otherwise specified.

Definition 6.2. A *character* of A is a non-zero algebra homomorphism from A to $\mathbb{C}$. The collection of all characters of A is denoted by $\Phi(A)$. A *maximal ideal* of A is an ideal I that is proper ($I \neq A$) and maximal with respect to inclusion: if J is an ideal such that $J \supseteq I$ then either $J = A$ or $J = I$.

Proposition 6.3. Every proper ideal of A contains no invertible element of A and is contained in a maximal ideal. A maximal ideal of A is closed.

Proof

If I is an ideal of A such that $I \cap G(A) \neq \emptyset$ then $1 = a^{-1}a \in I$ for some $a \in G(A)$ and so $I = A$, i.e., I is not proper.

To see the next claim, let I be a proper ideal of A, let $\mathcal{F}$ denote the collection of proper ideals of A that contain I, preordered by inclusion, and apply Zorn's lemma (note that no proper ideal contains an invertible element of A, so neither does the union of a chain of such).

Finally, let I be a maximal ideal; since the closure of an ideal is an ideal (an easy exercise), either $\bar{I} = A$ or $\bar{I} = I$; if the former then $\bar{I} \cap G(A) \neq \emptyset$, but $I \subseteq A \setminus G(A)$ implies that $\bar{I} \subseteq A \setminus G(A)$ since $G(A)$ is open. Hence $I = \bar{I}$, as claimed. $\square$

Before we can prove the fundamental connexion between characters and maximal ideals, we need a fact from algebra.

Lemma 6.4. Let A be a unital commutative algebra (or even a commutative ring with identity) and suppose that I is an ideal of A. Then A/I is a field if and only if I is maximal.

Proof

Suppose that I is maximal and let $a \in A \setminus I$. Then $aA + I$ is an ideal in A that properly contains I, so $aA + I = A$. Hence there exists $b \in A$ and $c \in I$ such that $ab + c = 1$, whence $[a][b] = [1]$ (where $[a] = a + I$ *et cetera*) and A/I is a field.

If I is not maximal then there exists $a \in A \setminus I$ such that $aA + I \neq A$, so $1 \notin aA + I$ and there exists no $b \in A$ such that $[a][b] = [1]$. Hence $[a]$ is not invertible and A/I is not a field. □

Theorem 6.5. The map $\phi \mapsto \ker \phi$ is a bijection between $\Phi(A)$ and the set of all maximal ideals of A.

Proof

Since $A/\ker\phi \cong \operatorname{im}\phi = \mathbb{C}$, $A/\ker\phi$ is a field for all $\phi \in \Phi(A)$ and hence $\ker\phi$ is maximal, by Lemma 6.4.

Suppose that $\phi, \psi \in \Phi(A)$ are such that $\ker\phi = \ker\psi$. For all $a \in A$ we have that $\phi(a - \phi(a)1) = 0$, so $a - \phi(a)1 \in \ker\phi = \ker\psi$ and $0 = \psi(a - \phi(a)1) = \psi(a) - \phi(a)$. Hence $\phi = \psi$ and $\phi \mapsto \ker\phi$ is injective.

Finally, if I is a maximal ideal of A then A/I is a field (by Lemma 6.4) and a Banach algebra (with respect to its quotient norm), so $A/I \cong \mathbb{C}$, by the Gelfand-Mazur theorem; let $i\colon A/I \to \mathbb{C}$ denote this isomorphism. Then $\phi = i \circ \pi$ is the desired character, where $\pi\colon A \to A/I$ is the quotient map. □

Characters and the Spectrum

Recall that A is a commutative, unital Banach algebra over $\mathbb{C}$, unless otherwise stated.

Corollary 6.6. Let $a \in A$. Then

(i) $a \in G(A)$ if and only if $\phi(a) \neq 0$ for all $\phi \in \Phi(A)$;

(ii) $\sigma(a) = \{\phi(a) : \phi \in \Phi(A)\}$;

(iii) $\nu(a) = \sup\{|\phi(a)| : \phi \in \Phi(A)\}$.

Proof

For (i), note that

$$\begin{aligned} a \in G(A) &\Leftrightarrow A = Aa \\ &\Leftrightarrow Aa \text{ is not contained in a maximal ideal of } A \\ &\Leftrightarrow Aa \not\subseteq \ker\phi \quad \forall\, \phi \in \Phi(A) \\ &\Leftrightarrow \phi(a) \neq 0 \quad \forall\, \phi \in \Phi(A). \end{aligned}$$

The other two claims are immediate. □

Corollary 6.7. If A is a (not necessarily commutative) unital Banach algebra and $a, b \in A$ commute, i.e., $ab = ba$, then

$$\nu(a+b) \leqslant \nu(a) + \nu(b) \qquad \text{and} \qquad \nu(ab) \leqslant \nu(a)\nu(b).$$

Proof

Let B be a maximal commutative subalgebra containing a and b. By Theorem 5.9, $\sigma_B(a+b) = \sigma_A(a+b)$ *et cetera*, and so

$$\begin{aligned}\nu(a+b) &= \sup\{|\phi(a+b)| : \phi \in \Phi(B)\} \\ &\leqslant \sup\{|\phi(a)| : \phi \in \Phi(B)\} + \sup\{|\phi(b)| : \phi \in \Phi(B)\} = \nu(a) + \nu(b).\end{aligned}$$

The other claim is proved in the same manner. □

Recall that an element a of a ring is said to be nilpotent if $a^n = 0$ for some $n \in \mathbb{N}$. If $a \in A$ is nilpotent then $\sigma(a) = 0$, by the spectral mapping theorem. More generally, we have the following definition.

Definition 6.8. An element $a \in A$ is *quasinilpotent* if $\nu(a) = 0$. The set of all quasinilpotent elements in A is the *Jacobson radical* of A, denoted by $J(A)$. An algebra is *semisimple* if $J(A) = \{0\}$.

Proposition 6.9. The Jacobson radical of A is an ideal; in fact,

$$J(A) = \bigcap_{\phi \in \Phi(A)} \ker \phi,$$

the intersection of all maximal ideals in A.

Proof

Since

$$\nu(a) = \sup\{|\phi(a)| : \phi \in \Phi(A)\},$$

we have that $\nu(a) = 0$ if and only if $a \in \ker \phi$ for all $\phi \in \Phi(A)$. □

The Gelfand Topology

Lemma 6.10. The character space $\Phi(A)$ is a compact, Hausdorff space when equipped with the *Gelfand topology*, i.e., the restriction of $\sigma(A^*, A)$, the weak* topology on A^*, to $\Phi(A)$. Equivalently, it is the coarsest topology to make the maps $\hat{a}|_{\Phi(A)} \colon \phi \mapsto \phi(a)$ continuous (for all $a \in A$).

Proof

Recall that A_1^* is a compact, Hausdorff space, by Theorem 3.22, and

$$\begin{aligned}\Phi(A) &= \{\phi \in A_1^* : \phi(1) = 1,\ \phi(ab) = \phi(a)\phi(b)\ \forall\, a, b \in A\} \\ &= A_1^* \cap \hat{1}^{-1}(1) \cap \bigcap_{a,b \in A} \left(\widehat{ab} - \hat{a}\hat{b}\right)^{-1}(0)\end{aligned}$$

is a closed subset of A_1^* (since $\hat{a}\colon A^* \to \mathbb{C};\ \phi \mapsto \phi(a)$ is continuous for all $a \in A$). [Note that the condition $\phi(1) = 1$ is necessary to rule out the zero homomorphism.] This gives the result. □

The Representation Theorem

From now on it is more convenient to let

$$\hat{a}\colon \Phi(A) \to \mathbb{C};\ \phi \mapsto \phi(a).$$

That is, $\hat{a}$ is the restriction to $\Phi(A)$ of the map $\phi \mapsto \phi(a)$ on A^*.

Theorem 6.11. If A is a commutative unital Banach algebra then the *Gelfand transform*

$$\hat{\cdot}\colon A \to C\big(\Phi(A)\big);\ a \mapsto \hat{a}$$

is a norm-decreasing homomorphism. Its kernel is $J(A)$ and its image $\hat{A}$ is a subalgebra of $C\big(\Phi(A)\big)$ that separates the points of $\Phi(A)$.

Proof

The Gelfand transform is a homomorphism because characters are; for example,

$$\widehat{ab}(\phi) = \phi(ab) = \phi(a)\phi(b) = \big(\hat{a}\hat{b}\big)(\phi) \qquad \forall\, \phi \in \Phi(A),\ a, b \in A.$$

Furthermore,

$$\|\hat{a}\|_\infty = \sup\{|\phi(a)| : \phi \in \Phi(A)\} = \nu(a) \leqslant \|a\| \qquad \forall\, a \in A,$$

so $a \mapsto \hat{a}$ is norm-decreasing; this calculation also shows that the kernel is as claimed. Finally, if $\phi, \psi \in \Phi(A)$ are such that $\hat{a}(\phi) = \hat{a}(\psi)$ for all $a \in A$ then $\phi = \psi$ (by definition), so $\hat{A}$ separates the points of $\Phi(A)$. □

Gelfand theory reaches its peak when the algebra is equipped with an involution, i.e., a conjugate-linear map $a \mapsto a^*$ such that

$$(ab)^* = b^*a^* \quad \text{and} \quad (a^*)^* = a \qquad \forall\, a, b \in A.$$

If the involution satisfies $\|a^*a\| = \|a\|^2$ for all $a \in A$ then we have a C^* *algebra*: for the theory of such, see [8, Chapter 4 *et seq.*] or [19, Chapter 11 *et seq.*].

Examples

Example 6.12. If X is a compact, Hausdorff space and $A = C(X)$ is the algebra of continuous functions on X then $\Phi(A) = \{\epsilon_x : x \in X\}$, where

$$\epsilon_x\colon C(X) \to \mathbb{C};\ \phi \mapsto \phi(x)$$

is the *evaluation homomorphism* at x.

Example 6.13. If $\mathbb{D} = B_1^{\mathbb{C}}(0) = \{z \in \mathbb{C} : |z| < 1\}$ and

$$A(\mathbb{D}) = \{f \in C(\bar{\mathbb{D}}) : f|_{\mathbb{D}} \text{ is holomorphic}\}$$

is the *disc algebra* then $A(\mathbb{D})$ is a Banach algebra (when equipped with the supremum norm: recall that uniform limits of holomorphic functions are holomorphic). In this case the character space of $A(\mathbb{D})$ is again just the set of evaluation homomorphisms, $\{\epsilon_z : z \in \bar{\mathbb{D}}\}$.

Example 6.14. Suppose that $n \in \mathbb{N}$ and let $\mathbb{Z}/n\mathbb{Z}$ denote the quotient group of integers with addition modulo n. The finite-dimensional Banach space

$$\ell^1(\mathbb{Z}/n\mathbb{Z}) = \big\{(x_j)_{j\in\mathbb{Z}} : x_j = x_{n+j}\ \forall\, j \in \mathbb{Z}\big\} = \big\{(x_{[j]})_{[j]\in\mathbb{Z}/n\mathbb{Z}}\big\}$$

becomes a commutative, unital Banach algebra when equipped with the convolution product

$$(x \star y)_{[j]} = \sum_{[k]\in\mathbb{Z}/n\mathbb{Z}} x_{[k]} y_{[j-k]} \quad \forall\, x, y \in \ell^1(\mathbb{Z}/n\mathbb{Z}).$$

Let

$$\delta := [j] \mapsto \begin{cases} 1 & [j] = [1], \\ 0 & [j] \neq [1] \end{cases}$$

and note that

$$x = \sum_{[j]\in\mathbb{Z}/n\mathbb{Z}} x_{[j]}\delta^j \qquad \forall\, x \in \ell^1(\mathbb{Z}/n\mathbb{Z}).$$

In particular, any $\phi \in \Phi\big(\ell^1(\mathbb{Z}/n\mathbb{Z})\big)$ is determined by $\lambda = \phi(\delta)$ and λ is an nth root of unity as $\lambda^n = \phi(\delta^n) = \phi(1) = 1$. Conversely, each $\lambda \in \{\omega^j : j = 0, 1, \ldots, n-1\}$, where $\omega = \exp(2\pi i/n)$, corresponds to a character, *via*

$$x \mapsto \sum_{[j]\in\mathbb{Z}/n\mathbb{Z}} x_{[j]}\lambda^j.$$

Hence the Gelfand theory of $\ell^1(\mathbb{Z}/n\mathbb{Z})$ corresponds to the theory of the discrete Fourier transform.

Before investigating the Gelfand theory of $L^1(\mathbb{R})$ we need a couple of preliminary results. (The fact that $L^1(\mathbb{R})^* = L^\infty(\mathbb{R})$ gives a simple proof of the following; as this has not been established the lemma is proved directly.)

Lemma 6.15. If $f \in L^1(\mathbb{R}^2)$ and $\phi \in L^1(\mathbb{R})^*$ then

$$\phi\Big(r \mapsto \int_{\mathbb{R}} f(r,s)\,\mathrm{d}s\Big) = \int_{\mathbb{R}} \phi\big(r \mapsto f(r,s)\big)\,\mathrm{d}s.$$

Proof

Note first that Fubini's theorem gives that $f(\cdot, s)\colon r \mapsto f(r,s) \in L^1(\mathbb{R})$ for almost every $s \in \mathbb{R}$ and that $\int_{\mathbb{R}} f(\cdot, s)\,\mathrm{d}s\colon r \mapsto \int_{\mathbb{R}} f(r,s)\,\mathrm{d}s \in L^1(\mathbb{R})$, so the quantities above are well defined; measurability of $s \mapsto \phi\big(f(\cdot, s)\big)$ will follow from the below. If $f = \chi_{A\times B}$ (where A, $B \subseteq \mathbb{R}$ are bounded intervals) then $s \mapsto \phi\big(f(\cdot, s)\big) = \phi(\chi_A)\chi_B$ and

$$\phi\Big(\int_{\mathbb{R}} f(\cdot, s)\,\mathrm{d}s\Big) = \phi\Big(\chi_A \int_{\mathbb{R}} \chi_B(s)\,\mathrm{d}s\Big) = \int_{\mathbb{R}} \phi(\chi_A)\chi_B(s)\,\mathrm{d}s = \int_{\mathbb{R}} \phi\big(f(\cdot, s)\big)\,\mathrm{d}s,$$

as claimed; linearity gives the result for all $f \in L^{\mathrm{step}}(\mathbb{R}^2)$. For general $f \in L^1(\mathbb{R}^2)$ take $(f_n)_{n\geqslant 1} \subseteq L^{\mathrm{step}}(\mathbb{R}^2)$ such that $f_n \to f$ in $L^1(\mathbb{R}^2)$. Note that $\int_{\mathbb{R}} f_n(\cdot, s)\,\mathrm{d}s \to \int_{\mathbb{R}} f(\cdot, s)\,\mathrm{d}s$ in $L^1(\mathbb{R})$ as

$$\int_{\mathbb{R}} \Big| \int_{\mathbb{R}} f_n(r, s)\,\mathrm{d}s - \int_{\mathbb{R}} f(r, s)\,\mathrm{d}s \Big|\,\mathrm{d}r \leqslant \int_{\mathbb{R}} \int_{\mathbb{R}} |f_n(r, s) - f(r, s)|\,\mathrm{d}s\,\mathrm{d}r = \|f_n - f\|_1,$$

so

$$\phi\Big(\int_{\mathbb{R}} f(\cdot, s)\,\mathrm{d}s\Big) = \lim_{n\to\infty} \phi\Big(\int_{\mathbb{R}} f_n(\cdot, s)\,\mathrm{d}s\Big) = \lim_{n\to\infty} \int_{\mathbb{R}} \phi\big(f_n(\cdot, s)\big)\,\mathrm{d}s = \int_{\mathbb{R}} \phi\big(f(\cdot, s)\big)\,\mathrm{d}s.$$

The last equality holds as

$$\Big| \int_{\mathbb{R}} \phi\big(f_n(\cdot, s)\big)\,\mathrm{d}s - \int_{\mathbb{R}} \phi\big(f(\cdot, s)\big)\,\mathrm{d}s \Big| \leqslant \int_{\mathbb{R}} \|\phi\|\,\|(f_n - f)(\cdot, s)\|_1\,\mathrm{d}s = \|\phi\|\,\|f_n - f\|_1.$$

(This calculation also shows that $s \mapsto \phi\big(f_n(\cdot, s)\big) \to s \mapsto \phi\big(f(\cdot, s)\big)$ in $L^1(\mathbb{R})$, so has there is a subsequence such that $\phi\big(f_{n_k}(\cdot, s)\big) \to \phi\big(f(\cdot, s)\big)$ for almost every $s \in \mathbb{R}$. In particular, $s \mapsto \phi\big(f(\cdot, s)\big)$ is the almost-everywhere limit of a sequence of measurable functions, so is measurable.) □

Lemma 6.16. If $\chi\colon \mathbb{R} \to \mathbb{C}$ is a continuous, bounded function such that $\chi(0) = 1$ and $\chi(s + t) = \chi(s)\chi(t)$ for all s, $t \in \mathbb{R}$ then there exists $\alpha \in \mathbb{R}$ such that $\chi(t) = e^{i\alpha t}$ for all $t \in \mathbb{R}$.

Proof

Since $\chi(0) = 1$ and χ is continuous, there exists $r > 0$ such that $c = \int_0^r \chi(x)\,\mathrm{d}x \neq 0$. Hence

$$c\chi(t) = \int_0^r \chi(x + t)\,\mathrm{d}x = \int_t^{t+r} \chi(y)\,\mathrm{d}y \qquad (t \in \mathbb{R})$$

and χ is differentiable; differentiating the equation $\chi(s+t) = \chi(s)\chi(t)$ with respect to s at 0 yields $\chi'(t) = \chi'(0)\chi(t)$ and so $\chi(t) = \exp(dt)$, where $d = \chi'(0)$. Since χ is bounded we must have that d is purely imaginary, as claimed. □

Example 6.17. Recall that $L^1(\mathbb{R})$ is a non-unital Banach algebra when equipped with the convolution product; let $A = L^1(\mathbb{R})^{\mathrm{u}}$ denote its unitization. If $\phi \in \Phi(A)$ then either $\ker\phi = L^1(\mathbb{R})$ (i.e., $\phi(\alpha 1 + f) = \alpha$ for all $\alpha \in \mathbb{C}$ and $f \in L^1(\mathbb{R})$) or there exists $f \in L^1(\mathbb{R})$ such that $\phi(f) = 1$. Suppose that the latter holds; as $C_{00}(\mathbb{R})$ is dense in $L^1(\mathbb{R})$ we may assume that f is continuous and has compact support. (If $g \in C_{00}(\mathbb{R})$ is such that $\|f - g\|_1 \leqslant 1/2$ then $|\phi(g)| \geqslant |\phi(f)| - |\phi(f - g)| \geqslant 1 - \|\phi\|\,\|f - g\|_1 \geqslant 1/2$; now replace f by $g/\phi(g)$.) If $g \in L^1(\mathbb{R})$ then Lemma 6.15 gives that

$$\phi(g) = \phi(f)\phi(g) = \phi(f \star g) = \phi\Big(s \mapsto \int_{\mathbb{R}} f_t(s)g(t)\,\mathrm{d}t\Big) = \int_{\mathbb{R}} \phi(f_t)g(t)\,\mathrm{d}t,$$

where $f_t(s) = f(s-t)$ for all $s, t \in \mathbb{R}$. Define

$$\chi\colon \mathbb{R} \to \mathbb{C};\ t \mapsto \phi(f_t)$$

and note that, since $\|f_t\|_1 = \|f\|_1$ for all $t \in \mathbb{R}$, $|\chi(t)| \leqslant \|\phi\| \, \|f_t\|_1 = \|f\|_1$, i.e., χ is bounded. Furthermore, $\chi(0) = \phi(f_0) = \phi(f) = 1$ and

$$|\chi(t+h) - \chi(t)| = |\phi(f_{t+h} - f_t)| \leqslant \|f_{t+h} - f_t\|_1 = \|f_h - f\|_1 \to 0$$

as $h \to 0$; this follows from the continuous form of the dominated-convergence theorem. Note also that, if $g, h \in L^1(\mathbb{R})$,

$$\big(g_{s+t} \star h\big)(r) = \int_{\mathbb{R}} g\big(r - p - (s+t)\big)h(p)\,\mathrm{d}p = \int_{\mathbb{R}} g(r-q-s)h(q-t)\,\mathrm{d}q = (g_s \star h_t)(r)$$

and so

$$\chi(s+t) = \phi(f_{s+t}) = \phi(f_{s+t})\phi(f) = \phi(f_{s+t} \star f) = \phi(f_s \star f_t) = \phi(f_s)\phi(f_t) = \chi(s)\chi(t)$$

for all $s, t \in \mathbb{R}$. By Lemma 6.16 we must have that $\chi(t) = e^{-ist}$ for all $t \in \mathbb{R}$, where $s \in \mathbb{R}$ is such that $e^{-is} = \chi(1)$. To see that $s \in \mathbb{R}$ is independent of the choice of f, let $g \in L^1(\mathbb{R})$ be such that $\phi(g) = 1$ and note that

$$\chi(t) = \phi(f_t)\phi(g) = \phi(f_t \star g) = \phi(f \star g_t) = \phi(f)\phi(g_t) = \phi(g_t) \qquad \forall\, t \in \mathbb{R}.$$

Hence

$$\phi(g) = \int_{\mathbb{R}} g(t)e^{-ist}\,\mathrm{d}t \qquad \forall\, g \in L^1(\mathbb{R})$$

and the Gelfand transform corresponds the classical Fourier transform for $L^1(\mathbb{R})$; the fact that $\alpha 1 + f \mapsto \alpha + \int_{\mathbb{R}} f(t)e^{-ist}\,\mathrm{d}t$ is a character for all $s \in \mathbb{R}$ is an immediate consequence of Fubini's theorem.

Example 6.18. If $L^1(\mathbb{R}^+)$ is equipped with the convolution product

$$(f \star g)(t) = \int_0^t f(t-s)g(s)\,\mathrm{d}s \qquad \forall\, f, g \in L^1(\mathbb{R}^+)$$

then it becomes a Banach algebra. The Gelfand theory here corresponds to the Laplace transform.

Example 6.19. Let $\mathbb{T} = \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ and $A = \{f \in C(\bar{\mathbb{D}}) : f|_{\mathbb{T}} \in A(\mathbb{D})|_{\mathbb{T}}\}$: A consists of those continuous functions on the closed unit disc $\bar{\mathbb{D}}$ that agree on the unit circle $\mathbb{T}$ with a continuous function on $\bar{\mathbb{D}}$ that is holomorphic in $\mathbb{D}$. The character space of A is homeomorphic to the sphere $S^2 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$.

Exercises 7

Exercise 7.1. Let $A = C(X)$, where X is a compact, Hausdorff space. Prove that the map $\epsilon\colon X \to \Phi(A);\ x \mapsto \epsilon_x$ is a homeomorphism, where $\epsilon_x(f) = f(x)$ for all $x \in X$ and $f \in C(X)$.

Exercise 7.2. Prove that if A is a unital Banach algebra generated by a single element (i.e., there exists $a \in A$ such that $\{p(a) : p(z) \in \mathbb{C}[z]\}$ is dense in A) then $\Phi(A)$ is homeomorphic to $\sigma(a)$. [Consider $\phi \mapsto \phi(a)$.] Deduce that $\Phi(A)$ is homeomorphic to $\bar{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leqslant 1\}$ if $A = A(\mathbb{D})$ is the disc algebra.

