Undergraduate Notes in Mathematics

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College Algebra for STEM

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2015 Edition

To my children

Amin & Nadia

Preface

From many years of teaching students pursuing education in the sciences, I have noticed that there are specific topics in introductory algebra that students keep encountering while taking upper level classes in mathematics. As a result, I decided to write this manuscript to include all these commonly encountered topics so that to enhance students' knowledge for these topics. This book is designed specifically as a College Algebra course for prospective STEM students.
A solution guide to the book is available to instructors by request. Email: mfinan@atu.edu

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Contents

Chapter 1

Review of Beginning/Intermediate Algebra

This chapter is devoted for a review of some basic high-school level algebra that deemed necessary for the remaining of this book.

1.1 A Brief Introduction to Sets

A **set** is simply a well-defined collection of **elements**. The elements of a set can be anything such as numbers, letters, points in the plane, and even sets themselves. We usually denote sets by upper-case letters and elements by lower-case letters. When an element x belongs to a set A, we write $x \in A$. Otherwise, we shall write $x \notin A$. A set with no elements is called the **empty set** and is denoted by $\emptyset$.
A set can be described by listing all its elements. In this case, we say that the set is given in **tabular form**. Conventionally, the elements of a set in tabular form are not repeated. For example, if A is the set of the English alphabet vowels then

$$A = \{a, e, i, o, u\}.$$

Examples of **sets of numbers** that will be used in this book are:

$$\mathbb{N} = \{1, 2, 3, \cdots\}(\textbf{Natural numbers} \text{ or counting numbers})$$
$$\mathbb{W} = \{0, 1, 2, \cdots\}(\textbf{Whole numbers})$$
$$\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}(\textbf{Integers}).$$

Example 1.1.1
Write the following sets in tabular form:
(a) A is the set of odd positive integers less than 15.
(b) B is the set of even integers between -10 and 10 inclusive.
(c) C is the set of integers satisfying the equation $x^2 - 3 = 0$.

Solution.
(a) $A = \{1, 3, 5, 7, 9, 11, 13\}$.
(b) $B = \{-10, -8, -6, -4, -2, 0, 2, 4, 6, 8, 10\}$.
(c) $C = \emptyset$ ■

Another way to represent a set is by describing a property characterizing the elements of the set. We refer to such representation as the **set-builder form**. For example, the set of rational numbers is the set

$$\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z} \text{ with } b \neq 0\}.$$

The colon symbol ":" means "such that".

Example 1.1.2
Write the following sets in set-builder form:
(a) A is the set of integer solution to the equation $x^2 - 4 = 0$.
(b) $B = \{a, e, i, o, u\}$.
(c) C is the set of negative integers.

Solution.
(a) $A = \{x \in \mathbb{Z} : x^2 - 4 = 0\}$.
(b) $B = \{x : x \text{ is a vowel}\}$.
(c) $C = \{x \in \mathbb{Z} : x < 0\}$ ■

Example 1.1.3
Any **repeating decimal** is a rational number. Show that the number $x = 5.\overline{13} = 5.1313\cdots$ is a rational number.

Solution.
The repeating part is 13, a two-digit number so we multiply x by 100 to obtain $100x = 513.\overline{13}$. Thus, $100x - x = 513 - 5 = 508$. That is, $99x = 508$. Solving for x, we find $x = \frac{508}{99} \in \mathbb{Q}$ ■

A number that can not be represented as the ratio of two integers or can not be represented as repeating or terminating decimal is called an **irrational number**. The collection of all irrational numbers will be denoted by $\mathbb{I}$. Examples of irrational numbers are $\sqrt{2}, \sqrt{3}, \pi$,etc.

Subsets
We say that a set A is a **subset** of a set B if every element of A is also an element of B. We write, $A \subseteq B$. A **Venn diagram** representing $A \subseteq B$ is shown in Figure 1.1.1.

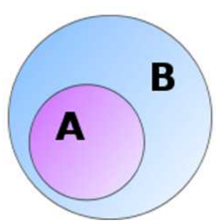


Figure 1.1.1

Note that since every element of a set A is in A, we can write $A \subseteq A$. In the notation $A \subseteq B$, the set B is called a **superset**. In case every element of a set A belongs to a set B but there is an element in B not in A we use the notation $A \subset B$ and we say that A is a **proper subset** of B.

Example 1.1.4
Using the symbol "$\subset$", complete the following:
(a) $\mathbb{N} \cdots \mathbb{Z}$ (b) $\mathbb{Z} \cdots \mathbb{Q}$ (c) $\mathbb{N} \cdots \mathbb{W}$ (d) $\mathbb{W} \cdots \mathbb{Z}$.

Solution.
(a) $\mathbb{N} \subset \mathbb{Z}$ (b) $\mathbb{Z} \subset \mathbb{Q}$ (c) $\mathbb{N} \subset \mathbb{W}$ (d) $\mathbb{W} \subset \mathbb{Z}$ ■

Union of Sets
The **union** of two sets A and B, denoted by $A \cup B$, is the set consisting of elements in A or in B or in both A and B. In set-builder notation, we have

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

A pictorial representation of $A \cup B$ is shown in Figure 1.1.2.

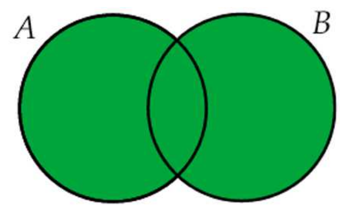


Figure 1.1.2

For example, the set of **real numbers** $\mathbb{R}$ is the union of the set of rational numbers $\mathbb{Q}$ and the set of irrationl numbers $\mathbb{I}$. That is, $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$.

Example 1.1.5
Find the union of the following sets:
(a) $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$.
(b) $A = \mathbb{N}$, $B = \{0\}$, and $C = \{\cdots, -2, -1, 0\}$.
(c) $A = \{a, b, c\}$ and $B = \emptyset$.

Solution.
(a) $A \cup B = \{1, 2, 3, 4\}$.
(b) $A \cup B \cup C = \mathbb{Z}$.
(c) $A \cup B = A$ ■

Intersection of Sets
The **intersection** of two sets A and B, denoted by $A \cap B$, is the set consisting of elements common to both A and B. A Venn diagram representing $A \cap B$ is shown in Figure 1.1.3.

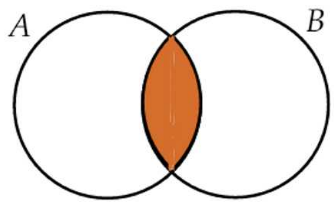


Figure 1.1.3

When $A \cap B = \emptyset$ we say that A and B are **disjoint.**

Example 1.1.6
Find the intersection of the following sets:
(a) $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$.
(b) $A = \mathbb{N}$ and $B = \mathbb{W}$.
(c) $A = \{a, b, c\}$ and $B = \{d\}$.

Solution.
(a) $A \cap B = \{2, 3\}$.
(b) $A \cap B = \mathbb{N}$.
(c) $A \cap B = \emptyset$ ■

Exercises

Exercise 1.1.1
A prime number is a natural number greater than 1 with only two divisors: 1 and the number itself. Let A be the set consisting of the first ten prime numbers. Write A in tabular form.

Exercise 1.1.2
Let $\mathbb{P}$ be the collection of all prime numbers. Let A be the set of prime numbers between 1 and 12. Write A in tabular form.

Exercise 1.1.3
Let $A = \{2, 3, 5, 7, 11\}$. Write A in set-builder notation.

Exercise 1.1.4
A **composite number** is a positive integer greater than 1 that is not prime. Let A be the set of the first six composite numbers. Write A in tabular form.

Exercise 1.1.5
Consider the set

$$A = \{n \in \mathbb{N} : 5n < 30\}.$$

Write A in tabular form.

Exercise 1.1.6
Write the set $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ in builder-set notation.

Exercise 1.1.7
Determine whether the following two sets have the same elements:

$$\begin{aligned} A =& \{x \in \mathbb{Z} : 0 < x < 5\} \\ B =& \{x \in \mathbb{W} : x^2 < 25\}. \end{aligned}$$

Exercise 1.1.8
Write the set $A = \{n \in \mathbb{N} : 10 < n^2 < 100\}$ in tabular form.

Exercise 1.1.9
Write the set $A = \{x \in \mathbb{P} : x^2 - 16 = 0\}$ in tabular form.

Exercise 1.1.10
Write the set $A = \{0, 2, 4, 6, \cdots\}$ in set-builder notation.

Exercise 1.1.11
Convert the number $x = 0.01\overline{83}$ into a fraction.

Exercise 1.1.12
Convert the number $x = 0.\overline{142857}$ into a fraction.

Exercise 1.1.13
Convert the number $x = 4.\overline{123}$ into a fraction.

Exercise 1.1.14
Convert the number $x = 2.87\overline{31}$ into a fraction.

Exercise 1.1.15
Convert the number $x = 2.13\overline{8}$ into a fraction.

Exercise 1.1.16
Suppose that $A = \{2, 4, 6\}$, $B = \{2, 6\}$, and $C = \{4, 6\}$. Determine which of these sets are subsets of which other(s) of these sets.

Exercise 1.1.17
Order the sets of numbers: $\mathbb{W}, \mathbb{Z}, \mathbb{R}, \mathbb{Q}, \mathbb{N}$ using $\subset$.

Exercise 1.1.18
Let $A = \{c, d, f, g\}, B = \{f, j\}$, and $C = \{d, g\}$. Answer each of the following questions. Give reasons for your answers.
(a) Is $B \subseteq A$?
(b) Is $C \subseteq A$?
(c) Is $C \subseteq C$?

Exercise 1.1.19
Find all possible non-empty subsets of $A = \{a, b, c\}$.

Exercise 1.1.20
Subway prepared 60 4-inch sandwiches for a birthday party. Among these sandwiches, 45 of them had tomatoes, 30 had both tomatoes and onions, and 5 had neither tomatoes nor onions. Using a Venn diagram, how many sandwiches did it make with
(a) tomatoes or onions?
(b) onions?
(c) onions but not tomatoes?

Exercise 1.1.21
A camp of international students has 110 students. Among these students,

75	speak English,
52	speak Spanish,
50	speak French,
33	speak English and Spanish,
30	speak English and French,
22	speak Spanish and French,
13	speak all three languages.

How many students speak
(a) English and Spanish, but not French,
(b) neither English, Spanish, nor French,
(c) French, but neither English nor Spanish,
(d) English, but not Spanish,
(e) only one of the three languages,
(f) exactly two of the three languages.

Exercise 1.1.22
The **relative complement** of A with respect to B is the set of members of A that are not in B. It is denoted by $A - B$. Suppose $A = \{x \in \mathbb{R} : x \leq 0\}$ and $B = \{x \in \mathbb{R} : x < -1 \text{ or } x > 1\}$. Find $A - B$.

Exercise 1.1.23
Let A be the set of the first five composite numbers and B be the set of positive integers less than or equal to 8. Find $A - B$ and $B - A$.

Exercise 1.1.24
Let A be the set of natural numbers less than 0 and $B = \{1, 3, 7\}$. Find $A \cup B$ and $A \cap B$.

Exercise 1.1.25
Let

$$A = \{x \in \mathbb{N} : 4 \leq x \leq 8\}$$
$$B = \{x \in \mathbb{N} : x \text{ even and } x \leq 10\}.$$

Find $A \cup B$ and $A \cap B$.

Exercise 1.1.26
Let $A = \{b, c, d, f, g\}$ and $B = \{a, b, c\}$. Find each of the following:
(a) $A \cup B$.
(b) $A \cap B$.
(c) $A - B$.
(d) $B - A$.

Exercise 1.1.27
Let $A = \{1, 2, 3, 5, 9, 10\}$ and $B = \{3, 4, 5, 10, 11\}$. Find $(A - B) \cap (B - A)$.

Exercise 1.1.28
Let $A = \{1, 3, 8, 9\}$, $B = \{2, 3, 4, 8\}$, and $C = \{1, 3, 4\}$. Find $(A \cup B) - C$.

Exercise 1.1.29
Let $A = \{1, 3, 8, 9\}$, $B = \{2, 3, 4, 8\}$, and $C = \{1, 3, 4\}$. Find $A \cup (B - C)$.

Exercise 1.1.30
Relate $\mathbb{R}, \mathbb{Q}$, and $\mathbb{I}$ using relative complement.

1.2 Arithmetic in $\mathbb{Q}$

The set of **rational numbers**, denoted by $\mathbb{Q}$ is the set

$$\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}.$$

That is, the set of rational numbers consists of all ratios (and thus the word rational) with their opposites. In the notation $\frac{a}{b}$ we call a the **numerator** and b the **denominator**.
Note that every integer is a rational number for if a is an integer then we can write $a = \frac{a}{1}$. Thus, $\mathbb{Z} \subset \mathbb{Q}$. Also, recall from Section 1.1 that any repeating decimal is a rational number.
A number that is not a repeating decimal or cannot be written as a ration of two integers is called an **irrational number**. The set of all irrational numbers is denoted by $\mathbb{I}$. Examples of rational numbers are $\sqrt{2}, \pi$, etc.

Example 1.2.1
Draw a Venn diagram to show the relationship between counting numbers, whole numbers, integers, and rational numbers.

Solution.
The relationship is shown in Figure 1.2.1 ■

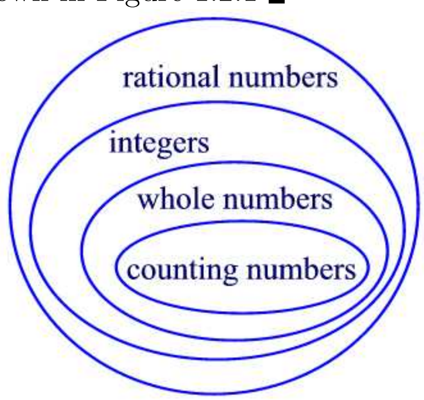


Figure 1.2.1

Equality of Rational Numbers
Let $\frac{a}{b}$ and $\frac{c}{d}$ be any two rational numbers. Then $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$ (Cross-multiplication). Equalities such as $\frac{a}{b} = \frac{c}{d}$ are called **proportions**.

Example 1.2.2
Determine if the following pairs are equal.
(a) $\frac{3}{-12}$ and $\frac{-36}{144}$.
(b) $\frac{-21}{86}$ and $\frac{-51}{215}$.

Solution.
(a) Since $3(144) = (-12)(-36)$, we have $\frac{3}{-12} = \frac{-36}{144}$.
(b) Since $(-21)(215) \neq (86)(-51)$, we have $\frac{-21}{86} \neq \frac{-51}{215}$ ■

The Fundamental Principle of Fractions
Let $\frac{a}{b}$ be any rational number and n be a non-zero integer then

$$\frac{a}{b} = \frac{an}{bn} = \frac{a \div n}{b \div n}.$$

As an important application of the Fundamental Law of Fractions we have

$$\frac{a}{-b} = \frac{(-1)a}{(-1)(-b)} = \frac{-a}{b}.$$

We also use the notation $-\frac{a}{b}$ for either $\frac{a}{-b}$ or $\frac{-a}{b}$.

Example 1.2.3
Write three rational numbers equal to $-\frac{2}{5}$.

Solution.
By the Fundamental Law of Fractions we have

$$-\frac{2}{5} = \frac{4}{-10} = -\frac{6}{15} = \frac{-8}{20} \blacksquare$$

Prime Factorization and the Least Common Multiple
Addition and subtraction of fractions involve the least common multiple of two positive integers. Let a and b be two positive integers. A result known as the **Fundamental Theorem of Arithmetic** allows us to write the **prime factorizations** of a and b, say

$$\begin{aligned} a =& p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k} \\ b =& p_1^{t_1} p_2^{t_2} \cdots p_k^{t_k}. \end{aligned}$$

We define the **least common multiple** of a and b to be the number

$$\mathrm{LCM}(a,b) = p_1^{\max\{s_1,t_1\}} p_2^{\max\{s_2,t_2\}} \cdots p_k^{\max\{s_k,t_k\}}$$

where the $p_1, p_2, \cdots, p_k$ are prime numbers and the s_i's and t_i's are whole numbers (i.e., elements of the ser $\mathbb{W}$).

Example 1.2.4
Find LCM(294, 84).

Solution.
Using the prime factorizations of 294 and 84 we find

$$\begin{array}{r|c} & 294 \\ 2 & 147 \\ 3 & 49 \\ 7 & 7 \\ 7 & 1 \end{array} \qquad \begin{array}{r|c} & 84 \\ 2 & 42 \\ 2 & 21 \\ 3 & 7 \\ 7 & 1 \end{array}$$

That is,

$$\begin{aligned} 294 =& 2 \cdot 3 \cdot 7^2 \\ 84 =& 2^2 \cdot 3 \cdot 7. \end{aligned}$$

Hence, $\mathrm{LCM}(294, 84) = 2^2 \cdot 3 \cdot 7^2 = 588$ ■

Rational Numbers in Simplest Form

A rational number $\frac{a}{b}$ is in **simplest form** or in **lowest terms** if a and b have no common factor greater than 1.

Example 1.2.5
Find the simplest form of the rational number $\frac{294}{-84}$.

Solution.
Using the prime factorizations of 294 and 84 we find

$$\frac{294}{-84} = \frac{2 \cdot 3 \cdot 7^2}{-2^2 \cdot 3 \cdot 7} = \frac{7}{-2} = \frac{-7}{2} = -\frac{7}{2} \ ■$$

Addition of Rational Numbers

The definition of **adding fractions** extends to rational numbers.

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b} \qquad \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}.$$

Example 1.2.6
Find each of the following sums.
(a) $\frac{2}{-3} + \frac{1}{5}$ (b) $\frac{-2}{5} + \frac{4}{-7}$ (c) $\frac{3}{7} + \frac{-5}{7}$.

Solution.
(a) $\frac{2}{-3} + \frac{1}{5} = \frac{2\cdot(-5)}{(-3)\cdot(-5)} + \frac{1\cdot 3}{5\cdot 3} = \frac{-10}{15} + \frac{3}{15} = \frac{(-10)+3}{15} = -\frac{7}{15}$.
(b) $\frac{-2}{5} + \frac{4}{-7} = \frac{(-2)\cdot 7}{5\cdot 7} + \frac{4\cdot(-5)}{(-5)\cdot(-7)} = \frac{-14}{35} + \frac{-20}{35} = \frac{(-14)+(-20)}{35} = -\frac{34}{35}$.
(c) $\frac{3}{7} + \frac{-5}{7} = \frac{3+(-5)}{7} = -\frac{2}{7}$ ■
Rational numbers have the following properties for addition.

Theorem 1.2.1
Let $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ be any rational numbers. Then we have the following:
Closure: $\frac{a}{b} + \frac{c}{d}$ is a unique rational number.
Commutative: $\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$.
Associative: $\left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f} = \frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right) = \frac{a}{b} + \frac{c}{d} + \frac{e}{f}$.
Identity Element: $\frac{a}{b} + 0 = \frac{a}{b}$.
Additive inverse: $\frac{a}{b} + \left(-\frac{a}{b}\right) = 0$.

Example 1.2.7
Find the additive inverse for each of the following:
(a) $\frac{3}{5}$ (b) $\frac{-5}{11}$ (c) $\frac{2}{-3}$ (d) $-\frac{2}{5}$.

Solution.
(a) $-\frac{3}{5} = \frac{-3}{5} = \frac{3}{-5}$ (b) $\frac{5}{11}$ (c) $\frac{2}{3}$ (d) $\frac{2}{5}$ ■

<u>Subtraction of Rational Numbers</u>

Subtraction of rational numbers can be defined in terms of addition as follows.

$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \left(-\frac{c}{d}\right).$$

Using the above result we obtain the following:

$$\begin{aligned}\frac{a}{b} - \frac{c}{d} &= \frac{a}{b} + \left(-\frac{c}{d}\right) = \frac{a}{b} + \frac{-c}{d} \\ &= \frac{ad + b(-c)}{bd} = \frac{ad - bc}{bd}.\end{aligned}$$

Example 1.2.8
Compute $\frac{103}{24} - \frac{-35}{16}$.

Solution.
Since $\text{LCM}(24, 16) = 48$, we have

$$\frac{103}{24} - \frac{-35}{16} = \frac{206}{48} - \frac{-105}{48} = \frac{206 - (-105)}{48}$$
$$= \frac{206 + 105}{48} = \frac{311}{48} \blacksquare$$

Multiplication of Rational Numbers
If $\frac{a}{b}$ and $\frac{c}{d}$ are any two rational numbers then their **product** is defined by

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

Properties of multiplication of rational numbers are summarized in the following theorem.

Theorem 1.2.2
Let $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ be any rational numbers. Then we have the following:
Closure: The product of two rational numbers is a unique rational number.
Commutativity: $\frac{a}{b} \cdot \frac{c}{d} = \frac{c}{d} \cdot \frac{a}{b}$.
Associativity: $\frac{a}{b} \cdot \left(\frac{c}{d} \cdot \frac{e}{f}\right) = \left(\frac{a}{b} \cdot \frac{c}{d}\right) \cdot \frac{e}{f} = \frac{a}{b}\frac{c}{d} \cdot \frac{e}{f}$.
Identity: $\frac{a}{b} \cdot 1 = \frac{a}{b} = 1 \cdot \frac{a}{b}$.
Inverse: $\frac{a}{b} \cdot \frac{b}{a} = 1$. We call $\frac{b}{a}$ the **reciprocal** of $\frac{a}{b}$ or the **multiplicative inverse** of $\frac{a}{b}$.
Distributivity: $\frac{a}{b} \cdot \left(\frac{c}{d} + \frac{e}{f}\right) = \frac{a}{b} \cdot \frac{c}{d} + \frac{a}{b} \cdot \frac{e}{f}$.

Example 1.2.9
Perform each of the following multiplications. Express your answer in simplest form: (a) $\frac{-5}{6} \cdot \frac{7}{3}$ (b) $\frac{-3}{10} \cdot \frac{-25}{27}$.

Solution.
(a) We have

$$\frac{-5}{6} \cdot \frac{7}{3} = \frac{(-5) \cdot 7}{6 \cdot 3} = -\frac{35}{18}.$$

(b)

$$\frac{-3}{10} \cdot \frac{-25}{27} = \frac{-1}{2} \cdot \frac{-5}{9} = \frac{(-1)(-5)}{2(9)} = \frac{5}{18} \blacksquare$$

Example 1.2.10
Use the properties of multiplication of rational numbers to compute the following.
(a) $-\frac{3}{5} \cdot \left(\frac{11}{17} \cdot \frac{5}{3}\right)$.
(b) $\frac{2}{3} \cdot \left(\frac{3}{2} + \frac{5}{7}\right)$.
(c) $\frac{5}{9} \cdot \frac{2}{7} + \frac{2}{7} \cdot \frac{4}{9}$.

Solution.
(a) $-\frac{3}{5} \cdot \left(\frac{11}{17} \cdot \frac{5}{3}\right) = \frac{-3}{5} \cdot \frac{11}{17} \cdot \frac{5}{3} = -\frac{11}{17}$.

(b) $\frac{2}{3} \cdot \left(\frac{3}{2} + \frac{5}{7}\right) = \frac{2}{3} \cdot \frac{3}{2} + \frac{2}{3} \cdot \frac{5}{7} = 1 + \frac{10}{21} = \frac{21+10}{21} = \frac{31}{21}$.

(c) $\frac{5}{9} \cdot \frac{2}{7} + \frac{2}{7} \cdot \frac{4}{9} = \frac{5}{9} \cdot \frac{2}{7} + \frac{4}{9} \cdot \frac{2}{7} = \left(\frac{5}{9} + \frac{4}{9}\right) \cdot \frac{2}{7} = \frac{2}{7}$ ■

Division of Rational Numbers
We define the **division** of rational numbers as an extension of the division of fractions. Let $\frac{a}{b}$ and $\frac{c}{d}$ be any rational numbers with $\frac{c}{d} \neq 0$. Then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}.$$

Using words, to find $\frac{a}{b} \div \frac{c}{d}$ multiply $\frac{a}{b}$ by the reciprocal of $\frac{c}{d}$.
By the above definition one gets the following two results.

$$\frac{a}{b} \div \frac{c}{b} = \frac{a}{c}$$

and

$$\frac{a}{b} \div \frac{c}{d} = \frac{a \div c}{b \div d}.$$

Remark 1.2.1
After inverting, it is often simplest to "cancel" before doing the multiplication. Cancelling is dividing one factor of the numerator and one factor of the denominator by the same number. For example: $\frac{2}{9} \div \frac{3}{12} = \frac{2}{9} \times \frac{12}{3} = \frac{2 \times 12}{9 \times 3} = \frac{2 \times 4}{3 \times 3} = \frac{8}{9}$.

Remark 1.2.2
Exponents and their properties are extended to rational numbers in a natural way. For example, if a is any rational number and n is a positive integer then

$$a^n = \underbrace{a \cdot a \cdots a}_{n \ factors} \quad \text{and} \quad a^{-n} = \frac{1}{a^n}.$$

Example 1.2.11
Compute the following and express the answers in simplest form.

(a) $\frac{-7}{4} \div \frac{2}{3}$ (b) $\frac{13}{17} \div \frac{-4}{9}$ (c) $\frac{-18}{23} \div \frac{-6}{23}$.

Solution.
(a) $\frac{-7}{4} \div \frac{2}{3} = \frac{-7}{4} \cdot \frac{3}{2} = \frac{(-7)(3)}{(4)(2)} = -\frac{21}{8}$.

(b) $\frac{13}{17} \div \frac{-4}{9} = \frac{13}{17} \cdot \frac{9}{-4} = \frac{13 \cdot 9}{17 \cdot (-4)} = -\frac{117}{68}$.

(c) $\frac{-18}{23} \div \frac{-6}{23} = \frac{-18}{23} \cdot \frac{23}{-6} = \frac{3}{1} \cdot \frac{1}{1} = 3$ ■

Comparison of Rational Numbers
Let $\frac{a}{b}$ and $\frac{c}{d}$ be two rational numbers. If $\frac{a}{b} < 0$ and $\frac{c}{d} > 0$ then $\frac{a}{b} < \frac{c}{d}$. Suppose that both numbers are of the same sign. We convert to fractions with the same denominator: $\frac{a}{b} = \frac{ad}{bd}$ and $\frac{c}{d} = \frac{bc}{bd}$. If $\frac{a}{b}$ and $\frac{c}{d}$ are positive with $ad < bc$ then $\frac{a}{b} < \frac{c}{d}$. If $\frac{a}{b}$ and $\frac{c}{d}$ are negative with $|ad| > |bc|$ then $\frac{a}{b} < \frac{c}{d}$.

Example 1.2.12
Compare: (a) $-\frac{3}{5}$ and $\frac{1}{2}$ (b) $\frac{4}{7}$ and $\frac{3}{4}$ (c) $-\frac{7}{8}$ and $-\frac{2}{5}$.

Solution.
(a) $-\frac{3}{5} < \frac{1}{2}$.
(b) $\frac{4}{7} = \frac{16}{28}$ and $\frac{3}{4} = \frac{21}{28}$. Hence, $\frac{4}{7} < \frac{3}{4}$.
(c) $-\frac{7}{8} = -\frac{35}{40}$ and $-\frac{2}{5} = -\frac{16}{40}$. Hence, $-\frac{7}{8} < -\frac{2}{5}$ ■

Example 1.2.13
Order the following numbers from largest to smallest:$-\frac{4}{5}$ 1.5 $-\frac{3}{8}$ 2 $\frac{7}{4}$.

Solution.
Note that $2 > 1.5$ and $2 = \frac{8}{4}$ so that $2 > \frac{7}{4}$. Also, $-\frac{4}{5} = -\frac{32}{40}$ and $-\frac{3}{8} = -\frac{15}{40}$ so that $-\frac{4}{5} < -\frac{3}{8}$. Hence,

$$2 > \frac{7}{4} > 1.5 > -\frac{3}{8} > -\frac{4}{5} \blacksquare$$

Remark 1.2.3
Mixed fractions such as $3\frac{1}{2} = 3 + \frac{1}{2} = \frac{7}{2}$ and percentages such as $3\% = \frac{3}{100}$ are also examples of rational numbers. The fractional part of a mixed number is a proper fraction, i.e., the numerator is less than the denominator.

Example 1.2.14
Compute each of the following:
(a) $5\frac{1}{3} + 2\frac{3}{4}$.

(b) $3\frac{2}{5} - 7\frac{3}{4}$.

(c) $6\frac{2}{7} \times 9\frac{3}{4}$.

(d) $4\frac{5}{6} \div 11\frac{1}{5}$.

Solution.
(a) $5\frac{1}{3} + 2\frac{3}{4} = 5 + 2 + \left(\frac{1}{3} + \frac{3}{4}\right) = 7 + \left(\frac{4}{12} + \frac{9}{12}\right) = 7 + \frac{13}{12} = 8\frac{1}{12}$.

(b) $3\frac{2}{5} - 7\frac{3}{4} = (3 - 7) + \left(\frac{2}{5} - \frac{3}{4}\right) = -4 + \left(\frac{8}{20} - \frac{15}{20}\right) = -4 - \frac{7}{20} = -4\frac{7}{20}$.

(c) $6\frac{2}{7} \times 9\frac{3}{4} = \frac{44}{7} \times \frac{39}{4} = \frac{11}{7} \times 39 = \frac{429}{7} = 61\frac{2}{7}$.

(d) $4\frac{5}{6} \div 11\frac{1}{5} = \frac{29}{6} \div \frac{56}{5} = \frac{29}{6} \times \frac{5}{65} = \frac{145}{390} = \frac{29}{78}$ ■

Exercises

Exercise 1.2.1
Show that each of the following numbers is a rational number.
(a) -3 (b) $4\frac{1}{2}$ (c) -5.6 (d) 25%.

Exercise 1.2.2
Which of the following are equal to -3?

$$\frac{-3}{1}, \frac{3}{-1}, \frac{3}{1}, -\frac{3}{1}, \frac{-3}{-1}, -\frac{-3}{1}, -\frac{-3}{-1}.$$

Exercise 1.2.3
Determine which of the following pairs of rational numbers are equal.
(a) $\frac{-3}{5}$ and $\frac{63}{-105}$.
(b) $\frac{-18}{-24}$ and $\frac{45}{60}$.

Exercise 1.2.4
Rewrite each of the following rational numbers in simplest form.
(a) $\frac{5}{-7}$ (b) $\frac{21}{-35}$ (c) $\frac{-8}{-20}$ (d) $\frac{-144}{180}$.

Exercise 1.2.5
How many different rational numbers are given in the following list?

$$\frac{2}{5}, 3, \frac{-4}{-10}, \frac{39}{13}, \frac{7}{4}.$$

Exercise 1.2.6
Find the value of x to make the statement a true one.

(a) $\frac{-7}{25} = \frac{x}{500}$ (b) $\frac{18}{3} = \frac{-5}{x}$.

Exercise 1.2.7
Find the prime factorizations of the numerator and the denominator and use them to express the fraction $\frac{247}{-77}$ in simplest form.

Exercise 1.2.8
(a) If $\frac{a}{b} = \frac{a}{c}$, what must be true?

(b) If $\frac{a}{c} = \frac{b}{c}$, what must be true?

Exercise 1.2.9
Perform the following additions. Express your answer in simplest form.
(a) $\frac{6}{8} - \frac{-25}{100}$ (b) $\frac{-57}{100} + \frac{13}{10}$.

Exercise 1.2.10
Perform the following subtractions. Express your answer in simplest form.
(a) $\frac{137}{214} - \frac{-1}{3}$ (b) $\frac{-23}{100} - \frac{198}{1000}$.

Exercise 1.2.11
Multiply the following rational numbers. Write your answers in simplest form.

(a) $\frac{3}{5} \cdot \frac{-10}{21}$ (b) $\frac{-6}{11} \cdot \frac{-33}{18}$ (c) $\frac{5}{12} \cdot \frac{48}{-15} \cdot \frac{-9}{8}$.

Exercise 1.2.12
Find the following quotients. Write your answers in simplest form.

(a) $\frac{-8}{9} \div \frac{2}{9}$ (b) $\frac{12}{15} \div \frac{-4}{3}$ (c) $\frac{-13}{24} \div \frac{-39}{-48}$.

Exercise 1.2.13
State the property that justifies each statement.

(a) $\left(\frac{5}{7} \cdot \frac{7}{8}\right) \cdot \frac{-8}{3} = \frac{5}{7} \cdot \left(\frac{7}{8} \cdot \frac{-8}{3}\right)$.
(b) $\frac{1}{4}\left(\frac{8}{3} + \frac{-5}{4}\right) = \frac{1}{4} \cdot \frac{8}{3} + \frac{1}{4} \cdot \frac{-5}{4}$.

Exercise 1.2.14
Compute the following and write your answers in simplest form.

(a) $\frac{-40}{27} \div \frac{-10}{9}$ (b) $\frac{21}{25} \div \frac{-3}{5}$ (c) $\frac{-10}{9} \div \frac{-9}{8}$.

Exercise 1.2.15
Find the reciprocals of the following rational numbers.

(a) $\frac{4}{-9}$ (b) 0 (c) $\frac{-3}{2}$ (d) $\frac{-4}{-9}$.

Exercise 1.2.16
Compute: $\left(\frac{-4}{7} \cdot \frac{2}{-5}\right) \div \frac{2}{-7}$.

Exercise 1.2.17
If $\frac{a}{b} \cdot \frac{-4}{7} = \frac{2}{3}$ what is $\frac{a}{b}$?

Exercise 1.2.18
Compute each of the following:

(a) $-\left(\frac{3}{4}\right)^2$ (b) $\left(-\frac{3}{4}\right)^2$ (c) $\left(\frac{3}{4}\right)^2 \cdot \left(\frac{3}{4}\right)^7$.

Exercise 1.2.19
Find a value of x so that $\frac{2}{3} = \frac{x-1}{-5}$.

Exercise 1.2.20
Reduce $-\frac{2940}{3150}$ to lowest terms.

Exercise 1.2.21
Simplify $\frac{882}{1680}$.

Exercise 1.2.22
Find $\mathrm{LCM}(2940, 3150)$.

Exercise 1.2.23
Find $\mathrm{LCM}(882, 1680)$.

Exercise 1.2.24
Find $\frac{3}{49} - \frac{5}{14} + \frac{1}{12}$.

Exercise 1.2.25
Find $-\frac{5}{12} - \frac{2}{3} + \frac{3}{8}$.

Exercise 1.2.26
Find $-\frac{5}{12} + \frac{13}{20} - \frac{17}{45}$. Simplify your answer.

Exercise 1.2.27
Multiply: $-\frac{5}{6} \cdot \frac{8}{9} \cdot \left(-\frac{12}{15}\right)$. Simplify your answer.

Exercise 1.2.28
Multiply: $\left(-\frac{3}{5}\right)\left(-\frac{5}{6}\right)\left(-\frac{6}{7}\right)$. Simplify your answer.

Exercise 1.2.29
Divide: $\frac{16}{21} \div \left(-\frac{4}{7}\right)$. Simplify your answer.

Exercise 1.2.30
Divide: $-\frac{2}{9} \div \left(-\frac{11}{15}\right)$. Simplify your answer.

Exercise 1.2.31
Compute:
(a) $3\frac{3}{4} + 5\frac{2}{3} - 7\frac{5}{12}$.

(b) $16 \times 7\frac{2}{9}$.

(c) $5\frac{2}{3} \times 8\frac{5}{7}$.

(d) $3\frac{2}{3} \div 2\frac{3}{5}$.

Exercise 1.2.32
Order the following numbers from greatest to least:

$$3\frac{2}{5} \qquad -\frac{2}{3} \qquad 1.7 \qquad \frac{3}{8}.$$

Exercise 1.2.33
Order the following numbers from least to greatest:

$$-8\frac{3}{4} \qquad 0.7 \qquad 4 \qquad -9.3.$$

1.3 Simplifying and Factoring Algebraic Expressions

An **algebraic expression** is the arithmetic combination of letters, the **variables**, and numbers, known as the **coefficients**. Examples of algebraic expressions are $-4x^3y+10x^2+4y$ and $\frac{a^3b}{a^2+3b^2}$. Terms with the same power of the variables but with different coefficients are called **like terms**. Thus, $-3x^4y^5$ and $10x^4y^5$ are like terms.
To **simplify** an algebraic expression is to reduce it into a simpler form. This is usually done by combining like terms, factoring and cancelling. When an expression is given as a product, by **expanding** it we mean carrying the multiplication process.

Example 1.3.1
Expand and simplify: (a) $(a+b)^2$ (b) $(a-b)^2$ (c) $(x+a)(x+b)$ (d) $(a-b)(a+b)$.

Solution.
(a) Using the fact that $ab = ba$, we have

$$(a+b)^2 = (a+b)(a+b) = a^2 + ab + ba + b^2 = a^2 + 2ab + b^2.$$

(b) Likewise, we have

$$(a-b)^2 = (a-b)(a-b) = a^2 - ab - ba + b^2 = a^2 - 2ab + b^2.$$

(c) We have

$$(x+a)(x+b) = x^2 + xb + ax + ab = x^2 + (a+b)x + ab.$$

(d) We have

$$(a-b)(a+b) = a^2 + ab - ab - b^2 = a^2 - b^2.$$

We refer to this equality as the **difference of two squares** ■

Example 1.3.2
Simplify: $(2x+5)^2 - 3(4-3x)^2$.

Solution.
We have

$$\begin{aligned}(2x+5)^2-4(4-3x)^2 =&(4x^2+20x+25)-4(16-24x+9x^2)\\ =&4x^2+20x+25-64+96x-36x^2\\ =&-39+116x-32x^2 \blacksquare\end{aligned}$$

An algebraic expression can be expressed as the square of a reduced expression is called a **complete square**. Examples are $a^2+2ab+b^2$ and $a^2-2ab+b^2$ since the first is just $(a+b)^2$ and the second is $(a-b)^2$.

Example 1.3.3
Show that each of the following is a complete square:
(a) $9x^2-6x+1$.
(b) $25x^2+40x+16$.
(c) $x^2y^2-6xy^2+9y^2$.

Solution.
(a) $9x^2-6x+1=(3x)^2-2(1)(3x)+1^2=(3x-1)^2$
(b) $25x^2+40x+16=(5x)^2+2(4)(5x)+4^2=(5x+4)^2$.
(c) $x^2y^2-xy^2+\frac{1}{4}y^2=(xy)^2-2\left(\frac{1}{2}y\right)(xy)+\left(\frac{1}{2}y\right)^2=\left(xy-\frac{1}{2}y\right)^2$ $\blacksquare$

The process of adding a term to an algebraic expression to convert it into a complete square is referred to as **completing the square**. Terms such as a^2+2ab and a^2-2ab are referred to as the **beginning of a complete square**.

Example 1.3.4
What is the missing term so that the given expression becomes a complete square?
(a) $36x^2-60x$.
(b) $9x^2+30x$.
(c) $5x^2+2\sqrt{5}x$.

Solution.
(a) $36x^2-60x=(6x)^2-2(5)(6x)$ so that $a=6x$ and $b=5$. The missing term is $b^2=25$.
(b) $9x^2+30x=(3x)^2+2(5)(3x)$ so that $a=3x$ and $b=5$. The missing term

is $b^2 = 25$.
(a) $5x^2 + 2\sqrt{5}x = (\sqrt{5}x)^2 + 2(1)(\sqrt{5}x)$ so that $a = \sqrt{5}x$ and $b = 1$. The missing term is $b^2 = 1$ ■

Factoring Algebraic Expressions

The process of writing an algebraic expression as a product is called **factoring**. This is the opposite of expanding. Factoring is useful when solving algebraic equations. We consider various forms of factoring.

Case 1: Factoring out the GCF

In this case, we look for the **greatest common factor**(GCF). We illustrate this in the next example.

Example 1.3.5

Factor each of the following expressions:
(a) $(2x+1)^2(3x-2) - (x-4)(2x+1) - (2x+1)^2$.
(b) $(5x+3)^4(2x+3)^3 + 3(5x+3)^3(2x+3)^4$.

Solution.

(a) We have

$$\begin{aligned}(2x+1)^2(3x-2) - (x-4)(2x+1) - (2x+1)^2 =&(2x+1)[(2x+1)(3x-2)\\ &-(x-4)-(2x+1)]\\ =&(2x+1)(6x^2-x-2-x+4\\ &-2x-1)\\ =&(2x+1)(6x^2-4x+1).\end{aligned}$$

(b) We have

$$\begin{aligned}(5x+3)^4(2x+3)^3 + 3(5x+3)^3(2x+3)^4 =&(5x+3)^3(2x+3)^3[(5x+3)+3(2x+3)]\\ =&(5x+3)^3(2x+3)^3(5x+3+6x+9)\\ =&(5x+3)^3(2x+3)^3(11x+12)\end{aligned}$$ ■

Case 2: Factoring a difference of two squares

This case of factoring is based upon the difference of squares identity $a^2-b^2 = (a-b)(a+b)$.

Example 1.3.6
Factor each of the following expressions:
(a) $(2x+3)^2 - (4x-1)^2$.
(b) $16x^2y^2 - 9z^2$.

Solution.
(a) We have

$$\begin{aligned}(2x+3)^2 - (4x-1)^2 =& [(2x+3)-(4x-1)][(2x+3)+(4x-1)]\\ =& (-2x+4)(6x+2) = 4(-x+2)(3x+1).\end{aligned}$$

(b) We have

$$16x^2y^2 - 9z^2 = (4xy)^2 - (3z)^2 = (4xy-3z)(4xy+3z) \blacksquare$$

Case 3: Factoring perfect square trinomials
Recall, the following perfect square identities

$$(a \pm b)^2 = a^2 \pm 2ab + b^2.$$

Example 1.3.7
Factor each of the following expressions:
(a) $9x^2 - 24x + 16$.
(b) $36x^4y^2 + 6x^2yz + \frac{z^2}{4}$.

Solution.
(a) We have

$$9x^2 - 24x + 16 = (3x)^2 - 2(3x)(4) + 4^2 = (3x-4)^2.$$

(b) We have

$$36x^4y^2 + 6x^2yz + \frac{z^2}{4} = (6x^2y)^2 + 2(6x^2y)\left(\frac{1}{2}z\right) + \left(\frac{1}{2}z\right)^2 = \left(6x^2y + \frac{z}{2}\right)^2 \blacksquare$$

Case 4: Factoring $x^2 + cx + d$
Recall that

$$(x+a)(x+b) = x^2 + (a+b)x + ab.$$

Thus, to factor a trinomial with a leading coefficient 1 such as $x^2 + cx + d$ we look for two numbers a and b that satisfy $a + b = c$ and $ab = d$.

Example 1.3.8
Factor each of the following expressions:
(a) $x^2 + 5x + 6$.
(b) $x^2 - 9x + 18$.

Solution.
(a) Two numbers that satisfy $a + b = 5$ and $ab = 6$ are 2 and 3. Thus,

$$x^2 + 5x + 6 = (x + 2)(x + 3).$$

(b) Two numbers that satisfy $a + b = -9$ and $ab = 18$ are -6 and -3. Thus,

$$x^2 - 9x + 18 = (x - 6)(x - 3) \blacksquare$$

Case 5: Factoring by grouping
We shall apply this method to algebraic expressions with an even number of terms. The idea is to group two terms together who have a greatest common factor. We next factor the GCF of each group. After this step, one notices that all groups have a common factor. We factor out front this common factor to end up with a product. We illustrate this process in the next examples.

Example 1.3.9
Factor by grouping terms:
(a) $xy + 3y - 5x - 15$.
(b) $2x^3 + 6x^2 + 4x + 12$.

Solution.
(a) We have

$$xy+3y-5x-15 = (xy+3y)+(-5x-15) = y(x+3)-5(x+3) = (x+3)(y-5).$$

(b) We have

$$\begin{aligned}2x^3 + 6x^2 + 4x + 12 =&(2x^3 + 6x^2) + (4x + 12)\\ =&2x^2(x + 3) + 4(x + 3) = (x + 3)(2x^2 + 4)\\ =&2(x + 3)(x^2 + 2) \blacksquare\end{aligned}$$

Example 1.3.10
Factor by grouping terms:
(a) $6xy + 4y - 21x - 14$.
(b) $x^3 + 3x^2 - 3x - 9$.

Solution.
(a) We have

$$\begin{aligned}6xy + 4y - 21x - 14 =& (6xy + 4y) - (21x + 14)\\ =& 2y(3x + 2) - 7(3x + 2)\\ =& (3x + 2)(2y - 7).\end{aligned}$$

(b) We have

$$\begin{aligned}x^3 + 3x^2 - 3x - 9 =& (x^3 + 3x^2) - (3x + 9)\\ =& x^2(x + 3) - 3(x + 3)\\ =& (x + 3)(x^2 - 3)\\ =& (x + 3)(x - \sqrt{3})(x + \sqrt{3}) \blacksquare\end{aligned}$$

Example 1.3.11
Factor by grouping: $a^3 + 3a^2b + 3ab^2 + b^3$.

Solution.
We have

$$\begin{aligned}a^3 + 3a^2b + 3ab^2 + b^3 =& (a^3 + a^2b) + (2a^2b + 2ab^2) + (ab^2 + b^3)\\ =& a^2(a + b) + 2ab(a + b) + b^2(a + b)\\ =& (a + b)(a^2 + 2ab + b^2)\\ =& (a + b)(a + b)^2 = (a + b)^3 \blacksquare\end{aligned}$$

Exercises

Exercise 1.3.1
Expand and simplify the expression: $(a+b)^3$.

Exercise 1.3.2
Expand and simplify the expression: $(a-b)^3$.

Exercise 1.3.3
Expand and simplify the expression: $(2x+5)^2$.

Exercise 1.3.4
Expand and simplify the expression: $(4-3x)^2$.

Exercise 1.3.5
Simplify the expression: $(x-2)^2-(2x-3)^2$.

Exercise 1.3.6
Simplify the expression: $(x^2-2x)^2+3x(x^3-5)$.

Exercise 1.3.7
Show that each of the following is a complete square:
(a) x^2+6x+9.
(b) $x^2-5x+\frac{25}{4}$.
(c) $3x^2+x+\frac{1}{3}$.

Exercise 1.3.8
Show that $2x^2+9x+\frac{81}{8}$ is a complete square.

Exercise 1.3.9
Expand using Example 1.3.1(c): (a) $(x-2)(x+3)$ (b) $(x+5)(x+7)$ (c) $(x-4)(x-8)$.

Exercise 1.3.10
Factor $18x^4y^5-6x^3y^4+24x^2y^2$.

Exercise 1.3.11
Factor $(2x+3y)^2+5(2x+3y)$.

Exercise 1.3.12
Factor $2(x-1)(3-2x)^2+8(1-x)^2(2x-3)^3$.

Exercise 1.3.13
Factor $3(x-1)(x-2)^2-(x-1)^2(2-x)+2(1-x)(x-2)$.

Exercise 1.3.14
Factor $28(x+2)^2-7(x-1)^2$.

Exercise 1.3.15
Factor $(x^2-4)^2+9(x+2)^2$.

Exercise 1.3.16
Factor $2x^2y^2-25z^2$.

Exercise 1.3.17
Factor $6x^4-24y^4$.

Exercise 1.3.18
Factor x^8-18x^4+81.

Exercise 1.3.19
Factor $x^{2n}-12x^n+36$.

Exercise 1.3.20
Factor $9x^2+42xy+49y^2$.

Exercise 1.3.21
Factor $x-2\sqrt{xy}+y$ where $x, y \geq 0$.

Exercise 1.3.22
Factor x^2+x-12.

Exercise 1.3.23
Factor x^2-x-6.

Exercise 1.3.24
Factor $x^2-8x+15$.

Exercise 1.3.25
Factor by grouping $24x^2y^2-18x^2y+60xy^2-45xy$.

Exercise 1.3.26
Factor by grouping $3xy + 2z + yz + 6x$.

Exercise 1.3.27
Factor by grouping $35x^3 - 10x^2 - 56x + 16$.

Exercise 1.3.28
Factor by grouping $42x^3 - 49x^2 + 18x - 21$.

Exercise 1.3.29
Factor by grouping $56ab + 14 - 49a - 16b$.

Exercise 1.3.30
Factor by grouping $28xy - 7z - 49x + 4yz$.

Exercise 1.3.31
Factor by grouping $45x^4 - 9x^3 + 30x^2 - 6x$.

Exercise 1.3.32
Factor by grouping $5x^2 + 15xy - 2xz - 6yz$.

Exercise 1.3.33
Factor by grouping $3x^5 - 15x^3 + 2x^2 - 10$.

Exercise 1.3.34
Factor by grouping $a^3 - 3a^2b + 3ab^2 - b^3$.

1.4 Solving Algebraic Equations

This section illustrates the processes of solving algebraic equations.

The Geometry of Real Numbers

In order to understand the mathematics of solving equations and inequalities a good understanding of the geometry of real numbers is deemed important. The various sets of numbers in increasing order are:

• The set of all positive integers or natural numbers

$$\mathbb{N} = \{1, 2, 3, \cdots\}.$$

• The set of whole numbers

$$\mathbb{W} = \{0, 1, 2, 3, \cdots\}.$$

• The set of all integers

$$\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}.$$

• The set of all rational numbers

$$\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z} \text{ with } b \neq 0\}.$$

• The set $\mathbb{R}$ of all real numbers.
• The set of complex numbers

$$\mathbb{C} = \{a + ib : a, b, \in \mathbb{R}\}.$$

We will pay close attention to the set of real numbers. Geometrically, $\mathbb{R}$ can be described by a **horizontal axis** pointing to the right as shown in Figure 1.4.1 with numbers being represented by points called **coordinates**. This line is referred to as the **real line.**

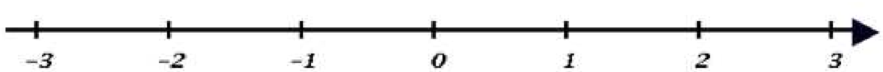


Figure 1.4.1

When the coordinate of a number a is to the left of that of a number b we say that a is **less than** b and we write $a < b$. We can also say that b is **greater than** a and write $b > a$. The symbols $<$ and $>$ are inequality symbols. Two other inequality symbols are $\leq$ and $\geq$.
Inequality symbols help us represent portions of the real lines by **intervals.** The various types of intervals and their graphs are shown in Figure 1.4.2.

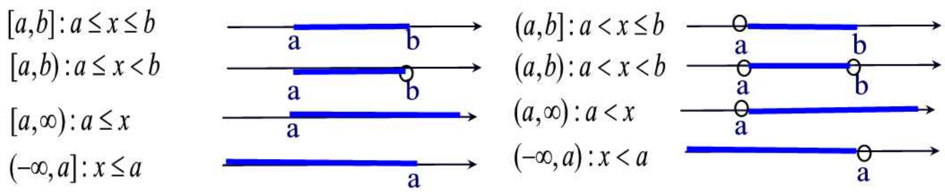


Figure 1.4.2

The **distance** between two numbers x and a on the real line is denoted by $|x - a|$. For example, the distance between the number x and 0 is $|x|$. We call $|x|$ the **absolute value** of x. Since it is a distance, it is a non-negative number. But x can be a non-negative or negative. So we define

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Thus, $|-3| = -(-3) = 3$ while $|3| = 3$. Some of the main properties of the absolute value of a number are:

$$|-x| = |x|$$

$$|x + y| \leq |x| + |y|$$

$$|xy| = |x||y|$$

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}, \ y \neq 0$$

$$|x^n| = |x|^n.$$

Solving Linear Equations

By a **linear equation** we mean an equation that can be converted to the form

$$ax + b = 0$$

where $a \neq 0$ and b are given numbers and x is the variable to be found, also called the **solution**, **zero**, or **root** of the equation. The process of finding x is referred to as **solving** the given equation.
To **solve** a linear equation in one variable, isolate the variable on one side of the equation. This can be done thanks to the following two properties of real numbers:

Property I: Adding or subtracting the same number to both sides of an equation does not change the solution to the equation.
Property II: Multiplying or dividing both sides of an equation by a nonzero number does not change the solution to the equation.

Remark 1.4.1
The above two properties apply to any equation and not only to linear equations.

Example 1.4.1
Solve the equation: $20 - 7x = 6x - 6$.

Solution.
To isolate x, add $7x+6$ to both sides of the given equation to obtain $13x = 26$. Now, divide both sides by 13 to obtain $x = 2$ ■

Solving Quadratic Equations
The second type of equations that we discuss here is the so called quadratic equations. By a **quadratic equation** we mean an equation of the form

$$ax^2 + bx + c = 0, \quad a \neq 0,$$

where a, b, and c are given numbers and x is the variable to be found.
There are two methods for finding x.

• **Solving by Factoring**
The process of factoring consists of rewriting the equation in the form

$$a(x - r)(x - s) = 0.$$

Now, by the **zero product property**, which states that if $u \cdot v = 0$ then either $u = 0$ or $v = 0$, we can conclude that either $x - r = 0$ or $x - s = 0$.

That is, $x = r$ or $x = s$.
To factor $ax^2 + bx + c$
1. find two integers that have a product equal to ac and a sum equal to b,
2. replace bx by two terms using the two new integers as coefficients,
3. then factor the resulting four-term polynomial by grouping. Thus, obtaining $a(x - r)(x - s) = 0$,
4. use the zero product property: $ab = 0 \Longrightarrow a = 0$ or $b = 0$.

Example 1.4.2
Solve $2x^2 + 9x + 4 = 0$ by factoring.

Solution.
We need two integers whose product is $ac = 8$ and sum equals to $b = 9$. Such two integers are 1 and 8. Thus,

$$\begin{aligned}
2x^2 + 9x + 4 =& 2x^2 + x + 8x + 4 \\
=& x(2x + 1) + 4(2x + 1) \\
=& (2x + 1)(x + 4) \\
=& 2(x + \frac{1}{2})(x + 4).
\end{aligned}$$

Hence, the zeros are $x = -\frac{1}{2}$ and $x = -4$ ■

• **Solving by Using the Method of Completing the Square**:
Many quadratic expressions are not easily factored by the method described above. For example, the expression $3x^2 - 7x - 7$. However, the zeros can be found by using the **method of completing the square**:

$$\begin{aligned}
ax^2 + bx + c =& 0 \quad \text{(subtract c from both sides)} \\
ax^2 + bx =& - c \text{ (multiply both sides by } 4a) \\
4a^2x^2 + 4abx =& - 4ac \text{ (add } b^2 \text{ to both sides)} \\
4a^2x^2 + 4abx + b^2 =& b^2 - 4ac \\
(2ax + b)^2 =& b^2 - 4ac \\
2ax + b =& \pm \sqrt{b^2 - 4ac} \\
x =& \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

provided that $b^2 - 4ac \geq 0$. This last formula is known as the **quadratic formula.** Note that if $b^2 - 4ac < 0$ then the equation $ax^2 + bx + c = 0$ has no real solutions (but has complex solutions if you are familiar with complex numbers).

Example 1.4.3
Solve by completing the square $x^2 + 3x + 2 = 0$.

Solution.
We have

$$\begin{aligned} x^2 + 3x + 2 =&(x^2 + 3x) + 2 \\ =&\left(x^2 + 3x + \frac{9}{4}\right) - \frac{9}{4} + 2 \\ =&\left(x + \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2 \\ =&\left(x + \frac{3}{2} - \frac{1}{2}\right)\left(x + \frac{3}{2} + \frac{1}{2}\right) \\ =&(x - 1)(x + 2). \end{aligned}$$

Hence, the zeros of the given equation are $x = 1$ and $x = -2$ ■

Example 1.4.4
Solve by using the quadratic formula $3x^2 - 7x - 7 = 0$.

Solution.
Letting $a = 3, b = -7$ and $c = -7$ in the quadratic formula we have

$$x = \frac{7 \pm \sqrt{133}}{6} \; ■$$

Example 1.4.5
Solve by using the quadratic formula $6x^2 - 2x + 5 = 0$.

Solution.
Letting $a = 6, b = -2$, and $c = 5$ in the quadratic formula we obtain

$$x = \frac{2 \pm \sqrt{-116}}{12}$$

But $\sqrt{-116}$ is not a real number. Hence, the function has no real zeros but complex zeros (See Section 1.8) ■

Solving equations involving radicals

In this subsection we are going to solve equations that contain one or more radical expressions. The plan is to isolate the radical expression, if possible, on one side of the equation so that we can raise both sides of the equation to a power that will eliminate the radical expression. We will restrict our attention to square roots.

Remark 1.4.2
Squaring both sides of an equation may results in extraneous solutions, i.e., "extra" solutions that will not check in the original problem. Hence, one must check each of the solutions in the original equation. This is the only way you can be sure you have a valid solution.

Example 1.4.6
Solve $\sqrt{3x-2} = 4, \ x \geq \frac{2}{3}$.

Solution.
Squaring both sides of the equation, we find $3x - 2 = 16$. Solving this equation, we find $x = 6$. Substituting this value in the equation shows that $\sqrt{3(6) - 2} = 4$ ■

Example 1.4.7
Solve $1 + \sqrt{4x+13} = 2x, \ x \geq 0$.

Solution.
Isolating the radical and then squaring both sides, we find $4x+13 = (2x-1)^2$. Simplifying this equation, we find $4x^2 - 8x - 12 = 0$ or after dividing through by 4, we have $x^2 - 2x - 3 = 0$. Hence, $(x-3)(x+1) = 0$ so that $x = 3$ and $x = -1$. Note that $x = -1 < 0$ so it is discarded. Hence, $x = 3$ is the only solution ■

Example 1.4.8
Solve $\sqrt{x+7} = 1 + \sqrt{x+2}, \ x \geq -2$.

Solution.
We have

$$\begin{aligned}(\sqrt{x+7})^2 =&(1+\sqrt{x+2})^2\\ x+7 =&x+3+2\sqrt{x+2}\\ 4 =&2\sqrt{x+2}\\ 2 =&\sqrt{x+2}\\ 4 =&x+2\\ 2 =&x.\end{aligned}$$

Substituting this value in the original equation, we find $\sqrt{2+7} = 1 + \sqrt{2+2} = 3$ ■

Solving rational equations

By a **rational equation** we mean an equation that involves ratios of algebraic expressions where the variables have whole integer exponents. To solve such an equation, we start by multiplying through by the least common denominator in order to clear the equation from all ratios and thus reducing the equation to an equation that is simple to solve. We illustrate this procedure in the next two examples.

Example 1.4.9
Solve $\frac{6}{x^2} + \frac{5}{x} = 6,\ x \neq 0.$

Solution.
The least common denominator is x^2. Multiplying through by x^2 and then reducing, we find

$$\begin{aligned}x^2\left(\frac{6}{x^2}+\frac{5}{x}\right) =&6x^2\\ 6+5x =&6x^2\\ 6x^2-5x-6 =&0.\end{aligned}$$

Solving this quadratic equation either by factoring or using the quadratic formula we find $x = \frac{3}{2}$ and $x = -\frac{2}{3}$ ■

Example 1.4.10
Solve $\frac{1}{x+2} + \frac{x}{x-2} = \frac{x+6}{x^2-4},\ x \neq \pm 2.$

Solution.
The least common denominator is x^2-4. Multiplying through by x^2-4 and then reducing, we find

$$\begin{aligned}
(x^2-4)\left(\frac{1}{x+2}+\frac{x}{x-2}\right) =& (x^2-4)\times\frac{x+6}{x^2-4}\\
x-2+x(x+2) =& x+6\\
x^2+2x-8 =& 0\\
(x+4)(x-2) =& 0.
\end{aligned}$$

Solving, we find $x=-4$ and $x=2$. Since $x=2$ results in a zero denominator in the original equation, it is discarded. Hence, the only solution is $x=-4$ ■

Solving absolute value equations

Absolute value equations are equations that involve absolute values of some algebraic expressions. Recall that $|x-a|$ gives the distance between x and a. Thus, if $b \geq 0$, an equation of the form $|x-a|=b$ says that the distance between x and a is b. There are two values from either side of a that are a distance b from a, namely $x=a-b$(from the left of a) and $x=a+b$ (from the right of a).

Example 1.4.11
Solve $|2x-3|=5$.

Solution.
We have

$$\begin{aligned}
|2x-3| =& 5\\
2x =& 3\pm 5\\
x =& \frac{3\pm 5}{2}.
\end{aligned}$$

Hence, $x=4$ or $x=-1$ ■

Example 1.4.12
Solve $|x-3|=|x+2|$.

Solution.
We have

$$\begin{aligned}|x-3| =&|x+2|\\ \left|\frac{x-3}{x+2}\right| =&1\\ x-3 =& \pm(x+2)\\ x =&3 \pm (x+2).\end{aligned}$$

If $x = 3 + x + 2$ then $0 = 5$, impossible. If $x = 3 - x - 2$ then $x = \frac{1}{2}$. Hence, the only solution is $x = \frac{1}{2}$ ■

Example 1.4.13
Solve $|x| = |x - 4| + 9$.

Solution.
We consider the following four cases:

Case 1: $x \geq 0, x - 4 \geq 0$, i.e., $x \geq 4$.
In this case, we have $x = x - 4 + 9$ which results in 5=0, impossible.

Case 2: $x \geq 0, x - 4 < 0$, that is $0 \leq x < 4$.
In this case, we obtain $x = -x + 4 + 9$. Solving, we find $x = \frac{13}{2} > 4$ so again no solutions.

Case 3: $x < 0, x - 4 \geq 0$. No such x exists.

Case 4: $x < 0, x - 4 < 0$
In this case, we obtain $-x = -x + 4 + 9$. Solving, we find $0 = 13$, impossible. Hence, the given equation has no solutions ■

Example 1.4.14
Solve: $|x - 2| = 3x + 1$.

Solution.
We note that $3x + 1 \geq 0$ or $x \geq -\frac{1}{3}$. If $-\frac{1}{3} \leq x \leq 2$ then $-x + 2 = 3x + 1$. Solving, we find $x = \frac{1}{4}$. If $x > 2$ then $x - 2 = 3x + 1$. Solving, we find $x = -\frac{3}{2} < 2$. Hence, the only solution is $x = \frac{1}{4}$ ■

Exercises

Exercise 1.4.1
Solve: $5x + 2 = 2x - 10$.

Exercise 1.4.2
Solve: $2(x - 3) - 5 = 4(x - 5)$.

Exercise 1.4.3
Solve: $\frac{3}{5}(n + 5) - \frac{3}{4}(n - 11) = 0$.

Exercise 1.4.4
Solve: $3(x + 5)(x - 1) = (3x + 4)(x - 2)$.

Exercise 1.4.5
Solve: $3x^2 = 27$.

Exercise 1.4.6
Solve: $x^2 = 12 - x$.

Exercise 1.4.7
Solve: $x^2 - 6x + 9 = 0$.

Exercise 1.4.8
Solve: $3x^2 - 5x + 1 = 0$.

Exercise 1.4.9
Solve: $3x^2 + 2 = 4x$.

Exercise 1.4.10
Solve: $x^2 + 5x + 6 = 0$.

Exercise 1.4.11
Solve: $x^2 + 2x - 15 = 0$.

Exercise 1.4.12
Solve: $x^2 - 4x + 3 = 0$.

Exercise 1.4.13
Solve by completing the square $x^2 - 6x - 5 = 0$.

Exercise 1.4.14
Solve by using the quadratic formula $x^2 + 3x + 1 = 0$.

Exercise 1.4.15
Solve by using the quadratic formula $2x^2 + 3x - 2 = 0$.

Exercise 1.4.16
Solve by using the quadratic formula $x^2 - 2\pi x + \pi^2 = 0$.

Exercise 1.4.17
Solve by using the quadratic formula $x^2 + \sqrt{2}x + \pi = 0$.

Exercise 1.4.18
Solve: $\sqrt{2x+3} = 7$.

Exercise 1.4.19
Solve: $\sqrt{x+4} = x - 2$.

Exercise 1.4.20
Solve: $\sqrt{3x-3} - \sqrt{x} = 1$.

Exercise 1.4.21
Solve: $\sqrt{2x} = 2 - \sqrt{x-2}$.

Exercise 1.4.22
Solve: $\frac{x(x-2)}{2} + \frac{x^2-4}{3} = \frac{x^2+x-6}{5}$.

Exercise 1.4.23
Solve: $\frac{x-1}{3x-2} + \frac{3x^2-3}{3x-2} = \frac{3x-3}{3x-2}$.

Exercise 1.4.24
Solve: $\frac{x}{x-5} = \frac{3x-10}{x-5}$.

Exercise 1.4.25
Solve: $\frac{1}{x+2} = \frac{4}{x} + \frac{1}{2}$.

Exercise 1.4.26
Solve: $\frac{1}{x} + \frac{1}{x+1} = \frac{-1}{x(x+1)}$.

Exercise 1.4.27
Solve: $|x-2|=5$.

Exercise 1.4.28
Solve: $|x+4|=13$.

Exercise 1.4.29
Solve: $|2x-5|=17$.

Exercise 1.4.30
Solve: $|2x+7|=|1-x|$.

Exercise 1.4.31
Solve: $|x-5|=-4$.

Exercise 1.4.32
Solve: $|4x+3|=3-x$.

Exercise 1.4.33
Solve: $|4+x|-2x=7-5x$.

Exercise 1.4.34
Solve: $|x-1|=|x-3|+1$.

1.5 Solving Algebraic Inequalities

This section illustrates the processes of solving linear, quadratic, rational, and absolute value inequalities.

Solving Linear Inequalities

By a **linear inequality** we mean an inequality that can be converted to the form

$$ax + b \square 0, \quad a \neq 0$$

where $\square$ can be any of the following: $<, >, \leq, \geq$.

A number that satisfies an inequality is called a **solution**. The set of all the solutions to an inequality is called the **solution set**. In most cases, a solution set is an interval or a collection of intervals.

To isolate the x, use the following two properties:

Property III: Adding or subtracting the same number to both sides of an inequality does not change the solution set of the inequality.

Property IV: Multiplying or dividing both sides of an inequality by a non-zero number does not change the solution set of the inequality. However, when you multiply or divide by a negative number make sure you reverse the inequality symbol.

Example 1.5.1

Solve the inequality: $x + 4 > 3x + 16$.

Solution.

Add $-x - 16$ to both sides of the inequality to obtain $-12 > 2x$ or $2x < -12$. Now divide both sides by 2 to obtain $x < -6$. The solution set is usually represented by an interval. Thus, the interval of solution to the given inequality is $(-\infty, -6)$ $\blacksquare$

Example 1.5.2

Solve the inequality: $-4(3x - 5) > 2(x - 4)$.

Solution.
We have

$$\begin{aligned}-4(3x-5) >&2(x-4)\\ -12x+20 >&2x-8\\ -14x >&-28\\ x <&2.\end{aligned}$$

Thus, the interval of solution to the given inequality is $(-\infty, 2)$ ■

Example 1.5.3
Solve the inequality: $-3(x+2) \leq 5x+7$.

Solution.
We have $-3(x+2) \leq 5x+7 \Leftrightarrow -3x-6 \leq 5x+7 \Leftrightarrow -8x \leq 13 \Leftrightarrow x \geq -\frac{13}{8}$. Thus, the interval of solution to the given inequality is $\left[-\frac{13}{8}, \infty\right)$ ■

Solving Quadratic Inequalities

By a **quadratic inequality** we mean an inequality of the form

$$ax^2+bx+c \square 0, \quad a \neq 0$$

where $\square$ can be any of the following:$<, >, \leq, \geq$.
The process of solving this type of inequalities consists of solving the quadratic equation $ax^2+bx+c=0$ so that we can locate the zeros and then construct a chart of signs which provide the solution interval to the inequality. This process of solution is referred to as the **critical value method.** We illustrate this method in the next examples.

Example 1.5.4
Solve the inequality $6x^2-4 \leq 5x$.

Solution.
Subtract $5x$ from both sides to obtain $6x^2-5x-4 \leq 0$. Factor $f(x) = 6x^2-5x-4 = (3x-4)(2x+1)$. Thus, the zeros (also known as the **critical values**) of $f(x)$ are $x = \frac{4}{3}$ and $x = -\frac{1}{2}$. These values separate the real line into three pieces. In each piece we select a **test value** to find the sign of f. Since $f(-1) = 7 > 0$, $f(0) = -4 < 0$, and $f(2) = 10 > 0$, we can construct the following chart of signs:

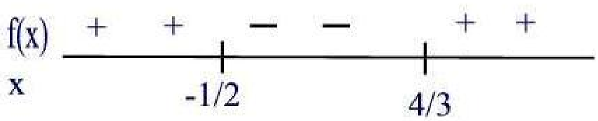


Figure 1.5.1

According to Figure 1.5.1, the interval of solution is given by $[-\frac{1}{2}, \frac{4}{3}]$ ■

Example 1.5.5
Solve the inequality $x^2 + 7x + 10 < 0$.

