# Investigations in Two-Dimensional Arithmetic Geometry

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# Abstract

THIS thesis explores a variety of topics in two-dimensional arithmetic geometry, including the further development of I. Fesenko's adèlic analysis and its relations with ramification theory, model-theoretic integration on valued fields, and Grothendieck duality on arithmetic surfaces.

I. Fesenko's theories of integration and harmonic analysis for higher dimensional local fields are extended to an arbitrary valuation field $F$ whose residue field is a local field; applications to local zeta integrals are considered.

The integral is extended to $F^n$, where a linear change of variables formula is proved, yielding a translation-invariant integral on $GL_n(F)$.

Non-linear changes of variables and Fubini's theorem are then examined. An interesting example is presented in which imperfectness of a positive characteristic local field causes Fubini's theorem to unexpectedly fail.

It is explained how the motivic integration theory of E. Hrushovski and D. Kazhdan can be modified to provide a model-theoretic approach to integration on two-dimensional local fields. The possible unification of this work with A. Abbes and T. Saito's ramification theory is explored.

Relationships between Fubini's theorem, ramification theory, and Riemann-Hurwitz formulae are established in the setting of curves and surfaces over an algebraically closed field.

A theory of residues for arithmetic surfaces is developed, and the reciprocity law around a point is established. The residue maps are used to explicitly construct the dualising sheaf of the surface.

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[1] http://codeinthehole.com/tutorials/thesisfile/index.html

# Contents

CHAPTER 1

# Introduction

## 1.1 Background, motivation, and brief summary

### 1.1.1 Zeta and $L$-functions

Let $K$ be a global field; that is, either a number field with ring of integers $\mathcal{O}_K$, or the function field of a smooth, projective curve $C$ over a finite field. To each point $x$ of $C$ or prime $x$ of $\mathcal{O}_K$, one associates the non-archimedean local field

$$K_x = \operatorname{Frac} \widehat{\mathcal{O}_x},$$

and it is now well accepted that one ought to study $K$ via the family of these completions:

$$\begin{tikzcd} \mathbb{Q}_2 \arrow[dr, dash] & \mathbb{R} \arrow[d, dash] & \\ \mathbb{Q}_3 \arrow[r, dash] & \mathbb{Q} \arrow[r, dotted, dash] \arrow[dr, dotted, dash] \arrow[d, dash] \arrow[dl, dash] & {} \\ \mathbb{Q}_5 & \mathbb{Q}_7 & {} \end{tikzcd}$$

Of course, we have included the completion $\mathbb{R}$ of $\mathbb{Q}$ at infinity in our diagram, and in general we must consider the archimedean places $x$ of a number field $K$, from which we form archimedean local fields $K_x$.

The ring of *adèles* of $K$ is the restricted product of these local fields; i.e.

$$\mathbb{A}_K := \{(a_x)_x \in \prod_x K_x : a_x \in \widehat{\mathcal{O}_x} \text{ for almost all } x\},$$

where 'almost all' means 'all but finitely many', and we ignore this condition at the infinite places if $K$ is a number field, for then $\mathcal{O}_x$ does not exist. The ring of adèles may be easily topologised to become a locally compact, Hausdorff ring, and one then has available the powerful tools of the theory of locally compact, abelian groups, including harmonic analysis and Pontryagin duality. Using these tools, K. Iwasawa [Iwa92] and J. Tate [Tat67] independently proved in the '50s that the zeta function of $K$,

$$\zeta_K(s) := \prod_x (1 - |k(x)|^{-s})^{-1},$$

(the product does not include archimedean $x$), or more generally the twist of the zeta function by a Hecke character, has a meromorphic continuation to the entire complex plane and satisfies a functional equation which relates $\zeta_K(s)$ to $\zeta_K(1-s)$ in terms of arithmetic and geometric data such as the discriminant and genus.

Of course, these results on $\zeta_K$ were already known. In the case of curves over global fields, they are due to E. Artin, F. K. Schmidt, and O. Teichmüller (see [Roq02] [Roq04] [Roq06] for a historical survey); for number fields, E. Hecke. However, the Tate-Iwasawa method is notable not only for its elegance, efficiency (Hecke's original proof for twisted zeta functions of number fields, using higher dimensional theta functions, and described in [Neu99], is very technical) and unification of the geometric and arithmetic worlds, but also for essentially providing the foundation of the Langlands programme: it establishes for the algebraic group $GL_1$ the otherwise conjectural and mysterious Langlands correspondence.

The Hasse-Weil zeta function $\zeta_X$ may be defined for an arbitrary scheme $X$ of finite type by

$$\zeta_X(s) = \prod_{x \in X_0} (1 - |k(x)|^{-s})^{-1},$$

where $x$ runs over the closed points of $X$. This infinite product converges for $\Re(s) > \dim X$, and thereby defines an analytic function on that half-plane. If $X$ is a curve over a finite field, or $X = \operatorname{Spec} \mathcal{O}_K$ with $K$ a number field, then we recover the aforementioned $\zeta_K$. So long as $X$ is regular, $\zeta_X$ is conjectured to have a meromorphic continuation to the entire complex plane, and to satisfy a precise functional equation, formulated by J.-P. Serre [Ser65], which relates $\zeta_X(s)$ to $\zeta_X(\dim X - s)$ in terms of arithmetic and geometric invariants of $X$.

Schemes of finite type are either geometric or arithmetic. The first are varieties over a finite field, while the second are models over $\mathcal{O}_K$ of a variety over a number field $K$. For example, an arithmetic surface $X$ can be obtained by starting with a curve over $\mathbb{Q}$, removing denominators in the equations defining the curve, and then allowing these equations to define curves $X_p$ over $\mathbb{F}_p$ for all primes $p$, simply by reducing the coefficients of the equation; $X$ should be imagined as the family of curves $(X_p)_p$, together with the original curve over $\mathbb{Q}$.

When $X$ is a smooth, projective variety over a finite field (the geometric case), then Serre's conjectures follow from Weil's conjectures, proved by A. Grothendieck, P. Deligne, and A. Weil using the beautiful theory of étale cohomology. However, it is here that the arithmetic and geometric worlds part. The tools of étale cohomology fail to apply properly to arithmetic varieties, for various mathematical and metamathematical reasons. Establishing Serre's conjectures for the zeta function of an arithmetic variety is perhaps the most significant open problem in arithmetic geometry.

The zeta function $\zeta_X$ even of an arithmetic surface $X$ is a mysterious object. In fact, since

$$\zeta_X(s) = \prod_p \zeta_{X_p}(s),$$

this zeta function encodes not only the geometric data of every reduction $X_p$, but also the arithmetic structure of how the reductions $X_p$ vary with $p$. It is astonishing that each $\zeta_{X_p}$ satisfies a functional equation relating $s$ to $1-s$, while the conjectural functional equation for $\zeta$ relates $s$ to $2-s$. If the generic fibre of $X$ over the number field $K$ is an elliptic curve $E$, then

$$\zeta_X(s) \sim \frac{\zeta_K(s)\zeta_K(s-1)}{L_E(s)},$$

where $L_E(s)$ is the $L$-function of $E$ and $\sim$ means 'equal up to some less interesting factors'. The study of the main conjectural properties of $L_E$ thus becomes equivalent to the investigation of $\zeta_X$.

### 1.1.2 Two-dimensional local fields

I was once asked, in response to a description of my research, “Why two?”, to which I replied “Because it is smaller than three, but bigger than one.”. My interlocutor received this with great delight. Flippancy aside, I ought at least to justify the title of this thesis. Many new problems appear when passing from one-dimensional arithmetic geometry, which is the study of number fields, to the case of arithmetic surfaces, which is dimension two. In climbing then to dimension three, similar, not new, but *similar*, problems reoccur. Undoubtedly, if we master arithmetic surfaces then we shall understand how to generalise our techniques to higher dimensional arithmetic varieties. So we shall often focus on arithmetic surfaces for the sake of concreteness.

A *two-dimensional local field* is a complete, discrete valuation field $F$ whose residue field $\overline{F}$ is a usual local field (which can be a called a one-dimensional local field). The reader who harbours the slightest doubt toward our arguments in the previous paragraph should now formulate for himself the definition of an $n$-dimensional local field. The simplest example of a two-dimensional local field is $\mathbb{Q}_p((t))$ with residue field $\mathbb{Q}_p$.

Just as local fields are used to study the local properties of global fields, two-dimensional local fields may be used to study two-dimensional schemes, as we now explain. Begin with a two-dimensional, domain $A$ which is finitely generated over $\mathbb{Z}$, with fields of fractions $F$. Let $0 \lhd \mathfrak{p} \lhd \mathfrak{m} \lhd A$ be a chain of primes in $A$ and consider the following sequence of localisations and completions:

$$\begin{array}{ccccccccccccc} A & \rightsquigarrow & A_\mathfrak{m} & \rightsquigarrow & \widehat{A_\mathfrak{m}} & \rightsquigarrow & \left(\widehat{A_\mathfrak{m}}\right)_{\mathfrak{p}'} & \rightsquigarrow & \widehat{\left(\widehat{A_\mathfrak{m}}\right)_{\mathfrak{p}'}} & \rightsquigarrow & \left(\widehat{\left(\widehat{A_\mathfrak{m}}\right)_{\mathfrak{p}'}}\right)_0 & = & \mathrm{Frac}\left(\widehat{\left(\widehat{A_\mathfrak{m}}\right)_{\mathfrak{p}'}}\right) \\ &&&&&&&& \| &&&& \| \\ &&&&&&&& A_{\mathfrak{m},\mathfrak{p}} &&&& F_{\mathfrak{m},\mathfrak{p}} \end{array}$$

which we now explain in greater detail. It follows from excellence of $A$ that $\mathfrak{p}' := \mathfrak{p}\widehat{A_\mathfrak{m}}$ is a radical ideal of $\widehat{A_\mathfrak{m}}$; we then localise and complete at $\mathfrak{p}'$, and again use excellence to deduce that $0$ is a radical ideal in the resulting ring, i.e. $A_{\mathfrak{m},\mathfrak{p}}$ is reduced. The total field of fractions $F_{\mathfrak{m},\mathfrak{p}}$ is therefore isomorphic to a finite direct sum of fields, and each is a two-dimensional local field.

Geometrically then, let $X$ be a two-dimension scheme of finite type (i.e. a surface over a finite field, or an arithmetic surface). Fix a closed point $x \in X$, and a curve (= irreducible, one-dimensional subscheme) $y$ containing $x$. Carrying out the above procedure, with $A = \mathcal{O}_{X,x}$ and $\mathfrak{p}$ being the local equation for $y$ at $x$, we obtain a finite direct sum of two-dimensional local fields $F_{x,y}$. Two-dimensional adèlic theory aims to study $X$ via the family $(F_{x,y})_{x,y}$. Chapter 7 is an adèlic study of Grothendieck duality of an arithmetic surface over its base; the more familiar methods using cohomology groups are replaced by explicit calculations involving two-dimensional local fields.

Moreover, just as one-dimensional local fields allowed us to *simultaneously* study both number fields and curves over finite fields, we hope that two-dimensional adèlic theory can give a uniform approach to arithmetic and geometric surfaces.

### 1.1.3 Integration on two-dimensional local fields

We may now explain the main content of this thesis: integration on two-dimensional local fields. Since the Tate-Iwasawa method allows us to so rapidly deduce the main properties of zeta functions in dimension one, but the zeta function of an arithmetic surface remains so perplexing, it is natural to ask if the Tate-Iwasawa method can be extended. S. Bloch, K. Kato, A. Parshin, and J. Tate have all dreamt of such a theory; we quote Parshin:

> " For a long time the author has been advocating the following:
>
> **Problem.** Extend Tate-Iwasawa's analytic method to higher dimensions.
>
> The higher adèles were introduced exactly for this purpose.
>
> **Problem.** Develop a measure theory and harmonic analysis on $n$-dimensional local fields.
>
> Note that $n$-dimensional local fields are not locally compact topological spaces for $n > 1$ and by Weil's theorem the existence of the Haar measure on a topological group implies its locally compactness. "
>
> – A. Parshin, *Higher dimensional local fields and $L$-functions*, in [FK00]:

As Parshin observes, two-dimensional local fields are not locally compact (in any reasonable topology), and therefore the powerful theory of harmonic analysis which Tate and Iwasawa used is no longer available. This thesis contributes towards the development of a suitable replacement.

I. Fesenko [Fes03] [Fes05] [Fes06] was the first to seriously develop theories of integration and harmonic analysis on higher-dimensional local fields, and there was later work by H. Kim and K.-H. Lee [KL04] [KL05]. Chapter 2 first presents my reinterpretation and generalisation of Fesenko's local theories, and this is then used to study zeta functions on two-dimensional local fields.

### 1.1.4 Non-commutative theory

In the study a global field $K$, it is now understood that a great deal of arithmetic information is contained not only in the adèle ring $\mathbb{A}_K$ and the idèle group $\mathbb{A}_K^\times$, but also in $G(\mathbb{A}_K)$, where $G$ is a (suitable) algebraic group over $K$. In stepping from $\mathbb{A}_K$ to $G(\mathbb{A}_K)$ we will almost always find ourselves in the non-commutative world, and the old tools of harmonic analysis must be replaced by those of representation theory.

The most immediate non-commutative generalisation of Tate-Iwasasa theory is due to R. Godement and H. Jacquet [GJ72], who proved that the $L$-function $L(\pi, s)$ associated to an automorphic, cuspidal representation $\pi$ of $GL_n(\mathbb{A}_K)$ has a meromorphic continuation to the whole complex plane and satisfies the functional equation $L(\pi, s) = \varepsilon(\pi, s)L(\widetilde{\pi}, 1 - s)$. According to Langlands' conjectures, this $L$-function is nothing other than the $L$-function associated to a Galois representation of $K$.

To generalise Godement and Jacquet's work, the representation theory of $p$-adic groups, and all other aspects of the Langlands programme to higher dimensions, a necessary first step is to extend the integration theory on a higher dimensional local field $F$ to produce a translation invariant integral on $G(F)$, with $G$ an algebraic group. This has been previously studied by Kim and Lee [KL04] [KL05] for $GL_n$ and $SL_n$, and is the main motivation of chapters 3 and 4.

In chapter 3, the integration theory of chapter 2 is extended to $GL_n(F)$. This requires proving a linear change of variables formula for integrals on $F^n$. Chapter 4 then considers certain non-linear changes of variables which might appear when generalising the theory to other algebraic groups.

### 1.1.5 Model theory of valued fields: a historical overview

The art of using model theory to study valued fields was initiated by J. Ax and S. Kochen [AK65a] [AK65b] [AK66] [Ax67] and Y. Ershov. Ax and Kochen used elementary ultraproduct methods to study Artin's conjecture on solutions to homogeneous equations. A field $F$ is said to be $C_2$ if and only if every homogeneous equation in $F$ of degree $d$ in $> d^2$ variables has a zero in $F$. The field $\mathbb{F}_p((t))$ is $C_2$, and Artin conjectured that the same was true for $\mathbb{Q}_p$. If $(A_p)_{p\in P}$ is a collection of non-empty sets indexed by an infinite set $P$, and $\mathcal{U}$ is a non-principal ultrafilter on $P$, recall that the *ultraproduct* of the $(A_p)_p$ with respect to $\mathcal{U}$ is

$$\prod_{p\in I} A_p \Big/ \mathcal{U} = \prod_{p\in I} A_p \Big/ \sim,$$

where $\sim$ denotes the equivalence relation

$$(a_p)_p \sim (a'_p)_p \Leftrightarrow \{p \in P : a_p = a'_p\} \in \mathcal{U}.$$

If each $A_p$ has some additional structure (e.g is a ring, group, etc. or is equipped with an order, valuation, etc), then the same will usually be true of the ultraproduct. Ax and Kochen took an ultraproduct $\mathcal{U}$ on the set of rational primes, and proved that the valued fields

$$\prod_p \mathbb{Q}_p \Big/ \mathcal{U}, \qquad \prod_p \mathbb{F}_p((t)) \Big/ \mathcal{U}$$

are isomorphic. One may then appeal to Łoš' theorem [BS69, 5.§2], which states that an elementary statement concerning the structures $A_p$ is true in the ultraproduct if and only it is true for almost all $p$, where 'almost all' means 'on a set belonging to $\mathcal{U}$'. Since the notion of 'being $C_2$ for a fixed $d$' can be expressed by an elementary statement, they deduce that, for any fixed degree $d$, there is $P(d) > 0$ such that for all primes $p > P(d)$, any homogeneous equation in $\mathbb{Q}_p$ with $> d^2$ variables has a zero.

The next history of interest to us is the quantifier elimination result of A. Macintyre [Mac76] for the $p$-adics. Macintyre studied $\mathbb{Q}_p$ as a model of the language $\mathcal{L}_{\text{Mac}}$ which now bears his name, which is the language of rings equipped with additional unary predicates $(P_n)_{n\geq 2}$ denoting the set of $n^{\text{th}}$ powers. He proved that this language is sufficient to eliminate quantifiers in the theory of $\mathbb{Q}_p$. The power of Macintyre's result is that it provides explicit information about the definable subsets of $\mathbb{Q}_p$. J. Denef [Den84] extended this study by proving a cell decomposition result, giving even further insight into the structure of such sets, and used it to show that the the Igusa local zeta function

$$\zeta_{\text{Ig}}(f, s) = \int_{\mathbb{Z}_p^n} |f(x)|^s \, dx$$

is a rational function of $p^{-s}$. Here $f \in \mathbb{Z}_p[X_1, \dots, X_n]$, $|\cdot|$ denotes the $p$-adic absolute value, and $dx$ is a Haar measure on $\mathbb{Q}_p^n$.

This rationality had previously been established by J. Igusa (see [Igu00] for the proof) using the resolution of singularities of $p$-adic manifolds. The importance of Igusa's result lies in the following interpretation. Letting $N_m$ denote the number of zeros of $f$ in $(\mathbb{Z}_p/p^m\mathbb{Z}_p)^n$, it had been conjectured by Z. Borevich and I.Shafarevich [BS66, 1.§5.ex9] that the associated Poincaré series

$$P(T) = \sum_{m=1}^{\infty} N_m T^m$$

was a rational function of $T$; but straightforward manipulations reveal that this is the case if and only the local zeta function $\zeta_{\text{Ig}}(f, s)$ is a rational function of $p^{-s}$.

A remaining problem with Igusa's local zeta function was to suppose that $f$ had $\mathbb{Z}$ coefficients and study the behaviour of the zeta functions $\zeta_{\text{Ig},p}(f, s)$ as $p$ varies. The first results in this direction were obtained by J. Pas [Pas89], who generalised the quantifier elimination and cell-decomposition results for $\mathbb{Q}_p$ to the case of a Henselian valuation field of residue characteristic zero. Pas applied this cell-decomposition to the ultraproduct $\prod_p \mathbb{Q}_p/\mathcal{U}$ and used Łoš' theorem to describe the Denef-type decompositions required to evaluate $\zeta_{\mathrm{Ig},p}(f,s)$ in a manner independent of $p$ (at least, for $p$ large enough). The final conclusion was that the degrees of the denominators and numerators of the $\zeta_{\mathrm{Ig},p}(f,s)$ (as rational functions in $p^{-s}$) were bounded independently of $p$.

A stronger uniformity result which one might expect to be true would be that the $\zeta_{\mathrm{Ig},p}(f,s)$ would even be uniformly rational; that is, that there exists a rational function $Q(T) \in \mathbb{Q}(T)$ such that $\zeta_{\mathrm{Ig},p}(f,s) = Q(p^{-s})$ for $p$ sufficiently large. This, however, is false, and we offer the following 'explanation'. The structure of $\zeta_{\mathrm{Ig},p}(f,s)$ is essentially encoding ramification information about singularities related to $f$, or about the action of Frobenius on certain cohomology groups with varying $p$; but the arithmetic aspect of this data means that it is controlled not by polynomials, but rather by congruences. To give a specific example, take $f(X) = X^2+1$; if $\zeta_{\mathrm{Ig},p}(f,s)$ were to be uniformly rational for large $p$, then it would follow that there exists $Q_0(T) \in \mathbb{Q}(T)$ such that

$$Q_0(p) = \#\{x \in \mathbb{F}_p : x^2+1=0\}$$

for $p \gg 0$. But the number of solutions to $X^2+1=0$ in $\mathbb{F}_p$ is determined by $p \mod 4$, so this is absurd.

Motivic integration has rapidly developed since it was introduced by M. Kontsevich in a lecture at Orsay in 1995, and has been subjected to several reincarnations due first to R. Cluckers, J. Denef, and F. Loeser [CL08] [DL98] [DL01] [DL02b] [DL02a], and then by E. Hrushovski and D. Kazhdan [HK06] [HK08]. The Cluckers-Denef-Loeser theory basically gives a geometric interpretation and unification of integration over different $p$-adic fields. Whereas Pas deduced his uniformity result for Igusa zeta functions at different $p$ via cell-decomposition in residue characteristic zero, the fundamental idea of motivic integration is that it is not only more efficient, but even more insightful, to directly integrate in residue characteristic zero.

Hrushovski and Kazhdan developed their theory of motivic integration partly in order to obtain uniformity results for $p$-adic integrals over towers of ramified extensions of $\mathbb{Q}_p$, which was lacking from the Cluckers-Denef-Loeser theory. Their theory is an incredible sophistication and formalisation of the Ax-Kochen-Ershov principle, which states that the entire theory of a valued field of residue characteristic zero reduces to the theory of the value group and residue field.

Hrushovski and Kazhdan only briefly mention the problem of integration on two-dimensional local fields [HK06, §9.4], and I have struggled to understand their long and difficult paper for sometime (in fact, Ivan gave me a copy in my PhD interview!). The major difficulty is that in usual motivic integration, the values of the integrals are varieties over the residue field, but in two-dimensional integration we wish to obtain Haar measurable sets over the residue field. With the kind help of Hrushovksi and Kazhdan during a trip a Jerusalem and subsequent ponderings while at Harvard, the main idea has become clear in recent months, and chapter 5 explains in detail how to apply their model theoretic techniques to two-dimensional integration. These results are only valid for two-dimensional local fields of residue characteristic zero, such as $\mathbb{Q}_p((t))$; extending this theory, as well as motivic integration, to finite residue characteristic is considered in subsection 6.1.3.

### 1.1.6 Ramification

One dimensional ramification theory, in which one studies the ramification properties of extensions of global and (one-dimensional) local fields, is a beautiful and complete theory (good references are [FV02] [Ser79] [Neu99]). Moreover, the passage from the local to the global is well understood, with global invariants typically expressed as products of the corresponding local invariants.

Extending the ramification theory to higher dimensional local fields, or, more generally, complete discrete valuation fields with imperfect residue field was, for a long time, a significant open problem. A theory has now been developed by A. Abbes and T. Saito [AS02] [AS03] using rigid geometry; some alternative approaches are due to J. Borger [Bor04b] [Bor04a], K. Kato [Kat89] [Kat94], and I. Zhukov [Zhu00] [Zhu03]. Subsection 6.1.2 provides a summary of Abbes and Saito's theory.

However, the global situation in higher dimensions remains mysterious, even for algebraic surfaces over a finite field. The Grothendieck-Ogg-Shafarevich formula for a curve expresses global information (the Euler characteristic of an $\ell$-adic sheaf on a dense open subset of the curve) in terms of the Euler characteristic of the curve and local ramification data. An open problem which has attracted many of the best arithmetic geometers including S. Bloch, P. Deligne, and K. Kato is that of finding a higher-dimensional generalisation. For arithmetic surfaces, partial results have been obtained by Saito [Sai91] for $\ell$-adic sheaves of dimension 1, using abelian ramification theory and two-dimensional class field theory, and by Abbes [Abb00], using the ramification theory he developed with Saito. Chapter 6 studies the Riemann-Hurwitz formula, which is a special case of Grothendieck-Ogg-Shafarevich, and investigates to what extent integration theory can be useful in understanding ramification.

In dimension one, the theories of Tate-Iwasawa and Godement-Jacquet capture ramification data such as the conductor using the properties of local zeta functions, and this was part of the motivation for studying the two-dimensional local zeta functions in chapter 2.

## 1.2 The writing and reading of this thesis

A few words on this thesis' history may be useful. The majority of my first year as a PhD student was occupied by the study of class field theory, automorphic representations, and model theory, the reading of various of Fesenko's papers, and research into higher-dimensional integration. This culminated in the writing of three papers [Mor08d] [Mor08c] [Mor08b], which, with only minor modifications (removal of introductions and summaries of earlier work, etc.) form chapters 2, 3, and 4.

I spent a significant portion of my second year learning Grothendieck-style algebraic geometry and motivic integration. Excluding section 6.1 on ramification, I wrote most of chapter 6 (as the paper [Mor08a]) during this period, while I was wondering about the importance of integration theory.

In my third year, thanks to the Cecil King Travel Scholarship, I visited the Institut des Hautes Études Scientifiques, Paris, for one month, the Hebrew University of Jerusalem for two weeks, and Harvard University, Boston, for six weeks. While at the IHÉS, C. Soulé suggested, as Fesenko had earlier, that an adèlic interpretation of duality was an interesting goal; although he had in mind $\ell$-adic duality, I was interested in Grothendieck duality at the time and this work is contained in chapter 7, which was not written in its final form until May 2009, initially as the article [Mor09].

Chapter 5 on model theoretic integration and most of section 6.1 were written during

June and July 2009 after, as I have already mentioned, a long personal battle with the subject.

The chapters were initially written as separate papers, and this will undoubtedly be clear to the astute reader. However, wishing not to frustrate the reader, I have removed duplicated material as far as is possible while simultaneously leaving the chapters largely independent. I hope that the reader notices a variation of mathematical maturity between chapters 2, 3, 4, 6 (minus 6.1) and chapters 5, 7, as they were written at least a year apart.

I would suggest to the reader that he begins with the introduction (where else?), including the summaries of the chapters and the basics of higher dimensional integration. Chapters 2, 3 and 4 could then be looked at briefly, to gain some intuition for two-dimensional integration. Sections 5.1, 6.1, 7.1 are quite discursive and therefore may be more enjoyable to read. The rest of chapter 5 is then probably only accessible to model theorists (sorry); the rest of chapters 6 and 7 are independent of the rest of the thesis (and of each other), and have a flavour closer to 'normal' algebraic/arithmetic geometry.

## 1.3 Detailed summaries

### 1.3.1 Chapter 2: Integration on valuation fields over local fields

Let $F$ be a valuation field with value group $\Gamma$ and ring of integers $\mathcal{O}_F$, whose residue field $\overline{F}$ is a non-discrete, locally compact field (i.e. a local field: $\mathbb{R}$, $\mathbb{C}$, or non-archimedean). Given a Haar integrable function $f:\overline{F}\to\mathbb{C}$, we consider the lift, denoted $f^{0,0}$, of $f$ to $\mathcal{O}_F$ by the residue map, as well as the functions of $F$ obtained by translating and scaling

$$x\mapsto f^{0,0}(\alpha x+a)$$

for $a\in F$, $\alpha\in F^\times$. We work with the space spanned by these function as $f$ varies. A simple linear independence result (proposition 2.1.5) is key to proving that an integral taking values in $\mathbb{C}\Gamma$ (the complex group algebra of $\Gamma$), under which $f^{0,0}$ has value $\int_{\overline{F}} f(u)\,du$, is well defined.

The integration yields a translation invariant measure, explained in section 2.2. For example, in the case of $\mathbb{C}((t))$, the set $St^n+t^{n+1}\mathbb{C}[[t]]$ is given measure $\mu(S)X^n$ in $\mathbb{R}[X,X^{-1}]$, where $S$ is a Lebesgue measurable subset of $\mathbb{C}$ of finite measure $\mu(S)$.

In section 2.3, the first elements of a theory of harmonic analysis are presented for fields which are self-dual in a certain sense. For this we must enlarge our space of integrable functions by allowing twists by a certain collection of additive characters; the central result is that the integral has a *unique* translation-invariant extension to this larger class of functions. A Fourier transform may then be defined in the usual way; a double transform formula is proved.

The short section 2.4 explains integration on the multiplicative group of $F$. Here we generalise the relationship $d\overset{\times}{x}=|x|^{-1}d^+x$ between the multiplicative and additive Haar measures of a local field.

If $F$ is a higher dimensional local field then the main results of the aforementioned sections reduce to results of Fesenko in [Fes03] and [Fes06]. However, the results here are both more general and abstract; in particular, if $\overline{F}$ is archimedean then we provide proofs of claims in [Fes03] regarding higher dimensional archimedean local fields, and whereas those papers work with complete fields, we require no topological conditions. This more abstract approach to the integration theory appears to be powerful; we will use it to deduce the existence of a translation invariant integral on $\mathrm{GL}_n(F)$ in chapter 3 and prove Fubini's theorem for certain repeated integrals over $F \times F$ in chapter 4.

In the final sections of the chapter, we consider various zeta integrals. In section 2.5, parts of the theory of local zeta integrals over $\overline{F}$ are lifted to $F$. In doing so we are led to consider certain divergent integrals related to quantum field theory and we suggest a method of obtaining epsilon constants from such integrals.

We then consider zeta integrals over the local field $\overline{F}$; a 'two-dimensional' Fourier transform $f \mapsto f^*$ is defined (following Weil [Wei95] and [Fes03] in the non-archimedean case) and we prove, following the approaches of Tate and Weil, that it leads to a local functional equation, with appropriate epsilon factor, with respect to $s$ goes to $2-s$:

$$Z(g^*, \omega^{-1}, 2-s) = \varepsilon_*(\omega, s) Z(g, \omega, s).$$

See proposition 2.6.17 for precise statements. After explicitly calculating some $^*$-transforms we use this functional equation to calculate the $^*$-epsilon factors for all quasi-characters $\omega$. These results on zeta integrals and epsilon factors are then used to prove that $^*$ is an automorphism of the Schwartz-Bruhat space $\mathcal{S}(\overline{F})$, which, though important, appears not to have been considered before. When $\overline{F}$ is archimedean we define a new $^*$-transform and consider some examples.

In section 2.7, zeta integrals over the two-dimensional local field $F$ are considered following [Fes03]. Lacking a measure theory on the topological $K$-group $K_2^{\mathrm{top}}(F)$ (the appropriate object for class field theory of $F$; see [Fes91]), a zeta integral over (a subgroup of) $F^\times \times F^\times$ is considered:

$$\zeta(f, \chi, s) = \int^{F^\times \times F^\times} f(x, y)\, \chi \circ \mathfrak{t}(x, y) |\mathfrak{t}(x, y)|^s \ \mathrm{char}_T(x, y)\, d\overset{\times}{x} d\overset{\times}{y}.$$

Meromorphic continuation and functional equation are established for certain 'tame enough' quasi-characters; in these cases the functional equation, and explicit L-functions and epsilon factors, follow from properties of the $^*$-transform on $\overline{F}$. Our results are compared with [Fes03].

The advantages of our new approach to the integration theory are apparent in these chapters on local zeta integrals. Our approach is to lift known results up from the local field $\overline{F}$, rather than try to generalise the proof for a local field to the two-dimensional field. For example, we therefore immediately know that many of our local zeta functions have meromorphic continuation. Apparently complicated integrals on $F$ reduce to familiar integrals over $\overline{F}$ where manipulations are easier; for example, we may work at the level of $\overline{F}$ even though we are calculating epsilon factors for *two-dimensional* zeta integrals. The weakness is that it does not seem to allow much wild ramification information to be obtained.

The appendices are used to discuss some results which would otherwise interrupt the chapter. Firstly, the set-theoretic manipulations in [Fes03] (used to prove that the measure is well-defined) are reproved here more abstractly. Secondly we discuss what we mean by a holomorphic function taking values in a complex vector space; this allows us to discuss meromorphic continuation of our zeta functions.

### 1.3.2 Chapter 3: Integration on product spaces and $GL_n$ of a valuation field over a local field

As discussed above, to generalise the non-commutative theory of local and global fields to higher dimensions, and particularly to generalise Godement-Jacquet theory, one must first develop a translation-invariant integration theory on $GL_n$ of higher dimensional local fields. That is the subject of this chapter.

$F$, $\Gamma$, etc. continue to be as in chapter 2. In section 3.1, the integral on $F$ developed in chapter 3 is extended to $F^n$ using repeated integration. So that Fubini's theorem holds, we consider $\mathbb{C}(\Gamma)$-valued functions $f$ on $F^n$ such that for any permutation $\sigma$ of $\{1,\dots,n\}$ the repeated integral

$$\int^F \dots \int^F f(x_1,\dots,x_n)\,dx_{\sigma(1)}\dots\,dx_{\sigma(n)}$$

is well defined, and its value does not depend on $\sigma$; such a function is called Fubini.

Now suppose that $g$ is a Schwartz-Bruhat function on $\overline{F}^n$; let $f$ be the complex-valued function on $F^n$ which vanishes off $\mathcal{O}_F^n$, and satisfies

$$f(x_1,\dots,x_n) = g(\overline{x}_1,\dots,\overline{x}_n)$$

for $x_1,\dots,x_n \in \mathcal{O}_F$. $f$ is shown to be Fubini in the second section. In proposition 3.2.12 it is shown that if $a \in F$ and $\tau \in GL_n(F)$, then $x \mapsto f(a+\tau x)$ is also Fubini and

$$\int^{F^n} f(a+\tau x)\,dx = |\det\tau|^{-1}\int^{F^n} f(x)\,dx \tag{$*$}$$

where $|\cdot|$ is an absolute value on $F$. The main result of the third section, theorem 3.2.4, easily follows: there exists a space of Fubini functions $\mathcal{L}(F^n, GL_n)$ such that $\mathcal{L}(F^n, GL_n)$ is closed under affine changes of variable, with $(*)$ holding for $f \in \mathcal{L}(F^n, GL_n)$.

Next, just as in the classical case of a local field, we look at $\mathbb{C}(\Gamma)$-valued functions $\phi$ on $GL_n(F)$, for which $\tau \mapsto \phi(\tau)|\det\tau|^{-n}$ belongs to $\mathcal{L}(F^{n^2})$, having identified $F^{n^2}$ with the space of $n\times n$ matrices over $F$. This leads to an integral on $GL_n(F)$ which is left and right translation invariant, and which lifts the Haar integral on $GL_n(\overline{F})$ in a certain sense.

Finally we discuss extending the theory to the case of an arbitrary algebraic group.

### 1.3.3 Chapter 4: Fubini's theorem and non-linear changes of variables over a two-dimensional local field

This chapter considers the issue of Fubini's theorem and non-linear polynomial changes of variables for integration over a two-dimensional local field.

To extend the approach in chapter 3 from $GL_n$ to an arbitrary algebraic group it is necessary to have a theory of integration on finite dimensional vector spaces over $F$ which behaves well under certain *non-linear* changes of variable (for the $GL_n$ theory, linear changes of variable sufficed). Moreover, for use in applications, it is essential that Fubini's theorem concerning repeated integrals is valid. This chapter considers the problem of establishing whether the equality

$$\int^F\int^F g(x,y-h(x))\,dydx = \int^F\int^F g(x,y-h(x))\,dxdy$$

holds for appropriate functions $g$ and polynomials $h$. Moreover, the methods used appear to be suitable for changes of variables much more general than $(x,y)\mapsto(x,y-h(x))$.

The chapter begins by describing the action of polynomials on $F$. Given a polynomial $h \in \mathcal{O}_F[X]$, and a translated fractional ideal $b + t^A\mathcal{O}_F \subseteq \mathcal{O}_F$, we show how to write

$\{x \in \mathcal{O}_F : h(x) \in b + t^A\mathcal{O}_F\}$ as a finite disjoint union of translated fractional ideals; here $t$ is a local parameter of $F$ as a discrete valuation field. If $a + t^c\mathcal{O}_F$ is one of these translated fractional ideals, it is also important to understand the behaviour of the function

$$h : a + t^c\mathcal{O}_F \to b + t^A\mathcal{O}_F.$$

The impetus of this chapter is conjecture 4.2.1, which we rapidly reduce to the following: if $f$ is a Schwartz-Bruhat function on $\overline{F} \times \overline{F}$, $f^0 = f^{0,0}$ is the lift of $f$ to $F \times F$, and $h \in F[X]$ is a polynomial, then surely

$$\int^F \int^F f^0(x, y - h(x))\, dydx = \int^F \int^F f^0(x, y - h(x))\, dxdy.$$

In section 4.2 the conjecture is shown to be true if $h$ is linear or if all coefficients of $h$ belong to $\mathcal{O}_F$.

The technically difficult case of when $h$ contains coefficients not in $\mathcal{O}_F$ is taken up in the next section. Introduce a polynomial $q \in F[X]$ and integer $R < 0$ by the three conditions $h(X) = h(0) + t^R q(X)$, $q \in \mathcal{O}_F[X]$, and $q \notin t\mathcal{O}_F[X]$. We give explicit expressions for the integral of $\int^F f^0(x, y - h(x))\, dx$ in terms of the decomposition of sets of the form $\{x \in \mathcal{O}_F : q(x) \in b + t^{-R}\mathcal{O}_F\}$; the conjecture easily follows if $R = -1$ so long as $\overline{q}$, the image of $q$ in $\overline{F}[X]$, is not a purely inseparable polynomial. When $R < -1$ calculations become difficult, and the function $y \mapsto \int^F f^0(x, y - h(x))\, dx$ can fail to be integrable, meaning that the conjecture fails; however, we present examples suggesting that the space of integrable functions could be extended so as to remedy this deficit.

We then consider the possibility that $\overline{F}$ has positive characteristic and $\overline{q}$ is purely inseparable. When $R = -1$ it is shown that

$$\int^F \int^F f^0(x, y - h(x))\, dydx = \int_{\overline{F}} \int_{\overline{F}} f(x, y)\, dydx$$

but

$$\int^F \int^F f(x, y - h(x))\, dxdy = 0.$$

So if $f$ has non-zero Haar integral over $K \times K$ then the conjecture drastically fails. This fascinating result provides an explicit example to show that the work of Hrushovski and Kazhdan really can fail in positive characteristic, and we discuss its relationship with ramification theory.

In the final section we summarise the results obtained and discuss possible future work in this direction.

### 1.3.4 Chapter 5: Two-dimensional integration à la Hrushovski-Kazhdan

Here we explain how Hrushovski and Kazhdan's model theoretic integration theory can be applied to two-dimensional integration.

The first section describes the main results of the chapter without model theory, for the reader unversed in the discipline; since it is thoroughly explained there, with motivation, we say no more about it here.

After a section on the possible first order languages which can be used to describe valued fields, and recalling standard results on which theories admit the elimination of quantifiers in their languages, the main content of the chapter begins with section 5.3, in which we analyse definable sets in a valued field of residue characteristic zero. We work in a theory of valued fields which eliminates field quantifiers, and we allow arbitrary structure on the residue field so that we can later specialise to the case when the residue field is $\mathbb{Q}_p$ or $\mathbb{R}$, say. In particular, we establish that definable sets without any topological interior are necessarily contained inside a proper Zariski closed set; this seemingly technical result has many useful consequences. For example, we use it to deduce that definable functions are smooth away from a proper Zariski closed set.

We then recall the notion of V-minimality for a theory of valued fields, which plays an important role in [HK06]. Finally, we generalise, from the algebraically closed situation to the case of a two-dimensional local field, Hrushovski and Kazhdan's main decomposition result which states that any definable subset of the valued field is isomorphic to lifts of sets from the residue field and value group.

### 1.3.5 Chapter 6: Ramification, Fubini's theorem, and Riemann-Hurwitz formulae

This chapter grew from the author's attempt to understand better the role of integration, particular Fubini's theorem, in geometry and ramification theory. The first section is really a continuation of the previous chapter. We first outline a possible methodology for using model theory to understand the ramification theory of complete discrete valuation fields of Abbes and Saito, and then explain why this gives hope that it will be possible to unify the Hrushovksi-Kazhdan integration theory with ramification theory, thereby developing a motivic integration theory which is valid in finite characteristic.

The main part of the chapter then begins with a section reviewing the concept of an Euler characteristic for a first order structure in model theory. The discussion is purely algebraic for the benefit of readers unfamiliar with model theory, and various examples are given.

Once an Euler characteristic is interpreted as an integral, it is natural to ask whether Fubini's theorem holds; that is, whether the order of integration can be interchanged in a repeated integral. In the second section we consider finite morphisms between smooth curves over any algebraically closed field, and show that Fubini's theorem is almost equivalent to the Riemann-Hurwitz formula. More precisely, in characteristic zero the two are equivalent and so Fubini's theorem is satisfied, whereas in finite characteristic the possible presence of wild ramification implies that, for any Euler characteristic, interchanging the order of integration is not always permitted.

Section 6.4 discusses a notion weaker than the full Fubini property: a so-called strong Euler characteristic [Kra00] [KS00]. We show that over an algebraically closed field of characteristic zero, there is exactly one strong Euler characteristic (over the complex numbers, this is the usual topological Euler characteristic).

We then return to finite morphisms between algebraic varieties, this time considering surfaces. Again, Fubini's theorem is related to a Riemann-Hurwitz formula, originally due to Iversen [Ive70]. Our methods provide a new proof of his result and we discuss the situation in finite characteristic.

### 1.3.6 Chapter 7: An explicit approach to residues on and canonical sheaves of arithmetic surfaces

This chapter studies arithmetic surfaces using two-dimensional local fields associated to the scheme, and thus further develops the adèlic approach to higher dimensional algebraic and arithmetic geometry. We study residues of differential forms and give an explicit construction of the dualising sheaf. While considerable work on these topics has been done for varieties over perfect fields by Lipman, Lomadze, Parshin, Osipov, Yekutieli, et al., the arithmetic case has been largely ignored. The chapter begins with a discussion of its relation to this earlier work, where we provide extensive references.

In section 7.2 we consider a two-dimensional local field $F$ of characteristic zero and a fixed local field $K \leq F$. We introduce a relative residue map

$$\mathrm{Res}_F : \Omega_{F/K}^{\mathrm{cts}} \to K,$$

where $\Omega_{F/K}^{\mathrm{cts}}$ is a suitable space of 'continuous' relative differential forms. In the case $F \cong K((t))$, this is the usual residue map; but if $F$ is of mixed characteristic, then our residue map is new (though essentially contained in Fesenko's adèlic analysis and Osipov's study of algebraic surfaces - see subsections 7.1.3 and 7.1.6). Functoriality of the residue map is established with respect to a finite extension $F'/F$, i.e.

$$\mathrm{Res}_F \mathrm{Tr}_{F'/F} = \mathrm{Res}_{F'}.$$

In section 7.3 we prove the reciprocity law for two-dimensional local rings, justifying our definition of the relative residue map for mixed characteristic fields. For example, suppose $A$ is a characteristic zero, two-dimensional, normal, complete local ring with finite residue field, and fix the ring of integers of a local field $\mathcal{O}_K \leq A$. To each height one prime $y \lhd A$ one associates the two-dimensional local field $\mathrm{Frac}\,\widehat{A_y}$ and thus obtains a residue map $\mathrm{Res}_y : \Omega_{\mathrm{Frac}\,A/K} \to K$. We prove

$$\sum_y \mathrm{Res}_y\,\omega = 0$$

for all $\omega \in \Omega_{\mathrm{Frac}\,A/K}$. The subsequent section restates these results in the geometric language.

Next we turn to the study of the canonical sheaf of an arithmetic surface. In section 7.5 we recall various results about local complete intersection curves from a perspective suitable for our work. Section 7.6 establishes an important local ramification result, generalising a classical formula for the different of an extension of local fields. Let $B$ be a Noetherian, normal ring, and

$$A = B[T_1, \dots, T_m]/\langle f_1, \dots, f_m \rangle$$

a normal, complete intersection over $B$ which is a finitely generated $B$-module; assume that the corresponding extension $F/M$ of fraction fields is separable. Letting $J \in A$ be the determinant of the Jacobian matrix of $f_1, \dots, f_m$, we prove that

$$\{x \in F : \mathrm{Tr}_{F/M}(xA) \subseteq B\} = J^{-1}A.$$

In other words, the canonical and dualising sheaves of $A/B$ are the same. The proof reduces to the case when $A$, $B$ are complete discrete valuation rings with an inseparable residue field extension; for more on the ramification theory of complete discrete valuation fields with imperfect residue field, see the discussion above and references therein.

Finally, in section 7.7, we use our local residue maps and results on complete intersections to explicitly construct the dualising sheaf of an arithmetic surface. Let $\mathcal{O}_K$ be a Dedekind domain of characteristic zero with finite residue fields; its field of fractions is $K$. Let $\pi : X \to S = \mathrm{Spec}\,\mathcal{O}_K$ be a flat, surjective, local complete intersection, with smooth, connected, generic fibre of dimension 1. To each closed point $x \in X$ and integral curve $y \subset X$ containing $x$, our local residue maps define $\mathrm{Res}_{x,y} : \Omega^1_{K(X)/K} \to K_{\pi(x)}$ ($= \pi(x)$-adic completion of $K$), and we prove

**Theorem 1.3.1.** *The canonical sheaf of $X \to S$ is explicitly given by, for open $U \subseteq X$,*

$$\omega_{X/S}(U) = \{\omega \in \Omega_{K(X)/K} : \operatorname{Res}_{x,y}(f\omega) \in \widehat{\mathcal{O}}_{K,\pi(x)} \text{ for all } x \in y \subset U \text{ and } f \in \mathcal{O}_{X,y}\}$$

*where $x$ runs over all closed points of $X$ inside $U$ and $y$ runs over all curves containing $x$.*

## 1.4 Precise basics of higher dimensional integration

Having informally discussed the problem of higher dimensional integration, we should present a precise summary of the basics of the theory so that the reader knows what is ahead.

Let $F$ be a valuation field with value group $\Gamma$ and ring of integers $\mathcal{O}_F$, whose residue field $\overline{F}$ is a (one-dimensional) local field. We assume that the valuation splits, and fix a splitting $t : \Gamma \to F^\times$. $\mathbb{C}(\Gamma)$ denotes the field of fractions of the complex group algebra $\mathbb{C}\Gamma$ of $\Gamma$; the basis element of the group algebra corresponding to $\gamma \in \Gamma$ shall be written as $X^\gamma$ rather than as $\gamma$. We fix a choice of Haar measure on $\overline{F}$.

### 1.4.1 Integration on $F$

Here we summarise the integration theory which will be developed in sections 2.1 and 2.4 of chapter 2.

**Definition 1.4.1.** For $g$ a function on $\overline{F}$ taking values in an abelian group $A$, set

$$\begin{aligned} g^0 &: F \to A \\ &x \mapsto \begin{cases} g(\overline{x}) & x \in \mathcal{O}_F \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

More generally, for $a \in F$, $\gamma \in \Gamma$, the *lift of $g$ at $a, \gamma$* is the $A$-valued function on $F$ defined by

$$g^{a,\gamma}(x) = \begin{cases} g(\overline{(x-a)t(-\gamma)}) & x \in a + t(\gamma)\mathcal{O}_F \\ 0 & \text{otherwise} \end{cases}$$

Note that $g^{0,0} = g^0$ and $g^{a,\gamma}(a + t(\gamma)x) = g^0(x)$ for all $x \in F$.

**Definition 1.4.2.** Let $\mathcal{L}$ denote the space of complex-valued Haar integrable functions on $\overline{F}$. A *simple* function on $F$ is a $\mathbb{C}(\Gamma)$-valued function of the form

$$x \mapsto g^{a,\gamma}(x)\, X^\delta$$

for some $g \in \mathcal{L}$, $a \in F$, $\gamma, \delta \in \Gamma$.

Let $\mathcal{L}(F)$ denote the $\mathbb{C}(\Gamma)$ space of all $\mathbb{C}(\Gamma)$-valued functions spanned by the simple functions; such functions are said to be *integrable* on $F$.

**Remark 1.4.3.** Note that the space of integrable functions is the smallest $\mathbb{C}(\Gamma)$ space of $\mathbb{C}(\Gamma)$-valued functions on $F$ with the following properties:

(i) If $g \in \mathcal{L}$, then $g^0 \in \mathcal{L}(F)$.

(ii) If $f \in \mathcal{L}(F)$ and $a \in F$ then $\mathcal{L}(F)$ contains $x \mapsto f(x + a)$.

(iii) If $f \in \mathcal{L}(F)$ and $\alpha \in F^\times$ then $\mathcal{L}(F)$ contains $x \mapsto f(\alpha x)$.

In fact, it is clear that if $f$ is simple then for $a \in F$ and $\alpha \in F^{\times}$, the functions $x \mapsto f(x+a)$ and $x \mapsto f(\alpha x)$ are also simple.

The basic result on existence and properties of an integral will be follows:

**Theorem 1.4.4.** *There is a unique $\mathbb{C}(\Gamma)$-linear functional $\int^F$ on $\mathcal{L}(F)$ which satisfies*

*(i) $\int^F$ lifts the Haar integral on $\overline{F}$: for $g \in \mathcal{L}$,*

$$\int^F (g^0) = \int g(u)\,du;$$

*(ii) Translation invariance: for $f \in \mathcal{L}(F)$, $a \in F$,*

$$\int^F f(x+a)\,dx = \int^F f(x)\,dx;$$

*(iii) Compatibility with multiplicative structure: for $f \in \mathcal{L}(F)$, $\alpha \in F^{\times}$,*

$$\int^F f(\alpha x)\,dx = |\alpha|^{-1} \int^F f(x)\,dx.$$

*Here the* absolute value *of $\alpha$ is defined by $|\alpha| = |\overline{\alpha t(-\nu(\alpha))}|X^{\nu(\alpha)}$, and we have adopted the customary integral notation $\int^F (f) = \int^F f(x)\,dx$.*

*Proof.* See chapter 2, especially proposition 2.1.12 and lemma 2.4.1. ☐

**Remark 1.4.5.** If $g^{a,\gamma}$ is the lift of a Haar integrable function, then

$$\int^F g^{a,\gamma}(x)\,dx = \int g(u)\,du\,X^{\gamma}.$$

### 1.4.2 Integration on $F \times F$

Now we summarise the integration theory for the product space $F \times F$. Proofs of this material may be found for the more general case of $F^n$ in section 3.1 of chapter 3.

**Definition 1.4.6.** A $\mathbb{C}(\Gamma)$-valued function $g$ on $F \times F$ is said to be *Fubini* if and only if both its repeated integrals exist and are equal. That is, we require:

(i) for all $x \in F$, the function $y \mapsto g(x,y)$ is integrable, and then that the function $x \mapsto \int^F g(x,y)\,dy$ is also integrable;

(ii) for all $y \in F$, the function $x \mapsto g(x,y)$ is integrable, and then that the function $y \mapsto \int^F g(x,y)\,dx$ is also integrable;

(iii) $\int^F \int^F g(x,y)\,dxdy = \int^F \int^F g(x,y)\,dydx$.

Similarly, an integrable complex valued function $f$ on $K \times K$ will be called *Fubini* if and only if both its repeated integrals exist and are equal.

**Remark 1.4.7.** Recall that the existence and equality of the repeated integrals of a complex valued function on $K \times K$ does not imply its integrability on $K \times K$ (see e.g. [Rud87, example 8.9c]) which is why we have separately imposed that condition. However, in our applications we will restrict to well enough behaved functions for this subtle problem to be irrelevant.

Fubini's theorem implies that almost all (in the sense of failing off a set of measure of zero) the horizontal and vertical sections of any integrable function on $K \times K$ are integrable. Therefore any integrable function on $K \times K$ differs off a null set from some Fubini function. However, there is no satisfactory theory of lifting null sets from $K$ to $F$, so we restrict attention to Fubini functions on $K \times K$.

Any function in the Schwartz-Bruhat space of $K \times K$ is Fubini; recall that if $K$ is archimedean these are the smooth functions of rapid decay at infinity, and if $K$ is non-archimedean these are the locally constant functions of compact support.

Also see remark 3.1.3.

The main properties of the collection of Fubini functions on $F \times F$ are the following:

**Proposition 1.4.8.** *The collection of Fubini functions on $F \times F$ is a $\mathbb{C}(\Gamma)$-space with the following properties:*

(i) *If $g$ is Fubini on $F \times F$, then so is $(x, y) \mapsto g(\alpha_1 x + a_1, \alpha_2 y + a_2)\, X^\gamma$ for any $a_i \in F$, $\alpha_i \in F^\times$, $\gamma \in \Gamma$, with repeated integral*

$$\int^F \int^F g(\alpha_1 x + a_1, \alpha_2 y + a_2)\, X^\gamma \, dx dy$$
$$= |\alpha_1|^{-1} |\alpha_2|^{-1} \int^F \int^F g(x, y)\, dx dy X^\gamma.$$

(ii) *If $f$ is Fubini on $K \times K$, then*

$$f^0(x, y) := \begin{cases} f(\overline{x}, \overline{y}) & x, y \in \mathcal{O}_F, \\ 0 & \textit{otherwise}, \end{cases}$$

*is Fubini on $F \times F$, with repeated integral*

$$\int^F \int^F f^0(x, y)\, dx dy = \int_K \int_K f(u, v)\, du dv.$$

*Proof.* See lemma 3.1.5 and proposition 3.1.8. □

**Remark 1.4.9.** The proposition implies that if $f$ is Fubini on $K \times K$, $a_1, a_2 \in F$, $\gamma_1, \gamma_2 \in \Gamma$, then the function $g = f^{(a_1, a_2), (\gamma_1, \gamma_2)}$ of $F \times F$ defined by

$$f^{(a_1, a_2), (\gamma_1, \gamma_2)}(a_1 + t(\gamma_1) x, a_2 + t(\gamma_2) y) = f^0(x, y)$$

for all $x, y \in F$ is Fubini. The function $g$ is said to be the *lift of $f$ at* $(a_1, a_2), (\gamma_1, \gamma_2)$.

Proposition 1.4.8 implies

$$\int^F \int^F g(x, y)\, dx dy = \int^F \int^F g(x, y)\, dy dx = \int_K \int_K f(u, v)\, du dv\, X^{\gamma_1 + \gamma_2}.$$

Also see remark 3.1.9.

## 1.5 Future directions

To finish the introduction, we mention several areas related to this thesis which demand investigation.

### 1.5.1 Mathematical physics

The field $\mathbb{R}(t)$, and certain subspaces of $\mathbb{R}((t))$, may be identified with spaces of functions. In particular, $t\mathbb{R}[t]$ may be identified with a subspace of the space of continuous paths $[0,1] \to \mathbb{R}$ which vanish at $0$ i.e. Wiener space. It would be interesting to understand relations between Wiener measure and the two-dimensional measure.

The Feynman integral is a mysterious tool of mathematical physics which can be used to make very accurate predictions in quantum field theory by computing integrals over certain infinite-dimensional spaces of paths. Finding a rigorous mathematical definition of these integrals is a major open problem in mathematical physics; see [JL00] for discussion of the problems. The archimedean two-dimensional local field $\mathbb{C}((t))$ contains many subspaces, such as $\mathbb{C}[t]$, which may be identified with spaces of continuous paths in the complex plane, and it is expected that two-dimensional integration will give new mathematical insights into Feynman's path integral. Evidence of the relations between quantum field theory and the measure on archimedean two-dimensional local fields may be found in sections 16, 18 of [Fes06] and example 2.5.6 of chapter 2.

The values of divergent integrals in quantum field theory, after renormalisation, appear as epsilon factors in our local zeta integrals (example 2.5.6). The duality provided by a functional equation would provide arithmetic arguments for the values of such integrals. It would be very interesting to investigate whether this arithmetic value coincides with the physical value.

There are relations between the geometric Langlands programme and conformal field theory (see e.g. [Fre07]). Hence suitable physical interpretations of this work and its extensions may provide insight into problems of field theory.

### 1.5.2 Model-theoretic integration

As discussed in subsection 5.1.4, there are remaining problems with the Hrushovski-Kazhdan style integration on two-dimensional local fields of residue characteristic zero. However, it seems that these difficulties are close to being resolved.

A different idea, which I did not manage to explore during the past three years, is Fesenko's idea of understanding higher dimensional integration using nonstandard (in the model-theoretic sense) techniques. The Haar measure on a locally-compact, abelian group may be interpreted as a hyperfinite counting measure [Gor97], and so perhaps it is possible to interpret the integral on a two-dimensional local field as a nonstandard limit of Haar measures in some sense.

### 1.5.3 Ramified zeta integrals

The proof of the functional equation in section 2.7 can surely be extended to a wider class of functions and characters. In particular there should be a theory for ramified characters which encodes interesting ramification data related to the Abbes and Saito theory.

### 1.5.4 Non-linear change of variables and Fubini's theorem

As discussed above regarding translation invariant integration on algebraic groups over two-dimensional local fields, it is important to understand the behaviour of the integral on $F^n$ with respect to non-linear changes of variables and to investigate the validity of Fubini's theorem. In residue characteristic zero I believe that continued work using the techniques of chapter 5 will produce all expected results. In finite residue characteristic the problem is more mysterious, as proposition 4.4.1 shows, and related to ramification theory; hopefully work on the programme outlined in subsection 6.1.3 on uniting integration theory with ramification theory will provide insight.

### 1.5.5 Integration on algebraic groups

See section 3.4.

### 1.5.6 Two-dimensional Langlands

Two-dimensional Langlands, if it exists, is deeply mysterious. Perhaps a study of suitable representations of $GL_n(F)$, with $F$ a two-dimensional local field, involving integration and non-commutative zeta integrals would be useful.

### 1.5.7 Arithmetic surfaces

See subsection 7.1.7.

CHAPTER 2

# Integration on valuation fields over local fields

This chapter develops the basic higher dimensional integration theory and harmonic analysis, and contains applications to two-dimensional local zeta functions.

### Notation

Let $\Gamma$ be a totally ordered abelian group and $F$ a field with a valuation $\nu : F^\times \to \Gamma$ with residue field $\overline{F}$, ring of integers $\mathcal{O}_F$ and residue map $\rho : \mathcal{O}_F \to \overline{F}$ (also denoted by an overline). Suppose further that the valuation is split; that is, there exists a homomorphism $t : \Gamma \to F^\times$ such that $\nu \circ t = \mathrm{id}_\Gamma$. The splitting of the valuation induces a homomorphism $\eta : F^\times \to \overline{F}^\times$ by $x \mapsto \overline{xt(-\nu(x))}$ (often called the angular component map). Assume also that $\Gamma$ contains a minimal positive element, denoted $1$ (this is not essential, but convenient for many examples).

Sets of the form $a + t(\gamma)\mathcal{O}_F$ are *translated fractional ideals*; $\gamma$ is referred to as the *height* of the set.

$\mathbb{C}(\Gamma)$ denotes the field of fractions of the complex group algebra $\mathbb{C}\Gamma$ of $\Gamma$; the basis element of the group algebra corresponding to $\gamma \in \Gamma$ shall be written as $X^\gamma$ rather than as $\gamma$. With this notation, $X^\gamma X^\delta = X^{\gamma+\delta}$. Note that if $\Gamma$ is a free abelian group of finite rank $n$, then $\mathbb{C}(\Gamma)$ is isomorphic to the rational function field $\mathbb{C}(X_1, \ldots, X_n)$.

The residue field $\overline{F}$ is assumed to be a non-discrete, locally compact field, i.e. a local field. We fix a choice of Haar measure on $\overline{F}$; occasionally, for convenience, we shall assume that $\mathcal{O}_{\overline{F}}$ has measure one. The measure on $\overline{F}^\times$ is chosen to satisfy $d\overset{\times}{x} = |x|^{-1}d^+x$.

These assumptions hold for a higher dimensional local field. For basic definitions and properties of such fields, see [FK00].

Indeed, suppose that $F = F_n$ is a higher dimensional local field of dimension $n \geq 2$: we allow the case in which $F_1$ is an archimedean local field. If $F_1$ is non-archimedean, instead of the usual rank $n$ valuation $\mathbf{v} : F^\times \to \mathbb{Z}^n$, let $\nu$ be the $n-1$ components of $\mathbf{v}$ corresponding to the fields $F_n, \ldots, F_2$; note that $\mathbf{v} = (\nu_{\overline{F}} \circ \eta, \nu)$. If $F_1$ is archimedean, then $F$ may be similarly viewed as a valuation field with value group $\mathbb{Z}^{n-1}$ and residue field $F_1$.

The residue field of $F$ with respect to $\nu$ is the local field $\overline{F} = F_1$. If $F$ is non-archimedean, then the ring of integers $O_F$ of $F$ with respect to the rank $n$ valuation is equal to $\rho^{-1}(\mathcal{O}_{\overline{F}})$, while the groups of units $O_F^\times$ with respect to the rank $n$ valuation is equal to $\rho^{-1}(\mathcal{O}_{\overline{F}}^\times)$.

## 2.1 Integration on $F$

In this section we explain the basic theory of integration on $F$; a summary of the main results can be found in subsection 1.4.1. The following definition is fundamental:

**Definition 2.1.1.** Let $f$ be a function on $\overline{F}$ taking values in an abelian group $A$; let $a \in F$, $\gamma \in \Gamma$. The *lift* of $f$ at $a, \gamma$ is the $A$-valued function on $F$ defined by

$$f^{a,\gamma}(x) = \begin{cases} f(\overline{(x-a)t(-\gamma)}) & x \in a + t(\gamma)\mathcal{O}_F, \\ 0 & \text{otherwise.} \end{cases}$$

In other words,

$$f^{0,0}(x) = \begin{cases} f(\overline{x}) & x \in \mathcal{O}_F, \\ 0 & \text{otherwise.} \end{cases}$$

and $f^{a,\gamma}(a + t(\gamma)x) = f^{0,0}(x)$ for all $x$.

It is useful to understand how lifted functions behave on translated fractional ideals:

**Lemma 2.1.2.** *Let $f^{a,\gamma}$ be a lifted function as in the definition; let $b \in F$, $\delta \in \Gamma$. Then for all $x$ in $\mathcal{O}_F$,*

**case $\delta > \gamma$:**

$$f^{a,\gamma}(b + t(\delta)x) = \begin{cases} f(\overline{(b-a)t(-\gamma)}) & b \in a + t(\gamma)\mathcal{O}_F, \\ 0 & \textit{otherwise.} \end{cases}$$

**case $\delta = \gamma$:**

$$f^{a,\gamma}(b + t(\delta)x) = \begin{cases} f(\overline{(b-a)t(-\gamma)} + \overline{x}) & b \in a + t(\gamma)\mathcal{O}_F, \\ 0 & \textit{otherwise.} \end{cases}$$

**case $\delta < \gamma$:**

$$f^{a,\gamma}(b + t(\delta)x) = \begin{cases} f(\overline{(b + t(\delta)x - a)t(-\gamma)}) & x \in (a-b)t(\delta)^{-1} + t(\gamma - \delta)\mathcal{O}_F, \\ 0 & \textit{otherwise.} \end{cases}$$

*In particular, in this final case, if $x, y \in \mathcal{O}_F$ are such that $f^{a,\gamma}(b + t(\delta)x)$ and $f^{a,\gamma}(b + t(\delta)y)$ are non-zero, then $\overline{x} = \overline{y}$.*

*Proof.* This follows from the definition of a lifted function by direct verification. □

Let $\mathcal{L}$ denote the space of complex-valued, Haar integrable functions on $\overline{F}$.

**Remark 2.1.3.**

(i) For $a \in F, \gamma \in \Gamma$, let $\mathcal{L}^{a,\gamma}$ denote the space of complex valued functions on $F$ of the form $f^{a,\gamma}$, for $f \in \mathcal{L}$. Suppose $a_1 + t(\gamma_1)\mathcal{O}_F = a_2 + t(\gamma_2)\mathcal{O}_F$. Then $\gamma_1 = \gamma_2$ and

$$f^{a_1,\gamma_1}(x) = f^{a_2,\gamma_2}(x + a_2 - a_1) = g^{a_2,\gamma_2}(x)$$

where $g \in \mathcal{L}$ is the function $g(y) = f(y + \overline{(a_2 - a_1)t(-\gamma_2)})$. Hence $\mathcal{L}^{a_1,\gamma_1} = \mathcal{L}^{a_2,\gamma_2}$.

(ii) Given a lifted function $f^{a,\gamma}$ and $\tau \in F$, the translated function $x \mapsto f^{a,\gamma}(x + \tau)$ is the lift of $f$ at $a - \tau, \gamma$

**Definition 2.1.4.** For $J = a + t(\gamma)\mathcal{O}_F$ a translated fractional ideal of $F$, define $\mathcal{L}(J)$ to be the space of complex-valued functions of $F$ of the form $f^{a,\gamma}$, for $f \in \mathcal{L}$. Introduce an integral on $\mathcal{L}(J)$ by

$$\int^J : \mathcal{L}(J) \to \mathbb{C}, \quad f^{a,\gamma} \mapsto \int_{\overline{F}} f(u)\, du.$$

By remarks 2.1.3 and translation invariance of the Haar integral on $\overline{F}$, the integral is well-defined (i.e. independent of $a, \gamma$).

**Proposition 2.1.5.** *The sum, inside the space of all complex-valued functions on $F$, of the spaces $\mathcal{L}(J)$, as $J$ varies over all translated fractional ideals, is a direct sum.*

*Proof.* Let $J_i$, for $i = 1 \dots, n$, be distinct translated fractional ideals, of height $\gamma_i$ say. Suppose $f_i \in \mathcal{L}(J_i)$ for each $i$, with $\sum_i f_i = 0$; we may suppose that $\gamma_1 \le \gamma_2 \le \cdots \le \gamma_n$.

Fix a value of $i$ satisfying $1 \le i < n$. If $\gamma_i = \gamma_n$, then $J_i$ and $J_n$ are disjoint translated fractional ideals, and so $f_i$ is constantly zero on $J_n$. Else $\gamma_i < \gamma_n$, and then the first case of lemma 2.1.2 implies that $f_i$ is constant on $J_n$.

Therefore $f_n = -\sum_{i=1}^{n-1} f_i$ is constant on $J_n$, implying that $f_n$ is the lift of a constant function, and therefore that it is zero (for $\mathcal{L}$ contains no other constant function). The proof now follows by induction. ☐

This linear independence result clearly allows us to extend the $\int^J$, as $J$ varies over all translated fractional ideals, to a single functional:

**Definition 2.1.6.** Let $\mathcal{L}(F)_\mathbb{C}$ be the space of complex-valued functions spanned by $\mathcal{L}(J)$ for all translated fractional ideals $J$. Let $\int^F : \mathcal{L}(F)_\mathbb{C} \to \mathbb{C}(\Gamma)$ denote the unique linear map such that if $f \in \mathcal{L}(J)$ for some $J$ of height $\gamma$, then $\int^F(f) = \int^J(f)\, X^\gamma$.

$\mathcal{L}(F)_\mathbb{C}$ will be referred to as the space of complex-valued, *integrable* functions on $F$.

Remarks 2.1.3 imply that $\mathcal{L}(F)_\mathbb{C}$ is closed under translation from $F$ and that $\int^F$ is translation invariant. We will of course usually write $\int^F f(x)\, dx$ in place of $\int^F(f)$

**Remark 2.1.7.** If $A$ were an arbitrary $\mathbb{C}$-algebra and elements $c_\gamma \in A$ were given for each $\gamma \in \Gamma$, we could define an $A$-valued linear translation invariant integral on $\mathcal{L}(F)$ by replacing $X^\gamma$ by $c_\gamma$ in the previous definition. However, using $X^\gamma$ ensures compatibility of the integral with the multiplicative group $F^\times$, in that it implies the existence of an absolute value with expected properties; see lemma 2.4.1.

This phenomenon also appears when extending the integration theory to $F^n$, $M_n(F)$, and $GL_n(F)$; one must take into account the action of $GL_n(F)$ on $F^n$ in order to develop a satisfactory theory; see section 3.4.

**Remark 2.1.8.** Let us check to what extent $\mathcal{L}(F)_\mathbb{C}$ and $\int^F$ depend on the choice of the splitting $t$.

Let $t'$ be another splitting of the valuation: that is, $t'$ is a homomorphism from $\Gamma$ to $F^\times$ with $\nu \circ t' = \mathrm{id}_\Gamma$. Then there is a homomorphism $u : \Gamma \to \mathcal{O}_F^\times$ which satisfies $t(\gamma) = u(\gamma)t'(\gamma)$ for $\gamma \in \Gamma$. Let $g \in \mathcal{L}$, $a \in F$, and $\gamma \in \Gamma$; let $f$ be the lift of $g$ at $a, \gamma$ with respect to $t$, and $f'$ the lift of $g$ at $a, \gamma$ with respect to $t'$. Thus, by definition, $f$ and $f'$ both vanish off $J = a + t(\gamma)\mathcal{O}_F = a + t'(\gamma)\mathcal{O}_F$, and for $x \in \mathcal{O}_F$,

$$f(a + t(\gamma)x) = g(\overline{x}), \qquad f'(a + t'(\gamma)x) = g(\overline{x}).$$

Therefore $f'(a + t(\gamma)x) = g(\overline{u(\gamma)}^{-1}\overline{x})$ and so $\int^J(f') = |\overline{u(\gamma)}| \int g(y)\, dy = |\overline{u(\gamma)}| \int^J(f)$.

Let $\int^{J,t'}$ (resp. $\int^{F,t'}$) denote the integral over $J$ (resp. $F$) with respect to $t'$; the previous paragraph proves that $\int^{J} = |\overline{u(\gamma)}| \int^{J,t'}$. Let $\sigma : \mathbb{C}(\Gamma) \to \mathbb{C}(\Gamma)$ be the $\mathbb{C}$-linear field automorphism of $\mathbb{C}(\Gamma)$ given by $\sigma(X^\gamma) = |\overline{u(\gamma)}|X^\gamma$, for $\gamma \in \Gamma$. Then for all $f \in \mathcal{L}(F)_{\mathbb{C}}$, the identity

$$\int^{F} f(x)\,dx = \sigma\left(\int^{F,t'} f(x)\,dx\right)$$

follows.

So the integral is well-defined up to an automorphism of $\mathbb{C}(\Gamma)$.

Regarding absolute values, we have the following attractive result:

**Proposition 2.1.9.** *If $f$ belongs to $\mathcal{L}(F)_{\mathbb{C}}$, then so does $x \mapsto |f(x)|$.*

*Proof.* We may write $f = \sum_{i=1}^{n} f_i$; here $J_i$, for $i = 1\dots,n$, are distinct translated fractional ideals, of height $\gamma_i$ say, and $f_i \in \mathcal{L}(J_i)$. We may also assume that $\gamma_1 \leq \dots \leq \gamma_n$.

The statement with $\mathcal{L}$ in place of $\mathcal{L}(F)_{\mathbb{C}}$ is true by definition of Haar integrability; hence the statement is true for $\mathcal{L}(J)$, where $J$ is any translated fractional ideal. So if $n = 1$ we are done, and we now assume $n > 1$, proving the result by induction.

In the same way as in the proof of proposition 2.1.5, each function $f_i$, for $1 \leq i < n$, is constant on $J_n$. Let $a$ be any element of $J_n$. Then the following identities hold:

$$\begin{aligned}
|f| &= |\sum_{i=1}^{n-1} f_i| + \left(|f| - |\sum_{i=1}^{n-1} f_i|\right) \operatorname{char}_{J_n}\\
&= |\sum_{i=1}^{n-1} f_i| + \left(|f_n + \sum_{i=1}^{n-1} f_i(a)| - |\sum_{i=1}^{n-1} f_i(a)|\right) \operatorname{char}_{J_n}\\
&= |\sum_{i=1}^{n-1} f_i| + |f + \sum_{i=1}^{n-1} f_i(a)| - |\sum_{i=1}^{n-1} f_i(a)|
\end{aligned}$$

The proof will be complete if we can show that

$$|f_n + \sum_{i=1}^{n-1} f_i(a)| - |\sum_{i=1}^{n-1} f_i(a)| \tag{$*$}$$

belongs to $\mathcal{L}(F)_{\mathbb{C}}$. Write $f_n = g^{a,\gamma_n}$ for some $g \in \mathcal{L}$; then the function $(*)$ is the lift at $a, \gamma_n$ of the Haar integrable function $|g + \sum_{i=1}^{n-1} f_i(a)| - |\sum_{i=1}^{n-1} f_i(a)|$. ☐

Although $\mathcal{L}(F)_{\mathbb{C}}$ is closed under taking absolute values, the following examples show that there is some unusual associated behaviour, and that there is no clear definition of a 'null function' on $F$:

**Example 2.1.10.** Introduce $f_1 = \operatorname{char}_{\{0\}}^{0,0}$, the characteristic function of $t(1)\mathcal{O}_F$, and $f_2 = -2\operatorname{char}_S^{0,\gamma}$ where $S$ is a Haar measurable subset of $\overline{F}$ with measure $1$ and $\gamma$ is a positive element of $\Gamma$. Let $f = f_1 + f_2$.

(i) Firstly we claim that the following hold:

$$\int^{F} |f(x)|\,dx = 0, \qquad \int^{F} f(x)\,dx = -2X^\gamma.$$

Indeed, the second identity is immediate from the definition of the integral. For the first identity, note that as in the proof of the previous proposition (with $n = 2$),

$$|f| = |f_1| + |f_2 + f_1(0)| - |f_1(0)|.$$

Further, $f_1(0) = 1$ and the function $|f_2 + 1|$ is identically 1. So $|f| = \mathrm{char}^{0,0}_{\{0\}}$, from which the first identity follows.

(ii) Secondly, the considerations above imply

$$\int^F |f(x)|\,dx = \int^F |f_1(x)|\,dx = 0, \qquad \int^F |f(x) - f_1(x)|\,dx = 2X^\gamma.$$

(iii) Finally, consider the translated function $f'(x) = f(x-a)$, where $a$ is any element of $F$ not in $\mathcal{O}_F$. Then $f'$ and $f$ have disjoint support and so

$$\begin{aligned}\int^F |f(x) - f'(x)|\,dx &= \int^F |f(x)| + |f'(x)|\,dx \\ &= \int^F |f(x)|\,dx + \int^F |f'(x)|\,dx = 0\end{aligned}$$

by translation invariance of the integral. Also, $\int^F f(x) - f'(x)\,dx = 0$. Thus $g = f - f'$ provides an example of a complex-valued integrable function on $F$ such that $\int^F |g(x)|\,dx = \int^F g(x)\,dx = 0$, but where the components of $g$ in $\mathcal{L}(J)$, for all $J$, are lifts of non-null functions.

As will become apparent in the study of harmonic analysis, it is more natural to integrate $\mathbb{C}(\Gamma)$-valued functions on $F$ than complex-valued ones, so we define our main class of functions as follows:

**Definition 2.1.11.** A $\mathbb{C}(\Gamma)$-valued function on $F$ will be said to be *integrable* if and only if it has the form $x \mapsto \sum_i f_i(x)\,p_i$ for finitely many $f_i \in \mathcal{L}(F)_\mathbb{C}$ and $p_i \in \mathbb{C}(\Gamma)$. The integral of such a function is defined to be

$$\int^F f(x)\,dx = \sum_i \int^F f_i(x)\,dx\,p_i.$$

This is well defined. The $\mathbb{C}(\Gamma)$ space of all such functions will be denoted $\mathcal{L}(F)$; the integral is a $\mathbb{C}(\Gamma)$-linear functional on this space.

In other words, $\mathcal{L}(F) = \mathcal{L}(F)_\mathbb{C} \otimes_\mathbb{C} \mathbb{C}(\Gamma)$ and the integral is extended in the natural way. The integrable functions which are complex-valued are precisely $\mathcal{L}(F)_\mathbb{C} \subset \mathcal{L}(F)$, so there is no ambiguity in the phrase 'complex-valued, integrable function'.

For the sake of completeness, we summarise this section as follows (also see subsection 1.4.1):

**Proposition 2.1.12.** *$\mathcal{L}(F)$ is the smallest $\mathbb{C}(\Gamma)$ space of $\mathbb{C}(\Gamma)$-valued functions on $F$ which contains $g^{a,\gamma}$ for all $g \in \mathcal{L}$, $a \in F$, $\gamma \in \Gamma$. There is a (necessarily unique) $\mathbb{C}(\Gamma)$-linear functional $\int^F$ on $\mathcal{L}(F)$ which satisfies*

$$\int^F g^{a,\gamma}(x)\,dx = \int_{\overline{F}} g(u)\,du\,X^\gamma.$$

*$\mathcal{L}(F)$ is closed under translation and $\int^F$ is translation invariant.*

**Remark 2.1.13.** Examination of the proofs in this section leads to the the following abstraction of the theory:

Let $F', \nu', t', \Gamma'$ satisfy the same conditions as $F, \nu, t, \Gamma$, except that we do not suppose $\overline{F}'$ is a local field. Let $L$ be an arbitrary field, and $\mathcal{L}'$ an $L$ space of $L$-valued functions on $\overline{F}'$, equipped with an $L$-linear functional $I$, with the following properties:

(i) $\mathcal{L}'$ is closed under translation from $\overline{F}'$ and $I$ is translation invariant (i.e. $f \in \mathcal{L}'$ and $a \in \overline{F}'$ implies $y \mapsto f(y+a)$ is in $\mathcal{L}'$ with image under $I$ equal to $I(f)$).

(ii) $\mathcal{L}'$ contains no non-zero constant functions.

Let $\mathcal{L}'(F')$ be the smallest $L(\Gamma')$ space of $L(\Gamma')$-valued functions on $F$ which contains $f^{a,\gamma}$ for $f \in \mathcal{L}'$, $a \in F'$, $\gamma \in \Gamma'$. Then there is a (necessarily) unique $L(\Gamma')$-linear functional $I^{F'}$ on $\mathcal{L}'(F')$ which satisfies $I^{F'}(f^{a,\gamma}) = I(f)\, X^{\gamma}$. Further, the pair $\mathcal{L}'(F'), I^{F'}$ satisfy (i) and (ii) with the field $L(\Gamma')$ in place of $L$.

In particular, suppose $F$ is a three dimensional local field, say, with first residue field $F_2$ (a two-dimensional local field), and $\overline{F} = F_1$ a local field. Then the integral on $F$ can be obtained either by lifting the Haar integral to $F_2$ and then (by using this remark) lifting again to $F$, or by following the arguments of this section and lifting the Haar integral directly to $F$.

This 'transitivity' of lifting the integral is also present in E. Hrushovski and D. Kazhdan's motivic integration theory; see [HK06, §12.2]

## 2.2 Measure theory

We now produce a measure theory from the integration theory; results of [Fes03] are recovered and extended.

