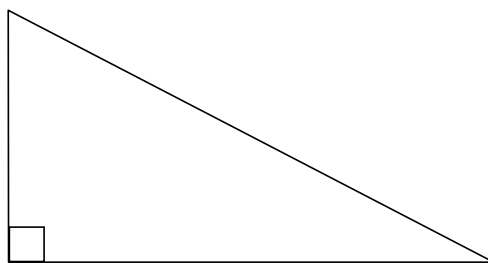


Chapter 26

Trigonometry

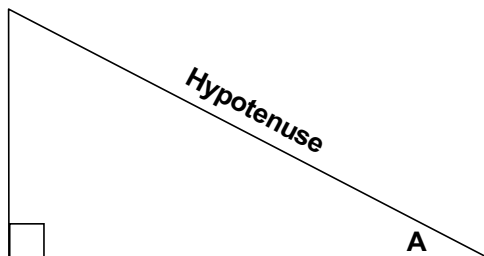
Trigonometry is one of those names you always hear about in mathematics. The topic of trigonometry starts off being about *right-angled triangles*. It's actually all about *ratios* between the side lengths in right-angled triangles. Let's start with a typical right-angled triangle:



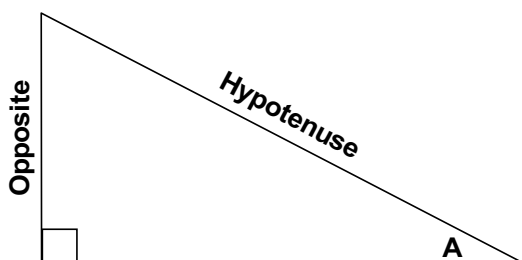
SECTION 26.1 - TRIANGLE SIDES

Naming the sides

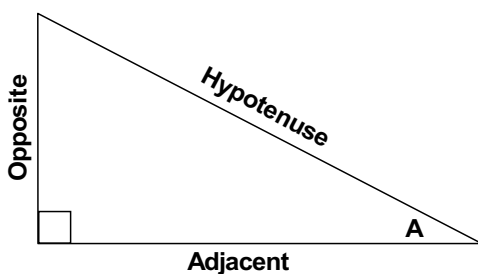
Now, first up, there are *three* different names for the three different sides of a right-angled triangle. The names are *opposite*, *hypotenuse* and *adjacent*. Now we've come across the hypotenuse before – it is the longest side of the triangle. This is always easy to spot. The other two names, opposite and adjacent, depend on which *angle* you're currently looking at in the triangle. For instance, say I was looking at angle A:



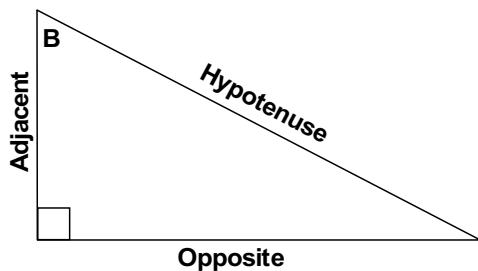
The opposite side is the side opposite the angle we're looking at. We're looking at angle A at the moment, so the side opposite it is the side on the left:



This leaves us with the adjacent side. The adjacent side is the side of the triangle that *touches* the angle we're looking at, but which is *not* the hypotenuse. There are two sides touching our angle A – one is the hypotenuse. The other side therefore is the adjacent side:



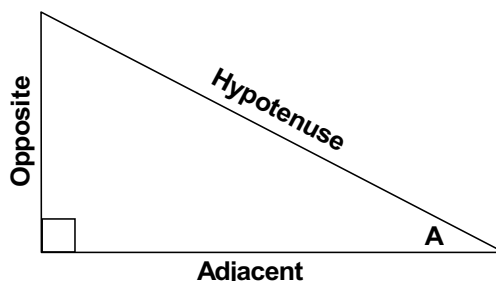
What about if we'd picked another angle, say angle B in the following diagram? Well, the hypotenuse would stay the same, but the adjacent and opposite sides would change, like this:



We don't usually have to worry about how to name the sides when the angle we're looking at is the 90 degree angle, so don't worry about that for the moment.