Solution.
We have

$$\begin{aligned} x^2 + 7x + 10 <&0 \\ (x+5)(x+2) <&0. \end{aligned}$$

The critical values are $x = -5$ and $x = -2$. Using the critical value method we obtain the chart

Figure 1.5.2

Thus, the final answer is the open interval $(-5, -2)$ ■

Example 1.5.6
Solve the inequality $x^3 + 25x \le 10x^2$.

Solution.
The given inequality reduces to

$$x^3 + 25x - 10x^2 = x(x^2 - 10x + 25) = x(x-5)^2 \le 0.$$

Since $(x-5)^2 \ge 0$ for all x, $x(x-5)^2 \le 0$ only if $x = 5$ or $x \le 0$. Hence, the solution set is $(-\infty, 0] \cup \{5\}$ ■

Rational inequalities
The critical value method can be used to solve **rational inequalities**.

Example 1.5.7
Solve: $\frac{x}{x+3}+2\le 0$.

Solution.
The given inequality can be written as

$$\frac{x}{x+3}+2=\frac{3(x+2)}{x+3}\le 0.$$

The solution set to this inequality is the same as the solution set of $3(x+2)(x+3)\le 0$ with the exception that $x\ne -3$. Solving this last inequality by the method of critical value and taking in consideration the fact that $x\ne -3$, we find the interval of solution $(-3,-2]$ ■

Example 1.5.8
Solve: $\frac{x}{x-1}\ge\frac{5}{x+5}$.

Solution.
We have

$$\begin{aligned}\frac{x}{x-1}&\ge\frac{5}{x+5}\\ \frac{x}{x-1}-\frac{5}{x+5}&\ge 0\\ \frac{x^2+5}{(x-1)(x+5)}&\ge 0.\end{aligned}$$

Thus, the last inequality is equivalent to $(x-1)(x+5)>0$ whose interval of solution is $(-\infty,-5)\cup(1,\infty)$ ■

Solving Absolute Value Inequalities

Recall from Section 1.4, the definition of the **absolute value** of a number x

$$|x|=\begin{cases}x, & \text{if } x\ge 0,\\ -x, & \text{if } x<0.\end{cases}$$

Geometrically, $|x|$ measures the distance from x to the origin. Thus, an inequality of the form $|x|>5$ indicates that x is more than five units from 0. Any number on the number line to the right of 5 or to the left of -5 is more than five units from 0. So $|x|>5$ is equivalent to $x<-5$ or $x>5$. Thus,

the interval of solution is given by the union of the intervals $(-\infty, -5)$ and $(5, \infty)$. Symbolically, we will write $(-\infty, -5) \cup (5, \infty)$.
Similarly, the inequality $|x - 9| < 2$ indicates that the distance from x to 9 is less than 2. On a number line, this happens when x is between 7 and 11. That is, the interval of solution is $(7, 11)$. As you can see, drawing the real line plays a major role in understanding how the solution interval(s) can be found.

Example 1.5.9
Solve $|5 - 3x| \le 6$.

Solution.
Let $u = 5-3x$. Then $|u| \le 6$. This means that the distance from u to 0 is less than or equal to 6. On a number line, this happens when $-6 \le u \le 6$. Thus, $-6 \le 5-3x \le 6$. Next, we have to isolate the x in this compound inequality. Subtract 5 from each part of the inequality to obtain $-11 \le -3x \le 1$. Now, divide through by -3 to obtain $-\frac{1}{3} \le x \le \frac{11}{3}$. Thus, the interval of solution is $[-\frac{1}{3}, \frac{11}{3}]$ ■

Example 1.5.10
Solve $|x - 4| \le 0$.

Solution.
Since there is no number x such that $|x-4| < 0$, the only solution to the given inequality exists only when $|x - 4| = 0$. Hence, $x = 4$ is the only solution ■

Example 1.5.11
Solve $|3x - 10| \le 14$.

Solution.
We have

$$\begin{aligned} -14 \le 3x - 10 \le 14 \\ -4 \le 3x \le 24 \\ -\frac{4}{3} \le x \le 8. \end{aligned}$$

The solution interval is the closed interval $\left[-\frac{4}{3}, 8\right]$ ■

Example 1.5.12
Solve: $|x - 5| < |x + 1|$.

Solution.
Squaring both sides, we obtain $(x-5)^2 < (x+1)^2$ or $x^2-10x+25 < x^2+2x+1$ and this reduces to $12x - 24 > 0$ or $x - 2 > 0$. Hence, the interval of solution is $(2, \infty)$ ■

Exercises

In the Problems 1.5.1 - 1.5.7, represent the solution sets as intervals.

Exercise 1.5.1
Solve: $-5x + 6 > 10$.

Exercise 1.5.2
Solve: $\frac{7}{3}x + 5 \leq \frac{1}{2}(3x - 7)$.

Exercise 1.5.3
Solve: $\frac{x}{2} + \frac{2x-1}{5} \geq \frac{x}{10}$.

Exercise 1.5.4
Solve: $x(x - 1) \geq x^2 + 2x - 5$.

Exercise 1.5.5
Solve: $\frac{3x-2}{5} + 3 \geq \frac{4x-1}{3}$.

Exercise 1.5.6
Solve: $4x + 7 \geq 2x - 3$.

Exercise 1.5.7
Solve the inequality: $-5 < 3x - 2 < 1$.

Exercise 1.5.8
An electric company charges for electricity is 10.494 cents per kilowatt-hour. In addition, each monthly bill contains a customer charge of \$9.36. If your bill ranged from a low of \$80.24 to a high of \$271.80, over what range did usage vary (in kilowatt-hour)?

Exercise 1.5.9
In your MATH 103 class, you have scores of $68, 82, 87$, and 89 on the first four of five tests. To get a grade of B, the average of the first five test scores must be greater than or equal to 80 and less than 90. Solve an inequality to find the range of the score you need on the last test to get a B.

Exercise 1.5.10
The percentage method of withholding for federal income tax (1998) states that a single person whose weekly wages, after subtracting withholding allowances, are over \$517, but not over \$1105, shall have \$69.90 plus 28% of the excess over \$517 withheld. Over what range does the amount withheld vary if the weekly wages vary from \$525 to \$600 inclusive?

Exercise 1.5.11
Solve: $x(x-1) > 6$.

Exercise 1.5.12
Solve: $(x-2)x \leq (x-2)x^2$.

Exercise 1.5.13
Solve: $x^2 + x - 12 > 0$.

Exercise 1.5.14
Solve: $x^2 \leq 4x + 12$.

Exercise 1.5.15
Solve: $\frac{(x+3)(2-x)}{(x-1)^2} > 0$.

Exercise 1.5.16
Solve: $\frac{4x+5}{x+2} \geq 3$.

Exercise 1.5.17
Solve: $2 < |3x-1| < 5$.

Exercise 1.5.18
Solve: $|x| > \frac{1}{x}$.

Exercise 1.5.19
Solve: $\frac{x^2-4}{1-|x|} \geq 0$.

Exercise 1.5.20
Solve: $|x+4| > 3$.

Exercise 1.5.21
Solve: $|x-2| \leq 5$.

Exercise 1.5.22
Solve: $|2x+3| < x-6$.

Exercise 1.5.23
Solve: $|2x+3| \geq x-6$.

Exercise 1.5.24
Solve: $|2x+3| < x+6$.

Exercise 1.5.25
Solve: $\frac{x-8}{x} \leq 3 - x$.

Exercise 1.5.26
Solve: $\frac{x+5}{x^2-4} < 0$.

Exercise 1.5.27
Solve: $\frac{x-2}{(x+1)^2} \leq 0$.

Exercise 1.5.28
Solve: $\frac{x^2+x-6}{x} \geq 0$.

Exercise 1.5.29
Solve: $x^2 + 6x + 9 \leq 0$.

Exercise 1.5.30
Solve: $\frac{3}{2x-7} \leq -1$.

1.6 Solving Linear Systems

In this section, we consider methods for solving systems with two linear equations in two unknowns and systems with three linear equations in three unknowns.

Linear Systems in two equations and two unknowns

A **linear equation** in the variables x and y is an equation of the form $ax + by = c$. A **linear system** in two equations in the variables x and y is a system of the form

$$\begin{cases} ax + by = c \\ a'x + b'y = c'. \end{cases}$$

Graphically, a linear system consists of two straight lines. But two lines can either intersect, coincide or parallel. When they intersect or coincide, we say that the system is **consistent**. When they are parallel, we say that the system is **inconsistent.** A consistent system can be **dependent** (i.e., the two lines coincide) or **independent** (i.e., the two lines has exactly one point of intersection). A point of intersection of the two lines is called a **solution** of the system.

To **solve** a system is to find all the solutions. We discuss two algebraic methods for solving a linear system in two equations and two unknowns:

Method of Elimination or Addition Method

As the name indicates, we add an appropriate multiples of the two equations together in order to eliminate one of the variables. By adding the equations we mean that we add the left sides of the two equations together, and we add the right sides together. This is legal because of the Addition Principle, which says that we can add the same amount to both sides of an equation. We illustrate this process in the next examples.

Example 1.6.1 *(Consistent and independent)*

Solve the system of linear equations

$$\begin{cases} 2x + 5y = 1 \\ -3x + 2y = 8. \end{cases}$$

Solution.

For this problem, to eliminate the x we have to multiply the first equation by 3 and the second equation by 2.

$$\begin{cases} 6x + 15y = 3 \\ -6x + \ \ 4y = 16. \end{cases}$$

Adding these equations results in the equation $19y = 19$. Solving for y, we find $y = 1$. Substituting this value into one of the given equations and then solving for x, we find $x = -2$. Thus, the system is consistent and independent ■

Example 1.6.2 *(Inconsistent)*
Solve the system

$$\begin{cases} 6x - 2y = 8 \\ 9x - 3y = 6. \end{cases}$$

Solution.
Multiply the first equation by -3 and the second equation by 2 and add the resulting equations we find $0 = -12$ which is not a true statement. Hence, the given system is inconsistent. Graphically, the system represents two parallel lines ■

Example 1.6.3 *(Consistent and dependent)*
Solve the system

$$\begin{cases} x - 2y = 4 \\ 2x - 4y = 8. \end{cases}$$

Solution.
Multiply the first equation by -2 and add the resulting equation to the second equation we find $0x + 0y = 0$ which is true regardless of the values of x and y. Hence, the system has an infinite number of solutions. Graphically, the system represents two lines that coincide. Points on the line can be represented by the **parametric equations**:

$$x = 2t + 4 \text{ and } y = t$$

where t is called a **parameter**. We conclude that the system is consistent and dependent ■

The Method of Substitution

With this method, we solve one of the equations for one variable in terms of the other, and then substitute that into the other equation. We illustrate this method by an example.

Example 1.6.4
Solve the system of linear equations

$$\begin{cases} x + 2y = 3 \\ -3x + 4y = 1. \end{cases}$$

Solution.
Solving the first equation for x we find $x = 3 - 2y$. Now we can use this result and substitute $x = 3 - 2y$ in for x in the second equation and find $4y - 3(3 - 2y) = 1$. Solving this equation we find $y = 1$. Hence, $x = 3 - 2y = 3 - 2 = 1$. The solution to the system is $(1, 1)$ so the system is consistent and independent ■

Linear Systems in three equations and three unknowns

Consider a linear system in three equations and three unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3. \end{aligned}$$

If $b_1 = b_2 = b_3 = 0$ then we call the system **homogeneous**. Otherwise, the system is called **non-homogeneous**

We define the **augmented matrix** to be the rectangular array

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$$

Two linear systems are said to be **equivalent** if and only if they have the same solution set. One can build equivalent systems that are easier to solve than the original system by applying the following **elementary row operations** on the augemented matrix:
(I) Multiply a row by a non-zero number.

(II) Replace a row by the sum of this row and another row multiplied by a number.
(III) Interchange two rows.
We apply these elementary row operations on the augmented matrix and reduces it to a triangular matrix, known as **elementary echelon form**. Then the corresponding system is triangular as well and is equivalent to the original system. Next, use either the backward-substitution or the forward-substitution technique to find the unknowns. We illustrate this technique in the following examples.

Example 1.6.5
Solve the following linear system using elementary row operations on the augmented matrix:

$$\begin{cases} x_1 - 2x_2 + \ \ x_3 = 0 \\ \qquad\quad 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9. \end{cases}$$

Solution.
In what follows, r_i denotes the i^{th} row of the augmented matrix. The augmented matrix for the system is

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

Step 1: The operations $r_2 \leftarrow \frac{1}{2}r_2$ and $r_3 \leftarrow r_3 + 4r_1$ give

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

Step 2: The operation $r_3 \leftarrow r_3 + 3r_2$ gives

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The corresponding system of equations is

$$\begin{cases} x_1 - 2x_2 + \ \ x_3 = 0 \\ \qquad\quad x_2 - 4x_3 = 4 \\ \qquad\qquad\quad\ \ x_3 = 3. \end{cases}$$

Using back-substitution we find the unique solution $x_1 = 29, x_2 = 16, x_3 = 3$. Thus, the system is consistent and independent ■

Example 1.6.6
Solve the following linear system using the method described above.

$$\begin{cases} \quad\quad x_2 + 5x_3 = -4 \\ x_1 + 4x_2 + 3x_3 = -2 \\ 2x_1 + 7x_2 + \;\; x_3 = -1. \end{cases}$$

Solution.
The augmented matrix for the system is

$$\begin{bmatrix} 0 & 1 & 5 & -4 \\ 1 & 4 & 3 & -2 \\ 2 & 7 & 1 & -1 \end{bmatrix}$$

Step 1: The operation $r_2 \leftrightarrow r_1$ gives

$$\begin{bmatrix} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 2 & 7 & 1 & -1 \end{bmatrix}$$

Step 2: The operation $r_3 \leftarrow r_3 - 2r_1$ gives the system

$$\begin{bmatrix} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 0 & -1 & -5 & 3 \end{bmatrix}$$

Step 3: The operation $r_3 \leftarrow r_3 + r_2$ gives

$$\begin{bmatrix} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The corresponding system of equations is

$$\begin{cases} x_1 + 4x_2 + 3x_3 = -2 \\ \quad\quad x_2 + 5x_3 = -4 \\ 0x_1 + 0x_2 + 0x_3 = -1. \end{cases}$$

From the last equation we conclude that the system is inconsistent ■

Example 1.6.7
Solve the following system by applying elementary row operations on the augmented matrix.

$$\begin{cases} x_1 + 2x_2 + \ \ x_3 = 0 \\ x_1 + 3x_2 + 6x_3 = 0 \\ 2x_1 + 3x_2 - 3x_3 = 0. \end{cases}$$

Solution.
The augmented matrix of the system is

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 3 & 6 & 0 \\ 2 & 3 & -3 & 0 \end{bmatrix}$$

Reducing this matrix to triangular form as follows.

Step 1: $r_2 \leftarrow r_2 - r_1$ and $r_3 \leftarrow r_3 - 2r_1$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & -1 & -5 & 0 \end{bmatrix}$$

Step 2: $r_3 \leftarrow r_3 + r_2$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding system is

$$\begin{cases} x_1 + 2x_2 + \ \ x_3 = 0 \\ \qquad\quad x_2 + 5x_3 = 0. \end{cases}$$

The system has infinitely many solutions. These solutions are given by the parametric equations: $x_1 = 9s,\ x_2 = -5s,\ x_3 = s$. Hence, the system is consistent and dependent ■

Exercises

Exercise 1.6.1
Solve by the method of elimination. Classify the system as consistent independent, consistent dependent, or inconsistent.

$$\begin{cases} -4x + 2y = 8 \\ \ \ 2x - \ \ y = 0. \end{cases}$$

Exercise 1.6.2
Solve by the method of elimination. Classify the system as consistent independent, consistent dependent, or inconsistent.

$$\begin{cases} \ \ x + y = 3 \\ 2x - y = 0. \end{cases}$$

Exercise 1.6.3
Solve by the method of elimination. Classify the system as consistent independent, consistent dependent, or inconsistent.

$$\begin{cases} \ \ x - 2y = 4 \\ 2x - 4y = 8. \end{cases}$$

Exercise 1.6.4
Solve by the method of elimination. Classify the system as consistent independent, consistent dependent, or inconsistent.

$$\begin{cases} \ \ 2x - 4y = -10 \\ -x + 2y = 5. \end{cases}$$

Exercise 1.6.5
Solve by the method of substitution. Classify the system as consistent independent, consistent dependent, or inconsistent.

$$\begin{cases} 2x - \ \ y = 3 \\ \ \ x + 2y = 14. \end{cases}$$

Exercise 1.6.6
Solve by the method of substitution. Classify the system as consistent independent, consistent dependent, or inconsistent.

$$\begin{cases} 2x - 3y = -2 \\ -4x + 6y = 7. \end{cases}$$

Exercise 1.6.7
Solve by the method of substitution. Classify the system as consistent independent, consistent dependent, or inconsistent.

$$\begin{cases} -6x + 10y = -30 \\ 3x - 5y = 15. \end{cases}$$

Exercise 1.6.8
Solve the following system. Classify the system as consistent independent, consistent dependent, or inconsistent.

$$\begin{cases} 20x + 24y = 10 \\ \frac{1}{3}x + \frac{4}{5}y = \frac{5}{6}. \end{cases}$$

Exercise 1.6.9
Solve the following system. Classify the system as consistent independent, consistent dependent, or inconsistent.

$$\begin{cases} 2x + 3y = 6 \\ -4x + y = 2. \end{cases}$$

Exercise 1.6.10
A chemist has two solutions of alcohol. One is 25% alcohol and the other is 45% alcohol. He wants to mix these two solutions to obtain 36 ounces of a 30% alcohol solution. How many ounces of each solution should be mixed together?

Exercise 1.6.11
You invested a total of $38,000 in two municipal bonds that have a yield of 4% and 6% interest per year, respectively. The interest you earned from the bonds was $1,930. How much did you invest in each bond?

Exercise 1.6.12
Determine if the following system is consistent.

$$\begin{cases} \phantom{2x_1 - {}} x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 5x_1 - 8x_2 + 7x_3 = 1. \end{cases}$$

Exercise 1.6.13
Solve the following linear system using elementary row operations on the augmented matrix. Classify the system as consistent independent, consistent dependent, or inconsistent.

$$\begin{cases} x_1 + 2x_2 \phantom{{}+3x_3} = 0 \\ -x_1 + 3x_2 + 3x_3 = -2 \\ \phantom{-x_1 + {}} x_2 + x_3 = 0. \end{cases}$$

Exercise 1.6.14
Solve the following system using elementary row operations on the augmented matrix. Classify the system as consistent independent, consistent dependent, or inconsistent.

$$\begin{cases} 5x_1 - 5x_2 - 15x_3 = 40 \\ 4x_1 - 2x_2 - 6x_3 = 19 \\ 3x_1 - 6x_2 - 17x_3 = 41. \end{cases}$$

Exercise 1.6.15
Solve the following system using elementary row operations on the augmented matrix. Classify the system as consistent independent, consistent dependent, or inconsistent.

$$\begin{cases} 2x_1 + x_2 + x_3 = -1 \\ x_1 + 2x_2 + x_3 = 0 \\ 3x_1 \phantom{{}+2x_2} - 2x_3 = 5. \end{cases}$$

Exercise 1.6.16
Solve the linear system whose augmented matrix is given by

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix}$$

Classify the system as consistent independent, consistent dependent, or inconsistent.

Exercise 1.6.17
Solve the following system using elementary row operations on the augmented matrix.

$$\begin{cases} x_1 - 2x_2 + 3x_3 = 7 \\ 2x_1 + x_2 + x_3 = 4 \\ -3x_1 + 2x_2 - 2x_3 = -10. \end{cases}$$

Exercise 1.6.18
Solve the following system using elementary row operations on the augmented matrix.

$$\begin{cases} 2x_1 - 4x_2 + 5x_3 = -33 \\ 4x_1 - x_2 = -5 \\ -2x_1 + 2x_2 - 3x_3 = 19. \end{cases}$$

Exercise 1.6.19
Solve the following system using elementary row operations on the augmented matrix.

$$\begin{cases} 6x - 2y = 12 \\ x + y = 18. \end{cases}$$

Exercise 1.6.20
Solve the following system by the substitution method.

$$\begin{cases} 12x - 7y = 106 \\ 8x + y = 82. \end{cases}$$

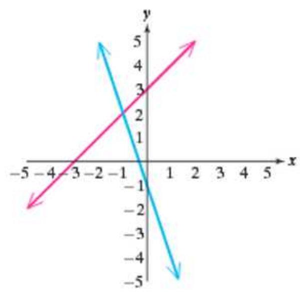
Exercise 1.6.21
Solve the following system..

$$\begin{cases} y = 5 + 3x \\ y = 3 - x. \end{cases}$$

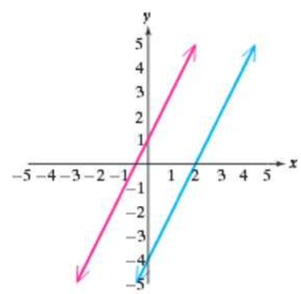
Exercise 1.6.22
Suppose there is a piggybank that contains 57 coins, which are only quarters and dimes. The total number of coins in the bank is 57, and the total value of these coins is \$9.45. Determine how many of the coins are quarters and how many are dimes.

In Exercises 1.6.23 - 1.6.25, the graph of a system of linear equations is given.
(a) Identify whether the system is consistent or inconsistent.
(b) Identify whether the system is dependent or independent.
(c) Identify the number of solutions to the system.

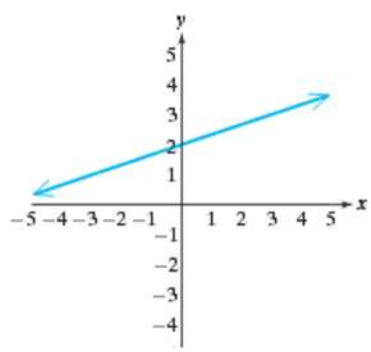
Exercise 1.6.23



Exercise 1.6.24



Exercise 1.6.25



Exercise 1.6.26
Solve the following system.

$$\begin{cases} 2x_1 + \ \ 5x_2 + 2x_3 = -1 \\ \ \ x_1 + \ \ 2x_2 - 3x_3 = 5 \\ 5x_1 + 12x_2 + \ \ x_3 = 10. \end{cases}$$

Exercise 1.6.27
Solve the following system.

$$\begin{cases} \ \ x_1 - \ \ 3x_2 + 2x_3 = 0 \\ 2x_1 - \ \ 5x_2 - 2x_3 = 0 \\ 4x_1 - 11x_2 + 2x_3 = 0. \end{cases}$$

Exercise 1.6.28
Two angles are complementary, i.e., their sum is $90°$. The measure of the first is five times the measure of the second. Find the degree measures of the two angles.

Exercise 1.6.29
James wants to mix a 50% saline solution with a 80% saline solution to make a 60% saline solution. He needs a total of 12 ounces in the mix. How many ounces of the 50% saline solution and the 80% saline solution should he use?

Exercise 1.6.30
The sum of two numbers is 2. Twice the first plus the second equals 8. Find the two numbers.

1.7 Geometry in the Cartesian System

This section is designed to familiarize students to the Cartesian coordinate system and its many uses in the world of mathematics.
The **Cartesian coordinate system**, also known as the **rectangular coordinate system** or the xy-plane, consists of two number scales, called the **x-axis** (a horizontal axis) and the **y-axis** (a vertical axis), that are perpendicular to each other at point O called the **origin.** Any point in the system is associated with an **ordered pair** of numbers (x, y) called the **coordinates** of the point. The number x is called the **abscissa** or the **x-coordinate** and the number y is called the **ordinate** or the **y-coordinate.** Positive values of the $x-$coordinate are measured to the right, negative values to the left. Positive values of the $y-$coordinate are measured up, negative values down. The origin is denoted as $(0, 0)$.
The axes divide the coordinate system into four regions called **quadrants** and are numbered counterclockwise as shown in Figure 1.7.1
To **plot a point** $P(a, b)$ means to draw a dot at its location in the $xy-$plane.

Example 1.7.1
Plot the point P with coordinates $(5, 2)$.

Solution.
Figure 1.7.1 shows the location of the point $P(5, 2)$ in the $xy-$plane ■

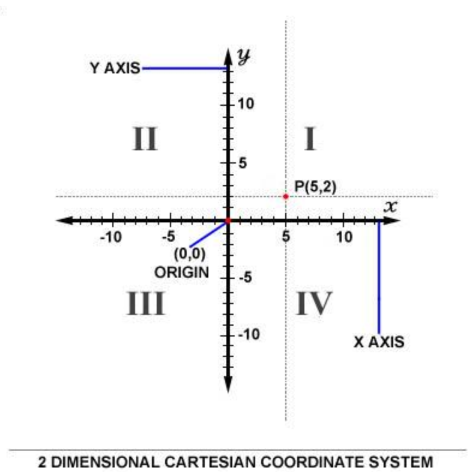


Figure 1.7.1

The following chart indicates the sign of each coordinate in the Cartesian system.

	x	y
Quadrant I	$+$	$+$
Quadrant II	$-$	$+$
Quadrant III	$-$	$-$
Quadrant IV	$+$	$-$
Positive x-axis	$+$	0
Negative x-axis	$-$	0
Positive y-axis	0	$+$
Negative y-axis	0	$-$

Distance Between Two Points

The **distance formula** is a variant of the Pythagorean formula that you used back in geometry. Here's how we get from one to the other: Given two points $A(x_1, y_1)$ and $B(x_2, y_2)$. Let d be the distance between the two points. Construct the right triangle as shown in Figure 1.7.2.

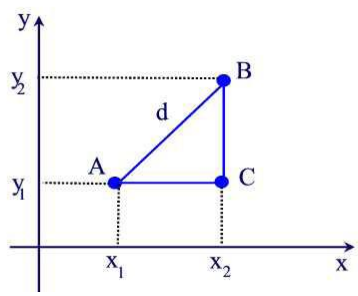


Figure 1.7.2

By the Pythagorean formula we have

$$d^2 = |AC|^2 + |CB|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

Taking the square root of both sides we obtain the **distance formula**

$$d = d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Example 1.7.2

Find the distance between the points $(-5, 8)$ and $(-10, 14)$.

Solution.
Applying the distance formula we find

$$d = \sqrt{(14-8)^2 + (-10-(-5))^2} = \sqrt{36+25} = \sqrt{61} \blacksquare$$

The Midpoint Formulas
The point halfway between the endpoints of a line segment is called the **midpoint**. Thus, a midpoint divides a line segment into two equal parts. Let $M(a,b)$ be the midpoint of the line segment with endpoints $A(x_1,y_1)$ and $B(x_2,y_2)$. See Figure 1.7.3.

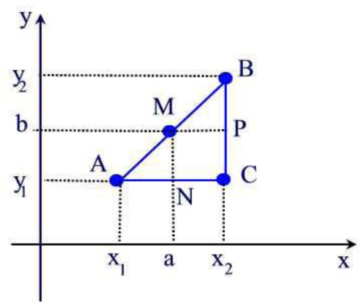


Figure 1.7.3

The triangles MAN and BMP are **similar** so that we can write

$$\frac{|MA|}{|BM|} = \frac{|AN|}{|MP|}.$$

But $|MA| = |BM|$ so that $|AN| = |MP|$. Also, $|MP| = |NC|$ so that $|AN| = |NC|$. Thus, N is the midpoint of the line segment with endpoints A and C. It follows that $a - x_1 = x_2 - a$ or $a = \frac{x_1+x_2}{2}$. A similar argument shows that P is the midpoint of the line segment with endpoints B and C and $b = \frac{y_1+y_2}{2}$. Thus, the midpoint M is given by the **midpoint formulas**

$$M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right).$$

Example 1.7.3
Find the midpoint of the line segment with endpoints $A(4,7)$ and $B(-10,7)$.

Solution.
Plugging into the midpoint formula we find

$$\begin{aligned} \text{Midpoint} &= \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) \\ &= \left(\frac{4+(-10)}{2}, \frac{7+7}{2}\right) \\ &= (-3, 7) \blacksquare \end{aligned}$$

Graph of an Equation

Given an equation involving the two variables x and y. The **graph** of an equation is the set of ordered pairs (x, y) that satisfy the equation.
A typical procedure for graphing an equation is to plot points and then connect them with a continuous curve as shown in the next examples.

Example 1.7.4
Graph the equation by plotting points: $2x + y = -1$.

Solution.
Writing y in terms of x we find $y = -1 - 2x$. The table below shows some points on the graph of the equation.

x	-2	-1	0	1	2
y	3	1	-1	-3	-5

Next, plot the points and draw a curve through them. See Figure 1.7.4 ■

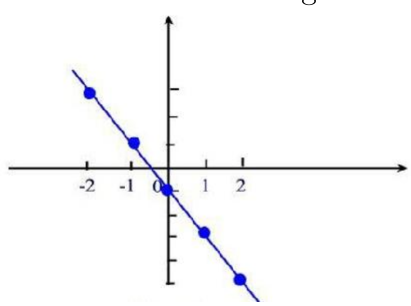


Figure 1.7.4

Example 1.7.5
Graph the equation by plotting points: $y = |x + 3| - 2$.

Solution.
The table below shows some points on the graph of the equation.

x	-6	-5	-4	-3	-2	-1	0
y	1	0	-1	-2	-1	0	1

Next, plot the points and draw a curve through them. See Figure 1.7.5 ■

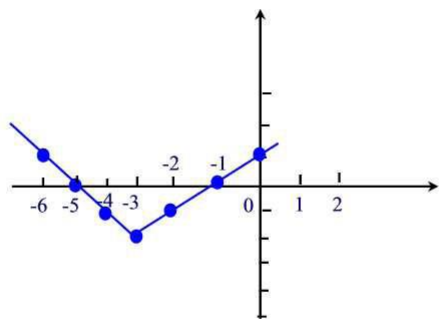


Figure 1.7.5

Example 1.7.6
Graph the equation $y = x^2 - 2x - 8$.

Solution.
The table below shows some points on the graph of the equation.

x	-3	- 2	-1	0	1	2	3	4	5
y	7	0	-5	-8	-9	-8	-5	0	7

Next, plot the points and draw a curve through them. See Figure 1.7.6 ■

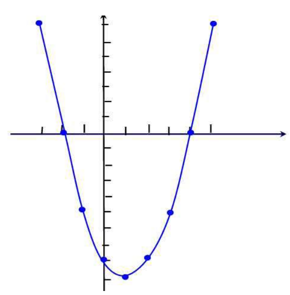


Figure 1.7.6

Intercepts

A point $(x, 0)$ on the graph of an equation is called an **x-intercept.** Geometrically, the $x-$intercept is the point where the graph crosses the $x-$axis. Similarly, a point of the form $(0, y)$ is called a **y-intercept**. This is the point where the graph crosses the $y-$axis.

Example 1.7.7
Find the $x-$ and $y-$intercepts of the graph of $x^2 + y^2 = 4$.

Solution.
Letting $y = 0$ in the given equation we find $x^2 = 4$. Solving for x to obtain $x = \pm 2$. Thus, the $x-$intercepts are the points $(-2, 0)$ and $(2, 0)$. Similarly, setting $x = 0$ to obtain $y^2 = 4$. Solving for y we obtain $y = \pm 2$. So the points $(0, 2)$ and $(0, -2)$ are the $y-$intercepts ■

The Equation of a Circle

By a **circle** we mean the collection of all points in the plane that are at an equal distance to a fixed point called the **center** of the circle. The distance of a point on a circle to its center is called the **radius**. The **diameter** of a circle is the length of a line segment crossing the center and with endpoints on the circle. Thus, the center is the midpoint and as a result a diameter is twice the radius.
Next, we want to find the equation of a circle with center $C(a, b)$ and radius r. For this, let $M(x, y)$ be an arbitrary point on the circle. Then $d(C, M) = r$. By the distance formula, we have

$$(x - a)^2 + (y - b)^2 = r^2.$$

This equation is called the **standard form** of the equation of a circle.

Example 1.7.8
Determine the center and the radius of the circle with equation: $(x - 2)^2 + (y + 4)^2 = 25$.

Solution.
The center is the point $(2, -4)$ and the radius is $r = \sqrt{25} = 5$ ■

Example 1.7.9
Find the equation of the circle with center $C(5, -3)$ and radius $r = 4$. Write the answer in standard form.

Solution.
The equation of the circle is given by

$$(x-5)^2 + (y+3)^2 = 16 \blacksquare$$

Example 1.7.10
Find the equation of the circle with center $C(-2,5)$ and passing through the point $M(1,7)$.

Solution.
The radius of the circle is $r = d(C,M) = \sqrt{(7-5)^2 + (1-(-2))^2} = \sqrt{13}$. Thus, the equation of the circle is

$$(x+2)^2 + (y-5)^2 = 13 \blacksquare$$

Another form of the equation of a circle is known as the **general form** and is given by the equation

$$x^2 + y^2 + Ax + By + C = 0.$$

To find the standard form from the general form we use the **process of completing the square** as shown in the following example.

Example 1.7.11
Find the center and the radius of the circle: $x^2 + y^2 - 6x - 4y + 12 = 0$.

Solution.
We use the method of completing the square:

$$\begin{aligned}(x^2 - 6x) + (y^2 - 4y) &= -12\\ (x^2 - 6x + 9) + (y^2 - 4y + 4) &= -12 + 9 + 4\\ (x-3)^2 + (y-2)^2 &= 1.\end{aligned}$$

Thus, the center is $(3,2)$ and the radius is $r = 1 \blacksquare$

Example 1.7.12
Find the equation of a circle that has diameter with endpoints $(7,-2)$ and $(-3,5)$. Write your answer in standard form.

Solution.
The center of the circle is the midpoint of the given diameter. By the midpoint formula, the coordinates of the center are $(\frac{7-3}{2}, \frac{-2+5}{2}) = (2, \frac{3}{2})$. The radius of the circle is the distance between the center and one of the endpoints. This can be found by using the distance formula

$$d = \sqrt{(2-7)^2 + (\frac{3}{2}+2)^2} = \frac{\sqrt{149}}{2}.$$

The equation of the circle is

$$(x-2)^2 + (y-\frac{3}{2})^2 = \frac{149}{4} \blacksquare$$

Example 1.7.13
Find an equation of a circle that has its center at $(-2, 3)$ and is tangent to the $y-$axis. Write your answer in standard form.

Solution.
The radius of the circle is the distance from the center to the $y-$axis which is the absolute value of the $x-$coordinate of the the center, i.e. $r = 2$. Hence, the equation of the circle is given by

$$(x+2)^2 + (y-3)^2 = 4 \blacksquare$$

Exercises

Exercise 1.7.1
Plot the points: $(2, 4)$, $(-2, 1)$, $(0, -3)$ and $(-5, -3)$.

Exercise 1.7.2
Plot the points: $(-3, -5), (-4, 3), (0, 2), (-2, 0)$.

Exercise 1.7.3
Find the distance between the points whose coordinates are given.
(a) $(6, 4), (-8, 11)$.
(b) $(5, -8), (0, 0)$.
(c) $(\sqrt{3}, \sqrt{8}), (\sqrt{12}, \sqrt{27})$.
(d) $(x, 4x), (-2x, 3x), \quad x < 0$.

Exercise 1.7.4
Find the midpoint of the line segment with the following endpoints.
(a) $(1, -1), (5, 5)$.
(b) $(6, -3), (6, 11)$.
(c) $(1.75, 2.25), (-3.5, 5.57)$.

Exercise 1.7.5
Find the $x-$ and $y-$intercepts of each equation.
(a) $2x + 5y = 12$.
(b) $x = |y| - 4$.
(c) $|x| + |y| = 4$.
(d) $|x - 4y| = 8$.

Exercise 1.7.6
Determine the value(s) of x if the distance between $(x, 0)$ and $(4, 6)$ is equal to 10.

Exercise 1.7.7
Graph the equation: $x - y = 4$.

Exercise 1.7.8
Graph the equation: $y = x^2$.

Exercise 1.7.9
Graph the equation: $y = 2 - x^2$.

Exercise 1.7.10
Find the center and the radius of the circle: $x^2 + y^2 = 36$.

Exercise 1.7.11
Find the center and the radius of the circle: $(x + 2)^2 + (y + 5)^2 = 25$.

Exercise 1.7.12
Find the center and the radius of the circle: $(x - 8)^2 + y^2 = \frac{1}{4}$.

Exercise 1.7.13
Find the equation of the circle centered at $(4, 1)$ and with radius 2. Write your answer in standard form.

Exercise 1.7.14
Find the equation of the circle with center $C(0, 0)$ and passing through the point $M(-3, 4)$.

Exercise 1.7.15
Find the equation of the circle with center $C(1, 3)$ and passing through the point $M(4, -1)$.

Exercise 1.7.16
Find the center and the radius of the circle: $x^2 + y^2 - 6x + 5 = 0$.

Exercise 1.7.17
Find the center and the radius of the circle: $4x^2 + 4y^2 + 4x - 63 = 0$.

Exercise 1.7.18
Find the center and the radius of the circle: $x^2 + y^2 - x + 3y - \frac{15}{4} = 0$.

Exercise 1.7.19
The midpoint of the line segment connecting the points (x, y) and $(-3, -8)$ is $(2, -7)$. Find x and y.

Exercise 1.7.20
Find the equation of the circle that has a diameter with endpoints $(2, 3)$ and $(-4, 11)$. Write your answer in standard form.

Exercise 1.7.21
Find the center and the radius of the circle with equation

$$x^2 + y^2 - 6x - 4y + 12 = 0.$$

Exercise 1.7.22
Find the equation of the circle centered at $(7, 11)$ and tangent to the $x-$axis. Write your answer in standard form.

Exercise 1.7.23
Find a formula for the set of all points (x, y) for which the distance from (x, y) to $(3, 4)$ is 5.

Exercise 1.7.24
Find an equation of a circle that is tangent to both axes, has its center in the second quadrant, and has a radius 3.

Exercise 1.7.25
Plot the points: $(-2, 11), (-1, 8), (0, 5), (1, 2)$, and $(2, -1)$.

Exercise 1.7.26
Find a point (x, x) such that the distance between this point and $(1, 2)$ is 1.

Exercise 1.7.27
Let $(-1, 1)$ be the midpoint of the line segment joining the points (x, y) and $(1, -3)$. Find x and y.

Exercise 1.7.28
Find the intercepts of the curve $x^2 + y^2 - 4x + 6y = 2$.

Exercise 1.7.29
A circle is centered at the point (x, y). The circle is tangent to the $x-$axis at $x = -1$. The circle is completely above the $x-$axis. Also, the radius of the circle is 2. Find x and y.

Exercise 1.7.30
Find the center and the radius of the circle $x^2 + y^2 - 4x + 10y + 20 = 0$.

1.8 Introduction to Complex Numbers

Up until now, you've been told that you can't take the square root of a negative number. That's because you had no numbers that, when squared, were negative. So an equation like $x^2 + 1 = 0$ has no real solutions. Trying to solve this last equation, we end up with $x = \pm\sqrt{-1}$. Thus, solving the equation involves using a new number called i, standing for "imaginary", such that $i = \sqrt{-1}$. It follows that $i^2 = -1$.
With the above definition, we are now in a position to find the square root of a negative number. If a is a positive number then $-a$ is negative and

$$\sqrt{-a} = \sqrt{(-1)a} = \sqrt{-1}\sqrt{a} = i\sqrt{a}.$$

Example 1.8.1
Simplify $\sqrt{-18}$.

Solution.
We have $\sqrt{-18} = \sqrt{9 \cdot 2 \cdot (-1)} = (3\sqrt{2})i$ ■

By a **complex number** we mean a number that can be written in the form $a + bi$. We call a the **real part** and b the **imaginary part**. When $a = 0$ we say that the number is **purely imaginary**.

Example 1.8.2
Write the number $\sqrt{-37} - 3$ in the form $a + bi$.

Solution.
Since $\sqrt{-37} = \sqrt{37(-1)} = (\sqrt{37})i$, we have $\sqrt{-37} - 3 = -3 + (\sqrt{37})i$ ■

The Arithmetic of Complex Numbers

When a number system is extended, the arithmetic operations must be defined for the new numbers, and the important properties of the operations should still hold. For example, addition of whole numbers is commutative. This means that we can change the order in which two whole numbers are added and the sum is the same: $3 + 5 = 8$ and $5 + 3 = 8$.
We need to define the four arithmetic operations on complex numbers.

- **Equality of Complex Numbers**

Two complex numbers $a + bi$ and $c + di$ are equal if and only if $a = c$ and $b = d$.

Example 1.8.3
Find x so that $3 + (4 - x)i = 3 + i$.

Solution.
By the equality of complex numbers, we must have $4 - x = 1$. Solving for x we find $x = 3$ ■

• Addition and Subtraction
To add or subtract two complex numbers, you add or subtract the real parts and the imaginary parts:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi) - (c + di) = (a - c) + (b - d)i.$$

Example 1.8.4
Perform the indicated operation (a) $(3 - 5i) + (6 + 7i)$ (b) $i - (3 - 4i)$.

Solution.
(a) $(3 - 5i) + (6 + 7i) = (3 + 6) + (-5 + 7)i = 9 + 2i$.
(b) $i - (3 - 4i) = (0 - 3) + (1 - (-4))i = -3 + 5i$ ■

Remark 1.8.1
The operations of addition and subtraction are the same as combining similar terms in expressions that have a variable. For example, if we were to simplify the expression $(3 - 5x) + (6 + 7x)$ by combining similar terms, then the constants 3 and 6 would be combined, and the terms $-5x$ and $7x$ would be combined to yield $9 + 2x$.

• Multiplication
The formula for multiplying two complex numbers is

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

You do not have to memorize this formula, because you can arrive at the same result by treating the complex numbers like expressions with a variable, multiply them as usual by using the FOIL method and then combine like terms. The only difference is that powers of i do simplify (using $i^2 = -1$), while powers of x do not.

Example 1.8.5
Multiply $(2+3i)(4+7i)$.

Solution.
We have

$$\begin{aligned}(2+3i)(4+7i) &= (2)(4)+(2)(7i)+(4)(3i)+(3i)(7i)\\ &= 8+14i+12i+21(-1)\\ &= (8-21)+(14+12)i = -13+26i \blacksquare\end{aligned}$$

• **Complex Conjugate**
The **conjugate** (or complex conjugate) of the complex number $a+bi$ is $a-bi$. We denote the conjugate of $a+bi$ by $\overline{a+bi} = a-bi$.

Multiplying $a+bi$ by its conjugate we find

$$(a+bi)(a-bi) = (a^2+b^2)+0i = a^2+b^2.$$

Thus, a complex number times its conjugate is always real; i.e., its imaginary part is zero.

Example 1.8.6
Find the conjugate of (a) $-3-4i$ and (b) $3+5i$.

Solution.
(a) $\overline{-3-4i} = -3+4i$ (b) $\overline{3+5i} = 3-5i \blacksquare$

• **Division of Complex Numbers**
By the ratio $\frac{a+bi}{c+di}$ we mean a complex number $\alpha+\beta i$ such that

$$\frac{a+bi}{c+di} = \alpha+\beta i \tag{1.8.1}$$

Cross multiply and simplify to obtain

$$\begin{aligned}a+bi &= (c+di)(\alpha+\beta i)\\ &= (c\alpha-d\beta)+(d\alpha+c\beta)i.\end{aligned}$$

Thus, $c\alpha - d\beta = a$ and $d\alpha + c\beta = b$. Solve this system of two equations for α and β, using the method of elimination, to obtain

$$\alpha = \frac{ac + bd}{c^2 + d^2} \quad and \quad \beta = \frac{bc - ad}{c^2 + d^2}.$$

One can easily see that the right hand side of (1.8.1) is obtained by multiplying $a + bi$ and $c + di$ by the conjugate $c - di$.(Prove that)

Example 1.8.7
Find $\frac{2+6i}{4+i}$.

Solution.

$$\frac{2 + 6i}{4 + i} = \frac{(2 + 6i)}{(4 + i)}\frac{(4 - i)}{(4 - i)} = \frac{14 + 22i}{17} = \frac{14}{17} + \frac{22}{17}i \blacksquare$$

Exercises

Exercise 1.8.1
Write the given complex number in the form $z = a + bi$.
(a) $2 + \sqrt{-9}$ (b) $4 - \sqrt{-121}$ (c) $-\sqrt{-100}$.

Exercise 1.8.2
Simplify and then write the complex number in the form $z = a + bi$.
(a) $(2 + 5i) + (3 + 7i)$.
(b) $(-5 - i) + (9 - 2i)$.
(c) $(-3 + i) - (-8 + 2i)$.

Exercise 1.8.3
Simplify and then write the complex number in the form $z = a + bi$.
(a) $8i - (2 - 3i)$.
(b) $(4i - 5) - 2$.
(c) $3(2 + 7i) + 5(2 - i)$.

Exercise 1.8.4
Simplify and then write the complex number in the form $z = a + bi$.
(a) $(2 + 3i)(4 - 5i)$.
(b) $(5 - 3i)(-2 - 4i)$.
(c) $(5 + 7i)(5 - 7i)$.

Exercise 1.8.5
Simplify and then write the complex number in the form $z = a + bi$.
(a) $(8i + 11)(-7 + 5i)$.
(b) $(9 - 12i)(15i + 7)$.

Exercise 1.8.6
Write each expression as a complex number in the form $z = a + bi$.
(a) $\frac{4+i}{3+5i}$ (b) $\frac{1}{-8+i}$ (c) $\frac{1}{7-3i}$.

Exercise 1.8.7
Write each expression as a complex number in the form $z = a + bi$.
(a) $\frac{2i}{11+i}$ (b) $\frac{6+i}{i}$ (c) $(-5 + 7i)^2$

Exercise 1.8.8
Write each expression as a complex number in the form $z = a + bi$.
(a) $(1 - i) - 2(4 + i)^2$ (b) $(1 - i)^3$ (c) $(2i)(8i)$ (d) $(-6i)(-5i)^2$.

Exercise 1.8.9
Simplify and write the complex number as $i, -i$, or -1.
(a) $-i^{40}$ (b) i^{223} (c) i^{2001} (d) i^0 (e) i^{-1}.

Exercise 1.8.10
Simplify each product.
(a) $\sqrt{-1}\sqrt{-4}$ (b) $\sqrt{-3}\sqrt{-121}$.

Exercise 1.8.11
Simplify each product.
(a) $(4+\sqrt{-81})(4-\sqrt{-81})$ (b) $(5+\sqrt{-16})^2$.

Exercise 1.8.12
The absolute value of a complex number $a+bi$ is the real number

$$|a+bi| = \sqrt{a^2+b^2}.$$

Find the indicated absolute value of each complex number.
(a) $|5+12i|$ (b) $|7-4i|$ (c) $|-3i|$.

Exercise 1.8.13
Establish that $|a+bi| = |a-bi|$. That is, the absolute value of a complex number and the absolute value of its conjugate are equal.

Exercise 1.8.14
Show that $z-\overline{z}$ is purely imaginary and $z+\overline{z}$ is a real number.

Exercise 1.8.15
Let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$. Show the following:
(a) $\overline{z_1+z_2} = \overline{z_1}+\overline{z_2}$.
(b) $\overline{z_1 \cdot z_2} = \overline{z_1}\cdot\overline{z_2}$.

Exercise 1.8.16
Show that if $x = 1+i\sqrt{3}$ then $x^2-2x+4=0$.

Exercise 1.8.17
Write the given complex number in the form $z = a+bi$.
(a) $(3+5i)+(4-2i)$ (b) $(3+5i)-(4-2i)$ (c) $(3+5i)(4+2i)$ (d) i^{23}.

Exercise 1.8.18
Simplify and then write the complex number in the form $z = a + bi$.
(a) $\frac{3+5i}{1-2i}$.
(b) $\frac{7+3i}{4i}$.

Exercise 1.8.19
Simplify and then write the complex number in the form $z = a + bi$.

$$(\sqrt{12} - \sqrt{-3})(3 + \sqrt{-4}).$$

Exercise 1.8.20
Find the indicated absolute value of each complex number.

(a) $|3 + 4i|$ (b) $|8 - 5i|$.

Exercise 1.8.21
Show that $z \cdot \overline{z}$ is a real number.

Exercise 1.8.22
(a) Show that if $z = \overline{z}$ then z is a real number.
(b) Show that if z is a real number then $z = \overline{z}$.

Chapter 2

Functions and Related Topics

Functions play an important role in mathematics. In fact, they play an important role in all forms of reality. Functions are a numerical description of the actions we do everyday such as throwing a ball, or driving a car, or figuring out the interest we have accrued in our savings accounts. Functions describe just about any action, reaction, or interaction through the use of defining variables, both dependent and independent. Everything and anything can be described by a function. Learning about functions is the primary, fundamental step to learning about anything in math that follows.

2.1 Functions and Function Notation

Functions play a crucial role in mathematics. A function describes how one quantity depends on others. More precisely, when we say that a quantity y is a **function** of a quantity x we mean a rule that assigns to every possible value of x exactly one value of y. We call x the **input** and y the **output**. In **function notation** we write

$$y = f(x).$$

Since y depends on x, it makes sense to call x the **independent variable** and y the **dependent variable**.
In applications of mathematics, functions are often representations of real world phenomena. Thus, the functions in this case are referred to as **mathematical models.** If the set of input values is a finite set then the models are known as **discrete** models. Otherwise, the models are known as **continuous** models. For example, if H represents the temperature after t hours for a specific day, then H is a discrete model. If A is the area of a circle of radius r then A is a continuous model.

The Four Different Ways of Representating Functions
There are four common ways in which functions are presented and used: By verbal descriptions, by tables, by graphs, and by formulas.

Example 2.1.1
The sales tax on an item is 6%. So if p denotes the price of the item and C the total cost of buying the item then an item that is sold at \$ 1 will cost is $C(1) = 1 + (0.06)(1) = \$1.06$. Likewise, if the item is sold at \$2 then the cost of buying the item is $2 + (0.06)(2) = \$2.12$, or $C(2) = \$2.12$, and so on. Thus, we have a relationship between the quantities C and p such that each value of p determines exactly one value of C. In this case, we say that C is a function of p. Describes this function using words, a table, a graph, and a formula.

Solution.
•**Words:** To find the total cost, multiply the price of the item by 0.06 and add the result to the price.
•**Table:** The chart below gives the total cost of buying an item at price p as a function of p for $1 \leq p \leq 6$.

p	1	2	3	4	5	6
C	1.06	2.12	3.18	4.24	5.30	6.36

•**Graph:** The graph of the function C is obtained by plotting the data in the above table. See Figure 2.1.1.
•**Formula:** The formula that describes the relationship between C and p is given by

$$C(p) = 1.06p \blacksquare$$

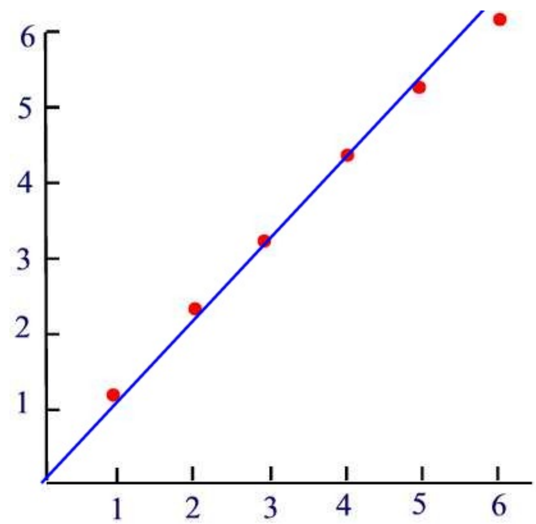


Figure 2.1.1

Recognizing a Function from a Table

A table can be viewed as a collection of ordered pairs (x, y). Thus, for a collection of data to define a function we need to show that every first component x corresponds to exactly one component y. Hence, if there are ordered pairs with the same x value but different y values then the collection of ordered pairs is *not* a function.

Example 2.1.2
Identify the set of ordered pairs (x, y) that define y as a function of x.
(a) $\{(5, 10), (3, -2), (4, 7), (5, 8)\}$.
(b) $\{(2, 2), (3, 3), (7, 2)\}$.

Solution.
(a) The first set does not define a function since the ordered pairs $(5, 10)$ and

$(5, 8)$ have the same first component with different second components.
(b) This set defines a function since all the first components are different ■

Recognizing a Function from an Equation

Suppose that an equation in the variables x and y is given. If for a given value of x, you solve the equation for y and you get exactly one value then the equation defines a function.

Example 2.1.3
Identify the equations that define y as a function of x.
(a) $x^2 - 2y = 2$.
(b) $x^2 + y^2 = 1$.

Solution.
(a) Solving the equation for y we find $y = \frac{x^2}{2} - 1$. Thus, each value of x yields exactly one value of y. This shows that y is a function of x.
(b) Solving for y to obtain $y = \pm\sqrt{1 - x^2}$. Thus, if we let $x = 0$ then $y = \pm 1$. Hence, y is not a function of x. Usually, when the output is raised to an even power, then the equation relating x and y is not a function ■

Recognizing a Function from a Graph

Next, suppose that the graph of a relationship between two quantities x and y is given. To say that y is a function of x means that for each value of x there is exactly one value of y. Graphically, this means that each vertical line must intersect the graph at most once. Hence, to determine if a graph represents a function one uses the following test:

Vertical Line Test: A graph is a function if and only if every vertical line crosses the graph at most once.

According to the vertical line test and the definition of a function, if a vertical line cuts the graph more than once, the graph could not be the graph of a function since we have multiple y values for the same $x-$value and this violates the definition of a function.

Example 2.1.4
Which of the graphs (a), (b), (c) in Figure 2.1.2 represent y as a function of

x?

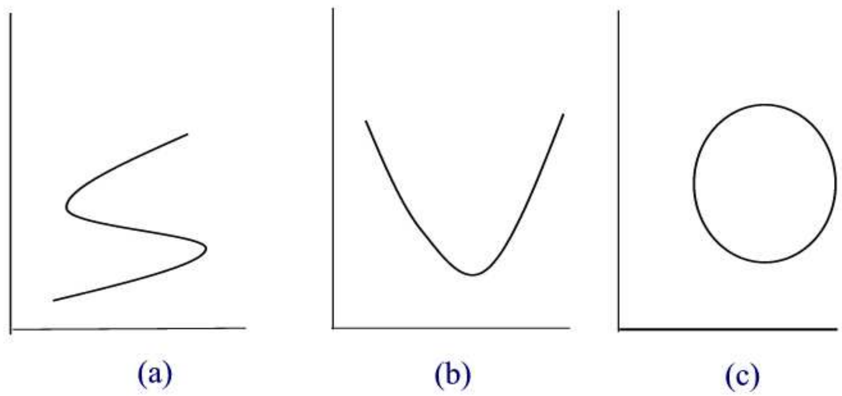


Figure 2.1.2

Solution.
By the vertical line test, (b) represents a function whereas (a) and (c) fail to represent functions since one can find a vertical line that intersects the graph more than once ■

Evaluating a Function

By **evaluating** a function, we mean figuring out the output value corresponding to a given input value. Thus, notation like $f(10) = 4$ means that the function's output, corresponding to the input 10, is equal to 4.
If the function is given by a formula, say of the form $y = f(x)$, then to find the output value corresponding to an input value a we replace the letter x in the formula of f by the input a and then perform the necessary algebraic operations to find the output value.

Example 2.1.5
Let $g(x) = \frac{x^2+1}{5+x}$. Evaluate the following expressions:

(a) $g(2)$ (b) $g(a)$ (c) $g(a) - 2$ (d) $g(a) - g(2)$.

Solution.
(a) $g(2) = \frac{2^2+1}{5+2} = \frac{5}{7}$

(b) $g(a) = \frac{a^2+1}{5+a}$
(c) $g(a) - 2 = \frac{a^2+1}{5+a} - 2\frac{5+a}{5+a} = \frac{a^2-2a-9}{5+a}$
(d) $g(a) - g(2) = \frac{a^2+1}{5+a} - \frac{5}{7} = \frac{7(a^2+1)}{7(5+a)} - \frac{5}{7}\frac{5+a}{5+a} = \frac{7a^2-5a-18}{7a+35}$ ■

Domain and Range of a Function

If we try to find the possible input values that can be used in the function $y = \sqrt{x-2}$ we see that we must restrict x to the interval $[2, \infty)$, that is $x \geq 2$. Similarly, the function $y = \frac{1}{x^2}$ takes only certain values for the output, namely, $y > 0$. Thus, a function is often defined for certain values of x and the dependent variable often takes certain values.
The above discussion leads to the following definitions: By the **domain** of a function we mean all possible input values that yield one output value. Graphically, the domain is part of the horizontal axis. The **range** of a function is the collection of all possible output values. The range is part of the vertical axis.
When finding the domain of a function, ask yourself what values can't be used. Your domain is everything else. There are simple basic rules to consider:
• The domain of all polynomial functions, i.e. functions of the form $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, where n is non-negative integer, is the set of all real numbers $\mathbb{R}$.
• Square root functions can not contain a negative underneath the radical. Set the expression under the radical greater than or equal to zero and solve for the variable. This will be your domain.
• Rational functions, i.e. ratios of two functions, determine for which input values the numerator and denominator are not defined and the domain is everything else. For example, make sure not to divide by zero!

Example 2.1.6
Find, algebraically, the domain and the range of each of the following functions. Write your answers in interval notation:
(a) $y = \pi x^2$ (b) $y = \frac{1}{\sqrt{x-4}}$ (c) $y = 2 + \frac{1}{x}$.

Solution.
(a) Since the function is a polynomial, its domain is the interval $(-\infty, \infty)$. To find the range, solve the given equation for x in terms of y obtaining $x = \pm\sqrt{\frac{y}{\pi}}$. Thus, x exists for $y \geq 0$. So the range is the interval $[0, \infty)$.
(b) The domain of $y = \frac{1}{\sqrt{x-4}}$ consists of all numbers x such that $x - 4 > 0$

or $x > 4$. That is, the interval $(4, \infty)$. Before finding the range, notice that $y \geq 0$. Now, to find the range, we solve for x in terms of $y > 0$ obtaining $x = 4 + \frac{1}{y^2}$. x exists for all $y > 0$. Thus, the range is the interval $(0, \infty)$.
(c) The domain of $y = 2 + \frac{1}{x}$ is the interval $(-\infty, 0) \cup (0, \infty)$. To find the range, write x in terms of y to obtain $x = \frac{1}{y-2}$. The values of y for which this later formula is defined is the range of the given function, that is, $(-\infty, 2) \cup (2, \infty)$ ■

Remark 2.1.1
A few words in order about the previous example. If $y = f(x)$ then solving for x in terms of y is equivalent to finding the inverse function f^{-1} of f, a concept that will be introduced in Section 2.4. For an inverse function, the domain of f^{-1} is the range of f.

Exercises

Exercise 2.1.1
Given $f(x) = 3x^2 - 1$, find
(a) $f(-4)$ (b) $f\left(\frac{1}{3}\right)$ (c) $f(-a)$ (d) $f(x+h)$ (e) $f(x+h) - f(x)$.

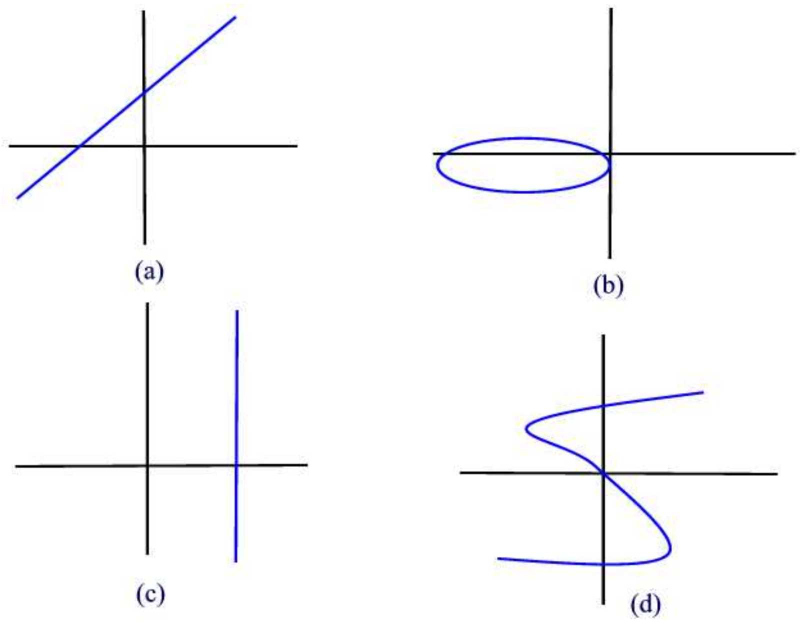
Exercise 2.1.2
Given $f(x) = \frac{x}{|x|}$, find
(a) $f(4)$ (b) $f(-2)$ (c) $f(x), x > 0$ (d) $f(x), x < 0$.

Exercise 2.1.3
Identify the equations that define y as a function of x.
(a) $2x + 3y = 7$.
(b) $-x + y^2 = 2$.
(c) $y = 4 \pm \sqrt{x}$.
(d) $y^2 = x^2$.

Exercise 2.1.4
Identify the collection of ordered pairs (x, y) that define y as a function of x.
(a) $\{(2,3), (5,1), (-4,3), (7,11)\}$.
(b) $\{(5,10), (3,-2), (4,7), (5,8)\}$.
(c) $\{(1,0), (2,0), (3,0)\}$.

Exercise 2.1.5
Determine the domain and the range of the function. Write answers in interval notation.
(a) $f(x) = 3x - 4$.
(b) $g(x) = x^2 + 2,\ x \geq 0$.
(c) $h(x) = \frac{4}{x+2}$.
(d) $i(x) = \sqrt{4 - x^2}$.
(e) $j(x) = \frac{1}{\sqrt{x+4}}$.

Exercise 2.1.6
Use the vertical line test to determine which of the following graphs are graphs of functions.



Exercise 2.1.7
A manufacturer produces a product at a cost of \$22.80 per unit. The manufacturer has a fixed cost of \$400,000 per day. Each unit retails for \$37.00. Let x represent the number of units produced in a 5-day period.
(a) Write the total cost C as a function of x.
(b) Write the revenue R as a function of x.
(c) Write the profit P as a function of x.

Exercise 2.1.8
An open box is to be made from a square piece of cardboard having dimensions 30 inches by 30 inches by cutting out squares of area x^2 from each corner.
(a) Express the volume V of the box as a function of x.
(b) State the domain of V.

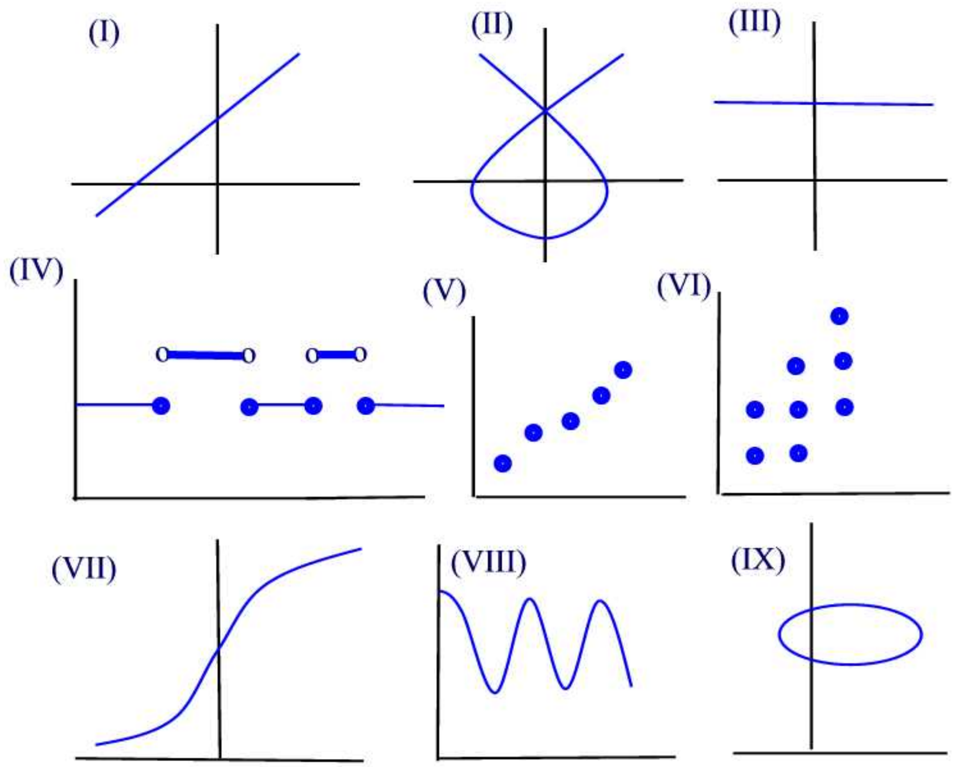
Exercise 2.1.9
If $f(x) = x^2 - x - 5$ and $f(c) = 1$, find the value of c.

Exercise 2.1.10
Determine whether 1 is in the range of $f(x) = \frac{x-1}{x+1}$.

Exercise 2.1.11
Determine whether 0 is in the range of $f(x) = \frac{1}{x-3}$.

Exercise 2.1.12
Suppose that weight w is a function of calorie intake c. Write the relationship in function notation.

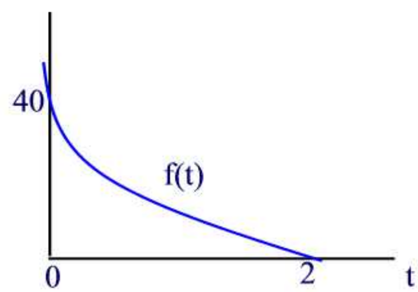
Exercise 2.1.13
Which of the graphs in the figure below represent y as a function of x?(Note that an open circle indicates a point that is not included in the graph; a solid dot indicates a point that is included in the graph.)



Exercise 2.1.14
Using the table below, graph $n = f(A)$, the number of gallons of paint needed to cover a house of area A. Identify the independent and dependent variables.

A	0	250	500	750	1000	1250	1500
n	0	1	2	3	4	5	6

Exercise 2.1.15
Use the figure below to fill in the missing values:
(a) $f(0) =?$ (b) $f(?) = 0$.



Exercise 2.1.16
(a) You are going to graph $p = f(w)$. Which variable goes on the horizontal axis?
(b) If $f(-4) = 10$, give the coordinates of a point on the graph of f.
(c) If 6 is a solution of the equation $f(w) = 1$, give a point on the graph of f.

Exercise 2.1.17
Label the axes for a sketch to illustrate the following statement:"Graph, the pressure, p, of a gas as a function of its volume, v, where p is in pounds per square inch and v is in cubic inches."

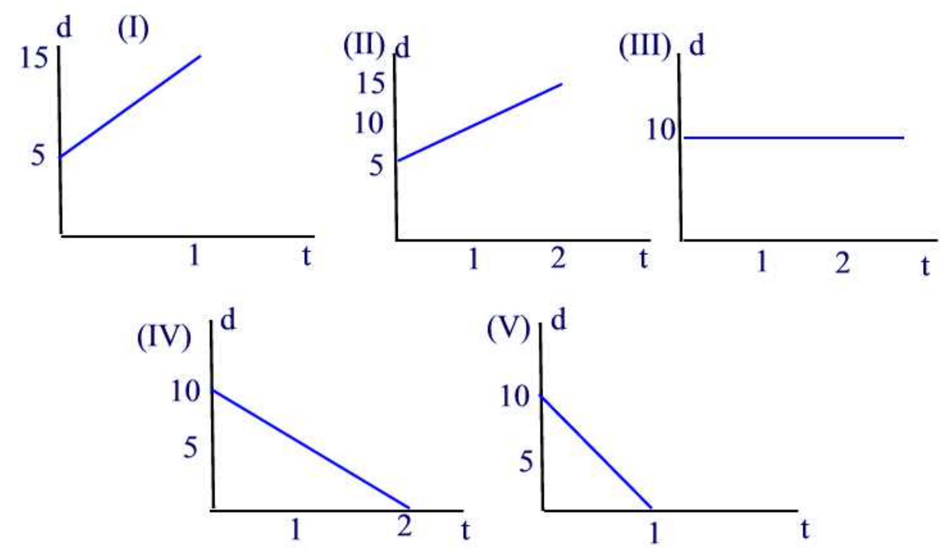
Exercise 2.1.18
You are looking at the graph of y, a function of x.
(a) What is the maximum number of times that the graph can intersect the $y-$axis? Explain.
(b) Can the graph intersect the $x-$axis an infinite number of times? Explain.

Exercise 2.1.19
Let $f(t)$ be the number of people, in millions, who own cell phones t years after 1990. Explain the meaning of the following statements:
(a) $f(10) = 100.3$ (b) $f(a) = 20$ (c) $f(20) = b$ (d) $f(t) = n$.

