Table of Contents

Chapter 3 Non-Euclidean Geometries

Chapter 4 Transformations

Chapter 5 Inversion

Chapter 1

GEOMETRY: Making a Start

1.1 INTRODUCTION. The focus of geometry continues to evolve with time. The renewed emphasis on geometry today is a response to the realization that visualization, problem-solving and deductive reasoning must be a part of everyone's education. Deductive reasoning has long been an integral part of geometry, but the introduction in recent years of inexpensive dynamic geometry software programs has added visualization and individual exploration to the study of geometry. All the constructions underlying Euclidean plane geometry can now be made accurately and conveniently. The dynamic nature of the construction process means that many possibilities can be considered, thereby encouraging exploration of a given problem or the formulation of conjectures. Thus geometry is ideally suited to the development of visualization and problem solving skills as well as deductive reasoning skills. Geometry itself hasn't changed: technology has simply added a powerful new tool for use while studying geometry.

So what is geometry? Meaning literally "earth measure", geometry began several thousand years ago for strictly utilitarian purposes in agriculture and building construction. The explicit 3-4-5 example of the Pythagorean Theorem, for instance, was used by the Egyptians in determining a square corner for a field or the base of a pyramid long before the theorem as we know it was established. But from the sixth through the fourth centuries BC, Greek scholars transformed empirical and quantitative geometry into a logically ordered body of knowledge. They sought irrefutable proof of abstract geometric truths, culminating in Euclid's Elements published around 300 BC. Euclid's treatment of the subject has had an enormous influence on mathematics ever since, so much so that deductive reasoning is the method of mathematical inquiry today. In fact, this is often interpreted as meaning "geometry is 2-column proofs". In other words geometry is a formal axiomatic structure – typically the axioms of Euclidean plane geometry - and one objective of this course is to develop the axiomatic approach to various geometries, including plane geometry. This is a very important, though limited, interpretation of the need to study geometry, as there is more to learn from geometry than formal axiomatic structure. Successful problem solving requires a deep knowledge of a large body of geometry and of different geometric techniques, whether or not these are acquired by emphasizing the ‘proving’ of theorems.

Evidence of geometry is found in all cultures. Geometric patterns have always been used to decorate buildings, utensils and weapons, reflecting the fact that geometry underlies the creation of design and structures. Patterns are visually appealing because they often contain some symmetry or sense of proportion. Symmetries are found throughout history, from dinosaur tracks to tire tracks. Buildings remain standing due to the rigidity of their triangular structures. Interest in the faithful representation of a three dimensional scene as a flat two-dimensional picture has led artists to study perspective. In turn perspective drawing led to the introduction of projective geometry, a different geometry from the plane geometry of Euclid. The need for better navigation as trading distances increased along with an ever more sophisticated understanding of astronomy led to the study of spherical geometry. But it wasn’t until the 19th century, as a result of a study examining the role of Euclid’s parallel postulate, that geometry came to represent the study of the geometry of surfaces, whether flat or curved. Finally, in the 20th century this view of geometry turned out to be a vital component of Einstein’s theory of relativity. Thus through practical, artistic and theoretical demands, geometry evolved from the flat geometry of Euclid describing one’s immediate neighborhood, to spherical geometry describing the world, and finally to the geometry needed for an understanding of the universe.

The most important contribution to this evolution was the linking of algebra and geometry in coordinate geometry. The combination meant that algebraic methods could be added to the synthetic methods of Euclid. It also allowed the use of calculus as well as trigonometry. The use of calculus in turn allowed geometric ideas to be used in real world problems as different as tossing a ball and understanding soap bubbles. The introduction of algebra also led eventually to an additional way of thinking of congruence and similarity in terms of groups of transformations. This group structure then provides the connection between geometry and the symmetries found in geometric decorations.

But what is the link with the plane geometry taught in high school which traditionally has been the study of congruent or similar triangles as well as properties of circles? Now congruence is the study of properties of figures whose size does not change when the figures are moved about the plane, while similarity studies properties of figures whose shape does not change. For instance, a pattern in wallpaper or in a floor covering is likely to be interesting when the pattern does not change under some reflection or rotation. Furthermore, the physical problem of actually papering a wall or laying a tile floor is made possible because the pattern repeats in directions parallel to the sides of the wall or floor, and thereby does not change under translations in two directions. In this way geometry becomes a study of properties that do not change under a family of transformations. Different families determine different geometries or different properties. The approach to geometry described above is known as Klein's Erlanger Program because it was introduced by Felix Klein in Erlangen, Germany, in 1872.

This course will develop all of these ideas, showing how geometry and geometric ideas are a part of everyone's life and experiences whether in the classroom, home, or workplace. To this is added one powerful new ingredient, technology. The software to be used is Geometer's Sketchpad. It will be available on the machines in this lab and in another lab on campus. Copies of the software can also be purchased for use on your own machines for approximately $45 (IBM or Macintosh). If you are 'uncertain' of your computer skills, don't be concerned - one of the objectives of this course will be to develop computer skills. There's no better way of doing this than by exploring geometry at the same time.

In the first chapter of the course notes we will cover a variety of geometric topics in order to illustrate the many features of Sketchpad. The four subsequent chapters cover the topics of Euclidean Geometry, Non-Euclidean Geometry, Transformations, and Inversion. Here we will use Sketchpad to discover results and explore geometry. However, the goal is not only to study some interesting topics and results, but to also give "proof" as to why the results are valid and to use Sketchpad as a part of the problem solving process.

1.2 EUCLID'S ELEMENTS. The *Elements* of Euclid were written around 300 BC. As Eves says in the opening chapter of his 'College Geometry' book,

> "this treatise by Euclid is rightfully regarded as the first great landmark in the history of mathematical thought and organization. No work, except the Bible, has been more widely used, edited, or studied. For more than two millennia it has dominated all teaching of geometry, and over a thousand editions of it have appeared since the first one was printed in 1482. ... It is no detraction that Euclid's work is largely a compilation of works of predecessors, for its chief merit lies precisely in the consummate skill with which the propositions were selected and arranged in a logical sequence ... following from a small handful of initial assumptions. Nor is it a detraction that ... modern criticism has revealed certain defects in the structure of the work."

The *Elements* is a collection of thirteen books. Of these, the first six may be categorized as dealing respectively with triangles, rectangles, circles, polygons, proportion and similarity. The next four deal with the theory of numbers. Book XI is an introduction to solid geometry, while XII deals with pyramids, cones and cylinders. The last book is concerned with the five regular solids. Book I begins with twenty three definitions in which Euclid attempts to define the notion of 'point', 'line', 'circle' *etc*. Then the fundamental idea is that all subsequent theorems – or Propositions as Euclid calls them – should be deduced logically from an initial set of assumptions. In all, Euclid proves 465 such propositions in the *Elements*. These are listed in detail in many texts and not surprisingly in this age of technology there are several web-sites devoted to them. For instance,

http://aleph0.clarku.edu/~djoyce/java/Geometry/Geometry.html

is a very interesting attempt at putting Euclid's *Elements* on-line using some very clever Java applets to allow real time manipulation of figures; it also contains links to other similar web-sites. The web-site

http://thales.vismath.org/euclid/

is a very ambitious one; it contains a number of interesting discussions of the *Elements*. Any initial set of assumptions should be as self-evident as possible and as few as possible so that if one accepts them, then one can believe everything that follows logically from them. In the *Elements* Euclid introduces two kinds of assumptions:

COMMON NOTIONS:

1. Things which are equal to the same thing are also equal to one another.
1. If equals be added to equals, the wholes are equal.
1. If equals be subtracted from equals, the remainders are equal.
1. Things which coincide with one another are equal to one another.
1. The whole is greater than the part.

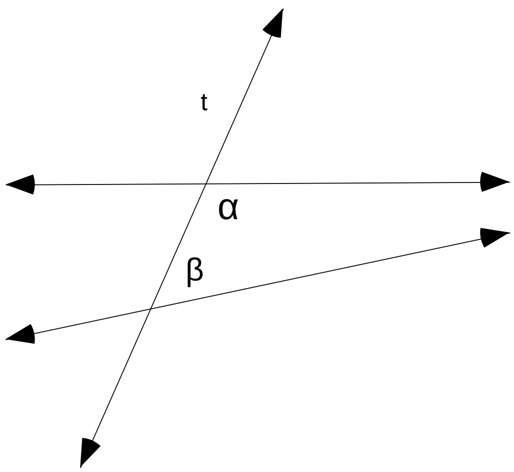
POSTULATES: Let the following be postulated.

1. To draw a straight line from any point to any point.
1. To produce a finite straight line continuously in a straight line.
1. To describe a circle with any center and distance.
1. That all right angles are equal to one another.
1. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, then the two straight lines if produced indefinitely, meet on that side on which are the angles less than two right angles.

Today we usually refer to all such assumptions as *axioms*. The common notions are surely self-evident since we use them all the time in many contexts not just in plane geometry – perhaps that's why Euclid distinguished them from the five postulates which are more geometric in character. The first four of these postulates too seem self-evident; one surely needs these constructions and the notion of perpendicularity in plane geometry. The Fifth postulate is of a more technical nature, however. To understand what it is saying we need the notion of parallel lines.

1.2.1 Definition. Two straight lines in a plane are said to be *parallel* if they do not intersect, *i.e.*, do not meet.

The Fifth postulate, therefore, means that straight lines in the plane are not parallel when there is a transversal t such that the sum $(\alpha + \beta)$ of the interior angles on one side is less than the sum of two right angles; in fact, the postulate states that the lines must meet on this side.



The figure above makes this clear. The need to assume this property, rather than showing that it is a consequence of more basic assumptions, was controversial even in Euclid's time. He himself evidently felt reluctant to use the Fifth postulate, since it is not used in any of the proofs of the first twenty-eight propositions in Book I. Thus one basic question from the time of Euclid was to decide if the Fifth Postulate is independent of the Common Notions and the first four Postulates or whether it could be deduced from them.

Attempts to deduce the Fifth postulate from the Common Notions and other postulates led to many statements logically equivalent to it. One of the best known is

1.2.2 Playfair's Axiom: Through a given point, not on a given line, exactly one line can be drawn parallel to the given line.

Its equivalence to the Fifth Postulate will be discussed in detail in Chapter 2. Thus the Fifth postulate would be a consequence of the Common notions and first four postulates if it could be shown that neither

ALTERNATIVE A: through a given point not on a given line, no line can be drawn parallel to the given line, nor

ALTERNATIVE B: through a given point not on a given line, more than one line can be drawn parallel to the given line

is possible once the five Common notions and first four postulates are accepted as axioms. Surprisingly, the first of these alternatives does occur in a geometry that was familiar already to the Greeks, replacing the plane by a sphere. On the surface of the earth, considered as a sphere, a great circle is the curve formed by the intersection of the earth's surface with a plane passing through the center of the earth. The arc between any two points on a great circle is the shortest distance between those two points. Great circles thus play the role of 'straight lines' on the sphere and arcs of great circles play the role of line segments. In practical terms, arcs of great circles are the most efficient paths for an airplane to fly in the absence of mountains or for a ship to follow in open water. Hence, if we interpret 'point' as having its usual meaning on a sphere and 'straight line' to mean great circle, then the resulting geometry satisfies Alternative A because two great circles must always intersect (why?). Notice that in this geometry 'straight lines' are finite in length though they can still be continued indefinitely as required by the second Postulate.

This still leaves open the possibility of Alternative B. In other words, there might be geometry in which Alternative B occurs, and hence a geometry in which Alternative B is a legitimate logical substitute for Playfair's axiom. If so, the familiar results of Euclidean geometry whose proofs rely on the Fifth postulate would not necessarily remain true in this geometry. In the early 19^{th} century Gauss, Lobachevsky, and Bolyai showed that there indeed exists such a logically reasonable geometry – what we now call hyperbolic geometry. It is based on Alternative B together with the five common notions and first four postulates of Euclid. Towards the end of the 19^{th} century simple 'models' of hyperbolic plane geometry were given by Poincaré and others in terms of two and three dimensional Euclidean geometry. As a result of this discovery of hyperbolic geometry, the mathematical world has been radically changed since Alternative B appears to run counter to all prior experiences. Thus Euclidean plane geometry is only one possible geometry - the one that follows by adopting the Fifth Postulate as an axiom. For this reason, the Fifth Postulate is often referred to as the Euclidean parallel postulate, and these notes will continue this convention. Some interesting consequences of the Euclidean Parallel postulate beyond those studied in high school will be developed in Chapter 2.

The first three postulates of Euclid reflect the growth of formal geometry from practical constructions – figures constructed from line segments and circles – and the same can be said for many of the subsequent propositions proved by Euclid. We will see that software will allow constructions to be made that Euclid could only describe in words or that previously one could draw only in a rudimentary fashion using ruler and compass. This software will provide a rapid and accurate means for constructing line-segments, lines, and circles, as well as constructions based upon these objects. It will enable us to construct accurate geometric configurations that in turn can be altered to new figures having the same construction constraints. This ability to drag the figure about has been available only within the past decade. It allows a student to carry out geometric experiments quickly, producing accurate sketches from which ‘conjectures’ can be made. These conjectures can then be in turn verified in whatever manner is deemed appropriate.

The *Geometer’s Sketchpad* referred to in these notes, is such a software program. It provides accurate constructions and measures of geometric configurations of points, line segments, circles, etc. and it has the ability to replay a given construction. The software can be used to provide visually compelling evidence of invariance properties such as concurrence of lines, the co linearity of points, or the ratios of particular measurements. In addition, Sketchpad allows translations, rotations, reflections and dilations of geometric constructions to be made either singly or recursively, permitting the study of transformations in a visually compelling way as will be seen in Chapters 2 and 4. Because the two-dimensional models of hyperbolic geometry – the so called *Poincaré disk* and *upper half-plane* models - make extensive use of circles and arcs of circles, Geometer’s Sketchpad is also particularly well-adapted to developing hyperbolic plane geometry as we shall see in Chapters 3 and 5.

1.3 GEOMETER’S SKETCHPAD. Successful use of any software requires a good working knowledge of its features and its possibilities. One objective of this course is the development of that working knowledge.

Basic geometric figures are constructed using the drawing tools in the toolbox and the dynamic aspect of Sketchpad can be exploited by using the selection arrow to drag any figure that has been constructed. The Measure menu allows us to measure properties of a figure. With the Edit and Display menus labels can be added to figures, and those figures can be animated. Using custom tools we also can replay complex geometric constructions in a single step. To start with we will use some of the more basic tools of Sketchpad - a more extensive listing is given in Appendix A.

General Instructions: The set of squares along the left-hand side of the screen comprises the toolbox. The tools in the toolbox are (from top to bottom):

- **Selection Arrow Tools:** Press and hold down the mouse clicker for Rotate and Dilate tools.
- **Point Tool:** Creates points.
- **Compass (Circle) Tool:** Creates circles

- **Straightedge (Segment) tool:** Press and hold down the mouse clicker for Ray and Line tools.
- **Text:** Click on an object to display or hide its label. Double click on a label, measurement or caption to edit or change the style. Double click in blank area to create caption. With the Selection arrow tool, labels can be repositioned by dragging.
- **Custom Tools**: Allows the user to create and access custom tools.

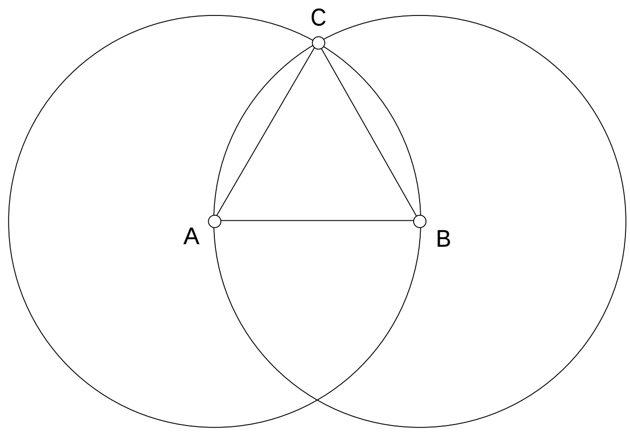
These notes contain several **Demonstrations**. In a **Demonstration**, a problem or task is proposed and the solution to the problem or task is described in the body of the **Demonstration**.

To get started using Sketchpad let's consider this **Demonstration**.

1.3.1 Demonstration: Construct an equilateral triangle using Geometer's Sketchpad.

In other words, using Sketchpad construct a triangle that remains equilateral no matter how we drag each of the vertices around the sketch using the Arrow tool. Here are the steps for one of several possible constructions.

- Open a new sketch. To create a new sketch, select "New Sketch" from under the **File** menu. Using the Segment tool, draw a line segment, and label its endpoints A, B. This defines one side of the equilateral triangle. The idea for our construction will be to construct the remaining sides so that they have length equal to that of $\overline{AB}$. To accomplish this we will construct a circle passing through A with radius $\overline{AB}$ as well as a circle passing through B with the same radius. Either point of intersection of these circles can then form the third vertex C of an equilateral triangle ΔABC. We proceed as follows:
- Using the Select arrow, select vertices A and B. Select "Circle By Center And Point" from under the **Construct** menu. Note that the order in which the points A and B are selected determines which is the center of the circle and which point lies on the circle. Repeat to construct a circle centered at the other endpoint.
- Using the Select arrow, select the two circles. Select "Intersections" from under the **Construct** menu. Using the Text tool, label one of the points C.
- To finish ΔABC, use the Segment tool to construct $\overline{AC}$ and $\overline{CB}$. The resulting figure should look similar to



To hide everything in this figure except the required equilateral triangle, first select the undesired objects and then choose “Hide Objects” from the **Display Menu**. You may click on objects individually with the Arrow Tool or you may use the Arrow Tool to drag over an area and select more than one object at once. If you selected too many objects, you can deselect an unwanted object with the Arrow Tool by simply clicking on it again.
Drag either *A* or *B* to verify that ΔABC remains equilateral. Does dragging vertex *C* have the same effect as dragging vertex *A*? The answer should be no. This is due to the fact that vertex *C* is not a *free point* because it was constructed *from A* and *B*. The vertex *A* is a free point so *A* might be thought of as an *independent* variable and *C* as a *dependent* variable. To save your figure select “Save” from under the **File** menu. The convention is to save sketches with the file extension .gsp.

End of Demonstration 1.3.1.

We can use measuring features of Sketchpad to confirm that we do have an equilateral triangle. Select the three sides of the equilateral triangle then select “Length” from the **Measure** menu. The lengths of the three segments should appear in the corner of your sketch. Drag a free vertex of the triangle. Of course, that fact that Sketchpad measures all sides with equal length does not provide a proof that your construction is correct. A proof would simply consist of the observation that both circles have the same radius and each edge of the triangle is a radius of one of the circles.

1.3.2 Exercise. Using Sketchpad, construct each of the following figures so that the figure retains its defining property when a free point on the figure is dragged:

a) a rectangle, given perpendicular segments $\overline{AB}$ and $\overline{AC}$;

b) a parallelogram, given two segments $\overline{AB}$ and $\overline{AC}$ with A, B, and C free points;

c) a rhombus, given two segments $\overline{AB}$ and $\overline{AC}$ with $\overline{AB}$ $\quad$ $\overline{AC}$;

d) a 30-60-90 triangle, given line segment $\overline{AB}$ as the hypotenuse of the triangle.

1.4 GETTING STARTED. Let's review briefly some of the principal ideas typically taught in high school geometry, keeping in mind the role of the Euclidean parallel postulate and the question of how one might incorporate the use of dynamic geometric. Many of the early propositions established by Euclid dealt with constructions which were a consequence of the first four postulates, so high school geometry often begins with the following constructions:

- construct a congruent copy of a given line segment (given angle)
- bisect a given line segment (given angle)
- construct the perpendicular bisector of a given line segment
- construct a line perpendicular to a given line through a point on the given line
- construct the perpendicular line to a given line from a point not on the given line

The **Construct** menu in Sketchpad allows us to do most of these constructions in one or two steps. If you haven't done so already, look at what is available under the **Construct** menu. It is worth noting that Euclid's constructions were originally accomplished with only a compass and straightedge. On Sketchpad this translates to using only the Circle and Segment tools. We will perform the compass and straightedge constructions once we have briefly reviewed the well-known short cuts to proving triangle congruences.

Although Euclid's fifth postulate is needed to prove many of his later theorems, he presents 28 propositions in The Elements before using that postulate for the first time. This will be important later because all these results remain valid in a geometry in which Alternative B is assumed and all but one of these remain valid in a geometry in which Alternative A is assumed. For this reason we will make careful note of the role of the fifth postulate while continuing to recall geometric ideas typically taught in high school geometry. For instance, the familiar congruence properties of triangles can be proved without the use of the Fifth postulate. In high school these may have been taught as 'facts' rather than as theorems, but it should be remembered that they could be deduced from the first four Postulates.

Recall that a triangle ΔABC is said to be congruent to ΔDEF , written $\Delta ABC \quad \Delta DEF$, when there is a correspondence $A \leftrightarrow D, B \leftrightarrow E, C \leftrightarrow F$ in which all three pairs of corresponding sides are congruent and all three pairs of corresponding angles are congruent. To establish congruence of triangles, however, it is not necessary to establish congruence of all sides and all angles.

1.4.1 Theorem (SAS). If two sides and the included angle of one triangle are congruent respectively to two sides and the included angle of another triangle, then the two triangles are congruent.

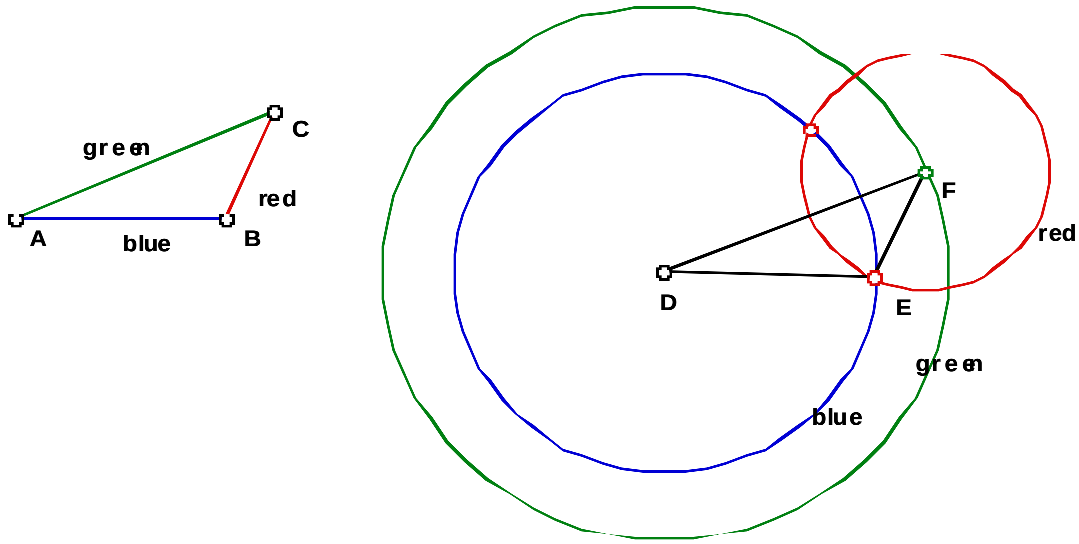
1.4.2 Theorem (ASA). If two angles and the included side of one triangle are congruent respectively to two angles and the included side of another triangle, then the two triangles are congruent.

1.4.3 Theorem (SSS). If three sides of one triangle are congruent respectively to three sides of another triangle, then the two triangles are congruent.

1.4.4 Theorem (HL). If the hypotenuse and a leg of one right triangle are congruent respectively to the hypotenuse and leg of another right triangle, then the two triangles are congruent.

These shortcuts to showing triangle congruence will be put to good use in the future. As an illustration of how we might implement them on Sketchpad consider the problem of constructing a triangle congruent to a given triangle. In more precise terms this can be formulated as follows.

1.4.5 Demonstration: Open a new sketch and construct ΔABC; now construct a new triangle in this sketch congruent to ΔABC. Here is one solution based on the SSS shortcut.



- Open a new sketch and construct ΔABC using the Segment tool in the toolbar on the left of the screen. Make certain that it is the segment tool showing, not the ray or line tool. To verify that the correct tool is selected look at the toolbar, the selected tool should be shaded. Now in the sketch window click down at the first vertex position, move the mouse to the second vertex and release the mouse clicker. At this same position, click down on the mouse, move the mouse to the third vertex, and release. Click down on the third vertex, and release on the first vertex. Label the vertices *A*, *B*, and *C* using the Text tool, re-labeling if necessary.
- Change the color of $\overline{AB}$ to blue, $\overline{BC}$ to red, and $\overline{AC}$ to green. To change the color of a line segment first select the segment then select "Color" from the **Display** menu and choose the desired color.
- Construct the point *D* elsewhere in your sketch. Now select the point *D* and the segment $\overline{AB}$. Using the **Construct** menu select "Circle By Center And Radius". Change the color of the circle to blue.
- Now select the point *D* and the segment $\overline{AC}$. Using the **Construct** menu select "Circle By Center And Radius". Change the color of the circle to green.
- Now construct any point on the green circle and label it *F*. Select that *F* and $\overline{BC}$. Using the **Construct** menu select "Circle By Center And Radius". Change the color of the circle to red.

- Construct one of the points of intersection between the red and the blue circle and label the point by *E*. To do this you may use the point tool to click on the intersection point directly. Alternatively, you can select both circles and using the **Construct** menu select "Point At Intersection".
- Finally, use the segment tool to construct $\overline{DE}$, $\overline{EF}$, and $\overline{DF}$. By SSS ΔABC is congruent to ΔDEF. Drag the vertices of ΔABC to observe the dynamic nature of your construction.

End of Demonstration 1.4.5.

Two important results follow from the previous theorems about triangle congruence.

1.4.6 Theorem. In an isosceles triangle, the angles opposite the congruent sides are congruent.

1.4.7 Corollary. In an isosceles triangle, the ray bisecting the angle included by the congruent sides bisects the side opposite to this angle and is perpendicular to it.

1.4.8 Exercise: Do the constructions below using only the Circle and Segment tools. (You can drag, label, hide etc.) In each case, prove that your construction works.

- construct a congruent copy of a given line segment (given angle)
- bisect a given line segment (given angle)
- construct the perpendicular bisector of a given line segment
- construct a line perpendicular to a given line through a point on the given line
- construct the perpendicular line to a given line from a point not on the given line

1.5 SIMILARITY AND TRIANGLE SPECIAL POINTS. One surprising discovery of a high school geometry course is the number of properties that the simplest of all geometric figures – a triangle – has. Many of these results rely on shortcuts to proving triangle similarity. The mathematical notion of similarity describes the idea of change of scale that is found in such forms as map making, perspective drawings, photographic enlargements and indirect measurements of distance. Recall from high school that geometric figures are ***similar*** when they have the *same shape*, *but not necessarily the same size*. More precisely, triangles ΔABC and ΔDEF are said to be ***similar***, written $\Delta ABC \sim \Delta DEF$, when all three pairs of corresponding angles are congruent and the lengths of all three pairs of corresponding sides are proportional. To establish similarity of triangles, however, it is not necessary to establish congruence of all pairs of angles and proportionality of all pairs of sides. The following results are part of high school geometry. It is important to note that unlike the shortcuts to triangle congruence, the shortcuts to triangle similarity do require Euclid's Fifth Postulate and therefore, any result that uses one of these shortcuts cannot be assumed to hold in a non-Euclidean geometry. *For the remainder of this chapter, we will work within Euclidean geometry, i.e., we will accept the vslidity of Euclid's Fifth Postulate.*

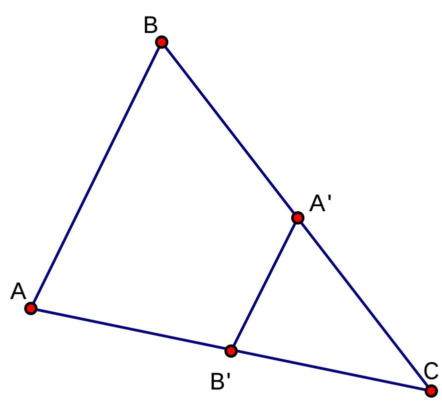
1.5.1 Theorem. (AA) If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.

1.5.2 Theorem. (SSS) If three sides of one triangle are proportional respectively to three sides of another triangle, then the triangles are similar.

1.5.3 Theorem. (SAS) If two sides of one triangle are proportional respectively to two sides of another triangle and the angles included by these sides are congruent, then the triangles are similar.

A very useful corollary of Theorem 1.4.11 is the following:

1.5.4 Corollary. Given ΔABC, let A' be the midpoint of $\overline{BC}$ and let B' be the midpoint of $\overline{AC}$. Then $\Delta B'CA' \sim \Delta ABC$ with ratio 1:2. Furthermore $\overline{AB}$ is parallel to $\overline{A'B'}$.

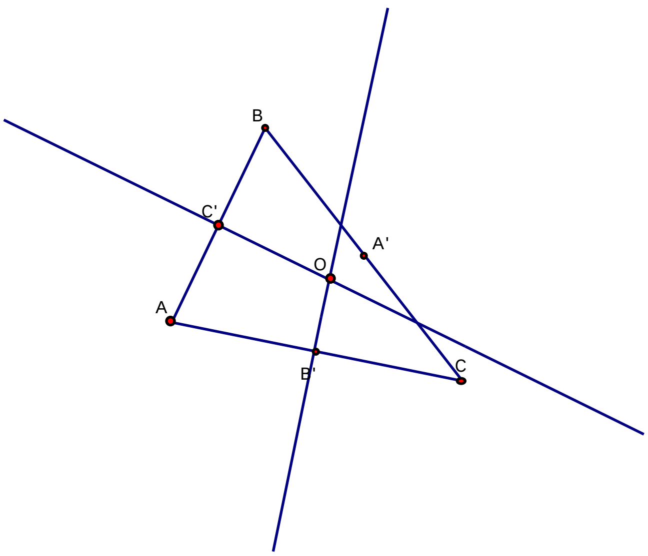


We now consider some special points related to a triangle. Recall first the definition of concurrent lines.