**Definition 2.2.1.** A *distinguished* subset of $F$ is a set of the form $a + t(\gamma)\rho^{-1}(S)$, where $a \in F$, $\gamma \in \Gamma$, and $S$ is a subset of $\overline{F}$ of finite Haar measure. $\gamma$ is said to be the *level* of the set.

Let $D$ denote the set of all distinguished subsets of $F$; let $R$ denote the ring of sets generated by $D$ (see appendix 2.A for the definition of 'ring').

**Remark 2.2.2.** Note that the characteristic function of a distinguished set $a+t(\gamma)\rho^{-1}(S)$ is precisely the lift of the characteristic function of $S$ at $a, \gamma$. Proposition 2.1.5 proves that if $a_1 + t(\gamma_1)\rho^{-1}(S_1) = a_2 + t(\gamma_2)\rho^{-1}(S_2)$, then $\gamma_1 = \gamma_2$ and $S_1$ is a translate of $S_2$. In particular, the level is well defined.

**Lemma 2.2.3.** *Let $A_i = a_i + t(\gamma_i)\rho^{-1}(S_i)$, $i = 1, 2$, be distinguished sets with non-empty intersection.*

*(i) If $\gamma_1 = \gamma_2$, then $A_1 \cap A_2$ and $A_1 \cup A_2$ are distinguished sets of level $\gamma_1$.*

*(ii) If $\gamma_1 \neq \gamma_2$, then $A_1 \subseteq A_2$ if $\gamma_1 > \gamma_2$, and $A_2 \subseteq A_1$ if $\gamma_2 > \gamma_1$.*

*Proof.* This is immediate from the definition of a distinguished set. $\square$

Referring to appendix 2.A, it has just been shown that $D$ is a d-class of sets. By proposition 2.A.9, the characteristic function of any set in $R$ may be written as the difference of two sums, each of characteristic functions of sets in $D$; therefore the characteristic function of any set in $R$ belongs to $\mathcal{L}(F)_{\mathbb{C}}$.

**Definition 2.2.4.** Define the measure $\mu^F(W)$ of a set $W$ in $R$ by

$$\mu^F(W) = \int^F \mathrm{char}_W(x)\,dx.$$

By the properties of the integral, $\mu^F$ is a translation-invariant, finitely additive set function $R \to \mathbb{R}\Gamma$ (the real group algebra of $\Gamma$). For a distinguished set $A = a + t(\gamma)\rho^{-1}(S)$, remark 2.2.2 implies

$$\mu^F(A) = \int^F (\mathrm{char}_A) = \int^F (\mathrm{char}_S^{a,\gamma}) = \mu(S)\,X^\gamma,$$

where $\mu$ denotes our choice of Haar measure on $\overline{F}$. The following examples demonstrate some unusual behaviour of this measure:

**Example 2.2.5.**

(i) For $\gamma \in \Gamma$, the set $t(\gamma)\mathcal{O}_F = t(\gamma-1)\rho^{-1}(\{0\})$ is distinguished, with measure zero.

(ii) Let $S$ be a subset of $\overline{F}$ of finite measure. The set $\rho^{-1}(\overline{F}\setminus S) = \mathcal{O}_F \setminus \rho^{-1}(S)$ belongs to $R$ and has measure $-\mu(S)$. Compare this with example 2.1.10

(iii) $\mu^F$ is not countably additive. Indeed, write $\overline{F}$ as a countable disjoint union of sets of finite measure; $\overline{F} = \bigsqcup_i S_i$ say. Then $\mathcal{O}_F = \bigsqcup_i \rho^{-1}(S_i)$ has measure zero, while $\sum_i \mu^F(\rho^{-1}(S_i)) = \infty$.

(iv) Suppose that $\overline{F} = \mathbb{R}$. Set $A_{2n-1} = nt(-1) + \rho^{-1}([0,1/n])$ and $A_{2n} = nt(-1) + \rho^{-1}(\mathbb{R}\setminus[0,1/n])$ for all natural numbers $n$. Then $\mu^F(A_{2n-1}) = 1/n$, $\mu^F(A_{2n}) = -1/n$, and $\bigsqcup_i A_i = \bigsqcup_n nt(-1) + \mathcal{O}_F = t(-1)\rho^{-1}(\mathbb{N})$ has measure 0.

The series $\sum_i \mu^F(A_i)$ is conditionally convergent in $\mathbb{R}$ (i.e. convergent, but not absolutely convergent). By a theorem of Riemann (see e.g. [Apo74, chapter 8.18]), there exists, for any real $q$, a permutation $\sigma$ of $\mathbb{N}$ such that $\sum_i \mu^F(A_{\sigma(i)})$ converges to $q$. But regardless of the permutation, $\mu^F(\bigsqcup_i A_{\sigma(i)}) = 0$.

Let us consider a couple of examples in greater detail and give a more explicit description of the measure:

**Example 2.2.6.**

(i) Suppose that $F$ is an $n$-dimensional, non-archimedean, local field, with local parameters $t_1, \dots, t_n$. We view $F$ as a valued field over the local field $\overline{F} = F_1$, rather than over the finite field $F_0$. The results of this section prove the existence of a finitely additive set function $\mu^F$ on the appropriate ring of sets, taking values in $\mathbb{R}[X_2^{\pm 1}, \dots, X_n^{\pm 1}]$, which satisfies

$$\mu^F(a + t_1^{r_1} \dots t_n^{r_n} O_F) = q^{-r_1} X_2^{r_2} \dots X_n^{r_n}$$

for $a \in F$ and integers $r_i$. Here $O_F$ denotes the ring of integers of $F$ with respect to the rank $n$ valuation, and $q$ is the cardinality of of $F_0$.

However, we have not made use of any topological property of $F$; in particular, this result holds for an arbitrary field with value group $\mathbb{Z}^{n-1}$ and a non-archimedean local field as residue field. This measure theory therefore extends that developed in [Fes03], while also providing proofs of statements in [Fes03] for the case in which the local field is archimedean.

Fesenko also extends his measure to be countably additive under certain hypotheses, a result which we will require in the model theoretic study of two-dimensional integration in section 5.

(ii) Suppose that $F = \overline{F}((t))$, the field of formal Laurent series over $\overline{F}$, or $F = \overline{F}(t)$, the rational function field (here we write $t = t(1)$). Then a typical distinguished set has the form

$$\begin{array}{rl} a(t) + St^n + t^{n+1}\overline{F}[[t]] & \text{(Laurent series case)} \\ a(t) + St^n + t^{n+1}\overline{F}[t] & \text{(rational functions case)} \end{array}$$

for $a(t) \in F$, and $S \subset \overline{F}$ of finite Haar measure. Such a set has measure $\mu(S)X^n$, where $\mu$ denotes our choice of Haar measure on $\overline{F}$.

## 2.3 Harmonic analysis on $F$

Now we develop elements of a theory of harmonic analysis on $F$.

**Definition 2.3.1.** Suppose that $\psi : F \to S^1$ is a homomorphism of the additive group of $F$ into the group of complex numbers of unit modulus. Then $\psi$ is said to be a *good character* if it is trivial, or if it satisfies the following two conditions:

(i) There exists $\mathfrak{f} \in \Gamma$ such that $\psi$ is trivial on $t(\mathfrak{f})\mathcal{O}_F$, but non-trivial on $t(\mathfrak{f} - 1)\mathcal{O}_F$; $\mathfrak{f}$ is said to be the *conductor* of $\psi$.

(ii) The resulting character $\overline{\psi}$ of the additive group of $\overline{F}$ defined by $\overline{\psi}(\overline{x}) = \psi(t(\mathfrak{f} - 1)x)$, for $x \in \mathcal{O}_F$, is continuous.

The conductor of the trivial character may be said to be $-\infty$. The induced character on $\overline{F}$ as in (ii) will always be denoted $\overline{\psi}$.

The definition of a good character is designed to replace the continuity assumption which would be imposed if $F$ had a suitable topology.

**Example 2.3.2.** Suppose that $F = \overline{F}((t))$, the field of formal Laurent series over $\overline{F}$ (here $t(1) = t$). Let $\psi_{\overline{F}}$ be a continuous character of $\overline{F}$. Then $\sum_i a_i t^i \mapsto \psi_{\overline{F}}(a_n)$ is a good character of $F$ of conductor $n + 1$ and induced character $\psi_{\overline{F}}$.

Recall that $\eta : F^\times \to \overline{F}^\times$ is the 'angular component map', defined by $\eta(\alpha) = \overline{\alpha t(-\nu(\alpha))}$.

**Lemma 2.3.3.** *Suppose that $\psi$ is a good character of $F$ of conductor $\mathfrak{f}$; let $\alpha \in F$. Then $x \mapsto \psi(\alpha x)$ is a good character of $F$, with conductor $\mathfrak{f} - \nu(\alpha)$; the character induced on $\overline{F}$ by $x \mapsto \psi(\alpha x)$ is $y \mapsto \overline{\psi}(\eta(\alpha)y)$ (assuming $\alpha \neq 0$).*

*Proof.* This is easily checked. □

Given $\psi, \alpha$ as in the previous lemma we will write $\psi_\alpha$ for the translated character $x \mapsto \psi(\alpha x)$ (and we employ similar notation for characters of $\overline{F}$).

Before proceeding, we must make a simple assumption:

*We assume that a non-trivial good character $\psi$ exists on $F$.*

By the previous lemma we may (and do) assume further that $\psi$ has conductor $1$, and we fix such a character for this section. With this choice of conductor, $x \in \mathcal{O}_F$ implies $\overline{\psi}(\overline{x}) = \psi(x)$. We will take Fourier transforms of integrable functions $g$ on $\overline{F}$ with respect to the character $\overline{\psi}$; that is, $\widehat{g}(x) = \int g(y)\overline{\psi}(xy)\,dy$.

### 2.3.1 Extending the integral to twisted functions

Let $\mathcal{L}(F,\psi)$ denote the $\mathbb{C}(\Gamma)$ space of $\mathbb{C}(\Gamma)$-valued functions on $F$ spanned by $f\psi_\alpha$, for $f \in \mathcal{L}(F)$, $\alpha \in F$; taking $\alpha = 0$ we see that $\mathcal{L}(F) \subseteq \mathcal{L}(F,\psi)$. Our immediate aim is proposition 2.3.7, which states that the integral on $F$ has a unique translation invariant extension to this space of functions.

**Remark 2.3.4.** Such a character certainly exists on a higher local field. Indeed, such a field is self-dual: if $\psi, \psi_1$ are good characters with $\psi$ non-trivial then there is $\alpha \in F^\times$ such that $\psi(x) = \psi_1(\alpha x)$ for all $x \in F$. For more details, see [Fes03] section 3.

It is convenient for the following results to write $\mathcal{L}^\gamma$ (where $\gamma \in \Gamma$) for the sum of the spaces $\mathcal{L}(J)$ over all translated fractional ideals of height $\gamma$; this sum is direct by proposition 2.1.5. Note that if $f \in \mathcal{L}^\gamma$ and $a \in F$ with $\nu(a) > \gamma$ then $f(x+a) = f(x)$ for all $x \in F$.

Certain products of an integrable function with a good character are still integrable:

**Lemma 2.3.5.** *Let $J = a + t(\gamma)\mathcal{O}_F$ be a translated fractional ideal and $\alpha \in F$. If $\gamma = -\nu(\alpha)$, then $\psi_\alpha \operatorname{char}_J$ is the lift of $\psi(\alpha a)\overline{\psi}_{\eta(\alpha)}$ at $a, \gamma$; if $\gamma > -\nu(\alpha)$, then $\psi_\alpha$ is constantly $\psi(\alpha a)$ on $J$.*

*Therefore, if $\gamma \geq -\nu(\alpha)$ and $f$ is in $\mathcal{L}^\gamma$ then $f\psi_\alpha$ is also in $\mathcal{L}^\gamma$.*

*Proof.* The identities may be easily verified by evaluating on $a + t(\gamma)\mathcal{O}_F$. The final statement follows by linearity. □

In contrast with the previous lemma we now consider the case $\gamma < -\nu(\alpha)$:

**Lemma 2.3.6.** *Let $\alpha_i, \gamma_i$ be finitely many ($1 \leq i \leq m$, say) elements of $F, \Gamma$ respectively, and let $f_i \in \mathcal{L}^{\gamma_i}$ for each $i$. Suppose further that $\nu(\alpha_i) < -\gamma_i$ for each $i$ and that $\sum_i f_i\psi_{\alpha_i}$ is integrable on $F$. Then $\int^F \sum_i f_i(x)\psi_{\alpha_i}(x)\,dx = 0$.*

*Proof.* The result is proved by induction on $m$. Let $y \in t(-\nu(\alpha_m))\mathcal{O}_F$ satisfy $\psi_{\alpha_m}(y) \neq 1$. The functions

$$x \mapsto \sum_i f_i(x+y)\psi_{\alpha_i}(x+y) = \sum_i \psi_{\alpha_i}(y) f_i(x+y)\psi_{\alpha_i}(x)$$
$$x \mapsto \sum_i \psi_{\alpha_m}(y) f_i(x)\psi_{\alpha_i}(x)$$

are integrable on $F$, the first having integral equal to that of $\sum_i f_i\psi_{\alpha_i}$ by translation invariance of $\int^F$. Taking the difference of the two functions, noting that $f_m(x+y) = f_m(x)$, and applying the inductive hypothesis, obtains

$$\int^F \sum_i f_i(x)\psi_{\alpha_i}(x)\,dx = \psi_{a_m}(y)\int^F \sum_i f_i(x)\psi_{\alpha_i}(x)\,dx,$$

which completes the proof. □

The first main result of this section may now be proved:

**Proposition 2.3.7.** *$\int^F$ has a unique extension to a translation-invariant, $\mathbb{C}(\Gamma)$-linear functional on $\mathcal{L}(F,\psi)$.*

*Proof.* To prove uniqueness, suppose that $I$ is a translation-invariant $\mathbb{C}(\Gamma)$-linear functional on $\mathcal{L}(F,\psi)$ which vanishes on $\mathcal{L}(F)$. We claim that $I$ is everywhere zero; by linearity it suffices to check that $I$ vanishes on $f\psi_\alpha$ for $f \in \mathcal{L}^\gamma$ (any $\gamma \in \Gamma$) and $\alpha \in F$. If $\gamma > -\nu(a)$, then $f\psi_\alpha$ is integrable by lemma 2.3.5 and so $I(f\psi_\alpha) = 0$. If $\gamma \leq -\nu(\alpha)$, then let $y \in t(-\nu(\alpha))\mathcal{O}_F$ satisfy $\psi_\alpha(y) \neq 1$; as in lemma 2.3.6 the identity $I(f\psi_\alpha) = \psi_\alpha(y)I(f\psi_\alpha)$ follows from translation invariance of $I$. This completes the proof of uniqueness.

To prove existence, suppose first that $f \in \mathcal{L}(F,\psi)$ is complex-valued, and write $f = \sum_i f_i\psi_{\alpha_i}$, for finitely many $\alpha_i \in F$, and $f_i \in \mathcal{L}^{\gamma_i}$ say. Attempt to define

$$I(f) = \sum_{i\,:\,\gamma_i \geq -\nu(\alpha_i)} \int^F f_i(x)\psi_{\alpha_i}(x)\,dx.$$

We claim that this is well-defined. Indeed, if $f = 0$, then function

$$\sum_{i\,:\,\gamma < -\nu(\alpha_i)} f_i\psi_{\alpha_i} = -\sum_{i\,:\,\gamma \geq -\nu(\alpha_i)} f_i\psi_{\alpha_i}$$

lies in $\mathcal{L}(F)$ by lemma 2.3.5. By lemma 2.3.6, the function has integral equal to zero, and so

$$0 = \int^F \sum_{i\,:\,\gamma \geq -\nu(\alpha_i)} f_i(x)\psi_{\alpha_i}(x)\,dx = \sum_{i\,:\,\gamma \geq -\nu(\alpha_i)} \int^F f_i(x)\psi_{\alpha_i}(x)\,dx.$$

This proves that $I$ is well-defined.

$I$ extends to $\mathcal{L}(F,\psi)$ by setting $I(\sum_j g_j\, X^{\gamma_j}) = \sum_j I(g_j)\, X^{\gamma_j}$ for finitely many complex-valued $g_j$ in $\mathcal{L}(F)$ and $\gamma_j$ in $\Gamma$. Translation invariance of $I$ follows from translation invariance of $\int^F$. $\square$

We shall denote the extension of $\int^F$ to $\mathcal{L}(F,\psi)$ by the same notation $\int^F$.

**Remark 2.3.8.** The previous results may be easily modified to prove that there is a unique extension of $\int^F$ to a translation-invariant $\mathbb{C}(\Gamma)$-linear function on the space spanned by $f\Psi$, for $f \in \mathcal{L}(F)$ and $\Psi$ varying over *all* good characters.

**Example 2.3.9.** Suppose that $\overline{F}$ is non-archimedean, with prime $\pi$ and residue field of cardinality $q$. Let $w = (\nu_{\overline{F}} \circ \eta, \nu)$ be the valuation on $F$ with value group $\mathbb{Z} \times \Gamma$ (ordered lexicographically from the right), with respect to which $F$ has residue field $\mathbb{F}_q$. Let $a \in F$, $\gamma \in \Gamma$, $j \in \mathbb{Z}$; then

$$\int^F \psi_a(x)\,\mathrm{char}_{t(\gamma)\rho^{-1}(\pi^j\mathcal{O}_{\overline{F}})}(x)\,dx = \begin{cases} 0 & \gamma < -\nu(a) \\ \int_{\pi^j\mathcal{O}_{\overline{F}}} \overline{\psi}(\eta(a)y)\,dyX^\gamma & \gamma = -\nu(a) \\ \int^F \mathrm{char}_{t(\gamma)\rho^{-1}(\pi^j\mathcal{O}_{\overline{F}})}(x)\,dx & \gamma > -\nu(a) \end{cases}$$

Suppose further, for simplicity, that $\overline{\psi}$ is trivial on $\pi\mathcal{O}_{\overline{F}}$ but not on $\mathcal{O}_{\overline{F}}$, and that the Haar measure on $\overline{F}$ has been chosen such that $\mathcal{O}_{\overline{F}}$ has measure 1; then

$$\int_{\pi^j\mathcal{O}_{\overline{F}}} \overline{\psi}(\eta(a)y)\,dy = \begin{cases} 0 & j \leq -\nu_{\overline{F}}(\eta(a)) \\ q^{-j} & j > -\nu_{\overline{F}}(\eta(a)) \end{cases}$$

Therefore

$$\int^F \psi_a(x)\,\mathrm{char}_{t(\gamma)\rho^{-1}(\pi^j\mathcal{O}_{\overline{F}})}(x) = \begin{cases} 0 & w(a) < (-j+1,-\gamma) \\ q^{-j}X^\gamma & w(a) \geq (-j+1,-\gamma), \end{cases}$$

Finally, as $\mathrm{char}_{t(\gamma)\rho^{-1}(\pi^j\mathcal{O}_{\overline{F}}^\times)} = \mathrm{char}_{t(\gamma)\rho^{-1}(\pi^j\mathcal{O}_{\overline{F}})} - \mathrm{char}_{t(\gamma)\rho^{-1}(\pi^{j+1}\mathcal{O}_{\overline{F}})}$, it follows that

$$\int^F \psi_a(x)\,\mathrm{char}_{t(\gamma)\rho^{-1}(\pi^j\mathcal{O}_{\overline{F}}^\times)}(x)\,dx = \begin{cases} 0 & w(a) < (-j,-\gamma) \\ -q^{-j-1}X^\gamma & w(a) = (-j,-\gamma) \\ q^{-j}(1-q^{-1})X^\gamma & w(a) > (-j,-\gamma) \end{cases}$$

Compare with the example in [Fes03, §7].

### 2.3.2 The Fourier transform

Now that we can integrate functions twisted by characters, we may define a Fourier transform on $F$:

**Definition 2.3.10.** Let $f$ be in $\mathcal{L}(F,\psi)$. The Fourier transform of $f$, denoted $\widehat{f}$, is the $\mathbb{C}(\Gamma)$-valued function on $F$ defined by $\widehat{f}(x) = \int^F f(y)\psi(xy)\,dy$.

The Fourier transforms on $F$ and $\overline{F}$ are related as follows:

**Proposition 2.3.11.** *Let $g$ be Haar integrable on $\overline{F}$, and $\gamma \in \Gamma$, $a,b \in F$; set $f = g^{a,\gamma}\psi_b$, the product of a lifted function with a good character. Then*

$$\widehat{f} = \psi(ab)\widehat{g}^{-b,-\gamma}\psi_a\,X^\gamma$$

*where $\widehat{g}$ is the Fourier transform of $g$ with respect to $\overline{\psi}$.*

*Proof.* By definition of the Fourier transform, $x \in F$ implies

$$\widehat{f}(x) = \int^F g^{a,\gamma}(y)\psi((b+x)y)\,dy. \tag{$*$}$$

This is zero if $\gamma < -\nu(b+x)$, i.e. if $x \notin -b + t(-\gamma)\mathcal{O}_F$. Conversely, suppose that $x = -b + t(-\gamma)x_0$, where $x_0 \in \mathcal{O}_F$; then the integrand in $(*)$ is

$$g^{a,\gamma}\psi_{t(-\gamma)x_0} = \psi(t(-\gamma)ax_0)g^{a,\gamma}\overline{\psi}_{\overline{x}_0}^{a,\gamma},$$

an identity which is easily checked by evaluating on $a + t(\gamma)\mathcal{O}_F$. So

$$\begin{aligned} \widehat{f}(x) &= \psi(t(-\gamma)ax_0)\int^F g^{a,\gamma}(y)\overline{\psi}_{\overline{x}_0}^{a,\gamma}(y)\,dy \\ &= \psi(t(-\gamma)ax_0)\widehat{g}(\overline{x}_0)\,X^\gamma \\ &= \psi(a(x+b))\widehat{g}(\overline{x}_0)\,X^\gamma, \end{aligned}$$

which completes the proof. □

Let $\mathcal{S}(F,\psi)$ denote the subspace of $\mathcal{L}(F,\psi)$ spanned over $\mathbb{C}(\Gamma)$ by functions of the form $g^{a,\gamma}\psi_b$, for $g$ a Schwartz-Bruhat function on $\overline{F}$, $\gamma \in \Gamma$, $a,b \in F$. Recall that the Schwartz-Bruhat space on $\overline{F}$ is invariant under the Fourier transform and that there exists a positive real $\lambda$ such that for any Schwartz-Bruhat function $g$, Fourier inversion holds: $\widehat{\widehat{g}}(x) = \lambda g(-x)$ for all $x \in \overline{F}$. The following proposition extends these results to $F$:

**Proposition 2.3.12.** *The space $\mathcal{S}(F,\psi)$ is invariant under the Fourier transform. For $f$ in $\mathcal{S}(F,\psi)$, a double transform formula holds: $\widehat{\widehat{f}}(x) = \lambda f(-x)$ for all $x \in F$.*

*Proof.* By linearity it suffices to consider the case $f = g^{a,\gamma}\psi_b$, for $\gamma \in \Gamma$, $a, b \in F$, and $g$ a Schwartz-Bruhat function on $\overline{F}$. Then $\widehat{f} = \psi(ab)\widehat{g}^{-b,-\gamma}\psi_a X^\gamma$ belongs to $\mathcal{S}(F,\psi)$ and so

$$\widehat{\widehat{f}} = \psi(ab)\,\psi(-ba)\widehat{\widehat{g}}^{-a,\gamma}\psi_{-b}X^{-\gamma}\,X^\gamma = (\widehat{\widehat{g}})^{-a,\gamma}\psi_{-b},$$

by proposition 2.3.11. Apply the inversion formula for $g$ to complete the proof. ☐

**Remark 2.3.13.** Let us consider the dependence of the theory on the choice of character $\psi$; let $\psi'$ be another good character of $F$. In the interesting case of a higher local field, self-duality implies that $\psi' = \psi_\alpha$ for some $\alpha \in F^\times$; so we will restrict to this case and assume henceforth $\psi' = \psi_\alpha$. Then $\mathcal{L}(F,\psi) = \mathcal{L}(F,\psi')$, where $\mathcal{L}(F,\psi')$ is defined in the same way as $\mathcal{L}(F,\psi)$ but replacing $\psi$ by $\psi'$; further, the uniqueness of the extension of $\int^F$ given by proposition 2.3.7 shows that this extension does not depend on $\psi$.

Let $\mathfrak{f}$ be the conductor of $\psi'$, and $\overline{\psi'}$ the induced character of $\overline{F}$; thus $\overline{\psi'}(\overline{x}) = \psi'(t(\mathfrak{f}-1)x)$ for $x \in \mathcal{O}_F$. By lemma 2.3.3, $\overline{\psi'} = \overline{\psi}_{\eta(\alpha)}$, and $\mathfrak{f} = 1 - \nu(\alpha)$.

Let $g$ be Haar integrable on $\overline{F}$, and $\gamma \in \Gamma$, $a, b \in F$; set $f = g^{a,\gamma}\psi'_b$. Let $\check{f}$ denote the Fourier transform of $f$ with respect to $\psi'$; then for $y \in F$,

$$\begin{aligned}
\check{f}(y) &= \int^F f(x)\psi'(yx)\,dx \\
&= \widehat{g^{a,\gamma}\psi_{\alpha b}}(\alpha y) \\
&= \psi(\alpha ab)\widehat{g}^{-\alpha b,-\gamma}(\alpha y)\psi_a(\alpha y)\,X^\gamma,
\end{aligned}$$

by proposition 2.3.11. Further, $y \mapsto \widehat{g}^{-\alpha b,-\gamma}(\alpha y)$ is the lift of $v \mapsto \widehat{g}(\eta(\alpha)v)$ at $-b, -\gamma - \nu(\alpha)$, an identity easily proved (or see the proof of lemma 2.4.1 below). Also, $\widehat{g}(\eta(\alpha)v) = \check{g}(v)$, where $\check{g}$ is the Fourier transform of $g$ with respect to $\overline{\psi}'$, and so the analogue of proposition 2.3.11 follows:

$$\check{f} = \psi'(ab)\check{g}^{-b,-\gamma-\nu(\alpha)}\psi'_a\,X^\gamma.$$

For $f$ in $\mathcal{S}(F,\psi') = \mathcal{S}(F,\psi)$, the analogue of proposition 2.3.12 now follows: $\check{\check{f}} = \check{\check{g}}^{-a,\gamma}\psi'_{-b}\,X^{-\nu(\alpha)}$. That is,

$$\check{\check{f}}(x) = \lambda' f(-x)\,X^{\mathfrak{f}-1}$$

for all $x \in F$, where $\lambda'$ is the double transform constant associated to $\overline{\psi'}$ (see the paragraph preceding proposition 2.3.12).

## 2.4 Integration on $F^\times$

In this section, we consider integration over the multiplicative group $F^\times$. By analogy with the case of a local field, we are interested in those functions $\phi$ of $F^\times$ for which $x \mapsto \phi(x)|x|^{-1}$ is integrable on $F$, where $|\cdot|$ is a certain modulus defined below.

Let $|\cdot| = |\cdot|_{\overline{F}}$ denote the absolute value on $\overline{F}$ normalised by the condition $\int g(\alpha x)\,dx = |\alpha|^{-1}\int g(x)\,dx$ for $g \in \mathcal{L}$, $\alpha \in F^\times$. First we lift this absolute value to $F$:

**Lemma 2.4.1.** *Let $f$ be a $\mathbb{C}(\Gamma)$-valued integrable function on $F$ and $\alpha \in F^\times$. Then the scaled function $x \mapsto f(\alpha x)$ also belongs to $\mathcal{L}(F)$, and*

$$\int^F f(\alpha x)\,dx = |\eta(\alpha)|^{-1}X^{-\nu(\alpha)}\int^F f(x)\,dx$$

*(for the definition of $\eta$ refer back to the notations introduced at the start of the chapter).*

*Proof.* By linearity we may assume that $f$ is the lift of a function from $\mathcal{L}$; $f = g^{a,\gamma}$ say. Then for all $x \in \alpha^{-1}(a + t(\gamma)\mathcal{O}_F)$,

$$f(\alpha x) = g(\overline{(\alpha x - a)t(-\gamma)}) = g(\overline{\eta(\alpha)}\ \overline{(x - \alpha^{-1}a)t(\nu(\alpha) - \gamma)})$$

So the function $x \mapsto f(\alpha x)$ is the lift of the function $y \mapsto g(\overline{\eta(\alpha)}y)$ at $\alpha^{-1}a, \gamma - \nu(\alpha)$. This has integral

$$\begin{aligned}\int_{\overline{F}} g(\eta(\alpha)y)\,dy\ X^{\gamma-\nu(\alpha)} &= |\eta(\alpha)|^{-1} \int_{\overline{F}} g(y)\,dy\,X^{\gamma}X^{-\nu(\alpha)} \\ &= |\eta(\alpha)|^{-1}X^{-\nu(\alpha)} \int^{F} f(x)\,dx,\end{aligned}$$

as required. □

**Remark 2.4.2.** Lemma 2.4.1 remains valid if $\mathcal{L}(F)$ is replaced by $\mathcal{L}(F,\psi)$.

The lemma and remark suggest the follows definition:

**Definition 2.4.3.** Let $\alpha$ be in $F^\times$; the *absolute value* of $\alpha$ is defined to be $|\alpha| = |\eta(\alpha)|X^{\nu(\alpha)}$.

Let $\mathcal{L}(F^\times,\psi)$ be the set of $\mathbb{C}(\Gamma)$-valued functions $\phi$ on $F^\times$ for which $x \mapsto \phi(x)|x|^{-1}$, a function of $F^\times$, may be extended to $F$ to give a function in $\mathcal{L}(F,\psi)$. The integral of such a function over $F^\times$ is defined to be

$$\int^{F^\times} \phi(x)\,d\overset{\times}{x} = \int^{F} \phi(x)|x|^{-1}\,dx,$$

where the integrand on the right is really the extension of the function to $F$.

**Remark 2.4.4.** There is no ambiguity in the definition of the integral over $F^\times$, for $x \mapsto \phi(x)|x|^{-1}$ can have at most one extension to $\mathcal{L}(F,\psi)$. This follows from the fact that $\mathcal{L}(F,\psi)$ does not contain $\mathrm{char}_{\{0\}}$.

$\mathcal{L}(F^\times,\psi)$ is a $\mathbb{C}(\Gamma)$-space of $\mathbb{C}(\Gamma)$-valued functions, and $\int^{F^\times}$ is a $\mathbb{C}(\Gamma)$-linear functional. Moreover, the integral is invariant under multiplication in the following sense:

**Proposition 2.4.5.** *If $\phi$ belongs to $\mathcal{L}(F^\times,\psi)$ and $\alpha$ is in $F^\times$, then $x \mapsto \phi(\alpha x)$ belongs to $\mathcal{L}(F^\times,\psi)$ and $\int^{F^\times} \phi(\alpha x)\,d\overset{\times}{x} = \int^{F^\times} \phi(x)\,d\overset{\times}{x}$.*

*Proof.* Let $x \mapsto \phi(x)|x|^{-1}$ be the restriction to $F^\times$ of $f \in \mathcal{L}(F,\psi)$, say. Then $x \mapsto \phi(\alpha x)|x|^{-1} = |\alpha|\phi(\alpha x)|\alpha x|^{-1}$ is the restriction to $F^\times$ of $x \mapsto |\alpha|f(\alpha x)$, which belongs to $\mathcal{L}(F,\psi)$ by lemma 2.4.1. By the same lemma,

$$\begin{aligned}\int^{F^\times} \phi(\alpha x)\,d\overset{\times}{x} &= \int^{F} |\alpha|f(\alpha x)\,dx \\ &= |\alpha||\alpha|^{-1}\int^{F} f(x)\,dx \\ &= \int^{F^\times} \phi(x)\,d\overset{\times}{x},\end{aligned}$$

as required. □

**Example 2.4.6.** We compute a couple of integrals on $F^\times$:

(i) Let $g$ be Haar integrable on $\overline{F}$, $a \in F$, $\gamma \in \Gamma$, and assume $0 \notin a + t(\gamma)\mathcal{O}_F$. Let $\phi$ be the restriction of $g^{a,\gamma}$ to $F^\times$. Then $\phi \in \mathcal{L}(F^\times, \psi)$, and

$$\int^{F^\times} \phi(x)\, d\overset{\times}{x} = |a|^{-1} \int^{F} g^{a,\gamma}(x)\, dx.$$

Indeed, $x \in a+t(\gamma)\mathcal{O}_F$ implies $\eta(x) = \eta(a)$, and so $x \mapsto \phi(x)|x|^{-1}$ is the restriction of $|a|^{-1}g^{a,\gamma}$ to $F^\times$.

(ii) Let $g$ be Haar integrable on $\overline{F}^\times$, and let $\phi$ be the function on $F^\times$ which vanishes off $\mathcal{O}_F^\times$ and satisfies $\phi(x) = g(\overline{x})$ for $x \in \mathcal{O}_F^\times$. Then $\phi \in \mathcal{L}(F^\times, \psi)$ and

$$\int^{F^\times} \phi(x)\, d\overset{\times}{x} = \int_{\overline{F}} g(x)|x|^{-1}\, dx.$$

Indeed, let $h$ be the extension of $x \mapsto g(x)|x|^{-1}$ to $\overline{F}$ defined by $h(0) = 0$. Then $h$ is Haar integrable on $F$, and $h^{0,0} \in \mathcal{L}(F)$ restricts to the function of $F^\times$ given by $x \mapsto \phi(x)|x|^{-1}$.

In this way, the integral on $F^\times$ lifts the Haar integral on $\overline{F}^\times$, just as integral on $F$ lifts the Haar integral on $\overline{F}$.

## 2.5 Local zeta integrals

In the remainder of this chapter we will discuss (generalisations of) local zeta integrals. We begin by summarising the main results of local zeta integrals for the local field $\overline{F}$; see [Mor05, chapter I.2]. Let $g$ be a Schwartz-Bruhat function on $\overline{F}$, $\omega$ a quasi-character of $\overline{F}^\times$, and $s$ complex. The associated local zeta integral on $\overline{F}$ is

$$\zeta_{\overline{F}}(g, \omega, s) = \int_{\overline{F}^\times} g(x)\omega(x)|x|^s\, d\overset{\times}{x};$$

this is well-defined (i.e. the integrand is integrable) for $\Re(s)$ sufficiently large. Associated to $\omega$ there is a meromorphic function $L(\omega, s)$, the local L-function, with the following properties:

(AC) Analytic continuation, with the poles 'bounded' by the L-function: for all Schwartz-Bruhat functions $g$, $\zeta_{\overline{F}}(g, \omega, s)/L(\omega, s)$, which initially only defines a holomorphic function for $\Re(s)$ sufficiently large, in fact has analytic continuation to an entire function

$$Z_{\overline{F}}(g, \omega, s)$$

of $s$.

(L) 'Minimality' of the L-function: there is a Schwartz-Bruhat function $g$ for which

$$Z_{\overline{F}}(g, \omega, s) = 1$$

for all $s$.

(FE) Functional equation: there is an entire function $\varepsilon(\omega, s)$, such that for all Schwartz-Bruhat functions $g$,

$$Z_{\overline{F}}(\widehat{g}, \omega^{-1}, 1 - s) = \varepsilon(\omega, s) Z_{\overline{F}}(g, \omega, s).$$

Moreover, $\varepsilon(\omega, s)$ is of exponential type, i.e. $\varepsilon(\omega, s) = aq^{bs}$ for some complex $a$ and integer $b$.

Having lifted aspects of additive measure, multiplicative measure, and harmonic analysis from the local field $\overline{F}$ up to $F$, we now turn to lifting these results for local zeta integrals. Later, in section 2.7, we will assume that $F$ is a two-dimensional local field and consider a different, more arithmetic, local zeta integral. To avoid confusion between the two we may later refer to those in this section as being *one-dimensional*; the terminology is justified by the fact that this section concerns lifting the usual (one-dimensional) zeta integrals on $\overline{F}$ up to $F$.

**Definition 2.5.1.** For $f$ in $\mathcal{S}(F,\psi)$, $\omega : \mathcal{O}_F^\times \to \mathbb{C}^\times$ a homomorphism, and $s$ complex, the associated *(one-dimensional) local zeta integral* is

$$\zeta_F^{1\mathrm{d}}(f,\omega,s) = \int^{F^\times} f(x)\omega(x)|x|^s \operatorname{char}_{\mathcal{O}_F^\times}(x) d\overset{\times}{x},$$

assuming that the integrand is integrable on $F^\times$.

**Remark 2.5.2.** The integral is taken over $\mathcal{O}_F^\times$, instead of the full multiplicative group of the field, because this will be more natural in the later study of two-dimensional zeta integrals.

We will focus on the situation where $\omega$ is trivial on $1 + t(1)\mathcal{O}_F$; that is, there is a homomorphism $\overline{\omega} : \overline{F}^\times \to \mathbb{C}^\times$ such that $\omega(x) = \overline{\omega}(\overline{x})$ for all $x \in \mathcal{O}_F^\times$. If this induced homomorphism $\overline{\omega}$ is actually a quasi-character (i.e. if it is continuous), then we will say that $\omega$ is a *good (multiplicative) character*; just as for additive characters, this imitates a continuity condition.

Restricting to such tame characters is a definite problem with the current theory. The difficult of twisting additive characters by ramified multiplicative characters also appears in motivic integration; for example, the current theories of motivic Igusa zeta functions [DL98] and motivic exponential sums [Clu08a] [Clu08b] do not apply to ramified characters.

### 2.5.1 Explicit calculations and analytic continuation

We perform explicit calculations to obtain formulae for local zeta integrals attached to a good character:

**Lemma 2.5.3.** *Let $\omega$ be a good character of $\mathcal{O}_F^\times$; let $f = g^{a,\gamma}\psi_b$ be the product of a lifted function and a character, where $g$ is Schwartz-Bruhat on $\overline{F}$, $a, b \in F$, $\gamma \in \Gamma$. Then we have explicit formulae for the local zeta integrals in the following cases:*

*(i) Suppose that $\nu(a) < \min(\gamma, 0)$; or that $0 < \nu(a) < \gamma$; or that $0 < \gamma \leq \nu(a)$. Then $f(x)\omega(x)|x|^s \operatorname{char}_{\mathcal{O}_F^\times}(x) = 0$ for all $x \in F$, $s \in \mathbb{C}$.*

*(ii) Suppose $0 = \nu(a) < \gamma$. Then $f(x)\omega(x)|x|^s \operatorname{char}_{\mathcal{O}_F^\times}(x) = \omega(a)|a|^s f(x)$ for all $x \in F$, $s \in \mathbb{C}$; the local zeta integral is well-defined for all $s$ and is given by*

$$\zeta_F^{1d}(f,\omega,s) = \omega(a)|a|^{s-1} \int^F f(x)\, dx.$$

*(iii) Suppose $0 = \gamma \leq \nu(a)$. Then the local zeta integral is well-defined for $\Re(s)$ sufficiently large, and is given by*

$$\zeta_F^{1d}(f,\omega,s) = \begin{cases} \zeta_{\overline{F}}(g_1\overline{\psi}_{\overline{b}}, \overline{\omega}, s) & \text{if } \nu(b) \geq 0 \\ 0 & \text{if } \nu(b) < 0 \end{cases}$$

*where $g_1$ is the Schwartz-Bruhat function on $\overline{F}$ given by $g_1(u) = g(u - a)$.*

*Proof.* In any of the cases in (i), $f$ vanishes on $\mathcal{O}_F^\times$; so $f(x)\,\mathrm{char}_{\mathcal{O}_F^\times}(x) = 0$ for all $x \in F$.

In case (ii), $a + t(\gamma)\mathcal{O}_F$ is contained in $\mathcal{O}_F^\times$, and $x \in a + t(\gamma)\mathcal{O}_F$ implies $\omega(x)|x|^s = \omega(a)|a|^s$; this implies that $f(x)\omega(x)|x|^s\,\mathrm{char}_{\mathcal{O}_F^\times}(x) = f(x)\omega(a)|a|^s\,\mathrm{char}_{\mathcal{O}_F^\times}(x)$ for all $x \in F$, $s \in \mathbb{C}$. Moreover, for all $x \in F$, these results again imply $f(x)|x|^{-1} = f(x)|a|^{-1}$; therefore $f$ is integrable over $F^\times$, with $\int^{F^\times} f(x)\,d\overset{\times}{x} = \int^F f(x)\,dx$.

Finally we turn to case (iii). First note that $g^{a,\gamma}\omega|\cdot|^{s-1}\,\mathrm{char}_{\mathcal{O}_F^\times}$ is the lift of $g_1\overline{\omega}|\cdot|^{s-1}\,\mathrm{char}_{\overline{F}^\times}$ at $0,0$. Now, if $\Re(s)$ is sufficiently large then the theory of local zeta integrals for $\overline{F}$ implies that $g_1\overline{\omega}|\cdot|^{s-1}\,\mathrm{char}_{\overline{F}^\times}$ is integrable on $\overline{F}$; thus $f\omega|\cdot|^{s-1}\,\mathrm{char}_{\mathcal{O}_F^\times}$ is the restriction to $F^\times$ of $(g_1\overline{\omega}|\cdot|^{s-1}\,\mathrm{char}_{\overline{F}^\times})^{0,0}\psi_b$, a function which belong to $\mathcal{L}(F,\psi)$.

By definition of the integral on $F^\times$ it follows that (for $\Re(s)$ sufficiently large) $f\omega|\cdot|^{s-1}\,\mathrm{char}_{\mathcal{O}_F^\times}$ belongs to $\mathcal{L}(F^\times,\psi)$, and

$$\begin{aligned}
\int^{F^\times} f(x)\omega(x)|x|^s\,\mathrm{char}_{\mathcal{O}_F^\times}(x)\,d\overset{\times}{x} &= \int^F (g_1\overline{\omega}|\cdot|^{s-1}\,\mathrm{char}_{\overline{F}^\times})^{0,0}(x)\psi_b(x)dx \\
&= \begin{cases} \int^F (g_1\overline{\omega}|\cdot|^{s-1}\,\mathrm{char}_{\overline{F}^\times})^{0,0}(x)\,dx & \text{if } \nu(b) > 0 \\ \int^F (g_1\overline{\omega}|\cdot|^{s-1}\,\mathrm{char}_{\overline{F}^\times}\,\overline{\psi_b})^{0,0}(x)\,dx & \text{if } \nu(b) = 0 \\ 0 & \text{if } \nu(b) < 0 \end{cases} \\
&= \begin{cases} \int g_1(u-a)\overline{\omega}(u)|u|^{s-1}\,\mathrm{char}_{\overline{F}^\times}(u)\,du & \text{if } \nu(b) > 0 \\ \int g_1(u-a)\overline{\omega}(u)|u|^{s-1}\,\mathrm{char}_{\overline{F}^\times}(u)\overline{\psi_b}(u)\,du & \text{if } \nu(b) = 0 \\ 0 & \text{if } \nu(b) < 0 \end{cases} \\
&= \begin{cases} \zeta(g_1,\overline{\omega},s) & \text{if } \nu(b) > 0 \\ \zeta(g_1\overline{\psi}_{\overline{b}},\overline{\omega},s) & \text{if } \nu(b) = 0 \\ 0 & \text{if } \nu(b) < 0 \end{cases}
\end{aligned}$$

as required. □

**Remark 2.5.4.** Let $\omega$ and $f = g^{a,\gamma}\psi_b$ be as in the statement of the previous lemma. The lemma treats all possible relations between $\nu(a)$, $\gamma$, and $0$ with the exception of $\nu(a) \geq \gamma < 0$. There are interesting complications in this case: since $f\,\mathrm{char}_{\mathcal{O}_F^\times} = f(0)\,\mathrm{char}_{\mathcal{O}_F^\times}$, we wish to calculate

$$\zeta_F^{\mathrm{1d}}(f,\omega,s) = f(0)\int^{F^\times} \psi_b(x)\omega(x)|x|\,\mathrm{char}_{\mathcal{O}_F^\times}(x)\,d\overset{\times}{x}.$$

For example, if $\psi_b$ has conductor $1$ then

$$\psi_b\omega|\cdot|^s\,\mathrm{char}_{\mathcal{O}_F^\times} = (\overline{\psi_b}\overline{\omega}|\cdot|^s\,\mathrm{char}_{\overline{F}^\times})^{0,0}$$

and so the zeta integral is formally given by

$$\zeta_F^{\mathrm{1d}}(f,\omega,s) = f(0)\int_{\overline{F}^\times} \overline{\psi_b}(x)\overline{\omega}(x)|x|^s\,d\overset{\times}{x}.$$

If $\overline{F}$ were finite then this would be a Gauss sum over a finite field, a standard ingredient of local zeta integrals; with $\overline{F}$ a local field it is unclear how to interpret this but the following examples provide insight.

**Example 2.5.5.** Suppose $K = \overline{F}$ is non-archimedean and consider the formal integral

$$\int_{\overline{F}^\times} \psi_K(x)\omega(x)\,d\overset{\times}{x}$$

with $\psi_K$ an additive character and $\omega$ a multiplicative quasi-character with $\Re(\omega) > 0$ (recall that this is defined by $|\omega(x)| = |x|^{\Re(\omega)}$ for all $x$). If $n$ is a sufficiently small integer, then we have a convergent integral

$$\int_{w^{-1}(n)} \psi_K(x)\omega(x)\, d\overset{\times}{x} = 0,$$

where $w$ is the discrete valuation of $\overline{F}$; so for $n$ sufficiently small the value of the integral

$$\int_{\{x:w(x)\geq n\}} \psi_K(x)\omega(x)\, d\overset{\times}{x}$$

does not depend on $n$. It seems reasonable to adopt this value as the meaning of the expression $\int_{\overline{F}^\times} \psi_K(x)\omega(x)\, d\overset{\times}{x}$.

**Example 2.5.6.** Suppose $\overline{F} = \mathbb{R}$ and we wish to understand the formal integral

$$\int_0^\infty e^{2\pi i x}\, dx.$$

Replacing $2\pi i$ by some complex $\lambda$ with $\Re(\lambda) < 0$ gives a true integral with value

$$\int_0^\infty e^{\lambda x}\, dx = -1/\lambda.$$

Similarly we have

$$\int_{-\infty}^0 e^{\lambda x}\, dx = 1/\lambda$$

for $\Re(\lambda) > 0$. This suggests that, formally,

$$\int_{\mathbb{R}} e^{2\pi i x}\, dx = -\int_\infty^0 e^{2\pi i x}\, dx + \int_0^\infty e^{2\pi i x}\, dx = 0$$

and

$$\int_{\mathbb{R}} e^{2\pi i x} \mathrm{sign}(x)\, dx = -\int_{-\infty}^0 e^{2\pi i x}\, dx + \int_0^\infty e^{2\pi i x}\, dx = -i/\pi$$

where sign$(x)$ is the sign ($\pm$) of $x$.

The first of these integrals is already taken into account by our measure theory: if $F = \mathbb{R}((t))$ and $\psi$ is the character defined by $\psi(\sum_n a_n t^n) = e^{2\pi i a_0}$ (see example 2.3.2), then $\psi\, \mathrm{char}_{\mathcal{O}_F}$ belongs to $\mathcal{L}(F, \psi)$ and $\int^F \psi(x)\, \mathrm{char}_{\mathcal{O}_F}(x)\, dx = 0$. But $\psi\, \mathrm{char}_{\mathcal{O}_F}$ is also the lift of $x \mapsto e^{2\pi x i}$ at $0, 0$ so formally $\int^F \psi(x)\, \mathrm{char}_{\mathcal{O}_F}(x)\, dx = \int_{\mathbb{R}} e^{2\pi i x}\, dx$.

Such manipulations of integrals are common in quantum field theory (see e.g. [JL00]) and I am grateful to Dr. Jorma Louko for discussions in this subject. That such integrals appear here further suggests a possible relation between this theory and Feynman path integrals. More evidence for such relations may be found in sections 16 and 18 of [Fes06].

Ignoring the complications caused by this difficult case we may now deduce the first main properties of some local zeta functions. Appendix 2.B explains what is meant by a $\mathbb{C}(\Gamma)$-valued holomorphic function.

**Proposition 2.5.7.** *Let $\omega$ be a good character of $\mathcal{O}_{\overline{F}}^\times$, and let $f$ be in $\mathcal{S}(F, \psi)$; assume that $f$ may be written as a finite sum of terms $f = \sum_i g_i^{a_i,\gamma_i} \psi_{b_i}\, p_i$ where each $g_i^{a_i,\gamma_i} \psi_{b_i}$ is treated by one of the cases of lemma 2.5.3 and $p_i \in \mathbb{C}(\Gamma)$. Then* *(i) For $\Re(s)$ sufficiently large, the integrand of the local zeta integral $\zeta_F^{1d}(f,\omega,s)$ is integrable over $F^\times$ and so the local zeta integral is well-defined.*

*(ii) $\zeta_F^{1d}(f,\omega,s)/L(\overline{\omega},s)$ has entire analytic continuation: that is, there is a $\mathbb{C}(\Gamma)$-valued holomorphic function $Z_F^1(f,\omega,s)$ on $\mathbb{C}$ which equals $\zeta_F^1(f,\omega,s)/L(\overline{\omega},s)$ for $\Re(s)$ sufficiently large.*

*(iii) There is some function $g \in \mathcal{S}(F,\psi)$ for which $Z_F^1(g,\omega,s) = 1$ for all complex $s$.*

*Proof.* The results follow by linearity, the previous lemma, and the main properties of local zeta integrals on $\overline{F}$. □

It is important to extend this result to all $f$ in $\mathcal{S}(F,\psi)$; therefore the complication discussed in remark 2.5.4 must be resolved.

**Remark 2.5.8.** We say a few words about functional equations. There is no result as satisfactory as for zeta functions of a one-dimensional local field, and there is no reason why there should be due to the $\mathrm{char}_{\mathcal{O}_{\overline{F}}^\times}$ factor appearing in our definition of the local zeta integrals. The most interesting issue here is making a functional equation compatible with the difficulties caused by remark 2.5.4; this should indicate correctness (or not) of examples 2.5.5 and 2.5.6.

## 2.6 Local functional equations with respect to $s$ goes to $2-s$

In this section we continue our study of local zeta functions, considering the problem of modifying the functional equation (FE) on $\overline{F}$ so that the symmetry is not $s$ goes to $1-s$, but instead $s$ goes to $2-s$. This is in anticipation of the next section on two-dimensional zeta integrals, where such a functional equation is natural.

Since this section is devoted to the residue field $\overline{F}$, we write $K = \overline{F}$. We fix an non-trivial additive character $\psi_K$ of $K$ (until proposition, 2.6.13 where we consider dependence on this choice). Fourier transforms of complex-valued functions are taken with respect to this character (and the measure which was fixed at the start of the chapter): $\widehat{g}(y) = \int g(x)\psi_K(xy)\,dx$.

The two main proofs of (FE) are Tate's [Tat67] using Fubini's theorem, and Weil's [Wei95] using distributions. For Weil, a fundamental identity in the non-archimedean case is

$$\widehat{g(\alpha\,\cdot)} = |\alpha|^{-1}\widehat{g}(\alpha^{-1}\,\cdot) \qquad (*)$$

for $\alpha \in K^\times$, where we write $g(\alpha\,\cdot)$ for the function $x \mapsto g(\alpha x)$, notation which we shall continue to use.

The aim of this section is to replace the Fourier transform with a new transform so that $(*)$ holds with $|\alpha|^{-2}$ in place of $|\alpha|^{-1}$. This leads to a modification of the local functional equation, with $|\cdot|^2$ in place of $|\cdot|$; see propositions 2.6.1 and 2.6.24.

### 2.6.1 Non-archimedean case

We assume first that $K$ is a non-archimedean local field, with residue field $\mathbb{F}_q$. The following proposition precisely explains the importance of the identity $\widehat{g(\alpha\,\cdot)} = |\alpha|^{-1}\widehat{g}(\alpha^{-1}\,\cdot)$:

**Proposition 2.6.1.** *Suppose that $g \mapsto g^*$ is a $\mathbb{C}$-linear endomorphism of the Schwartz-Bruhat space $\mathcal{S}(K)$ of $K$ which satisfies, for some fixed integer $n$,*

$$g(\alpha\,\cdot)^* = |\alpha|^{-n} g^*(\alpha^{-1}\,\cdot)$$

*for all $g \in \mathcal{S}(K)$, $\alpha \in K^\times$. Let $\omega$ be a quasi-character of $K^\times$. Then there is a unique entire function $\varepsilon_*(\omega, s)$ which satisfies*

$$Z_K(g^*, \omega^{-1}, n-s) = \varepsilon_*(\omega, s) Z_K(g, \omega, s)$$

*for all $g \in \mathcal{S}(K)$, $\alpha \in K^\times$.*

*Proof.* Let $g$ be a Schwartz-Bruhat function on $K$, and $\alpha \in K^\times$. Then for $\Re(s)$ sufficiently large to ensure integrability, the identity

$$\zeta_K(g(\alpha\cdot), \omega, s) = \omega(\alpha)^{-1}|\alpha|^{-s}\zeta_K(g, \omega, s)$$

holds. Conversely, for $\Re(s)$ sufficiently small, the assumed property of $^*$ implies that

$$\zeta_K(g(\alpha\cdot)^*, \omega^{-1}, n-s) = \omega(\alpha)^{-1}|\alpha|^{-s}\zeta_K(g^*, \omega^{-1}, n-s).$$

Therefore, for all complex $s$,

$$Z_K(g(\alpha\cdot), \omega, s) = \omega(\alpha)^{-1}|\alpha|^{-s} Z_K(g, \omega, s)$$

and

$$Z_K(g(\alpha\cdot)^*, \omega^{-1}, n-s) = \omega(\alpha)^{-1}|\alpha|^{-s} Z_K(g^*, \omega^{-1}, n-s).$$

Hence the $\mathbb{C}$-linear functionals $\Lambda$ on $\mathcal{S}(K)$ given by

$$g \mapsto Z_K(g, \omega, s)$$

and

$$g \mapsto Z_K(g^*, \omega^{-1}, n-s)$$

(for fixed $s$) each satisfy $\Lambda(g(\alpha\cdot)) = \omega(\alpha)^{-1}|\alpha|^{-s}\Lambda(g)$ for all $g \in \mathcal{S}(K)$, $\alpha \in K^\times$. But the space of such functionals is one-dimensional (see e.g. [Mor05, I.2]) (for $\omega \neq |\cdot|^{-s}$) and there is $f \in \mathcal{S}(K)$ such that $Z_K(f, \omega, s) = 1$ for all $s$ (property (L) of local zeta integrals; see beginning of section 2.5); this implies the existence of an entire function $\varepsilon_*(\omega, s)$ as required. $\square$

**Remark 2.6.2.** Suppose that $^*$ maps $\mathcal{S}(K)$ *onto* $\mathcal{S}(K)$. Then there is $g \in \mathcal{S}(K)$ such that $Z_K(g^*, \omega^{-1}, n-s) = 1$ for all $s$ and so $\varepsilon_*(\omega, s)$ is nowhere vanishing.

Our aim now is to investigate the epsilon factors attached to a particular transform $^*$ which satisfies $g(\alpha\,\cdot)^* = |\alpha|^{-2} g^*(\alpha^{-1}\,\cdot)$. Let $w : K^\times \to \mathbb{Z}$ be the discrete valuation of $K$ and $\pi \in K$ a fixed prime.

**Definition 2.6.3.** Define

$$\nabla : K \to K, \quad x \mapsto \pi^{w(x)} x$$

(and $\nabla(0) = 0$).

For $g$ a complex-valued function on $K$, denote by $Wg$ the function

$$Wg(x) = \begin{cases} g(\pi^{-w(x)/2} x) & \text{if } w(x) \text{ is even} \\ g(\pi^{(-w(x)-1)/2} x) & \text{if } w(x) \text{ is odd} \end{cases}$$

(and $Wg(0) = g(0)$).

Assuming that $Wg$ is integrable on $K$, define the *$^*$-transform* (with respect to $\pi$) of $g$ by

$$g^* = \widehat{Wg} \circ \nabla.$$

**Remark 2.6.4.** Compare this definition with [Wei95] and [Fes03, §15], where Fesenko defines the transform on two copies of a two-dimensional local field $F \times F$.

The $^*$-transform depends on choice of prime $\pi$. We may also denote by $\nabla$ the composition operator $\nabla(g) = g \circ \nabla$.

The space of Schwartz-Bruhat functions $\mathcal{S}(K)$ is closed under the $^*$-transform.

It is easy to verify that the $^*$-transform has the desired property:

**Lemma 2.6.5.** *Suppose that $g$ is a Schwartz-Bruhat on $K$ and that $\alpha \in K^\times$. Then*

$$g(\alpha\,\cdot)^* = |\alpha|^{-2} g^*(\alpha^{-1}\,\cdot).$$

*Proof.* If $x \in F^\times$, then $W(g(\alpha\,\cdot))(x) = W(g)(\pi^{w(\alpha)}\alpha x)$. Hence

$$\widehat{W(g(\alpha\cdot))} = |\pi^{w(\alpha)}\alpha|^{-1}\widehat{Wg}(\pi^{-w(\alpha)}\alpha^{-1}\,\cdot).$$

Evaluating this at $\nabla(x)$ yields

$$g(\alpha\,\cdot)^*(x) = |\alpha|^{-2}\widehat{Wg}(\pi^{-w(\alpha)}\alpha^{-1}\pi^{w(x)}x) = |\alpha|^{-2}g^*(\alpha^{-1}x).$$

□

**Remark 2.6.6.** More generally, the previous lemma holds for any complex valued $g$ for which $Wg$ and $W(g(\alpha\cdot))$ are both integrable.

We now $^*$-transform several functions. Let $\mu$ be the measure of $\mathcal{O}_K$ under our chosen Haar measure and let $d$ be the conductor of $\psi_K$.

**Example 2.6.7.** Suppose $g = \operatorname{char}_{\pi^r\mathcal{O}_K}$. Then $Wg = \operatorname{char}_{\pi^{2r}\mathcal{O}_K}$, which has Fourier transform $\mu q^{-2r}\operatorname{char}_{\pi^{d-2r}\mathcal{O}_K}$. So the $^*$-transform of $g$ is

$$g^* = \mu q^{-2r}\operatorname{char}_{\pi^{\lceil d/2\rceil - r}\mathcal{O}_K},$$

where $\lceil d/2\rceil$ denotes the least integer not strictly less than $d/2$. Compare this with the Fourier transform

$$\widehat{g} = \mu q^{-r}\operatorname{char}_{\pi^{d-r}\mathcal{O}_K}.$$

**Example 2.6.8.** Suppose $h = \operatorname{char}_{1+\pi^r\mathcal{O}_K}$ with $r \geq 1$. Let $x \in K^\times$. If $w(x)$ is even, then $Wh(x) = 1$ if and only if $x \in 1 + \pi^r\mathcal{O}_K$; if $w(x)$ is odd, then $Wh(x) = 1$ if and only if $\pi^{-1}x \in 1 + \pi^r\mathcal{O}_K$. So

$$Wh = \operatorname{char}_{1+\pi^r\mathcal{O}_K} + \operatorname{char}_{\pi(1+\pi^r\mathcal{O}_K)},$$

whence

$$\widehat{Wh} = \mu q^{-r}\operatorname{char}_{\pi^{d-r}\mathcal{O}_K}\psi_K + \mu q^{-r-1}\operatorname{char}_{\pi^{d-r-1}\mathcal{O}_K}\psi_K(\pi\,\cdot).$$

For the remainder of this example assume $\mu = 1$, $d = 0$, $r = 2$; we shall compute the double $^*$-transform $h^{**}$.

It may be easily checked that if $x \in K$, then

$$\operatorname{char}_{\pi^{-2}\mathcal{O}_K}(\nabla(x))\psi_K(\nabla(x)) = \begin{cases} 0 & \text{if } x \notin \pi^{-1}\mathcal{O}_K \\ \psi_K(\pi^{-1}x) & \text{if } x \in \pi^{-1}\mathcal{O}_K^\times \\ 1 & \text{if } x \in \mathcal{O}_K \end{cases}$$

and

$$\mathrm{char}_{\pi^{-3}\mathcal{O}_K}(\nabla(x))\psi_K(\pi\nabla(x)) = \begin{cases} 0 & \text{if } x \notin \pi^{-1}\mathcal{O}_K \\ \psi_K(x) & \text{if } x \in \pi^{-1}\mathcal{O}_K^\times \\ 1 & \text{if } x \in \mathcal{O}_K. \end{cases}$$

From the identity for $\widehat{Wh}$ it now follows that

$$h^* = q^{-2}(\psi_K(\pi^{-1}\cdot) + q^{-1}\psi_K)\ \mathrm{char}_{\pi^{-1}\mathcal{O}_K^\times} + q^{-2}(1+q^{-1})\,\mathrm{char}_{\mathcal{O}_K}.$$

Set $h_1 = \psi_K(\pi^{-1}\cdot)\ \mathrm{char}_{\pi^{-1}\mathcal{O}_K^\times}$, $h_2 = q^{-1}\psi_K\ \mathrm{char}_{\pi^{-1}\mathcal{O}_K^\times}$; it may be checked that

$$\begin{aligned} Wh_1 &= \psi_K(\pi^{-1}\cdot)\ \mathrm{char}_{\pi^{-1}\mathcal{O}_K^\times} + \psi_K\ \mathrm{char}_{\pi^{-2}\mathcal{O}_K^\times} \\ Wh_2 &= q^{-1}\psi_K\ \mathrm{char}_{\pi^{-1}\mathcal{O}_K^\times} + q^{-1}\psi_K(\pi\cdot)\ \mathrm{char}_{\pi^{-2}\mathcal{O}_K^\times} \end{aligned}$$

Standard Fourier transform calculations now yield

$$\begin{aligned} \widehat{Wh_1} &= q\ \mathrm{char}_{-\pi^{-1}+\pi\mathcal{O}_K} - \mathrm{char}_{-\pi^{-1}+\mathcal{O}_K} + q^2\ \mathrm{char}_{-1+\pi^2\mathcal{O}_K} - q\ \mathrm{char}_{-1+\pi\mathcal{O}_K} \\ \widehat{Wh_2} &= \mathrm{char}_{-1+\pi\mathcal{O}_K} - q^{-1}\ \mathrm{char}_{\mathcal{O}_K} + q\ \mathrm{char}_{-\pi^{-1}+\pi^2\mathcal{O}_K} - \mathrm{char}_{\pi\mathcal{O}_K}. \end{aligned}$$

Further, by example 2.6.7, $W(\widehat{\mathrm{char}_{\mathcal{O}_K}}) = \mathrm{char}_{\mathcal{O}_K}$, and so

$$\begin{aligned} q^2\widehat{W(h^*)} = {} & q\ \mathrm{char}_{-\pi^{-1}+\pi\mathcal{O}_K} - \mathrm{char}_{-\pi^{-1}+\mathcal{O}_K} + q^2\ \mathrm{char}_{-1+\pi^2\mathcal{O}_K} - q\ \mathrm{char}_{-1+\pi\mathcal{O}_K} \\ & + \mathrm{char}_{-1+\pi\mathcal{O}_K} + q\ \mathrm{char}_{-\pi^{-1}+\pi^2\mathcal{O}_K} - \mathrm{char}_{\pi\mathcal{O}_K} + \mathrm{char}_{\mathcal{O}_K}. \end{aligned}$$

Now, $x \in K^\times$ implies $w(\nabla x)$ is even, and so

$$\begin{aligned} q^2\widehat{W(h^*)} \circ \nabla &= q^2\ \mathrm{char}_{-1+\pi^2\mathcal{O}_K} \circ\nabla - q\ \mathrm{char}_{-1+\pi\mathcal{O}_K} \circ\nabla \\ &\quad + \mathrm{char}_{-1+\pi\mathcal{O}_K} \circ\nabla - \mathrm{char}_{\pi\mathcal{O}_K} \circ\nabla + \mathrm{char}_{\mathcal{O}_K} \circ\nabla \\ &= q^2\ \mathrm{char}_{-1+\pi^2\mathcal{O}_K} - q\ \mathrm{char}_{-1+\pi\mathcal{O}_K} \\ &\quad + \mathrm{char}_{-1+\pi\mathcal{O}_K} - \mathrm{char}_{\pi\mathcal{O}_K} + \mathrm{char}_{\mathcal{O}_K}. \end{aligned}$$

That is,

$$h^{**} = q^{-2}\ \mathrm{char}_{\mathcal{O}_K^\times} - q^{-1}(1-q^{-1})\,\mathrm{char}_{-1+\pi\mathcal{O}_K} + \mathrm{char}_{-1+\pi^2\mathcal{O}_K}.$$

Note that although the definition of the $^*$-transform depends on choice of prime $\pi$, the double $^*$-transform $h^{**}$ of $h$ does not. This will be proved in general below.

These examples were specifically chosen to allow us to compute explicit formulae for the epsilon factors $\varepsilon_*(\omega, s)$:

**Example 2.6.9.** We calculate the epsilon factor attached to the $^*$-transform for the trivial character $1$. Suppose for simplicity that $\mathcal{O}_K$ has measure $1$ under our chosen Haar measure.

Let $f = \mathrm{char}_{\mathcal{O}_K}$. Example 2.6.7 implies $f^* = \mathrm{char}_{\pi^{\lceil d/2\rceil - r}\mathcal{O}_K}$; it is a standard calculation that $Z_K(f, 1, s) = 1 - q^{-1}$ and $Z_K(f^*, 1, 2-s) = (1-q^{-1})q^{\lceil d/2\rceil(s-2)}$ for all $s$. Therefore

$$\varepsilon_*(1, s) = q^{\lceil d/2\rceil(s-2)}$$

for all $s$.

**Example 2.6.10.** We now calculate the epsilon factor attached to the $^*$-transform for ramified quasi-characters. Continue to suppose that that $\mathcal{O}_K$ has measure $1$.

Let $\omega$ be a quasi-character of $K^\times$ of conductor $r > 0$; that is, $\omega|_{1+\pi^r\mathcal{O}_K} = 1$ but $\omega|_{1+\pi^{r-1}\mathcal{O}_K} \neq 1$

Let $h = \mathrm{char}_{1+\pi^r\mathcal{O}_K}$; so $\zeta_K(h,\omega,s)$ is constantly $m$, the measure of $1+\pi^r\mathcal{O}_K$ under $d\overset{\times}{x} = |x|^{-1}dx$. The aim is now to calculate $\zeta_K(h^*,\omega^{-1},2-s)$ without calculating $h^*$. By example 2.6.8, $Wh = h + h(\pi^{-1}\cdot)$, and so $\widehat{Wh} = \widehat{h} + q^{-1}\widehat{h}(\pi\cdot)$. Therefore

$$\begin{aligned}\zeta_K(h^*,\omega^{-1},2-s) &= \int_{K^\times} \widehat{h}(\pi^{w(x)}x)\omega(x)^{-1}|x|^{2-s}d\overset{\times}{x} \\ &\qquad + q^{-1}\int_{K^\times} \widehat{h}(\pi^{w(x)+1}x)\omega(x)^{-1}|x|^{2-s}d\overset{\times}{x} \\ &= \sum_{n\in\mathbb{Z}} q^{n(s-2)} \int_{w^{-1}(n)} \widehat{h}(\pi^n x)\omega(x)^{-1}d\overset{\times}{x} \\ &\qquad + q^{-1}\sum_{n\in\mathbb{Z}} q^{n(s-2)} \int_{w^{-1}(n)} \widehat{h}(\pi^{n+1}x)\omega(x)^{-1}d\overset{\times}{x} \\ &= \sum_{n} q^{n(s-2)}\omega(\pi)^{-n} \int_{\mathcal{O}_K^\times} \widehat{h}(\pi^{2n}x)\omega(x)^{-1}d\overset{\times}{x} \\ &\qquad + q^{-1}\sum_{n} q^{n(s-2)}\omega(\pi)^{-n} \int_{\mathcal{O}_K^\times} \widehat{h}(\pi^{2n+1}x)\omega(x)^{-1}d\overset{\times}{x}\end{aligned}$$

But by Tate's calculation [Tat67] when calculating the epsilon factor in this same case,

$$\int_{\mathcal{O}_K^\times} \widehat{h}(\pi^N x)\omega(x)^{-1}d\overset{\times}{x} = \begin{cases} q^{-r/2}m\rho_0(\omega^{-1}) & \text{if } N = d-r \\ 0 & \text{otherwise,}\end{cases}$$

where $\rho_0(\omega^{-1})$ is the *root number* of absolute value one

$$\rho_0(\omega^{-1}) = q^{-r/2}\sum_{\theta} \omega^{-1}(\theta)\psi_K(\pi^{d-r}\theta),$$

the sum being taken over coset representatives of $1+\pi^r\mathcal{O}_K$ in $\mathcal{O}_K^\times$.

Therefore

$$\begin{aligned}\zeta_K(h^*,\omega^{-1},2-s) &= \begin{cases} q^{(d-r)(s-2)/2}\omega(\pi)^{(r-d)/2}q^{-r/2}m\rho_0(\omega^{-1}) & d-r \text{ even} \\ q^{(d-r-1)(s-2)/2-1}\omega(\pi)^{(1+r-d)/2}q^{-r/2}m\rho_0(\omega^{-1}) & d-r \text{ odd}\end{cases} \\ &= q^{\lceil (r-d)/2\rceil(2-s)}\omega(\pi)^{\lceil (r-d)/2\rceil}q^{-r/2}\delta_{d-r}m\rho_0(\omega^{-1})\end{aligned}$$

where $\delta_{d-r} = 1$ if $r-d$ is even and $= q^{-1}$ if $r-d$ is odd. Finally, as we have already observed that $\zeta_K(h,\omega,s) = m$ for all $s$, and $L(\omega,s) = 1$ for such a character, we obtain

$$\varepsilon_*(\omega,s) = q^{\lceil (r-d)/2\rceil(2-s)}\omega(\pi)^{\lceil (r-d)/2\rceil}q^{-r/2}\delta_{d-r}\rho_0(\omega^{-1}).$$

**Remark 2.6.11.** More generally, if $\mathcal{O}_K$ has measure $\mu$ under our chosen Haar measure, then each of the epsilon factors above is multiplied by a factor of $\mu$.

Let us now consider what happens when we take the double transform $f^{**}$. If $\omega$ is ramified with conductor $r$, then

$$\begin{aligned}\varepsilon_*(\omega,s)\varepsilon_*(\omega^{-1},2-s) &= \mu^2 q^{2\lceil (r-d)/2\rceil}\delta_{d-r}^2 q^{-r}\rho_0(\omega^{-1})\rho_0(\omega) \\ &= \mu^2 q^{2\lceil (r-d)/2\rceil}\delta_{d-r}^2 q^{-r}\omega(-1)\overline{\rho_0(\omega)}\rho_0(\omega) \\ &= \mu^2 q^{r-d}\delta_{d-r}q^{-r}\omega(-1) \\ &= \mu^2 q^{-d}\delta_{d-r}\omega(-1).\end{aligned}$$

If we declare the conductor of an unramified character to be 0 then this formula remains valid for unramified $\omega$.

Therefore two applications of the functional equation imply that for all $f \in \mathcal{S}(K)$, all characters $\omega$ of conductor $r \geq 0$, and all complex $s$,

$$\zeta_K(f^{**}, \omega, s) = \mu^2 q^{-d} \delta_{d-r} \omega(-1) \zeta_K(f, \omega, s). \tag{†}$$

We will now proceed to use our results on epsilon factors to deduce properties of the $^*$-transform; the idea is to use identities between zeta integrals to obtain identities between the functions. The following result is clearly of great importance in this method:

**Lemma 2.6.12.** *Let $f \in \mathcal{S}(K)$ and suppose that $\zeta_K(f, \omega, s) = 0$ for all quasi-characters $\omega$ and complex $s$; then $f = 0$.*

*Proof.* Let $f$ be in $\mathcal{S}(K)$. Then $f - f(0)\,\mathrm{char}_{\mathcal{O}_K}$ belongs to $\mathcal{S}(K^\times)$ and so the zeta integral $\zeta_K(f(0)\,\mathrm{char}_{\mathcal{O}_K}, \omega, s)$ is well-defined for all $s$ and belongs to $\mathbb{C}[q^s, q^{-s}]$. Indeed, it suffices to observe that $\mathcal{S}(K^\times)$ is spanned by $\mathrm{char}_{a+\pi^m\mathcal{O}_K}$ where $w(a) > m$, and $\zeta_K(\mathrm{char}_{a+\pi^m\mathcal{O}_K}, \omega, s) = q^{-w(a)s} \int_{a+\pi^m\mathcal{O}_K} \omega(s)\, d\overset{\times}{x}$.

However, for $\omega = 1$ the trivial character,

$$\zeta_K(f(0)\,\mathrm{char}_{\mathcal{O}_K}, 1, s) = f(0) m (1 - q^{-s})^{-1}$$

where $m$ is the multiplicative measure of $\mathcal{O}_K^\times$. So the assumption that $\zeta_K(f, 1, s) = 0$ implies $f(0)(1 - q^{-s})^{-1} \in \mathbb{C}[q^s, q^{-s}]$ as a function of $s$. This is false unless $f(0) = 0$; therefore $f(0) = 0$ and so $f \in \mathcal{S}(F^\times)$.

So now $\zeta_K(f, \omega, 1)$ is well-defined for all characters $\omega$ of $F^\times$ and equals $\widetilde{f}(\omega)$, where $\widetilde{\ }$ denotes Fourier transform on the group $K^\times$; so $\widetilde{f}$ is a function on the dual group of $X(K^\times)$ of $K^\times$. By the injectivity of the Fourier transform (see e.g. [GRS64, chapter IV]) from $L^1(K^\times)$ to $C(X(K^\times))$ our hypothesis implies that $f = 0$. □

We will now use the weak functional equation (†) to prove results about the $^*$-transform. Recall that the transform depends on the choice of both non-trivial additive character and prime; surprisingly, the double $^*$-transform does not depend on choice of prime:

**Proposition 2.6.13.** *The double $^*$-transform does not depend on choice of prime $\pi$. If the character $\psi_K$ is replaced by some other character, with conductor $d'$ say, and we assume that $d' \equiv d \mod 2$, then the double $^*$-transform is multiplied by a constant factor of $q^{d'-d}$.*

*Proof.* Write more generally $D_i$ for the double $^*$-transform with respect to prime $\pi_i$ and character $\psi_K^i$ for $i = 1, 2$; let $d_i$ be the conductor of $\psi_K^i$ and assume $d_1 \equiv d_2 \mod 2$. Equation (†) implies that for all $f \in \mathcal{S}(K)$, all characters $\omega$ of conductor $r \geq 0$, and all complex $s$,

$$\begin{aligned}
\zeta_K(D_1 f, \omega, s) &= \mu^2 q^{-d_1} \delta_{d_1 - r} \omega(-1) \zeta_K(f, \omega, s) \\
&= q^{d_2 - d_1} \zeta_K(D_2 f, \omega, s).
\end{aligned}$$

Lemma 2.6.12 implies now that $D_1 f = q^{d_2 - d_1} D_2 f$, revealing the independence from the prime and claimed dependence on the conductor of the character. □

We use (†) again, this time to prove that $^*$ is an automorphism of $\mathcal{S}(K)$. It is interesting that we are using properties of zeta integrals and epsilon factors to deduce properties of $^*$; one would usually work in the other direction but the author could find no direct proof and it is very satisfying to apply zeta integrals to such a problem!

**Proposition 2.6.14.** *The* $^*$*-transform is a linear automorphism of* $\mathcal{S}(K)$.

*Proof.* Let $D$ denote the double $^*$-transform on $\mathcal{S}(K)$ with respect to our chosen character (we have shown that it does not depend on choice of prime); let $D_1$ denote the double $^*$-transform on $\mathcal{S}(K)$ with respect to a character $\psi_K^1$ with conductor $d_1 \not\equiv d \mod 2$. Equation (†) implies that for all $f \in \mathcal{S}(K)$, all characters $\omega$ of conductor $r \geq 0$, and all complex $s$,

$$\begin{aligned}\zeta_K(D_1 Df, \omega, s) &= \mu^2 q^{-d_1} \delta_{d_1 - r} \omega(-1) \zeta_K(Df, \omega, s)\\ &= \mu^4 q^{-d-d_1} \delta_{d-r} \delta_{d_1-r} \omega(-1)^2 \zeta_K(f, \omega, s)\\ &= \mu^4 q^{-d-d_1} q^{-1} \zeta_K(f, \omega, s)\end{aligned}$$

as $\delta_{d-r}\delta_{d_1-r} = q^{-1}$ for all $r$.

Lemma 2.6.12 now implies that $D_1 Df = \mu^4 q^{-d-d_1} q^{-1} f$ for all $f \in \mathcal{S}(K)$. Therefore $^*$ is injective. Replacing $D_1 D$ by $DD_1$ in the argument similarly shows that $^*$ is surjective. □

**Remark 2.6.15.** The key to the previous proof is the identity $\delta_{d-r}\delta_{d_1-r} = q^{-1}$, which removes the dependence on the conductor $r$ of the multiplicative character. There is no clear way to relate zeta integrals of $f^{**}$ with those of $f$ in a manner independent of the character; so we were forced to transform four times!

The following result shows that if $\psi_K$ has conductor $0$ then the $^*$-transform and Fourier transform agree on functions lifted from the residue field $\overline{K}$:

**Proposition 2.6.16.** *Assume that the conductor of* $\psi_K$ *is* $0$. *Let* $h$ *be a complex-valued function on* $\overline{K}$ *and* $r$ *an integer; let* $f = h^{0,r}$ *be the lift of* $h$ *at* $0, r$ *(that is,* $f$ *vanishes off* $\pi^r \mathcal{O}_K$ *and satisfies* $f(\pi^r x) = h(\overline{x})$ *for* $x \in \mathcal{O}_K$*). Then* $f^* = q^{-r-1}\widehat{f}$.

*Proof.* Suppose initially that $r = -1$; to prove the assertion it suffices to consider functions $f = \mathrm{char}_{a+\mathcal{O}_K}$ for $a \in \pi^{-1}\mathcal{O}_K$. For such an $f$ it is easily checked that $W(f) = f$ and $f^* = \widehat{f}$.