Exercise 2.1.20
According to the Social Security Administration (SSA), the most popular name for male babies in 2001 was Jacob. However, this popularity is a recent phenomenon. The table below gives the popularity ranking r of the name Jacob for baby boys born t years after 1990 (so $t = 0$ is 1990). Ranking of 1 means most popular, a ranking of 2 means second most popular, and so forth.
(a) In what year did Jacob first become the most popular name?
(b) In what year did Jacob first enter the top ten most popular names?
(c) In what year(s) was Jacob among the top five most popular names?

t	0	1	2	3	4	5	6	7	8	9	10	11
r	20	16	15	11	7	5	3	2	2	1	1	1

Exercise 2.1.21
Match each story about a bike ride to one of the graphs (i)-(v), where d represents the distance from home and t is time in hours since the start of the ride.(A graph may be used more than once.)
(a) Starts 5 miles from home and rides 5 miles per hour away from home.
(b) Starts 5 miles from home and rides 10 miles per hour away from home.
(c) Starts 10 miles from home and arrives home one hour later.
(d) Starts 10 miles from home and is halfway home after one hour.
(e) Starts 5 miles from home and is 10 miles from home after one hour.



Exercise 2.1.22
A chemical company spends \$2 million to buy machinary before it starts producing chemicals. Then it spends \$0.5 million on raw materials for each million liters of chemical produced.
(a) The number of liters produced ranges from 0 to 5 million. Make a table showing the relationship between the number of million liters produced, l, and the total cost, C, in millions of dollars, to produce that number of million liters.
(b) Find a formula that expresses C as a function of l.

Exercise 2.1.23
A person leaves home and walks due west for a time and then walks due north.
(a) The person walks 10 miles in total. If w represents the (variable) distance west she walks, and D represents her (variable) distance from home at the end of her walk, is D a function of w? Why or why not?
(b) Suppose now that x is the distance that she walks in total. Is D a function of x? Why or why not?

Exercise 2.1.24
Determine the range of $y = f(x) = 1 + \frac{2}{x-1}$ algebraically.

Exercise 2.1.25
Find the $\frac{f(x+h)-f(x)}{h}$ if $f(x) = \frac{x}{x-1}$.

Exercise 2.1.26
Find the $x-$ and $y-$intercepts of $y^2 = x + 5$. Plot them and connect them with a curve. Does the curve represent a function?

Exercise 2.1.27
Show that $x = |y + 3|$ does not define y as a function of x.

Exercise 2.1.28
Determine whether the table below defines y as a function of x.

x	0	1	5	1	3
y	0	-1	0.5	5	7

Exercise 2.1.29
Find the domain of $f(x) = \sqrt{-x^2 + x + 6}$.

Exercise 2.1.30
Suppose that f has the domain $2 \leq x \leq 11$. For what values of x, $f(5x - 8)$ is defined?

2.2 The Algebra of Functions

In this section we are going to construct new functions from old ones using the operations of addition, subtraction, multiplication, division, and composition.
Let $f(x)$ and $g(x)$ be two given functions. Then for all x in the common domain of these two functions we define new functions as follows:

- **Sum**: $(f+g)(x) = f(x) + g(x)$.
- **Difference**: $(f-g)(x) = f(x) - g(x)$.
- **Product**: $(f \cdot g)(x) = f(x) \cdot g(x)$.
- **Quotient**: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ provided that $g(x) \neq 0$.

Example 2.2.1
Let $f(x) = x+1$ and $g(x) = \sqrt{x+3}$. Find the common domain and then find a formula for each of the functions $f+g, f-g, f \cdot g, \frac{f}{g}$.

Solution.
The domain of $f(x)$ consists of all real numbers whereas the domain of $g(x)$ consists of all numbers $x \geq -3$. Thus, the common domain is the interval $[-3, \infty)$. For any x in this domain we have

$$\begin{aligned}
(f+g)(x) &= x+1+\sqrt{x+3}.\\
(f-g)(x) &= x+1-\sqrt{x+3}.\\
(f \cdot g)(x) &= x\sqrt{x+3}+\sqrt{x+3}.\\
\left(\frac{f}{g}\right)(x) &= \frac{x+1}{\sqrt{x+3}} \text{ provided } x > -3 \blacksquare
\end{aligned}$$

Example 2.2.2
Let $f(x) = x^2 - 3x + 2$ and $g(x) = 2x - 4$. Evaluate the indicated function.

(a) $(f+g)\left(\frac{1}{2}\right)$ (b) $(f-g)(-1)$ (c) $(fg)\left(\frac{2}{5}\right)$ (d) $\left(\frac{f}{g}\right)(11)$.

Solution.
(a) $f(\frac{1}{2}) = \frac{3}{4}$ and $g(\frac{1}{2}) = -3$ so that $(f+g)\left(\frac{1}{2}\right) = \frac{3}{4} - 3 = -\frac{9}{4}$.
(b) $f(-1) = 6$ and $g(-1) = -6$ so that $(f-g)(-1) = 6 - (-6) = 12$.
(c) $f(\frac{2}{5}) = \frac{24}{25}$ and $g(\frac{2}{5}) = -\frac{16}{5}$ so that $(fg)\left(\frac{2}{5}\right) = -\frac{384}{125}$.
(d) $f(11) = 90$ and $g(11) = 18$ so that $\left(\frac{f}{g}\right)(11) = \frac{90}{18} = 5$ $\blacksquare$

Example 2.2.3
Suppose the functions f and g are given in numerical forms. Complete the following table:

x	-2	-1	0	1	2	3
$f(x)$	8	2	7	-1	-5	-3
$g(x)$	-1	-5	-11	7	8	9
$(f+g)(x)$						
$(f-g)(x)$						
$(f\cdot g)(x)$						
$(\frac{f}{g})(x)$						

Solution.

x	-2	-1	0	1	2	3
$f(x)$	8	2	7	-1	-5	-3
$g(x)$	-1	-5	-11	7	8	9
$(f+g)(x)$	7	-3	-4	6	3	6
$(f-g)(x)$	9	7	18	-8	-13	-12
$(f\cdot g)(x)$	-8	-10	-77	-7	-40	-27
$(\frac{f}{g})(x)$	-8	$-\frac{2}{5}$	$-\frac{7}{11}$	$-\frac{1}{7}$	$-\frac{5}{8}$	$-\frac{1}{3}$

Example 2.2.4
Using the graphs of the functions f and g given in Figure 2.2.1, evaluate, if possible, the following
(a) $(f+g)(-1)$ (b) $(f-g)(1)$ (c) $(f\cdot g)(2)$ (d) $\left(\frac{f}{g}\right)(0)$.

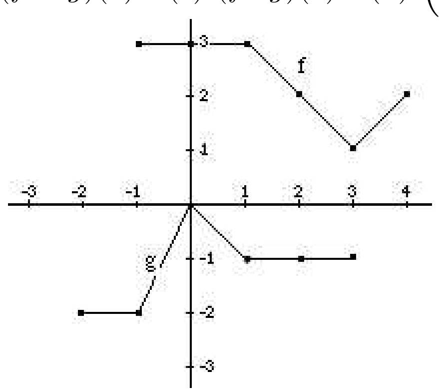


Figure 2.2.1

Solution.
(a) Since $f(-1) = 3$ and $g(-1) = -2$, $(f + g)(-1) = 3 - 2 = 1$.
(b) Since $f(1) = 3$ and $g(1) = -1$, $(f - g)(1) = 3 - (-1) = 4$.
(c) Since $f(2) = 2$ and $g(2) = -1$, $(f \cdot g)(2) = -2$.
(d) Since $f(0) = 3$ and $g(0) = 0$, $\left(\frac{f}{g}\right)(0)$ is undefined ■

Difference Quotient

Difference quotients are what they say they are. They involve a difference and a quotient. This is a common expression in calculus. It is used when introducing a concept called the derivative. Geometrically, a difference quotient is the slope of a secant line between two points on a curve. The formula for the difference quotient is:

$$\frac{f(x+h) - f(x)}{h}, \ h \neq 0.$$

Example 2.2.5
Find the difference quotient of the function $f(x) = x^2$.

Solution.
Since $f(x + h) = (x + h)^2 = x^2 + 2hx + h^2$, we obtain

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{(x^2 + 2hx + h^2) - x^2}{h} \\ &= \frac{h(2x + h)}{h} \\ &= 2x + h \ ■\end{aligned}$$

Example 2.2.6
Find the difference quotient of the function $f(x) = \frac{1}{x+2}$.

Solution.
We have

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{\frac{1}{x+h+2} - \frac{1}{x+2}}{h} = -\frac{\frac{h}{(x+h+2)(x+2)}}{h} \\ &= -\frac{1}{(x+h+2)(x+2)} \ ■\end{aligned}$$

Example 2.2.7
Find the difference quotient of the function $f(x) = \sqrt{x}$.

Solution.
We have

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} =& \frac{\sqrt{x+h}-\sqrt{x}}{h} \\
=& \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{(\sqrt{x+h}+\sqrt{x})h} \\
=& \frac{h}{(\sqrt{x+h}+\sqrt{x})h} = \frac{1}{\sqrt{x+h}+\sqrt{x}} \blacksquare
\end{aligned}$$

Composition of Functions

Suppose we are given two functions f and g such that the range of g is contained in the domain of f so that the output of g can be used as input for f. We define a new function, called the **composition** of f followed by g, by the formula

$$(g \circ f)(x) = g(f(x)).$$

Using a Venn diagram (See Figure 2.2.2) we have

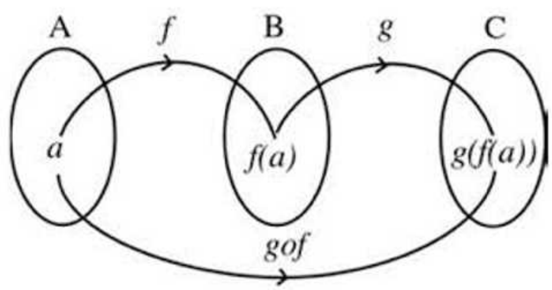


Figure 2.2.2

Example 2.2.8
Suppose that $f(x) = 2x + 1$ and $g(x) = x^2 - 3$.
(a) Find $f \circ g$ and $g \circ f$.
(b) Calculate $(f \circ g)(5)$ and $(g \circ f)(-3)$.
(c) Are $f \circ g$ and $g \circ f$ equal?

Solution.
(a) $(f \circ g)(x) = f(g(x)) = f(x^2 - 3) = 2(x^2 - 3) + 1 = 2x^2 - 5$. Similarly, $(g \circ f)(x) = g(f(x)) = g(2x + 1) = (2x + 1)^2 - 3 = 4x^2 + 4x - 2$.
(b) $(f \circ g)(5) = 2(5)^2 - 5 = 45$ and $(g \circ f)(-3) = 4(-3)^2 + 4(-3) - 2 = 22$.
(c) $f \circ g \neq g \circ f$ ■

With only one function you can build new functions using composition of the function with itself. Also, there is no limit on the number of functions that can be composed.

Example 2.2.9
Suppose that $f(x) = 2x + 1$ and $g(x) = x^2 - 3$.
(a) Find $(f \circ f)(x)$.
(b) Find $[f \circ (f \circ g)](x)$.

Solution.
(a) $(f \circ f)(x) = f(f(x)) = f(2x + 1) = 2(2x + 1) + 1 = 4x + 3$.
(b) $[f \circ (f \circ g)](x) = f(f(g(x))) = f(f(x^2 - 3)) = f(2x^2 - 5) = 2(2x^2 - 5) + 1 = 4x^2 - 9$ ■

Decomposing Functions

Next, we explore how we can **decompose** a composite function into two seperate functions. In general, there might be several ways for decomposing a function.

Example 2.2.10
Decompose $h(x) = \sqrt{x^2 + 1}$ into simpler functions.

Solution.
We want to find $f(x)$ and $g(x)$ such that $h(x) = f(g(x))$. One such an answer is $f(x) = \sqrt{x + 1}$ and $g(x) = x^2$. Another answer is $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$ ■

Increasing, Decreasing, and Constant Functions

A function f is said to be **increasing** on an interval I if $f(a) < f(b)$ whenever $a < b$ where $a, b \in I$. Geometrically, the graph of f goes up from left to right. A function f is said to be **decreasing** if $f(a) < f(b)$ whenever $a < b$ where $a, b \in I$. Geometrically, the graph of f falls down as x moves from left to right. A function f is said to be **constant** in I if $f(x) = k$ for all $x \in I$. In this case, the graph of f is a horizontal line on I.

Example 2.2.11
Determine the intervals where the function, given in Figure 2.2.3, is increasing and decreasing.

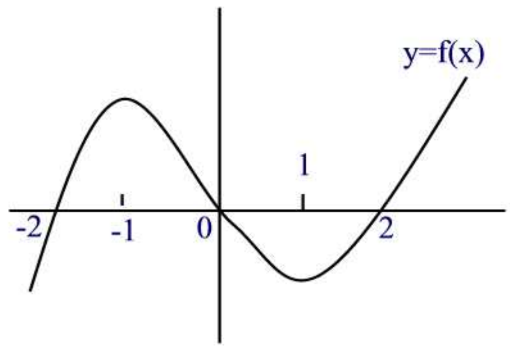


Figure 2.2.3

Solution.
The function is increasing on $(-\infty, -1) \cup (1, \infty)$ and decreasing on the interval $(-1, 1)$ ■

Example 2.2.12
Show that the function $f(x) = 2x - 3$ is always increasing by using the definition of increasing.

Solution.
Suppose $a < b$. Then $f(a) - f(b) = 2a - 3 - 2b + 3 = 2(a - b) < 0$. That is, $f(a) < f(b)$ which means that f is increasing ■

Example 2.2.13
Show that the function $f(x) = \frac{1}{x}$ is decreasing for $x > 0$ by using the definition of decreasing.

Solution.
Suppose $0 < a < b$. Then $f(a) - f(b) = \frac{1}{a} - \frac{1}{b} = \frac{b-a}{ab} > 0$. Hence, $f(a) > f(b)$ which means that f is decreasing ■

Exercises

Exercise 2.2.1
Use the given functions f and g to find $f+g, f-g, fg$, and $\frac{f}{g}$. State the domain of each.

$$f(x)=x^2-2x-15, g(x)=x+3.$$

Exercise 2.2.2
Use the given functions f and g to find $f+g, f-g, fg$, and $\frac{f}{g}$. State the domain of each.

$$f(x)=x^3-2x^2+7x, g(x)=x.$$

Exercise 2.2.3
Use the given functions f and g to find $f+g, f-g, fg$, and $\frac{f}{g}$. State the domain of each.

$$f(x)=2x^2+5x-7, g(x)=2x^2+3x-5.$$

Exercise 2.2.4
Use the given functions f and g to find $f+g, f-g, fg$, and $\frac{f}{g}$. State the domain of each.

$$f(x)=\sqrt{4-x^2}, g(x)=2+x.$$

Exercise 2.2.5
Evaluate the indicated function, where $f(x)=x^2-3x+2$ and $g(x)=2x-4$.
(a) $(f+g)(5)$ (b) $(f+g)\left(\frac{2}{3}\right)$ (c) $(f-g)(-3)$ (d) $(fg)\left(\frac{2}{5}\right)$ (e) $\left(\frac{f}{g}\right)\left(\frac{1}{2}\right)$.

Exercise 2.2.6
Find the difference quotient of the function $f(x)=3x+4$.

Exercise 2.2.7
Find the difference quotient of the function $f(x)=4x^2+3x-2$.

Exercise 2.2.8
Find the difference quotient of the function $f(x)=\sqrt{x+1}$.

Exercise 2.2.9
Find the difference quotient of the function $f(x)=\frac{3}{x+1}$.

Exercise 2.2.10
Find the difference quotient of the function $f(x) = \frac{x}{x+1}$.

Exercise 2.2.11
Find $f \circ g$ and $g \circ f$ where $f(x) = 3x + 5$ and $g(x) = 2x - 7$.

Exercise 2.2.12
Find $f \circ g$ and $g \circ f$ where $f(x) = x^3 + 2x$ and $g(x) = -5x$.

Exercise 2.2.13
Find $f \circ g$ and $g \circ f$ where $f(x) = \frac{2}{x+1}$ and $g(x) = 3x - 5$.

Exercise 2.2.14
Find $f \circ g$ and $g \circ f$ where $f(x) = \frac{1}{x^2}$ and $g(x) = \sqrt{x - 1}$.

Exercise 2.2.15
Find $f \circ g$ and $g \circ f$ where $f(x) = \frac{3}{|5-x|}$ and $g(x) = -\frac{2}{x}$.

Exercise 2.2.16
Evaluate each composite function where $f(x) = 2x + 3, g(x) = x^2 - 5x$, and $h(x) = 4 - 3x^2$.

(a) $(f \circ g)(-3)$ (b) $(h \circ g)\left(\frac{2}{5}\right)$ (c) $(g \circ f)(\sqrt{3})$ (d) $(h \circ f)(2c)$.

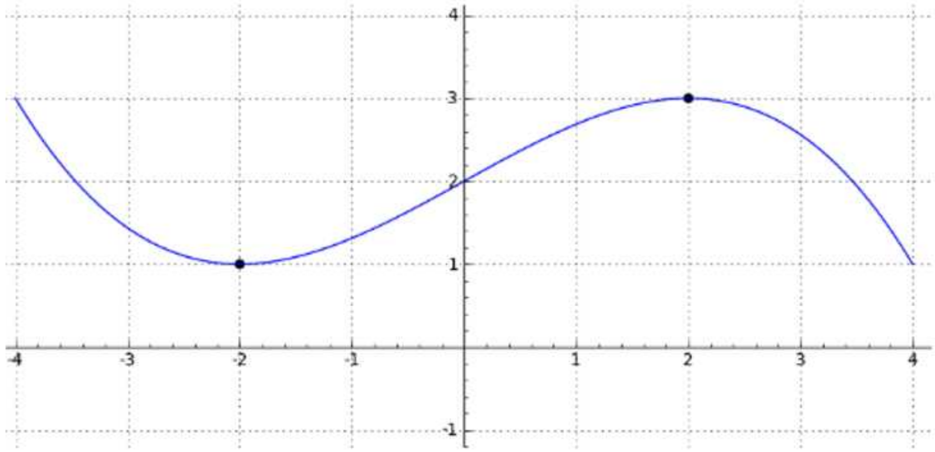
Exercise 2.2.17
Decompose the function $h(x) = \frac{2}{(x^2+3x+1)^3}$ into simpler functions.

Exercise 2.2.18
Decompose the function $h(x) = (2x + 1)^3$ into simpler functions.

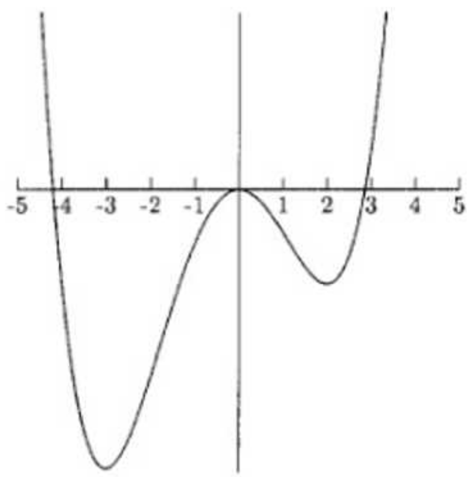
Exercise 2.2.19
Decompose the function $h(x) = \frac{4}{(5x^2+2)^2}$ into simpler functions.

Exercise 2.2.20
Decompose the function $h(x) = \sqrt[3]{x^2 - 1}$ into simpler functions.

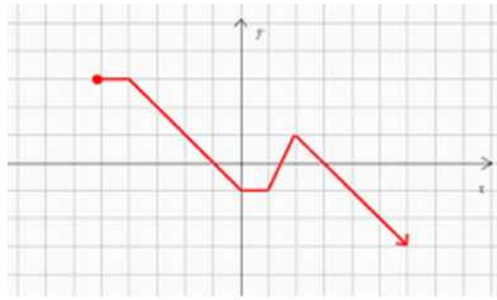
Exercise 2.2.21
Use the following graph of f to determine its intervals of increase and decrease.



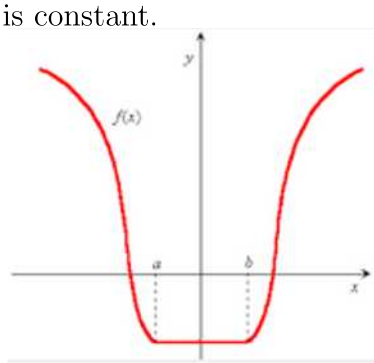
Exercise 2.2.22
Use the following graph of f to determine its intervals of increase and decrease.



Exercise 2.2.23
Use the following graph of f to determine its intervals of increase and decrease and where the function is constant.



Exercise 2.2.24
Use the following graph of f to determine its intervals of increase and decrease and where the function is constant.



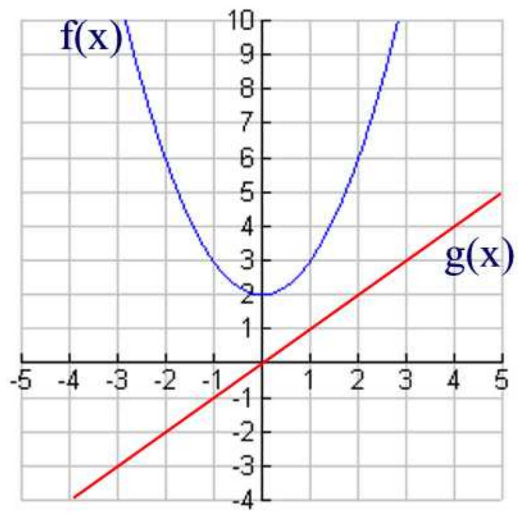
Exercise 2.2.25
Find each function and state its domain: $f(x) = \sqrt{x+1}$ and $g(x) = \sqrt{x-1}$.
(a) $(f+g)(x)$ (b) $(f-g)(x)$ (c) $(f \cdot g)(x)$ (d) $\left(\frac{f}{g}\right)(x)$.

Exercise 2.2.26
Suppose the functions f and g are given in numerical forms. Complete the following table:

x	-2	-1	0	1	2	3
$f(x)$	5	2	1	2	5	10
$g(x)$	-7	0	1	2	9	28
$(f+g)(x)$						
$(f-g)(x)$						
$(f \cdot g)(x)$						
$(\frac{f}{g})(x)$						

Exercise 2.2.27
Use the graphs of the functions f and g given below to Complete the following table:

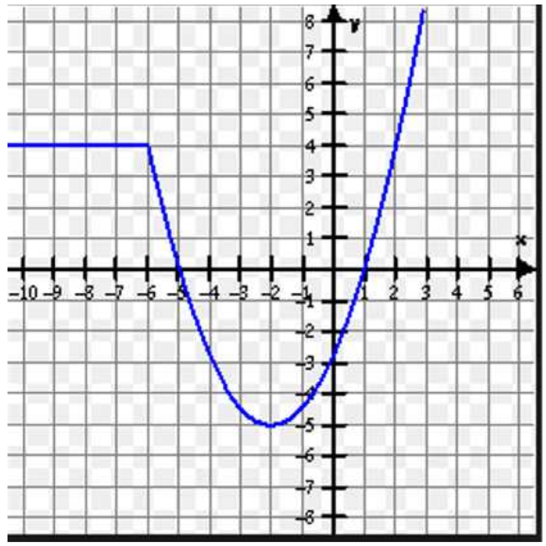
x	-2	-1	0	1	2
$f(x)$					
$g(x)$					
$(f+g)(x)$					
$(f-g)(x)$					
$(f \cdot g)(x)$					
$(\frac{f}{g})(x)$					



Exercise 2.2.28
Find the difference quotient of $f(x) = \frac{x-3}{5-x}$.

Exercise 2.2.29
(a) Find $(f \circ g)(x)$ where $f(x) = 3x$ and $g(x) = 2x^2 - 5$.
(b) Suppose that $h(x) = f(g(x)) = \sqrt[3]{2x+1}$. Find possible formulas for f and g.

Exercise 2.2.30
Use the following graph of f to determine its intervals of increase and decrease and where the function is constant.


y
x
8
7
6
5
4
3
2
1
-1
-2
-3
-4
-5
-6
-7
-8
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 1 2 3 4 5 6

2.3 Transformations of Functions

Throughout this section we consider the relationship between changes made to the formula of a function and the corresponding changes made to its graph. The resulting changes in the graph will consist of shifting, flipping, compressing, and stretching of the original graph.

Reflections and Symmetry

Reflections occur when either the input or the output of a function is multiplied by -1.

Reflection about the $x-$axis

For a given function $f(x)$, the points $(x, f(x))$ and $(x, -f(x))$ are on opposite sides of the $x-$axis. So the graph of the new function $-f(x)$ is the reflection of the graph of $f(x)$ about the $x-$axis.

Example 2.3.1

Graph the functions $f(x) = x^2$ and $-f(x) = -x^2$ on the same axes.

Solution.

The graphs of both $f(x) = x^2$ (in red) and $-f(x)$ (in green) are shown in Figure 2.3.1 ■

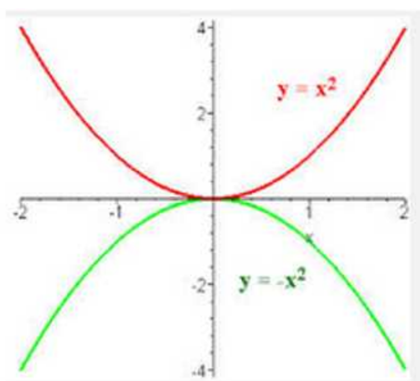


Figure 2.3.1

Reflection About the $y-$axis

We know that the points $(x, f(x))$ and $(-x, f(x))$ are on opposite sides of the $y-$axis. So the graph of the new function $f(-x)$ is the reflection of the graph of $f(x)$ about the $y-$axis.

Example 2.3.2
Graph the functions $f(x) = (x + 1)^3$ and $f(-x) = (-x + 1)^3$ on the same axes.

Solution.
The graphs of both $f(x)$ (in red) and $f(-x)$ (in green) are shown in Figure 2.3.2 ■

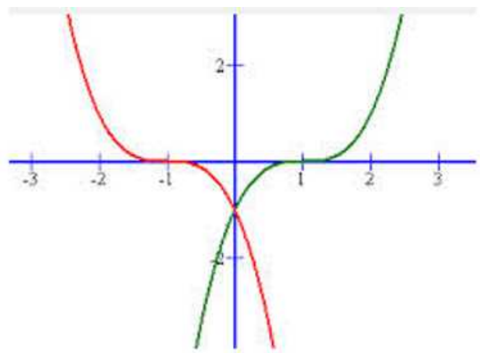


Figure 2.3.2

Symmetry About the $y-$axis
When constructing the graph of $f(-x)$ sometimes you will find that this new graph is the same as the graph of the original function. That is, the reflection of the graph of $f(x)$ about the y-axis is the same as the graph of $f(x)$,i.e., $f(-x) = f(x)$. In this case, we say that the graph of $f(x)$ is symmetric about the y-axis. We call such a function an **even** function.

Example 2.3.3
(a) Using a graphing calculator show that the function $f(x) = (x - x^3)^2$ is even.
(b) Now show that $f(x)$ is even algebraically.

Solution.
(a) The graph of $f(x)$ is symmetric about the $y-$axis so that $f(x)$ is even. See Figure 2.3.3.
(b) Since $f(-x) = (-x - (-x)^3)^2 = (-x + x^3)^2 = [-(x - x^3)]^2 = (x - x^3)^2 =$

$f(x)$, $f(x)$ is even ■

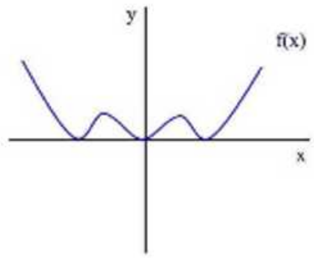


Figure 2.3.3

Symmetry About the Origin

Now, if the images $f(x)$ and $f(-x)$ are of opposite signs , i.e., $f(-x) = -f(x)$, then the graph of $f(x)$ is symmetric about the origin. In this case, we say that $f(x)$ is **odd.** Alternatively, since $f(x) = -f(-x)$, if the graph of a function is reflected first across the $y-$axis and then across the $x-$axis and you get the graph of $f(x)$ again then the function is odd.

Example 2.3.4

(a) Using a graphing calculator show that the function $f(x) = \frac{1+x^2}{x-x^3}$ is odd.
(b) Now show that $f(x)$ is odd algebraically.

Solution.

(a) The graph of $f(x)$ is symmetric about the origin so that $f(x)$ is odd. See Figure 2.3.4.
(b) Since $f(-x) = \frac{1+(-x)^2}{(-x)-(-x)^3} = \frac{1+x^2}{-x+x^3} = \frac{1+x^2}{-(x-x^3)} = -f(x)$, $f(x)$ is odd ■

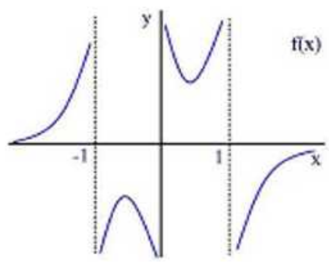


Figure 2.3.4

A function can be either even, odd, neither, or both.

Example 2.3.5
(a) Show that the function $f(x) = x^2$ is even but not odd.
(b) Show that the function $f(x) = x^3$ is odd but not even.
(c) Show that the function $f(x) = x + x^2$ is neither odd nor even.
(d) Is there a function that is both even and odd? Explain.

Solution.
(a) Since $f(-x) = (-x)^2 = x^2 = f(x)$ and $f(-x) \neq -f(x)$, $f(x)$ is even but not odd.
(b) Since $f(-x) = (-x)^3 = -x^3 = -f(x)$ and $f(-x) \neq f(x)$, $f(x)$ is odd but not even.
(c) Since $f(-x) = -x + x^2 \neq \pm f(x)$, $f(x)$ is neither even nor odd.
(d) We are looking for a function such that $f(-x) = f(x)$ and $f(-x) = -f(x)$. This implies that $f(x) = -f(x)$ or $2f(x) = 0$. Dividing by 2 to obtain $f(x) = 0$. This function is both even and odd. This is the only function that is both even and odd. Its graph is the $x-$axis ■

Vertical and Horizontal Shifts

Given the graph of a function, by shifting this graph vertically or horizontally one gets the graph of a new function. In this section we want to find the formula for this new function using the formula of the original function.

Vertical Shifts (Fixing x and Changing y)

We start with an example of a vertical shift.

Example 2.3.6
(a) Graph the functions $f(x) = x^2$ and $g(x) = x^2 + 1$. How does the graph of $g(x)$ compare to the graph of $f(x)$?
(b) Graph the functions $f(x) = x^2$ and $h(x) = x^2 - 1$.How does the graph of $h(x)$ compare to the graph of $f(x)$?

Solution.
(a) In Figure 2.3.5(a) we have included the graph of $g(x) = x^2 + 1 = f(x) + 1$. This shows that the graph of $g(x)$ is obtained by moving the graph of $f(x)$ up 1 unit.
(b) Figure 2.3.5(b) shows the graph of both $f(x)$ and $h(x) = f(x) - 1$. The graph of $h(x)$ is obtained by moving the graph of $f(x)$ 1 unit down ■

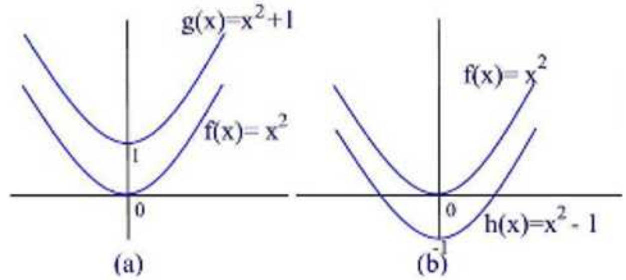


Figure 2.3.5

In general, if $c > 0$, the graph of $f(x) + c$ is obtained by shifting the graph of $f(x)$ upward a distance of c units. The graph of $f(x) - c$ is obtained by shifting the graph of $f(x)$ downward a distance of c units.

Horizontal Shifts (Fixing y and Changing x)

This discussion parallels the one about vertical shifts. Follow the same general directions.

Example 2.3.7
(a) Graph the functions $f(x) = x^2$ and $g(x) = (x+1)^2 = f(x+1)$. How does the graph of $g(x)$ compare to the graph of $f(x)$?
(b) Graph the functions $f(x) = x^2$ and $h(x) = (x-1)^2 = f(x-1)$. How does the graph of $h(x)$ compare to the graph of $f(x)$?

Solution.
(a) In Figure 2.3.6(a) we have included the graph of $g(x) = (x+1)^2$. We see that the new graph is obtained from the old one by shifting to the left 1 unit. This is as expected since the value of x^2 at $x = 0$ is the same as the value of $(x+1)^2$ at $x = -1$.
(b) Similar to (a), we see in Figure 2.3.6(b) that we get the graph of $h(x)$ by moving the graph of $f(x)$ to the right 1 unit ■

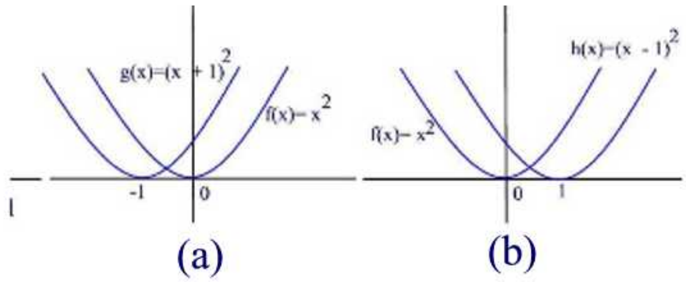


Figure 2.3.6

In general, if $c > 0$, the graph of $f(x+c)$ is obtained by shifting the graph of $f(x)$ to the left a distance of c units. The graph of $f(x-c)$ is obtained by shifting the graph of $f(x)$ to the right a distance of c units.

Remark 2.3.1
Be careful when translating graph horizontally. In determing the direction of horizontal shifts we look for the value of x that would cause the expression between parentheses equal to 0. For example, the graph of $f(x-5) = (x-5)^2$ is the graph of $f(x) = x^2$ shifted 5 units to the right since $+5$ would cause the quantity $x-5$ to equal 0. On the other hand, the graph of $f(x+5) = (x+5)^2$ is the graph of $f(x) = x^2$ shifted 5 units to the left since -5 would cause the expression $x+5$ to equal 0.

Combinations of Vertical and Horizontal Shifts

One can use a combination of both horizontal and vertical shifts to create new functions as shown in the next example.

Example 2.3.8
Let $f(x) = x^2$. Let $g(x)$ be the function obtained by shifting the graph of $f(x)$ two units to the right and then up three units. Find a formula for $g(x)$ and then draw its graph.

Solution.
The formula of $g(x)$ is $g(x) = f(x-2)+3 = (x-2)^2+3 = x^2-4x+7$. The graph of $g(x)$ consists of a horizontal shift of x^2 of two units to the right followed by a vertical shift of three units upward. See Figure 2.3.7 ■

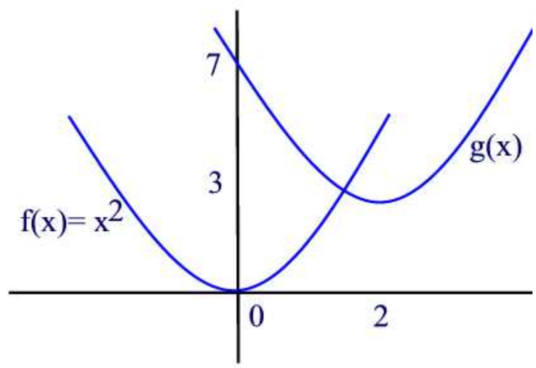


Figure 2.3.7

Combinations of Shifts and Reflections

We can obtain more complex functions by combining the horizontal and vertical shifts with the horizontal and vertical reflections.

Example 2.3.9
Let $f(x) = x^3$.
(a) Suppose that $g(x)$ is the function obtained from $f(x)$ by first reflecting about the $y-$axis, then translating down three units. Write a formula for $g(x)$.
(b) Suppose that $h(x)$ is the function obtained from $f(x)$ by first translating up two units and then reflecting about the $x-$axis. Write a formula for $h(x)$.

Solution.
(a) $g(x) = f(-x) - 3 = -x^3 - 3$.
(b) $h(x) = -(f(x) + 2) = -x^3 - 2$ ■

Vertical Stretches and Compressions

We have seen that for a positive k, the graph of $f(x) + k$ is a vertical shift of the graph of $f(x)$ upward and the graph of $f(x) - k$ is a vertical shift down. Next, we want to study the effect of multiplying a function by a constant k. This will result by either a vertical stretch or vertical compression.
A **vertical stretching** is the stretching of the graph away from the $x-$axis.
A **vertical compression** is the squeezing of the graph towards the $x-$axis.

Example 2.3.10
Let $f(x) = x^2$.
(a) Graph the functions $2f(x)$ and $3f(x)$ on the same axes. What do you notice?
(b) Graph the functions $\frac{1}{2}f(x)$ and $\frac{1}{3}f(x)$ on the same axes. What do you notice?

Solution.
(a) Figure 2.3.8(a) shows that the graphs of $2f(x)$ and $3f(x)$ are vertical stretches of the graph of $f(x)$ by a factor of 2 and 3 respectively.
(b) Figure 2.3.8(b) shows that the graphs of $\frac{1}{2}f(x)$ and $\frac{1}{3}f(x)$ are vertical compressions of the graph of $f(x)$ by a factor of $\frac{1}{2}$ and $\frac{1}{3}$ respectively ■

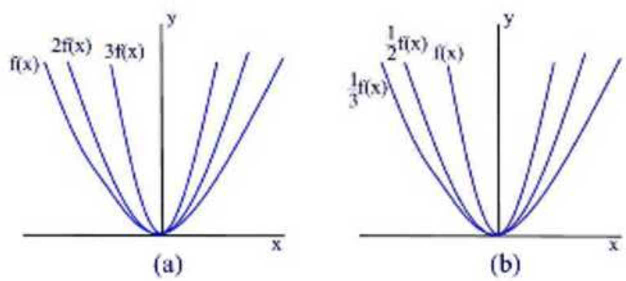


Figure 2.3.8

Summary
It follows that if a function $f(x)$ is given, then the graph of $kf(x)$ is a vertical stretch of the graph of $f(x)$ by a factor of k for $k > 1$, and a vertical compression for $0 < k < 1$.
What about $k < 0$? If $|k| > 1$ then the graph of $kf(x)$ is a vertical stretch of the graph of $f(x)$ followed by a reflection about the $x-$axis. If $0 < |k| < 1$ then the graph of $kf(x)$ is a vertical compression of the graph of $f(x)$ by a factor of k followed by a reflection about the $x-$axis.

Example 2.3.11
(a) Graph the functions $f(x) = x^2, -2f(x)$, and $-3f(x)$ on the same axes. How do they compare?
(b) Graph the functions $f(x) = x^2, -\frac{1}{2}f(x)$, and -$\frac{1}{3}f(x)$ on the same axes. How do they compare?

Solution.
(a) Figure 2.3.9(a) shows that the graphs of $-2f(x)$ and $-3f(x)$ are vertical stretches followed by reflections about the x-axis of the graph of $f(x)$
(b) Figure 2.3.9(b) shows that the graphs of $-\frac{1}{2}f(x)$ and $-\frac{1}{3}f(x)$ are vertical compressions of the graph of $f(x)$ ■

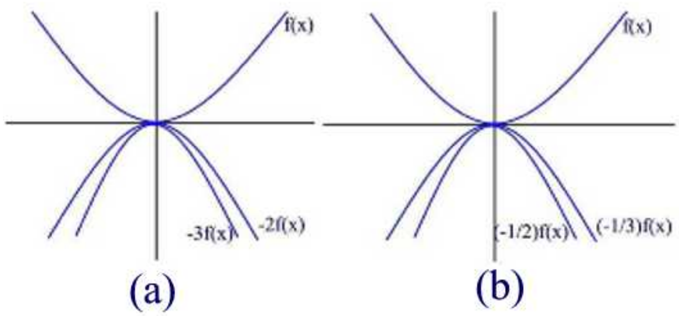


Figure 2.3.9

Combinations of Shifts
Any transformations of vertical, horizontal shifts, reflections, vertical stretches or compressions can be combined to generate new functions. In this case, always work from inside the parentheses outward.

Example 2.3.12
How do you obtain the graph of $g(x) = -\frac{1}{2}f(x+3) - 1$ from the graph of $f(x)$?

Solution.
The graph of $g(x)$ is obtained by first shifting the graph of $f(x)$ to the left by 3 units then the resulting graph is compressed vertically by a factor of $\frac{1}{2}$ followed by a reflection about the $x-$axis and finally moving the graph down by 1 unit ■

Horizontal Stretches and Compressions
A vertical stretch or compression results from multiplying the outside of a function by a constant k. We will see that multiplying the inside of a function by a constant k results in either a horizontal stretch or compression.
A **horizontal stretching** is the stretching of the graph away from the $y-$axis. A **horizontal compression** is the squeezing of the graph towards the $y-$axis. We consider first the effect of multiplying the input by $k > 1$.

Example 2.3.13
Consider the function $f(x) = x^2$.
(a) Graph the functiosn $f(x), f(2x)$, and $f(3x)$ on the same axes. How do they compare?
(a) Graph the functiosn $f(x), f(\frac{1}{2}x)$, and $f(\frac{1}{3}x)$ on the same axes. How do they compare?

Solution.
(a) Figure 2.3.10(a) shows that the graphs of $f(2x) = (2x)^2 = 4x^2$ and $f(3x) = (3x)^2 = 9x^2$ are horizontal compressions of the graph of $f(x)$ by a factor of $\frac{1}{2}$ and $\frac{1}{3}$ respectively.
(b) Figure 2.3.10(b) shows that the graphs of $f(\frac{x}{2})$ and $f(\frac{x}{3})$ are horizontal stretches of the graph of $f(x)$ by a factor of 2 and 3 respectively ■

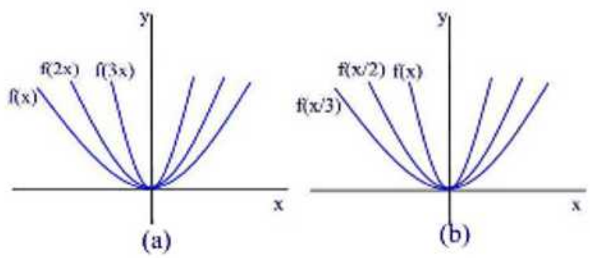


Figure 2.3.10

Summary
It follows from the above two examples that if a function $f(x)$ is given, then the graph of $f(kx)$ is a horizontal stretch of the graph of $f(x)$ by a factor of $\frac{1}{k}$ for $0 < k < 1$, and a horizontal compression for $k > 1$.
What about $k < 0$? If $|k| > 1$ then the graph of $f(kx)$ is a horizontal compression of the graph of $f(x)$ by a factor of $\frac{1}{|k|}$ followed by a reflection about the $y-$axis. If $0 < |k| < 1$ then the graph of $f(kx)$ is a horizontal stretch of the graph of $f(x)$ by a factor of $\frac{1}{|k|}$ followed by a reflection about the $y-$axis.

Example 2.3.14
(a) Graph the functions $f(x) = x^3, f(-2x)$, and $f(-3x)$ on the same axes. How do they compare?
(b) Graph the functions $f(x) = x^3, f(-\frac{x}{2})$, and $f(-\frac{x}{3})$ on the same axes. How do they compare?

Solution.
(a) Figure 2.3.11(a) shows that the graphs of $f(-2x)$ and $f(-3x)$ are vertical stretches followed by reflections about the $y-$axis of the graph of $f(x)$.
(b) Figure 2.3.11(b) shows that the graphs of $f(-\frac{x}{2})$ $f(-\frac{x}{3})$ are horizontal stretches of the graph of $f(x)$ ■

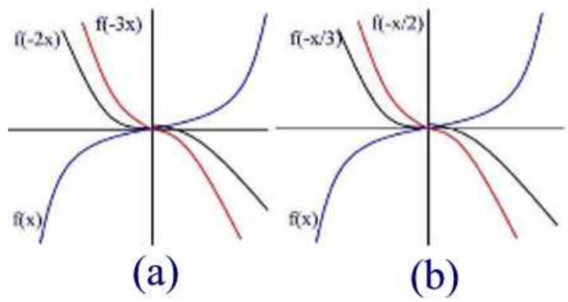


Figure 2.3.11

Exercises

Exercise 2.3.1
Determine whether the function $f(x) = x^4 - 3x^2 + 10$ is even, odd, or neither.

Exercise 2.3.2
Determine whether the function $f(x) = x^3 - 3x + 7$ is even, odd, or neither.

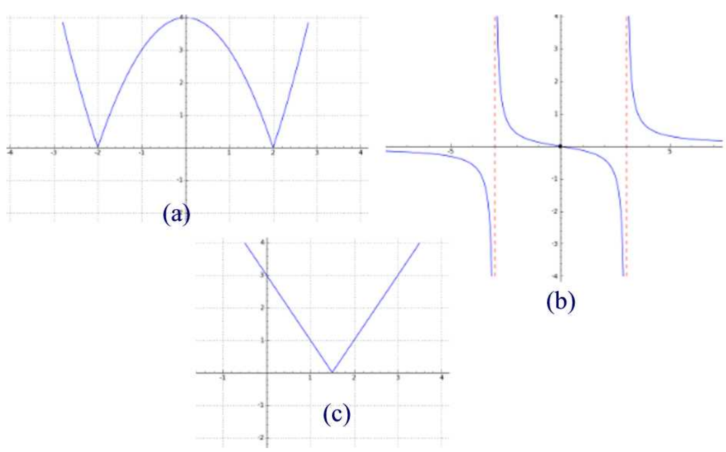
Exercise 2.3.3
Determine whether the function $f(x) = x^5 - 3x^3 + 7x$ is even, odd, or neither.

Exercise 2.3.4
Determine whether the function $f(x) = \frac{x^2+1}{x^3-x}$ is even, odd, or neither.

Exercise 2.3.5
Given a function $f(x)$. Show that $g(x) = \frac{1}{2}[f(x) + f(-x)]$ is even and $h(x) = \frac{1}{2}[f(x) - f(-x)]$ is odd.

Exercise 2.3.6
Show that any function $f(x)$ can be written as the sum of an even function and an odd function.

Exercise 2.3.7
Determine which of the following functions are even, odd, or neither by their graphs.



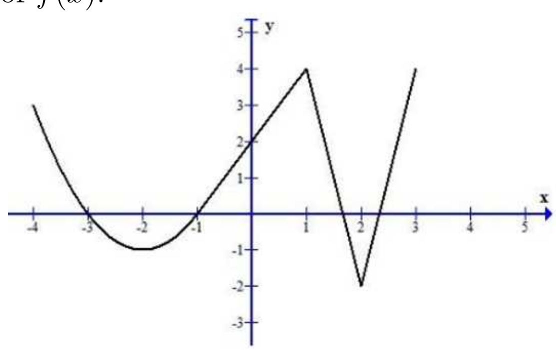
Exercise 2.3.8
Graph the function $f(x) = \sqrt{x}$. Using an appropriate transformation, graph each of the following functions:
(a) $y = \sqrt{x} - 2$.
(b) $y = \sqrt{x - 2}$.
(c) $y = -\sqrt{x}$.
(d) $y = 2\sqrt{x}$.
(e) $y = \sqrt{-x}$.

Exercise 2.3.9
Find an algebraic expression for the function that is found from $y = f(x)$ by shifting to the right by 5 units, then compressing horizontally by 3 units, and then reflecting about the $y-$axis.

Exercise 2.3.10
Find the formula for the graph obtained from the graph of $y = |x|$ using a shift left two units, then a vertical stretch by a factor of 2, and finally a shift down of 4 units.

Exercise 2.3.11
How do you obtain the graph of $y = f(3 - x)$ from the graph of $y = f(x)$?

Exercise 2.3.12
Given the graph of $f(x)$.

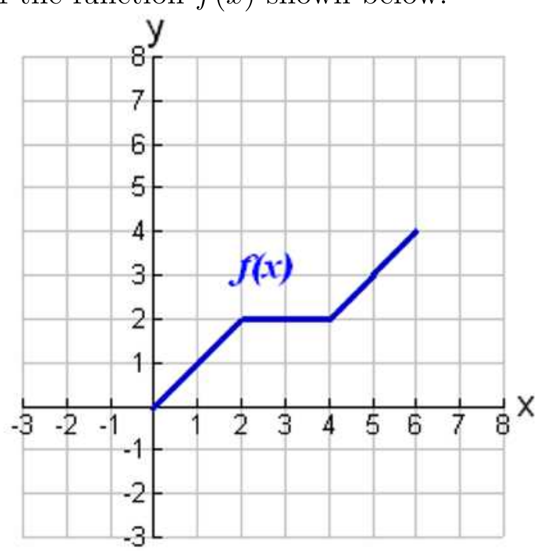


Sketch the graph of
(a) $y = f(x) - 3$.
(b) $y = f(x + 1)$.
(c) $y = f(x - 2) + 1$.
(d) $y = -f(x + 1)$.
(e) $y = f(-x) - 2$.

Exercise 2.3.13
Start with the function $f(x) = \sqrt{x}$, and write the function which results from the given transformations. Then decide if the results from parts (a) and (b) are equivalent.
(a) Shift upward 7 units, then right 2 units.
(b) Shift right 2 units, then upward 7 units.
(c) Do parts (a) and (b) yield the same function?

Exercise 2.3.14
Write the equation for the graph of function $g(x)$, obtained by shifting the graph of $f(x) = x^2$ three units left, stretching the graph vertically by a factor of two, reflecting that result over the $x-$axis, and then translating the graph up four units.

Exercise 2.3.15
Given the graph of the function $f(x)$ shown below.

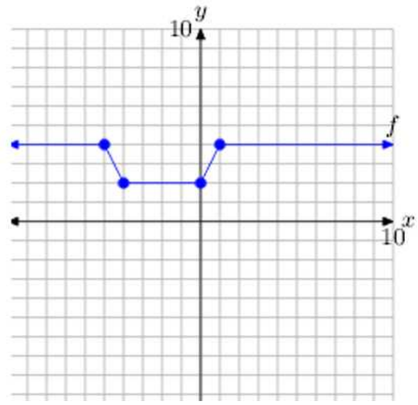


Sketch the graphs of:
(a) $y = f(x+1)$.
(b) $y = f(x) - 2$.
(c) $y = f(-x)$.
(d) $y = -f(x)$.
(e) $y = 2f(x)$.

Exercise 2.3.16
In what follows fill in the blank:
(a) To obtain the graph of ________, we translate the graph of $y = |x|$ to the left by 3 units.
(b) To obtain the graph of $y = -|x|$, we ________ of $y = |x|$ ________.
(c) To obtain the graph of $y = -x + 5$, we ________ the graph of $y = x + 5$ ________
(e) To obtain the graph of ________ we stretch the graph of $y = x + 5$ vertically by a factor of 3.
(f) To obtain the graph of $y = 4x^3 - 6$ we translate the graph of ________ downward by 6 units.
(g) To obtain the graph of $y = \frac{1}{x+5}$ we ________ the graph of $y = \frac{1}{x}$ ________.
(h) To obtain the graph of $y = 4x^2$ we compress the graph of ________ horizontally by a factor of $\frac{1}{2}$.
(i) To obtain the graph of $y = 4x^2$ we expand the graph of ________ vertically by a factor of 4.
(j) To obtain the graph of ________ we translate of $y = x^3$ to the left by 8 units.
(k) To obtain the graph of $y = |6x|$ we ________ the graph of $y = |x|$ ________
(l) To obtain the graph of $y = -x^2$ we ________ the graph of $y = x^2$ ________

Exercise 2.3.17
Let $f(x) = x^3$. Write the transformation in each case that follows:
(a) Shift right 7 units, then reflect in the $x-$axis, then stretch vertically by a factor of 5, then shift upward 1 unit.
(b) Reflect in the $x-$axis, then shift right 7 units, then stretch vertically by a factor of 5, then shift upward 1 unit.
(c) Stretch vertically by a factor of 5, then shift downward 1 unit, then shift right 7 units, then reflect in the $x-$axis.
(d) Shift right 7 units, then shift upward 1 unit, then reflect in the $x-$axis, then stretch vertically by a factor of 5.
(e) Reflect in the $x-$axis, then shift left 7 units, then stretch vertically by a factor of 5, then shift upward 1 unit.

Pictured below is the graph of a function $f(x)$.

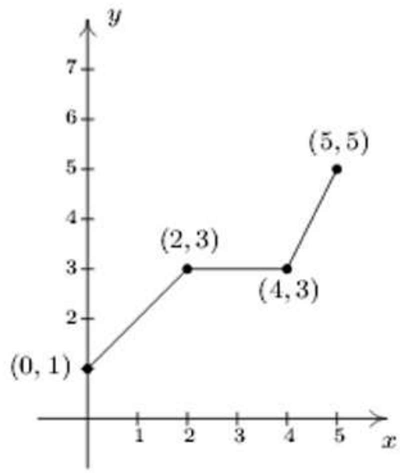


Use this graph to answer Exercises 2.3.18 - 2.3.19.

Exercise 2.3.18
Sketch the graph of
(a) $y = -\frac{1}{2}f(x)$.
(b) $y = -f(x) + 2$.

Exercise 2.3.19
Sketch the graph of $y = 2f(x) - 3$.

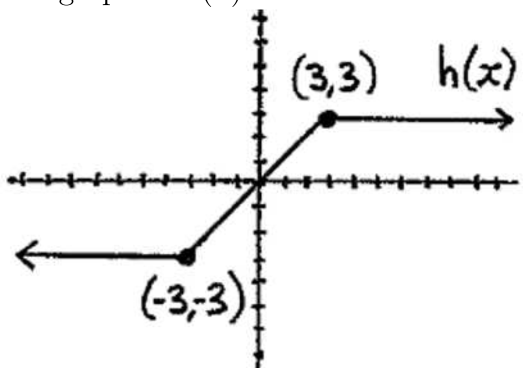
Exercise 2.3.20
Pictured below is the graph of $f(x)$.



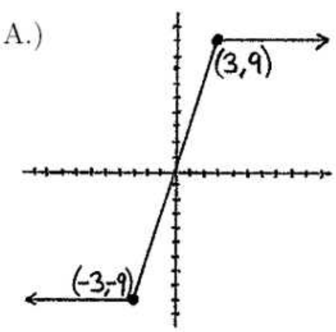
$y = f(x)$

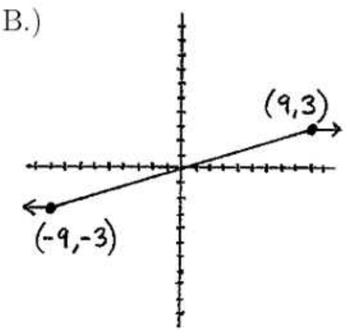
Sketch the graph of
(a) $y = f(x) + 2$.
(b) $y = f(x + 2)$.

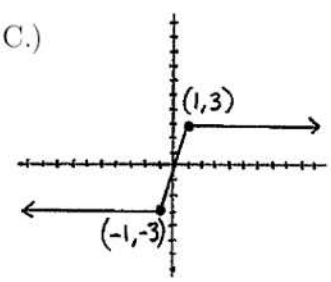
Exercise 2.3.21
Pictured below is the graph of $h(x)$.

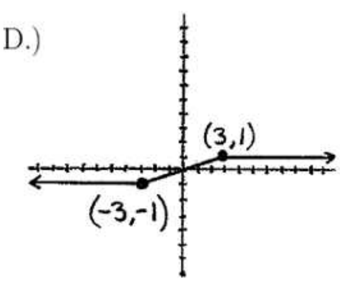


Match the following four functions with their graphs.
(a) $3h(x)$ (b) $\frac{1}{3}h(x)$ (c) $h(3x)$ (d) $h\left(\frac{1}{3}x\right)$.

A.)


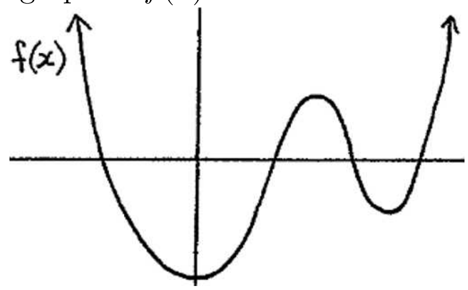
B.)


C.)


D.)


Exercise 2.3.22

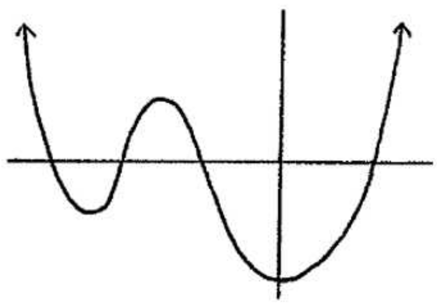
Pictured below is the graph of $f(x)$.



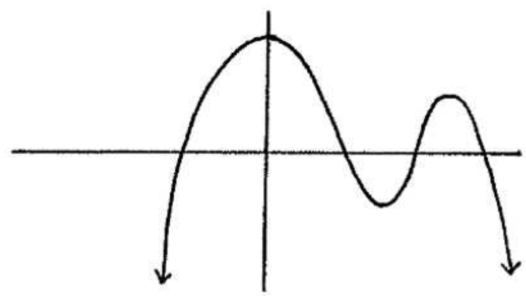
Match the following two functions with their graphs.

(a) $-f(x)$ (b) $f(-x)$

A.)



B.)



Exercise 2.3.23

Given the graph of $f(x) = \sqrt{x}$. Use transformations to graph $f(x) = 1 - \sqrt{1+x}$.

Exercise 2.3.24

Suppose you want to graph $f(x) = 3\sqrt{x+1} - 7$ using transformations. List a possible sequence of transformations that lead to the desired graph.

Exercise 2.3.25

Suppose you want to graph $f(x) = \sqrt{-x+2} + 7$ using transformations. List a possible sequence of transformations that lead to the desired graph.

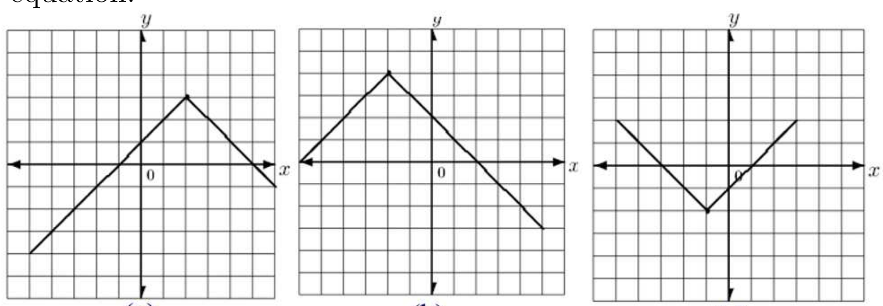
Exercise 2.3.26

Sketch the graph of $f(x) = 3\sqrt{-x+2} - 7$ using transformations.

Exercise 2.3.27
Sketch the graph of $f(x) = -|x-3| - 2$ using transformations.

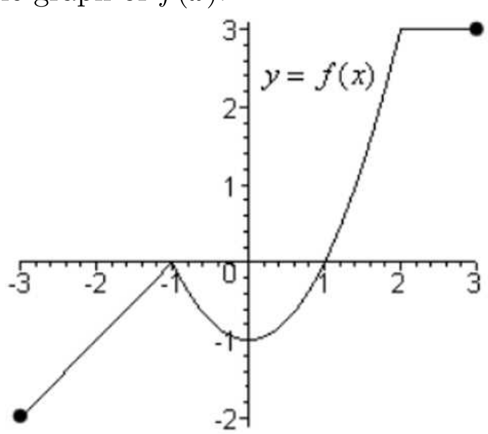
Exercise 2.3.28
Determine algebraically if the equation $x^2+y^2 = 9$ is symmetric with respect to the $x-$axis, the $y-$axis, or the origin.

Exercise 2.3.29
The following functions are transformations of $y = |x|$. Determine each function's equation.



(a) (b) (c)

Exercise 2.3.30
Pictured below is the graph of $f(x)$.



Sketch the graph of $y = f(x-1)+2$.

2.4 One-to-one and Inverse Functions

In Chapter 5, the logarithmic function is defined as the inverse function of the exponential function. The purpose of this section is to introduce the concept of an inverse function. A requirement for a function to have an inverse is that it is one-to-one, a concept that we introduce next.

One-To-One Functions

A function f is said to be **one-to-one** if no two or more inputs share the same image[1]. Algebraically, if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$. Geometrically, one can show that the graph of a function is one-to-one if every horizontal line crosses the graph at most once. We refer to this test as the **horizontal line test**.

Example 2.4.1

Decide whether or not the function is one-to-one.

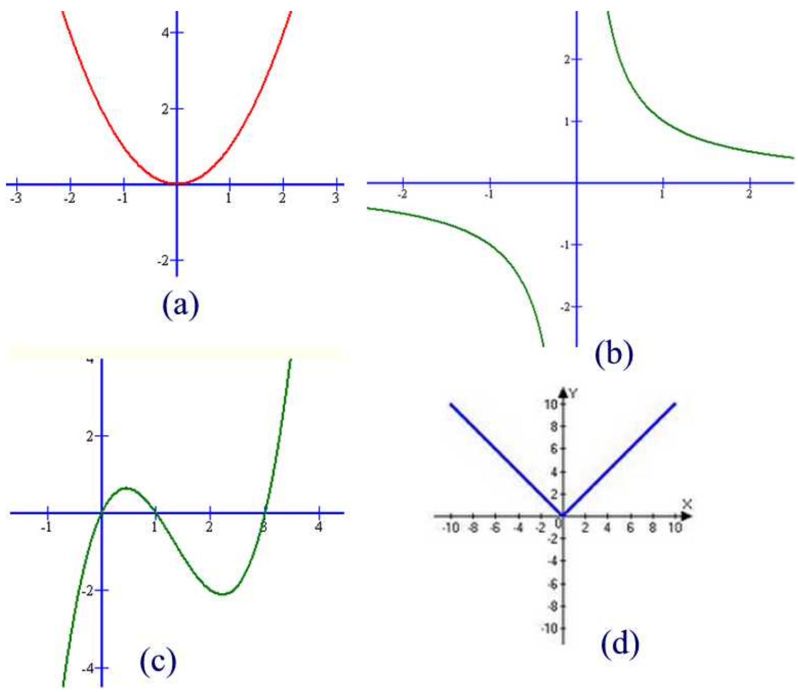


Figure 2.4.1

[1] Alternatively, every output corresponds to a unique input. Algebraically, this says that if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

Solution.
Only (b) is one-to-one function ■

Example 2.4.2
Show, algebraically, that the function $f(x) = 3x - 2$ is one-to-one.

Solution.
Suppose that $f(x_1) = f(x_2)$. Then $3x_1 - 2 = 3x_2 - 2$. Adding 2 to both sides and then dividing by 3 to obtain $x_1 = x_2$. That is, f is one-to-one ■

Example 2.4.3
Show that the function $f(x) = x^2$ is not one-to-one

Solution.
All we need to do is to find two different inputs with the same output. One such an example is $x_1 = -1$ and $x_2 = 1$. Note that $f(x_1) = f(x_2) = 1$. Hence, f is not one-to-one ■

Example 2.4.4
Show that if f is increasing (respectively decreasing) on its domain then f is one-to-one.

Solution.
Suppose that $x_1 \neq x_2$. For simplicity, suppose that $x_1 < x_2$. Since f is increasing, we have $f(x_1) < f(x_2)$. This implies that $f(x_1) \neq f(x_2)$. Hence, f is one-to-one ■

Example 2.4.5
Determine which function is one-to-one.
(a) $f = \{(1, 2), (3, 4), (6, 10), (8, 10)\}$.
(b) $f = \{(1, 2), (3, 4), (6, 10), (8, 11)\}$.

Solution.
(a) f is not one-to-one since the inputs 6 and 8 share the same output 10.
(b) f is one-to-one ■

Inverse Functions

An important feature of one-to-one functions is that they can be used to build new functions. So suppose that f is a one-to-one function. A new function, called the **inverse function** (denoted by f^{-1}), is defined such that if f takes an input x to an output y then f^{-1} takes y as its input and x as its output. That is,

$$f(x) = y \text{ if and only if } f^{-1}(y) = x.$$

When a function has an inverse then we say that the function is **invertible.** It follows from Example 2.4.4 that if a function is always increasing or always decreasing then it is invertible.

Example 2.4.6
Are the following functions invertible?

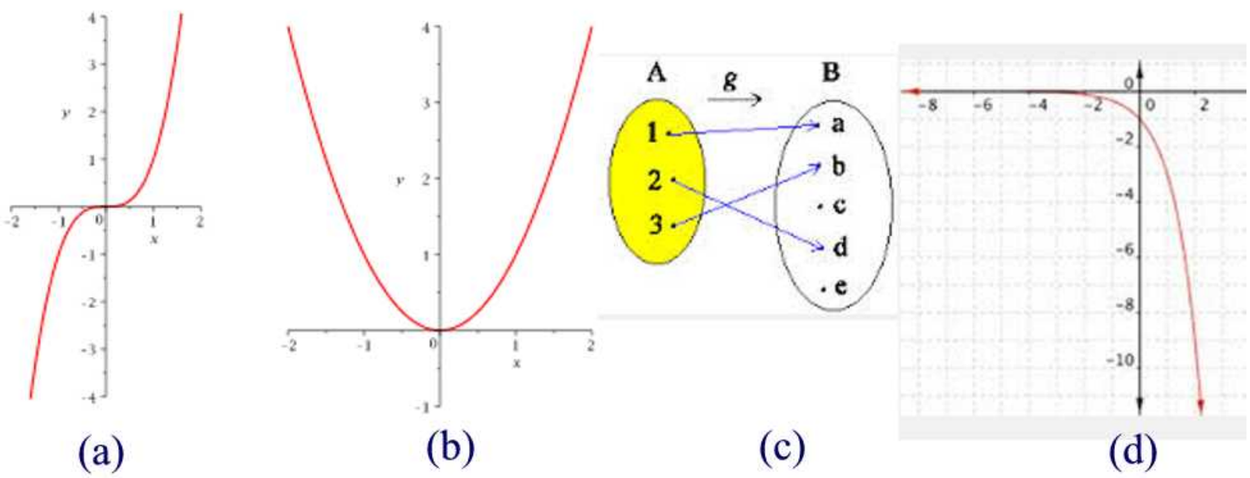


Figure 2.4.2

Solution.
(a), (c), and (d) are invertible functions ■

Remark 2.4.1
It is important not to confuse between $f^{-1}(x)$ and $(f(x))^{-1}$. The later is just the reciprocal of $f(x)$, that is, $(f(x))^{-1} = \frac{1}{f(x)}$ whereas the former is how the inverse function is represented.

Finding a Formula for the Inverse Function

How do you find the formula for f^{-1} from the formula of f? The procedure consists of the following steps:
1. Replace $f(x)$ with y.
2. Interchange the letters x and y.
3. Solve for y in terms of x.
4. Replace y with $f^{-1}(x)$.

Example 2.4.7
Find the formula for the inverse function of $f(x) = x^3 + 1$.

Solution.
$f(x)$ is invertible by Exercise 2.4.2. We find its inverse as follows:
1. Replace $f(x)$ with y to obtain $y = x^3 + 1$.
2. Interchange x and y to obtain $x = y^3 + 1$.
3. Solve for y to obtain $y^3 = x - 1$ or $y = \sqrt[3]{x-1}$.
4. Replace y with $f^{-1}(x)$ to obtain $f^{-1}(x) = \sqrt[3]{x-1}$ ■

Domain and Range of an Inverse Function
Figure 2.4.3 shows the relationship between f and f^{-1}.

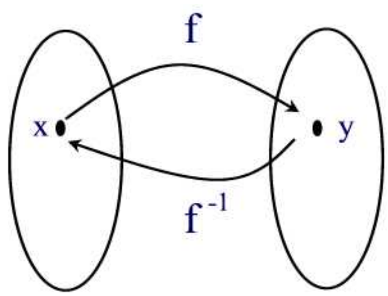


Figure 2.4.3

This figure shows that we get the inverse of a function by simply reversing the direction of the arrows. That is, the outputs of f are the inputs of f^{-1} and the outputs of f^{-1} are the inputs of f. It follows that

$$\text{Domain of } \mathrm{f}^{-1} = \text{Range of f} \quad \text{and} \quad \text{Range of } \mathrm{f}^{-1} = \text{Domain of f.}$$

Example 2.4.8
Consider the function $f(x) = \sqrt{x-4}$.
(a) Find the domain and the range of $f(x)$.
(b) Use the horizontal line test to show that $f(x)$ has an inverse.
(c) Find a formula for $f^{-1}(x)$.
(d) What are the domain and range of f^{-1}?

Solution.
(a) The function $f(x)$ is defined for all $x \geq 4$. That is, the domain is $[4, \infty)$.