Exercise 7.3. Let A be a unital Banach algebra that is generated by one element, a, and let $\lambda \notin \sigma(a)$. Show there exists a sequence of polynomials $(p_n)_{n\geqslant 1}$ such that $p_n(z) \to (\lambda - z)^{-1}$ uniformly for all $z \in \sigma(a)$. [Hint: $(\lambda 1 - a)^{-1} \in A$.] Deduce that the complement of $\sigma(a)$ is connected. [Prove that if C is a bounded, maximally connected component of $\mathbb{C} \setminus \sigma(a)$ then C is open and then employ the maximum-modulus theorem [16, Theorem 5.20].]

Exercise 7.4. Let A be a commutative, unital Banach algebra. Prove that the Gelfand transform on A is isometric if and only if $\|a^2\| = \|a\|^2$ for all $a \in A$.

Exercise 7.5. Let A be a Banach algebra and B a semisimple, commutative, unital Banach algebra. Prove that if $\phi\colon A \to B$ is a homomorphism then ϕ is continuous. [Use the closed-graph theorem.]

Exercise 7.6. Let $A = C^1[0,1]$, equipped with the norm $\|f\| := \|f\|_\infty + \|f'\|_\infty$. Prove that A is a semisimple, commutative, unital Banach algebra and find its character space. Prove that $I = \{f \in A : f(0) = f'(0) = 0\}$ is a closed ideal in A such that A/I is a two-dimensional algebra with one-dimensional radical. What do you notice about A and A/I?

Exercise 7.7. Prove that the Banach space $\ell^1(\mathbb{Z})$ is a commutative, unital Banach algebra when equipped with the multiplication

$$a \star b\colon \mathbb{Z} \to \mathbb{C};\ n \mapsto \sum_{m\in\mathbb{Z}} a_m b_{n-m} \qquad \bigl(a, b \in \ell^1(\mathbb{Z})\bigr).$$

[You may assume that $\sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} a_m b_n = \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} a_m b_n$ for all a, $b \in \ell^1(\mathbb{Z})$.]

Exercise 7.8. Let $\delta \in \ell^1(\mathbb{Z})$ be such that $\delta_1 = 1$ and $\delta_n = 0$ if $n \neq 1$. Prove that $a = \sum_{n\in\mathbb{Z}} a_n \delta^n$ for all $a \in \ell^1(\mathbb{Z})$. Deduce that the character space of $\ell^1(\mathbb{Z})$ is homeomorphic to $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and (with this identification) the Gelfand transform on $\ell^1(\mathbb{Z})$ is the map

$$\Gamma\colon \ell^1(\mathbb{Z}) \to C(\mathbb{T});\ \Gamma(a)(\lambda) = \sum_{n\in\mathbb{Z}} a_n \lambda^n \qquad \forall\, \lambda \in \mathbb{T},\ a \in \ell^1(\mathbb{Z}).$$

Exercise 7.9. Let $f\colon \mathbb{T} \to \mathbb{C}$ be continuous and such that $\sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty$, where

$$\hat{f}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) e^{-int}\, \mathrm{d}t \qquad (n \in \mathbb{Z}).$$

Prove that if $f(z) \neq 0$ for all $z \in \mathbb{T}$ then $g = 1/f : \mathbb{T} \to \mathbb{C};\ z \mapsto 1/f(z)$ satisfies $\sum_{n\in\mathbb{Z}} |\hat{g}(n)| < \infty$. [This result is known as *Wiener's lemma*.]

Exercise 7.10. Let $\mathbb{T} = \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ and $A = \{f \in C(\bar{\mathbb{D}}) : f|_{\mathbb{T}} \in A(\mathbb{D})|_{\mathbb{T}}\}$: A consists of those continuous functions on the closed unit disc $\bar{\mathbb{D}}$ that agree on the unit circle $\mathbb{T}$ with a continuous function on $\bar{\mathbb{D}}$ that is holomorphic in $\mathbb{D}$.

[A corollary of the maximum-modulus theorem [16, Theorem 5.20] will be useful: if $f \in A(\mathbb{D})$ then $\|f\|_\infty := \sup\{|f(z)| : |z| \leqslant 1\} = \sup\{|f(z)| : |z| = 1\} =: \|f|_{\mathbb{T}}\|_\infty$.]

(i) Show that A is a Banach algebra when equipped with the supremum norm.

(ii) Prove that $I = \{f \in A : f|_{\mathbb{T}} = 0\}$ is a closed ideal in A and that $A = A(\mathbb{D}) \oplus I$. [Consider a suitable map $j : A \to A(\mathbb{D})$.]

(iii) Prove that

$$i : I \to C_0(\mathbb{D});\ f \mapsto f|_{\mathbb{D}}$$

is an isometric isomorphism. Deduce that I^u is topologically isomorphic to $C(\dot{\mathbb{D}})$, where $\dot{\mathbb{D}}$ is the one-point compactification of $\mathbb{D}$.

(iv) Prove that if $\phi \in \Phi(A)$ is such that $\ker\phi \supseteq I$ then $\phi = \tilde{\phi} \circ j$, where $\tilde{\phi} \in \Phi(A(\mathbb{D}))$. Deduce that $\phi = \varepsilon_z \circ j$ for some $z \in \bar{\mathbb{D}}$ (where $\varepsilon_z : A(\mathbb{D}) \to \mathbb{C};\ f \mapsto f(z)$).

(v) Let $\bar{\mathbb{D}}_1$ and $\bar{\mathbb{D}}_2$ be two copies of the unit disc and let $S^2 = \bar{\mathbb{D}}_1 \cup \bar{\mathbb{D}}_2 \big/\!\sim$ be the sphere obtained by identifying each point on $\mathbb{T}_1 = \partial\mathbb{D}_1$ with the corresponding point on $\mathbb{T}_2 = \partial\mathbb{D}_2$. Define

$$T : S^2 \to \Phi(A);\ z \mapsto \begin{cases} \varepsilon_z \circ j & (z \in \bar{\mathbb{D}}_1), \\ \varepsilon_z & (z \in \bar{\mathbb{D}}_2) \end{cases}$$

and prove that this is a well-defined, continuous injection.

(vi) Prove that $\Phi(A)$ is homeomorphic to the sphere S^2.

A Tychonov *via* Nets

Compactness in metric spaces can be characterised by the behaviour of sequences (*viz* every sequence having a convergent subsequence: the Bolzano-Weierstrass property). Furthermore, a proof of Tychonov's theorem for countable products of metric spaces can be given using sequences, together with construction of a diagonal subsequence (of a sequence of sequences!).

The astute reader may suspect that compactness in general topological spaces is equivalent to a property involving nets and that a proof of Tychonov's theorem can be given using *subnets*, whatever they may be. This suspicion is well founded; first we need to define a subnet.

Definition A.1. Let $(A, \leqslant)$ be a directed set and $(x_a)_{a\in A}$ a net. A *subnet* of $(x_a)_{a\in A}$ is a net $(y_b)_{b\in B}$ (where $(B, \ll)$ is a directed set) and a map $j\colon B \to A$ such that

(i) $y_b = x_{j(b)}$ for all $b \in B$;

(ii) for all $a_0 \in A$ there exists $b_0 \in B$ such that if $b \gg b_0$ then $j(b) \geqslant a_0$.

Definition A.2. A net $(x_a)_{a\in A}$ is *eventually* in a set S if there exists $a_0 \in A$ such that $x_a \in S$ for all $a \geqslant a_0$. (A net in a topological space converges to a point if it is eventually in every open set containing that point.) A net $(x_a)_{a\in A}$ is *frequently* in S if for all $a_0 \in A$ there exists $a \in A$ such that $a \geqslant a_0$ and $x_a \in A$. (Note that a net is not frequently in a set if it is eventually in its complement *et cetera*.) If X is a topological space then $x \in X$ is an *accumulation point* for a net if that net is frequently in every open set containing x.

Lemma A.3. Let $(x_a)_{a\in A}$ be a net in X and let $\mathcal{F}$ be a non-empty family of subsets of X that forms a directed set under reverse inclusion, such that $(x_a)_{a\in A}$ is frequently in S for all $S \in \mathcal{F}$. There is a subnet of $(x_a)_{a\in A}$ that is eventually in every $S \in \mathcal{F}$.

Proof

Let

$$B = \{(c, S) \in A \times \mathcal{F} : x_c \in S\},$$

ordered by setting

$$(c, S) \ll (d, T) \quad \Longleftrightarrow \quad c \leqslant d \ \text{ and } \ S \supseteq T.$$

If (c, S), $(d, T) \in B$ then there exist $f \in A$ and $W \in \mathfrak{F}$ such that c, $d \leqslant f$ and S, $T \supseteq W$. Since $(x_a)_{a\in A}$ is frequently in W, there exists $g \geqslant f$ such that $x_g \in W$, and so (c, S), $(d, T) \ll (g, W) \in B$. This shows that $(B, \ll)$ is a directed set; we claim that defining

$$y_{(c,S)} = x_c \quad \forall\,(c, S) \in B \qquad \text{and} \qquad j\colon B \to A;\ (c, S) \mapsto c$$

makes $(y_b)_{b\in B}$ a subnet of $(x_a)_{a\in A}$ that is eventually in S for all $S \in \mathfrak{F}$.

Let $S \in \mathfrak{F}$ and $a_0 \in A$; there exists $c \in A$ such that $c \geqslant a_0$ and $x_c \in S$, so if $b_0 := (c, S)$ then $b = (d, T) \gg b_0$ implies that $j(b) \geqslant a_0$ (we have a subnet) and that $y_b = x_d \in T \subseteq S$ (so $(y_b)_{b\in B}$ is eventually in S). This gives the result. $\square$

Proposition A.4. Let $(x_a)_{a\in A}$ be a net in the topological space X. The point $x \in X$ is an accumulation point of $(x_a)_{a\in A}$ if and only if there is a subnet of $(x_a)_{a\in A}$ that converges to x.

Proof

Let $(y_b)_{b\in B}$ be a subnet of $(x_a)_{a\in A}$ that converges to x, let U be an open set containing x and let $a_0 \in A$. There exists $b_0 \in B$ such that $j(b) \geqslant a_0$ for all $b \gg b_0$ (from the definition of a subnet) and there exists $b_1 \in B$ such that $y_b \in U$ for all $b \gg b_1$ (as $(y_b)_{b\in B}$ converges to x). If $b \gg b_0$ and $b \gg b_1$ then $x_{j(b)} = y_b \in U$ and $j(b) \geqslant a_0$, as required.

Conversely, let x be an accumulation point of $(x_a)_{a\in A}$ and let $\mathfrak{F} = \{U \in \mathfrak{T} : x \in U\}$. Then $(x_a)_{a\in A}$ has a subnet that is eventually in every element of $\mathfrak{F}$, i.e., converges to x, by Lemma A.3. $\square$

From here we can prove the Bolzano-Weierstrass theorem in its full generality.

Theorem A.5. A topological space X is compact if and only if every net in X has a convergent subnet.

Proof

Suppose that every net in X has a convergent subnet, and so an accumulation point, by Proposition A.4. Suppose further that X has an open cover $\mathfrak{F}$ with no finite subcover, and let A denote the collection of finite subsets of $\mathfrak{F}$. Ordered by inclusion, A is a directed set; we define a net $(x_a)_{a\in A}$ by choosing $x_a \notin \bigcup_{U\in a} U$ for all $a \in A$. Let x be an accumulation point of $(x_a)_{a\in A}$; since $\mathfrak{F}$ is a cover, there exists $U_0 \in \mathfrak{F}$ such that $x \in U_0$. The net $(x_a)_{a\in A}$ is frequently in U_0 (as this set contains an accumulation point) and so there exists $a \in A$ such that $a \supseteq \{U_0\}$ and $x_a \in U_0$. Since $x_a \notin \bigcup_{U\in a} U \supseteq U_0$, this is a contradiction.

Conversely, suppose that X is compact and let $(x_a)_{a\in A}$ be a net with no accumulation points: for all $x \in X$ there exists an open set U_x containing x and $a_x \in A$ such that $x_b \notin U_x$ for all $b \geqslant a_x$. The sets $\{U_x : x \in X\}$ form an open cover of X, so there is a finite subcover $\{U_{x_1}, \ldots, U_{x_n}\}$, but if $a \in A$ is such that $a \geqslant a_{x_i}$ for $i = 1, \ldots, n$ then $x_a \notin \bigcup_{i=1}^n U_{x_i} = X$. This contradiction gives the result. $\square$

Definition A.6. A net in X is *universal* if, for every $S \subseteq X$, the net is eventually in S or eventually in $X \setminus S$.

A universal net can be thought of as being 'maximally refined'; the first part of the next proposition makes this idea rigorous.

Proposition A.7. A universal net converges to its accumulation points. The image of a universal net by any function is universal.

Proof

If x is an accumulation point of the universal net $(x_a)_{a \in A}$ then let U be an open set containing x. Since $(x_a)_{a \in A}$ is frequently in U, it is eventually in U (since it cannot eventually be in its complement). Hence $(x_a)_{a \in A}$ converges to x.

If $(x_a)_{a \in A}$ is a universal net in X and $f\colon X \to Y$ then let $S \subseteq Y$. The net $(x_a)_{a \in A}$ is eventually in either $f^{-1}(S)$ or $X \setminus f^{-1}(S) = f^{-1}(Y \setminus S)$, hence $\big(f(x_a)\big)_{a \in A}$ is eventually in either S or $Y \setminus S$, as required. (Note that $f(f^{-1}(B)) = B \cap f(X)$ for any $B \subseteq Y$.) □

The next lemma is the most technically involved; the proof involves construction of what a *Bourbakiste* would call a *filter* (indeed, an *ultrafilter*).

Lemma A.8. Every net has a universal subnet.

Proof

Let $(x_a)_{a \in A}$ be a net in X and let

$$\mathcal{C} := \big\{\mathcal{U} = \{U \subseteq X\} : U \in \mathcal{U} \Rightarrow (x_a) \text{ is frequently in } U;\ U, V \in \mathcal{U} \Rightarrow U \cap V \in \mathcal{U}\big\},$$

ordered by inclusion. By Zorn's lemma, $\mathcal{C}$ has a maximal element, $\mathcal{U}_0$, and Lemma A.3 implies that $(x_a)_{a \in A}$ has a subnet $(y_b)_{b \in B}$ which is eventually in every $U \in \mathcal{U}_0$.

If $S \subseteq X$ is such that $(y_b)_{b \in B}$ is not eventually in $X \setminus S$ then $(y_b)_{b \in B}$ is frequently in S and so $(x_a)_{a \in A}$ is frequently in $U \cap S$ for all $U \in \mathcal{U}_0$: by the construction in Lemma A.3, $(y_b)_{b \in B} = (y_{(a,U)})_{(a,U) \in B}$ and given $(a, U) \in B$ there exists $(b, V) \in B$ such that $b \geqslant a$, $V \subseteq U$ and $x_b = y_{(b,V)} \in S$, so as $x_b \in V$ (by construction) $x_b \in V \cap S \subseteq U \cap S$. Hence, by maximality, $S \in \mathcal{U}_0$ (as $\mathcal{U}_0 \cup \{S, S \cap U : U \in \mathcal{U}_0\} \in \mathcal{C}$) and so $(y_b)_{b \in B}$ is eventually in S. This shows that $(y_b)_{b \in B}$ is universal, as required. □

Corollary A.9. A space is compact if and only if every universal net is convergent.

Proof

By the Bolzano-Weierstrass theorem (Theorem A.5), if the space is compact then every universal net has a convergent subnet and so an accumulation point (by Proposition A.4). Since universal nets converge to their accumulation points (Proposition A.7), the universal net is convergent.

Conversely, if every universal net is convergent then, as every net has a universal subnet (Lemma A.8), every net has a convergent subnet. Theorem A.5 gives the result. □

We can now present a proof of Tychonov's theorem that has a beautiful simplicity (the dust having been swept under the rug that is Corollary A.9).

Alternative Proof of Tychonov's Theorem

Let $(X, \mathcal{T})$ be the product of the compact spaces $\{(X_b, \mathcal{T}_b) : b \in B\}$ and suppose that $(x_a)_{a \in A}$ is a universal net in X. Since the image of a universal net is universal

(Proposition A.7), $(\pi_b(x_a))_{a\in A}$ is universal in X_b, so convergent (by Corollary A.9), for all $b \in B$. Hence $x_a \to x$, where $x_b := \lim_{a\in A} \pi_b(x_a)$ (by a property of initial topologies), and X is compact, by Corollary A.9 again. $\square$

Exercises A

Exercise A.1. Let $X = \ell^\infty$ and for all $n \in \mathbb{N}$ define $\delta_n \in X^*$ by setting $\delta_n\big((x_k)_{k\geqslant 1}\big) = x_n$. Prove that $(\delta_n)_{n\geqslant 1}$ has no weak*-convergent subsequence but that $(\delta_n)_{n\geqslant 1}$ has a weak*-convergent subnet.

Solutions to Exercises

Solutions to Exercises 1

Exercise 1.1. Let X be a normed vector space and let M be a closed subspace of X. Prove that

$$\pi\{y \in X : \|y - x\| < \varepsilon\} = \big\{[y] \in X/M : \big\|[y] - [x]\big\| < \varepsilon\big\} \qquad \forall\, x \in X,\ \varepsilon > 0,$$

where $\pi\colon X \to X/M;\ x \mapsto [x]$ is the natural map from X onto X/M (the *quotient map*).

Let
$$B_\varepsilon^X(x) := \{y \in X : \|x - y\| < \varepsilon\}$$
denote the open ball in X with centre x and radius ε. If $y \in B_\varepsilon^X(x)$ then

$$\big\|[y] - [x]\big\| = \big\|[y - x]\big\| \leqslant \|y - x\| < \varepsilon$$

and $\pi(y) \in B_\varepsilon^{X/M}\big([x]\big)$. If $y \in X$ is such that $\big\|[y] - [x]\big\| < \varepsilon$ then there exists $m \in M$ such that $\|y - x - m\| < \varepsilon$, whence $[y] = \pi(y - m) \in \pi\big(B_\varepsilon^X(x)\big)$.

Deduce that the quotient norm yields the *quotient topology* on X/M given by

$$\mathcal{Q} := \{U \subseteq X/M : \pi^{-1}(U) \in \mathcal{T}\},$$

where $\mathcal{T}$ denotes the norm topology on X, and that the quotient map is open (i.e., sends open sets to open sets).

Let $\mathcal{T}_{\|\cdot\|}$ denote the topology on X/M given by the quotient norm. If $U \in \mathcal{Q}$ then $\pi^{-1}(U) \in \mathcal{T}$ and so $\pi^{-1}(U) = \bigcup_{x \in U} B_{\varepsilon_x}^X(x)$, whence

$$U = \pi\big(\pi^{-1}(U)\big) = \bigcup_{x \in U} \pi\big(B_{\varepsilon_x}^X(x)\big) = \bigcup_{x \in U} B_{\varepsilon_x}^{X/M}\big([x]\big) \in \mathcal{T}_{\|\cdot\|}.$$

(The first equality holds because π is surjective.) Conversely, since $\pi^{-1}\big(\pi(A)\big) = A + M$ for all $A \subseteq X$,

$$\pi^{-1}\Big(B_\varepsilon^{X/M}\big([x]\big)\Big) = \pi^{-1}\Big(\pi\big(B_\varepsilon^X(x)\big)\Big) = B_\varepsilon^X(x) + M = \bigcup_{m \in M} B_\varepsilon^X(x + m) \in \mathcal{T}.$$

Hence $B_\varepsilon^{X/M}\big([x]\big) \in \mathcal{Q}$ and we have the first result. The quotient map is open by the first part of this exercise.

Prove also that the quotient map is linear and continuous.

Linearity is immediate and continuity follows from the fact that $\big\|[x]\big\| \leqslant \|x\|$ for all $x \in X$.

Exercise 1.2. Prove directly that if E is a Banach space and M is a closed subspace of E then the quotient space $\big(E/M, \|\cdot\|_{E/M}\big)$ is complete.

Let $\big([x_n]\big)_{n\geqslant 1}$ be a sequence in E/M with $\sum_{n=1}^{\infty}\big\|[x_n]\big\|$ convergent. For all $n \geqslant 1$ there exists $m_n \in M$ such that $\|x_n - m_n\| \leqslant \big\|[x_n]\big\| + 2^{-n}$, by definition of the quotient norm, and by comparison $\sum_{n=1}^{\infty}\|x_n - m_n\|$ is convergent. Hence $\sum_{n=1}^{\infty} x_n - m_n$ converges, by the completeness of E, and, as the quotient map $\pi\colon E \to E/M;\ x \mapsto [x]$ is continuous, so does $\pi(\sum_{n=1}^{\infty} x_n - m_n) = \sum_{n=1}^{\infty}[x_n]$.

Exercise 1.3. Let M and N be subspaces of the normed space X. Prove that if M is finite dimensional and N is closed then $M + N$ is closed.

Note that $M + N = \pi^{-1}\big(\pi(M)\big)$ if $\pi\colon X \to X/N$ is the quotient map, and that if $\{x_1, \ldots, x_n\}$ is a basis for M then $\{\pi(x_1), \ldots, \pi(x_n)\}$ is a spanning set for $\pi(M)$, so $\pi(M)$ is finite-dimensional and therefore closed. Hence $M + N$ is the preimage of a closed set under a continuous map, so is itself closed.

Exercise 1.4. Prove that if $\{A^i_j : i \in I,\ j \in J\}$ and $\{B^k_l : k \in K,\ l \in L\}$ are families of sets, where the index sets I, J, K and L are arbitrary, then

$$\Big(\bigcup_{i\in I}\bigcap_{j\in J} A^i_j\Big) \cap \Big(\bigcup_{k\in K}\bigcap_{l\in L} B^k_l\Big) = \bigcup_{(i,k)\in I\times K}\ \bigcap_{(j,l)\in J\times L} A^i_j \cap B^k_l.$$

We have that

$$\begin{aligned} & x \in \Big(\bigcup_{i\in I}\bigcap_{j\in J} A^i_j\Big) \cap \Big(\bigcup_{k\in K}\bigcap_{l\in L} B^k_l\Big) \\ \iff\quad & \exists\, i_0 \in I,\ k_0 \in K \text{ such that } x \in A^{i_0}_j \text{ and } x \in B^{k_0}_l \quad \forall\, j \in J,\ l \in L \\ \iff\quad & \exists\,(i_0, k_0) \in I \times K \text{ such that } x \in A^{i_0}_j \cap B^{k_0}_l \quad \forall\,(j,l) \in J \times L \\ \iff\quad & x \in \bigcup_{(i,k)\in I\times K}\ \bigcap_{(j,l)\in J\times L} A^i_j \cap B^k_l, \end{aligned}$$

as claimed.

What does this have to do with initial topologies?

A consequence of this is that the collection of arbitrary unions of finite intersections of elements of a subbase is itself closed under finite intersections, and so initial topologies are as claimed in Definition 1.24.

Exercise 1.5. Prove that if $\mathcal{T}_F$ is the initial topology on X generated by a collection of functions F and $Y \subseteq X$ then $\mathcal{T}_F|_Y$, the relative initial topology on Y, is the initial topology generated by $F|_Y = \{f|_Y : f \in F\}$, the restrictions of the functions in F to Y.

Since $f|_Y^{-1}(U) = f^{-1}(U) \cap Y$, this is immediate.

Exercise 1.6. Let $(x_a)_{a\in A}$ be a family of non-negative real numbers and let $\mathcal{A}$ denote the collection of finite subsets of A. Prove that $(x_a)_{a\in A}$ is summable (with sum α) if and only if $\beta = \sup\{\sum_{a\in A_0} x_a : A_0 \in \mathcal{A}\} < \infty$ and in this case $\alpha = \beta$.