For arbitrary $r$, note that $x \mapsto f(\pi^{r+1}x)$ satisfies the hypotheses for the $r = -1$ case; lemma 2.6.5 and the corresponding result for the Fourier transform, namely $\widehat{f(\alpha\,\cdot)} = |\alpha|^{-1}\widehat{f}(\alpha\,\cdot)$ for $\alpha \in K^\times$, imply $f^* = q^{-r-1}\widehat{f}$. □

Let us summarise the main results of this section concerning local zeta integrals, the $^*$-transform, and related epsilon factors.

**Proposition 2.6.17.** *Let* $\omega$ *be a quasi-character of* $K^\times$. *Then*

*(AC*) Analytic continuation, with the poles 'bounded' by the L-function: for all Schwartz-Bruhat functions* $g$, $\zeta_K(g, \omega, s)/L(\omega, s)$, *which initially only defines a holomorphic function for* $\Re(s)$ *sufficiently large, in fact has analytic continuation to an entire function*

$$Z_K(g, \omega, s)$$

*of* $s$.

*(L*) 'Minimality' of the L-function: there is a Schwartz-Bruhat function* $g$ *for which*

$$Z_K(g, \omega, s) = 1$$

*for all* $s$.

*(FE*) Functional equation: there is an entire function $\varepsilon_*(\omega, s)$, such that for all Schwartz-Bruhat functions $g$,*

$$Z_K(g^*, \omega^{-1}, 2-s) = \varepsilon_*(\omega, s) Z_K(g, \omega, s).$$

*Moreover, $\varepsilon_*(\omega, s)$ is of exponential type; that is, $\varepsilon_*(\omega, s) = aq^{bs}$ for some complex $a$ and integer $b$.*

*Proof.* Properties (AC*) and (L*) are just (AC) and (L) because they are independent of the chosen transform. (FE*) is proposition 2.6.1 and the epsilon factors were shown to be of exponential type by explicit calculation in examples 2.6.9 and 2.6.10. □

**Remark 2.6.18.** For applications to zeta-integrals on two-dimensional local fields we will require the $^*$-transform and zeta integrals for functions defined on the product space $K \times K$. As $\mathcal{S}(K \times K) = \mathcal{S}(K) \otimes \mathcal{S}(K)$, we may just define the $^*$-transform on $\mathcal{S}(K \times K)$ by $(f \otimes g)^* = f^* \otimes g^*$ and linearity.

Suppose that $\omega$ is a quasi-character of $K^\times \times K^\times$; write $\omega(x, y) = \omega_1(x)\omega_2(y)$ for quasi-characters $\omega_i$ of $K^\times$. The decomposition $\mathcal{S}(K \times K) = \mathcal{S}(K) \otimes \mathcal{S}(K)$ and previous proposition imply

(i) For all $f \in \mathcal{S}(K \times K)$, the integral $\zeta_{K\times K}(f, \omega, s) = \int\int f(x, y)\omega(x, y)|x|^s|y|^s \, d\overset{\times}{x} d\overset{\times}{y}$ is well-defined for $\Re(s)$ large enough. Moreover, $s \mapsto \zeta_{K\times K}(f, \omega, s)/(L(\omega_1, s)L(\omega_2, s))$ has analytic continuation to an entire function $Z_{K\times K}(f, \omega, s)$.

(ii) There is $f \in \mathcal{S}(K \times K)$ such that $Z_{K\times K}(f, \omega, s) = 1$ for all $s$.

(iii) For all $f \in \mathcal{S}(K \times K)$, there is a functional equation:

$$Z_{K\times K}(f^*, \omega^{-1}, 2-s) = \varepsilon_*(\omega_1, s)\varepsilon_*(\omega_2, s) Z_{K\times K}(f, \omega, s)$$

for all $s$. Note that $\varepsilon_*(\omega_1, s)\varepsilon_*(\omega_2, s)$ is of exponential type.

### 2.6.2 Archimedean case

Now suppose that $K$ is an archimedean local field. Rather than present a version of proposition 2.6.1 using tempered distributions, we will just define and investigate an analogue of the $^*$-transform. The existence of an $s$ goes to $2-s$ functional equation will be shown as in [Tat67], via Fubini's theorem.

**Definition 2.6.19.** Introduce

$$\nabla : K \to K, \quad x \mapsto |x|x.$$

Note that this $\nabla$ is a bijection with inverse $x \mapsto x|x|^{-\frac{1}{2}}$ (for $x \in K^\times$). Given a complex-valued function $f$ on $K$, define its $^*$-transform by

$$f^* = \widehat{f \circ \nabla^{-1}} \circ \nabla,$$

assuming that $f \circ \nabla^{-1}$ is integrable.

**Remark 2.6.20.** Note that the archimedean and non-archimedean $\nabla$ maps have the same form: $\nabla x = \sigma(x)x$ where $\sigma$ is a splitting of the absolute value.

This archimedean $^*$-transform has an integral representation similar to the Fourier transform:

**Lemma 2.6.21.** *Let $g$ be a complex-valued function on $K$ such that $x \mapsto g(x)|x|$ is integrable. Then $g^*$ is well-defined and*

$$g^*(y) = 2\int_K g(x)\psi_K(\nabla(yx))|x|\,dx.$$

*Proof.* By definition of the $^*$-transform,

$$g^*(y) = \int g \circ \nabla^{-1}(u)\psi_K(u\nabla(y))\,du = \int g(u|u|^{-\frac{1}{2}})\psi_K(uy|y|)\,du.$$

To obtain the desired expression, change variables $x = u|u|^{1/2} = \nabla^{-1}(u)$ in the integral. □

**Remark 2.6.22.** The previous lemma is enough to prove that if $f$ is a Schwartz function on $K$, then both $f^*$ and $f^{**}$ are well-defined. Unfortunately, it is false that the $^*$-transform of a Schwartz function is again a Schwartz function, as the following example shows.

**Example 2.6.23.** We $^*$-transform the Schwartz function $g(x) = e^{-\pi x^2}$ on $\mathbb{R}$ with additive character $e^{2\pi i x}$. Firstly, $g \circ \nabla^{-1}(x) = e^{-\pi\,\mathrm{sign}(x)x}$, where $\mathrm{sign}(x)$ is the sign ($\pm$) of $x$, and so

$$\widehat{g \circ \nabla^{-1}}(y) = \int_0^\infty e^{-\pi x}e^{2\pi i x y}\,dx + \int_0^\infty e^{-\pi x}e^{-2\pi i x y}\,dx.$$

A standard calculation from the calculus of residues is $\int_0^\infty e^{-\alpha x}e^{ibx}dx = 1/(\alpha - ib)$ for real $\alpha, b$ with $\alpha > 0$. Therefore $\widehat{g \circ \nabla^{-1}}(y) = 2\pi/(\pi^2 + 4\pi^2 y^2)$ and so

$$g^*(y) = \frac{2\pi}{\pi^2 + 4\pi^2 y^4}$$

which does not decay rapidly enough to be a Schwartz function. Since $g \circ \nabla^{-1}$ is not differentiable at 0, this is in agreement with the duality provided by the Fourier transform between smoothness and rapid decrease.

We now prove an $s$ goes to $2 - s$ functional equation:

**Proposition 2.6.24.** *Suppose that $\omega$ is a quasi-character of $K^\times$. If $f, g$ are Schwartz functions on $K$ for which $f^*, g^*$ are also Schwartz, then*

$$\zeta_K(f, \omega, s)\zeta_K(g^*, \omega^{-1}, 2 - s) = \zeta_K(f^*, \omega^{-1}, 2 - s)\zeta_K(g, \omega, s)$$

*for all complex $s$. Here we write zeta functions where we strictly mean their meromorphic continuation.*

*Proof.* One imitates Tate's method, using the representation of the $^*$-transform given by lemma 2.6.21 to show that

$$\zeta_K(f, \omega, s)\zeta_K(g^*, \omega^{-1}, 2 - s) = 2\int\int\int_{K^3} f(x)g(z)\psi_K(\nabla(xyz))|xyz|\omega(y)^{-1}|y|^{-s}\,dxdydz$$

for $s$ with $\Re(s) = 1 - \Re(\omega)$; here $\Re(\omega)$ is the exponent of $\omega$, defined by $|\omega| = |\cdot|^{\Re(\omega)}$. This expression is symmetric in $f$ and $g$, from which follows

$$\zeta_K(f, \omega, s)\zeta_K(g^*, \omega^{-1}, 2 - s) = \zeta_K(f^*, \omega^{-1}, 2 - s)\zeta_K(g, \omega, s).$$

Apply the identity theorem to deduce that this holds for all complex $s$. □

**Example 2.6.25.** Suppose that $K = \mathbb{R}$; let $g(x) = e^{-\pi x^2}$, $f = g \circ \nabla$. Assume that $\psi_{\mathbb{R}}(x) = e^{2\pi i x}$, and that the chosen measure is Lebesgue measure; then $\widehat{g} = g$ which implies here that $f^* = f$. For $s$ complex of positive real part,

$$\zeta_K(f, 1, s) = \frac{1}{2}\pi^{-s/4}\Gamma(s/4) = \frac{1}{2}\zeta_K(g, 1, s/2).$$

The previous proposition implies that if $h, h^*$ are Schwartz on $\mathbb{R}$, then

$$\begin{aligned}\zeta_K(h^*, 1, 2-s) &= \frac{\pi^{(s-2)/4}\Gamma((2-s)/4)}{\pi^{-s/4}\Gamma(s/4)}\zeta_K(h, 1, s)\\ &= 2^{s/2-1}\pi^{s/2}\left(\cos\left(\frac{\pi s}{4}\right)\Gamma\left(\frac{s}{2}\right)\right)^{-1}\zeta_K(h, 1, s),\end{aligned}$$

by the same Gamma function identities used in [Tat67].

**Remark 2.6.26.** If $f$ is a Schwartz function and $\omega$ a quasi-character, then we know that $\zeta_K(f, \omega, s)/L(\omega, s)$ analytically continues to an entire function; also, $f$ may be chosen such that $\zeta_K(f, \omega, s) = L(\omega, s)$. *However*, as example 2.6.23 demonstrates, the standard choice of $f$ may be such that $f^*$ is not Schwartz.

The author suspects that if $f$ is a Schwartz function on $\mathbb{R}$ for which $f^*$ is also Schwartz, then $\zeta_K(f, 1, s)/(\pi^{-s/4}\Gamma(s/4))$ will analytically continue to an entire function; moreover, we have seen in the previous example that this denominator satisfies the 'minimality' condition (i.e. it occurs as a zeta function). This would justify calling $\pi^{-s/4}\Gamma(s/4)$ the local L-function for $^*$.

## 2.7 Two dimensional zeta integrals

In this, the final section of the chapter, we apply the integration theory to the study of two-dimensional local zeta integrals.

### 2.7.1 Non-archimedean case

$F$ is now a non-archimedean, two-dimensional local field. Thus $\Gamma = \mathbb{Z}$ and $F$ is complete with respect to the discrete valuation $\nu$, with residue field $\overline{F}$ a non-archimedean (one-dimensional) local field; the residue field of $\overline{F}$ is $\mathbb{F}_q$. The rank two ring of integers of $F$ is $O_F = \rho^{-1}(\mathcal{O}_{\overline{F}})$. Let $t_1, t_2$ be local parameters for $F$ which satisfy $t_2 = t(1)$ and $\overline{t}_1 = \pi$, where $\pi$ is the prime of $\overline{F}$ which was used to define the $^*$-transform on $K = \overline{F}$ in the previous section.

Let $K_2^{\mathrm{top}}(F)$ denote the second topological $K$-group of $F$ (see [Fes00]); recall that $K_2^{\mathrm{top}}(F)$ is the appropriate object for class field theory of $F$ (see [Fes91] for details). We recall those properties of $K_2^{\mathrm{top}}(F)$ which we shall use:

(i) A border map of $K$-theory defines a continuous map $\partial : K_2^{\mathrm{top}}(F) \to \overline{F}^{\times}$ which satisfies
$$\partial\{u, t_2\} = \overline{u}, \quad \partial\{u, v\} = 1 \qquad (\text{for } u, v \in \mathcal{O}_F^{\times}).$$
$\partial$ does not depend on choice of $t_1, t_2$. Introduce an absolute value
$$|\cdot| : K_2^{\mathrm{top}}(F) \to \mathbb{R}_{>0}, \quad \xi \mapsto |\partial(\xi)|_{\overline{F}}\,.$$

(ii) Let $U$ be the subgroup of $K_2^{\mathrm{top}}(F)$ whose elements have the form $\{u, t_1\} + \{v, t_2\}$, for $u, v \in O_F^{\times}$. $K_2^{\mathrm{top}}(F)$ decomposes as a direct sum $\mathbb{Z}\{t_1, t_2\} \oplus U$. Note that $|n\{t_1, t_2\} + u| = q^{-ns}$ for $n \in \mathbb{Z}$, $u \in U$.

(iii) For any quasi-character $\chi : K_2^{\mathrm{top}}(F) \to \mathbb{C}^\times$, there exist complex $s$ and a character $\chi_0 : U \to S^1$ such that

$$\chi(n\{t_1, t_2\} + u) = \chi_0(u) q^{-ns} \qquad (\text{for } n \in \mathbb{Z},\ u \in U).$$

The real part of $s$ is uniquely determined by $\chi$ and is said to be, as in the one-dimensional case, the exponent of $\chi$ (denoted $\Re(\chi)$).

**Definition 2.7.1.** Introduce $T = \mathcal{O}_F^\times \times \mathcal{O}_F^\times$, $T^+ = \mathcal{O}_F \times \mathcal{O}_F$, and a surjective homomorphism

$$\mathfrak{t} : T \to K_2^{\mathrm{top}}(F), \quad (\alpha, \beta) \mapsto \{\alpha, t_2\} + \{t_1, \beta\} + w(\overline{\beta})\{t_1, -t_2\}$$

for $\alpha, \beta \in \mathcal{O}_F^\times$.

Note that $u, v \in O_F^\times$ and $i, j \in \mathbb{Z}$ implies $\mathfrak{t}(t_1^i u, t_1^j v) = (i + j)\{t_1, t_2\} + \{t_1, v\} + \{u, t_2\}$.

**Remark 2.7.2.** Compare with [Fes03]. $\mathfrak{t}$ depends on the choice of local parameters $t_1, t_2$. $T^+$ is the closure of $T$ in the two-dimensional topology of $F$; its relation to $T$ is the same as $\overline{F}$ to $\overline{F}^\times$ in the one-dimensional local theory, the adèle group $\mathbb{A}$ to the idèle group $\mathbb{A}^\times$ in the one-dimensional global theory, or the matrix algebra $M_n$ to the group $GL_n$ in R. Godement and H. Jacquet's generalisation [GJ72] of Tate's thesis.

Note that $(x, y) \in T$ implies $|\mathfrak{t}(x, y)| = |\overline{x}|\,|\overline{y}| \in \mathbb{R}_{>0}$.

Given a $\mathbb{C}(X)$ $(= \mathbb{C}(\Gamma))$ -valued function $f$ on $T^+$, a quasi-character $\chi$ of $K_2^{\mathrm{top}}(F)$, and complex $s$, Fesenko suggests in [Fes03] the following definition for the associated (two-dimensional) local zeta integral:

$$\zeta(f, \chi, s) = \zeta_F^{\mathrm{2d}}(f, \chi, s) = \int^{F^\times \times F^\times} f(x, y)\, \chi \circ \mathfrak{t}(x, y) |\mathfrak{t}(x, y)|^s \ \mathrm{char}_T(x, y)\, d\overset{\times}{x} d\overset{\times}{y},$$

assuming that the integrand is integrable on $F^\times \times F^\times$; integration on this space is a simple union of the integration theory on $F^\times$ (section 2.4) and the basic theory for $F \times F$ (summarised in subsection 1.4.2).

We now prove analytic continuation, and moreover a functional equation, for a class of functions $f$ and characters $\chi$; we write $f^0$ for the lift of $f \in \mathcal{S}(\overline{F} \times \overline{F})$ at $(0, 0), (0, 0)$ (see 1.4.2 for the definition).

**Proposition 2.7.3.** *Let $\chi$ be a quasi-character of $K_2^{top}(F)$ and suppose that $\chi \circ \mathfrak{t}$ factors through the residue map $T \to \overline{F}^\times \times \overline{F}^\times$. Let $\omega_i$ be the quasi-characters of $\overline{F}^\times$ defined by $\chi \circ \mathfrak{t}(x, y) = \omega_1(\overline{x})\omega_2(\overline{y})$. Define $L_F(\chi, s) = L(\omega_1, s)L(\omega_2, s)$, a product of two L-functions for $\overline{F}$, and $\varepsilon_F(\chi, s) = \varepsilon_*(\omega_1, s)\varepsilon_*(\omega_2, s)$, a product of two epsilon factors for $\overline{F}$. Then*

*(AC2) For all $f \in \mathcal{S}(\overline{F} \times \overline{F})$, the zeta function $\zeta(f^0, \chi, s)$ is well-defined for $\Re(s)$ sufficiently large. Moreover,*

$$\zeta(f^0, \chi, s)/L_F(\chi, s)$$

*has analytic continuation to an entire function, $Z(f^0, \chi, s)$.*

*(L2) There is $f \in \mathcal{S}(\overline{F} \times \overline{F})$ such that $Z(f^0, \chi, s) = 1$ for all $s$.*

*(FE2) For all $f \in \mathcal{S}(\overline{F} \times \overline{F})$, a functional equation holds:*

$$Z(f^{*0}, \chi^{-1}, 2 - s) = \varepsilon_F(\chi, s) Z(f^0, \chi, s).$$

*for all $s$. Moreover, $\varepsilon_F(\chi, s)$ is of exponential type; that is $\varepsilon_F(\chi, s) = aq^{bs}$ for some complex $a$ and integer $b$.*

*Proof.* By definition of the integral on $F^\times \times F^\times$ and a similar argument to example 2.4.6 (i), we have

$$\zeta(f^0, \chi, s) = \int_{\overline{F}^\times} \int_{\overline{F}^\times} f(u,v)\omega_1(u)\omega_2(v)|u|^s|v|^s \, d\overset{\times}{u} d\overset{\times}{v},$$

which we denoted $\zeta_{\overline{F}\times\overline{F}}(f, \omega_1 \otimes \omega_2, s)$ in remark 2.6.18. That is, since we are only considering functions $f$ which lift from $\overline{F} \times \overline{F}$, the zeta integral over $\mathcal{O}_F \times \mathcal{O}_F$ reduces to a zeta integral over $\overline{F} \times \overline{F}$. All required results follow from that remark. $\square$

**Remark 2.7.4.** The previous example highlights the interest of lifting the $^*$-transform up to $F$ in a similar way to how we lifted the Fourier transform. Then it may be possible to generalise this proposition to more functions on $\mathcal{O}_F \times \mathcal{O}_F$ than simply those which lift from $\overline{F}\times\overline{F}$. However, it is unclear whether this would produce anything essentially new.

**Remark 2.7.5.** Having calculated epsilon factors for the $^*$-transformation in section 2.6, we have formulae for the *two-dimensional epsilon factors*

$$\varepsilon_F(\chi, s) = \varepsilon_*(\omega_1, s)\varepsilon_*(\omega_2, s).$$

For example, if $\omega_1$ is ramified with conductor $r > 0$ but $\omega_2$ is unramified, then

$$\varepsilon_F(\chi, s) = q^{(\lceil (r-d)/2\rceil - \lceil d/2\rceil)(2-s)}\chi(t_1, 1)^{\lceil (r-d)/2\rceil} q^{-r/2}\delta_{d-r}\rho_0(\omega_1^{-1})$$

where $d$ is the conductor of the additive character on $\overline{F}$ used to define the $^*$-transform.

There is another relation between zeta integrals on $F$ and $\overline{F}$ which we now discuss; first we need a lemma:

**Lemma 2.7.6.** *Let $g$ be a complex-valued function on $\overline{F}$ and $s$ complex such that $g|\cdot|^{2s}$ is integrable on $\overline{F}^\times$. Let $w : \overline{F}^\times \to \mathbb{Z}$ be the discrete valuation on $\overline{F}$; introduce*

$$g' : \overline{F}^\times \times \overline{F}^\times \to \mathbb{C}, \quad (x,y) \mapsto g(\pi^{\min(w(x),w(y))-w(x)}x)\,|xy|^s.$$

*Then $g'$ is integrable over $\overline{F}^\times \times \overline{F}^\times$, with integral*

$$\int\int g'(x,y)\, d\overset{\times}{x} d\overset{\times}{y} = \mu(\mathcal{O}_{\overline{F}}^\times)\frac{1+q^{-s}}{1-q^{-s}}\int g(x)|x|^{2s} d\overset{\times}{x},$$

*where $\mu$ is the multiplicative Haar measure on $\overline{F}^\times$.*

*Proof.* The integral of $g'$ over $\overline{F}^\times \times \overline{F}^\times$ is

$$\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\int_{w^{-1}(n)}\int_{w^{-1}(m)} g(\pi^{\min(n,m)-m}x)q^{-s(n+m)-2}\, dxdy.$$

Split the inner summation over $m < n$ and $m \geq n$, and then interchange the order of the double summation $\sum_n \sum_{m<n}$; elementary manipulations complete the proof. $\square$

**Definition 2.7.7.** Introduce a 'generalised residue map':

$$\rho_2 : T^+ \longrightarrow \overline{F}, \quad (t_1^{i_1}t_2^{i_2}u, t_1^{j_1}t_2^{j_2}v) \mapsto \overline{t_1^{\min(i_1,j_1)}t_2^{\min(i_2,j_2)}u}$$

where $u, v \in O_F^\times$ and $i_1, i_2, j_1, j_2 \in \mathbb{Z}$.

**Remark 2.7.8.** The map $\rho_2$, when restricted to $T$, factors through $K_2^{\mathrm{top}}(F)$:

$$\rho_2(t_1^i u, t_1^j v) = \partial\left(\min(i,j)\{t_1,t_2\} + \{t_1,v\} + \{u,t_2\}\right)$$

where $i,j \in \mathbb{Z}$, $u,v \in O_F^\times$.

$\rho_2$ provides a new method for lifting zeta integrals from $\overline{F}$ to $F$:

**Proposition 2.7.9.** *Let $\omega$ be a quasi-character of $\overline{F}^\times$, $s$ complex, and $g$ a complex-valued function on $\overline{F}$ such that $g\omega^2\,|\cdot|^{2s}$ is integrable on $\overline{F}^\times$; let $\chi = \omega \circ \partial$. Then the zeta integral $\zeta(g \circ \rho_2, \chi, s)$ is well-defined and*

$$\zeta(g \circ \rho_2, \chi, s) = \mu(\mathcal{O}_{\overline{F}}^\times)\frac{1+q^{-s-c}}{1-q^{-s-c}}\,\zeta_{\overline{F}}(g,\omega,2s+c),$$

*where $c \in \mathbb{C}$ is defined by $\omega = \omega_0\,|\cdot|^c$ with $\omega_0$ a character of $\overline{F}^\times$ trivial on $\pi$.*

*Proof.* For $(x,y) \in T$,

$$\begin{aligned} &g \circ \rho_2(x,y)\,\chi \circ \mathrm{t}(x,y)\,|\mathrm{t}(x,y)^s|\,|x|^{-1}\,|y|^{-1} \\ &\quad = g(\pi^{\min(w(\overline{x}),w(\overline{y}))-w(\overline{x})}\overline{x})\,\omega(\overline{x}\pi^{w(\overline{y})})\,|\overline{xy}|^{s-1} \\ &\quad = g(\pi^{\min(w(\overline{x}),w(\overline{y}))-w(\overline{x})}\overline{x})\,\omega_0(\overline{x})\,|\overline{xy}|^{s+c-1} \\ &\quad = g(\pi^{\min(w(\overline{x}),w(\overline{y}))-w(\overline{x})}\overline{x})\,\omega_0(\pi^{\min(w(\overline{x}),w(\overline{y}))-w(\overline{x})}\overline{x})\,|\overline{xy}|^{s+c-1}, \end{aligned}$$

so that $(x,y) \mapsto g \circ \rho_2(x,y)\,\chi \circ \mathrm{t}(x,y)\,|\mathrm{t}(x,y)^s|\,|x|^{-1}\,|y|^{-1}$ is the lift of

$$(u,v) \mapsto g(\pi^{\min(w(u),w(v))-w(v)}u)\,\omega_0(\pi^{\min(w(u),w(v))-w(u)}u)\,|uv|^{s+c-1}$$

at $(0,0),(0,0)$.

The result now follows from the previous lemma. □

This is enough to deduce analytic continuation of some more zeta functions:

**Corollary 2.7.10.** *Let $\omega$ be a quasi-character of $\overline{F}^\times$, $L(\omega,s)$ the associated $L$-function, and $g$ a Schwartz-Bruhat function on $\overline{F}$; let $\chi = \omega \circ \partial$. Then*

*(i) For $\Re(s)$ sufficiently large, the zeta integral $\zeta(g \circ \rho, \chi, s)$ is well-defined.*

*(ii) The holomorphic function $\zeta(g\circ\rho,\chi,s)/(L(\omega,s)(1-\chi(\{t_1,t_2\})q^{-s})^{-1})$, initially defined for $\Re(s)$ sufficiently large, has analytic continuation to an entire function.*

*Proof.* This follows from the corresponding results for local zeta functions on $\overline{F}$, the previous proposition, and the identity $\chi(\{t_1,t_2\}) = \omega(\pi) = q^{-c}$ where $c$ is as in the previous proposition. □

It has been useful throughout for $\chi\circ\mathrm{t}$ to factor through the residue map $T \to \overline{F}^\times \times \overline{F}^\times$. In the next two examples we consider some situations in which this happens. Let $L$, a two-dimensional local field, be a finite abelian extension of $F$ and let $\chi$ be a character of $K_2^{\mathrm{top}}(F)$ which vanishes on $N_{L/F}K_2^{\mathrm{top}}(L)$. So $\chi$ corresponds, via two-dimensional class field theory, to a character of Gal$(L/F)$.

**Example 2.7.11.** Suppose $\overline{L}/\overline{F}$ is separable with $|\overline{L}:\overline{F}| = |L:F|$; i.e. $L/F$ is unramified as an extension of complete discrete valuation fields.

Then $\partial$ induces a surjection $K_2^{\mathrm{top}}(F)/N_{L/F}K_2^{\mathrm{top}}(L) \to \overline{F}^\times/N_{L/F}\overline{L}^\times$. Further, the separability assumption implies $\overline{L}/\overline{F}$ is an abelian extension of local fields, so that $|\overline{F}^\times/N_{\overline{L}/\overline{F}}\overline{L}^\times| = |\overline{L}:\overline{F}| = |L:F| = |K_2^{\mathrm{top}}(F)/N_{L/F}K_2^{\mathrm{top}}(L)|$; thus the aforementioned induced surjection is an isomorphism. Therefore $\chi$ factors through $\partial$.

**Example 2.7.12.** Suppose $\overline{L} = \overline{F}$, $p \nmid |L : F|$, and $t_2 \in N_{L/F}L^\times$ ('a totally tamely ramified extension in the second parameter').

Then $(x, y) \in T$ implies $\mathfrak{t}(x, y) \equiv \{t_1, \Theta(y)\} \mod N_{L/F}K_2^{\mathrm{top}}(L)$ (see [Fes91]), where $\Theta$ is the projection

$$\Theta : F^\times = \langle t_1 \rangle \times \langle t_2 \rangle \times \mathbb{F}_q^\times \times V_F \to \mathbb{F}_q^\times.$$

Here $V_F$ is the two-dimensional group of principal units of $F$.

Therefore there exists a tamely ramified quasi-character $\omega$ of $\overline{F}^\times$ such that $\chi \circ \mathfrak{t}(x, y) = \omega(\overline{y})$ for $(x, y) \in T$.

These examples show that our functional equation applies to all 'sufficiently unramified' characters; but do observe that in example 2.7.11, the residue extension $\overline{L}/\overline{F}$ is allowed to be as ramified as desired. The proof of the functional equation in [Fes03] is valid whenever all relevant functions are integrable, and proposition 2.7.3 is certainly a special case. However, it appears that if $\chi$ is ramified then certain interesting functions fail to be integrable.

The failure of the integral to work in the ramified setting is a serious difficulty, which may only be overcome through a systematic comparison of the current theory with the ramification theory of two-dimensional local fields. See section 6.1 for some thoughts on the subject.

### 2.7.2 Archimedean case

Now suppose that $F$ is an archimedean, two-dimensional local field; that is, $\Gamma = \mathbb{Z}$, $F$ is complete with respect to the discrete valuation $\nu$, and the residue field $\overline{F}$ is an archimedean local field. The classification of complete discrete valuation fields (see e.g. [FV02, II.5]) implies that $F$ is isomorphic to a field of Laurent series $\mathbb{C}((t))$ or $\mathbb{R}((t))$, where we write $t = t(1)$.

The correct way to use topological $K$-groups for class field theory and zeta integrals of such fields is not clear, so we content ourselves with making a few remarks about generalising the results in the non-archimedean case without appealing to $K$-groups.

Given Schwartz functions $f, g$ on $\overline{F}$ for which $f^*, g^*$ are also Schwartz, and $\omega$ a quasi-character of $\mathcal{O}_F^\times$ which factors through the residue map $\mathcal{O}_F^\times \to \overline{F}^\times$, proposition 2.6.24 implies that

$$\int^{F^\times} f^{0,0}(x)\, \omega(x)|x|^s \, \mathrm{char}_{\mathcal{O}_F^\times}(x)\, d\overset{\times}{x} \int^{F^\times} (g^*)^{0,0}(x)\, \omega(x)^{-1}|x|^{2-s} \, \mathrm{char}_{\mathcal{O}_F^\times}(x)\, d\overset{\times}{x}$$

is invariant under interchanging $f$ and $g$. There is an analogous result for integrals over $\mathcal{O}_F^\times \times \mathcal{O}_F^\times$.

An extension of $F$ cannot be wildly ramified in any sense, and so by analogy with examples 2.7.11 and 2.7.12 we expect arithmetic characters on $\mathcal{O}_F^\times$ (or $\mathcal{O}_F^\times \times \mathcal{O}_F^\times$) to lift from $\overline{F}^\times$. Hence this functional equation may be satisfactory in the archimedean case.

Indeed, in the case $F = \mathbb{C}((t))$, the finite abelian extensions of $F$ have the form $\mathbb{C}((t^{1/n}))$ for natural $n$. A character attached to such an extension is surely a purely imaginary power of $|\cdot|$; this lifts to $\mathcal{O}_F^\times$ from $\overline{F}^\times$.

If $F = \mathbb{R}((t))$, then $F$ has maximal abelian extension $\mathbb{C}((t^{1/2}))$, with subextensions $\mathbb{R}((t^{1/2}))$ and $\mathbb{C}((t))$. A character attached to the extension $\mathbb{C}((t^{1/2}))$ is $\mathcal{O}_F^\times \to \{\pm 1\} : x \mapsto \mathrm{sign}(\overline{x})$, which again lifts from $\overline{F}^\times$.

## 2.A Rings generated by d-classes

This appendix gives a clear exposition of the calculations required to develop the measure theory of section 2.2 from the integration theory; many of the manipulations here are inspired by [Fes03] and [Hal50].

**Definition 2.A.1.** Let $\mathcal{A}$ be a collection of subsets of some set $\Omega$.

$\mathcal{A}$ is said to be a *ring* if it is closed under taking differences and finite unions.

$\mathcal{A}$ is said to be a *d-class* if it contains the empty set and satisfies the following: $A, B$ in $\mathcal{A}$ with non-trivial intersection implies $\mathcal{A}$ contains $A \cap B$ and $A \cup B$. Elements of a d-class are called d sets.

**Example 2.A.2.** The following are examples of d-classes.

(i) The collection of finite intervals of $\mathbb{R}$, open on the right and closed on the left, together with the empty set.

(ii) The collection of translates of some chain of subgroups of a group, together with the empty set.

We fix for the remainder of this appendix a d-class on some set.

**Lemma 2.A.3.** *Let $A_i$ be d sets, for $i = 1, \dots, n$. Then there exist disjoint d sets $B_j$, $j = 1, \dots, m$ such that each $B_j$ is a union of some of the $A_i$ and such that $\bigcup_i A_i = \bigsqcup_j B_j$*

*Proof.* A simple induction on $n$. $\square$

Informally, the result states that any finite union of d sets may be refined to a disjoint union.

**Definition 2.A.4.** A set of the form $A \setminus \bigsqcup_i A_i$ for some d sets $A, A_1, \dots, A_n$, with $A_i \subseteq A$ for each $i$, is said to be a *dd* set.

**Remark 2.A.5.**

(i) Consider a set of the form $X = A \setminus \bigcup_i A_i$ for d sets $A, A_1 \dots, A_m$, where we make no assumption on disjointness or inclusions. Then $X = A \setminus \bigcup_i A \cap A_i$; lemma 2.A.3 implies that $X$ is a dd set.

(ii) The identity $(A \setminus \bigsqcup_i A_i) \cap (B \setminus \bigsqcup_j B_j) = (A \cap B) \setminus (\bigsqcup_i A_i \cup \bigsqcup_j B_j)$ and lemma 2.A.3 imply that dd sets are closed under finite intersection.

**Definition 2.A.6.** A finite disjoint union of dd sets is said to be a *ddd* set.

**Lemma 2.A.7.** *The difference of two dd sets is a ddd set.*

*Proof.* For arbitrary sets $A, A_0, B, (B_j)_j$ with $B_j \subseteq B$, the identity

$$(A \setminus A_0) \setminus (B \setminus \bigsqcup_j B_j) = (A \setminus (B \cup A_0)) \sqcup \bigsqcup_j ((B_j \cap A) \setminus A_0)$$

is easily verified. Replace $A_0$ by a disjoint union of d sets and use remark 2.A.5 to complete the proof. $\square$

**Proposition 2.A.8.** *The difference of two ddd sets is a ddd set. The union of two ddd sets is a ddd set.*

*Proof.* The difference of two ddd sets may be written as a finite disjoint union of sets of the form $\bigcap E_i \setminus D_i$, a finite intersection of differences of dd sets; such a set is an intersection of ddd sets by lemma 2.A.7. By De Morgan's laws, this may be rewritten as a disjoint union of intersections of dd sets. Hence the difference of two ddd sets is again a ddd set.

Let $D_1, \dots, D_n$ and $E_1, \dots, E_m$ be disjoint dd sets. Then $\bigsqcup_i D_i \cup \bigsqcup_j E_j$ is the disjoint union of the following three sets:

$$W_1 = \bigsqcup_i D_i \cap \bigsqcup_j E_j$$
$$W_2 = \bigsqcup_i D_i \setminus \bigsqcup_j E_j$$
$$W_3 = \bigsqcup_j E_j \setminus \bigsqcup_i D_i.$$

$W_2$ and $W_3$ are ddd sets by lemma 2.A.3. Further, $W_1 = \bigsqcup_{i,j}(D_i \cap E_j)$ is a ddd set by remark 2.A.5. □

**Proposition 2.A.9.** *The collection of all ddd sets is a ring; indeed, it is the ring generated by the d-class.*

*Proof.* This is the content of the previous result. □

## 2.B $\mathbb{C}(\Gamma)$-valued holomorphic functions

We briefly explain the required theory of holomorphic functions from the complex plane to $\mathbb{C}(\Gamma)$, though $\mathbb{C}(\Gamma)$ could be replaced with an arbitrary complex vector space.

**Definition 2.B.1.** Suppose $f$ is a $\mathbb{C}(\Gamma)$-valued function defined on some open subset of the complex plane. We shall say that $f$ is holomorphic at a point of $U$ if and only if, in some neighbourhood $U_0$ of this point,

$$f(z) = \sum_{i=1}^{n} f_i(z) p_i,$$

for some $f_1, \dots, f_n$, complex-valued holomorphic functions of $U_0$, and $p_1, \dots, p_n$, elements of $\mathbb{C}(\Gamma)$.

Although the definition of holomorphicity is a local one, we can find a global representation of any such function on a connected set:

**Proposition 2.B.2.** *Let $(p_i)_{i \in I}$ be any basis for $\mathbb{C}(\Gamma)$ over $\mathbb{C}$, and let $(\pi_i)_{i \in I}$ be the associated coordinate projections to $\mathbb{C}$. Let $f$ be a $\mathbb{C}(\Gamma)$-valued holomorphic function on some open subset $U$ of $\mathbb{C}$. Then*

*(i) $\pi_i \circ f$ is a complex-valued holomorphic (in the usual sense) function of $U$.*

*(ii) If $U$ is connected then there is a finite subset $I_0$ of $I$ and complex-valued holomorphic functions $f_i$, for $i \in I_0$, of $U$ such that*

$$f(x) = \sum_{i \in I_0} f_i(z) p_i$$

*for all $z \in U$.*

*Proof.* Let us suppose that

$$f(z) = \sum_{j=1}^{n} f_j(z) q_j \tag{$*$}$$

for all $z$ in some open $U_0 \subset U$, where the $f_j$ are complex valued holomorphic functions of $U_0$ and $q_1, \dots, q_n \in \mathbb{C}(\Gamma)$. Then each $q_j$ is a linear sum (with complex coefficients) of finitely many $p_i$; therefore there is finite $I_0 \subset I$ such that $f(z) = \sum_{i \in I_0} f_i(z) p_i$ for all $z \in U_0$, where each $f_i$ is a sum of finitely many $f_j$. So for any $i \in I$,

$$\pi_i \circ f|_{U_0} = \begin{cases} f_i & i \in I_0 \\ 0 & i \notin I_0 \end{cases}$$

therefore $\pi_i \circ f$ is holomorphic on $U_0$.

But $f$ is holomorphic, so each point of $U$ has an open neighbourhood where $f$ can be written as in $(*)$; therefore $\pi_i \circ f$ is holomorphic on all of $U$. This proves (i).

(ii) follows from (i) as soon as it is known that there are only finitely many $i$ in $I$ for which $\pi_i \circ f$ is not identically zero on $U$. But the identity theorem of complex analysis implies that if $\pi_i \circ f$ is not identically zero on $U$, then it is not identically zero on any open set $U_0 \subset U$. So choose $U_0$ as at the start of the proof and write $f|_{U_0}$ as in $(*)$; if $\pi_i \circ f$ is not identically zero on $U_0$, then $i \in I_0$. So for all $z \in U$,

$$f(z) = \sum_{i \in I_0} \pi_i \circ f(z)\, p_i.$$

□

Although it is very easy to prove, the identity theorem here is fundamental, for else we would not be assured of the uniqueness of analytic continuations:

**Proposition 2.B.3.** *Suppose that $f$ is a $\mathbb{C}(\Gamma)$-valued holomorphic function on some connected open subset $U$ of $\mathbb{C}$. Suppose that the zeros of $f$ have a limit point in $U$; then $f$ is identically zero on $U$.*

*Proof.* Let $(p_i)_{i \in I}$ and $(\pi_i)_{i \in I}$ be as in the previous proposition. By the usual identity theorem of complex analysis, each $\pi_i \circ f$ vanishes everywhere; therefore the same is true of $f$. □

Enough has now been proved to discuss analytic continuation of $\mathbb{C}(\Gamma)$-valued functions as required in section 2.5.

CHAPTER 3

# Integration on product spaces and $GL_n$ of a valuation field over a local field

We work with the same notation as chapter 2; our aim is to extend the integration theory to finite dimensional vector spaces and $GL_n$ over $F$.

## 3.1 Repeated Integration on $F^n$

In this section we extend the integral on $F$ to the product space $F^n$ for $n$ a positive integer. We do this by using the integral over $F$ to define repeated integrals. The idea is simple, though the notation is not. A summary of the theory for $n = 2$ was given in subsection 1.4.2.

Given a sequence $x_1, \dots, x_n$ of $n$ terms, and $r$ such that $1 \leq r \leq n$, the notation

$$x_1, \dots, \dot{x}_r, \dots, x_n \quad = \quad x_1, \dots, x_{r-1}, x_{r+1}, \dots, x_n$$

denotes the sequence of $n-1$ terms obtained by removing the $r^{\text{th}}$ term.

We introduce the largest space of functions for which all repeated integrals exist and are equal:

**Definition 3.1.1.** Let $f$ be a $\mathbb{C}(\Gamma)$-valued function on $F^n$. The inductive definition of $f$ being *Fubini*, and the *repeated integral* of $f$, are as follows:

If $n = 1$, then $f$ is Fubini if and only if it is integrable, and the repeated integral of $f$ is defined to be its integral $\int^F f(x)\, dx$.

For $n > 1$, $f$ is Fubini if and only if it satisfies the following conditions:

(i) For each $r$ with $1 \leq r \leq n$, and all $x_1, \dots, \dot{x}_r, \dots, x_n$ in $F$, the function

$$x_r \mapsto f(x_1, \dots, x_n)$$

is required to be integrable on $F$, and then the function

$$(x_1, \dots, \dot{x}_r, \dots, x_n) \mapsto \int^F f(x_1, \dots, x_n)\, dx_r$$

is required to be Fubini on $F^{n-1}$.

(ii) Then we require that the repeated integral of

$$(x_1, \dots, \dot{x}_r, \dots, x_n) \mapsto \int^F f(x_1, \dots, x_n)\, dx_r$$

does not depend on $r$. The repeated integral of $f$ on $F^n$ is defined to be the common value of these $n$ repeated integrals on $F^{n-1}$.

The repeated integral of a Fubini function $f$ on $F^n$ will be denoted $\int^{F^n} f(x)\,dx$.

The repeated integral is a $\mathbb{C}(\Gamma)$-linear functional on the $\mathbb{C}(\Gamma)$-space of all Fubini functions on $F^n$.

**Remark 3.1.2.** Informally, a $\mathbb{C}(\Gamma)$-valued function $f$ is Fubini if and only if, for each permutation $\sigma$ of $\{1,\dots,n\}$, the expression

$$\int^F \dots \int^F f(x_1,\dots,x_n)\,dx_{\sigma(1)}\dots\,dx_{\sigma(n)}$$

is well defined and its value does not depend on $\sigma$. The repeated integral of $f$ is of course the common value of these $n!$ integrals.

**Remark 3.1.3.** We will also be interested in repeated integrals of complex-valued functions on $\overline{F}^n$. Since the integration theory on $F$ does not allow for functions on $\overline{F}$ which are perhaps only defined off a null set, we must ensure that such functions do not arise. Therefore we define a complex-valued function $g$ on $\overline{F}^n$ to be Fubini if it is Haar integrable and satisfies the obvious rewording of definition 3.1.1. Informally, such a function is Fubini if and only if it is Haar integrable and each partial integral

$$\int \dots \int g(u_1,\dots,u_n)\,du_{\sigma(1)}\dots du_{\sigma(r)}$$

is defined for *all* $u_{\sigma(r+1)},\dots,u_{\sigma(n)} \in \overline{F}$, where $\sigma$ is any permutation of $\{1,\dots,n\}$ and $1 \le r \le n$. Fubini's theorem then implies that the value of the repeated integral

$$\int^F \dots \int^F g(u_1,\dots,u_n)\,du_{\sigma(1)}\dots\,du_{\sigma(n)}$$

is independent of $\sigma$.

Fubini's theorem and induction on $n$ imply that any integrable function on $\overline{F}^n$ is almost everywhere equal to a Fubini function.

Any continuous complex-valued function on $\overline{F}$ with compact support is Fubini, as is any Schwartz function if $\overline{F}$ is archimedean. So the class of Fubini functions is still large enough for applications in representation theory, harmonic analysis, etc.

In fact, most Fubini functions on $F^n$ encountered in this paper will be of the following form, which is a generalisation of the notion of a simple function (see subsection 1.4.1)on $F$:

**Definition 3.1.4.** Let $f$ be a Fubini function on $F^n$; the inductive definition of $f$ being *strongly Fubini* is as follows:

If $n = 1$, then $g$ is strongly Fubini if and only if it is a simple function.

For $n > 1$, $g$ is strongly Fubini if and only if the following hold: For each $r$ with $1 \le r \le n$, and each $x_1,\dots,\dot{x}_r,\dots,x_n$ in $F$, we require that

$$x_r \mapsto f(x_1,\dots,x_n)$$

is a simple function on $F$, and then that

$$(x_1,\dots,\dot{x}_r,\dots,x_n) \mapsto \int^F f(x_1,\dots,x_n)\,dx_r$$

is strongly Fubini on $F^{n-1}$.

The property of being strongly Fubini is preserved under translation and scaling, as is the weaker property of being Fubini. For $\alpha = (\alpha_1, \dots, \alpha_n)$ in ${F^\times}^n$ ($n$ copies of $F^\times$, not the group of $n^{\text{th}}$ powers of $F^\times$), write $|\alpha| = \prod_i |\alpha_i|$, where $|\cdot|$ is the absolute value introduced in theorem 1.4.4; for $x \in F^n$ write $\alpha x$ to denote the coordinate-wise product $\alpha x = (\alpha_1 x_1, \dots, \alpha_n x_n)$.

**Lemma 3.1.5.** *Suppose $f$ is a strongly Fubini (resp. Fubini) function on $F^n$. For $a \in F^n$ and $\alpha \in {F^\times}^n$, the functions $x \mapsto f(x+a)$ and $x \mapsto f(\alpha x)$ are strongly Fubini (resp. Fubini), with repeated integrals*

$$\int^{F^n} f(x+a)\,dx = \int^{F^n} f(x)\,dx, \qquad \int^{F^n} f(\alpha x)\,dx = |\alpha|^{-1} \int^{F^n} f(x)\,dx.$$

*Proof.* This is a simple induction on $n$; the case $n = 1$ is remark 1.4.3. □

A continuing theme of this thesis is showing how integrals constructed at the level of $F$ lift Haar integrals on $\overline{F}$. For the integral on $F$, this is the identity

$$\int^F g^0(x)\,dx = \int g(u)\,du$$

for Haar integrable $g$ on $\overline{F}$.

We will denote by $t : \Gamma^n \to F^n$ the product of $n$ copies of $t$; the value of $n$ will be clear from the context. Similarly, we write $\rho$ or an overline for the the residue map $\mathcal{O}_F^n \to \overline{F}^n$. Given $a = (a_1, \dots, a_n) \in F^n$ and $\gamma = (\gamma_1, \dots, \gamma_n) \in \Gamma^n$, there is a product of translated fractional ideals given by

$$a + t(\gamma)\mathcal{O}_F^n = \prod_{i=1}^n a_i + t(\gamma_i)\mathcal{O}_F.$$

Now we may generalise the notion of lifting a function:

**Definition 3.1.6.** For $g$ a function on $\overline{F}^n$ taking values in an abelian group $A$, set

$$\begin{aligned} g^0 : F^n &\to A \\ x &\mapsto \begin{cases} g(\overline{x}) & x \in \mathcal{O}_F^n \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Again, more generally, for $a \in F^n$, $\gamma \in \Gamma^n$, the *lift of $g$ at $a, \gamma$* is the $A$-valued function on $F$ defined by

$$g^{a,\gamma}(x) = \begin{cases} g(\overline{(x-a)t(-\gamma)}) & x \in a + t(\gamma)\mathcal{O}_F^n \\ 0 & \text{otherwise.} \end{cases}$$

Of course, $g^0 = g^{0,0}$ and $g^{a,\gamma}(a + t(\gamma)x) = g^{0,0}(x)$ for all $x \in F^n$.

**Remark 3.1.7.** It is a straightforward observation that a section of a lifted function is again a lifted function. To be precise, suppose that $f = g^{a,\gamma}$ is a lifted function as in the definition, $r$ is such that $1 \le r \le n$, and $x_1, \dots, \dot{x}_r, \dots, x_n \in F$. Then the function

$$x_r \mapsto f(x_1, \dots, x_n)$$

of $F$ is identically zero unless $x_i \in a_i + t(\gamma_i)\mathcal{O}_F$ for all $i \neq r$.

If in fact $x_i \in a_i + t(\gamma_i)\mathcal{O}_F$ for all $i \neq r$, then

$$x_r \mapsto f(x_1, \dots, x_n)$$

is the lift of

$$u_r \mapsto g(\overline{\xi}_1, \dots, \overline{\xi}_{r-1}, u_r, \overline{\xi}_{r+1}, \dots, \overline{\xi}_n)$$

at $a_r, \gamma_r$, where $\xi_i := (x_i - a_i)t(-\gamma_i) \in \mathcal{O}_F$ for $i \neq r$.

This generalises to $s$-dimensional sections of $f$ for any $s$ with $1 \leq s \leq n$. We shall frequently employ the cases $s = 1$ and $s = 2$.

We may now prove the fundamental result that the repeated integral on $F^n$ lifts the Haar integral on $\overline{F}^n$:

**Proposition 3.1.8.** *Suppose $g$ is a Fubini function on $\overline{F}^n$. Then $g^0$ is strongly Fubini on $F^n$, with repeated integral*

$$\int^{F^n} g^0(x)\,dx = \int_{\overline{F}^n} g(u)\,du.$$

*Proof.* Let $r$ be such that $1 \leq r \leq n$, and fix $x_1, \dots, \dot{x}_r, \dots, x_n \in F$. The previous remark and the case $n = 1$ (contained in theorem 1.4.4) imply that $x_r \mapsto g^0(x_1, \dots, x_n)$ is simple and integrable on $F$ with integral

$$\begin{cases} \int g(\overline{x}_1, \dots, \overline{x}_{r-1}, u_r, \overline{x}_{r+1}, \dots, \overline{x}_n)\,du_r & x_i \in \mathcal{O}_F \text{ for all } i \neq r \\ 0 & \text{otherwise.} \end{cases}$$

That is,

$$(x_1, \dots, \dot{x}_r, \dots, x_n) \mapsto \int^F g^0(x_1, \dots, x_n)\,dx_r$$

is the lift of the everywhere defined Haar integrable function

$$(u_1, \dots, \dot{u}_r, \dots, u_n) \mapsto \int g(u_1, \dots, u_n)\,du_r$$

on $\overline{F}^{n-1}$.

The result now follows easily by induction on $n$. ☐

**Remark 3.1.9.** More generally, suppose $f = g^{a,\gamma}$ is the lift of a Fubini function to $F^n$; here $g$ is Fubini on $\overline{F}^n$, $a \in F^n$ and $\gamma \in \Gamma^n$. Then the proposition and the invariance of being strongly Fubini under translation and scaling (lemma 3.1.5) imply $f$ is strongly Fubini on $F^n$, with repeated integral

$$\int^{F^n} f(x)\,dx = \int_{\overline{F}^n} g(u)\,du\,X^{\sum_{i=1}^n \gamma_i}.$$

**Remark 3.1.10.** Using a similar inductive argument as in the previous proposition and the details of the proof in subsection 2.3 on harmonic analysis on $F$, there is no difficulty in showing that if $g$ is a Fubini function on $\overline{F}^n$ and $\psi : F \to S^1$ is a good character on $F$, then

$$x \mapsto g^{a,\gamma}(x)\psi(b_1 x_1 + \dots b_n x_n)$$

is Fubini on $F^n$, for any $b \in F^n$ (though, of course, one must replace the integrability condition in the definition of a Fubini function on $F^n$ by the condition that it belongs to the enlarged space $\mathcal{L}(F, \psi)$).

Similarly, it is straightforward to generalise the results of subsection 2.3.2 on Fourier transforms to $F^n$. Also see remark 3.2.14.

## 3.2 Linear changes of variables in repeated integrals

With the basics of repeated integrals in place, we turn to the interaction of the theory with $GL_n(F)$. We shall write the action of $GL_n(F)$ on $F^n$ as a left action, though we also write elements of $F^n$ as row vectors; given $\tau \in GL_n(F)$ and $x = (x_1, \dots, x_n) \in F^n$, $\tau x$ means

$$\tau x = \tau \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Given a function $f$ on $F^n$, we write $f \circ \tau$ for the function $x \mapsto f(\tau x)$. $SL_n(F)$ denotes the determinant $1$ subgroup of $GL_n(F)$. These notation also apply to $\overline{F}$ in place of $F$.

**Definition 3.2.1.** A complex-valued function $g$ on $\overline{F}^n$ is said to be *GL-Fubini* if and only if $g \circ \tau$ is Fubini for all $\tau \in GL_n(\overline{F})$.

**Remark 3.2.2.** Any continuous complex-valued function with compact support is $GL$-Fubini, as is any Schwartz function when $\overline{F}$ is archimedean; this follows from remark 3.1.3 and the invariance of these properties under $GL_n(\overline{F})$. In the following results this is the sort of function to have in mind.

**Definition 3.2.3.** Let $\mathcal{L}(F^n, GL_n)$ be the $\mathbb{C}(\Gamma)$ space of $\mathbb{C}(\Gamma)$-valued functions spanned by $g^{a,\gamma} \circ \tau$ for $g$ $GL$-Fubini, $\tau \in GL_n(F)$, $a \in F^n$, $\gamma \in \Gamma^n$.

The aim of this section is the following result:

**Theorem 3.2.4.** *Every function in $\mathcal{L}(F^n, GL_n)$ is Fubini on $F^n$. If $f \in \mathcal{L}(F^n, GL_n)$, $a \in F^n$, and $\tau \in GL_n(F)$, then the functions $x \mapsto f(x + a)$ and $x \mapsto f(\tau x)$ belong to $\mathcal{L}(F^n, GL_n)$, with repeated integrals given by*

$$\int^{F^n} f(x+a)\, dx = \int^{F^n} f(x)\, dx, \qquad \int^{F^n} f(\tau x)\, dx = |\det \tau|^{-1} \int^{F^n} f(x)\, dx$$

The theorem will be proved through several smaller results. First we recall the Iwasawa decomposition, where we abbreviate 'unipotent upper triangular' to u.u.t.

**Lemma 3.2.5.** *Let $\tau$ be in $GL_n(F)$. Then there exist $A$ in $GL_n(\mathcal{O}_F)$, a u.u.t. $U$ in $GL_n(F)$, and a diagonal $\Lambda$ in $GL_n(F)$ such that $\tau = AU\Lambda$.*

*Proof.* When $\Gamma = \mathbb{Z}$ and $F$ is complete with respect to the discrete valuation $\nu$, this is the standard Iwasawa decomposition. However, the standard proof is valid in the generality in which we require it (see e.g. [Bum97, Proposition 4.5.2]). □

This decomposition allows us to restrict attention to upper triangular matrices, for the $GL_n(\mathcal{O}_F)$ term can be 'absorbed' into the function:

**Lemma 3.2.6.** *$\mathcal{L}(F^n, GL_n)$ is spanned over $\mathbb{C}(\Gamma)$ by functions of the form $x \mapsto g^0 \circ U(\alpha x + a)$, for $g$ $GL$-Fubini on $\overline{F}^n$, $U$ a u.u.t. matrix, $\alpha \in F^{\times n}$, and $a \in F^n$.*

*Proof.* Let $g$ be $GL$-Fubini on $\overline{F}^n$, $\tau \in GL_n(F)$, $a \in F^n$ and $\gamma \in \Gamma^n$. Let $A, U, \Lambda$ be the Iwasawa decomposition of

$$\begin{pmatrix} t(-\gamma_1) & & \\ & \ddots & \\ & & t(-\gamma_n) \end{pmatrix} \tau,$$

as in lemma 3.2.5. For $x$ in $F^n$, the identity $g^{a,\gamma} \circ \tau(x) = g^0 \circ AU\Lambda(x - \tau^{-1}a)$ holds.

Now note that $g^0 \circ A = (g \circ \overline{A})^0$ where $\overline{A}$ is the image of $A$ in $GL_n(\overline{F})$. So $x \in F^n$ implies $g^{a,\gamma} \circ \tau(x) = (g \circ \overline{A})^0(U(\lambda x + b))$, where $\lambda \in F^{\times n}$ is defined by

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix},$$

and $b = -\lambda\tau^{-1}a$. We have written $g^{a,\gamma} \circ \tau$ in the required form, and this is enough to complete the proof. $\square$

We now prove special cases of the main theorem as well as some technical lemmas. Particular attention is given to the case $n = 2$, for it is required several times later in inductions.

**Lemma 3.2.7.** *Let $g$ be GL-Fubini on $\overline{F}^2$ and set $f = g^0$. Let $\alpha \in F$ and set $e = \alpha^{-1}t(\nu(\alpha))$ if $\alpha \neq 0$, and $e = 0$ otherwise; set $\delta_0 = \min(\nu(\alpha), 0)$.*

*There exists $\tau \in SL_2(\overline{F})$, independent of $g$, such that, for any $x \in F$, the function $y \mapsto f(x + \alpha y, y)$ equals*

$$\begin{cases} \text{the lift of } v \mapsto g \circ \tau(\overline{xt(-\delta_0)}, v) \text{ at } -xet(-\delta_0), -\delta_0 & \text{if } x \in t(\delta_0)\mathcal{O}_F \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* If $\alpha = 0$ then we are just considering a section of a Fubini function and so $\tau = \mathrm{id}$ suffices by remark 3.1.7. Henceforth assume that $\alpha \neq 0$.

We first consider the case $\alpha = t(\delta)$ for some $\delta \in \Gamma$; so $e = 1$. Consider, for any $x \in F$, the section

$$\begin{aligned} D_x : F &\to \mathbb{C} \\ y &\mapsto f(x + t(\delta)y, y). \end{aligned}$$

We make the following claim, dependent on the sign of $\delta$, regarding $D_x$:

**case:** $\delta < 0$.

$$D_x = \begin{cases} \text{lift of } v \mapsto g(v, -\overline{xt(-\delta)}) \text{ at } -xt(-\delta), -\delta & \text{if } x \in t(\delta)\mathcal{O}_F \\ 0 & \text{otherwise.} \end{cases}$$

**case:** $\delta = 0$.

$$D_x = \begin{cases} \text{lift of } v \mapsto g(v + \overline{x}, v) \text{ at } 0, 0 & \text{if } x \in \mathcal{O}_F \\ 0 & \text{otherwise} \end{cases}$$

**case:** $\delta > 0$.

$$D_x = \begin{cases} \text{lift of } v \mapsto g(\overline{x}, v) \text{ at } 0, 0 & \text{if } x \in \mathcal{O}_F \\ 0 & \text{otherwise.} \end{cases}$$

We shall prove the case $\delta = 0$. For any $x, y \in F$, $f(x + y, y)$ vanishes unless $x + y$ and $y$ both belong to $\mathcal{O}_F$; hence $D_x$ is identically zero unless $x \in \mathcal{O}_F$. Assuming that $x \in \mathcal{O}_F$, it remains to verify that

$$D_x = \text{lift of } v \mapsto g(v + \overline{x}, v) \text{ at } 0, 0.$$

Both sides vanish off $\mathcal{O}_F$ and are seen to agree on $\mathcal{O}_F$ by direct evaluation. This proves the claim in this case. The other cases are proved by similar arguments and we omit the details.

If $\delta \geq 0$ and $x \in \mathcal{O}_F$, then $D_x$ is also the lift of a function at $-x, 0$:
**case:** $\delta = 0$.

$$D_x = \text{lift of } v \mapsto g(v, v - \overline{x}) \text{ at } -x, 0$$

**case:** $\delta > 0$.

$$D_x = \text{lift of } v \mapsto g(\overline{x}, v - \overline{x}) \text{ at } -x, 0$$

The proof when $\alpha \in t(\Gamma)$ is completed by setting:
**case:** $\delta < 0$.

$$\tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

**case:** $\delta = 0$.

$$\tau = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

**case:** $\delta > 0$.

$$\tau = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

In the general case, write $\alpha = e^{-1}t(\delta)$, with $\delta = \nu(\alpha)$ and $e \in \mathcal{O}_F^\times$; let $\tau' = \begin{pmatrix} \overline{e}^{-1} & 0 \\ 0 & 1 \end{pmatrix}$. Also introduce $f'(x, y) = f(e^{-1}x, y)$, which is the lift of $(u, v) \mapsto g(\overline{e}^{-1}u, v) = g \circ \tau'(u, v)$ (a Fubini function on $\overline{F}^2$) at $0, 0$. By the case above, there exists $\tau \in SL_2(\overline{F})$ such that $x \in F$ implies $y \mapsto f'(x + t(\delta)y, y) = f(e^{-1}x + \alpha y, y)$ equals

$$\begin{cases} \text{the lift of } v \mapsto g \circ \tau'\tau(\overline{xt(-\delta_0)}, v) \text{ at } -xt(-\delta_0), -\delta_0 & \text{if } \nu(x) \geq \delta_0 \\ 0 & \text{otherwise.} \end{cases}$$

Hence $y \mapsto f(x + \alpha y, y) = f'(ex + t(\delta)y, y)$ equals

$$\begin{cases} \text{the lift of } v \mapsto g \circ \tau'\tau(\overline{e}\,\overline{xt(-\delta_0)}, v) \text{ at } -ext(-\delta_0), -\delta_0 & \text{if } \nu(x) \geq \delta_0 \\ 0 & \text{otherwise.} \end{cases}$$

As $\tau'\tau \begin{pmatrix} \overline{e} & 0 \\ 0 & 1 \end{pmatrix}$ has determinant $1$, this completes the proof. $\square$

Remaining with the case $n = 2$, we now extend the previous lemma slightly in preparation for the induction on $n$:

**Lemma 3.2.8.** *Let $g$ be GL-Fubini on $\overline{F}^2$, $a \in F$, $\gamma \in \Gamma$; set $f = g^{(0,a),(0,\gamma)}$. Let $\alpha \in F$ and set $\delta = \min(\nu(\alpha) + \gamma, 0)$.*

*There exist $b, c \in F$ (independent of $g$) and $\tau \in SL_2(\overline{F})$ (independent of $g$ and $a$) such that $x \in F$ implies that $y \mapsto f(x + \alpha y, y)$ equals*

$$\begin{cases} \textit{the lift of } v \mapsto g \circ \tau(\overline{(x - c)t(-\delta)}, v) \textit{ at } b, \gamma - \delta & \textit{if } x \in c + t(\delta)\mathcal{O}_F \\ 0 & \textit{otherwise} \end{cases}$$

*Proof.* Let $e = \alpha^{-1}t(\nu(\alpha))$ if $\alpha \neq 0$, and $e = 0$ otherwise. For $x$ in $F$ the previous lemma implies that $y \mapsto g^0(x + t(\gamma)\alpha y, y)$ equals

$$\begin{cases} \text{the lift of } v \mapsto g \circ \tau(\overline{xt(-\delta)}, v) \text{ at } -xet(-\delta), -\delta & \text{if } x \in t(\delta)\mathcal{O}_F \\ 0 & \text{otherwise} \end{cases}$$

for some $\tau \in SL_2(\overline{F})$ (independent of $g$ by the previous lemma, and clearly independent of $a$).

For $x, y \in F$, the identity

$$\begin{aligned} &f(x+\alpha y, y) \\ &\quad = g^0(x+\alpha y, (y-a)t(-\gamma)) \\ &\quad = g^0(x+\alpha a + t(\gamma)\alpha(y-a)t(-\gamma), (y-a)t(-\gamma)) \\ &\quad = \begin{cases} g \circ \tau(\overline{(x+\alpha a)t(-\delta)}, \overline{((y-a)t(-\gamma)+xet(-\delta))t(\delta)}) & \text{if } x+\alpha a \in t(\delta)\mathcal{O}_F \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

follows. Set $b = a - ext(\gamma - \delta)$ and $c = -\alpha a$ to complete the proof. □

**Remark 3.2.9.** The proper interpretation of the previous two lemmas is available through Hrushovski and Kazhdan's work [HK06]. They prove, in a precise sense which requires model theory and Grothendieck groups, that any bijection at the valued field level with Jacobian 1 (such as our $(x, y) \mapsto (x + \alpha y, y)$) descends to a bijection at the residue field level, also with Jacobian 1 (such as $u \mapsto \tau u$, with $\tau$ as in the statement of our lemmas). Their deeper result is the converse: bijections at the residue field level may be lifted.

However, our result is not entirely a special case of theirs, since their methods work only in residue characteristic zero, whereas the lemmas above hold in general.

The following result extends the previous lemma to the case of arbitrary $n \geq 2$; it is a slightly technical proof by induction:

**Lemma 3.2.10.** *Let $g$ be GL-Fubini on $\overline{F}^n$, $a \in F$, $\gamma \in \Gamma$; set $f = g^{(0,\ldots,0,a),(0,\ldots,0,\gamma)}$. Let $\alpha_i \in F$ for $1 \leq i \leq n-1$. Then*

*(i) For all $x_1, \ldots, x_{n-1} \in F$, the function of $F$*

$$x_n \mapsto f(x_1 + \alpha_1 x_n, \ldots, x_{n-1} + \alpha_{n-1} x_n, x_n)$$

*is integrable and simple.*

*(ii) Further, there exist $\tau \in SL_n(\overline{F})$, $\delta \in \Gamma^{n-1}$, and $c \in F^{n-1}$ such that the function of $F^{n-1}$*

$$(x_1, \ldots, x_{n-1}) \mapsto \int^F f(x_1 + \alpha_1 x_n, \ldots, x_{n-1} + \alpha_{n-1} x_n, x_n)\, dx_n$$

*is the lift of*

$$(u_1, \ldots, u_{n-1}) \mapsto \int g \circ \tau(u_1, \ldots, u_n)\, du_n\, X^{\gamma - \sum_{i=1}^{n-1} \delta_i}$$

*at $c, \delta$. Also, $\tau$ may be chosen to be independent of $g$ and $a$.*

*Proof.* The proof is by induction on $n$.

Let $\delta_{n-1} = \min(\nu(\alpha_{n-1}) + \gamma, 0)$. Let $\xi_1, \ldots, \xi_{n-2}$ be in $\mathcal{O}_F$; the function

$$(x_{n-1}, x_n) \mapsto f(\xi_1, \ldots, \xi_{n-2}, x_{n-1}, x_n)$$

is the lift of

$$(u_{n-1}, u_n) \mapsto g(\overline{\xi}_1, \ldots, \overline{\xi}_{n-2}, u_{n-1}, u_n),$$

which is $GL$-Fubini, at $(0,a),(0,\gamma)$; this is just a generalisation of remark 3.1.7 to a two dimensional section. By the previous lemma, there exist $b, c_{n-1} \in F$ and $\tau \in SL_2(\overline{F})$, all independent of $\xi_1,\dots,\xi_{n-2}$, such that for all $x_{n-1} \in F$,

$$x_n \mapsto f(\xi_1,\dots,\xi_{n-2},x_{n-1}+\alpha_{n-1}x_n,x_n)$$

equals the lift of

$$u_n \mapsto g(\overline{\xi}_1,\dots,\overline{\xi}_{n-2},\tau(\overline{(x_{n-1}-c_{n-1})t(-\delta_{n-1})},u_n))$$

at $b,\gamma-\delta_{n-1}$ if $x_{n-1} \in c_{n-1}+t(\delta_{n-1})\mathcal{O}_F$, and equals $0$ otherwise.

Also denote by $\tau$ the element of $SL_n(\overline{F})$ given by $\begin{pmatrix} I_{n-2} & 0 \\ 0 & \tau \end{pmatrix}$, where $I_{n-2}$ denotes the $n-2$ by $n-2$ identity matrix.

Now take $\xi_{n-1} \in c_{n-1}+t(\delta_{n-1})\mathcal{O}_F$; so $\xi_{n-1} = c_{n-1}+t(\delta_{n-1})\xi'_{n-1}$, say. We have just shown that

$$(x_1,\dots,x_{n-2},x_n) \mapsto f(x_1,\dots,x_{n-2},\xi_{n-1}+\alpha_{n-1}x_n,x_n)$$

is the lift of

$$(u_1,\dots,u_{n-2},u_n) \mapsto g\circ\tau(u_1,\dots,u_{n-2},\overline{\xi}'_{n-1},u_n),$$

which is $GL$-Fubini, at $(0,\dots,0,b),(0,\dots,0,\gamma-\delta_{n-1})$. By the inductive hypothesis, the following hold:

(i) For all $x_1,\dots,x_{n-2} \in F$,

$$x_n \mapsto f(x_1+\alpha_1x_n,\dots,\xi_{n-1}+\alpha_{n-1}x_n,x_n)$$

is a simple, integrable function.

(ii) There exists $\tau' \in SL_{n-1}(\overline{F})$ (independent of $\xi_{n-1}$, $g$, $b$) and $\delta_i \in \Gamma$, $c_i \in F$ ($1 \le i \le n-2$), such that

$$(x_1,\dots,x_{n-2}) \mapsto \int^F f(x_1+\alpha_1x_n,\dots,\xi_{n-1}+\alpha_{n-1}x_n,x_n)\,dx_n$$

is the lift of

$$(u_1,\dots,u_{n-2}) \mapsto \int g\circ\tau\tau'(u_1,\dots,u_{n-2},\overline{\xi}'_{n-1},u_n)\,du_n\ X^{\gamma-\delta_{n-1}-\sum_{i=1}^{n-2}\delta_i}$$

at $(c_1,\dots,c_{n-2}),(\delta_1,\dots,\delta_{n-2})$.

It follows that

(i) For any $x_1,\dots,x_{n-1}$ in $F$,

$$x_n \mapsto f(x_1+\alpha_1x_n,\dots,x_{n-1}+\alpha_{n-1}x_n,x_n)$$

is a simple, integrable function (this function is zero unless $x_{n-1} \in c_{n-1}+t(\delta_{n-1})\mathcal{O}_F$, in which case the statement follows from (i) above).

(ii) The function

$$(x_1,\dots,x_{n-1}) \mapsto \int^F f(x_1+\alpha_1x_n,\dots,x_{n-1}+\alpha_{n-1}x_n,x_n)\,dx_n$$

is the lift of

$$(u_1,\dots,u_{n-1}) \mapsto \int g\circ\tau\tau'(u_1,\dots,u_n)\,du_n\ X^{\gamma-\sum_{i=1}^{n-1}\delta_i}$$

at $(c_1,\dots,c_{n-1}),(\delta_1,\dots,\delta_{n-1})$.

This completes the proof. □

The previous lemma was concerned with the case of a matrix differing from the identity only along the left-most column. We now consider an arbitrary u.u.t. matrix:

**Proposition 3.2.11.** *Suppose $g$ is GL-Fubini on $\overline{F}^n$, $a \in F^n$, $\gamma \in \Gamma^n$, $\delta \in \Gamma$; set $f = g^{a,\gamma} X^\delta$. Let $U$ be a u.u.t. matrix in $GL_n(F)$. Then $f \circ U$ is strongly Fubini on $F^n$, with*

$$\int^{F^n} f \circ U(x)\, dx = \int^{F^n} f(x)\, dx.$$

*Proof.* The proof is by induction on $n$.

For any $n$, we claim that it suffices to prove the special case $a = 0$, $\gamma = 0$, $\delta = 0$. We may clearly assume $\delta = 0$ by linearity. For $x \in F^n$ the identity

$$\begin{aligned} f(Ux) = g^{a,\gamma}(Ux) &= g^{0,0}((Ux - a)t(-\gamma)) \\ &= g^0 \circ U_1(t(-\gamma)(x - U^{-1}a)) \end{aligned}$$

holds, where $U_1$ is the u.u.t. matrix

$$U_1 = \begin{pmatrix} t(-\gamma_1) & & \\ & \ddots & \\ & & t(-\gamma_n) \end{pmatrix} U \begin{pmatrix} t(\gamma_1) & & \\ & \ddots & \\ & & t(\gamma_n) \end{pmatrix}.$$

Assuming the special case, we may conclude that $g^0 \circ U_1$ is strongly Fubini, with repeated integral equal to that of $g^0$. Thus $f \circ U$ differs from a strongly Fubini function by translation and scaling and hence is itself strongly Fubini (lemma 3.1.5), while compatibility between the repeated integral on $F^n$ and Haar integral on $\overline{F}^n$ (proposition 3.1.8) implies

$$\begin{aligned} \int^{F^n} f \circ U(x)\, dx &= |t(\gamma)| \int^{F^n} g^0(x)\, dx \\ &= X^{\sum_{i=1}^n \gamma_i} \int_{\overline{F}^n} g(u)\, du \\ &= \int^{F^n} f(x)\, dx. \end{aligned}$$

This completes the proof of the claim; so now assume $a = 0$, $\gamma = 0$, $\delta = 0$.

For each $r$ with $1 \le r \le n$, we must now prove that

(i) For $x_1, \dots, \dot{x}_r, \dots, x_n \in F$, the function of $F$, $x_r \mapsto f \circ U(x_1, \dots, x_n)$, is simple and integrable.

(ii) The function of $F^{n-1}$

$$(x_1, \dots, \dot{x}_r, \dots, x_n) \mapsto \int^F f \circ U(x_1, \dots, x_n)\, dx_r$$

is strongly Fubini, with repeated integral equal to that of $f$.

The inductive step depends on decomposing $U$ in a certain way. Write

$$U = \begin{pmatrix} 1 & \alpha_{1,2} & \cdots & \alpha_{1,n} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \alpha_{n-1,n} \\ & & & 1 \end{pmatrix}$$

and observe that $U(x_1,\dots,x_n)=(x_1+\sum_{i=2}^n\alpha_{1,i}x_i,\dots,x_{n-1}+\alpha_{n-1,n}x_n,x_n)$. Let $V$ be the u.u.t. matrix obtained by setting to zero all entries in the $r^{\text{th}}$ row and $r^{\text{th}}$ column of $U$, apart from the $1$ in the $r,r$-place. Let $V'$ be the $n-1$ by $n-1$ u.u.t. matrix obtained by removing the $r^{\text{th}}$ row and $r^{\text{th}}$ column of $U$. Then there exist $\beta_{r+1},\dots,\beta_n\in F$ such that the u.u.t. matrix $P$ defined by

$$P(x_1,\dots,x_n)=(x_1+\alpha_{1,r}x_r,\dots,x_{r-1}+\alpha_{r-1,r}x_r,x_r+\sum_{i=r+1}^n\beta_ix_i,x_{r+1},\dots,x_n)$$

satisfies $U=PV$.

We are now equipped to begin the main part of the proof. The previous lemma (if $r>1$; it follows straight from the definition of a strongly Fubini function if $r=1$) implies that for fixed $x_1,\dots,\dot{x}_r,\dots,x_n\in F$, the function

$$\begin{aligned}x_r\mapsto f((x_1-\alpha_{1,r}\sum_{i=r+1}^n\beta_ix_i)+\alpha_{1,r}x_r,\dots\\ \dots,(x_{r-1}-\alpha_{r-1,r}\sum_{i=r+1}^n\beta_ix_i)+\alpha_{r-1,r}x_r,x_r,\dots,x_n)\end{aligned}$$

is simple and integrable on $F$. Therefore

$$\begin{aligned}x_r\mapsto &f(x_1+\alpha_{1,r}x_r,\dots,x_{r-1}+\alpha_{r-1,r}x_r,x_r+\sum_{i=r+1}^n\beta_ix_i,x_{r+1}\dots,x_n)\\ &=f\circ P(x_1,\dots,x_n)\end{aligned}$$

is a translate of a simple, integrable function and hence is itself simple and integrable by remark 1.4.3. Replacing $x_1,\dots,\dot{x}_r,\dots,x_n$ by $V'(x_1,\dots,\dot{x}_r,\dots,x_n)$ implies that the function

$$\begin{aligned}x_r\mapsto &f\circ PV(x_1,\dots,x_n)\\ &=f\circ U(x_1,\dots,x_n)\end{aligned}$$

is simple and integrable, proving (i).

The previous lemma (if $r>1$) and translation invariance (any $r$) of the integral also imply that

$$f':F^{n-1}\to\mathbb{C}(\Gamma),\quad(x_1,\dots,\dot{x}_r,\dots,x_n)\mapsto\int^F f\circ P(x_1,\dots,x_n)\,dx_r$$

is the lift of

$$(u_1,\dots,\dot{u}_r,\dots,u_n)\mapsto\int g\circ\tau(u_1,\dots,u_n)\,du_r\;X^{-\sum_{i=1}^{n-1}\delta_i}$$

at $b,\delta$ for some $b\in F^{n-1}$, $\delta=(\delta_i)\in\Gamma^{n-1}$, $\tau\in SL_n(\overline{F})$.

The inductive hypothesis with function $f'$ and matrix $V'$ implies that $f'\circ V'$ is strongly Fubini with repeated integral equal to that of $f'$. But the repeated integral of $f'$ is

$$\begin{aligned}\int_{\overline{F}^n}g\circ\tau(u)\,du\;X^{-\sum_{i=1}^{n-1}\delta_i}\;X^{\sum_{i=1}^{n-1}\delta_i}&=\int_{\overline{F}^n}g(u)\,du\\ &=\int^{F^n}f(x)\,dx\end{aligned}$$

by remark 3.1.9, and

$$\begin{aligned} f' \circ V'(x_1, \dots, \dot{x}_r, \dots, x_n) &= \int^F f \circ PV(x_1, \dots, x_n)\, dx_r \\ &= \int^F f \circ U(x_1, \dots, x_n)\, dx_r, \end{aligned}$$

which proves (ii). □

**Proposition 3.2.12.** *Let $g$ be GL-Fubini on $\overline{F}^n$, $a \in F^n$, $\gamma \in \Gamma^n$, $\delta \in \Gamma$; set $f = g^{a,\gamma} X^\delta$. Let $\tau \in GL_n(F)$; then $f \circ \tau$ is strongly Fubini on $F^n$, with*

$$\int^{F^n} f \circ \tau(x)\, dx = |\det \tau|^{-1} \int^{F^n} f(x)\, dx.$$

*Proof.* We claim that it suffices to prove the special case $a = 0$, $\gamma = 0$, $\delta = 0$. This claim follows in the same way as the beginning of proposition 3.2.11. Now assume $a = 0$, $\gamma = 0$, $\delta = 0$.

Write $\tau = AU\Lambda$ as in lemma 3.2.5. Then $f \circ A = (g \circ \overline{A})^0$ where $\overline{A}$ is the image of $A$ in $GL_n(\overline{F})$; proposition 3.1.8 implies

$$\begin{aligned} \int^{F^n} f \circ A(x)\, dx &= \int_{\overline{F}^n} g \circ \overline{A}(u)\, du \\ &= |\det \overline{A}|^{-1} \int_{\overline{F}^n} g(u)\, du \\ &= |\det A|^{-1} \int^{F^n} f(x)\, dx. \end{aligned}$$

Proposition 3.2.11 implies that $f \circ AU$ is strongly Fubini, with

$$\int^{F^n} f \circ AU(x)\, dx = \int^{F^n} f \circ A(x)\, dx.$$

Finally, lemma 3.1.5 implies that $f \circ AU\Lambda$ is strongly Fubini, with

$$\int^{F^n} f \circ AU\Lambda(x)\, dx = |\det \Lambda|^{-1} \int^{F^n} f \circ AU(x)\, dx.$$

Since $\det \tau = \det A \det \Lambda$, the proof is complete. □

The previous proposition extends by linearity to all of $\mathcal{L}(F^n, GL_n)$ and so the main theorem 3.2.4 is proved!