The range is the interval $[0, \infty)$.
(b) Graphing $f(x)$, we see that $f(x)$ satisfies the horizontal line test and so f has an inverse. See Figure 2.4.4.
(c) We have $y = \sqrt{x-4}$. Interchange the letters x and y to obtain $x = \sqrt{y-4}$. Square both sides and solve for y to find $y = f^{-1}(x) = x^2 + 4$.
(d) The domain of f^{-1} is the range of f, i.e. the interval $[0, \infty)$. The range of f^{-1} is the domain of f, that is, the interval $[4, \infty)$ ■

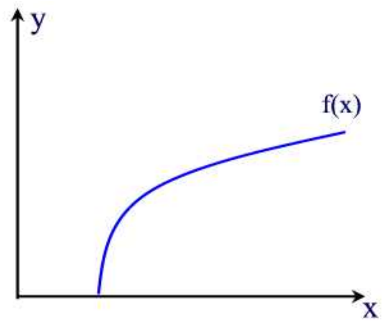


Figure 2.4.4

Compositions of f and its Inverse

Suppose that f is an invertible function. Then the expressions $y = f(x)$ and $x = f^{-1}(y)$ are equivalent. So if x is in the domain of f then

$$f^{-1}(f(x)) = f^{-1}(y) = x$$

and for y in the domain of f^{-1} we have

$$f(f^{-1}(y)) = f(x) = y.$$

It follows that for two functions f and g to be inverses of each other we must have $f(g(x)) = x$ for all x in the domain of g and $g(f(x)) = x$ for all x in the domain of f. That is, the two functions "undo" each other.

Example 2.4.9
Check that the pair of functions $f(x) = \frac{x}{4} - \frac{3}{2}$ and $g(x) = 4(x + \frac{3}{2})$ are inverses of each other.

Solution.
The domain and range of both functions consist of the set of all real numbers. Thus, for any real number x we have

$$f(g(x)) = f(4(x + \frac{3}{2})) = f(4x + 6) = \frac{4x + 6}{4} - \frac{3}{2} = x.$$

and

$$g(f(x)) = g(\frac{x}{4} - \frac{3}{2}) = 4(\frac{x}{4} - \frac{3}{2} + \frac{3}{2}) = x.$$

So f and g are inverses of each other ■

Graph of the inverse function

Given an invertible function f. If $A(x, y)$ belongs on the graph of f then $B(y, x)$ is on the graph of f^{-1}. If M is the midpoint of the line segment connecting A to B then $M\left(\frac{x+y}{2}, \frac{x+y}{2}\right)$. Hence, M is on the graph of the line $y = x$. It follows that the graph of f^{-1} is the reflection of the graph of f about the line $y = x$. See Figure 2.4.5.

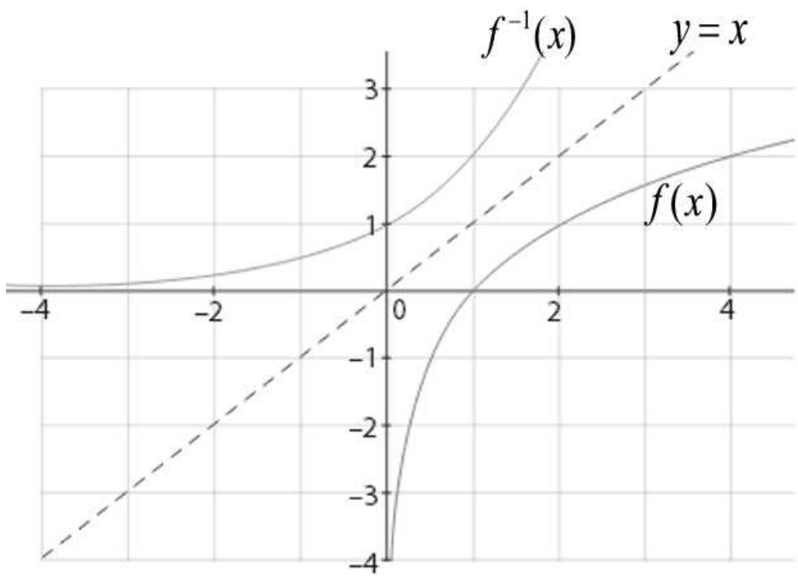


Figure 2.4.5

Example 2.4.10

Show that the function $f(x) = 2x + 1$ is invertible. Graph $f(x)$ and $f^{-1}(x)$ on the same window.

Solution.

Suppose that $f(x_1) = f(x_2)$. Then $2x_1 + 1 = 2x_2 + 1$. Subtracting 1 from both sides and then dividing by 2, we find $x_1 = x_2$. Hence, f is one-to-one and so invertible. The graphs of f and f^{-1} are shown in Figure 2.4.6 ■

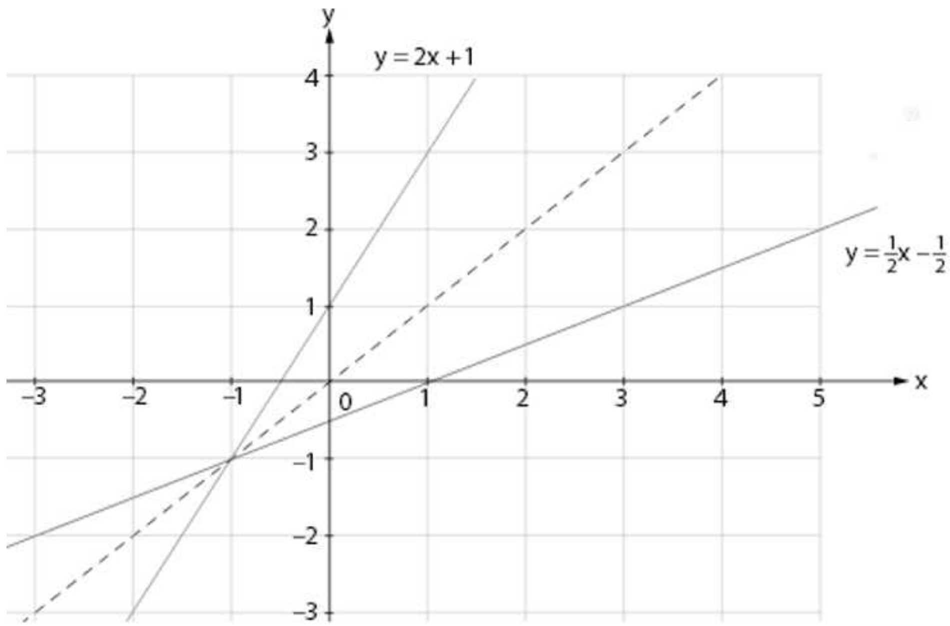


Figure 2.4.6

Exercises

Exercise 2.4.1
Show that $f(x) = x^n - x$, where $n > 0$ is not one-to-one

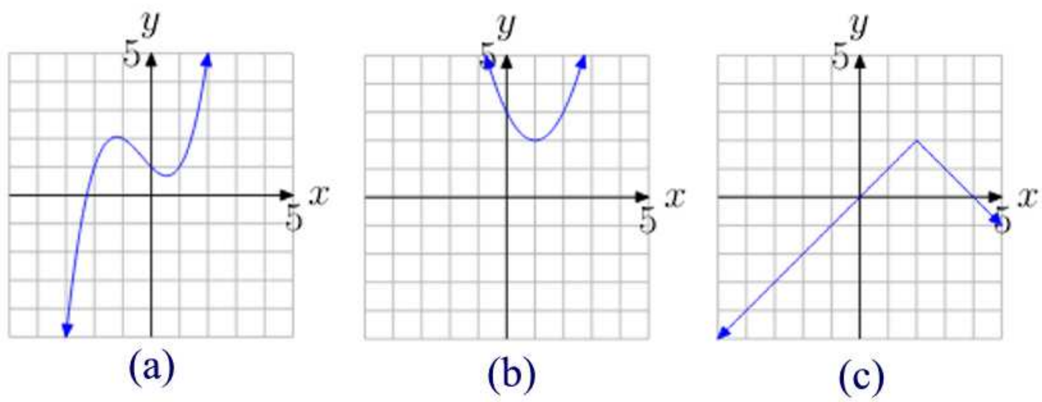
Exercise 2.4.2
Show, algebraically, that the function $f(x) = x^3 + 1$ is one-to-one.

Exercise 2.4.3
Show that if f and g are one-to-one then $f \circ g$ is also one-to-one.

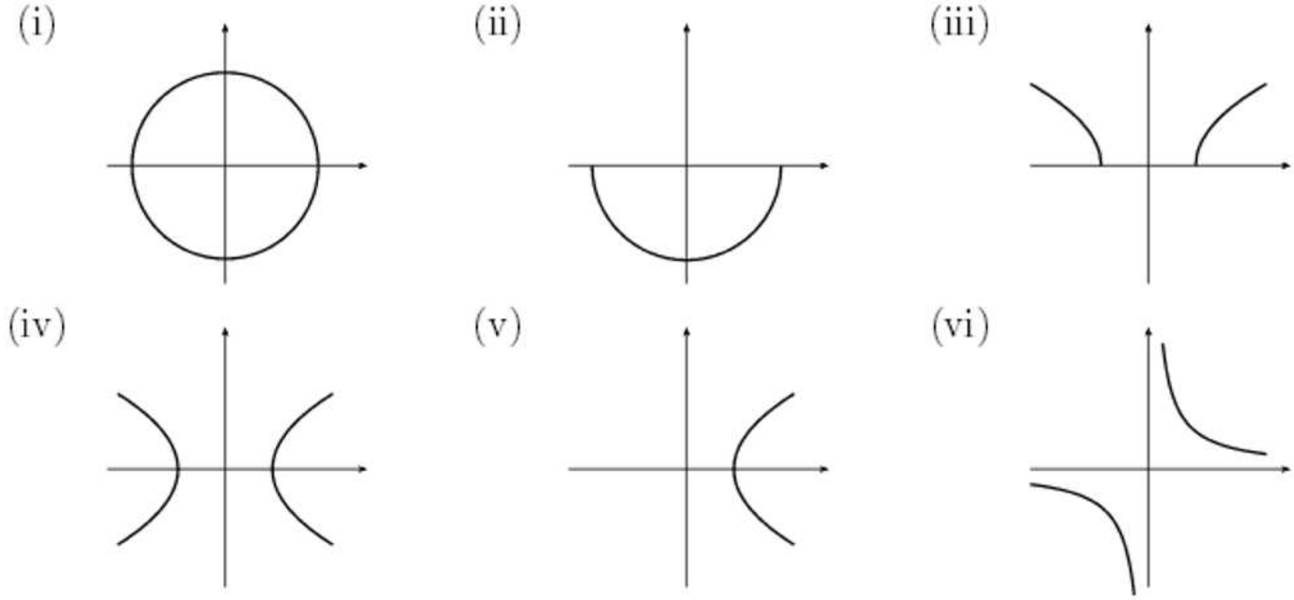
Exercise 2.4.4
Use the above exercise to show that $h(x) = (3x - 1)^3 + 1$ is one-to-one.

Exercise 2.4.5
Suppose $f = \{(-2, 4), (-1, 1), (0, 0), (1, 1)\}$. Is f one-to-one?

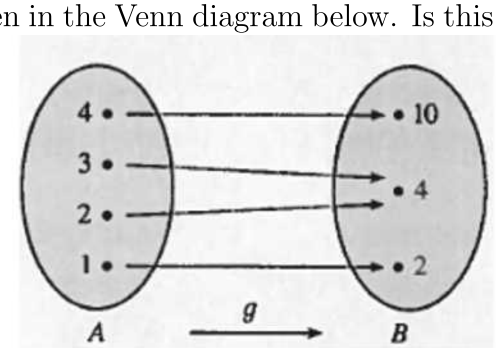
Exercise 2.4.6
Use the graph to determine whether the function is one-to-one.



Exercise 2.4.7
Which of the following graphs represent functions? one-to-one functions?



Exercise 2.4.8
A function g is given in the Venn diagram below. Is this function one-to-one?



Exercise 2.4.9
Show that the function $f(x) = x^2$ with $x \geq 0$ is one-to-one.

Exercise 2.4.10
Determine whether the following functions are one-to-one. Justify your answer with the technique of your choice.
(a) $y = x^2 - x$ (b) $y = 5x - 2$ (c) $y = -\sqrt{2-x}$ (d) $y = \frac{x}{x^2-9}$.

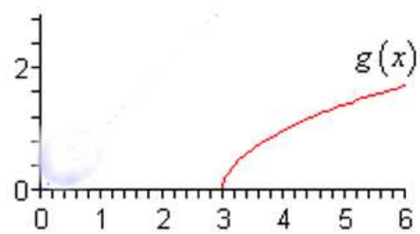
Exercise 2.4.11
Show that the function $f(x) = 2x^3 + 5$ is invertible and find its inverse.

Exercise 2.4.12
Show that the function $f(x) = (5x-1)^3$ is invertible and find its inverse.

Exercise 2.4.13
Using a graphing calculator, show that the function $f(x) = \frac{x+4}{3x-5}$ is invertible and find its inverse.

Exercise 2.4.14
Show that the function $f(x) = \frac{3}{2}\sqrt{x} - \frac{5}{2},\ x \geq 0$ is invertible and find its inverse.

Exercise 2.4.15
The graph of an invertible function g is shown below. Find the domain and the range of g^{-1}.



Exercise 2.4.16
Are the following functions inverses of each other: $f(x) = 2(x-2)^3$ and $g(x) = 2 + \frac{\sqrt[3]{4x}}{2}$?

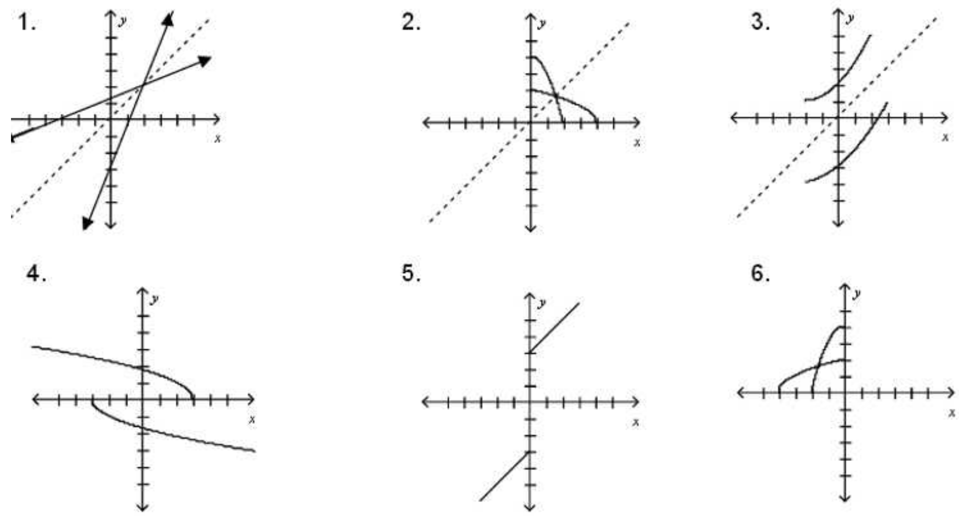
Exercise 2.4.17
Are the following functions inverses of each other: $f(x) = 4 - \frac{3}{2}x$ and $g(x) = \frac{x}{2} + \frac{3}{2}$?

Exercise 2.4.18
Consider the function $f(x) = \frac{1}{x-1}$.
(a) Find the domain and range of f.
(b) Find the inverse of f.
(c) Find the domain and range of f^{-1}.

Exercise 2.4.19
Consider the function $f(x) = \sqrt{x^3+5}$.
(a) Find the domain and range of f.
(b) Find the inverse of f.
(c) Find the domain and range of f^{-1}.

Exercise 2.4.20
Find the inverse of the function $f = \{(-10, 5), (-7, 9), (0, 6), (8, 12)\}$.

Exercise 2.4.21
Which of the following pairs of functions graphed below are inverses of each other?



Exercise 2.4.22
Suppose that $f^{-1}(5) = 3$. Find $f(3)$.

Exercise 2.4.23
Let $f(x) = \sqrt{4x + 4}$.
(a) Find the domain and range of $f(x)$.
(b) Show that f is one-to-one.
(c) Find $f^{-1}(x)$.
(d) Find the domain and range of f^{-1}.

Exercise 2.4.24
Show that $f(x) = \frac{2x+1}{x-3}$ and $g(x) = \frac{3x+1}{x-2}$ are inverses of each other.

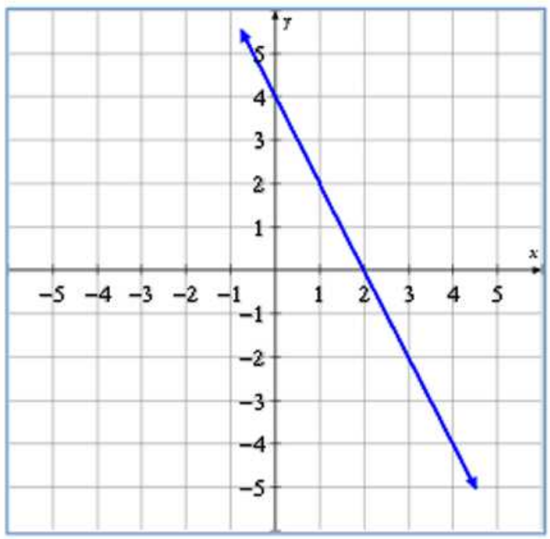
Exercise 2.4.25
Show that $f(x) = 7x - 2$ and $g(x) = \frac{x+2}{7}$ are inverses of each other.

Exercise 2.4.26
Show that the function $f(x) = \frac{2x}{7-5x}$ is one-to-one. Find its domain and range.

Exercise 2.4.27
Show that the function $f(x) = \frac{2x}{7-5x}$ is invertible. Find $f^{-1}(x)$ and then find the domain and range of $f^{-1}(x)$.

Exercise 2.4.28
The function $f(x) = \frac{1-2x}{5}$ is invertible. Use the graph of f to graph f^{-1}.

Exercise 2.4.29
Use the graph below to determine the following: (a) $f^{-1}(2)$ (b) $f^{-1}(0)$ (c) $f^{-1}(4)$.



Exercise 2.4.30
The function $f(x) = \sqrt{x+5}$, $x \geq -5$ is invertible. Sketch the graph of f and f^{-1} on the same window.

2.5 Piecewise Defined Functions

Piecewise-defined functions are functions defined by different formulas for different intervals of the independent variable.

Example 2.5.1 *(The Absolute Value Function)*
(a) Show that the function $f(x) = |x|$ is a piecewise defined function.
(b) Graph $f(x)$.

Solution.
(a) The absolute value function $f(x) = |x|$ is a piecewise defined function since

$$|x| = \begin{cases} x, & \text{for } x \geq 0 \\ -x, & \text{for } x < 0. \end{cases}$$

(b) The graph is given in Figure 2.5.1.■

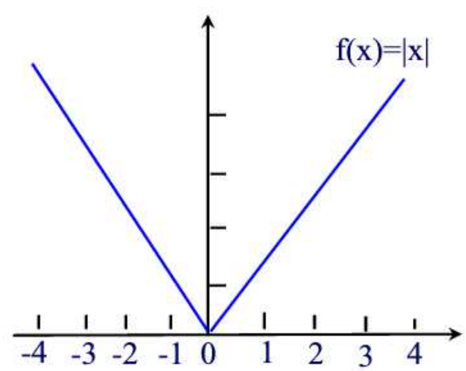


Figure 2.5.1

Example 2.5.2 *(The Ceiling Function)*
The ceiling function $f(x) = \lceil x \rceil$ is the piecewise defined function given by

$$\lceil x \rceil = \text{smallest integer greater than or equal to x.}$$

Sketch the graph of $f(x)$ on the interval $[-3, 3]$.

Solution.
The graph is given in Figure 2.5.2. An open circle represents a point which is not included ■

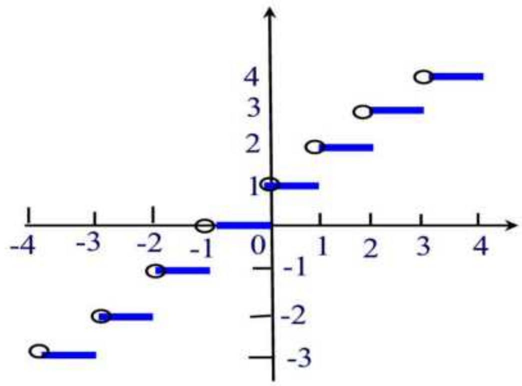


Figure 2.5.2

Example 2.5.3 *(The Floor Function)*
The floor function $f(x) = \lfloor x \rfloor$ is the piecewise defined function given by

$$\lfloor x \rfloor = \text{greatest integer less than or equal to x.}$$

Sketch the graph of $f(x)$ on the interval $[-3, 3]$.

Solution.
The graph is given in Figure 2.5.3.■

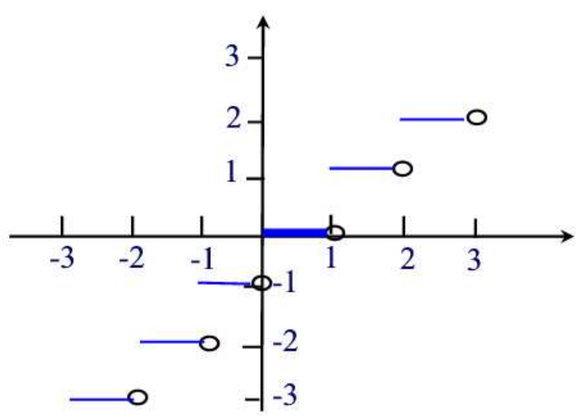


Figure 2.5.3

Example 2.5.4
Sketch the graph of the piecewise defined function given by

$$f(x) = \begin{cases} x+4, & \text{for } x \leq -2 \\ 2, & \text{for } -2 < x < 2 \\ 4-x, & \text{for } x \geq 2. \end{cases}$$

Solution.
The following table gives values of $f(x)$.

x	-3	-2	-1	0	1	2	3
f(x)	1	2	2	2	2	2	1

The graph of the function is given in Figure 2.5.4 ■

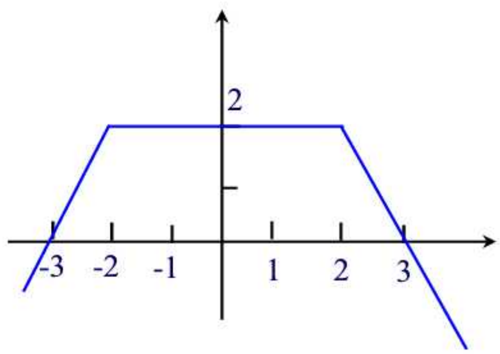


Figure 2.5.4

We conclude this section with the following real-world situation:

Example 2.5.5
The charge for a taxi ride is \$1.50 for the first $\frac{1}{5}$ of a mile, and \$0.25 for each additional $\frac{1}{5}$ of a mile (rounded up to the nearest $\frac{1}{5}$ mile).
(a) Sketch a graph of the cost function C as a function of the distance traveled x, assuming that $0 \leq x \leq 1$.
(b) Find a formula for C in terms of x on the interval $[0, 1]$.
(c) What is the cost for a $\frac{4}{5} - mile$ ride?

Solution.
(a) The graph is given in Figure 2.5.5.

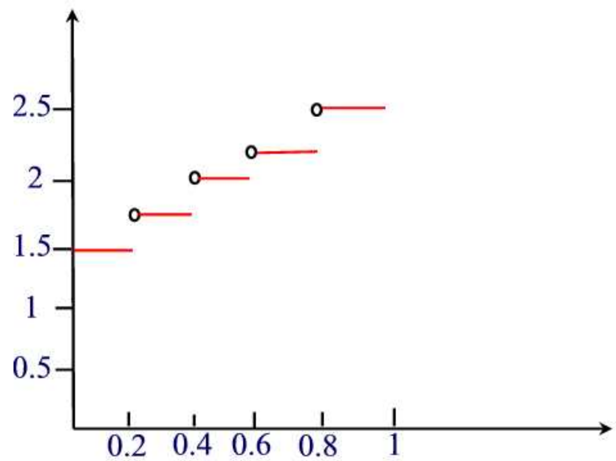


Figure 2.5.5

(b) A formula of $C(x)$ is

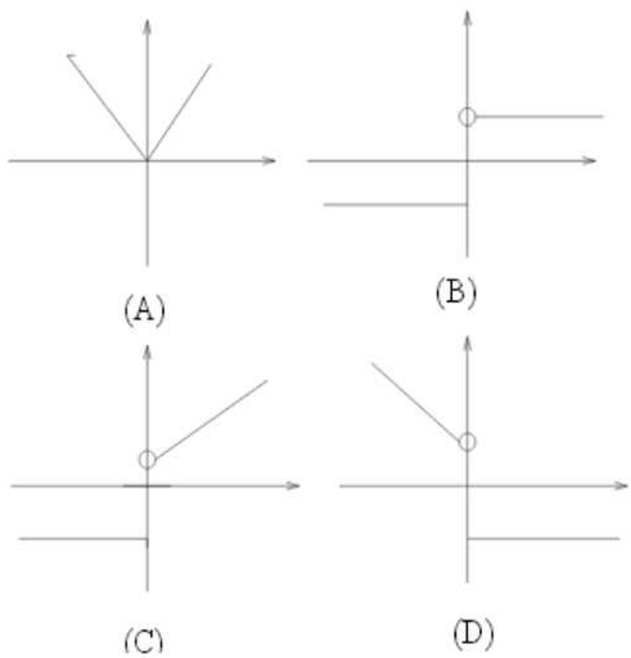
$$C(x) = \begin{cases} 1.50, & \text{if } 0 \le x \le \frac{1}{5} \\ 1.75, & \text{if } \frac{1}{5} < x \le \frac{2}{5} \\ 2.00, & \text{if } \frac{2}{5} < x \le \frac{3}{5} \\ 2.25, & \text{if } \frac{3}{5} < x \le \frac{4}{5} \\ 2.50, & \text{if } \frac{4}{5} < x \le 1. \end{cases}$$

(c) The cost for a $\frac{4}{5}$ ride is $C(\frac{4}{5}) = \$2.25$ ■

Exercises

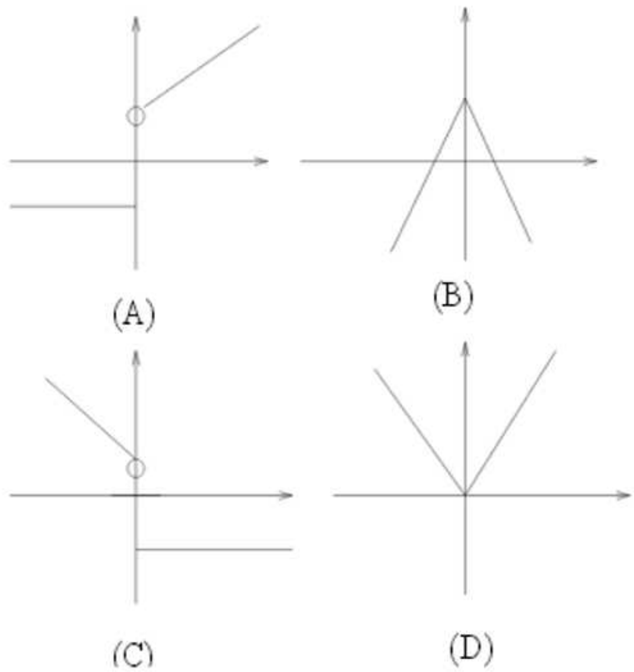
Exercise 2.5.1
Which graph corresponds to the piecewise defined function:

$$f(x) = \begin{cases} -1, & x \leq 0 \\ x+1, & x > 0. \end{cases}$$



Exercise 2.5.2
Which graph corresponds to the piecewise defined function:

$$f(x) = \begin{cases} x+1, & x \leq 0 \\ -x+1, & x > 0. \end{cases}$$

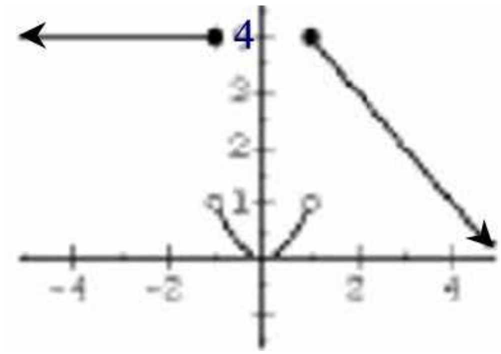


Exercise 2.5.3
Let $\lfloor x \rfloor$ denote the largest integer less than or equal to x. For example, $\lfloor -3 \rfloor = -3$, $\lfloor -3.2 \rfloor = -4$, $\lfloor 2.5 \rfloor = 2$. Calculate $\lfloor \pi \rfloor$.

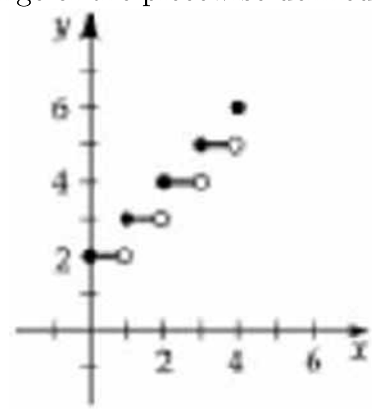
Exercise 2.5.4
Let $f(x) = \lfloor x \rfloor$ and $g(x) = \pi$. Find $(f \circ g)(x)$.

Exercise 2.5.5
What is the range of $f(x) = \lfloor x \rfloor$?

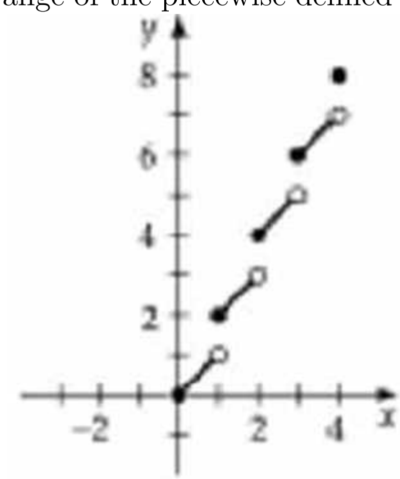
Exercise 2.5.6
Find the domain and range of the piecewise defined function pictured below.



Exercise 2.5.7
Find the domain and range of the piecewise defined function pictured below.



Exercise 2.5.8
Find the domain and range of the piecewise defined function pictured below.



Exercise 2.5.9
Sketch the graph of the piecewise defined function

$$f(x) = \begin{cases} x+3, & \text{for } x < -1 \\ x^2, & \text{for } -1 \le x \le 2 \\ 3, & \text{for } x > 2. \end{cases}$$

Exercise 2.5.10
Graph $g(x) = \lceil x \rceil$ for $-4 \le x \le 4$.

Exercise 2.5.11
Sketch the graph of the piecewise defined function

$$f(x) = \begin{cases} x^2 - 1, & \text{for } x \le 0 \\ x - 1, & \text{for } 0 \le x \le 4 \\ 3, & \text{for } x > 4. \end{cases}$$

Exercise 2.5.12
Consider the piecewise defined function

$$f(x) = \begin{cases} x^2, & \text{for } x < 2 \\ -1, & \text{for } x \ge 2. \end{cases}$$

Complete the following table.

x	-1	0	1	1.9	2	3
$f(x)$						

Exercise 2.5.13
Graph the piecewise defined function

$$f(x) = \begin{cases} 2x+2, & \text{for } x < -2 \\ 3, & \text{for } 2 \le x < 3 \\ -x, & \text{for } x \ge 3. \end{cases}$$

Exercise 2.5.14
The Heaviside step function is commonly occurred in the study of Laplace transforms. It is defined by the function

$$H(t) = \begin{cases} 1, & \text{for } t \ge 0 \\ 0, & \text{for } t < 0. \end{cases}$$

Sketch the graph of $H(x)$.

Exercise 2.5.15
Sketch the graph of the function $f(x) = x - \lfloor x \rfloor$. This function is called the **fractional part** function. It gives the fractional (noninteger) part of a real number x.

Exercise 2.5.16
Complete the following table.

x	-3	-1.5	0	1.5	2
$\lfloor x \rfloor$					
$\lceil x \rceil$					

Exercise 2.5.17
Sketch the graph of the piecewise defined function

$$f(x) = \begin{cases} x^2, & x < 1, \\ 6, & x = 1, \\ 10 - x, & x > 1. \end{cases}$$

Exercise 2.5.18
Sketch the graph of the piecewise defined function

$$f(x) = \begin{cases} -x + 3, & -3 < x < 0, \\ 3x + 1, & 0 < x < 3. \end{cases}$$

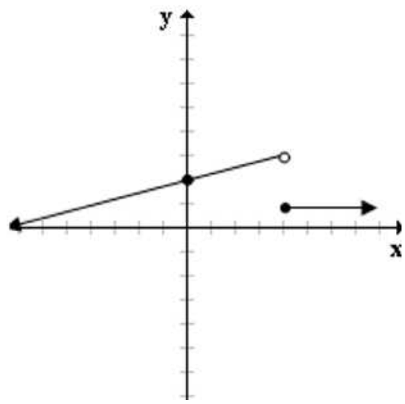
Exercise 2.5.19
Sketch the graph of the piecewise defined function

$$f(x) = \begin{cases} 0, & x < 0, \\ 1, & 0 \le x < 2, \\ 2, & x \ge 2. \end{cases}$$

Exercise 2.5.20
For a cellular phone billing plan, \$60 per month buys 450 minutes or less. Additional time costs \$0.35 per minute. Let the monthly cost $C(x)$ be a function of the time x. Write a formula of $C(x)$.

Exercise 2.5.21
In Missouri, income tax is 3.5% on the first \$9,000 of income or less, and 6% on any income in excess of \$9,000. Let the tax $T(x)$ be a function of the income x. Write a formula for $T(x)$.

Exercise 2.5.22
Give the piecewise function that is represented by the following graph.



Exercise 2.5.23
When a diabetic takes long-acting insulin, the insulin reaches its peak effect on the blood sugar level in about three hours. This effect remains fairly constant for 5 hours, then declines, and is very low until the next injection. In a typical patient, the level of insulin might be modeled by the following function.

$$f(t) = \begin{cases} 40t + 100 & 0 \le t \le 3, \\ 220 & 3 < t \le 8, \\ -80t + 860 & 8 < t \le 10, \\ 60 & 10 < t \le 24. \end{cases}$$

Here, $f(t)$ represents the blood sugar level at time t hours after the time of the injection. If a patient takes insulin at 6 am, find the blood sugar level at each of the following times.

(a) 7 am (b) 11 am (c) 3 pm (d) 5 pm.

Exercise 2.5.24
Consider the piecewise defined function

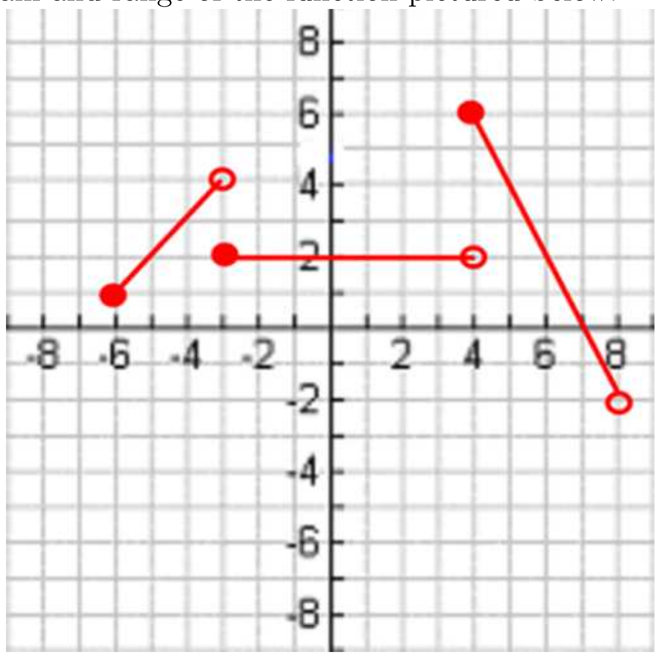
$$f(x) = \begin{cases} 2 + x & x < -4, \\ -x & -4 \le x \le 2, \\ \frac{x}{3} & x > 2. \end{cases}$$

Find (a) $f(2)$ (b) $f(3)$.

Exercise 2.5.25
Sketch the graph of

$$f(x) = \begin{cases} 2x+4 & x \leq 2, \\ -x+6 & x > 3. \end{cases}$$

Exercise 2.5.26
Find the domain and range of the function pictured below.



Exercise 2.5.27
Find the set of values where the graph of floor function has jumps.

Exercise 2.5.28
Rewrite the function $f(x) = 3|x-2| + x$ as a piecewise defined function.

Exercise 2.5.29
If x moves close to the number 2 from the right, what number does $\lfloor x \rfloor$ approach to? What if x moves from the left side toward ?

Exercise 2.5.30
The **signum function** is the piecewise defined function

$$\operatorname{sgn}(x) = \begin{cases} 1 & x > 0, \\ 0 & x = 0, \\ -1 & x < 0. \end{cases}$$

Write the function $f(x) = \operatorname{sgn}(x+2) - 3$ as a piecewise defined function.

2.6 Sequences and Summation Notation

In this section we study two important concepts that are used in calculus: Sequences and sigma notation.

Sequences

A **sequence** is a function f with domain the set of positive integers $\mathbb{N}$. Instead of using the $f(n)$ notation, however, a sequence is listed using the a_n notation. That is, $f(n) = a_n$. We call a_n the **general term** or the n^{th} term. If the range of f is finite, we say that the sequence is a **finite sequence**, otherwise the sequence is **infinite**
Several ways for denoting a sequence exists, namely,

$$(a_n)_{n\geq 1}, \quad \{a_n\}_{n=1}^{\infty}, \quad \{a_1, a_2, \cdots, a_n, \cdots\}.$$

Example 2.6.1
List the first five terms of the sequence $a_n = (1 + (-1)^n)_{n\geq 1}$.

Solution.
We have:

$$\begin{aligned}
a_1 =& 1 + (-1)^1 = 0\\
a_2 =& 1 + (-1)^2 = 2\\
a_3 =& 1 + (-1)^3 = 0\\
a_4 =& 1 + (-1)^4 = 2\\
a_5 =& 1 + (-1)^5 = 0 \blacksquare
\end{aligned}$$

There are two ways for generating the terms of a sequence: First, we can generate the terms of the sequence if the n^{th} term a_n is expressed as a function of n only. For example, the sequence $(3n - 2)_{n\geq 1}$. Another way, is to express one term in terms of one or more terms. For example, $a_n = a_{n-2} + a_{n-1}$. In this case, we say that the sequence is defined **recursively**.

Example 2.6.2
The **Fibonacci sequence** is defined recursively by: $a_1 = a_2 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$. List the first six terms of this sequence.

Solution.
We have:

$$\begin{aligned} a_1 =& 1 \\ a_2 =& 1 \\ a_3 =& a_1 + a_2 = 2 \\ a_4 =& a_2 + a_3 = 3 \\ a_5 =& a_3 + a_4 = 5 \\ a_6 =& a_4 + a_5 = 8 \blacksquare \end{aligned}$$

Example 2.6.3
Consider the **arithmetic sequence**

$$a_n = a_{n-1} + d, \quad n \geq 2$$

where a_1 is the initial value. Find an explicit formula for a_n.

Solution.
Listing the first four terms of the sequence after a_1 we find

$$\begin{aligned} a_2 =& a_1 + d \\ a_3 =& a_1 + 2d \\ a_4 =& a_1 + 3d \\ a_5 =& a_1 + 4d. \end{aligned}$$

Hence, a guess is $a_n = a_1 + (n-1)d$ $\blacksquare$

Sigma/Summation Notation

Summation is something that is done quite often in mathematics, and there is a symbol that means summation. That symbol is the capital Greek letter sigma, Σ, and so the notation is sometimes called **sigma notation** instead of summation notation. For example, a sum such as $a_1 + a_2 + \cdots + a_n$ can be shortened as

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

The value $k = 1$ is the **lower limit** of the sum and $k = n$ is the **upper limit**. The k is called the **index** of the summation.

Example 2.6.4
Find the value of the sum $\sum_{k=1}^{4}(3k-2)$.

Solution.
We have

$$\sum_{k=1}^{4}(3k-2) = [3(1)-2]+[3(2)-2]+[3(3)-2]+[3(4)-2] = 22 \blacksquare$$

Example 2.6.5
Suppose that $a_n = n$ for $n \geq 1$. Find the value of $\sum_{k=1}^{4} a_k^2$.

Solution.
We have

$$\sum_{k=1}^{4} a_k^2 = 1^2+2^2+3^2+4^2 = 30 \blacksquare$$

Properties of Summation
• You can factor a constant out of a sum: $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$.
• The summation symbol can be distributed over addition: $\sum_{k=1}^{n}(a_k+b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$.
• The summation symbol can be distributed over subtraction: $\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$.

Example 2.6.6
Suppose that $\sum_{k=1}^{n} a_k = -15$ and $\sum_{k=1}^{n} b_k = 7$. Find the value of $\sum_{k=1}^{n}(3a_k - 5b_k)$.

Solution.
We have

$$\sum_{k=1}^{n}(3a_k-5b_k) = 3\sum_{k=1}^{n} a_k - 5\sum_{k=1}^{n} b_k = 3(-15)-5(7) = -80 \blacksquare$$

Example 2.6.7
Let $(a_1+(n-1)d)_{n\geq 1}$ be the arithmetic sequence. Let S_n be the sum of the first n terms of the sequence. We call S_n the n^{th} **partial sum**. Find a formula of S_n in terms of n.

Solution.
We have

$$S_n = a_1 + a_2 + \cdots + a_n = a_1 + (a_1 + d) + \cdots + (a_1 + (n-2)d) + (a_1 + (n-1)d).$$

Writing this sum in reverse order, we find

$$S_n = (a_1 + (n-1)d) + (a_1 + (n-2)d) + \cdots + (a_1 + d) + a_1.$$

Adding the two sums, we find

$$\begin{aligned} 2S_n =&(2a_1 + (n-1)d) + (2a_1 + (n-1)d) + \cdots + (2a_1 + (n-1)d) + (2a_1 + (n-1)d) \\ =&n[2a_1 + (n-1)d] \\ S_n =&\frac{n}{2}[2a_1 + (n-1)d] \ \blacksquare \end{aligned}$$

Example 2.6.8
Find the value of the sum: $1 + 2 + 3 + \cdots + n$.

Solution.
Using the previous example with $a_1 = 1$ and $d = 1$, we have

$$1 + 2 + 3 + \cdots + n = \frac{n}{2}(2 + n - 1) = \frac{n(n+1)}{2} \ \blacksquare$$

Example 2.6.9
Let $(r^{n-1}a_1)_{n\geq 1}$ be the geometric sequence (Exercise 2.6.5) with $r \neq 1$. Find a formula of S_n in terms of n.

Solution.
We have

$$S_n = a_1 + ra_1 + r^2a_1 + \cdots + r^{n-2}a_1 + r^{n-1}a_1$$

and

$$rS_n = ra_1 + r^2a_1 + r^3a_1 + \cdots + r^{n-1}a_1 + r^na_1.$$

Hence,

$$S_n - rS_n = a_1 - r^na_1 = a_1(1 - r^n).$$

Solving for S_n, we find

$$S_n = a_1\frac{1 - r^n}{1 - r}, \ r \neq 1 \ \blacksquare$$

Example 2.6.10
Find the value of $2^n - 2^{n-1} + 2^{n-2} - 2^{n-3} + \cdots + (-1)^{n-1} \cdot 2 + (-1)^n, \quad n \geq 1.$

Solution.
We have

$$
\begin{aligned}
2^n - 2^{n-1} + 2^{n-2} - 2^{n-3} + \cdots + (-1)^{n-1} \cdot 2 + (-1)^n = \\
2^n \left[1 - \frac{1}{2} + \frac{1}{2^2} + \cdots + (-1)^{n-1}\frac{1}{2^{n-1}} + (-1)^n \frac{1}{2^n}\right] = \\
&= 2^n \frac{1 - (-2^{-1})^{n+1}}{1 - (-2^{-1})} \\
&= \frac{2^{n+1}}{3}(1 - (-\frac{1}{2})^{n+1}) \blacksquare
\end{aligned}
$$

Exercises

Exercise 2.6.1
The n^{th} term of a sequence is given by $a_n = 3n + 2$. Find the first five terms of the sequence.

Exercise 2.6.2
A sequence is defined recursively by: $a_1 = 2,\ a_2 = 1$, and $a_{n+1} = 3a_{n-1} - a_n$ for $n \geq 3$. List the first five terms of this sequence.

Exercise 2.6.3
Find the range of the sequence defined by $a_n = (-1)^n$ for $n \in \mathbb{N}$.

Exercise 2.6.4
A sequence is defined recursively by

$$\begin{aligned} a_1 =& 1 \\ a_n =& a_{n-1} + 2, \quad n \geq 2. \end{aligned}$$

Find a formula for a_n in terms of n.

Exercise 2.6.5
Consider the **geometric sequence**

$$a_n = ra_{n-1}, \quad n \geq 2$$

where a_1 is the initial value. Find an explicit formula for a_n.

Exercise 2.6.6
Write a_n in terms of n for the recursive sequence

$$\begin{aligned} a_1 =& 0 \\ a_n =& a_{n-1} + (n-1), \quad n \geq 2. \end{aligned}$$

Exercise 2.6.7
Find a formula for the sum:

$$1 + 2 + \cdots + (n-1), \quad n \geq 2.$$

Exercise 2.6.8
Find a formula for the sum:

$$3 + 2 + 4 + 6 + 8 + \cdots + 2n, \quad n \geq 1.$$

Exercise 2.6.9
Find a formula for the sum:

$$3 \cdot 1 + 3 \cdot 2 + 3 \cdot 3 + \cdots 3 \cdot n, \quad n \geq 1.$$

Exercise 2.6.10
Find a formula for the sum

$$1 + 2 + 2^2 + \cdots + 2^{n-1}, \quad n \geq 1.$$

Exercise 2.6.11
Find a formula for the sum

$$3^{n-1} + 3^{n-2} + \cdots + 3^2 + 3 + 1, \quad n \geq 1.$$

Exercise 2.6.12
Find a formula for the sum

$$3 \cdot 2^{n-1} + 3 \cdot 2^{n-2} + 3 \cdot 2^{n-3} + \cdots + 3 \cdot 2^2 + 3 \cdot 2 + 3, \quad n \geq 1.$$

Exercise 2.6.13
Use iteration to guess a formula for the following recursively defined sequence: $c_1 = 1, c_n = 3c_{n-1} + 1$, for all $n \geq 2$.

Exercise 2.6.14
Let $a_1, a_2, \cdots$ be the sequence defined by the explicit formula

$$a_n = C \cdot 2^n + D, \quad n \geq 1$$

where C and D are real numbers.
Find C and D so that $a_1 = 1$ and $a_2 = 3$. What is a_3 in this case?

Exercise 2.6.15
Let $a_1, a_2, \cdots$ be the sequence defined by the explicit formula

$$a_n = C \cdot 2^n + D, \quad n \geq 1$$

where C and D are real numbers.
Find C and D so that $a_1 = 0$ and $a_2 = 2$. What is a_3 in this case?

Exercise 2.6.16
Suppose that $\sum_{k=1}^{n} a_k = 8$ and $\sum_{k=1}^{n} b_k = -4$. Find the value of $\sum_{k=1}^{n}(7a_k - 2b_k)$.

Exercise 2.6.17
Find the value of $\sum_{k=1}^{4}(3k)$.

Exercise 2.6.18
We define the **factorial** of a whole number n by $n! = n(n-1)\cdots 2\cdot 1$ and $0! = 1$. Write the first five terms of the sequence with n^{th} term $a_n = n!$.

Exercise 2.6.19
Find the value of the sum $\sum_{k=1}^{n} a$ where a is a constant.

Exercise 2.6.20
Given that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ and $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. Find the value of $\sum_{k=1}^{n}(2k+3)^2$. Do not simplify your answer.

Exercise 2.6.21
Find the range of the sequence $\{1 + (-1)^n\}_{n\geq 1}$.

Exercise 2.6.22
Write the following sum in summation notation:

$$2 - 4 + 8 - 16 + 32 - 64.$$

Exercise 2.6.23
Let $a_n = \left(1 + \frac{1}{n}\right)^n$. Find $a_{10{,}000}, a_{100{,}000}$, and $a_{1{,}000{,}000}$ to the nearest thousandth. Use your calculator to approximate the number e to the nearest thousandth. What can you conclude?

Exercise 2.6.24
Find the value of the sum $\sum_{k=1}^{4} k\left(\frac{1}{k} + k\right)$.

Exercise 2.6.25
A sequence is defined recursively by $a_1 = 1$ and $a_{n+1} = (n+1)a_n$. Find a formula of a_n in terms of n.

Exercise 2.6.26
Find the value of the sum $10 + 20 + 30 + \cdots + 10n$.

Exercise 2.6.27
The 17^{th} term of an arithmetic sequence is 34 and the first term is 2, what is the value of the common difference d?

Exercise 2.6.28
The first term of an arithmetic sequence is equal to 6 and the common difference is equal to 3. Find a formula for the n^{th} term and the value of the 50^{th} term.

Exercise 2.6.29
The third term of a geometric sequence is 3 and the sixth term is $\frac{1}{9}$. Find the first term.

Exercise 2.6.30
A ball is dropped from a height of 8 feet. The ball bounces to 80% of its previous height with each bounce. How high (to the nearest tenth of a foot) does the ball bounce on the fifth bounce?

2.7 Average Rate of Change

Functions given by tables of values have their limitations in that nearly always leave gaps. One way to fill these gaps is by using the **average rate of change**. For example, Table 1 below gives the population of the United States between the years 1950 - 1990.

d(year)	1950	1960	1970	1980	1990
N(in millions)	151.87	179.98	203.98	227.23	249.40

Table 1

This table does not give the population in 1972. One way to estimate $N(1972)$, is to find the average yearly rate of change of N from 1970 to 1980 given by

$$\frac{227.23 - 203.98}{10} = 2.325 \textit{ million people per year.}$$

Then,

$$N(1972) = N(1970) + 2(2.325) = 208.63 \textit{ million.}$$

Average rates of change can be calculated not only for functions given by tables but also for functions given by formulas. The **average rate of change** of a function $y = f(x)$ from $x = a$ to $x = b$ is given by the **difference quotient**

$$\frac{\Delta y}{\Delta x} = \frac{\text{Change in function value}}{\text{Change in x value}} = \frac{f(b) - f(a)}{b - a}.$$

Geometrically, this quantity represents the slope of the secant line going through the points $(a, f(a))$ and $(b, f(b))$ on the graph of $f(x)$. See Figure 2.7.1.

The average rate of change of a function on an interval tells us how much the function changes, on average, per unit change of x within that interval. On some part of the interval, f may be changing rapidly, while on other parts f may be changing slowly. The average rate of change evens out these variations.

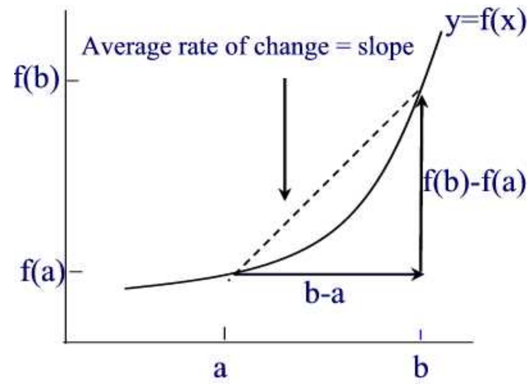


Figure 2.7.1

Example 2.7.1
Find the average rate of change of the function $f(x) = x^2$ from $x = 3$ to $x = 5$.

Solution.
The average rate of change is

$$\frac{\Delta y}{\Delta x} = \frac{f(5) - f(3)}{5 - 3} = \frac{25 - 9}{2} = 8 \blacksquare$$

Example 2.7.2 *(Average Speed)*
During a typical trip to school, your car will undergo a series of changes in its speed. If you were to inspect the speedometer readings at regular intervals, you would notice that it changes often. The speedometer of a car reveals information about the instantaneous speed of your car; that is, it shows your speed at a particular instant in time. The instantaneous speed of an object is not to be confused with the average speed. Average speed is a measure of the distance traveled in a given period of time. That is,

$$\text{Average Speed} = \frac{\text{Distance traveled}}{\text{Time elapsed}}.$$

If the trip to school takes 0.2 hours (i.e. 12 minutes) and the distance traveled is 5 miles then what is the average speed of your car?

Solution.
The average speed is given by

$$\text{Ave. Speed} = \frac{5 \text{ miles}}{0.2 \text{ hours}} = 25\text{miles/hour}.$$

This says that on the average, your car was moving with a speed of 25 miles per hour. During your trip, there may have been times that you were stopped and other times that your speedometer was reading 50 miles per hour; yet on the average you were moving with a speed of 25 miles per hour ■

Average Rate of Change and Increasing/Decreasing Functions

Now, we would like to use the concept of the average rate of change to test whether a function is increasing or decreasing on a specific interval. First, we recall the following definition: We say that a function is **increasing** if the function values increase as x increases, i.e., $f(a) < f(b)$ whenever $a < b$. Graphically, a function is increasing if its graph climbs as x moves from left to right. It is said to be **decreasing** if the function values decrease as x increases, i.e., $f(a) > f(b)$ whenever $a < b$. In this case, the graph of f falls as x moves from left to right.
As an application of the average rate of change, we can use such quantity to decide whether a function is increasing or decreasing. If a function f is increasing on an interval I then by taking any two points in the interval I, say $a < b$, we see that $f(a) < f(b)$ and in this case

$$\frac{f(b) - f(a)}{b - a} > 0.$$

Going backward with this argument we see that if the average rate of change is positive in an interval then the function is increasing in that interval. Similarly, if the average rate of change is negative in an interval I then the function is decreasing there and conversely if a function is decreasing on an interval I then its rate of change is negative.

Example 2.7.3
The table below gives values of a decreasing function $w = f(t)$. Verify that the rate of change is always negative.

t	0	4	8	12	16	20	24
w	100	58	32	24	20	18	17

Solution.
The average of w over the interval $[0, 4]$ is

$$\frac{w(4) - w(0)}{4 - 0} = \frac{58 - 100}{4 - 0} = -10.5.$$

The average rate of change of the remaining intervals are given in the chart below

time interval	[0,4]	[4,8]	[8,12]	[12,16]	[16, 20]	[20,24]
Average	-10.5	-6.5	-2	-1	-0.5	-0.25

Notice that the average rate of change is always negative on $[0, 24]$ ■

Rate of Change and Concavity

Some graphs of functions may bend up or down as shown in the following two examples.

Example 2.7.4
Consider the following two graphs in Figure 2.7.2.

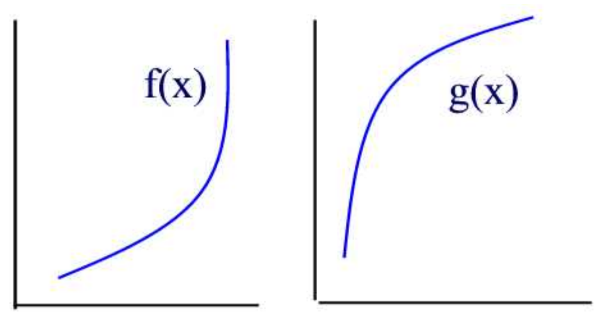


Figure 2.7.2

(a) What do the graphs above have in common?
(b) How are they different? Specifically, look at the rate of change of each.

Solution.
(a) Both graphs represent increasing functions.
(b) The rate of change of $f(x)$ is more and more positive so the graph bends up whereas the rate of change of $g(x)$ is less and less positive and so it bends down ■

The following example deals with version of the previous example for decreasing functions.

Example 2.7.5
Consider the following two graphs given in Figure 2.7.3.

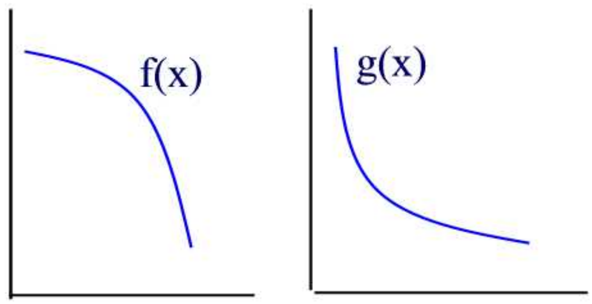


Figure 2.7.3

(a) What do the graphs above have in common?
(b) How are they different? Specifically, look at the rate of change of each.

Solution.
(a) Both functions are decreasing.
(b) The rate of change of $f(x)$ is more and more negative so the graph bends down, whereas the rate of change of $g(x)$ is less and less negative so the graph bends up.

Conclusions:
• When the rate of change of a function is increasing then the function is **concave up.** That is, the graph bends upward.
• When the rate of change of a function is decreasing then the function is **concave down**. That is, the graph bends downward.

The following example discusses the concavity of a function given by a table.

Example 2.7.6
Given below is the table for the function $H(x)$. Calculate the rate of change for successive pairs of points. Decide whether you expect the graph of $H(x)$ to concave up or concave down?

x	*12*	*15*	*18*	*21*
H(x)	*21.40*	*21.53*	*21.75*	*22.02*

Solution.

$$\begin{aligned}
\frac{H(15)-H(12)}{15-12} &= \frac{21.53-21.40}{3} \approx 0.043\\
\frac{H(18)-H(15)}{18-15} &= \frac{21.75-21.53}{3} \approx 0.073\\
\frac{H(21)-H(18)}{21-18} &= \frac{22.02-21.75}{3} \approx 0.09
\end{aligned}$$

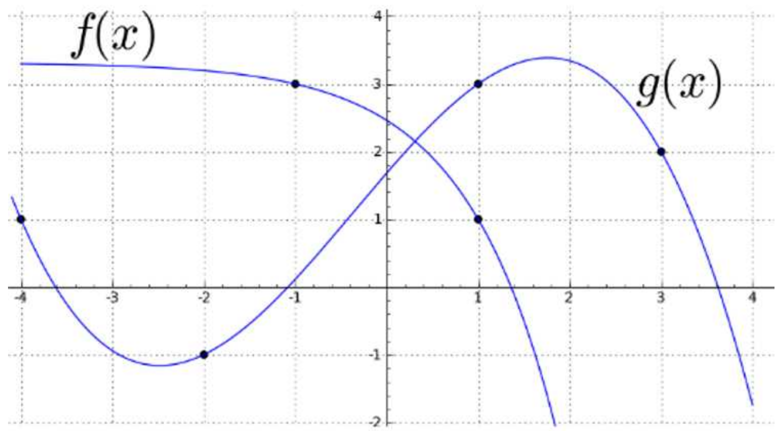
Since the rate of change of $H(x)$ is increasing, the function is concave up ■

Exercises

Exercise 2.7.1
Calculate the average rate of change of $f(x) = \sqrt{x}$ on the interval $[4, 9]$.

Exercise 2.7.2
Calculate the average rate of change of $f(x) = x^3 - x$ on the interval $[1, 2]$.

Exercise 2.7.3
Calculate the average rate of change of the functions whose graphs are shown below on the indicated intervals.
(a) f on $[-1, 1]$ (b) g on $[-4, 1]$ (c) g on $[1, 3]$.



Exercise 2.7.4
Compute the average rate of change of $f(x) = \sqrt{x-1}$ on the interval $[x, x+h]$. Simplify so that the substitution $h = 0$ does not yield $\frac{0}{0}$.

Exercise 2.7.5
Compute the average rate of change of $f(x) = \frac{1}{x-1}$ on the interval $[x, x+h]$. Simplify so that the substitution $h = 0$ does not yield $\frac{0}{0}$.

Exercise 2.7.6
Suppose an object's position (measured in meters, along a straight line) at time t (in seconds) is described by $s(t) = t^2 - 5t + 1$. What is the object's average speed between time 0 and 3 seconds later?

Exercise 2.7.7
A climber is on a hike. After 2 hours he is at an altitude of 400 feet. After 6 hours, he is at an altitude of 700 feet. What is the average rate of change?

Exercise 2.7.8
A teacher weighed 145 lbs in 1986 and weighs 190 lbs in 2007. What was the average rate of change in weight?

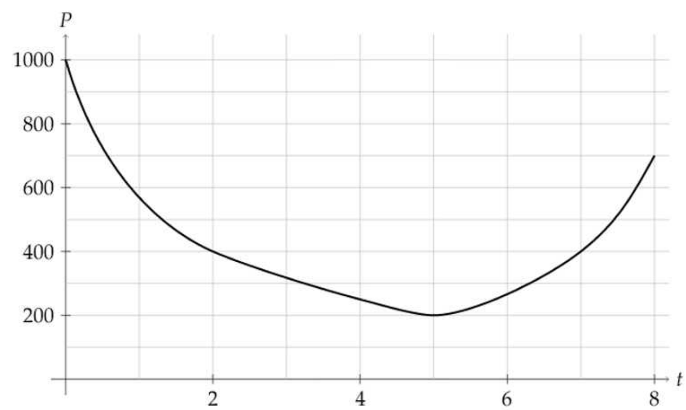
Exercise 2.7.9
Suppose $f(t) = t^3 + 1$. Find a value $A \geq 0$ such that the average rate of change of $f(t)$ from 0 to A equals 2.

Exercise 2.7.10
A train travels from city A to city B. The cities are 600 miles apart. The distance from city A at t hours after the train leaves A is given by

$$d(t) = 50t + t^2.$$

What is the average velocity of the train in miles per hour during the trip from A to B?

Exercise 2.7.11
Below is a graph of the population $P(t)$ of wolves in a forest t years after the year 2000.



(a) Over which time intervals is the graph increasing? decreasing? concave up? concave down?
(b) What is the average rate of change of the population from 2000 to 2002?
(c) What is the percentage change in the population from 2007 to 2008?

Exercise 2.7.12
A lead ball is tossed into the air by a person standing on a bridge. The height of the ball (in meters above the water) is modelled by the equation

$$s(t) = -4.9t^2 + 5t + 40.$$

(a) Calculate the average velocity for the intervals $\left[0, \frac{1}{2}\right]$ and $\left[\frac{1}{2}, 1\right]$.
(b) Using (a), what can you conclude about the flight of the ball?

Exercise 2.7.13
The total acreage of farms in the US has decreased since 1980.

Year	1980	1985	1990	1995	2000
Farmland (million acres)	1039	1012	987	963	945

Find the average rate of change in farm land between 1985 and 2000.

Exercise 2.7.14
The total acreage of farms in the US has decreased since 1980.

Year	1980	1985	1990	1995	2000
Farmland (million acres)	1039	1012	987	963	945

Let f denote the farm land in millions of acres. Is f concave down or up between 1980 and 2000?

Exercise 2.7.15
Let $f(x) = mx + b$. Find the average rate of change of f in each of the following intervals:
(a) $[1, 7]$ (b) $[-4, 4]$ (c) $[9, 100]$ (d) $[a, a + h]$.
What do you notice about the average rate of change?

Exercise 2.7.16
Values of a function f are given below.

x	1	2	3	4	5
$f(x)$	2	4	8	16	32

Determine whether the function is concave up or concave down.

Exercise 2.7.17
Values of a function f are given below.

x	1	2	3	4	5
$f(x)$	300	290	270	240	200

Determine whether the function is concave up or concave down.

Exercise 2.7.18
Values of a function f are given below.

x	1	2	3	4	5
$f(x)$	90	70	80	75	72

Determine whether the function is concave up or concave down.

Exercise 2.7.19
Find the slope of the line that passes through the points $(1, 5)$ and $(3, 9)$.

Exercise 2.7.20
Consider the function $f(x) = -16x^2 + 52x + 125$. Find the slope of the secant line that passes through the points $(3, f(3))$ and $(0, f(0))$.

Exercise 2.7.21
Let $f(x) = x^2$. Find an expression for $\frac{f(a+\Delta x)-f(a)}{\Delta x}$.

Exercise 2.7.22
The following table records the temperature at a particular location during a certain day.

Time	7am	8am	9am	10am	11am	noon	1pm	2pm	3pm	4pm	5pm
Temp.	49	58	66	72	76	79	80	80	78	74	69

(a) Describe the change in temperature from 7am to noon.
(b) Describe the percentage change in temperature between 11am and 4pm.
(c) Describe the average change in temperature between 9am and 5pm.

Exercise 2.7.23
A company determines that the cost in dollars of manufacturing x units of a certain item is $C(x) = 100 + 5x - x^2$. Find the average rate of change of cost per unit for manufacturing between 1 and 5 units.

Exercise 2.7.24
The height of a ball thrown upward at a speed of 30 ft/s from a height of 15 feet after t seconds is given by

$$h(t) = 15 + 30t - 16t^2.$$

Find the average velocity of the ball in the first 2 seconds after it is thrown.

Exercise 2.7.25
An oil slick has area $f(x) = 30x^3 + 100x$ square meters x minutes after a tanker explosion. Find the average rate of change in area with respect to time during the period from $x = 2$ to $x = 3$.

Exercise 2.7.26
A function f is described by the table below.

x	0	2	4	5
$f(x)$	26	17	5	1

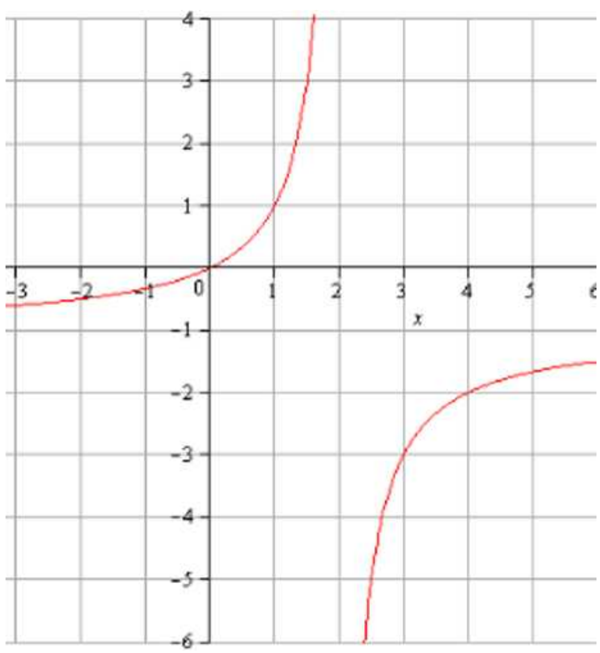
Without doing any computations, what is the sign of the average rate of change of $f(x)$ on any interval between 0 and 5?

Exercise 2.7.27
A function f is described by the table below.

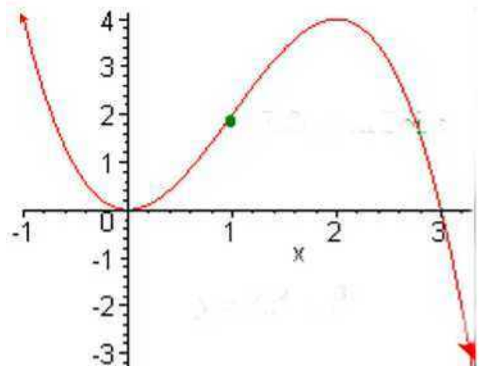
x	0	2	4	5
$f(x)$	26	17	5	-1.5

Determine whether the graph of f is concave down or up on the interval $[0, 5]$.

Exercise 2.7.28
Shown is the graph of $f(x)$. Find the average rate of change of f on: (a) $[0, 3]$ (b) $[1, 4]$.



Exercise 2.7.29
Shown is the graph of $f(x)$. Over what interval is the rate of change (a) increasing (b) decreasing?



Exercise 2.7.30
Determine the average rate of change of the function $f(x) = \frac{2}{x}$ from $x = a$ to $x = a + h$.

Chapter 3

Polynomial Functions

Polynomial functions are considered "nice" functions because of the following two properties:
• **Continuity**: By that we mean that the graph of the function can be drawn without picking up a pencil. There are no jumps or holes in the graph.
• **Smooth:** By that we mean that the graph has no sharp turns (like an absolute value).

This chapter is devoted for the study of polynomials.

3.1 Linear Functions

In Section 2.7, we introduced the average rate of change of a function. In general, the average rate of change of a function is different on different intervals. For example, consider the function $f(x) = x^2$. The average rate of change of $f(x)$ on the interval $[0, 1]$ is

$$\frac{f(1) - f(0)}{1 - 0} = 1.$$

The average rate of change of $f(x)$ on $[1, 2]$ is

$$\frac{f(2) - f(1)}{2 - 1} = 3.$$

A **linear** function is a function with the property that the average rate of change on any interval is the same. We say that y is changing at a constant rate with respect to x. Thus, y changes by the same amount for every unit change in x. Geometrically, the graph is a straight line.