Definition: Three or more lines that intersect in one point are called *concurrent lines*.

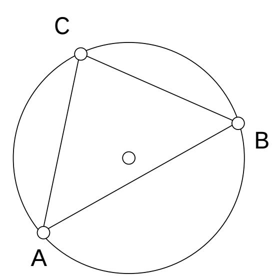
1.5.5 Theorem. The perpendicular bisectors of the sides of a triangle are concurrent at a point called the *circumcenter*r, denoted by *O*. Furthermore, *O* is equidistant from all three vertices of the triangle.

Proof: Consider ΔABC and label the midpoints of the sides A', B', and C'. Let O denote the point of intersection of the perpendicular bisectors of sides $\overline{AB}$ and $\overline{AC}$.



It suffices to prove that $\overleftrightarrow{OA'}\quad \overleftrightarrow{BC}$. First note that $\Delta OBC' \quad \Delta OAC'$ and $\Delta OAB' \quad \Delta OCB'$. Why? It follows that $OB = OA = OC$. Consequently, $\Delta OBA' \quad \Delta OCA'$. Why? Now, since corresponding angles are congruent, we have that $\quad OA'B \quad\quad OA'C$ and since the sum of their measures is 180°, each must be a right angle. Q.E.D.

Since the circumcenter *O* is equidistant from the vertices of the triangle, a circle centered at *O* will pass through all three vertices. Such a circle is called the *circumcircle* of the triangle.

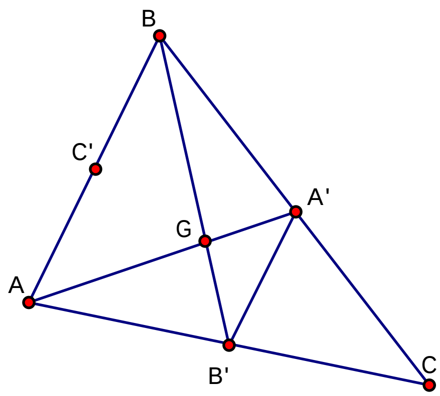


Thus every triangle in Euclidean geometry can be inscribed in a circle. The same is not true in non-Euclidean geometry. See if you can find where the Fifth Postulate was used in the proof. Don't worry if you can't. We will revisit this question in Chapter 3.

Definition: The segment connecting the vertex of a triangle and the midpoint of its opposite side is called a *median.*

1.5.6 Theorem. The medians of a triangle are concurrent, at a point called the *centroid*, denoted by G. Furthermore, the centroid trisects each of the medians.

Proof: Consider ΔABC and label the midpoints of the sides A', B', and C'. Let G denote the point of intersection of the medians $\overline{AA'}$ and $\overline{BB'}$.



We will show that $\Delta B'GA' \sim \Delta BGA$. By Corollary 1.5.4, $\overline{AB} \parallel \overline{A'B'}$ and therefore BAG $B'A'G$ since they form alternate interior angles. Similarly, GAB $GA'B'$. In addition, $A'B' = \frac{1}{2}(AB)$, again by Corollary 1.5.4. Thus the triangles in question are similar with ratio 1:2, by SAS. Consequently, $A'G = \frac{1}{2}(AG)$.

Now, let G' represent the intersection of $\overline{AA'}$ and $\overline{CC'}$. We can use the same argument to prove that $A'G' = \frac{1}{2}(AG')$. It follows that the two points coincide, and thus the three medians are concurrent at a point which trisects each median. Q.E.D.

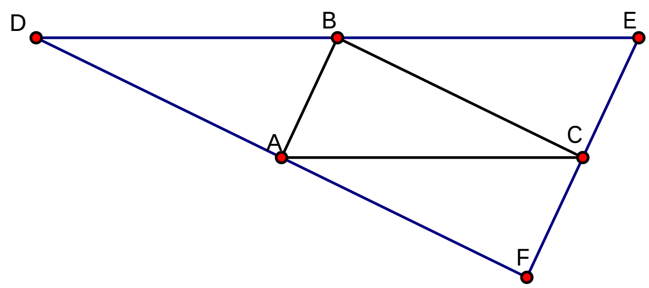
Since our proof used the shortcuts to triangle similarity, this proof cannot be used to establish the existence of the centroid of a triangle in non-Euclidean geometry. There are other proofs of the existence of the centroid and some of them are independent of Euclid's Fifth

Postulate. However, the proof that the centroid trisects each median is dependant on the Fifth Postulate and hence is not true in non-Euclidean geometry.

Definition: The segment connecting the vertex of a triangle and perpendicular to its opposite side is called an *altitude.*

1.5.7 Theorem. The altitudes of a triangle are concurrent at a point called the *orthocenter*, denoted by H.

Proof: In mathematics, one tries to use results that have already been established when possible. We can do so now, by relating the orthocenter of our triangle to the circumcenter of another triangle. We do so as follows. Through each vertex of ΔABC, draw a line parallel to the opposite side. Label the intersection points *D, E,* and *F*.



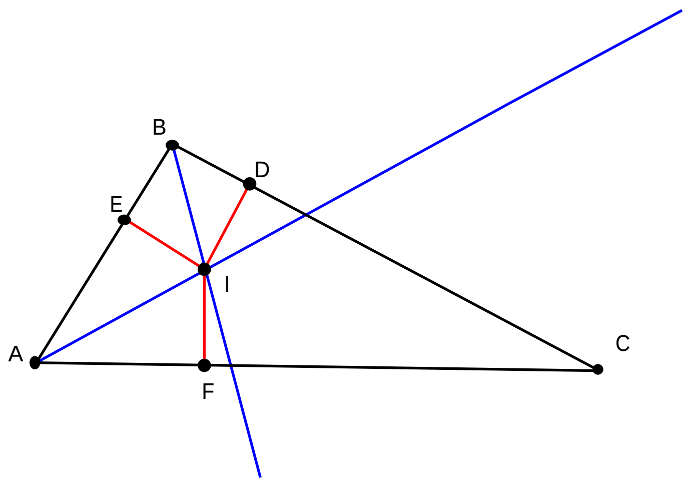
We claim that each altitude of ΔABC is a line segment lying on a perpendicular bisector of ΔDEF. Since we have already established that the perpendicular bisectors of a triangle are concurrent, it follows that as long as the altitudes intersect, they intersect in a single point. (Of course, you must convince yourself that the altitudes do intersect.) Let us prove that the altitude of ΔABC at B lies on the perpendicular bisector of $\overline{DE}$. By definition, the altitude of ΔABC at B is perpendicular to $\overline{AC}$ and hence to $\overline{DE}$, since $\overline{DE}$ is parallel to $\overline{AC}$. It remains to show that B is the midpoint of $\overline{DE}$. Note that $ABDC$ and $ACBF$ are both parallelograms. It follows, since opposite sides of a parallelogram are congruent, that $BD = AC$ and $EB = AC$. Thus B bisects $\overline{DE}$, and we are done. Q.E.D.

Question: Does the existence of the orthocenter depend on Euclid's Fifth Postulate?

Exercise 1.5.8 (a) Consider a set of 4 points consisting of 3 vertices of a triangle and the orthocenter of that triangle. Prove that any one point of this set is the orthocenter of the triangle formed by the remaining three points. Such a set is called an *orthocentric system*.

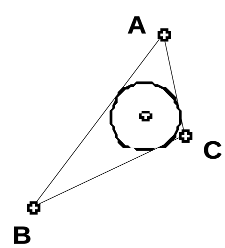
1.5.9 Theorem. The bisectors of the angles of a triangle are concurrent at a point called the incenter , denoted by *I*. Furthernnore, the incenter is equidistant from the three sides of the triangle, and thus is the center of the inscribed circle.

Proof: Let *I* denote the intersection of the angle bisectors of the angles at vertices *A* and *B*. We must show that $\overline{IC}$ bisects the angle at vertex *C*. Let *D*, *E*, and *F* denote the feet of the perpendicular lines from *I* to the sides of the triangle.



Note that $\Delta IDB \quad \Delta IEB$ and $\Delta IEA \quad \Delta IFA$. Why? It follows that $ID = IE = IF$. Consequently, $\Delta IDC \quad \Delta IFC$. Why? Therefore $ICD \quad ICF$, as we needed to show. Q.E.D.

The incenter is equidistant from all three sides of a triangle and so is the center of the unique circle, the *incircle* or *inscribing* circle, of a triangle.



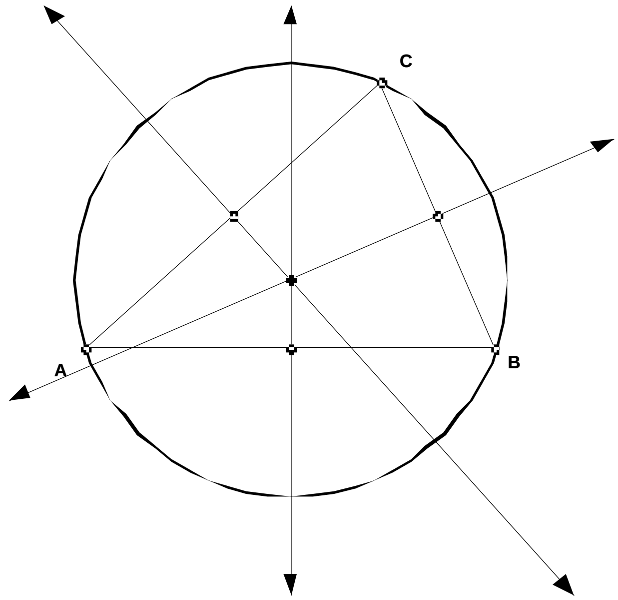
A close look at the proof above shows that it is independent of Euclid's Fifth postulate and hence every triangle, whether Euclidean or non-Euclidean, has an incenter and an inscribed circle.

It may come as an even greater surprise is that triangles have many more properties than the ones taught in high school. In fact, there are many special points and circles associated with triangles other than the ones previously listed. The web-site **http://www.evansville.edu/~ck6/tcenters/** lists a number of them; look also at **http://www.evansville.edu/~ck6/index.html**. Sketchpad explorations will be given or suggested in subsequent sections and chapters enabling the user to discover and exhibit many of these properties. First we will look at a Sketchpad construction for the circumcircle of a triangle.

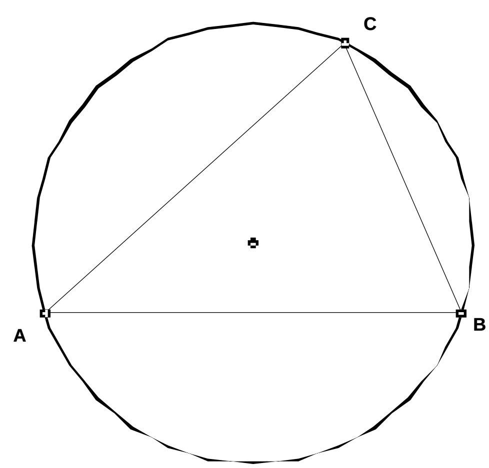
1.5.10 Demonstration: Construct the circumcircle of a given triangle.

- Open a new sketch. To construct ΔABC use the Segment tool in the toolbar on the left of the screen. Make certain that it is the segment tool showing, not the ray or line tool. Now in the sketch window click down at the first vertex position, move the mouse to the second vertex and release the mouse clicker. At this same position, click down on the mouse button, move the mouse to the third vertex, and release. Click down on the third vertex, and release on the first vertex. Re-label the vertices *A*, *B*, and *C* using the Text tool.
- To construct a midpoint of a segment, use the Select arrow tool from the toolbar. Select a segment on screen, say $\overline{AB}$, by pointing the arrow at it and clicking. Select “Point At Midpoint” from under the **Construct** menu. Upon releasing the mouse, the midpoint of $\overline{AB}$ will be constructed immediately as a highlighted small circle. Repeat this procedure for the remaining two sides of ΔABC. (Note that all three midpoints can be constructed simultaneously.)
- To construct a perpendicular bisector of a segment, use the Select arrow tool to select a segment and the midpoint of the segment . Select “Perpendicular Line” from under the **Construct** menu. Repeat this procedure for the remaining two sides of ΔABC.
- These perpendicular bisectors are concurrent at a point called the circumcenter of ΔABC, confirming visually Theorem 1.4.8.
- To identify this point as a specific point, use the arrow tool to select two of the perpendicular bisectors. Select “Point At Intersection” from under the **Construct** menu. In practice this means that only two perpendicular bisectors of a triangle are needed in order to find the circumcenter.

- To construct the circumcircle of a triangle, use the Select arrow to select the circumcenter and a vertex of the triangle, *in that order.* Select "Circle By Center+Point" from under the **Construct** menu. This sketch contains all parts of the construction.



- To hide all the objects other than the triangle ΔABC and its circumcircle, use the Select arrow tool to select all parts of the figure *except* the triangle and the circle. Select "Hide Objects" from under the **Display** menu. The result should look similar to the following figure.



The dynamic aspect of this construction can be demonstrated by using the 'drag' feature. Select one of the vertices of ΔABC using the Select arrow and 'drag' the vertex to another point on the screen while holding down on the mouse button. The triangle and its circumcenter remain a triangle with a circumcenter. In other words, the construction has the *ability to replay itself*. Secondly, once this construction is completed there will be no need to repeat it every time the circumcircle of a triangle is needed because a tool can be created for use whenever a circumcircle is needed. This feature will be presented in Section 1.8, once a greater familiarity with Sketchpad's basic features has been attained.

End of Demonstration 1.5.10.

1.6 Exercises. The following problems are designed to develop a working knowledge of Sketchpad as well as provide some indication of how one can gain a good understanding of plane geometry at the same time. It is important to stress, however, that use of Sketchpad is not the only way of studying geometry, nor is it always the best way. *For the exercises, in general, when a construction is called for your answer should include a description of the construction, an explanation of why the construction works and a print out of your sketches.*

Exercise 1.6.1, Particular figures I: In section 1.3 a construction of an equilateral triangle starting from one side was given. This problem will expand upon those ideas.

a) Draw a line segment and label its endpoints *A* and *B*. Construct a square having $\overline{AB}$ as one of its sides. Describe your construction and explain why it works.
a) Draw another line segment and label its endpoints *A* and *B*. Construct a triangle ΔABC having a right angle at *C* so that the triangle remains right-angled no matter which vertex is dragged. Explain your construction and why it works. Is the effect of dragging the same at each vertex in your construction? If not, why not?

Exercise 1.6.2, Particular figures II:

a) Construct a line segment and label it $\overline{CD}$. Now construct an isosceles triangle having $\overline{CD}$ as its base and altitude half the length of $\overline{CD}$. Describe your construction and explain why it works.
a) Modify the construction so that the altitude is twice the length of $\overline{CD}$. Describe your construction and explain why it works.

Exercise 1.6.3, Special points of triangles: For several triangles which are not equilateral, the incenter, orthocenter, circumcenter and centroid do not coincide and are four distinct points. For an equilateral triangle, however, the incenter, orthocenter, circumcenter and centroid all coincide at a unique point we'll label by *N*.

- Using Sketchpad, in a new sketch place a point and label it N. Construct an equilateral triangle ΔABC such that N is the common incenter, orthocenter, circumcenter and centroid of ΔABC. Describe your construction and explain why it works.

Exercise 1.6.4, Euclid's Constructions: Use only the segment and circle tools to construct the following objects. (You may drag, hide, and label objects.)

(a) Given a line segment $\overline{AB}$ and a point C above $\overline{AB}$ construct the point D on $\overline{AB}$ so that $\overline{CD}$ is perpendicular to $\overline{AB}$. We call D the *foot* of the perpendicular from C to $\overline{AB}$. Prove that your construction works.

(b) Construct the bisector of a given an angle ABC. Prove that your construction works.

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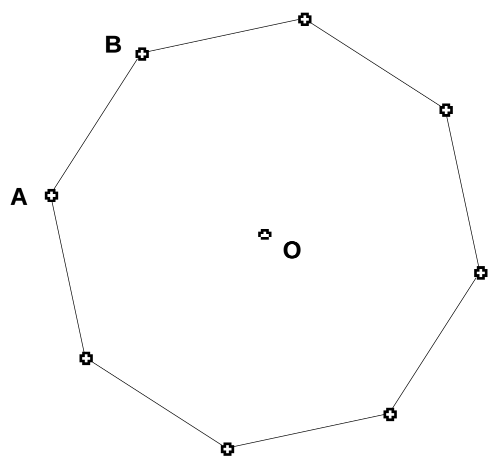
Exercise 1.6.5, Regular Octagons: By definition an octagon is a polygon having eight sides; a regular octagon, as shown below, is one whose sides are all congruent and whose interior angles are all congruent:



Think of all the properties of a regular octagon you know or can derive (you may assume that the sum of the angles of a triangle is 180 degrees). For instance, one property is that all the vertices lie on a circle centered at a point O. Use this property and others to complete the following.

(a) Using Sketchpad draw two points and label them O and A, respectively. Construct a regular octagon having O as center and A as one vertex. In other words, construct an octagon by center and point.

(b) Open a new sketch and draw a line segment $\overline{CD}$ (don't make it too long). Construct a regular octagon having $\overline{CD}$ as one side. In other words, construct an octagon by edge.

Exercise 1.6.6, Lost Center: Open a new sketch and select two points; label them O and A. Draw the circle centered at O and passing through A. Now hide the center O of the circle. How could you recover O? *EASY WAY:* if hiding O was the last keystroke, then “Undo hide point” can be used. Instead, devise a construction that will recover the center of the circle - in other words, given a circle, how can you find its center?

1.7 SKETCHPAD AND LOCUS PROBLEMS. The process of finding a set of points or its equation from a geometric characterization is called a locus problem. The 'Trace' and 'Locus' features of Sketchpad are particularly well adapted for this. The Greeks identified and studied the three types of conics: ellipses, parabolas, and hyperbolas. They are called conics because they each can be obtained by intersecting a cone with a plane. Here we shall use easier characterizations based on distance.

1.6.1 Demonstration: Determine the locus of a point P which moves so that

$$dist(P, A) = dist(P, B)$$

where A and B are fixed points.

The answer, of course, is that the locus of P is the perpendicular bisector of $\overline{AB}$. This can be proved synthetically using properties of isosceles triangles, as well as algebraically. But Sketchpad can be used to *exhibit* the locus by exploiting the ‘trace’ feature as follows.

- Open a new sketch and construct points A and B near the center of your sketch. Near the top of your sketch construct a segment $\overline{CD}$ whose length is a least one half the length of $\overline{AB}$ (by eyeballing).
- Construct a circle with center A and radius of length $\overline{CD}$. Construct another circle with center B and radius of length $\overline{CD}$.
- Construct the points of intersection between the two circles. (As long as your segment $\overline{CD}$ is long enough they will intersect). Label the points P and Q. Select both points and under the **Display** menu select Trace Intersections. You should see a √ next to it when you click and hold **Display**.
- Now drag C about the screen and then release the mouse. Think of the point C as the driver. What is the locus of P and Q?

- To erase the locus, select Erase Traces under the **Display** menu. We can also display the locus using the Locus command under the **Construct** menu. However, to use the 'locus' feature our driver must be constructed to lie on a path. An example to be discussed shortly will illustrate this.

End of Demonstration 1.7.1.

Now let's use Sketchpad on a locus problem where the answer is not so well known or so clear. Consider the case when the distances from P are not equal but whose ratio remains constant. To be specific, consider the following problem.

1.7.2 Exercise: Determine the locus of a point P which moves so that

$$dist(P, A) = 2\ dist(P, B)$$

where A and B are fixed points. (How might one modify the previous construction to answer this question?) Then, give the completion to **Conjecture 1.7.3** below.

1.7.3 Conjecture. Given points A and B, the locus of a point P which moves so that

$$dist(P, A) = 2\ dist(P, B)$$

is a/an _________________________.

A natural question to address at this point is: *How might one prove this conjecture*? More generally, what do we mean by a proof or what sort of proof suffices? Does it have to be a 'synthetic' proof, *i.e.* a two-column proof? What about a proof using algebra? Is a visual proof good enough? In what sense does Sketchpad provide a proof? A synthetic proof will be given in Chapter 2 once some results on similar triangles have been established, while providing an algebraic proof is part of a later exercise.

It is also natural to ask: *is there is something special about the ratio of the distances being equal to 2?*

1.7.4 Exercise: Use Sketchpad to determine the locus of a point P which moves so that

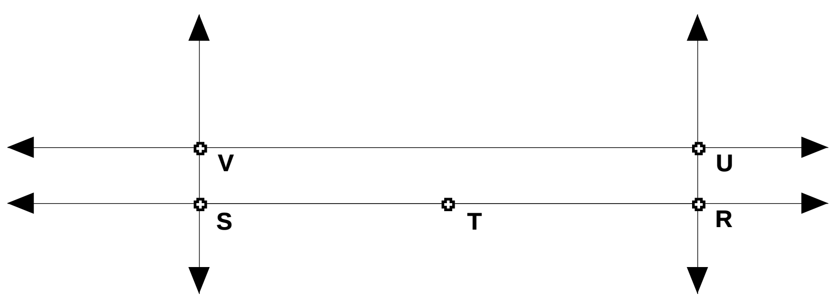
$$dist(P, A) = m\ dist(P, B)$$

where *A* and *B* are fixed points and $m = 3, 4, 5, \ldots, 1/2, 1/3, \ldots$. Use your answer to conjecture what will happen when *m* is an arbitrary positive number, $m \neq 1$? What's the effect of requiring $m > 1$? What happens when $m < 1$? How does the result of **Demonstration 1.7.1** fit into this conjecture?

1.7.5 Demonstration, A Locus Example: In this Demonstration, we give an alternate way to examine Exercise 1.7.2 through the use of the Locus Construction. Note: to use "Locus" our driver point must be constructed upon a track. Open a new sketch and make sure that the Segment tool is set at Line (arrows in both directions).

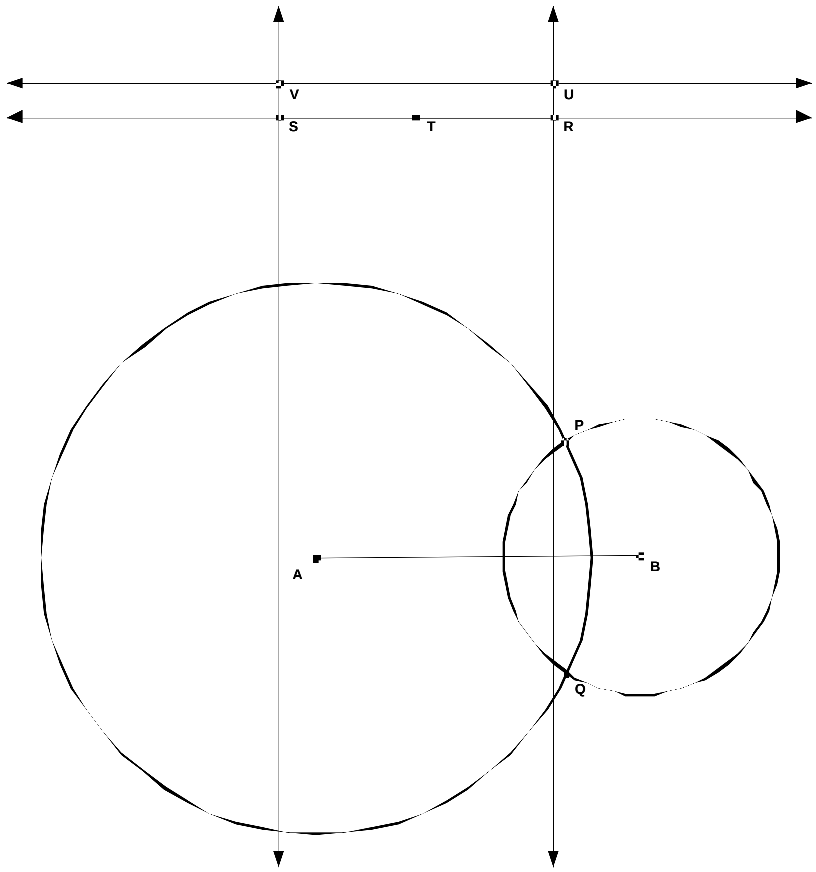
· Draw a line near the top of the screen using the Line tool. Hide any points that are drawn automatically on this line. Construct two points on this line using the Point tool by clicking on the line in two different positions. Using the Text tool, label and re-label these two points as *V* and *U* (with *V* to the left of *U*). Construct the lines through *U* and *V* perpendicular to $\overline{UV}$. Construct a point on the perpendicular line through *U*. Label it *R*.

· Construct a line through *R* parallel to the first line you drew. Construct the point of intersection of this line with the vertical line through *V* using "Point At Intersection" from under the **Construct** menu. Label this point S. Construct the midpoint $\overline{RS}$. Label this point T. A figure similar to the following figure should appear on near the top of the screen.



This figure will be used to specify radii of circles. Also, the "driver point" will be *U* and the track it moves along is the line containing $\overline{UV}$.

· Towards the middle of the screen, construct $\overline{AB}$ using the Segment tool. Construct the circle with center *A* and radius *UV* using "Circle By Center+Radius" from under the **Construct** menu. Construct the circle with center B and radius *RT* using "Circle By Center+Radius" from under the **Construct** menu. Construct both points of intersection of these two circles. Label or re-label these points *P* and *Q*. Both points have the property that the distance from *P* and *Q* to *A* is twice the distance from *P* and *Q* to *B* because the length of $\overline{UV}$ is twice that of the length of $\overline{RT}$. The figure on screen should be similar to:



- Hide everything except $\overline{AB}$, the points of intersection P and Q of the two circles and the point U.
  - Now select just the points P and U in that order. Go to "Locus" in the **Construct** menu. Release the mouse. What do you get? Repeat this construction with Q instead of P.

The "Locus" function causes the point U to move along the object it is on (here, line RS) and the resulting path of point P (and Q, in the second instance) is traced.

End of Demonstration 1.7.5.

Similar ideas can used to construct conic sections. First recall their definitions in terms of distances:

1.7.6 Definition.

(a) An ellipse is the locus a point P which moves so that

$$dist(P, A) + dist(P, B) = const$$

where A, B are two fixed points called the *foci* of the ellipse. Note: The word "*foci*" is the plural form of the word "*focus.*"

(b) A hyperbola is the locus of a point P which moves so that

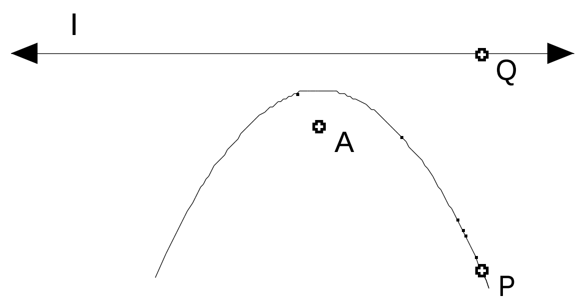
$$dist(P, A) - dist(P, B) = const$$

where A, B are two fixed points (the *foci* of the hyperbola).

(c) A parabola is the locus a point P which moves so that

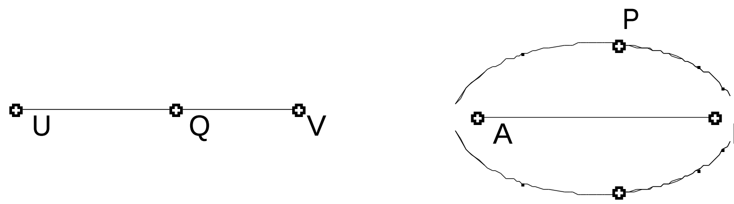
$$dist(P, A) = dist(P, l)$$

where A is a fixed point (the *focus*) and l is a fixed line (the *directrix*). Note: By $dist(P, l)$ we mean $dist(P, Q)$ where Q is on the line l and $\overline{PQ}$ is perpendicular to l. The points A and B are called the *foci* and the line l is called the *directerix*. The following figure illustrates the case of the parabola.



1.7.6a Demonstration: Construct an ellipse given points A, B for foci.

- Open a new sketch and construct points A, B. Near the top of your sketch construct a line segment $\overline{UV}$ of length greater than AB. Construct a random point Q on $\overline{UV}$.
- Construct a circle with center at A and radius $\overline{UQ}$. Construct also a circle with center at B and radius $\overline{VQ}$. Label one of the points of intersection of these two circles by P. Thus $dist(P, A) + dist(P, B) = \overline{UV}$ (why?).
- Construct the other point of intersection the two circles. Now trace both points as you drag the point Q. Your figure should like



Why is the locus of *P* an ellipse?

The corresponding constructions of a hyperbola and a parabola appear in later exercises.
End of Demonstration 1.7.6a.

1.8 CUSTOM TOOLS AND CLASSICAL TRIANGLE GEOMETRY. We will continue to explore geometric ideas as we exploit the “tool” feature of Sketchpad while looking at a sampling of geometry results from the 18th and 19th centuries. In fact, it's worth noting that many of the interesting properties of triangles were not discovered until the 18th, 19th, and 20th centuries despite the impression people have that geometry began and ended with the Greeks! Custom Tools will allow us to easily explore these geometric ideas by giving us the ability to repeat constructions without having to explicitly repeat each step.