Ratios between the side lengths

Let's go back to the 'A' angle triangle:



Trigonometry is all about the *ratio* of the side lengths in the triangle. For instance, when we're looking at the angle A, we could talk about the ratio between the length of the adjacent side and the length of the hypotenuse:

$$\text{Ratio} = \frac{\text{Length of adjacent side}}{\text{Length of hypotenuse side}}$$

Now when we talk about this ratio, we have to remember what angle we're currently looking at in the diagram – angle A. There is a special name in trigonometry for this ratio we have just looked at – it is known as 'cosine A'. When we say 'cosine A', what we mean is the ratio between the length of the adjacent side and the hypotenuse side. Often we use 'cos' instead of 'cosine' as a shorter name.

$$\cosine A = \cos A = \frac{\text{Length of adjacent side}}{\text{Length of hypotenuse side}}$$

There are two other ratios you need to know about. The first is 'tangent A' – it is the ratio between the length of the opposite side and the adjacent side. We use 'tan' for short. The other is 'sine A' – it is the ratio between the length of the opposite side and the hypotenuse. We use just 'sin' for short. Here's a little summary of the three ratios:

$$\cosine A = \cos A = \frac{\text{Length of adjacent side}}{\text{Length of hypotenuse side}}$$

$$\text{sine } A = \sin A = \frac{\text{Length of opposite side}}{\text{Length of hypotenuse side}}$$

$$\text{tangent } A = \tan A = \frac{\text{Length of opposite side}}{\text{Length of adjacent side}}$$

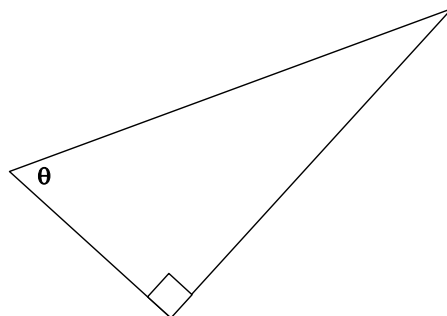
Now, there's an easy way to remember what all these ratios are – **SOH CAH TOA**. Say it out aloud – it is a word you can say easily and should be able to remember after saying it a few times. The way to use it is to look at each of the letters in it, which stand for the following:

SOH = (S)in : (O)pposite over (H)ypotenuse

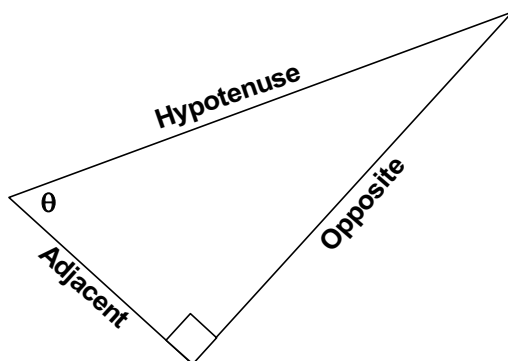
CAH = (C)os : (A)djacent over (H)ypotenuse

TOA = (T)an : (O)pposite over (A)djacent

So say I have a triangle like this one, and I'm interested in *tangent* θ :



First I need to label the names of the sides. The longest side is the hypotenuse. The side *opposite* the angle θ is the opposite side. The side touching the angle θ which is *not* the hypotenuse is the adjacent side:



Now, I remember my SOH CAH TOA. Which part am I interested in? Well the question is asking for *tangent* θ , or just *tan* θ for short. This means I'm interested in the last part of the word – the 'TOA' bit:

TOA = (T)an : (O)pposite over (A)djacent

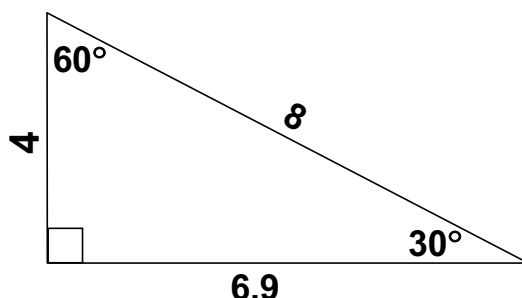
So the ratio I'm looking for is the length of the opposite side divided by the length of the adjacent side:

$$\tan(\theta) = \frac{\text{Length of opposite side}}{\text{Length of adjacent side}}$$

This is the basic procedure you need to follow whenever you need to find cos, sin or tan of an angle. Look at where the angle is in the triangle, and label the sides of the triangle. Then, using SOH CAH TOA, work out which ratio you need.