If $\sum_{a\in A_0} x_a \to \alpha$ then let $\varepsilon > 0$ and choose $A_1 \in \mathcal{A}$ such that $\left|\sum_{a\in A_0} x_a - \alpha\right| < \varepsilon$ for all $A_0 \in \mathcal{A}$ with $A_0 \supseteq A_1$. If $A_2 \in \mathcal{A}$ then

$$\sum_{a\in A_2} x_a \leqslant \sum_{a\in A_1\cup A_2} x_a \leqslant \Big|\sum_{a\in A_1\cup A_2} x_a - \alpha\Big| + |\alpha| < |\alpha| + \varepsilon,$$

so $\sup\{\sum_{a\in A_0} x_a : A_0 \in \mathcal{A}\} < \infty$.

Conversely, if $\beta = \sup\{\sum_{a\in A_0} x_a : A_0 \in \mathcal{A}\} < \infty$ then let $\varepsilon > 0$ and choose $A_1 \in \mathcal{A}$ such that $\sum_{a\in A_1} x_a > \beta - \varepsilon$. Then

$$\beta + \varepsilon > \sum_{a\in A_0} x_a \geqslant \sum_{a\in A_1} x_a > \beta - \varepsilon$$

for all $A_0 \in \mathcal{A}$ such that $A_0 \supseteq A_1$; hence $\sum_{a\in A_0} x_a \to \beta$. Since $\mathbb{R}$ is Hausdorff we must have $\alpha = \beta$.

Exercise 1.7. Let E be a Banach space and let $(x_a)_{a\in A}$ a family of vectors in E. Prove that if $\sum_{a\in A} \|x_a\|$ is convergent then $S := \{a \in A : x_a \neq 0\}$ is countable.

Let $S_n := \{a \in A : \|x_a\| > 1/n\}$ for $n \in \mathbb{N}$; applying the previous exercise to $\sum_{a\in A} \|x_a\|$ we see that $\beta = \sup\{\sum_{a\in A_0} \|x_a\| : A_0 \in \mathcal{A}\} < \infty$. If S_n is infinite for some $n \in \mathbb{N}$ then let $a_1, \ldots, a_m$ be distinct elements of S_n, where $m > n\beta$, and note that $\sum_{j=1}^m \|x_{a_j}\| > m/n > \beta$, a contradiction. Hence $S = \bigcup_{n=1}^\infty S_n$ is a countable union of finite sets, so countable.

Deduce that $(x_a)_{a\in A}$ is summable with sum

$$\textstyle\sum_{a\in A} x_a = \begin{cases} \sum_{a\in S} x_a & \text{if } S \text{ is finite,} \\ \sum_{j=1}^\infty x_{a_j} & \text{if } S \text{ is infinite,} \end{cases}$$

where (if S is infinite) $j \mapsto a_j$ is a bijection between $\mathbb{N}$ and S.

If S is finite then $\sum_{a\in A_0} x_a - \sum_{a\in S} x_a = 0$ for all $A_0 \in \mathcal{A}$ such that $A_0 \supseteq S$, hence $\sum_{a\in A_0} x_a \to \sum_{a\in S} x_a$. If S is countably infinite, $j \mapsto a_j$ is as above and $m, n \in \mathbb{N}$ are such that $m > n$ then

$$\Big\|\sum_{j=1}^m x_{a_j} - \sum_{j=1}^n x_{a_j}\Big\| \leqslant \sum_{j=n+1}^m \|x_{a_j}\| \leqslant \beta - \sum_{j=1}^n \|x_{a_j}\|. \tag{$\star$}$$

Let $\varepsilon > 0$ and choose $A_0 \in \mathcal{A}$ such that $\sum_{a\in A_0} \|x_a\| > \beta - \varepsilon$; if $n_0 := \max\{j : a_j \in A_0\}$ then $\sum_{j=1}^n \|x_{a_j}\| \geqslant \sum_{a\in A_0} \|x_a\| > \beta - \varepsilon$ for all $n \geqslant n_0$, so from ($\star$) we have that

$$\Big\|\sum_{j=1}^m x_{a_j} - \sum_{j=1}^n x_{a_j}\Big\| < \beta - (\beta - \varepsilon) = \varepsilon \qquad \forall\, m, n \geqslant n_0.$$

Thus $\left(\sum_{j=1}^n x_{a_j}\right)_{n\geqslant 1}$ is Cauchy, so convergent to $x \in E$, say. If $n \geqslant n_0$ is such that $\left\|\sum_{j=1}^n x_{a_j} - x\right\| < \varepsilon$ then, for all $A_0 \in \mathcal{A}$ such that $A_0 \supseteq \{a_1, \ldots, a_n\}$,

$$\begin{aligned} \Big\|\sum_{a\in A_0} x_a - x\Big\| &\leqslant \Big\|\sum_{a\in A_0\setminus\{a_1,\ldots,a_n\}} x_a\Big\| + \Big\|\sum_{j=1}^n x_{a_j} - x\Big\| \\ &< \sum_{a\in A_0}\|x_a\| - \sum_{j=1}^n \|x_a\| + \varepsilon \\ &< \beta - (\beta - \varepsilon) + \varepsilon \;=\; 2\varepsilon. \end{aligned}$$

Hence $\sum_{a\in A_0} x_a \to x$, as claimed.

Exercise 1.8. Prove that a family of complex numbers $(z_a)_{a\in A}$ is summable if and only if $\left(|z_a|\right)_{a\in A}$ is summable.

Note first that, since

$$\max\bigl\{|\operatorname{Re} z|, |\operatorname{Im} z|\bigr\} \leqslant |z| \leqslant |\operatorname{Re} z| + |\operatorname{Im} z| \qquad \forall\, z \in \mathbb{C},$$

$\left(|z_a|\right)_{a\in A}$ is summable if and only if $\left(|\operatorname{Re} z_a|\right)_{a\in A}$ and $\left(|\operatorname{Im} z_a|\right)_{a\in A}$ are summable, by Exercise 1.6. Note also that $(z_a)_{a\in A}$ is summable if and only if $\left(\operatorname{Re} z_a\right)_{a\in A}$ and $\left(\operatorname{Im} z_a\right)_{a\in A}$ are summable, because if A_0 is a finite subset of A then

$$\begin{aligned} &\max\Bigl\{\Big|\sum_{a\in A_0}\operatorname{Re} z_a - \operatorname{Re} z\Big|, \Big|\sum_{a\in A_0}\operatorname{Im} z_a - \operatorname{Im} z\Big|\Bigr\} \\ &\qquad\qquad \leqslant \Big|\sum_{a\in A_0} z_a - z\Big| \leqslant \Big|\sum_{a\in A_0}\operatorname{Re} z_a - \operatorname{Re} z\Big| + \Big|\sum_{a\in A_0}\operatorname{Im} z_a - \operatorname{Im} z\Big|. \end{aligned}$$

Hence it suffices to prove the real case of this proposition.

Given $(x_a)_{a\in A} \subseteq \mathbb{R}$ let $x_a^+ = \max\{x_a, 0\}$ and $x_a^- = -\min\{x_a, 0\}$ for all $a \in A$; note that

$$\sum_{a\in A_0}|x_a| = \sum_{a\in A_0} x_a^+ + \sum_{a\in A_0} x_a^- \qquad \forall\, A_0 \in \mathcal{A},$$

so $\left(|x_a|\right)_{a\in A}$ is summable if and only if $(x_a^+)_{a\in A}$ and $(x_a^-)_{a\in A}$ are summable. Since $(y_a)_{a\in A}$ is summable if and only if $(-y_a)_{a\in A}$ is, and $x_a = x_a^+ - x_a^-$, it suffices to prove that the summability of $(x_a)_{a\in A}$ implies that of $(x_a^+)_{a\in A}$.

For this we suppose otherwise; let $\sum_{a\in A} x_a = x$, take $\varepsilon > 0$ and choose $A_1 \in \mathcal{A}$ such that $\left|\sum_{a\in A_0} x_a - x\right| < \varepsilon$ for all $A_0 \in \mathcal{A}$ that contain A_1. Set $A_2 = \{a \in A_1 : x_a^- > 0\}$ and let $A_3 \in \mathcal{A}$ be such that

$$\sum_{a\in A_3} x_a^+ > \sum_{a\in A_2} x_a^- + |x| + \varepsilon;$$

this exists as $\sum_{a\in A} x_a^+$ is not convergent. If $A_0 = \{a \in A_3 : x_a^+ > 0\} \cup A_1$ then

$$\sum_{a\in A_3} x_a^+ \leqslant \sum_{a\in A_0} x_a^+ \leqslant \Big|\sum_{a\in A_0} x_a^+ - x_a^- - x\Big| + \Big|\sum_{a\in A_0} x_a^- + x\Big| < \varepsilon + \sum_{a\in A_2} x_a^- + |x|,$$

the desired contradiction.

Exercise 1.9. Find a Hilbert space H and a countable family of vectors $(x_n)_{n\in\mathbb{N}}$ in H that is summable but not absolutely summable (i.e., $\big(\|x_n\|\big)_{n\in\mathbb{N}}$ is not summable).

Let $H = \ell^2(\mathbb{N})$ and let $x_n = e_n/n$ for all $n \in \mathbb{N}$, where e_n is the standard basis vector, with 1 in nth position and 0 elsewhere. As the harmonic series is divergent,

$$\sup\Big\{\sum_{n\in N_0} \|x_n\| : N_0 \text{ is a finite subset of } \mathbb{N}\Big\} \geqslant \sup\Big\{\sum_{n=1}^{k} \frac{1}{n} : k \in \mathbb{N}\Big\} = \infty,$$

and Exercise 1.6 gives that $\big(\|x_n\|\big)_{n\in\mathbb{N}}$ is not summable. However, $(x_n)_{n\in\mathbb{N}}$ has sum $x := (1, 1/2, 1/3, \ldots)$: for $\varepsilon > 0$ take $k \in \mathbb{N}$ such that $\sum_{n=k+1}^{\infty} 1/n^2 < \varepsilon^2$ and note that $\|\sum_{n\in N_0} x_n - x\| < \varepsilon$ for every finite subset $N_0 \subseteq \mathbb{N}$ that contains $\{1, \ldots, k\}$.

Exercise 1.10. Prove the converse to Proposition 1.31, that in a space with a non-Hausdorff topology there exists a net that converges to two distinct points.

Suppose that $\mathcal{T}$ is not a Hausdorff topology on the set X: there exist distinct points x, $y \in X$ such that every pair of open sets (U, V) with $x \in U$ and $y \in V$ satisfies $U \cap V \neq \emptyset$. Let $\mathcal{A} \subseteq \mathcal{T} \times \mathcal{T}$ denote the aggregate of such pairs, with preorder $\leqslant$ defined by setting

$$(A, B) \leqslant (C, D) \quad \Longleftrightarrow \quad A \supseteq C \text{ and } B \supseteq D.$$

This is a directed set: if (A, B), $(C, D) \in \mathcal{A}$ then $(A \cap C, B \cap D) \in \mathcal{T} \times \mathcal{T}$ is an upper bound for (A, B) and (C, D), and since $x \in A \cap C$ and $y \in B \cap D$, $(A \cap C, B \cap D) \in \mathcal{A}$. For all $(A, B) \in \mathcal{A}$ let $x_{(A,B)} \in A \cap B$; we claim that the net $(x_{(A,B)})_{(A,B)\in\mathcal{A}}$ converges to x and to y. If $U \in \mathcal{T}$ is such that $x \in U$ then $x_{(A,B)} \in U$ for all $(A, B) \in \mathcal{A}$ such that $A \subseteq U$, hence $x_{(A,B)} \in U$ for all $(A, B) \geqslant (U, X)$ and $x_{(A,B)} \to x$. This same argument works for y and so we have the result.

Exercise 1.11. A sequence in a normed vector space that is convergent is necessarily bounded. Is the same true for nets?

For a simple counterexample, consider $\mathbb{Z}$ directed by its usual order and define a net $(x_n)_{n\in\mathbb{Z}}$ by setting $x_n = n$ if $n \leqslant 0$ and $x_n = 1/n$ if $n > 0$.

Solutions to Exercises 2

Exercise 2.1. Let X be a topological space and E a Banach space; recall that $C_{\mathrm{b}}(X, E)$, the space of E-valued, bounded, continuous functions on X, is complete with respect to the norm

$$f \mapsto \|f\|_\infty := \sup\{\|f(x)\|_E : x \in X\}.$$

Prove that $C_0(X, E)$, the continuous, E-valued functions on X that *vanish at infinity* (i.e., those $f \in C(X, E)$ such that $\{x \in X : \|f(x)\|_E \geqslant \varepsilon\}$ is compact for all $\varepsilon > 0$) is a closed subspace of $C_{\mathrm{b}}(X, E)$.

Note that any function $f \in C_0(X,E)$ is bounded, as f is continuous on the compact set $\{x \in X : \|f(x)\|_E \geqslant 1\}$ and so bounded there. Let $(f_n)_{n\geqslant 1} \subseteq C_0(X,E)$ converge to f; if $\varepsilon > 0$ and $n \in \mathbb{N}$ are such that $\|f_n - f\|_\infty < \varepsilon/2$ then

$$\|f_n(x)\| \geqslant \|f(x)\| - \|f_n(x) - f(x)\| > \|f(x)\| - \tfrac{1}{2}\varepsilon,$$

so $\{x \in X : \|f(x)\| \geqslant \varepsilon\}$ is a closed subset of the compact set $\{x \in X : \|f_n(x)\| \geqslant \varepsilon/2\}$ and therefore is itself compact. Hence $C_0(X,E)$ is closed.

If f, $g \in C_0(X,E)$ then $K = \{x \in X : \|f(x)\| \geqslant \varepsilon/2\}$ and $L = \{x \in X : \|g(x)\| \geqslant \varepsilon/2\}$ are compact, and if $x \notin K \cup L$ then

$$\|f(x) + g(x)\| \leqslant \|f(x)\| + \|g(x)\| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon,$$

whence $\{x \in X : \|(f+g)(x)\| \geqslant \varepsilon\} \subseteq K \cup L$ is compact. If $\alpha \in \mathbb{F}$ then either $\alpha = 0$, in which case $\alpha f = 0 \in C_0(X,E)$ trivially, or $\{x \in X : \|\alpha f\| \geqslant \varepsilon\} = \{x \in X : \|f\| \geqslant \varepsilon/|\alpha|\}$ is compact for all $\varepsilon > 0$. This shows that $C_0(X,E)$ is a subspace of $C_\mathrm{b}(X,E)$.

Exercise 2.2. Let $(X, \mathcal{T})$ be a Hausdorff, locally compact space and let ∞ denote a point not in X. Show that

$$\dot{\mathcal{T}} := \mathcal{T} \cup \{U \subseteq \dot{X} : \infty \in U,\ X \setminus U \text{ is compact}\}$$

is a Hausdorff, compact topology on $\dot{X} := X \cup \{\infty\}$.

It is routine to verify that $\dot{\mathcal{T}}$ is a topology; recall that compact sets are closed under finite unions and arbitrary intersections, and that compact sets are closed in Hausdorff spaces. If $\mathcal{C} \subseteq \dot{\mathcal{T}}$ is an open cover of $\dot{X}$ then there exists $U \in \mathcal{C}$ such that $\infty \in U$, and since $X \setminus U$ is compact and has open cover $\mathcal{C} \setminus \{U\}$, this has a finite subcover $\mathcal{C}_0$. Hence $\mathcal{C}_0 \cup \{U\}$ is a finite subcover of the cover $\mathcal{C}$, showing that $\dot{\mathcal{T}}$ is compact. Finally, since $\mathcal{T}$ is Hausdorff it suffices to take $x \in X$ and prove that there exist open sets separating ∞ and x. As $\mathcal{T}$ is locally compact there exists $U \in \mathcal{T}$ such that $x \in U$ and $\bar{U}$ is compact, whence U and $\{\infty\} \cup (X \setminus \bar{U})$ are elements of $\dot{\mathcal{T}}$ as required.

Prove that there is a natural correspondence between $C_0(X,E)$ and $\{f \in C(\dot{X},E) : f(\infty) = 0\}$.

For $f \in C_0(X,E)$ let

$$\dot{f} \colon \dot{X} \to E;\ x \mapsto \begin{cases} f(x) & \text{if } x \in X, \\ 0 & \text{if } x = \infty. \end{cases}$$

Let $U \subseteq E$ be open; if $0 \notin U$ then $\dot{f}^{-1}(U) = f^{-1}(U) \in \mathcal{T} \subseteq \dot{\mathcal{T}}$, and if $0 \in U$ then without lost of generality $U = B_\varepsilon^E(0)$ (as we may write U as the union of a set of this form and the open set $U \setminus \{0\} = (X \setminus \{0\}) \cap U$). Since

$$\dot{f}^{-1}\big(B_\varepsilon^E(0)\big) = \{\infty\} \cup \{x \in X : \|f(x)\| < \varepsilon\} = \{\infty\} \cup \big(X \setminus \{x \in X : \|f(x)\| \geqslant \varepsilon\}\big) \in \dot{\mathcal{T}},$$

we see that $\dot{f}$ is continuous.

Conversely, let $\dot{f} \in C(\dot{X}, E)$ be such that $\dot{f}(\infty) = 0$, and let $f = \dot{f}|_X$. Since the relative topology $\dot{\mathcal{T}}_X$ equals $\mathcal{T}$, f is continuous, and if $\varepsilon > 0$ then

$$\{x \in X : \|f(x)\| < \varepsilon\} = \{x \in \dot{X} : \|\dot{f}(x)\| < \varepsilon\} \cap X = \dot{f}^{-1}\big(B^E_\varepsilon(0)\big) \cap X = X \setminus K$$

for some compact set $K \subseteq X$, as required.

Exercise 2.3. Let X be a separable normed space. Prove that X_1 is separable (in the norm topology).

Let S be a countable, dense set in X and note that $\{qs : q \in \mathbb{Q},\ s \in S\}$ is countable (being the image of the countable set $\mathbb{Q} \times S$ under the mapping $(q, s) \mapsto qs$). Hence we may assume, without loss of generality, that S is closed under multiplication by rationals.

Let $x \in X_1$ and choose $(y_n)_{n\geqslant 1} \subseteq S$ such that $\|x - y_n\| \leqslant 1/n$ for all $n \geqslant 1$. Then $z_n := ny_n/(n + 1) \in S$ for all $n \geqslant 1$, $z_n \to x$ by the algebra of limits and

$$\|z_n\| \leqslant \Big\|\frac{n}{n+1}(y_n - x)\Big\| + \Big\|\frac{n}{n+1}x\Big\| \leqslant \frac{1}{n+1} + \frac{n}{n+1}\|x\| \leqslant 1,$$

as required.

Prove that any separable Banach space E is isometrically isomorphic to a quotient space of ℓ^1.

Let $(e_n)_{n\geqslant 1} \subseteq E_1$ be dense; such exists by the first part of the question. Define

$$T : \ell^1 \to E;\ x \mapsto Tx := \sum_{n=1}^{\infty} x_n e_n$$

and note that $T \in \mathcal{B}(\ell^1, E)$; linearity is obvious, as is the absolute convergence of Tx (and the fact that $\|T\| \leqslant 1$) because

$$\sum_{n=1}^{\infty} \|x_n e_n\| \leqslant \sum_{n=1}^{\infty} |x_n| = \|x\|_1.$$

Since $T\big((\ell^1)_1\big) \supseteq \{e_n : n \geqslant 1\}$, which is dense in E_1, the open-mapping lemma gives that T is surjective. Furthermore, this set is k-dense in E_1 for all $k \in (0, 1)$, so if $y \in E$ there exists $x_k \in \ell^1$ such that $y = Tx_k$ and $\|x_k\| \leqslant \|Tx_k\|/(1 - k)$. The identity $Tx_k = Tx_{k'}$ for all k, $k' \in (0, 1)$ gives that

$$\big\|[x_k]\big\| \leqslant \|x_k\| \leqslant \frac{\|Tx_k\|}{1-k} = \frac{\big\|\tilde{T}[x_k]\big\|}{1-k} \qquad \forall\, k \in (0, 1),$$

where $[x] = x + \ker T$ and $\tilde{T} : \ell^1/\ker T \to \operatorname{im} T = E$ is a bijection such that $Tx = \tilde{T}[x]$ for all $x \in \ell^1$. This shows that $\big\|[y]\big\| \leqslant \big\|\tilde{T}[y]\big\|$ for all $y \in \ell^1$, and the opposite inequality follows from the fact $\big\|\tilde{T}\big\| = \|T\| \leqslant 1$. Hence $\ell^1/\ker T$ is isometrically isomorphic to $\operatorname{im} T = E$, as required.

Exercise 2.4. Prove that no infinite-dimensional Banach space E has a countable Hamel basis (where a *Hamel basis* is a linearly independent set S such that every vector in E is a finite linear combination of elements of S).

Suppose for contradiction that $S = \{e_1, e_2, \ldots\}$ is a countable Hamel basis for the Banach space E, let $F_n = \mathbb{F}e_1 + \cdots \mathbb{F}e_n$ and note that $E = \bigcup_{n=1}^\infty F_n$. Each F_n is closed (being finite-dimensional) and has empty interior: if $U \subseteq F_n$ is open and non-empty then it contains $B_\varepsilon^E(u)$ for some $u \in U$ and $\varepsilon > 0$, but then $u + \frac{1}{2}\varepsilon \|e_{n+1}\|^{-1} e_{n+1} \in U \subseteq F_n$, whence $e_{n+1} \in F_n$, contradicting linear independence. This shows that E is a countable union of nowhere dense sets, a contradiction to the Baire category theorem.

Exercise 2.5. Let $T \colon X \to Y$ be a linear transformation from the normed space X onto the finite-dimensional normed space Y. Prove that T is continuous if and only if $\ker T$ is closed and that if T is continuous then T is open.

It is immediate that if T is continuous then $\ker T = T^{-1}\{0\}$ is closed. If $\ker T$ is closed then $X/\ker T$ is a normed space (with respect to the quotient norm) and the quotient map $\pi \colon X \to X/\ker T$ is open and continuous. Furthermore, $\tilde{T} : X/\ker T \to \operatorname{im} T = Y$ is a linear bijection between finite-dimensional normed spaces, so is a homeomorphism. (Recall than a linear transformation between normed space is continuous if its domain is finite dimensional: if $S \colon Y \to Z$ is a linear transformation and $\dim Y < \infty$ then the norm $y \mapsto \|y\|_Y + \|Sy\|_Z$ is equivalent to $\|\cdot\|_Y$, so there exists $M > 0$ such that $\|Sy\|_Z \leqslant \|y\|_Y + \|Sy\|_Z \leqslant M\|y\|_Y$ for all $y \in Y$.) As $T = \tilde{T} \circ \pi$ and both these maps are open and continuous, so is T.

Exercise 2.6. Let $X = C([0,1], \mathbb{R})$ denote the Banach space of continuous, real-valued functions on the unit interval and for all $k \in \mathbb{N}$ let

$$D_k := \{f \in X : \text{ there exists } t \in [0,1] \text{ such that } |f(s) - f(t)| \leqslant k|s-t| \text{ for all } s \in [0,1]\}.$$

Prove that D_k is closed.