**Remark 3.2.13.** Suppose $F$ is a two-dimensional local field, with $O_F = \rho^{-1}(\mathcal{O}_{\overline{F}})$ the rank two ring of integers. Assume that our chosen Haar measure on $\overline{F}$ assigns $\mathcal{O}_{\overline{F}}$ measure 1. Then for any $\tau \in GL_n(F)$ and $a \in F^n$, the characteristic function of $a + \tau(O_F^n)$ belongs to $\mathcal{L}(F^n, GL_n)$, and

$$\int^{F^n} \mathrm{char}_{a+\tau(O_F^n)}(x)\, dx = |\det \tau| \in \mathbb{C}(X) = \mathbb{C}(\Gamma).$$

Kim and Lee [KL05] have also developed a measure theory on $F^n$. Their measurable sets are the algebra of sets generated by $\varnothing$, $F^n$ and $a + \tau(O_F^n)$ for $a \in F^n$, $\tau \in GL_n(F)$; the measure assigned to $a + \tau(O_F^n)$ is $|\det \tau|$, as in the approach of this chapter.

However, the measure of Kim and Lee does not take values in $\mathbb{C}(X)$, but rather in an additive monoid consisting of elements 0 and $\lambda X^i$, $\lambda \in \mathbb{R}_{>0}$, $i \in \mathbb{Z}$; addition is defined by

$$\lambda X^i + \lambda' X^j = \begin{cases} \lambda X^i & \text{if } i < j \\ (\lambda + \lambda')X^i & \text{if } i = j \\ \lambda' X^j & \text{if } i > j. \end{cases}$$

If $S$ is a measurable set in the approach of Kim and Lee, then $\mathrm{char}_S$ will belong to $\mathcal{L}(F^n, GL_n)$; expanding the value of the integral in $\mathbb{R}((X))$ we may write

$$\int^{F^n} \mathrm{char}_S(x)\,dx = \sum_{i \geq I} \lambda_i X^i,$$

where $\lambda_i \in \mathbb{R}$ and $\lambda_I \neq 0$. Kim and Lee assign $S$ measure $\lambda_I X^I$; this truncation of the measure is suitable for defining a convolution of functions on $GL_n(F)$ and for ensuring $\sigma$-additivity.

**Remark 3.2.14.** Whether the extension of the integral to $\mathcal{L}(F^n, GL_n)$ is compatible with harmonic analysis on $F^n$ (remark 3.1.10) is indisputable; the integral surely extends to the $\mathbb{C}(\Gamma)$ space of functions on $F^n$ generated by

$$x \mapsto g^{a,\gamma} \circ \tau(x)\, \psi(b_1 x_1 + \cdots + b_n x_n)$$

with $g$ Schwartz-Bruhat on $\overline{F}^n$, $a, b \in F$, and $\tau \in GL_n(F)$.

Unfortunately, the author can see no easy way of proving this, except by simply modifying all the proofs of this section to include twisted functions.

## 3.3 Invariant integral on $GL_N(F)$

We will now consider integration on the space of matrices $M_N(F)$ and its unit group $GL_N(F)$.

Let $n = N^2$ and identify $M_N(F)$ with $F^n$ via an isomorphism $T : F^n \to M_N(F)$ of $F$ vector spaces. Let $\mathcal{L}(M_N(F))$ be the $\mathbb{C}(\Gamma)$ space of $\mathbb{C}(\Gamma)$-valued functions $f$ on $M_N(F)$ for which $fT$ belongs to $\mathcal{L}(F^n, GL_n)$; set

$$\int^{M_N(F)} f(x)\,dx = \int^{F^n} fT(x)\,dx.$$

**Remark 3.3.1.** The space $\mathcal{L}(M_N(F))$ does not depend on the choice of the isomorphism $T$ since $\mathcal{L}(F^n, GL_n)$ is invariant under the action of $GL_n(F)$, and the functional $\int^{M_N(F)}$ depends on $T$ only up to a scalar multiple from $|F^\times| = \{\lambda X^\gamma : \lambda \in |\overline{F}^\times|, \gamma \in \Gamma\}$.

$\mathcal{L}(M_N(F))$ is closed under translation, and $\int^{M_N(F)}$ is a translation invariant $\mathbb{C}(\Gamma)$-linear functional on the space.

Of course, integrating on $M_N(F)$ is no harder than integrating on $F^n$. We are really interested in $GL_N(F)$, for which we proceed by analogy with subsection 2.4

**Definition 3.3.2.** Let $\mathcal{L}(GL_N(F))$ denote the space of $\mathbb{C}(\Gamma)$-valued functions $\phi$ on $GL_N(F)$ such that $\tau \mapsto \phi(\tau)|\det \tau|^{-n}$ may be extended to all of $M_N(F)$ to give a function in $\mathcal{L}(M_N(F))$.

The integral of $\phi$ over $GL_N(F)$ is defined by

$$\int^{GL_N(F)} \phi(\tau)\, d\tau = \int^{M_N(F)} \phi(x)|\det x|^{-n}\, dx,$$

where the integrand on the right is really the extension of the function to $M_N(F)$.

**Remark 3.3.3.** For the previous definition of the integral to be well defined, we must show that if $f_1, f_2 \in \mathcal{L}(M_N(F))$ are equal when restricted to $GL_N(F)$ then $f_1 = f_2$.

It suffices to prove that if $f \in \mathcal{L}(F^n, GL_n)$ vanishes off some Zariski closed set (other than $F^n$), then $f$ is identically zero. By a *locally constant* function $g$ on $F^n$, we mean a function such that for each $a \in F^n$, there exists $\gamma \in \Gamma$ such that, if $\varepsilon_1, \dots, \varepsilon_n \in F$ have valuation greater than $\gamma$, then $f(a_1 + \epsilon_1, \dots, a_n + \epsilon_n) = f(a_1, \dots, a_n)$. If $g_1$, $g_2$ are locally constant, then so are $g_1 + g_2$ and $g_1 \circ A$ for any affine transformation of $F^n$. But a lifted function is locally constant and so any function in $\mathcal{L}(F^n, GL_n)$ is locally constant. It is now enough to show that if $p$ is a polynomial in $F[X_1, \dots, X_n]$, such that $p(\epsilon_1, \dots, \epsilon_n) = 0$ whenever $\varepsilon_1, \dots, \varepsilon_n \in F$ have large enough valuation, then $p$ is the zero polynomial. This is easily proved by induction on $n$.

This calculation even means that we may enlarge the space $\mathcal{L}(F^n, GL_n)$ by adjoining the characteristic functions of all proper Zariski closed sets, and extend the integral by insisting that such sets have zero measure. Ignoring proper Zariski closed sets is an essential part of the model-theoretic approach to integration in chapter 5.

The integral is translation invariant, as desired:

**Proposition 3.3.4.** *Suppose $\phi$ belongs to $\mathcal{L}(GL_N(F))$ and $\sigma \in GL_N(F)$. Then the functions $\tau \mapsto \phi(\sigma\tau)$ and $\tau \mapsto \phi(\tau\sigma)$ also belong to $\mathcal{L}(GL_N(F))$, with*

$$\int^{GL_N(F)} \phi(\sigma\tau)\, d\tau = \int^{GL_N(F)} \phi(\tau)\, d\tau = \int^{GL_N(F)} \phi(\tau\sigma)\, d\tau.$$

*Proof.* Let $r_\sigma$ (resp. $l_\sigma$) denote the element of $GL_n(F)$ (identified with $GL(M_N(F))$ via. $T$) defined by right (resp. left) multiplication by $\sigma$. Let $\tau \mapsto \phi(\tau)|\det\tau|^{-n}$ be the restriction of $f \in \mathcal{L}(M_N(F))$ to $GL_N(F)$, say. The function

$$\begin{aligned} \tau \mapsto\, & \phi(\tau\sigma)|\det\tau|^{-n} \\ =& |\det\sigma|^n\phi(\tau\sigma)|\det\tau\sigma|^{-n} \\ =& |\det\sigma|^n\phi\circ r_\sigma(\tau)|\det(r_\sigma\tau)|^{-n} \end{aligned}$$

is the restriction of $|\det\sigma|^n f \circ r_\sigma \in \mathcal{L}(M_N(F))$ to $GL_N(F)$.

Theorem 3.2.4 therefore implies that

$$\begin{aligned} \int^{GL_N(F)} \phi(\tau\sigma)\, d\tau &= \int^{M_N(F)} |\det\sigma|^n f\circ r_\sigma(x)\, dx \\ &= |\det\sigma|^n|\det r_\sigma|^{-1} \int^{M_N(F)} f(x)\, dx \\ &= |\det\sigma|^n|\det r_\sigma|^{-1} \int^{GL_N(F)} \phi(\tau)\, d\tau. \end{aligned}$$

Note that $\det\sigma$ is the determinant of $\sigma$ as an $N \times N$ matrix, and $\det r_\sigma$ is the determinant of $r_\sigma$ as an automorphism of the $N^2$-dimensional space $M_N(F)$.

To complete the proof for $r_\sigma$ it suffices to show that $\det r_\sigma = \det \sigma^n$. Let $e_{i,j}$ denote the $N \times N$ matrix with a 1 in the $i, j$ position and zeros elsewhere. With respect to the ordered basis

$$e_{1,1}, e_{1,2}, \dots, e_{1,N}, e_{2,1}, \dots, e_{2,N}, \dots, e_{N,1}, \dots, e_{N,N},$$

$r_\sigma$ acts as the block matrix

$$\begin{pmatrix} \sigma^t & & \\ & \ddots & \\ & & \sigma^t \end{pmatrix}$$

($^t$ denotes transpose), which has determinant $\det \sigma^n$, as required.

The proof with $l_\sigma$ in place of $r_\sigma$ differs only in notation, except that one should use the ordered basis

$$e_{1,1}, e_{2,1}, \dots, e_{N,1}, e_{1,2}, \dots, e_{N,2}, \dots, e_{1,N}, \dots, e_{N,N}$$

instead. □

So we have obtained a translation invariant integral on the algebraic group $GL_N(F)$. Just as the integrals on $F$ and $F^n$ lift the usual Haar integral on $\overline{F}$ and $\overline{F}^n$, so too does this integral incorporate the Haar integral on $GL_N(\overline{F})$. To demonstrate this most clearly, it is prudent to now assume that the chosen isomorphism $T$ restricts to an $\mathcal{O}_F$-linear isomorphism $\mathcal{O}_F^n \to M_N(\mathcal{O}_F)$. Thus $T$ descends to a $\overline{F}$-linear isomorphism $\overline{T} : \overline{F}^n \to M_N(\overline{F})$ which makes the diagram commute:

$$\begin{array}{ccc} \mathcal{O}_F^n & \xrightarrow{T} & M_N(\mathcal{O}_F) \\ \downarrow & & \downarrow \\ \overline{F}^n & \xrightarrow[\overline{T}]{} & M_N(\overline{F}), \end{array}$$

where the vertical arrows are coordinate-wise residue homomorphisms. This will ensure a functoriality between our algebraic groups at the level of $\overline{F}$ and at the level of $F$.

**Remark 3.3.5.** This assumption holds if we identify $M_N(F)$ with $F^{n^2}$ in the most natural way, via the standard basis of $F^{n^2}$ and the basis of $M_N(F)$ used in proposition 3.3.4.

Further, we now normalise the Haar measures on $M_N(\overline{F})$ and $GL_N(\overline{F})$ in the following way: give $M_N(\overline{F})$ the Haar measure obtained by pushing forward the product measure on $\overline{F}^n$ via $\overline{T}$, and then give $GL_N(\overline{F})$ the standard Haar measure $d_{GL_N} u = |\det u|^{-n} d_{M_N} u$. Such normalisations are not essential, but otherwise extraneous constants would appear in formulae below. It will be useful to call a complex-valued function on $M_N(\overline{F})$ *GL-Fubini* if its pull back to $\overline{F}^n$ via $\overline{T}$ is $GL$-Fubini in the sense already defined. Again, note that a Schwartz-Bruhat function on $M_N(\overline{F})$ is certainly $GL$-Fubini.

We have already defined what is meant by the lift of a Haar integrable from $\overline{F}$ or $\overline{F}^n$. The following is a trivial generalisation:

**Definition 3.3.6.** Let $G$ denote either of the algebraic groups $M_N, GL_N$. Given a complex valued function $g$ on $G(\overline{F})$, let $g^0$ be the complex valued function on $G(F)$ defined by

$$g^0 : G(F) \to \mathbb{C}$$
$$x \mapsto \begin{cases} g(\overline{x}) & x \in G(\mathcal{O}_F) \\ 0 & \text{otherwise.} \end{cases}$$

Then the compatibility between the integrals on $M_N$ at the level of $\overline{F}$ and $F$ is the following:

**Proposition 3.3.7.** *Suppose that $g$ is a complex-valued, GL-Fubini function on $M_N(\overline{F})$ (e.g. a Schwartz-Bruhat function on $M_N(\overline{F})$). Then $g^0$ belongs to $\mathcal{L}(M_N(F))$, and*

$$\int^{M_N(F)} g^0(x)\,dx = \int_{M_N(\overline{F})} g(u)\,du.$$

*Proof.* By the existence of $\overline{T}$ and its compatibility with $T$ we have an equality of functions on $M_N(F)$:

$$(g\overline{T}^{-1})^0 T = g^0.$$

The definition of the integral on $M_N(F)$ therefore implies

$$\int^{M_N(F)} g^0(x)\,dx = \int^{F^n} (g\overline{T}^{-1})^0(x)\,dx.$$

Taking $G$ to be $n$ copies of the additive group, we showed in proposition 3.1.8 that the result corresponding to this one holds; so

$$\int^{F^n} (g\overline{T}^{-1})^0(x)\,dx = \int_{\overline{F}^n} g\overline{T}^{-1}(u)\,du.$$

Finally, our normalisation of the Haar measure on $M_N(\overline{F})$ implies

$$\int_{\overline{F}^n} g\overline{T}^{-1}(u)\,du = \int_{M_N(\overline{F})} g(u)\,du,$$

which completes the proof. □

And now we prove the same result for $GL_N$:

**Proposition 3.3.8.** *Suppose that $g$ is a complex-valued, Schwartz-Bruhat function on $GL_N(\overline{F})$ such that*

$$f(u) := \begin{cases} g(u)|\det u|^{-n} & u \in GL_N(\overline{F}), \\ 0 & \det u = 0 \end{cases}$$

*is GL-Fubini on $M_N(\overline{F})$. Then $g^0$ belongs to $\mathcal{L}(GL_N(F))$, and*

$$\int^{GL_N(F)} g^0(\tau)\,d\tau = \int_{GL_N(\overline{F})} g(u)\,du.$$

*Proof.* The assumption on $f$ and the previous proposition imply that $f^0$ belongs to $\mathcal{L}(M_N(F))$. Moreover, $\tau \in GL_N(\mathcal{O}_F)$ implies

$$f^0(\tau) = g(\overline{\tau})|\det \overline{\tau}|^{-n} = g^0(\tau)|\det \tau|^{-n},$$

so that $f^0$ is an extension of $\tau \mapsto g^0(\tau)|\det \tau|^{-n}$ from $GL_N(F)$ to a function in $\mathcal{L}(M_N(F))$.

Therefore $g^0$ belongs to $\mathcal{L}(GL_N(F))$, and

$$\begin{aligned}\int^{GL_N(F)} g^0(\tau)\,d\tau &= \int^{M_N(F)} f^0(x)\,dx \\ &= \int_{M_N(\overline{F})} f(u)\,du \\ &= \int_{GL_N(\overline{F})} g(u)\,du\end{aligned}$$

where the second equality follows from the previous proposition. □

**Remark 3.3.9.** If $g$ decreases sufficiently rapidly towards the boundary of $GL_N(\overline{F})$ in $M_N(\overline{F})$ then the hypothesis in the previous proposition will hold, i.e. $f$ will be $GL$-Fubini on $M_N(\overline{F})$. In particular, if $g$ is the restriction to $GL_N(\overline{F})$ of a Schwartz-Bruhat function on $M_N(\overline{F})$ then (replacing $g$ by $g|\det\cdot|^s$) the function

$$f(u) := \begin{cases} g(u)|\det u|^{s-n} & u \in GL_N(\overline{F}), \\ 0 & \det u = 0 \end{cases}$$

is $GL$-Fubini on $M_N(\overline{F})$, for $s \in \mathbb{C}$ and $\Re(s)$ sufficiently large. The previous result now implies

$$\int^{GL_N(F)} g^0(\tau)|\det\tau|^s\,d\tau = \int_{GL_N(\overline{F})} g(u)|\det u|^s\,du$$

(note that for any $\tau \in GL_N(F)$ in the support of $g^0$, one has $|\det\tau|^s = |\det\overline{\tau}|^s \in \mathbb{C}^\times$).

Thus we can lift Godement-Jacquet zeta functions [GJ72] to $GL_N(F)$ in the same way as we lifted zeta integrals from $\overline{F}^\times$ to $F^\times$ in section 2.5, though more work in this direction is required.

## 3.4 Other algebraic groups and related problems

### 3.4.1 Integration over an arbitrary algebraic group

Having established an integral on $GL_N(F)$, it would be useful also to be able to integrate on algebraic subgroups such as $SL_N(F)$ or $B_N(F)$, the group of invertible upper triangular matrices. Arguments similar to those above will surely provide such an integral, but to establish such results for an arbitrary reductive algebraic group $G$ we require a more general abstract approach.

The author suspects that to each reductive, algebraic group $G$ there is a space of $\mathbb{C}(\Gamma)$-valued functions $\mathcal{L}(G(F))$ on $G(F)$ and a linear functional $\int^{G(F)}$ on these functions with the following properties:

(i) Compatibility between $\overline{F}$ and $F$: if $g$ is a 'nice' (e.g. Schwartz-Bruhat) Haar integrable function on $G(\overline{F})$, then $g^0$ (an obvious generalisation of definition 3.3.6) belongs to $\mathcal{L}(G(F))$ and

$$\int^{G(F)} g^0(x)\,dx = \int_{G(\overline{F})} g(u)\,du.$$

(ii) Translation invariance: if $f \in \mathcal{L}(G(F))$ and $\tau \in G(F)$, then $x \mapsto g(x\tau)$ is in $\mathcal{L}(G(F))$, and

$$\int^{G(F)} f(x\tau)\,dx = \int^{G(F)} f(x)\,dx.$$

There should also be a left translation-invariant integral on $G(F)$, and this would coincide with the right-invariant integral if $G(\overline{F})$ is unimodular.

Even in the simplest case $G =$'additive group' these conditions are not enough to make the integral unique in a reasonable way; this is discussed in remark 2.1.7 of chapter 2. However, if we assume the existence of an absolute value which relates the integrals on $F^\times$ and $F$, the uniqueness does follow. We have observed a similar phenomenon in this paper where we constructed the integral on $F^n$ to be compatible with change of variables from $GL_n(F)$. So to ensure uniqueness we should add to the list the informal statement

(iii) Compatibility between the integrals over different algebraic groups.

### 3.4.2 Subgroups of $GL_N$

Once integration over algebraic subgroups of $GL_N(F)$ has been established, there are certain formulae which are expected to hold by analogy with the case of a local field. We quote two examples from [Car79]; for $f$ a complex-valued, integrable function on $GL_N(\overline{F})$ (resp. on $B_N(\overline{F})$),

$$\int_{GL_N(\overline{F})} f(g)\,dg = \int_{GL_N(\mathcal{O}_{\overline{F}})} \int_{B_N(\overline{F})} f(kb)\,dk\,d_R b$$
$$\int_{B_N(\overline{F})} f(b)\,d_R b = \int_{\Delta_N(\overline{F})} \int_{U_N(\overline{F})} f(u\lambda)\,du\,d\lambda,$$

where $U_N$ denotes the u.u.t. matrices, $\Delta_N$ the diagonal matrices, and $d_R$ right Haar measure (apart from $B_N$, these groups are unimodular).

Writing these identities explicitly, one sees that these formulae require the class of integrable functions on $GL_N(F)$ to be invariant under certain polynomial changes of variables. It is therefore also important to extend the class of functions $\mathcal{L}(F^n, GL_n)$ so that it is closed under the required changes of variables.

This is also precisely the sort of compatibility which may be important in (iii).

### 3.4.3 Non-linear change of variables

To develop integration on arbitrary algebraic groups and prove compatibility between them we are lead to investigate non-linear change of variables on $F^n$. Steps in this direction are taken in chapter 4 in the case of a two-dimensional local field (that is, $F$ is a complete discrete valuation field whose residue field is a local field). It is proved that if $f = g^{a,\gamma}$ is the lift to $F^2$ of a Schwartz-Bruhat function on $\overline{F}^2$ and $h$ is a polynomial over $F$ then, assuming certain conditions, $(x, y) \mapsto f(x, y - h(x))$ is Fubini on $F^2$, and so

$$\int^{F^2} f(x, y - h(x))\,dxdy = \int^{F^2} f(x, y - h(x))\,dydx = \int^{F^2} f(x, y)\,dydx.$$

Note that the second equality follows simply from translation invariance of the integral.

However, it is essential to make some assumptions on the singularities of $h$, for we will also see in proposition 4.4.1 that:

**Proposition 3.4.1.** *Suppose $F$ is a two-dimensional local field and $\overline{F}$ has finite characteristic $p$. Let $h(X) = t^{-1}X^p$, where $t$ is a uniformiser of $F$, and let $g$ be* any *Schwartz-Bruhat function on $\overline{F} \times \overline{F}$. Then for all $y \in F$, the function $x \mapsto g^0(x, y - h(x))$ is integrable, with $\int^F g^0(x, y - h(x))\,dx = 0$. Therefore*

$$\int^F \int^F g^0(x, y - h(x))\,dxdy = 0,$$

*whereas*

$$\int^F \int^F g^0(x, y - h(x))\,dydx = \int \int g(u, v)\,dvdu,$$

*which need not be zero.*

Whether this failure of Fubini's theorem will cause a problem in verifying existence of integrals on algebraic groups is unclear. If such "wild" changes of variable do not appear when changing charts on one's algebraic group, then this may not be too serious (preliminary work on $SL_2(F)$ suggests something interesting does happen in residue characteristic 2...). However, it is certainly an unexpected result; it appears to be a measure-theoretic consequence of the characteristic $p$ local field $\overline{F}$ being imperfect. See remark 4.4.2 and subsection 6.1.3 for further discussion.

### 3.4.4 Godement-Jacquet theory

To generalise Godement-Jacquet theory to a higher local field $F$, the immediate question to ask is "What is a smooth representation of $GL_n(F)$?", and the second is "Are the matrix coefficients of a smooth representation integrable?".

Whatever the answer to the first question, the answer to the second is surely "No, the space of integrable functions on $GL_n(F)$ is too small.". In the residue characteristic zero case (e.g. $\mathbb{Q}_p((t))$), the methods of chapter 5 should produce a more extensive integration theory, and hopefully this will help to answer the first question. Unfortunately, developing a Godement-Jacquet theory in this case may not produce any significant new insights into two-dimensional Langlands, because all the representations will be tame and the theory will reduce entirely to $GL_n(\overline{F})$.

CHAPTER 4

# Fubini's theorem and non-linear changes of variables over a two-dimensional local field

We consider non-linear changes of variables and Fubini's theorem for certain integrals over a two-dimensional local field. An interesting example is presented in which imperfectness of a positive characteristic local field causes Fubini's theorem to unexpectedly fail.

### Notation

In this chapter $F$ is a two-dimensional local field, i.e. a complete discrete valuation field whose residue field $K = \overline{F}$ is a local field ($\mathbb{R}$, $\mathbb{C}$, or non-archimedean). We fix a prime $t$ of $F$ and denote its ring of integers by $\mathcal{O}_F$. The residue map $\mathcal{O}_F \to K$ is denoted $x \mapsto \overline{x}$; the discrete valuation is denoted $\nu : F \to \mathbb{Z} \cup \{\infty\}$. We fix a Haar measure on $K$.

The reason we work with a discrete valuation in this chapter, rather than the arbitrary valuation in chapters 2 and 3, is that several arguments proceed by induction on the value group. By modifying the arguments it is likely that this restriction can be eliminated.

The fixed prime $t$ induces a splitting of the valuation given by

$$\mathbb{Z} \to F^{\times}, \quad n \mapsto t(n) = t^n,$$

and therefore the integration theory developed in chapters 2 and 3 can be applied with respect to this splitting. We recommend that the reader consult the summary of the integration theories on $F$ and $F \times F$ found in subsections 1.4.1 and 1.4.2 respectively, everywhere replacing "$\gamma \in \Gamma$" and "$t(\gamma)$" by "$n \in \mathbb{Z}$" and "$t^n$".

## 4.1 Decomposition results

We begin by examining the action of polynomials on $F$; the results hold for any Henselian discrete valuation field $F$ with infinite residue field.

**Lemma 4.1.1.** *Suppose $h(X)$ is a polynomial over $F$, that $a + t^c\mathcal{O}_F$, $b + t^A\mathcal{O}_F$ are two translated fractional ideals, and that $h(a + t^c\mathcal{O}_F) \subseteq b + t^A\mathcal{O}_F$. Then there is a unique polynomial $\psi \in K[X]$ which gives a commutative diagram*

$$\begin{tikzcd} a + t^c\mathcal{O}_F \arrow[r, "h"] \arrow[d, "a+t^c x \mapsto \overline{x}"'] & b + t^A\mathcal{O}_F \arrow[d, "b+t^A x \mapsto \overline{x}"] \\ K \arrow[r, "\psi"'] & K. \end{tikzcd}$$

*Moreover,* $\deg \psi \leq \deg h$.

*Proof.* There is certainly at most one function $\psi$ making the diagram commute; but $K$ is an infinite field so if two polynomials are equal as functions then they are equal as polynomials. So there can be at most one polynomial $\psi$.

We may write $h(a+t^cX) = h(a) + t^RH(X)$ where $H \in \mathcal{O}_F[X]$ is a polynomial with integer coefficients, no constant term, and with non-zero image in $K[X]$ (i.e. not all coefficients of $H$ are in $t\mathcal{O}_F$). We clearly have a commutative diagram

$$\begin{tikzcd} a+t^c\mathcal{O}_F \arrow[r, "h"] \arrow[d, "a+t^cx\mapsto\overline{x}"'] & h(a)+t^R\mathcal{O}_F \arrow[d, "h(a)+t^Rx\mapsto\overline{x}"] \\ K \arrow[r, "\overline{H}"'] & K, \end{tikzcd}$$

where $\overline{H}$ denotes the image of $H$ in $K[X]$.

If $A > R$ then the inclusion $h(a+t^c\mathcal{O}_F) \subseteq b + t^A\mathcal{O}_F$ implies $\overline{H}$ is everywhere equal to $\overline{(b-h(a))t^{-R}}$; but $K$ infinite then implies $\overline{H}$ is a constant polynomial and hence is zero (since $H$ has no constant term), which is a contradiction. Hence $A \le R$, and we may easily complete the proof:

If $A = R$ then the desired commutative diagram is

$$\begin{tikzcd} a+t^c\mathcal{O}_F \arrow[r, "h"] \arrow[d, "a+t^cx\mapsto\overline{x}"'] & b+t^A\mathcal{O}_F \arrow[d, "b+t^Ax\mapsto\overline{x}"] \\ K \arrow[r, "\overline{H}+\overline{(h(a)-b)t^{-A}}"'] & K, \end{tikzcd}$$

where the lower horizontal map is the function $u \mapsto \overline{H}(u) + \overline{(h(a)-b)t^{-A}}$. If $A < R$ then the desired diagram is

$$\begin{tikzcd} a+t^c\mathcal{O}_F \arrow[r, "h"] \arrow[d, "a+t^cx\mapsto\overline{x}"'] & b+t^A\mathcal{O}_F \arrow[d, "b+t^Ax\mapsto\overline{x}"] \\ K \arrow[r, "\overline{(h(a)-b)t^{-A}}"'] & K, \end{tikzcd}$$

where the lower horizontal map is the constant function $u \mapsto \overline{(h(a)-b)t^{-A}}$. $\square$

**Definition 4.1.2.** Suppose $h(X)$ is a polynomial over $F$, that $a+t^c\mathcal{O}_F$, $b+t^A\mathcal{O}_F$ are two translated fractional ideals, and that $h(a+t^c\mathcal{O}_F) \subseteq b+t^A\mathcal{O}_F$. The unique polynomial $\psi \in K[X]$ which gives a commutative diagram

$$\begin{tikzcd} a+t^c\mathcal{O}_F \arrow[r, "h"] \arrow[d, "a+t^cx\mapsto\overline{x}"'] & b+t^A\mathcal{O}_F \arrow[d, "b+t^Ax\mapsto\overline{x}"] \\ K \arrow[r, "\psi"'] & K. \end{tikzcd}$$

is said to be the *residue field approximation* of $h$ with respect to the translated fractional ideals.

**Remark 4.1.3.** Regarding the previous definition, the translated fractional ideals will usually be clear from the context. The constant term of $\psi$ depends not only on the sets $a+t^c\mathcal{O}_F$ and $b+t^A\mathcal{O}_F$, but on the representatives $a, b$ we choose.

When drawing the diagram above, we will henceforth omit the vertical maps, even though they do depend on the choice of $a, b$. We will also follow the habit used in the previous lemma of denoting a constant function on $K$ by the value it assumes.

Much of this chapter is concerned with the problem of explicitly decomposing the preimage of a set under a polynomial and describing the resulting residue field approximations. Here is a example to illustrate the ideas:

**Example 4.1.4.** Set $q(X) = X^3 + X^2 + t^2$ and assume $\operatorname{char} K \neq 2$. The aim of this example is to give explicit descriptions of the sets $\{x \in \mathcal{O}_F : q(x) \in t^A\mathcal{O}_F\}$ for $A = 2, 3$, as well as all associated residue field approximations.

Direct calculations easily show that if $x \in \mathcal{O}_F$, then $q(x) \in t^2\mathcal{O}_F$ if and only if $x \in t\mathcal{O}_F$ or $x \in -1 + t^2\mathcal{O}_F$. Further, the residue field approximations are

$$\begin{tikzcd} t\mathcal{O}_F \arrow[r, "q"] \arrow[d] & t^2\mathcal{O}_F \arrow[d] \\ K \arrow[r, "X^2+1"'] & K \end{tikzcd} \qquad \begin{tikzcd} -1+t^2\mathcal{O}_F \arrow[r, "q"] \arrow[d] & t^2\mathcal{O}_F \arrow[d] \\ K \arrow[r, "X+1"'] & K \end{tikzcd}$$

Similarly, if we suppose $x \in \mathcal{O}_F$ then $q(tx) \in t^3\mathcal{O}_F$ if and only if $x^2 + 1 \in t\mathcal{O}_F$; and $q(-1 + t^2x) \in t^3\mathcal{O}_F$ if and only if $x \in -1 + t\mathcal{O}_F$.

If $K$ contains a square root of $-1$, let $i$ denote a lift of it to $\mathcal{O}_F$; then

$$\{x \in \mathcal{O}_F : q(x) \in t^3\mathcal{O}_F\} = it + t^2\mathcal{O}_F \sqcup -it + t^2\mathcal{O}_F \sqcup -1 - t^2 + t^3\mathcal{O}_F,$$

with residue field approximations

$$\begin{tikzcd} it+t^2\mathcal{O}_F \arrow[r, "q"] \arrow[d] & t^3\mathcal{O}_F \arrow[d] \\ K \arrow[r, "2iX+\overline{(i^2+1)t^{-1}}-\overline{i}"'] & K \end{tikzcd} \qquad \begin{tikzcd} -it+t^2\mathcal{O}_F \arrow[r, "q"] \arrow[d] & t^3\mathcal{O}_F \arrow[d] \\ K \arrow[r, "-2iX+\overline{(i^2+1)t^{-1}}+\overline{i}"'] & K \end{tikzcd}$$

$$\begin{tikzcd} -1-t^2+t^3\mathcal{O}_F \arrow[r, "q"] \arrow[d] & t^3\mathcal{O}_F \arrow[d] \\ K \arrow[r, "X"'] & K \end{tikzcd}$$

If $K$ does not contain a square root of $-1$, then $\{x \in \mathcal{O}_F : q(x) \in t^3\mathcal{O}_F\} = -1 - t^2 + t^2\mathcal{O}_F$, with the residue field approximation given by the third diagram above.

We now turn to generalising the example to an arbitrary polynomial; for later applications to integration the following results will allow us to reduce calculations to the residue field, where we change variable according to the residue field approximation polynomials for example, and then return to $F$.

The first decomposition result treats the non-singular part of the polynomial, and is really just a rephrasing of Hensel's lemma:

**Proposition 4.1.5.** *Let $q(X)$ be a polynomial with coefficients in $\mathcal{O}_F$, of degree $\geq 1$ and with non-zero image in $K[X]$; let $b \in F$.*

*(i) Suppose that $q(a) = b$ for some $a \in \mathcal{O}_F$ and that $\overline{q}'(\overline{a}) \neq 0$. Then for any $A \geq 1$,*

$$\{x \in \mathcal{O}_F \,:\, \overline{x} = \overline{a} \text{ and } q(x) \in b + t^A\mathcal{O}_F\} = a + t^A\mathcal{O}_F,$$

*and the residue field approximation is 'multiplication by $\overline{q}'(\overline{a})$':*

$$\begin{array}{ccc} a+t^A\mathcal{O}_F & \xrightarrow{q} & b+t^A\mathcal{O}_F \\ \downarrow & & \downarrow \\ K & \xrightarrow[\overline{q}'(\overline{a})X]{} & K \end{array}$$

*(ii) Let $\omega_1,\dots,\omega_r$ be the simple (i.e. $\overline{q}'(\omega_i)\neq 0$) solutions in $K$ to $\overline{q}(X)=\overline{b}$; let $\check{\omega}_i$ be a lift by Hensel of $\omega_i$ to $\mathcal{O}_F$; that is, $q(\omega_i)=b$. Then for any $A\geq 1$,*

$$\{x\in\mathcal{O}_F\,:\,\overline{q}'(\overline{x})\neq 0 \text{ and } q(x)\in b+t^A\mathcal{O}_F\}=\bigsqcup_{i=1}^{r}\check{\omega}_i+t^A\mathcal{O}_F.$$

*Proof.* (i) is essentially contained in the proof of Hensel's lemma and so we omit it. (ii) easily follows. $\square$

We now consider the singular part, which is much more interesting and will be the root of future difficulties:

**Proposition 4.1.6.** *Let $q(X)$ be a polynomial with coefficients in $\mathcal{O}_F$, of degree $\geq 1$ and with non-zero image in $K[X]$; let $b\in F$. For $A\geq 1$ there is a decomposition*

$$\{x\in\mathcal{O}_F\,:\,\overline{q}'(\overline{x})=0 \text{ and } q(x)\in b+t^A\mathcal{O}_F\}=\bigsqcup_{j=1}^{N}a_j+t^{c_j}\mathcal{O}_F$$

*(assuming this set is non-empty i.e. that $\overline{q}(X)-\overline{b}$ has a repeated root in $K$), where $a_1,\dots,a_N\in\mathcal{O}_F$, and $c_1,\dots,c_N\geq 1$ are positive integers.*

*Proof.* First suppose $A=1$. Let $a_1,\dots,a_N$ be lifts to $\mathcal{O}_F$ of the distinct solutions in $K$ to $\overline{q}(X)=\overline{b}$ and $\overline{q}'(X)=0$, and set $c_j=1$ for each $j$. Then the required decomposition is

$$\bigsqcup_{j=1}^{N}a_j+t^{c_j}\mathcal{O}_F.$$

We now determine the residue field approximation of $q$ on each $a_j+t^{c_j}\mathcal{O}_F$ as it will be used later in corollary 4.3.5. So, for each $j$, consider the Taylor expansion

$$q(a_j+tX)=q(a_j)+q'(a_j)tX+q_2(a_j)t^2X^2+\dots+q_d(a_j)t^dX^d$$

where $d=\deg q$. But $q'(a_j)\in t\mathcal{O}_F$ implies $q(a_j+tx)\in q(a_j)+t^2\mathcal{O}_F$ for all $x$ in $\mathcal{O}_F$, which is to say that

$$\begin{array}{ccc} a_j+t^{c_j}\mathcal{O}_F & \xrightarrow{q} & b+t\mathcal{O}_F \\ \downarrow & & \downarrow \\ K & \xrightarrow[\overline{(q(a_j)-b)t^{-1}}]{} & K \end{array}$$

commutes, where the lower horizontal map is constant i.e. each residue field approximation associated to the decomposition is constant.

We now suppose $A>1$ and proceed by induction on $A$. Let $u_1,\dots,u_N\in K$ be the distinct solutions to $\overline{q}(X)=\overline{b}$ and $\overline{q}'(X)=0$, and write

$$W_j=\{x\in\mathcal{O}_F\,:\,\overline{x}=u_j \text{ and } q(x)\in b+t^A\mathcal{O}_F\}$$

for $j = 1, \dots, N$. Since

$$\{x \in \mathcal{O}_F \,:\, \overline{q}'(\overline{x}) = 0 \text{ and } q(x) \in b + t^A\mathcal{O}_F\} = \bigsqcup_{j=1}^{N} W_j,$$

it is enough to decompose each $W_j$ in the required manner, so we now fix a value of $j$, writing $W = W_j$ and $u = u_j$.

If $W$ is empty then we are done; else $u$ has a lift to $a \in \mathcal{O}_F$ such that $q(a) \in b + t^A\mathcal{O}_F$, and we now fix such an $a$. Using the same Taylor expansion as above, there exist $\rho \geq 1$ and $Q \in \mathcal{O}_F[X]$ such that $q(a+tX) = q(a) + t^\rho Q(X)$ and $\overline{Q}(X) \neq 0$; in fact, $q'(a) \in t\mathcal{O}_F$ implies $\rho \geq 2$, though we will not use this. Therefore

$$W = a + t\{x \in \mathcal{O}_F \,:\, Q(x) \in (b - q(a))t^{-\rho} + t^{A-\rho}\mathcal{O}_F\},$$

but also note that

$$(b - q(a))t^{-\rho} + t^{A-\rho}\mathcal{O}_F = t^{A-\rho}((b - q(a))t^{-A} + \mathcal{O}_F) = t^{A-\rho}\mathcal{O}_F$$

by choice of $a$. Therefore $W = a + t\{x \in \mathcal{O}_F \,:\, Q(x) \in t^{A-\rho}\mathcal{O}_F\}$, and it becomes clear how the induction should proceed.

In fact, we must consider three cases, depending on the relative magnitudes of $\rho$ and $A$:

(i) $A - \rho < 0$. Then $\{x \in \mathcal{O}_F \,:\, Q(x) \in t^{A-\rho}\mathcal{O}_F\} = \mathcal{O}_F$ and $Q(\mathcal{O}_F) \subseteq \mathcal{O}_F \subset t^{A-\rho}\mathcal{O}_F$; therefore the residue field approximation is constant, given by the diagram

$$\begin{tikzcd} \mathcal{O}_F \arrow[r, "Q"] \arrow[d] & t^{A-\rho}\mathcal{O}_F \arrow[d] \\ K \arrow[r, "0"'] & K \end{tikzcd}$$

This implies $W = a + t\mathcal{O}_F$ with a constant residue field approximation:

$$\begin{tikzcd} a + t\mathcal{O}_F \arrow[r, "q"] \arrow[d] & b + t^A\mathcal{O}_F \arrow[d] \\ K \arrow[r, "\overline{(q(a)-b)t^{-A}}"'] & K. \end{tikzcd}$$

(ii) $A - \rho = 0$. Again, $\{x \in \mathcal{O}_F \,:\, Q(x) \in t^{A-\rho}\mathcal{O}_F\} = \mathcal{O}_F$; the residue field approximation is clearly

$$\begin{tikzcd} \mathcal{O}_F \arrow[r, "Q"] \arrow[d] & \mathcal{O}_F \arrow[d] \\ K \arrow[r, "\overline{Q}"'] & K \end{tikzcd}$$

Therefore $W = a + t\mathcal{O}_F$, with residue field approximation

$$\begin{tikzcd} a + t\mathcal{O}_F \arrow[r, "q"] \arrow[d] & b + t^A\mathcal{O}_F \arrow[d] \\ K \arrow[r, "\overline{Q}(X)+\overline{(q(a)-b)t^{-A}}"'] & K. \end{tikzcd}$$

(iii) $A - \rho > 0$. Here we may use the inductive hypothesis and proposition 4.1.5 to write
$$\{x \in \mathcal{O}_F \,:\, Q(x) \in t^{A-\rho}\mathcal{O}_F\} = \bigsqcup_i d_i + t^{e_i}\mathcal{O}_F,$$
with residue field approximations $\psi_i(X)$, say:
$$\begin{array}{ccc} d_i + t^{e_i}\mathcal{O}_F & \xrightarrow{Q} & t^{A-\rho}\mathcal{O}_F \\ \downarrow & & \downarrow \\ K & \xrightarrow[\psi_i]{} & K \end{array}$$
Therefore $W = \bigsqcup_i a + d_i t + t^{e_i+1}\mathcal{O}_F$, with residue field approximations
$$\begin{array}{ccc} a + d_i t + t^{e_i+1}\mathcal{O}_F & \xrightarrow{q} & b + t^A\mathcal{O}_F \\ \downarrow & & \downarrow \\ K & \xrightarrow[\psi_i(X)+\overline{(q(a)-b)t^{-A}}]{} & K. \end{array}$$

□

For $q(X)$ as in the previous two propositions, these two decomposition results completely describe $\{x \in \mathcal{O}_F : q(x) \in b + t^A\mathcal{O}_F\}$ in terms of $\leq (\deg q)^A$ translated fractional ideals equipped with polynomial residue field approximations. Moreover, the proof of the second result gives some insight into how structure of the polynomial $q$ effects the resulting residue field approximations. For applications beyond those described in this chapter, it will be necessary to better understand how the decomposition varies with $b$ and $A$. For small $A$ we have the following result:

**Lemma 4.1.7.** *Let $q(X)$ be a polynomial with coefficients in $\mathcal{O}_F$, of degree $\geq 1$ and such that $q'$ has non-zero image in $K[X]$; let $A = 1$ or $2$. There are finitely many $b_1, \dots, b_m \in \mathcal{O}_F$ such that if $b \in \mathcal{O}_F$ and $\{x \in \mathcal{O}_F : q(x) \in b + t^A\mathcal{O}_F,\ \overline{q}'(\overline{x}) = 0\}$ is non-empty, then $b \equiv b_i \mod t^A\mathcal{O}_F$ for some $i \in \{1, \dots, m\}$.*

*Proof.* First suppose $A = 1$. Then $\{x \in \mathcal{O}_F : q(x) \in b + t^A\mathcal{O}_F, \overline{q}'(\overline{x}) = 0\}$ being non-empty implies that $\overline{b}$ is the image under $\overline{q}$ of one of the finitely many roots of $\overline{q}'$.

Now suppose that $A = 2$. Then the argument is just the same as for $A = 1$, except it is important to observe the following: if $a_1, a_2 \in \mathcal{O}_F$ are equal modulo $t\mathcal{O}_F$, and $\overline{q}'(\overline{a}_i) = 0$ for $i = 1, 2$, then $q(a_1) = q(a_2) \mod t^2\mathcal{O}_F$. This follows from the Taylor expansion and the fact that $q'(a_i) \in t\mathcal{O}_F$. □

**Remark 4.1.8.** Decomposition results similar to the previous ones are common in model theory; for example, in the theory of algebraically closed valued fields [Rob77], every definable subset of the field is a finite disjoint union of points and 'Swiss cheeses'. Further, these decompositions are related to ramification theory and rigid geometry through the Abbes-Saito theory; see subsection 6.1.2.

## 4.2 Non-linear changes of variables

In this section we investigate the behaviour of Fubini functions on $F \times F$ under certain non-linear changes of variables. More precisely, we consider the following:

**Conjecture 4.2.1.** Let $a_1, a_2 \in F$, $n_1, n_2 \in \mathbb{Z}$, and let $h(X)$ be a polynomial over $F$. Then for any Schwartz-Bruhat function $f$ on $K \times K$, letting $g = f^{(a_1,a_2),(n_1,n_2)}$ be the lift of $f$ at $(a_1, a_2), (n_1, n_2)$, the function

$$\Phi(x, y) = g(x, y - h(x))$$

is Fubini on $F \times F$, with repeated integral equal to that of $f$.

The conjecture is false in the generality in which we have stated it, though an important special case has already been treated in chapter 3:

**Theorem 4.2.2.** *With notation as in the conjecture, if* $\deg h \leq 1$ *then the conjecture is true.*

*Proof.* According to theorem 3.2.4, with $n = 2$, the function $(x, y) \mapsto g(\tau(x, y))$ is Fubini on $F \times F$ for any $\tau \in GL_2(F)$. If $\deg h = 1$ then the conjecture is a special case of that result; in fact, it essentially follows from lemma 3.2.7.

If $\deg h = 0$ then the conjecture follows from translation invariance of the integral; see proposition 1.4.8 and remark 1.4.9. □

Because of the previous theorem, we will have in mind polynomials $h(X)$ of degree at least 2, though our results are equally valid for lower degree. We will be interested in conditions on the data $a_1, a_2, n_1, n_2, h$ such that the conjecture is true for all Schwartz-Bruhat functions $f$. We assign to the data two invariants as follows:

**Definition 4.2.3.** Let $a_1, a_2, n_1, n_2, h$ be data for the conjecture, and write $h(a_1 + t^{n_1} X) = h(a_1) + t^R q(X)$, where $R \in \mathbb{Z}$, $q \in \mathcal{O}_F[X]$, and the image of $q$ in $K[X]$ is non-zero. Note that $q(0) = 0$.

The *depth* and *normalised polynomial* associated to the data are defined to be $R - n_2$ and $q(X)$ respectively.

A summary of what we know about the validity of the conjecture, classified by the depth and normalised polynomial, may be found in section 4.5. The sense in which the depth and normalised polynomial are invariants, and why they are useful, is given by the following lemma in which we reduce the conjecture to a special case:

**Lemma 4.2.4.** *Fix a polynomial* $q \in \mathcal{O}_F[X]$ *with nonzero image in* $K[X]$ *and no constant term, and an integer* $R \in \mathbb{Z}$. *Then the following are equivalent:*

*(i) the conjecture is true for all data* $a_1, a_2, n_1, n_2, h$ *with depth* $R$ *and normalised polynomial* $q$*;*

*(ii) the conjecture is true for all data of the form* $0, 0, 0, 0, h$ *with depth* $R$*, normalised polynomial* $q$*, and such that* $h(0) = 0$*;*

*(iii) for all Schwartz-Bruhat functions* $f$ *on* $K \times K$*, the function*

$$(x, y) \mapsto f^0(x, y - t^R q(x))$$

*is Fubini;*

*(iv) for all Schwartz-Bruhat functions* $f$ *on* $K \times K$*, the following hold: for each* $y \in F$*, the function* $x \mapsto f^0(x, y - t^R q(x))$ *is integrable, then that* $y \mapsto \int^F f^0(x, y - t^R q(x))\, dx$ *is integrable, and finally that*

$$\int^F \int^F f^0(x, y - t^R q(x))\, dx dy = \int_K \int_K f(u, v)\, du dv.$$

*Proof.* Clearly (i)$\Rightarrow$(ii). The only data satisfying the conditions of (ii) are $0,0,0,0,t^Rq$, and so (ii)$\Leftrightarrow$(iii).

(iii)$\Rightarrow$(i): So assume (iii), letting $a_1,a_2,n_1,n_2,h$ be data for the conjecture with depth $R$ and normalised polynomial $q$. Let $f$ be Schwartz-Bruhat on $K\times K$ and write $g = f^{(a_1,a_2),(n_1,n_2)}$. Note that $h(a_1+t^{n_1}X) = h(a_1)+t^{R+n_2}q(X)$, and that therefore for all $x,y\in F$,

$$\begin{aligned} g(a_1+t^{n_2}x, a_2+t^{n_2}y-h(a_1+t^{n_1}x)) &= f^0(x, y-t^{-n_2}h(a_1+t^{n_1}x)) \\ &= f^0(x,(y-t^{-n_2}h(a_1))-t^Rq(x)). \end{aligned}$$

By (iii), this final function of $(x,y)$ differs from a Fubini function by translation. So $(x,y)\mapsto g(x,y-h(x))$ differs from a Fubini function only by translation and scaling, and hence is itself Fubini, by proposition 1.4.8. Therefore we have proved (i).

(iii)$\Leftrightarrow$(iv): First note that for any $x\in F$, the function $y\mapsto f^0(x,y-t^Rq(x))$ is just the translation of $y\mapsto f^0(x,y)$ by $t^Rq(x)$; since $f^0$ is Fubini this is integrable, and translation invariance of the integral implies

$$\int^F f^0(x,y-t^Rq(x))\,dy = \int^F f^0(x,y)\,dy.$$

But as a function of $x$ this is integrable, again since $f^0$ is Fubini, and

$$\int^F\int^F f^0(x,y-t^Rq(x))\,dydx = \int^F\int^F f^0(x,y)\,dydx.$$

Now by remark 1.4.9 and Fubini's theorem for $K\times K$,

$$\int^F\int^F f^0(x,y)\,dydx = \int_K\int_K f(u,v)\,dudv.$$

By the definition of a Fubini function, it now follows that $(x,y)\mapsto f^0(x,y-t^Rq(x))$ is Fubini if and only if the $dxdy$ repeated integral is well-defined and equals $\int_K\int_K f(u,v)dudv$, which is precisely what is stated in (iv). $\square$

With these reductions at hand it is straightforward to establish the conjecture in the case of non-negative depth:

**Theorem 4.2.5.** *Let $a_1,a_2,n_1,n_2,h$ be data for the conjecture, and suppose that the associated depth is non-negative. Then the conjecture is true.*

*Proof.* By the reductions, we suppose that $q\in\mathcal{O}_F[X]$ is a polynomial with no constant term and non-zero image in $K[X]$, that $R\geq 0$ is an integer, and we will prove condition (iv) of the lemma above. Write $h(X)=t^Rq(X)$, and let $f$ be Schwartz-Bruhat on $K\times K$.

The assumption on $R$ implies that all coefficients of $h$ are integral, and for $y\in F$ we have

$$\{x\in\mathcal{O}_F\,:\,y-h(x)\in\mathcal{O}_F\} = \begin{cases}\mathcal{O}_F & y\in\mathcal{O}_F,\\ \varnothing & y\notin\mathcal{O}_F.\end{cases}$$

Hence if $y\in\mathcal{O}_F$, we see that $x\mapsto f^0(x,y-h(x))$ is the lift of

$$u\mapsto f(u,\overline{y}-\overline{h}(u))$$

at $0,0$, where $\overline{h}$ is the image of $h$ in $K[X]$. If $y\notin\mathcal{O}_F$, then $f^0(x,y-h(x))=0$ for all $x$ in $F$.

Integrating with respect to $x$ therefore obtains

$$\int^F f^0(x, y - h(x))\, dx = \begin{cases} \int_K f(u, \overline{y} - \overline{h}(u))\, du & y \in \mathcal{O}_F, \\ 0 & y \notin \mathcal{O}_F, \end{cases}$$

which simply says that $y \mapsto \int^F f^0(x, y - h(x))\, dx$ is the lift of

$$v \mapsto \int_K f(u, v - \overline{h}(u))\, du$$

at $0, 0$.

Hence we may integrate with respect to $y$ to get

$$\begin{aligned} \int^F \int^F f^0(x, y - h(x))\, dx dy &= \int_K \int_K f(u, v - \overline{h}(u))\, du dv \\ &= \int_K \int_K f(u, v - \overline{h}(u))\, dv du \end{aligned}$$

where the second line follows from the first by Fubini's theorem on $K \times K$. The result now follows by translation invariance of the measure on $K$ and lemma 4.2.4. □

## 4.3 Negative depth

Having reduced the problem as far as possible and treated the relatively easy case, we discuss the case of negative depth in this section and the following section 4.4.

For this section and the next we fix the following notation: $R < 0$ a negative integer as the depth; a polynomial $q \in \mathcal{O}_F[X]$ without constant term and with non-zero image in $K[X]$ as the normalised polynomial; and a Schwartz-Bruhat function $f$ on $K \times K$. Write $\Phi$ for the function of $F \times F$ given by $\Phi(x, y) = f^0(x, y - h(x))$, and $\overline{q}$ for the image of $q$ in $K[X]$.

In this section, we also assume that $\overline{q}$ does not have everywhere vanishing derivative; since $\overline{q}$ is non-zero and without constant term, this condition can only fail to be satisfied if $K$ has positive characteristic $p$ and $q(X)$ is a purely inseparable polynomial i.e. a polynomial in $X^p$. We shall drop this assumption in section 4.4 and see that conjecture 4.2.1 fails for such highly singular $q$.

We will study the conjecture for data of depth $R$ and normalised polynomial $q$ through condition (iv) of lemma 4.2.4. We will establish various conditions under which the conjecture holds.

Introduce two sets: the non-singular part of $q$

$$W_{\text{ns}} = \{x \in \mathcal{O}_F : \overline{q}'(\overline{x}) \neq 0\} = \{x \in \mathcal{O}_F : q'(x) \in \mathcal{O}_F^\times\},$$

and the singular part

$$W_{\text{sing}} = \{x \in \mathcal{O}_F : \overline{q}'(\overline{x}) = 0\} = \{x \in \mathcal{O}_F : q'(x) \in t\mathcal{O}_F\}.$$

By our assumption on $q$, the non-singular part $W_{\text{ns}}$ is non-empty. The corresponding singular and non-singular parts of $\Phi$ are the restriction of $\Phi$ to these sets extended by zero elsewhere:

$$\begin{aligned} \Phi_{\text{ns}} &= \Phi \ \operatorname{char}_{W_{\text{ns}} \times F} \\ \Phi_{\text{sing}} &= \Phi \ \operatorname{char}_{W_{\text{sing}} \times F}. \end{aligned}$$

Note that $\Phi = \Phi_{\text{ns}} + \Phi_{\text{sing}}$.

The singular and non-singular parts are treated separately. Using the decomposition result 4.1.5, we will now explicitly evaluate $x \mapsto \Phi_{\text{ns}}(x, y)$ for any $y \in F$:

**Proposition 4.3.1.** *For all $y \in F$, the function $x \mapsto \Phi_{ns}(x,y)$ is integrable, and $y \mapsto \int^F \Phi_{ns}(x,y)\,dx$ is the lift of*

$$v \mapsto \sum_{\substack{\omega\in K\\ \overline{q}(\omega)=v\\ \overline{q}'(\omega)\neq 0}} \int_K f(\omega, -\overline{q}'(\omega)u)\,du\,X^{-R}$$

*at $0, R$; the sum is taken over all simple solutions $\omega$ to $\overline{q}(\omega)=v$.*

*Moreover, this function $y \mapsto \int^F \Phi_{ns}(x,y)\,dx$ is integrable on $F$, with*

$$\int^F\int^F \Phi_{ns}(x,y)\,dxdy = \int_K\int_K f(\omega,u)\,d\omega du.$$

*Proof.* Firstly, for $y \notin t^R\mathcal{O}_F$, we have $\Phi(x,y)=0$ for all $x\in F$. Now fix $y = t^R y_0 \in t^R\mathcal{O}_F$.

Then for $\Phi_{\mathrm{ns}}(x,y)$ to be non-zero, $x$ must belong to

$$\begin{aligned}\{x\in W_{\mathrm{ns}} : y - t^R q(x)\in\mathcal{O}_F\} &= \{x\in W_{\mathrm{ns}} : q(x)\in y_0 + t^{-R}\mathcal{O}_F\}\\ &= \{x\in\mathcal{O}_F : q(x)\in y_0+t^{-R}\mathcal{O}_F,\ \overline{q}'(\overline{x})\neq 0\}\\ &= \bigsqcup_{i=1}^r \breve{\omega}_i + t^{-R}\mathcal{O}_F,\end{aligned}$$

where $\breve{\omega}_i$ are lifts by Hensel of the simple solutions $\omega_i$ in $K$ to $q(\omega)=\overline{y}_0$ and the decomposition is provided by the decomposition result 4.1.5; that proposition also implies that there are commutative diagrams

$$\begin{array}{ccc}\breve{\omega}_i + t^{-R}\mathcal{O}_F & \xrightarrow{q} & y_0 + t^{-R}\mathcal{O}_F\\ \downarrow & & \downarrow\\ K & \xrightarrow[\overline{q}'(\omega_i)X]{} & K.\end{array}$$

So we write $\Phi_{\mathrm{ns}}(x,y)=\sum_{i=1}^r g_i(x)$, where $g_i$ is the restriction of $x\mapsto\Phi_{\mathrm{ns}}(x,y)$ to $\breve{\omega}_i+t^{-R}\mathcal{O}_F$, extended by zero elsewhere; if $x=\breve{\omega}_i+t^{-R}x_0$ belongs to $\breve{\omega}_i+t^{-R}\mathcal{O}_F$ then the commutative diagram implies

$$\Phi_{\mathrm{ns}}(x,y) = g_i(x) = f(\omega_i, -\overline{q}'(\omega_i)\overline{x}_0).$$

Therefore $g_i$ is the lift of the Haar integrable function

$$u\mapsto f(\omega_i,-\overline{q}'(\omega_i)u)$$

at $\breve{\omega}_i, -R$, the integral of which is

$$\int^F g_i(x)\,dx = \int_K f(\omega_i,-\overline{q}'(\omega_i)u)\,du\,X^{-R}$$

by remark 1.4.5. By linearity, $x\mapsto\Phi_{\mathrm{ns}}(x,y)$ is integrable, with

$$\int^F \Phi_{\mathrm{ns}}(x,y)\,dx = \sum_{i=1}^r\int_K f(\omega_i,-\overline{q}'(\omega_i)u)\,du\,X^{-R}. \qquad (*)$$

The previous paragraph considered a fixed value of $y = t^R y_0$ in $t^R\mathcal{O}_F$. We now consider the integral $(*)$ as a function of $y$; that is,

$$y \mapsto \int^F \Phi_{\mathrm{ns}}(x,y)\,dx.$$

Recall that $\omega_1,\dots,\omega_r$ are the simple solutions in $K$ to $q(\omega)=\overline{y}_0$. So we may rewrite the integral as

$$\int^F \Phi_{\mathrm{ns}}(x,y)\,dx = \sum_\omega \int_K f(\omega,-\overline{q}'(\omega)u)\,du\,X^{-R},$$

where the sum is over the finitely many $\omega$ in $K$ which satisfy $\overline{q}(\omega)=\overline{y}_0$ and $\overline{q}'(\omega)\neq 0$.

Finally, by appendix 4.A, the function $v \mapsto \sum_{\substack{\omega:\,\overline{q}(\omega)=v\\ \overline{q}'(\omega)\neq 0}} \int_K f(\omega,-\overline{q}'(\omega)u)\,du$ is in fact Haar integrable on $K$ with integral

$$\int_K \sum_{\substack{\omega:\,\overline{q}(\omega)=v\\ \overline{q}'(\omega)\neq 0}} \int_K f(\omega,-\overline{q}'(\omega)u)\,dudv = \int_K\int_K f(\omega,u)\,d\omega du.$$

Therefore $y\mapsto \int^F \Phi_{\mathrm{ns}}(x,y)\,dx$ is integrable on $F$, with

$$\int^F\int^F \Phi_{\mathrm{ns}}(x,y)\,dxdy = \int_K\int_K f(\omega,u)\,d\omega du.$$

□

The proposition has an immediate corollary:

**Corollary 4.3.2.** *If $\overline{q}'(X)$ is no-where vanishing on $K$, then $\Phi$ is Fubini.*

*Proof.* If $\overline{q}'(X)$ has no roots in $K$, then $\Phi=\Phi_{\mathrm{ns}}$, so the previous proposition and lemma 4.2.4 imply $\Phi$ is Fubini. □

More generally, the proposition reduces the problem to showing that the singularities of $\overline{q}$ give no contribution to the integrals:

**Corollary 4.3.3.** *The function $\Phi$ is Fubini if and only if the following hold: for each $y\in F$, the function $x\mapsto \Phi_{sing}(x,y)$ is integrable, then that $y\mapsto\int^F \Phi_{sing}(x,y)\,dx$ is integrable, and finally that*

$$\int^F\int^F \Phi_{sing}(x,y)\,dxdy = 0.$$

*Proof.* This follows immediately from the identity $\Phi=\Phi_{\mathrm{ns}}+\Phi_{\mathrm{sing}}$, the previous proposition, lemma 4.2.4, and linearity. □

We may verify the first requirement of corollary 4.3.3 using the decomposition result 4.1.6:

**Proposition 4.3.4.** *For each $y\in F$, the function $x\mapsto\Phi_{sing}(x,y)$ is integrable, and we have the following explicit descriptions of its integral:*

*If $y\notin t^R\mathcal{O}_F$, or if $y\in t^R\mathcal{O}_F$ but $\{x\in\mathcal{O}_F\,:\,q(x)\in t^{-R}y+t^{-R}\mathcal{O}_F,\ \overline{q}'(\overline{x})=0\}$ is empty, then $\int^F\Phi_{sing}(x,y)\,dx=0$.*

*Otherwise we have $y \in t^R\mathcal{O}_F$ and write*

$$\{x \in \mathcal{O}_F \,:\, q(x) \in t^{-R}y + t^{-R}\mathcal{O}_F,\, \overline{q}'(\overline{x}) = 0\} = \bigsqcup_{j=1}^{N} a_j + t^{c_j}\mathcal{O}_F,$$

*where the decomposition (which depends on $y$) is provided by the decomposition result 4.1.6; let $\psi_j \in K[X]$ for $j = 1, \dots, N$ denote the corresponding residue field actions i.e.*

$$\begin{array}{ccc} a_j + t^{c_j}\mathcal{O}_F & \xrightarrow{q} & t^{-R}y + t^{-R}\mathcal{O}_F \\ \downarrow & & \downarrow \\ K & \xrightarrow[\psi_j]{} & K. \end{array}$$

*commutes. Then*

$$\int^F \Phi_{sing}(x, y)\, dx = \sum_j{}' \int_K f(\overline{a}_j, -\psi_j(u))\, du\, X^{c_j},$$

*where the summation $\sum'$ is over those $j \in \{1, \dots, N\}$ for which $\psi_j$ is not a constant polynomial.*

*Proof.* By the definition of a lifted function, $f^0$ vanishes off $\mathcal{O}_F \times \mathcal{O}_F$. So if $\{x \in \mathcal{O}_F \,:\, q(x) \in t^{-R}y + t^{-R}\mathcal{O}_F,\, \overline{q}'(\overline{x}) = 0\}$ is empty for some $y$ then $x \mapsto \Phi_{\text{sing}}(x, y)$ is everywhere zero and hence integrable; note that this set is certainly empty if $y \notin t^R\mathcal{O}_F$.

Now fix $y = t^R y_0 \in t^R\mathcal{O}_F$ for the remainder of the proof. Then for $x \in F$, $\Phi_{\text{sing}}(x, y)$ vanishes unless $x$ belongs to

$$\begin{aligned} \{x \in W_{\text{sing}} \,:\, y - t^R q(x) \in \mathcal{O}_F\} &= \{x \in W_{\text{sing}} \,:\, q(x) \in y_0 + t^{-R}\mathcal{O}_F\} \\ &= \{x \in \mathcal{O}_F \,:\, q(x) \in y_0 + t^{-R}\mathcal{O}_F,\, \overline{q}'(\overline{x}) = 0\} \\ &= \bigsqcup_{j=1}^{N} a_j + t^{c_j}\mathcal{O}_F, \end{aligned}$$

where the decomposition is as in the statement of the proposition; let $\psi_j$ be the corresponding residue field approximations. Denote by $g_j$ the restriction of $x \mapsto \Phi_{\text{sing}}(x, y)$ to $a_j + t^{c_j}\mathcal{O}_F$, extended by zero elsewhere. We shall now prove that each $g_j$ is an integrable function. Indeed, $g_j$ vanishes off $a_j + t^{c_j}\mathcal{O}_F$, and if $x = a_j + t^{c_j}x_0 \in a_j + t^{c_j}\mathcal{O}_F$, then

$$\begin{aligned} g_j(x) &= f^0(a_j + t^{c_j}x_0, t^R y_0 - t^R q(a_j + t^{c_j}x_0)) \\ &= f^0(a_j + t^{c_j}x_0, t^R(y_0 - q(a_j + t^{c_j}x_0)) \\ &= f(\overline{a_j + t^{c_j}x_0}, \overline{t^R(y_0 - q(a_j + t^{c_j}x_0))}) \\ &= f(\overline{a}_j, -\psi_j(\overline{x}_0)) \end{aligned}$$

by definition of the residue field approximation $\psi_j$. Therefore $g_j$ is a lifted function: it is the lift of $u \mapsto f(\overline{a}_j, -\psi_j(u))$ at $a_j, c_j$. Further, since we assumed $f$ is Schwartz-Bruhat, this function of $u$ is Haar integrable on $K$ *so long as $\psi_j$ is not constant*, and therefore $g_j$ is integrable on $F$, with

$$\int^F g_j(x)\, dx = \int_K f(\overline{a}_j, -\psi_j(u))\, du\, X^{c_j}.$$

However, if $\psi_j$ is a constant polynomial, then $g_j = g_j(a_j)\,\mathrm{char}_{a_j+t^{c_j}\mathcal{O}_F}$, which is integrable with zero integral by example 2.1.10.

By linearity, $x \mapsto \Phi_{\text{sing}}(x,y)$ is integrable, with

$$\int^F \Phi_{\text{sing}}(x,y)\,dx = \sum_j{}' \int_K f(\overline{a}_j, -\psi_j(u))\,du\,X^{c_j},$$

as required. We emphasise again that the decomposition $a_j, c_j, \psi_j$ which we have used to express the integral *depends on* $y$. □

**Corollary 4.3.5.** *If $R = -1$ then $\Phi$ is Fubini.*

*Proof.* Looking at the proof of decomposition result 4.1.6, we see that if $R = -1$ (i.e. $A = 1$ in the notation of that result), then all the residue field approximations are constant. So by the previous proposition, $\int^F \Phi_{\text{sing}}(x,y)\,dx = 0$ for all $y \in F$. Corollary 4.3.3 implies $\Phi$ is Fubini. □

By proposition 4.3.4 we now have a well defined function $y \mapsto \int^F \Phi_{\text{sing}}(x,y)\,dx$; to establish the validity of the conditions of corollary 4.3.3 the next step is to prove that this function of $y$ is integrable. The complication in establishing its integrability is that we lack explicit information on the variation of the sets

$$\{x \in \mathcal{O}_F \,:\, q(x) \in y_0 + t^{-R}\mathcal{O}_F,\ \overline{q}'(\overline{x}) = 0\}$$

as $y_0$ runs though $\mathcal{O}_F$.

We now present some results and calculations which reveal considerable insight into why $y \mapsto \int^F \Phi_{\text{sing}}(x,y)\,dx$ can in fact fail to be integrable. We shall also give evidence that this phenomenon is merely a result of the integration theory not yet being sufficiently developed.

**Proposition 4.3.6.** *Assume that there exist $b_1, \dots, b_m \in \mathcal{O}_F$ such that if $b \in \mathcal{O}_F$ and $\{x \in \mathcal{O}_F \,:\, q(x) \in b + t^{-R}\mathcal{O}_F,\ \overline{q}'(\overline{x}) = 0\}$ is non-empty, then $b \equiv b_i \mod t^{-R}\mathcal{O}_F$ for some $i \in \{1, \dots, m\}$. Note that this is satisfied if $R = -1$ or $-2$, by corollary 4.1.7.*

*Then $y \mapsto \int^F \Phi_{\textit{sing}}(x,y)\,dx$ is a finite sum of lifts of functions of the form*

$$v \mapsto \int_K f(a, -\psi(u) - v)\,du\,X^c$$

*for $\psi \in K[X]$ non-constant, $a \in K$, and $c \geq 1$.*

*Proof.* Let $b_1, \dots, b_m$ be as in the statement of the proposition; we also assume that $b_1, \dots, b_m$ are distinct modulo $t^{-R}\mathcal{O}_F$.

By proposition 4.3.4, if $y \in F$ is not in $b_i t^R + \mathcal{O}_F$ for some $i$, then $\int^F \Phi_{\text{sing}}(x,y)\,dx = 0$. So letting $G_i$ be the restriction of $y \mapsto \int^F \Phi_{\text{sing}}(x,y)\,dx$ to $b_i t^R + \mathcal{O}_F$, extended by zero elsewhere, we have an equality of functions of $y$:

$$\int^F \Phi_{\text{sing}}(x,y)\,dx = \sum_{i=1}^m G_i(y).$$

For convenience of notation, we now fix some $i$ and write $G = G_i$, $b = b_i$. Write

$$\{x \in \mathcal{O}_F : q(x) \in b + t^{-R}\mathcal{O}_F,\ \overline{q}'(\overline{x}) = 0\} = \bigsqcup_{j=1}^N a_j + t^{c_j}\mathcal{O}_F,$$

with residue field approximations $\psi_j$. We claim that $G$ is the lift of

$$v \mapsto {\sum_{j=1}^{N}}' \int_K f(\overline{a}_j, -\psi_j(u) - v)\, du\, X^{c_j}$$

at $b^{-R}, 0$ (the sum $\sum'$ is restricted to those $j$ such that $\psi_j$ is not constant). So suppose $y = bt^R + y_0 \in bt^R + \mathcal{O}_F$. Then of course $yt^{-R} + t^{-R}\mathcal{O}_F = b + t^{-R}\mathcal{O}_F$, and so

$$\{x \in \mathcal{O}_F : q(x) \in yt^{-R} + t^{-R}\mathcal{O}_F, \overline{q}'(\overline{x}) = 0\} = \bigsqcup_{j=1}^{N} a_j + t^{c_j}\mathcal{O}_F,$$

with the residue field approximations of this decomposition given by

$$\begin{array}{ccc} a_j + t^{c_j}\mathcal{O}_F & \xrightarrow{q} & t^{-R}y + t^{-R}\mathcal{O}_F \\ \downarrow & & \downarrow \\ K & \xrightarrow[\psi_j(X) - \overline{y}_0]{} & K. \end{array}$$

Proposition 4.3.4 implies

$$G(y) = \int^F \Phi_{\text{sing}}(x,y)\, dx = {\sum_{j=1}^{N}}' \int_K f(\overline{a}_j, -\psi_j(u) - \overline{y}_0)\, du\, X^{c_j},$$

proving the claim, and completing the proof. $\square$

**Remark 4.3.7.** Suppose that the assumption of the previous proposition is satisfied. Then to establish integrability of $y \mapsto \int^F \Phi_{\text{sing}}(x,y)\, dx$ and prove it has zero integral, it is enough to prove that for any $a \in K$, non-constant $\psi \in K[X]$, and $g$ Schwartz-Bruhat on $K$, the lift of $v \mapsto \int_K g(-\psi(u) - v)\, du$ at $0, 0$ is integrable and has zero integral; let $G$ denote this function of $F$, that is,

$$G : F \to \mathbb{C}$$
$$y \mapsto \begin{cases} \int_K g(-\psi(u) - \overline{y})\, du & y \in \mathcal{O}_F, \\ 0 & \text{otherwise.} \end{cases}$$

*Then $G$ may not be integrable on $F$.* Indeed, it is not hard to show that if $G$ were to belong to $\mathcal{L}(F)$, the space of integrable functions, then $G$ would be the lift at $0, 0$ of a Haar integrable function on $K$; this Haar integrable function would then have to be $v \mapsto \int_K g(-\psi(u) - v)\, du$, but the arguments to follow reveal that this function is Haar integrable if and only if $g = 0$.

We now offer the following nonsense argument for why $G$ should be integrable, and why $\int^F G(y)\, dy$ should be zero. As a lifted function, we evaluate the integral of $G$ by theorem 1.4.4 to give

$$\int^F G(y)\, dy = \int_K \int_K g(-\psi(u) - v)\, du dv$$

and then apply Fubini's theorem for $K$ and translation invariance of the integral to deduce

$$\begin{aligned}\int^F G(y)\,dy &= \int_K \int_K g(-\psi(u)-v)\,dvdu\\ &= \int_K \int_K g(-v)\,dvdu\\ &= \int_K du \int_K g(v)dv.\end{aligned}$$

At this point it is clear why our arguments are not valid: the function $v \mapsto \int_K g(-\psi(u)-v)\,du$ is not integrable on $K$. However, we may apply similar nonsense to the function $\mathrm{char}_{\mathcal{O}_F}$, which is the lift of $\mathrm{char}_K$, to deduce

$$\int^F \mathrm{char}_{\mathcal{O}_F}(x)\,dx = \int_K du.$$

Finally, example 2.1.10(i) implies $\int^F \mathrm{char}_{\mathcal{O}_F}(x)\,dx = 0$ and so

$$\begin{aligned}\int^F G(y)\,dy &= \int_K du \int_K g(v)dv\\ &= \int^F \mathrm{char}_{\mathcal{O}_F}(x)\,dx \int_K g(v)dv\\ &= 0.\end{aligned}$$

It should be possible to extend the measure theory on $F$ so that these manipulations become rigorous. The key idea is that from the vantage point of $F$, the residue field $K$ truly has zero measure, as used above; so one expects Fubini's theorem on $K$ to hold for certain functions which, though not Haar integrable, are integrable in some sense after imposing the condition $\int_K du = 0$. Once this is properly incorporated into the measure, the theory should become considerably richer. It should also yield new methods to treat divergent integrals on $K$ by lifting them to $F$, applying Fubini theorem there, and then pulling the results back down to $K$; this would be a refreshing contrast to the main techniques so far, which have centred around reducing integrals on $F$ down to $K$.

**Example 4.3.8.** Now we treat an example of depth $-3$ in which the assumption of proposition 4.3.6 is not satisfied. We assume $R = -3$, $q(X) = X^2$, and $\mathrm{char}\, K \neq 2$. The decompositions required for the proposition are given by

$$\begin{aligned}&\{x \in \mathcal{O}_F : q(x) \in b + t^{-R}\mathcal{O}_F,\ \overline{q}'(\overline{x}) = 0\}\\ &\qquad = \{x \in \mathcal{O}_F : x^2 \in b + t^3\mathcal{O}_F,\ \overline{x} = 0\}\\ &\qquad = \begin{cases} \varnothing & b \notin t^2\mathcal{O}_F,\\ \varnothing & b \in t^2\mathcal{O}_F \text{ but } \overline{t^{-2}b} \notin K^2,\\ b^{1/2} + t^2\mathcal{O}_F \sqcup -b^{1/2} + t^2\mathcal{O}_F & b \in t^2\mathcal{O}_F \text{ and } \overline{t^{-2}b} \in {K^{\times}}^2,\\ t^2\mathcal{O}_F & b \in t^3\mathcal{O}_F,\end{cases}\end{aligned}$$

where we use Hensel's lemma to take a square root in the third case. The associated residue field approximations in the final two cases are given by

$$\begin{tikzcd} b^{1/2}+t^2\mathcal{O}_F \arrow[r, "X^2"] \arrow[d] & b+t^3\mathcal{O}_F \arrow[d] \\ K \arrow[r, "\overline{2b^{1/2}t^{-1}}X"'] & K \end{tikzcd} \qquad \begin{tikzcd} -b^{1/2}+t^2\mathcal{O}_F \arrow[r, "X^2"] \arrow[d] & b+t^3\mathcal{O}_F \arrow[d] \\ K \arrow[r, "-\overline{2b^{1/2}t^{-1}}X"'] & K \end{tikzcd}$$

$$\begin{tikzcd} t^2\mathcal{O}_F \arrow[r, "X^2"] \arrow[d] & b+t^3\mathcal{O}_F \arrow[d] \\ K \arrow[r, "-\overline{bt^{-3}}"'] & K. \end{tikzcd}$$

Proposition 4.3.4 therefore implies that for $y \in F$,

$$\int^F \Phi_{\text{sing}}(x,y)\,dx$$
$$= \begin{cases} 0 & y \notin t^{-1}\mathcal{O}_F, \\ 0 & y \in t^{-1}\mathcal{O}_F \text{ but } \overline{ty} \notin K^2, \\ \int_K f(0,-2\overline{(yt)^{1/2}}u)\,du\,X^2 + \int_K f(0,2\overline{(yt)^{1/2}}u)\,du\,X^2 & y \in t^{-1}\mathcal{O}_F \text{ and } \overline{ty} \in {K^\times}^2, \\ 0 & y \in \mathcal{O}_F. \end{cases}$$

Therefore $y \mapsto \int^F \Phi_{\text{sing}}(x,y)\,dx$ is the lift of

$$v \mapsto \int_K f(0,-2v^{1/2}u) + f(0,2v^{1/2}u)\,du\,X^2 \ \operatorname{char}_{K^{\times 2}}(v)$$

at $0,-1$.

This function of $F$ need not be integrable, but as in the previous remark, there is a good argument to suggest that it should be, and why its integral should be zero:

Indeed, the function on the residue field has the form

$$J(v) = \sum_{\substack{\omega \in K \\ \overline{q}(\omega)=v \\ \overline{q}'(\omega)\neq 0}} \int_K g(-\overline{q}'(\omega)u)\,du$$

where $g$ is a Schwartz-Bruhat function on $K$. Now replace the integrand by $g(-\overline{q}'(\omega)u)\operatorname{char}_K(\omega)$ and appeal to appendix 4.A to deduce

$$\int_K J(v)dv = \int_K \int_K g(u)\operatorname{char}_K(\omega)\,d\omega du.$$

But arguing as in the proceeding remark, $\int_K d\omega = 0$, and so $\int_K J(v)\,dv = 0$. Of course, the argument is nonsense because $J$ is not integrable, but it should be after a suitable extension of the measure.

## 4.4 Negative depth with $\overline{q}$ purely inseparably

We maintain all notation introduced at the beginning of the previous section but drop the additional hypothesis that $\overline{q}'$ is not everywhere zero. Instead, we now assume $K$ has positive characteristic $p$ and that $\overline{q}(X)$ is purely inseparable.

Whereas in the previous section conjecture 4.2.1 could fail to hold because the integration theory is not yet sufficiently developed, causing functions not to be integrable, we will present a result now to show that if $\overline{q}$ is purely inseparable then all required functions are integrable, but the conjecture is simply false!