SECTION 26.2 - USING ACTUAL VALUES IN TRIGONOMETRIC RATIOS

So far, all we've done is work out *which* two sides are involved in each ratio, given a triangle and an angle we're interested in. We haven't done any calculations with actual values yet though. If we're given a triangle and told what the side lengths are, we can actually write a ratio with values in it. Take this triangle for instance:



Now, say we want to write the *tangent* of the 30° angle. Using SOH CAH TOA, we want the 'TOA' bit, which tells us that tangent is the (O)pposite side divided by the (A)djacent side. The opposite side to the 30° angle is the side of length 4. The adjacent side is the side touching the 30° angle *which is not* the hypotenuse, so it is the side of length 6.9. But now we have actual real numbers we can write in our ratio:

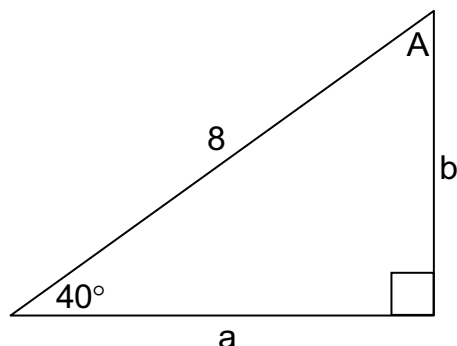
$$\tan 30^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{4}{6.9} \approx 0.5797$$

Perhaps we're interested in the *cosine* of the 60° angle? Well, this means we want the "CAH" part – (C)osine is the (A)djacent side divided by the (H)ypotenuse:

$$\cos 60^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{4}{8} = 0.5$$

Using trigonometry to work out side lengths

Trigonometry becomes very useful when you have a right-angled triangle which you know *some* side lengths and angles for. You can use these three ratios – sin, cos and tan – to help you find the unknown side lengths and unknown angles. Take this right-angled triangle as an example:



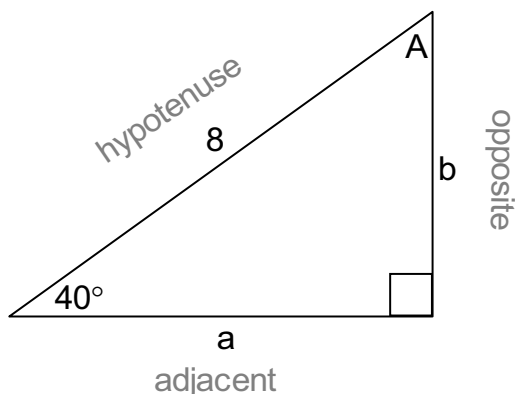
There are two unknown side lengths in this triangle, and one unknown angle. I've labelled the three unknowns using letters, using a capital letter for the angle, and lowercase letters for the sides. Often when you have an unknown angle and an unknown side *opposite* it, you use the same letter for each – a capital for the angle and a lowercase letter for the side opposite it.

The unknown angle is easy to work out – since we know angles add up to 180° inside a triangle, and we've already got a 40° and 90° angle, the unknown angle must be 50° .

The two unknown side lengths though are harder to calculate. We can't use Pythagoras' Theorem because we only know *one* side length, and we'd need to know *two* to use Pythagoras' Theorem. What we can use is our trigonometric ratios, but we're going to need a calculator to do it.

Your calculator can give you the answer to *any* trigonometric ratio you want. Say we want to find how long side 'a' is. What we can do is write a trigonometric ratio which has 'a' in it. To do this, we're going to have to pick one of the angles (apart from the right angle) in the triangle. Let's pick the 40° angle.

Now, we only know one side length – we know the hypotenuse is 8 long. What we want is a trigonometric ratio that involves both the side we're trying to work out (side a) and the side we already know.