1.8.1 Question: Given ΔABC construct the circumcenter, the centroid, the orthocenter, and the incenter. What special relationship do three of these four points share?

To explore this question via Sketchpad we need to start with a triangle and construct the required points. As we know how to construct the circumcenter and the other triangle points it would be nice if we did not have to repeat all of the steps again. Custom Tools will provide the capability to repeat all of the steps quickly and easily. Now we will make a slight detour to learn about tools then we will return to our problem.

To create a tool, we first perform the desired construction. Our construction will have certain independent objects (givens) which are usually points, and some objects produced by our construction (results). Once the construction is complete, we select the givens as well as the results. The order in which the givens are selected determines the order in which the tool will match the givens each time it is used. Objects in the construction that are not selected will not be reproduced when the tool is used. Now select **Create New Tool** from the Custom Tools menu. A dialogue box will appear which allows you to name your tool. Once the tool has been created, it is available for use each time the sketch in which it was created is open.

1.8.2 Custom Tool Demonstration: Create a custom tool that will construct a Square-By-Edge.

- Start with a sketch that contains the desired construction, in this case a Square-By-Edge.
- Use the Arrow Tool to select all the objects from which you want to make a script, namely the two vertices that define the edge, and the four sides of the square. Remember you can click and drag using the Arrow Tool to select more than one object at once. Of course, if you do this, you must hide all intermediate steps.
- Choose "Create New Tool" from the **Custom Tools** menu. The dialogue box will open, allowing you to name your tool. If you click on the square next to "Show Script View" in the dialogue box, you will see a script which contains a list of givens as well as the steps performed when the tool is used. At this point, you may also add comments your script, describing the construction and the relationship between the givens and the constructed object (Note: Once your tool has been created, you can access the script by choosing **Show Script View** from the **Custom Tools** menu.)
- In order to save your tool, you must save the sketch in which it was created. As long as that sketch is open, the tools created in that sketch will be available for use. It is important that the sketch be given a descriptive name, so that the tools will be easily found.

To use your tool, you can do the following.

- Open a sketch.
- Create objects that match the Givens in the script in the order they are listed.
- From the **Custom Tools** menu, select the desired tool. Match the givens in the order listed and the constructed object appears in the sketch.
- If you would like to see the construction performed step-by-step, you can do so as follows: Once you have selected tool you wish to use, select **Show Script View** from the **Custom Tools** menu. Select all the given objects simultaneously and two buttons will appear at the bottom of the script window: "Next Step" and "All steps". If you click successively on "Next Step", you can walk through the steps of the script one at a time. If you click on "All Steps", the script is played out step by step without stopping, until it is complete.

End of Custom Tool Demonstration 1.8.2.

In order to make a tool available at all times, you must save the sketch in which it was created in the Tool Folder, which is located in the Sketchpad Folder. There are two ways to do this. When we first create the tool, we can save the sketch in which it was created in the Tool Folder, by using the dialogue box that appears when selecting **Save** or **Save as** under the **File** menu. Alternatively, if our sketch was saved elsewhere, we can drag it into the Tool Folder. *In either case, Sketchpad must be restarted before the tools will appear in the* **Custom Tools** *menu.*

IMPORTANT: It is worth noting at this time, that there are a number of useful tools already available for use. To access these tools, go to the Sketchpad Folder. There you will see a folder called Samples. Inside the Samples folder, you will find a folder called Custom Tools. The Custom Tools folder contains several documents, each of which contains a number of useful tools. You can move one or more of these documents, or even the entire Custom Tools folder, into the Tool Folder to make them generally available. Remember to restart Sketchpad before trying to access the tools. (You may have to click on the Custom Tools icon two or three times before all the custom tools appear in the toolbar.)

1.8.2a Exercise: Open a sketch and name it "Triangle Special Points.gsp" . Within this sketch, create tools to construct each of the following, given the vertices *A*, *B*, and *C* of ΔABC :

a) the circumcenter of ΔABC

b) the centroid of ΔABC

c) the orthocenter of ΔABC

d) the incenter of ΔABC .

Save your sketch in the Tool Folder and restart Sketchpad.

1.8.2b Exercise: In a new sketch draw triangle ΔABC . Construct the circumcenter of ΔABC and label it *O*. Construct the centroid of ΔABC and label it *G*. Construct the orthocenter of ΔABC and label it *H*. What do you notice? Confirm your observation by dragging each of the vertices *A*, *B*, and *C*. Complete **Conjectures 1.8.3** and **1.8.4** and also answer the questions posed in the text between them.

1.8.3 Conjecture. (Attributed to Leonhard Euler in 1765) For any ΔABC the circumcenter, orthocenter, and centroid are

________________________________.

Hopefully you will not be satisfied to stop there! Conjecture 1.8.3 suggested $O, G,$ and H are collinear, that is they lie on the so-called **Euler Line** of a triangle. What else do you notice about $O, G,$ and H? Don't forget about your ability to measure lengths and other quantities. What happens when ΔABC becomes obtuse? When will the Euler line pass through a vertex of ΔABC?

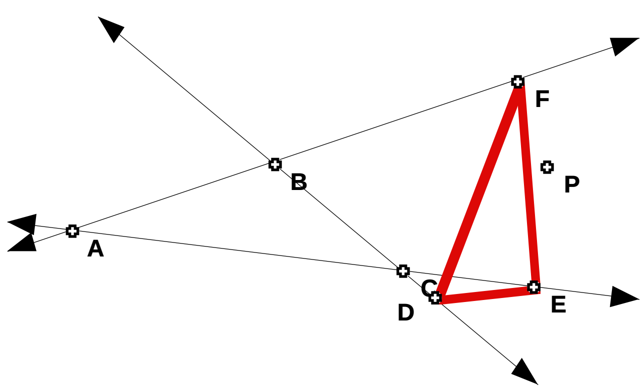
1.8.4 Conjecture. The centroid of a triangle _bisects / trisects (Circle one)_ the segment joining the circumcenter and the orthocenter.

End of Exercise 1.8.2b.

Another classical theorem in geometry is the so-called Simson Line of a triangle, named after the English mathematician Robert Simson (1687-1768). The following illustrates well how Sketchpad can be used to discover such results instead of being given them as accepted facts. We begin by exploring Pedal triangles.

1.8.5 Demonstration on the Pedal Triangle:

- In a new sketch construct three non-collinear points labeled A, B, and C and then construct the three lines containing segments $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$. (We want to construct a triangle but with the sides extended.) Construct a free point P anywhere in your sketch.
- Construct the perpendicular from P to the line containing $\overline{BC}$ and label the foot of the perpendicular as D. Construct the perpendicular from P to the line containing $\overline{AC}$ then and the foot of the perpendicular as E. Construct the perpendicular from P to the line containing $\overline{AB}$ and label the foot of the perpendicular as F.
- Construct ΔDEF. Change the color of the sides to red. ΔDEF is called the pedal triangle of ΔABC with respect to the point P.



End of Demonstration 1.8.5.

1.8.5a Exercise: Make a Script which constructs the Pedal Triangle ΔDEF for a given point P and the triangle with three given vertices A, B and C. (Essentially, save the script constructed in **Demonstration 1.8.5** as follows: Hide everything except the points A, B, C, and P and the pedal triangle ΔDEF. Select those objects in that order with the Selection tool. Then choose "Create New Tool" from the **Custom Tool** menu.)

Now you can start exploring with your pedal triangle tool.

1.8.5b Exercise: When is ΔDEF similar to ΔABC? Can you find a position for P for which ΔDEF is equilateral? Construct the circumcircle of ΔABC and place P close to or even on the circumcircle. Complete **Conjecture 1.8.6** below.

1.8.6 Conjecture. P lies on the circumcircle of ΔABC if and only if the pedal triangle is ________________________.

We shall turn to the proof of some of these results in Chapter 2.

1.9 Exercises. In these exercises we continue to work with Sketchpad, including the use of scripts. We will look at some problems introduced in the last few sections as well as discover some new results. Later on we'll see how Yaglom's Theorem and Napoleon's Theorem both relate to the subject of tilings.

Exercise 1.9.1, Some algebra: Write down the formula for the distance between two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ in the coordinate plane. Now use coordinate geometry to prove the assertion in Conjecture 1.6.3 (regarding the locus of P when $dist(P, A) = 2\ dist(P, B)$) that the locus is a circle. To keep the algebra as simple as possible assume that $A = (-a, 0)$ and $B = (a, 0)$ where a is fixed. Set $P = (x, y)$ and compute $dist(P, A)$ and $dist(P, B)$. Then use the condition $dist(P, A) = 2\ dist(P, B)$ to derive a relation between x and y. This relation should verify that the locus of P is the conjectured figure.

Exercise 1.9.2, Locus Problems.

(a) Given points A, B in the plane, use Sketchpad to construct the locus of the point P which moves so that

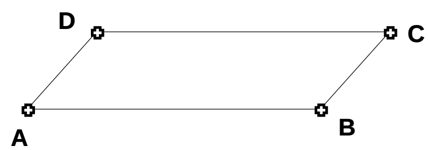
$$dist(P, A) - dist(P, B) = \text{constant.}$$

(b) Given point A and line l in the plane, use Sketchpad to construct the locus of the point P which moves so that

$$dist(P, A) = dist(P, l).$$

Hint: Construct a random point Q on the line l. Then think about relationship between Q and A to P and use that to find the corresponding point P on the parabola.

Exercise 1.9.3, Yaglom's Theorem. In a new sketch construct any parallelogram *ABCD.*



- On side $\overline{AB}$ construct the outward pointing square having $\overline{AB}$ as one of its sides. Construct the center of this square and label it Z.
- Construct corresponding squares on the other sides $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, and label their centers X, U, and V respectively.

What do you notice? Confirm your observation(s) by dragging the vertices of the original parallelogram.

Exercise 1.9.4, Miquel Point. In a new sketch draw an acute triangle ΔABC.

- On side $\overline{AB}$ select a point and label it D. Construct a point E on side $\overline{BC}$, and a point F on side $\overline{CA}$.
- Construct the circumcircles of ΔADF, ΔBDE, and ΔCEF.

What do you notice? Confirm your observation(s) by dragging each of the points $A, B, C, D, E,$ and F. Now drag vertex A so that ΔABC becomes obtuse. Do your observation(s) remain valid or do they change for obtuse triangles? What happens if the three points $D, E,$ and F are collinear (allow $D, E,$ and F to be on the extended sides of the triangle)?

Exercise 1.9.5, Napoleon's Theorem. In a new sketch draw any acute triangle ΔABC.

- On side $\overline{AB}$ construct the outward pointing equilateral triangle having $\overline{AB}$ as one of its sides. Construct the corresponding equilateral triangle on each of $\overline{BC}$, and $\overline{CA}$.
- Construct the circumcircle of each of the equilateral triangles just constructed.

What do you notice? Confirm your observation(s) by dragging the vertices of ΔABC.

Exercise 1.9.6. Open a new sketch and construct an equilateral triangle ΔABC. Select any point *P* on the triangle or in its interior.

- Construct the perpendicular segment from *P* to each of the sides of ΔABC.
- Measure the length of the segment from *P* to $\overline{BC}$; call it r_a. Measure the length of the segment from *P* to $\overline{CA}$; call it by r_b. Measure the length of the segment from P to $\overline{AB}$; call it r_c. Compute the sum $r_a+r_b+r_c$.

Drag *P* around to see how the value of $r_a+r_b+r_c$ changes as *P* varies. What do you notice? Explain your answer by relating the value you have obtained to some property of ΔABC. (Hint: look first at some special locations for *P*.)

Exercise 1.9.7. Confirm your observation in Conjecture 1.8.3 regarding the Euler Line for the special case of the triangle ΔABC in which *A* = (-2, -1), *B* = (2, -1), and *C* = (1, 2). That is, find the coordinates of the circumcenter *O*, the centroid *G* and the orthocenter H using coordinate geometry and show that they all lie on a particular straight line.

1.10 SKETCHPAD AND COORDINATE GEOMETRY. Somewhat surprisingly perhaps, use of coordinate geometry and some algebra is possible with Sketchpad. For instance, graphs defined by $y = f(x)$ or parametrically by $(x(t), y(t))$ can be drawn by regarding the respective variable *x* or *t* as a parameter on a fixed curve. Graphs can even be drawn in polar coordinates.

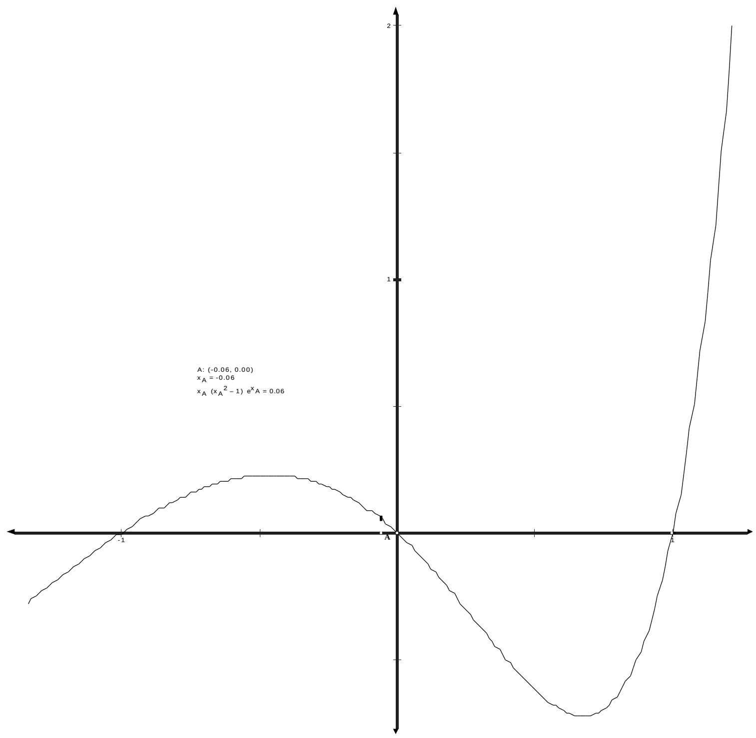
1.10.1 Demonstration: As an illustration let's consider the problem of drawing the graph of *y* = 2*x* + 1; it is a straight line having slope 2 and y-intercept 1 and the points on the graph have the form $(x, 2x+1)$. Sketchpad draws this graph by constructing the locus of $(x, 2x+1)$ as *x* varies over a portion of the *x*-axis. This can be done via the **Trace Point** or **Locus** feature described earlier, but is can also be done using the **Animation** feature as follows.

- Open a new sketch and from the **Graph** menu and choose "Define Coordinate System". Select the *x*-axis and construct a point on this axis using the "Construct Point on Object" tool from the **Construct** menu. Label this point *A*.

- To graph the function we want to let *A* vary along the *x*-axis so let's illustrate the animation feature first. Select *A* and from the **Edit** menu choose "Action Button" and then move the cursor over to the right and select "Animation". A menu will appear - by default the menu will read "Point *A* moves bidirectionally along the *x*-axis at medium speed". Say "O.K." , and an animation button will appear in the sketch. Double click on it to start or stop the animation. Try this.
- Select point *A* and then select "Abscissa (x)" from the **Measure** menu. The x-coordinate of A will appear on the screen.
- You are now ready to begin graphing. Select "Calculate" from the **Measure** menu. This is used to define whatever function is to be graphed, say the function $2x + 1$. Type in the box on the calculator whatever function of x you want to graph, clicking on the x_A-coordinate on the screen for the *x*-variable in your function. The expression will appear on the sketch.
- To plot a point on the graph of $y = 2x + 1$ select x_A and $2x_A + 1$ from the screen and then select "Plot as (x, y)" from the **Graph** menu. This plots a point on the coordinate plane on your screen. Select this point and then select "Trace Point" from the **Display** menu. If you want, you can color the point so that the graph will be colored when you run the animation! Now double click on the "Animate" button on screen and watch the graph evolve.

End of Demonstration 1.10.1.

1.10.1a Exercise. Use the construction detailed above to draw the graph of $y = x(x^2 - 1)e^x$ as shown below.



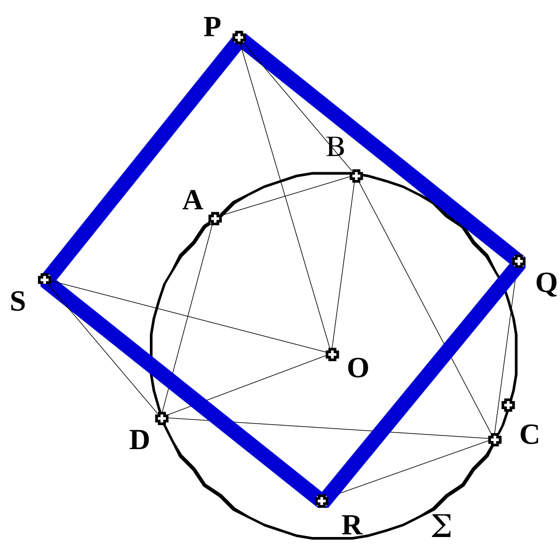
Further examples are given in the later exercises.

1.11 AN INVESTIGATION VIA SKETCHPAD. As a final illustration of the possibilities for using Sketchpad before we actually begin the study of various geometries, let us see how it might be used in problem-solving to arrive at a conjecture which we then prove by traditional coordinate geometry and synthetic methods.

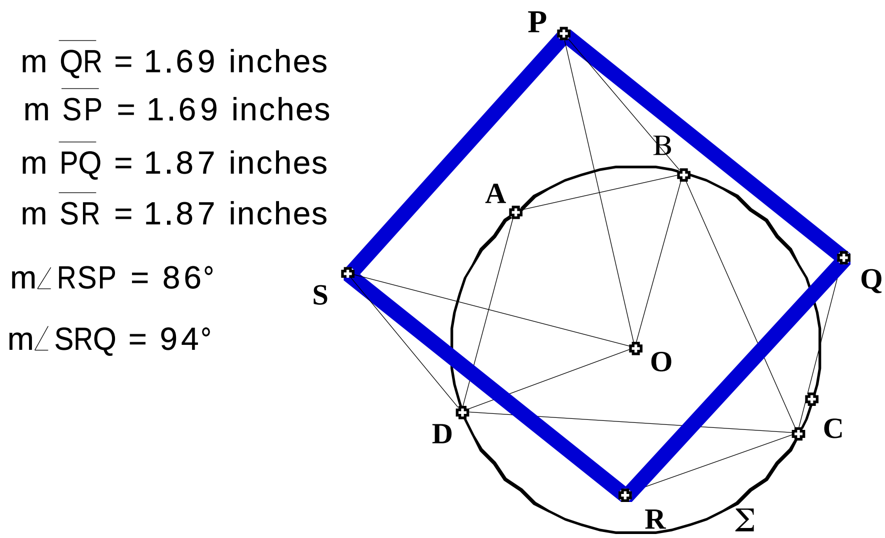
1.11.1 Demonstration. Let A, B, C and D be four distinct points on a circle Σ whose center is O. Now let P, Q, R and S be the mirror images of O in the respective chords $\overline{AB}, \overline{BC}, \overline{CD}$ and $\overline{DA}$ of Σ. Investigate the properties of the quadrilateral $PQRS$. Justify algebraically or synthetically any conjecture you make. Investigate the properties of the corresponding triangle ΔPQR when there are only three distinct points on Σ.

One natural first step in any problem-solving situation is to draw a picture if at all possible - in other words to realize the problem as a visual one.

- Open a new sketch. Draw a circle and label it Σ. Note the point on the circle which when dragged allows the radius of the circle to be varied - this will be useful in checking if a conjecture is independent of a particular Σ.
- Construct four random points on Σ and label them A, B, C and D. Construct the corresponding chords $\overline{AB}, \overline{BC}, \overline{CD}$ and $\overline{DA}$ of Σ. Make sure that A, B, C and D can be moved freely and independently of each other - this will be important in testing if a conjecture is independent of the location of A, B, C and D.
  - To construct the mirror image P of O in $\overline{AB}$ select $\overline{AB}$ and then double click on it - the small squares denoting that $\overline{AB}$ has been selected should 'star-burst'. Alternatively, drag down on the **Transform** menu and select "mark mirror" indicating that reflections can be made with respect to $\overline{AB}$.
- Select O and drag down on the **Transform** menu until "Reflect" is highlighted. The mirror image of O in the mirror $\overline{AB}$ will be constructed. Label it P. Repeat this to construct the respective mirror images Q, R and S. Draw line segments to complete the construction of the quadrilateral *PQRS*. Your figure should look similar to the following



The problem is to decide what properties quadrilateral *PQRS* has. Sketchpad is a particularly good tool for investigating various possibilities. For example, as drawn, it looks as if its side-lengths are all equal. To check this, measure the lengths of all four sides of *PQRS*. Immediately we see that adjacent sides do not have the same length, but opposite sides do. Drag each of A, B, C and D as well as the point specifying the radius of Σ to check if the equality $PQ=RS$ does not depend on the location of these points or the radius of Σ. In the figure as drawn the side $\overline{SP}$ looks to be parallel to the opposite side $\overline{RQ}$. To check this measure angles RSP and SRQ. Your figure should now look like:



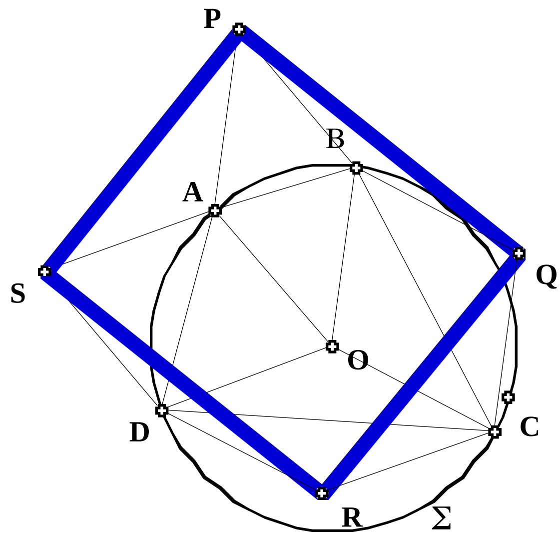
Since $m \quad RSP + m \quad SRQ = 180\,°$, this provides evidence that $\overline{SP}$ *is* parallel to $\overline{RQ}$ though it does not prove it of course (why?). To check if the sum is always 180°, drag each of A, B, C and D as well as the point specifying the radius of Σ.

End of Demonstration 1.11.1.

All this Sketchpad activity thus suggests the following result.

1.11.2 Theorem. Let A, B, C and D be four distinct points on a circle Σ whose center is O. Then the mirror images P,Q,R and S of O in the respective chords $\overline{AB}, \overline{BC}, \overline{CD}$ and $\overline{DA}$ of Σ are always the vertices of a parallelogram *PQRS.*

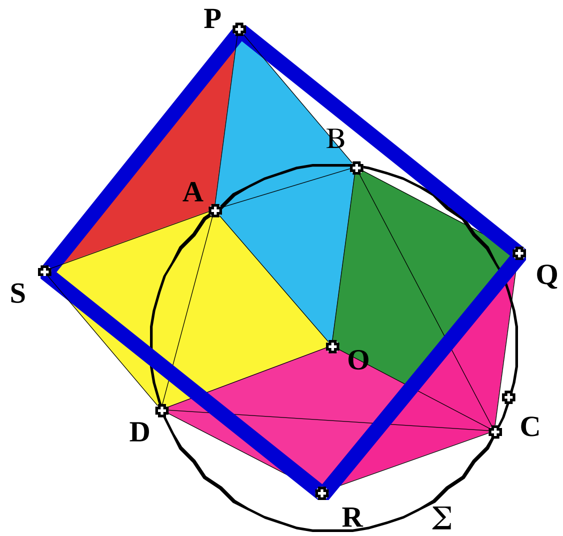
While Sketchpad has provided very strong visual support for the truth of Theorem 1.11.2, it hasn't supplied a complete proof (why?). For that we have to use synthetic methods or coordinate geometry. Nonetheless, preliminary use of Sketchpad often indicates the path that a formal proof may follow. For instance, in the figure below



a number of line segments have been added. In particular, the lengths of the line segments $\overline{OA}, \overline{OB}, \overline{OC}$ and $\overline{OD}$ are equal because each is a radial line of Σ. This plus visual inspection suggests that each of

$$OAPB, \quad OBQC, \quad OCRD, \quad ODSA$$

is a rhombus and that they all have the same side length, namely the radius of Σ. Assuming that this is true, how might it be used to prove Theorem 1.11.2? Observe first that to prove that *PQRS* is a parallelogram it is enough to show that $PS=QR$ and $PQ=SR$ (why?). To prove that $PS=QR$ it is enough to show that $\Delta SAP \quad \Delta RCQ$. At this point a clear diagram illustrating what has been discussed is helpful. In general, a good diagram isolating the key features of a figure often helps with a proof.



One may color the interior of each rhombus for example, by selecting the vertices in order then by choosing "Polygon Interior" from the **Construct** menu. You may construct the interior of any polygon in this manner. You can change the color of the interior by selecting the interior and then by using the **Display** menu. Indeed, to show that $\Delta SAP \quad \Delta RCQ$ we can use (SAS) because

$$SA=RC, \quad AP=CQ$$

since all four lengths are equal to the radius of Σ, while

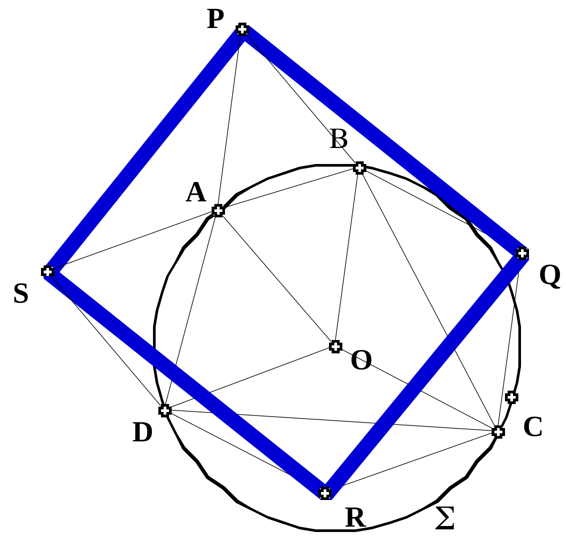
$$m \quad SAP = m \quad DOB = m \quad RCQ\,.$$

Consequently, the key property needed in proving Theorem 1.10.2 is the fact that each of

$$OAPB, \quad OBQC, \quad OCRD, \quad ODSA$$

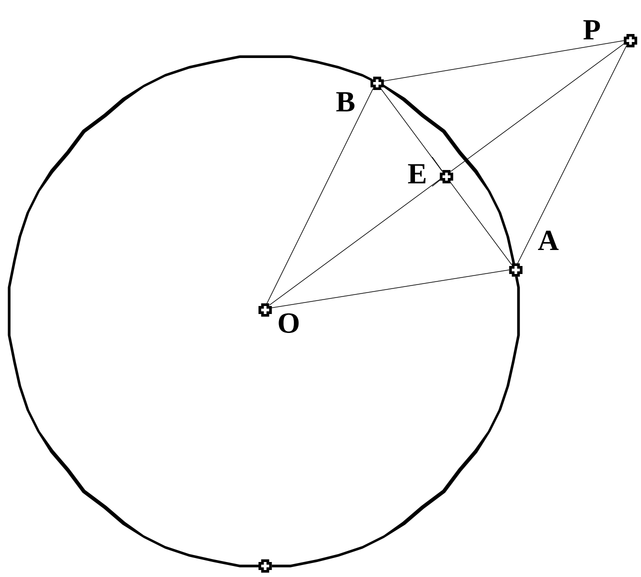
is a rhombus. Although this still doesn't constitute a complete proof (why?), it does illustrate how drawing accurate figures with Sketchpad can help greatly in visualizing the steps needed in a proof.

Proof of Theorem 1.11.2. (Synthetic) Recall the earlier figure



To prove that *PQRS* is a parallelogram it is enough to show that *PS*=*QR* and *PQ*=*SR*. We prove first that *PS*=*QR* by showing that $\Delta SAP = \Delta RCQ$.

Step 1. The construction of *P* ensures that *OAPB* is a rhombus. Indeed, in the figure



OA=*OB*, $m\angle AEO = 90°$ and *OE*=*EP*. Thus, by (HL),

$$\Delta AEO = \Delta BEO = \Delta AEP = \Delta BEP.$$

Consequently, $OA=OB = AP=AQ$ and so $OAPB$ is a rhombus. Similarly, each of $OBQC$, $OCRD$ and $ODSA$ is rhombus; in addition, they all have the same side length since $OA=OB=OC=OD$.

Step 2. Now consider ΔSAP and ΔRCQ. By step 1, $SA=AP$ and $RC=CQ$. On the other hand, because Step 1 ensures that $\overline{SA}$ is parallel to both $\overline{DO}$ and $\overline{RC}$, while $\overline{PA}$ is parallel to $\overline{BO}$ and $\overline{QC}$, it follows that

$$m \quad SAP = m \quad DOB = m \quad RCQ\,.$$

Hence $\Delta SAP \quad \Delta RCQ$ by (SAS).