Let $(f_n)_{n \geqslant 1} \subseteq D_k$ be convergent, with limit f, and for each f_n let t_n be as in the definition of D_k. Then $(t_n)_{n \geqslant 1}$ is a bounded sequence, so (by the Bolzano-Weierstrass theorem, passing to a subsequence if necessary) we may assume that $t_n \to t \in [0,1]$. Let $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ be such that $|f(t_n) - f(t)| < \varepsilon$ for all $n \geqslant n_0$; for such n,

$$\begin{aligned} |f(s) - f(t)| &\leqslant |f(s) - f_n(s)| + |f_n(s) - f_n(t_n)| + |f_n(t_n) - f(t_n)| + |f(t_n) - f(t)| \\ &< 2\|f - f_n\|_\infty + k|s - t_n| + \varepsilon \\ &\to k|s-t| + \varepsilon \end{aligned}$$

as $n \to \infty$, and since this holds for all $\varepsilon > 0$ we see that $f \in D_k$.

Prove further that D_k is nowhere dense.

Let $f \in X$ and let $\varepsilon > 0$; since f is uniformly continuous on $[0,1]$ we may find $\delta > 0$ such that $|s-t| < \delta$ implies that $|f(s) - f(t)| < \varepsilon/2$. Choose $0 = t_0 < t_1 < \cdots < t_n = 1$

such that $t_i - t_{i-1} < \delta$ for $i = 1, 2, \ldots, n$ and let

$$g\colon [0,1] \to \mathbb{R};\ t \mapsto \frac{t_i - t}{t_i - t_{i-1}} f(t_{i-1}) + \frac{t - t_{i-1}}{t_i - t_{i-1}} f(t_i) \quad (\text{if } t \in [t_{i-1}, t_i],\ i = 1, 2, \ldots, n).$$

The function g is piecewise-linear (so continuous) and if $t \in [t_{i-1}, t_i]$ then

$$|f(t) - g(t)| \leqslant \frac{t_i - t}{t_i - t_{i-1}} |f(t) - f(t_{i-1})| + \frac{t - t_{i-1}}{t_i - t_{i-1}} |f(t) - f(t_i)| < \frac{\varepsilon}{2},$$

so $\|f - g\|_\infty < \varepsilon/2$; furthermore,

$$\begin{aligned} M &:= \inf\{m \in \mathbb{R}^+ : |g(s) - g(t)| \leqslant m|s - t| \text{ for all } s, t \in [0,1]\} \\ &= \sup\{|g(s) - g(t)|/|s - t| : s, t \in [0,1],\ s \neq t\} \\ &< \infty, \end{aligned}$$

as if $0 \leqslant s < t \leqslant 1$ then either s, $t \in [t_{i-1}, t_i]$, so $\big(g(t) - g(s)\big)/(t - s) = g'\big((t + s)/2\big)$, or $s \in [t_{i-1}, t_i]$ and $t \in [t_{j-1}, t_j]$ for $j > i$, so

$$\left|\frac{g(t) - g(s)}{t - s}\right| \leqslant \left|\frac{g(t) - g(t_{j-1})}{t - t_{j-1}}\right| + \left|\frac{g(t_{j-1}) - g(t_{j-2})}{t_{j-1} - t_{j-2}}\right| + \cdots + \left|\frac{g(t_i) - g(s)}{t_i - s}\right|$$

(since $t - s \geqslant t - t_{j-1}$, $t - s \geqslant t_i - s$ and $t - s \geqslant t_k - t_{k-1}$ if k lies between j and i).

Next, define the saw-tooth function (draw a picture)

$$h\colon \mathbb{R} \to \mathbb{R};\ t \mapsto \varepsilon|\tfrac{1}{2} - (t - n)| \qquad (t \in [n, n+1],\ n \in \mathbb{Z})$$

and let $g_m\colon [0,1] \to \mathbb{R};\ x \mapsto g(x) + h(mx)$ for $m \in \mathbb{N}$. Since $\|h\|_\infty \leqslant \varepsilon/2$ we have that $\|f - g_m\|_\infty < \varepsilon$, so if $m > (M + k)/\varepsilon$, $t \in [0,1]$ and s is sufficiently near to t then

$$\begin{aligned} \left|\frac{g_m(s) - g_m(t)}{t - s}\right| &\geqslant m\left|\frac{h(ms) - h(mt)}{ms - mt}\right| - \left|\frac{g(s) - g(t)}{s - t}\right| \\ &> \frac{M + k}{\varepsilon}\varepsilon - M = k, \end{aligned}$$

which shows that f lies in the closure of $X \setminus D_k$. Thus D_k has empty interior and is nowhere dense.

Deduce that there exist continuous functions on $[0,1]$ that are differentiable at no point in $(0,1)$.

If $f \in X$ is differentiable at $t \in (0,1)$ then

$$g\colon [0,1] \to \mathbb{R};\ s \mapsto \begin{cases} \dfrac{f(s) - f(t)}{s - t} & s \neq t, \\ f'(t) & s = t \end{cases}$$

is continuous on $[0,1]$, so bounded. Hence $|f(s) - f(t)| \leqslant \|g\|_\infty |t - s|$ for all $s \in [0,1]$, and $f \in D_k$ for all $k \geqslant \|g\|_\infty$. Since $X \neq \bigcup_{k=1}^\infty D_k$ (by the Baire category theorem) the result follows.

Exercise 2.7. Let H be an infinite-dimensional, separable Hilbert space. Prove that $\mathcal{B}(H)_1$ is not separable in the norm topology.

Let $\{e_1, e_2, \ldots\}$ be an orthornormal basis for H, let $H_0 = \operatorname{lin}\{e_1, e_2, \ldots\}$ denote the linear span of this basis and for each subset $A \subseteq \mathbb{N}$ define

$$P_0(A)\colon H_0 \to H_0; \ \sum_{n\in\mathbb{N}} \alpha_n e_n \mapsto \sum_{n\in A} \alpha_n e_n.$$

(In H_0 only finitely many coefficients in $\sum_{n\in\mathbb{N}} \alpha_n e_n$ are non-zero, so this map is well defined.) It is immediate that $\|P_0(A)\| \leqslant 1$, so we may extend $P_0(A)$ to $P(A) \in \mathcal{B}(H)$ such that $P(A) = P_0(A)|_{H_0}$ and $\|P(A)\| \leqslant 1$. If A, $B \subseteq \mathbb{N}$ are distinct then $\|P(A) - P(B)\| \geqslant 1$: let $n \in (A \setminus B) \cup (B \setminus A)$ and consider $\big\|\big(P(A) - P(B)\big)e_n\big\|$. Suppose that $S \subseteq \mathcal{B}(H)_1$ is dense and let $S_A \in S$ be such that $\|S_A - P(A)\| < 1/2$ for all $A \subseteq \mathbb{N}$; if A, $B \subseteq \mathbb{N}$ are distinct then

$$1 \leqslant \|P(A) - P(B)\| \leqslant \|P(A) - S_A\| + \|S_A - S_B\| + \|S_B - P(B)\| < 1 + \|S_A - S_B\|,$$

hence $A \mapsto S_A$ is injective and thus S is uncountable.

Solutions to Exercises 3

Exercise 3.1. Let H be a separable Hilbert space with orthonormal basis $\{e_1, e_2, \ldots\}$. For $n \geqslant 1$ let P_n denote the orthogonal projection onto $\mathbb{F}e_1 + \cdots + \mathbb{F}e_n$; prove that $P_n T P_n x \to Tx$ as $n \to \infty$ for all $T \in \mathcal{B}(H)$ and $x \in H$.

Note that

$$\begin{aligned} \|P_n T P_n x - Tx\| &\leqslant \|P_n T\| \, \|(P_n - I)x\| + \|(P_n - I)Tx\| \\ &\leqslant \|T\| \, \|(P_n - I)x\| + \|(P_n - I)Tx\|, \end{aligned}$$

so it suffices to prove that $P_n y \to y$ for all $y \in H$. To see this, note that Parseval's equality gives that

$$\|(P_n - I)y\|^2 = \langle (P_n - I)y, (P_n - I)y \rangle = \|y\|^2 - \sum_{k=1}^{n} |\langle e_k, y \rangle|^2 \to 0$$

as $n \to \infty$.

Deduce that $\mathcal{B}(H)$ is separable in the strong operator topology.

Note that $\bigcup_{n\geqslant 1} P_n \mathcal{B}(H) P_n$ is strong operator dense in $\mathcal{B}(H)$ and that a countable union of countable sets is countable, so it suffices to prove that, for all $n \geqslant 1$, $P_n \mathcal{B}(H) P_n$ contains a countable, strong-operator-dense set. Next, note that $P_n \mathcal{B}(H) P_n$ is isomorphic to $\mathbb{F}^{n^2}$ and all norm topologies on finite-dimensional spaces coincide, so $P_n \mathcal{B}(H) P_n$ is norm separable. As the norm topology is finer than the strong operator topology the result follows.

Exercise 3.2. Prove that if E is a Banach space with respect to two different norms then they are either equivalent or non-comparable (i.e., neither is coarser than the other).

If the norms $\|\cdot\|$ and $\|\cdot\|'$ are comparable then, without loss of generality, the topology $\mathcal{T}_{\|\cdot\|}$ generated by the first is finer than $\mathcal{T}_{\|\cdot\|'}$, the topology generated by the second. In particular, the identity map is continuous from $(E, \mathcal{T}_{\|\cdot\|})$ to $(E, \mathcal{T}_{\|\cdot\|'})$ (as $\mathcal{T}_{\|\cdot\|'} \subseteq \mathcal{T}_{\|\cdot\|}$). Since every continuous, linear bijection between Banach spaces has continuous inverse, we have the result.

Exercise 3.3. Prove the following extension of Tietze's theorem to complex-valued functions: if X is a normal space, Y a closed subset of X and $f \in C_{\mathrm{b}}(Y)$ then there exists $F \in C_{\mathrm{b}}(X)$ such that $F|_Y = f$ and $\|F\|_\infty = \|f\|_\infty$.

Apply Tietze's theorem to obtain g, $h \in C_{\mathrm{b}}(X, \mathbb{R})$ such that $g|_Y = \operatorname{Re} f$ and $h|_Y = \operatorname{Im} f$. Let $k = g + ih$, so that $k|_Y = f$ and let

$$l \colon \mathbb{C} \to \mathbb{C};\ z \mapsto \begin{cases} z & \text{if } |z| \leqslant \|f\|_\infty, \\ z\|f\|_\infty/|z| & \text{if } |z| > \|f\|_\infty. \end{cases}$$

Then $F = l \circ k$ is as required: if $y \in Y$ then $F(y) = l(k(y)) = l(f(y)) = f(y)$ and $\|F\|_\infty = \|l\|_\infty \leqslant \|f\|_\infty$.

Prove also that Tietze's theorem applies to unbounded, real-valued functions: if X and Y are as above and $f \colon Y \to \mathbb{R}$ is continuous then there exists $F \colon X \to \mathbb{R}$ such that $F|_Y = f$.

Let $g = \arctan \circ f$, so that g takes values in $(-\pi/2, \pi/2)$; by Tietze's theorem there exists $G \in C_{\mathrm{b}}(X)$ such that $\|G\|_\infty \leqslant \pi/2$ and $G|_Y = g$. The set $C = G^{-1}\{\pm\pi/2\}$ is closed and, by Urysohn's lemma, there exists $H \in C_{\mathrm{b}}(X)$ such that $\|H\|_\infty \leqslant 1$, $H|_C = 0$ and $H|_Y = 1$. The function $F = \tan \circ GH$ is as required: $|GH| < \pi/2$ and

$$F(y) = \tan(G(y)H(y)) = \tan(\arctan f(y)) = f(y) \qquad \forall\, y \in Y.$$

Exercise 3.4. Let E be a Banach space, Y a normed vector space and suppose that $(T_n)_{n \geqslant 1} \subseteq \mathcal{B}(E, Y)$ is such that $\lim_{n\to\infty} T_n x$ exists for all $x \in E$. Prove that there exists $T \in \mathcal{B}(E, Y)$ such that $T_n \to T$ in the strong operator topology.

It is immediate that

$$T \colon E \to Y;\ x \mapsto \lim_{n\to\infty} T_n x$$

is a linear operator, by the continuity of vector addition and scalar multiplication in a normed space. As convergent sequences in normed spaces are bounded, $\{\|T_n x\| : n \geqslant 1\}$ is bounded for all $x \in E$, and, as E is complete, the principle of uniform boundedness implies that $M := \sup\{\|T_n\| : n \geqslant 1\}$ is finite. Hence $\|T\| \leqslant M$, because

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| \leqslant M\|x\| \qquad \forall\, x \in E.$$

What can be said about the norm of T?

The working above shows that $\big(\|T_n\|\big)_{n\geqslant 1}$ is a bounded sequence, so there exists a subsequence $\big(\|T_{n_k}\|\big)_{k\geqslant 1}$ such that $\|T_{n_k}\| \to \underline{\lim}_{n\to\infty} \|T_n\| := \sup_{n\geqslant 1} \inf_{m\geqslant n} \|T_m\|$. Since a subsequence of a convergent sequence converges to the same limit,

$$\|Tx\| = \lim_{k\to\infty} \|T_{n_k}x\| \leqslant \lim_{k\to\infty} \|T_{n_k}\| \, \|x\| \to \underline{\lim}_{n\to\infty} \|T_n\| \, \|x\| \qquad \forall\, x \in E.$$

Hence $\|T\| \leqslant \underline{\lim}_{n\to\infty} \|T_n\|$.

Exercise 3.5. Let $x = (x_n)_{n\geqslant 1}$ be a sequence of complex numbers such that the series $\sum_{n=1}^{\infty} x_n y_n$ is convergent for all $y \in c_0$. Prove that $x \in \ell^1$.

For $n \geqslant 1$ define the linear operator

$$f_n \colon c_0 \to \mathbb{C};\ y \mapsto \sum_{j=1}^{n} x_j y_j.$$

Since $|f_n(y)| \leqslant \sum_{j=1}^{n} |x_j|\,|y_j| \leqslant \|y\|_\infty \sum_{j=1}^{n} |x_j|$ it follows that f_n is bounded, with $\|f_n\| \leqslant \sum_{j=1}^{n} |x_j|$. This is actually an equality: let $y_j \in \mathbb{T} := \{\alpha \in \mathbb{C} : |\alpha| = 1\}$ be such that $y_j x_j = |x_j|$ and note that $z = (y_1, \ldots, y_n, 0, 0, \ldots) \in (c_0)_1$ is such that $|f_n(z)| = \sum_{j=1}^{n} |x_j|$. For all $y \in c_0$ the sequence $(f_n(y))_{n\geqslant 1}$ is convergent, so

$$\sup\{|f_n(y)| : n \geqslant 1\} < \infty$$

and, since c_0 is a Banach space, the principle of uniform boundedness implies that

$$\|x\|_1 = \sum_{n=1}^{\infty} |x_n| = \sup\{\|f_n\| : n \geqslant 1\} < \infty.$$

Exercise 3.6. Let E be a Banach space with closed subspaces F and G such that $E = F \oplus G$ (i.e., every element of E can be expressed uniquely as the sum of an element of F and an element of G). Define P_F and P_G by setting

$$P_F \colon E \to E;\ f + g \mapsto f \quad \text{and} \quad P_G \colon E \to E;\ f + g \mapsto g \qquad \forall\, f \in F,\ g \in G.$$

Prove that P_F and P_G are bounded linear operators such that $P_F^2 = P_F$, $P_G^2 = P_G$ and $P_F P_G = P_G P_F = 0$.

The algebraic facts are easily verified; we prove only that P_F is bounded, by applying the closed-graph theorem. Let $(x_n)_{n\geqslant 1} \subseteq E$ be such that $x_n \to x$ and $P_F x_n \to y$; since $P_F x_n \in F$ for all n and F is closed, $y \in F$, and similarly $x - y = \lim_{n\to\infty} x_n - P_F x_n \in G$. Thus $x = y + (x - y) \in F + G$ and by uniqueness $P_F x = y$, as required.

Exercise 3.7. Find a Banach space E with closed subspaces F and G such that $E = F \oplus G$ and

$$P : E \to E;\ f + g \mapsto f \qquad \forall\, f \in F,\ g \in G$$

has norm strictly greater than one.

Let $E = \mathbb{R}^3$ equipped with the norm $\|\mathbf{x}\| = \max\{|x_1|, |x_2|, |x_3|\}$ (writing vectors in bold and denoting their coordinates in the obvious manner) and let

$$f : \mathbb{R}^3 \to \mathbb{R};\ (x_1, x_2, x_3) \mapsto 2x_1 + 2x_2 - 3x_3,$$

$F := \ker f$ and $G := \mathbb{R}\mathbf{u}$, where $\mathbf{u} := (1, 1, 1)$; since $\mathbf{x} = (\mathbf{x} - f(\mathbf{x})\mathbf{u}) + f(\mathbf{x})\mathbf{u}$ for all $\mathbf{x} \in \mathbb{R}^3$ and $f(\mathbf{u}) = 1$, E, F and G are as required. Suppose for contradiction that $\|P\| = 1$ and let $\mathbf{v} := P\mathbf{u}$; note that $\|\mathbf{v}\| = \|P\mathbf{u}\| \leqslant \|P\|\,\|\mathbf{u}\| = 1$, so $|v_i| \leqslant 1$ for $i = 1, 2, 3$.

If $\mathbf{x} := (1, -1, 1)$ and $\mathbf{y} := \mathbf{x} + t(\mathbf{u} - P\mathbf{u})$, where $t \in \mathbb{R}$ is chosen so that $\mathbf{y} \in F$, whence $\mathbf{y} = P\mathbf{x}$, then

$$\mathbf{y} \in F \iff \mathbf{x} + t\mathbf{u} \in F \iff t = 3,$$

by direct calculation, so $\mathbf{y} = (4, 2, 4) - 3P\mathbf{u} = (4 - 3v_1, 2 - 3v_2, 4 - 3v_3)$. Now, $\|\mathbf{y}\| = \|P\mathbf{x}\| \leqslant 1$, so $|4 - 3v_1| \leqslant 1$ and $v_1 \in [1, 5/2]$; since $|v_1| \leqslant 1$, $v_1 = 1 = v_3$ and, as $\mathbf{y} \in F$, $v_2 = 1/2$. Thus $\mathbf{v} = P\mathbf{u} = (1, 1/2, 1)$.

Now let $\mathbf{z} := (-1, 1, 1)$ and $\mathbf{w} := \mathbf{z} + s(\mathbf{u} - P\mathbf{u})$, where $s \in \mathbb{R}$ is chosen so that $\mathbf{w} \in F$ and so $\mathbf{w} = P\mathbf{z}$. As $\mathbf{w} = (-1, 1 + s/2, 1)$, it follows that $s = 3$ and $\mathbf{w} = (-1, 5/2, 1)$, contradicting the fact that $\|\mathbf{w}\| = \|P\mathbf{z}\| \leqslant 1$.

[This example is due to Goodner [6].]

Exercise 3.8. Let E be a Banach space with closed subspaces F and G such that $F \cap G = \{0\}$. Prove that $F \oplus G$ is closed if and only if there exists $C > 0$ such that

$$\|f\| \leqslant C\|f + g\| \qquad \forall\, f \in F,\ g \in G.$$

If $F \oplus G$ is closed then the projection map $P_F \colon F \oplus G \to F;\ f + g \mapsto f$ is bounded (by Exercise 3.6). Hence

$$\|f\| = \|P_F(f + g)\| \leqslant \|P_F\|\,\|f + g\| \qquad \forall\, f \in F,\ g \in G,$$

as required.

Conversely, suppose that such $C > 0$ exists and let $(x_n)_{n\geqslant 1} \subseteq F \oplus G$ be such that $x_n \to x$ for some $x \in E$. Let $x_n = f_n + g_n$ for all $n \geqslant 1$, where $f_n \in F$ and $g_n \in G$, and note that

$$\|f_n - f_m\| \leqslant C\|(f_n - f_m) + (g_n - g_m)\| = C\|x_n - x_m\|,$$

so $(f_n)_{n\geqslant 1}$ is Cauchy and hence convergent, say $f_n \to f \in F$. Furthermore, since G is closed,

$$\lim_{n\to\infty} g_n = \lim_{n\to\infty} x_n - f_n = x - f \in G$$

and $x = f + (x - f) \in F \oplus G$, as required.

Deduce that $F \oplus G$ is closed if and only if

$$c := \inf\{\|f - g\| : f \in F,\ g \in G,\ \|f\| = \|g\| = 1\} > 0.$$

If $F \oplus G$ is closed then, by the previous part, there exists $C > 0$ such that $\|f\| \leqslant C\|f+g\|$ for all $f \in F$ and $g \in G$. Hence (replacing g by $-g$) $\|f-g\| \geqslant C^{-1}\|f\| = C^{-1}$ if $f \in F$, $g \in G$ and $\|f\| = \|g\| = 1$, whence $c \geqslant C^{-1} > 0$.

Conversely, if $F \oplus G$ is not closed then there exists no such C, so for all $n \geqslant 1$ there exist $f_n \in F$ and $g_n \in G$ such that $\|f_n\| > n\|f_n + g_n\|$; replacing f_n by $f_n/\|f_n\|$ and g_n by $g_n/\|f_n\|$ we may assume that $\|f_n\| = 1$ and $\|f_n + g_n\| < 1/n$. Since

$$\big|1 - \|g_n\|\big| = \big|\|f_n\| - \|-g_n\|\big| \leqslant \|f_n + g_n\| < 1/n$$

we see that

$$\big\|f_n + \|g_n\|^{-1}g_n\big\| \leqslant \|f_n + g_n\| + \big\|-g_n + \|g_n\|^{-1}g_n\big\| < \frac{1}{n} + \big|-1 + \|g_n\|^{-1}\big|\,\|g_n\| < \frac{2}{n};$$

thus $c < 2/n$ for all $n \geqslant 1$ and so $c = 0$, as claimed.

Solutions to Exercises 4

Exercise 4.1. A closed subspace M of the normed space X is *complemented in* X if there exists a closed subspace N such that $M \oplus N = X$, i.e., $M + N = X$ and $M \cap N = \{0\}$. Prove that M is complemented in X if M is finite dimensional.

Let $\{x_1, \ldots, x_n\}$ be a basis for M, let $\{\phi_1, \ldots, \phi_n\} \subseteq M^*$ be the dual basis, use the Hahn-Banach theorem to extend ϕ_i to $\tilde{\phi}_i \in X^*$ and then let $N = \bigcap_{i=1}^n \ker \tilde{\phi}_i$. The fact that N is closed is immediate and if $x \in M \cap N$ then $\phi_i(x) = 0$ for all i, so $x = 0$. Finally, note that $x - \sum_{i=1}^n \tilde{\phi}_i(x)x_i \in N$ for all $x \in X$.

Prove also that M is complemented in X if M has finite codimension, i.e., $\dim X/M < \infty$.

Let X/M have basis $\big\{[x_1], \ldots, [x_n]\big\}$ and suppose that N is the vector space spanned by $\{x_1, \ldots, x_n\}$. Then N is closed (being finite-dimensional), $X = \pi^{-1}(X/M) = \pi^{-1}\big(\pi(N)\big) = M + N$ (where $\pi\colon X \to X/M$ is the quotient map) and if $\sum_{i=1}^n \alpha_i x_i \in M \cap N$ then $[0] = \sum_{i=1}^n \alpha_i [x_i]$, whence $\alpha_1 = \cdots = \alpha_n = 0$ by linear independence of $\big\{[x_1], \ldots, [x_n]\big\}$.

Exercise 4.2. Let M be a finite-dimensional subspace of the normed space X and let N be a closed subspace of X such that $X = M \oplus N$. Prove that if ϕ_0 is a linear functional on M then

$$\phi\colon M \oplus N \to \mathbb{F};\ m + n \mapsto \phi_0(m) \qquad \forall\, m \in M,\ n \in N$$

is an element of the dual space X^*.