So since we're using the 40° angle, we want a ratio which involves the adjacent side and the hypotenuse. Let's go through our SOH CAH TOA:

(S)in is the (O)pposite over the (H)ypotenuse, so it won't work.

(C)os is the (A)djacent over the (H)ypotenuse – bingo, that's what we want.

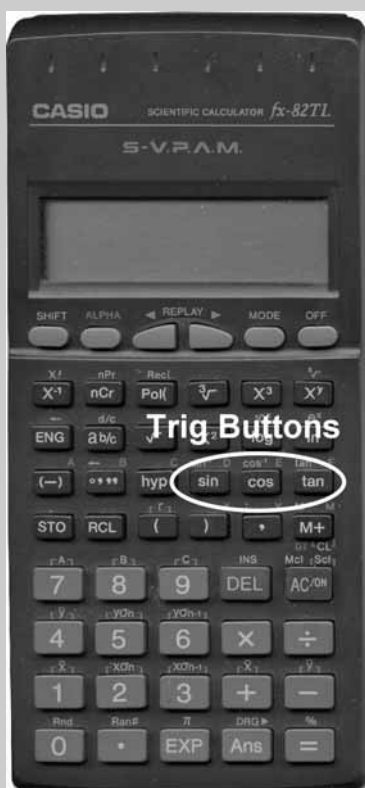
So we can write a trigonometric ratio using *cosine*:

$$\cos 40^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{8}$$

$$\cos 40^\circ = \frac{a}{8}$$

So we have two unknowns in our equation – 'cos 40°' and 'a'. This is where our calculator comes in handy. Your calculator can give you the value of any trigonometric ratio you want. So if we want to find out what cos 40° is equal to, we just type it into our calculator:

Trigonometry on the calculator



First we need to specify which of the three trig ratios we're using. In this case it's cosine, so press the **cos** button.

Next we need to tell the calculator what angle we want the ratio for. In this case, it's



First we need to tell the calculator what angle we want the trigonometric ratio for. In this case, it's 40°, so just type in '40'.

Now we need to tell the calculator which trigonometric ratio we're using. In this case,

40°. So type in '40'.

Press the $=$ button and you should get an answer of 0.766044443 on your screen.

it's cosine, so press the $\cos$ button.

You should get an answer of 0.766044443 on your screen.

So because we know what $\cos 40^\circ$ is equal to, we can rewrite our equation, and solve it, since we've only got one unknown – 'a':

$$\cos 40^\circ = \frac{a}{8}$$

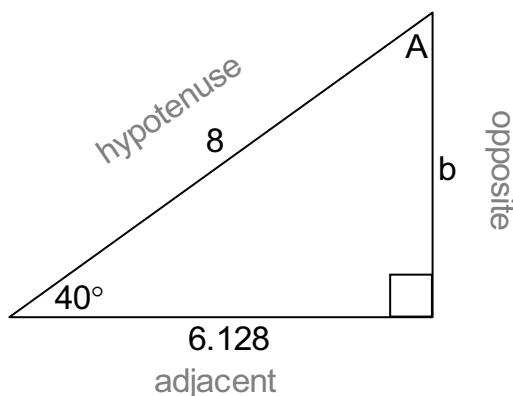
$$0.766044443 = \frac{a}{8}$$

$$a = 6.128355545$$

$$a \approx 6.1$$

So we've managed to find the length of one of the unknown sides. It's generally a good idea to have a diagram which is *roughly* to scale – that way you can check whether the side lengths you calculate make sense. In this case, side 'a' seems to be shorter than the hypotenuse, but not by too much – it's more than half as long. So we'd expect an answer below 8, but above 4 – 6.1 sounds quite reasonable.

We can work out the length of the other unknown side 'b' as well. We can use either the 40° angle or the 50° angle. Let's use the 40° one again. Now, once again, we want our ratio to have both our unknown side in it and also a known side.



Here's the updated diagram. As you can see, we now know *all* the side lengths except for the 'b' side. Which ratios could we use – SOH CAH TOA?

- $\sin 40^\circ$ is the ratio of the opposite side over the hypotenuse – this involves our unknown side and a known side, so this would be OK.
- $\cos 40^\circ$ is the ratio of the adjacent side over the hypotenuse – this isn't any use, because this ratio doesn't involve the side we're trying to find the length of – side 'b'. So we can't use $\cos 40^\circ$.