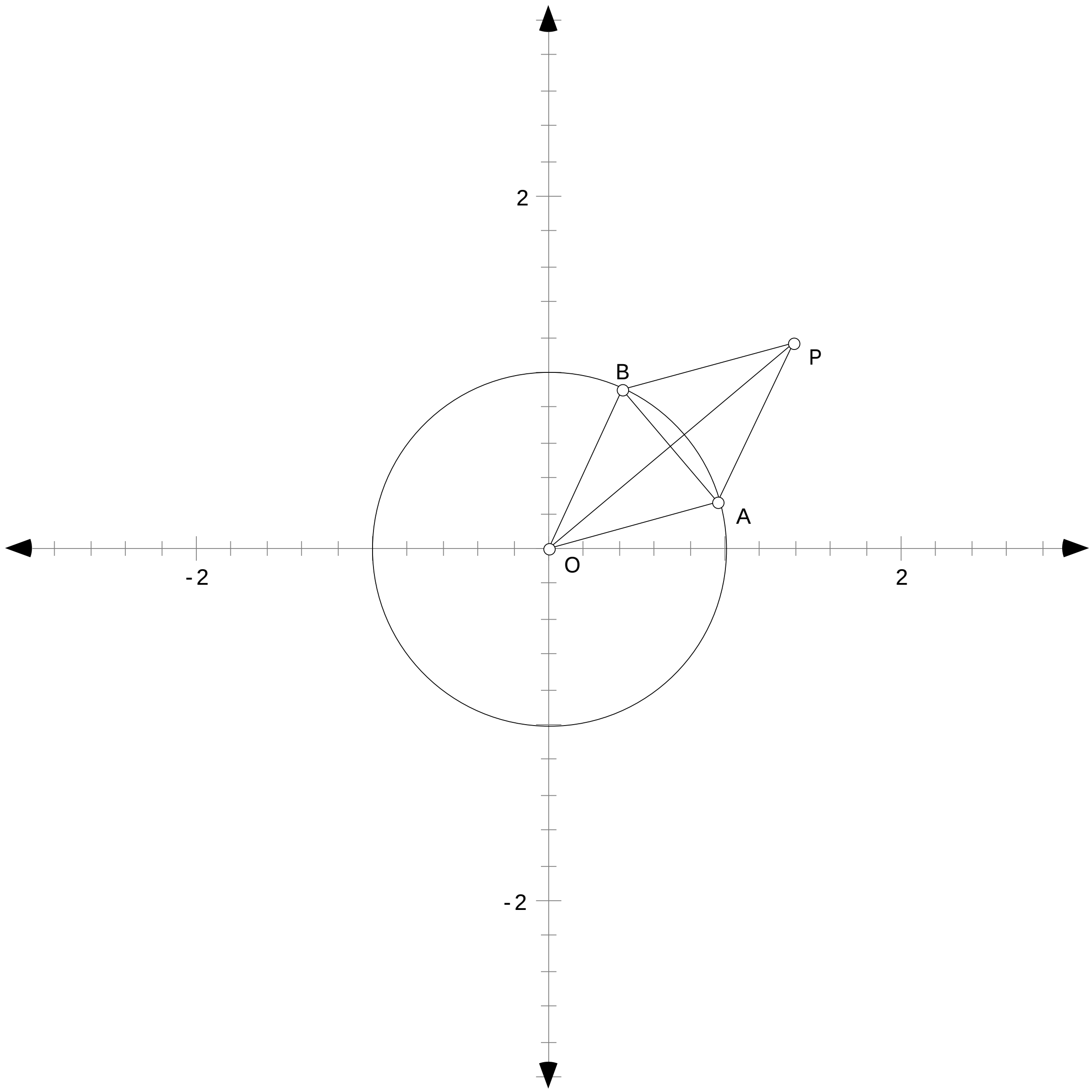
Step 3. In exactly the same way as in Step 2, by showing that $\Delta PBQ \quad \Delta SCR$, we see that $PQ=SR$. Hence $PQRS$ is a parallelogram, completing the proof of Theorem 1.10.2. **QED**

As often happens, a coordinate geometry proof of Theorem 1.11.2 is shorter than a synthetic one - that's one reason why it's often a smart idea to try first using algebra when proving a given result! Nonetheless, an algebraic proof often calls for good algebra skills!

Proof of Theorem 1.11.2. (Algebraic) Now the idea is to set up the algebra in as simple a form as possible. So take the unit circle $x^2 + y^2 = 1$ for Σ; the center O of Σ is then the origin. As any point on Σ has the form $(\cos\theta, \sin\theta)$ for a choice of θ with $0 \le \theta < 2\pi$, we can assume that

$$A = \left(\cos\theta_1, \sin\theta_1\right), \quad B = (\cos\theta_2, \sin\theta_2) \quad C = (\cos\theta_3, \sin\theta_3) \quad D = (\cos\theta_4, \sin\theta_4)\,,$$

and if we assume that $0 \le \theta_1 < \theta_2 < \theta_3 < \theta_4 < 2\pi$, then points on Σ will be in counter-clockwise order. The first thing to do now is calculate the coordinates of the mirror images of the origin in the respective chords $\overline{AB}, \overline{BC}, \overline{CD}$ and $\overline{DA}$ of Σ; again a picture helps:



Then

$$E = (\tfrac{1}{2}\cos\theta_1 + \tfrac{1}{2}\cos\theta_2,\ \ \tfrac{1}{2}\sin\theta_1 + \tfrac{1}{2}\sin\theta_2)\,,$$

since E is the midpoint of $\overline{AB}$, while

$$P = (\cos\theta_1 + \cos\theta_2,\ \ \sin\theta_1 + \sin\theta_2)\,,$$

since *OE*=*EP*. (Notice that here again we are really using the fact *OAPB* is a rhombus.) Similarly,

$$Q = (\cos\theta_2 + \cos\theta_3,\ \sin\theta_2 + \sin\theta_3), \qquad R = (\cos\theta_3 + \cos\theta_4,\ \sin\theta_3 + \sin\theta_4)$$

and

$$S = (\cos\theta_4 + \cos\theta_1,\ \sin\theta_4 + \sin\theta_1).$$

To show that *PQRS* is a parallelogram it is enough to show that $\overline{PQ}$ and $\overline{SR}$ have equal slope, and that $\overline{QR}$ and $\overline{PS}$ have equal slope. But

$$\text{slope}\,\overline{PQ} = \frac{\sin\theta_3 - \sin\theta_1}{\cos\theta_3 - \cos\theta_1} = \text{slope}\,\overline{SR}.$$

Similarly,

$$\text{slope}\,\overline{QR} = \frac{\sin\theta_4 - \sin\theta_2}{\cos\theta_4 - \cos\theta_2} = \text{slope}\,\overline{PS}.$$

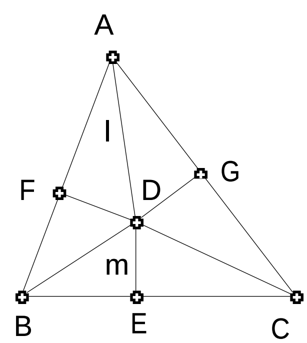
Hence *PQRS* is a parallelogram completing an algebraic proof of Theorem 1.11.2. **QED**

The algebraic proof is undoubtedly simpler than the synthetic proof, but neither gives an explanation of *why* the result is true *which is what a good proof should do*. For this we shall have to wait until Chapter 2. Other investigative problems of a similar nature are given in the problems for this chapter.

1.12 FALSE THEOREMS We all know the power a figure can provide when trying to understand why certain geometry results are true. In fact, figures seem to be essential for understanding and providing proofs. One of the most crucial features of dynamic geometry software is the possibility it provides for ‘dragging’ a given construction to provide many different views of the same setting. This can be used for investigation of a given problem with a view to formulating a conjecture or it can be used in the proof of a given conjecture. There is, however, a danger in relying on sketches, extra assumptions may be added by relying on the sketch, special cases may be omitted, or absurd results can be derived from an inaccurate sketch. We’ll look at two examples of where figures can be deceiving.

1.12.1 False Theorem: All triangles are isosceles.

Proof: Let ΔABC be a triangle with l the angle bisector of $\quad A$, m the perpendicular bisector of BC cutting BC at midpoint E, and D the intersection of l and m. From D, draw perpendiculars to AB and AC, cutting them at F and G, respectively. Finally, draw DB and DC. The following figure shows a sketch of the situation.

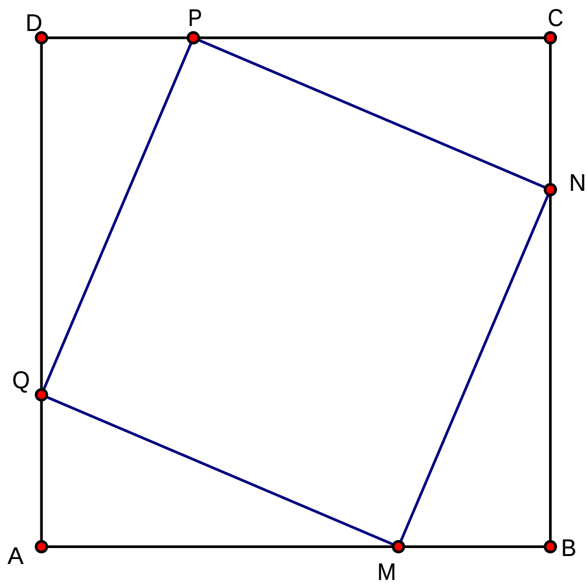


$\Delta ADF \quad \Delta ADG$ (AAS), so $AF \quad AG$ and $DF \quad DG$. $\Delta BDE \quad \Delta CDE$ (SAS), so $BD \quad CD$. This implies $\Delta BDF \quad \Delta CDG$ (HL), so $FB \quad GC$. Thus $AB \quad AF + FB \quad AG + GC = AC$, and so ΔABC is isosceles. **QED**

1.12.2 Exercise: What is wrong with the proof of Theorem 1.12.1? Absolutely nothing is wrong with the chain of reasoning, so where does the problem lie? Try constructing the given configuration on Sketchpad. What do you observe?

1.12.3 False Theorem: Any rectangle inscribed in a square must itself be a square.

Proof. Consider the following picture of a rectangle *MNPQ* inscribed in a square *ABCD*

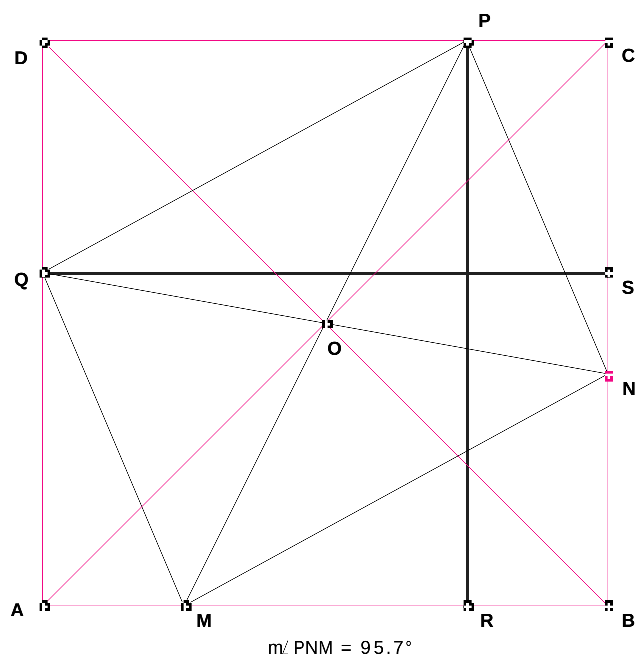


The rectangle *PQMN* certainly looks like a square doesn't it? To prove that it is we'll show that the diagonals of *PQMN* are perpendicular since the only rectangles having perpendicular diagonals are squares. Construct the point of intersection of the diagonals of rectangle *MNPQ*; label it *O*. Construct the perpendicular $\overline{PR}$ to $\overline{AB}$; construct also the perpendicular $\overline{QS}$ to $\overline{BC}$. Then $\overline{PR}$ $\overline{QS}$. Since the diagonals of any rectangle are congruent, $\overline{PM}$ $\overline{QN}$. So ΔPMR ΔQNS, and hence PMR QNS. Now consider the quadrilateral MBNO. Its exterior angle at vertex *N* is congruent to the interior angle at vertex *M*, so the two interior angles at vertices *N* and *M* are supplementary. Thus the interior angles at vertices *B* and *O* must be supplementary. But ABC is a right angle and hence NOM must also be a right angle. Therefore the diagonals of the rectangle *MNPQ* are perpendicular; ensuring that *MNPQ* is a square.

QED

1.12.4 Exercise. What is wrong with the proof of Theorem 1.12.4? Nothing actually; every step is logically correct! We could use Sketchpad to explore the various possibilities for inscribing a rectangle in a square by using the 'dragging' feature.

1.12.5 Demonstration: Construct a figure to reveal the error in the proof of Theorem 1.12.1. Open a new sketch and draw a square *ABCD*; draw also the diagonals of this square and label their point of intersection by *O*. To vary the inscribed rectangle *MNPQ* of the proof by dragging we want to construct fixed opposite vertices, say *P* and *Q*, but construct opposite vertices *M* and *N* so that they vary as we drag *N*. Select a point *P* on side $\overline{CD}$ and draw the line passing through *O* and *P*; label by *M* its point of intersection with $\overline{AB}$. Hide the line and then construct $\overline{PM}$. Now select a point *N* on $\overline{BC}$. The idea now is that everything constructed starting from *N* will move as the point *N* is dragged along $\overline{BC}$, while nothing that was constructed before will move. Draw the line through *N* and *O* to determine the remaining vertex *Q* on $\overline{DA}$; hide the line through *N* and *O*. Construct $\overline{NQ}$; this ensures that vertex *Q* will move as *N* moves. Finally, draw the line segments $\overline{PN}$, $\overline{NM}$, $\overline{QP}$, and $\overline{QM}$ as well as the perpendiculars $\overline{PR}$ and $\overline{QS}$. This gives the following figure



To check if the inscribed figure is a rectangle, measure the angle *PNM.*

Drag the point *N* along $\overline{BC}$. Does the figure *MNPQ* move? Investigate when *MNPQ* is a square. When is it a rectangle, but not a square? When is it neither a square nor a rectangle?

Formulate a conjecture describing conditions under which *MNPQ* is a rectangle. Prove your conjecture! Why are all the steps in the proof of the theorem above correct, yet the result is incorrect?

End of Demonstration 1.12.5.

The previous two examples should not suggest the banning of any figures but instead stress the need for accurately drawn figures, rather than quick sketches. By using Sketchpad we construct accurate figures and consider many different cases, thus reducing the likelihood of overlooking something. The first false theorem (all triangles are isosceles) could have been totally avoided if we had started with an accurately drawn figure. The second false theorem (any rectangle inscribed in a square is itself a square) can be avoided by the use of the dragging feature.

1.13 Exercises. The following provide multiple viewpoints on geometry: synthetic, algebraic and dynamic.

The various circles associated with a triangle may seem to involve interesting but non-practical ideas. This is not the case as the following problems show. Indeed, one of the points we shall be emphasizing throughout this course is that the study of geometry is important as a study in logical analysis, but it is also very important for the uses that can be made of geometry.

Exercise 1.13.1. What is the largest sphere that will pass through a triangular hole whose sides are 7 in., 8 in., and 9 in. long?

Exercise 1.13.2. A thin triangular-shaped iron plate is accidentally dropped into a hemispherical tank, which is 10 ft. deep and full of water. It is noticed that the iron triangle is lying parallel to the surface of the water, so it is proposed to retrieve the triangle by lowering a powerful magnet into the tank at the end of a rope. What is the minimum length of rope needed if the shortest side of the triangle is 10 ft. long and the angles of the triangle are 45, 60, and 75 degrees

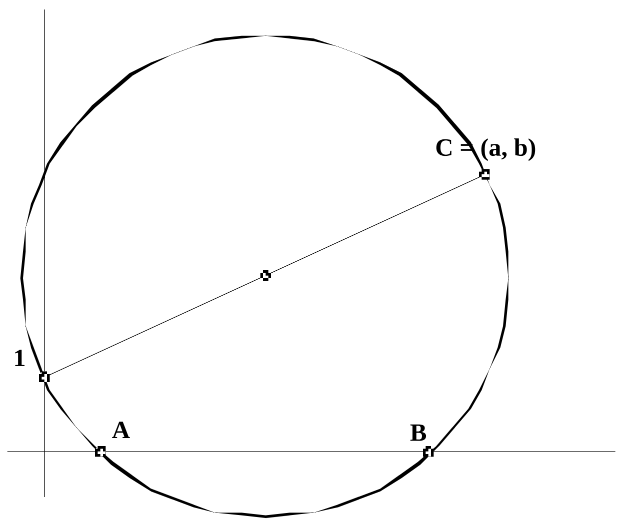
Exercise 1.13.3. Using Sketchpad, open a new sketch and draw a triangle ΔABC.

- On side $\overline{BC}$ construct the outward pointing square having $\overline{BC}$ as one of its sides; construct the center of this square and label it *X*.
- Construct the corresponding outward pointing squares on $\overline{CA}$ and $\overline{AB}$; label their respective centers *Y* and *Z*.

· Construct the segments $\overline{AX}$ and $\overline{YZ}$.

What properties do $\overline{AX}$ and $\overline{YZ}$ have? Check your conjecture by dragging the vertices of ΔABC around.

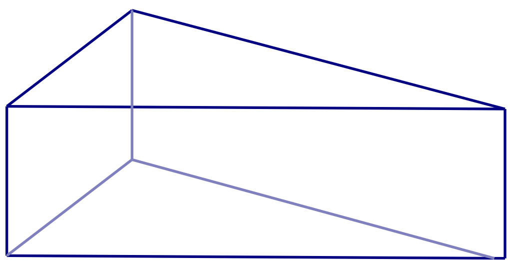
Exercise 1.13.4. Quadratic equations: The Greeks used geometry where now we would use algebra. For instance, they knew how to construct the roots of the quadratic equation $x^2 - ax + b = 0$ for given values of a, b when $a^2 > 4b$. Let's use Sketchpad to illustrate their method. Draw a pair of perpendicular lines, which we'll think of as the *x*- and *y*-axes. On the *y*-axis choose a fixed point and label it 1; the distance of this point from the point of intersection of the two perpendicular lines is to be thought of as specifying what unit length means. Given *a*, *b* draw the point *C* having (*a*, *b*) as coordinates as well as the point on the *y*-axis having *y*-coordinate 1. Now draw the circle having the line segment from this point on the *y* -axis to *C* as a diameter. Finally, label the points of intersection of this circle with the *x*-axis by *A* and *B*. You should have a figure looking like



Show that the *x*-coordinates of *A*, *B* are the roots of the equation $x^2 - ax + b = 0$. Where was the condition $a^2 > 4b$ used? What is the equation of the circle you drew? This all looks pretty straightforward to us now that we have the analytic geometry of circles available to us, but it should be remembered that almost 2,000 years elapsed after the Elements were written before Descartes combined algebra with geometry!

Exercise 1.13.5. A gardener cut a piece of sod to fill a hole in the shape of an acute triangle in a grass lawn. When he came to put the grass sod in the hole he found that it fit perfectly, but only with the wrong side up. To fit the sod in the triangular hole with the right side up he had to cut it. How did he cut it into three pieces so that the shape of each piece was unchanged when he turned it over?

Exercise 1.13.6. Birthday Cake: For her birthday party, Sally's father baked a chocolate cake in the shape of a triangular prism. Sally will have eight of her friends at her birthday party, and everyone likes chocolate cake and icing. How is Sally to cut the cake efficiently so that she and each of her friends get equal shares of cake and icing?



Exercise 1.13.7. Using Sketchpad, in a new sketch draw any convex quadrilateral *ABCD*. Recall that a convex quadrilateral is one that has all interior angles less than 180°.

- On side $\overline{AB}$ construct the outward pointing square having $\overline{AB}$ as one of its sides. Construct the center of this square and label it *Z*.
- Construct corresponding squares on the other sides $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, and label their centers *X*, *U* and *V* respectively.
- Draw the line segments $\overline{ZU}$ and $\overline{XV}$. Make a conjecture about the properties of $\overline{ZU}$ and $\overline{XV}$. Check these properties by dragging the vertices of the quadrilateral *ABCD*. Drag one of the vertices so that the quadrilateral becomes concave. Do the properties of $\overline{ZU}$ and $\overline{XV}$ still hold true or do they change for concave quadrilaterals?

Chapter 2

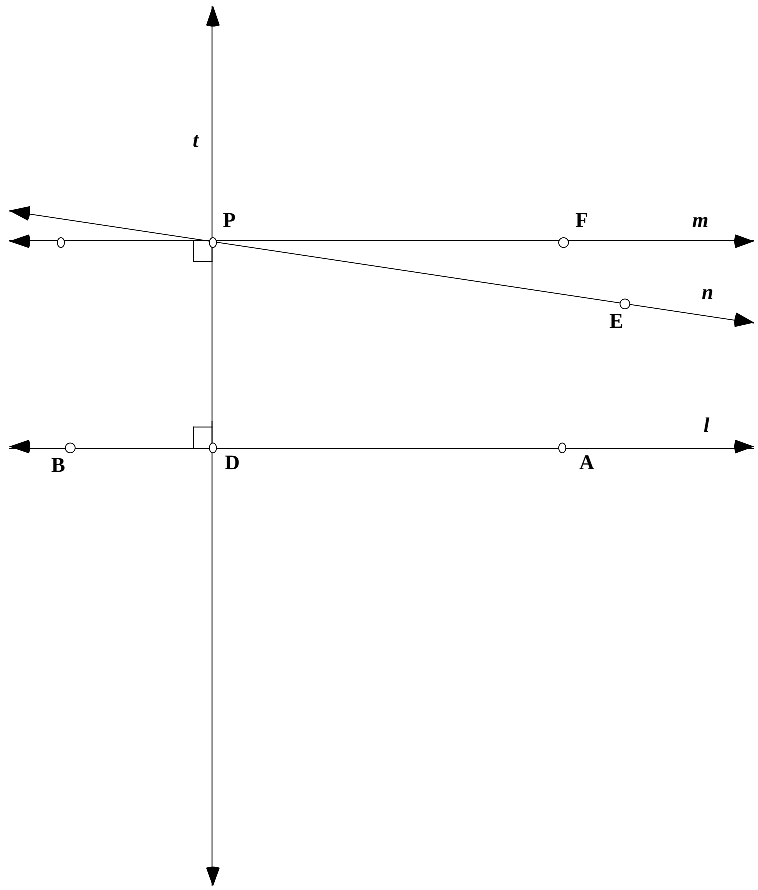
EUCLIDEAN PARALLEL POSTULATE

2.1 INTRODUCTION. There is a well-developed theory for a geometry based solely on the five Common Notions and first four Postulates of Euclid. In other words, there is a geometry in which neither the Fifth Postulate nor any of its alternatives is taken as an axiom. This geometry is called *Absolute Geometry,* and an account of it can be found in several textbooks - in Coxeter's book "Introduction to Geometry", for instance, - or in many college textbooks where the focus is on developing geometry within an axiomatic system. Because nothing is assumed about the existence or multiplicity of parallel lines, however, Absolute Geometry is not very interesting or rich. A geometry becomes a lot more interesting when some Parallel Postulate is added as an axiom! In this chapter we shall add the Euclidean Parallel Postulate to the five Common Notions and first four Postulates of Euclid and so build on the geometry of the Euclidean plane taught in high school. It is more instructive to begin with an axiom different from the Fifth Postulate.

2.1.1 Playfair's Axiom. Through a given point, not on a given line, exactly one line can be drawn parallel to the given line.

Playfair's Axiom is equivalent to the Fifth Postulate in the sense that it can be deduced from Euclid's five postulates and common notions, while, conversely, the Fifth Postulate can deduced from Playfair's Axiom together with the common notions and first four postulates.

2.1.2 Theorem. Euclid's five Postulates and common notions imply Playfair's Axiom.
Proof. First it has to be shown that if P is a given point not on a given line l, then there is *at least one* line through P that is parallel to l. By Euclid's Proposition I 12, it is possible to draw a line t through P perpendicular to l. In the figure below let D be the intersection of l with t.



By Euclid's Proposition I 11, we can construct a line *m* through *P* perpendicular to *t* . Thus by construction *t* is a transversal to *l* and *m* such that the interior angles on the same side at *P* and *D* are both right angles. Thus *m* is parallel to *l* because the sum of the interior angles is 180°. (Note: Although we used the Fifth Postulate in the last statement of this proof, we could have used instead Euclid's Propositions I 27 and I 28. Since Euclid was able to prove the first 28 propositions without using his Fifth Postulate, it follows that the existence of *at least one* line through *P* that is parallel to *l*, can be deduced from the first four postulates. For a complete list of Euclid's propositions, see “College Geometry” by H. Eves, Appendix B.)

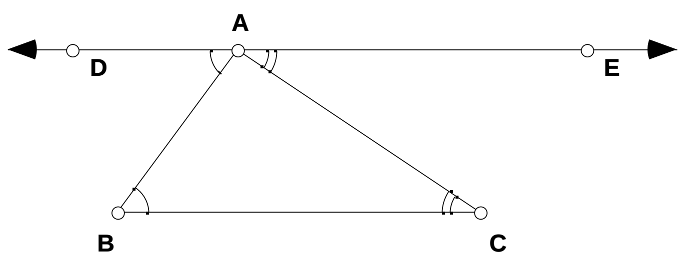
To complete the proof of 2.1.2, we have to show that *m* is the only line through *P* that is parallel to *l*. So let *n* be a line through *P* with $m \neq n$ and let E ≠ P be a point on *n*. Since $m \neq n$, *EPD* cannot be a right angle. If m *EPD* < 90°, as shown in the drawing, then m *EPD* + m *PDA* is less than 180°. Hence by Euclid’s fifth postulate, the line *n* must intersect *l* on the same side of transversal *t* as E, and so *n* is not parallel to *l*. If m *EPD* > 90°, then a similar argument shows that *n* and *l* must intersect on the side of *l* opposite E. Thus, *m* is the one and only line through *P* that is parallel to *l*. **QED**

A proof that Playfair's axiom implies Euclid's fifth postulate can be found in most geometry texts. On page 219 of his "College Geometry" book, Eves lists eight axioms other than Playfair's axiom each of which is logically equivalent to Euclid's fifth Postulate, *i.e.*, to the Euclidean Parallel Postulate. A geometry based on the Common Notions, the first four Postulates and the Euclidean Parallel Postulate will thus be called *Euclidean* (plane) geometry. In the next chapter *Hyperbolic* (plane) geometry will be developed substituting Alternative B for the Euclidean Parallel Postulate (see text following Axiom 1.2.2)..

2.2 SUM OF ANGLES. One consequence of the Euclidean Parallel Postulate is the well-known fact that the sum of the interior angles of a triangle in Euclidean geometry is constant whatever the shape of the triangle.

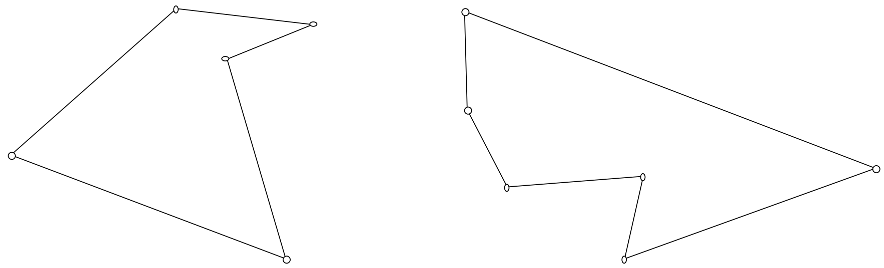
2.2.1 Theorem. In Euclidean geometry the sum of the interior angles of any triangle is always 180°.