Note that $\ker \phi = N + \ker \phi_0$ is closed, as N is closed and $\ker \phi_0$ is finite-dimensional. The result follows from the fact that surjective linear transformations with finite-dimensional range and closed kernel are continuous (Exercise 2.5).

Exercise 4.3. Prove that a normed vector space X is separable if its dual X^* is.

Let $(\phi_n)_{n\geqslant 1}$ be dense in X^*, let $(x_n)_{n\geqslant 1} \subseteq X_1$ be such that $|\phi_n(x_n)| \geqslant \|\phi_n\|/2$ for all $n \geqslant 1$ and let $M = \mathbb{Q} - \mathrm{lin}\{x_n\}$ (or $(\mathbb{Q}+i\mathbb{Q}) - \mathrm{lin}\{x_n\}$ if $\mathbb{F} = \mathbb{C}$). If $\bar{M} \neq X$ then let $x_0 \in X \setminus \bar{M}$; by a corollary to the separation theorem there exists $\phi \in X^*$ such that $\phi|_M = 0$ and $\phi(x_0) = 1$. Let $(\phi_{n_k})_{k\geqslant 1}$ converge to ϕ and note that

$$\tfrac{1}{2}\|\phi_{n_k}\| \leqslant |\phi_{n_k}(x_{n_k})| \leqslant |\phi_{n_k}(x_{n_k}) - \phi(x_{n_k})| + |\phi(x_{n_k})| \leqslant \|\phi_{n_k} - \phi\|\,\|x_{n_k}\| \leqslant \|\phi_{n_k} - \phi\| \to 0$$

as $k \to \infty$, implying that $\phi = 0$. This contradiction gives the result.

Find a separable Banach space E such that E^* is not separable.

Note that the separable Banach space ℓ^1 has non-separable dual space ℓ^∞.

Prove that a reflexive Banach space E is separable if and only if E^* is.

The first part gives one implication and the other follows immediately by applying the first part with E^* in place of E; note that E and E^{**} are isometrically isomorphic.

Exercise 4.4. Prove that a Banach space E is reflexive if and only its dual E^* is reflexive.

If E is reflexive then the canonical embedding $\Gamma\colon E \to E^{**}$ is an isometric isomorphism: for all $Z \in E^{***}$ we wish to find $\phi \in E^*$ such that

$$Z(\Psi) = \Psi(\phi) \qquad \forall\, \Psi \in E^{**}.$$

Consider $\phi = Z \circ \Gamma$; it is immediate that $\phi \in E^*$, and if $\Psi \in E^{**}$ then $\Psi = \Gamma(x)$ for some $x \in E$, so

$$\Psi(Z \circ \Gamma) = \Gamma(x)(Z \circ \Gamma) = Z(\Gamma(x)) = Z(\Psi),$$

as required.

Conversely, suppose that E^* is reflexive but E is not, so that $\Gamma(E)$ is a proper subspace of E^{**} (which is closed because E is complete). Exercise 4.2 and Theorem 3.9 give $Z \in E^{***}$ such that $Z|_{\Gamma(E)} = 0$ but $Z \neq 0$; as E^* is reflexive, $Z = \Gamma^*(\phi)$ for some $\phi \in E^*$ (where $\Gamma^*\colon E^* \to E^{***}$ is the canonical embedding) but then

$$0 = Z(\Gamma(x)) = \Gamma^*(\phi)(\Gamma(x)) = \Gamma(x)(\phi) = \phi(x) \qquad \forall\, x \in E,$$

contradicting the fact that $Z = \Gamma^*(\phi) \neq 0$.

Exercise 4.5. Prove that any infinite-dimensional normed space has a discontinuous linear functional defined on it.

First use Zorn's lemma to prove that the space X has a Hamel basis $\{e_a : a \in A\}$ with $\|e_a\| = 1$ for all $a \in A$. Define ϕ by choosing an infinite set of distinct elements $\{a_n : n \in \mathbb{N}\}$ and setting

$$\phi\Big(\sum_{a\in A} \lambda_a e_a\Big) = \sum_{n=1}^{\infty} n\lambda_{a_n} \qquad \forall\, x = \sum_{a\in A} \lambda_a e_a \in X;$$

it is easy to verify that ϕ is a well-defined linear functional on X and since $\phi(e_{a_n}) = n \to \infty$ as $n \to \infty$, ϕ is not bounded.

Exercise 4.6. Let A be a subset of the normed vector space X. Prove that A is norm bounded (there exists $r \in \mathbb{R}^+$ such that $\|a\| \leqslant r$ for all $a \in A$) if and only if it is weakly bounded (for all $\phi \in X^*$ there exists $r_\phi \in \mathbb{R}^+$ such that $|\phi(a)| \leqslant r_\phi$ for all $a \in A$).

If A is norm bounded then there exists $r \in \mathbb{R}^+$ such that $\|a\| \leqslant r$ for all $a \in A$. If $\phi \in X^*$ then $|\phi(a)| \leqslant \|\phi\| \, \|a\| \leqslant \|\phi\| r$ for all $a \in A$, showing that A is weakly bounded. Conversely, if A is weakly bounded then $\{\phi(a) : a \in A\} = \{\hat{a}(\phi) : a \in A\}$ is bounded for all $\phi \in X^*$, and the principle of uniform boundedness yields boundedness of $\{\|\hat{a}\| : a \in A\}$. Since $\|\hat{a}\| = \|a\|$ we see that A is norm bounded.

Deduce that a weakly holomorphic function is (strongly) continuous.

Suppose that $f \colon U \to X$ be weakly holomorphic, where U is an open subset of $\mathbb{C}$ and X is a complex normed space. Let $a \in U$ and choose $\varepsilon > 0$ such that $B^{\mathbb{C}}_{2\varepsilon}(a) \subseteq U$; if $\phi \in X^*$ then, by Cauchy's integral formula,

$$\Big|\frac{(\phi \circ f)(z) - (\phi \circ f)(a)}{z - a}\Big| = \Big|\frac{1}{2\pi i}\oint_\gamma \frac{(\phi \circ f)(w)}{(w - z)(w - a)} \, \mathrm{d}w\Big| \leqslant \frac{M_\phi}{\varepsilon/2} \qquad \forall\, z \in B^{\mathbb{C}}_{\varepsilon/2}(a) \setminus \{a\},$$

where $\gamma \colon [0, 2\pi] \to \mathbb{C};\ t \mapsto a + \varepsilon e^{it}$ and $M_\phi = \sup\big\{|(\phi \circ f)(w)| : |w - a| = \varepsilon\big\}$. Hence

$$\Big\{\frac{f(z) - f(a)}{z - a} : 0 < |z - a| < \varepsilon/2\Big\}$$

is weakly bounded, so norm bounded: there exists $r \in \mathbb{R}^+$ such that

$$\|f(z) - f(a)\| \leqslant r|z - a| \qquad \forall\, z \in B^{\mathbb{C}}_{\varepsilon/2}(a),$$

whence $f(z) \to f(a)$ in X as $z \to a$.
[This is the first step in proving Dunford's theorem, that weakly holomorphic functions are strongly holomorphic [19, Theorem 3.31].]

Exercise 4.7. Let H be a Hilbert space. Prove that the adjoint $T \mapsto T^*$ is continuous with respect to the weak operator topology on $\mathcal{B}(H)$, but not necessarily with respect to the strong operator topology.

Note that $T_a \to T$ in the weak operator topology on $\mathcal{B}(H)$ if and only if $\langle x, T_a y\rangle \to \langle x, Ty\rangle$ for all x, $y \in H$. Since $\langle x, Sy\rangle = \langle S^*x, y\rangle = \overline{\langle y, S^*x\rangle}$ and $z \mapsto \bar{z}$ is continuous on $\mathbb{C}$ we have the first claim. For the second, let $\{e_n : n \in \mathbb{N}\}$ denote the standard orthonormal basis of ℓ^2 and let $T_n = |e_n\rangle\langle e_1|$ for all $n \geqslant 1$, i.e.,

$$T_n \colon \ell^2 \to \ell^2;\ x \mapsto \langle e_1, x\rangle e_n.$$

It is readily verified that $T_n \in \mathcal{B}(H)$ (with $\|T_n\| \leqslant 1$) and $\|T_n e_1\| = 1$. Furthermore, $T_n^* = |e_1\rangle\langle e_n|$ and $\|T_n^* x\| = |\langle e_n, x\rangle| \to 0$ as $n \to \infty$ (by Parseval's equality) so $T_n^* \to 0$ in the strong operator topology. This gives the result.

Exercise 4.8. Let E and F be Banach spaces. Show that if $T \colon E \to F$ and $S \colon F^* \to E^*$ are linear transformations that satisfy

$$\phi(Tx) = (S\phi)(x) \qquad \forall\, x \in E,\ \phi \in F^* \tag{$\star$}$$

then S and T are bounded, with $S = T^*$.

Let (x, y) be a limit point of $\mathcal{G}(T)$, the graph of T, and suppose that $(x_n)_{n\geqslant 1} \subseteq E$ is such that $x_n \to x$ and $Tx_n \to y$. For all $\phi \in F^*$ we have that $S\phi \in E^*$ and so

$$\phi(y) = \lim_{n\to\infty} \phi(Tx_n) = \lim_{n\to\infty} (S\phi)(x_n) = (S\phi)(x) = \phi(Tx),$$

hence $Tx = y$ (since F^* separates points in F); by the closed-graph theorem $T \in \mathcal{B}(E, F)$. Furthermore, $T^* \in \mathcal{B}(F^*, E^*)$ is such that $S = T^*$, by $(\star)$, as E certainly separates points in E^*.
[This is the Hellinger-Toeplitz theorem.]

Exercise 4.9. Let E and F be Banach spaces and suppose that $T \in \mathcal{B}(E, F)$ has closed range, i.e., $\operatorname{im} T$ is closed in F. Prove that $\operatorname{im} T^* = (\ker T)^\perp$ (where

$$M^\perp := \{\phi \in E^* : \phi(x) = 0 \text{ for all } x \in M\}$$

is the annihilator of the subspace $M \subseteq E$).

If $\phi \in F^*$ then

$$(T^*\phi)(x) = \phi(Tx) = \phi(0) = 0 \qquad \forall\, x \in \ker T,$$

so that $\operatorname{im} T^* \subseteq (\ker T)^\perp$. Conversely, suppose that $\psi \in (\ker T)^\perp$, i.e., $\psi \in E^*$ satisfies $\psi(x) = 0$ for all $x \in \ker T$, and define

$$\theta_0 \colon E/\ker T \to \mathbb{F};\ [x] \mapsto \psi(x).$$

This is a good definition (because $\ker T \subseteq \ker \psi$) and

$$|\theta_0[x]| = |\psi(x)| = |\psi(x+m)| \leqslant \|\psi\|\,\|x+m\| \qquad \forall\, m \in \ker T,$$

so $|\theta_0[x]| \leqslant \|\psi\|\,\|[x]\|$ for all $x \in E$ and θ_0 is continuous. Since $\operatorname{im} T$ is closed, the bounded linear operator $\tilde{T} \colon E/\ker T \to \operatorname{im} T$ has continuous inverse (by the open-mapping theorem); extending $\theta_0 \circ \tilde{T}^{-1}$ to $\theta \in F^*$ by the Hahn-Banach theorem we see that

$$(T^*\theta)(x) = \theta(Tx) = \theta_0[x] = \psi(x) \qquad \forall\, x \in E$$

so $\psi \in \operatorname{im} T^*$ and the result follows.

Exercise 4.10. Let $E = c_0$, so that $E^* = \ell^1$ and $E^{**} = \ell^\infty$. Prove that $x \mapsto \sum_{n=1}^\infty x_n$ is weakly continuous on ℓ^1 but is not weak* continuous.

The weak topology on ℓ^1 is such that $\hat{y} \colon x \mapsto \sum_{n=1}^\infty y_n x_n$ is continuous for all $y \in \ell^\infty$, so if $1 := (1, 1, \ldots)$ then $\hat{1} \colon x \mapsto \sum_{n=1}^\infty x_n$ is weakly continuous. If $\left(x^{(n)}\right)_{n\geqslant 1} \subseteq \ell^1$ is defined by setting $x_n^{(n)} = 1$ and $x_k^{(n)} = 0$ if $k \neq n$ then $\hat{1}\left(x^{(n)}\right) = 1$ for all $n \geqslant 1$ but $\hat{y}\left(x^{(n)}\right) = y_n \to 0$ as $n \to \infty$ for all $y \in c_0$, i.e., $x^{(n)} \to 0$ in the weak* topology. Hence $\hat{1}$ is not weak* continuous.

Exercise 4.11. Prove that a compact metric space is separable.

Let (X, d) be a compact metric space. For $n \in \mathbb{N}$ the set $\{B_{1/n}(x) : x \in X\}$ is an open cover of X, so there exist $x_1^{(n)}, \ldots, x_{m_n}^{(n)}$ such that $X = \bigcup_{k=1}^{m_n} B_{1/n}\bigl(x_k^{(n)}\bigr)$. We claim that $S = \bigcup_{n\in\mathbb{N}}\{x_1^{(n)}, \ldots, x_{m_n}^{(n)}\}$ is countable (being a countable union of finite sets) and dense in X. For the latter claim, let $\varepsilon > 0$ and $x \in X$; there exists $n \in \mathbb{N}$ such that $\varepsilon > 1/n$ and some $k \in \{1, \ldots, m_n\}$ such that $x \in B_{1/n}\bigl(x_k^{(n)}\bigr)$, whence $d\bigl(x, x_k^{(n)}\bigr) < \varepsilon$ and $S \cap B_\varepsilon(x) \neq \emptyset$. The result follows.

Prove that if X is a separable normed space then X_1^*, the closed unit ball of the dual space X^*, is metrizable when equipped with the weak* topology.

Let $(x_n)_{n\geqslant 1} \subseteq X_1$ be dense in X_1 and define

$$d\colon X_1^* \times X_1^* \to \mathbb{R}^+;\ (\phi, \psi) \mapsto \sum_{n=1}^{\infty} 2^{-n}|\phi(x_n) - \psi(x_n)|. \tag{$\dagger$}$$

Note that the series is convergent (by comparison with $\sum_{n=1}^{\infty} 2^{-n}\|\phi - \psi\|$) so d is well defined. Symmetry and the triangle inequality are immediate and if $d(\phi, \psi) = 0$ then $(\phi - \psi)(x_n) = 0$ for all $n \geqslant 1$, whence $\phi = \psi$. Hence d is a metric on X_1^*; it remains to prove that $\mathcal{T}_d = \sigma(X^*, X)|_{X_1^*}$. Note first that $(\phi, \psi) \mapsto |\phi(x_n) - \psi(x_n)| = |\hat{x}_n(\phi - \psi)|$ is a continuous function on $X_1^* \times X_1^*$ (where each factor is equipped with the weak* topology) and the series ($\dagger$) is uniformly convergent on this set, so continuous. In particular, the balls

$$\{\psi \in X_1^* : d(\psi, \phi) < \varepsilon\} \qquad (\phi \in X_1^*,\ \varepsilon > 0)$$

are $\sigma(X^*, X)|_{X_1^*}$-open, so $\mathcal{T}_d \subseteq \sigma(X^*, X)|_{X_1^*}$. As $\mathcal{T}_d$ is Hausdorff and $\sigma(X^*, X)|_{X_1^*}$ is compact, these topologies are equal. (Recall that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism.)

Deduce that X^* is separable in the weak* topology.

This follows because X_n^* is separable for all $n \in \mathbb{N}$ (being the image of the weak*-separable space X_1^* under the homeomorphism $x \mapsto nx$) and $X^* = \bigcup_{n\in\mathbb{N}} X_n^*$.

Exercise 4.12. Let X and Y be normed spaces and for all $x \in X$ and $y \in Y$ let

$$x \otimes y\colon \mathcal{B}(X, Y^*) \to \mathbb{F};\ T \mapsto (Tx)(y).$$

Prove that $x \otimes y \in \mathcal{B}(X, Y^*)^*$, with $\|x \otimes y\| = \|x\|\,\|y\|$, and that the mapping

$$X \times Y \to \mathcal{B}(X, Y^*); (x, y) \mapsto x \otimes y$$

is bilinear.

It is straightforward to see that $x \otimes y \in \mathcal{B}(X, Y^*)$ with $\|x \otimes y\| \leqslant \|x\|\,\|y\|$; for the reverse inequality, suppose that x and y are non-zero and let $\phi \in X^*$ and $\psi \in Y^*$ be such that $\|\phi\| = \|\psi\| = 1$, $\phi(x) = \|x\|$ and $\psi(y) = \|y\|$. (These exist by Theorem 3.10). If $T := z \mapsto \phi(z)\psi$ then $T \in \mathcal{B}(X, Y^*)$, with $\|T\| \leqslant \|\phi\|\,\|\psi\| = 1$, and $(x \otimes y)(T) = \phi(x)\psi(y) = \|x\|\,\|y\|$, whence $\|x \otimes y\| \geqslant \|x\|\,\|y\|$, as required. Verification of bilinearity is routine.

If Z is the closed linear span of $\{x \otimes y : x \in X,\ y \in Y\}$ in $\mathcal{B}(X, Y^*)^*$, prove that

$$j \colon \mathcal{B}(X, Y^*) \to Z^*;\ j(T)z = z(T)$$

is an isometric isomorphism.

Clearly, j is a linear map and $\|j(T)\| \leqslant \|T\|$ for all $T \in \mathcal{B}(X, Y^*)$. For the reverse inequality, let $\varepsilon > 0$ and choose $x \in X$ such that $\|x\| = 1$ and $\|Tx\| > \|T\| - \varepsilon/2$, then choose $y \in Y$ such that $\|y\| = 1$ and $\|(Tx)(y)\| > \|Tx\| - \varepsilon/2$. It follows that $\|x \otimes y\| = 1$ and $\|(x \otimes y)T\| = \|(Tx)(y)\| > \|T\| - \varepsilon$, whence $\|j(T)\| \geqslant \|T\|$. Finally, let $\phi \in Z^*$ and let $\phi_x : Y \to \mathbb{F};\ y \mapsto \phi(x \otimes y)$. Since $|\phi(x \otimes y)| \leqslant \|\phi\|\,\|x\|\,\|y\|$, $\phi_x \in Y^*$, and if $T := x \mapsto \phi_x$ then $\|T\| \leqslant \|\phi\|$, so $T \in \mathcal{B}(X, Y^*)$ and $j(T) = \phi$; hence j is surjective.

[This idea, which allows one to put a weak* topology on $\mathcal{B}(X, Y^*)$, was used by Arveson in [1] and dates back to Schatten [20, Theorem 3.2].]

Solutions to Exercises 5

Exercise 5.1. Prove that if X is a vector space with separating subspace $M \subseteq X'$ and $\phi \in X'$ is a linear functional that is $\sigma(X, M)$-continuous then there exist $\phi_1, \ldots, \phi_n \in M$ such that

$$|\phi(x)| \leqslant \max_{1 \leqslant i \leqslant n} |\phi_i(x)| \qquad \forall\, x \in X.$$

By the definition of $\sigma(X, M)$,

$$|\phi|^{-1}[0, 1) \supseteq \bigcap_{i=1}^{n} |\phi_i|^{-1}[0, \varepsilon_i),$$

where $\phi_1, \ldots, \phi_n \in M$ and $\varepsilon_1, \ldots, \varepsilon_n > 0$. Replacing ϕ_i by ϕ_i/ε_i if necessary, without loss of generality $\varepsilon_1 = \cdots = \varepsilon_n = 1$. If $x \in X$ is such that $\alpha := |\phi(x)| > \max_{1 \leqslant i \leqslant n} |\phi_i(x)|$ then $|\phi(\alpha^{-1}x)| = 1$ but

$$\max_{1 \leqslant i \leqslant n} |\phi_i(\alpha^{-1}x)| = \max_{1 \leqslant i \leqslant n} |\phi_i(x)|/|\phi(x)| < 1,$$

a contradiction.

Deduce that $\bigcap_{i=1}^{n} \ker \phi_i \subseteq \ker \phi$ and that there exists $f \in (\mathbb{F}^n)^*$ such that

$$f\big(\phi_1(x), \ldots, \phi_n(x)\big) = \phi(x) \qquad \forall\, x \in X.$$

The first deduction is immediate; for the next, define $\Phi \colon X \to \mathbb{F}^n$ by setting

$$\Phi(x) := \big(\phi_1(x), \ldots, \phi_n(x)\big) \qquad \forall\, x \in X$$

and note that $\ker \Phi = \bigcap_{i=1}^{n} \ker \phi_i \subseteq \ker \phi$, so that if $\Phi(x) = \Phi(y)$ then $\phi(x) = \phi(y)$. Hence

$$f_0 \colon \Phi(X) \to \mathbb{F};\ \big(\phi_1(x), \ldots, \phi_n(x)\big) \mapsto \phi(x)$$

is well defined, and we extend f_0 to $f \in (\mathbb{F}^n)^*$ by setting $f|_{\Phi(X)} = f_0$ and $f|_{\Phi(X)^\perp} = 0$.

Conclude that $\phi \in M$.

Let $f(0, \dots, 1, \dots, 0) = \alpha_i$ (where the 1 is in the ith place) so that

$$\phi(x) = f\big(\phi_1(x), \dots, \phi_n(x)\big) = \sum_{i=1}^{n} \phi_i(x)\alpha_i = \Bigg(\sum_{i=1}^{n} \alpha_i \phi_i\Bigg)(x) \qquad \forall\, x \in X,$$

i.e., $\phi = \sum_{i=1}^{n} \alpha_i \phi_i \in M$.

Exercise 5.2. Let X be an infinite-dimensional normed space and let $V \subseteq X$ be a weakly open set containing the origin. Show that V contains a closed subspace of finite codimension in X.

Since V is a weakly open set containing 0 there exist $\phi_1, \dots, \phi_n \in X^*$ such that V contains $\bigcap_{j=1}^{n} |\phi_j|^{-1}[0, 1)$. The linear transformation

$$\Phi \colon X \to \mathbb{F}^n;\; x \mapsto \big(\phi_1(x), \dots, \phi_n(x)\big)$$

has kernel $\ker \Phi = \bigcap_{i=1}^{n} \ker \phi_i \subseteq V$, which is closed because $\ker \phi_i$ is closed for $i = 1, \dots, n$. Since $X/\ker \Phi \cong \operatorname{im} \Phi \subseteq \mathbb{F}^n$, $\ker \Phi$ has finite codimension, as desired.

Deduce that the weak topology on X is strictly coarser than the norm topology.

Note that the open unit ball $B_1^X(0)$ is open with respect to the norm topology on X, but it contains no subspace of X other than $\{0\}$, which does not have finite codimension. Hence $\sigma(X, X^*)$ has fewer open sets than the norm topology.

(To see that these topologies are comparable, note that $x \mapsto \phi(x)$ is norm continuous for all $\phi \in X^*$, so $\phi^{-1}(U)$ is open (with respect to the norm topology) for all open $U \subseteq \mathbb{F}$.)

Exercise 5.3. Let X be a topological vector space. Prove that every $\phi \in X' \setminus \{0\}$ is open.

Let $x_0 \in X$ be such that $\phi(x_0) = 1$ and let $A \subseteq X$ be open. If $x \in A$ then $A - x$ is an open set containing 0 (by the continuity of vector addition) and so $m_{x_0}^{-1}(A - x)$ is an open set containing 0, where $m_{x_0} \colon \mathbb{F} \to X;\; \alpha \mapsto \alpha x_0$ (by the continuity of scalar multiplication). Hence there exists $\delta > 0$ such that $\alpha x_0 \in A - x$ if $|\alpha| < \delta$, and so $\alpha + \phi(x) = \phi(\alpha x_0 + x) \in \phi(A)$ for such α. This shows that $\phi(A)$ is open and gives the result.