Example 3.1.1
Suppose you pay \$192 to rent a booth for selling necklaces at an art fair. The necklaces sell for \$32. Explain why the function that shows your net income (revenue from sales minus rental fees) as a function of the number of necklaces sold is a linear function.

Solution.
Let $P(n)$ denote the net income from selling n necklaces. Each time a necklace is sold, that is, each time n is increased by 1, the net income P is increased by the same constant, \$32. Thus the rate of change for P is always the same, and hence P is a linear function ■

Testing Data for Linearity
Next, we will consider the question of recognizing a linear function given by a table.
Let f be a linear function given by a table. Then the rate of change is the same for all pairs of points in the table. In particular, when the x values are evenly spaced the change in y is constant.

Example 3.1.2
Which of the following tables could represent a linear function?

x	0	5	10	15
$f(x)$	10	20	30	40

x	0	10	20	30
$g(x)$	20	40	50	55

Solution.
Since equal increments in x yield equal increments in y, $f(x)$ is a linear function. On the contrary, equal increments in x do not result in equal increments in $g(x)$. Thus, $g(x)$ is not linear ■

It is possible to have a table of linear data in which neither the $x-$values nor the $y-$values go up by equal amounts. However, the rate of change of any pairs of points in the table is constant.

Example 3.1.3
The following table contains linear data, but some data points are missing. Find the missing data points.

x	2	5		8	
y	5		17	23	29

Solution.
Consider the points $(2,5),(5,a),(b,17),(8,23)$, and $(c,29)$. Since the data is linear, we must have $\frac{a-5}{5-2}=\frac{23-5}{8-2}$. That is, $\frac{a-5}{3}=3$. Cross multiplying to obtain $a-5=9$ and by adding 5 to both sides we find $a=14$. It follows that when x is increased by 1, y increases by 3. Hence, $b=6$ and $c=10$ ■

Now, suppose that $f(x)$ is a linear function of x. Then f changes at a constant rate m. That is, if we pick two points $(0,f(0))$ and $(x,f(x))$ then

$$m=\frac{f(x)-f(0)}{x-0}.$$

That is, $f(x)=mx+f(0)$. This is the function notation of the linear function $f(x)$. Another notation is the equation notation, $y=mx+f(0)$. We will denote the number $f(0)$ by b. In this case, the linear function will be written as $f(x)=mx+b$ or $y=mx+b$. Since $b=f(0)$, the point $(0,b)$ is the point where the line crosses the vertical line. We call it the **y-intercept**. So the $y-$intercept is the output corresponding to the input $x=0$, sometimes known as the **initial value** of y.

If we pick any two points (x_1, y_1) and (x_2, y_2) on the graph of $f(x) = mx + b$ then we must have

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}}.$$

We call m the **slope** of the line.

Example 3.1.4
The value of a new computer equipment is \$20,000 and the value drops at a constant rate so that it is worth \$ 0 after five years. Let $V(t)$ be the value of the computer equipment t years after the equipment is purchased.
(a) Find the slope m and the $y-$intercept b.
(b) Find a formula for $V(t)$.

Solution.
(a) Since $V(0) = 20,000$ and $V(5) = 0$, the slope of $V(t)$ is

$$m = \frac{0 - 20,000}{5 - 0} = -4,000$$

and the vertical intercept is $V(0) = 20,000$.
(b) A formula of $V(t)$ is $V(t) = -4,000t + 20,000$. In financial terms, the function $V(t)$ is known as the **straight-line depreciation** function ■

Formulas for Linear Functions
Next, we will discuss ways for finding the formulas for linear functions. Recall that f is linear if and only if $f(x)$ can be written in the form $f(x) = mx + b$. So the problem of finding the formula of f is equivalent to finding the slope m and the vertical intercept b.
Suppose that we know two points on the graph of $f(x)$, say $(x_1, f(x_1))$ and $(x_2, f(x_2))$. Since the slope m is just the average rate of change of $f(x)$ on the interval $[x_1, x_2]$, we have

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

To find b, we use one of the points in the formula of $f(x)$; say we use the first point. Then $f(x_1) = mx_1 + b$. Solving for b we find

$$b = f(x_1) - mx_1.$$

Example 3.1.5
Let's find the formula of a linear function given by a table of data values. The table below gives data for a linear function. Find the formula.

x	1.2	1.3	1.4	1.5
f(x)	0.736	0.614	0.492	0.37

Solution.
We use the first two points to find the value of m :

$$m = \frac{f(1.3) - f(1.2)}{1.3 - 1.2} = \frac{0.614 - 0.736}{1.3 - 1.2} = -1.22.$$

To find b we can use the first point to obtain

$$0.736 = -1.22(1.2) + b.$$

Solving for b we find $b = 2.2$. Thus,

$$f(x) = -1.22x + 2.2 \blacksquare$$

Example 3.1.6
Suppose that the graph of a linear function is given and two points on the graph are known. For example, Figure 3.1.1 is the graph of a linear function going through the points $(100, 1)$ and $(160, 6)$. Find the formula.

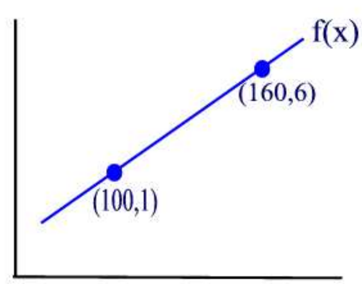


Figure 3.1.1

Solution.
The slope m is found as follows:

$$m = \frac{6 - 1}{160 - 100} = 0.083.$$

To find b we use the first point to obtain $1 = 0.083(100) + b$. Solving for b we find $b = -7.3$. So the formula for the line is $f(x) = -7.3 + 0.083x$ ■

Example 3.1.7
Sometimes a linear function is given by a verbal description as in the following problem: In a college meal plan you pay a membership fee; then all your meals are at a fixed price per meal. If 30 meals cost \$152.50 and 60 meals cost \$250 then find the formula for the cost C of a meal plan in terms of the number of meals n.

Solution.
We find m first:

$$m = \frac{250 - 152.50}{60 - 30} = \$3.25/meal.$$

To find b or the membership fee we use the point $(30, 152.50)$ in the formula $C = mn + b$ to obtain $152.50 = 3.25(30) + b$. Solving for b we find $b = \$55$. Thus, $C = 3.25n + 55$ ■

So far we have represented a linear function by the expression $y = mx + b$. This is known as the **slope-intercept form** of the equation of a line. Now, if the slope m of a line is known and one point (x_0, y_0) is given then by taking any point (x, y) on the line and using the definition of m we find

$$\frac{y - y_0}{x - x_0} = m.$$

Cross multiply to obtain: $y - y_0 = m(x - x_0)$. This is known as the **point-slope form** of a line.

Example 3.1.8
Find the equation of the line passing through the point $(100, 1)$ and with slope $m = 0.01$.

Solution.
Using the above formula we have: $y - 1 = 0.01(x - 100)$ or $y = 0.01x$ ■

Note that the form $y = mx + b$ can be rewritten in the form

$$Ax + By + C = 0 \tag{3.1.1}$$

where $A = m, B = -1$, and $C = b$. The form (3.1.1) is known as the **standard form** of a linear function.

Example 3.1.9
Rewrite in standard form: $3x + 2y + 40 = x - y$.

Solution.
Subtracting $x - y$ from both sides to obtain $2x + 3y + 40 = 0$ ■

Exercises

Exercise 3.1.1
Find the slope and the $x-$ and $y-$intercepts of the line $y + 3 = -2(x - 5)$.

Exercise 3.1.2
In economics the **demand function** relates the price per unit of an item to the number of units that consumers will buy at that price. The demand, q, is considered to be the independent variable, while the price, p, is considered to be the dependent variable.
Suppose that in a certain market, the demand function for widgets is a linear function

$$p = -0.75q + 54,$$

where p is the price in dollars and q is the number of units (hundreds of widgets in this case).
(a) What is the slope of this function? Explain the meaning of the sign of the slope in practical terms.
(b) Find the $p-$ and $q-$ intercepts for this function. What is the significance of these intercepts in the context of the problem?

Exercise 3.1.3
A small college has 2546 students in 1994 and 2702 students in 1996. Assume that the enrollment follows a linear growth pattern. Let $t = 0$ correspond to 1990 and let $y(t)$ represent the enrollment in year t. Assume that $y(t)$ is linear.
(a) Using the data given, find the slope of $y(t)$.
(b) What does the slope of $y(t)$ signify in terms of enrollment growth?
(c) Find an equation for $y(t)$ and use it predict the enrollment of the college in 1999.

Exercise 3.1.4
Find the equation of the line with slope $-\frac{2}{3}$ and crossing the point $(6, -1)$.

Exercise 3.1.5
Find the equation of the line containing the points $(7, -1)$ and $(4, 5)$.

Exercise 3.1.6
Find the equation of the line with slope 6 and crossing the graph of $f(x) = x^2$ at $x = 3$.

Exercise 3.1.7
Find the equation of the line passing through $(4,0)$ and the graph of $f(x) = x^{\frac{2}{3}}$ at $x = -8$.

Exercise 3.1.8
Find the slope and the $y-$ intercept of the line $9x + 2y = 10$.

Exercise 3.1.9
Find the slope and the $y-$ intercept of the line $y = 2$.

Exercise 3.1.10
A company has fixed costs of \$7,000 for plant and equuipment and variable costs of \$600 for each unit of output. What is total cost at varying levels of output, assuming linear costs?

Exercise 3.1.11
A company receives \$45 for each unit of output sold. It has a variable cost of \$25 per item and a fixed cost of \$1600. What is its profit if it sells (a) 75 items, (b)150 items?

Exercise 3.1.12
Sketch the graph of $3x + 2y = -4$.

Exercise 3.1.13
Sketch the graph of $2x - y = -1$.

Exercise 3.1.14
A copy shop can produce a course reader at a cost of \$25 per copy. The monthly fixed costs are \$10,000.
(a) Determine the total monthly cost as a function of the number of copies produced.
(b) Graph the total monthly cost function.

Exercise 3.1.15
Find the equation of a line in standard form that passes through the points $(-1,4)$ and $(3,2)$.

Exercise 3.1.16
Write in standard form the equation of the line passing through the point $(-3,-4)$ and with slope equals to $\frac{3}{2}$.

Exercise 3.1.17
Identify the true statement below.
(a) All linear graphs are functions
(b) The graph of $y = 4$ is a function
(c) To find the $y-$ intercept of a function, let y equal to zero and solve for x
(d) The graph of $x = 4$ is a function

Exercise 3.1.18
Identify the false statement below.
(a) The standard form of a linear equation is $y = ax + b$.
(b) The point $(4, 0)$ is on the graph of $3x - 4y = 12$.
(c) The graph of $3x - 4y = 12$ passes the vertical line test for functions.
(d) The $y-$intercept of $3x - 4y = 12$ is $(0, -3)$.

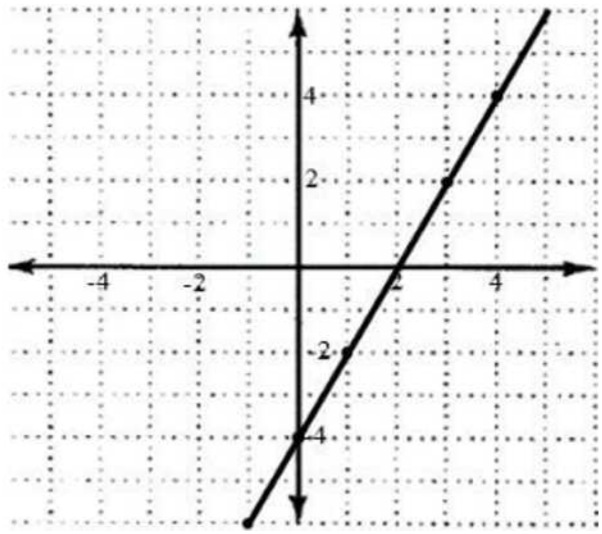
Exercise 3.1.19
A school's funding for free and reduced lunches increases by \$700 for each additional economically disadvantaged (e.d.) student enrolled. Last year, 250 e.d. students were enrolled and the school collected \$250,000. What is the amount collected if n economically disadvantaged students are enrolled?

Exercise 3.1.20
Let C denote a temperature on the Celsius scale, and let F denote the corresponding temperature on the Fahrenheit scale. Find a linear function relating C and F; use the facts that 32°F corresponds to 0°C and 212°F corresponds to 100°C.

Exercise 3.1.21
Give the formula of the linear function that generates the following table

x	-5	-2	1	4	9
$f(x)$	91	67	43	19	-21

Exercise 3.1.22
Give the formula of the linear function that generates the following graph.



Exercise 3.1.23
Give the equation of the horizontal line passing through $(1, 2)$.

Exercise 3.1.24
Give the equation of the vertical line passing through $(-3, 5)$.

Exercise 3.1.25
Give the equation of (a) the $x-$axis (b) the $y-$axis.

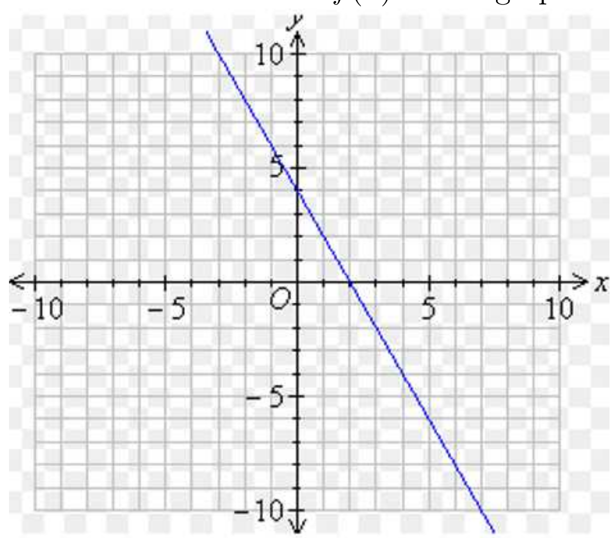
Exercise 3.1.26
A gym charges \$175 initiation fee plus \$35 per month. Write the cost function $C(x)$ where x in the number of months of enrollment.

Exercise 3.1.27
A computer that cost \$1200 new is expected to depreciate linearly at a rate of \$300 per year.
(a) Determine the linear depreciation function f.
(b) Explain why the domain of f is $[0, 4]$?

Exercise 3.1.28
The length of a steel beam is a linear function of the temperature. When temperature is 40°F a certain steel beam is 50 feet long and at 95°F the same beam is 50.12 feet long. What is the length of the beam at 70°F? At what temperature is the beam 49.9 feet long?

Exercise 3.1.29
Find the slope and the $y-$intercept of the line with the equation $10x+2y=7$.

Exercise 3.1.30
Find the formula of the linear function $f(x)$ whose graph is pictured below.



3.2 Geometric Properties of Linear Functions

In this section we discuss four geometric related questions of linear functions. The first question considers the significance of the parameters m and b in the equation $f(x) = mx + b$.
We have seen that the graph of a linear function $f(x) = mx + b$ is a straight line. But a line can be horizontal, vertical, rising to the right or falling to the right. The slope is the parameter that provides information about the steepness of a straight line.
• If $m = 0$ then $f(x) = b$ is a constant function whose graph is a horizontal line at $(0, b)$.
• For a vertical line, the slope is undefined since any two points on the line have the same $x-$value and this leads to a division by zero in the formula for the slope. The equation of a vertical line has the form $x = a$. Note that a vertical line is not a function.
• Suppose that the line is neither horizontal nor vertical. If $m > 0$ then by Section 2.7, $f(x)$ is increasing. That is, the line is rising to the right.
• If $m < 0$ then $f(x)$ is decreasing. That is, the line is falling to the right.
• The slope, m, tells us how fast the line is climbing or falling. The larger the value of m the more the line rises and the smaller the value of m the more the line falls.
The parameter b tells us where the line crosses the vertical axis.

Example 3.2.1
Arrange the slopes of the lines in the Figure 3.2.1 from largest to smallest.

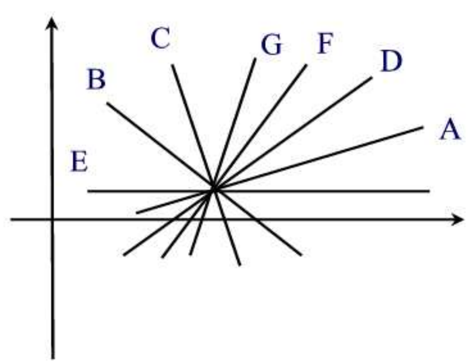


Figure 3.2.1

Solution.
According to Figure 3.2.1 we have

$$m_G > m_F > m_D > m_A > m_E > m_B > m_C \blacksquare$$

The second question of this section is the question of finding the point of intersection of two lines. The point of intersection of two lines is basically the solution to a system of two linear equations. This system can be solved by either the method of elimination or the method of substitution as discussed in Section 1.6.

Example 3.2.2
Find the point of intersection of the two lines $y + \frac{x}{2} = 3$ and $2(x+y) = 1-y$.

Solution.
We will use the method of substitution. Solving the first equation for y we obtain $y = 3 - \frac{x}{2}$. Substituting this expression in the second equation to obtain

$$2\left(x + \left(3 - \frac{x}{2}\right)\right) = 1 - \left(3 - \frac{x}{2}\right).$$

Thus,

$$\begin{aligned} 2x + 6 - x &= -2 + \frac{x}{2} \\ x + 6 &= -2 + \frac{x}{2} \\ 2x + 12 &= -4 + x \\ x &= -16. \end{aligned}$$

Using this value of x in the first equation to obtain $y = 3 - \left(\frac{-16}{2}\right) = 11 \blacksquare$

Our third question in this section is the question of determining when two lines are parallel, i.e., they have no points in common. As we noted earlier in this section, the slope of a line determines the direction in which it points. Thus, if two lines have the same slope then the two lines are either parallel (if they have different vertical intercepts) or coincident (if they have same $y-$intercept). Also, note that any two vertical lines are parallel even though their slopes are undefined.

Example 3.2.3
Line l in Figure 3.2.2 is parallel to the line $y = 2x + 1$. Find the coordinates of the point P.

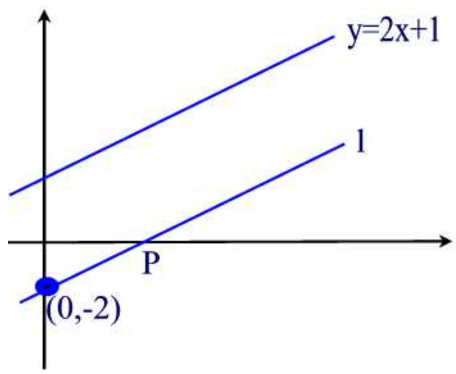


Figure 3.2.2

Solution.
Since the two lines are parallel, the slope of the line l is 2. Since the vertical intercept of l is -2, the equation of l is $y = 2x - 2$. The point P is the $x-$intercept of the line l, i.e., $P(x, 0)$. To find x, we set $2x - 2 = 0$. Solving for x we find $x = 1$. Thus, $P(1, 0)$ ■

Example 3.2.4
Find the equation of the line l passing through the point $(6, 5)$ and parallel to the line $y = 3 - \frac{2}{3}x$.

Solution.
The slope of l is $m = -\frac{2}{3}$ since the two lines are parallel. Thus, the equation of l is $y = -\frac{2}{3}x + b$. To find the value of b, we use the given point. Replacing y by 5 and x by 6 to obtain, $5 = -\frac{2}{3}(6) + b$. Solving for b we find $b = 9$. Hence, $y = 9 - \frac{2}{3}x$ ■

The fourth and final question of this section is the question of determining when two lines are perpendicular.
It is clear that if one line is horizontal and the second is vertical then the two lines are perpendicular. So we assume that neither of the two lines is horizontal or vertical. Hence, their slopes are defined and nonzero. Let's see how the slopes of lines that are perpendicular compare. Call the two lines l_1

and l_2 and let A be the point where they intersect. From A take a horizontal segment of length 1 and from the right endpoint C of that segment construct a vertical line that intersect l_1 at B and l_2 at D. See Figure 3.2.3.

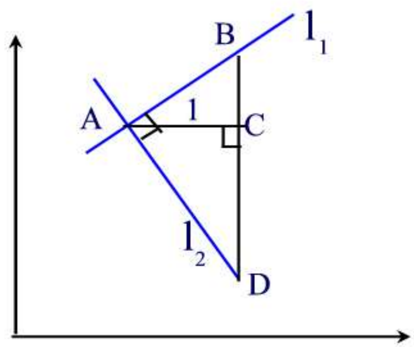


Figure 3.2.3

It follows from this construction that if m_1 is the slope of l_1 then

$$m_1 = \frac{|CB|}{|AC|} = |CB| \text{ since } |AC| = 1.$$

Similarly, the slope of l_2 is

$$m_2 = -\frac{|CD|}{|AC|} = -|CD|.$$

Since ΔABD is a right triangle at A, $\angle DAC = 90° - \angle BAC$. Similarly, $\angle ABC = 90° - \angle BAC$. Thus, $\angle DAC = \angle ABC$. A similar argument shows that $\angle CDA = \angle CAB$. Hence, the triangles ΔACB and ΔDCA are similar. As a consequence of this similarity we can write

$$\frac{|CB|}{|CA|} = \frac{|AC|}{|DC|}$$

or

$$m_1 = -\frac{1}{m_2}.$$

Thus, if two lines are perpendicular, then the slope of one is the negative reciprocal of the slope of the other.

Example 3.2.5
Find the equation of the line l_2 in Figure 3.2.4.

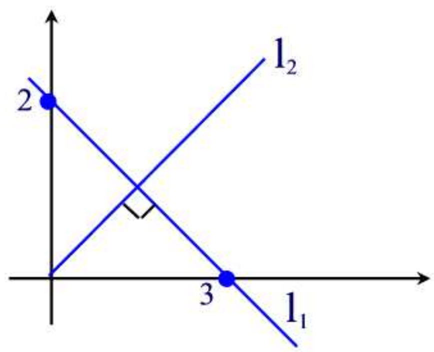


Figure 3.2.4

Solution.
The slope of l_1 is $m_1 = -\frac{2}{3}$. Since the two lines are perpendicular then the slope of l_2 is $m_2 = \frac{3}{2}$. The horizontal intercept of l_2 is 0. Hence, the equation of the line l_2 is $y = \frac{3}{2}x$ ■

Exercises

Exercise 3.2.1
Find the equation of the line passing through the point $\left(15, \frac{5}{2}\right)$ and perpendicular to the line $3x + y = 17$.

Exercise 3.2.2
State whether the following pairs of lines are parallel, perpendicular or neither:
(a) $y = \frac{3}{2}x - 7$ and $3x - 2y = 4$.
(b) $5x - 3y = 12$ and $3x + 5y = 10$.
(c) $x - 2y = 1$ and $2x - y = 5$.
(d) $x = 10$ and $y = 5$.

Exercise 3.2.3
Find the points of intersection of the two line $2x - y = 10$ and $x + y = -1$.

Exercise 3.2.4
Find the points of intersection of the two line $x - 3y = 5$ and $-2x + 6y = 8$.

Exercise 3.2.5
Find the points of intersection of the two line $2x + y = 5$ and $4x + 2y = 10$.

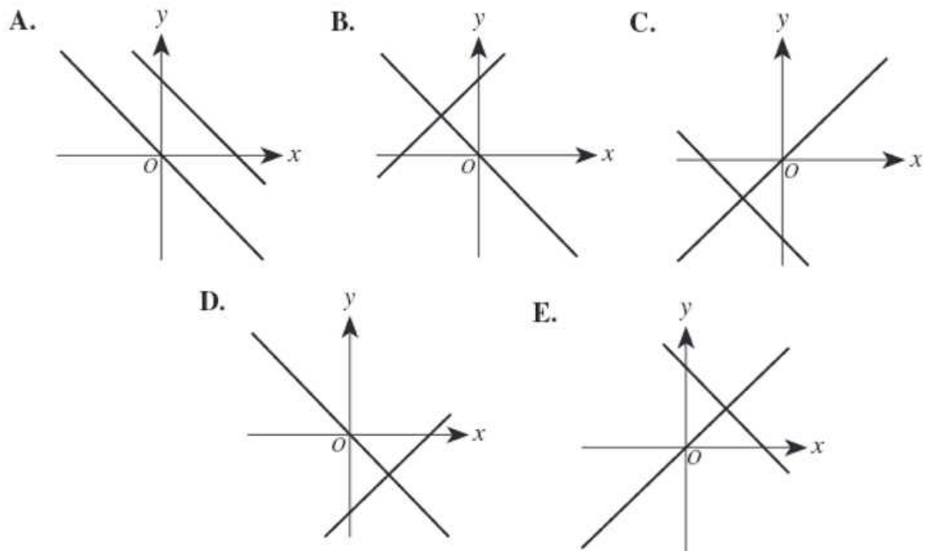
Exercise 3.2.6
Give the equation of the line passing through the point $(10, 3)$ that is perpendicular to $2x - 5y = -1$.

Exercise 3.2.7
Give the equation of the line passing through the point $(6, 0)$ that is parallel to $y = 2 - \frac{9}{2}x$.

Exercise 3.2.8
Find the point of intersection between the line that passes through the two points $(1, 2)$ and $(3, 7)$, and the line perpendicular to this first line that passes through the point $(2, 4)$.

Exercise 3.2.9
Which graph best presents the following system of equations and its solution?

$$\begin{cases} x - y = 0 \\ x + y = 4 \end{cases}$$

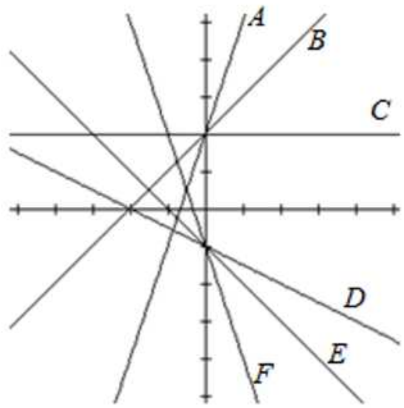


Exercise 3.2.10
Find an equation of the perpendicular bisector of the line segment between the points $(6, -3)$ and $(2, 5)$.

Exercise 3.2.11
Find an equation of the tangent line to the circle $x^2 + y^2 = 25$ at the point $(4, -3)$.

Exercise 3.2.12
Find the point of intersection of the lines with equations $x - 2y = 5$ and $2x - 5y = 11$.

Exercise 3.2.13
Match each linear function with its graph.
(1) $f(x) = -x - 1$.
(2) $f(x) = -2x - 1$.
(3) $f(x) = -\frac{1}{2}x - 1$.
(4) $f(x) = 2$.
(5) $f(x) = x + 2$.
(6) $f(x) = 3x + 2$.



Exercise 3.2.14
Find the value of c so that the two lines $f(x) = 2 - \frac{1}{2}x$ and $g(x) = 1 + cx$ intersect at a point with $x-$coordinate 10.

Exercise 3.2.15
Find the parameters m and b so that the two lines $y = mx+2$ and $y = 2x+b$ intersect at the point $(-3, -7)$.

Exercise 3.2.16
Find the equation of the line that contains the point $(16, 25)$ and is perpendicular to the line $x = 20$.

Exercise 3.2.17
Write an equation in slope-intercept form for the line that is perpendicular to the graph of $3x + 2y = 8$ and passes through the $y-$intercept of that line.

Exercise 3.2.18
What is the point of intersection of the lines with equations $x = 7$ and $y = -9$?

Exercise 3.2.19
Write the equation of the line parallel to the line $x = 5$ and passing through the point $(3, -10)$.

Exercise 3.2.20
Find the distance from the point $(0, 0)$ to the line $y = -x + 2$. Hint: The bisector of an isosceles triangle is perpendicular to opposite side.

Exercise 3.2.21
Find the point of intersection of the two lines $3x - y = 7$ and $2x + 3y = 1$.

Exercise 3.2.22
Find the point of intersection of the two lines $8x+7y = 38$ and $3x-5y = -1$.

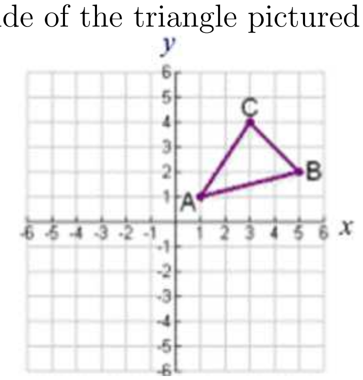
Exercise 3.2.23
Find the equation of the line parallel to $3x = 4y+5$ and passing through the point $(2, -3)$.

Exercise 3.2.24
Show that the two lines $2x + 5y = -9$ and $6x + 15y = 3$ are parallel.

Exercise 3.2.25
Find the equation of the line that contains $(0, -2)$ and is perpendicular to $y = 5x + 3$.

Exercise 3.2.26
What is the slope of the line perpendicular to the line $x = -2$?

Exercise 3.2.27
Find the slope of each side of the triangle pictured below.



Exercise 3.2.28
Find the equation of the line which goes through the point $(1, 2)$ and is parallel to the line through the points $(0, 1)$ and $(-2, 7)$.

Exercise 3.2.29
Let L_1 be the line $y = \frac{1}{2}x - 5$ and L_2 be the line perpendicular to L_1 and passing through $(1, 3)$. Find the point of intersection of L_1 and L_2.

Exercise 3.2.30
Determine whether the lines are parallel, perpendicular, or neither: $5x-6y = 25$ and $6x + 5y = 0$.

3.3 Quadratic Functions

You recall that a linear function is a function that involves a first power of x. A function of the form

$$f(x) = ax^2 + bx + c, \quad a \neq 0$$

is called a **quadratic function**. The word "quadratus" is the latin word for a square.
Quadratic functions are useful in many applications in mathematics when a linear function is not sufficient. For example, the motion of an object thrown either upward or downward is modeled by a quadratic function.
The graph of a quadratic function is known as a **parabola** and has a distinctive shape that occurs in nature. Geometrical discussion of quadratic functions will be covered below.

Finding the Zeros of a Quadratic Function
In many applications one is interested in finding the zeros or the $x-$intercepts of a quadratic function. This means we wish to find all possible values of x for which

$$ax^2 + bx + c = 0.$$

For example, if $v(t) = t^2 - 4t + 4$ is the velocity of an object in meters per second then one may be interested in finding the time when the object is not moving.
Finding the zeros of a quadratic function can be accomplished in two ways:

Factoring:
To factor $ax^2 + bx + c$
(1) find two integers that have a product equal to ac and a sum equal to b,
(2) replace bx by two terms using the two new integers as coefficients,
(3) then factor the resulting four-term polynomial by grouping. Thus, obtaining $(c_1x - d_1)(c_2x - d_2) = 0$. By the zero product property, we find $x = \frac{d_1}{c_1}$ or $x = \frac{d_2}{c_2}$.

Example 3.3.1
Find the zeros of $f(x) = x^2 - 2x - 8$ using factoring.

Solution.
We need two numbers whose product is -8 and sum is -2. Such two integers

are -4 and 2. Thus,

$$\begin{aligned} x^2 - 2x - 8 &= x^2 + 2x - 4x - 8 \\ &= x(x+2) - 4(x+2) \\ &= (x+2)(x-4) = 0. \end{aligned}$$

Thus, either $x = -2$ or $x = 4$ ■

Example 3.3.2
Find the zeros of $f(x) = 2x^2 + 9x + 4$ using factoring.

Solution.
We need two integers whose product is $ac = 8$ and sum equals to $b = 9$. Such two integers are 1 and 8. Thus,

$$\begin{aligned} 2x^2 + 9x + 4 =& 2x^2 + x + 8x + 4 \\ =& x(2x+1) + 4(2x+1) \\ =& (2x+1)(x+4). \end{aligned}$$

Hence, the zeros are $x = -\frac{1}{2}$ and $x = -4$ ■

Quadratic Formula:
Many quadratic functions are not easily factored. For example, the function $f(x) = 3x^2 - 7x - 7$. However, the zeros can be found by using the quadratic formula which we derive next:

$$\begin{aligned} ax^2 + bx + c &= 0 \text{ (subtract c from both sides)} \\ ax^2 + bx &= -c \text{ (multiply both sides by 4a)} \\ 4a^2x^2 + 4abx &= -4ac \text{ (add b}^2 \text{ to both sides)} \\ 4a^2x^2 + 4abx + b^2 &= b^2 - 4ac \\ (2ax + b)^2 &= b^2 - 4ac \\ 2ax + b &= \pm\sqrt{b^2 - 4ac} \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

provided that the **discriminant** $\Delta = b^2 - 4ac \geq 0$. This last formula is known as the **quadratic formula.** Geometrically, when $\Delta > 0$, the graph crosses the $x-$axis twice. When $\Delta = 0$, the graph is tangent to the $x-$axis. If $\Delta < 0$ then equation $ax^2 + bx + c = 0$ has not real solutions. However, it has complex solutions. Geometrically, the graph of $f(x) = ax^2 + bx + c$ does not cross the $x-$axis.

Example 3.3.3
Find the zeros of $f(x) = 3x^2 - 7x - 7$ using the quadratic formula.

Solution.
Letting $a = 3, b = -7$ and $c = -7$ in the quadratic formula we have

$$x = \frac{7 \pm \sqrt{133}}{6} \blacksquare$$

Example 3.3.4
Find the zeros of the function $f(x) = 6x^2 - 2x + 5$ using the quadratic formula.

Solution.
Letting $a = 6, b = -2$, and $c = 5$ in the quadratic formula we obtain

$$x = \frac{2 \pm \sqrt{-116}}{12}$$

But $\sqrt{-116}$ is not a real number. Hence, the function has no zeros. Its graph does not cross the $x-$axis $\blacksquare$

Example 3.3.5
What are the solutions of the quadratics: $x^2 + x + 1 = 0$ and $x^2 - 2x + 3 = 0$.

Solution.
Using the usual quadratic formula, we get the solutions as follows

$$\frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

and

$$\frac{2 \pm \sqrt{4-12}}{2} = \frac{2 \pm i\sqrt{8}}{2} = 1 \pm i\sqrt{2} \blacksquare$$

Graphs of Quadratic Functions

The graph of a quadratic function opens upward for $a > 0$ and opens downward for $a < 0$.

Example 3.3.6
Determine the concavity of $f(x) = -x^2 + 4$ from $x = -1$ to $x = 5$ using rates of change over intervals of length 2. Graph $f(x)$.

Solution.
Calculating the rates of change we find

$$\frac{f(1) - f(-1)}{1 - (-1)} = 0$$
$$\frac{f(3) - f(1)}{3 - 1} = -4$$
$$\frac{f(5) - f(3)}{5 - 3} = -8$$

Since the rates of change are getting smaller and smaller, the graph is concave down from $x = -1$ to $x = 5$. See Figure 3.3.1 ■

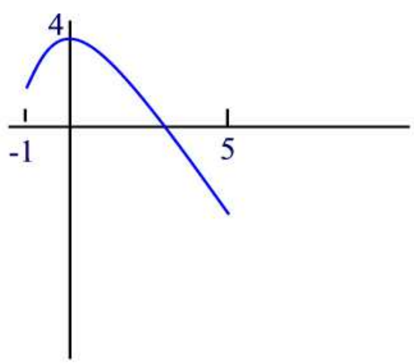


Figure 3.3.1

For $a > 0$ the graph is concave up so the graph has a lowest point (global minimum) and for $a < 0$ the graph open down so it has a heighest point (global maximum). Either point is called the **vertex.**

The Vertex Form of a Quadratic Function

Using the method of completing the square we can rewrite the standard form of a quadratic function into the form

$$f(x) = a(x - h)^2 + k \tag{3.3.1}$$

where $h = -\frac{b}{2a}$ and $k = f(-\frac{b}{2a}) = \frac{4ac-b^2}{4a^2}$. To see this:

$$\begin{aligned} f(x) = ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\ &= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right) \\ &= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) + \frac{4ac-b^2}{4a^2} \\ &= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2} = a(x-h)^2 + k. \end{aligned}$$

Form (3.3.1) is known as the **vertex form** for a quadratic function. The point (h, k) is the **vertex** of the parabola.

It follows from the vertex form that the graph of a quadratic function is obtained from the graph of $y = x^2$ by shifting horizontally h units, stretching or compressing vertically by a factor of a (and reflecting about the $x-$axis if $a < 0$), and shifting vertically $|k|$ units. Thus, if $a > 0$ then the parabola opens up and the vertex in this case is the minimum point whereas for $a < 0$ the parabola opens down and the vertex is the maximum point. Also, note that a parabola is symmetric about the line through the vertex. That is, the line $x = -\frac{b}{2a}$. This line is called the **axis of symmetry**.

Example 3.3.7

Find the vertex of the parabola $f(x) = -4x^2 - 12x - 8$ by first finding the vertex form.

Solution.

Using the method of completing the square we find

$$\begin{aligned} f(x) = -4x^2 - 12x + 8 &= -4(x^2 + 3x) + 8 \\ &= -4(x^2 + 3x + \frac{9}{4} - \frac{9}{4}) + 8 \\ &= -4(x^2 + 3x + \frac{9}{4}) - 9 + 8 = -4(x + \frac{3}{2})^2 - 1. \end{aligned}$$

Thus, the vertex is the point $(-\frac{3}{2}, -1)$ ■

Next, we discuss some techniques for finding the formula for a quadratic function.

Example 3.3.8
Find the formula for a quadratic function with vertex $(-3, 2)$ and passing through the point $(0, 5)$.

Solution.
Using the vertex form, we have $h = -3$ and $k = 2$. It remains to find a. Since the graph crosses the point $(0, 5)$, we have $5 = a(0 + 3)^2 + 2$. Solving for a we find $a = \frac{1}{3}$. Thus, $f(x) = \frac{1}{3}(x + 3)^2 + 2 = \frac{1}{3}x^2 + 2x + 5$ ■

Example 3.3.9
Find the formula for a quadratic function with vertical intercept $(0, 6)$ and $x-$intercepts $(1, 0)$ and $(3, 0)$.

Solution.
Since $x = 1$ and $x = 3$ are the $x-$intercepts, we have $f(x) = a(x-1)(x-3)$. But $f(0) = 6$ so that $6 = 3a$ or $a = 2$. Thus, $f(x) = 2(x - 1)(x - 3) = 2x^2 - 8x + 6$ ■

We end this section by an application problem.

Example 3.3.10
A rancher has 1200 meters of fence to enclose a rectangular corral with another fence dividing it in the middle as shown in Figure 3.3.2.

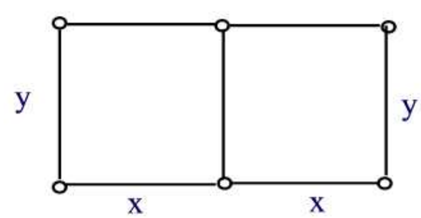


Figure 3.3.2

What is the largest area that can be enclosed by the given fence?

Solution.
The two rectangles each have area xy, so we have

$$A = 2xy$$

Next, we rewrite A in terms of x. Since $3y+4x = 1200$, solving for y we find $y = 400 - \frac{4}{3}x$. Substitute this expression for y in the formula for total area A to obtain

$$A = 2x(400 - \frac{4}{3}x) = 800x - \frac{8}{3}x^2.$$

This is a parabola that opens down so that its vertex yields the maximum area. But in this case, $x = -\frac{b}{2a} = -\frac{800}{-\frac{16}{3}} = 150$ meters.
Now that we know the value of x corresponding to the largest area, we can find the value of y by going back to the equation relating x and y :

$$y = 400 - \frac{4}{3}(150) = 200 \blacksquare$$

Exercises

Exercise 3.3.1
Find the vertex of $f(x) = -3x^2 + 6x + 1$.

Exercise 3.3.2
Find the $x-$intercepts of the function $f(x) = x^2 - 8x + 16$.

Exercise 3.3.3
Determine the extreme point of $f(x) = -2x^2 + 8x + 3$ and state whether it is a minimum or a maximum point.

Exercise 3.3.4
Sketch the graph of $f(x) = 2x^2 + 3x - 2$.

Exercise 3.3.5
Sketch the graph of $f(x) = -3x^2 + 6x + 1$.

Exercise 3.3.6
Sketch the graph of $f(x) = x^2 - 6x + 9$.

Exercise 3.3.7
Sketch the graph of $f(x) = 2x^2 + x + 1$.

Exercise 3.3.8
If the graph of $f(x) = ax^2 + 2x + 3$ contains the point $(1, -2)$, what is a?

Exercise 3.3.9
If the graph of $f(x) = x^2 + bx + 1$ has an $x-$intercept at $x = -2$, what is b?

Exercise 3.3.10
Suppose that the manufacturer of a gas clothes dryer has found that, when the unit price is p dollars, the revenue R is

$$R(p) = -4p^2 + 4,000p.$$

What unit price should be established for the dryer to maximize revenue?

Exercise 3.3.11
What is the largest rectangular area that can be enclosed with 400 ft of fencing?

Exercise 3.3.12
Consider the function $f(x) = x^2 - 4x + 5$.
(a) Find the coordinates of the vertex.
(b) Find the axis of symmetry.
(c) Find the $x-$ and $y-$ intercepts.
(d) Graph $f(x)$.

Exercise 3.3.13
A projectile is fired from a cliff 200 ft above the water at an inclination of 45° to the horizontal, with an initial velocity of 50 ft per second. The height h of the projectile above the water is given by

$$h(x) = -\frac{32}{2500}x^2 + x + 200$$

where x is the horizontal distance of the projectile from the base of the cliff. How far from the base of the cliff is the height of the projectile a maximum?

Exercise 3.3.14
Find two numbers whose sum is 7 and whose product is maximum.

Exercise 3.3.15
A projectile is shot upward. It's distance, in ft, above the ground after t seconds is $h(t) = -16t^2 + 400t$. Calculate the time it takes for the projectile to hit the ground and find the maximum altitude achieved by the projectile.

Exercise 3.3.16
Write the function $y = 2x^2 + 6x - 10$ in vertex form.

Exercise 3.3.17
Determine the equation of a quadratic function whose graph passes through the point $(1, -2)$ and has vertex $(4, 5)$.

Exercise 3.3.18
Let $f(x)$ be the function whose graph is obtained by translating the graph of $g(x) = 2x^2 - 3x + 1$ two units to the left and 5 units down. Find the expression of $f(x)$.

Exercise 3.3.19
Find the maximum value: $f(x) = -5(x - 1)^2 + 3$.

Exercise 3.3.20
Find the minimum value: $f(x) = 12(x-5)^2 + 2$.

Exercise 3.3.21
Let x be the amount (in hundreds of dollars) a company spends on advertising, and let P be the profit, where $P(x) = 230 + 20x - 0.5x^2$. What expenditure for advertising results in the maximum profit?

Exercise 3.3.22
The number of bacteria in a refrigerated food is given by

$$N(T) = 20T^2 - 80T + 55, \quad 2 \leq T \leq 14$$

where T is the Celsius temperature of the food. When the food is removed from refrigeration, the temperature is given by

$$T(t) = 4t + 2, \quad 0 \leq t \leq 3$$

where t is the time in hours. The composition function $(N \circ T)(t)$ represents the number of bacteria as a function of the amount of time the food has been out of refrigeration. Find a formula for $N(T(t))$.

Exercise 3.3.23
A company produces a certain type of shirts. If the price x dollars for each shirt, then the number actually sold (in thousands) is given by the expression:

$$N = 57 - x.$$

How much revenue does the company get by selling shirts at x dollars?

Exercise 3.3.24
The formula $h(t) = -16t^2 + 64t + 960$ gives the height h (in ft) of an object thrown upward from the roof of a 960 ft building at an initial velocity of $64ft/sec$. For what times t will the height be greater than $992ft$?

Exercise 3.3.25
Solve by factoring: $2x^2 + 13x - 24 = 0$.

Exercise 3.3.26
Solve using the quadratic formula: $3x^2 - 10x + 5 = 0$.

Exercise 3.3.27
Find the vertext form: $y = -3x^2 - 12x - 8$.

Exercise 3.3.28
Find the vertex and the axis of symmetry: $y = 2x^2 + 12x + 19$.

Exercise 3.3.29
Mark jumped off of a cliff into the ocean in Hawai while vacationing with some friends. His height as a function of time could be modeled by the function $h(t) = -16t^2 + 16t + 480$, where t is the time in seconds and h is the height in feet.
(a) How long did it take for Mark to reach his maximum height?
(b) What was the highest point that Mark reached?
(c) Mark hit the water after how many seconds?

Exercise 3.3.30
Solve the given quadratic equation and write the solutions in the form $z = a + bi$.
(a) $z^2 + 2z + 2 = 0$
(b) $6z^2 - 5z + 5 = 0$.

Exercise 3.3.31
Solve the given quadratic equation and write the solutions in the form $z = a + bi$.
(a) $2z^2 + z + 3 = 0$
(b) $3z^2 + 2z + 4 = 0$.

Exercise 3.3.32
Solve the given quadratic equation and write the solutions in the form $z = a + bi$.
(a) $z^2 + 9 = 0$
(b) $x^2 + 4x + 5 = 0$.

Exercise 3.3.33
Show that the solutions of the equation

$$4x^2 - 24x + 37 = 0$$

are complex conjugate of each other.

3.4 Polynomial Functions

In addition to linear and quadratic functions, many other types of functions occur in mathematics and its applications. In this section, we will study polynomial functions.

Polynomial functions are among the simplest, most important, and most commonly used mathematical functions. These functions consist of one or more terms of variables with whole number exponents. (Whole numbers are positive integers and zero.) All such functions in one variable (usually x) can be written in the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0$$

where $a_n, a_{n-1}, \cdots, a_1, a_0$ are all real numbers, called the **coefficients** of $f(x)$. The number n is a non-negative integer. It is called the **degree** of the polynomial. A polynomial of degree zero is just a constant function. A polynomial of degree one is a linear function, of degree two a quadratic function, etc. The number a_n is called the **leading coefficient** and a_0 is called the **constant term**.

Note that the terms in a polynomial are written in descending order of the exponents. Polynomials are defined for all values of x. That is, the domain of a polynomial is the interval $(-\infty, \infty) = \mathbb{R}$.

Example 3.4.1

Find the leading coefficient, the constant term and the degreee of the polynomial $f(x) = 4x^5 - x^3 + 3x^2 + x + 1$.

Solution.

The given polynomial is of degree 5, leading coefficient 4, and constant term 1 ■

A polynomial function will never involve terms where the variable occurs in a denominator, underneath a radical, as an input of either an exponential, logarithmic, or trigonometric function.

Example 3.4.2

Determine whether the function is a polynomial function or not:

(a) $f(x) = 3x^4 - 4x^2 + 5x - 10$

(b) $g(x) = x^3 - e^x + 3$

(c) $h(x) = x^2 - 3x + \frac{1}{x} + 4$
(d) $i(x) = x^2 - \sqrt{x} - 5$
(e) $j(x) = x^3 - 3x^2 + 2x - 5\ln x - 3.$
(f) $k(x) = x - \sin x.$

Solution.
(a) $f(x)$ is a polynomial function of degree 4.
(b) $g(x)$ is not a ploynomial because one of the terms is an exponential function.
(c) $h(x)$ is not a polynomial because x is in the denominator of a fraction.
(d) $i(x)$ is not a polynomial because it contains a radical sign.
(e) $j(x)$ is not a polynomial because one of the terms is a logarithm of x.
(f) $k(x)$ is not a polynomial function because it involves a trigonometric function ■

Graph of a Polynomial Function

Polynomials are continuous and smooth everywhere:
• A **continuous** function means that it can be drawn without picking up your pencil. There are no jumps or holes in the graph of a polynomial function.
• A **smooth curve** means that there are no sharp turns (like an absolute value) in the graph of the function.
• The $y-$intercept of the polynomial is the constant term a_0.

Long-Run Behavior of a Polynomial Function: Leading Coefficient Test

The behaviour of a polynomial in the long-run depends on the degree and leading coefficient:
(i) If n is even and the leading coefficient $a_n > 0$ then $f(x) \to \infty$ as $x \to \pm\infty$.
(ii) If n is even and the leading coefficient $a_n < 0$ then $f(x) \to -\infty$ as $x \to \pm\infty$.
(iii) If n is odd and the leading coefficient $a_n > 0$ then $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$.
(iv) If n is odd and the leading coefficient $a_n < 0$ then $f(x) \to \infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to \infty$.

Example 3.4.3
According to the graphs in Figure 3.4.1, indicate the sign of a_n and the parity of n for each curve.

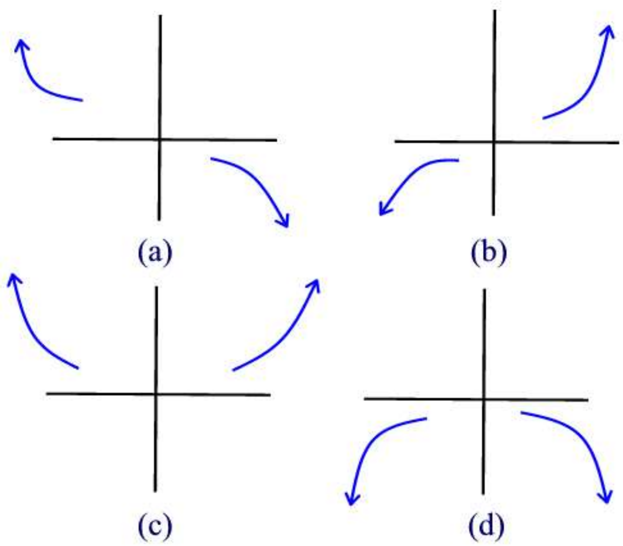


Figure 3.4.1

Solution.
(a) $a_n < 0$ and n is odd.
(b) $a_n > 0$ and n is odd.
(c) $a_n > 0$ and n is even.
(d) $a_n < 0$ and n is even ■

Zeros of Polynomial Functions

We next discuss methods for solving a polynomial equation of the form

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0, \quad a_n \neq 0.$$

A number a is said to be a **solution** or a **zero** to the above equation if the equality is true when x is replaced by the number a. Denoting the polynomial by $f(x)$, a solution a of the polynomial equation satisfies $f(a) = 0$. Geometrically, solutions to a polynomial equation are just the $x-$intercepts of the function $f(x)$. That is, the points where the graph of the function crosses the $x-$axis. Two important results concerning the zeros of a polynomial:

The Fundamental Theorem of Algebra: A polynomial function of degree n has exactly n roots counting repeated roots and complex roots. Complex roots occur in conjugate pair. The number of time a zero is repeated is called the **algebraic multiplicity** of the zero. At a zero with repeated multiplicity, the graph is tangent to the $x-$axis at that point.

Example 3.4.4
Find the zeros with their corresponding multiplicities: $f(x) = 3(x-2)^4(3x+5)^3(x+3)$.

Solution.
$x = 2$ is a zero of multiplicity 4; $x = -\frac{5}{3}$ is of multiplicity 3; and $x = -3$ is of multiplicity 1 ■

Intermediate value theorem: If f is a polynomial and a and b are two numbers such that $f(a)$ and $f(b)$ are of opposite signs, then there is a number c between a and b such that $f(c) = 0$.

Example 3.4.5
Given the function $f(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0,\ a_5 > 0$. Suppose that: $f(-4) = f(4) = 0, f(-1) > 0, f(0) < 0$ and $f(1) > 0$. How many real zeros does the function $f(x)$ have?

Solution.
Since $f(-4) = f(4) = 0$, ± 4 are two zeros. Since $f(-1) > 0$ and $f(0) < 0$, by the intermediate value theorem there is a zero between -1 and 0. Likewise there is a zero between 0 and 1. Since there are four real roots, by the fundamental theorem of algebra, the fifth must be also real (remember that complex zeros occur in pair). Hence, there are a total of five real zeros ■

We next discuss ways for finding the roots.

Solving by Factoring
This process consists of factoring the polynomial and then using the zero product rule which says that if a product of several numbers is zero then at least one of the factors must be zero. We illustrate this process in the next couple of examples.

Example 3.4.6
Solve: $(x-1)(x-2)(x+2) = 0$.

Solution.
By the zero product rule we have $x - 1 = 0,\ x - 2 = 0$ or $x + 2 = 0$. Thus, the solutions are $x = 1,\ x = 2$, and $x = -2$ ■

Example 3.4.7
Find the $x-$intercepts of the polynomial $f(x) = x^3 - x^2 - 6x$.

Solution.
Factoring the given function to obtain

$$\begin{aligned} f(x) &= x(x^2 - x - 6) \\ &= x(x-3)(x+2). \end{aligned}$$

Thus, the $x-$intercepts are the zeros of the equation

$$x(x-3)(x+2) = 0.$$

That is, $x = 0, x = 3$, or $x = -2$ ■

Factoring by Grouping

We have already encountered the factoring by grouping when solving quadratic equations. We illsutrate this method for non-quadratic equations.

Example 3.4.8
Use factoring by grouping to solve the equation : $x^3 - x^2 - 4x + 4 = 0$.

Solution.
We factor the given polynomial as follows:

$$\begin{aligned} x^3 - x^2 - 4x + 4 =&(x^3 - 3x^2) - (4x - 4) \\ =&x^2(x-1) - 4(x-1) \\ =&(x-1)(x^2-4) = (x-1)(x-2)(x+2). \end{aligned}$$

Thus, the problem reduces to solving the equation

$$(x-1)(x-2)(x+2) = 0.$$

By the zero product rule we find the solutions $x = 1,\ x = 2$, and $x = -2$ ■

The n^{th} Root Method

This method is used to solve equations of the form $x^n = a$. If n is odd then there is only one solution which is $x = \sqrt[n]{a}$. If n is even and $a > 0$ then there are two solutions, namely, $x = -\sqrt[n]{a}$ and $x = \sqrt[n]{a}$. If n is even and $a < 0$ then the equation has complex solutions.

Example 3.4.9
Solve: (a) $x^3 + 27 = 0$ (b) $5x^4 = 80$ (c) $x^2 + 1 = 0$.

Solution.
(a) We have $x^3 = -27 = -3^3$ and this implies that $x = -3$.
(b) The given equation simplifies to $x^4 = 16 = 2^4$ so that $x = -2$ or $x = 2$.
(c) Since $x^2 = -1 = i^2$, we have the conjugate solutions $x = \pm i$ ■

Graphical Method

When algebraic methods fail to work, a remedy is to solve graphically using a graphing calculator or a computer software. This method is known as the $x-$**intercept method** as illustrated in the example below.

Example 3.4.10
Solve graphically the equation: $-2.56x^3 + 53.73x^2 - 356.47x + 757.74 = 0$.

Solution.
We use a calculator to graph the function $f(x) = -2.56x^3 + 53.73x^2 - 356.47x + 757.74$. The graph shows three $x-$intercepts. You can find the $x-$intercepts by either using the key **TRACE** or the graphical solver **ZERO**. The $x-$intercepts found by this method are decimal approximations of the exact solutions rather than the exact solutions. TI83 will show that the approximate $x-$intercepts to the given function are $x \approx 5$, $x \approx 6$, or $x \approx 10.1$ ■

Finding a Formula for a Polynomial from its Graph

Example 3.4.11
Find a formula of the function whose graph is given Figure 3.4.2.

Solution.
From the graph we see that $f(x)$ has the form

$$f(x) = k(x + 1)(x - 3)^2.$$

Since $f(0) = -3$, we find $k(0 + 1)(0 - 3)^2 = -3$ or $k = -\frac{1}{3}$. Thus,

$$f(x) = -\frac{1}{3}(x + 1)(x - 3)^2 \blacksquare$$

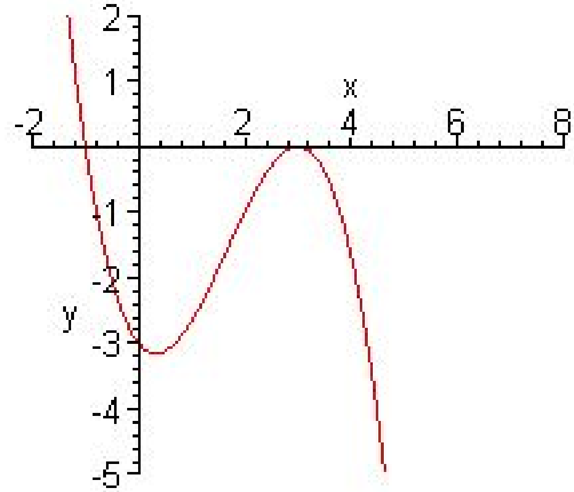


Figure 3.4.2

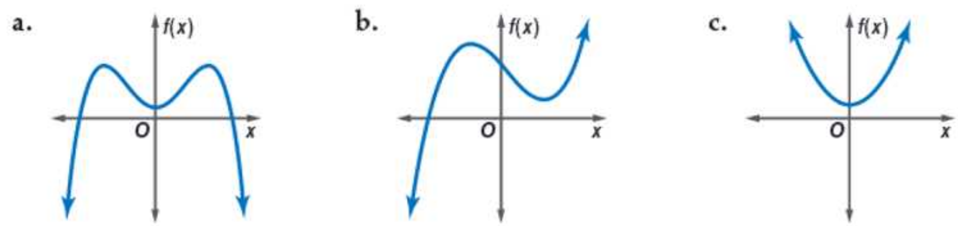
Exercises

Exercise 3.4.1
What are the leading term, the constant term, and the degree of the polynomial function $f(x) = -3x^4 + x^3 + 7x^2 - 10$?

Exercise 3.4.2
Given the function $f(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0,\ a_5 > 0$. Determine the right-hand and left-hand behavior of the graph of $f(x)$.

Exercise 3.4.3
Use a graphing calculator, find the $x-$ intercepts of the function $f(x) = -x^4 - 3x^3 + 6x^2 + 8x$.

Exercise 3.4.4
Each graph below is the graph of a polynomial function:
(1) Describe the long run behaviour;
(2) determine whether the polynomial is of odd or even degree;
(3) state the number of real zeros.



Exercise 3.4.5
What is the largest number of real roots that a 7$^{\text{th}}$ degree polynomial could have? What is the smallest?

Exercise 3.4.6
What is the long-run behavior of $f(x) = -x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + g$?

Exercise 3.4.7
Use factoring by grouping to find the zeros of $f(x) = x^3 - x^2 - 5x + 5$.

Exercise 3.4.8
Subtract: $(3x^3 - 4x^2 + 3) - (x - 2x^2 - x^3 + 1)$.

Exercise 3.4.9
Find the three zeros of $f(x) = x^3 - x^2 - 2x$ by factoring.

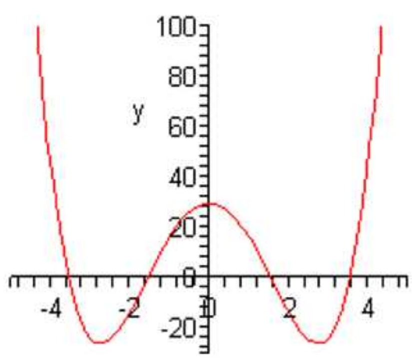
Exercise 3.4.10
Find a polynomial of degree three, with leading coefficient -2 and having the real roots $-3, 2$ and 5.

Exercise 3.4.11
Use direct substitution to determine whether the number -2 is a zero of the polynomial $f(x) = -2x^3 + x^2 + 2x + 3$.

Exercise 3.4.12
Find the zeros of the polynomial function $f(x) = 5(x+9)^2(x-3)^3$ and state the multiplicity of each.

Exercise 3.4.13
Find a polynomial of degree 4, leading coefficient 1, and zeros:$-\sqrt{10}, \sqrt{10}$, and -9 with multiplicity 2.

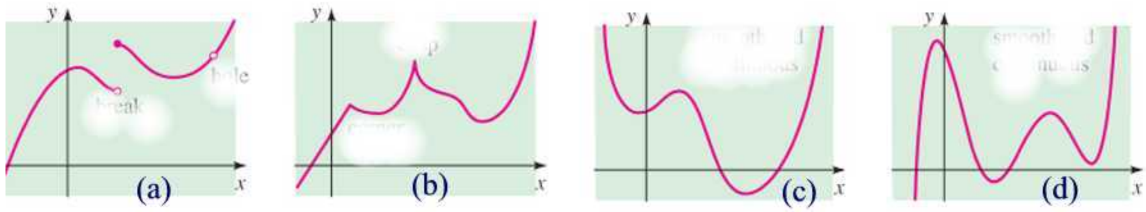
Exercise 3.4.14
Find a polynomial of degree 4 whose graph is given below.



Exercise 3.4.15
Find the real zeros of $f(x) = x^4 - 5x^2 + 4$.

Exercise 3.4.16
Which of the following are polynomial functions?
(a) $f(x) = -x^3 + 2x + 4$.
(b) $f(x) = (\sqrt{x})^3 - 2(\sqrt{x})^2 + 5\sqrt{x} - 3$.
(c) $f(x) = \frac{x^2-5}{x^3+2x^2-1}$.
(d) $f(x) = (\sqrt{x})^4 + 2(\sqrt{x})^2 - 10$.

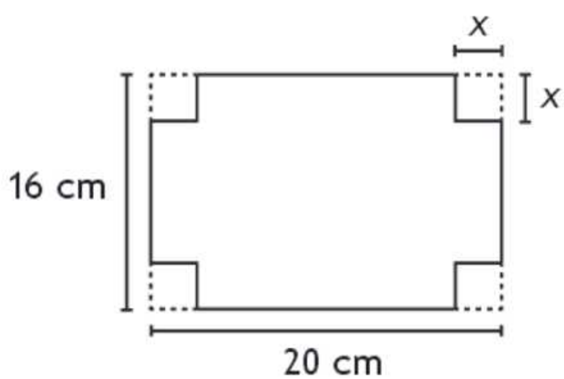
Exercise 3.4.17
Which of the following is the graph of a polynomial function?



Exercise 3.4.18
Find the zeros and their corresponding multiplicities for $f(x) = x^3 - 2x^2 - 4x + 8$.

Exercise 3.4.19
Starting with the function $f(x) = x^3$, find the formula of the function $g(x)$ obtained from $f(x)$ by a shift 3 units to the left, followed by a vertical stretch by a factor of 4 and then shifted down by 7 units.

Exercise 3.4.20
A box with no lid can be made by cutting out squares from each corner of a rectangular piece of cardboard and folding up the sides. A particular piece of cardboard has a length of 20 cm and a width of 16 cm. The side length of a corner square is x. Write the function expressing the volume of the box and find its domain.



Exercise 3.4.21
Show that the polynomial function $f(x) = x^5 - 2x^3 - 2 = 0$ has a real solution between 0 and 2.

Exercise 3.4.22
Sketch a graph of a 3rd degree polynomial function with two positive real zeros, 1 negative real zero, and a negative leading coefficient.

Exercise 3.4.23
Sketch the graph of $f(x) = x^2(x-4)(x+2)$.

Exercise 3.4.24
Without using a graphing tool, make a sketch of the polynomial $f(x) = (x+1)(x-3)(x-4)$ by using the end-behavior roots and degree.

Exercise 3.4.25
Without using a graphing tool, make a sketch of the polynomial $f(x) = (x-6)^2(x+1)$ by using the end-behavior roots and degree.

Exercise 3.4.26
Without using a graphing tool, make a sketch of the polynomial $f(x) = x(x+1)^2(x-4)^2$ by using the end-behavior roots and degree.

Exercise 3.4.27
Without using a graphing tool, make a sketch of the polynomial $f(x) = -(x+1)^3(x-1)^2$ by using the end-behavior roots and degree.

Exercise 3.4.28
Show that every polynomial of odd degree has a zero.

Exercise 3.4.29
Solve using factoring: $x^3 + 2x^2 - 25x - 50 = 0$.

Exercise 3.4.30
A designer is making a rectangular prism box with maximum volume, with the sum of its length, width and height equal to 8 inches. The length must be twice the width. What should each dimension be? Use graphical method for drawing the volume function. Round to the nearest tenth of an inch.

3.5 Division of Polynomials

In this section we consider the algebraic operation of division of polynomials. Two methods to be discussed: the long division and the synthetic division.

Long Division

Just as we can divide two numbers using long division, we can also use the same process to divide two polynomials. To divide a polynomial $f(x)$ by a polynomial $g(x)$ is to find a polynomial function $q(x)$, called the **quotient**, and a polynomial function $r(x)$, called the **remainder**, such that $f(x) = q(x)g(x) + r(x)$ where the degree of $r(x)$ is less than the degree of $g(x)$.

Example 3.5.1

Divide $x^2 + x - 1$ by $x + 2$.

Solution.

We set this up in the same manner as long division of numbers,

$$\begin{array}{r|rrr}
\multicolumn{1}{r}{} & & x & -\,1 \\ \cline{2-4}
x+2 & x^2 & +\,x & -\,1 \\
 & -\,x^2 & -\,2x & \\ \cline{2-3}
 & & -\,x & -\,1 \\
 & & x & +\,2 \\ \cline{3-4}
 & & & 1
\end{array}$$

Thus, the quotient is $q(x) = x$ and the remainder is $r(x) = 1$ ■

Example 3.5.2

Divide $3x^5 - 2x^4 + 6x^3 - 4x^2 - 24x + 16$ by $x^2 + 4$.

Solution.
We set this up in the same manner as long division of numbers,

$$
\begin{array}{rrrrrrr}
 & 3x^3 & -2x^2 & & -6x & +4 & \\
\cline{2-7}
x^2+4\big) & 3x^5 & -2x^4 & +6x^3 & -4x^2 & -24x & +16 \\
 & -3x^5 & & -12x^3 & & & \\
\cline{2-4}
 & & -2x^4 & -6x^3 & -4x^2 & & \\
 & & 2x^4 & & +8x^2 & & \\
\cline{3-5}
 & & & -6x^3 & +4x^2 & -24x & \\
 & & & 6x^3 & & +24x & \\
\cline{4-6}
 & & & & 4x^2 & & +16 \\
 & & & & -4x^2 & & -16 \\
\cline{5-7}
 & & & & & & 0
\end{array}
$$

Thus, the quotient is $q(x) = 3x^3 - 2x^2 - 6x + 4$ and the remainder is $r(x) = 0$ ■

Example 3.5.3
Divide $3x^3 + 4x + 11$ by $x^2 - 3x + 2$.

Solution.
We set this up in the same manner as long division of numbers,

$$
\begin{array}{rrrrr}
 & & & 3x & +9 \\
\cline{2-5}
x^2-3x+2\big) & 3x^3 & & +4x & +11 \\
 & -3x^3 & +9x^2 & -6x & \\
\cline{2-4}
 & & 9x^2 & -2x & +11 \\
 & & -9x^2 & +27x & -18 \\
\cline{3-5}
 & & & 25x & -7
\end{array}
$$

Thus, the quotient is $q(x) = 3x + 9$ and the remainder is $r(x) = 25x - 7$ ■

Now, when we divide a polynomial function $f(x)$ by $x - a$ we obtain two functions $q(x)$ and $r(x)$ such that

$$f(x) = (x - a)q(x) + r(x).$$

Letting $x = a$, we find $r(x) = f(a)$. We refer to this result as the **remainder theorem**

Example 3.5.4
Use the remainder theorem to find the remainder of the division of $f(x) = x^4 + x^3 - 7x - 10$ by $x - 2$.

Solution.
By the remainder theorem, we have

$$r(x) = f(2) = 2^4 + 2^3 - 7(2) - 10 = 0 \blacksquare$$

As a result of the remainder theorem we have the **factor theorem** which states that if $f(a) = 0$ then $x - a$ is a factor of $f(x)$. Indeed, if $r(x) = f(a) = 0$ then by the remainder theorem, we have $f(x) = (x - a)q(x)$. Thus, $x - a$ is a factor of $f(x)$. Also, since $f(a) = 0$, a is an $x-$intercept of $f(x)$.

Example 3.5.5
Show that $x - 2$ is a factor of $f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18$.

Solution.
By the remainder theorem, we have $f(2) = 2(2)^4 + 7(2)^3 - 4(2)^2 - 27(2) - 18 = 0$. Hence, $x - 2$ is a factor of $f(x)$ $\blacksquare$

Example 3.5.6
Use the factor theorem to determine whether $f(x) = 2x^3 - x^2 - 16x + 15$ is divisible by $x - 2$ and $x + 3$.

Solution.
We have $f(2) = 2(2)^3 - (2)^2 - 16(2) + 15 = -5 \neq 0$ so $f(x)$ is not divisible by $x - 2$. Likewise, $f(-3) = 0$ so that $f(x)$ is divisible by $x + 3$ $\blacksquare$

Synthetic Division
A quicker method for finding the quotient and the remainder of the division of a polynomial function $f(x)$ by $x - a$ is the method of **synthetic division** which we explore in the next examples.

Example 3.5.7
Find the quotient and the remainder of the division of $f(x) = x^3 + x^2 - 1$ by $x - 1$.