Proof: Let ΔABC be any triangle and construct the unique line $\overleftrightarrow{DE}$ through A, parallel to the side $\overline{BC}$, as shown in the figure

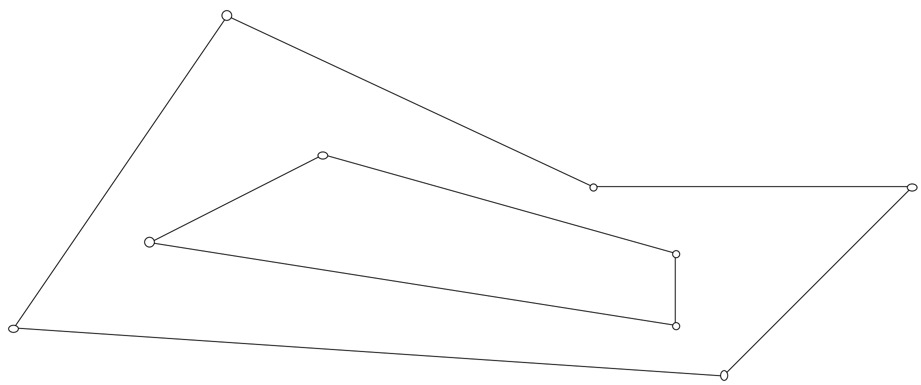


Then $m\angle EAC = m\angle ACB$ and $m\angle DAB = m\angle ABC$ by the alternate angles property of parallel lines, found in most geometry textbooks. Thus $m\angle ACB + m\angle ABC + m\angle BAC = 180°$. **QED**

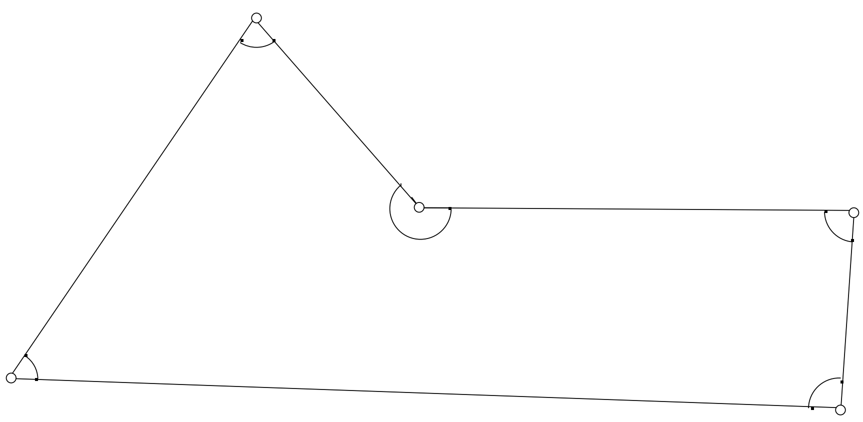
Equipped with Theorem 2.2.1 we can now try to determine the sum of the interior angles of figures in the Euclidean plane that are composed of a finite number of line segments, not just three line segments as in the case of a triangle. Recall that a ***polygon*** is a figure in the Euclidean plane consisting of points $P_1, P_2, ..., P_n$, called ***vertices***, together with line segments $\overline{P_1P_2}$, $\overline{P_2P_3}$, ..., $\overline{P_nP_1}$, called ***edges*** or ***sides***. More generally, a figure consisting of the union of a finite number of non-overlapping polygons will be said to be a ***piecewise linear figure***. Thus



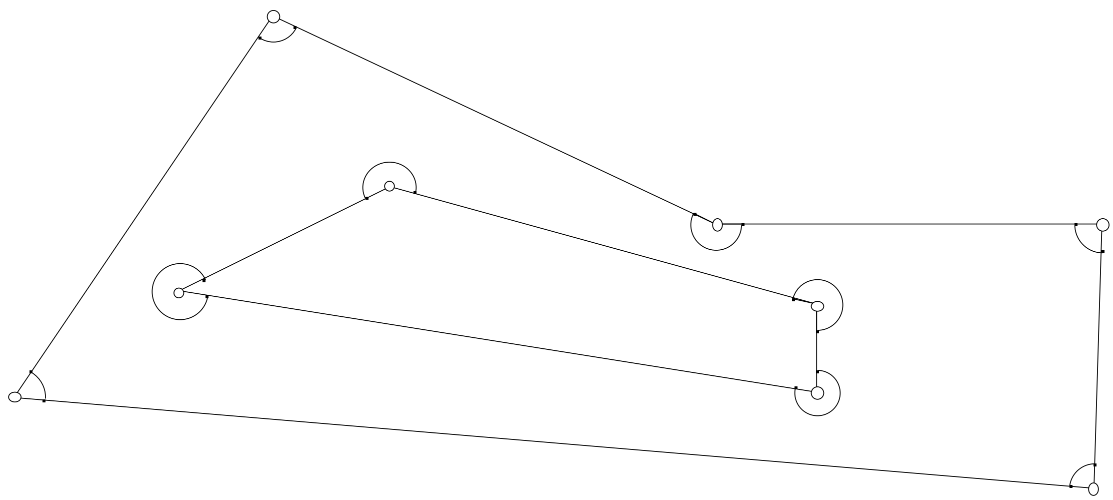
are piecewise linear figures as is the example of nested polygons below.



This example is a particularly interesting one because we can think of it as a figure containing a ‘hole’. But is it clear what is meant by the interior angles of such figures? For such a polygon as the following:



we obviously mean the angles indicated. But what about a piecewise linear figure containing holes? For the example above of nested polygons, we shall mean the angles indicated below



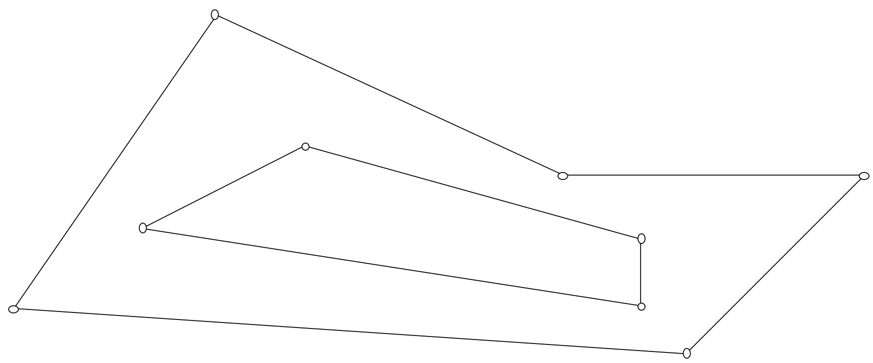
This makes sense because we are really thinking of the two polygons as enclosing a region so that *interior angle* then refers to the angle lying between two adjacent sides and inside the enclosed region. What this suggests is that for piecewise linear figures we will also need to specify what is meant by its *interior.*

The likely formula for the sum of the interior angles of piecewise linear figures can be obtained from Theorem 2.2.1 in conjunction with Sketchpad. In the case of polygons this was probably done in high school. For instance, the sum of the angles of any quadrilateral, *i.e.*, any four-sided figure, is 360°. To see this draw any diagonal of the quadrilateral thereby dividing the quadrilateral into two triangles. The sum of the angles of the quadrilateral is the sum of the angles of each of the two triangles and thus totals 360°. If the polygon has *n* sides, then it can be divided into *n*-2 triangles and the sum of the angles of the polygon is equal to the sum of the angles of the *n*-2 triangles. This proves the following result.

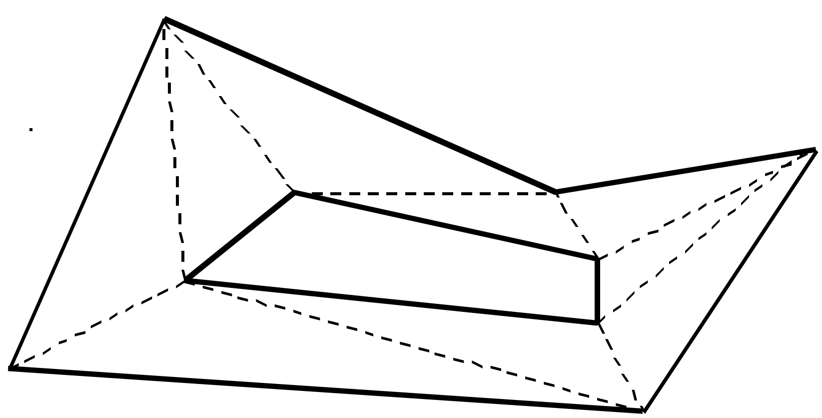
2.2.2 Theorem. The sum of the interior angles of an *n*-sided polygon, $n \geq 3$, is $(n-2)\ 180°$.

2.2.2a Demonstration.

We can use a similar method to determine the sum of the angles of the more complicated piecewise linear figures. One such figure is a polygon having "holes", that is, a polygon having other non-overlapping polygons (the holes) contained totally within its interior. Open a new sketch and draw a figure such as



An interesting computer graphics problem is to color in the piecewise linear figure between the two polygons. Unfortunately, computer graphics programs will only fill polygons and the interior of the figure is not a polygon. Furthermore, Sketchpad measures angles greater than 180° by using directed measurements. Thus Sketchpad would give a measurement of -90° for a 270° angle. To overcome the problem we use the same strategy as in the case of a polygon: *join enough of the vertices of the outer polygon to vertices on the inner polygon so that the region is sub-divided into polygons. Continue joining vertices until all of the polygons are triangles* as in the figure below. Color each of these triangles in a different color so that you can distinguish them easily.



We call this a *triangular tiling* of the figure. Now use Theorem 2.2.2 to compute the total sum of the angles of all these new polygons. Construct a different triangular tiling of the same figure and compute the total sum of angles again. Do you get the same value? Hence complete the following result.

2.2.3 Theorem. When an n-sided piecewise linear figure consists of a polygon with one polygonal hole inside it then the sum of its interior angles is ________________________.
Note: Here, n equals the number of sides of the outer polygon plus the number of sides of the polygonal hole.

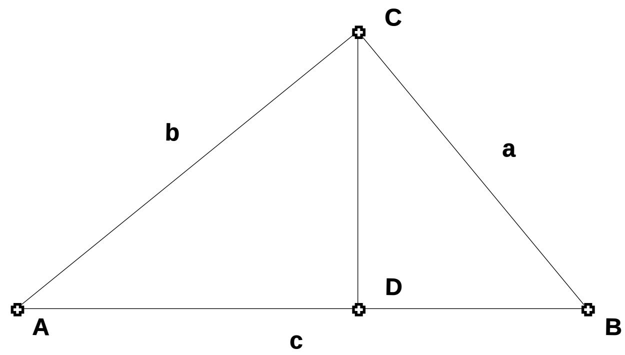
End of Demonstration 2.2.2a.

Try to prove Theorem 2.2.3 algebraically using Theorem 2.2.2. The case of a polygon containing h polygonal holes is discussed in Exercise 2.5.1.

2.3 SIMILARITY AND THE PYTHAGOREAN THEOREM

Of the many important applications of similarity, there are two that we shall need on many occasions in the future. The first is perhaps the best known of all results in Euclidean plane geometry, namely Pythagoras' theorem. This is frequently stated in purely algebraic terms as $a^2 + b^2 = c^2$, whereas in more geometrically descriptive terms it can be interpreted as saying that, *in area, the square built upon the hypotenuse of a right-angled triangle is equal to the sum of the squares built upon the other two sides*. There are many proofs of Pythagoras' theorem, some synthetic, some algebraic, and some visual as well as many combinations of these. Here you will discover an algebraic/synthetic proof based on the notion of similarity. Applications of Pythagoras' theorem and of its isosceles triangle version to decorative tilings of the plane will be made later in this chapter.

2.3.4 Theorem. (The Pythagorean Theorem) In any triangle containing a right angle, the square of the length of the side opposite to the right angle is equal to the sum of the squares of the lengths of the sides containing the right angle. In other words, if the length of the hypotenuse is c and the lengths of the other two sides are a and b, then $a^2 + b^2 = c^2$.
Proof: Let ΔABC be a right-angled triangle with right angle at C, and let $\overline{CD}$ be the perpendicular from C to the hypotenuse $\overline{AB}$ as shown in the diagram below.



- Show ΔCAB is similar to ΔDAC.
- Show ΔCAB is similar to ΔDCB.
- Now let $\overline{BD}$ have length x, so that $\overline{AD}$ has length $c - x$. By similar triangles,

$$\frac{x}{a} = \frac{a}{c} \quad \text{and} \quad \frac{c-x}{b} = ?$$

- Now eliminate x from the two equations to show $a^2 + b^2 = c^2$.

There is an important converse to the Pythagorean theorem that is often used.

2.3.5 Theorem. (Pythagorean Converse) Let ΔABC be a triangle such that $a^2 + b^2 = c^2$. Then ΔABC is right-angled with ACB a right angle.

2.3.5a Demonstration (Pythagorean Theorem with Areas)

You may be familiar with the geometric interpretation of Pythagoras' theorem. If we build squares on each side of ΔABC then Pythagoras' theorem relates the area of the squares.

- Open a new sketch and draw a right-angled triangle ΔABC. Using the 'Square By Edge' tool construct an outward square on each edge of the triangle having the same edge length as the side of the triangle on which it is drawn.
  - Measure the areas of these 3 squares: to do this select the vertices of a square and then construct its interior using "Construct Polygon Interior" tool. Now compute the area of each of these squares and then use the calculator to check that Pythagoras' theorem is valid for the right-angled triangle you have drawn.

End of Demonstration 2.3.5a.

This suggests a problem for further study because the squares on the three sides can be thought of as *similar* copies of the same piecewise linear figure with the lengths of the sides determining the edge length of each copy. So what does Pythagoras' theorem become when the squares on each side are replaced by, say, equilateral triangles or regular pentagons? In order to investigate, we will need tools to construct other regular polygons given one edge. If you haven't already done so, move the document called Polygons.gsp into the Tool Folder and restart Sketchpad or simply open the document to make its tools available.

2.3.5b Demonstration (Generalization of Pythagorean Theorem)

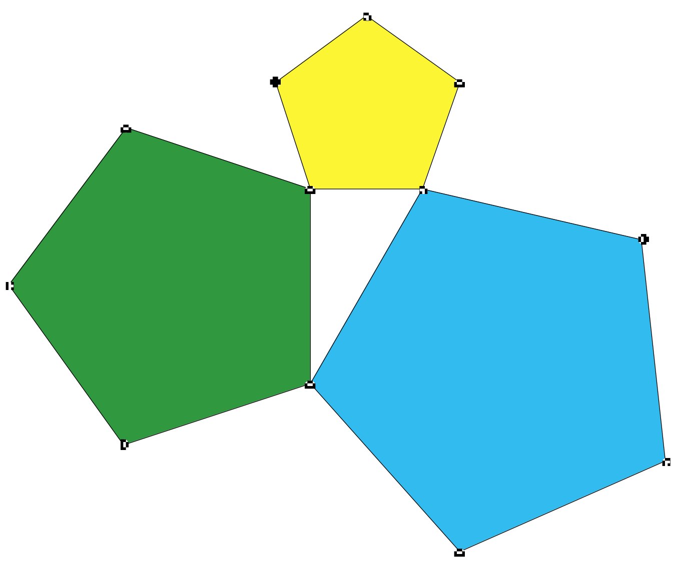
- Draw a new right-angled triangle ΔABC and use the '5/Pentagon (By Edge)' script to construct an outward regular pentagon on each side having the same edge length as the side of the triangle on which it is drawn. As before measure the area of each pentagon. What do you notice about these areas?
- Repeat these constructions for an octagon instead of a pentagon. (Note: You can create an "Octagon By Edge" script from your construction for **Exercise 1.3.5(b)**.) What do you notice about the areas in this case? Now complete the statement of **Theorem 2.3.6** below for regular n-gons.

End of Demonstration 2.3.5b.

2.3.6 Theorem. (Generalization of Pythagoras' theorem) When similar copies of a regular n-gon, $n \geq 3$, are constructed on the sides of a right-angled triangle, each n-gon having the same edge length as the side of the triangle on which it sits, then ____________________

____.

The figure below illustrates the case of regular pentagons.



2.3.7 Demonstration. Reformulate the result corresponding to Theorem 2.3.6 when the regular n-gons constructed on each side of a right-angled triangle are replaced by similar triangles.

This demonstration presents an opportunity to explain another feature of Custom Tools called Auto-Matching. We will be using this feature in Chapter 3 when we use Sketchpad to explore the Poincaré Disk model of the hyperbolic plane. In this problem we can construct the first isosceles triangle and then we would like to construct two other similar copies of the original one. Here we will construct a "similar triangle script" based on the AA criteria for similarity.

Tool Composition using Auto-Matching

- Open a new sketch and construct ΔABC with the vertices labeled.
- Next construct the line (not a segment) $\overleftrightarrow{DE}$.
- Select the vertices *B*-*A*-*C* in order and choose "Mark Angle *B*-*A*-C" from the **Transform Menu.** Click the mouse to deselect those points and then select the point *D*. Choose "Mark Center *D*" from the **Transform Menu.** Deselect the point and then select the line $\overleftrightarrow{DE}$. Choose "Rotate…" from the **Transform Menu** and then rotate by Angle *B*-*A*-*C*.
- Select the vertices *A*-*B*-*C* in order and choose "Mark Angle *A*-*B*-*C*" from the **Transform Menu.** Click the mouse to deselect those points and then select the point *E*. Choose "Mark Center *E*" from the **Transform Menu.** Deselect the point and then select the line $\overleftrightarrow{DE}$. Choose "Rotate…" from the **Transform Menu** and rotate by Angle *A*-*B*-*C*.
- Construct the point of intersection between the two rotated lines and label it *F*. ΔDEF is similar to ΔABC. Hide the three lines connecting the points *D*, *E*, and *F* and replace them with line segments.
- Now from the **Custom Tools** menu, choose **Create New Tool** and in the dialogue box, name your tool and check **Show Script View**. In the Script View, double click on the Given "Point *A*" and a dialogue box will appear. Check the box labeled **Automatically Match Sketch Object**. Repeat the process for points *B* and *C*.

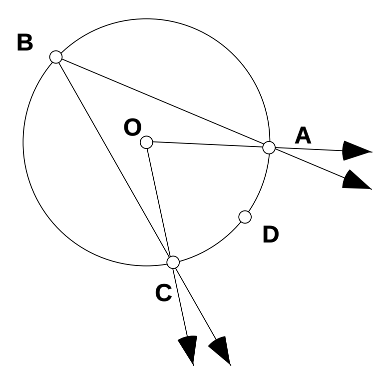
In the future, to use your tool, you need to have three points labeled *A*, *B*, and *C* already constructed in your sketch where you want to construct the similar triangle. Then you only need to click on or construct the points corresponding to *D* and *E* each time you want to use the script. Your script will automatically match the points labeled *A*, *B*, and *C* in your sketch with those that it needs to run the script. Notice in the Script View that the objects which are automatically matched are now listed under “Assuming” rather than under “Given Objects”. If there are no objects in the sketch with labels that match those in the Assuming section, then Sketchpad will require you to match those objects manually, as if they were “Given Objects.”

· Now open a new sketch and construct a triangle with vertices labeled *A*, *B*, and *C*.

· In the same sketch, construct a right triangle. Use the “similar triangle” tool to build triangles similar to ΔABC on each side of the right triangle. For each similar triangle, select the three vertices and then in the Construct menu, choose “construct polygon interior”. Measure the areas of the similar triangles and see how they are related.

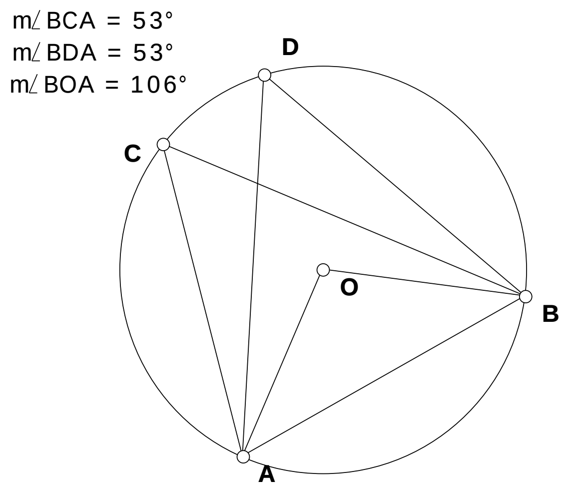
End of Demonstration 2.3.7.

2.4 INSCRIBED ANGLE THEOREM: One of the most useful properties of a circle is related to an angle that is inscribed in the circle and the corresponding subtended arc. In the figure below, ABC is inscribed in the circle and *Arc ADC* is the subtended arc. We will say that AOC is a central angle of the circle because the vertex is located at the center O. The measure of *Arc ADC* is defined to be the angle measure of the central angle, AOC.



2.4.0 Demonstration. Investigate the relationship between an angle inscribed in a circle and the arc it intercepts (subtends) on the circle.

· Open a new script in *Sketchpad* and draw a circle, labeling the center of the circle by *O*.



- Select an arbitrary pair of points A, B on the circle. These points specify two possible arcs - let's choose the shorter one in the figure above, that is, the arc which is subtended by a central angle of measure less than 180°. Now select another pair of points C, D on the circle and draw line segments to form $\angle BCA$ and $\angle BDA$. Measure these angles. What do you observe?
- If you drag C or D what do you observe about the angle measures? Now find the angle measure of $\angle BOA$. What do you observe about its value?
- Drag B until the line segment $\overline{AB}$ passes through the center of the circle. What do you now observe about the three angle measures you have found?

Use your observations to complete the following statements; proving them will be part of later exercises.

2.4.1 Theorem. (Inscribed Angle Theorem): The measure of an inscribed angle of a circle equals ______________________ that of its intercepted (or subtended) arc.

2.4.2 Corollary. A diameter of a circle always inscribes ______________________ at any point on the circumference of the circle.

2.4.3 Corollary. Given a line segment $\overline{AB}$, the locus of a point P such that $\angle APB = 90^\circ$ is a circle having $\overline{AB}$ as diameter.

End of Demonstration 2.4.0.

The result you have discovered in Corollary 2.4.2 is a very useful one, especially in constructions, since it gives another way of constructing right-angled triangles. Exercises 2.5.4 and 2.5.5 below are good illustrations of this. The Inscribed Angle Theorem can also be used to prove the following theorem, which is useful for proving more advanced theorems.

2.4.4 Theorem. A quadrilateral is inscribed in a circle if and only if the opposite angles are supplementary. (A quadrilateral that is inscribed in a circle is called a *cyclic quadrilateral*.)

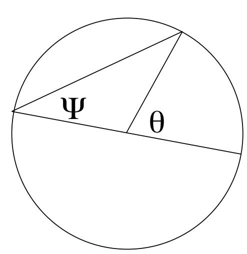
2.5 Exercises

Exercise 2.5.1. Consider a piecewise linear figure consisting of a polygon containing h holes (non-overlapping polygons in the interior of the outer polygon) has a total of n edges, where n includes both the interior and the exterior edges. Express the sum of the interior angles as a function of n and h. Prove your result is true.

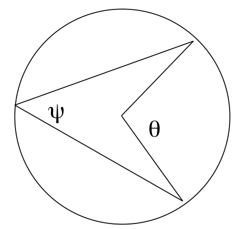
Exercise 2.5.2. Prove that if a quadrilateral is cyclic, then the opposite angles of the quadrilateral are supplementary, *i.e.*, the sum of opposite angles is 180°. [This will provide half of the proof of Theorem 2.4.4.]

Exercise 2.5.3. Give a synthetic proof of the Inscribed Angle Theorem 2.4.1 using the properties of isosceles triangles in Theorem 1.4.6. Hint: there are three cases to consider: here ψ is the angle subtended by the arc and θ is the angle subtended at the center of the circle. The problem is to relate ψ to θ.

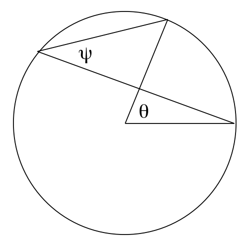
Case 1: The center of the circle lies on the subtended angle.



Case 2: The center of the circle lies within the interior of the inscribed circle.



Case 3: The center of the circle lies in the exterior of the inscribed angle.

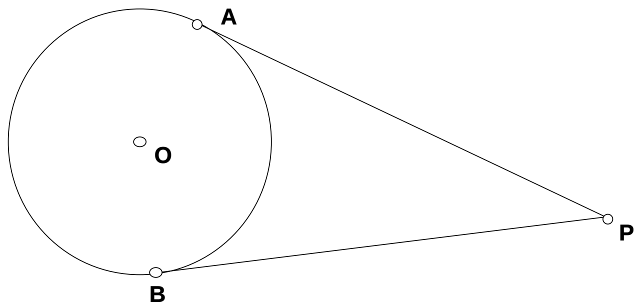


End of Exercise 2.5.3.

For Exercises 2.5.4, 2.5.5, and 2.5.6, recall that any line tangent to a circle at a particular point must be perpendicular to the line connecting the center and that same point. For all three of these exercises, the Inscribed Angle Theorem is useful.

Exercise 2.5.4. Use the Inscribed Angle Theorem to devise a Sketchpad construction that will construct the tangents to a given circle from a given point P outside the circle. Carry out your construction. (Hint: Remember Corollary 2.4.2).

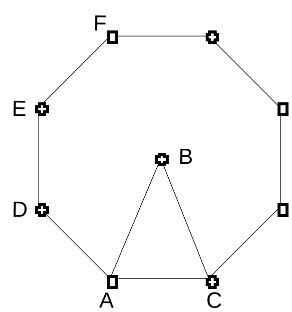
Exercise 2.5.5. In the following figure



the line segments $\overline{PA}$ and $\overline{PB}$ are the tangents to a circle centered at O from a point P outside the circle. Prove that $\overline{PA}$ and $\overline{PB}$ are congruent.

Exercise 2.5.6. Let l and m be lines intersecting at some point P and let Q be a point on l. Use the result of Exercise 2.5.5 to devise a Sketchpad construction that constructs a circle tangential to l and m that passes through Q. Carry out your construction.

For Exercises 2.5.7 and 2.5.8, we consider regular polygons again, that is, polygons with all sides congruent and all interior angles congruent. If a regular polygon has n sides we shall say it is a regular n-gon. For instance, the following figure



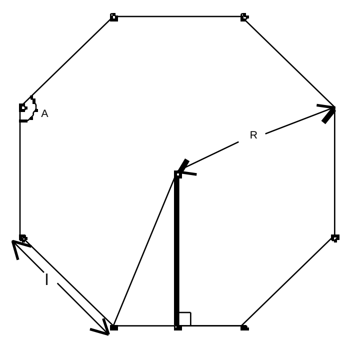
is a regular octagon above, *i.e.*, a regular 8-gon. By Theorem 2.2.2 the interior angle of a regular n-gon is

$$\left(\frac{n-2}{n}\right)180^{\circ}.$$

The measure of any central angle is $\frac{360°}{n}$. In the figure DEF is an interior angle and ABC is a central angle.

Exercise 2.5.7. Prove that the vertices of a regular polygon always lie on a circumscribing circle. (Be careful! Don't assume that your polygon has a center; you must *prove* that there is a point equidistant from all the vertices of the regular polygon.)

Exercise 2.5.8. Now suppose that the edge length of a regular n-gon is l and let R be the *radius* of the circumscribing circle for the n-gon. The *Apothem* of the n-gon is the perpendicular distance from the center of the circumscribing circle to a side of the n-gon.



The Apothem

(a) With this notation and terminology and using some trigonometry complete the following

$R = l$ ______________ , $l = R$ ______________, *Apothem* $= R$___________ .

Use this to deduce

(b) area of n-gon $= \frac{1}{2} nR^2 \sin\left(\frac{2\pi}{n}\right)$, (c) perimeter of n-gon $= 2nR \sin\left(\frac{\pi}{n}\right)$.

(d) Then use the well-known fact from calculus that

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

to derive the formulas for the area of a circle of radius R as well as the circumference of such a circle.

Exercise 2.5.9. Use Exercise 2.5.8 together with the usual version of Pythagoras' theorem to give an algebraic proof of the generalized Pythagorean Theorem (Theorem 2.3.6).

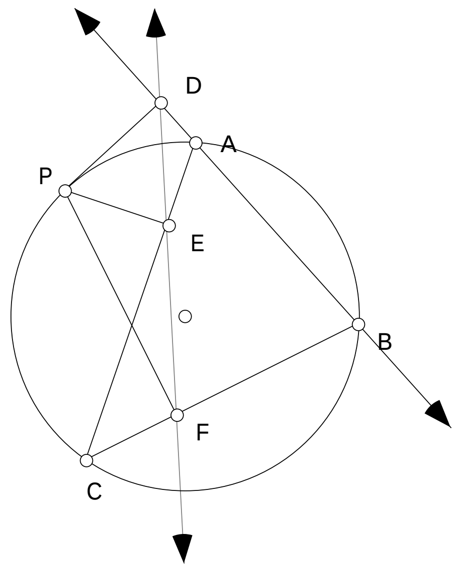
Exercise 2.5.10 Prove the converse to the Pythagorean Theorem stated in Theorem 2.3.5.

2.6 RESULTS REVISITED. In this section we will see how the Inscribed Angle Theorem can be used to prove results involving the Simson Line, the Miquel Point, and the Euler Line. Recall that we discovered the Simson Line in Section 1.8 while exploring Pedal Triangles.

2.6.1 Theorem (Simson Line). If P lies on the circumcircle of ΔABC, then the perpendiculars from P to the three sides of the triangle intersect the sides in three collinear points.

Proof. Use the notation in the figure below.

- Why do $P, D, A,$ and E all lie on the same circle? Why do $P, A, C,$ and B all lie on another circle? Why do P, D, B and F all lie on a third circle? Verify all three of these statements using Sketchpad.



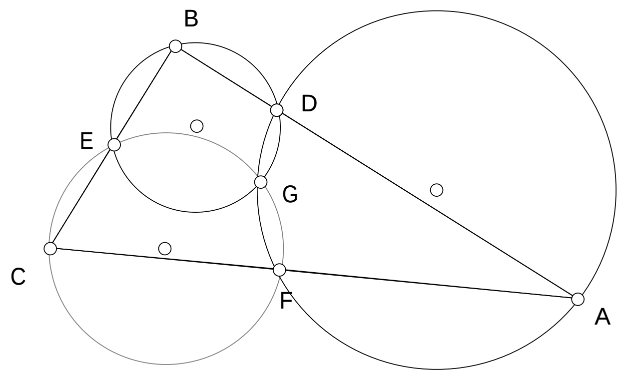
- In circle $PDAE$, $m \quad PDE \quad m \quad PAE = m \quad PAC$. Why?
- In circle $PACB$, $m \quad PAC \quad m \quad PBC = m \quad PBF$. Why?
- In circle $PDBF$, $m \quad PBF \quad m \quad PDF$. Why?

Since $m \quad PDE \quad m \quad PDF$, points $D, E,$ and F must be collinear. **QED**

In Exercise 1.9.4, the Miquel Points of a triangle were constructed.

2.6.2 Theorem (Miquel Point) If three points are chosen, one on each side of a triangle, then the three circles determined by a vertex and the two points on the adjacent sides meet at a point called the Miquel Point.

Proof. Refer to the notation in the figure below.



Let D, E and F be arbitrary points on the sides of ΔABC. Construct the three circumcircles. Suppose the circumcircles for ΔAFD and ΔBDE intersect at point G. We need to show the third circumcircle also passes through G. Now, G may lie inside ΔABC, on ΔABC, or outside ΔABC. We prove the theorem here in the case that G lies inside ΔABC, and leave the other two cases for you (see Exercise 2.8.1).