- Tan 40° is the ratio of the opposite side over the hypotenuse – this involves our unknown side and a known side, so it's OK too.

So we've got two options – sin or tan. Now, it's always better to use original values given in the question in your calculations if possible, instead of values you've calculated as you've worked through it.

$$\sin 40^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{b}{8}$$

This is a good ratio to use because it uses an *original* value given in the question

This ratio's not as good to use, because it uses an *intermediate* value you've calculated. This intermediate value is not *exact*, and also, if it's incorrect, the error will *carry through* and ruin your answer for the length of 'b' as well.

$$\tan 40^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{b}{6.128355545}$$

So it's best if we use the *sin* ratio:

$$\sin 40^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{b}{8}$$

Now, rather than working out what $\sin 40^\circ$ is straightaway, let's *rearrange* this equation so we get it in the form "b = something...":

$$\begin{aligned}\sin 40^\circ &= \frac{b}{8} \\ b &= 8 \times \sin 40^\circ\end{aligned}$$

Now once we work out $\sin 40^\circ$ we can immediately multiply it by 8 to get our answer for 'b', rather than having to write it down and type it back into our calculator, which is a lot more *error prone*. Now we just calculate what $\sin 40^\circ$ is and find 'b':

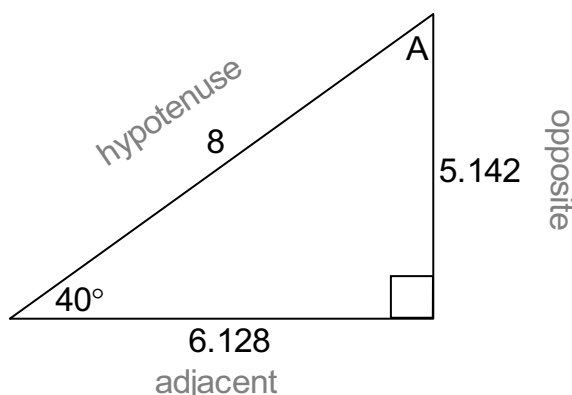
$$\begin{aligned}b &= 8 \times \sin 40^\circ \\ b &= 8 \times 0.6428 \\ b &= 5.142300878 \\ b &\approx 5.1\end{aligned}$$

From the diagram 'b' looks like it's around the same length as 'a', but perhaps a little bit shorter – 5.1 makes sense as an answer.

Using Pythagoras' Theorem to check your triangle side lengths

If you've worked out all the side lengths of a right-angled triangle using trigonometry, there's one quick check you can do to see if your answers make sense. All you need to do is check whether Pythagoras' Theorem holds true for your triangle and its side lengths.

We can do it for the triangle we worked with in the last section:



So Pythagoras' Theorem is that the square of the hypotenuse is equal to the sum of the squares of the other two sides:

$$\text{Hypotenuse}^2 = \text{opposite}^2 + \text{adjacent}^2 ???$$

$$8^2 = 5.142^2 + 6.128^2 ???$$

$$64 = 26.44 + 37.55 ???$$

$$64 = 63.99 ???$$

$$64 \approx 63.99$$

So Pythagoras' Theorem works for the side lengths we've calculated, which is a good sign for our answers being right.

Deciding whether you have enough information

Now, even though trigonometry allows you to find side lengths which you couldn't solve with just Pythagoras' Theorem, there's a limit to what even it can do. In order to be able to work out what all the side lengths and angles in a right-angled triangle are, you need to know at least the following:

- One of the angles in the triangle *apart* from the right angle
- One of the side lengths

So if you've just got one angle, you've got no chance. Same for if you only know the length of one side – no chance. But remember, sometimes even though the diagram doesn't have enough information in it, there might be extra information *written* in the question.