First note that, in the notation of the previous section, $\overline{q}'$ being everywhere zero implies $\Phi = \Phi_{\text{sing}}$. Secondly, proposition 4.3.4 remains valid, so that $x \mapsto \Phi(x, y)$ is integrable for any $y \in F$ and we have an explicit description of its integral.

**Proposition 4.4.1.** *Suppose $R = -1$. Then both repeated integrals of $\Phi$ are well-defined, but $f$ may be chosen so that*

$$\int^F \int^F \Phi(x, y)\, dxdy \neq \int^F \int^F \Phi(x, y)\, dydx.$$

*Proof.* Arguing exactly as in corollary 4.3.5 it follows that $\int^F \Phi_{\text{sing}}(x, y)\, dx = 0$ for all $y \in F$, and therefore $y \mapsto \int^F \Phi_{\text{sing}}(x, y)\, dx = 0$ is certainly integrable, with integral $0$. That is,

$$\int^F \int^F \Phi(x, y)\, dxdy = 0.$$

The $dydx$ integral of $\Phi$ was showed to make sense in lemma 4.2.4 and have value

$$\int^F \int^F \Phi(x, y)\, dxdy = \int_K \int_K f(u, v)\, dudv.$$

To complete the proof simply choose $f$ to be any Schwartz-Bruhat function on $K \times K$ such that $\int_K \int_K f(u, v) dudv$ is non-zero. $\square$

**Remark 4.4.2.** The integration theory of chapter 2 is easily modified to allow integration on a complete discrete valuation field whose residue field is any infinite field equipped with discrete counting measure; this is an elementary form of motivic integration. In that situation one may ask similar questions about changes of variables and Fubini's theorem; results are generally easier to prove and closer to the analogous results for a usual local field. In particular, if the residue field is perfect, then the pathologies exhibited in this section no longer exist.

The failure of Fubini's theorem appears therefore to be a measure-theoretic consequence of the local field $K$ being imperfect. Note that the set of $p^{\text{th}}$ powers of $K$ have zero measure, in stark contrast with in a perfect field. The approach to ramification theory for complete discrete valuation fields with imperfect residue fields by A. Abbes and T. Saito [AS02] [AS03] is based on rigid algebraic geometry and uses decomposition results similar to 4.1.6 and 4.1.5; see subsection 6.1.2 for a more detailed discussion. A deeper understanding of this failure of Fubini's theorem will undoubtedly lead to progress in the ramification theory of two-dimensional local fields.

## 4.5 Summary and future work

Let us summarise our main results on conjecture 4.2.1. Given data $a_1, a_2, n_1, n_2, h$ for the conjecture, let $q$ be the associated normalised polynomial and $R$ the depth; then:

(i) If $\deg h \,(= \deg q) \leq 1$ then the conjecture is true (theorem 4.2.2).

(ii) If $R \geq 0$ then the conjecture is true (theorem 4.2.5)

(iii) If $\overline{q}'$ is no-where vanishing on $K$ then the conjecture is true (corollary 4.3.2).

(iv) If $R = -1$ and $\overline{q}$ is not purely inseparable, then the conjecture is true (lemma 4.2.4 + corollary 4.3.5).

(v) If $R < -1$ and $\overline{q}$ is not purely inseparable, then $y \mapsto \int^F \Phi(x,y)\,dx$ may fail to be integrable and so the conjecture may fail; it appears that it is possible to increase the space of integrable functions so that the conjecture becomes true (remark 4.3.7 + example 4.3.8).

(vi) If $R = -1$ but $\overline{q}$ is purely inseparable, then the conjecture fails and would continue to fail even if we increased the scope of the integral (section 4.4).

(vii) If $R < -1$ but $\overline{q}$ is purely inseparable, then similarly to case (v) calculations become difficult. We have not included examples, but in all cases which the author can explicitly evaluate, $\int^F \int^F \Phi(x,y)\,dxdy = 0$, Thus the conjecture seems to fail as in (vi).

The immediate task is evident: the integral must be extended to a wider class of functions so that the nonsense manipulations in remark 4.3.7 and example 4.3.8 become valid.

Secondly, we should consider more general changes of coordinates on $F \times F$ than $(x,y) \mapsto (x, y - h(x))$. Similar methods to those in this chapter will be required: firstly one needs to approximate the transformation at the level of $K \times K$ and find a suitable decomposition. This will lead to integrals over $K$ which can be explicitly evaluated as well as some functions on $F$; these functions on $F$ will perhaps be within the scope of the integral, or instead will provide further impetus for extending the integral.

## 4.A Evaluation of an important integral on $K$

Let $K$ be a local field, $f$ a Fubini function of $K \times K$, and $\psi \in K[X]$ a polynomial with $\psi'$ not everywhere zero. We discuss the function of $K$ given by

$$J(v) = \sum_{\substack{\omega \in K \\ \psi(\omega)=v \\ \psi'(\omega)\neq 0}} \int_K f(\omega, -\psi'(\omega)u)\,du.$$

Note that the assumption that $f$ is Fubini implies that $J$ is defined (i.e. not infinite) for all $v$. We will prove the following:

**Proposition 4.A.1.** *The function $J$ is integrable on $K$, with*

$$\int_K J(v)\,dv = \int_K \int_K f(\omega, u)\,d\omega du.$$

*Proof.* The proof is an exercise in analysis over a local field. Let $\Sigma = \{x : \psi'(x) = 0\}$ be the finite set of singular points of $\psi$.

Let $v_0 \in K$ and assume that there is a non-singular solution to $\psi(Y) = v_0$. The inverse function theorem for complete fields (see e.g. [Igu00]) implies that there exists an open disc $V \ni v_0$, open discs $\Omega_1, \dots, \Omega_n$, and $K$-analytic maps $\lambda_i : V \to \Omega_i$, $i = 1, \dots, n$ (that is, representable in $V$ by a convergent power series) with the following properties:

(i) $\Omega_1, \dots, \Omega_n$ are pairwise disjoint;

(ii) $\psi(\Omega_i) = V$ for each $i$; moreover, $\psi|_{\Omega_i}$ and $\lambda_i$ are inverse diffeomorphisms between $\Omega_i$ and $V$;

(iii) the non-singular solutions in $K$ to $\psi(Y) = v_0$ are $Y = \lambda_1(v_0), \dots, \lambda_n(v_0)$.

Moreover, we claim that, possibly after shrinking the sets $\Omega_i$, $V$, we may further assume

(iv) for any $v \in V$, the non-singular solutions in $K$ to $\psi(Y) = v$ are $Y = \lambda_1(v), \dots, \lambda_n(v)$.

For if not, then there would exist a sequence $(x_n)_n$ in $K$ such that $x_n \notin \bigcup_i \Omega_i$ for all $n$ and $\psi(x_n) \to v_0$; the relative compactness of $\psi^{-1}(V)$ now allows us to pass to a convergent subsequence of $(x_n)$, giving an element $x \in K \setminus \bigcup_i \Omega_i$ which satisfies $\psi(x) = v_0$. But this contradicts (iii) and so proves our claim. Informally, the $\lambda_i$ parametrise the non-singular solutions of $\psi(Y) = v$, for $v \in V$.

For $v \in V$, we deduce that

$$J(v) = \int \sum_{i=1}^{n} f(\lambda_i(v), -\psi'(\lambda_i(v))u)\, du$$

and so

$$\begin{aligned}
\int_V J(v)\, dv &= \sum_{i=1}^{n} \int_K \int_V f(\lambda_i(v), -\psi'(\lambda_i(v))u)\, dv du \\
&= \sum_i \int_K \int_V |\psi'(\lambda_i(v))|^{-1} f(\lambda_i(v), u)\, dv du \\
&= \sum_i \int_K \int_{\Omega_i} f(\omega, u)\, d\omega du \\
&= \int_K \int_{\psi^{-1}(V)} f(\omega, u)\, d\omega du
\end{aligned}$$

by Fubini's theorem and an analytic change of variables $v = \psi(\omega)$. An elementary introduction to change of variables in integrals over non-archimedean fields may be found in [VVZ94].

If $J$ is replaced by $J \operatorname{char}_A$ for any measurable subset $A \subseteq V$ then this working is easily modified to show

$$\int_A J(v)\, dv = \int_K \int_{\psi^{-1}(A)} f(\omega, u)\, d\omega du. \qquad (*)$$

It is now easy to see that $\psi(K \setminus \Sigma)$ admits a partition into countably many Borel sets $(A_j)_{j=1}^{\infty}$ where $(*)$ holds with $A_j$ in place of $A$ for each $A$. Therefore

$$\begin{aligned}
\int_K J(v)\, dv &= \sum_j \int_{A_j} J(v)\, dv \\
&= \sum_j \int_K \int_{\psi^{-1}(A_j)} f(\omega, u)\, d\omega du \\
&= \int_K \int_{\Omega} f(\omega, u)\, d\omega du
\end{aligned}$$

where $\Omega = \psi^{-1}(\psi(K \setminus \Sigma)) = K \setminus \psi^{-1}(\psi(\Sigma))$ differs from $K$ only by a finite set. So we have reached the desired result:

$$\int_K J(v)\, dv = \int_K \int_K f(\omega, u)\, d\omega du. \qquad \square$$

CHAPTER 5

# Two-dimensional integration à la Hrushovski-Kazhdan

We explain how the results of Hrushovski and Kazhdan apply to integration on two-dimensional local fields of residue characteristic zero.

## 5.1 Summary, without model theory

We now explain rigorously exactly how the model theoretic approach to integration developed by E. Hrushovski and D. Kazhdan in [HK06] applies to the problem of integration on two-dimensional local fields. We focus on the case of dimension two, but there would be no essential difference caused by considering higher dimensional local fields.

The results here are based on the model theoretic calculations of the subsequent sections, but we are going to begin by presenting our main results avoiding model theory as far as possible, so that this section remains accessible to the reader unversed in that theory. As a result, a few technical issues are omitted. The model theoretically inclined reader will have no difficulty in remoulding this discussion to his preferred shape, and will hopefully feel nothing worse than slight satisfaction if he notices one of the omissions.

For the remainder of this section we fix a two-dimensional local field $F$, and a uniformiser $t \in F$. We set

$$\mathrm{RV}(F) = F^{\times}/1 + \mathfrak{p}_F \sqcup \{\infty\},$$

where $\mathfrak{p}_F$ is the prime ideal of $\mathcal{O}_F$. The natural map $F \to \mathrm{RV}(F)$, sending $0$ to $\infty$, is denoted $\mathrm{rv}$. Our choice of $t$ induces an isomorphism

$$\mathrm{RV}(F)^{\times} := F^{\times}/1 + \mathfrak{p}_F \cong \overline{F}^{\times} \times \mathbb{Z}, \quad ut^r \mapsto (\overline{u}, r),$$

which will be essential.

### 5.1.1 Motivation

A recurring idea in the development of the integration theory on a two-dimensional local field $F$ has been that the integral ought to reduce to Haar integration on the local field $\overline{F}$. Explicit phenomena of this appeared in the original definition in chapter 2, the reduction in section 4.3 of an integral on $F$ to one of $\overline{F}$ which was then calculated in appendix 4.A, and the way in which the invariant integral on $GL_n(F)$ lifted the Haar integral on $GL_n(\overline{F})$ (proposition 3.3.8). We saw in remark 4.3.7 that there seem not to be enough integrable functions, and a major component of chapter 3 involved proving that linear changes of variables of determinant 1 preserve the two-dimensional measure. We will mainly address the first two-problems here: reducing the integral to the residue field and increasing the scope of the integral. Understanding its behaviour under changes of variables is covered by one of the difficulties discussed in subsection 5.1.4 below.

A subset of $F$ (or of $F^n$) will be called bounded if and only if it is bounded with respect to the discrete valuation on $F$. A more subtle notion of boundedness is the following:

**Definition 5.1.1.** Firstly, call a subset $Y \subseteq \mathrm{RV}(F)$ bounded if and only if $\mathrm{rv}^{-1}(Y) \subseteq F$ is bounded. Now, $Y$ is *two-dimensionally bounded* if and only if it is not only bounded, but also each section

$$Y_k := \{y \in \overline{F} : (y, k) \in Y\}$$

is bounded in the local field $\overline{F}$.

Here we have identified $F^\times/1 + \mathfrak{p}_F$ with $\overline{F}^\times \times \mathbb{Z}$, but the notion of two-dimensional boundedness is easily seen to be independent of the choice of uniformiser $t$.

The two notions of boundedness for $Y \subseteq \mathrm{RV}(F)^n$ are defined similarly.

Let $\mu^F$ denote the measure on $F$ introduced in section 2.2. As a reminder, $\mu^F$ is characterised by

$$\mu^F(a + t^k\{x \in \mathcal{O}_F : \overline{x} \in S\}) = \mu(S)T^k \in \mathbb{R}(T)$$

for $a \in F$, $k \in \mathbb{Z}$, and $S \subseteq \overline{F}$ of finite Haar measure. Here $\mu$ is a fixed Haar measure on $\overline{F}$, and we have replaced the $X$ variable used in earlier chapters by $T$, to avoid confusion as $X$ is always used to denote certain sets in this chapter.

Suppose $Y \subseteq \mathrm{RV}(F)^\times$; then it is easy to see that

$$\mathrm{rv}^{-1}(Y) = \bigsqcup_{k \in \mathbb{Z}} t^k\{x \in \mathcal{O}_F : \overline{x} \in Y_k\},$$

where $Y_k$ is the section of $Y$ which appeared in the previous definition. If $Y$ is bounded, then $Y_k = \varnothing$ for $k \ll 0$, and if $Y$ is moreover two-dimensionally bounded, then

$$\mu^F(t^k\{x \in \mathcal{O}_F : \overline{x} \in Y_k\}) = \mu(Y_k)\, T^k.$$

Although we mentioned in section 2.2 that $\mu^F$ is not always countably additive, Fesenko has shown in [Fes03, §6] that it can be consistently extended to certain countable disjoint unions; if $Y$ is two-dimensionally bounded, then $\bigsqcup_{k \in \mathbb{Z}} t^k\{x \in \mathcal{O}_F : \overline{x} \in Y_k\}$ will be such a union, and so

$$\mu^F(\mathrm{rv}^{-1}(Y)) = \sum_{k \in \mathbb{Z}} \mu(Y_k)\, T^k \in \mathbb{R}((T)).$$

This all easily extends to two-dimensionally bounded $Y \subseteq (\mathrm{RV}(F)^\times)^n$, with

$$\mu^F(\mathrm{rv}^{-1}(Y)) = \sum_{\underline{k} \in \mathbb{Z}^n} \mu(Y_{\underline{k}}) X^{k_1 + \cdots + k_n} \in \mathbb{R}((T)).$$

However, the class of subsets of $F$ of the form $\mathrm{rv}^{-1}(Y)$ for $Y \subseteq \mathrm{RV}(F)^\times$ is very small; it is not even closed under translations. But if we also allow 'measure-preserving' maps, then we shall soon see in theorem 5.1.5 that the situation is much better.

### 5.1.2 Semi-algebraic sets

The reader should look forward at the notion of 'structure' which will be used in section 6.2; indeed, it would be profitable to skim that entire section before proceeding. We assume here for simplicity that $\overline{F}$ is a non-archimedean local field; for the archimedean case, see remark 5.1.6.

**Definition 5.1.2.** Let $\mathcal{A}$ be the smallest structure on $F$ satisfying the following properties:

(i) $\mathcal{A}(F^n)$ contains any Zariski closed set;

(ii) $\mathcal{A}(F)$ contains both $\mathcal{O}_F$ and $O_F := \{x \in \mathcal{O}_F : \overline{x} \in \mathcal{O}_{\overline{F}}\}$.

A subset of $F^n$ belonging to $\mathcal{A}(F^n)$ will be called *semi-algebraic*; a function between subsets of $F^n$ and $F^m$ will be called semi-algebraic if and only if its graph $\Gamma \subseteq F^{n+m}$ is semi-algebraic.

In other words, the semi-algebraic sets are those which are definable with respect to the structure $\mathcal{A}$.

We define semi-algebraic subsets of $\overline{F}^n$ in a similar way as for $F^n$, by taking the smallest structure which contains all Zariski closed sets and $\mathcal{O}_{\overline{F}}$.

**Example 5.1.3.** Hopefully a few examples will convince the reader that semi-algebraic sets are not too daunting:

(i) Any single point $a \in F^n$ is semi-algebraic, because it is the image of a constant polynomial.

(ii) If $f \in F[x_1, \dots, x_n]$, then $f^{-1}(O_F) \subseteq F^n$ is semi-algebraic. Indeed, it is the preimage of a semi-algebraic set under a semi-algebraic function (the function $f$ is semi-algebraic because its graph is Zariski closed, hence is semi-algebraic).

(iii) If $S$ is a compact open subset of $\overline{F}^n$, then $X := a + \tau\{x \in \mathcal{O}_F^n : \overline{x} \in S\}$ is semi-algebraic, for any $a \in F^n$, $\tau \in GL_n(F)$. Indeed, decomposing $S$ into a finite, disjoint union of products of translated fractional ideals, we see that $X$ is a finite, disjoint union of sets of the form $a' + \tau' O_F^n$, with $a' \in F^n$, $\tau' \in GL_n(F)$; but such sets are the image under a semi-algebraic map of a semi-algebraic set, hence are semi-algebraic.

**Example 5.1.4.** As well as polynomial maps, semi-algebraic functions can include the inverse of polynomial maps. For example, the group of principal units $U_F^1 = 1 + \mathfrak{p}_F$ is uniquely $l$-divisible for any $l$ not divisible by char $\overline{F}$, so that the map

$$f : U_F^1 \to U_F^1, \quad x \mapsto x^{1/l}$$

is well-defined. Moreover, $f$ is semi-algebraic, for the following reasons:

(i) by the previous example $\mathfrak{p}_F^1$, hence $U_F^1$, is semi-algebraic;

(ii) the polynomial $x^l$ is a semi-algebraic function, meaning that

$$\Gamma = \{(x, y) \in F^2 : x^l = y\}$$

is semi-algebraic;

(iii) by permuting coordinates and intersection with $U_F^1 \times U_F^1$, we see that

$$\Gamma' = \{(y, x) \in U_F^1 \times U_F^1 : x^l = y\}$$

is semi-algebraic; but $\Gamma'$ is exactly the graph of $f$, and therefore $f$ is semi-algebraic.

The advantage of working with the class of semi-algebraic objects is that it is large enough to include all interesting sets and functions, while not so large that pathologies appear. Here are several particularly attractive limitations, which are true in the case char$\overline{F} = 0$:

(i) Call a subset of $F^n$ *null* if and only if it is contained in a proper Zariski closed subset of $F^n$. Then a semi-algebraic set $X \subseteq F^n$ is null if and only if it has no interior in the valuation topology on $F$. Hence the boundary $\partial X$ of any semi-algebraic set $X \subseteq F^n$ is null. See proposition 5.3.12 and the subsequent corollaries.

(ii) Let $X \subseteq F^n$ be semi-algebraic, and $f : X \to F^n$ a semi-algebraic function. Then $f$ is almost everywhere smooth; here 'smooth' means infinitely differentiable in the usual sense for valued fields, and 'almost everywhere' means that we are permitted to ignore a semi-algebraic null set. See subsection 5.3.2.

So if $X \subseteq F^n$ and $f : X \to F^n$ are semi-algebraic, then there is a proper Zariski closed set $V \subseteq X$ such that $X \setminus V$ is open and $f|_{X\setminus V}$ is smooth!

In fact, the class of semi-algebraic sets is slightly too large for integration theory, because one rarely computes the measure of something like the set of squares (which is semi-algebraic). Therefore we say that $X \subseteq F^n$ is *quantifier-free semi-algebraic* if and only if it belongs to the algebra of subsets of $F^n$ generated by $f^{-1}(\mathcal{O}_F)$ and $f^{-1}(O_F)$, where $f$ varies over $F[x_1, \dots, x_n]$. In fact, the examples of semi-algebraic sets presented above are all quantifier-free semi-algebraic.

Finally, we call a subset $Y$ of $(\mathrm{RV}(F)^\times)^n$ semi-algebraic if and only if each section $Y_{\underline{k}} \subseteq (\overline{F}^\times)^n$ is semi-algebraic, for all $\underline{k} \in \mathbb{Z}^n$. This is easily seen not to depend on the choice of $t$ which induces the splitting $\mathrm{RV}(F)^\times \cong \overline{F}^\times \times \mathbb{Z}$.

### 5.1.3 Descent to RV

We may now precisely state the main result:

**Theorem 5.1.5.** *Assume char $\overline{F} = 0$. Let $X \subseteq F^n$ be a bounded, quantifier-free semi-algebraic set. Then $X$ may be written as a disjoint union of semi-algebraic sets $X = \bigsqcup_{i=0}^s X_i$ such that $X_0$ is null, and the remaining $X_i$ are of the following form: there are a semi-algebraic subset $Y_i \subseteq (RV(F)^\times)^n$, an integer $N_i \geq 1$, and a semi-algebraic $N_i$-to-1 map $f_i : X_i \to \mathrm{rv}^{-1}(Y_i)$ which is almost everywhere smooth with Jacobian $= 1$.*

*Proof.* This result is obtained by modifying a similar result for algebraically closed valued fields due to Hrushovski and Kazhdan. The precise argument, for the model theorists, is as follows:

Let $\mathcal{L}$ be the RV-expansion of the language $\mathcal{L}_{\mathrm{RV}}$ obtained by adjoining a unary predicate to the RV-sort to denote a valuation subring of the residue field. Then $F$ is an $\mathcal{L}$ structure and we set $T^+ = T(F)$ and $\mathcal{L}^+ = \mathcal{L}_F$. This theory is an RV-expansion of $\mathrm{H}(0,0)_F$, the parameter-extension of the theory of Henselian fields. Hence we may apply corollary 5.5.10, and the result immediately follows (since semi-algebraic really means $T^+$-definable). ☐

If each $Y_i$ in the previous lemma is actually two-dimensionally bounded, then, as explained in subsection 5.1.1, we know exactly what the two-dimensional measure of $\mathrm{rv}^{-1}(Y_i)$ is:

$$\mu^F(\mathrm{rv}^{-1}(Y_i)) = \sum_{\underline{k}\in\mathbb{Z}^n} \mu(\{y \in (\overline{F}^\times)^n : (y,\underline{k}) \in Y_i\})\, T^{|\underline{k}|} \in \mathbb{R}((T)).$$

Therefore, *assuming* that $\mu^F$ extends to a finitely additive measure which ignores proper Zariski closed sets and which is preserved under Jacobian 1 smooth maps, we *deduce*

$$\mu^F(X) = \sum_{i=1}^{s} N_i\, \mu^F(\mathrm{rv}^{-1}(Y_i)) \in \mathbb{R}((T)). \tag{†}$$

Conversely, we would like to *use* the theorem to extend $\mu^F$ by using (†) as a definition.

That is, the theorem not only proves that the two-dimensional measure can be extended in at most one reasonable way, it also offers a definition of the measure for a wide class of sets. Of course, the reader will already have noticed various difficulties, which we are compelled to discuss next.

**Remark 5.1.6.** There is no difficulty in extending these results to archimedean two-dimensional local fields, i.e. $\mathbb{R}((t))$ or $\mathbb{C}((t))$. One must modify the definitions of semi-algebraic sets for both $F$ and $\overline{F}$, and in the proof of theorem 5.1.5 one must use a different first order language.

### 5.1.4 The remaining problems

There are two problems which prevent us from immediately offering (†) as a *definition* of $\mu^F(X)$:

(i) the sets $Y_i$ may not be two-dimensionally bounded, and therefore the definition of $\mu^F(\mathrm{rv}^{-1}(Y_i))$ does not make sense;

(ii) even if the $Y_i$ are two-dimensionally bounded, perhaps there is a different decomposition of $X$, as $\bigsqcup_j X'_j$ say, with the corresponding $Y'_j$ also two-dimensionally bounded; then we need to show that $\sum_i N_i\mu^F(\mathrm{rv}^{-1}(Y_i)) = \sum_j N'_j\mu^F(\mathrm{rv}^{-1}(Y'_j))$.

**Example 5.1.7.** This example is fundamental in Hrushovski and Kazhdan's theory. Set $X = \mathfrak{p}_F$; we will offer two decompositions of $X$.

Firstly, let $Y = \{(y,n) \in \mathrm{RV}(F) : n > 0\}$. Then $\mathrm{rv}^{-1}(Y) = \mathfrak{p}_F \setminus \{0\}$, so we have

$$X = \{0\} \sqcup \mathrm{rv}^{-1}(Y),$$

which is a valid decomposition since $\{0\}$ is Zariski closed.

Secondly, let $Y' = \{(1,0)\} \subseteq \mathrm{RV}(F)$, so that $\mathrm{rv}^{-1}(Y') = 1 + \mathfrak{p}_F$. Let $f$ be the Jacobian 1 bijection $x \mapsto x+1$. Then $X = f^{-1}(\mathrm{rv}^{-1}(Y'))$ is also a valid decomposition for $X$.

In a sense which we will not make precise, Hrushovski and Kazhdan explain that, in their setting of an algebraically closed valued field, this is the *only* ambiguity which can arise in the decomposition of any set into RV lifts.

Now consider how the previous example interacts with the two-dimensional measure. In the first decomposition, $Y$ is not two-dimensionally bounded (indeed, each section $Y_n$ for $n > 0$ is all of $\overline{F}^\times$), and so we cannot use this decomposition to define $\mu^F(X)$. But in the second decomposition, $Y'$ is two-dimensionally bounded, with

$\mu^F(\mathrm{rv}^{-1}(Y')) = 0$; hence we expect $\mu^F(X) = 0$, which is indeed true according to example 2.2.5(i).

In this way, the non-uniqueness of a decomposition appears to be 'orthogonal' to the condition that the $Y_i$ appearing in the decomposition are two-dimensionally bounded. The author is confident that further examination of Hrushovski and Kazhdan's proof of their corresponding result will lead to the elimination of problem (ii).

Problem (i) is more subtle; it is unclear how to provide an intrinsic characterisation of which semi-algebraic sets $X$ admit a decomposition with all the $Y_i$ two-dimensionally bounded. It is not even clear if the class of such sets is closed under unions (it is certainly closed under disjoint unions) and intersections. Hopefully resolving problem (ii) will lead to further insights.

## 5.2 Languages and known results

The remainder of this chapter is essentially a proof of theorem 5.5.9 below, from which corollary 5.5.10 and the aforementioned theorem 5.1.5 then follow. The remainder of this chapter is presented in the language of model theory; we begin by collecting together some basic results pertaining to the model theory of valued fields.

### Fields

Let $T_{\text{ring}}, \mathcal{L}_{\text{ring}}$ denote the theory and language of rings. This language has binary operations $+, -, \times$ and constants $0, 1$; the theory contains the obvious sentences such as $\forall x\, \forall y\, (x + y = y + x)$ so that the models of $T_{\text{ring}}$ are precisely commutative, associative rings with unit. Adjoining to $T_{\text{ring}}$ the sentence $\forall x\, \exists y\, (x \neq 0 \to xy = 1)$ obtains the theory of fields $T_{\text{field}}$, formulated in the language of rings.

For algebraically closed fields, one adds to $T_{\text{field}}$ a countable collection of sentences

$$\forall a_0 \dots \forall a_{n-1}\, \exists x\, (x^n + a_{n-1}x^{n-1} \dots + a_0 = 0) \qquad \text{(all } n \geq 2\text{)}$$

to obtain the theory ACF. This can be further augmented by

$$\underbrace{1 + \dots + 1}_{p \text{ times}} = 0$$

to give ACF($p$), the theory of algebraically closed fields of characteristic $p$, for some rational prime $p > 0$; alternatively, adding the negation of all these sentences gives ACF$(0)$, the theory of algebraically closed fields of characteristic zero.

A. Tarski established that ACF admits elimination of quantifiers in the language $\mathcal{L}_{\text{ring}}$. Moreover, each theory ACF($p$) ($p \geq 0$) is complete and model complete.

### Ordered groups

Let $T_{\text{oag}}, \mathcal{L}_{\text{oag}}$ denote the theory and language of ordered abelian groups. This language has binary operations $+, -$, a binary relation $\leq$, and a constant $0$; the models of $T_{\text{oag}}$ are precisely abelian groups equipped with a total ordering which is compatible with the group operation.

Adding to $T_{\text{oag}}$ the collection of sentences

$$\forall x\, \exists y\, (\underbrace{x + \dots + x}_{n \text{ times}} = y) \qquad \text{(all } n \geq 2\text{)}$$

yields $T_{\text{doag}}$, the theory of divisible ordered abelian groups. This is complete and admits elimination of quantifiers [Rob77].

## Valued fields

There are many different languages for valued fields, and although they are all essentially the same, some are more convenient. The most basic language is obtained by adding to $\mathcal{L}_{\text{ring}}$ a single unary predicate $\mathcal{O}$ and to $T_{\text{field}}$ an additional sentence

$$\forall x\,(x \notin \mathcal{O} \to x^{-1} \in \mathcal{O}),$$

so that $\mathcal{O}$ is interpreted as a (possibly trivial) valuation subring of the field. One can add further sentences such as

$$\underbrace{1+\cdots+1}_{p \text{ times}} \neq 0 \wedge (\underbrace{1+\cdots+1}_{p \text{ times}})^{-1} \notin \mathcal{O}$$

to obtain the theory of valued fields of characteristic $0$ and residue characteristic $p$.

Even using the simple language $\mathcal{L}_{\text{ring}} \cup \{\mathcal{O}\}$, one can interpret the residue field $\overline{F}$ and value group $\Gamma(F)$ of any valued field $F$. Indeed,

$$\Gamma(F) \cong F^{\times}/\mathcal{O}^{\times}$$

and

$$\overline{F} = \mathcal{O}/\mathfrak{m},$$

where $\mathfrak{m} = \{X \in F : X = 0 \vee X^{-1} \notin \mathcal{O}\}$ is the maximal ideal of $\mathcal{O}$. Therefore there is little change to the model theoretic nature of the situation if we add an extra sort or two to be interpreted as the residue field or value group, together with additional function symbols to represent the residue map and valuation.

However, the main theme of the model theory of valued fields is understanding how properties of the field $F$ reduce to properties of the value group and residue field. The convenient object which suits this purpose is

$$F^{\times}/1+\mathfrak{m}.$$

Indeed, there is a natural exact sequence

$$1 \to \overline{F}^{\times} \to F^{\times}/1+\mathfrak{m} \to \Gamma(F) \to 0,$$

so that $F^{\times}/1+\mathfrak{m}$ wraps together the value group and residue field; in particular, if the valuation is discrete, then a choice of a uniformiser will induce an isomorphism $F^{\times}/1+\mathfrak{m} \cong \overline{F} \times \mathbb{Z}$. Following Hrushovski and Kazhdan, we shall therefore work in the two-sorted *RV-language* $\mathcal{L}_{\text{RV}}$, which we now describe. The first sort, denoted VF, uses the language $\mathcal{L}_{\text{ring}}$. The second sort, denoted RV, uses the language obtained by augmenting $\mathcal{L}_{\text{oag}}$ with a unary predicate $k^{\times}$, a constant $\infty$, and a binary operation $+ : k \times k \to k$, where $k$ is the union of $k^{\times}$ and an imaginary constant $0$. There is also a function symbol $\text{rv} : \text{VF} \to \text{RV}$. The theory $T_{\text{RV}}$ contains all required sentences to ensure that if $F = (\text{VF}(F), \text{RV}(F))$ is a model of $T_{\text{RV}}$, then $\text{VF}(F)$ is a valued field, $\text{RV}(F) = F^{\times}/1+\mathfrak{m} \sqcup \{\infty\}$, and rv is the natural quotient map, extended to all of $\text{VF}(F)$ by setting $\text{rv}(0) = \infty$; the ordering $\leq$ on $\text{RV}(F)$ is the partial ordering $x \leq y \Leftrightarrow yx^{-1} \in \mathcal{O}_F$, with $\infty$ being maximal. We write $\text{RV}(F)^{\times} = F^{\times}/1+\mathfrak{m}$. One can of course augment the theory $T_{\text{RV}}$ to ensure that the models have appropriate characteristic, are Henselian, are algebraically closed, etc. We shall be particularly interested in RV-expansions, in which one adds additional structure only to the RV sort.

Formulated in any of these languages, the theory of algebraically closed valued fields ACVF admits elimination of quantifiers (essentially follows from A. Robinson's work

[Rob77]), and the theories ACVF$(p,p)$, ACVF$(0,p)$, ACVF$(0,0)$ of algebraically closed valued fields with specified characteristic and residue characteristic are complete. Further, the theory H$(0,0)$ of Henselian valued fields of residue characteristic zero admits elimination of field quantifiers (see e.g. [Pas89], [HK06, Prop. 12.9]).

## 5.3 Structure results for definable sets in a valued field

In this section we establish a variety of results describing the structure of definable sets and maps in valued fields. Our main tool is explicit, syntactical analysis of formulae, similarly to Y. Yin's reworking of the Hrushovski-Kazhdan theory for ACVF$(0,0)$ in [Yin08].

Let $(T,\mathcal{L})$ be a theory of valued fields formulated in an extension-by-parameters of the language $\mathcal{L}_{\mathrm{RV}}$; assume $(T,\mathcal{L})$ admits the elimination of VF quantifiers. The examples to have in mind are when $T$ is an extension of ACVF$(0,0)$ or H$(0,0)$ by parameters. Let $(T^+,\mathcal{L}^+)$ be an RV-expansion of $(T,\mathcal{L})$; we shall see later that $(T^+,\mathcal{L}^+)$ also admits the elimination of VF-quantifiers (lemma 5.3.9).

These languages have two types of terms: the VF terms, i.e those terms interpreted in each model as an element of the VF sort, and the RV terms, defined analogously. The VF terms of $\mathcal{L}^+$ which do not include any variables are the same as those of $\mathcal{L}$, namely terms of the form

$$g(c_1,\dots,c_n), \tag{†}$$

with $g\in\mathbb{Z}[x_1,\dots,x_n]$ and $c_1,\dots,c_n$ VF constants of $\mathcal{L}$. The VF terms of $\mathcal{L}^+$ which do include variables (also the same as those of $\mathcal{L}$) are of the form $f(\underline{x})$, where $\underline{x}$ are VF variables and $f$ is a polynomial whose coefficients are all of the form (†). Since they will appear often, we shall call polynomials like $f(\underline{x})$ *$\mathcal{L}$-polynomials* (it would be equally correct to call them $\mathcal{L}^+$-polynomials, but we wish to emphasise that they are already definable in the weaker language $\mathcal{L}$).

Expressions such as 'definable', 'equivalent', etc. mean '$T^+$-definable', '$T^+$-equivalent', etc. unless a prefix is included to indicate otherwise.

**Remark 5.3.1.** Although we are only really interested in subsets of $\mathrm{VF}^n$ for all $n$, it is more convenient to work with subsets of $\mathrm{VF}^n\times\mathrm{RV}^m$ for all $n,m$, not least so that we can follow Hrushovski and Kazhdan as closely as possible.

From the perspective of measure theory, proper Zariski closed sets are negligible, and so the following definition is convenient:

**Definition 5.3.2.** A definable subset $X\subseteq\mathrm{VF}^n\times\mathrm{RV}^m$ will be said to be *$T^+$-null* (or simply *null*) if and only if there is a non-zero $\mathcal{L}$-polynomial $g(\underline{x})$ such that $X\subseteq g^{-1}(0)\times\mathrm{RV}^m$, i.e.

$$T^+\vdash(\underline{x},\underline{y})\in X\to g(\underline{x})=0.$$

Note that this notion depends on the ambient space $\mathrm{VF}^n\times\mathrm{RV}^m$. Definable subsets of $\mathrm{RV}^m$ are not null, unless empty, either by convention or degeneracy of the definition. We will sometimes say 'almost everywhere' to mean 'away from a null set'. We will see in proposition 5.3.12 that a set is null if and only if it has no interior.

**Lemma 5.3.3.** *Any definable subset of a null set is null, and a finite union of null sets is null.*

*Proof.* Obvious. □

### 5.3.1 Syntactical analysis of $T^+$

We begin our syntactical analysis of formulae of $(T^+, \mathcal{L}^+)$ with some simple results:

**Lemma 5.3.4.** *Let $\tau = \tau(\underline{x}, \underline{y})$ be an RV term of $\mathcal{L}^+$, where $\underline{x}$ are VF variables and $\underline{y}$ are RV variables. Then we can write*

$$\tau = \tau'(\mathrm{rv}(f_1(\underline{x})), \dots, \mathrm{rv}(f_s(\underline{x})), \underline{y})$$

*where $\tau'$ is an RV term of $\mathcal{L}^+$ all of whose variables are of the RV sort, and the $f_i$ are non-zero $\mathcal{L}$-polynomials (more precisely, $\tau$ is equivalent to a term of the given form).*

*Proof.* If $\tau$ is atomic, then $\tau$ is either a single RV variable or an RV constant; such expressions are certainly of the required form. Now assume that $\tau$ is not atomic. Then we may write $\tau = g(\tau_1, \dots, \tau_m)$ for a function symbol $g$ and terms $\tau_1, \dots, \tau_m$; note that either $g$ is a function symbol $\mathrm{RV}^l \to \mathrm{RV}$ for some $l \geq 0$, or $g = \mathrm{rv}$, because there are no other function symbols with values in RV.

It now follows by induction on the length of $\tau$ that $\tau = \tau'(\mathrm{rv}(\tau_1(\underline{x})), \dots, \mathrm{rv}(\tau_s(\underline{x})), \underline{y})$, where each $\tau_i$ is a VF term and $\tau'$ is an RV term all of whose variables are of the RV sort. But we observed above that any VF term $\tau_i(\underline{x})$ is an $\mathcal{L}$-polynomial $f_i(\underline{x})$. Moreover, if any of the $f_i$ are identically zero, then $T^+ \vdash \mathrm{rv}(f_i(\underline{x})) = \infty$, so we may replace $\mathrm{rv}(f_i(\underline{x}))$ by the constant $\infty$ and absorb it into $\tau'$. □

**Corollary 5.3.5.** *Let $\tau = \tau(\underline{x}, \underline{y})$ be an RV term of $\mathcal{L}^+$, where $\underline{x}$ are VF variables and $\underline{y}$ are RV variables. Then there is a null set $N \subset \mathrm{VF}^n$ such that for any model $F \models T^+$ and $\underline{a} \in \mathrm{VF}(F)^n \setminus N(F)$, there is an open neighbourhood $U$ of $\underline{a}$ such that for all $\underline{b} \in \mathrm{RV}(F)^m$ and all $\varepsilon \in U$,*

$$\tau(\underline{a}, \underline{b}) = \tau(\underline{a} + \varepsilon, \underline{b}).$$

*i.e. away from a null set, the term $\tau(\underline{x}, \underline{y})$ is locally constant in $\underline{x}$, independently of $\underline{y}$.*

*Proof.* We write $\tau = \tau'(\mathrm{rv}(f_1(\underline{x})), \dots, \mathrm{rv}(f_s(\underline{x})), \underline{y})$ satisfying the conditions of the previous lemma, set $f = f_1 \dots f_s$, and put $N = f^{-1}(0)$, which is a null set of $\mathrm{VF}^n$. Let $F \models T^+$ and $\underline{a} \in \mathrm{VF}(F)^n \setminus N(F)$. Since $\mathrm{rv} : \mathrm{VF}(F)^\times \to \mathrm{RV}(F)^\times$ is continuous with respect to the valuation topology on $\mathrm{VF}(F)^\times$ and discrete topology on $\mathrm{RV}(F)^\times$, there is an open neighbourhood $U$ of $\underline{a}$ on which $f$ does not vanish and on which $\mathrm{rv}\, f_i$ is constant for all $i$. This is exactly what is required. □

The following classification of atomic formulae is absolutely fundamental for the subsequent results:

**Lemma 5.3.6.** *Let $\phi(\underline{x}, \underline{y})$ be an atomic formula of $\mathcal{L}^+$, or the negation of an atomic formula; assume $\phi$ is not a tautology or a contradiction. Then either*

*(i) $\phi$ is $T$-equivalent to a formula '$g(\underline{x}) = 0$' for some non-constant, $\mathcal{L}$-polynomial $g$; or*

*(ii) $\phi$ is $T$-equivalent to a formula '$g(\underline{x}) \neq 0$' for some non-constant, $\mathcal{L}$-polynomial $g$; or*

*(iii) $\phi$ is $T^+$-equivalent to a formula of the form*

$$\phi'(\mathrm{rv}(f_1(\underline{x})), \dots, \mathrm{rv}(f_s(\underline{x})), \underline{y}),$$

*where $\phi'(w_1, \dots, w_s, \underline{y})$ is an atomic formula of $\mathcal{L}^+$ (or the negation of an atomic formula), all of whose variables are of the RV sort, and $f_i$ are non-constant, $\mathcal{L}$-polynomials.*

*Proof.* By the very definition of an atomic formula, $\phi$ is equal to $R(\tau_1, \dots, \tau_r)$ for some relation symbol $R$ and terms $\tau_i$ (or the negation of such an expression).

Case: At least one $\tau_i$ is a VF term. Well, the only relation symbol of $\mathcal{L}^+$ accepting any VF terms is the binary relation of equality $=$, and so $\phi$ is '$\tau_1(\underline{x}) = \tau_2(\underline{x})$' with VF-terms $\tau_1$, $\tau_2$ (or it is the negation of this formula). So $\tau_1$ and $\tau_2$ are both $\mathcal{L}$-polynomials, and we set $g(\underline{x}) = \tau_1(\underline{x}) - \tau_1(\underline{x})$. Then $\phi$ is $T$-equivalent to '$g(\underline{x}) = 0$' (in which case $\phi$ is of type (i)) or to '$g(\underline{x}) \neq 0$' (in which case $\phi$ is of type (ii)).

Case: All the $\tau_i$ are RV terms. Then according to lemma 5.3.4, each $\tau_i$ is $T^+$-equivalent to a term of the form

$$\tau_i'(\mathrm{rv}(f_i^{(1)}(\underline{x})), \dots, \mathrm{rv}(f_i^{s(i)}(\underline{x})), \underline{y})$$

where the $f_i^k$ are non-zero $\mathcal{L}$-polynomials and $\tau_i'$ is an RV term all of whose variables are of the RV sort. It easily follows that $\phi$ is of type (iii), with

$$\phi' = R(\tau_1'(w_1^{(1)}, \dots, w_1^{s(i)}, \underline{y}), \dots, \tau_r'(w_r^{(1)}, \dots, w_r^{s(r)}, \underline{y})).$$

□

**Definition 5.3.7.** Let $\phi(\underline{x}, \underline{y})$ be an atomic formula of $\mathcal{L}^+$, or the negation of an atomic formula; assume $\phi$ is not a tautology or a contradiction. We will say that $\phi$ is *of type (i), (ii), (iii)* according as which of the three cases $\phi$ satisfies in the previous lemma.

**Corollary 5.3.8.** *Let $\phi(\underline{x}, \underline{y})$ be an atomic formula of $\mathcal{L}^+$, or the negation of an atomic formula; assume $\phi$ is not a tautology or a contradiction. If $\phi$ is of type (ii) or (iii) then there is a null set $N$ such that for any model $F \models T^+$ and $\underline{a} \in \mathrm{VF}(F)^n \setminus N(F)$, there is an open neighbourhood $U$ of $\underline{a}$ such that for all $\underline{b} \in \mathrm{RV}(F)^m$ and all $\varepsilon \in U$,*

$$F \models \phi(\underline{a}, \underline{b}) \iff F \models \phi(\underline{a} + \varepsilon, \underline{b}).$$

*Proof.* If $\phi$ is $T$-equivalent to '$g(\underline{x}) \neq 0$' then we may take $N = g^{-1}(0)$. Else $\phi$ is of type (iii), and we proceed exactly as in corollary 5.3.5. □

The following result seems to be well-known among model theorists, but a reference is hard to find, and so for the sake of completeness we present a proof in the same style as our other results. Recall that we are assuming $(T, \mathcal{L})$ has elimination of VF quantifiers.

**Lemma 5.3.9.** *$T^+$ has elimination of VF quantifiers in the language $\mathcal{L}^+$.*

*Proof.* Letting $v$ denote a single VF variable; it is enough to take a formula $\phi(v, \underline{x}, \underline{y})$ of $\mathcal{L}^+$ with no VF quantifiers and to rewrite

$$\exists v\, \phi(v, \underline{x}, \underline{y})$$

without any VF quantifiers.

By (the proof of) prenex normal form, $\phi$ is $T^+$-equivalent to a formula of the form

$$Q_1 z_1 \cdots Q_m z_s \bigvee_i \bigwedge_j \phi_{i,j}(v, \underline{x}, \underline{y}, \underline{z})$$

where $\underline{z}$ are RV variables, each $Q_i$ is $\forall$ or $\exists$, and each $\phi_{i,j}$ is an atomic formula of $\mathcal{L}^+$ or the negation of an atomic formula (and we may clearly assume that each $\phi_{i,j}$ is neither a tautology nor a contradiction, unless $\phi$ itself is one, in which case we are done). Let $I$ denote the set of those $(i,j)$ for which $\phi_{i,j}$ is of type (i) or (ii), and $I'$ those $(i,j)$ for which $\phi_{i,j}$ is of type (iii).

For each $(i,j) \in I$, lemma 5.3.6 implies that $\phi_{i,j}$ is $T$-equivalent to a formula '$g_{i,j}(v,\underline{x}) \diamond_{i,j} 0$', where $\diamond_{i,j}$ is either $=$ or $\neq$, and $g_{i,j}$ is an $\mathcal{L}$-polynomial. We will use $\star$ to denote either 'no symbol' or 'negation', so that $\star\phi_{i,j}$ is either $\phi_{i,j}$ or $\neg\phi_{i,j}$; in fact, for each $(i,j) \in I$, choose conditions $\star = (\star_{i,j})_{(i,j)\in I}$ ($2^{|I|}$ such possibilities), and set

$$\psi_\star(v,\underline{x}) = \bigwedge_{(i,j)\in I} \star_{i,j}\phi_{i,j}.$$

So the sentence

$$\bigvee_\star \psi_\star(v,\underline{x})$$

is a tautology; here $\star$ varies over the $2^{|I|}$ combinations of 'no symbol' or 'negation'.

Now, $\exists v\, \phi(v,\underline{x},\underline{y})$ is $T^+$-equivalent to the sentence

$$\bigvee_\star \exists v\, (\psi_\star(v,\underline{x}) \wedge \phi(v,\underline{x},\underline{y})),$$

and it is therefore enough to eliminate the $v$ quantifier from $\phi_\star :=$ '$\exists v\, (\psi_\star(v,\underline{x}) \wedge \phi(v,\underline{x},\underline{y}))$' for some fixed $\star$ (now fixed for the remainder of the proof); further, since $\psi_\star$ is independent of the variables $\underline{z}$, we have

$$\phi_\star(\underline{x},\underline{y}) \equiv \exists v\, Q_1 z_1 \cdots Q_m z_s \bigvee_i \bigwedge_j (\psi_\star(v,\underline{x}) \wedge \phi_{i,j}(v,\underline{x},\underline{y},\underline{z})). \tag{†}$$

Momentarily fix $(i,j) \in I$. If $\star_{i,j}$ is 'negation', then the formula $\psi_\star \wedge \phi_{i,j}$ is a contradiction. On the other hand, if $\star_{i,j}$ is 'no symbol', then $\psi_\star \wedge \phi_{i,j} \equiv \psi_\star$. Introducing

$$I'' = \{(i,j) \in I' : \text{for all } j_0 \text{ such that } (i,j_0) \in I,\ \star_{i,j_0} \text{ is 'no symbol'}\},$$

it follows that

$$\begin{aligned}\bigvee_i \bigwedge_j (\psi_\star(v,\underline{x}) \wedge \phi_{i,j}(v,\underline{x},\underline{y},\underline{z})) &\equiv \bigvee\bigwedge_{(i,j)\in I''} (\psi_\star(v,\underline{x}) \wedge \phi_{i,j}(v,\underline{x},\underline{y},\underline{z})) \\ &\equiv \psi_\star(v,\underline{x}) \wedge \bigvee\bigwedge_{(i,j)\in I''} \phi_{i,j}(v,\underline{x},\underline{y},\underline{z}).\end{aligned}$$

Therefore $\phi_\star$ is $T^+$-equivalent to

$$\exists v \left( \psi_\star(v,\underline{x}) \wedge Q_1 z_1 \cdots Q_m z_s \bigvee\bigwedge_{(i,j)\in I''} \phi_{i,j}(v,\underline{x},\underline{y},\underline{z}) \right).$$

But now recall that for each $(i,j) \in I''$, $\phi_{i,j}$ is of type (iii) and hence is $T^+$-equivalent to a formula

$$\phi_{i,j} \equiv \phi'_{i,j}(\mathrm{rv}(f^{(1)}_{i,j}(v,\underline{x})), \ldots, \mathrm{rv}(f^{(s(i,j))}_{i,j}(v,\underline{x})), \underline{y}, \underline{z}),$$

where $\phi'(w^{(1)}_{i,j}, \ldots, w^{(s(i,j))}_{i,j}, \underline{y})$ is an atomic formula of $\mathcal{L}^+$ (or the negation of an atomic formula), all of whose variables are of the RV sort, and the $f^k_{i,j}$ are non-constant $\mathcal{L}$-polynomials. It is clear that $\phi_\star$ is $T^+$-equivalent to

$$\begin{aligned}\exists_{\substack{(i,j)\in I'' \\ 1\le k\le s(i,j)}} w^k_{i,j} \Bigg( \exists v & \bigwedge_{\substack{(i,j)\in I'' \\ 1\le k\le s(i,j)}} w^k_{i,j} = \mathrm{rv}(f^{(k)}_{i,j}(v,\underline{x})) \\ & \wedge Q_1 z_1 \ldots Q_s z_s \bigvee\bigwedge_{(i,j)\in I''} \phi'_{i,j}(w^{(1)}_{i,j}, \ldots, w^{(s(i,j))}_{i,j}, \underline{y}, \underline{z}) \Bigg).\end{aligned}$$

Finally,

$$\exists v \bigwedge_{\substack{(i,j)\in I''\\ 1\le k\le s(i,j)}} w_{i,j}^k = \mathrm{rv}(f_{i,j}^{(k)}(v,\underline{x}))$$

is a formula in $\mathcal{L}$, and therefore is $T$-equivalent to a formula without VF-quantifiers, which completes the proof. □

**Remark 5.3.10.** Now that we know $T^+$ has elimination of VF-quantifiers in $\mathcal{L}^+$, the usual proof of prenex normal form implies that any formula $\phi(\underline{x},\underline{y})$ of $\mathcal{L}^+$ is equivalent to one of the form

$$Q_1 z_1,\cdots Q_m z_s \bigvee_i \bigwedge_j \phi_{i,j}(\underline{x},\underline{y},\underline{z})$$

where $\underline{z}$ are RV variables, each $Q_i$ is $\forall$ or $\exists$, and each $\phi_{i,j}$ is an atomic formula of $\mathcal{L}^+$ or the negation of an atomic formula (and we may clearly assume that each $\phi_{i,j}$ is neither a tautology nor a contradiction, unless $\phi$ itself is one).

Applying lemma 5.3.6 on the structure of atomic formulae to the $\phi_{i,j}$ appearing in the remark, we will now begin to derive the promised structural results for definable sets and functions.

**Proposition 5.3.11.** *Let $X \subseteq \mathrm{VF}^n \times \mathrm{RV}^m$ be a $T^+$-definable set; then there exists a quantifier-free, $T$-definable function $h : \mathrm{VF}^n \times \mathrm{RV}^m \to \mathrm{RV}^l$ (some $l \ge 0$) such that $X$ consists of fibres of $h$, i.e. $T^+ \vdash h(\underline{x},\underline{y}) = h(\underline{x}',\underline{y}') \to ((\underline{x},\underline{y}) \in X \leftrightarrow (\underline{x}',\underline{y}') \in X)$.*

*Proof.* We write

$$X = \left\{ (\underline{x},\underline{y}) \in \mathrm{VF}^n \times \mathrm{RV}^m : Q_1 z_1,\cdots Q_m z_s \bigvee_i \bigwedge_j \phi_{i,j}(\underline{x},\underline{y},\underline{z}) \right\}$$

according to the previous remark. Let $I$ denote the set of pairs $(i,j)$ for which $\phi_{i,j}$ is of type (i) or (ii), and $I'$ those $(i,j)$ for which $\phi_{i,j}$ is of type (iii).

For $(i,j) \in I'$, $\phi_{i,j}$ is $T^+$-equivalent to a formula

$$\phi'_{i,j}(\mathrm{rv}(f_{i,j}^{(1)}(\underline{x})),\dots,\mathrm{rv}(f_{i,j}^{s(i,j)}(\underline{x})),\underline{y},\underline{z}),$$

where $\phi'(w_{i,j}^{(1)},\dots,w_{i,j}^{(s(i,j))},\underline{y})$ is an atomic formula of $\mathcal{L}^+$ (or the negation of an atomic formula), all of whose variables are of the RV sort, and the $f_{i,j}^k$ are non-constant $\mathcal{L}$-polynomials. Introduce

$$h_{i,j} : \mathrm{VF}^n \times \mathrm{RV}^m \to \mathrm{RV}^{s(i,j)} \times \mathrm{RV}^m, \quad (\underline{x},\underline{y}) \mapsto (\mathrm{rv}(f_{i,j}^{(1)}(\underline{x})),\dots,\mathrm{rv}(f_{i,j}^{s(i,j)}(\underline{x})),\underline{y}).$$

Then $h_{i,j}$ is quantifier-free, $T$-definable since the same is true of each polynomial $f_{i,j}^k$, and further

$$T^+ \vdash h_{i,j}(\underline{x},\underline{y}) = h_{i,j}(\underline{x}',\underline{y}') \to (\phi_{i,j}(\underline{x},\underline{y},\underline{z}) \leftrightarrow \phi_{i,j}(\underline{x}',\underline{y}',\underline{z}))$$

Secondly, for $(i,j) \in I$, $\phi_{i,j}$ is $T$-equivalent to a formula $g_{i,j}(\underline{x})\diamond_{i,j}0$, where $\diamond_{i,j}$ is either $=$ or $\neq$, and $g_{i,j}$ is an $\mathcal{L}$-polynomial. For each such $(i,j)$, introduce a 'test function'

$$\delta_{i,j} : \mathrm{VF}^n \times \mathrm{RV}^m \to \mathrm{RV}, \quad \underline{x} \mapsto \begin{cases} 1 & \text{if } g_{i,j}(\underline{x}) \diamond_{i,j} 0 \\ \infty & \text{if } g_{i,j}(\underline{x}) \not\diamond_{i,j} 0. \end{cases}$$

Then $\delta_{i,j}$ is quantifier-free $T$-definable, and the formula $\phi_{i,j}(\underline{x}, \underline{y}, \underline{z})$ is $T$-equivalent to '$\delta_{i,j}(\underline{x}) = 1$'.

Finally, set

$$h = \prod_{(i,j)\in I'} h_{i,j} \times \prod_{(i,j)\in I} \delta_{i,j} : \mathrm{VF}^n \times \mathrm{RV}^m \to \prod_{(i,j)\in I} (\mathrm{RV}^{s(i,j)} \times \mathrm{RV}^m) \times \prod_{(i,j)\in I'} \mathrm{RV}\,.$$

Then $h$ is quantifier-free, $T$-definable and evidently satisfies

$$T^+ \vdash h(\underline{x}, \underline{y}) = h(\underline{x}', \underline{y}') \to (\phi_{i,j}(\underline{x}, \underline{y}, \underline{z}) \leftrightarrow \phi_{i,j}(\underline{x}', \underline{y}', \underline{z}))$$

for all $i, j$. This completes the proof ☐

We now reach the topological characterisation of nullity (note that RV carries the discrete topology):

**Proposition 5.3.12.** *Let $X \subseteq \mathrm{VF}^n \times \mathrm{RV}^m$ be definable. Then $X$ is null if and only if it has empty interior.*

*Proof.* The interior of the $X$ is the definable set

$$X^o = \{(\underline{x}, \underline{y}) \in X : \exists \gamma \in \mathrm{RV}^\times\ \forall \varepsilon_1, \dots, \varepsilon_n \in \mathrm{VF}\ ((\bigwedge_i \mathrm{rv}(\varepsilon_i) > \gamma) \to (\underline{x} + \underline{\varepsilon}, \underline{y}) \in X\}.$$

The interior of a null set is certainly empty, so we need only consider the converse assertion. Suppose $X^o = \emptyset$; we may assume that $X \neq \emptyset$.

Write $X$ in prenex normal form

$$X = \left\{ (\underline{x}, \underline{y}) \in \mathrm{VF}^n \times \mathrm{RV}^m : Q_1 z_1, \cdots Q_m z_m \bigvee_i \bigwedge_j \phi_{i,j}(\underline{x}, \underline{y}, \underline{z}) \right\}$$

as in remark 5.3.10. Suppose for a contradiction that every $\phi_{i,j}$ is of type (ii) or (iii). Then corollary 5.3.8 implies that there is a null set $N$ with the following property: for any $F \models T^+$ and $\underline{a} \in \mathrm{VF}(F)^n \setminus N(F)$ there is an open neighbourhood $U$ of $\underline{a}$ such that for all $\underline{b} \in \mathrm{RV}(F)^m$, all $\underline{c} \in \mathrm{RV}(F)^s$, and all $\varepsilon \in U$,

$$F \models \phi_{i,j}(\underline{a}, \underline{b}, \underline{c}) \Leftrightarrow F \models \phi_{i,j}(\underline{a} + \varepsilon, \underline{b}, \underline{c})$$

for all $i, j$, and so

$$(\underline{a}, \underline{b}) \in X(F) \Leftrightarrow (\underline{a} + \varepsilon, \underline{b}) \in X(F).$$

But this implies $X(F)$ is open, contradicting $X^o = \emptyset$. We conclude that at least one $\phi_{i,j}$ is of type (i), i.e. $\phi_{i_0,j_0} \equiv$ '$g(\underline{x}) = 0$', say, for some non-zero $\mathcal{L}$-polynomial $g$.

Now set $X' = X \setminus g^{-1}(0)$; if $X' = \emptyset$ then we are done, so suppose not. Since we have equivalent formulae

$$g(\underline{x}) \neq 0 \wedge \bigvee_i \bigwedge_j \phi_{i,j}(\underline{x}, \underline{y}, \underline{z}) \equiv \bigvee_{i \neq i_0} \left( g(\underline{x}) \neq 0 \wedge \bigwedge_j \phi_{i,j}(\underline{x}, \underline{y}, \underline{z}) \right),$$

we see that $X'$ is defined by

$$X' = \left\{ (\underline{x}, \underline{y}) \in \mathrm{VF}^n \times \mathrm{RV}^m : Q_1 z_1, \cdots Q_m z_m \bigvee_{i \neq i_0} \left( g(\underline{x}) \neq 0 \wedge \bigwedge_j \phi_{i,j}(\underline{x}, \underline{y}, \underline{z}) \right) \right\}.$$

So, by shrinking $X$ to $X'$ we have decreased the number of disjunctions appearing in the prenex normal form, and inserted a new formula of type (ii) into each conjunction.

$X'$ also has empty interior and so by an induction on the number of disjunctions in the prenex formal form, we may assume that it is a null set. Hence $X$ is contained in the null set $X' \cup g^{-1}(0)$ and therefore is itself null. □

Many useful results follow:

**Corollary 5.3.13.** *Let $X \subseteq \mathrm{VF}^n \times \mathrm{RV}^m$ be a definable set. Then the boundary of $X$, namely $\partial X := X \setminus A^o$, is null. Hence $X$ is the disjoint union of an open set and a null set.*

*Proof.* Since $\partial X$ has no interior, this is an immediate consequence of the previous proposition. □

**Corollary 5.3.14.** *Let $X \subseteq \mathrm{VF}^n \times \mathrm{RV}^m$ be definable and let $f : X \to \mathrm{VF}^{n'} \times \mathrm{RV}^{m'}$ be a definable function, with $n' > 0$. Then there are non-zero VF-polynomials $g_1(\underline{x}, x_1'), \dots, g_{n'}(\underline{x}, x_{n'}')$ such that*

$$T^+ \vdash f(\underline{x}, \underline{y}) = (\underline{x}', \underline{y}') \to g_i(\underline{x}, x_i') = 0$$

*for all $i = 1, \dots, n'$.*

*Proof.* The graph of the function $f$ cannot have any interior (since $n' > 0$) and hence the graph is null by the previous proposition; this implies the existence of a non-zero $\mathcal{L}$-polynomial $g$ such that

$$T^+ \vdash f(\underline{x}, \underline{y}) = (\underline{x}', \underline{y}') \to g(\underline{x}, \underline{x}') = 0.$$

Now just apply this result to each function $X \to \mathrm{VF}^{n'} \times \mathrm{RV}^{m'} \stackrel{\text{proj}}{\to} \mathrm{VF} \times \mathrm{RV}^{m'}$, where the second arrow varies over the $n'$ projection maps. □

**Corollary 5.3.15.** *The sorts VF and RV are 'orthogonal' in the following ways:*

*(i) Let $Y \subseteq \mathrm{RV}^m$ be definable and let $f : Y \to \mathrm{VF}^n$ be a definable function. Then $f(Y)$ is a finite set.*

*(ii) Suppose that a definable set $X \subseteq \mathrm{VF}^n$ admits a finite-to-one, definable map $f : X \to \mathrm{RV}^l$ for some $l \geq 0$. Then $X$ is finite.*

*Proof.* (i): By the previous corollary, there are non-zero $\mathcal{L}$-polynomials $g_1(x_1), \dots, g_n(x_n)$ such that $f(Y) \subseteq \{\underline{x} \in \mathrm{VF}^n : g_i(x_i) = 0 \text{ for all i}\}$; this is enough.

(ii): Let $\Gamma$ be the graph of $f$. Then $\Gamma$ cannot have any interior, for else there would be an open ball $B \subseteq X$ and $\underline{y} \in f(X)$ such that $B \times \{\underline{y}\} \subseteq \Gamma$, contradicting that $f$ has finite fibres. By the previous proposition, there is a non-zero $\mathcal{L}$-polynomial $g$ such that $\underline{x} \in X$ implies $g(\underline{x}) = 0$.

In fact, for each $i = 1, \dots, n$, let $\Gamma_i$ be the image of $\Gamma$ under the projection

$$(\text{projection to } i^{\text{th}} \text{ VF-coordinate}) : \mathrm{VF}^n \times \mathrm{RV}^l \to \mathrm{VF} \times \mathrm{RV}^l .$$

Although $\Gamma_i$ is not necessarily the graph of a function, each section

$$\{x \in \mathrm{VF} : (x, \underline{y}) \in \Gamma_i\},$$

for $\underline{y} \in \mathrm{RV}^l$, is still finite and therefore has no interior. So, just as in the previous paragraph, there is a non-zero, one variable, $\mathcal{L}$-polynomial $g_i$ such that $\underline{x} \in X$ implies $g_i(x_i) = 0$. As this holds for all $i$, we deduce that $X$ is a finite set. □

**Corollary 5.3.16.** *Let $X \subseteq \mathrm{VF}^n \times \mathrm{RV}^m$, $Y \subseteq \mathrm{VF}^{n'} \times \mathrm{RV}^{m'}$, and $f : X \to Y$ be definable. Then*

*(i) the set of $\underline{y} \in Y$ for which the fibre $X_y$ is null is a definable set;*

*(ii) if $n' = 0$, i.e. $Y \subseteq \mathrm{RV}^{m'}$, then $X$ is null if and only if every fibre $X_y$, for $y \in Y$, is null.*

*Proof.* (i): The set of $\underline{y} \in Y$ for which the fibre $X_y$ has non-empty interior is clearly definable; by the previous proposition we are done.

(ii): The implication $\Rightarrow$ follows at once from the fact that a subset of a null set is null. Conversely, suppose that every fibre is null. Then, arguing similarly to the proof of (ii) in the previous corollary, we see that the graph $\Gamma$ of $f$ contains no interior, and hence is null. So there is a non-zero VF-polynomial $g(\underline{x})$ such that $f(\underline{x}) = \underline{y}$ implies $g(\underline{x}) = 0$; i.e. $X \subseteq g^{-1}(0)$, as required. ☐

### 5.3.2 Structure of definable functions

We now turn our attention to definable functions. Our aim is to show that the class of definable functions is not too large, and that any such function, at least off a null set, is essentially of the following form:

**Definition 5.3.17.** Let $U \subseteq \mathrm{VF}^n$ be a definable, non-empty open set, and $f : U \to \mathrm{VF}^m$ a definable function. Then we shall say that $f$ is an *implicit polynomial function* if and only if there are non-zero VF-polynomials $g_1(\underline{x}, \underline{y}), \dots, g_m(\underline{x}, \underline{y})$ and an open set $V \subseteq \mathrm{VF}^{n'}$ with the following properties:

(i) for all $\underline{x} \in U$, $g_i(\underline{x}, f(\underline{x})) = 0$ for all $i$;

(ii) for all $\underline{x} \in U$, $f(\underline{x})$ is the unique $\underline{y} \in V$ satisfying $g_i(\underline{x}, \underline{y})$ for all $i$;

(iii) the determinant of the Jacobian matrix

$$\left(\frac{\partial g_i}{\partial y_j}\right)_{1 \leq i,j \leq m}$$

is non-zero at $(\underline{x}, f(\underline{x}))$, for all $\underline{x} \in U$.

In other words, $f$ is the implicit function defined by the polynomials $(g_i)_i$ on $U \times V$.

Before we can prove our desired classification, we must discuss some ideas of classical elimination theory which are closely related to elimination of quantifiers. See [Lan02, IV, §8].

Let $k$ be a field of characteristic zero, and $g(y)$ a polynomial in $k[y]$. The discriminant $D_g \in k$ of $g$ is obtained by evaluating a certain polynomial with integer coefficients on the coefficients of $g$; also $D_g$ vanishes if and only if $g$ and $g'$ have a common zero in $k^{\mathrm{alg}}$.

Now replace $k$ by $k(x_1, \dots, x_n)$ and suppose that $g \in k[\underline{x}, y]$. Then the discriminant $D_g$ belongs to $k[\underline{x}]$. We may factor $g$ into a product of non-associated, irreducible polynomials in $k[\underline{x}, y]$ as

$$g = g_1^{n_1} \dots g_s^{n_s}$$

and we henceforth assume that

(i) none of the irreducible factors of $g$ belong to $k[\underline{x}]$;

(ii) $g$ has no multiple factors, i.e. $n_i = 1$ for all $i$.

The first assumption implies that each $g_i$ remains irreducible when viewed as a polynomial in $k(\underline{x})[y]$, and therefore $g_1 \dots g_s$ is the decomposition of $g$ into irreducible factors in $k(\underline{x})[y]$. By assumption (ii), $g$ therefore has no repeated roots in $k(\underline{x})^{\mathrm{alg}}$ (this is where we use the characteristic zero assumption), and so $D_g(\underline{x})$ is not the zero polynomial. Further, the zeros $\underline{\xi}$ of $D_g(\underline{x})$ are exactly those $\underline{\xi}$ for which $g(\underline{\xi}, y)$ and $\frac{\partial g}{\partial y}(\underline{\xi}, y)$ have a common zero in $k^{\mathrm{alg}}$. In other words,

$$\{\underline{x} : \exists y \text{ such that } g(\underline{x}, y) = \frac{\partial g}{\partial y}(\underline{x}, y) = 0\} = \{\underline{x} : D_g(\underline{x}) = 0\},$$

and our assumptions imply that this is a proper Zariski closed set.

**Proposition 5.3.18.** *Let $X \subseteq \mathrm{VF}^n$ be definable, and $f : X \to \mathrm{VF}^m$ a definable function. Then there exist finitely many disjoint open sets $X_r \subseteq X$ such that $f|_{X_r}$ is an implicit polynomial function for each $r$, and such that $X \setminus \bigsqcup_r X_r$ is null.*

*Proof.* According to corollary 5.3.14, there are non-zero VF-polynomials $g_1(\underline{x}, y_1), \dots, g_m(\underline{x}, y_m)$ such that

$$g_i(\underline{x}, f(\underline{x})_i) = 0 \text{ for all } \underline{x} \in X \tag{†}$$

for all $i$. We may decompose each $g_i$ into a product of irreducible VF-polynomials in $k[\underline{x}, y_i]$, where $k$ is the field of fractions of the constant VF-terms; i.e.

$$g_i = g_{i,1}^{n(i,1)} \dots g_{i,s(i)}^{n(i,s(i))}.$$

If any $g_i$ is divisible by a non-zero polynomial in $k[\underline{x}]$, then (†) implies that $g_i(\underline{x}) = 0$ for all $\underline{x} \in X$, so that $X$ is a null set and there is nothing more to show. Further, we may replace each exponent $n(i, j)$ by 1 without affecting (†).

In conclusion, we may now suppose that the VF-polynomials $g_1, \dots, g_m$ satisfying (†) also satisfy (i) and (ii) above.

The associated Jacobian matrix

$$\left(\frac{\partial g_i}{\partial y_j}\right)_{1 \le i,j \le m} = \begin{pmatrix} \frac{\partial g_1}{\partial y_1} & & \\ & \ddots & \\ & & \frac{\partial g_m}{\partial y_m} \end{pmatrix}$$

is diagonal, and each $\frac{\partial g_i}{\partial y_i}$ is not the zero polynomial, for else $g_i$ would be a polynomial in $\underline{x}$. Let $J(\underline{x}, \underline{y}) = \prod_{i=1}^{n'} \frac{\partial g_i}{\partial y_i}$ be the determinant of the Jacobian.

Set

$$N_i = \left\{\underline{x} \in \mathrm{VF}^n : \exists y \in \mathrm{VF} \text{ such that } g_i(\underline{x}, y) = \frac{\partial g_i}{\partial y_i}(\underline{x}, y) = 0\right\}.$$

By the elimination theory discussed above, $N_i$ is a null set; set $N = \bigcup_{i=1}^m N_i$. The importance of $N$ is that if $F \models T$ and $\underline{a} \in X(F) \setminus N(F)$, then each $g_i(\underline{a}, y_i)$ is not the zero polynomial in $y_i$, and hence it has only finitely many solutions; therefore there are only finitely many $\underline{y}$ for which

$$g_1(\underline{a}, y_1) = \dots = g_m(\underline{x}, y_m) = 0.$$

Continuing with this fixed model $F$ and $a \in X(F) \setminus N(F)$, the usual arguments used in the implicit function theorem for a (usually complete) valued field imply the following: there is an open neighbourhood $U$ of $\underline{a}$, and disjoint open $V_1, \dots, V_l$, such that if $\underline{x} \in U$ then, for each $r = 1, \dots, l$, there is at most one $\underline{y} \in V_r$ which satisfies

$g_i(\underline{x}, y_i) = 0$ for all $i$. $U$ and each $V_r$ are defined in terms of $\underline{a}$ and the coefficients of each $g_i$; hence they are $T^+_a$-definable. However, since the conditions we wish for them to satisfy are expressible without $\underline{a}$, we may, possibly after shrinking them, assume they are $T^+$-definable.

For each $r = 1, \dots, l$ introduce $U_r = \{\underline{x} \in U : f(\underline{x}) \in V_r\}$; the $(U_r)_r$ are $T^+$-definable and form a disjoint cover of the open set $U$. By construction, the restriction of $f$ to the restriction of the interior of each $U_r$ is an implicit polynomial function (even on all of $U_r$, but we have only defined implicit polynomial functions on open sets). Thus we obtain a definable decomposition of $U$ into a disjoint union of null sets (since the boundary of each $U_r$ is null by corollary 5.3.13) and open sets, such that the restriction of $f$ to each of the opens is an implicit polynomial function.

Apply compactness to complete the proof. □

**Corollary 5.3.19.** *Let $X \subseteq \mathrm{VF}^n$ be definable, and $f : X \to \mathrm{VF}^m$ a definable function. Then, away from a null set, $f$ is smooth (i.e. infinity differentiable).*

*Proof.* This follows from the previous proposition, since the usual calculations from analysis show that an implicit polynomial function is smooth. □

**Corollary 5.3.20.** *Suppose that $X \subseteq \mathrm{VF}^n \times \mathrm{RV}^m$ and $Y \subseteq \mathrm{VF}^n \times \mathrm{RV}^{m'}$ are definably isomorphic sets. Then $X$ is null if and only if $Y$ is null.*

*Proof.* Using similar arguments to those found in corollary 5.3.16, this may be reduced to the case of $X \subseteq \mathrm{VF}^n$, $Y \subseteq \mathrm{VF}^{n'}$. If $X$ is not null then the previous corollary implies that $f$ is a smooth injection on some non-empty, open ball $B \subseteq \mathrm{VF}^n$. Familiar estimates from analysis imply that $f(B)$ is open in $\mathrm{VF}^{n'}$ and therefore $Y$ has interior; so $Y$ is not null. □

### 5.3.3 Dimension theory

There is a very satisfactory dimension theory for $T^+$; as we shall not require it, we content ourselves with a summary. For more information see [HK06, §3.8] and [vdD89].

**Definition 5.3.21.** Let $X$ be a $T^+$-definable subset of $\mathrm{VF}^n \times \mathrm{RV}^m$. The $T^+$-dimension (or simply dimension) of $X$, denoted $\dim_{T^+} X$, is the smallest integer $d$ such that for some $l \geq 0$ there is a finite-to-one, definable map $X \to \mathrm{VF}^d \times \mathrm{RV}^{m'}$.

**Remark 5.3.22.** Hrushovski and Kazhdan call this the VF-dimension, since they also introduce an RV-dimension; we have no need of the latter.

**Lemma 5.3.23.** *Let $f : X \to Y$ be a definable surjection between definable sets $X \subseteq \mathrm{VF}^n \times \mathrm{RV}^m$, $Y \subseteq \mathrm{VF}^{n'} \times \mathrm{RV}^{m'}$. Suppose that for each $F \models T^+$ and $b \in Y(F)$, the fibre $X_b = f^{-1}(b)$ has $T^+_b$-dimension $\leq d$; then $\dim_{T^+} X \leq d + \dim_{T^+} Y$.*

*If $\dim_{T^+} Y = 0$, then $\dim_{T^+} X = \max_{F,b} \dim_{T^+_b} X_b$, where $F$ ranges over all models of $T^+$ and $b \in Y(F)$.*

*Proof.* A 'fibre and compactness' argument lets us construct a $T^+$-definable map $g : X \to \mathrm{VF}^d \times \mathrm{RV}^l$ (for some $l \geq 0$) such that the restriction of $g$ to each fibre of $f$ is finite-to-one. Hence $f \times g$ is finite-to-one and the first claim follows.

The second claim is now immediate, since each fibre certainly has dimension no greater than that of $X$. □

According to corollary 5.3.15, a subset of $\mathrm{VF}^n$ with zero dimension is necessarily finite. Using this, and the previous lemma for an induction using fibrations, one can prove that the dimension of a definable set $X \subseteq \mathrm{VF}^n$ is equal to the Zariski dimension of its Zariski closure. Moreover, $X$ is null if and only if its dimension is $< n$.

## 5.4 V-minimality

Hrushovski and Kazhdan introduce a condition called V-minimality, which a theory of valued fields may or may not satisfy. This notion only concerns us in that we must ensure that our theories of interest possess it; for much more information, see [HK06, §3].

**Definition 5.4.1.** Let $(T, \mathcal{L})$ be an extension of $(\mathrm{ACVF}, \mathcal{L}_{\mathrm{RV}})$. Then $T$ is said to be *C-minimal* if and only if for every $F \models T$ and every $T_F$-definable set $X \subseteq \mathrm{VF}$, the set $X(F)$ is a finite Boolean combination of open balls, closed balls, and points.

Further, $T$ is said to be *V-minimal* if and only if it is C-minimal and satisfies the following conditions:

(i) $T$ extends $\mathrm{ACVF}(0,0)$ and every parametrically $T$-definable relation on RV is already parametrically definable in $\mathrm{ACVF}(0,0)$;

(ii) if $F \models T$ then $\mathrm{VF}(F)$ is 'definably complete';

(iii) if $F \models T$, $A \subseteq \mathrm{VF}(F)$, and $B$ is an almost $T_A$-definable closed ball, then $B$ contains a $T_A$-algebraic point.

Finally, $T$ is said to be *effective* if and only if every finite, disjoint union of balls contains a definable set which has exactly one point in each ball.

The following summarises everything we need to know:

**Proposition 5.4.2.** $(\mathrm{ACVF}(0,0), \mathcal{L}_{RV})$ *is V-minimal and effective. If $T$ is V-minimal (resp. V-minimal and effective), $F \models T$, and $A \subseteq \mathrm{VF}(F) \sqcup \mathrm{RV}(F)$, then $T_A$ is also V-minimal (resp. V-minimal and effective).*

*Proof.* V-minimality of $\mathrm{ACVF}(0,0)$ essentially follows from well-known properties of the theory; see [HK06, Lemma 3.33] and also [Hol97]. The preservation of V-minimality and effectivity under base change is discussed in [HK06, 6.0.1]. ☐

## 5.5 Descent to RV

Now we describe the main result of [HK06, §4] and then extend it to a wider class of valued fields. We work with a theory $(T, \mathcal{L})$ of valued fields formulated in a parameter- and RV-expansion of the language $\mathcal{L}_{\mathrm{RV}}$, and we assume that it has elimination of VF-quantifiers; so both of the theories $T$ and $T^+$ which appeared in section 5.3 are valid, and the results we derived for $T^+$ in that section apply to $T$ in this section. There are three possibilities for $(T, \mathcal{L})$ which interest us:

(i) parameter-expansions of $(\mathrm{ACVF}(0,0), \mathcal{L}_{\mathrm{RV}})$;

(ii) parameter-expansions of $(\mathrm{H}(0,0), \mathcal{L}_{\mathrm{RV}})$, the theory of Henselian valued fields of residue characteristic zero;

(iii) RV-expansions of (ii).

We will treat each case in turn; 'definable', etc. means '$T$-definable'.