Solution.
Using synthetic division as shown in Figure 3.5.1, we write the coefficients of f in decreasing order at the top of the chart. If a power of x is missing, we use the coefficient 0. On the second row, the 1 on the left is the zero of $x-1$. A blank is left under the leading coefficient in the chart. The first entry on the third row is always the leading coefficient of f. We multiply this entry by the zero of $x-1$ which is in this case 1. We carry the result up into the next column as the arrows indicate. Next, add down the column and repeat the process.

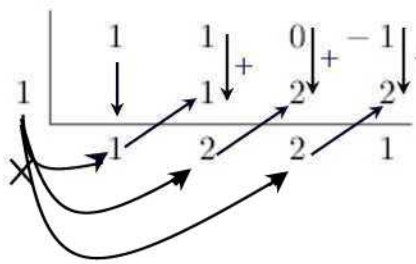


Figure 3.5.1

The numbers along the bottom row are the coefficients of the quotient with the powers of x in descending order. The last coefficient is the remainder. The first power is one less than the highest power of the polynomial that was being divided. Hence, $q(x) = x^2 + 2x + 2$ and $r(x) = 1$ ■

Example 3.5.8
Let $f(x) = x^5 + 2x^3 - x + 1$. Use the Remainder Theorem and synthetic division to compute $f(-2)$.

Solution.
Using synthetic division, we find

$$\begin{array}{r|rrrrrr} & 1 & 0 & 2 & 0 & -1 & 1 \\ -2 & & -2 & 4 & -12 & 24 & -46 \\ \hline & 1 & -2 & 6 & -12 & 23 & -45 \end{array}$$

Hence, $f(-2) = -45$ ■

Rational Root Test

The **rational root test** is a useful test in finding the roots of a polynomial with integer coefficients. The test provides a listing of possible rational roots of the polynomial.

Rational Root Test: If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial with integer coefficients and if $\frac{p}{q}$ is a rational root of f then p divides a_0 and q divides a_n.
Hence, according to this test, a potential candidate for a root of f is a ratio of a divisor of a_0 over a divisor of a_n. We illustrate the use of this test next.

Example 3.5.9
Use the rational root test to solve $x^3 - 6x^2 + 11x - 6 = 0$.

Solution.
In this case, $a_3 = 1$ and $a_0 = -6$. Divisors of a_1 are: ± 1. Divisors of a_0 are: $\pm 1, \pm 2, \pm 3, \pm 6$. Hence, possible rational roots are

$$\pm 1, \pm 2, \pm 3, \pm 6.$$

It must be stressed that these are not necessarily roots of our polynomial. In fact it may turn out that none of these is a root of our polynomial, in which case the polynomial has only complex or irrational roots, but if our polynomial has any rational roots whatsoever, those rational roots must be one of the rational numbers in our list.
The next step is to check each of the values on the list. This is usually done by using synthetic division. The reader can check that the first rational root on the list is $x = 1$

$$\begin{array}{c|cccc} & 1 & -6 & 11 & -6 \\ 1 & & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

In this case, the given equation can be factored into $(x-1)(x^2 - 5x + 6) = (x-1)(x-2)(x-3)$. Hence, the roots of the given equation are $1, 2$, and 3 ■

Example 3.5.10
Use the rational root test to find all the real zeros of $f(x) = 10x^4 - 3x^3 - 29x^2 + 5x + 12$.

Solution.
We have $a_4 = 10$ and $a_0 = 12$. The possible rational roots are

$$\pm 1, \pm 2, \pm 3, \pm 4, \pm 12, \pm\frac{3}{2}, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{3}{5}, \pm\frac{6}{5}, \pm\frac{12}{5}, \pm\frac{1}{10}, \pm\frac{3}{10}.$$

Checking these value, we find

$$\begin{array}{r|rrrrr} & 10 & -3 & -29 & 5 & 12 \\ -\frac{3}{2} & & -15 & 27 & 3 & -12 \\ \hline & 10 & -18 & -2 & 8 & 0 \end{array}$$

Hence, $f(x) = \left(x + \frac{3}{2}\right)(10x^3 - 18x^2 - 2x + 8) = (2x+3)(5x^3 - 9x^2 - x + 4)$. Next, we apply the rational root test to the function $g(x) = 5x^3 - 9x^2 - x + 4$. We have $a_3 = 5$ and $a_0 = 4$ so that the possible rational roots are

$$\pm 1, \pm 2, \pm 4, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{4}{5}.$$

We find

$$\begin{array}{r|rrrr} & 5 & -9 & -1 & 4 \\ \frac{4}{5} & & 4 & -4 & -4 \\ \hline & 5 & -5 & -5 & 0 \end{array}$$

Thus, $f(x) = (2x+3)\left(x - \frac{4}{5}\right)(5x^2 - 5x - 5) = (2x+3)(5x-4)(x^2 - x - 1)$. We solve the equation $x^2 - x - 1 = 0$ using the quadratic formula, we find $x = \frac{1 \pm \sqrt{5}}{2}$. Hence, all the real zeros of $f(x)$ are: $-\frac{3}{2}, \frac{4}{5}, \frac{1\pm\sqrt{5}}{2}$ ■

Exercises

Exercise 3.5.1
Find the quotient and the remainder of the division of $f(x) = 3x^3 - 2x + 4$ by $g(x) = x^2 + x + 1$.

Exercise 3.5.2
Find the quotient and the remainder of the division of $f(x) = 2x^5 - x^4 + 2x^2 - 1$ by $g(x) = x^3 - x^2 + 1$.

Exercise 3.5.3
Find the quotient and the remainder of the division of $f(x) = 6x^4 + 5x^3 - 3x^2 + x + 3$ by $g(x) = 2x^2 - x + 1$.

Exercise 3.5.4
Find the quotient and the remainder of the division of $f(x) = 3x^3 - 8x - 4$ by $g(x) = x^2 + 2x + 1$.

Exercise 3.5.5
Find the quotient and the remainder of the division of $f(x) = 2x^5 - 3x^3 + 2x^2 - x + 3$ by $g(x) = x^3 + 1$.

Exercise 3.5.6
Find the quotient and the remainder of the division of $f(x) = x^3 + x^2 - 4$ by $g(x) = x^4 + 1$.

Exercise 3.5.7
Find the quotient and the remainder of the division of $f(x) = 2x^3 + 7x^2 - 4x + 7$ by $g(x) = x^2 + 2x - 1$.

Exercise 3.5.8
Find the quotient and the remainder of the division of $f(x) = 6x^4 + 15x^3 - 5x^2 - 4$ by $g(x) = 3x^2 - 4$.

Exercise 3.5.9
Find the quotient and the remainder of the division of $f(x) = x^4 - x^3 - x^2 + 4x - 2$ by $g(x) = x^2 + x - 1$.

Exercise 3.5.10
Find the quotient and the remainder of the division of $f(x) = 5x^5 - 24x^3 - 7x^2 - 5x + 35$ by $g(x) = x^3 + 2x - 1$.

Exercise 3.5.11
Use the remainder theorem to find the remainder of the division of $f(x) = x^5 + 2x^3 - x + 1$ by $x + 2$.

Exercise 3.5.12
Use the remainder theorem to find the remainder of the division of $f(x) = 5x^4 - 2x^2 - 1$ by $x - \frac{1}{3}$.

Exercise 3.5.13
Find k such that when $f(x) = 2x^3 + x^2 - 5x + 2k$ is divided by $x + 1$ the remainder is 6.

Exercise 3.5.14
Use the remainder theorem to find the remainder of the division of $f(x) = -x^4 - 5x^3 + 4x^2 - 9x + 10$ by $x + 3$.

Exercise 3.5.15
Show that $x + 4$ is a factor of $f(x) = 5x^3 + 12x^2 - 20x + 48$.

Exercise 3.5.16
Find the value of c so that $x+3$ is a factor of $f(x) = 2x^4 - x^3 - 9x^2 + 22x + c$.

Exercise 3.5.17
Find the value of c so that $x-2$ is a factor of $f(x) = x^5 + 5x^3 - 6x^2 + cx - 64$.

Exercise 3.5.18
Show that $x + 1$ is not a factor of $f(x) = -5x^3 + 4x^2 - 3x + 9$.

Exercise 3.5.19
Is $x - 5$ a factor of $f(x) = x^3 + x^2 - 27x - 15$?

Exercise 3.5.20
Given that $x = -2$ is a zero of $f(x) = 2x^3 - 3x^2 - 11x + 6$. Find the remaining zeros.

Exercise 3.5.21
Use the remainder theorem and synthetic division to show that $x = 2$ is a solution to the equation: $x^3 - 3x^2 + 3x - 2 = 0$.

Exercise 3.5.22
Let $f(x) = 5x^4 - 2x^2 - 1$. Use the Remainder Theorem and synthetic division to compute $f(\frac{1}{3})$.

Exercise 3.5.23
Use synthetic division to determine $f(-3)$ where $f(x) = -x^4 - 5x^3 + 4x^2 - 9x + 10$.

Exercise 3.5.24
Use synthetic division to find the quotient and the remainder of the division of $f(x) = 3x^3 - 5x^2 + 4x + 2$ by $3x + 1$.

Exercise 3.5.25
Use synthetic division to find the quotient and the remainder of the division of $f(x) = 3x^3 - 2x^2 - 7x + 6$ by $x + 1$.

Exercise 3.5.26
Using the rational root test, obtain all the real roots of $f(x) = 8x^3 + 2x^2 + 4x + 1$.

Exercise 3.5.27
Using the rational root test, obtain all the real roots of $f(x) = x^3 + 2x^2 - 11x - 12$.

Exercise 3.5.28
Using the rational root test, obtain all the real roots of $f(x) = x^3 + 6x^2 - x - 6$.

Exercise 3.5.29
Using the rational root test, obtain all the real roots of $f(x) = 15x^3 + 14x^2 - 3x - 2$.

Exercise 3.5.30
Using the rational root test, obtain all the real roots of $f(x) = x^4 - 6x^2 - 8x + 24$.

Chapter 4

Rational Functions

In this chapter we will introduce rational functions, study their graphs, and learn how to decompose them into simpler fractions.

4.1 Graphs and Asymptotes

A **rational function** is a function that is the ratio of two polynomial functions $\frac{f(x)}{g(x)}$. The **domain** consists of all real numbers x such that $g(x) \neq 0$.

Example 4.1.1
Find the domain of the function $f(x) = \frac{x-2}{x^2-x-6}$. Write your answer in interval notation.

Solution.
The domain consists of all numbers x such that $x^2 - x - 6 \neq 0$. But this last quadratic expression is 0 when $x = -2$ or $x = 3$. Thus, the domain is the set $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$ ■

The Long-Run Behavior: Horizontal and Oblique Asymptotes
Given a rational function

$$f(x) = \frac{a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0}.$$

We consider the following three cases:
Case 1: $m = n$
In this case, we can write

$$f(x) = \frac{x^n}{x^n} \cdot \frac{a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}}{b_n + \frac{b_{n-1}}{x} + \cdots + \frac{b_1}{x^{n-1}} + \frac{b_0}{x^n}}.$$

Now, for $1 \leq k \leq n$ we know that

$$\frac{1}{x^n} \to 0 \text{ as } x \to \pm\infty.$$

Thus, $f(x) \to \frac{a_n}{b_n}$ as $x \to \pm\infty$. We call the line $y = \frac{a_n}{b_n}$ a **horizontal asymptote**. Geometrically, the graph flatten out as $x \to \pm\infty$.

Example 4.1.2
Find the horizontal asymptote of $f(x) = \frac{3x^2+2x-4}{2x^2-x+1}$.

Solution.
As $x \to \pm\infty$, we have

$$\begin{aligned} f(x) =& \frac{3x^2 + 2x - 4}{2x^2 - x + 1} \\ =& \frac{x^2}{x^2} \cdot \frac{3 + \frac{2}{x} - \frac{4}{x^2}}{2 - \frac{1}{x} + \frac{1}{x^2}} \to \frac{3}{2} \end{aligned}$$

Hence, $y = \frac{3}{2}$ is a horizontal asymptote ■

Case 2: $m < n$

In this case, we can write

$$f(x) = \frac{1}{x^{n-m}} \frac{a_m + \frac{a_{m-1}}{x^{m-1}} + \cdots + \frac{a_1}{x^{m-1}} + \frac{a_0}{x^m}}{b_n + \frac{b_{n-1}}{x} + \cdots + \frac{b_1}{x^{n-1}} + \frac{b_0}{x^n}}.$$

Hence, $f(x) \to 0$ as $x \to \pm\infty$. In this case, the $x-$ axis is the horizontal asymptote.

Example 4.1.3

Find the horizontal asymptote of $f(x) = \frac{2x+3}{x^3-2x^2+4}$.

Solution.

As $x \to \pm\infty$, we have

$$\begin{aligned} f(x) &= \frac{2x+3}{x^3-2x^2+4} \\ &= \frac{1}{x^2} \cdot \frac{2+\frac{3}{x}}{1-\frac{2}{x}+\frac{4}{x^3}} \to 0 \end{aligned}$$

so the $x-$axis is the horizontal asymptote ■

Case 3: $m > n$

In this case, we can use long division of polynomial to write $\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$ where the degree of $r(x)$ is less than that of $g(x)$. As in case 2, $\frac{r(x)}{g(x)} \to 0$ as $x \to \pm\infty$ so that $\frac{f(x)}{g(x)} - q(x) \to 0$ as $x \to \pm\infty$. We call $y = q(x)$ an **oblique asymptote**. Geometrically, $f(x)$ gets level out close to the oblique line as $x \to \pm\infty$.

Example 4.1.4

Find the oblique asymptote of $f(x) = \frac{2x^2-3x-1}{x-2}$.

Solution.

Using long division of polynomials, we find

$$f(x) = 2x + 1 + \frac{1}{x-2}.$$

Hence, $f(x) - (2x+1) \to 0$ as $x \to \pm\infty$. Hence, $y = 2x + 1$ is the oblique asymptote ■

Remark 4.1.1
It is possible for the graph to cross either the horizontal asymptote or the oblique asymptote.

The Short-Run Behavior: Horizontal Intercepts/Vertical Asymptotes
We next study the local behavior of rational functions which includes the zeros and the vertical asymptotes.

The Zeros of a Rational Function
The **zeros of a rational function** are its $x-$intercepts. They are those numbers that make the numerator zero and the denominator non-zero.

Example 4.1.5
Find the zeros of each of the following functions:

(a) $f(x) = \frac{x^2+x-2}{x-3}$ (b) $g(x) = \frac{x^2+x-2}{x-1}$.

Solution.
(a) Factoring the numerator we find $x^2 + x - 2 = (x - 1)(x + 2)$. Thus, the zeros of the numerator are 1 and -2. Since the denominator is different from zero at these values, the zeros of $f(x)$ are 1 and -2.
(b) The zeros of the numerator are 1 and -2. Since 1 is also a zero of the denominator, $g(x)$ has -2 as the only zero ■

Vertical Asymptotes
When the graph of a function either grows without bounds or decay without bounds as $x \to a$ from either sides, then we say that $x = a$ is a **vertical asymptote**. For rational functions, the vertical asymptotes are the zeros of the denominator. Thus, if $x = a$ is a vertical asymptote then as x approaches a from either sides, the function either increases without bounds or decreases without bounds. The graph of a function never crosses its vertical asymptotes since the function is not defined there.

Example 4.1.6
Find the vertical asymptotes of the function $f(x) = \frac{2x-11}{x^2+2x-8}$

Solution.
Factoring $x^2 + 2x - 8 = 0$ we find $(x - 2)(x + 4) = 0$. Thus, the vertical

asymptotes are the lines $x = 2$ and $x = -4$ ■

Graphing Rational Functions

To graph a rational function $h(x) = \frac{f(x)}{g(x)}$:

1. Find the domain of $h(x)$ and therefore sketch the vertical asymptotes of $h(x)$.
2. Sketch the horizontal or the oblique asymptotes if they exist.
3. Find the $x-$intercepts of $h(x)$ by solving the equation $f(x) = 0$.
4. Find the $y-$intercept, if it exists: $h(0)$.
5. Draw the graph.

Example 4.1.7
Sketch the graph of the function $f(x) = \frac{x(4-x)}{x^2-6x+5}$.

Solution.

1. *Domain* $= (-\infty, 1) \cup (1, 5) \cup (5, \infty)$. The vertical asymptotes are $x = 1$ and $x = 5$.
2. As $x \to \pm\infty$, $f(x) \to -1$ so the line $y = -1$ is the horizontal asymptote.
3. The $x-$intercepts are at $x = 0$ and $x = 4$.
4. The $y-$intercept is $y = 0$.
5. The graph is given in Figure 4.1.1 ■

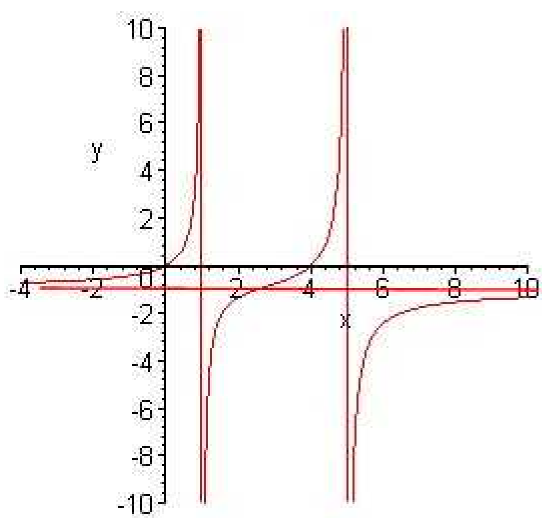


Figure 4.1.1

Case when numerator and denominator have common zeros

We have seen in Example 4.1.5, that the function $g(x) = \frac{x^2+x-2}{x-1}$ has a common zero at $x = 1$. You might wonder what the graph looks like. For $x \neq 1$, the function reduces to $g(x) = x + 2$. Thus, the graph of $g(x)$ is a straight line with a hole at $x = 1$ as shown in Figure 4.1.2.

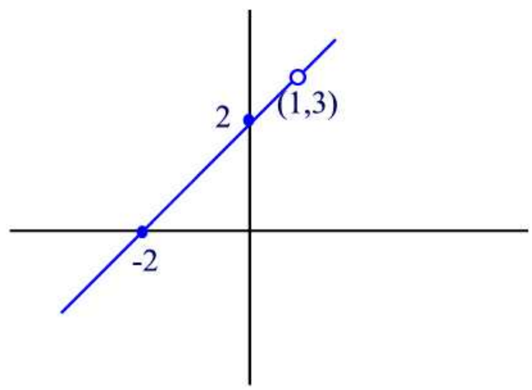


Figure 4.1.2

Exercises

In Exercises 4.1.1 - 4.1.9 answer the following questions:
(a) Find the domain of existence.
(b) Find the horizontal/oblique asymptotes, if they exist.
(c) Find the vertical asymptote(s), if they exist.
(d) Find the intercepts.
(e) Graph.

Exercise 4.1.1

$$f(x) = \frac{1}{x^2}.$$

Exercise 4.1.2

$$f(x) = \frac{2}{x+3}.$$

Exercise 4.1.3

$$f(x) = \frac{-3}{(x-1)^2}.$$

Exercise 4.1.4

$$f(x) = \frac{x}{x^2-1}.$$

Exercise 4.1.5

$$f(x) = \frac{3x}{x+1}.$$

Exercise 4.1.6

$$f(x) = \frac{4}{x^2+1}.$$

Exercise 4.1.7

$$f(x) = \frac{2x+1}{x+1}.$$

Exercise 4.1.8

$$f(x) = \frac{2x^2}{3x^2 + 1}.$$

Exercise 4.1.9

$$f(x) = \frac{x^2 - x}{x + 1}.$$

Exercise 4.1.10
Find the oblique asymptote of $f(x) = \frac{2x^3-1}{x^2-1}$.

Exercise 4.1.11
Write a rational function satisfying the following criteria:
Vertical asymptote: $x = -1$.
Horizontal asymptote: $y = 2$.
$y-$intercept: $y = 3$.
$x-$intercept: $x = -\frac{3}{2}$.

Exercise 4.1.12
Find the zeros of the rational function $f(x) = \frac{x^2+x-2}{x+1}$.

Exercise 4.1.13
Find the $y-$intercept of the function $f(x) = \frac{3}{x-2}$.

Exercise 4.1.14
Write a rational function with vertical asymptotes $x = -2$ and $x = 1$.

Exercise 4.1.15
Find the horizontal asymptote of $f(x) = \frac{2x-1}{x^2+1}$.

Exercise 4.1.16
Find the domain of the function $f(x) = \frac{x+4}{x^2+x-6}$.

Exercise 4.1.17
Find the horizontal asymptote of $f(x) = \frac{x^2}{3x^2-4x-1}$.

Exercise 4.1.18
Find the oblique asymptote of $f(x) = \frac{x^2-1}{2x}$.

Exercise 4.1.19
Find the domain of $(f \circ g)(x)$ if $f(x) = \frac{1}{x+2}$ and $g(x) = \frac{4}{x-1}$.

Exercise 4.1.20
Find the domain of $f(x) = \frac{2x-9}{x^3+2x^2-8x}$.

Exercise 4.1.21
Find the vertical asymptotes of $f(x) = \frac{x^2+1}{x^3+2x^2-25x-50}$.

Exercise 4.1.22
Find the horizontal asymptote of $f(x) = \frac{5x^2-x+2}{2x^2+3x-7}$.

Exercise 4.1.23
Find the horizontal asymptote of $f(x) = \frac{5x^3-x^2+2}{2x^4+3x^3-7}$.

Exercise 4.1.24
Find the oblique asymptote of $f(x) = \frac{10x^2+7x+2}{2x-3}$.

Exercise 4.1.25
Find the $x-$intercepts of $f(x) = \frac{x^3+2x^2-25x+50}{x^2+x+1}$.

Exercise 4.1.26
Sketch the graph of $f(x) = \frac{x^2}{x^2-1}$.

Exercise 4.1.27
Sketch the graph of $f(x) = \frac{1-x^2}{x+1}$.

Exercise 4.1.28
Sketch the graph of $f(x) = \frac{x^2-x+2}{x-3}$.

Exercise 4.1.29
The concentration C(in mg/dl), of a certain antibiotic in a patient's bloodstream is given by

$$C(t) = \frac{50t}{t^2+25}$$

where t is the time (in hours) after taking the antibiotic.
(a) What is the concentration 4 hours after taking the antibiotic?
(b) In order for the antibiotic to be effective, 4 or more mg/dl must be present in the bloodstream. When do you have to take the antibiotic again?

Exercise 4.1.30

A rare species of insect was discovered in the rain forest of Costa Rica. Environmentalists transplant the insect into a protected area. The population of the insect t months after being transplanted is

$$P(t) = \frac{45(1+0.6t)}{(3+0.02t)}.$$

(a) What was the population when $t = 0$?
(b) What will the population be after 10 years?
(c) When will there be 549 insects?

4.2 Partial Fractions Decomposition

The **method of partial fractions decomposition** consists of writing a rational function, i.e., a function of the form

$$R(s) = \frac{N(s)}{D(s)}$$

where $N(s)$ and $D(s)$ are polynomials, as a sum of simpler fractions called **partial fractions**. This can be done in the following way:

Step 1. Use long division to find two polynomials $r(s)$ and $q(s)$ such that

$$\frac{N(s)}{D(s)} = q(s) + \frac{r(s)}{D(s)}.$$

Note that if the degree of $N(s)$ is smaller than that of $D(s)$ then $q(s) = 0$ and $r(s) = N(s)$.

Step 2. Write $D(s)$ as a product of factors of the form $(as + b)^n$ or $(as^2 + bs+c)^n$ where as^2+bs+c is irreducible, i.e., $as^2+bs+c = 0$ has no real zeros.

Step 3. Decompose $\frac{r(s)}{D(s)}$ into a sum of partial fractions in the following way:
(1) For each factor of the form $(as + b)^k$ write

$$\frac{A_1}{as+b} + \frac{A_2}{(as+b)^2} + \cdots + \frac{A_k}{(as+b)^k},$$

where the numbers $A_1, A_2, \cdots, A_k$ are to be determined.
(2) For each factor of the form $(as^2 + bs + c)^k$ write

$$\frac{B_1 s + C_1}{as^2+bs+c} + \frac{B_2 s + C_2}{(as^2+bs+c)^2} + \cdots + \frac{B_k s + C_k}{(as^2+bs+c)^k},$$

where the numbers $B_1, B_2, \cdots, B_k$ and $C_1, C_2, \cdots, C_k$ are to be determined.

Step 4. Multiply both sides of $\frac{r(s)}{D(s)}$ by $D(s)$ and simplify. This leads to an expression of the form

$r(s) =$ a polynomial whose coefficients are combinations of $A_i, B_i,$ and C_i.

Finally, we find the constants, A_i, B_i, and C_i by equating the coefficients of like powers of x on both sides of the last equation.

Example 4.2.1
Decompose into partial fractions $R(s) = \frac{s^3+s^2+2}{s^2-1}$.

Solution.
Step 1. $\frac{s^3+s^2+2}{s^2-1} = s+1+\frac{s+3}{s^2-1}$.
Step 2. $s^2-1=(s-1)(s+1)$.
Step 3. $\frac{s+3}{(s+1)(s-1)} = \frac{A}{s+1}+\frac{B}{s-1}$.
Step 4. Multiply both sides of the last equation by $(s-1)(s+1)$ to obtain

$$s+3 = A(s-1)+B(s+1).$$

Expand the right hand side, collect terms with the same power of s, and identify coefficients of the polynomials obtained on both sides:

$$s+3=(A+B)s+(B-A).$$

Hence, $A+B=1$ and $B-A=3$. Adding these two equations gives $B=2$. Thus, $A=-1$ and so

$$\frac{s^3+s^2+2}{s^2-1} = s+1-\frac{1}{s+1}+\frac{2}{s-1} \blacksquare$$

Example 4.2.2
Decompose $\frac{1}{s(s-3)}$ into partial fractions.

Solution.
We write

$$\frac{1}{s(s-3)} = \frac{A}{s}+\frac{B}{s-3}.$$

Multiply both sides by $s(s-3)$ and simplify to obtain

$$1 = A(s-3)+Bs$$

or

$$1=(A+B)s-3A.$$

Now equating the coefficients of like powers of s to obtain $-3A=1$ and $A+B=0$. Solving for A and B we find $A=-\frac{1}{3}$ and $B=\frac{1}{3}$. Thus,

$$\frac{1}{s(s-3)} = \frac{1}{3}\cdot\frac{1}{s}+\frac{1}{3}\cdot\frac{1}{s-3} \blacksquare$$

Example 4.2.3
Decompose $\frac{1}{(s+1)(s^2+3s+2)}$ into partial fractions.

Solution.
We factor the denominator and split the rational function into partial fractions:

$$\frac{1}{(s+1)^2(s+2)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+2)}.$$

Multiplying both sides by $(s+1)^2(s+2)$ and simplifying to obtain

$$\begin{aligned} 1 =& A(s+1)(s+2) + B(s+2) + C(s+1)^2 \\ =& (A+C)s^2 + (3A+B+2C)s + 2A + 2B + C. \end{aligned}$$

Equating coefficients of like powers of s we find $A+C=0, 3A+B+2C=0$ and $2A+2B+C=1$. Applying the first equation to the second and the third, the second equation reduces to $A+B=0$ and the third reduces to $A+2B=1$. Solving this system, we find $A=-1, B=1$. Thus, $C=1$. Hence,

$$\frac{1}{(s+1)(s^2+3s+2)} = -\frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{s+2} \blacksquare$$

Example 4.2.4
Decompose $\frac{1}{(s+2)(s^2+9)}$ into partial fractions.

Solution.
Using the partial fraction decomposition we can write

$$\frac{1}{(s+2)(s^2+9)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+9}.$$

Multipliying both sides by $(s+2)(s^2+9)$ to obtain

$$\begin{aligned} 1 &= A(s^2+9) + (Bs+C)(s+2) \\ &= (A+B)s^2 + (2B+C)s + 9A + 2C \end{aligned}$$

Equating coefficients of like powers of s we find $A+B=0,\ 2B+C=0,$ and $9A+2C=1$. Substituting $B=-A$ in the second equation and solving the resulting equation and the third equation we find $B=-\frac{1}{13}$, and $C=\frac{2}{13}$. Hence, $A=\frac{1}{13}$. Thus,

$$\frac{1}{(s+2)(s^2+9)} = \frac{1}{13(s+2)} - \frac{s}{13(s^2+9)} + \frac{2}{13(s^2+9)^2} \blacksquare$$

Example 4.2.5
Decompose $\frac{10}{(x-1)(x^2+9)}$ into partial fractions.

Solution.
Using the method of partial fractions decomposition, we have

$$\begin{aligned}
\frac{10}{(x-1)(x^2+9)} =& \frac{A}{x-1} + \frac{Bx+C}{x^2+9}\\
10 =& (A+B)x^2 + (-B+C)x + 9A - C\\
A+B =& 0\\
-B+C =& 0\\
9A - C =& 10.
\end{aligned}$$

The second equation implies $B = C$. Substituting this into the other two equations to obtain the system

$$\begin{aligned}
A+C =& 0\\
9A - C =& 10.
\end{aligned}$$

Solving this system, we find $A = 1$ and $C = -1 = B$. Hence,

$$\frac{10}{(x-1)(x^2+9)} = \frac{1}{x-1} - \frac{x}{x^2+9} - \frac{1}{x^2+9} \blacksquare$$

Example 4.2.6
Decompose $\frac{1}{x(x^2+4)^2}$ into partial fractions.

Solution.
Using the method of partial fractions decomposition, we have

$$\begin{aligned}
\frac{1}{x(x^2+4)^2} =& \frac{A}{x} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}\\
1 =& (A+B)x^4 + Cx^3 + (8A+4B+D)x^2 + (4C+E)x + 16A.
\end{aligned}$$

Equating coefficients of like powers of x, and solving the resulting system, we find $A = \frac{1}{16}$, $B = -\frac{1}{16}$, $C = E = 0$ and $D = -\frac{1}{4}$. Thus,

$$\frac{1}{x(x^2+4)^2} = \frac{1}{16}\frac{1}{x} - \frac{1}{16}\frac{x}{x^2+4} - \frac{1}{4}\frac{x}{(x^2+4)^2} \blacksquare$$

Exercise

In Problems 4.2.1 - 4.2.4, give the form of the partial fraction expansion for the given fraction. You need not evaluate the constants in the expansion.

Exercise 4.2.1

$$\frac{s^3+3s+1}{(s-1)^3(s-2)^2}.$$

Exercise 4.2.2

$$\frac{s^2+5s-3}{(s^2+16)(s-2)}.$$

Exercise 4.2.3

$$\frac{s^3-1}{(s^2+1)^2(s+4)^2}.$$

Exercise 4.2.4

$$\frac{s^4+5s^2+2s-9}{(s^2+8s+17)(s-2)^2}.$$

Exercise 4.2.5
Decompose $\frac{3s+6}{s^2+3s}$ into partial fractions.

Exercise 4.2.6
Decompose $\frac{s^2+1}{s(s+1)^2}$ into partial fractions.

Exercise 4.2.7
Decompose $\frac{2s-3}{s^2-3s+2}$ into partial fractions.

Exercise 4.2.8
Decompose $\frac{4s^2+s+1}{s^3+s}$ into partial fractions.

Exercise 4.2.9
Decompose $\frac{s^2+6s+8}{s^4+8s^2+16}$ into partial fractions.

Exercise 4.2.10
Decompose $\frac{1}{(s+2)s^2}$ into partial fractions.

Exercise 4.2.11
Decompose $\frac{s^2+6s+11}{(s+1)^2(s+2)}$ into partial fractions.

Exercise 4.2.12
Decompose $\frac{1}{(s-1)^2(s-2)}$ into partial fractions.

Exercise 4.2.13
Decompose $\frac{6}{s(s^2+9)}$ into partial fractions.

Exercise 4.2.14
Give the form of the partial fraction expansion for the given fractions. You need not evaluate the constants in the expansion.
(a) $\frac{x^4+1}{x^5+4x^3}$.
(b) $\frac{1}{(x^2-9)^2}$.

Exercise 4.2.15
Give the form of the partial fraction expansion for the given fractions. You need not evaluate the constants in the expansion.
(a) $\frac{t^6+1}{t^6+t^3}$.
(b) $\frac{x^5+1}{(x^2-x)(x^4+2x^2+1)}$.

Exercise 4.2.16
Decompose $\frac{2}{2x^2+3x+1}$ into partial fractions.

Exercise 4.2.17
Decompose $\frac{4y^2-7y-12}{y(y+2)(y-3)}$ into partial fractions.

Exercise 4.2.18
Decompose $\frac{u}{(u+1)(u+2)}$ into partial fractions.

Exercise 4.2.19
Decompose into partial fractions: $\frac{3x-37}{x^2-3x-4}$.

Exercise 4.2.20
Decompose into partial fractions: $\frac{4x^2}{(x-1)(x-2)^2}$.

Exercise 4.2.21
Decompose into partial fractions: $\frac{9x+25}{(x+3)^2}$.

Exercise 4.2.22
Decompose into partial fractions: $\frac{4x^3-3x+5}{x^2-2x}$.

Exercise 4.2.23
Decompose into partial fractions: $\frac{2x-3}{x^3+x}$.

Exercise 4.2.24
Give the form of the partial fraction expansion for the given fraction. You need not evaluate the constants in the expansion.

$$\frac{2x^3+5x-1}{(x+1)^3(x^2+1)^2}.$$

Exercise 4.2.25
Decompose into partial fractions: $\frac{x}{(x^2+1)(x^2+2)}$.

Exercise 4.2.26
Decompose into partial fractions: $\frac{x^2+1}{x^3-6x^2+11x-6}$.

Exercise 4.2.27
Decompose into partial fractions: $\frac{x+3}{(x+5)(x^2+4x+5)}$.

Exercise 4.2.28
Decompose into partial fractions: $\frac{x^4+2x^3+6x^2+20x+6}{x^3+2x^2+x}$.

Exercise 4.2.29
Decompose into partial fractions: $\frac{2x^4+4x^3+x^2+4x+1}{x^5+x^4+x^3+x^2+x}$.

Exercise 4.2.30
Decompose into partial fractions: $\frac{3x^2-3x-8}{(x-5)(x^2-x+4)}$.

Chapter 5

Exponential and Logarithmic Functions

Two commonly encountered functions in applications are the exponential function and its inverse function, the logarithmic function which are the topics of this chapter.

5.1 Exponential Functions and their Graphs

Exponential functions appear in many applications such as population growth, radioactive decay, and interest on bank loans.
Recall that linear functions are functions that change at a constant rate. For example, if $f(x) = mx + b$ then $f(x+1) = m(x+1) + b = f(x) + m$. So when x increases by 1, the y value increases by m. In contrast, an exponential function with base a is one that changes by constant multiples of a. That is, $f(x+1) = af(x)$. Writing $a = 1 + r$ we obtain $f(x+1) = f(x) + rf(x)$. Thus, an exponential function is a function that changes at a constant percent rate.
Exponential functions are used to model increasing quantities such as **population growth** problems.

Example 5.1.1
Suppose that you are observing the behavior of cell duplication in a lab. In one experiment, you started with one cell and the cells doubled every minute. That is, the population cell is increasing at the constant rate of 100%. Write an equation to determine the number (population) of cells after one hour.

Solution.
Table 1 below shows the number of cells for the first 5 minutes. Let $P(t)$ be the number of cells after t minutes.

t	0	1	2	3	4	5
P(t)	1	2	4	8	16	32

Table 1

At time 0, the number of cells is 1 or $2^0 = 1$. After 1 minute, when $t = 1$, there are two cells or $2^1 = 2$. After 2 minutes, when $t = 2$, there are 4 cells or $2^2 = 4$.
Therefore, one formula to estimate the number of cells (size of population) after t minutes is the equation (model)

$$f(t) = 2^t.$$

It follows that $f(t)$ is an increasing function. Computing the rates of change to obtain

$$\begin{aligned}
\tfrac{f(1)-f(0)}{1-0} &= 1\\
\tfrac{f(2)-f(1)}{2-1} &= 2\\
\tfrac{f(3)-f(2)}{3-2} &= 4\\
\tfrac{f(4)-f(3)}{4-3} &= 8\\
\tfrac{f(5)-f(4)}{5-4} &= 16.
\end{aligned}$$

Thus, the rate of change is increasing. Geometrically, this means that the graph of $f(t)$ is concave up. See Figure 5.1.1.

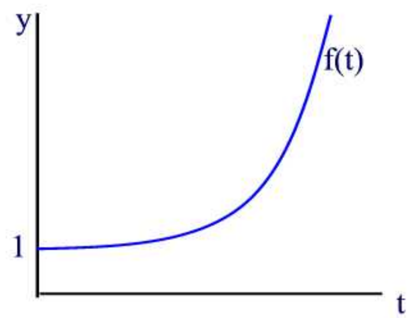


Figure 5.1.1

Now, to determine the number of cells after one hour we convert to minutes to obtain $t = 60$ minutes so that $f(60) = 2^{60} \approx 1.15 \times 10^{18}$ cells ■

Exponential functions can also model decreasing quantities known as **decay models.**

Example 5.1.2
If you start a biology experiment with 5,000,000 cells and 45% of the cells are dying every minute, how long will it take to have less than 50,000 cells?

Solution.
Let $P(t)$ be the number of cells after t minutes. Then $P(t+1) = P(t) - 0.45P(t)$ or $P(t+1) = 0.55P(t)$. By constructing a table of data we find

t	P(t)
0	5,000,000
1	2,750,000
2	1,512,500
3	831,875
4	457,531.25
5	251,642.19
6	138,403.20
7	76,121.76
8	41,866.97

So it takes 8 minutes for the population to reduce to less than 50,000 cells. A formula of $P(t)$ is $P(t) = 5,000,000(0.55)^t$. The graph of $P(t)$ is given in Figure 5.1.2 ■

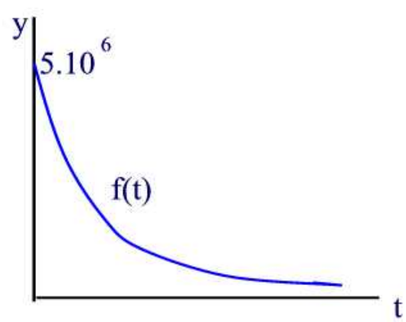


Figure 5.1.2

From the previous two examples, we see that an **exponential function** has the general form

$$P(t) = b \cdot a^t, a > 0\ a \neq 1.$$

Since $b = P(0)$, we call b the **initial value.** We call a the **base** of $P(t)$.
If $a > 1$, then $P(t)$ shows exponential growth with **growth factor** a. The graph of P will be similar in shape to that in Figure 5.1.1.
If $0 < a < 1$, then P shows exponential decay with **decay factor** a. The graph of P will be similar in shape to that in Figure 5.1.2.
Since $P(t+1) = aP(t)$, $P(t+1) = P(t) + rP(t)$ where $r = a - 1$. We call r the **percent growth rate**.

Remark 5.1.1
Why a is restricted to $a > 0$ and $a \neq 1$? Since t is allowed to have any value,

a negative a will create meaningless expressions such as $\sqrt{a}$ (if $t = \frac{1}{2}$). Also, for $a = 1$ the function $P(t) = b$ is called a **constant function** and its graph is a horizontal line.

The Effect of the Parameters a and b

Recall that an exponential function with base a and initial value b is a function of the form $f(x) = b \cdot a^x$. We will assume that $b > 0$. Since $b = f(0)$, $(0, b)$ is the vertical intercept of $f(x)$.
Let's see the effect of the parameter b on the graph of $f(x) = ba^x$.

Example 5.1.3

Graph, on the same axes, the exponential functions $f_1(x) = 2\cdot(1.1)^x$, $f_2(x) = (1.1)^x$, and $f_3(x) = 0.75(1.1)^x$.

Solution.

The three functions are shown in Figure 5.1.3.

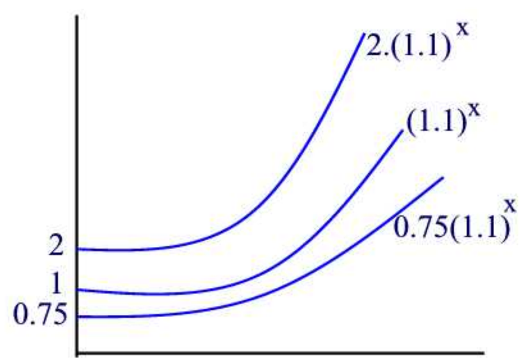


Figure 5.1.3

Note that these functions have the same growth factor but different b and therefore different vertical intercepts ■

We know that the slope of a linear function measures the steepness of the graph. Similarly, the parameter a measures the steepness of the graph of an exponential function. First, we consider the effect of the growth factor on the graph.

Example 5.1.4
Graph, on the same axes, the exponential functions $f_1(x) = 4^x$, $f_2(x) = 3^x$, and $f_3(x) = 2^x$.

Solution.
Using a graphing calculator we find

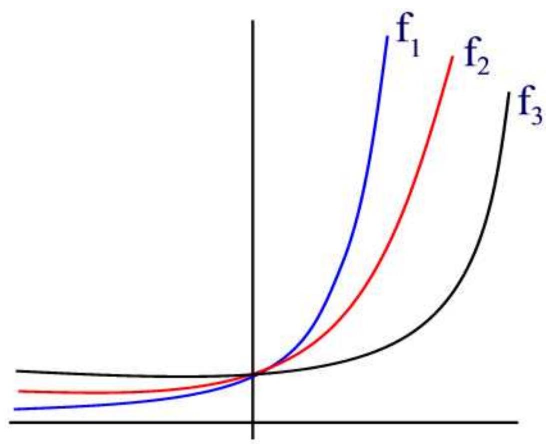


Figure 5.1.4

It follows that the greater the value of a, the more rapidly the graph rises. That is, the growth factor a affects the steepness of an exponential function. Also note that as x decreases, the function values approach the $x-$axis. Symbolically, as $x \to -\infty, y \to 0$ and the $x-$axis is a horizontal asymptote ■

Next, we study the effect of the decay factor on the graph.

Example 5.1.5
Graph, on the same axes, the exponential functions $f_1(x) = 2^{-x} = \left(\frac{1}{2}\right)^x$, $f_2(x) = 3^{-x}$, and $f_3(x) = 4^{-x}$.

Solution.
Using a graphing calculator we find

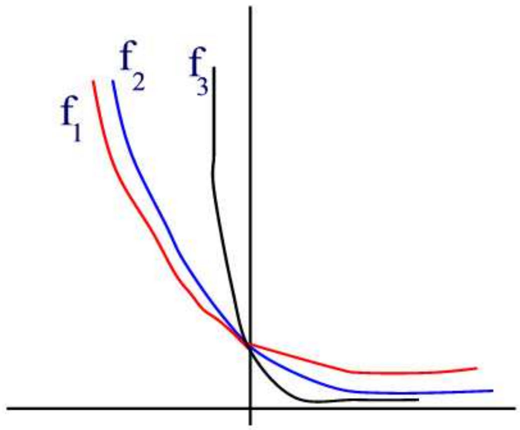


Figure 5.1.5

It follows that the smaller the value of a, the more rapidly the graph falls. Also as x increases, the function values approach the $x-$axis. Symbolically, as $x \to \infty, y \to 0$ ■

Exponential Functions with Base e

When writing $y = be^t$ then we say that y is an exponential function with base e. Now, suppose that $Q(t) = ba^t$. Then there is a number k such that $a = e^k$. Thus,

$$Q(t) = b(e^k)^t = be^{kt}.$$

Note that if $k > 0$ then $e^k > 1$ so that $Q(t)$ represents an exponential growth and if $k < 0$ then $e^k < 1$ so that $Q(t)$ is an exponential decay.
We call the constant k the **continuous growth rate**.

Example 5.1.6
If $f(t) = 3(1.072)^t$ is rewritten as $f(t) = 3e^{kt}$, find k.

Solution.
By comparison of the two functions we find $e^k = 1.072$. Solving this equation graphically (e.g. using a calculator) we find $k \approx 0.695$ ■

Example 5.1.7
A population increases from its initial level of 7.3 million at the continuous rate of 2.2% per year. Find a formula for the population $P(t)$ as a function of the year t. When does the population reach 10 million?

Solution.
We are given the initial value 7.3 million and the continuous growth rate $k = 0.022$. Therefore, $P(t) = 7.3e^{0.022t}$. Next,we want to find the time when $P(t) = 10$. That is , $7.3e^{0.022t} = 10$. Divide both sides by 7.3 to obtain $e^{0.022t} \approx 1.37$. Solving this equation graphically to obtain $t \approx 14.3$ ■

Next, in order to convert from $Q(t) = be^{kt}$ to $Q(t) = ba^t$ we let $a = e^k$. For example, to convert the formula $Q(t) = 7e^{0.3t}$ to the form $Q(t) = ba^t$ we let $b = 7$ and $a = e^{0.3} \approx 1.35$. Thus, $Q(t) = 7(1.35)^t$.

Example 5.1.8
Find the annual percent rate and the continuous percent growth rate of $Q(t) = 200(0.886)^t$.

Solution.
The annual percent of decrease is $r = a - 1 = 0.886 - 1 = -0.114 = -11.4\%$. To find the continuous percent growth rate we let $e^k = 0.886$ and solve for k graphically to obtain $k \approx -0.121 = -12.1\%$ ■

Continuous Compound Interest:
Continuous compounding occurs when the rate of increase of a balance is a fixed percent of the balance. That is, if $B(t)$ denotes the balance in an account after t years with compound interest rate k then

$$\frac{dB}{dt} = kB.$$

Thus, the formula for the continuous compound interest is given by

$$B(t) = B_0 e^{kt}.$$

We call k the **continuous growth rate.** The **annual effective interest rate** is given by $e^k - 1$.

Example 5.1.9
A certain investment grows from a balance of \$2500 to \$4132 in 12 years. The investment account offers continuously compounded interest. What is the annual effective interest rate ?

Solution.
Since $B(t) = 2500e^{kt}$, we find $4132 = 2500e^{12k}$. Thus, $e^k = \left(\frac{4132}{2500}\right)^{\frac{1}{12}}$ so that the annual interest rate is

$$e^k - 1 = \left(\frac{4132}{2500}\right)^{\frac{1}{12}} - 1 \approx 4.28\% \blacksquare$$

Newton's Law of Cooling

Imagine that you are really hungry and in one minute the pizza that you are cooking in the oven will be finished and ready to eat. But it is going to be very hot coming out of the oven. How long will it take for the pizza, which is in an oven heated to 450 degrees Fahrenheit, to cool down to a temperature comfortable enough to eat and enjoy without burning your mouth?
Have you ever wondered how forensic examiners can provide detectives with a time of death (or at least an approximation of the time of death) based on the temperature of the body when it was first discovered?
All of these situations have answers because of Newton's Law of Heating or Cooling. The general idea is that *over time an object will heat up or cool down to the temperature of its surroundings.* The cooling model is given by

$$\frac{dH}{dt} = k(H - S), \ k < 0$$

where S is the temperature of the surroundings. Letting $W = H - S$, the above equation becomes

$$\frac{dW}{dt} = kW$$

whose solution is

$$W(t) = W(0)e^{kt}$$

or

$$H(t) = S + (H(0) - S)e^{kt}.$$

Example 5.1.10
The temperature of a cup of coffee is initially $150°F$. After two minutes its temperature cools to $130°F$. If the surrounding temperature of the room remains constant at $70°F$, how much longer must I wait until the coffee cools to $110°F$?

Solution.
We have
$$H = 70 + (150 - 70)e^{kt} = 70 + 80e^{kt}.$$
To find k we use the fact that $H(2) = 130$. In this case, $130 = 70 + 80e^{2k}$ or $e^{2k} = \frac{3}{4}$. Solving this equation with a calculator, we find $k \approx -0.144$. To finish the problem we must solve for t in the equation
$$110 = 70 + 80e^{-0.144t}.$$
From this equation, we find $e^{-0.144t} = 0.5$ or $t \approx 4.81$ minutes. Thus, I need to wait an additional 2.81 minutes ■

Exercises

Exercise 5.1.1
Identify the initial value and the growth factor of the exponential function $f(t) = 0.75(0.2)^t$.

Exercise 5.1.2
Suppose you are offered a job at a starting salary of \$40,000 per year. To strengthen the offer, the company promises annual raises of 6% per year for the first 10 years. Let $P(t)$ be your salary after t years. Find a formula for $P(t)$ and then compute your projected salary after 4 years from now.

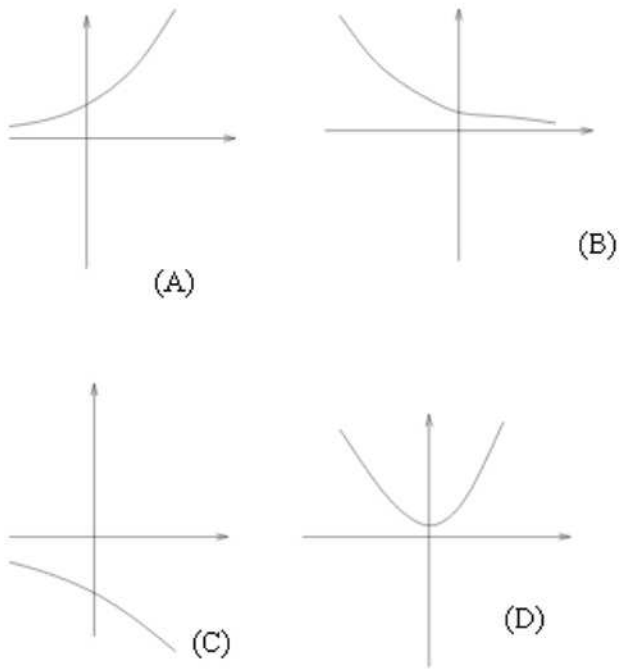
Exercise 5.1.3
The amount in milligrams of a drug in the body t hours after taking a pill is given by $A(t) = 25(0.85)^t$.

(a) What is the initial dose given?
(b) What percent of the drug leaves the body each hour?
(c) What is the amount of drug left after 10 hours?

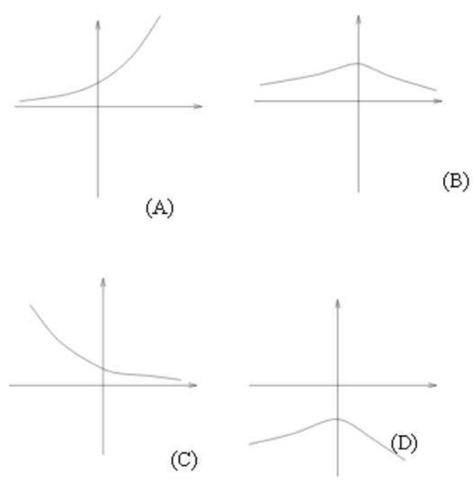
Exercise 5.1.4
The future value B of an investment of P dollars over a period of t years with a continuous compound interest r is given by the formula $B = Pe^{rt}$. If an investment of \$10,000 is invested for 15 years at an annual interest rate of 10% compounded continuously, what is the future value of the investment?

Exercise 5.1.5
Find the annual effective rate if \$1000 is deposited at 5% annual interest rate compounded continuously.

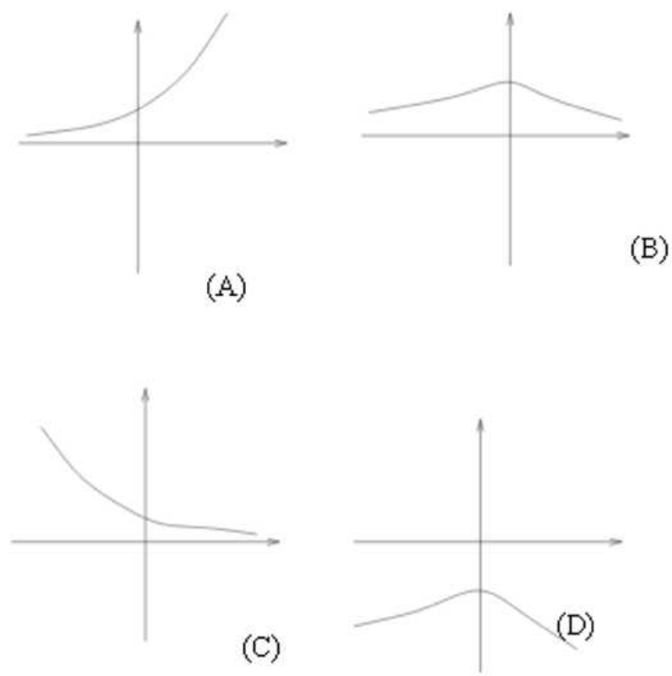
Exercise 5.1.6
Which of the following is the graph of $f(x) = 3^x$.



Exercise 5.1.7
Which of the following is the graph of $f(x) = 4^{-x}$.



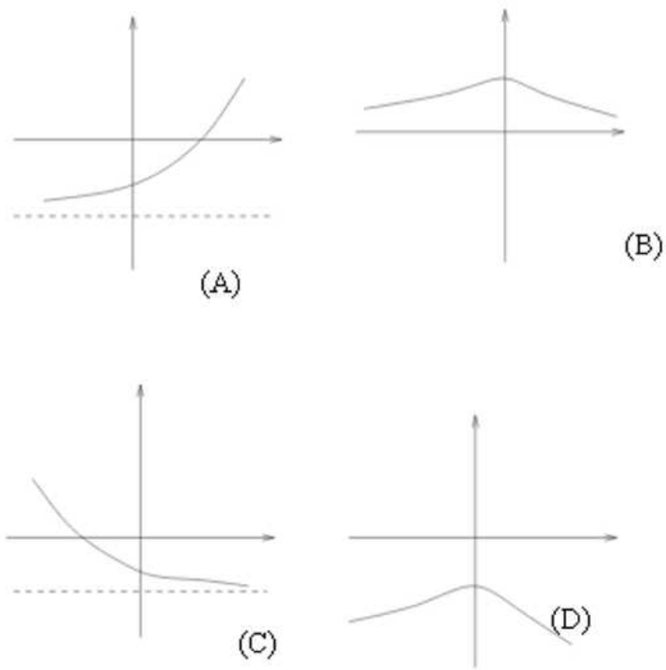
Exercise 5.1.8
Which of the following is the graph of $f(x) = e^{-|x|}$.



Exercise 5.1.9
Simplify: $(e^x + e^{-x})^2 - (e^x - e^{-x})^2$.

Exercise 5.1.10
Using a calculator evaluate $2^{1.4}$ to the nearest thousandth.

Exercise 5.1.11
Which of the following is the graph of $f(x) = 2^{-x} - 3$.



Exercise 5.1.12
What is the horizontal asymptote of $f(x) = 2^{-x} - 3$?

Exercise 5.1.13
What is the horizontal asymptote of $f(x) = -e^{x-3}$?

Exercise 5.1.14
Sketch the graph of the piecewise defined function

$$f(x) = \begin{cases} e^{-x}, & x < 0 \\ e^{x}, & x \geq 0. \end{cases}$$

Exercise 5.1.15
If $2^x = 3$ what does 4^{-x} equal to?

Exercise 5.1.16
Suppose that P dollars are deposited in a bank account paying annual interest

at a rate r and compounded n times per year. After a length of time t, in years, the amount $A(t)$ in the account is given by the formula

$$A(t) = P(1 + \frac{r}{n})^{nt}.$$

Suppose that $n = 1$, $r = 10\%$, $P = \$10,000$ and $t = 10$ years. Find $A(10)$. Round the answer to the nearest penny.

Exercise 5.1.17
Simplify: $(e^x + 1)(e^x - 4)$.

Exercise 5.1.18
Suppose that at time t (in hours), the number $N(t)$ of $E-$coli bacteria in a culture is given by the formula $N(t) = 5000e^{0.1t}$. How many bacteria are in the culture at time 5 hours? Approximate the answer to a whole integer.

Exercise 5.1.19
Sketch the graph of $f(x) = e^{-x} + 2$.

Exercise 5.1.20
Simplify: $(e^x + e^{-x})(e^x - e^{-x})$.

Exercise 5.1.21
Which of the following is an exponential function?
(a) $f(x) = e^{2x}$.
(b) $f(x) = x^{\pi}$.
(c) $f(x) = \pi^x$.
(d) $f(x) = \frac{1}{7^x}$.

Exercise 5.1.22
The population of bacteria in a culture is growing exponentially. At 12:00PM there were 80 bacteria present and by 4:00 PM there were 500 bacteria. Find an exponential function $f(t) = ke^{rt}$ that models this growth, and use it to predict the size of the population at 8:00 PM.

Exercise 5.1.23
Using transformations, sketch the graph of $f(x) = 2^{x+3} - 4$.

Exercise 5.1.24
Find the horizontal asymptote of $f(x) = 2 - e^{-x}$.

Exercise 5.1.25
A father sets up a savings account for his daughter. He puts \$1000 in an account that is compounded quarterly at an annual interest rate of 8%. How much money will be in the account at the end of 10 years? (Assume no other deposits or withdrawals were made after the original one.)

Exercise 5.1.26
The population of a certain city has a continuous growth rate of 9% per year. The population in 1978 was 24,000. Find the projected population of the city for the year 1998.

Exercise 5.1.27
Sketch the graph of $f(x) = 3^x$ and $g(x) = 3^{x+1}$ on the same window.

Exercise 5.1.28
You want to invest \$8,000 for 6 years, and you have a choice between two accounts. The first pays 7% per year, compounded monthly. The second pays 6.85% per year, compounded continuously. Which is the better investment?

Exercise 5.1.29
What is the vertical asymptote of the function $f(x) = \frac{1}{2^{1-x}-1}$?

Exercise 5.1.30
Find the oblique asymptote to the function $f(x) = \frac{2-3e^{2x}}{e^x-1}$. Hint: let $u = e^x$.

5.2 Logarithmic Functions

An equation of the form $a^x = b$ can be solved graphically. That is, using a calculator we graph the horizontal line $y = b$ and the exponential function $y = a^x$ and then find the point of intersection.
In this section we discuss an algebraic way to solve equations of the form $a^x = b$ where a and b are positive constants. For this, we introduce two functions that are found in today's calculators, namely, the functions $\log x$ and $\ln x$.

If $x > 0$, the function $f(x) = 10^x$ is increasing and hence is one-to-one function (Example 2.4.4). Its inverse function is denoted by $f^{-1}(x) = \log x$. We call $\log x$ the **common logarithm of x**. Thus,

$$y = f(x) = 10^x \text{ if and only if } x = f^{-1}(y) = \log y.$$

In words, we define $\log y$ to be a number x that satisfies the equality $10^x = y$. For example, $\log 100 = 2$ since $10^2 = 100$. Similarly, $\log 0.01 = -2$ since $10^{-2} = 0.01$.
Likewise, we have

$$y = \ln x \text{ if and only if } e^y = x.$$

We call $\ln x$ the **natural logarithm** of x.

Example 5.2.1
(a) Rewrite $\log 30 = 1.477$ using exponents instead of logarithms.
(b) Rewrite $10^{0.8} = 6.3096$ using logarithms instead of exponents.

Solution.
(a) $\log 30 = 1.477$ is equivalent to $10^{1.477} = 30$.
(b) $10^{0.8} = 6.3096$ is equivalent to $\log 6.3096 = 0.8$ ■

Example 5.2.2
Without a calculator evaluate the following expressions:

(a) $\log 1$ (b) $\log 10^0$ (c) $\log\left(\frac{1}{\sqrt{10}}\right)$ (d) $10^{\log 100}$ (e) $10^{\log(0.01)}$.

Solution.
(a) $\log 1 = 0$ since $10^0 = 1$.

(b) $\log 10^0 = \log 1 = 0$ by (a).
(c) $\log\left(\frac{1}{\sqrt{10}}\right) = \log 10^{-\frac{1}{2}} = -\frac{1}{2}$.
(d) $10^{\log 100} = 10^2 = 100$.
(e) $10^{\log(0.01)} = 10^{-2} = 0.01$ ■

Properties of Logarithms

(i) Since $10^x = 10^x$ we can write

$$\log 10^x = x.$$

(ii) Since $\log x = \log x$, we have

$$10^{\log x} = x.$$

(iii) $\log 1 = 0$ since $10^0 = 1$.
(iv) $\log 10 = 1$ since $10^1 = 10$.
(v) Suppose that $m = \log a$ and $n = \log b$. Then $a = 10^m$ and $b = 10^n$. Thus, $a \cdot b = 10^m \cdot 10^n = 10^{m+n}$. Rewriting this using logs instead of exponents, we see that

$$\log(a \cdot b) = m + n = \log a + \log b.$$

(vi) If, in (v), instead of multiplying we divide, that is $\frac{a}{b} = \frac{10^m}{10^n} = 10^{m-n}$ then using logs again we find

$$\log\left(\frac{a}{b}\right) = \log a - \log b.$$

(vii) It follows from (vi) that if $a = b$ then $\log a - \log b = \log 1 = 0$ that is $\log a = \log b$. Conversely, if $\log a = \log b$ then $\log\left(\frac{a}{b}\right) = 0$ so that $\frac{a}{b} = 1$. Hence, $a = b$.
(viii) Now, if $n = \log b$ then $b = 10^n$. Taking both sides to the power k we find $b^k = (10^n)^k = 10^{nk}$. Using logs instead of exponents we see that $\log b^k = nk = k \log b$ that is

$$\log b^k = k \log b.$$

Example 5.2.3
Solve the equation: $4(1.171)^x = 7(1.088)^x$.

Solution.
Rewriting the equation into the form $\left(\frac{1.171}{1.088}\right)^x = \frac{7}{4}$ and then using properties (vii) and (viii) to obtain

$$x \log\left(\frac{1.171}{1.088}\right) = \log\left(\frac{7}{4}\right).$$

Thus,

$$x = \frac{\log\left(\frac{7}{4}\right)}{\log\left(\frac{1.171}{1.088}\right)} \blacksquare$$

Example 5.2.4
Solve the equation $\log(2x+1) + 3 = 0$.

Solution.
Subtract 3 from both sides to obtain $\log(2x+1) = -3$. Switch to exponential form to get $2x+1 = 10^{-3} = 0.001$. Subtract 1 and then divide by 2 to obtain $x = -0.4995$ $\blacksquare$

Remark 5.2.1
• All of the above arguments are valid for the function $\ln x$ for which we replace the number 10 by the number $e = 2.718\cdots$. That is, $\ln(a \cdot b) = \ln a + \ln b, \ln\frac{a}{b} = \ln a - \ln b$ etc.
• Keep in mind the following:
$\log(a+b) \neq \log a + \log b$. For example, $\log 1 + 1 \neq \log 1 + \log 1 = 0$.
$\log(a-b) \neq \log a - \log b$. For example, $\log(2-1) = \log 1 = 0$ whereas $\log 2 - \log 1 = \log 2 \neq 0$.
$\log(ab) \neq \log a \cdot \log b$. For example, $\log 1 = \log(2 \cdot \frac{1}{2}) = 0$ whereas $\log 2 \cdot \log\frac{1}{2} = -(\log 2)^2 \neq 0$.
$\log\left(\frac{a}{b}\right) \neq \frac{\log a}{\log b}$. For example, letting $a = b = 2$ we find that $\log\frac{a}{b} = \log 1 = 0$ whereas $\frac{\log a}{\log b} = 1$.
$\log\left(\frac{1}{a}\right) \neq \frac{1}{\log a}$. For example, $\log\frac{1}{\frac{1}{2}} = \log 2$ whereas $\frac{1}{\log\frac{1}{2}} = -\frac{1}{\log 2}$.

Logarithmic Functions and their Graphs

Since $y = \log x$ is the inverse function of the $y = 10^x$, its graph is the reflection of the graph of $y = 10^x$ with respect to the line $y = x$ as shown in

Figure 5.2.1.

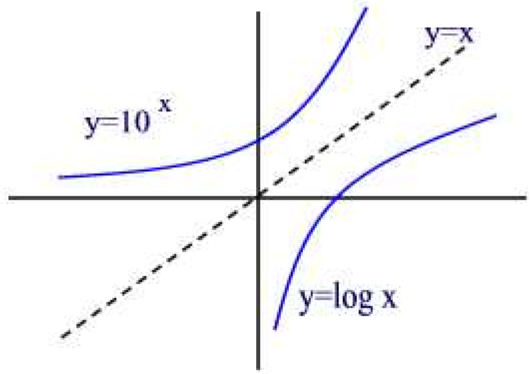


Figure 5.2.1

From the graph we observe the following properties:
(a) The range of $\log x$ consists of all real numbers.
(b) The graph never crosses the $y-$axis since a positive number raised to any power is always positive.
(c) The graph crosses the $x-$axis at $x = 1$.
(d) As x gets closer and closer to 0 from the right the function $\log x$ decreases without bound. That is, as $x \to 0^+, x \to -\infty$. We call the $y-$axis a **vertical asymptote**.

Example 5.2.5
Sketch the graphs of the functions $y = \ln x$ and $y = e^x$ on the same axes.

Solution.
The functions $y = \ln x$ and $y = e^x$ are inverses of each like the functions $y = \log x$ and $y = 10^x$. So their graphs are reflections of one another across the line $y = x$ as shown in Figure 5.2.2.■

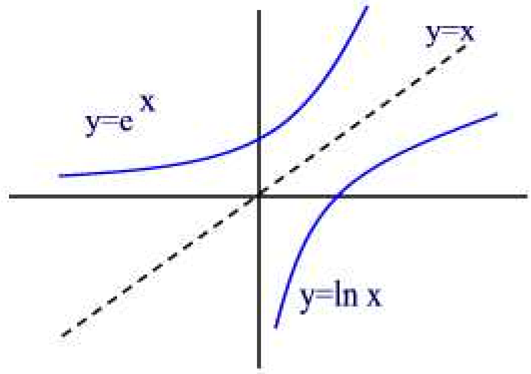


Figure 5.2.2

Doubling Time and Half-Life

In some exponential models one is interested in finding the time for an exponential growing quantity to double. We call this time the **doubling time**. To find it, we start with the equation $b \cdot a^t = 2b$ or $a^t = 2$. Solving for t we find $t = \frac{\ln 2}{\ln a}$.

Example 5.2.6

Find the doubling time of a population growing according to $P = P_0 e^{0.2t}$.

Solution.

Setting the equation $P_0 e^{0.2t} = 2P_0$ and dividing both sides by P_0 to obtain $e^{0.2t} = 2$. The logarithmic form of this equation is $0.2t = \ln 2$. Thus, $t = \frac{\ln 2}{0.2} \approx 3.47$ ■

On the other hand, if a quantity is decaying exponentially then the time required for the initial quantity to reduce into half is called the **half-life**. To find it, we start with the equation $ba^t = \frac{b}{2}$ and we divide both sides by b to obtain $a^t = 0.5$. Solving for t we find $t = \frac{\log(0.5)}{\log a}$.

Example 5.2.7

The half-life of Iodine-123 is about 13 hours. You begin with 50 grams of this substance. What is a formula for the amount of Iodine-123 remaining after t hours?

Solution.
Since the problem involves exponential decay, if $Q(t)$ is the quantity remaining after t hours then $Q(t) = 50a^t$ with $0 < a < 1$. But $Q(13) = 25$. That is, $50a^{13} = 25$ or $a^{13} = 0.5$. Thus $a = (0.5)^{\frac{1}{13}} \approx 0.95$ and $Q(t) = 50(0.95)^t$ ■

Change of base formula
Most calculators have the keys $\log x$ and $\ln x$. What if one encounters a number of the form $\log_a x$ where $a \neq 10$ or $a \neq e$. In this case, one uses the **change of base formula**

$$\log_a x = \frac{\ln x}{\ln a}.$$

Example 5.2.8
Use a calculator to find the value of $\log_2 5$ to the nearest thousandth.

Solution.
Using a calculator we find $\log_2 5 = \frac{\ln 5}{\ln 2} \approx 2.322$ ■

Exercises

Exercise 5.2.1
Show that $\log \frac{1}{b} = -\log b$.

Exercise 5.2.2
Write $\log x^2 y^3 z^5$ as the sum of logarithms.

Exercise 5.2.3
Write the logarithmic form of $x^\pi = e$.

Exercise 5.2.4
Write the exponential form of $\log_\pi x = \frac{1}{2}$.

Exercise 5.2.5
Find the exact value of $\log_{\sqrt{3}} 9$.

Exercise 5.2.6
Find the domain of the function $f(x) = \log_5 \frac{x+1}{x}$.

Exercise 5.2.7
Find k such that the graph of $\log_k x$ contains the point $(2, 2)$.

Exercise 5.2.8
Sketch the graph of $f(x) = \ln(4 - x)$.

Exercise 5.2.9
Sketch the graph of $f(x) = 2 - \ln x$.

Exercise 5.2.10
Sketch the graph of

$$f(x) = \begin{cases} -\ln x, & 0 < x < 1 \\ \ln x, & x \geq 1. \end{cases}$$

Exercise 5.2.11
Sketch the graph of $f(x) = |\log_2 x|$.