Exercise 5.4. Suppose that X is a vector space equipped with a topology that makes vector addition and scalar multiplication, i.e., the maps

$$X \times X \to X;\; (x, y) \mapsto x + y \qquad \text{and} \qquad \mathbb{F} \times X \to X;\; (\alpha, x) \mapsto \alpha x,$$

continuous. Show that if this topology is such that singleton sets are closed (i.e., $\{x\}$ is closed for all $x \in X$) then the topology is Hausdorff (so X is a topological vector space).

Note that, for fixed $y \in X$, the map $x \mapsto x+y$ is a homeomorphism of X with itself, so it suffices take $x \in X \setminus \{0\}$ and find an open set U such that $0 \in U$ and $U \cap (x+U) = \emptyset$. Note that the map

$$X \times X \to X;\ (y,z) \mapsto y - z$$

is continuous and so $\{(y,z) \in X \times X : y - z \neq x\}$ is an open subset of $X \times X$ containing $(0,0)$. Hence there exist open sets V, $W \subseteq X$ such that $0 \in V \cap W$ and $x \notin V - W$; taking $U = V \cap W$ gives the result.

Exercise 5.5. Let X be a topological vector space. Prove that every open set containing the origin contains a non-empty open set which is *balanced*: a set B is balanced if $\lambda b \in B$ for all $b \in B$ and $\lambda \in \mathbb{F}_1$.

Let $U \subseteq X$ be an open set containing 0. As $m : \mathbb{F} \times X \to X;\ (\lambda, x) \mapsto \lambda x$ is continuous, $m^{-1}(U)$ is an open set in $\mathbb{F} \times X$ which contains $(0,0)$; by the definition of the product topology, there exist $\varepsilon > 0$ and an open set $V \subseteq X$ such that $0 \in V$ and $B_\varepsilon^{\mathbb{F}}(0) \times V \subseteq m^{-1}(U)$. It is easily verified that $m(B_\varepsilon^{\mathbb{F}}(0) \times V) = \{\lambda v : |\lambda| < \varepsilon,\ v \in V\}$ is balanced, contained in U and open, since it is the union of λV over all $\lambda \in \mathbb{F}$ such that $0 < |\lambda| < \varepsilon$ and the map $v \mapsto \lambda v$ is a homeomorphism for all $\lambda \neq 0$.

Deduce that if $C \subseteq X$ is compact and does not contain the origin then there exist disjoint open sets A, $B \subseteq X$ such that $C \subseteq A$ and B is a balanced set containing 0.

As X is Hausdorff, for all $x \in C$ there exist disjoint open sets A_x, $B_x \subseteq X$ such that $x \in A_x$ and $0 \in B_x$; by the first part of this exercise, B_x may be taken to be balanced. Since C is compact, there exist $x_1, \ldots, x_n \in C$ such that $A := A_{x_1} \cup \cdots \cup A_{x_n} \subseteq X$, and if $B := B_{x_1} \cap \cdots \cap B_{x_n}$ then A and B are as required.

Show that a balanced set is connected and give an example to show that a balanced set need not be convex.

For any point x in a balanced set B, the path $t \mapsto (1-t)x$ connects x to the origin; thus B is path connected, so connected. The cross-shaped set $\{(x,0),(0,y) : x, y \in [-1,1]\} \subseteq \mathbb{R}^2$ is balanced but not convex.

Exercise 5.6. Let $p \in (0,1)$,

$$\mathcal{L}^p[0,1] := \big\{ f \colon [0,1] \to \mathbb{C} \ \big|\ f \text{ is measurable and } \Delta(f) < \infty \big\},$$

where $\Delta(f) := \int_0^1 |f(x)|^p \,\mathrm{d}x$, and let $L^p[0,1] := \mathcal{L}^p[0,1]/\mathcal{N}$, where

$$\mathcal{N} := \big\{ f \colon [0,1] \to \mathbb{C} \ \big|\ f \text{ is measurable and zero almost everywhere} \big\}.$$

Prove that $d([f],[g]) := \Delta(f-g)$ is a metric on $L^p[0,1]$ and that $L^p[0,1]$ is a topological vector space (when equipped with this topology).

The mean-value theorem may be used to establish the inequality $(1+z)^p - z^p \leqslant 1$ for all $z \geqslant 0$, from which it follows that $(x+y)^p \leqslant x^p + y^p$ for all x, $y \geqslant 0$. This is enough to see that $\mathcal{L}^p[0,1]$ is closed under sums and so is a vector space; it also gives the triangle inequality for d. The other two requirements of a metric are obviously satisfied and the inequalities

$$d([f+g],[f_a+g_a]) \leqslant d([f],[f_a]) + d([g],[g_a])$$

and

$$d(\lambda f, \lambda_a f_a) \leqslant |\lambda - \lambda_a|^p \int_0^1 |f(t)|^p \,\mathrm{d}t + |\lambda_a|^p\, d([f],[f_a])$$

show that vector addition and scalar multiplication are suitably continuous.

Prove further that $L^p[0,1]$ has no convex, open sets other than $\emptyset$ and $L^p[0,1]$.

Let $V \neq \emptyset$ be convex and open; without loss of generality $0 \in V$ and so there exists $\varepsilon > 0$ such that $\{f \in L^p[0,1] : d(f,0) < \varepsilon\} \subseteq V$. Let $f \in L^p[0,1]$ and choose $n \in \mathbb{N}$ such that $n^{p-1}d(f,0) < \varepsilon$. By the intermediate-value theorem, there exist $0 = x_0 < x_1 < \cdots < x_n = 1$ such that

$$\int_{x_{i-1}}^{x_i} |f(t)|^p \,\mathrm{d}t = d(f,0)/n \qquad (i = 1, \ldots, n);$$

let $g_{i,n} := nf$ on $[x_{i-1}, x_i)$ and 0 elsewhere, so that $f = (g_{1,n} + \cdots + g_{n,n})/n$ and

$$d(g_{i,n}, 0) = n^p \int_{x_{i-1}}^{x_i} |f(t)|^p \,\mathrm{d}t = n^{p-1} d(f,0) < \varepsilon$$

for all i. It follows immediately that $f \in V$ and $V = L^p[0,1]$.

Deduce that the only continuous linear functional on $L^p[0,1]$ is the zero functional.

If $\phi \in L^p[0,1]'$ is continuous then $\phi^{-1}\big(B_\varepsilon^{\mathbb{C}}(0)\big)$ is convex, open and non-empty, for all $\varepsilon > 0$. Hence $\phi(L^p[0,1]) \subseteq B_\varepsilon^{\mathbb{C}}(0)$ for all $\varepsilon > 0$ and so $\phi = 0$.

[This argument is (essentially) due to Tychonov [23], who worked with the sequence space $\{(x_n)_{n\in\mathbb{N}} : \sum_{n=1}^\infty |x_n|^{1/2} < \infty\}$.]

Exercise 5.7. Let X be a topological vector space over $\mathbb{F}$ and let M be a finite-dimensional subspace of X. Prove that M is linearly homeomorphic to $\mathbb{F}^n$, where n is the dimension of M.

Let $\{e_1, \ldots, e_n\}$ be a basis for M and let

$$T\colon \mathbb{F}^n \to M;\ (\lambda_1, \ldots, \lambda_n) \mapsto \lambda_1 e_1 + \cdots + \lambda_n e_n.$$

It is immediate that T is a linear bijection which is continuous, because scalar multiplication and vector addition are; it remains to show that T is open. Let

$$S := \{(\lambda_1, \ldots, \lambda_n) : |\lambda_1|^2 + \cdots + |\lambda_n|^2 = 1\}$$

be the unit sphere in $\mathbb{F}^n$; this is compact and therefore so is its image, $T(S)$. By Exercise 5.5, there exist disjoint open sets U, $V \subseteq M$ such that $0 \in U$, which is balanced, and $T(S) \subseteq V$. It follows that $U \subseteq T(B_1^{\mathbb{F}}(0))$: otherwise, there exists $u \in U$ such that $u = T(\lambda_1, \ldots, \lambda_n)$ with $\alpha := |\lambda_1|^2 + \cdots + |\lambda_n|^2 \geqslant 1$, but then $\alpha^{-1/2}(\lambda_1, \ldots, \lambda_n) \in S$ and $\alpha^{-1/2} u \in T(S) \cap U$, a contradiction. Hence if $W \subseteq \mathbb{F}^n$ is open and $w \in W$ then there exists $\varepsilon > 0$ such that $B_\varepsilon^{\mathbb{F}}(0) \subseteq W - w$, so

$$Tw + \varepsilon U \subseteq Tw + \varepsilon T(B_1^{\mathbb{F}}(0)) \subseteq T(w + B_\varepsilon^{\mathbb{F}}(0)) \subseteq T(W);$$

since $Tw + \varepsilon U$ is open and contains Tw, the set $T(W)$ is open and the result follows.

Prove also that M is closed in X.

Let T be the map above, now considered to have codomain X; clearly T is (still) continuous and the same working as above yields an open set $U \subseteq X$ such that $0 \in U$ and $U \cap M \subseteq T(B_1^{\mathbb{F}}(0))$. Let x be in the closure of M and note that $tx \to 0$ as $t \to 0+$, by the continuity of scalar multiplication, so there exists $r > 0$ such that $x \in rU$. Hence

$$x \in rU \cap \overline{M} \subseteq \overline{rU \cap M} \subseteq \overline{rT(B_1^{\mathbb{F}}(0))} \subseteq \overline{T((\mathbb{F}^n)_r)} = T((\mathbb{F}^n)_r) \subseteq M,$$

since $(\mathbb{F}^n)_r$ is compact and T is continuous. (The first inclusion holds as rU is open.)

Exercise 5.8. Prove that a topological vector space with topology given by a separating family of linear functionals is locally convex.

Let X be such a topological vector space, with separating family M. If U is an open subset of X containing the origin, there exist $\phi_1, \ldots, \phi_n \in M$ and $\varepsilon_1, \ldots, \varepsilon_n > 0$ such that

$$\bigcap_{i=1}^{n} \phi_i^{-1}\big((B_{\varepsilon_i}^{\mathbb{F}}(0)\big) \subseteq U,$$

so it suffices to prove that $\phi^{-1}(B_\varepsilon^{\mathbb{F}}(0))$ is convex for any $\phi \in M$ and $\varepsilon > 0$. To see this, let $t \in (0,1)$ and suppose u and $v \in X$ are such that $|\phi(u)| < \varepsilon$ and $|\phi(v)| < \varepsilon$. Then $|\phi(tu + (1-t)v)| \leqslant t|\phi(u)| + (1-t)|\phi(v)| < \varepsilon$, as required.

Exercise 5.9. Suppose that X is a locally convex topological vector space and M is the collection of continuous linear functionals on X. Prove that a convex subset of X is closed (with respect to the original topology) if and only if it is closed with respect to $\sigma(X, M)$, the initial topology generated by M.

As $\sigma(X, M)$ is the coarsest topology to make every element of M continuous, by definition, $\sigma(X, M)$ is coarser than $\mathcal{T}_X$, the original topology on X; in particular, every $\sigma(X, M)$-closed set is $\mathcal{T}_X$-closed. Conversely, if $B \subseteq X$ is convex and $\mathcal{T}_X$-closed then let $x_0 \in X \setminus B$; by Theorem 3.32(ii), there exists $\phi \in M$ such that $\operatorname{Re} \phi(x_0) < \inf_{y \in B} \operatorname{Re} \phi(y)$. Thus B is $\sigma(X, M)$-closed, as required.

Need this hold if X is not locally convex?

No: if $X = L^p[0,1]$ for some $p \in (0,1)$ then, by Exercise 5.6, $F = \{0\}$ and $\mathcal{T}_F = \{0, X\}$, the trivial topology. However, the one-dimensional subspace of X consisting of the constant functions is closed, convex, non-empty and not the whole of X. (For the first of these claims, either use Exercise 5.7 or prove it directly: if the sequence of constant functions $(\lambda_n 1)_{n\geqslant 1}$ converges to $f \in X$ then it is Cauchy, but $d(\lambda_n 1, \lambda_m 1) = |\lambda_n - \lambda_n|^p$ and so $\lambda_n \to \lambda$ for some $\lambda \in \mathbb{C}$. Furthermore, $d(f, \lambda 1) \leqslant d(f, \lambda_n 1) + d(\lambda_n 1, \lambda) \to 0$, so $f = \lambda 1$, as required.)

Exercise 5.10. Let X be a locally convex topological vector space. Show that if N is a non-empty subspace of X and $x_0 \in X \setminus \bar{N}$ then there exists a continuous linear functional $\phi \in X'$ such that $\phi|_N = 0$ and $\phi(x_0) = 1$.

Let $B = \bar{N}$; by the separation theorem (Theorem 3.32) there exists a continuous $\phi_0 \in X'$ such that

$$\operatorname{Re}\phi_0(x_0) < \inf\{\operatorname{Re}\phi_0(y) : y \in B\}.$$

In particular, $\phi_0\big(\{x_0\}\big) \cap \phi_0(B) = \emptyset$, and as $\phi_0(B)$ is a subspace of $\phi_0(X) = \mathbb{F}$ we must have that $\phi_0(B) = \{0\}$. Hence $\phi_0(x_0) \neq 0$ and $\phi\colon x \mapsto \phi_0(x)/\phi_0(x_0)$ is as required.

Exercise 5.11. Let X be a topological vector space and suppose V is an open set containing 0. Prove there exists an open set U containing 0 such that $U + U \subseteq V$.

Since $a : X \times X \to X$; $(x, y) \mapsto x + y$ is continuous and $0 + 0 = 0 \in V$, the pre-image $a^{-1}(V)$ is an open set containing $(0, 0)$; hence there exist open sets W_1, $W_2 \subseteq X$ such that $0 \in W_1 \cap W_2$ and $W_1 + W_2 \subseteq V$. Letting $U := W_1 \cap W_2$ establishes the claim.

Deduce or prove otherwise that if $A \subseteq B \subseteq X$, where A is compact and B is open, then there exists an open set $U \subseteq X$ such that $0 \in U$ and $A + U \subseteq B$.

If $x \in A$ then $B - x$ is an open set containing 0 so, by the first part, there exists an open set U_x which contains 0 and satisfies $U_x + U_x \subseteq B - x$. As $x + U_x$ is open for all $x \in A$, compactness yields $x_1, \ldots, x_n \in A$ such that $A \subseteq (x_1 + U_{x_1}) \cup \cdots \cup (x_n + U_{x_n})$; if $U := U_{x_1} \cap \cdots \cap U_{x_n}$ then U is an open set containing 0 and

$$A + U \subseteq \bigcup_{i=1}^{n} (x_i + U_{x_i} + U) \subseteq \bigcup_{i=1}^{n} (x_i + U_{x_i} + U_{x_i}) \subseteq B,$$

as required.

The Appendix provides the means for an alternative proof. Suppose, for contradiction, that if $U \subseteq X$ is open and contains 0 then there exist $x_U \in A$ and $y_U \in U$ such that $x_U + y_U \notin B$. Ordering such U by reverse inclusion, (y_U) converges to 0 and (x_U) is contained in the compact set A, so has a convergent subnet, by Theorem A.5. Hence $(x_U + y_U)$ has a convergent subnet, with limit z in A, but $x_U + y_U \in X \setminus B$ for all U, which is closed, so $z \in X \setminus B \subseteq X \setminus A$. This contradiction gives the result.

Exercise 5.12. Suppose that X is a topological vector space such that the continuous elements of X' separate points. Prove that given disjoint, non-empty, compact, convex A, $B \subseteq X$ there exists a continuous $\phi \in X'$ such that

$$\sup_{x \in A} \operatorname{Re}\phi(x) < \inf_{x \in B} \operatorname{Re}\phi(x).$$

Since $\sigma(X, M)$ is coarser than the original topology on X, the sets A and B are $\sigma(X, M)$-compact; furthermore, $\sigma(X, M)$ is Hausdorff, because M separates points, (Proposition 1.26) and both vector addition and scalar multiplication are $\sigma(X, M)$-continuous, by Propositions 1.25, 1.35 and 1.37 together with the continuity of addition and multiplication in $\mathbb{F}$. Hence X is a topological vector space when equipped with $\sigma(X, M)$; we work with this topology from now on, which has the benefit of being locally convex, by Exercise 5.8, and also gives rise to the same collection of continuous linear functionals, M, by Exercise 5.1. Applying Exercise 5.11 to A and $X \setminus B$ yields an open set U containing 0 such that $A + U \subseteq X \setminus B$, whence $A + U$ and B are disjoint. Moreover, the set U may be taken to be convex, so $A + U$ is convex and Theorem 3.32(i) yields $\phi \in M$ and $s \in \mathbb{R}$ such that $\operatorname{Re}\phi(x) < s \leqslant \inf_{y \in B} \operatorname{Re}\phi(y)$ for all $x \in A + U$; the result follows.

Deduce that Theorem 3.40 is true for topological spaces with continuous dual spaces that separate points.

The proof given in Section 3.12 applies with the following changes: Corollary 3.33 is not required, since the continuous dual space is assumed to separate points, and the result just established is a generalisation of Theorem 3.32(ii).

Exercise 5.13. Let X be a topological vector space and suppose that C is a non-empty, compact, convex subset of X. Prove that $(\mathcal{F}, \supseteq)$, the collection of closed faces of C ordered by reverse inclusion, is a non-empty, partially ordered set such that every chain in $\mathcal{F}$ has an upper bound.

Since C is trivially a face of itself, $\mathcal{F}$ is non-empty, and reverse inclusion is easily seen to be reflexive, transitive and antisymmetric, i.e., a partial order. If $\mathcal{C}$ is a chain in $\mathcal{F}$ then consider F, the intersection of all the elements of $\mathcal{C}$; it is immediate that this is an upper bound for $\mathcal{C}$ if it is a face. The set F is closed and also non-empty: C is compact, so every collection of closed subsets of C with the finite-intersection property has non-empty intersection and $\mathcal{C}$ has this property because it is a chain. If $x, y \in C$ and $t \in (0, 1)$ are such that $tx + (1-t)y \in F$ then $tx + (1-t)y$ lies in every element of $\mathcal{C}$, so x and y lie in every element of $\mathcal{C}$, as these elements are faces. Hence x and y are in F and F is a face.

Exercise 5.14. Prove that the closed unit ball of c_0 has no extreme points.

Let $x \in (c_0)_1$ and note that there exists $n_0 \in \mathbb{N}$ such that $|x_{n_0}| < \frac{1}{2}$. Define y and $z \in c_0$ by setting $y_n = z_n = x_n$ for all $n \neq n_0$, $y_{n_0} = x_{n_0} - \frac{1}{2}$ and $z_{n_0} = x_{n_0} + \frac{1}{2}$. It is immediate that $y, z \in (c_0)_1$ and $x = \frac{1}{2}(y + z)$, showing that x is not an extreme point of $(c_0)_1$.

Exercise 5.15. Let H be a Hilbert space. Prove that every unit vector in H is an extreme point of the closed unit ball H_1.

Suppose that x is a unit vector in H and let $y, z \in H_1$ and $t \in (0, 1)$ be such that

$x = ty + (1-t)z$. Then

$$1 = \|x\|^2 = t\langle y, x\rangle + (1-t)\langle z, x\rangle,$$

which implies that $t = \frac{1}{2}$ and $\langle y, x\rangle = \langle z, x\rangle = 1$ (because 1 is an extreme point in $\mathbb{F}_1$). As $1 = |\langle y, x\rangle| \leqslant \|y\|\,\|x\| \leqslant 1$ we have equality in the Cauchy-Schwarz inequality and so x and y are linearly dependent; since $\langle y, x\rangle = 1$ we must have $y = x$. Hence x is an extreme point of H_1.

Deduce that every isometry in $\mathcal{B}(H)$ is an extreme point of the closed unit ball $\mathcal{B}(H)_1$.

Let $V \in \mathcal{B}(H)$ be a isometry; it is immediate that $\|V\| = 1$ and so $V \in \mathcal{B}(H)_1$. Let $S, T \in \mathcal{B}(H)_1$ and $t \in (0,1)$ be such that $V = tS + (1-t)T$ and let $x \in H$ be a unit vector. Then $\|Vx\| = \|x\| = 1$, so Vx is a unit vector, and $\|Sx\|$, $\|Tx\| \leqslant 1$, so the identity $Vx = tSx + (1-t)Tx$ implies that $Sx = Tx = Vx$, by the first part. As this holds for all unit vectors in H we have that $S = T = V$, so V is an extreme point of $\mathcal{B}(H)_1$, as claimed.

Exercise 5.16. Let C be a convex subset of a topological vector space X. Prove that if $x \in C$ and $y \in C^\circ$, the interior of C, then $tx + (1-t)y \in C^\circ$ for all $t \in [0,1)$.

Since $y \in C^\circ$, there exists an open set U such that $y \in U \subseteq C$; if $t \in [0,1)$ then $tx + (1-t)U$ is open (since $z \mapsto z + tx$ and $z \mapsto (1-t)z$ are homeomorphisms) and contained in C (since C is convex).

[It follows that C is the sequential closure of C°, if this set is non-empty: for every $x \in C$ there exists a sequence $(x_n)_{n\geqslant 1} \subseteq C^\circ$ such that $x_n \to x$.]

Prove also that the interior C° and the extremal boundary $\partial_{\mathrm{e}} C$ are disjoint (as long as $X \neq \{0\}$).

Suppose for contradiction that $x \in C^\circ \cap \partial_{\mathrm{e}} C$ and let U be an open set with $x \in U \subseteq C$. Let $u \in X$ be a non-zero vector and note that function $f : \mathbb{R} \to X;\ \lambda \mapsto x + \lambda u$ is continuous, so $f^{-1}(U)$ is an open set containing 0. Hence there exists $\delta > 0$ such that $x \pm \delta u \in U$, but then, since $x \in \partial_{\mathrm{e}} C$,

$$x = \tfrac{1}{2}(x + \delta u) + \tfrac{1}{2}(x - \delta u) \Longrightarrow x + \delta u = x - \delta u \Longrightarrow u = 0;$$

this contradiction give the result.

Exercise 5.17. Let $C \subseteq \mathbb{R}^n$ be compact and convex. Prove that every element of C can be written as a convex combination of at most $(n+1)$ elements of $\partial_{\mathrm{e}} C$.

If $n = 1$ then C is connected, so is a compact interval and the claim is clear. Now suppose the result holds for some n and let $C \subseteq \mathbb{R}^{n+1}$ be compact and convex. If every set of $n+1$ points in C is linearly dependent then $\operatorname{lin} C$ is at most n-dimensional (since C is a spanning set which contains no more than n linearly independent points) so $C \subseteq \operatorname{lin} C$ and the result follows by the inductive hypothesis. Otherwise, there exists a linearly independent subset of C containing $n+1$ points; the convex hull of this set, being homeomorphic to the standard $n+1$-simplex

$$\{(\lambda_1, \ldots, \lambda_{n+1}) \in \mathbb{R}^{n+1} : \lambda_1, \ldots, \lambda_{n+1} \geqslant 0, \sum_{i=1}^{n+1} \lambda_i = 1\},$$

has non-empty interior, and therefore so does C. Suppose now that $x \in C \setminus C^\circ$; Theorem 3.32(i) gives a continuous linear functional f such that $f(c) < f(x)$ for all $c \in C^\circ$ and so $f(c) \leqslant f(x)$ for all $c \in C$, by Exercise 5.16. If

$$D := \{c \in C : f(c) = f(x)\} = (x + \ker f) \cap C$$

then D is a face of C which lies in an n-dimensional hyperplane, so the result follows by the inductive hypothesis; recall that $\partial_e D \subseteq \partial_e C$. Finally, if $x \in C^\circ$ then let $y \in \partial_e C$ and consider the line L through x and y; this is a closed set and so $C \cap L$ is compact and convex, so is an interval. As y is an extreme point of C, it must be one end-point of this interval; let z be the other and note that $z \in C \setminus C^\circ$, since $C^\circ \cap L \subseteq (C \cap L)^\circ$ (where the second interior is taken with respect to the relative topology on $C \cap L$). Hence x is an interior point of the interval, so there exists $t \in (0,1)$ such that $x = ty + (1-t)z$; furthermore, by the previous working, z is a convex combination of at most n points in $\partial_e C$ and the result follows.