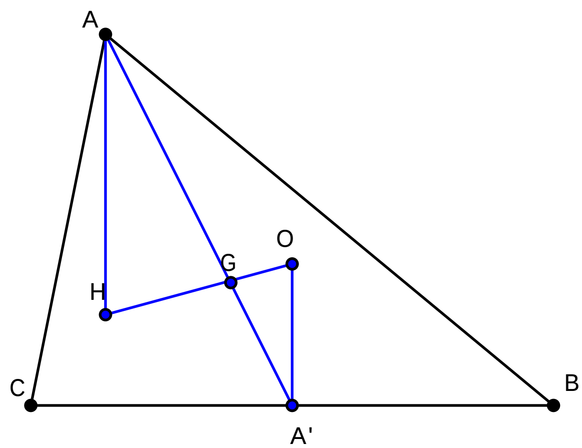
- FGD and DAF are supplementary. Why?
- EGD and DBE are supplementary. Why?

Notice $m\ \ FGD + m\ \ DGE + m\ \ EGF = 360°$. Combining these facts we see the following. $(180° - m\ \ A) + (180° - m\ \ B) + m\ \ EGF = 360°$. So $m\ \ EGF = 180 - m\ \ C$ or C and EGF are supplementary. Thus $C, E, G,$ and F all lie on a circle and the third circumcircle must pass through G. **QED**

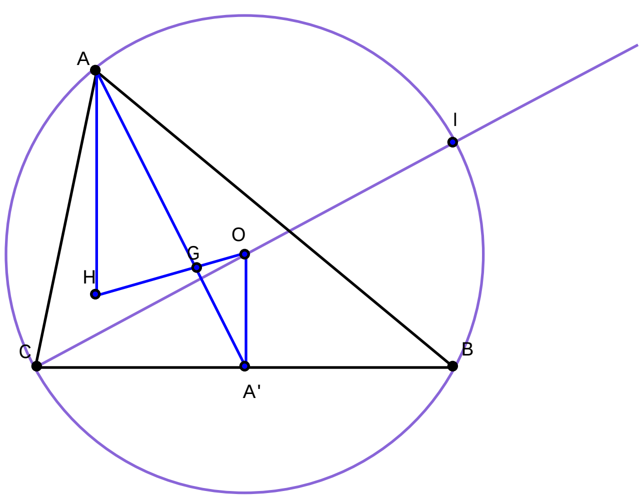
The proof of Theorem 2.6.3 below uses two results on the geometry of triangles, which were proven in Chapter 1. The first result states that the line segment between the midpoints of two sides of a triangle is parallel to the third side of the triangle and it is half the length of the third side (see **Corollary 1.5.4**). The second results states that the point which is 2/3 the distance from a vertex (along a median) to the midpoint of the opposite side is the centroid of the triangle (see **Theorem 1.5.6**).

2.6.3 Theorem (Euler Line). For any triangle, the centroid, the orthocenter, and the circumcenter are collinear, and the centriod trisects the segment joining the orthocenter and the circumcenter. The line containing the centroid, orthocenter, and circumcenter of a triangle is called the **Euler Line**.

Proof. In the diagram below, A' is the midpoint of the side opposite to A and O, G, and H are the circumcenter, centroid, and orthocenter, respectively. Since A, G, and A' are collinear, we can show that O, G, and H are also collinear, by showing that $AGH \quad A'GO$. To do this, it suffices to show that $\Delta AHG \sim \Delta A'OG$. If we also show that the ratio of similiarity is 2:1, then we will also prove that G trisects $\overline{OH}$.



The proof that $\Delta AHG \sim \Delta A'OG$ with ratio 2:1 proceeds as follows: Let I be the point where the ray $\overrightarrow{CO}$ intersects the circumcircle of ΔABC. Then $\overline{IB} \quad \overline{CB}$ (why?). It follows that $\Delta BCI \sim \Delta A'CO$ with ratio 2:1 (why?) It is also true that $AIBH$ is a parallelogram (why?) and



hence $AH = IB = 2(OA')$. Since G is the median, we know that $AG = 2(GA')$. Thus we have two corresponding sides proportional. The included angles are congruent because they are alternate interior angles formed by the parallel lines $\overleftrightarrow{AH}$ and $\overleftrightarrow{OA'}$ and the transversal $\overleftrightarrow{AA'}$. (Why are $\overleftrightarrow{AH}$ and $\overleftrightarrow{OA'}$ parallel?) Thus, $\Delta AHG \sim \Delta A'OG$ with ratio 2:1 by SAS.

Of course, as we noted in Chapter 1, we must be careful not to rely too much on a picture when proving a theorem. Use Sketchpad to find examples of triangles for which our proof breaks down, i.e. triangles in which we can't form the triangles ΔAHG and $\Delta A'OG$. What sorts of triangles arise? You should find two special cases. Finish the proof of Theorem 2.6.3 by proving the result for each of these cases (see **Exercise 2.8.2**).

2.7 THE NINE POINT CIRCLE. Another surprising triangle property is the so-called Nine-Point Circle, sometimes credited to K.W. Feuerbach (1822). Sketchpad is particularly well adapted to its study. The following Demonstration will lead you to its discovery.

2.7.0 Demonstration: Investigate the nine points on the Nine Point Circle.

The First set of Three points:

- In a new sketch construct ΔABC. Construct the midpoints of each of its sides; label these midpoints *D*, *E*, and *F*.
- Construct the circle that passes through *D*, *E*, and *F*. (You know how to do this!)
- This circle is called the *Nine-Point Circle*. Complete the statement: The nine-point circle passes through ________________________________.

The Second set of Three points: In general the nine-point circle will intersect ΔABC in three more points. If yours does not, drag one of the vertices around until the circle does intersect ΔABC in three other points. Label these points *J*, *K*, and *L*.

- Construct the line segment joining *J* and the vertex opposite *J*. Change the color of this segment to red. What is the relationship between the red segment and the side of the triangle containing *J*? What is an appropriate name for the red segment?
- Construct the corresponding segment joining *K* and the vertex opposite *K* and the segment joining *L* to the vertex opposite *L*. Color each segment red. What can you say about the three red segments?
- Place a point where the red segments meet; label this point *M* and complete the following statement: The nine-point circle also passes through ____________________________.

.

The Third set of Three points: The red segments intersect the circle at their respective endpoints ($J, K,$ or L). For each segment there exists a second point where the segment intersects the circle. Label them N, O and P.

- To describe these points measure the distance between M and each of $A, B,$ and C. Measure also the distance between M and each of $N, O,$ and P. What do you observe? Confirm your observation by dragging the vertices of ΔABC.
- Complete the following statement: The nine-point circle also passes through ___________

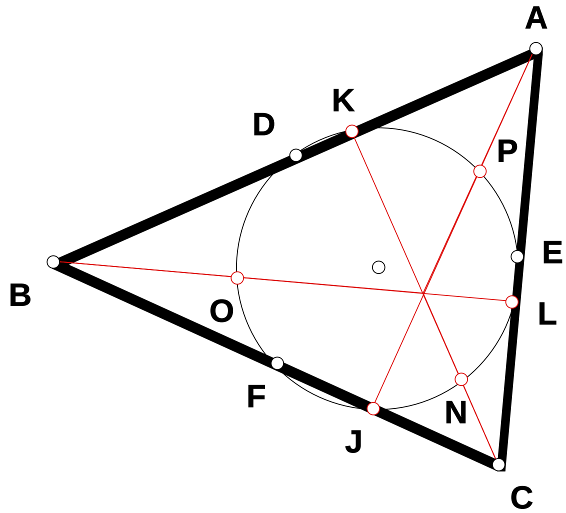
You should create a Nine Point Circle tool from this sketch and save it for future use.

End of Demonstration 2.7.0.

To understand the proof of Theorem 2.7.1 below, it is helpful to recall some results discussed earlier. As in the proof of the existence of the Euler Line, it is necessary to use the fact that the segment connecting the midpoints of two sides of a triangle is parallel to the third side of the triangle. Also, we recall that a quadrilateral can be inscribed in a circle if and only if the opposite angles in the quadrilateral are supplementary. It is not difficult to show that an isosceles trapezoid has this property. Finally, recall that a triangle can be inscribed in a circle with a side of the triangle coinciding with a diameter of the circle if and only if the triangle is a right triangle.

2.7.1 Theorem (The Nine-point Circle) The midpoints of the sides of a triangle, the points of intersection of the altitudes and the sides, and the midpoints of the segments joining the orthocenter and the vertices all lie on a circle called the nine-point circle.

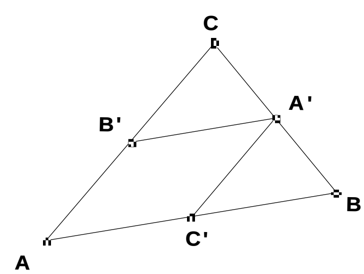
Your final figure should be similar to



Proof:

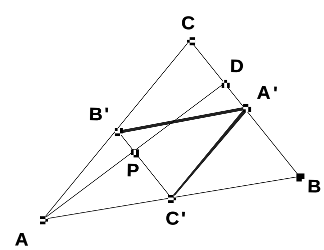
(See Figure 1) In ΔABC label the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, by A', B' and C' respectively. There is a circle containing A', B' and C'. In addition, we know $A'C'AB'$ is a parallelogram, and so $A'C' = AB'$.

Figure 1



(See Figure 2) Construct the altitude from A intersecting $\overline{BC}$ at D. As $\overline{C'B'}$ is parallel to $\overline{BC}$ and $\overline{AD}$ is perpendicular to $\overline{BC}$, then $\overline{AD}$ must be perpendicular to $\overline{B'C'}$. Denote the intersection of $\overline{B'C'}$ and $\overline{AD}$ by P. Then APB' DPB', $\overline{PB'}$ $\overline{PB'}$ and $\overline{AP}$ $\overline{DP}$.

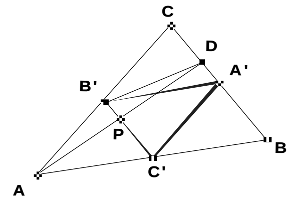
Figure 2



(See Figure 3) Consequently, $\Delta APB'$ $\Delta DPB'$ by SAS. So $AB' = B'D$. By transitivity with $A'C' = AB'$ we have $B'D = A'C'$. Thus $A'C'B'D$ is an isosceles

Figure 3

trapezoid. Hence, by the remarks preceding this theorem, A', C', B', and D are points which lie on one circle. (See Figure 4)



By a similar argument, the feet of the other two altitudes belong to this circle.

Figure 4

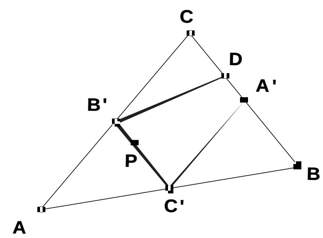
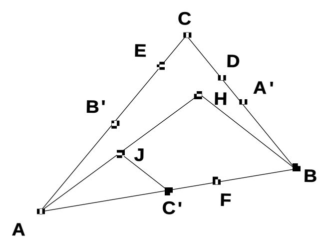


Figure 5

(See Figure 5) Let J denote the midpoint of the segment joining vertex A and the orthocenter H. Then, again by the connection of midpoints of the sides of a triangle, $\overline{C'J}$ is parallel to $\overline{BH}$.



(See Figure 6) Now $\overline{C'A'} \,||\, \overline{AC}$ *and* $\overline{AC}$ $\overline{BH}$ but $\overline{\text{BH}} \,||\, \overline{\text{C' J}}$. Hence $\overline{C'A'}$ $\overline{C'J}$.

Figure 6

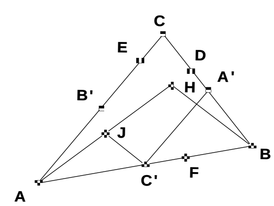
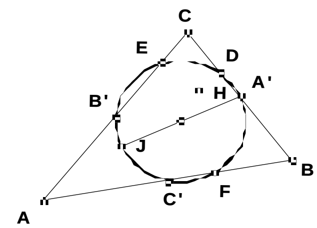


Figure 7

(See Figure 7) Therefore C' lies on a circle with diameter $\overline{A'J}$.

A similar argument shows that B' lies on the circle with diameter $\overline{A'J}$, and hence J lies on the circle determined by A', B', and C'. Likewise, the other two midpoints of the segments joining the vertices with the orthocenter lie on the same circle.



QED

2.8 Exercises. In this exercise set, Exercise 2.8.3 – 2.8.7 are related to the nine point circle.

Exercise 2.8.1. Using Sketchpad, illustrate a case where the Miquel Point lies outside the triangle. Prove Theorem 2.6.2 in this case.

Exercise 2.8.2. Prove Theorem 2.6.3 for the two special cases:

(a) The triangle is isosceles.

(b) The triangle is a right triangle.

Exercise 2.8.3. For special triangles some points of the nine-point circle coincide. Open a new sketch and draw an arbitrary ΔABC. Explore the various possibilities by dragging the vertices of ΔABC. Describe the type of triangle (if it exists) for which the nine points of the nine-point circle reduce to:

4 points: ______________________ 5 points: _________________________

6 points: ______________________ 7 points: _________________________

8 points: ______________________

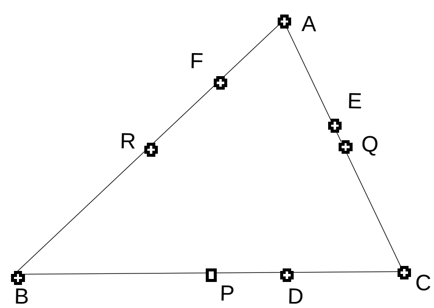
Exercise 2.8.4. Open a new sketch and draw an arbitrary triangle ΔABC.

- Construct the circumcenter O, the centroid G, the orthocenter H, and the center of the nine-point circle N for this triangle. What do you notice?
- Measure the length of $\overline{ON}$, $\overline{NH}$, $\overline{NG}$, and $\overline{OH}$. What results for a general triangle do your calculations suggest?
- Measure the radius of the nine-point circle of ΔABC. Measure the radius of the circumcircle of ΔABC. What results for a general triangle do your calculations suggest?

Drag the vertices of the triangle around. Do your conjectures still remain valid?

Exercise 2.8.5. Open a new sketch and draw an arbitrary ΔABC. Let H be the orthocenter and O be the circumcenter of ΔABC. Construct the nine-point circles for ΔOHA, ΔOHB, and ΔOHC. Use sketchpad to show that these nine-point circles have two points in common. Can you identify these points? Check your observation by dragging the vertices A, B, and C around

If one starts with given vertices A, B, and C, then the locations of the midpoints P, Q, and R of the sides of ΔABC are uniquely determined. Similarly, the locations of the feet of the altitudes D, E, and F will be determined once A, B, and C are given. The remaining two problems in this exercise set use the geometric properties we have developed so far to reverse this process, *i.e.*, we construct the vertices A, B, and C knowing the midpoints or the feet of the altitudes. Use the notation from the following figure.



Exercise 2.8.6.

(a) Prove the line segment $\overline{PQ}$ is parallel to side $\overline{AB}$.

(b) Given points P, Q, and R, show how to construct points A, B, and C so that P, Q, and R are the midpoints of the sides of ΔABC.

(c) Formulate a conjecture concerning the relation between the centroid G of ΔABC and the centroid of ΔPQR.

Exercise 2.8.7.

(a) Assume ΔABC is acute (to ensure the feet of the altitudes lie on the sides of the triangle). Prove that $PC = PB = PE = PF$ and that P lies on the perpendicular bisector of the line segment $\overline{EF}$.

(b) Given points D, E, and F, show how to construct points A, B, and C so that D, E, and F are the feet of the altitudes from the vertices of ΔABC to the opposite sides. (Hint: remember the nine-point circle).

2.9 THE POWER OF A POINT AND SYNTHESIZING APOLLONIUS. Another application of similarity will be to a set of ideas involving what is often called the *power of a point with respect to a circle*. The principal result will be decidedly useful later in connection with the theory of inversion and its relation to hyperbolic geometry.

Demonstration 2.9.0. Discover the formula for *the power of point P with respect to a given circle*.

- Open a new sketch and draw a circle. Select any point P outside the circle and let A, B be the points of intersection on the circle of a line l through P.
- Compute the lengths PA, PB of $\overline{PA}$, and $\overline{PB}$ respectively; then compute the product $PA{\cdot}PB$ of PA and PB. Drag l while keeping P fixed. What do you observe?
- Investigate further by considering the case when l is tangential to the given circle. Use this to explain your previous observation.
- What happens to the product $PA{\cdot}PB$ when P is taken as a point on the circle?
- Now let P be a point inside the circle, l a line through P and A, B its points of intersection with the circle. Again compute the product $PA{\cdot}PB$ of PA and PB. Now vary l.
- Investigate further by considering the case when l passes through the center of the given circle.
- Can you reconcile the three values of the product $PA{\cdot}PB$ for P outside, on and inside the given circle? Hint: consider the value of $OP^2 - r^2$ where O is the center of the given circle and r is its radius.

End of Demonstration 2.9.0.

The value of $PA{\cdot}PB$ in Demonstration 2.9.0 is often called the *power of P with respect to the given circle.* Now complete the following statement.

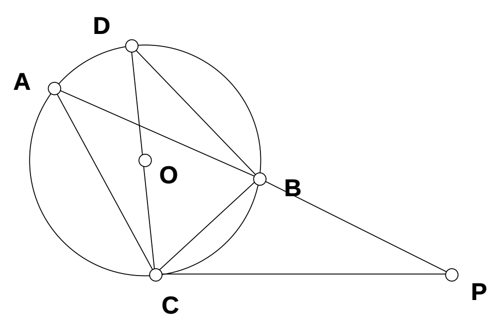
2.9.1 Theorem. Let P be a given point, Σ a given circle, and l a line through P intersecting Σ at A and B. Then

1. the product $PA{\cdot}PB$ of the distances from P to A and B is ________________ whenever P is outside, whenever it is inside or when it is on Σ;
2. the value of the product $PA{\cdot}PB$ is equal to _________________ where O is the center of Σ and r is the radius of Σ.

The proof of part 2 of Theorem 2.9.1 in the case when P is outside the given circle is an interesting use of similarity and the inscribed angle theorem. In the diagram below let C be a point on the circle such that $\overline{PC}$ is a tangent to the circle. By the Pythagorean Theorem $OP^2 - r^2 = PC^2$ so it suffices to show that $PA{\cdot}PB = PC^2$.

2.9.2 Theorem. Given a circle Σ and a point P outside Σ, let l be a ray through P intersecting Σ at points A and B. If C is a point on the circle such that $\overline{PC}$ is a tangent to Σ at C then $PA{\cdot}PB = PC^2$.

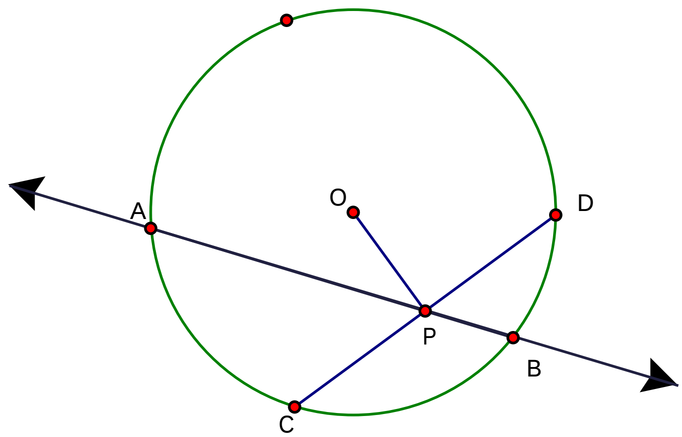
Proof. The equation $PA{\cdot}PB = PC^2$ suggests use of similar triangles, but which ones?



Let $\overline{CD}$ be a diameter of the circle. By the Inscribed Angle Theorem $m\ \ PAC = m\ \ BDC$ and CBD is a right angle. Thus $m\ \ BDC + m\ \ DCB = 90°$ and as $\overline{PC}$ is tangent to the circle $m\ \ DCB + m\ \ PCB = 90°$. Therefore, $m\ \ PAC = m\ \ PCB$. By AA similarity ΔPAC is similar to ΔPCB proving $PA/PC = PC/PB$ or $PA{\cdot}PB = PC^2$. **QED**

Theorem 2.9.3. Given a circle Σ and a point P inside Σ, let *l* be a line through *P* intersecting Σ at points *A* and *B*. Let $\overline{CD}$ be the chord perpendicular to the segment $\overline{OP}$. Then the value of the product $PA{\cdot}PB$ is equal to $r^2 - OP^2 = PC^2$ where *O* is the center of Σ and *r* is the radius of Σ.

Proof.



By AA similarity ΔACP is similar to ΔDBP so that $PA/PC = PD/PB$. Thus $PA \cdot PB = PC \cdot PD$. By HL, ΔCPO is congruent to ΔDPO so that $PC = PD$. By the Pythagorean Theorem $PD^2 + OP^2 = OD^2$. Re-arranging and substituting, we obtain $PC\ \ PD = r^2 - OP^2$. Therefore, $PA\ \ PB = r^2 - OP^2$ as desired. **QED**

There is a converse to theorem 2.9.2 that also will be useful later. You will be asked to provide the proof in Exercise 2.11.1 below.

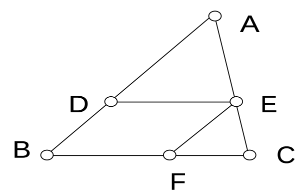
2.9.4 Theorem. Given a circle Σ and a point *P* outside Σ, let *l* be a ray through *P* intersecting Σ at points *A* and *B*. If *C* is a point on Σ such that $PA{\cdot}PB = PC^2$, then $\overline{PC}$ is a tangent to Σ at *C*.

In Chapter 1 we used Sketchpad to discover that when a point *P* moves so that the distance from *P* to two fixed points *A*, *B* satisfies the condition $PA = 2PB$ then the path traced out by *P* is a circle. In fact, the locus of a point *P* such that $PA = mPB$ is always a circle, when *m* is any positive constant not equal to one. From restorations of Apollonius' work '*Plane Loci* ' we infer that he considered this locus problem, now called the "Circle of Apollonius". However, this is a misnomer since Aristotle who had used it to give a mathematical justification of the semicircular form of the rainbow had already known the result.

That this locus is a circle was confirmed algebraically using coordinate geometry in Chapter 1. However, it can be also be proven by synthetic methods and the synthetic proof exploits properties of similar triangles and properties of circles. Since the synthetic proof will suggest how we can construct the *Circle of Apollonius* with respect to fixed points A, B through an arbitrary point P we shall go through the proof now. The proof requires several lemmas, which we consider below.

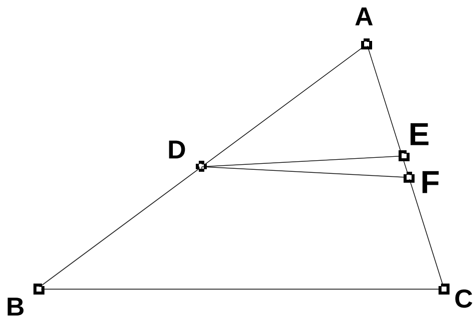
2.9.5 Lemma Given ΔABC, let D be on $\overline{AB}$, and E on $\overline{AC}$ such that $\overline{DE}$ is parallel to $\overline{BC}$. Then

$$\frac{AD}{DB} = \frac{AE}{EC} \text{ and } \frac{AB}{DB} = \frac{AC}{EC}.$$



Proof. Let F be the intersection of $\overline{BC}$ with the line parallel to $\overline{AB}$ passing through E. Then $\Delta AED \sim \Delta ECF$ by AA similarity and $\frac{AD}{EF} = \frac{AE}{EC}$. The quadrilateral $EFBD$ is a parallelogram, therefore $EF=DB$ and $\frac{AD}{DB} = \frac{AE}{EC}$. A similar argument shows $\frac{AB}{DB} = \frac{AC}{EC}$. **QED**

2.9.5a Lemma (Converse of Lemma 2.9.5). Given ΔABC, let D be on $\overline{AB}$, and E on $\overline{AC}$ such that $\frac{AD}{DB} = \frac{AE}{EC}$ or $\frac{AB}{DB} = \frac{AC}{EC}$ (see figure below), then $\overline{DE}$ is parallel to $\overline{BC}$.



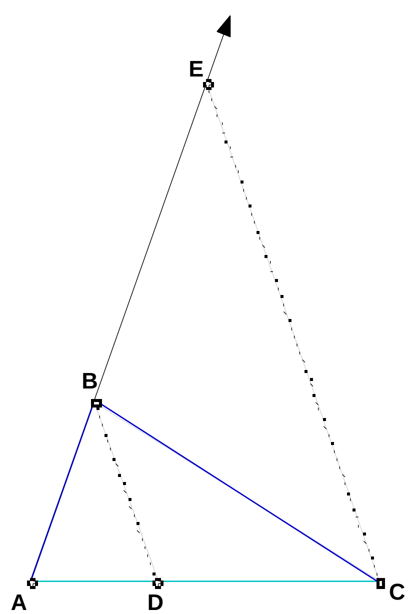
Proof. Assume $\frac{AB}{DB} = \frac{AC}{EC}$. The line through D parallel to $\overline{BC}$ intersects $\overline{AC}$ at point F with $\overline{DF}$ parallel to $\overline{BC}$. By Lemma 2.9.5, $\frac{AB}{DB} = \frac{AC}{FC}$. But $\frac{AB}{DB} = \frac{AC}{EC}$ also, so $\frac{AC}{FC} = \frac{AC}{EC}$ which implies that $F = E$. Thus $\overline{DE} = \overline{DF}$ is parallel to $\overline{BC}$.

If $\frac{AD}{DB} = \frac{AE}{EC}$, the proof is similar. **QED**

2.9.6 Theorem The bisector of the internal angle ABC of ΔABC divides the opposite side $\overline{AC}$ in the ratio of the adjacent sides $\overline{BA}$ and $\overline{BC}$. In other words,

$$\frac{AD}{DC} = \frac{AB}{BC}.$$

Proof. Suppose $\overline{BD}$ bisects ABC in ΔABC. At C construct a line parallel to $\overline{BD}$, intersecting $\overline{AB}$ at E, producing the figure below.



But then ABD CBD and BEC ABD since they are corresponding angles of parallel lines. In addition, BCE CBD since they are alternate interior angles of parallel lines. Hence ΔCBE is isosceles and $BE = BC$. By the previous lemma

$$\frac{AB}{BE} = \frac{AD}{DC}.$$

But $BE = BC$, so

$$\frac{AB}{BC} = \frac{AD}{DC}.$$

This completes the proof.

QED

2.9.7 Exercise. The converse to Theorem 2.9.6 states that if

$$\frac{AB}{BC} = \frac{AD}{DC},$$

then $\overline{BD}$ bisects ABC in the figure above. Prove this converse. You may use the converse to Lemma 2.9.5, proven in Lemma 2.9.5a.

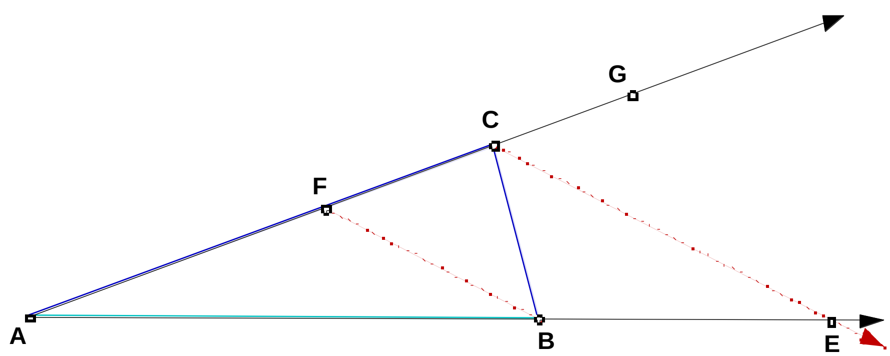
2.9.8. Theorem The bisector of an external angle of ΔABC cuts the extended opposite side at a point determined by the ratio of the adjacent sides. That is to say, if $\overline{AB}$ is extended and intersects the line containing the bisector of the exterior angle of C at E, then

$$\frac{AC}{BC} = \frac{AE}{BE}.$$

Proof: There are two cases to consider.

Either m BAC < m ABC or m BAC > m ABC. (If m BAC =m ABC, then the bisector of the exterior angle at C is parallel to $\overline{AB}$.)

Assume that m BAC < m ABC. Then (as shown in the figure) the bisector of BCG will intersect the extension of $\overline{AB}$ at E, and $AE > AB$. At B, construct a line parallel to $\overline{CE}$, intersecting $\overline{AC}$ at F.



Then BFC ECG since they are corresponding angles of parallel lines;

And ECG BCE since $\overline{CE}$ bisects BCG; and BCE CBF since they are alternate interior angles of parallel lines.

Hence ΔBFC is isosceles and $FC = BC$. Now by a previous lemma, $\dfrac{AC}{FC} = \dfrac{AE}{BE}$. But $FC = BC$; so $\dfrac{AC}{BC} = \dfrac{AE}{BE}$.

This proves the assertion for the case when m BAC < m ABC.