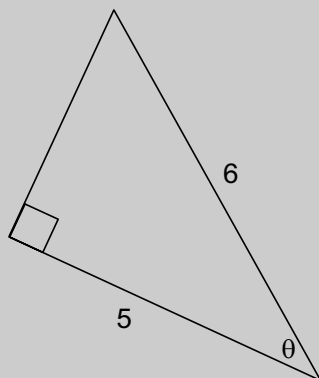
SECTION 26.3 - WORKING BACKWARDS – FINDING THE ANGLE USING TRIAL AND ERROR

Sometimes you'll get given a diagram of a triangle with two of the side lengths labelled,

but not the angle. The question will ask you to find the angle, usually to a certain degree of accuracy. In these cases, start with a guess at the angle, and see if it works using the appropriate cos, sin or tan ratio. If it doesn't, try increasing the angle. If this gets you closer keep going, otherwise try decreasing the angle.

Finding angle question

Find θ to the nearest degree:



Solution

If the diagram is roughly to scale, we can make an initial guess at what θ is – let's say it's about 40° . Now, to check whether that's a good guess, we can use a trigonometric ratio. But which one to use? Well, relative to the θ angle, the '5' side is the adjacent side, and the '6' side is the hypotenuse. So we want the cos ratio:

$$\cos \theta = \frac{5}{6} = 0.8333$$

So what we are trying to do is find a value of θ that will make this equation true. Now, we've guessed that θ is 40° . We can use our calculator to work out what cos of 40° is:

$$\cos 40^\circ = 0.7660$$

So our guess of 40° gives us an answer of 0.7660, which is smaller than 0.8333. Let's try increasing our guess for θ to 41° :

$$\cos 41^\circ = 0.7547$$

Uh-oh! 0.7547 is even further away from 0.8333, so perhaps we should be *reducing* our guess for θ , down to 39° :

$$\cos 39^\circ = 0.7771$$

Aha! We're getting closer to 0.8333 now, so we should continue *decreasing* our angle. Now, once you get used to this sort of thing, you may be able to jump around a little with your guesses. For instance, going from $\cos 40^\circ$ to $\cos 39^\circ$, the result only changed from 0.7660 to 0.7771, and we want to get all the way to 0.8333. So let's try skipping a few degrees, perhaps go all the way down to 35° :

$$\cos 35^\circ = 0.8192$$

That's a lot closer to 0.8333, but we're still a bit short. So let's drop another 2 degrees:

$$\cos 33^\circ = 0.8387$$

Now, 0.8387 is *larger* than 0.8333, so we've probably gone a bit too far. Let's go back to 34°:

$$\cos 34^\circ = 0.8290$$

So the *cosine* of 33° is 0.8387, which is *larger* than 0.8333, and the *cosine* of 34° is 0.8290, which is *smaller* than 0.8333. So the actual angle we're looking for must be somewhere between 33° and 34°. The question asked for the angle *to the nearest degree*, so we're going to have to pick one of them. One approximate way to pick the closer angle is to see which of their *cosines* is closer to 0.8333:

cos 33°:

$$\begin{aligned}\text{Distance from } 0.8333 &= 0.8387 - 0.8333 \\ &= 0.0054\end{aligned}$$

cos 34°:

$$\begin{aligned}\text{Distance from } 0.8333 &= 0.8333 - 0.8290 \\ &= 0.0043\end{aligned}$$

So the result of $\cos 34^\circ$ is a smaller distance from what we want – 0.8333, so we'll use it as our answer – $\theta = 34^\circ$. There is a much faster way to do this question using the *inverse cosine* function on your calculator, which is covered later. However, it's good to have an idea of how to approach a problem like this using trial and error.

Handy Hint #27 - Application problems in trigonometry

One of the common questions you get in trigonometry is one about a ship, person or plane etc... travelling a certain distance at a certain angle. You're then usually asked to find out how far in a certain direction, north for instance, the person is of where they started. There are a lot of variations on this type of question, but the general procedure for solving them is the same:

- Draw a diagram with as much information labelled on it as possible
- Work out what you're trying to find, and label it on the diagram