**Definition 5.5.1.** Suppose that $Y$ is a definable subset of $\mathrm{RV}^m$ and that $\pi : Y \to \mathrm{RV}^n$ is a definable map; to this data we associate the definable subset of $\mathrm{VF}^n \times \mathrm{RV}^m$

$$\mathbb{L}(Y, \pi) = \{(\underline{x}, \underline{y}) \in \mathrm{VF}^n \times Y : \mathrm{rv}(\underline{x}) = \pi(\underline{y})\},$$

and call it the *lift* of $Y, f$.

**Definition 5.5.2.** Fix $n \geq 0$. A *($T$-) elementary admissible transformation* is, for any $m \geq 0$ a definable map $\mathrm{VF}^n \times \mathrm{RV}^m \to \mathrm{VF}^n \times \mathrm{RV}^m$ of the form

$$(\underline{x}, \underline{y}) \mapsto (x_1, \dots, x_{i-1}, x_i + a(x_1, \dots, x_{i-1}, \underline{y}), x_{i+1}, \dots, x_n, \underline{y}),$$

where $a : \mathrm{VF}^{i-1} \times \mathrm{RV}^m \to \mathrm{VF}$ is some definable map. We also call the map

$$\mathrm{VF}^n \times \mathrm{RV}^m \to \mathrm{VF}^n \times \mathrm{RV}^{m+1}, \quad (\underline{x}, \underline{y}) \mapsto (\underline{x}, \underline{y}, \mathrm{rv}(x_i))$$

for any $m \geq 0$ and $1 \leq i \leq n$ an elementary admissible map.

A *($T$-) admissible transformation* is any composition of elementary admissible transformations

$$\mathrm{VF}^n \times \mathrm{RV}^m \to \mathrm{VF}^n \times \mathrm{RV}^{m'}$$

(necessarily $m' \geq m$); note that such a map is injective.

Admissible transformations are also 'measure-preserving', in the following sense:

**Remark 5.5.3.** Before the lemma we should say a word about differentiation. Suppose that $X \subseteq \mathrm{VF}^n$ and $f : X \to \mathrm{VF}^{n'}$ are definable. Then the partial derivatives of $f$, if they exist, are definable (and the set on which they exist is definable). Since they will typically only exist away from a null set anyway, it is sensible only to consider their existence on the interior of $X$ (recall that the boundary is null by corollary 5.3.13) so that there are no issues with forming $f(a + \varepsilon)$ for small $\varepsilon$. If all the partial derivatives exist, then we say that the Jacobian matrix exists. Corollary 5.3.19 implies that the Jacobian does exist away from a a null set.

**Lemma 5.5.4.** *Let $f : \mathrm{VF}^n \to \mathrm{VF}^n$ be the composition of an admissible transformation $\mathrm{VF}^n \to \mathrm{VF}^n \times \mathrm{RV}^{m'}$ followed by the projection map $\mathrm{VF}^n \times \mathrm{RV}^{m'} \overset{proj}{\to} \mathrm{VF}^n$. Then, away from a null set of $\mathrm{VF}^n$, the Jacobian matrix of $f$ exists and has determinant $= 1$.*

*Proof.* By adding extra variables and arguing by induction, it is essentially enough to suppose that $f$ is given by

$$f : \mathrm{VF}^2 \to \mathrm{VF}^2, \quad (x_1, x_2) \mapsto (x_1, x_2 + a(x_1, \mathrm{rv}(x_2))),$$

for some definable function $a : \mathrm{VF} \times \mathrm{RV} \to \mathrm{VF}$; write $f = (f_1, f_2)$. Firstly, $\frac{\partial f_1}{\partial x_1} \equiv 1$ and $\frac{\partial f_1}{\partial x_2} \equiv 0$. Further, away from $x_2 = 0$, the function $x_2 \mapsto a(x_1, \mathrm{rv}(x_2))$ is locally constant and so $\frac{\partial f_2}{\partial x_2}(x_1, x_2) = 1$. It remains only to consider $\frac{\partial f_2}{\partial x_1}$.

Let $F \models T$ and take $b \in \mathrm{RV}(F)$. According to corollary 5.3.19, there is a $T_b$-definable null set $N_b \subseteq \mathrm{VF}$ such that $x \mapsto a(x, b)$ is differentiable for $x \notin N_b$. Then $N_b$ is the zero set of a polynomial with coefficients which are $T_b$-definable, constant VF terms; but adding $b$ to the language does not increase the constant VF terms, and so $N_b$ is already $T$-definable. It follows that there is a $T$-definable set $A \subseteq \mathrm{RV}$ such that $b \in A(F)$ and

$$\mathrm{rv}(x_2) \notin A \Longrightarrow x_1 \mapsto a(x_1, \mathrm{rv}(x_2)) \text{ is differentiable for } x_1 \notin N_b.$$

It follows by compactness that there is a null set $N \subseteq \mathrm{VF}$ such that, for any $x_2 \in \mathrm{VF}$, $x_1 \mapsto a(x_1, \mathrm{rv}(x_2))$ is differentiable for $x_1 \notin N$; but then $\frac{\partial f_2}{\partial x_1}(x_1, x_2)$ exists. This completes the proof that the Jacobian exists off a null set.

Finally, off this null set, the Jacobian is a triangular matrix with 1s on the diagonal; hence its determinant is 1. ☐

Hrushovski and Kazhdan's main decomposition result is as follows; recall that any parameter extension of $(\mathrm{ACVF}(0,0), \mathcal{L}_{\mathrm{RV}})$ is V-minimal, by proposition 5.4.2.

**Proposition 5.5.5.** *Suppose that $(T, \mathcal{L})$ is V-minimal, and fix $n \geq 0$. Let $X$ be a definable subset of $\mathrm{VF}^n \times \mathrm{RV}^m$. Then $X$ is a finite disjoint union of definable sets, $X = \bigsqcup_{i=1}^s X_i$, each of the following form: there are a definable set $Y_i \subseteq \mathrm{RV}^{m_i}$ (some $m_i \geq 0$), a generalised projection (see below) $\pi_i : Y_i \to \mathrm{RV}^n$, and an admissible transformation $\tau_i : \mathrm{VF}^n \times \mathrm{RV}^m \to \mathrm{VF}^n \times \mathrm{RV}^{m_i}$ such that*

$$\tau_i(X_i) = \mathbb{L}(Y_i, \pi_i).$$

*Moreover, if the projection map $X \to \mathrm{VF}^n$ is finite-to-one, then each $\pi_i$ is finite-to-one; if $X$ is bounded, then each $Y_i$ is bounded.*

*Proof.* This is the content of [HK06, §4]. ☐

By 'generalised projection' in the statement of the previous proposition, we mean the restriction to $Y_i$ of a map $\mathrm{RV}^{m_i} \to \mathrm{RV}^n$ of the form $\underline{y} \mapsto (y_{\sigma(1)}, \dots, y_{\sigma(n)})$, for some $\sigma : \{1, \dots, n\} \to \{1, \dots, m_i\}$, e.g. $(y_1, y_2, y_3, y_4) \mapsto (y_3, y_1, y_1)$.

Following ideas found in [HK06, §12.4], our immediate aim now is to extend their decomposition result from algebraically closed valued fields to Henselian ones.

**Lemma 5.5.6.** *Suppose that $F \models \mathrm{ACVF}(0,0)$ (in the language $\mathcal{L}_{RV}$), and that $F_0$ is a subfield of $\mathrm{VF}(F)$ which is Henselian under the restriction of the valuation. Then the $\mathcal{L}_{RV}$ structure $(F_0, \mathrm{RV}(F_0))$ is definably closed in $F$. In particular, any $\mathrm{ACVF}(0,0)_{F_0}$-definable function preserves $F_0$-points.*

*Proof.* We follow [HK06, Example 12.8]. Since $F_0^{\mathrm{alg}}$ is an elementary submodel of $F$, we may replace $F$ by $F_0^{\mathrm{alg}}$. Let $\mathrm{Aut}(F/F_0)$ denote the automorphisms of the $\mathcal{L}_{\mathrm{RV}}$ structure $F$ which fix the substructure $(F_0, \mathrm{RV}(F_0))$. By the Henselian property of $F_0$, any field automorphism of $\mathrm{VF}(F)/\mathrm{VF}(F_0)$ automatically preserves the valuation and therefore belongs to $\mathrm{Aut}(F/F_0)$. By Galois theory, $\mathrm{VF}(F_0)$ is therefore the VF sort of the fixed substructure of $\mathrm{Aut}(F/F_0)$, and so $\mathrm{VF}(F_0) \supseteq \mathrm{VF}(\mathrm{dcl}(F_0))$; hence $\mathrm{VF}(F_0) = \mathrm{VF}(\mathrm{dcl}(F_0))$.

Secondly, suppose that $y \in \mathrm{RV}(\mathrm{dcl}(F_0))$. Then $\mathrm{rv}^{-1}(y)$ is an $\mathrm{ACVF}(0,0)_{F_0}$-definable closed ball of $F$; but since $\mathrm{ACVF}(0,0)_{F_0}$ is V-minimal (by proposition 5.4.2), this ball contains a $\mathrm{ACVF}(0,0)_{F_0}$-definable point $x$. We have just proved that this means $x \in \mathrm{VF}(F_0)$, and therefore $y = \mathrm{rv}(x) \in \mathrm{RV}(F_0)$, as required. ☐

This is enough to pass from the V-minimal case to the Henselian case:

**Proposition 5.5.7.** *Suppose that $(T, \mathcal{L})$ is a parameter-expansion of $(H(0,0), \mathcal{L}_{RV})$. Then proposition 5.5.5 continues to hold if the V-minimal theory is replaced by $T$, so long as $X$ is quantifier-free definable; further, each $X_i, Y_i, \tau_i$ may be assumed to be quantifier-free definable.*

*Proof.* We begin a few general remarks on the relation between the theories $\mathrm{H}(0,0)$ and $\mathrm{ACVF}(0,0)$.

By the hypothesis, there is a Henselian field $F$ and $A \subseteq \mathrm{VF}(F) \bigsqcup \mathrm{RV}(F)$ such that $T = \mathrm{H}(0,0)_A$. The valuation on $F$ extends uniquely to $F^{\mathrm{alg}}$, making $F^{\mathrm{alg}}$ into a model of $\mathrm{ACVF}(0,0)$, since $\mathrm{H}(0,0)$ and $\mathrm{ACVF}(0,0)$ are formulated in the same language. Thus we may add the parameters $A$ to $\mathrm{ACVF}(0,0)$ to obtain the theory $\mathrm{ACVF}(0,0)_A$, so that if $L \models T$ then $L^{\mathrm{alg}} \models \mathrm{ACVF}(0,0)_A$.

If $X \subseteq \mathrm{VF}^n \times \mathrm{RV}^m$ is a $T$-definable set, then let $X^{\mathrm{alg}}$ denote the $\mathrm{ACVF}(0,0)_A$-definable set given by the same formula. Assuming that $X$ is defined without quantifiers, then

$$X^{\mathrm{alg}}(L^{\mathrm{alg}}) \cap (\mathrm{VF}(L)^n \times \mathrm{RV}(L)^m) = X(L)$$

for any $L \models T$. Conversely, if $S \subseteq \mathrm{VF}^n \times \mathrm{RV}^m$ is an $\mathrm{ACVF}(0,0)_A$-definable set, then we may assume that $S$ is defined by a formula in $\mathcal{L}_{\mathrm{RV},A}$ without quantifiers and let

$S_{\mathrm{Hens}}$ be the $T$-definable set defined by the same formula; then $S_{\mathrm{Hens}}$ is defined without quantifiers and $(S_{\mathrm{Hens}})^{\mathrm{alg}} = S$.

We now begin the proof. By proposition 5.5.5, we may decompose $X^{\mathrm{alg}}$ as $X^{\mathrm{alg}} = \bigsqcup_{i=1}^{s} X_i^{\mathrm{alg}}$, with $Y_i, \pi_i, \tau_i$ as described in that proposition. By the previous lemma, the $\tau_i$ restrict to $T$-definable admissible transformations, and by the previous paragraph the $Y_i$ may be restricted to give the required $T$-definable sets $(Y_i)_{\mathrm{Hens}}$. In short, everything restricts from $L^{\mathrm{alg}}$ to $L$. $\square$

**Remark 5.5.8.** Since our main aim is to develop a theory of integration, it is quite reasonable to restrict to quantifier-free definable sets. Indeed, the projection (i.e. insertion of a existential quantifier) of a Lebesgue measurable (resp. Borel) subset of $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ can be extremely unpleasant, and certainly need not be Lebesgue measurable (resp. Borel); though, in fact, the projection of a Borel will be Lebesgue measurable. The study of such problems leads to the theory of analytic sets and Polish spaces; see e.g. [Chr74].

Having restricted to the case of a Henselian field, a standard 'fibre and compactness' argument lets us add additional structure at the RV level, following an outline in [HK06, §12.1]. We abuse notation slightly by talking of the Jacobian of maps $\mathrm{VF}^n \to \mathrm{VF}^n \times \mathrm{RV}^m$; this really means the Jacobian of the composition $\mathrm{VF}^n \to \mathrm{VF}^n \times \mathrm{RV}^m \stackrel{\mathrm{proj}}{\to} \mathrm{VF}^n$.

**Theorem 5.5.9.** *Let $(T, \mathcal{L})$ be as in the previous proposition, and let $(T^+, \mathcal{L}^+)$ be an RV-expansion of $(T, \mathcal{L})$. Then proposition 5.5.5 holds for $T^+$ in place of the V-minimal theory, as long as $X$ is quantifier-free $T^+$-definable.*

*Proof.* For simplicity, we are actually going to prove the following slightly weaker result (the full result can be proved using similar arguments):

*Let $X \subseteq \mathrm{VF}^n$ be $T^+$-definable; then $X$ can be written as a disjoint union, $X = \bigsqcup_{i=1}^{s} X_i$ of $T^+$-definable sets, each of the following form: there are a $T^+$-definable $Y_i \subseteq \mathrm{RV}^{m_i}$, a generalised projection $\pi_i : \mathrm{RV}^{m_i} \to \mathrm{RV}^n$, and a $T^+$-definable bijection $\tau_i : X_i \to \mathbb{L}(Y_i, \pi_i)$ with Jacobian $= 1$ off a null set.*

First recall proposition 5.3.11: there is a quantifier-free, $T$-definable map $h : \mathrm{VF}^n \to \mathrm{RV}^l$ for some $l \geq 0$ such that $X$ consists of fibres of $h$.

Let $F \models T^+$ and $b \in h(X)(F)$. Then the fibre $X_b = h^{-1}(b)$ is quantifier-free, $T_b$-definable, and so, by the previous proposition, it is a disjoint union, $X_b = \bigsqcup_{i=1}^{s} X_i$ of $T_b$-definable sets, each of the following form: there are a $T_b$-definable set $Y_i \subseteq \mathrm{RV}^{m_i}$, a generalised projection $\pi_i : Y_i \to \mathrm{RV}^n$, and a $T_b$-definable bijection $\tau_i : X_i \to \mathbb{L}(Y_i, \pi_i)$ such that the Jacobian of the composition $X_i \stackrel{\tau_i}{\to} \mathbb{L}(Y_i, \pi_i) \stackrel{\mathrm{proj}}{\to} \mathrm{VF}^n$ equals 1 away from a null set (because we saw in lemma 5.5.4 that this is true for any admissible transformation).

Fix some $i$. In the usual way, $\tau_i$ extends to a $T^+$-definable map $\check{\tau}_i : U \to \mathrm{VF}^n \times \mathrm{RV}^{m_i}$, where $U$ is some $T^+$-definable subset of $X$ which contains $X_i$. Possibly after shrinking $U$, we may also suppose, for each $y \in h(U)$, that the following hold:

(i) the restriction of $\check{\tau}_i$ to the fibre $h^{-1}(y) \cap U$ is injective and has Jacobian equal to 1 off a null set;

(ii) the image $\check{\tau}_i(h^{-1}(y) \cap U)$ is of the form $\mathbb{L}(Y, \pi_i)$ for some $T^+$-definable $Y \subseteq \mathrm{RV}^{m_i}$; here $\pi_i$ is the generalised projection associated to $Y_i$, but we view it as defined on all of $\mathrm{RV}^{m_i}$; note that this condition is definable because necessarily $Y$ is the image of $\check{\tau}_i(h^{-1}(y) \cap U)$ under $\mathrm{VF}^n \times \mathrm{RV}^{m_i} \stackrel{\text{proj}}{\to} \mathrm{RV}^{m_i}$.

We now consider

$$\widetilde{\tau}_i := \check{\tau}_i \times h : U \to \mathrm{VF}^n \times \mathrm{RV}^{m_i} \times \mathrm{RV}^l,$$

which is certainly $T^+$-definable and injective; moreover, using corollary 5.3.16, we see that the Jacobian equalling $1$ off a null set on any fixed fibre $h^{-1}(y) \cap U$ is enough to imply that it $= 1$ off a null set on all of $U$. Let $Y$ be the image of $\widetilde{\tau}_i(U)$ under $\mathrm{VF}^n \times \mathrm{RV}^{m_i+l} \stackrel{\text{proj}}{\to} \mathrm{RV}^{m_i+l}$, and let $\widetilde{\pi}$ denote the generalised projection given by the composition

$$\mathrm{RV}^{m_i+l} \stackrel{\text{proj}}{\to} \mathrm{RV}^{m_i} \stackrel{\pi_i}{\to} \mathrm{RV}^n .$$

Since each $\check{\tau}_i(h^{-1}(y) \cap U)$ is a lift, it is easy to check that

$$\widetilde{\tau}(h^{-1}(U)) = \mathbb{L}(Y, \widetilde{\pi}).$$

In fact, even more is true:

(†) If $V$ is any definable subset of $h(U)$, then $\widetilde{\tau}(h^{-1}(V) \cap U) = \mathbb{L}(Y', \widetilde{\pi})$, where $Y'$ is the image of $\widetilde{\tau}_i(h^{-1}(V) \cap U)$ under $\mathrm{VF}^n \times \mathrm{RV}^{m_i+l} \stackrel{\text{proj}}{\to} \mathrm{RV}^{m_i+l}$.

Now vary $i$ over $1, \dots, s$, writing $U_i = U$, $\widetilde{Y_i} = Y$, $\widetilde{\pi}_i = \widetilde{\pi}$. Using (†) we may shrink the $(U_i)_i$ to ensure both that they are disjoint and that $\bigsqcup_{i=1}^s U_i$ is a family of fibres of $h$ which contains $h^{-1}(b)$; set $V = h(\bigsqcup_{i=1}^s U_i)$. To summarise:

(‡) There is a $T^+$-definable set $V \subseteq h(X)$ containing $b$, such that $h^{-1}(V)$ is a disjoint union of definable sets $U_1, \dots, U_s$, each of the following form: there are a $T^+$-definable $\widetilde{Y_i} \subseteq \mathrm{RV}^{m_i+l}$, a generalised projection $\widetilde{\pi}_i : Y_i \to \mathrm{RV}^n$, and a $T^+$-definable bijection $\widetilde{\tau}_i : U_i \to \mathbb{L}(\widetilde{Y_i}, \widetilde{\tau}_i)$ with Jacobian $= 1$ away from a null set.

By compactness, there are finitely many $\{V\}$ as in (‡) which cover $h(X)$. If $V$ and $V'$, say, overlap then (†) allows us to replace $V'$ by $V' \setminus V$ without affecting (‡). The required decomposition follows. □

These decomposition results in terms of lifts of the form $\mathbb{L}(Y, \pi)$ are, as we have just seen, extremely convenient for model-theoretic manipulations, but for the concrete applications there is a more aesthetic reinterpretation:

**Corollary 5.5.10.** *Let $(T^+, \mathcal{L}^+)$ be as in the previous proposition, but assume further that $T^+$ is a complete theory. Let $X \subseteq \mathrm{VF}^n$ be $T^+$-definable. Then $X$ is a disjoint union of definable sets, $X = \bigsqcup_{i=0}^s X_i$, with $X_0$ null and the remaining $X_i$ of the following form: there are a definable $Y_i \subseteq (\mathrm{RV}^\times)^n$, an integer $N_i \geq 1$, and a definable map $f_i : X_i \to \mathrm{rv}^{-1}(Y_i)$ which is everywhere $N_i$-to-$1$ and has Jacobian $= 1$ away from a null set.*

*Proof.* By first decomposing $X$ as proved in the previous proposition, we may suppose that there are a definable $Y \subseteq \mathrm{RV}^m$, a generalised projection $\pi : Y \to \mathrm{RV}^n$, and a definable bijection $\tau : X \to \mathbb{L}(Y, \pi)$ with Jacobian $= 1$ off a null set.

We claim first that $\pi$ is finite-to-one on $Y$. Let $\underline{x} \in \mathrm{VF}^n$; then

$$\mathbb{L}(Y, \pi) \cap \{\underline{x}\} \times \mathrm{RV}^m = \{\underline{x}\} \times \pi^{-1}(\mathrm{rv}(\underline{x})).$$

Hence $\pi^{-1}(\mathrm{rv}(\underline{x}))$ is $T^+_{\underline{x}}$-isomorphic, via the restriction of $\tau^{-1}$, to a subset of $X$; but according to corollary 5.3.15, this forces $\pi^{-1}(\mathrm{rv}(\underline{x}))$ to be finite. This completes the proof of the claim.

Set $Y_0 = Y \setminus \pi^{-1}((\mathrm{RV}^\times)^n)$. If $\underline{x} \in \mathbb{L}(Y_0, \pi)$, then at least one coordinate of $\underline{x}$ is zero; hence $\mathbb{L}(Y_0, \pi)$ is a null set, and therefore $X_0 := \tau^{-1}(\mathbb{L}(Y_0, \pi))$ is also null by corollary 5.3.20. Since

$$\mathbb{L}(Y, \pi) = \mathbb{L}(Y_0, \pi) \sqcup \mathbb{L}(Y \setminus Y_0, \pi),$$

we may now replace $X, Y$ by $X \setminus X_0$, $Y \setminus Y_0$ to assume that $\pi(Y) \subseteq (\mathrm{RV}^\times)^n$.

Let $N \geq 0$ be big enough so that all fibres of $\pi$ have cardinality $\leq N$, and for $j = 1, \ldots, N$, put

$$Y_i = \{\underline{y} \in \mathrm{RV}^n : |Y_{\underline{y}}| = j\}.$$

The fact that $Y_1, \ldots, Y_N$ form a disjoint, definable cover of $\pi(Y)$ requires the completeness of $T^+$. So

$$\mathbb{L}(Y_j, \pi) = \bigsqcup_{j=1}^{N} \mathbb{L}(\pi^{-1}(Y_j), \pi)$$

and we set $X_j = \tau^{-1}(Y_j)$; clearly $X = \bigsqcup_j X_j$.

Let $\rho : \mathrm{VF}^n \times \mathrm{RV}^m \to \mathrm{VF}^n$ be the projection map. Its restriction induces a surjection

$$\mathbb{L}(\pi^{-1}(Y_j), \pi)) \to \mathrm{rv}^{-1}(Y_j)$$

with fibres of cardinality $j$. Hence

$$f_j := \rho \circ \tau : X_j \to \mathbb{L}(\pi^{-1}(Y_j), \pi)) \to \mathrm{rv}^{-1}(Y_j)$$

is everywhere $j$-to-1 and has Jacobian $= 1$ off a null set. We have produced the required decomposition. $\square$

**Remark 5.5.11.** It appears to be possible to assume further in the statement of the previous corollary that each $X_i$ (apart from $X_0$) is open and that $f_i$ is a smooth cover $X_i \to \mathrm{rv}^{-1}(Y_i)$. The proof of this does not seem to easily follow from the decomposition results which we have stated, but rather from Hrushovski and Kazhdan's proof of their result. The idea is basically, at each stage of the construction of the decomposition, to throw out the sets on which certain maps fail to be differentiable, continuous, etc.; such sets will all be null, and these will form $X_0$.

The previous theorem and corollary were the essential results required to complete our proof of theorem 5.1.5; for the concrete applications to two-dimensional integration, refer back to section 5.1.

CHAPTER 6

# Ramification, Fubini's theorem, and Riemann-Hurwitz formulae

We consider various relations between integration and ramification theories.

## 6.1 Ramification of local fields

From section 6.2 onwards, we will be considering ramification theory for geometric objects. The analogous problems in the local setting are closely related with the previous chapter and can be discussed independently from the remaining material, so this initial section focuses on local ramification theory. We begin with a reminder of the theory in the perfect residue field case:

### 6.1.1 Perfect residue field

Fix a complete discrete valuation field $F$ with perfect residue field $\overline{F}$, and let $F^{\mathrm{alg}}$ denote its algebraic closure. Fix a finite Galois extension $L/F$ with Galois group $G$, and define the usual ramification objects as follows:

$$\begin{aligned} i_{L/F}(\sigma) &= \min\{\nu_F(\sigma(x)-x) : x\in\mathcal{O}_F\},\\ G_a &= \{\sigma\in G : i_{L/F}(\sigma)\geq a+1\}\quad (a\geq -1),\\ \eta_{L/F}(a) &= e_{L/F}^{-1}\int_0^a |G_x|\,dx\quad (a\geq -1)\\ &= -1+e_{L/F}^{-1}\sum_{\sigma\in G}\min\{i_{L/F}(\sigma),a+1\}. \end{aligned}$$

One proves that $\eta_{L/F}$ is a strictly increasing, piecewise linear, function $[-1,\infty)\to[-1,\infty)$, and defines the *Hasse-Herbrand* function $\psi_{L/F}:[-1,\infty)\to[-1,\infty)$ to be its inverse. The *upper ramification filtration* on the Galois group is defined by $G^a = G_{\psi_{L/F}(a)}$, for $a\geq -1$.

The central results of the theory are the following (see e.g. [FV02, Chapter III] or [Ser79, Part 2]):

**Theorem 6.1.1 (Herbrand).** *Let $M/F$ be a Galois subextension of $L/F$. Then, for any $a\geq -1$, the image of Gal$(L/F)^a$ under the restriction map Gal$(L/F)\to$ Gal$(M/F)$ is exactly Gal$(M/F)^a$.*

Let $k$ be an algebraically closed field of characteristic $0$; in arithmetic applications this will be $\mathbb{Q}_l^{\mathrm{alg}}$ or $\mathbb{C}$.

**Theorem 6.1.2 (Artin).** *The* Artin character

$$a_{L/F} : G \to k, \quad \sigma \mapsto \begin{cases} -f_{L/F}\, i_{L/F}(\sigma) & \sigma \neq id \\ f_{L/K} \sum_{\sigma \in G \setminus \{id\}} i_{L/F}(\sigma) & \sigma = id \end{cases}$$

*is the character of a finite-dimensional, $k$ representation of $G$.*

**Theorem 6.1.3 (Non-commutative Hasse-Arf).** *Let $(V, \pi)$ be a finite-dimensional, $k$ representation of $G$. Then the* conductor *of $\pi$,*

$$\mathfrak{f}(\pi) := \sum_{i=0}^{\infty} |G_0 : G_i|^{-1} \dim V/V^{G_i},$$

*is a positive integer.*

The non-commutative Hasse-Arf theorem and Artin's theorem can be easily deduced from one another, because $\mathfrak{f}(\pi) = \langle \chi_\pi, a_{L/F} \rangle$, where $\langle \cdot, \cdot \rangle$ is the inner product on the space of class functions on $G$, and $\chi_\pi$ is the character of $\pi$. Using R. Brauer's theorem on characters, one reduces the Hasse-Arf theorem to the case $\dim V = 1$; Herbrand's theorem then implies that it is enough to show that the upper ramification breaks of an abelian extension $L/F$ occur at integers. This is proved by explicit, local calculations.

### 6.1.2 Arbitrary residue field

Until the work of A. Abbes and T. Saito [AS02] [AS03] it was a significant open problem to generalise the ramification theory above to the case of non-perfect residue field. Geometrically, the importance of this lies in the following situation. If $\phi : S_1 \to S_2$ is a finite morphism between smooth, projective surfaces, over a field $k$ which is allowed to be perfect, then according to section 6.5, the ramification of $\phi$ occurs along curves. Let $B \subset S_1$ be an irreducible curve with generic point $y$, and set

$$K(S_1)_y = \operatorname{Frac} \widehat{\mathcal{O}_{S_1,y}}\,;$$

this is a complete discrete valuation field whose residue field is $k(B)$. Moreover, we have a finite extension

$$K(S_1)_y / K(S_2)_{\phi(y)},$$

whose ramification properties reflect the local ramification of $\phi$ along $B$. But $k(B)$ will be imperfect and $K(S_1)_y/K(S_2)_{\phi(y)}$ may have an inseparable residue field extension.

We now give a summary of the basics of Abbes and Saito's theory. There is a more extensive overview by L. Xiao [Xia07]. Let $L/F$ be a finite, Galois extension of complete discrete valuation fields with arbitrary residue fields. Then $\mathcal{O}_L$ is a complete intersection algebra over $\mathcal{O}_F$ (since they are both regular local rings) and we may therefore write

$$\mathcal{O}_L = \mathcal{O}_F[T_1, \dots, T_n]/\langle f_1, \dots, f_n \rangle$$

for a regular sequence $f_1, \dots, f_n$.

Now, for any real $a \geq 1$, one introduces the rigid space

$$X^a_{L/F} = \{\underline{x} \in (F^{\mathrm{alg}})^n : \nu_F(x_i) \geq 0 \text{ all } i,\ \nu_F(f_i(\underline{x})) \geq a \text{ all } i\},$$

where $\nu_F : F^{\mathrm{alg}} \to \mathbb{Q} \cup \{\infty\}$ is the extension of the discrete valuation on $F$. By some rigid geometry, model theory, or explicit calculations, it is known that $X^a_{L/F}$ may be written in a unique way as a disjoint union of closed balls (if $n = 1$, which one may assume if $\overline{L}/\overline{F}$ is separable, this follows from our decomposition results 4.1.5 and 4.1.6); let $\pi_0(X^a_{L/F})$ denote this set of balls. As $a \to \infty$, $X^a_{L/F}$ will consist of $|L : F|$ small balls; conversely, $X^0_{L/F}$ is a single large ball. A central idea of Abbes and Saito's theory is to analyse the behaviour of $X^a_{L/F}$ as $a$ varies; in particular, when it breaks into $|L : F|$ balls. This will soon be made precise.

The natural action of the absolute Galois group $\mathrm{Gal}(F^{\mathrm{alg}}/F)$ on $X^a_{L/F}$ induces an action on $\pi_0(X^a_{L/F})$, which then factors transitively through $G = \mathrm{Gal}(L/F)$.

**Remark 6.1.4.** To motivate what follows, let us briefly suppose that $F$ has perfect residue field. Then it is not hard to prove:

*For $a \geq -1$, $\sigma \in G$ acts trivially on $\pi_0(X_{L/F}^{\eta_{L/F}(a)+1})$ if and only if $\sigma \in G_a$.*

(A nice sketch is given in [Xia07]). So, for any $a \geq -1$, the kernel of the action of $G$ on $\pi_0(X^{a+1}_{L/F})$ is $G^a$.

Abbes and Saito take the final observation in this remark as the definition of the upper filtration in their theory:

**Definition 6.1.5.** Let $L/F$ be a finite, Galois extension of complete discrete valuation fields. The *upper ramification filtration* on $G = \mathrm{Gal}(L/F)$, is defined, for $a \geq -1$, by

$$G^a = \mathrm{Ker}(G \to \mathrm{Aut}(X^{a+1}_{L/F})).$$

Starting from this definition of the upper ramification filtration, Abbes and Saito develop fully a ramification theory for $F$. Xiao has extended their work by establishing the Hasse-Arf integrality theorem for certain conductors [Xia08a] [Xia08b].

**Remark 6.1.6.** Again suppose that $\overline{F}$ is perfect. The definition of the Hasse-Herbrand function implies that

$$\frac{d\psi_{L/F}}{da}(a) = e^{-1}_{L/F}|G^a|^{-1},$$

at least away from the ramification breaks, and therefore that

$$\psi_{L/F}(a) = e^{-1}_{L/F}\int_{-1}^{a} |G^x|^{-1}\,dx - 1,$$

since both sides are $= -1$ at $a = -1$. But $|G : G^x| = |\pi_0(X^{x+1}_{L/F})|$, and so

$$\psi_{L/F}(a) = f^{-1}_{L/F}\int_{0}^{a+1} |\pi_0(X^x_{L/F})|\,dx - 1 \tag{$*$}$$

for all $a \geq -1$.

If we think of 'the number of connected components' as a measure, then $(*)$ is a repeated integral taken over certain fibres, and it is exactly the variation of the fibres which contributes to the interesting structure of the Hasse-Herbrand function. Whether this repeated integral interpretation of ramification can be more systematically exploited in the local setting is an interesting question.

### 6.1.3 Model-theoretic integration in finite residue characteristic

We finish this discussion on ramification theory with some conjectural remarks on how the Abbes-Saito approach to ramification may be compatible with the Hrushovski-Kazhdan integration theory. We work in a model theoretic setting, as in chapter 5: $T$ is a theory of algebraically closed valued fields in a language $\mathcal{L}$ obtained by adding parameters to $\mathcal{L}_{\mathrm{RV}}$.

If $T$ is a theory of residue characteristic zero, then we saw in 5.5 that every definable subset of $\mathrm{VF}^n$ was isomorphic, by 'measure-preserving' bijections, to a disjoint union of sets lifted from the RV-level (=the Residue field and Value group.) In finite residue characteristic this is known to fail. For example. suppose that $T = \mathrm{ACVF}(0,p)$ for some prime $p > 0$; so $\mathbb{Q}_p^{\mathrm{alg}}$ is a model of $T$, and we consider

$$X^0 = \{x \in \mathbb{Q}_p^{\mathrm{alg}} : \nu_p(x^p - x - p^{-1}) \geq 0\}.$$

It is easy to check that $X^0$ contains no rational points (i.e. $X^0 \cap \mathbb{Q}_p = \varnothing$), and it is essentially this which prevents it from being realised as a lift from RV.

Note that the roots of $T^p - T - p^{-1}$ generate a wild, totally ramified extension $L$ of $\mathbb{Q}_p$ of degree $p$ and conductor 1. Moreover, if $\xi$ is a root of $T^p - T - p^{-1}$ then $\xi^{-1}$ is a prime of $L$, with minimal polynomial $T^p + pX^{p-1}T - p$; hence $\mathcal{O}_L = \mathbb{Z}_p[T]/\langle T^p + pX^{p-1}T - p\rangle$. Now consider the family of sets

$$X^a_{L/\mathbb{Q}_p} = \{x \in \mathbb{Q}_p^{\mathrm{alg}} : \nu_p(x^p + px^{p-1} - p) \geq a\}$$

which arise in the Abbes-Saito theory; then $X^0_{L/F}$ contains rational points, while $X^1_{L/F}$ does not, because when $a$ passes from 0 to 1 the rigid space splits into separable balls.

Hence we may detect the conductor of $L/\mathbb{Q}_p$ by examining existence of rational points in families of definable sets. Although we worked with a specific example, the ideas appear to generalise to arbitrary extensions of valued fields. The following therefore seems to be an important programme of study, which the author intends to pursue:

#### Develop a model theoretic approach to ramification theory

Perhaps Abbes and Saito's theory, currently based on rigid geometry, can be redeveloped using the model theory of algebraically closed valued fields. The existence of definable points and numbers of definable components will replace their arguments using rigid spaces, and model theory provides an ideal tool for the many fibration arguments which appear in their work.

Moreover, model theory may give a more refined ramification theory for higher dimensional local fields, because it is often straightforward to 'add additional structure' to the residue field (e.g. insist the residue field is a local field), as we saw in section 5.5.

#### Unify the ramification theory with Hrushovski-Kazhdan integration theory

According to theorem 5.1.5, definable subsets of a valued field of residue characteristic zero all 'come from' the RV-level. As we just discussed, this fails in characteristic $p$, with the main problem being the appearance of definable sets related to ramification theory. Perhaps a theory can be developed in which objects associated with the valued field can be split apart, with one component coming from RV and the other encoding ramification data. This ramification component will allow ramification invariants to be associated to the original object, which can provide 'correction factors' for integrals over the RV component. This may lead to the proper understanding of proposition 4.4.1, a more powerful theory of local zeta integrals, and a theory of motivic integration in residue characteristic $p$.

## 6.2 Structures and Euler characteristics

Before we can tackle the main problems of this chapter, we must present some elementary objects from model theory from a perspective suitable for this work. We must understand what sort of sets we can measure and what it means to measure them. This material is well-known but hopefully this explicit exposition will appeal to those unfamiliar with model theory.

### 6.2.1 Structures

Given a set $\Omega$, a *ring* of subsets of $\Omega$ is defined to be a non-empty collection of subsets $\mathcal{R}$ of $\Omega$ such that

$$A, B \in \Omega \Longrightarrow A \setminus B,\ A \cup B,\ A \cap B \in \Omega.$$

It is enough to assume that $\mathcal{R}$ is closed under differences and unions for this implies it is closed under intersections. A ring of sets is said to be an *algebra* if and only if it contains $\Omega$.

Following van den Dries [vdD98] we define a *structure* $\mathcal{A} = (\mathcal{A}(\Omega^n))_{n=0}^{\infty}$ on $\Omega$ to be an algebra $\mathcal{A}(\Omega^n)$ of subsets of $\Omega^n$ for each $n \geq 0$ such that

(i) if $A \in \mathcal{A}(\Omega^n)$ then $A \times \Omega, \Omega \times A \in \mathcal{A}(\Omega^{n+1})$;

(ii) $\{(x_1, \ldots, x_n) \in \Omega^n : x_1 = x_n\} \in \mathcal{A}(\Omega^n)$;

(iii) if $\pi : \Omega^{n+1} \to \Omega^n$ is the projection map to the first $n$ coordinates, then $A \in \mathcal{A}(\Omega^{n+1})$ implies $\pi(A) \in \mathcal{A}(\Omega^n)$.

Given a structure, one refers to the sets in $\mathcal{A}(\Omega^n)$ as being the *definable* subsets of $\Omega^n$. If $A \subseteq \Omega^n$ and $f : A \to \Omega^m$ then $f$ is said to be definable if and only if its graph belongs to $\mathcal{A}(\Omega^{n+m})$.

**Proposition 6.2.1.** *Let $\mathcal{A}$ be a structure on a set $\Omega$. Then*

*(i) if $A \in \mathcal{A}(\Omega^n), B \in \mathcal{A}(\Omega^m)$ then $A \times B \in \mathcal{A}(\Omega^{n+m})$;*

*(ii) if $1 \leq i < j \leq n$, then $\{(x_1, \ldots, x_n) \in \Omega^n : x_i = x_j\}$ is in $\mathcal{A}(\Omega^n)$;*

*(iii) if $\sigma$ is a permutation of $\{1, \ldots, n\}$, then the function $\Omega^n \to \Omega^n$ given by permuting the indices of the coordinates by $\sigma$ is definable.*

*Moreover, if $A \subseteq \Omega^n$ and $f : A \to \Omega^m$ is definable, then*

*(i) $A$ is definable;*

*(ii) if $B \subseteq A$ is definable, then $f(B)$ is definable, and the function given by restricting $f$ to $B$ is definable;*

*(iii) if $B \in \mathcal{A}(\Omega^m)$, then $f^{-1}(B) \in \mathcal{A}(\Omega^n)$;*

*(iv) if $f$ is injective, then its inverse is definable;*

*(v) if $B \supseteq f(A)$ and $g : B \to \Omega^l$ is definable, then $g \circ f : A \to \Omega^l$ is definable.*

*Proof.* These are straightforward to check; proofs may be found in [vdD98]. $\square$

**Remark 6.2.2.** If $\mathcal{L}$ is a first order language of logic, and $\Omega$ is an $\mathcal{L}$-structure, then there is a structure on $\Omega$ in which $\mathcal{A}(\Omega^n)$ consists of precisely those sets of the form

$$\{x \in \Omega^n : \Omega \models \phi(x, b)\}$$

where $\phi(x, y)$ is a formula of $\mathcal{L}$ in variables $x_1, \ldots, x_n, y_1, \ldots, y_m$ and $b \in \Omega^m$; that is, those sets which are definable with parameters in the sense of model theory.

Realistically, any structure in which we will be interested will arise in this way as the parameter-definable sets of some language. But for the reader less familiar with logic, the axiomatic approach above is more immediately appealing, though ultimately less satisfying.

**Example 6.2.3.** We present some examples to explain what we can and cannot study using structures. All are well-known.

(i) If $\Omega$ is an arbitrary set, we may take $\mathcal{A}(\Omega^n)$ to be the collection of all subsets of $\Omega^n$; that is, every set is definable.

(ii) If $k$ is an algebraically closed field, let $\mathcal{A}(k^n)$ be the ring of sets generated by the Zariski closed subsets of $k^n$; such sets are called constructible. It is known that $(\mathcal{A}(k^n))_n$ forms a structure on $k$. The difficulty is establishing that such sets are closed under projection; this may either be proved in a model theoretic setting, where it is equivalent to establishing that the theory of algebraically closed fields admits quantifier elimination, or it may be seen as a special case of a result of algebraic geometry concerning constructible subsets of Noetherian schemes (see e.g. [Har77] exercises 3.17-3.19).

(iii) If $k$ is an arbitrary field, then an affine subset of $k^n$ is a set of the form $a + X$ where $a \in k^n$ and $X$ is a $k$-subspace of $k^n$. Letting $\mathcal{A}(k^n)$ be the ring of sets generated by affine subsets of $k^n$ gives a structure on $k$.

(iv) If $\mathbb{R}$ is the real line, then let $\mathcal{A}(\mathbb{R}^n)$ be the ring of sets generated by $\{x \in \mathbb{R}^n : p(x) \geq 0\}$ for $p \in \mathbb{R}[X_1, \ldots, X_n]$; the sets in $\mathcal{A}(\mathbb{R}^n)$ are called semi-algebraic subsets of $\mathbb{R}^n$. This gives a structure for $\mathbb{R}$. Again, the difficulty is verifying that such sets are closed under projection.

(v) None of the following give structures on the real line: the Borel sets, the Lebesgue measurable sets, the Suslin sets.

So structures are typically quite coarse from the point of view of classical analysis and measure theory.

### 6.2.2 Euler characteristics and the Grothendieck ring of a structure

Having introduced the sets of interest we now discuss what it means to take the measure of such a set.

**Definition 6.2.4.** Let $\Omega$ be a set with a structure $\mathcal{A}$. An *Euler characteristic* is a map $\chi$ from the definable sets to some commutative ring $R$, i.e.

$$\chi : \bigsqcup_{n=0}^{\infty} \mathcal{A}(\Omega^n) \to R,$$

which satisfies

(i) if $A, B \in \mathcal{A}(\Omega^n)$ are disjoint, then $\chi(A \sqcup B) = \chi(A) + \chi(B)$;

(ii) if $A \in \mathcal{A}(\Omega^n)$, $B \in \mathcal{A}(\Omega^m)$, then $\chi(A \times B) = \chi(A)\chi(B)$;

(iii) if $A \in \mathcal{A}(\Omega^n)$, $B \in \mathcal{A}(\Omega^m)$ and there is a definable bijection $f : A \to B$, then $\chi(A) = \chi(B)$.

**Remark 6.2.5.** From the additivity of $\chi$, one might think that an Euler characteristic is similar to a measure in the classical sense. The vast difference between the two is the invariance of $\chi$ under definable bijections. For example, if $\Omega$ is a field $k$, $\alpha \in k^\times$, and multiplication by $\alpha$ is a definable map from $k$ to itself, then $\chi(A) = \chi(\alpha A)$ for definable $A \subseteq k$; in other words, scaling a set does not affect its size. Or if $\Omega$ is the real line and $x \mapsto x^2$ is definable, then for any definable $A$ of the positive reals, $\chi(\{x^2 : x \in A\}) = \chi(A)$.

Some authors prefer the term *generalised Euler characteristic* or *additive invariant*, to avoid possible confusion with the *topological Euler characteristic* $\chi_{\text{top}}$ for complex projective manifolds, defined as the alternating sum of the Betti numbers.

**Example 6.2.6.** The easiest example of an Euler characteristic is counting measure: let $\Omega$ be a finite set, $\mathcal{A}(\Omega^n)$ the algebra of all subsets of $\Omega^n$, and set $\chi(A) = |A|$ to define a $\mathbb{Z}$-valued Euler characteristic.

Explicitly exhibiting more interesting Euler characteristics requires some work, so we present here without proof some known examples using the structures of example 6.2.3.

(i) Let $k$ be a field, equipped with the structure generated by the affine subsets. If $k$ is infinite then there is a unique $\mathbb{Z}[t]$-valued Euler characteristic $\chi$ which satisfies

$$\chi(a + X) = t^{\dim_k X}$$

where $a \in k^n$ and $X$ is a $k$-subspace of $k^n$.

(ii) Give $\mathbb{R}$ the structure of semi-algebraic sets. Then there is a unique $\mathbb{Z}$-valued Euler characteristic $\chi$ which satisfies

$$\chi((0,1)) = -1,$$

sometimes called the *combinatorial* Euler characteristic.

(iii) Give $\mathbb{C}$ the structure of constructible sets; then there is a unique Euler characteristic $\chi_{\text{top}}$ which agrees with the topological Euler characteristic for any projective manifold.

**Definition 6.2.7.** Let $\Omega$ be a set with structure $\mathcal{A}$. The associated *Grothendieck ring*, denoted $K_0(\Omega)$ (though it does of course depend on the structure, not just the set $\Omega$), is defined to be the free commutative unital ring generated by symbols $[A]$ for $A$ a definable subset of $\Omega^n$, any $n \geq 0$, modulo the following relations

(i) if $A, B \in \mathcal{A}(\Omega^n)$ are disjoint, then $[A \sqcup B] = [A] + [B]$;

(ii) if $A \in \mathcal{A}(\Omega^n)$, $B \in \mathcal{A}(\Omega^m)$, then $[A \times B] = [A][B]$;

(iii) if $A \in \mathcal{A}(\Omega^n)$, $B \in \mathcal{A}(\Omega^m)$ and there is a definable bijection $f : A \to B$, then $[A] = [B]$.

**Remark 6.2.8.** The map $A \mapsto [A]$ defines a $K_0(\Omega)$-valued Euler characteristic on $\Omega$, which is universal in the sense that if $\chi : \bigsqcup_{n=0}^{\infty} \mathcal{A}(\Omega^n) \to R$ is an Euler characteristic, then there is a unique ring homomorphism $\chi' : K_0(\Omega) \to R$ such that $\chi(A) = \chi'([A])$ for any definable $A$. Thus $A \mapsto [A]$ is the most general Euler characteristic of a structure.

Note that if $\{x\} \subseteq \Omega^n$ is a single point, and $A \subseteq \Omega^m$ is definable, then projection induces a definable isomorphism $\{x\} \times A \to A$. So $[\{x\}][A] = [A]$ for all definable $A$ and therefore $[\{x\}]$=1; more generally, $[B] = |B|$ for any finite definable set $B$.

**Remark 6.2.9.** *Extending the Euler characteristic to varieties.* Assume that $\Omega = k$ is an algebraically-closed field with the structure $\mathcal{A}$ of constructible subsets. Let $V$ be a separated algebraic variety over $k$ (our varieties in this chapter usually consist only of the closed points of the corresponding scheme) and let $\mathcal{A}(V)$ be the ring generated by the Zariski closed subsets of $V$, i.e. the algebra of constructible subsets of $V$.

It is straightforward to prove that $\chi$ uniquely extends to $\mathcal{A}(V)$ in such a way that if $U \subseteq V$ is an affine open or closed subset, $C \subseteq U$ is constructible, and $i : U \to \mathbb{A}_k^d$ is an open or closed embedding for some $d$, then $\chi(C) = \chi(i(C))$.

**Remark 6.2.10.** *Extending the measure to an integral.* If $\Omega$ is a set equipped with a structure and Euler characteristic $\chi$, then there is a unique $R$-linear map $\int d\chi$ from the space of functions spanned by characteristic functions of definable sets to $R$ which satisfies $\int \mathrm{char}\, A\, d\chi = \chi(A)$ for any definable $A$. We will allow ourselves to use typical notation for integrals, writing $\int f(x)\, d\chi(x)$.

## 6.3 Riemann-Hurwitz and Fubini's theorem for curves

Here we relate Fubini's theorem for Euler characteristics to the Riemann-Hurwitz formula for morphisms between curves; then we produce a startling result implying that in finite characteristic it is always possible for Fubini's theorem to fail.

Throughout this section $k$ is an algebraically closed field of arbitrary characteristic, $\mathcal{A}$ is the structure of constructible sets, and $\chi$ is a fixed $R$-valued Euler characteristic on $\mathcal{A}$. By a curve $C$ over $k$, in this section, we mean a smooth, one-dimensional, irreducible algebraic variety over $k$; we only consider the closed points of $C$. Following remarks 6.2.9 and 6.2.10, the space of integrable functions on $C$ is the $R$-module generated by characteristic functions of constructible sets; the integral on this space will be denoted $\int_C \cdot\, d\chi$.

Let $\phi : C_1 \to C_2$ be a non-constant morphism of curves. We will study whether Fubini's theorem holds for the morphism $\phi$, which is to say that for each $y \in C_2$, the fibre $\phi^{-1}(y)$ is constructible, that $y \mapsto \chi(\phi^{-1}(y))$ is integrable, and finally that $\chi(C_1) = \int_{C_2} \chi(\phi^{-1}(y))\, d\chi(y)$. The problem immediately simplifies:

**Lemma 6.3.1.** *Fubini's theorem holds for a separable morphism $\phi : C_1 \to C_2$ of projective curves if and only if the following formula relating the Euler characteristics of $C_1$ and $C_2$ is satisfied:*

$$\chi(C_1) = \chi(C_2) \deg \phi - \sum_{x \in C_1} (e_x(\phi) - 1),$$

*where $e_x(\phi)$ is the ramification degree of $\phi$ at $x$.*

*Proof.* Let $\Sigma \subseteq C_1$ be the finite set of points at which $\phi$ is ramified. Let $y$ be a point of $C_2$. The fibre $\phi^{-1}(y)$ is finite; moreover, it contains exactly $\deg \phi$ points when $y \notin \phi(\Sigma)$. So each fibre is certainly constructible and $\chi(\phi^{-1}(y)) = |\phi^{-1}(y)|$. Thus $y \mapsto \chi(\phi^{-1}(y))$

is constant off the finite set $\phi(\Sigma)$ and hence is integrable on $C_2$; integrating obtains

$$\int_{C_2} \chi(\phi^{-1}(y))\, d\chi(y) = \chi(C_2 \setminus \phi(\Sigma)) \deg \phi + \sum_{y \in \phi(\Sigma)} |\phi^{-1}(y)|.$$

The fundamental ramification equality $\sum_{x \in \phi^{-1}(y)} e_x(\phi) = \deg \phi$ transforms this into

$$\chi(C_2) \deg \phi - \sum_{y \in \phi(\Sigma)} \sum_{x \in \phi^{-1}(y)} (e_x(\phi) - 1),$$

which completes the proof. □

**Remark 6.3.2.** More generally, if $\operatorname{char} k = p > 0$ and $\phi : C_1 \to C_2$ is a morphism of projective curves which is not necessarily separable, then we decompose $\phi$ as $\phi = \phi_{\text{sep}} \circ F^m$; here $F$ is the Frobenius morphism of $C_1$, $\phi_{\text{sep}} : C_1 \to C_2$ is a separable morphism, and $m$ is a non-negative integer. The previous proof shows that Fubini holds for $\phi$ if and only if

$$\chi(C_1) = \chi(C_2) \deg \phi_{\text{sep}} - \sum_{x \in C_1} (e_x(\phi_{\text{sep}}) - 1).$$

So Fubini holds for $\phi$ if and only if it holds for the separable part $\phi_{\text{sep}}$; in particular, Fubini holds for any purely inseparable morphism of projective curves

For this reason we are justified in focusing our attention on separable morphisms.

**Remark 6.3.3.** More usually Fubini's theorem is concerned with measuring subsets of product space via repeated integrals; let us show that this is the same as our current activity considering fibres of morphisms between projective curves.

Suppose $\phi : C_1 \to C_2$ is a separable morphism of projective curves over $k$. Then $\phi$ is a finite morphism, so that if $U_2 \subseteq C_2$ is a non-empty, affine, open subset then the same is true of $U_1 = \phi^{-1}(U_2)$. Choose closed embeddings $U_1 \hookrightarrow \mathbb{A}^n_k$, $U_2 \hookrightarrow \mathbb{A}^m_k$ and let $\Gamma = \{(x, \phi(x)) \in \mathbb{A}^{n+m}_k : x \in U_1\}$ be the graph of $\phi|_{U_1}$.

It is immediate that the integral $\int_{k^n} \int_{k^m} \operatorname{char}_\Gamma(x, y)\, d\chi(y) d\chi(x)$ is well-defined and equal to $\chi(U_1)$. Conversely, if we fix $y \in U_2$ then $\int_{k^n} \operatorname{char}_\Gamma(x, y)\, d\chi(x) = \chi(\phi^{-1}(y))$; arguing as in the previous lemma now obtains

$$\int_{k^m} \int_{k^n} \operatorname{char}_\Gamma(x, y)\, d\chi(x) d\chi(y) = \chi(U_2) \deg \phi - \sum_{x \in U_1} (e_x(\phi) - 1).$$

So interchanging the order of integration preserves the value of the integral if and only if

$$\chi(U_1) = \chi(U_2) \deg \phi - \sum_{x \in U_1} (e_x(\phi) - 1).$$

Further, $C_2 \setminus U_2$ and $\phi^{-1}(C_2 \setminus U_2) = C_1 \setminus U_1$ are finite sets and it is straightforward to verify, similarly to the previous lemma, that

$$|C_1 \setminus U_1| = |C_2 \setminus U_2| \deg \phi - \sum_{x \in C_1 \setminus U_1} (e_x(\phi) - 1).$$

Taking the sum of the previous two formulae shows that Fubini's theorem holds for $\phi : C_1 \to C_2$ if and only if the repeated integrals of $\operatorname{char}_\Gamma$ are equal.

Recall that the Riemann-Hurwitz formula states that if $\phi : C_1 \to C_2$ is a non-constant morphism of projective curves, then there are integers $\widetilde{e}_x(\phi)$ for each $x \in C_1$ (which we shall call the *Riemann-Hurwitz ramification degrees*) such that $\widetilde{e}_x(\phi) \geq e_x(\phi)$, with equality if and only if $\phi$ is tamely ramified at $x$, and such that

$$2(1-g_2) = 2(1-g_1)\deg\phi - \sum_{x \in C_1} (\widetilde{e}_x(\phi) - 1),$$

where $g_i$ is the genus of $C_i$. It is apparent that Fubini's theorem and the Riemann-Hurwitz formula are related.

**Remark 6.3.4.** The non-negative integer $\widetilde{e}_x(\phi) - 1$ is equal to the different of the extension $\mathcal{O}_{C_1,x}/\mathcal{O}_{C_2,\phi(x)}$ of discrete valuation rings, though we will not use this fact.

**Remark 6.3.5.** It is useful to have some explicit examples of morphisms between projective curves. Let $f(t)$ be a polynomial over $k$ and let $\Gamma_f$ be the algebraic variety over $k$ which is the graph of $f$, i.e.

$$\Gamma_f = \{(x,y) \in \mathbb{A}^2_k : y = f(x)\}.$$

Let $F : \mathbb{A}^1_k \to \Gamma_f$ be the morphism $F(x) = (x, f(x))$ and let $\pi : \Gamma_f \to \mathbb{A}^1_k$ be the projection map $\pi(x,y) = y$. Note that $F$ is an isomorphism of algebraic varieties and that $\pi \circ F = f$; here we abuse notation and write $f$ for the morphism $\mathbb{A}^1_k \to \mathbb{A}^1_k$ induced by the polynomial $f(t)$. Let $\Gamma^*_f$ denote the projective closure of $\Gamma_f$, obtained by adding a single point at infinity. The morphisms $F, \pi, f$ extend to morphisms $F : \mathbb{P}^1_k \xrightarrow{\cong} \Gamma^*_f$, $\pi : \Gamma^*_f \to \mathbb{P}^1_k$, $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$.

Remark 6.3.3 implies that the following are all equivalent:

(i) Fubini holds for $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$;

(ii) Fubini holds for $f : \mathbb{A}^1_k \to \mathbb{A}^1_k$;

(iii) The repeated integrals of $\mathrm{char}_{\Gamma_f}$ are equal.

To make use of the examples afforded by the previous remark we now calculate the ramification degrees:

**Lemma 6.3.6.** *We retain the notation of the previous remark. The ramification degrees of $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ are*

$$e_a(f) = \begin{cases} \nu_{t-a}(f(t) - f(a)) & a \in k = \mathbb{A}^1_k \\ \deg f & a = \infty, \end{cases}$$

*and the Riemann-Hurwitz ramification degrees are*

$$\widetilde{e}_a(f) = \begin{cases} 1 + \nu_{t-a}(f'(t)) & a \in k = \mathbb{A}^1_k \\ \deg f + (\deg f - \deg f' - 1) & a = \infty. \end{cases}$$

Here $\nu_{t-a}$ denotes the $t-a$-adic valuation on $k(t)$.

*Proof.* The ramification degrees are clear so we only consider the Riemann-Hurwitz degrees.

Write $s = f(t)$ so that $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ corresponds to the extension of function fields $K(s) \leq K(t)$. A local coordinate $t_a \in K(t)$ at $a \in k$ is $t - a$; a local coordinate $s_b \in K(s)$ at $b = f(a)$ is $s - b$. By definition of the Riemann-Hurwitz ramification degree,

$$\widetilde{e}_a(f) - 1 = \nu_{t-a}\left(\frac{d}{dt_a} s_b\right);$$

writing $f(t) - b = g(t-a)$ for some polynomial $g$ gives

$$\nu_{t-a}\left(\frac{d}{dt_a}s_b\right) = \nu_{t-a}(g'(t-a)) = \nu_{t-a}(f'(t)).$$

Secondly, $f(\infty) = \infty$ and local parameters there are given by $t^{-1}$, $s^{-1}$; therefore the Riemann-Hurwitz ramification degree at infinity is given by

$$\widetilde{e}_\infty(f) = \nu_{t^{-1}}\left(\frac{1}{f(t)}\right) + 1 = \deg f + (\deg f - \deg f' - 1).$$

$\square$

**Example 6.3.7.** For any integer $m > 1$ not divisible by $\operatorname{char} k$, let $f(t) = t^m$ in remark 6.3.5. Then $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ is unramified away from 0 and infinity, with $e_0(f) = e_\infty(f) = m$. Thus Fubini's theorem holds for $f$ (or, equivalently, for the set $\Gamma_f \subseteq k \times k$) if and only if $\chi(\mathbb{P}^1_k) = m\chi(\mathbb{P}^1_k) - 2(m-1)$; that is, if and only if $(\chi(\mathbb{P}^1_k) - 2)(m-1) = 0$.

However, now assume $\operatorname{char} k = p > 0$ and set $f(t) = t^p - t$. Then $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ is unramified outside infinity, where it is wildly ramified of degree $p$. Thus Fubini's theorem holds for $f$ (or, equivalently, for the set $\Gamma_f \subseteq k \times k$) if and only if $\chi(\mathbb{P}^1_k) = p\chi(\mathbb{P}^1_k) - (p-1)$; that is, if and only if $(\chi(\mathbb{P}^1_k) - 1)(p-1) = 0$.

Taking $m = p+1$ in the previous two paragraphs shows that Fubini fails for one of the sets $\Gamma_{t^p - t}$, $\Gamma_{t^p}$ or that $p$ is an idempotent in $R$.

The example shows that Fubini's theorem can fail when in finite characteristic:

**Theorem 6.3.8.** *Assume $\operatorname{char} k = p > 2$ and that $p \neq 1$ in $R$. Then there exists a subset of $k \times k$ for which Fubini's theorem does not hold.*

*Proof.* If Fubini does hold for the sets $\Gamma_{t^{p+1}}$ and $\Gamma_{t^{p+2}}$ of the previous example then it follows that $\chi(\mathbb{P}^1_k) = 2$. But then Fubini does not hold for $\Gamma_{t^p - X}$, unless $p - 1 = 0$ in $R$. $\square$

Now we prove the next main result, namely that Fubini's theorem forces $\chi$, our arbitrary Euler characteristic on the algebra of constructible sets, to be the usual Euler characteristic of a curve:

**Theorem 6.3.9.** *Suppose that $\operatorname{char} k \neq 2$ and that Fubini's theorem is true for any non-constant, separable, tame morphism $\phi : C \to \mathbb{P}^1_k$ from a projective curve to the projective line. Then for any projective curve $C$ we have $\chi(C) = 2(1-g)$, where $g$ is the genus of $C$.*

*Proof.* For any integer $m > 1$ not divisible by $\operatorname{char} k$, the morphism $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ induced by $f(t) = t^m$ is separable and tame; therefore we may apply Fubini's theorem to deduce $(\chi(\mathbb{P}^1_k) - 2)(m-1) = 0$. Therefore $\chi(\mathbb{P}^1_k) = 2$, which agrees with the desired genus formula.

Now let $C$ be a projective curve over $k$. By a classical result of algebraic geometry [Ful69, prop 8.1] there is, for any $n$ sufficiently large (depending on the genus $g$ of $C$), a non-constant morphism $\phi : C \to \mathbb{P}^1_k$ of degree $n$ with the property that any fibre contains at least $n-1$ points. For $n$ not divisible by $\operatorname{char} k$ such a morphism is separable and tame; therefore we are permitted to apply Fubini's theorem, deducing

$$\chi(C) = 2\deg\phi - \sum_{x \in C}(e_x(\phi) - 1).$$

But this is nothing other than the Riemann-Hurwitz formula for the morphism $\phi$; so we obtain $\chi(C) = 2(1-g)$ as claimed. $\square$

This allows us to strengthen the observation that Fubini fails in finite characteristic:

**Theorem 6.3.10.** *Suppose that char $k \neq 2$ and that Fubini's theorem is true for any non-constant, separable, tame morphism between projective curves. Then Fubini's theorem holds for a separable morphism between projective curves if and only if the morphism is tame.*

*Proof.* The previous result implies that $\chi(C) = 2(1-g)$ is the usual Euler characteristic of any projective curve $C$. Suppose that $\phi : C_1 \to C_2$ is a separable morphism of projective curves which is not everywhere tame. Then the Riemann-Hurwitz formula tells us that

$$\chi(C_1) = \chi(C_2)\deg\phi - \sum_{x\in C_1}(\widetilde{e}_x(\phi) - 1),$$

which is incompatible with Fubini's theorem for $\phi$ as $\widetilde{e}_x(\phi) \geq e_x(\phi)$ for all $x \in C_1$ with at least one value of $x$ for which we do not have equality. ☐

**Remark 6.3.11.** More precisely, in the situation of the previous result, we have

$$\chi(C_1) - \int_{C_2}\chi(\phi^{-1}(y))\,d\chi(y) = \sum_{x\in C_1} d_x(\phi),$$

where $d_x(\phi)$ is defined by $\mathfrak{D}_x(\phi) = e_x(\phi) - 1 + d_x(\phi)$; here $\mathfrak{D}_x(\phi)$ denotes the different of the extension $\mathcal{O}_{C_1,x}/\mathcal{O}_{C_2,\phi(x)}$ of discrete valuation rings (see also remark 6.3.4). $d_x(\phi)$ measures the wild ramification at $x$.

An Euler characteristic is typically considered an object of 'tame' mathematics [vdD98], and so this formula is slightly surprising in that it expresses wild information purely in terms of tame.

**Remark 6.3.12.** In proposition 4.4.1 we saw that if $F$ is a two-dimensional local field, then the characteristic function of

$$\Gamma = \{(x,y) \in F : (x, y - t^{-1}x^p) \in \mathcal{O}_F \times \mathcal{O}_F\}$$

fails to satisfy Fubini's theorem with respect to the two-dimensional integral; in fact,

$$\int_F\int_F \mathrm{char}_\Gamma(x,y)\,dxdy = 0$$

and

$$\int_F\int_F \mathrm{char}_\Gamma(x,y)\,dxdy = 1.$$

This is similar phenomenon to what we have just observed for the Euler characteristic $\chi$.

These results suggest interpreting the Riemann-Hurwitz formula as a modified 'repeated integral', adjusted in a suitable way to ensure that Fubini's theorem holds. Perhaps it is possible to modify the two-dimensional integration theory in a similar way by taking into account additional ramification data as suggested in section 6.1.3 above.

## 6.4 Strong Euler characteristics

In the previous section, we in fact only considered interchanging the order of integration in morphisms all of whose fibres were finite. This brief section is a study of the possible Euler characteristics which do satisfy this restricted version of Fubini's theorem.

**Definition 6.4.1.** Let $\Omega$ be a set with structure $\mathcal{A}$. An Euler characteristic $\chi$ is said to be *strong* if and only if whenever $f : A \to B$ is a definable function between two definable sets such that there exists a positive integer $n$, with $|\chi(f^{-1}(b))| = n$ for all $b \in B$, then $\chi(A) = n\chi(B)$.

**Remark 6.4.2.** A strong Euler characteristic satisfies Fubini's theorem in a very weak sense. For suppose $\chi$ is an Euler characteristic, $A \subseteq \Omega^n$, $B \subseteq \Omega^m$ are definable, and $f : A \to B$ is an $n$-to-1 mapping as in the definition; set $\Gamma = \{(x, y) \in \Omega^n \times \Omega^m : x \in A,\ y \in B,\ f(x) = y\}$. Then Fubini's theorem holds for $\mathrm{char}_\Gamma$ if and only if $\chi(A) = n\chi(B)$.

It is straightforward to establish non-existence in certain cases and uniqueness in others:

**Theorem 6.4.3.** *Suppose $k$ is an algebraically closed field, of finite characteristic $> 2$, with the structure of constructible sets; then no strong Euler characteristic exists.*

*Proof.* This is just a restatement of theorem 6.3.8, where the counterexample did not require $\chi$ to satisfy the full Fubini property, but merely be strong. □

**Theorem 6.4.4.** *Suppose $k$ is an algebraically closed field, of characteristic zero, with the structure of constructible sets; then at most one strong Euler characteristic exists, and it is $\mathbb{Z}$ valued.*

*Proof.* Let $\chi_i$ be strong Euler characteristics, for $i = 1, 2$. The algebra of constructible subsets of $k^n$ is generated by the irreducible closed subsets, and therefore it is enough to establish $\chi_1(V) = \chi_2(V)$ for any irreducible closed $V \subseteq k^n$; this we do by induction on the dimension $d$ of $V$. Let $V'$ be the closure of $V$ in $\mathbb{P}^n_k$; then $V' \setminus V$ has dimension strictly less than that of $V$, and so, by the inductive hypothesis, it is enough to establish $\chi_1(V') = \chi_2(V')$.

Let $f : V' \to \mathbb{P}^d_k$ be a finite projective morphism; this always exists (see e.g. [Liu02, Lem. 6.4.27]). Let $\Sigma \subset V'$ denote the points at which $V'$ is non-singular or at which $f$ is not étale; this is closed in $V'$ by [Liu02, Prop. 4.2.24, Cor. 4.4.12]. Since morphisms of finite type are closed, $U := \mathbb{P}^d_k \setminus f(\Sigma)$ is an open subset of $\mathbb{P}^d_k$, and it is non-empty because it contains the generic point (here it is important to observe that $K(V')/K(\mathbb{P}^1_k)$ is a *separable* extension of fields).

Hence the restriction of $f$ to $f^{-1}(U)$ is a finite étale morphism to $\mathbb{P}^1_k$, i.e. an étale cover, of degree $m = |K(V') : K(\mathbb{P}^1_k)|$; the assumption that each $\chi_i$ is strong implies

$$\chi_i(f^{-1}(U)) = m\chi_i(U)$$

for $i = 1, 2$. Moreover, $\dim(V' \setminus f^{-1}(U))$ and $\dim(f(\Sigma))$ are both $< d$, and therefore the inductive hypothesis lets us deduce

$$\begin{aligned}
\chi_1(V') &= \chi_1(f^{-1}(U)) + \chi_1(V' \setminus f^{-1}(U)) \\
&= m(\chi_1(\mathbb{P}^d_k) - \chi_1(f(\Sigma))) + \chi_2(V' \setminus f^{-1}(U)) \\
&= m(\chi_1(\mathbb{P}^d_k) - \chi_2(f(\Sigma))) + \chi_2(V' \setminus f^{-1}(U)) \\
&= m(\chi_1(\mathbb{P}^d_k) - \chi_2(\mathbb{P}^d_k)) + \chi_2(V').
\end{aligned}$$

It remains only to prove that our two Euler characteristics agree on $\mathbb{P}^d_k$. Decomposing projective space into a disjoint union of constructible sets $\mathbb{P}^d_k = \bigsqcup_{i=0}^d \mathbb{A}^i_k$ and using multiplicativity of each $\chi_i$ on products, we have finally reduced the problem to proving that $\chi_1(\mathbb{A}^1_k) = \chi_2(\mathbb{A}^1_k)$.

But the argument of the first paragraph of theorem 6.3.9, which is valid for any strong Euler characteristic, establishes that $\chi_i(\mathbb{A}^1_k) = 1$ for $i = 1, 2$. □

**Remark 6.4.5.** If $k = \mathbb{C}$ then a strong Euler characteristic does exist on the structure of constructible sets, namely the topological Euler characteristic. This follows from the classical result that if $\widetilde{X} \to X$ is an $n$-sheeted covering of a CW-complex $X$, then $\chi_{\text{top}}(\widetilde{X}) = n\chi_{\text{top}}(X)$.

The Lefschetz principle (i.e. that the first order theory of algebraically closed fields of characteristic zero is complete; see [Che76] for a classical discussion of this principle) now implies that a strong Euler characteristic exists for any algebraically closed field of characteristic zero.

**Remark 6.4.6.** The inclusion of this material is inspired by [Kra00] and [KS00], where strong Euler characteristics (in fact, the definition of 'strong' in these papers is slightly stronger than the definition we have used) are discussed from the perspective of model theory. In [KS00], it is proved that a universal strong Euler characteristic $\bigsqcup_{n=0}^{\infty} \mathcal{A}(\Omega^n) \to K_0^{\mathrm{s}}(\Omega)$ exists, and so our previous theorem and remark prove that if $k$ is an algebraically closed field of characteristic zero, with the structure of constructible sets, then $K_0^{\mathrm{s}}(k) = \mathbb{Z}$.

## 6.5 Riemann-Hurwitz and Fubini's theorem for surfaces

Now we generalise the results of section 6.3 from curves to surfaces. $k$ continues to be an algebraically closed field, and $\chi$ is a fixed $R$-valued Euler characteristic on the structure of constructible sets. In this section, 'surface' means a smooth, two-dimensional, irreducible algebraic variety over $k$, whereas a 'curve' is merely a one-dimensional, reduced, algebraic variety over $k$.

If $\phi : S_1 \to S_2$ is a finite, separable morphism between projective surfaces of degree $n$, then let $B \subseteq S_2$ be the set of $y \in S_2$ such that $\phi^{-1}(y)$ does not contain $n$ points. Zariski's purity theorem (see e.g. [Liu02, ex. 8.2.15] or [Zar58]) states that $B$ is equidimensional of dimension one; let $B_1, \ldots, B_r$ be its irreducible components, and let $n_i$ be the degree of the morphism $\phi|_{\phi^{-1}(B_i)} : \phi^{-1}(B_i) \to B_i$ (note that the degree is well-defined, as the base curve is irreducible, though the covering curve $\phi^{-1}(B_i)$ may be reducible). Using this data we may prove an analogue of lemma 6.3.1:

**Theorem 6.5.1.** *Let $\phi : S_1 \to S_2$ be a finite, separable morphism between projective surfaces, with notation as in the previous paragraph. Then Fubini holds for $\phi$ (in the same sense as section 6.3) if and only if the following formula relating $\chi(S_1)$ and $\chi(S_2)$ is satisfied:*

$$\chi(S_1) = \chi(S_2)\ \deg\phi - \sum_{i=1}^{r}(n-n_i)\chi(B_i) + \sum_{y\in B}\left(|\phi^{-1}(y)| - n + \sum_{i=1}^{r}(n-n_i)m_i(y)\right),$$

*where $m_i(y)$ denotes the number of local branches of $B_i$ at $y$. If $\chi$ is a strong Euler characteristic then this formula holds.*

*Proof.* We must show that the right hand side of the formula is equal to the fibre integral $\int_{S_2} |\phi^{-1}(y)|\, d\chi(y)$.

The normalisation of $B$ is by definition $\pi_B : \widetilde{B} = \bigsqcup_{i=1}^{r} \widetilde{B}_i \to B$, where $\pi_i : \widetilde{B}_i \to B_i$ is the normalisation of the irreducible curve $B_i$. Write $D = \phi^{-1}(B)$, and let $\pi_D : \widetilde{D} \to D$ be its normalisation in the same way as $B$; the functoriality of normalising implies that there is an induced morphism $\widetilde{\phi} : \widetilde{D} \to \widetilde{B}$ such that $\pi_B\widetilde{\phi} = \phi|_D\pi_D$.

Let $Z \subset B$ be a large enough finite set of points such that $Z$ includes all singular points of the curve $B$, $\phi^{-1}(Z)$ includes all singular points of the curve $\phi^{-1}(B)$, and

$\widetilde{\phi}^{-1}(\pi_B^{-1}(Z))$ includes all points of ramification of $\widetilde{\phi}$. Then $\pi_D$ and $\pi_B$ induce isomorphisms $\widetilde{D}\setminus\widetilde{\phi}^{-1}(\pi_B^{-1}(Z)\cong D\setminus\phi^{-1}(Z)$ and $\widetilde{B}\setminus\pi_B^{-1}(Z)\cong B\setminus Z$; therefore

$$\begin{aligned}\int_{B\setminus Z}|\phi^{-1}(y)|\,d\chi(y)&=\int_{\widetilde{B}\setminus\pi_B^{-1}(Z)}|\widetilde{\phi}^{-1}(y)|\,d\chi(y)\\&=\int_{\widetilde{B}}|\widetilde{\phi}^{-1}(y)|\,d\chi(y)-\int_{\pi_B^{-1}(Z)}|\widetilde{\phi}^{-1}(y)|\,d\chi(y)\\&=\sum_{i=1}^{r}\int_{\widetilde{B}_i}|\widetilde{\phi}^{-1}(y)|\,d\chi(y)-\sum_{y\in\pi_B^{-1}(Z)}|\widetilde{\phi}^{-1}(y)|.\end{aligned}$$

Further, as we saw in the proof of lemma 6.3.1,

$$\int_{\widetilde{B}_i}|\widetilde{\phi}^{-1}(y)|\,d\chi(y)=n_i\chi(\widetilde{B}_i)+\sum_{y\in\widetilde{B}_i\cap\pi_B^{-1}(Z)}(|\widetilde{\phi}^{-1}(y)|-n_i).$$

Since $\widetilde{B}_i\setminus\pi_B^{-1}(Z)\cap\widetilde{B}_i\cong B_i\setminus Z\cap B_i$, we have $\chi(\widetilde{B}_i)=\chi(B_i)+\sum_{y\in Z\cap B_i}(m_i(y)-1)$; combining the last few identities therefore gives

$$\begin{aligned}\int_B|\phi^{-1}(y)|\,d\chi(y)=&\sum_i n_i\chi(B_i)+\sum_i n_i\sum_{y\in B_i\cap Z}m_i(y)-\sum_i|B_i\cap Z|\\&-\sum_i\sum_{y\in\widetilde{B}_i\cap\pi_B^{-1}(Z)}n_i+\sum_{y\in Z}|\phi^{-1}(y)|.\end{aligned}$$

To complete the proof, combine this formula with

$$\begin{aligned}\int_{S_2}|\phi^{-1}(y)|\,d\chi(y)&=n\chi(S_2\setminus B)+\int_B|\phi^{-1}(y)|\,d\chi(y)\\&=n\chi(S_2)-n(\sum_i\chi(B_i)-\sum_{y\in Z}(c(y)-1))+\int_B|\phi^{-1}(y)|\,d\chi(y),\end{aligned}$$

where $c(y)$ denotes the number of irreducible components of $B$ which pass through $y$ (note that $\sum_{y\in Z}c(y)=\sum_i|B_i\cap Z|$). $\square$

**Remark 6.5.2.** When $k=\mathbb{C}$ and $\chi=\chi_{\text{top}}$ is the topological Euler characteristic, which we have remarked earlier (remark 6.4.5) is a strong Euler characteristic, then the theorem proves that

$$\chi_{\text{top}}(S_1)=\chi_{\text{top}}(S_2)\,\deg\phi-\sum_{i=1}^{r}(n-n_i)\chi(B_i)+\sum_{y\in B}\left(|\phi^{-1}(y)|-n+\sum_{i=1}^{r}(n-n_i)m_i(y)\right).$$

The Lefschetz principle now implies that the formula remains true if we replace $k$ by any algebraically closed field of characteristic zero, and $\chi_{\text{top}}(S_i)$ by the $l$-adic Euler characteristic (=alternating sum of Betti numbers of $l$-adic étale cohomology of $S_i$, =degree of the second Chern class of $S_i$).

This generalisation of the Riemann-Hurwitz formula to surfaces is due to B. Iversen [Ive70], who established it with purely algebraic techniques by studying pencils of curves on the surfaces. Iversen remarks in his paper that a more topological proof should be possible when $k=\mathbb{C}$, and our approach provides that.

**Remark 6.5.3.** A natural question now to ask is whether an analogue of the theorem holds in higher dimensions. If $X_1 \to X_2$ is a finite morphism between $d$-dimensional smooth projective varieties over $k$, then the branch locus will be pure of dimension $d - 1$, so one can hope to obtain results by induction on dimension. The difficulty which appears when the branch locus has dimension $> 1$ is that there is no functorial way to desingularise. It is unclear to the author at present how significant a problem this is. The resulting formulae may even be too elaborate to be useful.

**Remark 6.5.4.** Another interesting question concerns the situation in characteristic $p$. We noted in remark 6.3.11 that, for curves, the difference between the Euler characteristic and the integral over the fibres was a measure of the wild ramification. For surfaces, the situation is more complex, since the wild ramification of surfaces is not fully understood. However, assuming that there is no ferocious ramification present (this is when inseparable morphisms between curves appear), I. Zhukov [Zhu05] has successfully generalised Iversen's formula by defining appropriate ramification invariants; this provides an explicit formula for

$$\chi(S_1) - \int_{S_2} |\phi^{-1}(y)| \, dy$$

in terms of the wild ramification of the cover.