If m BAC > m ABC, then the line containing the bisector of BCG intersects the extension of $\overline{AB}$ at point E on the other side of A, with A between E and B. A similar argument proves the assertion for this case as well and the theorem is proved. **QED**

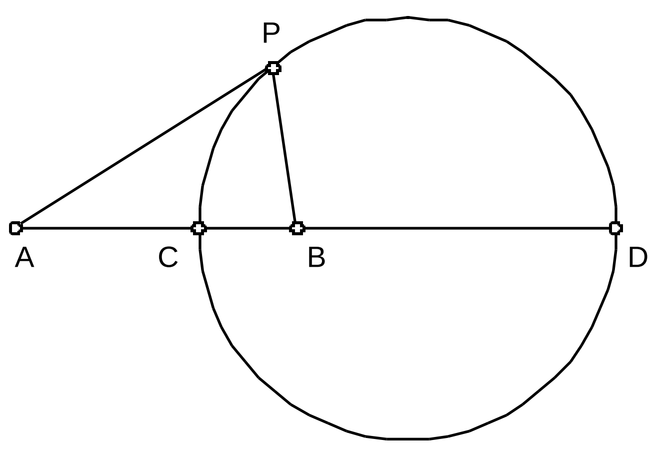
2.9.9 Exercise. The converse to Theorem 2.9.8 states that if

$$\frac{AC}{BC} = \frac{AE}{BE}$$

in the figure above, then $\overline{CE}$ bisects the external angle of ΔABC at C. Prove this conjecture.

We are now able to complete the proof of the main theorem.

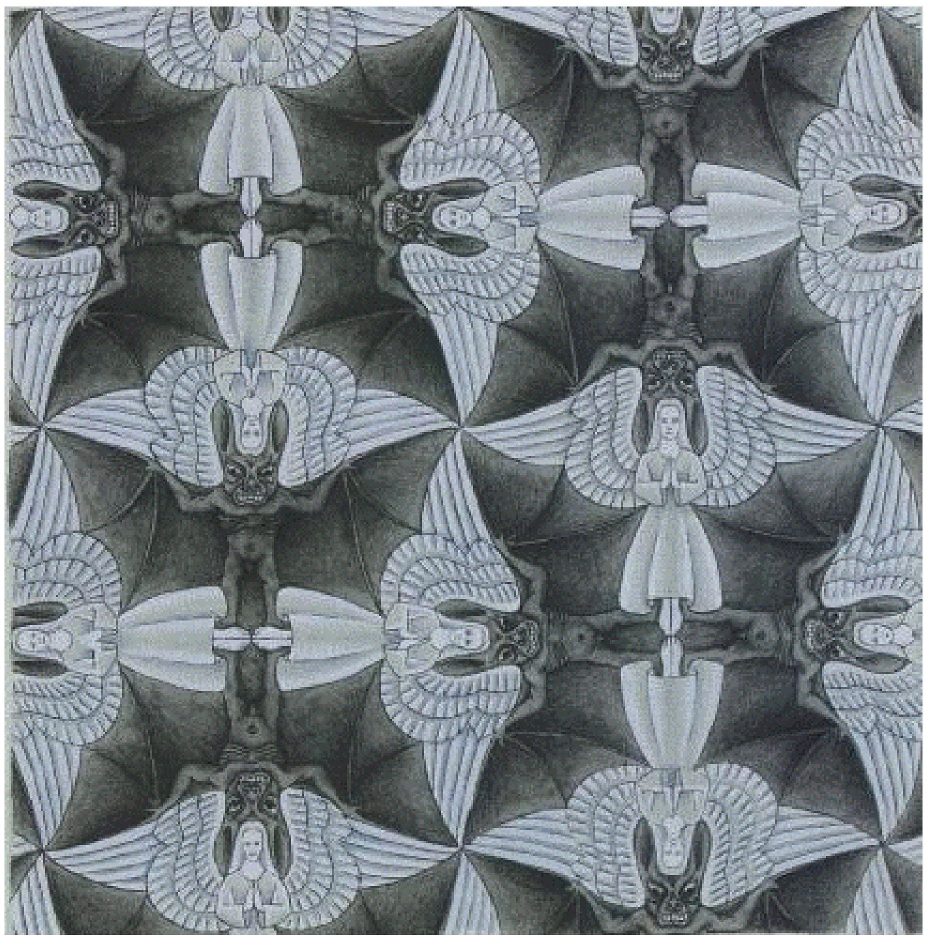
2.9.10 Theorem (Circle of Apollonius). The set of all points P such that the ratio of the distances to two fixed points A and B (that is PA/PB) is constant (but not equal to 1) is a circle.

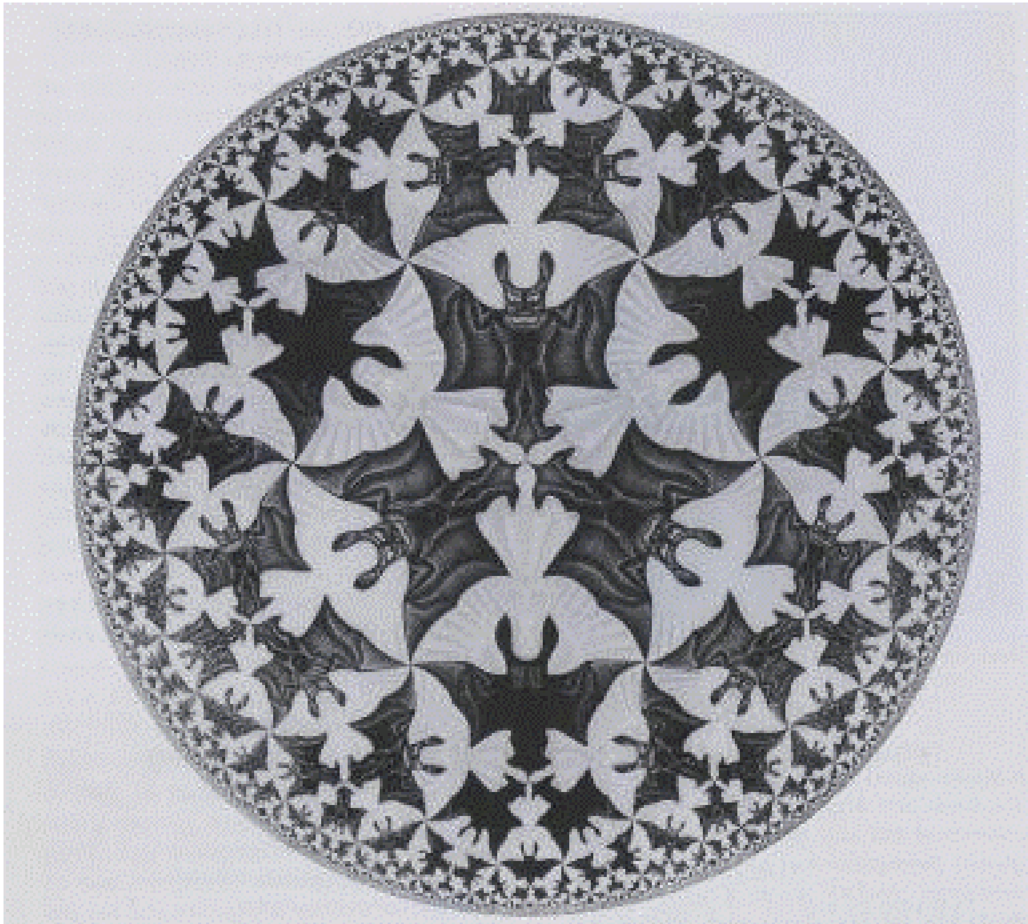


Proof: Assume the notation above and that $PA = mPB$ where $m > 1$ is a constant. There are two points on $\overleftrightarrow{AB}$ indicated by C and D in the figure with the desired ratio. By the converse to Theorem 2.9.6 and the converse to Theorem 2.9.8, $\overline{PC}$ and $\overline{PD}$ are the internal and external angle bisectors of the angle at P. Thus they are perpendicular (why?), so CPD is a right angle. This means that P lies on a circle with diameter $\overline{CD}$. **QED**

In the previous proof what happens in the case where $m < 1$? Also, see **Exercise 2.11.2** .

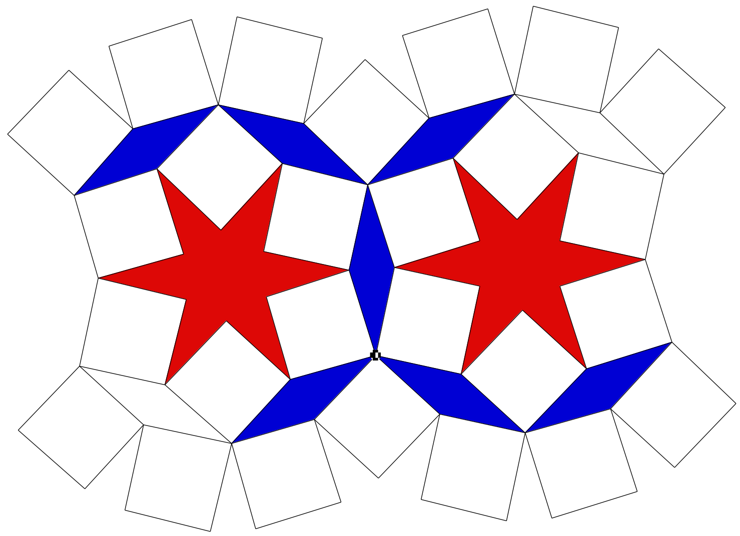
2.10 TILINGS OF THE EUCLIDEAN PLANE. The appeal of many of the most interesting decorations or constructions we see around us, whether manufactured or in nature, is due to underlying symmetries. Two good illustrations of this are the so-called ‘Devils and Angels’ designs by the Dutch graphic artist M. C. Escher. Underlying both is the idea of tilings of the plane, in the first example the Euclidean plane, in the second example the hyperbolic plane.





But examples can be found everywhere from floor coverings, to wallpaper, to the mosaics of Roman villas and to decorations of structures as varied as Highway 183 in Austin and Islamic mosques. An understanding of the geometry underlying these designs and their symmetries increases our understanding and appreciation of the artistic design as well as geometry itself. The classification of these symmetries is actually a fascinating problem linking both algebra and geometry, as we shall see later.

Some of the simplest, yet most striking designs come from 'tilings' by regular polygons or by congruent polygons. Examples can be found everywhere in Islamic art because of the ban imposed by the Koran on the use of living forms in decoration and art. This style of ornamentation is especially adapted to surface decoration since it is strongly rooted in Euclidean plane geometry. Sketchpad will enable us to reproduce these complicated and colorful designs. Once the underlying geometry has been understood, however, we can make our own designs and so learn a lot of Euclidean plane geometry in the process. Four examples illustrate some of the basic ideas.

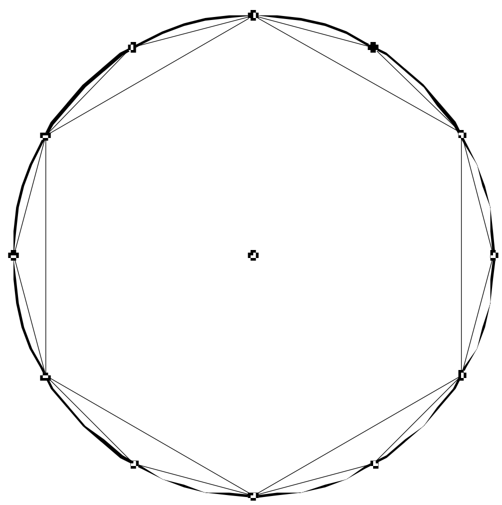


Example 1

The above example shows a typical Arabic design. This was drawn starting from a regular hexagon inscribed in a circle.

Demonstration 2.10.0.   Construct the design in Example 1 using Sketchpad.

- First draw a regular hexagon and its circumscribing circle. Now construct a regular 12-sided regular polygon having the same circumscribing circle to give a figure like the one below.

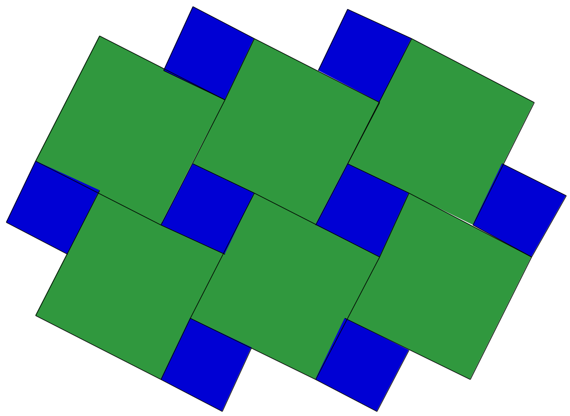


- To construct a second 12-sided regular polygon having one side adjacent to the first regular hexagon, reflect your figure in one of the sides of the first regular hexagon. Now complete the construction of the previous Arabic design.

End of Demonstration 2.10.0.

2.10.1 Exercise. If the radius of the circumscribing circle of the initial regular hexagon is R, determine algebraically the area of the six-pointed star inside one of the circles.

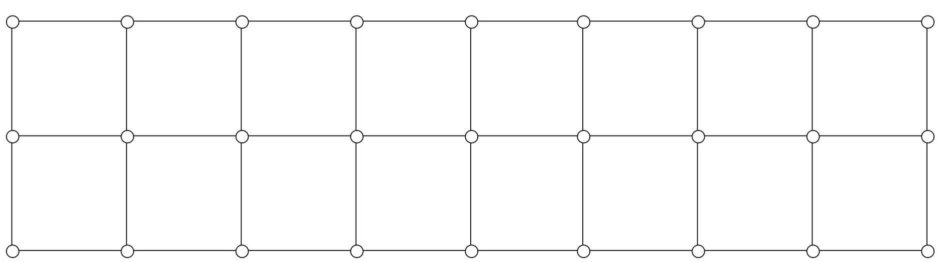
Continuing this example indefinitely will produce a covering of the plane by congruent copies of three polygons - a square, a rhombus and a six-pointed star. Notice that all these congruent copies have the same edge length and adjacent polygons meet only at their edges, *i.e.*, the polygons do not overlap. The second example



Example 2

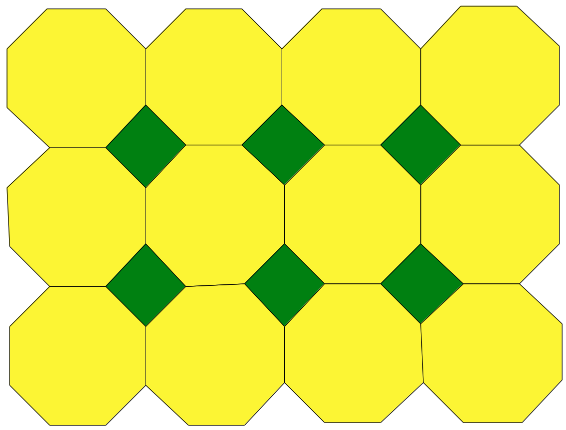
if continued indefinitely also will provide a covering of the plane by congruent copies of two regular polygons - two squares, in fact. Again adjacent polygons do not overlap, but now the individual tiles do not meet along full edges.

The next example



Example 3

is one very familiar one from floor coverings or ceiling tiles; when continued indefinitely it provides a covering of the plane by congruent copies of a single, regular polygon - a square. But now adjacent polygons meet along the full extent of their edges. Finally, notice that continuations of the fourth example



Example 4

produce a covering of the plane by congruent copies of two regular polygons, one a square the other an octagon; again the covering is edge-to-edge.

To describe all these possibilities at once what we want is a general definition of coverings of the plane by polygons without overlaps. Specializations of this definition can then be made when the polygons have special features such as the ones in the first four examples.

2.10.2. Definition. A *tiling* or *tessellation* of the Euclidean plane is a collection $T_1, T_2, \ldots, T_n$, of polygons and their interiors such that

- no two of the tiles have any interior points in common,
- the collection of tiles completely covers the plane.

When all the tiles in a plane tiling are congruent to a single polygon, the tiling is said to have *order one*, and the single region is called the *fundamental region* of the tiling. If each tile is congruent to one of n different tiles, also called *fundamental regions*, the tiling is said to have *order n*.

Now we can add in special conditions on the polygons. For instance, when the polygons are all regular we say that the tiling is a *regular tiling*. Both the second, third and fourth examples above are regular tilings, but the first is not regular since neither the six-pointed polygon nor the rhombus is regular. To distinguish the second example from the others we shall make a crucial distinction.

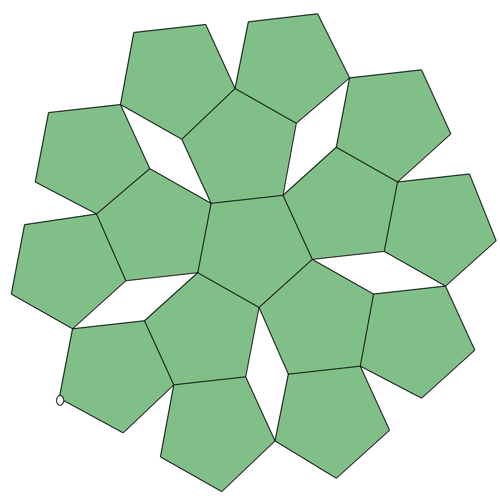
2.10.3. Definition. A tessellation is said to be *edge-to-edge* if two tiles intersect along a full common edge, only at a common vertex, or not at all.

Thus examples one, three and four are edge-to-edge, whereas example two is not edge-to-edge. The point of this edge-to-edge condition is that it reduces the study of regular tilings to combinatorial problems for the interior angles of the regular polygons meeting at a vertex. It is in this way that the Euclidean plane geometry of this chapter, particularly the sums of angles of polygons, comes into play. So from now on a tiling will always mean an edge-to-edge tiling unless it is explicitly stated otherwise.

A major problem in the theory is to determine whether a given polygon can serve as fundamental region for a tiling of order one, or if a collection of *n* polygons can serve as fundamental regions for a tiling of order *n*. The case of a square is well-known from floor coverings and was given already in example 3 above.

2.10.4. Demonstration. Investigate which regular polygons could be used to create an edge-to-edge regular tiling of order one.

Use the '3/Triangle (By Edge)' script to show that an equilateral triangle can tile the plane meaning that it can serve as fundamental region for a regular tiling of order one. Try the same with a regular hexagon using the '6/Hexagon (By Edge)' script - what in nature does your picture remind you of? Now use the '5/Pentagon (By Edge) to check if a regular pentagon can be used a fundamental region for a regular tiling of order one. Experiment to see what patterns you can make. One example is given below; can you find others?



End of Demonstration 2.10.4.

Can you tile the plane with a regular pentagon? To see why the answer is no we prove the following result.

2.10.5. Theorem. The only regular polygons that tile the plane are equilateral triangles, squares and regular hexagons. In particular, a regular pentagon does not tile the plane.

Proof. Suppose a regular *p*-sided polygon tiles the plane with *q* tiles meeting at each vertex. Since the interior angle of a regular *p*-sided polygon has measure $180\left(\frac{p-2}{p}\right)$, it follows that $q180(1-2/p) = 360$. But then

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \quad i.e., (p-2)(q-2) = 4\,.$$

The only integer solutions of this last equation that make geometric sense are the pairs

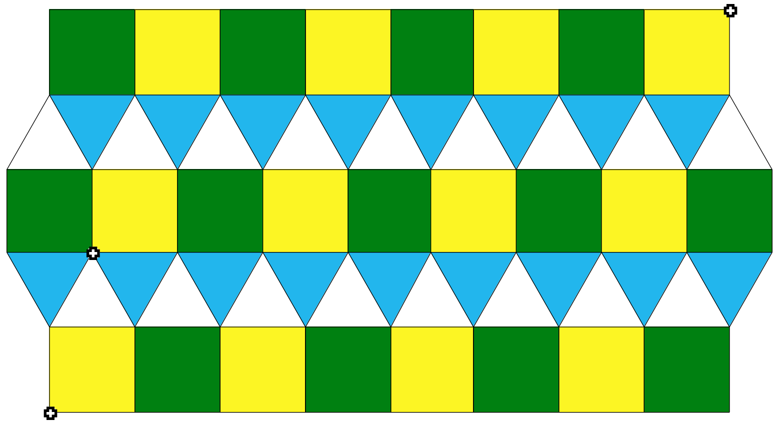
$$(p,q) = (3,6),\ (4,4),\ \text{or}\ (6,3).$$

These correspond to the case of equilateral triangles meeting 3 at each vertex, squares meeting 4 at each vertex and regular hexagons meeting 3 at each vertex. **QED**

Tilings of the plane by congruent copies of a regular polygon does not make a very attractive design unless some pattern is superimposed on each polygon - that's a design problem we shall return to later. What we shall do first is try to make the tiling more attractive by using more than one regular polygon or by using polygons that need not be regular. Let's look first at the case of an equilateral triangle and a square each having the same edge length.

Demonstration 2.10.5a. Construct a regular tiling of order 2 where the order of the polygons is preserved at each vertex.

- Open a new sketch and draw a square (not too big since this is the starting point) and draw an equilateral triangle on one of its sides so that the side lengths of the triangle and the square are congruent. Use the scripts to see if these two regular polygons can serve as the fundamental regions of a regular tiling of order 2 where the order of the polygons is preserved at each vertex. Here's one such example.



Notice that the use of colors can bring out a pattern to the ordering of the polygons at each vertex. As we move in counter-clockwise order around each vertex we go from

$$S(\text{green}) \to S(\text{yellow}) \to T(\text{white}) \to T(\text{blue}) \to T(\text{white})$$

(and then back to S(green)) where S = square and T = equilateral triangle. This is one example of an edge-to-edge regular tiling of order two. Consider how many are there.

End of Demonstration 2.10.5a.

2.10.6 Theorem. Up to similarity there are exactly eight edge-edge regular tilings of order at least 2, where the cyclic order of the polygons is preserved at each vertex.

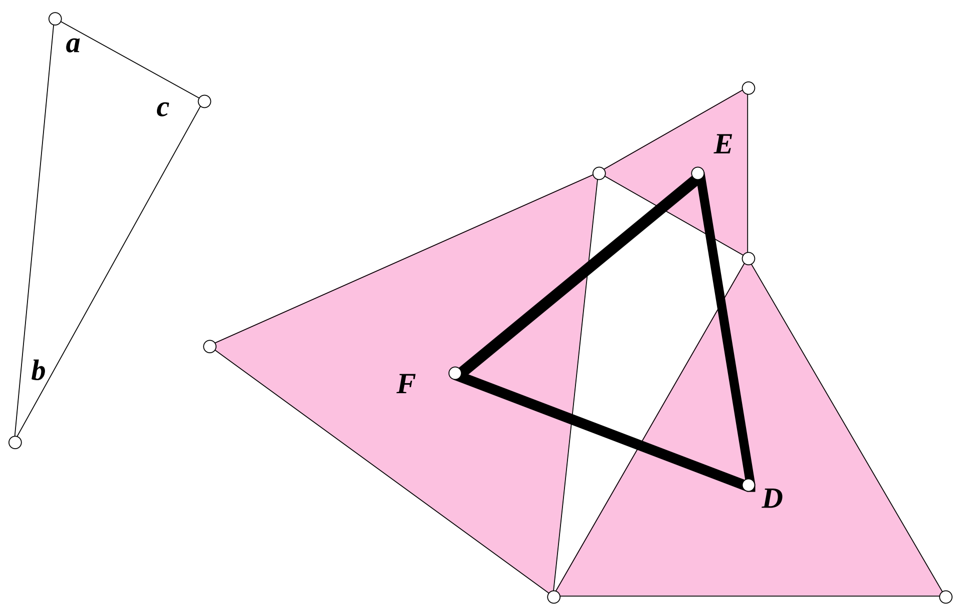
Keeping the order $S \to S \to T \to T \to T$ of squares and triangles produced one such tiling. Convince yourself that $S \to T \to T \to S \to T$ produces a different tiling. Why are these the only two possible orderings for two squares and three triangles? How many permutations are possible for the letters S, S, T, T, and T?

What are the other six tilings? Algebraic conditions limit drastically the possible patterns so long as the tiling is edge-to-edge and that the order of the polygons is the same at each vertex. Using the angle sum formulas for regular polygons one can easily see that you need at least three polygons around a vertex, but can have no more than six. In the case of a p-gon, a q-gon, and an r-gon at each vertex, you get the equation

$$180\left(\frac{p-2}{p}\right)+180\left(\frac{q-2}{q}\right)+180\left(\frac{r-2}{r}\right)=360$$

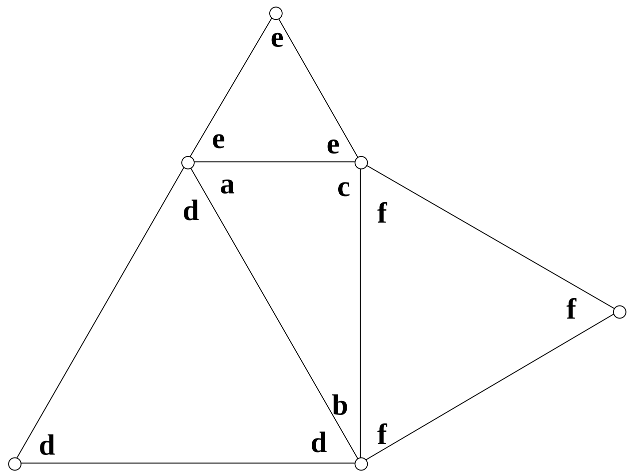
You can check that (4,8,8), (4,6,12), and (3,12,12) are solutions. (There are a few other solutions as well, but they will not make geometric sense.) Thus $S \to O \to O$, $S \to H \to D$, and $T \to D \to D$ all produce tilings, where O stands for Octagon, H for hexagon, and D for Dodecagon. We are still missing three tilings, but you can have fun looking for them! (See **Exercise 2.11.3**.) Now we will take a look at some less regular tilings.

It is surprising how much of geometry can be related to tilings of the plane. Let's consider two instances of this, the second being Pythagoras' theorem. The first instance is a theorem known familiarly as Napoleon's theorem after the famous French general though there is no evidence that he actually had anything to do with the theorem bearing his name! Recall that earlier we proved the form of Pythagoras' theorem saying that the area of the equilateral triangle on the hypotenuse is equal to the sum of the areas of the equilateral triangles on the other two sides. On the other hand, Napoleon's theorem says that the centers of these three equilateral triangles themselves form an equilateral triangle, as we saw in Exercise 1.8.5. The figure below makes this result clearer.



Here $D, E,$ and F are the centers of the three equilateral triangles where by *center* is meant the common circumcenter, centroid and orthocenter of an equilateral triangle. Napoleon's theorem says that ΔDEF is equilateral - it certainly looks as if its sides are congruent and measuring them on Sketchpad will establish congruence. You will provide a proof of the result in Exercise 2.11.5. The question we consider here is how all this relates to tilings of the plane. Notice now that we have labeled the interior angles of the triangle because we are going to allow polygons which are not necessarily regular. Since the interior can then be different, the particular interior angle of polygons that appears at a vertex is going to be just as important as which polygon appears. Now we will see how we can continue the figure above indefinitely and thus tile the plane.

One should notice that the edge-to-edge condition imposes severe restrictions on the angles that can occur at a vertex. Label the angles in the original figure as follows.



Of course, the angles of the equilateral triangles are all the same but we have used different letters to indicate that they are the interior angles of equilateral triangles of different size. Since $a + b + c + d + e + f = 360^\circ$, three copies of the right-angled triangle and one copy of each of the three different sizes of equilateral triangle will fit around a vertex with no gaps or overlaps. The figure can thus be constructed indefinitely by maintaining the same counter-clockwise order $a \to e \to c \to f \to b \to d$ at each vertex. Now draw the figure for yourself! It may be instructive to use a different color for each equilateral triangle to highlight the fact that the equilateral triangles are not necessarily congruent.

2.10.6a Demonstration.

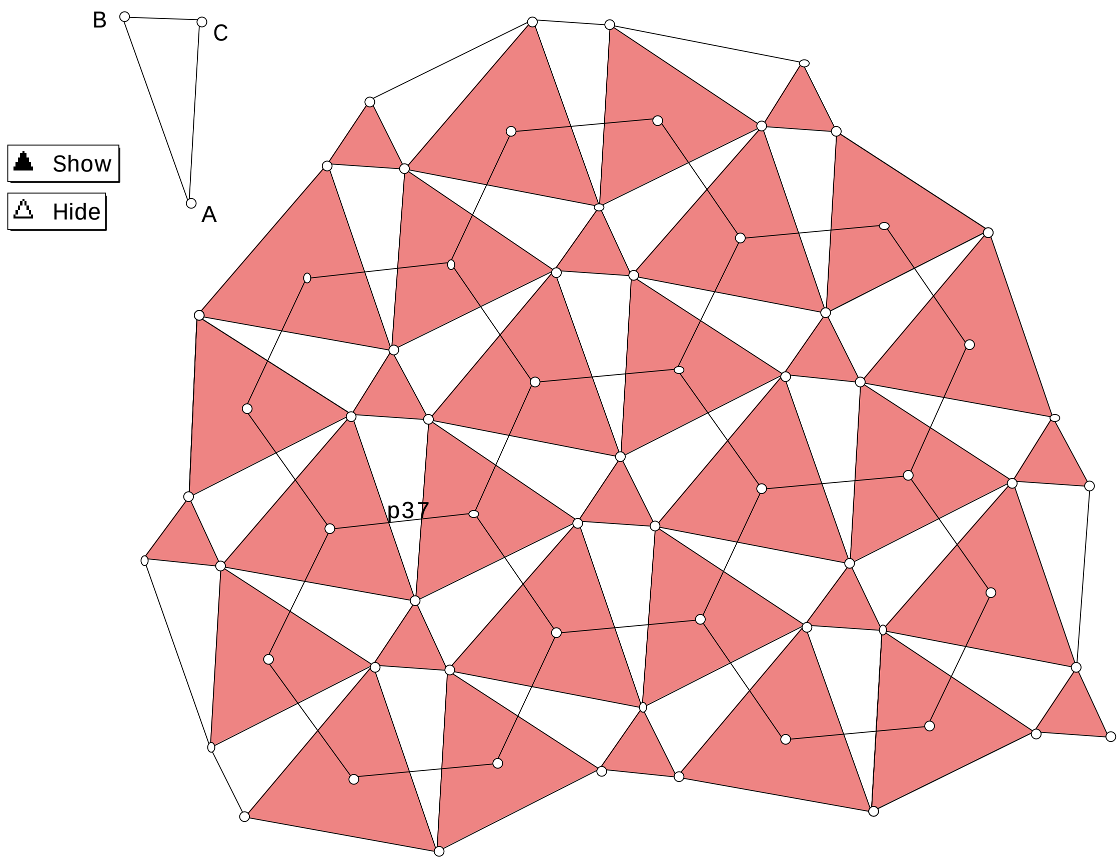
· Open a new sketch and in the top left-hand corner of the screen draw a right-angled triangle as shown in the figure above. Make sure that your construction is dynamic in the sense that the triangle remains right-angled whenever any one of the vertices is dragged.