[This result is due to Carathéodory [4].]

Solutions to Exercises 6

Exercise 6.1. Prove that the vector space $L^1(\mathbb{R})$ is a commutative Banach algebra when equipped with the convolution product and the norm $\|f\|_1 := \int |f|$.

We take it for granted that $L^1(\mathbb{R})$ is a normed space and that $(f,g) \mapsto f \star g$ is a well-defined, bilinear map. It remains to prove that this multiplication is commutative and associative and that $L^1(\mathbb{R})$ is complete.

For commutativity and associativity note that the Fourier transform $f \mapsto \hat{f}$ is injective on $L^1(\mathbb{R})$, that

$$\widehat{f \star g} = \hat{f}\hat{g} = \hat{g}\hat{f} = \widehat{g \star f}$$

and that

$$\widehat{f \star (g \star h)} = \hat{f}\widehat{g \star h} = \hat{f}\big(\hat{g}\hat{h}\big) = \big(\hat{f}\hat{g}\big)\hat{h} = \widehat{f \star g}\hat{h} = \widehat{(f \star g) \star h}$$

by the convolution theorem. (This theorem, which asserts the existence of $f \star g$, is an easy application of the theorems of Fubini and Tonelli; see [17, §§ 26.15–26.16]).

For the proof of completeness we use Banach's criterion. If $(f_k)_{k\geqslant 1} \subseteq L^1(\mathbb{R})$ is such that $\sum_{k=1}^\infty \int |f_k| < \infty$ then the monotone-convergence theorem applied to $(\sum_{k=1}^n |f_k|)_{n\geqslant 1}$ yields the convergence almost everywhere of $\sum_{k=1}^\infty |f_k(x)|$ to an integrable function and the fact that $\sum_{k=1}^\infty \int |f_k| = \int \sum_{k=1}^\infty |f_k|$. Hence $f(x) = \sum_{k=1}^\infty f_k(x)$ is (absolutely) convergent almost everywhere, and applying the dominated-convergence theorem to $(\sum_{k=1}^n f_k)_{n\geqslant 1}$ (with dominating function $\sum_{k=1}^\infty |f_k|$) gives that $f \in L^1(\mathbb{R})$, as required.

Prove further that this algebra is not unital.

For $n \geqslant 1$ let

$$f_n \colon \mathbb{R} \to \mathbb{R};\ x \mapsto \frac{n}{\sqrt{2\pi}} e^{-\frac{1}{2}n^2x^2}$$

and note that $\int f_n = 1$. The continuous version of the dominated-convergence theorem gives that $\|g_t - g\|_1 \to 0$ as $t \to 0$ for all $g \in C_{00}(\mathbb{R})$, where $g_t \colon s \mapsto g(s-t)$. As $C_{00}(\mathbb{R})$ is dense in $L^1(\mathbb{R})$ it follows that $\|g_t - g\|_1 \to 0$ as $t \to 0$ for all $g \in L^1(\mathbb{R})$; note that $t \mapsto g_t$ maps $L^1(\mathbb{R})$ to itself, that $\|g_t - h_t\|_1 = \|g - h\|_1$ and that

$$\|g_t - g\|_1 \leqslant \|g_t - h_t\|_1 + \|h_t - h\|_1 + \|h - g\|_1 = 2\|h - g\|_1 + \|h_t - h\|_1.$$

If $g \in L^1(\mathbb{R})$ then

$$\begin{aligned}\|g \star f_n - g\|_1 &= \int_{\mathbb{R}} \Big|\int_{\mathbb{R}} g(t-s)f_n(s) - g(t)f_n(s)\,\mathrm{d}s\Big|\,\mathrm{d}t \\ &\leqslant \int_{\mathbb{R}}\int_{\mathbb{R}} |g_s - g|(t)\,\mathrm{d}t\, n f_1(ns)\,\mathrm{d}s = \int_{\mathbb{R}} \|g_{r/n} - g\|_1 f_1(r)\,\mathrm{d}r \to 0\end{aligned}$$

as $n \to \infty$, by the dominated-convergence theorem: the map $t \mapsto \|g_t - g\|_1$ is bounded and $f_1 \in L^1(\mathbb{R})$.

Suppose that there exists $1 \in L^1(\mathbb{R})$ such that $1 \star f = f$ for all $f \in L^1(\mathbb{R})$. Then $1 \star f_n \to 1$ as $n \to \infty$ (in $L^1(\mathbb{R})$) but $(1 \star f_n)(x) = f_n(x) \to 0$ as $n \to \infty$ almost everywhere, which implies that $1(x) = 0$ almost everywhere. This is clearly impossible.

Exercise 6.2. Let X be a Hausdorff, locally compact space. Prove that $C_0(X)^{\mathrm{u}}$, the unitization of the algebra of continuous functions on X that vanish at infinity, is topologically isomorphic to $C(\dot{X})$, the algebra of continuous functions on $\dot{X}$, the one-point compactification of X.

Recall from the solution of Exercise 2.2 that the map $f \mapsto f|_X$ is a bijection between $I := \{f \in C(\dot{X}) : f(\infty) = 0\}$ and $C_0(X)$. Since $f - f(\infty)1 \in I$ for all $f \in C(\dot{X})$ the map

$$C(\dot{X}) \to C_0(X)^{\mathrm{u}};\ f \mapsto (f - f(\infty)1)|_X + f(\infty)1$$

is a bijection, and it is readily verified that it is an algebra homomorphism. Finally,

$$\|(f - f(\infty)1)|_X\|_\infty + |f(\infty)| \leqslant \|f|_X\|_\infty + 2|f(\infty)| \leqslant 3\|f\|_\infty$$

so this map is a continuous bijection between Banach spaces, hence a homeomorphism (by Theorem 2.14).

Exercise 6.3. Let $A = \mathbb{C}[z]$ denote the unital algebra of complex polynomials and let $\|p\| := \sup\{|p(\alpha)| : |\alpha| \leqslant 1\}$ for all $p \in A$. Show that $(A, \|\cdot\|)$ is a unital, normed algebra which is not complete.

Proving that $\|\cdot\|$ is a submultiplicative norm on A is trivial.

Consideration of degree shows that $G(A) = (\mathbb{C} \setminus \{0\})1$, so $p_n \in A \setminus G(A)$ for all $n \geqslant 1$, where $p_n(z) := 1 + z/n$. If A is complete then $G(A)$ is open, but $\|p_n - 1\| \to 0$ as $n \to \infty$ and $1 \in G(A)$, so $A \setminus G(A)$ is not closed. This gives the result.

Exercise 6.4. Let A be a (non-unital) Banach algebra such that every element is nilpotent (i.e., for all $a \in A$ there exists $n \in \mathbb{N}$ such that $a^n = 0$). Prove that A is uniformly nilpotent: there exists $N \in \mathbb{N}$ such that $a^N = 0$ for all $a \in A$.

Let $E_n = \{a \in A : a^n = 0\}$ for all $n \in \mathbb{N}$; these sets are closed and $A = \bigcup_{n \in \mathbb{N}} E_n$ so, by the Baire category theorem, $E_N^\circ \neq \emptyset$ for some $N \in \mathbb{N}$. Hence there exists $a_0 \in A$ and $\varepsilon > 0$ such that $\|a - a_0\| < \varepsilon$ implies that $a^N = 0$. Let $b \in A \setminus \{0\}$ and note that

$$p(t) := (a_0 + tb)^N = a_0^N + \cdots + t^N b^N = 0 \qquad \forall\, t \in \big(-\varepsilon/\|b\|, \varepsilon/\|b\|\big)$$

because $\|(a_0 + tb) - a_0\| = |t|\|b\| < \varepsilon$. Hence $0 = p^{(N)}(0) = N!\, b^N$, where $p^{(N)}$ is the Nth derivative of p, and so $b^N = 0$, as required.

Exercise 6.5. Let A be a unital Banach algebra over $\mathbb{C}$ and let $e^a := \sum_{n=0}^\infty a^n/n!$ for all $a \in A$. Prove that $e^{a+b} = e^a e^b$ if a and b commute.

Note first that e^a converges absolutely, so converges. As

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k} = \frac{1}{n!} \sum_{k=0}^n \frac{a^k}{k!} \frac{b^{n-k}}{(n-k)!}$$

if a, $b \in A$ commute, it suffices to prove that if $a = \sum_{n=0}^\infty a_n$ and $b = \sum_{n=0}^\infty b_n$ are absolutely convergent in A then $ab = \sum_{n=0}^\infty c_n$, where $c_n = \sum_{k=0}^n a_k b_{n-k}$.

To see this, note that

$$\begin{aligned}
\sum_{k=0}^n \sum_{l=0}^k a_l b_{k-l} &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \cdots + (a_0 b_n + \cdots + a_n b_0) \\
&= a_0 \sum_{l=0}^n b_l + a_1 \sum_{l=0}^{n-1} b_l + \cdots + a_n b_0 \;=\; \Big(\sum_{k=0}^n a_k\Big) b + \sum_{k=0}^n a_k \Big(\sum_{l=0}^{n-k} b_l - b\Big)
\end{aligned}$$

so it suffices to prove that $r_n = \sum_{k=0}^n a_k d_{n-k} = \sum_{k=0}^n a_{n-k} d_k \to 0$ as $n \to \infty$, where $d_k = \sum_{l=0}^k b_l - b$. Let $\varepsilon > 0$ and choose N such that $\|d_n\| < \varepsilon$ for all $n \geqslant N$; if $n > N$ then

$$\begin{aligned}
\Big\|\sum_{k=0}^n a_{n-k} d_k\Big\| &\leqslant \Big\|\sum_{k=0}^N a_{n-k} d_k\Big\| + \sum_{k=N+1}^n \|a_{n-k}\|\,\|d_k\| \\
&\leqslant \Big\|\sum_{k=0}^N a_{n-k} d_k\Big\| + \varepsilon \sum_{k=1}^\infty \|a_k\| \;\;\to\;\; \varepsilon \sum_{k=1}^\infty \|a_k\|
\end{aligned}$$

as $n \to \infty$. As $\varepsilon > 0$ is arbitrary we have the result.

Deduce that e^a is invertible.

Since $e^0 = 1$ we have that $e^a e^{-a} = 1 = e^{-a} e^a$, i.e., $(e^a)^{-1} = e^{-a}$, for all $a \in A$.

Prove further that $f\colon \lambda \mapsto e^{\lambda a}$ is holomorphic everywhere, with $f'(\lambda) = af(\lambda) = f(\lambda)a$, for all $a \in A$.

Let $a \in A$, $\lambda \in \mathbb{C}$ and $h \in \mathbb{C} \setminus \{0\}$. Then

$$\frac{e^{(\lambda+h)a} - e^{\lambda a}}{h} - ae^{\lambda a} = \frac{e^{ha} - 1 - ha}{h} e^{\lambda a} \to 0$$

as $h \to 0$, because

$$\left\| \frac{e^{ha} - 1 - ha}{h} \right\| \leqslant \sum_{n=2}^{\infty} \frac{|h|^{n-1}\|a\|^n}{n!} = \frac{e^{|h|\,\|a\|} - 1}{|h|} - \|a\|$$

and $\displaystyle \frac{\mathrm{d}}{\mathrm{d}z} e^{z\|a\|} \bigg|_{z=0} = \|a\|$.

Exercise 6.6. Let A be a unital Banach algebra over $\mathbb{C}$ and let a, $b \in A$. Use the identity $(ab)^n = a(ba)^{n-1}b$ to prove that ab and ba have the same spectral radius.

Note that

$$\begin{aligned} \|(ab)^n\|^{1/n} &\leqslant \|a\|^{1/n} \|(ba)^{n-1}\|^{1/n} \|b\|^{1/n} \\ &= \|a\|^{1/n} \big(\|(ba)^{n-1}\|^{1/(n-1)}\big)^{(n-1)/n} \|b\|^{1/n} \;\to\; \nu(ba) \end{aligned}$$

and so $\nu(ab) \leqslant \nu(ba)$. Exchanging the rôles of a and b gives the result.

Exercise 6.7. Let A be a unital Banach algebra over $\mathbb{C}$. Suppose that there exists $K > 0$ such that $\|a\| \leqslant K\nu(a)$ for all $a \in A$, where $\nu(a)$ denotes the spectral radius of a. Prove that A is commutative.

The function $g \colon \mathbb{C} \to A$; $\lambda \mapsto e^{\lambda a} b e^{-\lambda a}$ is holomorphic everywhere and bounded, because

$$\|g(\lambda)\| \leqslant K\nu(e^{\lambda a} b e^{-\lambda a}) = K\nu(b e^{\lambda a} e^{-\lambda a}) = K\nu(b).$$

Hence g is constant, so $0 = g'(0) = ag(0) - g(0)a = ab - ba$, as required.

Exercise 6.8. Let A be a Banach algebra and suppose that $(x_p)_{p \in P}$, $(y_q)_{q \in Q} \subseteq A$ are absolutely summable. Prove that

$$\sum_{p \in P} \sum_{q \in Q} x_p y_q = \sum_{(p,q) \in P \times Q} x_p y_q = \sum_{q \in Q} \sum_{p \in P} x_p y_q.$$

We prove only the first equality; the second follows by symmetry. Let $\varepsilon > 0$ and note that $(x_p y_q)_{(p,q) \in P \times Q}$ is absolutely summable (as $\|x_p y_q\| \leqslant \|x_p\|\,\|y_q\|$) so there exist finite sets $R_0 \subseteq P$, $S_0 \subseteq Q$ and $T_0 \subseteq P \times Q$ such that

$$\Big\|\sum_{p \in R} x_p - \sum_{p \in P} x_p\Big\| < \frac{\varepsilon}{1 + \|y\|_1} \qquad \text{for all finite } R \subseteq P \text{ such that } R \supseteq R_0$$

(where $\|y\|_1 = \sum_{q \in Q} \|y_q\|$)

$$\Big\|\sum_{q \in S} y_q - \sum_{q \in Q} y_q\Big\| < \frac{\varepsilon}{1 + \|x\|_1} \qquad \text{for all finite } S \subseteq Q \text{ such that } S \supseteq S_0,$$

(where $\|x\|_1 = \sum_{p\in P} \|x_p\|$) and

$$\Big\| \sum_{(p,q)\in T} x_p y_q - \sum_{(p,q)\in P\times Q} x_p y_q \Big\| < \varepsilon \qquad \text{for all finite } T \subseteq P\times Q \text{ such that } T \supseteq T_0.$$

If $R \subseteq P$ and $S \subseteq Q$ are finite sets such that $T = R \times S \supseteq T_0 \cup (R_0 \times S_0)$ then

$$\begin{aligned}
&\Big\| \sum_{(p,q)\in P\times Q} x_p y_q - \sum_{p\in P} x_p \sum_{q\in Q} y_q \Big\| \\
&\quad \leqslant \Big\| \sum_{(p,q)\in P\times Q} x_p y_q - \sum_{(p,q)\in T} x_p y_q \Big\| \\
&\qquad\qquad + \sum_{p\in R} \|x_p\| \, \Big\| \sum_{q\in S} y_q - \sum_{q\in Q} y_q \Big\| + \Big\| \sum_{p\in R} x_p - \sum_{p\in P} x_p \Big\| \, \|y\|_1 \\
&\quad < \varepsilon + \varepsilon + \varepsilon.
\end{aligned}$$

As $\varepsilon > 0$ is arbitrary, we have the result.

Solutions to Exercises 7

Exercise 7.1. Let $A = C(X)$, where X is a compact, Hausdorff space. Prove that the map $\epsilon\colon X \to \Phi(A);\ x \mapsto \epsilon_x$ is a homeomorphism, where $\epsilon_x(f) = f(x)$ for all $x \in X$ and $f \in C(X)$.

Note that ϵ is continuous if and only if $\hat{f} \circ \epsilon\colon x \mapsto \epsilon_x(f) = f(x)$ is continuous for all $f \in A$, by definition of the Gelfand topology. Furthermore, if x, $y \in X$ are distinct then there exists $f \in A$ such that $f(x) \neq f(y)$, i.e., $\epsilon_x(f) \neq \epsilon_y(f)$, by Urysohn's lemma. As X is compact and $\Phi(A)$ is Hausdorff, ϵ is a homeomorphism between X and $\epsilon(X)$; it remains to prove that ϵ is surjective.

If $\phi \in \Phi(A) \setminus \epsilon(X)$ then for all $x \in X$ there exists $f \in A$ such that $f \in \ker\phi$ but $f \notin \ker\epsilon_x$ (i.e., $f(x) \neq 0$). By compactness there exist $f_1, \ldots, f_n \in \ker\phi$ such that $\bigcup_{i=1}^n \{x \in X : f_i(x) \neq 0\} = X$, but then $f := \sum_{i=1}^n \bar{f}_i f_i \in \ker\phi$ and $f(x) > 0$ for all $x \in X$, whence $\ker\phi = A$, a contradiction.

Exercise 7.2. Prove that if A is a unital Banach algebra generated by a single element (i.e., there exists $a \in A$ such that $\{p(a) : p(z) \in \mathbb{C}[z]\}$ is dense in A) then $\Phi(A)$ is homeomorphic to $\sigma(a)$.

Note that $\hat{a}\colon \Phi(A) \to \sigma(a)$ is a continuous function from a compact space onto a Hausdorff space. If ϕ, $\psi \in \Phi(A)$ are such that $\hat{a}(\phi) = \hat{a}(\psi)$ then $\phi(p(a)) = \psi(p(a))$ for any complex polynomial p (as $\phi(1) = 1$ and $\phi(a^n) = \phi(a)^n$ for all $n \in \mathbb{N}$ so $\phi(p(a)) = p(\phi(a)) = p(\hat{a}(\phi))$). Hence $\phi = \psi$, by continuity and the density of $\{p(a) : p(z) \in \mathbb{C}[z]\}$ in A, so $\hat{a}$ is a homeomorphism.

Deduce that $\Phi(A)$ is homeomorphic to $\bar{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leqslant 1\}$ if $A = A(\mathbb{D})$ is the disc algebra.

If $f \in A(\mathbb{D})$ then f is uniformly continuous, so if $\varepsilon > 0$ there exists $r \in (0,1)$ such that $|f(z) - f(rz)| < \varepsilon/2$ for all $z \in \bar{\mathbb{D}}$ (because $|z - rz| \leqslant 1 - r$). Since $z \mapsto f(rz)$ is holomorphic on $\{z \in \mathbb{C} : |z| < r^{-1}\}$, its Taylor series is uniformly convergent on $\bar{\mathbb{D}}$, so there exists a polynomial p such that $|p(z) - f(rz)| < \varepsilon/2$ for all $z \in \bar{\mathbb{D}}$. Hence $\|f - p\|_\infty < \varepsilon$ and polynomials in $\mathrm{id}\colon \bar{\mathbb{D}} \to \mathbb{C};\ z \mapsto z$ are dense in A.

Exercise 7.3. Let A be a unital Banach algebra that is generated by one element, a, and let $\lambda \notin \sigma(a)$. Show there exists a sequence of polynomials $(p_n)_{n\geqslant 1}$ such that $p_n(z) \to (\lambda - z)^{-1}$ uniformly for all $z \in \sigma(a)$.

Since $\lambda \notin \sigma(a)$, the inverse $(\lambda 1 - a)^{-1}$ is an element of A, so there exists a sequence of polynomials $(p_n)_{n\geqslant 1}$ such that $p_n(a) \to (\lambda 1 - a)^{-1}$. If $z \in \sigma(a)$ then, by Exercise 7.2, there exists $\phi \in \Phi(A)$ such that $\phi(a) = z$; since ϕ is a unital algebra homomorphism, it follows that

$$|p_n(z) - (\lambda - z)^{-1}| = |\phi(p_n(a) - (\lambda 1 - a)^{-1})| \leqslant \|p_n(a) - (\lambda 1 - a)^{-1}\|$$

and therefore $p_n(z) \to (\lambda - z)^{-1}$ uniformly on $\sigma(a)$ as $n \to \infty$.

Deduce that the complement of $\sigma(a)$ is connected.

Suppose for contradiction that $\mathbb{C} \setminus \sigma(a)$ has a maximally connected component, C, which is bounded (and non-empty); fix $\lambda \in C$ and choose $(p_n)_{n\geqslant 1}$ as above. Note first that $\partial C := \overline{C} \setminus C^\circ$, the (topological) boundary of C, is contained in $\sigma(a)$: if $x \in \overline{C} \cap (\mathbb{C} \setminus \sigma(a))$ then there exist a sequence $(x_n)_{n\geqslant 1} \subseteq C$ such that $x_n \to x$ and $\varepsilon > 0$ such that $B_\varepsilon^{\mathbb{C}}(x) \cap \sigma(a) = \emptyset$, as $\sigma(a)$ is closed. Hence $B_\varepsilon^{\mathbb{C}}(x) \cap C \neq \emptyset$ and therefore $B_\varepsilon^{\mathbb{C}}(x) \subseteq C$, as C is maximally connected, so $x \in C^\circ$. It follows that C is a bounded region in $\mathbb{C}$ (i.e., an open, connected set): if $x \in C$ then $x \notin \sigma(a)$, so $x \notin \partial C$ and thus $x \in C^\circ$.

By the maximum-modulus theorem [16, Theorem 5.20],

$$\begin{aligned}\sup\{|p_n(z) - p_m(z)| : z \in \overline{C}\} &= \sup\{|p_n(z) - p_m(z)| : z \in \partial C\} \\ &\leqslant \sup\{|p_n(z) - p_m(z)| : z \in \sigma(a)\} \to 0\end{aligned}$$

as $m, n \to \infty$, so $(p_n)_{n\geqslant 1}$ is uniformly convergent on $\overline{C}$ and its limit, f, is continuous there and holomorphic in C. Then $g\colon z \mapsto (\lambda - z)f(z) - 1$ is holomorphic in C and $g(z) = 0$ for all $z \in \partial C$, so $g \equiv 0$, by the identity theorem, contradicting the fact that $g(\lambda) = -1$.

Exercise 7.4. Let A be a commutative, unital Banach algebra. Prove that the Gelfand transform on A is isometric if and only if $\|a^2\| = \|a\|^2$ for all $a \in A$.

If $\|a^2\| = \|a\|^2$ then $\|a^{2^n}\| = \|a\|^{2^n}$ for all $n \in \mathbb{N}$, and so

$$\|\hat{a}\|_\infty = \nu(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \lim_{n\to\infty} \|a^{2^n}\|^{1/2^n} = \|a\|.$$

Conversely, if $\nu(a) = \|\hat{a}\|_\infty = \|a\|$ then there exists $\lambda \in \sigma(a)$ such that $|\lambda| = \|a\|$. Hence $\lambda^2 \in \sigma(a^2)$ (by the spectral-mapping theorem for polynomials) and so

$$\|a^2\| \leqslant \|a\|^2 = |\lambda|^2 = |\lambda^2| \leqslant \nu(a^2) \leqslant \|a^2\|.$$

Exercise 7.5. Let A be a Banach algebra and B a semisimple, commutative, unital Banach algebra. Prove that if $\phi\colon A \to B$ is a homomorphism then ϕ is continuous.