Exercise 5.2.12
Find the domain of the function $f(x) = \log_2 (2 - x - x^2)$.

Exercise 5.2.13
Simplify: $a^{\log_{a^2} b}$.

Exercise 5.2.14
Simplify: $e^{\ln 2 - 3\ln 5}$.

Exercise 5.2.15
Let $\ln 2 = a$ and $\ln 3 = b$. Write $\ln \sqrt[4]{48}$ in terms of a and b.

Exercise 5.2.16
Write the following expression as a sum/difference of logarithms.

$$\ln \frac{5x^2 \sqrt[3]{1-x}}{4(x+1)^2}.$$

Exercise 5.2.17
Write the following expression as a single logarithm.

$$\ln\left(\frac{x}{x-1}\right) + \ln\left(\frac{x+1}{x}\right) - \ln(x^2-1).$$

Exercise 5.2.18
Find the exact value of :$5^{\log_5 6 + \log_5 7}$.

Exercise 5.2.19
Use the change of base formula and a calculator to evaluate $\log_{\frac{1}{2}} 15$ to the nearest thousandth.

Exercise 5.2.20
Simplify: $\log_a(x + \sqrt{x^2-1}) + \log_a(x - \sqrt{x^2-1})$.

Exercise 5.2.21
Given: $2\ln y = -\frac{1}{2}\ln x + \frac{1}{3}\ln(x^2+1) + \ln C$. Express y in terms of x and C.

Exercise 5.2.22
A reservoir has become polluted due to an industrial waste spill. The pollution has caused a buildup of algae. The number of algae $N(t)$ present per 1000 gallons of water t days after the spill is given by the formula: $N(t) = 100e^{2.1t}$. How long will it take before the algae count reaches $20,000$ per 1000 gallons? Write answer to the nearest thousandth.

Exercise 5.2.23
Let $f(x) = \ln x$. Express the difference quotient $\frac{f(x+h)-f(x)}{h}$ as a single logarithm.

Exercise 5.2.24
Express the product $(\log_x a)(\log_a b)$ as a single logarithm.

Exercise 5.2.25
After t years the value of a car that originally cost \$14,000 is given by: $V(t) = 14,000(\frac{3}{4})^t$. Find the value of the car two years after it was purchased. Write your answer to the nearest thousandth.

Exercise 5.2.26
On the day a child is born, a deposit of \$50,000 is made in a trust fund that pays 8.75% interest compounded continuously. Determine the balance in the account after 35 years. Write your answer to the nearest hundredth. Recall that $A(t) = Pe^{rt}$.

Exercise 5.2.27
Find a constant k such that $\log_2 x = k \log_8 x$.

Exercise 5.2.28
Solve for $x : \log(\log_5(\log_2 x)) = 0$.

Exercise 5.2.29
Write as a single logarithm: $2\log_a x - 3\log_a y + \log_a(x+y)$.

Exercise 5.2.30
Solve for x: $\log_2 x = \log_2 3$.

Exercise 5.2.31
Find x such that $2 \cdot 5^x = 4$.

Exercise 5.2.32
Find x such that $5\log_2 x = 20$.

5.3 Solving Exponential and Logarithmic Equations

Equations that involve exponential functions are referred to as **exponential equations**. Equations involving logarithmic functions are called **logarithmic equations**. The purpose of this section is to study ways for solving these equations.

Solving Exponential Equations

In order to solve an exponential equation, we use algebra to reduce the equation into the form $a^x = b$ where a and $b > 0$ are constants and x is the unknown variable. Taking the common logarithm of both sides and using the property $\log(a^x) = x \log a$ we find $x = \frac{\log b}{\log a}$.

Example 5.3.1
Solve the equation: $4(1.171)^x = 7(1.088)^x$.

Solution.
Rewriting the equation into the form $\left(\frac{1.171}{1.088}\right)^x = \frac{7}{4}$ and then using properties (vii) and (viii) to obtain

$$x \log\left(\frac{1.171}{1.088}\right) = \log\frac{7}{4}.$$

Thus,

$$x = \frac{\log\frac{7}{4}}{\log\left(\frac{1.171}{1.088}\right)} \blacksquare$$

Example 5.3.2
Solve the equation $200(0.886)^x = 25$ algebraically.

Solution.
Dividing both sides by 200 to obtain $(0.886)^x = 0.125$. Take the log of both sides to obtain $x \log(0.886) = \log 0.125$. Thus, $x = \frac{\log(0.125)}{\log(0.886)} \approx 17.18$ $\blacksquare$

Example 5.3.3
Solve the equation $50,000(1.035)^x = 250,000(1.016)^x$.

Solution.
Divide both sides by $50,000(1.016)^x$ to obtain

$$\left(\frac{1.035}{1.016}\right)^x = 5.$$

Take log of both sides to obtain

$$x \log\left(\frac{1.035}{1.016}\right) = \log 5.$$

Divide both sides by the coefficient of x to obtain

$$x = \frac{\log 5}{\log\left(\frac{1.035}{1.016}\right)} \approx 86.9 \blacksquare$$

Solving Logarithmic Equations

We next describe a method for solving logarithmic equations. The method consists of rewriting the equation into the form $\log x = a$ or $\ln x = a$ and then find the exponential form to obtain $x = 10^a$ or $x = e^a$. Also, you must check these values in the original equation for extraneous solutions.

Example 5.3.4
Solve the equation: $\log(x-2) - \log(x+2) = \log(x-1)$.

Solution.
Using the property of the logarithm of a quotient we can rewrite the given equation into the form $\log\left(\frac{x-2}{x+2}\right) = \log(x-1)$. Thus, $\frac{x-2}{x+2} = x - 1$. Cross multiply and then foil to obtain $(x+2)(x-1) = x-2$ or $x^2 = 0$. Solving we find $x = 0$. However, this is not a solution because it yields logarithms of negative numbers when plugged into the original equation $\blacksquare$

Example 5.3.5
Solve the equation: $\ln(x-2) + \ln(2x-3) = 2\ln x$.

Solution.
Using the property $\ln(ab) = \ln a + \ln b$ we can rewrite the given equation into the form $\ln(x-2)(2x-3) = \ln x^2$. Thus,$(x-2)(2x-3) = x^2$ or $x^2 - 7x + 6 = 0$. Factoring to obtain $(x-1)(x-6) = 0$. Solving we find $x = 1$ or $x = 6$. The value $x = 1$ must be discarded since it yields a logarithm of a negative number $\blacksquare$

Example 5.3.6
Solve the equation $\log(2x+1)+3=0$.

Solution.
Subtract 3 from both sides to obtain $\log(2x+1)=-3$. Switch to exponential form to get $2x+1=10^{-3}=0.001$. Subtract 1 and then divide by 2 to obtain $x=-0.4995$ ■

Can all exponential equations be solved using logarithms?
The answer is no. For example, the only way to solve the equation $x+2=2^x$ is by graphical methods which give the solutions $x\approx -1.69$ and $x=2$.

Example 5.3.7
Solve the equation $2(1.02)^t=4+0.5t$.

Solution.
Using a calculator, we graph the functions $y=2(1.02)^t$ and $y=4+0.5t$ as shown in Figure 5.3.1.

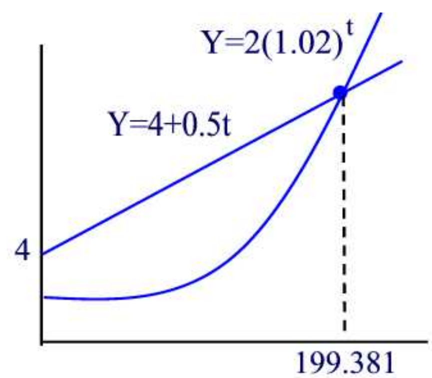


Figure 5.3.1

Using the key INTERSECTION one finds $t\approx 199.381$ ■

Exercises

Exercise 5.3.1
Solve: $\ln\left(\frac{x-1}{2x}\right) = 4$.

Exercise 5.3.2
Solve: $2e^{2x} - e^x - 2 = 0$.

Exercise 5.3.3
Solve: $\ln(x^2 + 2x) = 0$.

Exercise 5.3.4
Solve: $\log_4(x+5) + \log_4(x-5) = 2$.

Exercise 5.3.5
Solve: $\log_3(3x-5) - \log_3(3x+1) = 4$.

Exercise 5.3.6
Solve: $\frac{\log(x+1)}{\log x} = 2$.

Exercise 5.3.7
Solve: $xa^{3\log_a x} = 16$.

Exercise 5.3.8
Solve: $\frac{e^x+e^{-x}}{2} = 1$.

Exercise 5.3.9
Solve: $\frac{e^x+e^{-x}}{e^x-e^{-x}} = 1$.

Exercise 5.3.10
Solve: $\ln(x-2) + \ln(2x-3) = 2\ln x$.

Exercise 5.3.11
The demand equation for a certain product is given by

$$p = 500 - 0.5e^{0.004x}.$$

Find the demands x for the price $p = \$350$. Round answer to a whole integer.

Exercise 5.3.12
Solve: $e^{x-1} = e^x$.

Exercise 5.3.13
Solve: $\log(x-2) - \log(x+2) = \log(x-1)$.

Exercise 5.3.14
Solve: $4^x - 2^x - 12 = 0$.

Exercise 5.3.15
Solve: $5^{x-2} = 3^{x+2}$.

Exercise 5.3.16
Solve: $5e^{2x+4} = 8$. Round to the nearest thousandth.

Exercise 5.3.17
Solve: $\ln 10 - \ln(7-x) = \ln x$.

Exercise 5.3.18
Solve: $8^{2x-3} = \left(\frac{1}{16}\right)^{x-2}$.

Exercise 5.3.19
Solve: $\log_2(x+1) + \log_2 x = 1$.

Exercise 5.3.20
Solve: $3^x = 4^{x+1}$.

Exercise 5.3.21
Solve: $\log_4(3x-2) - \log_4(4x+1) = 2$.

Exercise 5.3.22
Solve: $e^{4x} + 7e^{2x} - 18 = 0$.

Exercise 5.3.23
Solve: $\log_3(x^2 - 6x) = 3$.

Exercise 5.3.24
Solve: $4e^{3x} - 8e^{7x} = 0$.

Exercise 5.3.25
Solve: $\log_3 2 \cdot \log_2(4x-3) = \log_3(2x+1)$.

Exercise 5.3.26
Solve: $3^{2x-11} = 7^{4x-5}$.

Exercise 5.3.27
Solve: $\log_2(2x+1) \geq \log_2(x+4)$.

Exercise 5.3.28
If you deposit \$1000 in an account paying 6% interest compounded continuously, how long will it take for you to have \$1500 in your account?

Exercise 5.3.29
Solve: $(\ln x)^2 = \ln(x^2)$.

Exercise 5.3.30
During its exponential growth phase, a certain bacterium can grow from 5,000 cells to 12,000 cells in 10 hours. At this rate how long will it take to grow to 50,000 cells?

Answer Key

Section 1.1

1.1.1 $A = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$.

1.1.2 $A = \{2, 3, 5, 7, 11\}$.

1.1.3 $A = \{x \in \mathbb{P} : 1 \leq x \leq 12\}$.

1.1.4 $A = \{4, 6, 8, 9, 10, 12\}$.

1.1.5 $A = \{1, 2, 3, 4, 5\}$.

1.1.6 $A = \{x \in \mathbb{N} : x < 10\}$.

1.1.7 $A = B = \{1, 2, 3, 4\}$.

1.1.8 $A = \{4, 5, 6, 7, 8, 9\}$.

1.1.9 $A = \emptyset$.

1.1.10 $A = \{x \in \mathbb{W} : x \text{ is even}\}$.

1.1.11 $x = \frac{91}{4950}$.

1.1.12 $x = \frac{1}{7}$.

1.1.13 $x = \frac{1373}{333}$.

1.1.14 $x = \frac{7111}{2475}$.

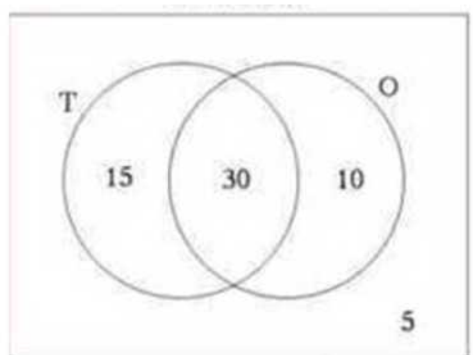
1.1.15 $x = \frac{77}{36}$.

1.1.16 $B \subseteq A$ and $C \subseteq A$.

1.1.17 $\mathbb{N} \subset \mathbb{W} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

1.1.18 (a) B is not a subset of A since $j \in B$ but $j \notin A$. (b) $C \subseteq A$. (c) $C \subseteq C$.

1.1.19 $\{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, A\}$.

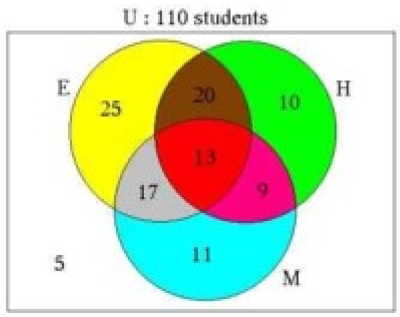


1.1.20

(a) 55 sandwiches with tomatoes or onions.
(b) There are 40 sandwiches with onions.
(c) There are 10 sandwiches with onions but not tomatoes.

1.21 (a) Students speaking English and Spanish, but not French, are in the brown region; there are 20 of these.
(b) Students speaking none of the three languages are outside the three circles; there are 5 of these.
(c) Students speaking French, but neither English nor Spanish, are in the blue region; there are 11 of these.
(d) Students speaking English, but not Spanish, are in the yellow and gray regions; there are 25 + 17 = 42 of these. (e) Students speaking only one of

the three languages are in the yellow, green, and blue regions; there are 25 + 10 + 11 = 46 of these.
(f) Students speaking two of the three languages are in the pink, gray, and brown regions; there are 9 + 17 + 20 = 46 of these.



1.1.22 The answer is $A - B = \{x \in \mathbb{R} : -1 \leq x \leq 0\}$.

1.1.23 First, writing the tabular form of each set, we find $A = \{4, 6, 8, 9, 10\}$ and $B = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Thus, $A - B = \{4, 6, 8\}$ and $B = \{1, 2, 3, 5, 7\}$.

1.1.24 Note that $A = \emptyset$ so that $A \cup B = B$ and $A \cap B = \emptyset$.

1.1.25 We have $A \cup B = \{2, 4, 5, 6, 7, 8, 10\}$ and $A \cap B = \{4, 6, 8\}$.

1.1.26 (a) $A \cup B = \{a, b, c, d, f, g\}$ (b) $A \cap B = \{b, c\}$ (c) $A - B = \{d, f, g\}$ (d) $B - A = \{a\}$.

1.1.27 $\emptyset$.

1.1.28 $A \cup B - C = \{1, 2, 3, 4, 8, 9\} - \{1, 3, 4\} = \{2, 8, 9\}$.

1.1.29 $A \cup (B - C) = \{1, 3, 8, 9\} \cup \{2, 8\} = \{1, 2, 3, 8, 9\}$.

1.1.30 $\mathbb{R} - \mathbb{Q} = \mathbb{I}$.

Section 1.2

1.2.1 (a) $-3 = \frac{-3}{1}$ (b) $\frac{9}{2}$ (c) $-\frac{28}{5}$ (d) $\frac{1}{4}$.

1.2.2 $-3 = \frac{-3}{1} = \frac{3}{-1} = -\frac{3}{1} = -\frac{-3}{-1}$.

1.2.3 (a) $\frac{-3}{5} = \frac{63}{-105}$ (b) $\frac{-18}{-24} = \frac{45}{60}$.

1.2.4 (a) Already in simpest form (b) $-\frac{3}{5}$ (c) $\frac{2}{5}$ (d) $-\frac{4}{5}$.

1.2.5 $\frac{2}{5} \neq 3 \neq \frac{7}{4}$.

1.2.6 (a) $x = -140$ (b) $x = -\frac{5}{6}$.

1.2.7 $\frac{247}{-77} == \frac{19\cdot 13}{7\cdot 11}$.

1.2.8 (a) $b = c$ (b) $a = b$.

1.2.9 (a) 1 (b) $\frac{73}{100}$.

1.2.10 (a) $\frac{625}{642}$ (b) $-\frac{107}{250}$.

1.2.11 (a) $-\frac{2}{7}$ (b) 1 (c) $\frac{3}{2}$.

1.2.12 (a) -4 (b) -1 (c) $-\frac{2}{3}$.

1.2.13 (a) Associativity (b) Distributivity.

1.2.14 (a) $\frac{4}{3}$ (b) $-\frac{7}{5}$ (c) $\frac{80}{81}$.

1.2.15 (a) $\frac{-9}{4}$ (b) Does not exist (c) $-\frac{2}{3}$ (d) $\frac{9}{4}$.

1.2.16 $-\frac{4}{5}$.

1.2.17 $-\frac{7}{6}$.

1.2.18 (a) $-\frac{9}{16}$ (b) $\frac{9}{16}$ (c) $\left(\frac{3}{4}\right)^9$.

1.2.19 $-\frac{7}{3}$.

1.2.20 $-\frac{14}{15}$.

1.2.21 $\frac{21}{40}$.

1.2.22 44,100.

1.2.23 35,280.

1.2.24 $-\frac{125}{588}$.

1.2.25 $-\frac{17}{24}$.

1.2.26 $-\frac{13}{90}$.

1.2.27 $\frac{16}{45}$.

1.2.28 $-\frac{3}{7}$.

1.2.29 $-\frac{4}{3}$.

1.2.30 $\frac{10}{33}$.

1.2.31 (a) 2 (b) $115\frac{5}{9}$ (c) $49\frac{8}{21}$ (d) $1\frac{16}{39}$.

1.2.32 $3\frac{2}{5} > 1.7 > \frac{3}{8} > -\frac{2}{3}$.

1.2.33 $-9.3 < -8\frac{3}{4} < 0.7 < 4$.

Section 1.3

1.3.1 $a^3 + 3a^2b + 3ab^2 + b^3$.

1.3.2 b$a^3 - 3a^2b + 3ab^2 - b^3$.

1.3.3 $4x^2 + 20x + 25$.

1.3.4 $16 - 24x + 9x^2$.

1.3.5 $-5 + 8x - 3x^2$.

1.3.6 $4x^4 - 4x^3 + 4x^2 - 15x$.

1.3.7 (a) $(x+3)^2$ (b) $\left(x - \frac{5}{2}\right)^2$ (c) $\left(\sqrt{3}x + \frac{1}{\sqrt{3}}\right)^2$.

1.3.8 $\left(\sqrt{2}x + \frac{9}{2\sqrt{2}}\right)^2$.

1.3.9 (a) $x^2 + x - 6$ (b) $x^2 + 12x + 35$ (c) $x^2 - 12x + 32$.

1.3.10 $6x^2y^2(3x^2y^3 - xy^2 + 4)$.

1.3.11 $(2x + 3y)(2x + 3y + 5)$.

1.3.12 $2(x-1)(3-2x)^2(8x^2 - 20x + 13)$.

1.3.13 $(x-1)(x-2)(4x-9)$.

1.3.14 $21(x+5)(x+1)$.

1.3.15 $(x+2)^2(x^2 - 4x + 13)$.

1.3.16 $(\sqrt{2}xy - 5z)(\sqrt{2}xy + 5z)$.

1.3.17 $6(x - \sqrt{2}y)(x + \sqrt{2}y)(x^2 + 2y^2)$.

1.3.18 $(x - \sqrt{3})^2(x + \sqrt{3})^2(x^2 + 3)^2$.

1.3.19 $(x^n - 6)^2$.

1.3.20 $(3x + 7y)^2$.

1.3.21 $(\sqrt{x} - \sqrt{y})^2$.

1.3.22 $(x + 4)(x - 3)$.

1.3.23 $(x - 3)(x + 2)$.

1.3.24 $(x - 3)(x - 5)$.

1.3.25 $3xy(4y - 3)(2x + 5)$.

1.3.26 $(3x + z)(y + 2)$.

1.3.27 $(7x - 2)(\sqrt{5}x - 2\sqrt{2})(\sqrt{5}x + 2\sqrt{2})$.

1.3.28 $(6x - 7)(7x^2 + 3)$.

1.3.29 $(7a - 2)(8b - 7)$.

1.3.30 $(4y - 7)(7x + z)$.

1.3.31 $3x(5x - 1)(x^2 + 2)$.

1.3.32 $(x + 3y)(5x - 2z)$.

1.3.33 $(x - \sqrt{5})(x + \sqrt{5})(3x^3 + 2)$.

1.3.34 $(a - b)^3$.

Section 1.4

1.4.1 -4.

1.4.2 $\frac{9}{2}$.

1.4.3 75.

1.4.4 $\frac{1}{2}$.

1.4.5 ± 3.

1.4.6 $x = -4$ or $x = 3$.

1.4.7 $x = 3$.

1.4.8 $x = \frac{5 \pm \sqrt{13}}{6}$.

1.4.9 No real solutions.

1.4.10 $x = -2$ or $x = -3$.

1.4.11 $x = -5$ or $x = 3$.

1.4.12 $x = 1$ or $x = 3$.

1.4.13 $x = 3 - \sqrt{14}$ or $x = 3 + \sqrt{14}$.

1.4.14 $x = \frac{-3 \pm \sqrt{5}}{2}$.

1.4.15 $x = -4$ or $x = 1$.

1.4.16 $x = \frac{2\pi}{2} = \pi$.

1.4.17 No real roots.

1.4.18 23.

1.4.19 $x = 5$.

1.4.20 $x = 4$.

1.4.21 $x = 2$.

1.4.22 $x = 2$ or $x = -\frac{2}{19}$.

1.4.23 $x = 1$ or $x = -\frac{1}{3}$.

1.4.24 No real solutions.

1.4.25 $x = -4$.

1.4.26 No real solutions.

1.4.27 $x = -3$ or $x = 7$.

1.4.28 $x = -17$ or $x = 9$.

1.4.29 $x = -6$ or $x = 11$.

1.4.30 $x = -2$ or $x = -8$.

1.4.31 No real solutions.

1.4.32 $x = 0$ or $x = -2$.

1.4.33 $x = \frac{3}{4}$.

1.4.34 $x = \frac{5}{2}$.

Section 1.5

1.5.1 $\left(-\infty, -\frac{4}{5}\right)$.

1.5.2 $\left(-\infty, -\frac{51}{5}\right]$.

1.5.3 $\left[\frac{1}{4}, \infty\right)$.

1.5.4 $\left(-\infty, \frac{5}{3}\right]$.

1.5.5 $(-\infty, 4]$.

1.5.6 $[-5, \infty)$.

1.5.7 $(-1, 1)$.

1.5.8 $[675.43, 2500.86]$.

1.5.9 At least 74.

1.5.10 $[72.14, 93.14]$.

1.5.11 $(-\infty, -2) \cup (3, \infty)$.

1.5.12 $[0, 1] \cup [2, \infty)$.

1.5.13 $(-\infty, -4) \cup (3, \infty)$.

1.5.14 $[-2, 6]$.

1.5.15 $(-3, 1) \cup (1, 2)$.

1.5.16 $(-\infty, -2) \cup [1, \infty)$.

1.5.17 $(1, 2) \cup \left(-\frac{4}{3}, -\frac{1}{3}\right)$.

1.5.18 $(-\infty, 0) \cup (0, \infty)$.

1.5.19 $(-\infty,-1)\cup(1,\infty)$.

1.5.20 $(-\infty,-7)\cup(-1,\infty)$.

1.5.21 $[-3,7]$.

1.5.22 No solutions.

1.5.23 $(-\infty,\infty)$.

1.5.24 $(-3,3)$.

1.5.25 $(-\infty,-2]\cup(0,4]$.

1.5.26 $(-\infty,-5)\cup(-2,2)$.

1.5.27 $(-\infty,-1)\cup(-1,2]$.

1.5.28 $[-3,0)\cup[2,\infty)$.

1.5.29 $x=-3$.

1.5.30 $\left[2,\frac{7}{2}\right)$.

Section 1.6

1.6.1 The system is inconsistent.

1.6.2 $x = 1$ and $y = 2$. The system is consistent and independent.

1.6.3 Parametric equations: $x = 4 + 2t,\ y = t$. The system is consistent and dependent.

1.6.4 The system is consistent and dependent. The parametric equations are given by $x = -5 + 2t$ and $y = t$.

1.6.5 $x = 4$ and $y = 5$. The system is consistent and independent.

1.6.6 The system is inconsistent.

1.6.7 The system is consistent and dependent. The parametric equations are $x = 5 + \frac{5}{3}t$ and $y = t$.

1.6.8 $x = -\frac{3}{2}$ and $y = \frac{5}{3}$. Hence, the system is consistent and independent.

1.6.9 $x = 0$ and $y = 2$. The system is consistent and independent.

1.6.10 27 ounces of solution with 25% alcohol and 9 ounces of solution with 45% alcohol.

1.6.11 33,900 in bonds with yield 4% and 4,100 in bonds with yield 6%.

1.6.12 The system is inconsistent.

1.6.13 $x_1 = 2, x_2 = -1, x_3 = 1.$

1.6.14 $x_1 = \frac{3}{2},\ x_2 = 1,\ x_3 = -\frac{5}{2}.$

1.6.15 $x_1 = \frac{1}{9}, x_2 = \frac{10}{9}, x_3 = -\frac{7}{3}.$

1.6.16 $x_1 = 3, x_2 = 1, x_3 = 2.$

1.6.17 $x_1 = 2, x_2 = -1, x_3 = 1.$

1.6.18 $x_1 = -\frac{1}{2}, x_2 = 3, x_3 = -4.$

1.6.19 $x = 6, y = 12.$

1.6.20 $x = 10, y = 2.$

1.6.21 $x = -\frac{1}{2},\ y = \frac{7}{2}.$

1.6.22 32 dimes and 25 quarters.

1.6.23 (a) consistent (b) independent (c) one solution.

1.6.24 (a) inconsistent (b) neither (c) no solutions.

1.6.25 (a) consistent (b) dependent (c) infinite number of solutions.

1.6.26 System is inconsistent.

1.6.27 $x_1 = 16s,\ x_2 = 6s,\ x_3 = s.$

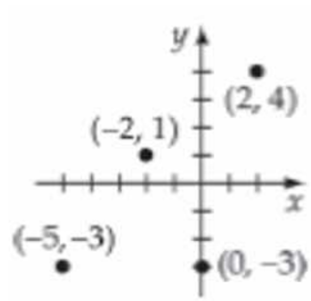
1.6.28 $x_1 = 75^\circ$ and $x_2 = 15^\circ$.

1.6.29 8 ounces of 50% and 4 ounces of 80%.

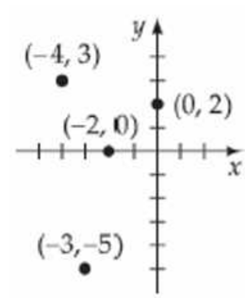
1.6.30 $x_1 = 6$ and $x_2 = -4$.

Section 1.7

1.7.1



1.7.2



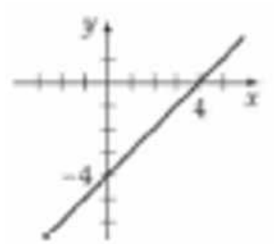
1.7.3 (a) $7\sqrt{5}$ (b) $\sqrt{89}$ (c) $\sqrt{38-12\sqrt{6}}$ (d) $-x\sqrt{10}$ since $x<0$.

1.7.4 (a) $(3,2)$ (b) $(6,4)$ (c) $(-0.875, 3.91)$.

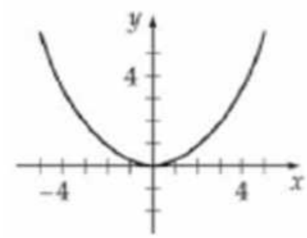
1.7.5 (a) $(6,0)$ and $\left(0, \frac{12}{5}\right)$ (b) $(-4,0),\ (0,-4)$ and $(0,4)$ (c) $(-4,0),\ (4,0),\ (0,-4)$ and $(0,4)$ (d) $(-8,0), (8,0),\ (0,-2)$ and $(0,2)$.

1.7.6 $x = 12$ or $x = -4$.

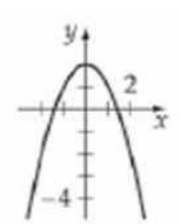
1.7.7



1.7.8



1.7.9



1.7.10 The center is $(0, 0)$ and the radius is $= 6$.

1.7.11 The center is $(-2, -5)$ and the radius is $r = 5$.

1.7.12 The center is $(8, 0)$ and the radius is $r = \frac{1}{2}$.

1.7.13$(x - 4)^2 + (y - 1)^2 = 4$.

1.7.14 $x^2 + y^2 = 25$.

1.7.15 $(x - 1)^2 + (y - 3)^2 = 25$.

1.7.16 The center of the circle is $(3, 0)$ and the radius is 2.

1.7.17 The center of the circle is $(-\frac{1}{2}, 0)$ and the radius is 4.

1.7.18 The center is $(1/2, -3/2)$ and the radius is $5/2$.

1.7.19 $x = 7$ and $y = -6$.

1.7.20 $(x+1)^2 + (y-7)^2 = 25$.

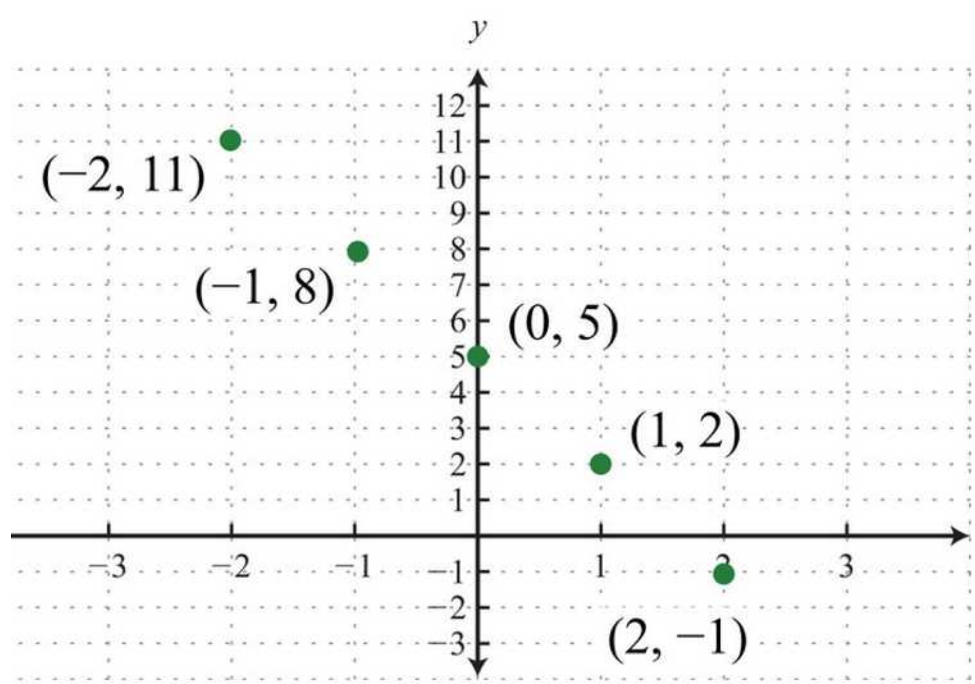
1.7.21 The center of the circle is $(3, 2)$ and the radius is 1.

1.7.22 $(x-7)^2 + (y-11)^2 = 121$.

1.7.23 $(x-3)^2 + (y-4)^2 = 25$.

1.7.24 $(x+3)^2 + (y-3)^2 = 9$.

1.7.25



1.7.26 $(1, 1)$ and $(2, 2)$.

1.7.27 $x = -3$ and $y = 5$.

1.7.28 $(2 - \sqrt{6}, 0), (2 + \sqrt{6}, 0), (0, -3 - \sqrt{10}), (0, -3 + \sqrt{10})$.

1.7.29 $x = -1$ and $y = 2$.

1.7.30 The center of the circle is $(2, -5)$ and the radius is 3.

Section 1.8

1.8.1 (a) $2+3i$ (b) $4-11i$ (c) $-10i$.

1.8.2 (a) $5+12i$ (b) $4-3i$ (c) $5-i$.

1.8.3 (a) $-2+11i$ (b) $-7+4i$ (c) $16+16i$.

1.8.4 (a) $23+2i$ (b) $-22-14i$ (c) 74.

1.8.5 (a) $-117-i$ (b) $243+51i$.

1.8.6 (a) $\frac{1}{2}-\frac{1}{2}i$ (b) $-\frac{8}{65}-\frac{1}{65}i$ (c) $\frac{7}{58}-\frac{3}{58}i$.

1.8.7 (a) $-\frac{1}{61}+\frac{11}{61}i$ (b) $1-6i$ (c) $-24-70i$.

1.8.8 (a) $-29-17i$ (b) $-2-2i$ (c) -16 (d) $150i$.

1.8.9 (a) -1 (b) $-i$ (c) i (d) 1 (e) $-i$.

1.8.10 (a) -2 (b) -33.

1.8.11 (a) 97 (b) $9+40i$.

1.8.12 (a) 13 (b) $\sqrt{85}$ (c) 3.

1.8.13 $|a+bi|=\sqrt{a^2+b^2}=\sqrt{a^2+(-b)^2}=|a-bi|$.

1.8.14 $z-\overline{z}=(a+bi)-(a-bi)=a+bi-a+bi=2bi$ and $z+\overline{z}=(a+bi)+(a-bi)=a+bi+a-bi=2a$.

1.8.15 (a) $\overline{z_1+z_2}=\overline{a_1+b_1i+a_2+b_2i}=\overline{(a_1+a_2)+(b_1+b_2)i}=(a_1+a_2)-(b_1+b_2)i=(a_1-b_1i)+(a_2-b_2i)=\overline{z_1}+\overline{z_2}$.
(b) $\overline{z_1\cdot z_2}=\overline{(a_1+b_1i)(a_2+b_2i)}=\overline{(a_1a_2-b_1b_2)+(a_1b_2+b_1a_2)i}=(a_1a_2-b_1b_2)-(a_1b_2+b_1a_2)i=(a_1-b_1i)(a_2-b_2i)=\overline{z_1}\cdot\overline{z_2}$.

1.8.16 We have $x^2-2x+4=(1+i\sqrt{3})^2-2(1+i\sqrt{3})+4=1+2i\sqrt{3}-3-2-2i\sqrt{3}+4=0$.

1.8.17 (a) $7 + 3i$ (b) $-1 + 7i$ (c) $2 + 26i$ (d) $-i$.

1.8.18 (a) $-\frac{7}{5} + \frac{11}{5}i$ (b) $\frac{3}{4} - \frac{7}{4}i$.

1.8.19 $8\sqrt{3} + i\sqrt{3}$.

1.8.20 (a) 5 (b) $\sqrt{89}$.

1.8.21 $z \cdot \overline{z} = (a + bi)(a - bi) = a^2 + b^2$ which is a real number.

1.8.22 (a) If $z = \overline{z}$ then $a + bi = a - bi$. Thus, $b = -b$ or $2b = 0$. Hence, $b = 0$ and $z = a$ is a real number (b) If z is a real number then it has no imaginary part and in this case $z = \overline{z}$.

Section 2.1

2.1.1 (a) $f(-4) = 47$ (b) $f(1/3) = -\frac{2}{3}$ (c) $f(-a) = 3a^2 - 1$ (d) $f(x+h) = 3x^2 + 6xh + 3h^2 - 1$ (e) $f(x+h) - f(x) = 6xh + 3h^2$.

2.1.2 (a) $f(4) = 1$ (b) $f(-2) = -1$ (c) $f(x) = 1$ (d) $f(x) = -1$.

2.1.3 (a) The given equation defines a function (b) The given equation does not define a function (c) The given equation does not define a function (d)The given equation does not define

2.1.4 (a) y is a function of x (b) y is not a function of x (c) y is a function of x.

2.1.5 (a) $(-\infty, \infty),\ (-\infty, \infty)$(b) $[0, \infty),\ [2, \infty)$ (c) $(-\infty, -2) \cup (-2, \infty),\ (-\infty, 0) \cup (0, \infty)$ (d) $[-2, 2],\ [0, 2]$ (e) $(-4, \infty),\ (0, \infty)$.

2.1.6 Only (a).

2.1.7 (a) $C(x) = 22.80x + 400,000$ (b) $R(x) = 37x$ (c) $P(x) = R(x) - C(x) = 14.2x - 400,000$.

2.1.8 (a) $V(x) = x(30 - 2x)^2$ (b) $0 \leq x \leq 15$.

2.1.9 $c = 3$ or $c = -2$.

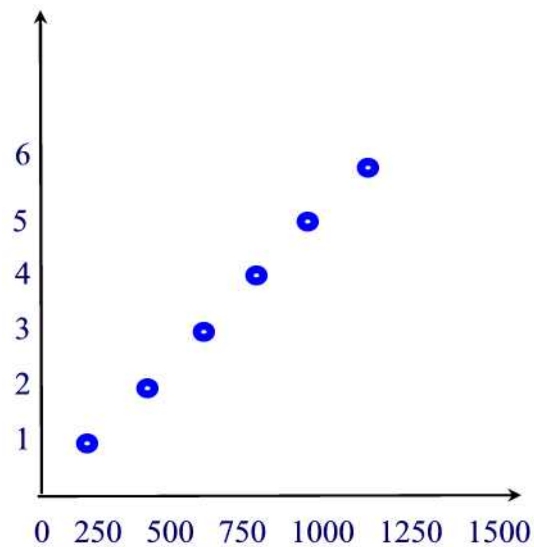
2.1.10 1 is not in the range of f.

2.1.11 0 is not in the range of f.

2.1.12 $w = f(c)$.

2.1.13 (I),(III),(IV),(V),(VII),(VIII).

2.1.14 A is the independent variable and n is the dependent variable.



2.1.15 (a) $f(0) = 40$ (b) $f(2) = 0$.

2.1.16 (a) w (b) $(-4, 10)$ (c) $(6, 1)$.

2.1.17 The variable v goes on the horizontal axis while the variable p goes on the vertical axis.

2.1.18 (a) The graph will cross the $y-$axis at most once (b) Yes. For example, a periodic function such as the sine function.

2.1.19 (a) 100.3 million people owned a cell phone in 2000 (b) 20 million people owned a cell phone a years after 1990 (c) b million people owned a cell phone in 2010 (d) n million people owned a cell phone t years after 1990.

2.1.20 (a) 1999 (b) 1994 (c) 1995-2001.

2.1.21 (a) (II) (b) (I) (c) (V) (d) (IV) (e) (I).

2.1.22 (a) We have

L	0	1	2	3	4	5
C	2	2.5	3	3.5	4	4.5

(b) $C(L) = 2 + 0.5L$.

2.1.23 (a) D is a function of w (b) D is not a function of x.

2.1.24 $(-\infty, 1) \cup (1, \infty)$.

2.1.25 $-\frac{1}{(x+h-1)(x+h)}$.

2.1.26 Not a function.

2.1.27 For $x = 1$ we have $y = -2$ and $y = -4$. Hence, y is not a function of x.

2.1.28 y does not define a function of x.

2.1.29 $[-2, 3] = \{x \in \mathbb{R} : -2 \leq x \leq 3\}$.

2.1.30 $2 \leq x \leq \frac{19}{5}$.

Section 2.2

2.2.1

$$\begin{aligned}
(f+g)(x) &= x^2 - x - 12,\ D = (-\infty, \infty) \\
(f-g)(x) &= x^2 - 3x - 18,\ D = (-\infty, \infty) \\
(f \cdot g)(x) &= x^3 + x^2 - 21x - 45,\ D = (-\infty, \infty) \\
\left(\frac{f}{g}\right)(x) &= \frac{x^2 - 2x - 15}{x+3} = x - 5,\ D = (-\infty, -3) \cup (-3, \infty)
\end{aligned}$$

2.2.2

$$\begin{aligned}
(f+g)(x) &= x^3 - 2x^2 + 8x,\ D = (-\infty, \infty) \\
(f-g)(x) &= x^3 - 2x^2 + 6x,\ D = (-\infty, \infty) \\
(f \cdot g)(x) &= x^4 - 2x^3 + 7x^2,\ D = (-\infty, \infty) \\
\left(\frac{f}{g}\right)(x) &= \frac{x^3 - 2x^2 + 7x}{x} = x^2 - 2x + 7,\ D = (-\infty, 0) \cup (0, \infty)
\end{aligned}$$

2.2.3

$$\begin{aligned}
(f+g)(x) &= 4x^2 + 7x - 12,\ D = (-\infty, \infty) \\
(f-g)(x) &= x - 2,\ D = (-\infty, \infty) \\
(f \cdot g)(x) &= 4x^4 + 14x^3 - 12x^2 - 41x + 35,\ D = (-\infty, \infty) \\
\left(\frac{f}{g}\right)(x) &= \frac{2x^2 + 5x - 7}{2x^2 + 3x - 5} = \frac{2x+7}{2x+5},\ D = (-\infty, -5/2) \cup (-5/2, 1) \cup (1, \infty)
\end{aligned}$$

2.2.4

$$\begin{aligned}
(f+g)(x) &= \sqrt{4 - x^2} + 2 + x,\ D = [-2, 2] \\
(f-g)(x) &= \sqrt{4 - x^2} - 2 - x,\ D = [-2, 2] \\
(f \cdot g)(x) &= x\sqrt{4 - x^2} + 2\sqrt{4 - x^2},\quad D = [-2, 2] \\
\left(\frac{f}{g}\right)(x) &= \frac{\sqrt{4 - x^2}}{2 + x},\ D = (-2, 2]
\end{aligned}$$

2.2.5 (a) 18 (b) $-\frac{20}{9}$ (c) 30 (d) $-\frac{384}{125}$ (e) $-\frac{1}{4}$.

2.2.6 3.

2.2.7 $8x + 4h + 3$.

2.2.8 $\frac{1}{\sqrt{x+h+1}+\sqrt{x+1}}$.

2.2.9 $-\frac{3}{(x+h+1)(x+1)}$.

2.2.10 $\frac{1}{(x+h+1)(x+1)}$.

2.2.11 $(f \circ g)(x) = 6x - 16$ and $(g \circ f)(x) = 6x + 3$.

2.2.12 $(f \circ g)(x) = -125x^3 - 10x$ and $(g \circ f)(x) = -5x^3 - 10x$.

2.2.13 $(f \circ g)(x) = \frac{2}{3x-4}$ and $(g \circ f)(x) = \frac{-5x+1}{x+1}$.

2.2.14 $(f \circ g)(x) = \frac{1}{x-1}$ and $(g \circ f)(x) = \frac{\sqrt{1-x^2}}{|x|}$.

2.2.15 $(f \circ g)(x) = \frac{3|x|}{|5x+2|}$ and $(g \circ f)(x) = -\frac{2}{3}|5 - x|$.

2.2.16 (a) 51 (b) $-\frac{3848}{625}$ (c) $6 + \sqrt{3}$ (d) $-48c^2 - 72c - 23$.

2.2.17 We assume that $h(x) = f(g(x))$. One correct answer is $f(x) = \frac{2}{x^3}$ and $g(x) = x^2+3x+1$. Another correct answer is $f(x) = 2x$ and $g(x) = \frac{1}{(x^2+3x+1)^3}$.

2.2.18 We assume that $h(x) = f(g(x))$. One correct answer is $f(x) = x^3$ and $g(x) = 2x + 1$.

2.2.19 We assume that $h(x) = f(g(x))$. One correct answer is $f(x) = \frac{4}{x^2}$ and $g(x) = 5x^2+2$. Another correct answer is $f(x) = \frac{4}{x}$ and $g(x) = (5x^2+2)^2$.

2.2.20 We assume that $h(x) = f(g(x))$. One correct answer is $f(x) = \sqrt[3]{x}$ and $g(x) = x^2 - 1$. Another correct answer is $f(x) = \sqrt[3]{x-1}$ and $g(x) = x^2$.

2.2.21 The function is decreasing in $(-4, -2) \cup (2, 4)$ and increasing in

$(-2, 2)$.

2.2.22 The function is decreasing in $(-\infty, -3) \cup (0, 2)$ and increasing in $(-3, 0) \cup (2, \infty)$.

2.2.23 The function is decreasing in $(-4, 0) \cup (2, \infty)$ and increasing in $(1, 2)$. The function is constant in $(-5, -4) \cup (0, 1)$.

2.2.24 The function is decreasing in $(-\infty, a)$ and increasing in (b, ∞). The function is constant in (a, b).

2.2.25

function	formula	domain
$f + g$	$\sqrt{x+1} + \sqrt{x-1}$	$[1, \infty)$
$f - g$	$\sqrt{x+1} - \sqrt{x-1}$	$[1, \infty)$
$f \cdot g$	$\sqrt{(x+1)(x-1)}$	$[1, \infty)$
$\frac{f}{g}$	$\sqrt{\frac{x+1}{x-1}}$	$(1, \infty)$

2.2.26 We have

x	-2	-1	0	1	2	3
$f(x)$	5	2	1	2	5	10
$g(x)$	-7	0	1	2	9	28
$(f+g)(x)$	-2	2	2	4	14	38
$(f-g)(x)$	12	2	0	0	-4	-18
$(f \cdot g)(x)$	-35	0	1	4	45	280
$(\frac{f}{g})(x)$	$-\frac{5}{7}$	undefined	1	1	$\frac{5}{9}$	$\frac{5}{14}$

2.2.27

x	-2	-1	0	1	2
$f(x)$	6	3	2	3	6
$g(x)$	-2	-1	0	1	2
$(f+g)(x)$	4	2	2	4	8
$(f-g)(x)$	8	4	2	2	4
$(f \cdot g)(x)$	-12	-3	0	3	12
$(\frac{f}{g})(x)$	-3	-3	undefined	3	3

2.2.28 $\frac{2}{(5-x-h)(5-x)}$.

2.2.29 (a) $6x^2 - 15$ (b) One answer is $f(x) = \sqrt[3]{x}$ and $g(x) = 2x + 1$.

2.2.30 The function is increasing in $(-2, \infty)$, decreasing is $(-6, -2)$ and constant in $(-\infty, -6)$.

Section 2.3

2.3.1 even.

2.3.2 neither.

2.3.3 odd.

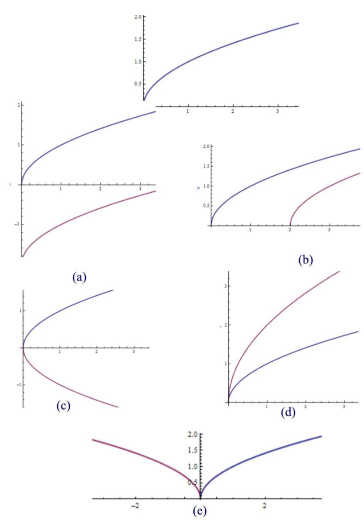
2.3.4 odd.

2.3.5 We have $g(-x) = \frac{1}{2}[f(-x) + f(x)] = \frac{1}{2}[f(x) + f(-x)] = g(x)$ so that $g(x)$ is even. Likewise, $h(-x) = \frac{1}{2}[f(-x) - f(x)] = -\frac{1}{2}[f(x) - f(-x)] = -h(x)$ so that $h(x)$ is odd.

2.3.6 This follows from $f(x) = g(x) + h(x)$ where $g(x)$ and $h(x)$ are defined in the previous problem.

2.3.7 (a) even (b) odd (c) neither.

2.3.8

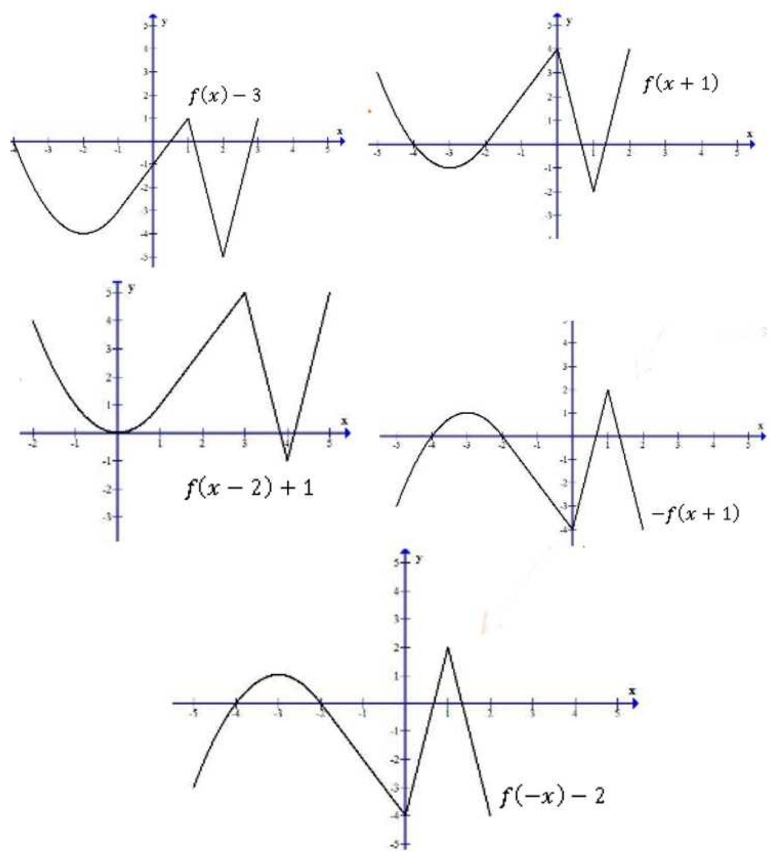


2.3.9 $y = f[-\frac{1}{3}(x - 5)]$.

2.3.10 $y = 2|x+2| - 4$.

2.3.11 Shift the graph of $y = f(x)$ 3 units to the right and then flip the resulting graph about the $y-$axis.

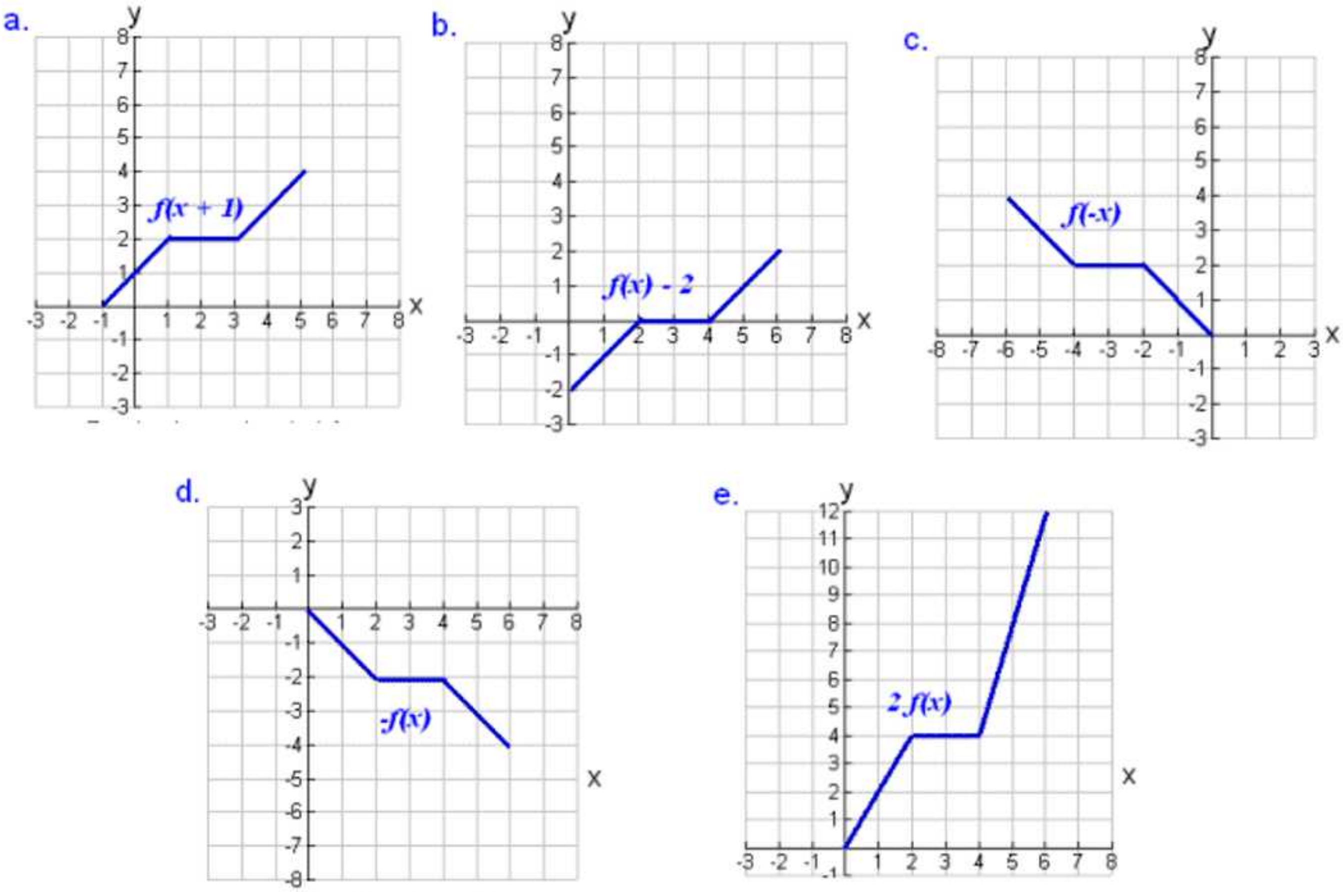
2.3.12



2.3.13 (a) $y = \sqrt{x-2} + 7$ (b) $y = \sqrt{x-2} + 7$ (c) Yes.

2.3.14 $g(x) = -2(x+3)^2 + 4$.

2.3.15

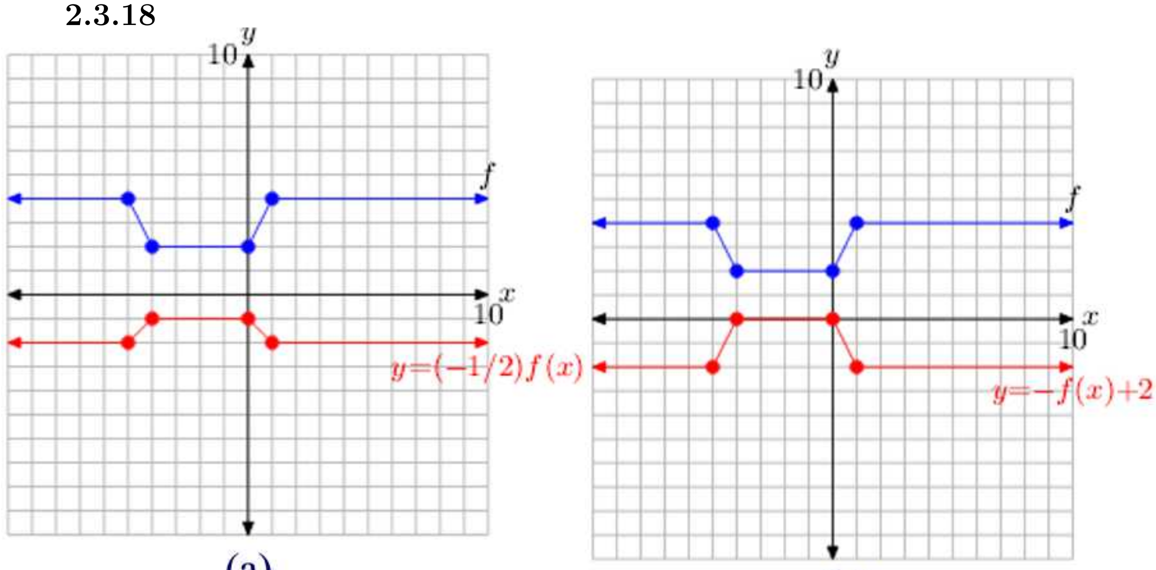


2.3.16
(a) To obtain the graph of $y = |x + 3|$, we translate the graph of $y = |x|$ to the left by 3 units.
(b) To obtain the graph of $y = -|x|$, we flip the graph of $y = |x|$ about the $x-$axis.
(c) To obtain the graph of $y = -x + 5$, we flip the graph of $y = x + 5$ about the $y-$axis.
(e) To obtain the graph of $y = 3x + 5$ we expand the graph of $y = x + 5$ vertically by a factor of 3.
(f) To obtain the graph of $y = 4x^3 - 6$ we translate the graph of $y = 4x^3$

downward by 6 units.
(g) To obtain the graph of $y = \frac{1}{x+5}$ we shift the graph of $y = \frac{1}{x}$ to the left by 5 units.
(h) To obtain the graph of $y = 4x^2$ we compress the graph of $y = x^2$ horizontally by a factor of $\frac{1}{2}$.
(i) To obtain the graph of $y = 4x^2$ we expand the graph of $y = x^2$ vertically by a factor of 4.
(j) To obtain the graph of $y = (x+8)^3$ we translate of $y = x^3$ to the left by 8 units.
(k) To obtain the graph of $y = |6x|$ we compress horizontally the graph of $y = |x|$ by a factor of $\frac{1}{6}$.
(l) To obtain the graph of $y = -x^2$ we flip the graph of $y = x^2$ about the $x-$axis.

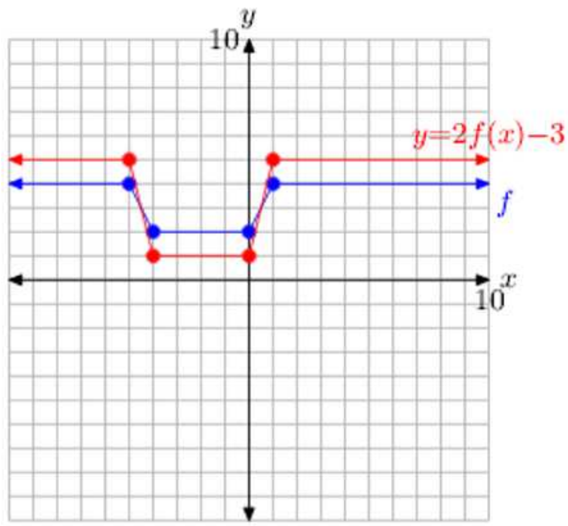
2.3.17 (a) $y = -5f(x-7)+1$ (b) $y = -5f(x-7)+1$ (c) $y = -[5f(x-7)-1]$ (d) $y = -5[f(x-7)+1]$ (e) $y = -5f(x+7)+1$.

2.3.18

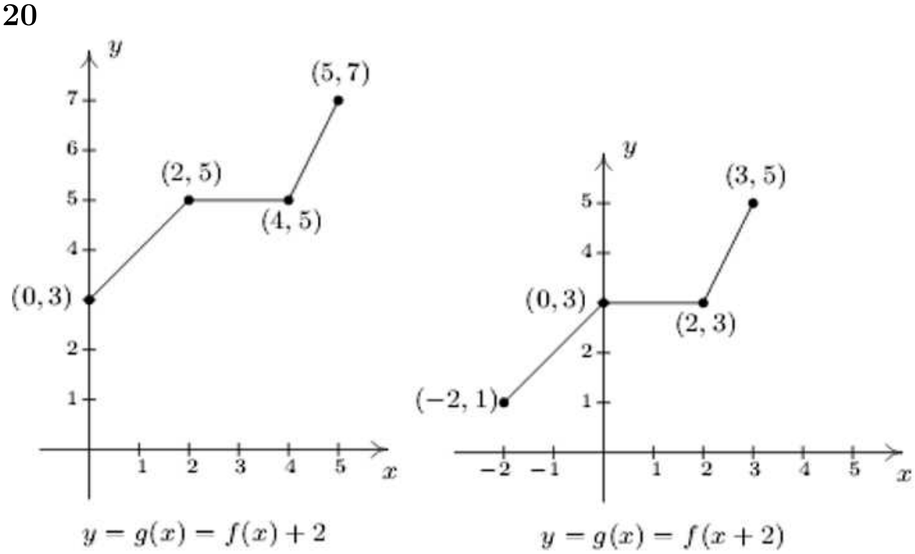


(a) (b)

2.3.19



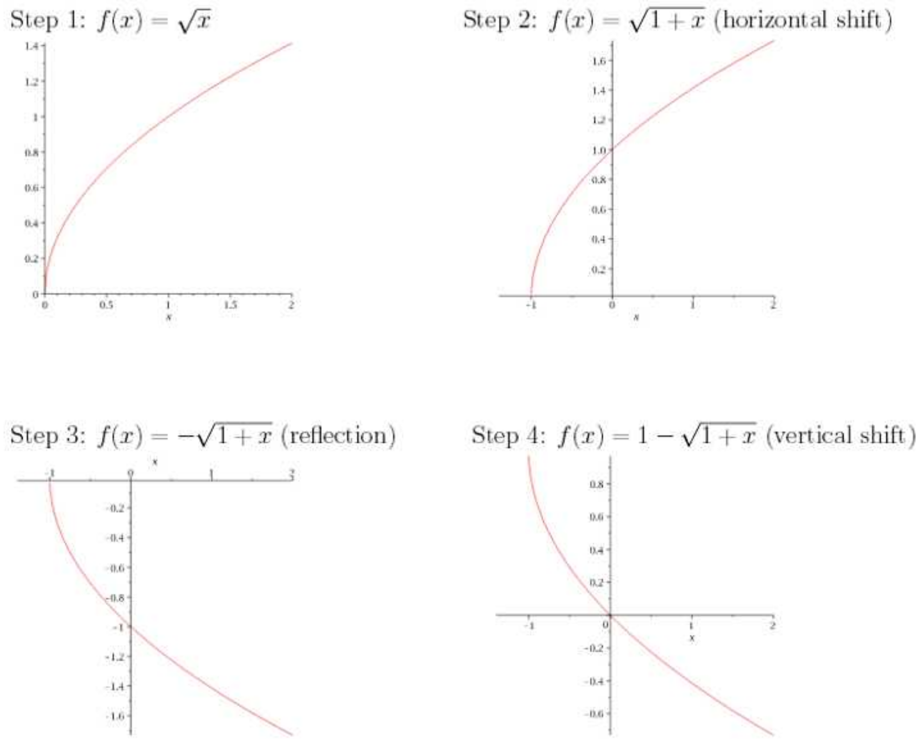
2.3.20



2.3.21 (a) - (A); (b) - (D); (c) - (B); (d) - (C).

2.3.22 (a) - (B); (b) - (A).

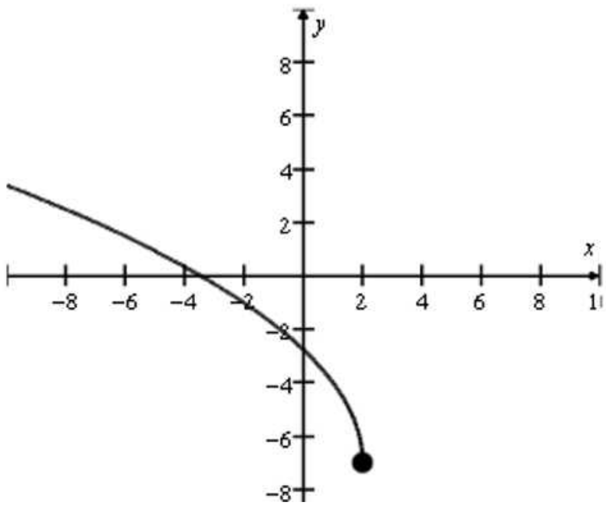
2.3.23



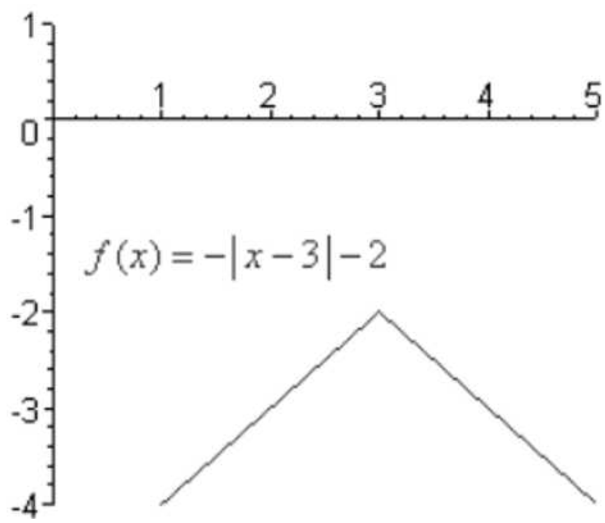
2.3.24 One way is to start by shifting the graph of $f(x) = \sqrt{x}$, 1 unit to the left, followed by a vertical stretch by a factor of 3 and ending with a vertical shift, 7 units down.

2.3.25 Note that $f(x) = \sqrt{-(x-2)} + 7$. One way is to start by shifting the graph of $f(x) = \sqrt{x}$, 2 units to the right; followed by a reflection about the $y-$axis and ending by a vertical shift of 7 units upward.

2.3.26 Note first that $f(x) = 3\sqrt{-(x-2)} - 7$. One way is to start by shifting the graph of $f(x) = \sqrt{x}$, 2 units to the right; followed by a reflection about the $y-$axis; followed by a vertical stretch by a factor of 3; and ending by a vertical shift of 7 units downward. We obtain the following graph.



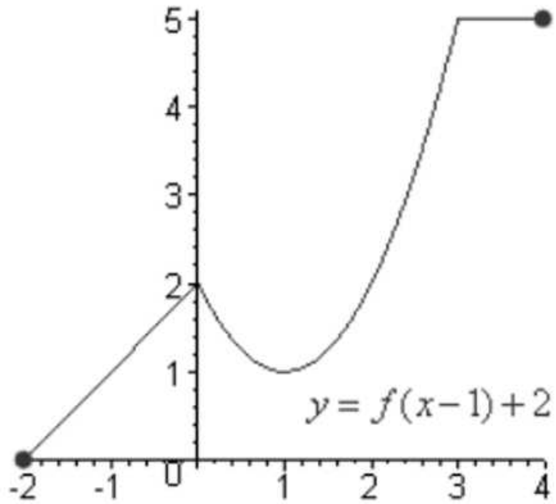
2.3.27 One way to graph $f(x)$ is to start by shifting the graph of $f(x) = |x|$ to the right by 3 units followed by a reflection about the $x-$axis and ending with a vertical shift of 2 units downward. We obtain the following graph.



2.3.28 Graph is symmetric about $x-$axis, $y-$axis, and the origin.

2.3.29 (a) $y = -|x-2|+3$ (b) $y = -|x+2|+4$ (c) $y = |x+1|-2$.

2.3.30



Section 2.4

2.4.1 Let $x_1 = 0$ and $x_2 = 1$. Then $f(x_1) = f(x_2) = 0$. Thus, f is not one-to-one.

2.4.2 Suppose $f(x_1) = f(x_2)$. Then $x_1^3 + 1 = x_2^3 + 1$. Subtracting 1 from both sides and then taking the cube root of both sides, we find $x_1 = x_2$. Hence, f is one-to-one.

2.4.3 Suppose that $f(g(x_1)) = f(g(x_2))$. Since f is one-to-one, we have $g(x_1) = g(x_2)$. Likewise, since g is one-to-one, we have $x_1 = x_2$. Hence, $f \circ g$ is one-to-one.

2.4.4 Note that $h(x) = f(g(x))$ where $g(x) = 3x - 1$ and $f(x) = x^3 + 1$. Since both f and g are one-to-one, h is also one-to-one.

2.4.5 f is not one-to-one.

2.4.6 None.

2.4.7 (i) Not a function (ii) A function that is not one-to-one (iii) A function that is not one-to-one (iv) Not a function (v) Not a function (vi) A function that is one-to-one.

2.4.8 No. The inputs 2 and 3 share the same output 4.

2.4.9 Suppose $f(x_1) = f(x_2)$ where $x_1, x_2 \geq 0$. Then $x_1^2 = x_2^2$. Taking the square root of both sides and using the fact that both x_1 and x_2 are non-negative, we obtain $x_1 = x_2$. Hence, f is one-to-one.

2.4.10 Only (b) and (c) are one-to-one.

2.4.11 $f^{-1}(x) = \sqrt[3]{\frac{x-5}{2}}$.

2.4.12 $f^{-1}(x) = \frac{1}{5}(\sqrt[3]{x} + 1)$.

2.4.13 $f^{-1}(x) = \frac{5x+4}{3x-1}$.

2.4.14 $f^{-1}(x) = \left(\frac{2x+5}{3}\right)^2$.

2.4.15 $\text{Dom}(g^{-1}) = \text{Range}(g) = [0,\infty)$ and $\text{Range}(g^{-1}) = \text{Dom}(g) = [3,\infty)$.