The Riemann-Hurwitz formula for curves is a special case of the Grothendieck-Ogg-Shafarevich formula for $\ell$-adic shaves, and the problem of understanding Riemann-Hurwitz for surfaces is a special case of extending Grothendieck-Ogg-Shavarevich to higher dimensional varieties. Assuming that a two-dimensional integration theory can be developed which encodes local ramification data, as suggested in subsection 6.1.3, then it may be possible to reproduce the arguments of theorem 6.5.1 with a similarly refined Euler characterstic, in such a way as to prove Riemann-Hurwitz for surfaces without any restrictions on the ramification.

CHAPTER 7

# An explicit approach to residues on and canonical sheaves of arithmetic surfaces

We develop a theory of residues for arithmetic surfaces, establish the reciprocity law around a point and use the residue maps to explicitly construct the dualising sheaf of our surface. These are generalisations of known results for surfaces over a perfect field.

## 7.1 Introduction

As much for author's benefit as that of the reader, we say a few words about the relation of this work to previous results of others:

### 7.1.1 An introduction to the higher adèlic method

We begin with a reminder of some material already contained in the introduction to the thesis. A two-dimensional local field is a compete discrete valuation field whose residue field is a local field (e.g. $\mathbb{Q}_p((t))$); for an introduction to such fields, see [FK00]. If $A$ is a two-dimensional domain, finitely generated over $\mathbb{Z}$, with fields of fractions $F$ and $0 \lhd \mathfrak{p} \lhd \mathfrak{m} \lhd A$ is a chain of primes in $A$, then consider the following sequence of localisations and completions:

$$\begin{array}{ccccccccccccc} A & \rightsquigarrow & A_{\mathfrak{m}} & \rightsquigarrow & \widehat{A_{\mathfrak{m}}} & \rightsquigarrow & \left(\widehat{A_{\mathfrak{m}}}\right)_{\mathfrak{p}'} & \rightsquigarrow & \widehat{\left(\widehat{A_{\mathfrak{m}}}\right)_{\mathfrak{p}'}} & \rightsquigarrow & \left(\widehat{\left(\widehat{A_{\mathfrak{m}}}\right)_{\mathfrak{p}'}}\right)_0 & = & \operatorname{Frac}\left(\widehat{\left(\widehat{A_{\mathfrak{m}}}\right)_{\mathfrak{p}'}}\right) \\ &&&&&&&& \| &&&& \| \\ &&&&&&&& A_{\mathfrak{m},\mathfrak{p}} &&&& F_{\mathfrak{m},\mathfrak{p}} \end{array}$$

which we now explain in greater detail. It follows from excellence of $A$ that $\mathfrak{p}' := \mathfrak{p}\widehat{A_{\mathfrak{m}}}$ is a radical ideal of $\widehat{A_{\mathfrak{m}}}$; we may localise and complete at $\mathfrak{p}'$ and again use excellence to deduce that $0$ is a radical ideal in the resulting ring i.e. $A_{\mathfrak{m},\mathfrak{p}}$ is reduced. The total field of fractions $F_{\mathfrak{m},\mathfrak{p}}$ is therefore isomorphic to a finite direct sum of fields, and each is a two-dimensional local field.

Geometrically, if $X$ is a two-dimensional, integral scheme of finite type over $\operatorname{Spec}\mathbb{Z}$ with function field $F$, then to each closed point $x \in X$ and integral curve $y \subset X$ which contains $x$, one obtains a finite direct sum of two-dimensional local fields $F_{x,y}$. Two-dimensional adèlic theory aims to study $X$ via the family $(F_{x,y})_{x,y}$, in the same way as one studies a curve or number field via its completions. Analogous constructions exist in higher dimensions. Useful references are [HY96] [Par83, §1].

### 7.1.2 The classical case of a curve over a perfect field

This chapter is based closely on similar classical results for curves and it will be useful to give a detailed account of that theory.

#### Smooth curves

Firstly, let $C$ be a smooth, connected, projective curve over a perfect field $k$ (of finite characteristic, to avoid complications with differential forms). We follow the discussion in [Har77, III.7.14]. For each closed point $x \in C$ one defines the residue map $\mathrm{Res}_x : \Omega^1_{K(C)/k} \to k$, and one then proves the reciprocity law

$$\sum_{x \in C_0} \mathrm{Res}_x(\omega) = 0,$$

for all $\omega \in \Omega^1_{K(C)/k}$. Consider $\Omega^1_{K(C)/k}$ as a constant sheaf on $C$; then

$$0 \to \Omega^1_{C/k} \to \Omega^1_{K(C)/k} \to \Omega^1_{K(C)/k}/\Omega^1_{C/k} \to 0$$

is a flasque resolution of $\Omega^1_{C/k}$, and the corresponding long exact sequence of Čech cohomology is

$$0 \to \Omega^1_{C/k}(C) \to \Omega^1_{K(C)/k} \to \bigoplus_{x \in C_0} \frac{\Omega^1_{K(C)/k}}{\Omega^1_{\mathcal{O}_{C,x}/k}} \to H^1(C, \Omega^1_{C/k}) \to 0. \qquad (\dagger)$$

Now, the map $\sum_x \mathrm{Res}_x : \bigoplus_{x \in C_0} \Omega^1_{K(C)/k}/\Omega^1_{\mathcal{O}_{C,x}/k} \to k$ vanishes on the image of $\Omega^1_{K(C)/k}$ (by the reciprocity law), and so induces

$$\mathrm{tr}_{C/k} : H^1(C, \Omega^1_{C/k}) \to k,$$

which is the trace map of $C/k$ with respect to the dualising sheaf $\Omega^1_{C/k}$.

Moreover, duality of $C$ may be interpreted (and proved) adèlically as follows; see [Ser88, II.§8]. For each $x \in C_0$, let $K(C)_x$ be the completion of $K(C)$ at the discrete valuation $\nu_x$ associated to $x$, and let

$$\mathbb{A}_C = \{(f_x) \in \prod_{x \in C_0} K(C)_x : \nu_x(f_x) \geq 0 \text{ for all but finitely many } x\}$$

be the adèlic space of $C$. Also, let

$$\mathbb{A}(\Omega^1_{C/k}) = \{(\omega_x) \in \prod_{x \in C_0} \Omega^1_{K(C)_x/k} : \nu_x(\omega_x) \geq 0 \text{ for all but finitely many } x\}$$

be the differential adèlic space of $C$. Then, under the pairing

$$\mathbb{A}_C \times \mathbb{A}(\Omega^1_{C/k}) \to k, \quad ((f_x), (\omega_x)) \mapsto \sum_{x \in C_0} \mathrm{Res}_x(f_x \omega_x),$$

the orthogonal complement of $\mathbb{A}(\Omega^1_{C/k}(D))$ is

$$\mathbb{A}(\Omega^1_{C/k}(D))^{\perp} = \mathbb{A}_C(D).$$

Here $D$ is a divisor on $C$, and $\mathbb{A}_C(D)$ (resp. $\mathbb{A}(\Omega^1_{C/k}(D))$) is the subgroup of $\mathbb{A}_C$ (resp. $\mathbb{A}(\Omega^1_{C/k})$ for which $\nu_x(f_x) \geq -\nu_x(D)$ (resp. $\nu_x(\omega_x) \geq \nu_x(D)$) for all $x$. Moreover, the global elements, embedded diagonally, are self-dual:

$$K(C)^\perp = \Omega^1_{K(C)/k}.$$

The exact sequence (†) generalises to the twisted sheaf $\Omega^1_{C/k}(D)$, and thereby provides an isomorphism $\mathbb{A}(\Omega^1_{C/k})/(\Omega^1_{K(C)/k} + \mathbb{A}(\Omega^1_{C/k}(D))) \cong H^1(C, \Omega^1_{C/k}(D))$; combining this with the aforementioned adèlic dualities yields the non-degenerate pairing

$$\mathcal{L}(D) \times H^1(C, \Omega^1_{C/k}(D)) \to k,$$

where

$$\mathcal{L}(D) := K(C) \cap \mathbb{A}_C(D) = \{f \in K(C) : \nu_x(f) \geq -\nu_x(D) \text{ for all } x \in C_0\}.$$

This is exactly duality of $C/k$.

### Singular curves

Secondly, suppose that $C$ is allowed to have singularities; we now follow [Ser88, IV.§3]. One may still define a residue map at each closed point $x$; in fact, if $\pi : \widetilde{C} \to C$ is the normalisation of $C$, then

$$\operatorname{Res}_x = \sum_{x' \in \pi^{-1}(x)} \operatorname{Res}_{x'}.$$

The sheaf of *regular differentials* $\Omega'_{C/k}$ is defined, for open $U \subseteq X$, by

$$\Omega'_{C/k}(U) = \{\omega \in \Omega^1_{K(C)/k} : \operatorname{Res}_x(f\omega) = 0 \text{ for all closed points } x \in U \text{ and all } f \in \mathcal{O}_{C,x}\}.$$

If $U$ contains no singular points of $C$, then $\Omega'_{C/k}|_U = \Omega^1_{U/k}$. By establishing a Riemann-Roch type result, it follows that $\Omega'_{C/k}$ is the dualising sheaf of $C/k$. Analogously to the smooth case, one explicitly constructs the trace map

$$\operatorname{tr}_{C/k} : H^1(C, \Omega'_{C/k}) \to k,$$

and, as in [Gre88], uses it and adèlic spaces to prove duality. See [Stö93] for more on the theory of regular differentials on curves.

### 7.1.3 The case of a surface over a perfect field

There is also a theory of residues on algebraic surfaces, developed by A. Parshin [Par83] [Par00], the founder of the higher dimensional adèlic approach to algebraic geometry. Let $X$ be a connected, smooth, projective surface over a perfect field $k$. To each closed point $x \in X$ and curve $y \subset X$ containing $x$, he defined a two-dimensional residue map

$$\operatorname{Res}_{x,y} : \Omega^2_{K(X)/k} \to k$$

and proved the reciprocity laws both around a point

$$\sum_{\substack{y \subset X \\ y \ni x}} \operatorname{Res}_{x,y} \omega = 0$$

(for fixed $x \in X_0$ and $\omega \in \Omega^2_{K(X)/k}$) and along a curve

$$\sum_{\substack{x \in X_0 \\ x \in y}} \operatorname{Res}_{x,y} \omega = 0$$

(for fixed $y \subset X$ and $\omega \in \Omega^2_{K(X)/k}$). By interpreting the Čech cohomology of $X$ adèlically and proceeding analogously to the case of a curve, these residue maps may be used to explicitly construct the trace map

$$\operatorname{tr}_{X/k} : H^2(X, \Omega^2_{X/k}) \to k$$

and, using two-dimensional adèlic spaces, prove duality.

D. Osipov [Osi00] considers the algebraic analogue of our setting, with a smooth, projective surface $X$ over a perfect field $k$ and a projective morphism $f : X \to S$ to a smooth curve. To each closed point $x \in X$ and curve $y \subset X$ containing $x$, he constructs a 'direct image map'

$$f_*^{x,y} : \Omega^2_{K(X)/k} \to \Omega^1_{K(S)_s/k},$$

where $s = f(x)$ and $K(S)_s$ is the $s$-adic completion of $K(S)$. He establishes the reciprocity law around a point, analogous to our theorem 7.4.1, and the reciprocity law along a fibre. He uses the $(f_*^{x,y})_{x,y}$ to construct $f_* : H^2(X, \Omega^2_{X/k}) \to H^1(S, \Omega^1_{S/k})$, which he proves is the trace map.

Osipov then considers multiplicative theory. Let $K_2(X)$ denote the sheafification of $X \supseteq U \mapsto K_2(\mathcal{O}_X(U))$; then $H^2(X, K_2(X)) \cong \mathrm{CH}_2(X)$. Osipov defines, for each $x \in y \subset X$, homomorphisms

$$f_*(\ ,\ )_{x,y} : K_2(K(X)) \to K(S)_s^\times,$$

and establishes the reciprocity laws around a point and along a fibre. At least when char $k = 0$, these are then used to construct a map

$$\mathrm{CH}^2(X) = H^2(X, K_2(X)) \to H^1(C, \mathcal{O}_C^\times) = \operatorname{Pic}(C),$$

which is proved to be the usual push-forward of cycles [Ful98, §1].

### 7.1.4 Higher dimensions

The theory of residues for surfaces was extended to higher dimensional varieties by V. G. Lomadze [Lom81]. Let $X$ be a $d$-dimensional, integral scheme of finite type over a field $k$. To each complete flag of integral subvarieties

$$\underline{x} = \langle x_0 \subset \cdots \subset x_d \rangle,$$

Lomadze associates a residue map $\operatorname{Res}_{\underline{x}} : \Omega^d_{K(X)/k} \to K$ and proves the reciprocity law

$$\sum_{x_i} \operatorname{Res}_{\underline{x}} \omega = 0.$$

Here we have fixed a flag $x_0 \subset \cdots \subset x_{i-1} \subset x_{i+1} \subset \cdots \subset x_n$ and vary the sum over all $i$-dimensional integral subvarieties $x_i$ sitting between $x_{i-1}$ and $x_{i+1}$ (if $i = n$ then we must assume $X$ is projective).

Lomadze also develops a higher dimensional relative theory, analogous to Osipov's study of a surface over a curve.

### 7.1.5 Explicit Grothendieck duality

It is an interesting problem whether Grothendieck duality [AK70] [Har66] can be made more explicit. The guiding example is that of a curve over a finite field which we discussed above, where the trace map may be constructed via residues. The duality theorem is even equivalent to Poisson summation on the ring of adèles of the curve; the simplest exposition of duality is probably that of [Mor91]. Using the Parshin-Lomadze theory of residues, A. Yekutieli [Yek92] has explicitly constructed the Grothendieck residue complex of an arbitrary reduced scheme of finite type over a field.

For a far better summary of this problem than the author can provide, the reader should consult the introduction to [Yek92] and others of Yekutieli's papers, e.g. [HY96] [SY95].

### 7.1.6 Adèlic analysis

This chapter has many connections to I. Fesenko's programme of two-dimensional adèlic analysis [Fes06] [Fes03] [Fes08b] [Fes08a], and is part of the author's attempt to understand the connection between adèlic analysis and more familiar methods in algebraic geometry.

Two-dimensional adèlic analysis aims to generalise the current rich theories of topology, measure, and harmonic analysis which exist for local fields, by which mathematicians study curves and number fields, to dimension two. In particular, Fesenko generalises the Tate-Iwasawa [Iwa92] [Tat67] method of studying the zeta function of a global field to dimension two, giving a new approach to the study of the $L$-function of an elliptic curve over a global field. The author hopes that the reader is satisfied to hear only the most immediate relations between this fascinating subject and the current chapter.

Let $E$ be an elliptic curve over a number field $K$, with function field $F = K(E)$, and let $\mathcal{E}$ be a regular, proper model of $E$ over the ring of integers $\mathcal{O}_K$. Then $\mathcal{E}$ satisfies the same assumptions of $X$ in our main theorem 7.7.5 below. Let $\psi = \otimes_{s\in S}\psi_s : \mathbb{A}_K \to S^1$ be an additive character on the adèle group of $K$, and let $\omega \in \Omega^1_{F/K}$ be a fixed, non-zero differential form. For $x \in y \subset \mathcal{E}$ a point contained in a curve as usual, with $x$ sitting over $s \in S$, introduce an additive character

$$\psi_{x,y} : F_{x,y} \to S^1, \quad a \mapsto \psi_s(\operatorname{Res}_{x,y}(a\omega)),$$

where $\operatorname{Res}_{x,y}$ is the relative residue map which we will construct in section 7.4. If $x$ is a fixed point, then our reciprocity law will imply

$$\sum_{\substack{y\subset X\\ y\ni x}} \psi_{x,y}(a) = 0$$

for any $a \in F$.

Moreover, suppose that $\psi$ is trivial on global elements and that $y$ is a fixed horizontal curve; then Fesenko also proves [Fes08b, §27 Proposition]

$$\sum_{\substack{x\in X_0\\ x\in y\cup\{\mathrm{arch}\}}} \psi_{x,y}(a) = 0.$$

We are deliberately vague here. Let us just say that we must adjoin archimedean points to $y$, consider two-dimensional archimedean local fields such as $\mathbb{R}((t))$, and define suitable additive characters at these places; once these have been suitably introduced, this reciprocity law follows from adèlic reciprocity for the number field $k(y)$.

### 7.1.7 Future work

The author is thinking about several topics related to this chapter which may interest the reader. Let $X \to \mathcal{O}_K$ be an arithmetic surface.

**Reciprocity along vertical curves**

There is surely a reciprocity law for the residue maps $(\mathrm{Res}_{x,y})_x$ along any fixed vertical curve $y \subset X$. The author can currently only prove it for certain special cases, such as when $y$ is an entire irreducible vertical fibre.

**Grothendieck duality**

The canonical sheaf $\omega_{X/S}$ is the dualising sheaf. It should be possible to use our residue maps $(\mathrm{Res}_{x,y})_{x,y}$ to construct the relative trace map

$$\mathrm{tr}_{X/S} : H^1(X, \omega_{X/S}) \to \mathcal{O}_K,$$

and give an explicit adèlic proof of Grothendieck duality, similar to the existing work of Yekutieli for varieties. This should follow relatively easily from the contents of this chapter.

**Horizontal reciprocity**

If $y$ is horizontal then such a reciprocity law does not make sense naively, since the residues $\mathrm{Res}_{x,y}\,\omega$ belong to different fields as $x$ varies across $y$. Of course, this is the familiar problem that $\mathrm{Spec}\,\mathcal{O}_K$ is not a relative curve. As explained in the discussion of Fesenko's work above, this is fixed by taking into account the archimedean data. Such results live outside the realm of algebraic geometry, and need to be better understood.

**Two-dimensional Poisson summation**

Perhaps it is possible to find a global duality result on $X$ which incorporates not only Grothendieck duality of $X$ relative to $S$, but also the arithmetic duality on the base i.e. Poisson summation. Such a duality would necessarily incorporate archimedean data and perhaps be most easily expressed adèlically. In the case of a regular, proper model of an elliptic curve, this may already be provided by one of Fesenko's additive dualities [Fes08b, §32, Proposition].

**Multiplicative theory**

We have focused on additive theory, but as we mentioned while discussing Osipov's work, there are natural multiplicative analogues. In fact, the 'multiplicative residue map' for mixed characteristic two-dimensional local fields has been defined by K. Kato [Kat83]. Fesenko's work includes an adèlic interpretation of the conductors of the special fibres of $\mathcal{E}$, but only under the assumption that the reduced part of each fibre is semi-stable [Fes08b, §40, Remark 2]; similar results surely hold in greater generality and are related to 'conductor = discriminant' formulae [KS04] [LS00] [Sai88].

Moreover, Fesenko's two-dimensional theta formula [Fes08b, 3.6] is an adèlic duality which takes into account the interplay between the additive and multiplicative structures. It is important to understand better its geometric interpretation, at least in the case of an algebraic surface.

Perhaps it is also possible to study vanishing cycles [Sai87] using similar techniques.

### 7.1.8 Notation

If $A$ is a (always commutative) ring, then we write $\mathfrak{p} \lhd A$ to denote that $\mathfrak{p}$ is an ideal of $A$; this notation seems to be common to those educated in Oxford, and less familiar to others. We write $\mathfrak{p} \lhd^1 A$ to indicate that the height of $\mathfrak{p}$ is $1$. If $\mathfrak{p}$ is prime, then $k(\mathfrak{p}) = \operatorname{Frac} A/\mathfrak{p}$ is the residue field at $\mathfrak{p}$. If $A$ is a local ring, then the maximal ideal is $\mathfrak{m}_A$.

If $F$ is a complete discrete valuation field, then its ring of integers is $\mathcal{O}_F$, with maximal ideal $\mathfrak{p}_F$. The residue field $k(\mathfrak{p}_F)$ will be denoted $\overline{F}$; this notation seems to be common among those affected by the Russian school of arithmetic geometry. Discrete valuations are denoted $\nu$, usually with an appropriate subscript to avoid confusion.

If $A$ is a $B$-algebra, the the space of relative Kahler differentials is $\Omega_{A/B} = \Omega^1_{A/B}$.

Injective maps are often denoted by $\hookrightarrow$, and surjective maps by $\twoheadrightarrow$.

## 7.2 Local relative residues

Here we develop a theory of residues of differential forms on two-dimensional local fields. Recall that a two-dimensional local field is a complete discrete valuation field $F$ whose residue field $\overline{F}$ is a (non-archimedean, in this chapter) local field. We will be interested in such fields $F$ of characteristic zero; when the local field $\overline{F}$ also has characteristic zero then we say that $F$ has *equal characteristic zero*; when $\overline{F}$ has finite characteristic, then $F$ is said to be of *mixed characteristic*.

### 7.2.1 Continuous differential forms

We begin by explaining how to construct suitable spaces of 'continuous' differential forms.

For any Noetherian, local ring $A$ and $A$-module $N$, we will denote by $N^{\mathrm{sep}}$ the maximal Hausdorff (=separated) quotient for the $\mathfrak{m}_A$-adic topology, i.e.

$$N^{\mathrm{sep}} = N \Big/ \bigcap_{n=1}^{\infty} \mathfrak{m}_A^n N \ .$$

**Remark 7.2.1.** Suppose that $A/B$ is a finite extension of Noetherian, local domains. Then $\mathfrak{m}_A \cap B = \mathfrak{m}_B$. Also, the fibre $A \otimes_B k(\mathfrak{m}_B)$ is a finite dimensional $k(\mathfrak{m}_B)$-vector space, and is therefore Artinian; hence $\mathfrak{m}_B A$ contains $\mathfrak{m}_A^n$ for $n \gg 0$. So for any $B$-module $N$,

$$N^{\mathrm{sep}} \otimes_B A = (N \otimes_B A)^{\mathrm{sep}}.$$

**Lemma 7.2.2.** *Let $A/B$ be a finite extension of Noetherian, local domains, which are $R$ algebras, where $R$ is a Noetherian domain. Assume that $\Omega^{sep}_{B/R}$ is a free $B$-module, and that $\operatorname{Frac} A/\operatorname{Frac} B$ is a separable extension. Then there is an exact sequence*

$$0 \to \Omega^{sep}_{B/R} \otimes_B A \to \Omega^{sep}_{A/R} \to \Omega_{A/B} \to 0$$

*of $A$-modules.*

*Proof.* The standard exact sequence of differential forms is

$$\Omega_{B/R} \otimes_B A \to \Omega_{A/R} \to \Omega_{A/B} \to 0.$$

Since $A$ is a finite $B$-module, the space of differentials $\Omega_{A/B}$ is a finitely generated, torsion $A$-module. Apply $^{\mathrm{sep}}$ to the sequence to obtain, using remark 7.2.1,

$$\Omega^{\mathrm{sep}}_{B/R} \otimes_B R \xrightarrow{j} \Omega^{\mathrm{sep}}_{B/R} \to \Omega_{A/B} \to 0,$$

which is exact. It remains to prove that $j$ is injective.

Let $F$, $M$, $K$ be the fields of fractions of $A$, $B$, $R$ respectively, and let $\omega \in \Omega^{\mathrm{sep}}_{B/R}$ be an element of some chosen $B$-basis for this free module. Let $D_\omega : \Omega^{\mathrm{sep}}_{B/R} \to B$ send $\omega$ to $1$ and vanish on all other elements of the chosen basis. This homomorphism extends first to an $M$-linear map $D_M : \Omega_{M/K} \to M$, and then to an $F$-linear map $D_F : \Omega_{F/K} \to F$; this follows from the identifications $\Omega_{B/R} \otimes_B M \cong \Omega_{M/K}$ and $\Omega_{M/K} \otimes_M F \cong \Omega_{F/K}$. Finally, it induces an $R$-linear derivation $D : A \to F$ by $D(a) = D_F(d(a))$, where $d : F \to \Omega_{F/K}$ is the universal derivation.

Let $N \subseteq F$ be the $A$-module spanned by $D(a)$, for $a \in A$. This is a finitely generated $A$-module, for if $a_1, \dots, a_n$ generate $A$ as a $B$-module, then $N$ is contained in the $A$-module spanned by $a_1, \dots, a_n, D(a_1), \dots, D(a_n)$. Thus the non-zero homomorphism $\widetilde{D} : \Omega_{A/R} \to N$ induced by $D$ factors through $\Omega^{\mathrm{sep}}_{A/R}$ (by Nakayama's lemma). Further, $\widetilde{D}$ sends $j(\omega) \in \Omega^{\mathrm{sep}}_{A/R}$ to $1$ and vanishes on the images under $j$ of the other basis elements. It follows that $j$ is injective. $\square$

**Remark 7.2.3.** Whether $\Omega^{\mathrm{sep}}_{B/R}$ is free is closely related to whether $B$ is a formally smooth algebra over $R$; see [Gro64, Théorème 20.5.7]. M. Kurihara uses such relations more systematically in his study of complete discrete valuation fields of mixed characteristic [Kur87].

**Remark 7.2.4.** Suppose that $R$ is a Noetherian ring and $A$ is a finitely generated $R$-algebra. Let $\mathfrak{p} \lhd A$ be a prime ideal. Then $\Omega_{A_\mathfrak{p}/R} = \Omega_{A/R} \otimes_A A_\mathfrak{p}$ is a finitely generated $A_\mathfrak{p}$-module, and the natural map $\Omega_{A_\mathfrak{p}/R} \otimes_{A_\mathfrak{p}} \widehat{A_\mathfrak{p}} \to \Omega_{\widehat{A_\mathfrak{p}}/R}$ gives rise to an isomorphism

$$\Omega_{A_\mathfrak{p}/R} \otimes_{A_\mathfrak{p}} \widehat{A_\mathfrak{p}} \cong \varprojlim_n \Omega_{\widehat{A_\mathfrak{p}}/R}/\mathfrak{p}^n \Omega_{\widehat{A_\mathfrak{p}}/R} = \widehat{\Omega_{\widehat{A_\mathfrak{p}}/R}}$$

(see e.g. [Liu02, exercise 6.1.13]).

Therefore $\Omega^{\mathrm{sep}}_{\widehat{A_\mathfrak{p}}/R}$ is a finitely generated $\widehat{A_\mathfrak{p}}$-module (since it embeds into $\widehat{\Omega_{\widehat{A_\mathfrak{p}}/R}}$), and it is therefore complete; so the embedding $\Omega^{\mathrm{sep}}_{\widehat{A_\mathfrak{p}}/R} \hookrightarrow \widehat{\Omega_{\widehat{A_\mathfrak{p}}/R}}$ is actually an isomorphism. Thus we have a natural isomorphism

$$\Omega_{A/R} \otimes_A \widehat{A_\mathfrak{p}} \cong \Omega^{\mathrm{sep}}_{\widehat{A_\mathfrak{p}}/R}.$$

We will occasionally give explicit proofs of results which could otherwise be deduced from this remark.

**Definition 7.2.5.** Let $F$ be a complete discrete valuation field, and let $K$ be a subfield of $F$ such that $\mathrm{Frac}(K \cap \mathcal{O}_F) = K$. The space of *continuous relative differentials* is

$$\Omega^{\mathrm{cts}}_{F/K} := \Omega^{\mathrm{sep}}_{\mathcal{O}_F/K \cap \mathcal{O}_F} \otimes_{\mathcal{O}_F} F.$$

It is vector space over $F$ and there is a natural surjection $\Omega_{F/K} \twoheadrightarrow \Omega^{\mathrm{cts}}_{F/K}$.

**Remark 7.2.6.** Suppose that $F$, $K$ are as in the previous definition, and that $F'$ is a finite, separable extension of $F$. Using remark 7.2.1, one shows $\Omega^{\mathrm{cts}}_{F'/K} = \Omega^{\mathrm{cts}}_{F/K} \otimes_F F'$, and therefore there is a well-defined trace map $\mathrm{Tr}_{F'/F} : \Omega^{\mathrm{cts}}_{F'/K} \to \Omega^{\mathrm{cts}}_{F/K}$.

### 7.2.2 Equal characteristic

We begin with residues in the equal characteristic case; this material is well-known (see e.g. [Ser88]) so we are brief. Let $F$ be a two-dimensional local field of equal-characteristic zero. We assume that an embedding of a local field $K$ (necessarily of characteristic zero) into $F$ is given; such an embedding will be natural in our applications. The valuation $\nu_F|_K$ must be trivial, for else it would be a multiple of $\nu_K$ (a complete discrete valuation field has a unique normalised discrete valuation) which would imply $\overline{K} \hookrightarrow \overline{F}$, contradicting our hypothesis on the characteristic of $F$; so $K \subseteq \mathcal{O}_F$ and $K \hookrightarrow \overline{F}$, making $\overline{F}$ into a finite extension of $K$.

**Lemma 7.2.7.** *$F$ has a unique coefficient field which contains $K$.*

*Proof.* Set $n = |\overline{F} : K|$. Suppose first that $K'/K$ is any finite subextension of $F/K$. Then $K' \subseteq \mathcal{O}_F$ and so the residue map restricts to a $K$-linear injection $K' \hookrightarrow \overline{F}$, proving that $|K' : K| \leq n$. This establishes that $K$ has at most one extension of degree $n$ inside $F$ (for if there were two extensions then we could take their composite), and that if such an extension exists then it is the desired coefficient field (for then the residue map $K' \hookrightarrow \overline{F}$ must be an isomorphism).

Since $K$ is perfect, apply Hensel's lemma to lift to $\mathcal{O}_F$ a generator for $\overline{F}/K$; the subextension of $F/K$ generated by this element has degree $n$, completing the proof. □

This unique coefficient field will be denoted $k_F$; it depends on the image of the embedding $K \hookrightarrow \mathcal{O}_F$, though the notation does not reflect that. $k_F$ is a finite extension of $K$; moreover, it is simply the algebraic closure of $K$ inside $F$. When the local field $K \subseteq F$ has been fixed, we will refer to $k_F$ as *the* coefficient field of $F$ (*with respect to* $K$, if we want to be more precise). Standard structure theory implies that choosing a uniformiser $t \in F$ induces a $k_F$-isomorphism $F \cong k_F((t))$.

**Lemma 7.2.8.** *$\Omega^{sep}_{\mathcal{O}_F/\mathcal{O}_K}$ is a free $\mathcal{O}_F$-module of rank 1, with basis $dt$, where $t$ is any uniformiser of $F$. Hence $\Omega^{cts}_{F/K}$ is a one-dimensional vector space over $F$ with basis $dt$.*

*Proof.* Any derivation on $\mathcal{O}_F$ which vanishes on $\mathcal{O}_K$ also vanishes on $K$, and it even vanishes on $k_F$ since $k_F/K$ is a finite, separable extension. Hence $\Omega_{\mathcal{O}_F/\mathcal{O}_K} = \Omega_{\mathcal{O}_F/K} = \Omega_{\mathcal{O}_F/k_F}$.

Fix a uniformiser $t \in F$, to induce an isomorphism $\mathcal{O}_F \cong k_F[[t]]$. Then for any $f \in \mathcal{O}_F$ and $n \geq 0$, we may write $f = \sum_{i=0}^{n} a_i t^i + g t^{n+1}$, with $a_0, \dots, a_n \in k_F$ and $g \in \mathcal{O}_F$; let $d : \mathcal{O}_F \to \Omega_{\mathcal{O}_F/\mathcal{O}_K}$ be the universal derivation and apply $d$ to obtain

$$d(f) = \sum_{i=0}^{n} a_i i t^{i-1} d(t) + g(n+1) t^n d(t) + t^{n+1} d(g).$$

It follows that $d(f) - \frac{df}{dt} d(t) \in \bigcap_{n=1}^{\infty} t^n \Omega_{\mathcal{O}_F/k_F}$. Taking the separated quotient shows that $dt$ generates $\Omega^{\mathrm{sep}}_{\mathcal{O}_F/k_F}$; the existence of the derivation $\frac{d}{dt}$ implies that $dt$ is not torsion. □

The *residue map of $F$, relative to $K$* is defined by

$$\mathrm{res}_F : \Omega^{\mathrm{cts}}_{F/K} \to k_F, \quad \omega = f dt \mapsto \mathrm{coeft}_{t^{-1}}(f),$$

where the notation means that we take the coefficient of $t^{-1}$ in the expansion of $f$. Implicit in the definition is the choice of a $k_F$-isomorphism $F \cong k_F((t))$.

It is well-known that the residue map does not depend on the choice of uniformiser $t$. Since the proof is straightforward in residue characteristic zero, we recall it. Any other uniformiser $T$ has the form $T = \sum_{i=1}^{\infty} a_i t^i$ with $a_i \in k_F$ and $a_1 \neq 0$; for $j \in \mathbb{Z} \setminus \{-1\}$, we have

$$\mathrm{coeft}_{t^{-1}}\left(T^j \frac{dT}{dt}\right) = \mathrm{coeft}_{t^{-1}}\left(\frac{1}{j+1}\frac{dT^{j+1}}{dt}\right) = 0.$$

When $j = -1$, we instead calculate as follows:

$$\mathrm{coeft}_{t^{-1}}\left(T^{-1}\frac{dT}{dt}\right) = \mathrm{coeft}_{t^{-1}}((a_1^{-1}t^{-1} - a_1^{-2}a_2 + \dots)(a_1 + 2a_2 t + \dots)) = 1.$$

Finally, since the residue is continuous with respect to the discrete valuation topology on $\Omega^{\mathrm{cts}}_{F/K} = F dt$ and the discrete topology on $k_F$, we have

$$\mathrm{coeft}_{t^{-1}}\left(\sum_{j \gg -\infty} b_j T^j \frac{dT}{dt}\right) = b_{-1},$$

and it follows that the residue map may also be defined with respect to the isomorphism $F \cong k_F((T))$.

Now we prove functoriality of the residue map. Note that if $F'$ is a finite extension of $F$, then there is a corresponding finite extension $k_{F'}/k_F$ of the coefficient fields.

**Proposition 7.2.9.** *Let $F'$ be a finite extension of $F$. Then the following diagram commutes:*

$$\begin{tikzcd} \Omega^{cts}_{F'/K} \arrow[r, "\mathrm{res}_{F'}"] \arrow[d, "\mathrm{Tr}_{F'/F}"'] & k_{F'} \arrow[d, "\mathrm{Tr}_{k_{F'}/k_F}"] \\ \Omega^{cts}_{F/K} \arrow[r, "\mathrm{res}_F"] & k_F \end{tikzcd}$$

*Proof.* This is another well-known result, whose proof we give since it is easy in the characteristic zero case. It suffices to consider two separate cases: when $F'/F$ is unramified, and when $F'/F$ is totally ramified (as extensions of complete discrete valuation fields).

In the unramified case, $|k_{F'} : k_F| = |F' : F|$ and we may choose compatible isomorphisms $F \cong k_F((t))$, $F' \cong k_{F'}((t))$; the result easily follows in this case.

In the totally ramified case, $F'/F$ is only tamely ramified, $k_{F'} = k_F$, and we may choose compatible isomorphisms $F \cong k_F((t))$, $F' \cong k_{F'}((T))$, where $T^e = t$. We may now follow the argument of [Ser88, II.13]. $\square$

### 7.2.3 Mixed characteristic

Now we introduce relative residue maps for two-dimensional local fields of mixed characteristic. We take a local, explicit approach, with possible future applications to higher local class field theory and ramification theory in mind. This residue map is used by Fesenko [Fes03, §3] to define additive characters in his two-dimensional harmonic analysis.

#### Two-dimensional local fields of mixed characteristic

We begin with a review of this class of fields.

**Example 7.2.10.** Let $K$ be a complete discrete valuation field. Let $K\{\{t\}\}$ be the following collection of formal series

$$K\{\{t\}\} = \left\{ \sum_{i=-\infty}^{\infty} a_i t^i : a_i \in K \text{ for all } i, \inf_i \nu_K(a_i) > -\infty, \text{ and } a_i \to 0 \text{ as } i \to -\infty \right\}.$$

Define addition, multiplication, and a discrete valuation by

$$\sum_{i=-\infty}^{\infty} a_i t^i + \sum_{j=-\infty}^{\infty} a_j t^j = \sum_{i=-\infty}^{\infty} (a_i + b_i) t^i$$
$$\sum_{i=-\infty}^{\infty} a_i t^i \cdot \sum_{j=-\infty}^{\infty} a_j t^j = \sum_{i=-\infty}^{\infty} \left( \sum_{r=-\infty}^{\infty} a_r b_{i-r} \right) t^i$$
$$\nu \left( \sum_{i=-\infty}^{\infty} a_i t^i \right) = \inf_i \nu_K(a_i)$$

Note that there is nothing formal about the sum over $r$ in the definition of multiplication; rather it is a convergent double series in the complete discrete valuation field $K$. These operations are well-defined, make $K\{\{t\}\}$ into a field, and $\nu$ is a discrete valuation under which $K\{\{t\}\}$ is complete. Note that $K\{\{t\}\}$ is an extension of $K$, and that $\nu|_K = \nu_K$, i.e. $e(K\{\{t\}\}/K) = 1$.

The ring of integers of $K\{\{t\}\}$ and its maximal ideal are given by

$$\mathcal{O}_{K\{\{t\}\}} = \left\{ \sum_i a_i t^i : a_i \in \mathcal{O}_K \text{ for all } i \text{ and } a_i \to \infty \text{ as } i \to -\infty \right\},$$
$$\mathfrak{p}_{K\{\{t\}\}} = \left\{ \sum_i a_i t^i : a_i \in \mathfrak{p}_K \text{ for all } i \text{ and } a_i \to \infty \text{ as } i \to -\infty \right\}.$$

The surjective homomorphism

$$\mathcal{O}_{K\{\{t\}\}} \to \overline{K}((t)), \quad \sum_i a_i t^i \mapsto \sum_i \overline{a}_i t^i$$

identifies the residue field of $K\{\{t\}\}$ with $\overline{K}((t))$.

The alternative description of $K\{\{t\}\}$ is as follows. It is the completion of $\mathrm{Frac}(\mathcal{O}_K[[t]])$ with respect to the discrete valuation associated to the height one prime $\pi_K \mathcal{O}_K[[t]]$.

We will be interested in the previous example when $K$ is a local field of characteristic $0$. In this case, $K\{\{t\}\}$ is a two-dimensional local field of mixed characteristic.

Now suppose $L$ is any two-dimensional local field of mixed characteristic of residue characteristic $p$. Then $L$ contains $\mathbb{Q}$, and the restriction of $\nu_L$ to $\mathbb{Q}$ is a valuation which is equivalent to $\nu_p$, since $\nu_L(p) > 0$; since $L$ is complete, we may topologically close $\mathbb{Q}$ to see that $L$ contains a copy of $\mathbb{Q}_p$. It is not hard to see that this is the unique embedding of $\mathbb{Q}_p$ into $L$, and that $L/\mathbb{Q}_p$ is an (infinite) extension of discrete valuation fields. The corresponding extension of residue fields is $\overline{L}/\mathbb{F}_p$, where $\overline{L}$ is a local field of characteristic $p$.

The analogue of the coefficient field in the equal characteristic case is the following:

**Definition 7.2.11.** The *constant* subfield of $L$, denoted $k_L$, is the algebraic closure of $\mathbb{Q}_p$ inside $L$.

**Lemma 7.2.12.** *If $K$ is an arbitrary field then $K$ is relatively algebraically closed in $K((t))$. If $K$ is a complete discrete valuation field then $K$ is relatively algebraically closed in $K\{\{t\}\}$; so if $K$ is a local field of characteristic zero, then the constant subfield of $K\{\{t\}\}$ is $K$.*

*Proof.* Suppose that there is an intermediate extension $K((t)) \geq L \geq K$ with $L$ finite over $K$. Then each element of $L$ is integral over $K[[t]]$, hence belongs to $K[[t]]$. The residue map $K[[t]] \to K$ is non-zero on $L$, hence restricts to a $K$-algebra injection $L \hookrightarrow K$. This implies $L = K$.

Now suppose $K$ is a complete discrete valuation field and that we have an intermediate extension $K\{\{t\}\} \geq M \geq K$ with $M$ finite over $K$. Then $M$ is a complete discrete valuation field with $e(M/K) = 1$, since $e(K\{\{T\}\}/K) = 1$. Passing to the residue fields and applying the first part of the proof to $\overline{K}((t))$ implies $f(M/K) = 1$. Therefore $|M : K| = 1$, as required. $\square$

Let $L$ be a two-dimensional local field of mixed characteristic. The algebraic closure of $\mathbb{F}_p$ inside $\overline{L}$ is finite over $\mathbb{F}_p$ (it is the coefficient subfield of $\overline{L}$); so, if $k$ is any finite extension of $\mathbb{Q}_p$ inside $L$, then $f(\overline{k}/\mathbb{F}_p)$ is bounded above. But also $e(k/\mathbb{Q}_p) < e(L/\mathbb{Q}_p) < \infty$ is bounded above. It follows that $k_L$ is a finite extension of $\mathbb{Q}_p$.

Thus the process of taking constant subfields canonically associates to any two-dimensional local field $L$ of mixed characteristic a finite extension $k_L$ of $\mathbb{Q}_p$.

**Lemma 7.2.13.** *Suppose $K$ is a complete discrete valuation field and $\Omega/K$ is a field extension with subextensions $F, K'$ such that $K'/K$ is finite and separable, and $F$ is $K$-isomorphic to $K\{\{T\}\}$. Then the composite extension $FK'$ is $K$-isomorphic to $K'\{\{T\}\}$.*

*Proof.* Let $K''$ be the Galois closure of $K'$ over $K$ (enlarging $\Omega$ if necessary); then the previous lemma implies that $K'' \cap F = K$ and therefore the extensions $K'', F$ are linearly disjoint over $K$ (here it is essential that $K''/K$ is Galois). This implies that $FK''$ is $K$-isomorphic to $F \otimes_K K''$, which is easily seen to be $K$-isomorphic to $K''\{\{T\}\}$. The resulting isomorphism $\sigma : FK'' \to K''\{\{T\}\}$ restricts to an isomorphism $FK' \to \sigma(K')\{\{T\}\}$, and this final field is isomorphic to $K'\{\{T\}\}$. $\square$

**Lemma 7.2.14.** *Suppose $L$ is a two-dimensional local field of mixed characteristic. Then there is a two-dimensional local field $M$ contained inside $L$, such that $L/M$ is a finite extension and*

*(i)* $\overline{M} = \overline{L}$;

*(ii)* $k_M = k_L$;

*(iii)* *$M$ is $k_M$-isomorphic to $k_M\{\{T\}\}$.*

*Proof.* The residue field of $L$ is a local field of characteristic $p$, and therefore there is an isomorphism $\overline{L} \cong \mathbb{F}_q((t))$; using this we may define an embedding $\mathbb{F}_p((t)) \hookrightarrow \overline{L}$, such that $\overline{L}/\mathbb{F}_p((t))$ is an unramified, separable extension. Since $\mathbb{Q}_p\{\{t\}\}$ is an absolutely unramified discrete valuation field with residue field $\mathbb{F}_p((t))$, a standard structure theorem of complete discrete valuation fields [FV02, Proposition 5.6] implies that there is an embedding of complete discrete valuation fields $j : \mathbb{Q}_p\{\{t\}\} \hookrightarrow L$ which lifts the chosen embedding of residue fields. Set $F = j(\mathbb{Q}_p\{\{t\}\})$, and note that $f(L/F) = |\overline{L} : \mathbb{F}_p((t))| = \log_p(q)$ and $e(L/F) = \nu_L(p) < \infty$; so $L/F$ is a finite extension.

Now apply the previous lemma with $K = \mathbb{Q}_p$ and $K' = k_L$ to obtain $M = FK' \cong k_L\{\{t\}\}$. Moreover, Hensel's lemma implies that $L$, and therefore $k_L$, contains the $q-1$ roots of unity; so $\overline{k}_L\overline{F} = \mathbb{F}_q \cdot \mathbb{F}_p((t)) = \overline{L}$, and therefore $\overline{M} = \overline{L}$. $\square$

We will frequently use arguments similar to those of the previous lemma in order to obtain suitable subfields of $L$.

**Definition 7.2.15.** A two-dimensional local field $L$ of mixed characteristic is said to be *standard* if and only if $e(L/k_L) = 1$.

The purpose of the definition is to provide a 'co-ordinate'-free definition of the class of fields we have already considered:

**Corollary 7.2.16.** *$L$ is standard if and only if there is a $k_L$-isomorphism $L \cong k_L\{\{t\}\}$. If $L$ is standard and $k'$ is a finite extension of $k_L$, then $Lk'$ is also standard, with constant subfield $k'$.*

*Proof.* Since $e(k_L\{\{t\}\}/k_L) = 1$, the field $L$ is standard if it is isomorphic to $k_L\{\{t\}\}$. Conversely, by the previous lemma, there is a standard subfield $M \leq L$ with $k_M = k_L$ and $\overline{M} = \overline{L}$; then $e(M/k_M) = 1$ and $e(L/k_L) = 1$ (since we are assuming $L$ is standard), so that $e(L/M) = 1$) and therefore $L = M$.

The second claim follows from lemma 7.2.13. $\square$

**Remark 7.2.17.** A *first local parameter* of a two-dimensional local field $L$ is an element $t \in \mathcal{O}_L$ such that $\overline{t}$ is a uniformiser for the local field $\overline{L}$. For example, $t$ is a first local parameter of $K\{\{t\}\}$. More importantly, if $L$ is standard, then any isomorphism $k_L\{\{t\}\} \xrightarrow{\cong} L$ is determined by the image of $t$, and conversely, $t$ may be sent to any first local parameter of $L$. This follows from similar arguments to those found in lemma 7.2.14 above and 7.2.18 below; see e.g. [FV02, Proposition 5.6] and [MZ95]. We will abuse notation in a standard way, by choosing a first local parameter $t \in L$ and then identifying $L$ with $k_L\{\{t\}\}$.

**The residue map for standard fields.**

Here we define a residue map for standard two-dimensional fields and investigate its main properties. As in the equal characteristic case, we work in the relative situation, with a fixed standard two-dimensional local field $L$ of mixed characteristic and a chosen (one-dimensional) local field $K \leq L$. It follows that $K$ is intermediate between $\mathbb{Q}_p$ and the constant subfield $k_L$.

We start by studying spaces of differential forms. Note that if we choose a first local parameter $t \in L$ to induce an isomorphism $L \cong k_L\{\{t\}\}$, then the derivative $\frac{d}{dt} : L \to L$ is well-defined.

**Lemma 7.2.18.** *Let $t$ be any first local parameter of $L$. Then $\Omega^{sep}_{\mathcal{O}_L/\mathcal{O}_K}$ decomposes as a direct sum*

$$\Omega^{sep}_{\mathcal{O}_L/\mathcal{O}_K} = \mathcal{O}_L dt \oplus \mathrm{Tors}(\Omega^{sep}_{\mathcal{O}_L/\mathcal{O}_K})$$

*with $\mathcal{O}_L dt$ free, and $\mathrm{Tors}(\Omega^{sep}_{\mathcal{O}_L/\mathcal{O}_K}) \cong \Omega_{\mathcal{O}_{k_L}/\mathcal{O}_K} \otimes_{\mathcal{O}_{k_L}} \mathcal{O}_L$. Hence $\Omega^{cts}_{L/K}$ is a one-dimensional vector space over $L$ with basis $dt$.*

*Proof.* First suppose that $K = k_L$ is the constant subfield of $L$. Then we claim that for any $f \in \mathcal{O}_L$, one has $df = \frac{df}{dt}dt$ in $\Omega^{\mathrm{sep}}_{\mathcal{O}_L/\mathcal{O}_K}$.

Standard theory of complete discrete valuation fields (see e.g. [MZ95]) implies that there exists a map $H : \overline{L} \to \mathcal{O}_L^\times \cup \{0\}$ with the following properties:

(i) $H$ is a lifting, i.e. $\overline{H}(a) = a$ for all $a \in \overline{L}$;

(ii) $H(\overline{t}) = t$;

(iii) for any $a_0, \dots, a_{p-1} \in \overline{F}$, one has $H(\sum_{i=0}^{p-1} a_i^p \overline{t}^i) = \sum_{i=0}^{p-1} H(a_i)^p t^i$.

The final condition replaces the Teichmuller identity $H(a^p) = a^p$ which ones sees in the perfect residue field case. We will first prove our claim for elements of the form $f = H(a)$. Indeed, for any $n > 0$, we expand $a$ using the $p$-basis $\overline{t}$ to write

$$a = \sum_{i=0}^{p^n-1} a^{p^n} \overline{t}^i$$

for some $a_0, \ldots, a_{p^{n-1}} \in \overline{L}$. Lifting, and using the Teichmuller-type property of $H$ $n$ times, obtains

$$f = \sum_{i=0}^{p^n-1} H(a_i)^{p^n} t^i.$$

Now apply the universal derivative to reveal that

$$df = \sum_{i=0}^{p^n-1} H(a_i)^{p^n} i t^{i-1} dt + p^n H(a_i)^{p^n-1} t^i d(H(a_i)).$$

We may apply $\frac{d}{dt}$ in a similar way, and it follows that $df - \frac{df}{dt} dt \in p^n \Omega_{\mathcal{O}_L/\mathcal{O}_K}$. Letting $n \to \infty$ gives us $df = \frac{df}{dt} dt$ in $\Omega^{\text{sep}}_{\mathcal{O}_L/\mathcal{O}_K}$.

Now suppose that $f \in \mathcal{O}_L$ is not necessarily in the image of $H$. For any $n$, we may expand $f$ as a sum

$$f = \sum_{i=0}^{n} f_i \pi^i + g\pi^{n+1}$$

where $\pi$ is a uniformiser of $K$ (also a uniformiser of $L$), $f_0, \ldots, f_n$ belong to the image of $H$, and $g \in \mathcal{O}_L$. Applying the universal derivative obtains

$$df = \sum_{i=0}^{n} \frac{df_i}{dt} \pi^i dt + \pi^{n+1} dg,$$

and computing $\frac{df}{dt}$ gives something similar. We again let $n \to \infty$ to deduce that $df = \frac{df}{dt} dt$ in $\Omega^{\text{sep}}_{\mathcal{O}_L/\mathcal{O}_K}$. This completes the proof of our claim.

This proves that $dt$ generates $\Omega^{\text{sep}}_{\mathcal{O}_L/\mathcal{O}_K}$, so we must now prove that it is not torsion. But the derivative $\frac{d}{dt}$ induces an $\mathcal{O}_L$-linear map $\Omega_{\mathcal{O}_L/\mathcal{O}_K} \to \mathcal{O}_L$ which descends to the maximal separated quotient and send $dt$ to 1; this is enough. This completes the proof in the case $k_L = K$.

Now consider the general case $k_L \geq K$. Using the isomorphism $L \cong k_L\{\{t\}\}$, we set $M = K\{\{t\}\}$. The inclusions $\mathcal{O}_K \leq \mathcal{O}_M \leq \mathcal{O}_L$, lemma 7.2.2, and the first case of this proof applied to $K = k_M$, give an exact sequence of differential forms

$$0 \to \Omega^{\text{sep}}_{\mathcal{O}_M/\mathcal{O}_K} \otimes_{\mathcal{O}_M} \mathcal{O}_L \to \Omega^{\text{sep}}_{\mathcal{O}_L/\mathcal{O}_K} \to \Omega_{\mathcal{O}_L/\mathcal{O}_M} \to 0. \qquad (*)$$

Furthermore, the isomorphism $L \cong M \otimes_K k_L$ restricts to an isomorphism $\mathcal{O}_L \cong \mathcal{O}_M \otimes_{\mathcal{O}_K} \mathcal{O}_{k_L}$, and base change for differential forms gives $\Omega_{\mathcal{O}_L/\mathcal{O}_M} \cong \Omega_{\mathcal{O}_{k_L}/\mathcal{O}_K} \otimes_{\mathcal{O}_{k_L}} \mathcal{O}_L$; this isomorphism is given by the composition

$$\Omega_{\mathcal{O}_{k_L}/\mathcal{O}_K} \otimes_{\mathcal{O}_{k_L}} \mathcal{O}_L \to \Omega_{\mathcal{O}_L/\mathcal{O}_K} \to \Omega_{\mathcal{O}_L/\mathcal{O}_M}.$$

But this factors through $\Omega^{\text{sep}}_{\mathcal{O}_L/\mathcal{O}_K}$, which splits $(*)$ and completes the proof. $\square$

We may now define the *relative residue map for $L/K$* similarly to the equal characteristic case:

$$\mathrm{res}_L : \Omega^{\mathrm{cts}}_{L/K} \to k_L, \quad \omega = f\,dt \mapsto -\mathrm{coeft}_{t^{-1}}(f)$$

where the notation means that we expand $f$ in $k_L\{\{t\}\}$ and take the coefficient of $t^{-1}$. Implicit in the definition is the choice of an isomorphism $L \cong k_L\{\{t\}\}$ fixing $k_L$. The twist by $-1$ is necessary for the future reciprocity laws.

**Proposition 7.2.19.** $\mathrm{res}_L$ *is well-defined, i.e. it does not depend on the chosen isomorphism* $L \cong k_L\{\{t\}\}$.

*Proof.* As we noted in remark 7.2.17, the chosen isomorphism is determined uniquely by the choice of first local parameter. Let $T \in \mathcal{O}_L$ be another local parameter. Using a similar lifting argument (which simulates continuity) to that in the first half of the previous lemma, it is enough to prove

$$\mathrm{coeft}_{t^{-1}}\left(T^i \frac{dT}{dt}\right) = \begin{cases} 1 & i = -1 \\ 0 & i \neq -1. \end{cases}$$

Well, when $i \neq -1$, then $T^i \frac{dT}{dt} = \frac{d}{dt}(i^{-1}T^{i+1})$, which has $t^{-1}$ coefficient $0$, since this is true for the derivative of any element.

Now, the image of $T$ in $\overline{L}$ has the form $\overline{T} = \sum_{i=1}^\infty \theta_i \overline{t}^i$, with $\theta_i \in \overline{k}_L$ and $\theta_1 \neq 0$. Hence $T \equiv \sum_{i=1}^\infty a_i t^i \mod \mathfrak{p}_L$, where each $a_i \in k_L$ is a lift of $\theta_i$. Expanding the difference, a principal unit, as an infinite product obtains

$$T = \left(\sum_{i=1}^\infty a_i t^i\right) \prod_{j=1}^\infty (1 + b_j \pi^j),$$

for some $b_j \in \mathcal{O}_L$, with $\pi$ a uniformiser of $k_L$ (also a uniformiser of $L$); we should remark that the above summation is a formal sum in $L \cong k_L\{\{t\}\}$, while the product is a genuinely convergent product in the valuation topology on $L$.

The map

$$L^\times \to k_L, \quad \alpha \mapsto \mathrm{coeft}_{t^{-1}}\left(\alpha^{-1}\frac{d\alpha}{dt}\right)$$

is a continuous (with respect to the valuation topologies) homomorphism, so to complete the proof it is enough to verify the identities

$$\mathrm{coeft}_{t^{-1}}\left(\alpha^{-1}\frac{d\alpha}{dt}\right) = \begin{cases} 1 & \alpha = \sum_{i=1}^\infty a_i t^i, \\ 0 & \alpha = 1 + b_j \pi^j. \end{cases}$$

The first of these identities follows exactly as in the equal characteristic case of subsection 7.2.2. For the second case, we compute

$$\begin{aligned}(1 + b_j\pi^j)^{-1}\frac{d}{dt}(1 + b_j\pi^j) &= (1 - b_j\pi^j + b_j^2\pi^{2j} + \dots)\frac{db_j}{dt}\pi^j \\ &= \frac{db_j}{dt}\pi^j - \frac{d(2^{-1}b_j^2)}{dt}\pi^{2j} + \frac{d(3^{-1}b_j^3)}{dt}\pi^{3j} + \dots\end{aligned}$$

This is a convergent sum, each term of which has no $t^{-1}$ coefficient; the proof is complete. □

We now establish the functoriality of residues with respect to the trace map:

**Proposition 7.2.20.** *Suppose that $L'$ is a finite extension of $L$, and that $L'$ is also standard. Then the following diagram commutes:*

$$\begin{tikzcd} \Omega^{cts}_{L'/K} \arrow[r, "\mathrm{res}_{L'}"] \arrow[d, "\mathrm{Tr}_{L'/L}"'] & k_{L'} \arrow[d, "\mathrm{Tr}_{k_{L'}/k_L}"] \\ \Omega^{cts}_{L/K} \arrow[r, "\mathrm{res}_{L'}"] & k_L \end{tikzcd}$$

*Proof.* Using the intermediate extension $Lk_{L'}$, we may reduce this to two cases: when we have compatible isomorphisms $L \cong k_L\{\{t\}\}$, $L \cong k_{L'}\{\{t\}\}$, or when $k_L = k_{L'}$. The first case is straightforward, so we only treat the second.

By the usual 'principle of prolongation of algebraic identities' trick [Ser88, II.13] we may reduce to the case $L \cong k_L\{\{t\}\}$, $L \cong k_L\{\{T\}\}$ with $t = T^e$. The same argument as in the equal characteristic case [loc. cit.] is then easily modified. □

**Extending the residue map to non-standard fields.**

Now suppose that $L$ is a two-dimensional local field of mixed characteristic which is not necessarily standard, and as usual fix a local field $K \leq L$. Choose a standard subfield $M$ of $L$ with the same constant field as $L$ and of which $L$ is a finite extension; this is possible by lemma 7.2.14. Attempt to define the *relative residue map for $L/K$* to be composition

$$\mathrm{res}_L : \Omega^{\mathrm{cts}}_{L/K} \stackrel{\mathrm{Tr}_{L/M}}{\to} \Omega^{\mathrm{cts}}_{M/K} \stackrel{\mathrm{res}_M}{\to} k_M = k_L.$$

**Lemma 7.2.21.** $\mathrm{res}_L$ *is independent of choice of $M$.*

*Proof.* Suppose that $M'$ is another field with the same properties as $M$, and let $\omega \in \Omega^{\mathrm{cts}}_{L/K}$. By an important structure result for two-dimensional local fields of mixed characteristic [Zhu95, Theorem 2.1] there is a finite extension $L'$ of $L$ such that $L'$ is standard. Using functoriality for standard fields, we have

$$\mathrm{res}_M(\mathrm{Tr}_{L/M}\,\omega) = |L' : L|^{-1}\,\mathrm{res}_M(\mathrm{Tr}_{L'/M}\,\omega) = |L' : L|^{-1}\,\mathrm{Tr}_{k_{L'}/k_L}(\mathrm{res}_{L'}(\omega))$$

(here we have identified $\omega$ with its image in $\Omega^{\mathrm{cts}}_{L'/K}$). Since this expression is equally valid for $M'$ in place of $M$, we are done. □

The definition of the residue in the general case is chosen to ensure that functoriality still holds:

**Proposition 7.2.22.** *Let $L'/L$ be a finite extension of two-dimensional local fields of mixed characteristic; then the following diagram commutes*

$$\begin{tikzcd} \Omega^{cts}_{L'/K} \arrow[r, "\mathrm{res}_{L'}"] \arrow[d, "\mathrm{Tr}_{L'/L}"'] & k_{L'} \arrow[d, "\mathrm{Tr}_{k_{L'}/k_L}"] \\ \Omega^{cts}_{L/K} \arrow[r, "\mathrm{res}_{L}"'] & k_L \end{tikzcd}$$

*Proof.* Let $M$ be a standard subfield of $L$ used to define $\mathrm{res}_L$; then $M' = Mk_{L'}$ may be used to define $\mathrm{res}_{L'}$. For $\omega \in \Omega^{\mathrm{cts}}_{L'/K}$, we have

$$\mathrm{res}_L(\mathrm{Tr}_{L'/L}\,\omega) = \mathrm{res}_M(\mathrm{Tr}_{L/M}\,\mathrm{Tr}_{L'/L}\,\omega) = \mathrm{res}_M(\mathrm{Tr}_{M'/M}\,\mathrm{Tr}_{L'/M'}\,\omega).$$

Apply functoriality for standard fields to see that this equals

$$\mathrm{Tr}_{k_{L'}/k_L}(\mathrm{res}_{M'}(\mathrm{Tr}_{L'/M'}\,\omega)) = \mathrm{Tr}_{k_{L'}/k_L}\,\mathrm{res}_{L'}(\omega),$$

as required. □

**Relation of the residue map to that of the residue fields.**

We finish this study of residues by proving that the residue map on a mixed characteristic, two-dimensional local field $L$ lifts the residue map of the residue field $\overline{L}$. More precisely, we claim that the following diagram commutes

$$\begin{array}{ccc} \Omega^{\text{sep}}_{\mathcal{O}_L/\mathcal{O}_K} & \xrightarrow{\operatorname{res}_L} & \mathcal{O}_{k_L} \\ \Big\downarrow & & \Big\downarrow \\ \Omega_{\overline{L}/\overline{K}} & \xrightarrow[e(L/k_L)\operatorname{res}_{\overline{L}}]{} & \overline{k}_L \end{array}$$

where some of the arrows deserve further explanation. The lower horizontal arrow is $e(L/k_L)$ times the usual residue map for $\overline{L}$ (a local field of finite characteristic); note that $\overline{K}$ is a finite subfield of $\overline{L}$, and that $\overline{k}_L$ is the constant subfield of $\overline{L}$, which we identify with the residue field of $\overline{L}$. Also, the top horizontal arrow is really the composition $\Omega^{\text{sep}}_{\mathcal{O}_L/\mathcal{O}_K} \xrightarrow{j} \Omega^{\text{cts}}_{L/K} \xrightarrow{\operatorname{res}_L} k_L$; part of our claim is that $\operatorname{res}_L \circ j$ has image in $\mathcal{O}_{k_L}$.

Combining the identifications $\Omega_{\overline{L}/\overline{K}} = \Omega_{\overline{L}/\overline{k}_L}$ and $\Omega^{\text{cts}}_{L/K} = \Omega^{\text{cts}}_{L/k_L}$ with the natural surjection $\Omega^{\text{sep}}_{\mathcal{O}_L/\mathcal{O}_K} \twoheadrightarrow \Omega^{\text{sep}}_{\mathcal{O}_L/\mathcal{O}_{k_L}}$, the problem is easily reduced to the case $K = k_L$, which we now assume to be true.

Let us first suppose that $L$ is a standard field (so that $e(L/K) = 1$); write $L = M$ for later clarity, and let $t \in M$ be a first local parameter. Then $\Omega^{\text{sep}}_{\mathcal{O}_M/\mathcal{O}_K} = \mathcal{O}_M\, dt$ by lemma 7.2.18 and so the image of $\Omega^{\text{sep}}_{\mathcal{O}_M/\mathcal{O}_K}$ inside $\Omega^{\text{cts}}_{M/K} = M\, dt$ is $\mathcal{O}_M\, dt$. We need to show that $\operatorname{res}_{\overline{M}}(\overline{f}\, d\overline{t}) = \overline{\operatorname{res}_M(f\, dt)}$ for all $f \in \mathcal{O}_M$; this is clear from the explicit definition of the residue map for $M = K\{\{t\}\}$.

Now suppose $L$ is arbitrary, choose a first local parameter $t \in \mathcal{O}_L$, and then choose a standard subfield $M$ such that $\overline{M} = \overline{L}$, $k_M = K$, and $t \in M$ (see lemma 7.2.14). To continue the proof, we must better understand the structure of $\Omega^{\text{sep}}_{\mathcal{O}_L/\mathcal{O}_K}$. Let $\pi_L$ denote a uniformiser of $L$, so that $\mathcal{O}_L = \mathcal{O}_M[\pi_L]$; let $f(X) \in \mathcal{O}_M[X]$ be the minimal polynomial of $\pi_L$, and write $f(X) = \sum_{i=0}^n b_i X^i$. We have our usual exact sequence

$$0 \to \Omega^{\text{sep}}_{\mathcal{O}_M/\mathcal{O}_K} \otimes_{\mathcal{O}_M} \mathcal{O}_L \to \Omega^{\text{sep}}_{\mathcal{O}_L/\mathcal{O}_K} \to \Omega_{\mathcal{O}_L/\mathcal{O}_M} \to 0,$$

so that $\Omega^{\text{sep}}_{\mathcal{O}_L/\mathcal{O}_K}$ is generated by $dt$ and $d\pi_L$. Moreover,

$$0 = d(f(\pi_L)) = f'(\pi_L)\, d\pi_L + c\, dt, \tag{†}$$

where $c = \sum_{i=0}^n \frac{db_i}{dt}\pi_L^i$. Further, using our exact sequence to see that $dt$ is not torsion, and from the fact that $\Omega_{\mathcal{O}_L/\mathcal{O}_M} \cong \mathcal{O}_L/\langle f'(\pi_L)\rangle$ (using the generator $d\pi_L$), it is easy to check that (†) is the only relation between the generators $dt$ and $d\pi_L$.

We now define a trace map $\operatorname{Tr}_{\mathcal{O}_L/\mathcal{O}_M} : \Omega^{\text{sep}}_{\mathcal{O}_L/\mathcal{O}_K} \to \Omega^{\text{sep}}_{\mathcal{O}_M/\mathcal{O}_K}$ as follows:

$$\operatorname{Tr}_{\mathcal{O}_L/\mathcal{O}_M}(a\, d\pi_L) = \operatorname{Tr}_{L/M}(-acf'(\pi_L)^{-1})\, dt$$
$$\operatorname{Tr}_{\mathcal{O}_L/\mathcal{O}_M}(b\, dt) = \operatorname{Tr}_{L/M}(b)\, dt$$

for $a, b \in \mathcal{O}_L$. It is important to recall the classical different formula ([Neu99, III.2]; also see section 7.6)

$$f'(\pi)^{-1}\mathcal{O}_L = \mathfrak{C}(\mathcal{O}_L/\mathcal{O}_M) \ \ (= \{x \in L : \operatorname{Tr}_{L/M}(x\mathcal{O}_L) \subseteq \mathcal{O}_F\}),$$

to see that this is well-defined. Furthermore, if we base change $- \otimes_{\mathcal{O}_L} L$, then we obtain the usual trace map $\operatorname{Tr}_{L/M} : \Omega^{\text{cts}}_{L/K} \to \Omega^{\text{cts}}_{M/K}$.

By definition of the residue map on $L$, it is now enough to show that the diagram

$$\begin{tikzcd} \Omega^{\mathrm{sep}}_{\mathcal{O}_L/\mathcal{O}_K} \arrow[r, "\mathrm{Tr}_{\mathcal{O}_L/\mathcal{O}_M}"] \arrow[d] & \Omega^{\mathrm{sep}}_{\mathcal{O}_M/\mathcal{O}_K} \arrow[d] \\ \Omega_{\overline{L}/\overline{K}} \arrow[r, "\times|L:M|"'] & \Omega_{\overline{M}/\overline{K}} \end{tikzcd}$$

commutes. Well, for an element of the form $b\,dt \in \Omega^{\mathrm{sep}}_{\mathcal{O}_L/\mathcal{O}_K}$ with $b \in \mathcal{O}_L$, commutativity is clear. Now consider an element $a\,d\pi_L \in \Omega^{\mathrm{sep}}_{\mathcal{O}_L/\mathcal{O}_K}$; the image of this in $\Omega_{\overline{L}/\overline{K}}$ is zero, so we must show that $\overline{\mathrm{Tr}_{\mathcal{O}_L/\mathcal{O}_M}(a\,d\pi_L)} = 0$ in $\Omega_{\overline{M}/\overline{K}}$. For this we recall another formula relating the trace map and different, namely

$$\mathrm{Tr}_{L/M}(\pi_L^i f'(\pi_L)^{-1}\mathcal{O}_L) = \pi_M^{\lfloor \frac{i}{e} \rfloor}\mathcal{O}_M,$$

where $i \in \mathbb{Z}$, $e = |L : M|$, and $\lfloor\ \rfloor$ denotes the greatest integer below (see e.g. [FV02, Proposition III.1.4]). Since $f$ is an Eisenstein polynomial, $\nu_L(\frac{da_i}{dt}) \geq e$ for all $i$, and so $\nu_L(c) \geq e$; by the aforementioned formula, $\mathrm{Tr}_{L/M}(cf'(\pi_L)^{-1}\mathcal{O}_L) \subseteq \pi_M\mathcal{O}_M$. This is what we needed to show, and completes the proof of compatibility between $\mathrm{res}_L$ and $\mathrm{res}_{\overline{L}}$.

**Corollary 7.2.23.** *Let $L$ be a two-dimensional local field of mixed characteristic, and $K \leq L$ a local field. Then the following diagram commutes:*

$$\begin{tikzcd} \Omega^{sep}_{\mathcal{O}_L/\mathcal{O}_K} \arrow[r, "\mathrm{Tr}_{k_L/K} \circ \mathrm{res}_L"] \arrow[d] & \mathcal{O}_K \arrow[d] \\ \Omega_{\overline{L}/\overline{K}} \arrow[r, "e(L/K)\,\mathrm{Tr}_{\overline{k}_L/\overline{K}} \circ \mathrm{res}_{\overline{L}}"'] & \overline{K} \end{tikzcd}$$

*Proof.* It is enough to combine what we have just proved with the commutativity of

$$\begin{tikzcd} \mathcal{O}_{k_L} \arrow[r, "\mathrm{Tr}_{k_L/K}"] \arrow[d] & \mathcal{O}_K \arrow[d] \\ \overline{k}_L \arrow[r, "e(k_L/K)\,\mathrm{Tr}_{\overline{k}_L/\overline{K}}"'] & \overline{K} \end{tikzcd}$$

□

## 7.3 Reciprocity for two-dimensional, normal, local rings

Now we consider a semi-local situation and prove the promised reciprocity law.

Let $A$ be a two-dimensional, normal, complete, local ring of characteristic zero, with finite residue field of characteristic $p$; for the remainder of this section, we will refer to these collective conditions as (†). Denote by $F$ the field of fractions of $A$ and by $\mathfrak{m}_A$ the maximal ideal. For each height one prime $y \lhd A$ (we will sometimes write $y \lhd^1 A$), the localisation $A_y$ is a discrete valuation ring, and we denote by $F_y = \mathrm{Frac}\,\widehat{A_y}$ the corresponding complete discrete valuation field. The residue field of $F_y$ is $\overline{F}_y = \mathrm{Frac}\,A/y$. Moreover, $A/y$ is a one-dimensional, complete, local domain, and so its field of fractions is a complete discrete valuation field whose residue field is a finite extension of the residue field of $A/y$, which is the same as the residue field of $A$. Therefore $F_y$ is a two-dimensional local field of characteristic zero.

Since $A$ is already complete, there is no confusion caused by writing $\widehat{A}_y$ instead of $\widehat{A_y}$ (note the different sized hats).

**Lemma 7.3.1.** *There is a unique ring homomorphism $\mathbb{Z}_p \to A$, and it is a closed embedding.*

*Proof.* The natural embedding $j : \mathbb{Z} \hookrightarrow A$ is continuous with respect to the $p$-adic topology on $\mathbb{Z}$ and the $\mathfrak{m}_A$-adic topology on $A$ since $p^n\mathbb{Z}_p \subseteq j^{-1}(\mathfrak{m}_A^n)$ for all $n \geq 0$. Therefore $j$ extends to a continuous injection $j : \mathbb{Z}_p \hookrightarrow A$, which is a closed embedding since $\mathbb{Z}_p$ is compact and $A$ is Hausdorff.

Now suppose that $\phi : \mathbb{Z}_p \to A$ is an arbitrary ring homomorphism. Then $\phi^{-1}(\mathfrak{m}_A^n)$ is an ideal of $\mathbb{Z}_p$ which contains $p^n\mathbb{Z}$; but every ideal of $\mathbb{Z}_p$ is closed, and so it contains $p^n\mathbb{Z}_p$. Therefore $\phi$ is continuous; since $\phi$ agrees with $j$ on $\mathbb{Z}$, they are equal. □

We fix a finite extension $\mathcal{O}_K$ of $\mathbb{Z}_p$ inside $A$, where $\mathcal{O}_K$ is the ring of integers of a finite extension $K$ of $\mathbb{Q}_p$. For each height one prime $y \lhd A$, we have $K \leq F_y$, and the constant/coefficient field $k_y = k_{F_y}$ of $F_y$ is a finite extension of $K$. There is a natural map $\Omega_{F/K} \to \Omega^{\text{cts}}_{F_y/K}$, so we may define the *residue map at $y$* by

$$\mathrm{res}_y : \Omega_{F/K} \to k_y, \quad \omega \mapsto \mathrm{res}_{F_y}(\omega).$$

It is a nuisance having the residue maps associated to different primes taking values in different finite extensions of $K$, so we also introduce

$$\mathrm{Res}_y = \mathrm{Tr}_{k_y/K}\,\mathrm{res}_y : \Omega_{F/K} \to K.$$

Our immediate aim, to be deduced in several stages, is the following reciprocity law:

**Theorem 7.3.2.** *Let $\omega \in \Omega_{F/K}$; then for all but finitely many height one primes $y \lhd A$ the residue $\mathrm{res}_y(\omega)$ is zero, and*

$$\sum_{y \lhd^1 A} \mathrm{Res}_y(\omega) = 0$$

*in $K$.*

We will also prove an analogous result without the assumption that $A$ is complete; see theorem 7.3.13.

### 7.3.1 Reciprocity for $\mathcal{O}_K[[T]]$

We begin by establishing reciprocity for $B = \mathcal{O}_K[[T]]$. More precisely, we shall consider $B \cong \mathcal{O}_K[[T]]$; although this may seem to be a insignificant difference, it is important to understand the intrinsic role of $T$, especially for the proof of proposition 7.2.19.

**Lemma 7.3.3.** *Let $B$ satisfy conditions (†) and also be regular; let $\mathcal{O}_K \leq B$ be the ring of integers of a local field, and assume that $\overline{K} = k(\mathfrak{m}_B)$ and that $B/\mathcal{O}_K$ is unramified (i.e. $\mathfrak{p}_K B = \mathfrak{m}_B$). Let $\pi_K$ be any prime of $K$.*

*Then there exists $t \in \mathfrak{m}_B$ such that $\mathfrak{m}_B = \langle \pi_K, t\rangle$. If $t$ is any such element, then each $f \in B$ may be uniquely written as a convergent series $f = \sum_{i=0}^{\infty} a_i t^i$, with $a_i \in \mathcal{O}_K$, and this defines an $\mathcal{O}_K$-isomorphism $B \cong \mathcal{O}_K[[T]]$, with $t \mapsto T$.*

*Proof.* Since $\pi_K$ is non-zero in the $k(\mathfrak{m}_B)$-vector space $\mathfrak{m}_B/\mathfrak{m}_B^2$, which has dimension two by regularity, there is $t \in B$ such that (the images of) $\pi_K, t$ are a basis for this space; hence $\mathfrak{m}_A = \langle \pi_K, t\rangle$.

Now, $B/tB$ is a one-dimensional, complete, regular, local ring, i.e. a complete discrete valuation ring, in which $\pi_K$ is prime. Since $tB$ is prime, $tB \cap \mathcal{O}_K = \{0\}$ and so $\mathcal{O}_K \hookrightarrow B/tB$; but these two complete discrete valuation fields have the same prime and same residue field, hence are equal.

Any series of the given form converges in $B$ because $B$ is complete and $a_i t^i$ belongs to $\mathfrak{m}_B^i$. Conversely, for any $f \in B$ we may write $f \equiv a_0 \mod tB$ for some $a_0 \in \mathcal{O}_K$ (since $B/tB = \mathcal{O}_K$); then replace $f$ by $t^{-1}(f - a_0)$ and repeat the process to obtain the desired expansion for $f$. If a series $\sum_{i \geq I} a_i t^i$ is zero, with $a_I \neq 0$, then we get $a_I t^I \in t^{I+1}B$, which contradicts the identity $tB \cap \mathcal{O}_K = \{0\}$. □

Now let $B$, $\mathcal{O}_K$, $\pi_K$, $t$ satisfy the conditions of the previous lemma; set $M = \operatorname{Frac} B$. Using the isomorphism $B \cong \mathcal{O}_K[[T]]$, we may describe the height one primes $y$ of $B$ (see e.g. [NSW00, Lemma 5.3.7]):

(i) $p \in y$. Then $y = \pi_K B$, and $M_y$ is a two-dimensional local field of mixed characteristic which is $K$-isomorphic to $K\{\{t\}\}$ and has constant field $k_y = K$.

(ii) $p \notin y$. Then $y = hB$, where $h \in \mathcal{O}_K[t]$ is an irreducible, Weierstrass polynomials (i.e. $h = t^d + a_{d-1}t^{d-1} + \cdots + a_0$, with $a_i \in \mathfrak{p}_K$), and $M_y$ is a two-dimensional local field of equal characteristic. The coefficient field $k_y$ is the finite extension of $K$ generated by a root of $h$. Finally, $M_y$ is $k_y$-isomorphic to $k_y((t_y))$, where $t_y$ is a uniformiser at $y$, e.g. $t_y = h$.

We need a convenient set of additive generators of $M$:

**Lemma 7.3.4.** *Each element of $M$ is a finite sum of elements of the form*

$$\frac{\pi_K^n g}{h^r},$$

*with $h \in \mathcal{O}_K[t]$ an irreducible, Weierstass polynomial, $r > 0$, $n \in \mathbb{Z}$, and $g \in B$.*

*Proof.* We begin with an element of $M$ of the form $1/(\pi_1^{r_1}\pi_2^{r_2})$, with $\pi_1, \pi_2$ distinct irreducible elements of $A$, and $r_1, r_2 \geq 1$. Set $I = \pi_1^{r_1}A + \pi_2^{r_2}A$; a standard lemma of intersection theory is that $\mathfrak{m}_A^m \subseteq I$ for $m \gg 0$. Thus we may write $\pi_K^m = g_1\pi_1^{r_1} + g_2\pi_2^{r_2}$ for some $g_1, g_2 \in B$, and we deduce

$$\frac{1}{\pi_1^{r_1}\pi_2^{r_2}} = \frac{\pi_K^m g_1}{\pi_2^{r_2}} + \frac{\pi_K^m g_2}{\pi_1^{r_1}}.$$

Now, a typical element of $M$ has the form $a/b$, with $a, b \in A$. Since $B$ is a unique factorisation domain whose prime ideals are as described above, we may write $b = u\pi_K^r h_1^{r_1}\cdots h_s^{r_s}$ where $u \in B^\times$, the $h_i$ are irreducible Weierstass polynomials, and all the exponents are $> 0$. Replacing $a$ with $u^{-1}a$, we may suppose $u = 1$. Applying the first part of the proof repeatedly decomposes $a/b$ into a sum of the required form. □

We also need to understand the space of relative differential forms:

**Lemma 7.3.5.** *$\Omega^{sep}_{B/\mathcal{O}_K}$ is a free $B$-module of rank one, with basis $dt$. For each height one prime $y \lhd B$, the natural map $\Omega_{B/\mathcal{O}_K} \otimes_B \widehat{B}_y \to \Omega_{\widehat{B}_y/\mathcal{O}_K}$ descends to an isomorphism*

$$\Omega^{sep}_{B/\mathcal{O}_K} \otimes_B \widehat{B}_y \xrightarrow{\simeq} \Omega^{sep}_{\widehat{B}_y/\mathcal{O}_K}.$$

*Hence there is an induced isomorphism $\Omega^{sep}_{B/\mathcal{O}_K} \otimes_B M_y \xrightarrow{\simeq} \Omega^{cts}_{M_y/K}$.*

*Proof.* The first claim may be proved in an identical way to lemma 7.2.8. Alternatively, use remark 7.2.4 to deduce that $\Omega^{\text{sep}}_{B/\mathcal{O}_K} = \Omega_{\mathcal{O}_K[t]/\mathcal{O}_K} \otimes_{\mathcal{O}_K[t]} B$.