By the closed-graph theorem, it suffices to take a sequence $(a_n)_{n\geqslant 1} \subseteq A$ such that $a_n \to a$ and $\phi(a_n) \to b$ and prove that $b = \phi(a)$. Let $\psi \in \Phi(B)$ and note that ψ and $\psi \circ \phi$ are continuous, by Proposition 6.1; note that ϕ extends to A^{u} by setting $\phi(\alpha 1 + a) = \alpha + \phi(a)$ so we may assume without loss of generality that A is unital. Hence

$$\psi(b) = \lim_{n\to\infty} \psi(\phi(a_n)) = \psi(\phi(a))$$

for all $\psi \in \Phi(B)$, and therefore $b - \phi(a) \in J(B)$, which gives the result.

Exercise 7.6. Let $A = C^1[0,1]$, equipped with the norm $\|f\| := \|f\|_\infty + \|f'\|_\infty$. Prove that A is a semisimple, commutative, unital Banach algebra and find its character space.

To prove completeness, let $(f_n)_{n\geqslant 1} \subseteq A$ be Cauchy, which implies that $(f_n)_{n\geqslant 1}$ and $(f_n')_{n\geqslant 1}$ are Cauchy, so convergent, sequences in $C[0,1]$. Let $f = \lim_{n\to\infty} f_n$, $g = \lim_{n\to\infty} f_n'$ and note that

$$\int_0^t g = \int_0^t \lim_{n\to\infty} f_n' = \lim_{n\to\infty} \int_0^t f_n' = \lim_{n\to\infty} f_n(t) - f_n(0) = f(t) - f(0),$$

so $f'(t) = g(t)$ for all $t \in [0,1]$ and

$$\|f - f_n\| = \|f - f_n\|_\infty + \|f' - f_n'\|_\infty = \|f - f_n\|_\infty + \|g - f_n'\|_\infty \to 0,$$

as required.

To see that A is semisimple, note that $\Phi(A) = \{\varepsilon_x : x \in X\}$ (which may be proved as for $C(X)$), so $J(A) = \bigcap_{x\in X}\{f \in A : f(x) = 0\} = \{0\}$.

Prove that $I = \{f \in A : f(0) = f'(0) = 0\}$ is a closed ideal in A such that A/I is a two-dimensional algebra with one-dimensional radical.

Note that

$$j\colon C^1[0,1] \to M_2(\mathbb{C});\ f \mapsto \begin{pmatrix} f(0) & f'(0) \\ 0 & f(0) \end{pmatrix}$$

is a continuous algebra homomorphism with kernel I and 2-dimensional image. Every linear functional on this image must have the form

$$\phi\colon \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mapsto \lambda a + \mu b \qquad \forall\, a, b \in \mathbb{C},$$

where λ, $\mu \in \mathbb{C}$. If $\phi(1) = 1$ then $\lambda = 1$, and multiplicativity forces $\mu = 0$, so A/I has one maximal ideal (viz. $\mathrm{id} + I$).

What do you notice about A and A/I?

This example shows that a quotient of a semisimple algebra need not be semisimple.

Exercise 7.7. Prove that the Banach space $\ell^1(\mathbb{Z})$ is a commutative, unital Banach algebra when equipped with the multiplication

$$a \star b\colon \mathbb{Z} \to \mathbb{C};\ n \mapsto \sum_{m\in\mathbb{Z}} a_m b_{n-m} \qquad \big(a, b \in \ell^1(\mathbb{Z})\big).$$

Note that $\sum_{n\in\mathbb{Z}} |b_n| < \infty$ implies that $(b_n)_{n\in\mathbb{Z}}$ is bounded, and so if a, $b \in \ell^1(\mathbb{Z})$ then $\sum_{m\in\mathbb{Z}} a_m b_{n-m}$ is (absolutely) summable for all $n \in \mathbb{Z}$. Furthermore,

$$\begin{aligned} \|a\|_1\|b\|_1 &= \sum_{m\in\mathbb{Z}} |a_m| \sum_{n\in\mathbb{Z}} |b_n| = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |a_m|\,|b_n| \\ &= \sum_{m\in\mathbb{Z}}\sum_{p\in\mathbb{Z}} |a_m|\,|b_{p-m}| \geqslant \sum_{p\in\mathbb{Z}} |(a\star b)_p| = \|a\star b\|_1 \end{aligned}$$

which shows that $a \star b \in \ell^1(\mathbb{Z})$ and $\|\cdot\|_1$ is submultiplicative on $\ell^1(\mathbb{Z})$. (To see that the reversal of order is valid, note that $\sum_{r\in R}\sum_{s\in S} x_r y_s = \sum_{(r,s)\in R\times S} x_r y_s$ for any absolutely summable $(x_r)_{r\in R}$, $(y_s)_{s\in S}$ in a Banach algebra.) Furthermore,

$$(a\star b)_n = \sum_{m\in\mathbb{Z}} a_m b_{n-m} = \sum_{p\in\mathbb{Z}} b_p a_{n-p} = (b\star a)_n$$

so $\star$ is commutative; bilinearity is readily verified, and if a, b, $c \in \ell^1(\mathbb{Z})$ and $p \in \mathbb{Z}$ then

$$\begin{aligned} \big((a\star b)\star c\big)_p &= \sum_{n\in\mathbb{Z}}\Big(\sum_{m\in\mathbb{Z}} a_m b_{n-m}\Big) c_{p-n} = \sum_{m\in\mathbb{Z}} a_m \sum_{n\in\mathbb{Z}} b_{n-m} c_{p-n} \\ &= \sum_{m\in\mathbb{Z}} a_m \sum_{r\in\mathbb{Z}} b_r c_{p-m-r} = \sum_{m\in\mathbb{Z}} a_m (b\star c)_{p-m} = \big(a\star(b\star c)\big)_p. \end{aligned}$$

If $1 \in \ell^1(\mathbb{Z})$ is defined by setting $1_0 = 1$ and $1_n = 0$ for all $n \in \mathbb{Z}\setminus\{0\}$ then $1 \in \ell^1(\mathbb{Z})$, $\|1\|_1 = 1$ and

$$(1\star a)_n = \sum_{m\in\mathbb{Z}} 1_m a_{n-m} = a_n \qquad \forall\, n\in\mathbb{Z},\ a\in\ell^1(\mathbb{Z}).$$

Exercise 7.8. Let $\delta \in \ell^1(\mathbb{Z})$ be such that $\delta_1 = 1$ and $\delta_n = 0$ if $n \neq 1$. Prove that $a = \sum_{n\in\mathbb{Z}} a_n \delta^n$ for all $a \in \ell^1(\mathbb{Z})$.

If $\varepsilon > 0$ choose a finite set $A_0 \subseteq \mathbb{Z}$ such that $\sum_{n\in A_0} |a_n| > \|a\|_1 - \varepsilon$ and note that

$$\|a - \sum_{n\in A} a_n\delta^n\|_1 = \sup\Big\{ \sum_{n\in B\setminus A} |a_n| : B \text{ is a finite subset of } \mathbb{Z}\Big\} = \|a\|_1 - \sum_{n\in A} |a_n| < \varepsilon$$

for any finite set $A \subseteq \mathbb{Z}$ such that $A \supseteq A_0$.

Deduce that the character space of $\ell^1(\mathbb{Z})$ is homeomorphic to $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and (with this identification) the Gelfand transform on $\ell^1(\mathbb{Z})$ is the map

$$\Gamma\colon \ell^1(\mathbb{Z}) \to C(\mathbb{T});\ \Gamma(a)(\lambda) = \sum_{n\in\mathbb{Z}} a_n \lambda^n \qquad \forall\, \lambda \in \mathbb{T},\ a \in \ell^1(\mathbb{Z}).$$

Let $\phi \in \Phi(\ell^1(\mathbb{Z}))$ and note that $\phi(\delta^n) = \phi(\delta)^n$ for all $n \in \mathbb{Z}$, so

$$\phi(a) = \sum_{n\in\mathbb{Z}} a_n \phi(\delta^n) = \sum_{n\in\mathbb{Z}} a_n \lambda^n \qquad \forall\, a \in \ell^1(\mathbb{Z}),$$

where $\lambda = \phi(\delta)$; as ϕ is continuous, if $x_s \to x$ then $\phi(x_s) \to \phi(x)$. Furthermore, $\lambda \in \mathbb{T}$ because

$$|\phi(\delta)| \leqslant \|\phi\|\,\|\delta\|_1 = 1 \qquad \text{and} \qquad |\phi(\delta)^{-1}| = |\phi(\delta^{-1})| \leqslant \|\phi\|\,\|\delta^{-1}\|_1 = 1.$$

Conversely, if $\lambda \in \mathbb{T}$ then $\phi\colon a \mapsto \sum_{n\in\mathbb{Z}} a_n\lambda^n$ defines a character. To see this, note that this series is absolutely summable, ϕ is linear and bounded (because $|\phi(a)| \leqslant \|a\|_1$) and

$$\phi(\delta^m \star \delta^n) = \phi(\delta^{m+n}) = \lambda^{m+n} = \lambda^m\lambda^n = \phi(\delta^m)\phi(\delta^n) \qquad \forall\, m, n \in \mathbb{Z};$$

it follows by continuity that $\phi(a)\phi(b) = \phi(a \star b)$ for all a, $b \in \ell^1(\mathbb{Z})$. Hence

$$\hat{\delta}\colon \Phi(\ell^1(\mathbb{Z})) \to \mathbb{T};\ \phi \mapsto \phi(\delta)$$

is a continuous bijection from a compact space to a Hausdorff space, so a homeomorphism.

Finally, note that $f \in C\big(\Phi(\ell^1(\mathbb{Z}))\big)$ corresponds to $f \circ \hat{\delta}^{-1} \in C(\mathbb{T})$ and so the Gelfand map becomes

$$\Gamma\colon \ell^1(\mathbb{Z}) \to C(\mathbb{T});\ a \mapsto \hat{a} \circ \hat{\delta}^{-1},$$

i.e.,

$$\Gamma(a)(\lambda) = \big(\hat{a} \circ \hat{\delta}^{-1}\big)(\lambda) = \hat{a}\big(\hat{\delta}^{-1}(\lambda)\big) = \hat{\delta}^{-1}(\lambda)(a) = \sum_{n\in\mathbb{Z}} a_n\lambda^n.$$

Exercise 7.9. Let $f\colon \mathbb{T} \to \mathbb{C}$ be continuous and such that $\sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty$, where

$$\hat{f}(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})e^{-int}\,\mathrm{d}t \qquad (n \in \mathbb{Z}).$$

Prove that if $f(z) \neq 0$ for all $z \in \mathbb{T}$ then $g = 1/f\colon \mathbb{T} \to \mathbb{C};\ z \mapsto 1/f(z)$ satisfies $\sum_{n\in\mathbb{Z}} |\hat{g}(n)| < \infty$.

Exercise 7.8 implies that $A = \Gamma\big(\ell^1(\mathbb{Z})\big)$ consists of those $g \in C(\mathbb{T})$ such that

$$g(z) = \sum_{n\in\mathbb{Z}} a_n z^n \qquad (z \in \mathbb{T}) \qquad \text{and} \qquad \sum_{n\in\mathbb{Z}} |a_n| < \infty.$$

Since $\sum_{n\in\mathbb{Z}} a_n z^n$ is uniformly convergent on $\mathbb{T}$ for such g we see that

$$\hat{g}(m) = \sum_{n\in\mathbb{Z}} \frac{a_n}{2\pi} \int_{-\pi}^{\pi} e^{int} e^{-imt}\,\mathrm{d}t = a_m \quad (m \in \mathbb{Z})$$

and so A consists of those $g \in C(\mathbb{T})$ such that $\sum_{n\in\mathbb{Z}} |\hat{g}(n)| < \infty$. In particular, $f \in A$ and the condition $f(z) \neq 0$ for all $z \in \mathbb{T}$ is equivalent to the fact that $\phi(f) \neq 0$ for all $\phi \in \Phi(A)$, i.e., $0 \notin \sigma(f)$. Hence $1/f \in A$, as required.

Exercise 7.10. Let $\mathbb{T} = \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ and $A = \{f \in C(\bar{\mathbb{D}}) : f|_{\mathbb{T}} \in A(\mathbb{D})|_{\mathbb{T}}\}$: A consists of those continuous functions on the closed unit disc $\bar{\mathbb{D}}$ that agree on the unit circle $\mathbb{T}$ with a continuous function on $\bar{\mathbb{D}}$ that is holomorphic in $\mathbb{D}$.

[A corollary of the maximum-modulus theorem [16, Theorem 5.20] will be useful: if $f \in A(\mathbb{D})$ then $\|f\|_\infty := \sup\{|f(z)| : |z| \leqslant 1\} = \sup\{|f(z)| : |z| = 1\} =: \|f|_{\mathbb{T}}\|_\infty$.]

(i) Show that A is a Banach algebra when equipped with the supremum norm.

Let $(f_n)_{n\geqslant 1} \subseteq A$ be such that $f_n \to f \in C(\bar{\mathbb{D}})$. For $n \geqslant 1$ let $g_n \in A(\mathbb{D})$ be such that $f_n|_{\mathbb{T}} = g_n|_{\mathbb{T}}$ and note that

$$\|g_n - g_m\|_\infty = \|(g_n - g_m)|_{\mathbb{T}}\|_\infty = \|(f_n - f_m)|_{\mathbb{T}}\|_\infty \leqslant \|f_n - f_m\|_\infty \to 0$$

as $m, n \to \infty$, so $(g_n)_{n\geqslant 1}$ converges to $g \in A(\mathbb{D})$. Since

$$g|_{\mathbb{T}} = \lim_{n\to\infty} g_n|_{\mathbb{T}} = \lim_{n\to\infty} f_n|_{\mathbb{T}} = f|_{\mathbb{T}},$$

we see that $f \in A$, as required.

(ii) Prove that $I = \{f \in A : f|_{\mathbb{T}} = 0\}$ is a closed ideal in A and that $A = A(\mathbb{D}) \oplus I$.

Suppose that $f \in A$; if $g, h \in A(\mathbb{D})$ are such that $f|_{\mathbb{T}} = g|_{\mathbb{T}} = h|_{\mathbb{T}}$ then $\|g-h\|_\infty = \|(g-h)|_{\mathbb{T}}\|_\infty = 0$ and so $g = h$. Thus to every $f \in A$ there exists a unique $j(f) \in A(\mathbb{D})$ such that $f|_{\mathbb{T}} = j(f)|_{\mathbb{T}}$; the map

$$j\colon A \to A(\mathbb{D});\ f \mapsto j(f)$$

is a norm-decreasing algebra homomorphism (as $\|j(f)\|_\infty = \|f|_{\mathbb{T}}\|_\infty \leqslant \|f\|_\infty$). In particular, $I = \{f \in A : f|_{\mathbb{T}} = 0\} = \ker j$ is a closed ideal. Furthermore, if $f \in A$ then $f - j(f) \in I$ so $f \in I + A(\mathbb{D})$ and if $f \in A(\mathbb{D}) \cap I$ then $\|f\|_\infty = \|f|_{\mathbb{T}}\|_\infty = 0$; thus $A = A(\mathbb{D}) \oplus I$.

(iii) Prove that

$$i\colon I \to C_0(\mathbb{D});\ f \mapsto f|_{\mathbb{D}}$$

is an isometric isomorphism. Deduce that I^u is topologically isomorphic to $C(\dot{\mathbb{D}})$, where $\dot{\mathbb{D}}$ is the one-point compactification of $\mathbb{D}$.

If $f \in I$ and $\varepsilon > 0$ then

$$\{z \in \mathbb{D} : |f(z)| \geqslant \varepsilon\} = \{z \in \bar{\mathbb{D}} : |f(z)| \geqslant \varepsilon\}$$

is a closed subset of $\bar{\mathbb{D}}$ and is therefore compact; hence $f|_{\mathbb{D}} \in C_0(\mathbb{D})$. Since $\|f|_{\mathbb{D}}\|_\infty = \|f\|_\infty$ the map i is isometric; it remains to prove that i is surjective. Let $f \in C_0(\mathbb{D})$ and define $g\colon \bar{\mathbb{D}} \to \mathbb{C}$ by setting $g|_{\mathbb{D}} = f$ and $g|_{\mathbb{T}} = 0$. Let $U \subseteq \mathbb{C}$ be open; if $0 \notin U$ then $g^{-1}(U) = f^{-1}(U)$ is an open subset of $\mathbb{D}$, and therefore of $\bar{\mathbb{D}}$, so suppose that $0 \in U$. Then $g^{-1}(U) = f^{-1}(U) \cup \mathbb{T}$ and, since $0 \in U$, there exists $\varepsilon > 0$ such that $B_\varepsilon^{\mathbb{C}}(0) \subseteq U$. Thus $f^{-1}(U) \supseteq f^{-1}\big(B_\varepsilon^{\mathbb{C}}(0)\big)$ and $f^{-1}\big(B_\varepsilon^{\mathbb{C}}(0)\big)$ has compact complement, hence

$$\mathbb{D} \setminus f^{-1}(U) \subseteq \mathbb{D} \setminus f^{-1}\big(B_\varepsilon^{\mathbb{C}}(0)\big) \subseteq B_r^{\mathbb{C}}[0]$$

for some $r \in (0,1)$. From this we see that $f^{-1}(U) \supseteq \mathbb{D} \setminus B_r^{\mathbb{C}}[0]$ and therefore $g^{-1}(U) = f^{-1}(U) \cup \big(\bar{\mathbb{D}} \setminus B_r^{\mathbb{C}}[0]\big)$ is open. This shows that g is continuous, as required.

The deduction follows immediately from Exercise 6.2.

(iv) Prove that if $\phi \in \Phi(A)$ is such that $\ker\phi \supseteq I$ then $\phi = \tilde{\phi} \circ j$, where $\tilde{\phi} \in \Phi(A(\mathbb{D}))$. Deduce that $\phi = \varepsilon_z \circ j$ for some $z \in \bar{\mathbb{D}}$ (where $\varepsilon_z\colon A(\mathbb{D}) \to \mathbb{C};\ f \mapsto f(z)$).

Note that if f, $g \in A$ are such that $j(f) = j(g)$ then $f - g \in I$ so $\phi(f) = \phi(g)$. Hence

$$\tilde{\phi}\colon A(\mathbb{D}) \to \mathbb{C};\ j(f) \mapsto \phi(f)$$

is well defined. It is immediate that $\tilde{\phi}$ is an algebra homomorphism such that $\phi = \tilde{\phi} \circ j$, and since ϕ is non-zero, so is $\tilde{\phi}$. The deduction is immediate, as $\Phi(A(\mathbb{D})) = \{\varepsilon_z : z \in \bar{\mathbb{D}}\}$ by the solution to Exercise 7.2.

(v) Let $\bar{\mathbb{D}}_1$ and $\bar{\mathbb{D}}_2$ be two copies of the unit disc and let $S^2 = \bar{\mathbb{D}}_1 \cup \bar{\mathbb{D}}_2 \big/\!\sim$ be the sphere obtained by identifying each point on $\mathbb{T}_1 = \partial\mathbb{D}_1$ with the corresponding point on $\mathbb{T}_2 = \partial\mathbb{D}_2$. Define

$$T\colon S^2 \to \Phi(A);\ z \mapsto \begin{cases} \varepsilon_z \circ j & (z \in \bar{\mathbb{D}}_1), \\ \varepsilon_z & (z \in \bar{\mathbb{D}}_2) \end{cases}$$

and prove that this is a well-defined, continuous injection.

If $z \in \mathbb{T}_1 \cup \mathbb{T}_2$ then $\varepsilon_z(j(f)) = j(f)(z) = f(z) = \varepsilon_z(f)$ for all $f \in A$, so this is a good definition. As $\Phi(A)$ has the weak* topology, T is continuous if $\hat{f} \circ T$ is continuous for all $f \in A$, and since

$$\big(\hat{f} \circ T\big)(z) = \begin{cases} j(f)(z) & (z \in \bar{\mathbb{D}}_1), \\ f(z) & (z \in \bar{\mathbb{D}}_2), \end{cases}$$

this follows because $f|_{\mathbb{T}} = j(f)|_{\mathbb{T}}$. Since the identity function lies in $A(\mathbb{D})$, $\varepsilon_z \neq \varepsilon_w$ and $\varepsilon_z \circ j \neq \varepsilon_w \circ j$ for all distinct z, $w \in \bar{\mathbb{D}}$. Finally, to see that $\varepsilon_z \neq \varepsilon_w \circ j$ for all z, $w \in \mathbb{D}$, note that there exists $f \in I$ such that $f(z) = 1$ (by Urysohn's lemma) and so $\varepsilon_z(f) = 1 \neq 0 = (\varepsilon_w \circ j)(f)$. Hence T is an injection.

(vi) Prove that $\Phi(A)$ is homeomorphic to the sphere S^2.

If $\phi \in \Phi(A)$ is such that $\ker\phi \supseteq I$ then $\phi = \varepsilon_z \circ j$ for some $z \in \bar{\mathbb{D}}$. Otherwise $\phi|_I \neq 0$; note that

$$\dot{\phi}\colon I^{\mathrm{u}} \to \mathbb{C};\ f + \alpha 1 \mapsto \phi(f) + \alpha$$

is a character of $I^{\mathrm{u}} \cong C(\dot{\mathbb{D}})$ and so has the form ε_z for some $z \in \dot{\mathbb{D}}$. Since $\phi|_I = \dot{\phi}|_I$ is non-zero, $z \neq \infty$ and $\phi|_I = \varepsilon_z$ for some $z \in \mathbb{D}$. Choose $f \in I$ such that $f(z) = 1$ (such exists by Urysohn's lemma) and note that $f(g - \phi(g)1) \in I$ for all $g \in A$, so

$$g(z) - \phi(g) = \varepsilon_z\big(f(g - \phi(g)1)\big) = \phi\big(f(g - \phi(g)1)\big) = \phi(f)\phi(g - \phi(g)1) = 0,$$

i.e., $\phi(g) = \varepsilon_z(g)$. Hence T is surjective and we have the result (since $\Phi(A)$ and S^2 are both compact, Hausdorff spaces).

Solutions to Exercises A

Exercise A.1. Let $X = \ell^\infty$ and for all $n \in \mathbb{N}$ define $\delta_n \in X^*$ by setting $\delta_n\big((x_k)_{k\geqslant 1}\big) = x_n$. Prove that $(\delta_n)_{n\geqslant 1}$ has no weak*-convergent subsequence but that $(\delta_n)_{n\geqslant 1}$ has a weak*-convergent subnet.

Suppose for contradiction that (δ_n) has a weak*-convergent subsequence, say $(\delta_{n_k})_{k\geqslant 1}$. Define $x \in \ell^\infty$ by setting

$$x_l = \begin{cases} -1 & l = n_{2k-1}, \\ 1 & l = n_{2k}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\big(\delta_{n_k}(x_l)_{l\geqslant 1}\big)_{k\geqslant 1} = (-1, 1, -1, 1, \ldots)$ is not convergent, the desired contradiction.

For the second claim, note that $(\delta_n)_{n\geqslant 1} \subseteq X_1^*$, which is weak* compact by the Banach-Alaoglu theorem (Theorem 3.22). Hence the existence of a convergent subnet follows from the (generalised) Bolzano-Weierstrass theorem, Theorem A.5.

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