2.4.16 Yes.

2.4.17 No.

2.4.18 (a) $\text{Dom}(f) = (-\infty,1)\cup(1,\infty)$ and $\text{Range}(f) = (-\infty,0)\cup(0,\infty)$ (b) $f^{-1}(x) = \frac{1}{x}+1$ (c) $\text{Dom}(f^{-1}) = \text{Range}(f) = (-\infty,0)\cup(0,\infty)$ and $\text{Range}(f^{-1}) = \text{Dom}(f) = (-\infty,1)\cup(1,\infty)$.

2.4.19 (a) $\text{Dom}(f) = [-\sqrt[3]{5},\infty)$ and $\text{Range}(f) = [0,\infty)$ (b) $f^{-1}(x) = \sqrt[3]{x^2-5}$ (c) $\text{Dom}(f^{-1}) = \text{Range}(f) = [0,\infty)$ and $\text{Range}(f^{-1}) = \text{Dom}(f) = [-\sqrt[3]{5},\infty)$.

2.4.20 $f^{-1} = \{(5,-10),(9,-7),(6,0),(12,8)\}$.

2.4.21 Only (1) and (2).

2.4.22 $f(3) = 5$.

(**2.4.23** (a) $\text{Dom}(f) = [-1,\infty)$ and $\text{Range}(f) = [0,\infty)$ (b) Suppose that $f(x_1) = f(x_2)$. That is, $\sqrt{4x_1+4} = \sqrt{4x_2+4}$. Hence, $(\sqrt{4x_1+4})^2 = (\sqrt{4x_2+4})^2 \Longrightarrow 4x_1+4 = 4x_2+4 \Longrightarrow 4x_1+4-4 = 4x_2+4-4 \Longrightarrow 4x_1 = 4x_2 \Longrightarrow x_1 = x_2$ (c) $f^{-1}(x) = \frac{x^2-4}{4}$ (d) $\text{Dom}(f^{-1}) = \text{Range}(f) = [0,\infty)$ and $\text{Range}(f^{-1}) = \text{Dom}(f) = [-1,\infty)$.

2.4.24 We have

$$\begin{aligned}
f(g(x)) =& f\left(\frac{3x+1}{x-2}\right) = \frac{2\left(\frac{3x+1}{x-2}\right)+1}{\left(\frac{3x+1}{x-2}\right)-3}\\
=& \frac{\frac{6x+2+x-2}{x-2}}{\frac{3x+1-3x+6}{x-2}} = \frac{7x}{7} = x\\
g(f(x)) =& g\left(\frac{2x+1}{x-3}\right) = \frac{3\left(\frac{2x+1}{x-3}\right)+1}{\left(\frac{2x+1}{x-3}\right)-2}\\
=& \frac{\frac{6x+3+x-3}{x-3}}{\frac{-2x+6+2x+1}{x-3}} = \frac{7x}{7} = x.
\end{aligned}$$

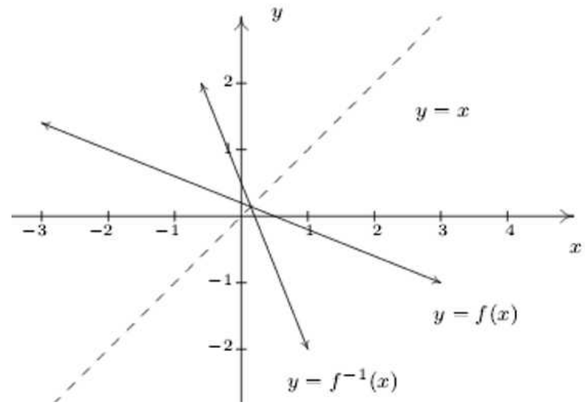
2.4.25 We have

$$\begin{aligned}
f(g(x)) =& f\left(\frac{x+2}{7}\right) = 7\left(\frac{x+2}{7}\right) - 2 = x+2-2 = x\\
g(f(x)) =& g(7x-2) = \frac{7x-2+2}{7} = \frac{7x}{7} = x.
\end{aligned}$$

2.4.26 Suppose $f(x_1) = f(x_2)$. Then $\frac{2x_1}{7-5x_1} = \frac{2x_2}{7-5x_2} \Longrightarrow x_1(7-5x_2) = x_2(7-5x_1) \Longrightarrow 7x_1 - 5x_1x_2 = 7x_2 - 5x_1x_2 \Longrightarrow 7x_1 = 7x_2 \Longrightarrow x_1 = x_2$. Hence, f is one-to-one. $\text{Dom}(f) = (-\infty, \frac{7}{5}) \cup (\frac{7}{5}, \infty)$ and $\text{Range}(f) = (-\infty, -\frac{2}{5}) \cup (-\frac{2}{5}, \infty)$.

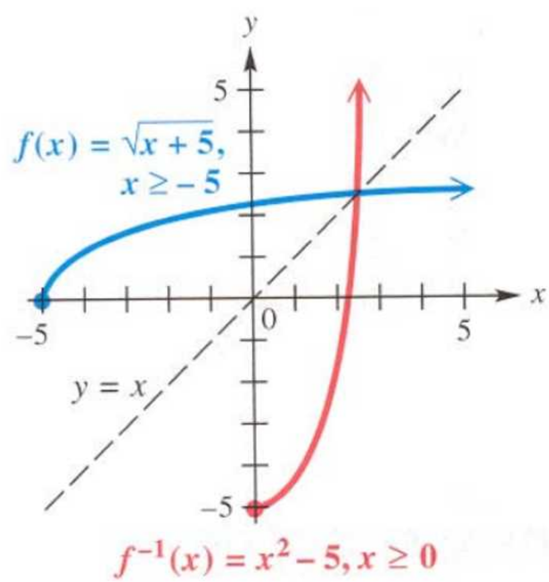
2.4.27 By the previous probem f is one-to-one so that f is invertible. To find f^{-1} we proceed as follows $y = \frac{2x}{7-5x} \Longrightarrow x = \frac{2y}{7-5y} \Longrightarrow 7x - 5xy = 2y \Longrightarrow y(5x+2) = 7x \Longrightarrow y = \frac{7x}{5x+2} \Longrightarrow f^{-1}(x) = \frac{7x}{5x+2}$. Finally, $\text{Dom}(f^{-1}) = \text{Range}(f) = (-\infty, -\frac{2}{5}) \cup (-\frac{2}{5}, \infty)$ and $\text{Range}(f^{-1}) = \text{Dom}(f) = (-\infty, \frac{7}{5}) \cup (\frac{7}{5}, \infty)$.

2.4.28



2.4.29 (a) $f^{-1}(1) = 2$ (b) $f^{-1}(0) = 2$ (c) $f^{-1}(4) = 0$.

2.4.30



Section 2.5

2.5.1 (C).

2.5.2 (B).

2.5.3 3.

2.5.4 3.

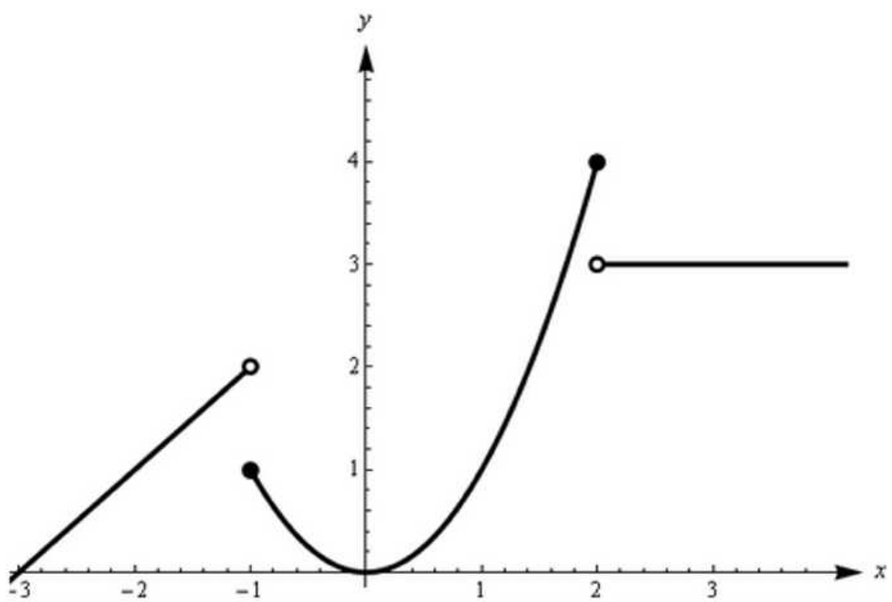
2.5.5 The set $\mathbb{Z}$ of all integers.

2.5.6 $\text{Dom}(f) = (-\infty, \infty)$ and $\text{Range}(f) = (-\infty, 4]$.

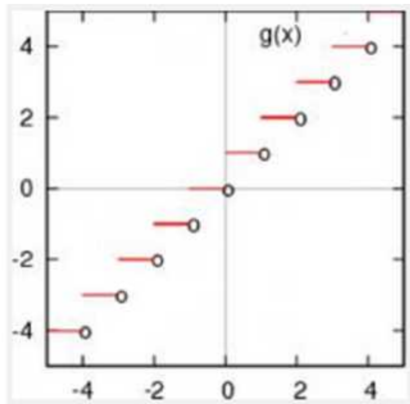
2.5.7 Dom $= [0, 4]$ and Range $= \{2, 3, 4, 5, 6\}$.

2.5.8 Dom $= [0, 4]$ and Range $= [0, 1) \cup [2, 3) \cup [4, 5) \cup [6, 7) \cup \{8\}$.

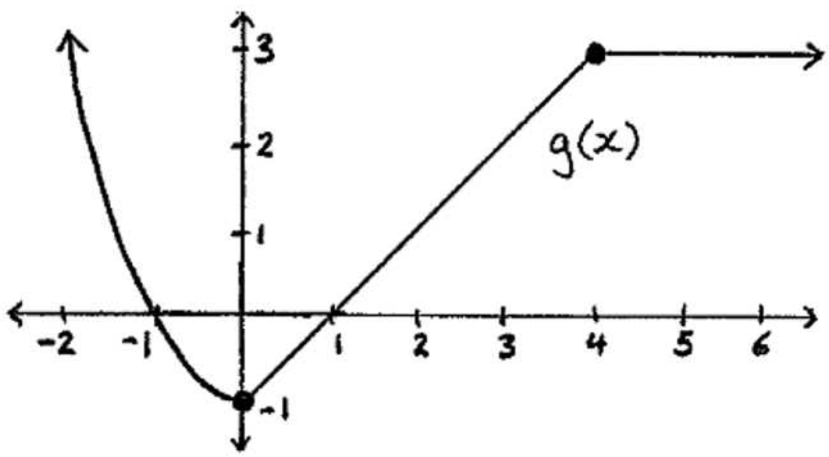
2.5.9



2.5.10



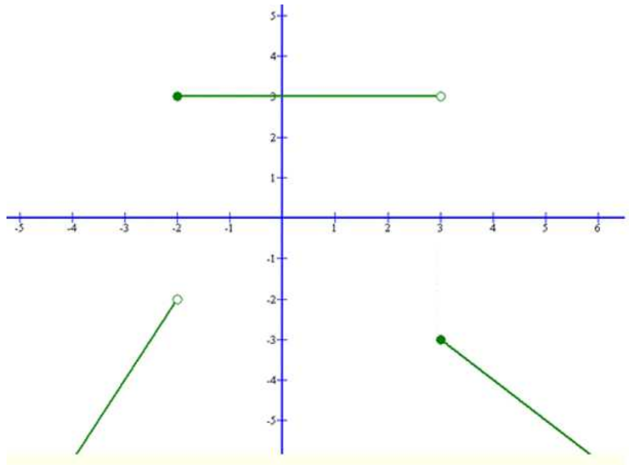
2.5.11



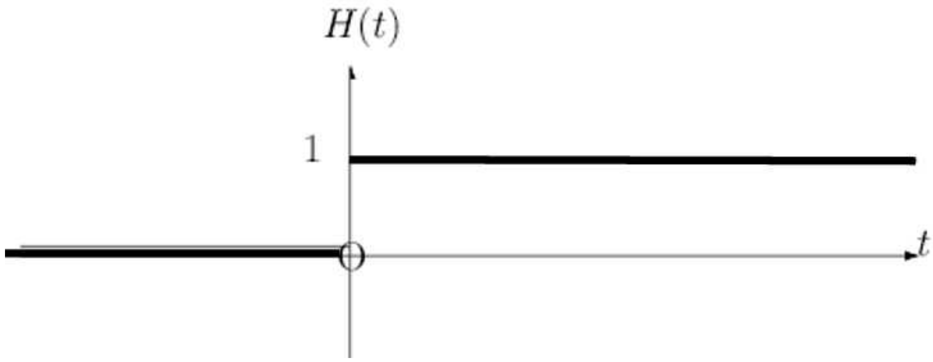
2.5.12

x	-1	0	1	1.9	2	3
$f(x)$	1	0	1	3.61	-1	-1

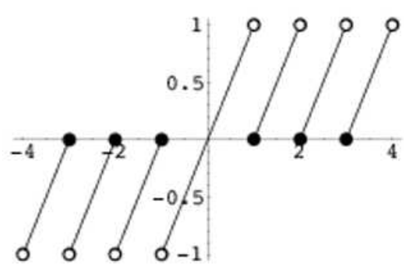
2.5.13



2.5.14



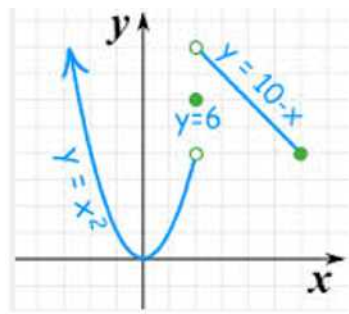
2.5.15



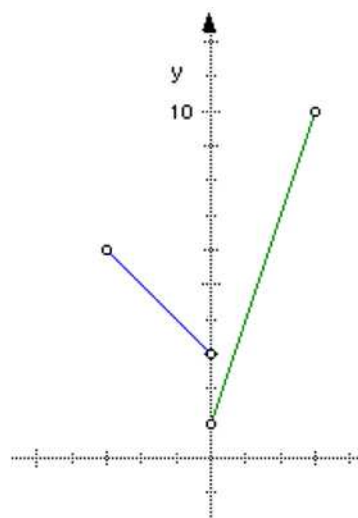
2.5.16

x	-3	-1.5	0	1.5	2
$\lfloor x \rfloor$	-3	-2	0	2	1
$\lceil x \rceil$	-3	-1	0	1	2

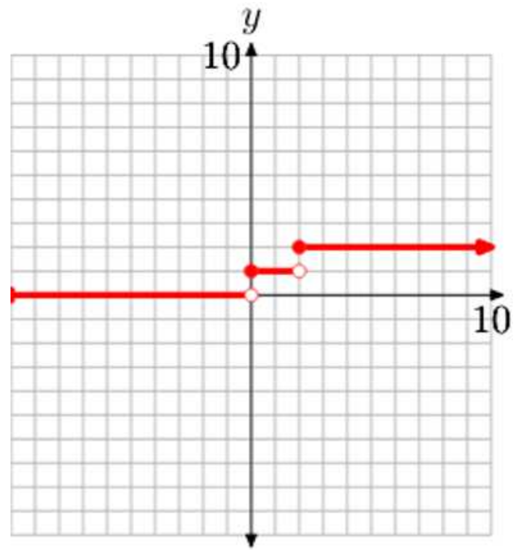
2.5.17



2.5.18



2.5.19



2.5.20

$$C(x) = \begin{cases} 60 & 0 \le x \le 450, \\ 60 + 0.35(60 - x) & x > 450. \end{cases}$$

2.5.21

$$C(x) = \begin{cases} 0.035x & 0 \le x \le 9000, \\ 0.035(9000) + 0.06(x - 9000) & x > 9000. \end{cases}$$

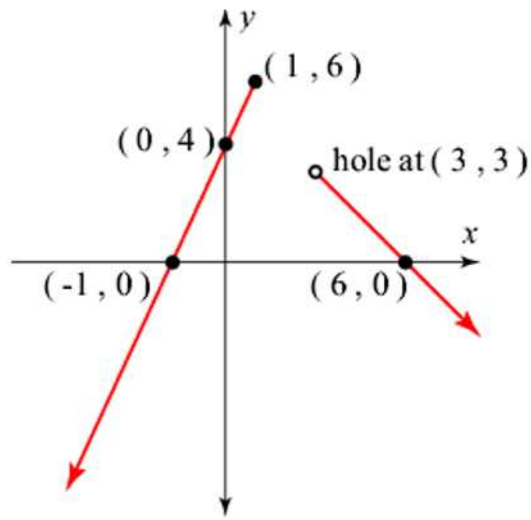
2.5.22

$$f(x) = \begin{cases} \frac{x}{4} + 2 & x < 4, \\ 1 & x \ge 4. \end{cases}$$

2.5.23 (a) $f(1) = 40 + 100 = 140$ (b) $f(5) = 220$ (c) $f(9) = -80(9) + 860 = 140$ (d) $f(11) = 60$.

2.5.24 (a) $f(2) = -2$ (b) $f(3) = 1$.

2.5.25



2.5.26 The domain is $[-6, 8)$ and the range is $[-2, 6]$.

2.5.27 $\mathbb{Z}$.

2.5.28

$$f(x) = \begin{cases} 4x - 6 & x \geq 2, \\ -2x + 6 & x < 2. \end{cases}$$

2.5.29 The graph of $\lfloor x \rfloor$ shows that as x moves toward 2 from the right, $\lfloor x \rfloor$ approaches the number 2. If x approaches 2 from the left, $\lfloor x \rfloor$ approaches the number 1.

2.5.30

$$\text{sgn}(x) = \begin{cases} -2 & x > -2, \\ -3 & x = -2, \\ -4 & x < -2. \end{cases}$$

Section 2.6

2.6.1

$$\begin{aligned} a_1 =& 3(1) + 2 = 5 \\ a_2 =& 3(2) + 2 = 8 \\ a_3 =& 3(3) + 2 = 11 \\ a_4 =& 3(4) + 2 = 14 \\ a_5 =& 3(5) + 2 = 17. \end{aligned}$$

2.6.2

$$\begin{aligned} a_1 =& 2 \\ a_2 =& 1 \\ a_3 =& 3(2) - 1 = 5 \\ a_4 =& 3(1) - 5 = -2 \\ a_5 =& 3(5) + 2 = 17. \end{aligned}$$

2.6.3 $\{-1, 1\}$.

2.6.4 $a_n = 2(n-1) + 1, n \geq 1$.

2.6.5 $a_n = r^{n-1}a_1$.

2.6.6 $\frac{n(n-1)}{2}$.

2.6.7 $\frac{n(n-1)}{2}$.

2.6.8 $3 + n(n+1)$.

2.6.9 $\frac{3}{2}n(n+1)$.

2.6.10 $2^n - 1$.

2.6.11 $\frac{3^n-1}{2}$.

2.6.12 $3 \cdot 2^n - 3$.

2.6.13 $\frac{3^n-1}{2}$.

2.6.14 $C = 1$ and $D = -1$. Hence, $a_n = 2^n - 1$. In particular, $a_3 = 7$.

2.6.15 $C = 1$ and $D = -2$. Hence, $a_n = 2^n - 2$. In particular, $a_3 = 6$.

2.6.16 64.

2.6.17 30.

2.6.18 $a_1 = 1; a_2 = 2; a_3 = 6; a_4 = 24; a_5 = 120$.

2.6.19 na.

2.6.20 $\frac{n(n+1)(2n+1)}{6} + 6n(n+1) + 9n$.

2.6.21 $\{0, 2\}$.

2.6.22 $\sum_{k=1}^{6}(-1)^{k-1}2^k$.

2.6.23

$$
\begin{aligned}
a_{10,000} &= \left(1 + \frac{1}{10,000}\right)^{10,000} \approx 2.718\\
a_{100,000} &= \left(1 + \frac{1}{100,000}\right)^{100,000} \approx 2.718\\
a_{1,000,000} &= \left(1 + \frac{1}{1,000,000}\right)^{1,000,000} \approx 2.718.
\end{aligned}
$$

We conclude that $\left(1 + \frac{1}{n}\right)^n \approx e$ for large values of n.

2.6.24 34.

2.6.25 $a_n = n!$.

2.6.26 $5n(n+1)$.

2.6.27 2.

2.6.28 153.

2.6.29 27.

2.6.30 2.6 ft.

Section 2.7

2.7.1 $\frac{1}{5}$.

2.7.2 6.

2.7.3 (a) -1 (b) $\frac{2}{5}$ (c) $-\frac{1}{2}$.

2.7.4 $\frac{1}{(\sqrt{x+h-1}+\sqrt{x-1})}$.

2.7.5 $-\frac{1}{(x+h-1)(x-1)}$.

2.7.6 2 m/s.

2.7.7 75 ft/hr.

2.7.8 $\frac{15}{7}$ lb/year.

2.7.9 $A=\sqrt{2}$.

2.7.10 60 mi/hr.

2.7.11 (a) The graph is increasing on $[5,8]$ and decreasing on $[0,5]$. It is concave up on $[0,8]$ and is concave down nowhere.
(b) -300 wolves per year.
(c) The percentage change in the population from 2007 to 2008 is

$$\frac{P(2008)-P(2007)}{P(2007)}=\frac{700-400}{400}=75\%.$$

2.7.12 (a) On $\left[0,\frac{1}{2}\right]$, the average velocity is

$$\frac{s(1/2)-s(0)}{1/2}=2.55 \text{ m/s}.$$

On $\left[\frac{1}{2},1\right]$, the average velocity is

$$\frac{s(1)-s(1/2)}{1/2}=-2.35 \text{ m/s}.$$

(b) In the time interval from $t = 0$ to $t = 1/2$, the ball is rising, but from $t = 1/2$ to $t = 1$ it has fallen.

2.7.13 -4.467 million acres per year.

2.7.14 concave up.

2.7.15 The average rate of change is constant.

2.7.16 concave up.

2.7.17 concave down.

2.7.18 concave up on [1,3] and $[4, 5]$ and concave down on $[3, 4]$.

2.7.19 2.

2.7.20 4.

2.7.21 $2a + \Delta x$.

2.7.22 (a) 30 (b) -2.6% (c) $\frac{3}{8}$.

2.7.23 $-\$1$ per unit.

2.7.24 -2 ft$/s$.

2.7.25 670 m^2/min.

2.7.26 Negative.

2.7.27 Concave down.

2.7.28 (a) -1 (b) -1.

2.7.29 (a) $(-\infty, 1]$ (b) $[1, \infty)$.

2.7.30 $-\frac{2}{(a+h)(a)}$.

Section 3.1

3.1.1 slope$= -2$, $x-$intercept$= \frac{7}{2}$ and $y-$intercept$= 7$.

3.1.2 (a) The slope is -0.75 dollar per hundred widgets. Since the slope is negative, the price decreases as the number of items bought by consumers increases. You could also say that the cheaper the item, the greater the demand.
(b) The $p-$intercept is $p = \$54$. Since $q = 0$, consumers will buy no widgets when the price is \$54. The $q-$intercept is $q = \frac{54}{0.75} = 72$. Since price is 0, 7200 widgets could be given away for free.

3.1.3 (a) The slope is 78 students/year (b) This means that the enrollment of the college is increasing by about 78 students per year (c) 2936 students.

3.1.4 $y = -\frac{2}{3}x + 3$.

3.1.5 $y = -2x + 13$.

3.1.6 $y = 6x - 9$.

3.1.7 $y = -\frac{1}{3}x + \frac{4}{3}$.

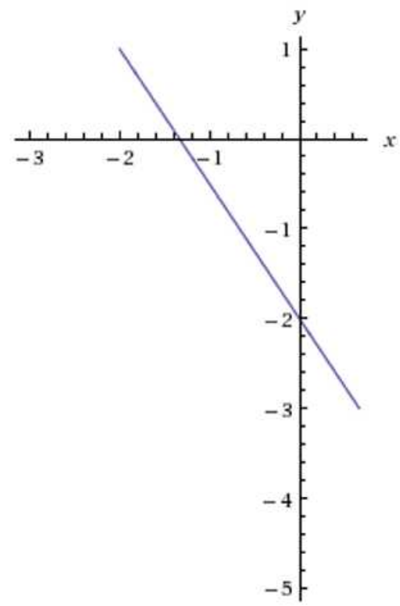
3.1.8 $m = -\frac{9}{2}$ and $b = 5$.

3.1.9 $m = 0$ and $b = 2$.

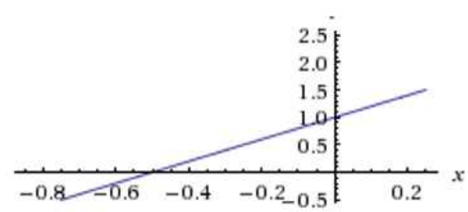
3.1.10 $C(x) = 600x + 7,000$.

3.1.11 $P(75) = -100$. A loss of \$100 (b) $P(150) = 1400$.

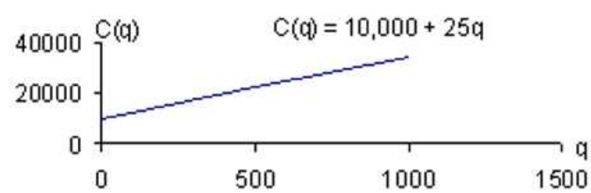
3.1.12



3.1.13



3.1.14 (a) $C(q) = 25q + 10,000$ (b)



3.1.15 $x + 2y - 7 = 0$.

3.1.16 $3x - 2y + 1 = 0$.

3.1.17 (b)

3.1.18 (a)

3.1.19 $f(n) = 75,000 + 700n$.

3.1.20 $F = \frac{9}{5}C + 32$.

3.1.21 $y = -8x + 51$.

3.1.22 $y = 2x - 4$.

3.1.23 $y = 1$.

3.1.24 $x = -3$.

3.1.25 (a) $y = 0$ (b) $x = 0$.

3.1.26 $C(x) = 175 + 35x$.

3.1.27 (a) $f(x) = -300x + 1200$.
(b) The domain of f is $[0, 4]$ since at 4 years the computer will be worth nothing and past four years f becomes negative which does not make sense.

3.1.28 50.065 ft; -5.83°F.

3.1.29 The slope is -5 and the $y-$intercept is 3.5.

3.1.30 $f(x) = -2x + 4$.

Section 3.2

3.2.1 $y = \frac{1}{3}x - \frac{5}{2}$.

3.2.2 (a) Parallel (b) Perpendicular (c) Neither (d) Perpendicular.

3.2.3 $(3, -4)$.

3.2.4 Lines are parallel.

3.2.5 Lines coincide.

3.2.6 $y = -\frac{5}{2}x + 28$.

3.2.7 $y = -\frac{9}{2}x + 27$.

3.2.8 $x = \frac{53}{29},\ y = \frac{118}{29}$.

3.2.9 (E).

3.2.10 $y = \frac{1}{2}x - 1$.

3.2.11 $y = \frac{4}{3}x - \frac{25}{3}$.

3.2.12 $(3, -1)$.

3.2.13 (1) (E) (2) (F) (3) (D) (4) (C) (5) (B) (6) (A).

3.2.14 $c = -\frac{2}{5}$.

3.2.15 $m = 3$ and $b = -1$.

3.2.16 $y = 25$.

3.2.17 $y = \frac{2}{3}x + 4$.

3.2.18 $(7, -9)$.

3.2.19 $x = 3$.

3.2.20 $\sqrt{2}$.

3.2.21 $x = 2$ and $y = -1$.

3.2.22 $x = 3$ and $y = 2$.

3.2.23 $y = \frac{3}{4}x - \frac{9}{2}$.

3.2.24 $m_1 = m_2 = -\frac{2}{5}$.

3.2.25 $y = -\frac{1}{5}x - 2$.

3.2.26 0.

3.2.27 $m_{AB} = \frac{1}{4}; m_{AC} = \frac{3}{2}; m_{BC} = -1$.

3.2.28 $y = -3x + 5$.

3.2.29 $x = 4$ and $y = -3$.

3.2.30 The two lines are perpendicular.

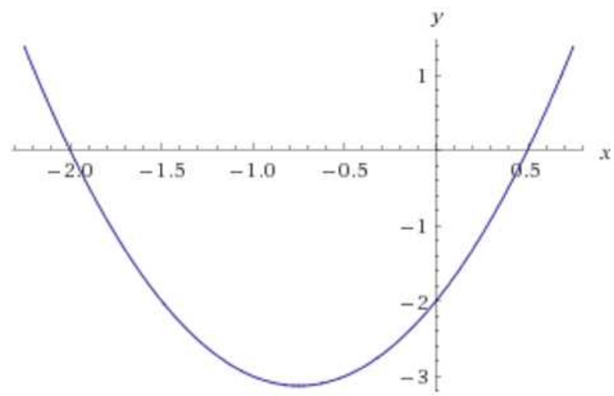
Section 3.3

3.3.1 $(1, 4)$.

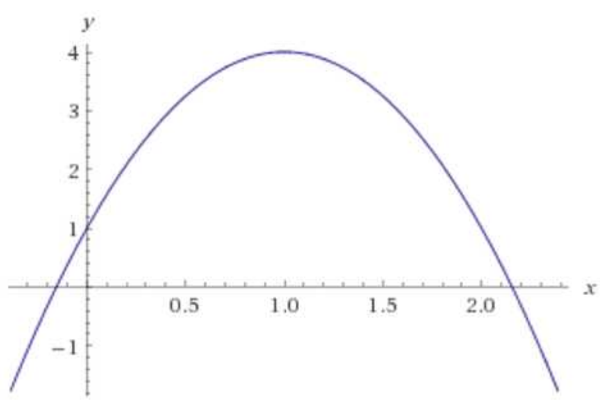
3.3.2 4.

3.3.3 $(2, 11)$, a global maximum.

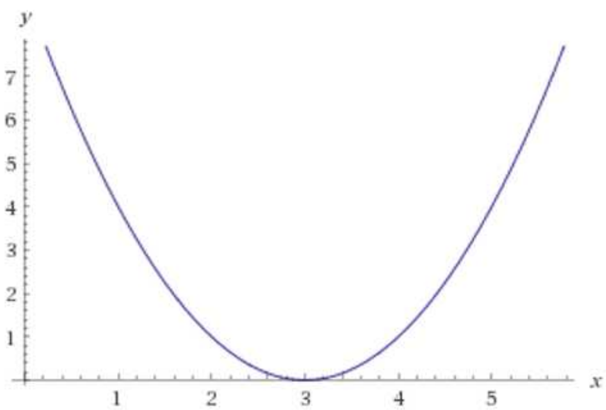
3.3.4



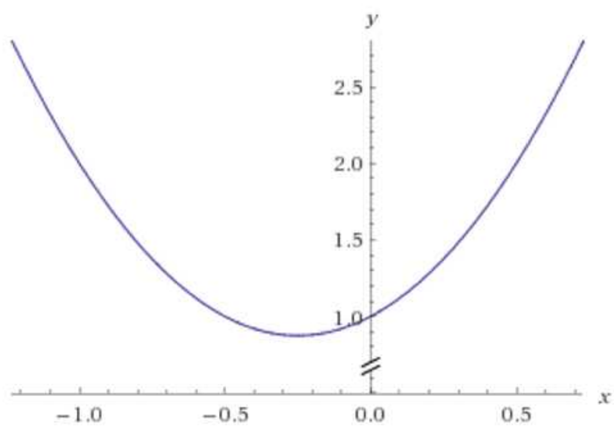
3.3.5



3.3.6



3.3.7



3.3.8 -7.

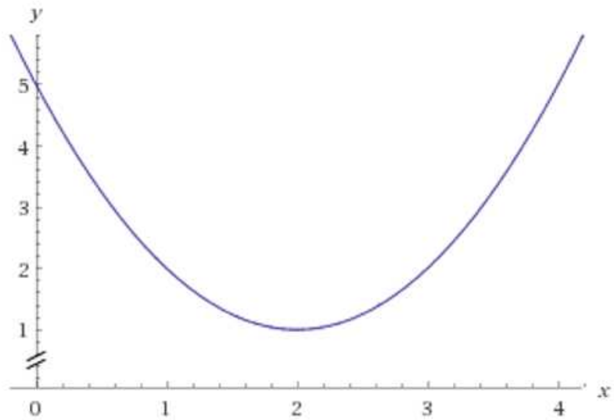
3.3.9 $\frac{5}{2}$.

3.3.10 \$500.

3.3.11 A square with side 100 with area 10,000 ft^2.

3.3.12 (a) $(2,1)$ (b) $x = 2$ (c) no $x-$intercept by $y-$intercept equals to

5.



3.3.13 $x = \frac{1250}{32}$ ft.

3.3.14 3.5 and 3.5

3.3.15 The projectile hits the ground after 25 seconds. The maximum altitude achieved by the projectile is 2500 ft.

3.3.16 $y = 2\left(x + \frac{3}{2}\right)^2 - \frac{29}{2}$.

3.3.17 $y = -\frac{7}{9}(x-4)^2 + 5$.

3.3.18 $f(x) = 2x^2 + 5x - 2$.

3.3.19 3.

3.3.20 2.

3.3.21 \$2,000.

3.3.22 $320t^2 - 25$.

3.3.23 $57x - x^2$.

3.3.24 $2 - \sqrt{2} < t < 2 + \sqrt{2}$.

3.3.25 $x = -8$ and $x = \frac{3}{2}$.

3.3.26 $\frac{5}{3} \pm \frac{\sqrt{10}}{3}$.

3.3.27 $y = -3(x + 2)^2 + 4$.

3.3.28 The axis of symmetry is $x = -3$ and the vertex is $(-3, 1)$.

3.3.29 (a) 0.5 seconds (b) 484 ft (c) 30.25 seconds.

3.3.30 (a) $-1 \pm i$ (b) $\frac{1}{19} \pm \frac{\sqrt{95}}{12}i$.

3.3.31 (a) $-\frac{1}{4} \pm \frac{\sqrt{6}}{2}i$ (b) $-\frac{1}{3} \pm \frac{\sqrt{5}}{3}i$.

3.3.32 (a) $z = \pm 3i$ (b) $x = -2 \pm i$.

3.3.33 $3 \pm \frac{1}{2}i$.

Section 3.4

3.4.1 The leading term is -3. The constant term is -10 and the degree is 4.

3.4.2 $f \to -\infty$ as $x \to -\infty$ and $f \to \infty$ as $x \to \infty$.

3.4.3 The zeros are: $-4, -1, 0, 2$.

3.4.4 (a) (1) $f(x) \to -\infty$ as $x \to \pm\infty$ (2) even degree (3) Two real zeros.
(b) (1) $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$ (2) odd degree (3) One real zeros.
(c) (1) $f(x) \to \infty$ as $x \to \pm\infty$ (2) even degree (3) No real zeros.

3.4.5 Since the degree is 7, the end behaviour act in an opposite way. Thus, the smallest number of real zeros is 1. By the fundamental theorem of algebra, the largest number of real zeros is 7.

3.4.6 $f(x) \to -\infty$ as $x \to \pm\infty$.

3.4.7 $x = \pm\sqrt{5}$ and $x = 1$.

3.4.8 $4x^3 - 2x^2 - x + 2$.

3.4.9 $-1, 0, 2$.

3.4.10 $-2x^3 + 8x^2 + 22x - 60$.

3.4.11 No.

3.4.12 $x = -9$ is of multiplicity 2 and $x = 3$ is of multiplicity 3.

3.4.13 $x^4 + 18x^3 + 71x^2 - 180x - 810$.

3.4.14 $f(x) = \frac{5}{16}(x+4)(x+2)(x-2)(x-4)$.

3.4.15 $-2, -1, 1, 2$.

3.4.16 (a) and (d).

3.4.17 (a) curve not continuous (b) curve is not smooth (sharp point) (c) polynomial function (d) polynomial function.

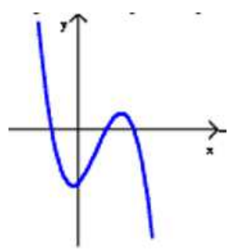
3.4.18 $x = 2$ is of multiplicity 2 and $x = -2$ is of multiplicity 1.

3.4.19 $g(x) = 4(x+3)^3 - 7$.

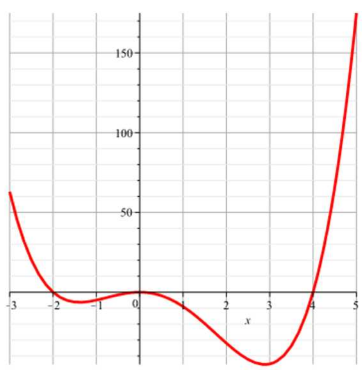
3.4.20 The volume is $V(x) = x(16-2x)(20-2x)$. The domain is $[0, 8]$.

3.4.21 Since $f(0) = -2 < 0$ and $f(2) = 14 > 0$, the intermediate value theorem guarantees the existence a real zero between 0 and 2.

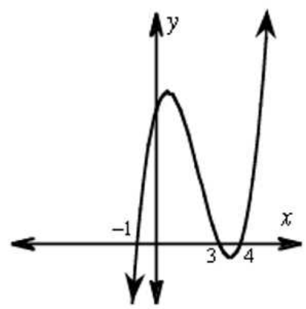
3.4.22



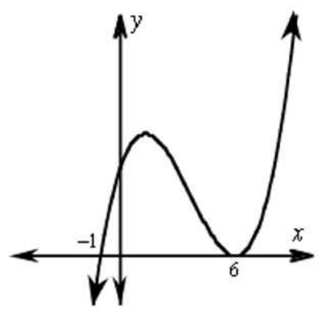
3.4.23



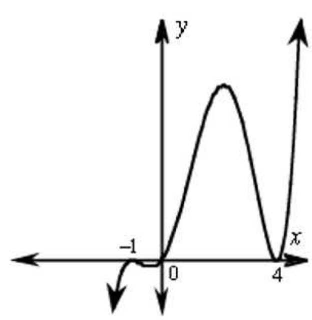
3.4.24



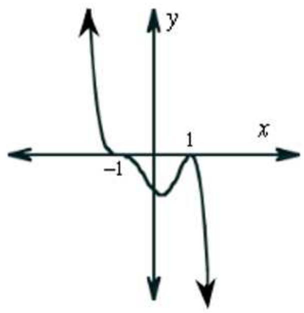
3.4.25



3.4.26



3.4.27



3.4.28 When a polynomial has odd degree its end behavior move in an opposite directions. By the intermediate value theorem, the graph must cross the $x-$axis.

3.4.29 $-5, -2, 5.$

3.4.30 $w = 1.78,\ l = 3.56,\ h = 2.66.$

Section 3.5

3.5.1 $q(x) = 3x - 3$ and $r(x) = 7 - 2x$.

3.5.2 $q(x) = 2x^2 + x + 1$ and $r(x) = x^2 - x - 2$.

3.5.3 $q(x) = 3x^2 + 4x - 1$ and $r(x) = -4x + 4$.

3.5.4 $q(x) = 3x - 6$ and $r(x) = x + 2$.

3.5.5 $q(x) = 2x^2 - 3$ and $r(x) = -x + 6$.

3.5.6 $q(x) = 0$ and $r(x) = x^3 + x^2 - 4$.

3.5.7 $q(x) = 2x + 3$ and $r(x) = -8x + 10$.

3.5.8 $q(x) = 2x^2 + 5x + 1$ and $r(x) = 20x$.

3.5.9 $q(x) = x^2 - 2x + 2$ and $r(x) = 0$.

3.5.10 $q(x) = 5x^2 - 34$ and $r(x) = -2x^2 + 63x + 1$.

3.5.11 -45.

3.5.12 $-\frac{94}{81}$.

3.5.13 $k = 1$.

3.5.14 127.

3.5.15 $f(-4) = 0$.

3.5.16 $c = -42$.

3.5.17 $c = 8$.

3.5.18 $f(-1) = 21 \neq 0$.

3.5.19 $f(5) = 0$.

3.5.20 The zeros of f are :$-2, \frac{1}{2}, 3$.

3.5.21 Let $f(x) = x^3 - 3x^2 + 3x - 2$. Using synthetic division, we find

$$\begin{array}{c|cccc} & 1 & -3 & 3 & -2 \\ 2 & & 2 & -2 & 2 \\ \hline & 1 & -1 & 1 & 0 \end{array}$$

Since $f(2) = 0$, 2 is a solution to the given equation.

3.5.22 $-\frac{94}{81}$.

3.5.23 127.

3.5.24 $q(x) = 3x^2 - 6x + 6$ and $r(x) = 0$.

3.5.25 $q(x) = 3x^2 - 5x - 2$ and $r(x) = 8$.

3.5.26 $-\frac{1}{4}$.

3.5.27 $-4, -1, 3$.

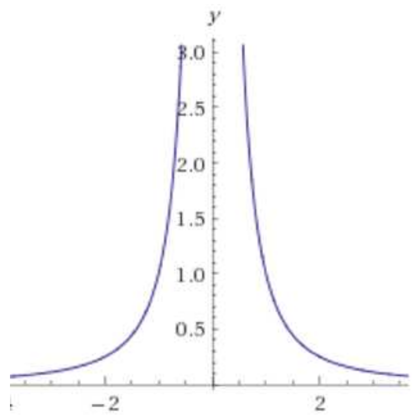
3.5.28 $-6, -1, 1$.

3.5.29 $-1, -\frac{1}{3}, \frac{2}{5}$.

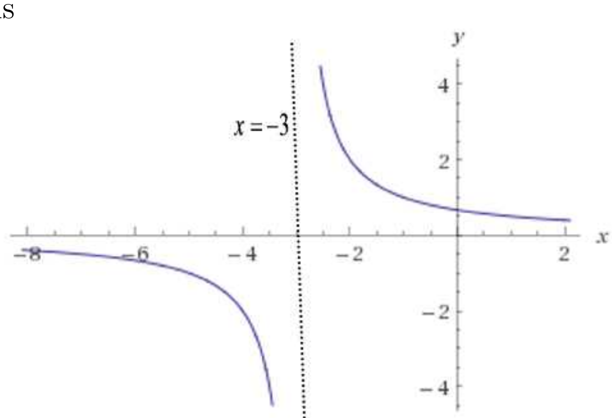
3.5.30 2.

Section 4.1

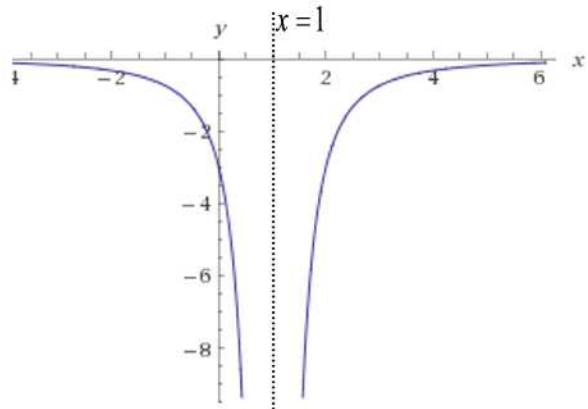
4.1.1 (a) Dom$(f) = (-\infty, 0) \cup (0, \infty)$.
(b) $f(x) \to 0$ as $x \to \pm\infty$. Hence, $y = 0$ is the horizontal asymptote.
(c) $x = 0$ is the vertical asymptote.
(d) No intercepts.
(e) The graph is



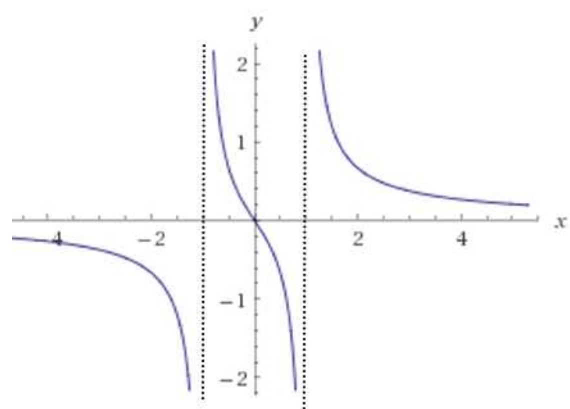
4.1.2 Solution.
(a) Dom$(f) = (-\infty, -3) \cup (-3, \infty)$.
(b) $f(x) \to 0$ as $x \to \pm\infty$. Hence, $y = 0$ is the horizontal asymptote.
(c) $x = -3$ is the vertical asymptote.
(d) No $x-$intercepts. $y = \frac{2}{3}$ is the $y-$intercept.
(e) The graph is



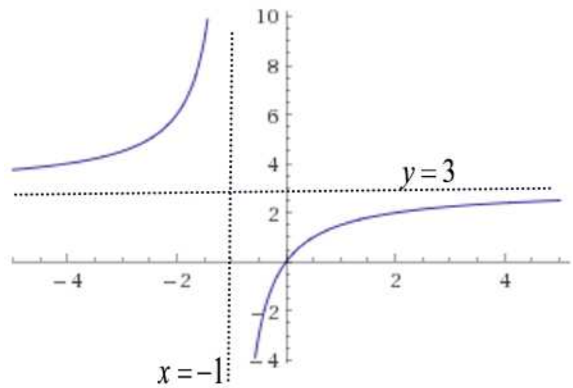
4.1.3 (a) Dom$(f) = (-\infty, 1) \cup (1, \infty)$.
(b) $f(x) \to 0$ as $x \to \pm\infty$. Hence, $y = 0$ is the horizontal asymptote.
(c) $x = 1$ is the vertical asymptote.
(d) No $x-$intercepts. $y = -3$ is the $y-$intercept.
(e) The graph is



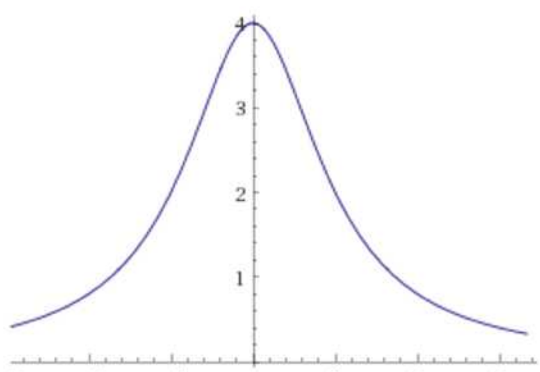
4.1.4 (a) Dom$(f) = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.
(b) $f(x) \to 0$ as $x \to \pm\infty$. Hence, $y = 0$ is the horizontal asymptote.
(c) $x = \pm 1$ are the vertical asymptotes.
(d) $(0, 0)$ is the only intercept.
(e) The graph is



4.1.5 (a) $\mathrm{Dom}(f) = (-\infty, -1) \cup (-1, \infty)$.
(b) $f(x) \to 3$ as $x \to \pm\infty$. Hence, $y = 3$ is the horizontal asymptote.
(c) $x = -1$ is the vertical asymptote.
(d) $(0, 0)$ is the only intercept.
(e) The graph is

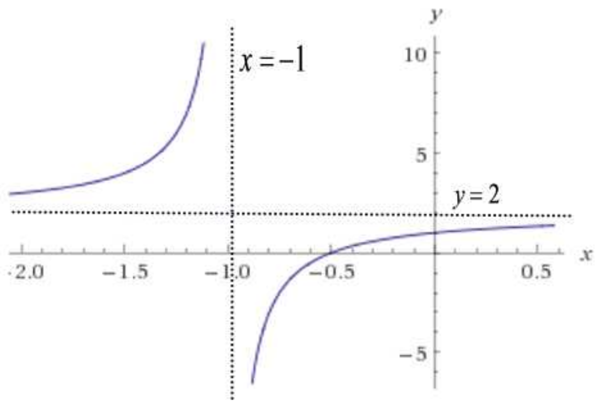


4.1.6 (a) $\mathrm{Dom}(f) = (-\infty, \infty)$.
(b) $f(x) \to 0$ as $x \to \pm\infty$. Hence, $y = 0$ is the horizontal asymptote.
(c) No vertical asymptotes.
(d) $y-$intercept at $y = 4$.
(e) The graph is

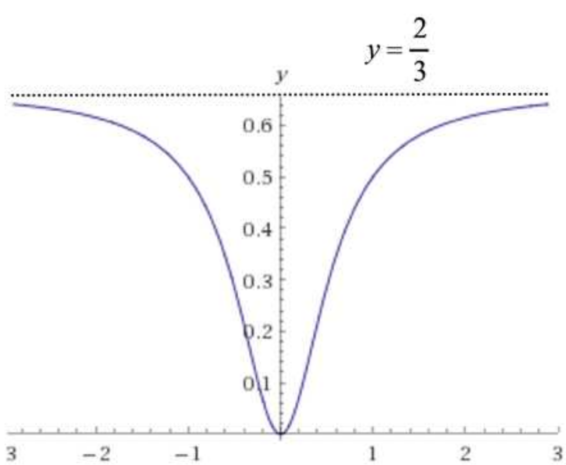


4.1.7 (a) $\mathrm{Dom}(f) = (-\infty, -1) \cup (-1, \infty)$.
(b) $f(x) \to 2$ as $x \to \pm\infty$. Hence, $y = 2$ is the horizontal asymptote.

(c) $x = -1$ is the vertical asymptote.
(d) $(0, 1)$ and $(-\frac{1}{2}, 0)$.
(e) The graph is

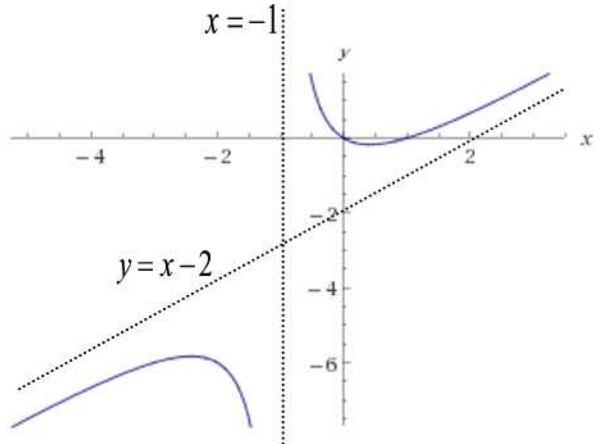


4.1.8 (a) Dom$(f) = (-\infty, \infty)$.
(b) $f(x) \to \frac{2}{3}$ as $x \to \pm\infty$. Hence, $y = \frac{2}{3}$ is the horizontal asymptote.
(c) No vertical asymptotes.
(d) $(0, 0)$.
(e) The graph is



4.1.9 (a) Dom$(f) = (-\infty, -1) \cup (-1, \infty)$.
(b) $f(x) - (x - 2) = \frac{2}{x+1} \to 0$ as $x \to \pm\infty$. Hence, $y = x - 2$ is the oblique asymptote.

(c) $x = -1$ is the vertical asymptote.
(d) $(0,0), (1,0)$
(e) The graph is



4.1.10 $y = 2x$.

4.1.11 $f(x) = \frac{2x+3}{x+1}$.

4.1.12 $x = 1$ and $x = -2$.

4.1.13 $y = -\frac{3}{2}$.

4.1.14 $f(x) = \frac{1}{x^2+x-2}$.

4.1.15 $y = 0$.

4.1.16 $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$.

4.1.17 $y = \frac{1}{3}$.

4.1.18 $y = \frac{x}{2}$.

4.1.19 $(-\infty, -1) \cup (-1, \infty)$.

4.1.20 $(-\infty, -4) \cup (-4, 0) \cup (0, 2) \cup (2, \infty)$.

4.1.21 $-5, -2, 5.$

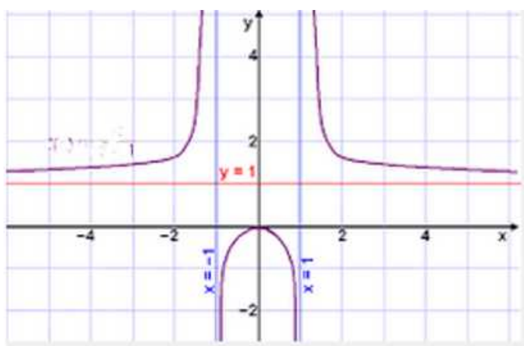
4.1.22 $y = \frac{5}{2}.$

4.1.23 $y = 0.$

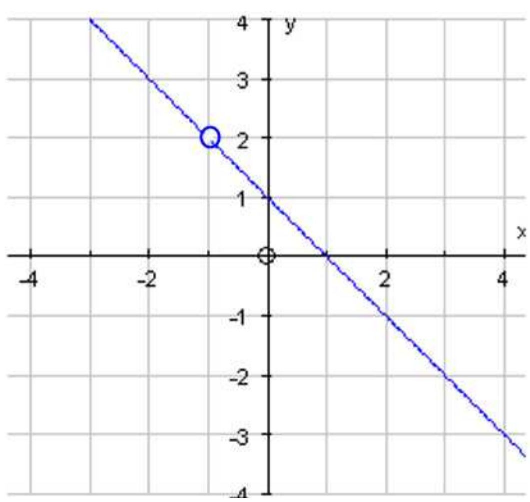
4.1.24 $y = 5x + 11.$

4.1.25 $-5, -2, 5.$

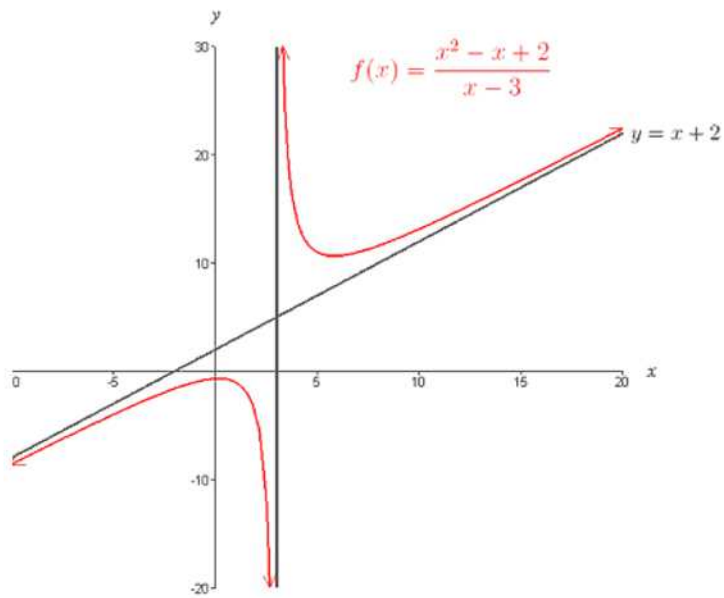
4.1.26



4.1.27



4.1.28



4.1.29 (a) 4.88 mg/dl (b) 10 hours later.

4.1.30 (a) 15 insects (b) 608 insects (c) 100 months later.

Section 4.2

4.2.1 $\frac{A_1}{(s-1)^3} + \frac{A_2}{(s-1)^2} + \frac{A_3}{s-1} + \frac{B_1}{(s-2)^2} + \frac{B_2}{s-2}$.

4.2.2 $\frac{A_1s+A_2}{s^2+16} + \frac{B_1}{s-2}$.

4.2.3 $\frac{A_1s+A_2}{(s^2+1)^2} + \frac{A_3s+A_4}{s^2+1} + \frac{B_1}{(s+4)^2} + \frac{B_2}{s+4}$.

4.2.4 $\frac{A_1}{(s-2)^2} + \frac{A_2}{s-2} + \frac{B_1s+B_2}{s^2+8s+17}$.

4.2.5 $\frac{3s+6}{s^2+3s} = \frac{2}{s} + \frac{1}{s+3}$.

4.2.6 $\frac{s^2+1}{s(s+1)^2} = \frac{1}{s} - \frac{2}{(s+1)^2}$.

4.2.7 $\frac{2s-3}{(s-1)(s-2)} = \frac{1}{s-1} + \frac{1}{s-2}$.

4.2.8 $\frac{4s^2+s+1}{s(s^2+1)} = \frac{1}{s} + \frac{3s}{s^2+1} + \frac{1}{s^2+1}$.

4.2.9 $\frac{s^2+6s+8}{(s^2+4)^2} = \frac{1}{s^2+4} + \frac{6s}{(s^2+4)^2} + \frac{4}{(s^2+4)^2}$.

4.2.10 $\frac{1}{(s+2)s^2} = \frac{1}{4(s+2)} - \frac{1}{4s} + \frac{1}{2s^2}$.

4.2.11 $\frac{s^2+6s+11}{(s+1)^2(s+2)} = \frac{3}{s+2} - \frac{2}{s+1} + \frac{6}{(s+1)^2}$.

4.2.12 $\frac{1}{(s-1)^2(s-2)} = -\frac{1}{s-1} - \frac{1}{(s-1)^2} + \frac{1}{s-2}$.

4.2.13 $\frac{6}{s(s^2+9)} = \frac{2}{3s} - \frac{2s}{3(s^2+9)}$.

4.2.14 (a) $\frac{x^4+1}{x^5+4x^3} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x^3} + \frac{Ax+B}{x^2+4}$.
(b) $\frac{1}{(x^2-9)^2} = \frac{A_1}{x-3} + \frac{A_2}{(x-3)^2} + \frac{A_3}{x+3} + \frac{A_4}{(x+3)^2}$.

4.2.15 (a) $\frac{t^6+1}{t^6+t^3} = \frac{A_1}{t} + \frac{A_2}{t^2} + \frac{A_3}{t^3} + \frac{At^2+Bt+C}{t^3+1}$.
(b) $\frac{x^5+1}{(x^2-x)(x^4+2x^2+1)} = \frac{x^5+1}{x(x-1)(x^2+1)^2} = \frac{A_1}{x} + \frac{A_2}{x-1} + \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$.

4.2.16 $\frac{2}{2x^2+3x+1} = \frac{4}{2x+1} - \frac{2}{x+1}$.

4.2.17 $\frac{4y^2-7y-12}{y(y+2)(y-3)} = \frac{2}{y} + \frac{9}{5}\frac{1}{y+2} + \frac{1}{5}\frac{1}{y-3}$.

4.2.18 $\frac{u}{(u+1)(u+2)} = -\frac{1}{u+1} + \frac{2}{u+2}$.

4.2.19 $\frac{3x-37}{x^2-3x-4} = \frac{8}{x+1} - \frac{5}{x-4}$.

4.2.20 $\frac{4x^2}{(x-1)(x-2)^2} = \frac{4}{x-1} + \frac{16}{(x-2)^2}$.

4.2.21 $\frac{9x+25}{(x+3)^2} = \frac{9}{x+3} - \frac{2}{(x+3)^2}$.

4.2.22 $\frac{4x^3-3x+5}{x^2-2x} = 4x + 8 - \frac{5}{2}\frac{1}{x} + \frac{31}{2}\frac{1}{x-2}$.

4.2.23 $\frac{2x-3}{x^3+x} = -\frac{3}{x} + \frac{3x+2}{x^2+1}$.

4.2.24 $\frac{2x^3+5x-1}{(x+1)^3(x^2+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{Dx+E}{x^2+1} + \frac{Fx+G}{(x^2+1)^2}$.

4.2.25 $\frac{x}{(x^2+1)(x^2+2)} = \frac{1}{x^2+1} - \frac{1}{x^2+2}$.

4.2.26 $\frac{x^2+1}{x^3-6x^2+11x-6} = \frac{1}{x-1} - \frac{5}{x-2} + \frac{5}{x-3}$.

4.2.27 $\frac{x+3}{(x+5)(x^2+4x+5)} = \frac{-1}{5}\frac{1}{x+5} + \frac{1}{5}\frac{x+4}{x^2+4x+5}$.

4.2.28 $\frac{x^4+2x^3+6x^2+20x+6}{x^3+2x^2+x} = x + \frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2}$.

4.2.29 $\frac{2x^4+4x^3+x^2+4x+1}{x^5+x^4+x^3+x^2+x} = \frac{3}{x} + \frac{1}{x^2} - \frac{2}{x+1} + \frac{x-1}{x^2+1}$.

4.2.30 $\frac{3x^2-3x-8}{(x-5)(x^2-x+4)} = \frac{13}{6}\frac{1}{x-5} + \frac{1}{6}\frac{5x+20}{x^2-x+4}$.

Section 5.1

5.1.1 The initial value is $b = 0.75$ and the growth factor is $a = 0.2$.

5.1.2 $P(t) = 40,000(1.06)^t$ and $P(4) \approx \$50,499.08$.

5.1.3 (a) Initial dose given is $A(0) = 25$ mg (b) $r = -0.15$ so that 15% of the drug leaves the body each hour (c) $A(10) \approx 4.92$ mg.

5.1.4 \$44,816.89.

5.1.5 5.127%

5.1.6 (A)

5.1.7 (C)

5.1.8 (B)

5.1.9 4

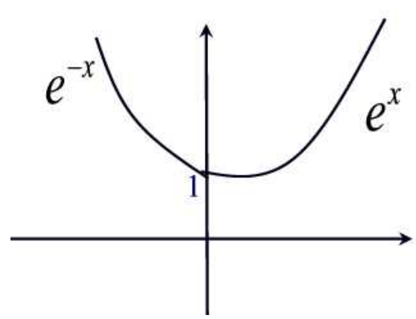
5.1.10 2.639

5.1.11 (C)

5.1.12 $y = -3$.

5.1.13 $x-$axis.

5.1.14



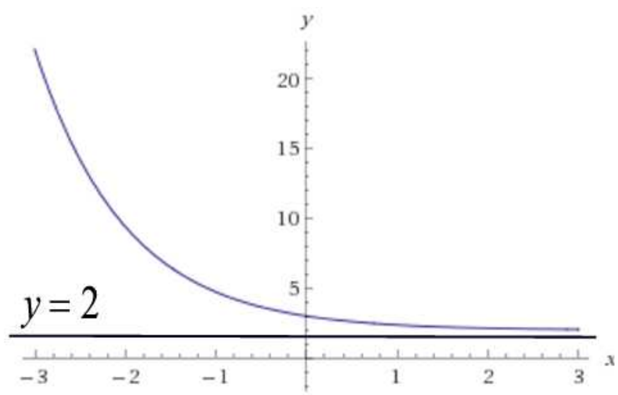
5.1.15 $\frac{1}{9}$.

5.1.16 \$25,937.42.

5.1.17 $e^{2x} - 3e^x - 4$.

5.1.18 8244.

5.1.19

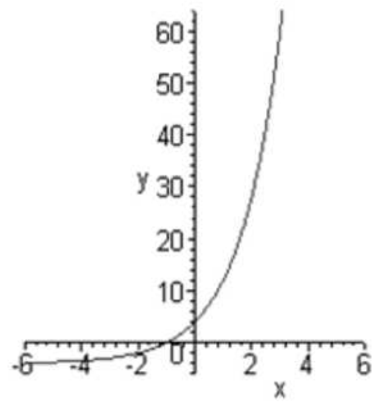


5.1.20 $e^{2x} - e^{-2x}$.

5.1.21 (a), (c) and (d).

5.1.22 $f(t) = 80e^{0.4581t}$ and $f(8) \approx 3124$ bacteria.

5.1.23

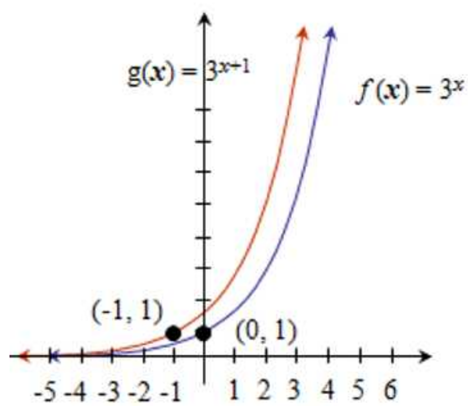


5.1.24 $y = 2$.

5.1.25 \$2,208.04.

5.1.26 145,192 people.

5.1.27



5.1.28 The first option is the better investment.

5.1.29 $x = 1$.

5.1.30 $y = -3e^x - 3$.

Section 5.2

5.2.1 $\log \frac{1}{b} = \log 1 - \log b = 0 - \log b = -\log b.$

5.2.2 $2\log x + 3\log y + 5\log z.$

5.2.3 $\log_x e = \pi.$

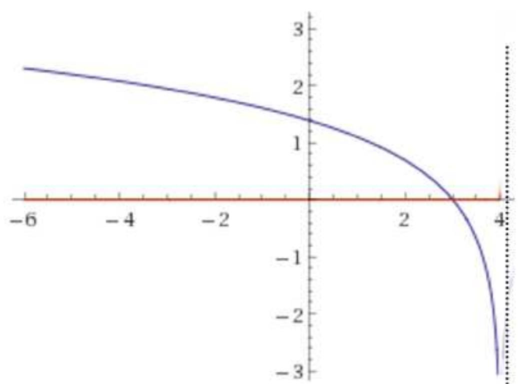
5.2.4 $\sqrt{\pi} = x.$

5.2.5 $x = 4.$

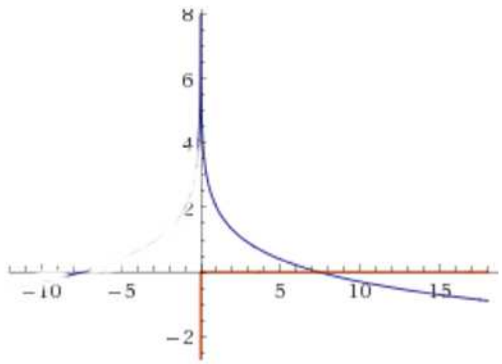
5.2.6 $(-\infty, -1) \cup (0, \infty).$

5.2.7 $k = \sqrt{2}.$

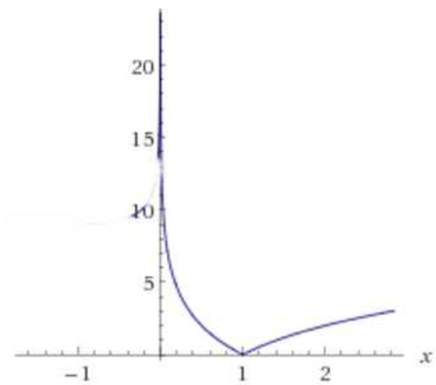
5.2.8



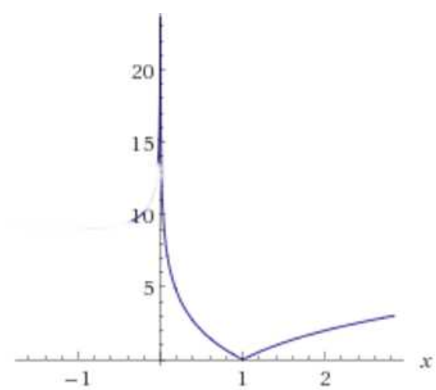
5.2.9



5.2.10



5.2.11



5.2.12 $-2 < x < 1$.

5.2.13 $\sqrt{b}$.

5.2.14 $\frac{2}{125}$.

5.2.15 $a + \frac{b}{4}$.

5.2.16 $2\ln x + \frac{1}{3}\ln(1-x) - 2\ln(x+1) + \ln 5 - 2\ln 2$.

5.2.17 $-2\ln(x-1)$.

5.2.18 42.

5.2.19 -3.075.

5.2.20 0.

5.2.21 $y = \left(\frac{C\sqrt[3]{x^2+1}}{\sqrt{x}}\right)^{\frac{1}{2}}$.

5.2.22 2.523 days.

5.2.23 $\ln\left((1+\frac{h}{x})^{\frac{1}{h}}\right)$.

5.2.24 $\log_x b$.

5.2.25 \$7,875.00

5.2.26 \$1,069,047.14.

5.2.27 $k = 3$.

5.2.28 32.

5.2.29 $\log_a \frac{x^2(x+y)}{y^3}$.

5.2.30 3.

5.2.31 $\log_5 2$.

5.2.32 16.

Section 5.3

5.3.1 $\frac{1}{1-2e^4}$.

5.3.2 $\ln\left(\frac{1+\sqrt{17}}{4}\right)$.

5.3.3 $-1 \pm \sqrt{2}$.

5.3.4 $\sqrt{41}$.

5.3.5 No solutions.

5.3.6 $\frac{1+\sqrt{5}}{2}$.

5.3.7 2.

5.3.8 $x = 0$.

5.3.9 No real solutions.

5.3.10 6.

5.3.11 1426.

5.3.12 No solutions.

5.3.13 No solutions.

5.3.14 2.

5.3.15 $\log_{\frac{5}{3}} 225$.

5.3.16 -1.765.

5.3.17 $x = 2$ and $x = 5$.

5.3.18 $x = \frac{17}{10}$.

5.3.19 $x = 1$.

5.3.20 $x = \log_{\frac{3}{4}} 4$.

5.3.21 No solutions.

5.3.22 $x = \frac{\ln 2}{2}$.

5.3.23 $x = -3$ and $x = 9$.

5.3.24 $x = -\frac{\ln 2}{4}$.

5.3.25 $x = 2$.

5.3.26 $x = \frac{11 \ln 3 - 5 \ln 7}{2 \ln 3 - 4 \ln 7}$.

5.3.27 The interval of solution is $[3, \infty)$.

5.3.28 6.76 years.

5.3.29 $x = 1$ or $x = e^2$.

5.3.30 26.3.

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