If $y$ is a height one prime of $B$ then there is a natural map

$$\phi : \Omega^{\text{sep}}_{B/\mathcal{O}_K} \otimes_B \widehat{B}_y = (\Omega_{\mathcal{O}_K[t]/\mathcal{O}_K} \otimes_{\mathcal{O}_K[t]} B) \otimes_B \widehat{B}_y = \Omega_{\mathcal{O}_K[t]/\mathcal{O}_K} \otimes_{\mathcal{O}_K[t]} \widehat{B}_y \to \Omega_{\widehat{B}_y/\mathcal{O}_K} \twoheadrightarrow \Omega^{\text{sep}}_{\widehat{B}_y/\mathcal{O}_K},$$

and we shall now construct the inverse of $\phi$. Define an $\mathcal{O}_K$-derivation of $B_y$ by

$$d_1 : B_y \to \Omega^{\text{sep}}_{B/\mathcal{O}_K} \otimes_B \widehat{B}_y \quad b/s \mapsto db \otimes s^{-1} - b\,ds \otimes s^{-2}$$

where $b \in B$, $s \in B \setminus y$ (this is well-defined). Moreover, the right hand side is a finite $\widehat{B}_y$-module, hence is complete and separated for the $y$-adic topology; so $d_1$ (which is easily seen to be $y$-adically continuous) extends from $B_y$ to $\widehat{B}_y$. This derivation then induces a homomorphism of $\widehat{B}_y$-modules $\Omega_{\widehat{B}_y/\mathcal{O}_K} \to \Omega^{\text{sep}}_{B/\mathcal{O}_K} \otimes_B \widehat{B}_y$, and this descends to

$$\psi : \Omega^{\text{sep}}_{\widehat{B}_y/\mathcal{O}_K} \to \Omega^{\text{sep}}_{B/\mathcal{O}_K} \otimes_B \widehat{B}_y$$

since $\Omega^{\text{sep}}_{B/\mathcal{O}_K} \otimes_B \widehat{B}_y$ is a finitely generated $\widehat{B}_y$-module.

It is immediate that $\psi\phi = \text{id}$. It is also easy to see that $\phi\psi(db) = db$ for any $b \in B_y$; since such elements are dense in the Hausdorff space $\Omega^{\text{sep}}_{\widehat{B}_y/\mathcal{O}_K}$, we deduce $\phi\psi = \text{id}$. ☐

In particular, we now know that the residue map at $y$, initially defined on $\Omega_{M/K}$, factors through its quotient $\Omega^{\text{sep}}_{B/\mathcal{O}_K} \otimes_B M$. We may now prove reciprocity for $B$:

**Theorem 7.3.6.** *For each $\omega \in \Omega^{sep}_{B/\mathcal{O}_K} \otimes_B M$, the local residue $\operatorname{res}_y \omega$ is zero for all but finitely many $y \lhd^1 B$, and*

$$\sum_{y \lhd^1 B} \operatorname{Res}_y \omega = 0$$

*in $K$.*

*Proof.* By lemmas 7.3.4 and 7.3.5, it is enough to consider the case $\omega = f\,dt$ with

$$f = \frac{\pi_K^n g}{h^r},$$

where $h, r, n, g$ are as in lemma 7.3.4.

Let $y = t_y B$ be a prime with $t_y$ an irreducible, Weierstrass polynomial. If $t_y \neq h$, then $\pi_K^n g/h^r$ and $t$ both belong to $B_y$, and so

$$\operatorname{coeft}_{t_y^{-1}} \left( \frac{\pi_K^n g}{h^r} \frac{dt}{dt_y} \right) = 0$$

by a basic property of the residue map; i.e. $\operatorname{res}_y(\omega) = 0$. This establishes our first assertion. For the remainder of the proof, set $y = hA$; we must prove that

$$\operatorname{Res}_y(\omega) + \operatorname{Res}_{\pi_K B}(\omega) = 0. \tag{$*$}$$

Suppose for a moment that $g$ belongs to $\mathcal{O}_K[t]$, and consider the rational function field $K(t) \leq M$. For any point $x$ of $\mathbb{P}^1_K$, let $K(t)_x$ be the completion of $K(t)$ at the place $x$; then $K(t)_x$ is a two-dimensional local field of equal characteristic. Let $k_x$ denote its unique coefficient field containing $K$, and let $\operatorname{res}_x : \Omega^{\text{cts}}_{L_x/K} \to k_x$ denote the residue map. By the assumption on $g$ we have $\omega \in \Omega_{K(t)/K}$, and global reciprocity for $\mathbb{P}^1_K$ implies that $\sum_{x\in\mathbb{P}^1_K} \mathrm{Tr}_{k_x/K}\,\mathrm{res}_x(\omega) = 0$.

Further, an argument as at the start of this proof proves that $\mathrm{res}_x(\omega) = 0$ unless $x$ corresponds to the irreducible polynomial $h$, or $x = \infty$. Moreover, in the first case, $K(t)_x = M_y$, $k_x = k_y$, and $\mathrm{res}_x(\omega) = \mathrm{res}_y(\omega)$. Therefore to complete the proof (with $g$ still a polynomial) it is necessary and sufficient to show that

$$\mathrm{res}_{\pi_K B}(\omega) = \mathrm{res}_\infty(\omega). \qquad (**)$$

Note that the residue map on the left is for a two-dimensional local field of mixed characteristic, while that on the right is for one of equal characteristic. This passage between different characteristics is the key to the proof.

To prove $(**)$, write $t_\infty = t^{-1}$, which is a local parameter at $\infty$, and expand $h^{-r}$ in $K(t)_\infty = K((t_\infty))$ as $h^{-r} = \sum_{i\geq I} a_i t_\infty^i$, say. Since $h^r$ is a Weierstrass polynomial, it is easily checked that $a_i \to 0$ in $K$ as $i \to \infty$; therefore the series $\sum_{i\leq -I} a_{-i}t^i$ is a well-defined element of $M_{\pi_k B} = K\{\{t\}\}$. Moreover, since multiplication in both $K\{\{t\}\}$ and $K((t_\infty))$ are given by formal multiplication of series, we deduce

$$h^r \sum_{i\leq -I} a_{-i}t^i = 1,$$

i.e. $\sum_{i\leq -I} a_{-i}t^i$ is the series expansion of $h^{-r}$ in $M_{\pi_k B} = K\{\{t\}\}$. Now let $\sum_i b_i t_\infty^i$ be the expansion of $\pi_K^n g/h^r$ of $K(t)_\infty$; then $\sum_i b_{-i}t^i$ is the formal expansion of $\pi_K^n g/h^r$ in $M_{\pi_k B}$, and so

$$\begin{aligned}
\mathrm{res}_\infty\left(\frac{\pi_K^n g}{h^r}dt\right) &= \text{coeft}_{t_\infty^{-1}}\left(\frac{\pi_K^n g}{h^r}\frac{dt}{dt_\infty}\right)\\
&= \text{coeft}_{t_\infty^{-1}}\left(-t_\infty^{-2}\sum_i b_i t_\infty^i\right)\\
&= -b_1\\
&= -\text{coeft}_{t^{-1}}\sum_i b_{-i}t^i\\
&= \mathrm{res}_{\pi_K B}\left(\frac{\pi_K^n g}{h^r}dt\right).
\end{aligned}$$

This completes the proof of identity $(*)$ for $g \in \mathcal{O}_K[t]$; to prove it in general and complete the proof, it is enough to check that both sides of $(*)$ are continuous functions of $g$, with respect to the $\mathfrak{m}_B$-adic topology on $B$ and the discrete valuation topology on $K$. This is straightforward, though tedious, and so we omit it. □

### 7.3.2 Reciprocity for complete rings

Now we extend the reciprocity law to the general case. Fix both a ring $A$ satisfying conditions (†) and the ring of integers of a local field $\mathcal{O}_K \leq A$. Reciprocity for $A$ will follows in the usual way by realising $A$ as a finite extension of $\mathcal{O}_K[[T]]$:

**Lemma 7.3.7.** *There is a ring $B$ between $\mathcal{O}_K$ and $A$ which is $\mathcal{O}_K$-isomorphic to $\mathcal{O}_K[[T]]$, and such that $A$ is a finite $B$-module.*

*Proof.* By [Coh46, Theorem 16], $A$ contains a subring $B_0$, over which it is a finitely generated module, and such that $B_0$ is a two-dimensional, $p$-adic ring with residue field equal to that of $A$. Supposing that this residue field is $\mathbb{F}_q$, we therefore have an isomorphism $i : \mathbb{Z}_q[[T]] \xrightarrow{\simeq} B_0$. By the uniqueness of the embedding $\mathbb{Z}_p \hookrightarrow A$, it follows that $i(\mathbb{Z}_p) \subseteq \mathcal{O}_K$. Define

$$j : \mathcal{O}_K[[T]] = \mathbb{Z}_p[[T]] \otimes_{\mathbb{Z}_p} \mathcal{O}_K \to A, \quad f \otimes \alpha \mapsto i(f)\alpha.$$

The kernel of $j$ is a prime ideal of $\mathcal{O}_K[[T]]$ whose contraction to $\mathcal{O}_K$ is zero. If the kernel is non-zero then there is an Eisenstein polynomial $h \in \mathcal{O}_K[T]$ such that that $h(i(T)) = 0$ (this follows from the classification of prime ideals in $\mathcal{O}_K[[T]]$ discussed earlier), suggesting that $i(T)$ is algebraic over $\mathcal{O}_K$ and hence over $\mathbb{Z}_p$; this contradicts the injectivity of $i$. Hence $j$ is an isomorphism onto its image, as desired. □

Let $B$ be as given by the previous lemma, and write $M = \operatorname{Frac} B$, $F = \operatorname{Frac} A$. We now generalise lemma 7.3.5. However, note that if $A$ can be written as the completion of a localisation of a finitely generated $\mathcal{O}_K$-algebra, then the following proof can be significantly simplified, simply by imitating the proof of lemma 7.3.5; see also lemma 7.3.11.

**Lemma 7.3.8.** $\Omega^{sep}_{A/\mathcal{O}_K}$ *is a finitely generated* $A$*-module of rank* 1*. For each height one prime* $y \lhd A$*, the natural map* $\Omega_{A/\mathcal{O}_K} \otimes_A \widehat{A}_y \to \Omega_{\widehat{A}_y/\mathcal{O}_K}$ *descends to an isomorphism*

$$\Omega^{sep}_{A/\mathcal{O}_K} \otimes_A \widehat{A}_y \xrightarrow{\simeq} \Omega^{sep}_{\widehat{A}_y/\mathcal{O}_K}.$$

*Hence there is an induced isomorphism* $\Omega^{sep}_{A/\mathcal{O}_K} \otimes_A F_y \xrightarrow{\simeq} \Omega^{cts}_{F_y/K}$.

*Proof.* Lemmas 7.3.5 and 7.2.2 imply that there is a natural exact sequence

$$0 \to \Omega^{\text{sep}}_{B/\mathcal{O}_K} \otimes_B A \to \Omega^{\text{sep}}_{A/\mathcal{O}_K} \to \Omega_{A/B} \to 0, \tag{$*$}$$

which proves the first claim since $\Omega_{A/B}$ is a finitely generated, torsion $A$-module.

Now we are going to construct a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^{\text{sep}}_{B/\mathcal{O}_K} \otimes_B \widehat{A}_y \arrow[r] & \Omega^{\text{sep}}_{A/\mathcal{O}_K} \otimes_A \widehat{A}_y \arrow[r] & \Omega_{A/B} \otimes_A \widehat{A}_y \arrow[r] & 0 \\
0 \arrow[r] & \Omega^{\text{sep}}_{\widehat{B}_{y'}/\mathcal{O}_K} \otimes_{\widehat{B}_{y'}} \widehat{A}_y \arrow[r] \arrow[u, "\psi'_B", "\cong"'] & \Omega^{\text{sep}}_{\widehat{A}_y/\mathcal{O}_K} \arrow[r] \arrow[u, "\psi_A"] & \Omega_{\widehat{A}_y/\widehat{B}_y} \arrow[r] \arrow[u, "\cong"'] & 0
\end{tikzcd}$$

The top line is obtained by tensoring $(*)$ with $\widehat{A}_y$. For the bottom row, set $y' = y \cap B$, use lemma 7.3.5 to see that $\Omega^{\text{sep}}_{B_{y'}/\mathcal{O}_K}$ is free and that we may therefore apply lemma 7.2.2 to the tower of rings $\widehat{A}_y \geq \widehat{B}_{y'} \geq \mathcal{O}_K$. In lemma 7.3.5 we also constructed a natural map

$$\psi_B = \psi : \Omega^{\text{sep}}_{\widehat{B}_y/\mathcal{O}_K} \to \Omega^{\text{sep}}_{B/\mathcal{O}_K} \otimes_B \widehat{B}_y;$$

its definition did not use any special properties of $B$ and so we may similarly define $\psi_A$. Base change $\psi_B$ by $\widehat{A}_y$ to obtain the isomorphism $\psi'_B$ in the diagram. Finally, one may see in a number of different ways that there is an isomorphism $\Omega_{A/B} \otimes_A \widehat{A}_y \cong \Omega_{\widehat{A}_y/\widehat{B}_y}$ which is natural enough so that the diagram will commute.

It follows that $\psi_A$ is an isomorphism, as required. □

The previous lemma implies that $\Omega^{\text{sep}}_{B/\mathcal{O}_K} \otimes_B F \cong \Omega^{\text{sep}}_{A/\mathcal{O}_K} \otimes_A F$, and so we have natural trace maps

$$\begin{aligned} \operatorname{Tr}_{F/M} &: \Omega^{\text{sep}}_{A/\mathcal{O}_K} \otimes_A F \to \Omega^{\text{sep}}_{B/\mathcal{O}_K} \otimes_B M \\ \operatorname{Tr}_{F_Y/M_y} &: \Omega^{\text{sep}}_{B/\mathcal{O}_K} \otimes_B F_Y \to \Omega^{\text{sep}}_{A/\mathcal{O}_K} \otimes_A M_y, \end{aligned}$$

where $Y$ is a height one prime of $A$ and $y = Y \cap B$. Using these we establish the expected functoriality for our residue maps:

**Proposition 7.3.9.** *Let $y$ be a fixed height one prime of $B$. Then for all $\omega \in \Omega^{sep}_{A/\mathcal{O}_K} \otimes_A F$, we have*

$$\operatorname{Res}_y \operatorname{Tr}_{F/M}(\omega) = \sum_{Y|y} \operatorname{Res}_Y(\omega),$$

*where $Y$ ranges over the (finitely many) height one primes of $A$ sitting over $y$.*

*Proof.* Set $A_y = A \otimes_B B_y = (B \setminus y)^{-1} A \subseteq F$. Then $A_y/B_{y'}$ is a finite extension of Dedekind domains, with the maximal ideals of $A_y$ corresponding to the primes $Y$ of $A$ (necessarily of height one) sitting over $y$. Therefore, for any $x \in L$, one has the usual local-global trace formula $\operatorname{Tr}_{F/M}(x) = \sum_{Y|y} \operatorname{Tr}_{F_Y/M_y}(x)$. In terms of differential forms,

$$\operatorname{Tr}_{F/M} \omega = \sum_{Y|y} \operatorname{Tr}_{F_Y/M_y} \omega$$

for all $\omega \in \Omega^{\text{sep}}_{A/\mathcal{O}_K} \otimes_A F$. Applying $\operatorname{res}_y$ to each side of this expression and using propositions 7.2.9 and 7.2.22 obtains

$$\operatorname{res}_y \operatorname{Tr}_{F/M}(\omega) = \sum_{Y|y} \operatorname{Tr}_{k_Y/k_y} \operatorname{res}_Y(\omega).$$

Apply $\operatorname{Tr}_{k_y/K}$ to complete the proof. □

Our desired reciprocity for $A$ follows:

**Theorem 7.3.10.** *For each $\omega \in \Omega^{sep}_{A/\mathcal{O}_K} \otimes_A F$, the local residue $\operatorname{res}_y \omega$ is zero for all but finitely many $y \lhd^1 A$, and*

$$\sum_{y \lhd^1 A} \operatorname{Res}_y \omega = 0.$$

*Proof.* Standard divisor theory implies that any $f \in F^\times$ belongs to $A_y$ for all but finitely many $y \lhd^1 A$. If $f dg$ is a nonzero element of $\Omega^{\text{sep}}_{A/\mathcal{O}_K} \otimes_A F$, then $\operatorname{res}_Y f dg = 0$ for any $Y \lhd^1 A$ which satisfies the following conditions: $p \notin Y$ and $f, g \in A_Y$. Since all but finitely many $Y$ satisfy these conditions, we have proved the first claim.

We may now complete the proof with the usual calculation, by reducing reciprocity via the previous proposition to the already-proved reciprocity for $B$:

$$\begin{aligned} \sum_{Y \lhd^1 A} \operatorname{Res}_y \omega &= \sum_{y \lhd^1 B} \sum_{Y|y} \operatorname{Res}_Y \omega \\ &= \sum_{y \lhd^1 B} \operatorname{Res}_y(\operatorname{Tr}_{F/M} \omega) \\ &= 0. \end{aligned}$$

□

### 7.3.3 Reciprocity for incomplete rings

We have thus far restricted out attention to complete local rings; we will now remove the completeness hypothesis. We do not prove reciprocity in the fullest generality, but restrict to those rings which will later arise from an arithmetic surface. Let $\mathcal{O}_K$ be a discrete valuation ring of characteristic zero and with finite residue field, and $A \geq \mathcal{O}_K$ a two-dimensional, normal, local ring with finite residue field of characteristic $p$; assume further that $A$ is the localisation of a finitely generated $\mathcal{O}_K$-algebra.

Since $A$ is excellent, its completion $\widehat{A}$ is also normal; therefore $\widehat{A}$ satisfies conditions (†), and $\widehat{\mathcal{O}}_K \leq \widehat{A}$ is the ring of integers of a local field, as has appeared in the previous subsections. Write $F = \mathrm{Frac}\,\widehat{A}$ and $\widehat{K} = \mathrm{Frac}\,\widehat{\mathcal{O}}_K$.

The following global-to-local isomorphism is extremely useful for explicit calculations. Since the notation can look confusing, let us mention that if $Y$ is a height one prime of $\widehat{A}$, then the completion of the discrete valuation ring $(\widehat{A})_Y$ is denoted $\widehat{\widehat{A}_Y}$.

**Lemma 7.3.11.** *Let $Y$ be a height one prime of $\widehat{A}$; then the natural map*

$$\Omega_{A/\mathcal{O}_K} \otimes_A \widehat{\widehat{A}_Y} \to \Omega^{sep}_{\widehat{\widehat{A}_Y}/\widehat{\mathcal{O}}_K}$$

*is an isomorphism.*

*Proof.* One follows the proof of lemma 7.3.5 almost exactly, replacing $B$ by $\widehat{A}$ and $\mathcal{O}_K[t]$ by $A$. The only additional observation which needs to be made is that the universal derivation $d : \widehat{A} \to \Omega^{\mathrm{sep}}_{\widehat{A}/\mathcal{O}_K}$ must be trivial on $\widehat{\mathcal{O}}_K$, and therefore $\Omega^{\mathrm{sep}}_{\widehat{A}/\mathcal{O}_K} = \Omega^{\mathrm{sep}}_{\widehat{A}/\widehat{\mathcal{O}}_K}$. □

For $Y \lhd^1 \widehat{A}$, the previous lemma gives us a natural isomorphism

$$\Omega_{A/\mathcal{O}_K} \otimes_A F_Y \xrightarrow{\simeq} \Omega^{\mathrm{cts}}_{F_Y/\widehat{K}},$$

and we thus pull back the relative residue map of $F_Y/\widehat{K}$ to get

$$\mathrm{res}_Y : \Omega_{\mathrm{Frac}\,A/K} = \Omega_{A/\mathcal{O}_K} \otimes_A \mathrm{Frac}\,A \to k_Y,$$

where, as usual, $k_Y$ denote the coefficient/constant field of $F_Y$.

More importantly, if $y$ is instead a height one prime of $A$, then set

$$\mathrm{Res}_y = \sum_{Y|y} \mathrm{Tr}_{k_Y/\widehat{K}}\, \mathrm{res}_Y : \Omega_{\mathrm{Frac}\,A/K} \to \widehat{K}$$

where $Y$ ranges over the finitely many height one primes of $\widehat{A}$ sitting over $y$.

We need a small lemma. We shall say that a prime of $\widehat{A}$ is *transcendental* if and only if its contraction to $A$ is zero; such a prime has height one and does not contain $p$. The transcendental primes are artificial in a sense; they have pathological properties (e.g. if $Y$ is transcendental then $\mathrm{Frac}\,A \leq \widehat{A}_Y$) and do not contain interesting information about $A$.

**Lemma 7.3.12.** *Let $Y$ be a height one prime of $\widehat{A}$. If $Y$ is not transcendental then it is a prime minimal over $y\widehat{A}$, where $y = A \cap Y$. On the other hand, if $Y$ is transcendental, then $\mathrm{res}_Y(\omega) = 0$ for all $\omega \in \Omega_{\mathrm{Frac}\,A/K}$.*

*Proof.* Since $y = A \cap Y$ is non-zero by assumption, so is $y\widehat{A}$. Since $y\widehat{A}$ is contained in $Y$ there is a prime $P \lhd \widehat{A}$ which is minimal over $y\widehat{A}$ and which is contained in $Y$. But then $P \neq 0$ and we have a chain of primes $0 \lhd P \unlhd Y$; since $Y$ has height 1 in $\widehat{A}$, we deduce $Y = P$, i.e. $Y$ is minimal over $y\widehat{A}$.

If $f dg$ is a element of $\Omega_{\mathrm{Frac}\,A/K}$, then as we remarked above, $f$ and $g$ belong to $\widehat{A}_Y$; therefore $\mathrm{res}_Y(f dg) = 0$, just as in the proof of theorem 7.3.10. □

The reciprocity law for $A$ follows:

**Theorem 7.3.13.** *For any $\omega \in \Omega_{\mathrm{Frac}\,A/K}$, the residue $\mathrm{Res}_y(\omega)$ is non-zero for only finitely many $y \lhd^1 A$, and*

$$\sum_{y \lhd^1 A} \mathrm{Res}_y(\omega) = 0$$

*in $\widehat{K}$.*

*Proof.* Immediate from theorem 7.3.10 and the previous lemma. □

## 7.4 Reciprocity laws for arithmetic surfaces

Let $\mathcal{O}_K$ be a Dedekind domain of characteristic zero and with finite residue fields; denote by $K$ its field of fractions. Let $X$ be a curve over $\mathcal{O}_K$; more precisely, $X$ is a normal scheme, flat and projective over $S = \mathrm{Spec}\,\mathcal{O}_K$, whose generic fibre is one dimensional and irreducible. These assumptions are enough to imply that each special fibre of $X$ is equidimensional of dimension one. Let $\eta$ be the generic point of $\mathrm{Spec}\,\mathcal{O}_K$; closed points will be denoted by $s$, and we set $K_s = \mathrm{Frac}\,\widehat{\mathcal{O}_{K,s}}$, which is a local field of characteristic zero. Let $\Omega_{X/S}$ denote the coherent sheaf of relative differential (one-)forms.

Let $x \in X$ be a closed point, and $y \subset X$ a curve containing $x$; let $s$ be the closed point of $S$ under $x$. Then $A := \mathcal{O}_{X,x}$ satisfies the conditions at the start of subsection 7.3.3, and contains the discrete valuation ring $\mathcal{O}_{K,s}$. Also denote by $y \lhd \mathcal{O}_{X,x}$ the local equation for $y$ at $x$; then $y$ is a height one prime of $A$, and we denote by

$$\mathrm{Res}_{x,y} : \Omega_{K(X)/K} \to K_s$$

the residue map $\mathrm{Res}_y : \Omega_{\mathrm{Frac}\,A/K} \to K_s$.

**Theorem 7.4.1.** *Let $\omega \in \Omega_{K(X)/K}$, and let $x \in X$ be a closed point sitting over $s \in S$. Then for all but finitely many curves $y \subset X$ containing $x$, the residue $\mathrm{Res}_{x,y}(\omega)$ is zero, and*

$$\sum_{\substack{y \subset X \\ y \ni x}} \mathrm{Res}_{x,y}(\omega) = 0$$

*in the local field $K_s$.*

*Proof.* This is the simply the geometric statement of theorem 7.3.13. □

## 7.5 Local complete intersection curves

The second part of the chapter now begins, in which we study the relative canonical sheaf of an arithmetic surface. First we collect together several results about complete intersections and relative canonical sheaves for relative curves, many of which I learnt from Q. Liu's wonderful book [Liu02]. Let $\pi : X \to S$ be a flat, surjective, local complete intersection, between irreducible, Noetherian, excellent schemes; assume that $\pi$ is smooth at the generic point of $X$. These assumptions are enough to imply that each fibre $X_s$ is equidimensional of dimension

$$\dim X_s = \dim X - \dim S$$

and we assume that these fibre are 1 dimensional, i.e. $X$ is a relative curve over $S$. The main example to have in mind is an arithmetic surface $X$ over $\operatorname{Spec} \mathcal{O}_K$, with $\mathcal{O}_K$ a Dedekind domain.

Locally, $X \to S$ is given by

$$R \hookrightarrow A = R[T_1, \dots, T_r]/I$$

where $R$, $A$ are excellent domains, and $I$ is an ideal generated by a regular sequence $f_1, \dots, f_{r-1}$. There are essentially two different ways to study the behaviour of $\operatorname{Spec} A \to \operatorname{Spec} R$, either by embedding $X$ into $r$-dimensional affine space over $\operatorname{Spec} R$, or by viewing $\operatorname{Spec} A$ as a finite cover of the affine line over $\operatorname{Spec} R$. These will both be important to us, and will give different explicit information about the canonical sheaf.

Set $F = K(X) = \operatorname{Frac} A$, $K = K(S) = \operatorname{Frac} R$.

### 7.5.1 Embedding the canonical sheaf into $\Omega_{K(X)/K}$

The $A$-module $I/I^2$ is free of rank $r-1$, with basis $f_1, \dots, f_{r-1}$ (or rather, the images of these mod $I^2$), and there is a natural exact sequence of $A$-modules

$$I/I^2 \xrightarrow{\delta} \Omega_{R[\underline{T}]/R} \otimes_{R[\underline{T}]} A \to \Omega_{A/R} \to 0$$

(in fact, the leftmost arrow is also injective, as we shall see below in corollary 7.5.2).

The *relative canonical module* $\omega_{A/R}$ is

$$\begin{aligned}\omega_{A/R} &= \operatorname{Hom}_A(\det I/I^2, A) \otimes_A \det(\Omega_{R[\underline{T}]/R} \otimes_{R[\underline{T}]} A)\\ &= \operatorname{Hom}_A(\det I/I^2, \det(\Omega_{R[\underline{T}]/R} \otimes_{R[\underline{T}]} A))\end{aligned}$$

where $\det I/I^2 = \bigwedge_A^{r-1} I/I^2$ and $\det(\Omega_{R[\underline{T}]/R} \otimes_{R[\underline{T}]} A) = \bigwedge_A^r(\Omega_{R[\underline{T}]/R} \otimes_{R[\underline{T}]} A)$.

Since the generality elucidates the situation, suppose that $P_1 \xrightarrow{j} P_2 \to P \to 0$ is an exact sequence of $A$-modules, where $P_1$, $P_2$ are free of ranks $r-1$, $r$ respectively. Then there is a natural map

$$P \to \operatorname{Hom}_R\left(\bigwedge\nolimits_A^{r-1} P_1, \bigwedge\nolimits_A^r P_2\right), \quad p \mapsto \langle n_1 \wedge \dots \wedge n_{r-1} \mapsto j(n_1) \wedge \dots \wedge j(n_{r-1}) \wedge \check{p}\rangle$$

where $\check{p} \in P_2$ denotes any lift of $p$. The fact that $\bigwedge_A^r P_1 = 0$ implies that this is well-defined.

Applying this to our situation gives a map of $A$-modules

$$c_{A/R} : \Omega_{A/R} \to \omega_{A/R}$$

which we will now examine in greater detail. Denote by $t_l$ the image of $T_l$ in $A$; the differentials $dt_1, \dots, dt_r$ generate $\Omega_{A/R}$ as a $A$-module, so it is enough to understand $c_{A/R}(dt_l)$ for each $l$. Further, since $\det I/I^2$ is an invertible $A$-module with basis $f_1 \wedge \dots \wedge f_{r-1}$ (we still identify the $f_l$ with their images mod $I^2$), it is enough to compute

$$c_{A/R}(dt_l)(f_1 \wedge \dots \wedge f_{r-1}) \in \det(\Omega_{R[\underline{T}]/R} \otimes_{R[\underline{T}]} A).$$

Well, chasing the definitions,

$$\begin{aligned} c_{A/R}(dt_l)(f_1 \wedge \cdots \wedge f_{r-1}) &= \delta(f_1) \wedge \cdots \wedge \delta(f_{r-1}) \wedge dT_l \\ &= df_1 \wedge \cdots \wedge df_{r-1} \wedge dT_l \\ &= \det(D_l)\, dT_1 \wedge \cdots \wedge dT_r, \end{aligned}$$

where

$$D_l = \begin{pmatrix} \frac{\partial f_1}{\partial T_1} & \cdots & \cdots & \cdots & \frac{\partial f_1}{\partial T_r} \\ \vdots & & & & \vdots \\ \frac{\partial f_{r-1}}{\partial T_1} & \cdots & \cdots & \cdots & \frac{\partial f_{r-1}}{\partial T_r} \\ 0 & \cdots & 1 & \cdots & 0 \end{pmatrix}$$

(the single 1 in the final row occurs in the $l^{\text{th}}$ column). Elementary matrix theory implies that $(-1)^{r+l}\det D_l$ is equal to the determinant of the matrix obtained from $D_l$ after removing the final row and the $l^{\text{th}}$ column; denote this matrix by $\Delta_l$. We have proved that

$$c_{A/R}(dt_l)(f_1 \wedge \cdots \wedge f_{r-1}) = (-1)^{r+l}\det(\Delta_l)\, dT_1 \wedge \cdots \wedge dT_r,$$

where $\Delta_l$ is the $r-1$ by $r-1$ matrix obtained by discarding the $l^{\text{th}}$ column (i.e. the $\frac{\partial}{\partial T_l}$ terms) from the Jacobian matrix $(\frac{\partial f_i}{\partial T_j})_{i,j}$.

The following fact about the matrices $\Delta_l$, $l = 1, \dots, r-1$ is very important to us:

**Lemma 7.5.1.** *There exists $l$ in the range $1 \leq l \leq r-1$ such that $\det \Delta_l$ is non-zero in $A$.*

*Proof.* We have assumed that the algebraic variety $\operatorname{Spec}(A \otimes_R K)$ is generically smooth over $K$, and therefore it contains a smooth closed point $x \in \operatorname{Spec}(A \otimes_R K)$. The Jacobian condition for smoothness asserts that

$$\operatorname{rank} J = r - \dim \mathcal{O}_{X,x} = r - 1,$$

where $J = (\frac{\partial f_i}{\partial T_j})_{i,j}$ is the Jacobian matrix inside $k(x)$ (a finite extension of $K$). This means that there is $l$ such that the matrix obtained by removing the $l^{\text{th}}$ column from $J$ is non-singular in $k(x)$; that is, $\det \Delta_l \notin \mathfrak{m}_{X,x}$, which is enough. ☐

**Corollary 7.5.2.** *The map $\delta : I/I^2 \to \Omega_{R[\underline{T}]/R} \otimes_{R[\underline{T}]} A$ is injective.*

*Proof.* Let $l$ be as provided by the previous lemma. It significantly simplifies the notation with matrices if we assume $l = r$, without making any essential difference to the proof. Recall that $\delta$ is given by

$$\delta : I/I^2 \to \Omega_{A[\underline{T}]/A} \otimes_{A[\underline{T}]} B, \quad b \operatorname{mod} I \mapsto db.$$

Since $\delta(f_i) = \sum_{j=1}^r \frac{\partial f_i}{\partial T_j} dT_j$, the matrix of $\delta$ with respect to the bases $f_1, \dots, f_{r-1}$ and $dT_1, \dots, dT_r$ is

$$\left(\frac{\partial f_i}{\partial t_j}\right)_{\substack{1 \leq i \leq r-1 \\ 1 \leq j \leq r}} = \begin{pmatrix} & \frac{\partial f_1}{\partial T_r} \\ \Delta_r & \vdots \\ & \frac{\partial f_{r-1}}{\partial T_r} \end{pmatrix}$$

(our matrices act on row vectors on the right, rather than column vectors on the left). If $v = \sum_{i=1}^{r-1} a_i f^i$ is a typical element of $I/I^2$, then we see that the identity $\delta(v) = 0$ implies $(a_1, \dots, a_{r-1})\Delta_r = 0$, implying that $v = 0$ by assumption on $\Delta_r$. ☐

More generally, if $x$ is any smooth, closed point of a fibre of $\operatorname{Spec} A \to \operatorname{Spec} R$, then the argument of the previous lemma shows that one can find $l$ (depending on $x$) such that $\det \Delta_l$ is non-zero in $k(x)$, i.e. $\det \Delta_l$ is a unit in $A_x = \mathcal{O}_{X,x}$. The explicit description of $c_{A/R}$ then implies that

$$c_{A/R,x} : \Omega_{A/R} \otimes_A A_x \to \omega_{A/R} \otimes_A A_x$$

is surjective. Further, since $\pi$ is smooth at $x$, it is well-known that $\Omega_{A/R} \otimes_A A_x$ is an invertible $A_x$-module. It follows that $c_{A/R,x}$ is an isomorphism. Localising further at the generic point $\eta$ of $X$ reveals that

$$c_{A/R,\eta} : \Omega_{A/R} \otimes_A F \to \omega_{A/R} \otimes_A F$$

is an isomorphism. Since $\omega_{A/R}$ is an invertible $A$-module, it embeds into $\omega_{A/R} \otimes_A F$ and we thus obtain a canonical embedding

$$\omega_{A/R} \hookrightarrow \omega_{A/R} \otimes_A F \cong \Omega_{A/R} \otimes_A F$$

of $\omega_{A/R}$ into the one-dimensional $F$-vector space $\Omega_{A/R} \otimes_A F = \Omega_{F/K}$.

Conversely, if $l$ satisfies $\det \Delta_l \neq 0$, then since $A \otimes_R K$ is reduced, there is a closed point $x$ of $\operatorname{Spec}(A \otimes_R K)$ for which $\det \Delta_l \notin \mathfrak{m}_{X,x}$, and so $x$ is a smooth point of the variety $\operatorname{Spec}(A \otimes_R K)$ and the previous argument applies. We summarise:

**Proposition 7.5.3.** *There is a canonical embedding of $\omega_{A/R}$ into $\Omega_{F/K}$ induced by $c_{A/R}$. If $l$ satisfies $\det \Delta_l \neq 0$, then the embedding is explicitly given by*

$$\omega_{A/R} \hookrightarrow \Omega_{F/K}, \quad \langle f_1 \wedge \cdots \wedge f_{r-1} \mapsto dT_1 \wedge \cdots \wedge dT_r \rangle \mapsto (-1)^{r+l} \det(\Delta_l)^{-1} dt_l.$$

*Proof.* This follows from the previous discussion and explicit description of $c_{A/F}$. Note that $\langle f_1 \wedge \cdots \wedge f_{r-1} \mapsto dT_1 \wedge \cdots \wedge dT_r \rangle$ is a basis for the invertible $A$-module $\omega_{A/F}$ and that $\Omega_{F/K}$ is a one-dimensional $F$-space with basis $dt_l$. ☐

### 7.5.2 Realising $\operatorname{Spec} A$ as a finite cover of $\mathbb{A}^1_R$

From the perspective of ramification theory, it is useful to realise $X$, at least locally, as a finite cover of the projective line over $S$. We now explain how this is done. Let $l$, in the range $1 \leq l \leq r-1$, satisfy $\det \Delta_l \neq 0$ (this exists by lemma 7.5.1).

**Lemma 7.5.4.** *With $l$ as above, $I \cap R[T_l] = 0$, and so the surjection $R[\underline{T}] \twoheadrightarrow A$ restricts to an embedding $R[T_l] \hookrightarrow A$; this makes $A$ into a finitely generated, flat $R[T_l]$-module.*

*Proof.* Denote by $t_l$ the image of $T_l$ in $A$. Just as above, we have an exact sequence of $A$-modules

$$I/I^2 \xrightarrow{\delta} \Omega_{R[\underline{T}]/R[T_l]} \otimes_{R[\underline{T}]} A \to \Omega_{A/R[T_l]} \to 0,$$

where $\delta$ is the $A$-linear map with matrix $\Delta_l$, with respect to the bases $f_1, \ldots, f_{r-1}$ and $dT_1, \ldots, dT_{l-1}, dT_{l+1}, \ldots, dT_r$. By assumption, this matrix is non-singular over $F$, and so $\delta$ is injective. Localising obtains an exact sequence

$$0 \to I/I^2 \otimes_A F \xrightarrow{\delta_F} \Omega_{R[\underline{T}]/R[T_l]} \otimes_{R[\underline{T}]} F \to \Omega_{A/R[T_l]} \otimes_A F \to 0,$$

and then counting dimensions reveals that $\delta_F$ is an isomorphism and $\Omega_{A/R[T_l]} \otimes_A F = 0$. But $\Omega_{A/R[T_l]} \otimes_A F = \Omega_{F/\operatorname{Frac} R[t_l]}$, and so $F$ is a separable, algebraic extension of $\operatorname{Frac} R[t_r]$. Since $F$ is finitely generated over $\operatorname{Frac} R$, we now see that $F/\operatorname{Frac} R[t_l]$ is a finite, separable extension.

Let $C'$ denote the integral closure of $R[t_l]$ inside $F$; since $R$ is excellent, $R[t_l]$ is Nagata and therefore $C'$ is a finitely generated $R[t_l]$-module. Since $C'$ is integrally closed in $F$, we have $\operatorname{Frac} C' = F$. For any height one prime $\mathfrak{p}$ of $C'$, the localisation $C'_\mathfrak{p}$ is a discrete valuation ring. Let $C'_\mathfrak{p}A$ be the $C'_\mathfrak{p}$ subalgebra of $F$ generated by $A$; it is easy to see, simply because $C'_\mathfrak{p}$ is a discrete valuation ring, that it is impossible to have proper inclusions $F \supset C'_\mathfrak{p}B \supset C'_\mathfrak{p}$. Therefore $A \subseteq C'_\mathfrak{p}$; but $C'$ is integrally closed, so $\bigcap_{\mathfrak{p} \lhd C'} C'_\mathfrak{p} = C'$ and therefore $A \subseteq C'$. Hence $A$ is a finitely generated $R[t_l]$-module.

As remarked in proposition 7.5.3, $dt_l$ is not $A$-torsion in $\Omega_{A/R}$; using the natural maps

$$(\Omega_{R[T_l]/R} \otimes_{R[T_l]} R[t_l]) \otimes_{R[t_l]} A \to \Omega_{R[t_l]/R} \otimes_{R[t_l]} A \to \Omega_{A/R},$$

we see that $dt_l$ is not $R[t_l]$-torsion in $\Omega_{R[T_l]/R} \otimes_{R[T_l]} R[t_l]$. Explicitly, this means that if $g$ is a polynomial with coefficients in $R$ such that $g(t_l) = 0$, then $g'(t_l) = 0$. Now suppose for a contradiction that $R[T_l] \to R[t_l]$ is not injective. Then $t_l$ is algebraic over $K$; further, $\Omega_{F/K}$ is a one-dimensional $F$-vector space, and so $F$ is a finite, separable extension of a degree $1$ purely transcendental extension of $K$. This means that the minimal polynomial $g$ of $t_l$ over $K$ is separable. Now take $a \in R$ so that $ag$ has coefficients in $R$. Then $ag$ is nonzero, $ag(t_l)$ vanishes, but $ag'(t_l) \neq 0$, giving the required contradiction.

Flatness of $R[t_l] \to A$ is proved below; see lemma 7.6.4, taking $B = R[t_l]$. ☐

We continue this study of finite morphisms in the next section.

## 7.6 Finite morphisms, differents and Jacobians

Suppose that $A/B$ is a finite extension of rings, with corresponding fraction fields $F/M$ (assumed to be separable). The associated *codifferent* is the $A$-module

$$\mathfrak{C}(A/B) = \{x \in F : \operatorname{Tr}_{F/M}(xA) \subseteq B\}.$$

The aim of this section is to prove that if $A$ is a complete intersection over $B$, then the codifferent is a free $A$-module generated by the determinant of a certain Jacobian matrix.

I am grateful to L. Xiao for some interesting discussions related to this section.

### 7.6.1 The case of complete discrete valuation rings

We begin by treating the case of complete discrete valuation rings. Let $F/M$ be a finite, separable extension of complete discrete valuation fields, with rings of integers $\mathcal{O}_F/\mathcal{O}_M$. In place of the codifferent, one usually considers the *different* $\mathfrak{D}(\mathcal{O}_F/\mathcal{O}_M)$, which is the $\mathcal{O}_F$-fractional ideal defined by

$$\mathfrak{C}(\mathcal{O}_F/\mathcal{O}_M)\,\mathfrak{D}(\mathcal{O}_F/\mathcal{O}_M) = \mathcal{O}_F$$

i.e. the complement of the codifferent. Since $\mathcal{O}_F/\mathcal{O}_M$ is a finite extension of regular, local rings, it is a complete intersection

$$\mathcal{O}_F = \mathcal{O}_M[T_1, \dots, T_m]/\langle f_1, \dots, f_m \rangle,$$

and we set

$$\mathfrak{J}(\mathcal{O}_F/\mathcal{O}_M) = \det\left(\frac{\partial f_i}{\partial T_j}\right)_{i,j} \mathcal{O}_F,$$

which we may as well call the *Jacobian ideal*. The fact that $F/M$ is separable implies that the Jacobian ideal is non-zero, and as argued several times in the previous section, we have an exact sequence

$$0 \to \langle f_1, \dots, f_m \rangle / \langle f_1, \dots, f_m \rangle^2 \xrightarrow{\delta} \Omega_{\mathcal{O}_M[\underline{T}]/\mathcal{O}_M} \otimes_{\mathcal{O}_M[\underline{T}]} \mathcal{O}_F \to \Omega_{\mathcal{O}_F/\mathcal{O}_M} \to 0.$$

The matrix of $\delta$ with respect to the bases $f_1, \dots, f_m$ and $dT_1, \dots, dT_m$ is the Jacobian matrix, and it easily follows (using the Iwasawa decomposition of $GL_m(F)$) that $\mathfrak{J}(\mathcal{O}_F/\mathcal{O}_M) = \mathfrak{p}_F^l$, where $l = \mathrm{length}_{\mathcal{O}_F} \Omega_{\mathcal{O}_F/\mathcal{O}_M}$; in particular, the Jacobian ideal does not depend on how we write $\mathcal{O}_F$ as a complete intersection over $\mathcal{O}_M$.

We are going to prove that

$$\mathfrak{J}(\mathcal{O}_F/\mathcal{O}_M) = \mathfrak{D}(\mathcal{O}_F/\mathcal{O}_M). \tag{$\mathfrak{J} = \mathfrak{D}$}$$

When $F/M$ is monogenic (i.e. we may write $\mathcal{O}_F = \mathcal{O}_M[\alpha]$ for some $\alpha \in \mathcal{O}_F$), which is the case whenever the residue field extension of $F/M$ is separable, the equality $\mathfrak{J} = \mathfrak{D}$ is well-known; it states that $\mathfrak{D}(\mathcal{O}_F/\mathcal{O}_M) = g'(\alpha)$, where $g$ is the minimal polynomial of $\alpha$ over $M$. A proof may be found in [Neu99, III.2] (this reference assumes throughout that the residue field extensions are separable, but the proof remains valid in the general case).

Several easy lemmas are required, firstly a product formula:

**Lemma 7.6.1.** *Let $F'$ be a finite, separable extension of $F$; then*

$$\mathfrak{D}(\mathcal{O}_{F'}/\mathcal{O}_M) = \mathfrak{D}(\mathcal{O}_F/\mathcal{O}_M)\mathfrak{D}(\mathcal{O}_{F'}/\mathcal{O}_F)$$

*and*

$$\mathfrak{J}(\mathcal{O}_{F'}/\mathcal{O}_M) = \mathfrak{J}(\mathcal{O}_F/\mathcal{O}_M)\mathfrak{J}(\mathcal{O}_{F'}/\mathcal{O}_F).$$

*Proof.* The different result is well-known; see e.g. [Neu99, III.2]. We will prove the Jacobian result. Write $\mathcal{O}_{F'}$ as a complete intersection over $\mathcal{O}_F$

$$\mathcal{O}_{F'} = \mathcal{O}_F[T_{m+1}, \dots, T_{m+n}]/\langle f_{m+1}, \dots, f_{m+n} \rangle,$$

and denote by $\widetilde{f_i}$ a lift of the $\mathcal{O}_F$ polynomials $f_i$ to $\mathcal{O}_M[T_1, \dots, T_{m+n}]$, for $i = m+1, \dots, m+n$. Then

$$\mathcal{O}_{F'} = \mathcal{O}_M[T_1, \dots, T_{m+n}]/\langle f_1, \dots, f_m, \widetilde{f}_{m+1}, \dots, \widetilde{f}_{m+n} \rangle$$

represents $\mathcal{O}_{F'}$ as a complete intersection over $\mathcal{O}_M$, and the Jacobian matrix in $\mathcal{O}_{F'}$ associated to this complete intersection is

$$\begin{pmatrix} \left(\frac{\partial f_i}{\partial T_j}\right)_{i,j=1,\dots,m} & 0 \\ \left(\frac{\partial f_i}{\partial T_j}\right)_{\substack{i=m+1,\dots,m+n \\ j=1,\dots,m}} & \left(\frac{\partial f_i}{\partial T_j}\right)_{i,j=m+1,\dots,m+n} \end{pmatrix}.$$

Since the determinant of this is the product of the determinants of the two square matrices, we are done. ☐

**Lemma 7.6.2.** *Suppose further that $F/M$ is Galois. Then there exists a sequence of intermediate extensions $F = F_s > \dots > F_{-1} = M$ such that each extension $F_i/F_{i-1}$ is monogenic.*

*Proof.* Let $F_0$ denote the maximal unramified subextension of $M$ inside $F$, and $F_1$ the maximal tamely ramified subextension (and set $F_{-1} = M$). Then $F/F_1$ is an extension whose residue field extension is purely inseparable, and whose ramification degree is a power of $p$ (= the residue characteristic, which we assume is $> 0$, for else we are done); therefore $\mathrm{Gal}(F/F_1)$ is a $p$-group, hence nilpotent, and so there is a sequence of intermediate fields $F = F_m > \cdots > F_1$ such that each $F_i$ is a normal extension of $F_1$ and such that each step is a degree $p$ extension.

Then $\mathcal{O}_{F_0} = \mathcal{O}_{F_{-1}}[\theta]$ where $\theta \in \mathcal{O}_{F_0}$ is a lift of a generator of $\overline{F}_0/\overline{M}$. Also, $\mathcal{O}_{F_1} = \mathcal{O}_{F_0}[\pi]$ where $\pi$ is a uniformiser of $F_1$. It remains to observe that any extension of prime degree $F_i/F_{i-1}$ is monogenic. Indeed, it is either totally ramified in which case $\mathcal{O}_{F_i} = \mathcal{O}_{F_{i-1}}[\pi']$ where $\pi'$ is a uniformiser of $F_i$; or else the ramification degree is 1 and $\mathcal{O}_{F_i} = \mathcal{O}_{F_{i-1}}[\theta']$ where $\theta' \in \mathcal{O}_{F_i}$ is a lift of a generator of the degree $p$ extension $\overline{F}_i/\overline{F}_{i-1}$ (which may be inseparable). ☐

Combining the previous two lemmas with the validity of $\mathfrak{J} = \mathfrak{D}$ in the monogenic case, we have proved that $\mathfrak{J} = \mathfrak{D}$ for any finite, Galois extension $F/M$. Now suppose that $F/M$ is finite and separable, but not necessarily normal, and let $F'$ be the normal closure of $F$ over $M$. The product formula gives us

$$\nu_{F'}(\mathfrak{D}(\mathcal{O}_{F'}/\mathcal{O}_M)) = e_{F'/F}\nu_F(\mathfrak{D}(\mathcal{O}_F/\mathcal{O}_M)) + \nu_{F'}(\mathfrak{D}(\mathcal{O}_{F'}/\mathcal{O}_F)),$$

and similarly for $\mathfrak{J}$. But the Galois case implies that $\mathfrak{J} = \mathfrak{D}$ for $F'/M$ and $F'/F$. We deduce $\mathfrak{J}(\mathcal{O}_F/\mathcal{O}_M) = \mathfrak{D}(\mathcal{O}_F/\mathcal{O}_M)$, which establishes our desired result. To summarise:

**Theorem 7.6.3.** *Let $F/M$ be a finite, separable extension of complete discrete valuation fields. Write $\mathcal{O}_F$ as a complete intersection over $\mathcal{O}_M$ as above, and let $J \in \mathcal{O}_F$ be the determinant of the Jacobian matrix. Then $J \neq 0$ and*

$$\mathfrak{C}(\mathcal{O}_F/\mathcal{O}_M) = J^{-1}\mathcal{O}_F.$$

*Proof.* Replacing $\mathfrak{C}(\mathcal{O}_F/\mathcal{O}_M)$ by its complementary ideal $\mathfrak{D}(\mathcal{O}_F/\mathcal{O}_M)$, this is what we have just proved. ☐

The previous theorem is really an elementary result concerning the ramification theory of complete discrete valuation fields with imperfect residue fields.

### 7.6.2 The higher dimensional case

We now generalise from complete discrete valuation rings to the general case. Let $B$ be a Noetherian, normal ring, and

$$A = B[T_1, \dots, T_m]/\langle f_1, \dots, f_m\rangle$$

a complete intersection over $B$ which is a finitely generated $B$-module; assume that $A$ is normal (this is automatic if $B$ is regular by Serre's criterion [Liu02, Corollary 8.2.24]). Set $F = \mathrm{Frac}\, A$, $M = \mathrm{Frac}\, B$, and assume that $F/M$ is separable. For a height one prime $y \lhd B$, the localisation $B_y$ is a discrete valuation ring, and we set $M_y = \mathrm{Frac}\, \widehat{B_y}$; use similar notation for $A$.

For any $y \lhd^1 B$, it is clear that $\mathfrak{C}(A_y/B_y) = \mathfrak{C}(A/B)A_y$ where $A_y = (B\setminus y)^{-1}A$, which is a Dedekind domain. A standard formula for extensions of Dedekind domains [Neu99] states

$$\mathfrak{C}(A_y/B_y) = \prod_{0 \neq Y \lhd A_y} Y^{-d_{Y/y}},$$

where $d_{Y/y} = \nu_Y(\mathfrak{D}(\widehat{A}_Y/\widehat{B}_y))$ (here $\nu_Y$ denotes the discrete valuation on $F_Y$). But by theorem 7.6.3, $d_{Y/y} = \nu_Y(J)$, where $J \in A$ is the determinant of the Jacobian matrix $\left(\frac{\partial f_i}{\partial T_j}\right)_{i,j}$. Therefore

$$\mathfrak{C}(A_y/B_y) = J^{-1}A_y.$$

To proceed further, we need the following result, which I learned from [Liu02, exercise 6.3.5]:

**Lemma 7.6.4.** *$A$ is flat over $B$.*

*Proof.* Let $\mathfrak{q} \lhd A$ be a maximal ideal of $A$, and let $\mathfrak{p}$, $Q$ denote its pullbacks to $A$, $k(\mathfrak{p})[T_1, \dots, T_m]$ respectively; we will first prove that (the images of) $f_1, \dots, f_m$ form a regular sequence in $k(\mathfrak{p})[T_1, \dots, T_m]_Q$. Well, if they do not, then pick $s$ minimally so that $f_s$ is a zero divisor in $k(\mathfrak{p})[T_1, \dots, T_m]_Q/\langle f_1, \dots, f_{s-1}\rangle$. This latter ring (call it $R$) is the quotient of a regular, local ring by a regular sequence (by minimality of $s$), and hence is Cohen-Macaulay [Mat89, §21]. Any Cohen-Macaulay local ring contains no embedded primes (and so the zero-divisor $f_s$ belongs to a minimal prime of $R$) and is equi-dimensional [Eis95, Corollaries 18.10 and 18.11]; together these imply that $\dim R = \dim R/\langle f_s\rangle$. Quotienting out by any other $f_i$ drops the dimension by at most one (by Krull's principal ideal theorem), so we deduce

$$\dim k(\mathfrak{p})[T_1, \dots, T_m]_Q/\langle f_1, \dots, f_m\rangle \geq \dim k(\mathfrak{p})[T_1, \dots, T_m]_Q - (m-1).$$

But the ring on the left is a localisation of the fibre $A \otimes_B k(\mathfrak{p})$, which is a finite dimensional $k(\mathfrak{p})$-algebra, and so is zero-dimensional. Hence $\dim k(\mathfrak{p})[T_1, \dots, T_m]_Q \leq m-1$, contradicting the fact that $Q$ is a maximal ideal of $k(\mathfrak{p})[T_1, \dots, T_m]$.

Secondly, since $B_\mathfrak{p} \to B[T_1, \dots, T_m]_Q$ is a flat map of local rings, and $f_1$ is not a zero-divisor in $k(\mathfrak{p})[T_1, \dots, T_m]_Q$, a standard criterion implies that $B_\mathfrak{p} \to B[T_1, \dots, T_m]_Q/\langle f_1\rangle$ is flat. Applying this criterion another $m-1$ times, we deduce that

$$B_\mathfrak{p} \to B[T_1, \dots, T_m]_Q/\langle f_1, \dots, f_m\rangle = A_\mathfrak{q}$$

is flat.

It is enough to check flatness at the maximal ideals of $A$, so we are done. □

There is a natural map

$$\mathfrak{C}(A/B) \to \mathrm{Hom}_B(A, B), \quad x \mapsto \mathrm{Tr}_{F/M}(x\,\cdot)$$

and non-degeneracy of the trace map for $F/M$ implies that this is an isomorphism of $A$-modules, where $A$ acts on $\mathrm{Hom}_B(A, B)$ by $a\,\phi := \phi(a\,\cdot)$. For any maximal ideal $\mathfrak{m} \lhd B$, the localisation $A_\mathfrak{m}$ is a flat (by the previous lemma), hence free, $B_m$-module of rank $n = |F : M|$; the importance of this is that it implies that $\mathfrak{C}(A/B)A_\mathfrak{m}$ is a free $B_m$-module of rank $n$. Using this, we will deduce our main 'different=Jacobian' result:

**Theorem 7.6.5.** *The codifferent is an invertible $A$ module, with basis $J^{-1}$, i.e.*

$$\mathfrak{C}(A/B) = J^{-1}A.$$

*Proof.* It is enough to prove $\mathfrak{C}(A/B)A_\mathfrak{m} = J^{-1}A_\mathfrak{m}$ for each maximal ideal $\mathfrak{m} \lhd B$, and therefore we will simply assume that $B$ is a local ring; as remarked above, this implies that $\mathfrak{C}(A/B)$ is free of rank $n$. Moreover, $J^{-1}A$ is also free of rank $n$, and so, by picking a basis of $F \cong M^n$ and identifying our two free submodules with submodules of $M^n$, there is $\tau \in GL_n(M)$ such that $\tau\mathfrak{C}(A/B) = J^{-1}A$.

Further, for any height one-prime $y \lhd B$, theorem 7.6.3 implies that $(B \backslash y)^{-1}\mathfrak{C}(A/B) = (B \setminus y)^{-1}J^{-1}A$, implying that $\tau \in GL_n(B_y)$. Since $B$ was assumed to be normal, $B = \bigcap_{y \lhd B} B_y$ and so $\tau \in GL_n(B)$; therefore $\tau\mathfrak{C}(A/B) = \mathfrak{C}(A/B)$, which completes the proof. □

**Remark 7.6.6.** If $P$ is any $A$-module, then the natural pairing

$$\mathrm{Hom}_A(P, \mathfrak{C}(A/B)) \times P \to \mathfrak{C}(A/B) \overset{\mathrm{Tr}_{F/M}}{\to} B$$

induces a $B$-linear map

$$\mathrm{Hom}_A(P, \mathfrak{C}(A/B)) \to \mathrm{Hom}_B(P, B),$$

which is easily checked to be an isomorphism (using non-degeneracy of $\mathrm{Tr}_{F/M}$). Thus $\mathfrak{C}(A/B)$ is exactly the Grothendieck dualising module of $\operatorname{Spec} A \to \operatorname{Spec} B$ (which is projective since it is finite, and flat by lemma 7.6.4).

One also has the relative canonical module

$$\omega_{A/B} = \mathrm{Hom}_A\left(\bigwedge_A^m \langle f_1, \dots, f_m\rangle/\langle f_1, \dots, f_m\rangle^2, \bigwedge_A^m (\Omega_{B[\underline{T}]/B} \otimes_{B[\underline{T}]} A)\right),$$

and a natural map

$$A \to \omega_{A/B}, \quad b \mapsto b\,\delta^{\wedge m} \tag{$*$}$$

where $\delta$ is the map in the exact sequence

$$0 \to \langle f_1, \dots, f_m\rangle/\langle f_1, \dots, f_m\rangle^2 \overset{\delta}{\to} \Omega_{B[\underline{T}]/B} \otimes_{B[\underline{T}]} A \to \Omega_{A/B} \to 0.$$

Moreover, $(*)$ is an isomorphism at any point $x \in \operatorname{Spec} A$ at which $\operatorname{Spec} A \to \operatorname{Spec} B$ is smooth, such as the generic point since $F/M$ is separable. This therefore defines a natural embedding of $A$-modules $\omega_{A/B} \hookrightarrow F$ given by

$$\langle f_1 \wedge \dots \wedge f_m \mapsto dT_1, \dots \wedge dT_m\rangle \mapsto J^{-1}$$

i.e. $\omega_{A/B} \cong J^{-1}A$. This is the generalisation of subsection 7.5.1 to the case of a finite extension, rather than one of relative dimension one.

In conjunction with theorem 7.6.5, we have produced a reasonably natural (though it depends on how we write $A$ as a complete intersection over $B$) isomorphism between the dualising and canonical sheaves.

This material is surely known to experts, and there are similar results in [Kle80]; a comprehensive discussion must be buried somewhere in SGA or EGA[1].

## 7.7 Explicit construction of the canonical sheaf for arithmetic surfaces

Now we turn to the main global content of this chapter, namely using the residue maps to construct the canonical sheaf of an arithmetic surface. We begin with the affine case.

[1] J.-P. Serre gave a talk at Harvard's 'Basic Notions' seminar, 10 November 2003, entitled "Writing Mathematics?", in which he explains how to write mathematics badly. He explains that if you wish to give a reference which can not be checked by the reader, then you should ideally refer, without any page references, to the complete works of Euler, but "if you refer to SGA or EGA, you have a good chance also". The reader interested in verifying *this* reference should consult timeframe 4.11–4.20 of the video at `http://modular.fas.harvard.edu/edu/basic/serre/`.

Let $\mathcal{O}_K$ be a Dedekind domain of characteristic zero with finite residue fields; its field of fractions is $K$. We suppose that we are given a finitely generated, flat $\mathcal{O}_K$-algebra $A$, which is normal and two-dimensional. Assume further that there is an intermediate ring $B$

$$\mathcal{O}_K \leq B \leq A$$

such that $B \cong \mathcal{O}_K[T]$ and such that $A$ is a finitely generated, flat $B$-module. Finally, set $F = \operatorname{Frac} A$, $M = \operatorname{Frac} B$, and assume that $F/M$ is separable. It follows that $\Omega_{F/K}$ is a one-dimensional $F$-vector space, with basis $dT$.

If $0 \lhd y \lhd x$ is a chain of primes in $A$, then $A_x$ is a two-dimensional, normal, local ring containing the discrete valuation ring $\mathcal{O}_{K,s(x)}$, where $s(x) = \mathcal{O}_K \cap x$. Therefore, as in section 7.4, we have the residue map $\operatorname{Res}_{x,y} : \Omega_{F/K} \to K_{s(x)}$ where $K_{s(x)} = \operatorname{Frac} \widehat{\mathcal{O}_{K,s(x)}}$. The situation is similar with $B$ in place of $A$.

We begin by establishing a functoriality result which we could have proved in section 7.3:

**Proposition 7.7.1.** *Let $\omega \in \Omega_{F/K}$, and let $0 \lhd y \lhd x \lhd B$ be a chain of primes in $B$. Then*

$$\operatorname{Res}_{x,y} \operatorname{Tr}_{F/M}(\omega) = \sum_{x',y'} \operatorname{Res}_{x',y'}(\omega)$$

*where the sum is taken over all chains $0 \lhd y' \lhd x' \lhd A$ such that $x'$ sits over $x$ and $y'$ sits over $y$.*

*Proof.* Let $x$ be a fixed maximal ideal of $B$; then $A \otimes_B \widehat{B_x} = \bigoplus_{x'|x} \widehat{A_{x'}}$ where $x'$ ranges over the finitely many maximal ideals of $A$ sitting over $x$. The $\widehat{B_x}$-modules $\widehat{A_{x'}}$ are flat, hence free, and so by choosing bases for them we may define trace maps $\operatorname{Tr}_{\widehat{A_{x'}}/\widehat{B_x}}$ in the usual way. Passing to the fields of fractions obtains

$$\operatorname{Tr}_{F/M} = \sum_{x'|x} \operatorname{Tr}_{\operatorname{Frac} \widehat{A_{x'}} / \operatorname{Frac} \widehat{B_x}},$$

a result which is of course very well known for Dedekind domains.

Let $Y$ be a height one prime of $\widehat{B_x}$. Then, for $\omega \in \Omega_{F/K}$,

$$\begin{aligned} \operatorname{Res}_Y \operatorname{Tr}_{F/M} \omega &= \sum_{x'|x} \operatorname{Res}_Y \operatorname{Tr}_{\operatorname{Frac} \widehat{A_{x'}} / \operatorname{Frac} \widehat{B_x}} \omega \\ &\overset{(*)}{=} \sum_{x'|x} \sum_{Y'|Y} \operatorname{Res}_{Y'} \omega \end{aligned}$$

where $Y'$ ranges over the height one primes of $\widehat{A_{x'}}$ sitting over $Y$, and equality $(*)$ follows from proposition 7.3.9. Now fix a height one prime $y$ of $B$ contained inside $x$;

then

$$\begin{aligned}
\operatorname{Res}_{x,y}\operatorname{Tr}_{F/M}\omega &= \sum_{\substack{Y\lhd\widehat{B_x}\\ Y|y}}\operatorname{Res}_Y\operatorname{Tr}_{F/M}\omega\\
&= \sum_{\substack{Y\lhd\widehat{B_x}\\ Y|y}}\sum_{x'|x}\sum_{Y'|Y}\operatorname{Res}_{Y'}\omega\\
&= \sum_{x'|x}\sum_{\substack{y'\lhd A_{x'}\\ y'|y}}\sum_{\substack{Y'\lhd\widehat{A_{x'}}\\ Y'|y'}}\sum_{Y'|Y}\operatorname{Res}_{Y'}\omega\\
&= \sum_{x'|x}\sum_{\substack{y'\lhd A_{x'}\\ y'|y}}\operatorname{Res}_{y'}\omega,
\end{aligned}$$

which is the required result. □

We now introduce the following $A$-submodule of $\Omega_{F/K}$ defined in terms of residues

$$W_{A/\mathcal{O}_K} = \{\omega\in\Omega_{F/K} : \operatorname{Res}_{x,y}(f\omega)\in\widehat{\mathcal{O}}_{K,s(x)} \text{ for all } 0\lhd y\lhd x\lhd A \text{ and } f\in A_y\}.$$

Similarly define $W_{B/\mathcal{O}_K}$.

Suppose that $\omega\in W_{A/\mathcal{O}_K}$ and $y\lhd x$ is a chain in $A$. We remarked at the end of the proof of theorem 7.3.6 that each residue map on a two-dimensional, complete, normal local ring is continuous with respect to the adic topology on the ring and the discrete valuation topology on the local field (this is easy to prove for $\mathcal{O}_K[[T]]$ and follows in the general case using functoriality). Therefore $\operatorname{Res}_{x,y}(f\omega)\in\widehat{\mathcal{O}}_{K,s(x)}$ for all $f\in\widehat{A_x}$. Another continuity argument even implies that this remains true for $f\in(\widehat{A_x})_y$.

Now, $y\widehat{A_x}$ is a radical ideal of $\widehat{A_x}$; localising and completing with respect to this ideal obtains

$$\widehat{((\widehat{A_x})_y} = \bigoplus_{Y|y}\widehat{(\widehat{A_x})_Y}$$

where $Y$ ranges over the height one primes of $\widehat{A_x}$ sitting over $y$. Each $\mathcal{O}_{x,Y} := \widehat{(\widehat{A_x})_Y}$ is a complete discrete valuation ring whose field of fractions is a two-dimensional local field, which we will denote $F_{x,Y}$. Note that $\operatorname{Res}_{x,y} = \sum_Y \operatorname{Res}_{F_{x,Y}}$ by definition.

Fix a particular height one prime $Y_0$ of $\widehat{A_x}$ over $y$. Since $(\widehat{A_x})_y$ is dense in $\bigoplus_{Y|y}\mathcal{O}_{x,Y}$ with respect to the discrete valuation topologies, there is $h\in(\widehat{A_x})_y$ which is $Y_0$-adically close to $1$ and $Y$-adically close to $0$ for $Y\neq Y_0$. More precisely, since each residue map $\operatorname{Res}_{F_{x,Y}}$ is continuous with respect to the discrete valuation topologies on $F_{x,Y}$ and $K_{s(x)}$, we may take $h$ to satisfy

(i) $\operatorname{Res}_{F_{x,Y}}(h\mathcal{O}_{x,Y}\omega)\subseteq\widehat{\mathcal{O}}_{K,s(x)}$ for $Y\neq Y_0$;

(ii) $\operatorname{Res}_{F_{x,Y_0}}((h-1)\mathcal{O}_{x,Y_0})\omega)\subseteq\widehat{\mathcal{O}}_{K,s(x)}$.

Replacing $f$ by $hf$, it follows that $\operatorname{Res}_{F_{x,Y_0}}(f\omega)\in\widehat{\mathcal{O}}_{K,s(x)}$ for all $f\in(\widehat{A_x})_y$, and therefore for all $f\in\mathcal{O}_{x,Y_0}$ by continuity. To summarise:

**Lemma 7.7.2.** *Let $\omega\in\Omega_{F/K}$; then $\omega\in W_{A/\mathcal{O}_K}$ if and only if $\operatorname{Res}_{F_{x,Y}}(f\omega)\in\widehat{\mathcal{O}}_{K,s(x)}$ for all maximal ideals $x\lhd A$, all height one primes $Y\lhd\widehat{A_x}$, and all $f\in\mathcal{O}_{x,Y}$.*

*Proof.* The implication $\Leftarrow$ is trivial, and we have just proved $\Rightarrow$. $\square$

Next we reduce the calculation of $W_{A/\mathcal{O}_K}$ to that of $W_{B/\mathcal{O}_K}$:

**Lemma 7.7.3.** *Let $\omega \in \Omega_{F/K}$; then $\omega \in W_{A/\mathcal{O}_K}$ if and only if $\mathrm{Tr}_{F/M}(g\omega) \in W_{B/\mathcal{O}_K}$ for all $g \in A$.*

*Proof.* The implication $\Leftarrow$ follows from proposition 7.7.1. Let us fix a chain $y \lhd x$ in $B$ and suppose that $\mathrm{Res}_{x,y}(f\,\mathrm{Tr}_{F/M}(g\omega)) \in \widehat{\mathcal{O}}_{K,s(x)}$ for all $g \in A$, $f \in B_y$; so

$$\sum_{x',y'} \mathrm{Res}_{x',y'}(g\omega) \in \widehat{\mathcal{O}}_{K,s(x)} \qquad (*)$$

for all $g \in A_y$ by proposition 7.7.1. Since we have $\widehat{A_x} = \oplus_{x'|x}\widehat{A_{x'}}$, it follows that if $\xi$ is a fixed maximal ideal of $A$ over $x$, then there is $h \in A$ which is close to $1$ $\xi$-adically and close to $0$ $x'$-adically for any other maximal ideal $x' \neq \xi$ over $x$. More precisely, as we remarked at the end of the proof of theorem 7.3.6, each residue map on a two-dimensional, complete, normal local ring is continuous with respect to the adic topology on the ring and the discrete valuation topology on the local field (this is easy to prove for $\mathcal{O}_K[[T]]$ and follows in the general case using functoriality); we may find $h$ such that

(i) $\mathrm{Res}_{x',y'}(hA_{x'}\omega) \subseteq \mathcal{O}_{K,s(x)}$ for $x' \neq \xi$ and $y' \lhd x'$ over $y$;

(ii) $\mathrm{Res}_{\xi,y'}((h-1)A_\xi\omega) \subseteq \mathcal{O}_{K,s(x)}$ for $y' \lhd \xi$ over $y$.

Replacing $g$ by $gh$ in $(*)$ obtains

$$\sum_{\substack{y' \lhd \xi \\ y'|y}} \mathrm{Res}_{\xi,y'}(g\omega) \in \widehat{\mathcal{O}}_{K,s(x)}$$

for all $g \in A$. This sum is equal to

$$\sum_{\substack{y' \lhd \widehat{A_\xi} \\ y'|y}} \mathrm{Res}_{y'}(g\omega),$$

and we may now repeat the argument, similarly to how we proved the previous lemma, by completing at $y$ instead of $x$, and using the fact that the residue map on a two-dimensional local field is continuous with respect to the valuation topology. This gives $\mathrm{Res}_{\xi,y'}(g\omega) \in \widehat{\mathcal{O}}_{K,s(x)}$ for all $g \in A_{y'}$, for any $y' \lhd \xi$ over $y$. This completes the proof. $\square$

We may now establish our main result in the affine case, relating $W_{A/\mathcal{O}_K}$ to the codifferent of $A/B$. The proof requires explicit arguments using residues, and uses the results and notation of sections 7.2 and 7.3.

**Theorem 7.7.4.** *We have $W_{A/\mathcal{O}_K} = \mathfrak{C}(A/B)dT$.*

*Proof.* Since $\Omega_{F/K} = F\,dT$ it is enough, by the previous lemma, to prove that $W_{B/\mathcal{O}_K} = B\,dT$. Let $\omega = h\,dT \in \Omega_{M/K}$, where $h \in M$; we wish to prove $h \in B$. As it makes the argument slightly more conceptual, we shall prove this merely under the assumption that $B$ is smooth over $\mathcal{O}_K$ (which is certainly true for $B = \mathcal{O}_K[T]$). Fix a maximal ideal $x \lhd B$ and write $s = s(x)$, $C = \widehat{B_x}$, $N = \mathrm{Frac}\,C$ for simplicity; let $\pi \in \mathcal{O}_{K,s}$ be a uniformiser at $s$.

If $y$ is a height one prime of $C$ which does not contain $\pi$, then $\pi^{-1} \in C_y$ and so

$$\mathrm{Res}_y(f\omega) \in \widehat{\mathcal{O}}_{K,s} \text{ for all } f \in \widehat{C_y} \iff \mathrm{Res}_y(f\omega) = 0 \text{ for all } f \in \widehat{C_y}.$$

Note that in the notation earlier in this section, $\widehat{C_y} = \mathcal{O}_{x,y}$. Further, non-degeneracy of the trace map from the coefficient field $k_y$ to $K_s$ implies

$$\mathrm{Res}_y(f\omega) = 0 \text{ for all } f \in \widehat{C_y} \iff \mathrm{res}_y(f\omega) = 0 \text{ for all } f \in \widehat{C_y}.$$

Let $t \in C_y$ be a uniformiser at $y$; then $\omega = h\frac{dT}{dt}\,dt$ and it easily follows from the definition of the residue map on the equi-characteristic two-dimensional field $N_{y\widehat{C}} = M_{x,y} \cong k_y((t))$ that

$$\mathrm{res}_y(f\omega) = 0 \text{ for all } f \in \widehat{C_y} \iff h\frac{dT}{dt} \in \widehat{C_y}.$$

Finally, we have identifications

$$\widehat{C_y}\,dT = \Omega_{B/\mathcal{O}_K} \otimes_B \widehat{C_y} \cong \Omega^{\mathrm{sep}}_{\widehat{C_y}/\widehat{\mathcal{O}}_{K,s}} = \widehat{C_y}\,dt,$$

with the isomorphism coming from lemma 7.3.11, and $dT$ corresponding to $\frac{dT}{dt}\,dt$. Hence $\frac{dT}{dt}$ is a unit in $\widehat{C_y}$, and so

$$\mathrm{res}_y(f\omega) = 0 \text{ for all } f \in \widehat{C_y} \iff h \in C_y.$$

Now we consider the prime(s) containing $\pi$. The special fibre $B/\pi B$ is smooth, and so $C/\pi C$ is a complete, regular, one-dimensional local ring, i.e. a complete discrete valuation ring, and $\pi C$ is prime in $C$. Therefore $\pi C$ is the only height one prime of $C$ which contains $\pi$. Further, $\pi$ is a uniformiser in the two-dimensional local field $N_{\pi A} = M_{x,\pi A}$, and therefore by corollary 7.2.16 there is an isomorphism $F_{\pi C} \cong k_{\pi C}\{\{t\}\}$, and moreover $k_{\pi C}$ is an unramified extension of $K_s$. It easily follows from the definition of the residue map in this case that

$$\mathrm{res}_{\pi C}(f\omega) \in \mathcal{O}_{k_{\pi\widehat{C}}} \text{ for all } f \in \widehat{C_{\pi C}} \iff h \in C_{\pi C}.$$

The fact that the extension $k_{\pi C}/K$ of local fields is unramified now implies

$$\mathrm{Res}_{\pi C}(f\omega) \in \mathcal{O}_{k_{\pi\widehat{C}}} \text{ for all } f \in \widehat{C_{\pi C}} \iff h \in C_{\pi C}.$$

Hence,

$$\mathrm{Res}_{M_{x,y}}(f\omega) \in \widehat{\mathcal{O}}_{K,s} \text{ for all } y \lhd^1 \widehat{B_x} \text{ and } f \in \mathcal{O}_{x,y} \iff h \in (\widehat{B_x})_y \text{ for all } y \lhd^1 \widehat{B_x}.$$

But $\widehat{B_x}$ is normal, so $\bigcap_{y \lhd^1 \widehat{B_x}} (\widehat{B_x})_y = \widehat{B_x}$. We deduce that $\omega$ belongs to $W_{B/\mathcal{O}_K}$ if and only if $h \in B_x$ for all $x$, which holds if and only if $h \in B$. This completes the proof. □

### 7.7.1 The main global result

All the required results have been established, and we now may now present the proof of our main theorem. Let $\mathcal{O}_K$ be a Dedekind domain of characteristic zero with finite residue fields; its field of fractions is $K$. Let $\pi : X \to S = \operatorname{Spec} \mathcal{O}_K$ be a flat, surjective, local complete intersection, with smooth, connected, generic fibre of dimension 1.

**Theorem 7.7.5.** *The canonical sheaf $\omega_{X/S}$ of $X \to S$ is explicitly given by, for open $U \subseteq X$,*

$$\omega_{X/S}(U) = \{\omega \in \Omega_{K(X)/K} : \mathrm{Res}_{x,y}(f\omega) \in \widehat{\mathcal{O}}_{K,\pi(x)} \text{ for all } x \in y \subset U \text{ and } f \in \mathcal{O}_{X,y}\}$$

*where $x$ runs over all closed points of $X$ inside $U$ and $y$ runs over all curves containing $x$.*

*Proof.* This reduces to the affine situation of $U = \mathrm{Spec}\, A$, with

$$A = \mathcal{O}_K[T_1, \dots, T_r]/I$$

where $I$ is an ideal generated by a regular sequence $f_1, \dots, f_{r-1}$ (we may also need to localise $\mathcal{O}_K$ away from finitely many primes, but we will continue to write $\mathcal{O}_K$).

By subsection 7.5.1, we can choose $l$ so that, setting $B = \mathcal{O}_K[t_l]$, the extension $A/B$ is a finite complete intersection with a separable fraction field extension. Further, $\omega_{A/\mathcal{O}_K}$ was identified with $\det \Delta_l \, dt_l \subseteq \Omega_{K(X)/K}$, where $\Delta_l$ is the matrix obtained by discarding the $l^{\text{th}}$ column (i.e. the $\frac{\partial}{\partial T_l}$ terms) from the Jacobian matrix $(\frac{\partial f_i}{\partial T_j})_{i,j}$. Therefore $\Delta_l$ is exactly the Jacobian of the complete intersection $A/B$, and so $\det \Delta_l = J$ in the notation of section 7.6; moreover, by theorem 7.6.5, we have $J^{-1}A = \mathfrak{C}(A/B)$. Combining this with theorem 7.7.4 completes the proof. $\square$

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