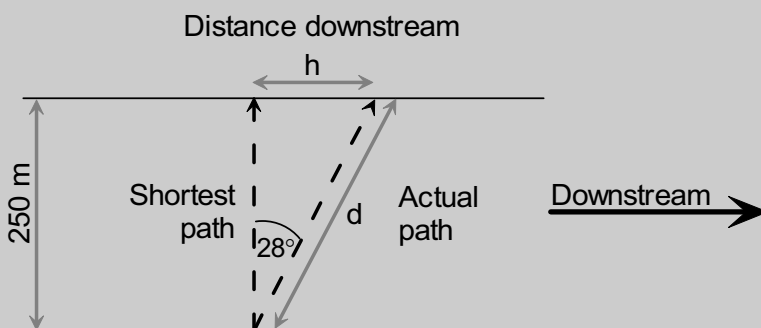
- Work out whether you have enough information to finish the problem straightaway, or whether you're going to need to find out some other values first before you can find the final answer
- Work through the problem. Each time you find out some more information, draw another diagram if possible and label it on the new diagram, or add it to the original diagram.

River rowing question

Bobby rows his boat across a wide river. He aims straight at the other bank, but because of the current, his actual path is a straight line 28° away from the shortest path across the river. The river is 250 metres wide. How far did Bobby actually travel? How much further downstream is he then if there had been no current?

Solution

First up, let's draw a diagram. We need a river, 250 metres wide. We need to draw Bobby's path across the river, which is a *straight line*, but 28° away from what the shortest path across the river is. So we could also draw this shortest possible path in the diagram as well.

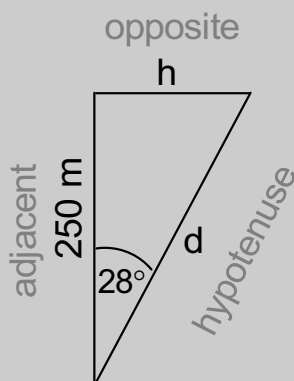


So in this diagram I have drawn the river by drawing two parallel straight lines horizontally – these lines represent the banks (sides) of the river. In between them is the water, where I've drawn the two paths across the river – the shortest possible path and the path that Bobby actually took. I've labelled these paths as well.

I've put numerical information from the question into the diagram – the river is 250 metres wide, and Bobby's path is 28° off the shortest path direction. I've also labelled the things that we want to try and find. The distance Bobby actually travelled is labelled using a 'd'. The distance further downstream between where he would have been had he gone straight across the river, and where he did go because of the current, is labelled using a 'h'. *Downstream* is the direction that the current is flowing in.

So we've got a diagram, now we need to find some answers. The central part of the diagram you should quickly realise is just a right-angled triangle. And we know one of the angles in it – 28° , and also one of the side lengths – 250 metres. This means we can

use trigonometry to find the lengths of the other two sides – which happen to be the two things we're trying to find out.



First up, let's find the length of 'd'. Relative to the angle 28° , I have labelled which sides are the adjacent, opposite and hypotenuse. To find 'd', we want a trigonometric ratio that involves both the unknown 'd', and the known side which is 250 metres long. These are the hypotenuse and adjacent sides, so we want to use *cos*:

$$\cos 28^\circ = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 28^\circ = \frac{250m}{d}$$

$$\frac{1}{\cos 28^\circ} = \frac{d}{250m}$$

$$\frac{250m}{\cos 28^\circ} = d$$

After a bit of rearranging, all we need to do now is use our calculator to work out what $\cos 28^\circ$ is, and find d:

$$d = \frac{250m}{\cos 28^\circ}$$

$$d \approx \frac{250m}{0.8829}$$

$$d \approx 283.1m$$

Looking at the diagram, this distance seems to make sense. Let's move on to finding 'h'.

Once again we want a trigonometric ratio that involves the unknown side 'h' and the known 250 metre side (we use the 250 metre side instead of the 283.1 metre side because it's an *original* value, not one we just calculated). The two sides are the opposite and the adjacent, so we need *tan*:

$$\tan 28^\circ = \frac{h}{250m}$$

$$250m \times \tan 28^\circ = h$$

Now we just use our calculator:

$$h = 250m \times \tan 28^\circ$$

$$h \approx 250m \times 0.5317$$

$$h \approx 132.9m$$

Notice I've used $\approx$ symbols in the final steps of finding each answer, because once you actually use your calculator to find the value of a trigonometric ratio, the decimal answer you get out is not exact, it's quite accurate, but not exact. Also, the final answer is rounded to only one decimal place.