· Use the ‘Circle By Center + Radius’ construction to construct a congruent copy of your triangle in the center of the screen. Draw an outwardly pointing equilateral triangle on each side of this right-angled triangle.

- Continue adding congruent copies of the right-angled triangle and the equilateral triangles to the sides of the triangles already in your figure. (One way to add congruent copies of the
- right triangle is to use your ‘Auto-Matching’ similar triangle script. Just label your original right triangle appropriately.)
- Experiment a little to see what figures can be produced. Check that your construction is dynamic by dragging the vertices of the first right-angled triangle you drew.

End of Demonstration 2.10.6a.

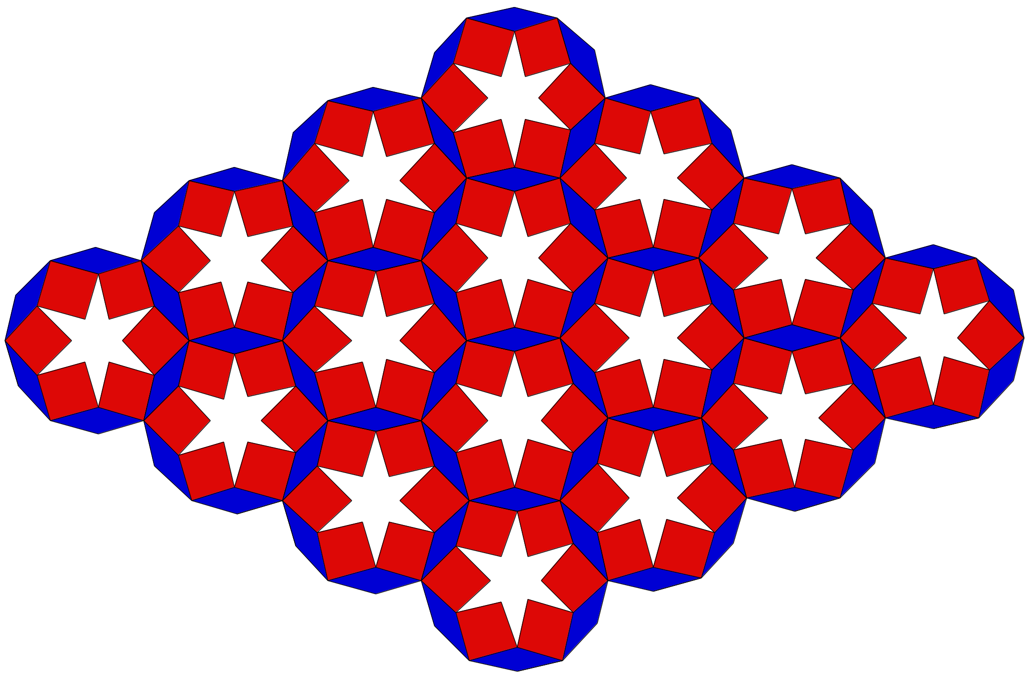
Here’s one that looks as if it might tile the plane if continued indefinitely.



Napoleon Tiling

The figure above of the Napoleon Tiling has an overlay of hexagons over it. To see where it came from, apply Napoleon’s Theorem to the tiling. That is around each right triangle connect the centers of the equilateral triangle to create a new equilateral triangle. Six of those new equilateral triangles make up each hexagon above. Thus Napoleon’s theorem brings out an *underlying* symmetry in the design because it showed that a regular tiling of the plane by regular hexagons could be overlaid on the figure. The same design could have been obtained by putting a *design* on each regular hexagon and then tiling the plane with these patterned regular hexagons.

This brings out a crucial connection between *tilings* and the sort of *designs* that are used for covering walls, floors, ceilings or any flat surface. A design is said to be *wallpaper* design if a polygonal portion of it provides a tiling of the plane by translations in two different directions. Thus all the examples obtained in this section are wallpaper designs. It is very clear that the Islamic design in problem 2.10.1 is a wall-paper design because the portion of the design inside the initial regular hexagon will tile the plane as the figure below clearly shows.



2.10.7. Exercise. Find a square portion of Example 4 in Seection 2.10 that tiles the plane. In other words, show that that example is a wallpaper design.

Example 2 is sometimes called the “Pythagorean Tiling”. It is created by a translation of two adjacent non-congruent squares. This tiling occurs often in architectural and decorative designs as seen in this sidewalk tiling. To see why this tiling might be called a “Pythagorean Tiling” open a new sketch and draw the tiling as it appears in example 2 using two squares of different sizes. Construct an overlaying of this design by a tiling, which consists of congruent copies of a single square. What is the area of this square? Use Pythagoras’ theorem to relate this area to the area of the two original squares you used to construct your pattern.

2.11 Exercises.

Exercise 2.11.1. Prove Theorem 2.9.4. Given a circle Σ and a point P outside Σ, let l be a ray through P intersecting Σ at points A and B. If C is a point on Σ such that $PA{\cdot}PB = PC^2$, then $\overline{PC}$ is tangent to Σ at C.

Exercise 2.11.2: Given points A, B and P use Sketchpad to construct the Circle of Apollonius passing through P. In other words, construct the set of points Q such that $QA = mQB$ where $PA/PB = m$.

Exercise 2.11.3. Produce two different order-preserving edge-to-edge regular tilings of order 2, just using triangles and hexagons. Produce an order-preserving edge-to-edge regular tiling of order 3 using triangles, squares, and hexagons. (We now have the eight tilings mentioned in Theorem 2.10.5!)

Exercise 2.11.4. Using Sketchpad construct the Napoleon Tiling. Choose a regular hexagon in your figure and describe its area in terms of the original triangle and the three equilateral triangles constructed on its sides. Now choose a different (larger or smaller area) regular hexagon having a different area and describe the area of this hexagon in terms of the original triangle and the three equilateral triangles.

Exercise 2.11.5. While the tiling above makes a very convincing case for the truth of Napoleon's theorem it doesn't prove it in the usual meaning of 'proof'. Here is a coordinate geometry proof based on the figure on the following page and on the notation in that figure.

(a) The points D, E, and F are the centers of the equilateral triangles constructed on the sides of the right-angled triangle ΔABC. Show that length $\overline{BF} = c/\sqrt{3}$. Determine also the lengths of $\overline{AD}$ and $\overline{BE}$.

(b) If $ABC = \theta$ and $CAB = \phi$, write the values of $\sin\theta$, $\cos\theta$, $\sin\phi$, and $\cos\phi$ in terms of a, b, and c.

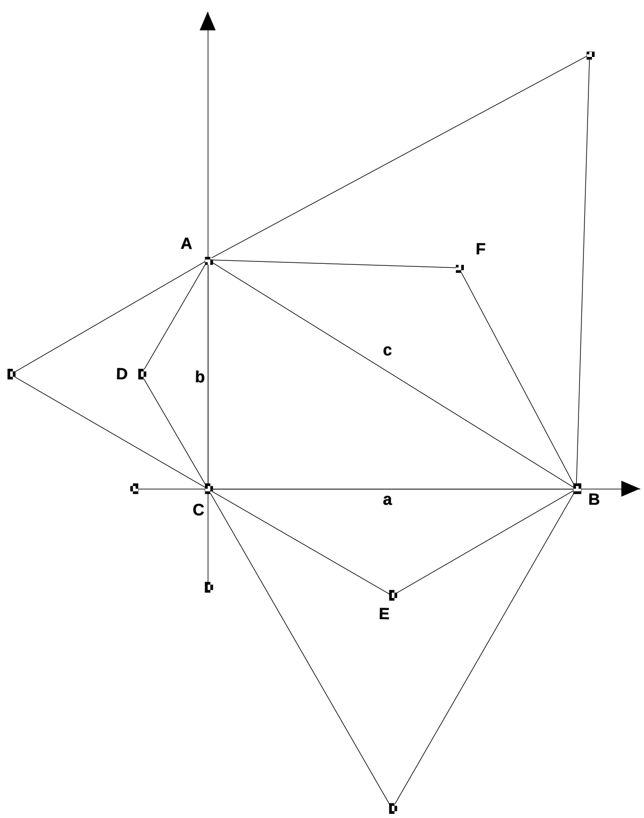
(c) Write down the addition formulas for sine and cosine.

$$\cos(u+v) = \qquad\qquad\qquad\qquad , \ \sin(u+v) = \qquad\qquad\qquad\qquad .$$

(d) Let the lengths of $\overline{FE}$, $\overline{DF}$, and $\overline{DE}$ be x, y and z respectively. Use the Law of Cosines to show that

$$z^2 = \frac{1}{3}\left(a^2 + b^2 + 2ab\cos 30^\circ\right).$$

Determine corresponding values for x and y. Deduce that $x = y = z$.

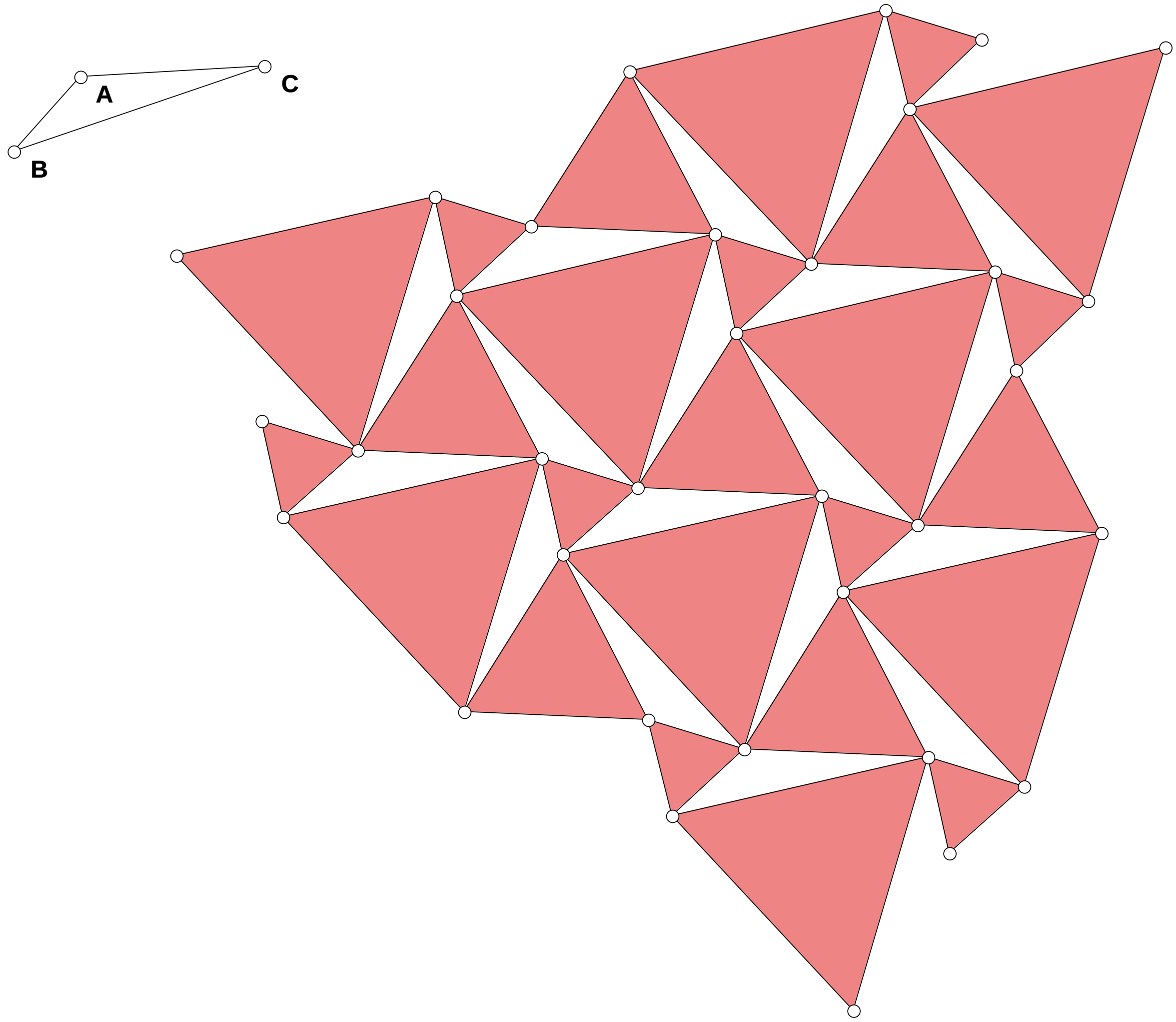


Use all the previous results to finish off a coordinate geometry proof of Napoleon's theorem.

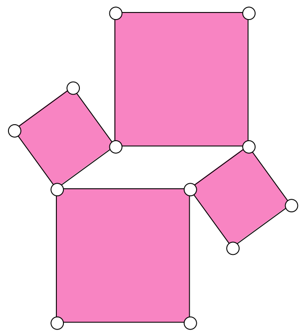
Exercise 2.11.6. Instead of starting with a right-angled triangle, start with an arbitrary ΔABC and draw equilateral triangles on each of its sides and repeat the previous construction.

- Open a new sketch and draw a small triangle near the top corner of the screen; label the vertices A, B, and C. By using the 'Circle By Center+Radius' tool you can construct congruent copies of this triangle.
- Draw one congruent copy of ΔABC in the center of the screen. Draw an equilateral triangle on each of its sides.

- Continue this construction preserving cyclic order at each vertex to obtain a tiling of the plane. The following figure is one such example.
- Construct the centers of all the equilateral triangles and draw hexagons as in the case of right-angled triangles. Do you think Napoleon's theorem remains valid for any triangle, not just right-angled triangles?

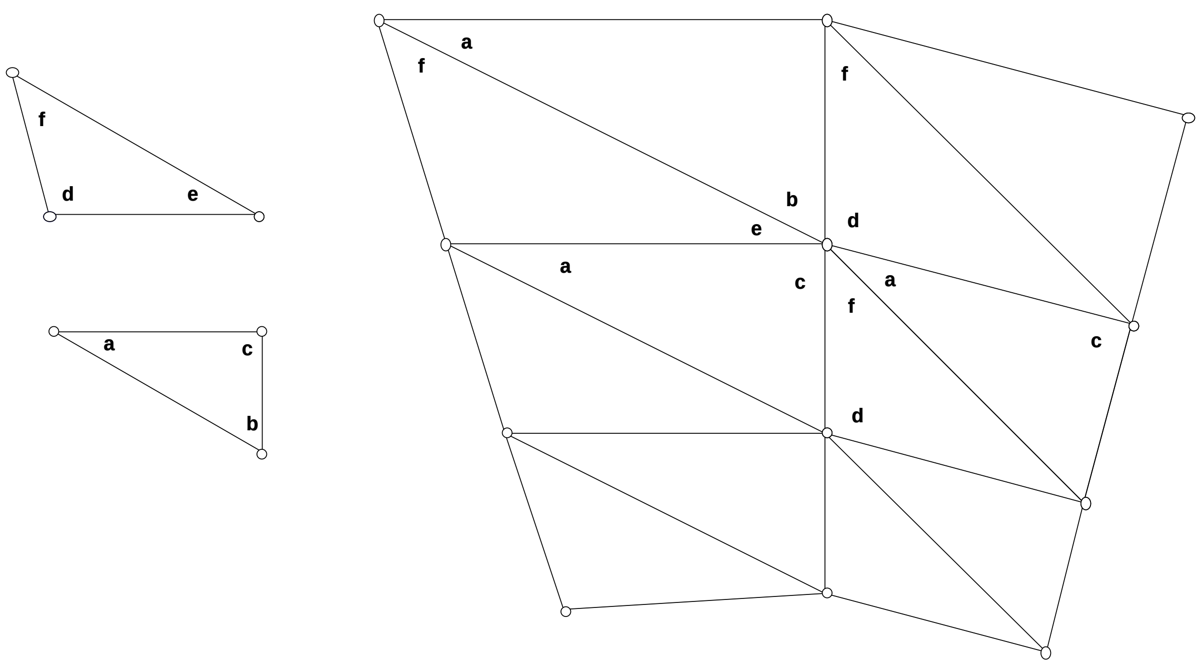


Exercise 2.11.7. Can the plane be tiled by copies of the diagram for Yaglom's Theorem (given below) as in the manner of the tiling corresponding to Napoleon's Theorem? If so, produce the tiling using Sketchpad. Recall that Yaglom's Theorem said if we place squares on the sides of a parallelogram, the centers of the squares also form a square.



2.12 One final Exercise.

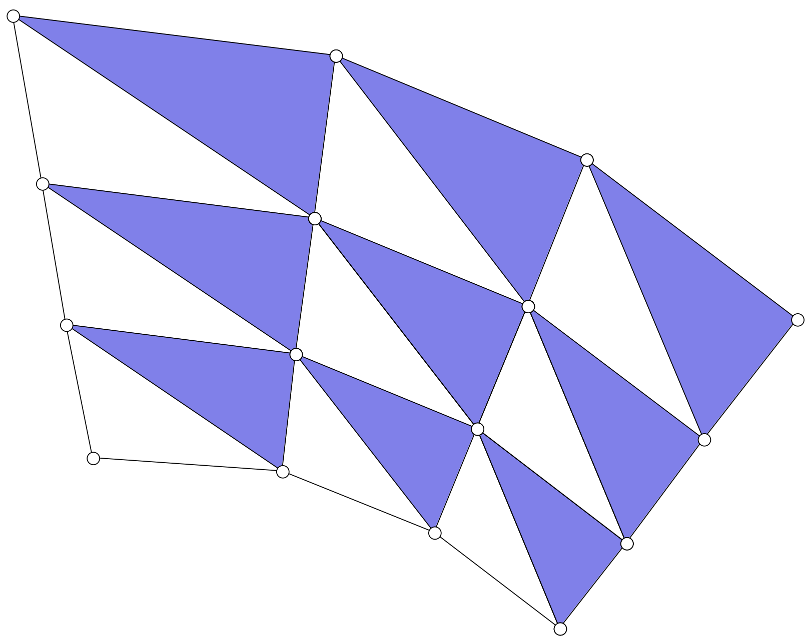
Exercise 2.12.1. To the left in the figure below are two triangles, one obtuse, the other right-angled. The interior angles of the two triangles have been labeled. Since the sum of these six angles is 360˚ there should be a tiling of the plane by congruent copies of these two triangles in which the cyclic order of the angles at each vertex is the same as the one shown in the figure to the right.



- Open a new sketch and continue this construction to provide a tiling of the plane. Unlike the previous tilings, the triangles in this tiling are not congruent. Explain why this tiling is more like a Nautilus Shell.

- Construct the circumcenters of the three outwardly pointing obtuse triangles on the sides of one of the right-angled triangles and join these circumcenters by line segments. What, if any, is the relation of the triangle having these three circumcenters as vertices and the original obtuse triangle? Is there any relation with the original right-angled triangle? Use Sketchpad if necessary to check any conjecture you make. (Don't forget to drag!)

Investigate what happens if you construct instead the three circumcenters of the right-angled triangles on the sides of one of the obtuse triangles? Draw the triangle having these circumcenters as vertices. What, if any, is the relation between the original right-angled triangle and the triangle having the three circumcenters as vertices? Is there any relation with the original obtuse triangle? Again use Sketchpad if necessary to check visually any conjecture you make. (Don't forget to drag!)



Chapter 3
NON-EUCLIDEAN GEOMETRIES

In the previous chapter we began by adding Euclid’s Fifth Postulate to his five common notions and first four postulates. This produced the familiar geometry of the ‘Euclidean’ plane in which there exists precisely one line through a given point parallel to a given line not containing that point. In particular, the sum of the interior angles of any triangle was always 180° no matter the size or shape of the triangle. In this chapter we shall study various geometries in which parallel lines need not exist, or where there might be more than one line through a given point parallel to a given line not containing that point. For such geometries the sum of the interior angles of a triangle is then always greater than 180° or always less than 180°. This in turn is reflected in the area of a triangle which turns out to be proportional to the difference between 180° and the sum of the interior angles.

First we need to specify what we mean by a geometry. This is the idea of an *Abstract Geometry* introduced in Section 3.1 along with several very important examples based on the notion of *projective geometries*, which first arose in Renaissance art in attempts to represent three-dimensional scenes on a two-dimensional canvas. Both Euclidean and hyperbolic geometry can be realized in this way, as later sections will show.

3.1 ABSTRACT AND LINE GEOMETRIES. One of the weaknesses of Euclid’s development of plane geometry was his ‘definition’ of points and lines. He defined a point as “... that which has no part” and a line as “... breadthless length”. These really don’t make much sense, yet for over 2,000 years everything he built on these definitions has been regarded as one of the great achievements in mathematical and intellectual history! Because Euclid’s definitions are not very satisfactory in this regard, more modern developments of geometry regard points and lines as undefined terms. A *model* of a modern geometry then consists of specifications of points and lines.

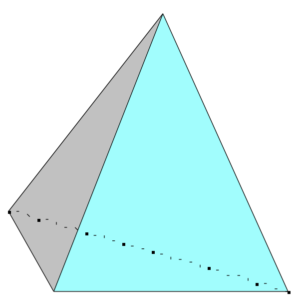
3.1.1 Definition. An *Abstract Geometry* G consists of a pair $\{\mathcal{P}, \mathcal{L}\}$ where $\mathcal{P}$ is a set and $\mathcal{L}$ is a collection of subsets of $\mathcal{P}$. The elements of $\mathcal{P}$ are called *Points* and the elements of $\mathcal{L}$ are called *Lines*. We will assume that certain statements regarding these points and lines are true at the outset. Statements like these which are assumed true for a geometry are called *Axioms* of the geometry. Two Axioms we require are that each pair of points P, Q in $\mathcal{P}$ belongs to at least one line l in $\mathcal{L}$, and that each line l in $\mathcal{L}$ contains at least two elements of $\mathcal{P}$.

We can impose further geometric structure by adding other axioms to this definition as the following example of a *finite* geometry - finite because it contains only finitely many points - illustrates. (Here we have added a third axiom and slightly modified the two mentioned above.)

3.1.2 Definition. A 4-POINT geometry is an abstract geometry $\mathcal{G} = \{\mathcal{P}, \mathcal{L}\}$ in which the following axioms are assumed true:

- **Axiom 1:** $\mathcal{P}$ contains exactly four points;
- **Axiom 2:** each pair of distinct points in $\mathcal{P}$ belongs to exactly one line;
- **Axiom 3:** each line in $\mathcal{L}$ contains exactly two distinct points.

The definition doesn't indicate what objects points and lines are in a 4-Point geometry, it simply imposes restrictions on them. Only by considering a model of a 4-Point geometry can we get an explicit description. Look at a tetrahedron.



It has 4 vertices and 6 edges. Each pair of vertices lies on exactly one edge, and each edge contains exactly 2 vertices. Thus we get the following result.

3.1.3 Example. A tetrahedron contains a model of a 4-Point geometry in which

$\mathcal{P}$ = {vertices of the tetrahedron} and $\mathcal{L}$ = {edges of the tetrahedron}.

This example is consistent with our usual thinking of what a point in a geometry should be and what a line should be. But points and lines in a 4-Point geometry can be anything so long as they satisfy all the axioms. Exercise 3.3.2 provides a very different model of a 4-Point geometry in which the points are opposite faces of an octahedron and the lines are the vertices of the octahedron!

Why do we bother with models? Well, they give us something concrete to look at or think about when we try to prove theorems about a geometry.

3.1.4 Theorem. In a 4-Point geometry there are exactly 6 lines.

To prove this theorem synthetically all we can do is use the axioms and argue logically from those. A model helps us determine what the steps in the proof should be. Consider the tetrahedron model of a 4-Point geometry. It has 6 edges, and the edges are the lines in the geometry, so the theorem is correct for this model. But there might be a different model of a 4-Point geometry in which there are more than 6 lines, or fewer than 6 lines. We have to show that there will be exactly 6 lines whatever the model might be. Let's use the tetrahedron model again to see how to prove this.

- Label the vertices *A*, *B*, *C*, and *D*. These are the 4 points in the geometry.
- Concentrate first on *A*. There are 3 edges passing through *A*, one containing *B*, one containing *C*, and one containing *D*; these are obviously distinct edges. This exhibits 3 distinct lines containing *A*.
- Now concentrate on vertex *B*. Again there are 3 distinct edges passing through *B*, but we have already counted the one passing also through *A*. So there are only 2 new lines containing *B*.
- Now concentrate on vertex *C*. Only the edge passing through *C* and *D* has not been counted already, so there is only one new line containing *C*.
- Finally concentrate on *D*. Every edge through *D* has been counted already, so there are no new lines containing *D*.

Since we have looked at all 4 points, there are a total of 6 lines in all. This proof applies to any 4-Point geometry if we label the four points *A*, *B*, *C*, and *D*, whatever those points are. Axiom 2 says there must be one line containing *A* and *B*, one containing *A* and *C* and one containing *A* and *D*. But the Axiom 3 says that the line containing *A* and *B* must be distinct from the line containing *A* and *C*, as well as the line containing *A* and *D*. Thus there will always be 3 distinct lines containing *A*. By the same argument, there will be 3 distinct lines containing *B*, but one of these will contain *A*, so there are only 2 new lines containing *B*. Similarly, there will be 1 new line containing *C* and no new lines containing *D*. Hence in any 4-Point geometry there will be exactly 6 lines.

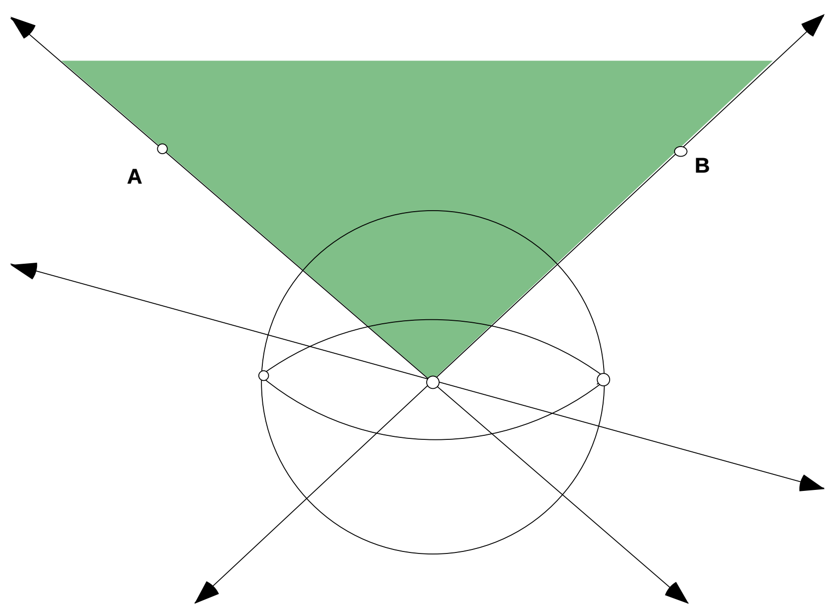
This is usually how we prove theorems in Axiomatic Geometry: look at a model, check that the theorem is true for the model, then use the axioms and theorems that follow from these axioms to give a logically reasoned proof. For Euclidean plane geometry that model is always the familiar geometry of the plane with the familiar notion of point and line. But it is not be the only model of Euclidean plane geometry we could consider! To illustrate the variety of forms that geometries can take consider the following example.

3.1.5 Example. Denote by $\mathbf{P}^2$ the geometry in which the 'points' (here called P-points) consist of all the Euclidean lines through the origin in 3-space and the P-lines consist of all Euclidean planes through the origin in 3-space.

Since exactly one plane can contain two given lines through the origin, there exists exactly one P-line through each pair of P-points in $\mathbf{P}^2$ just as in Euclidean plane geometry. But what about parallel P-lines? For an abstract geometry $\mathcal{G}$ we shall say that two lines m, and l in $\mathcal{G}$ are *parallel* when l and m contain no common points. This makes good sense and is consistent with our usual idea of what parallel means. Since any two planes through the origin in 3-space must always intersect in a line in 3-space we obtain the following result.

3.1.6 Theorem. In $\mathbf{P}^2$ there are no parallel P-lines.

Actually, $\mathbf{P}^2$ is a model of Projective plane geometry. The following figure illustrates some of the basic ideas about $\mathbf{P}^2$.



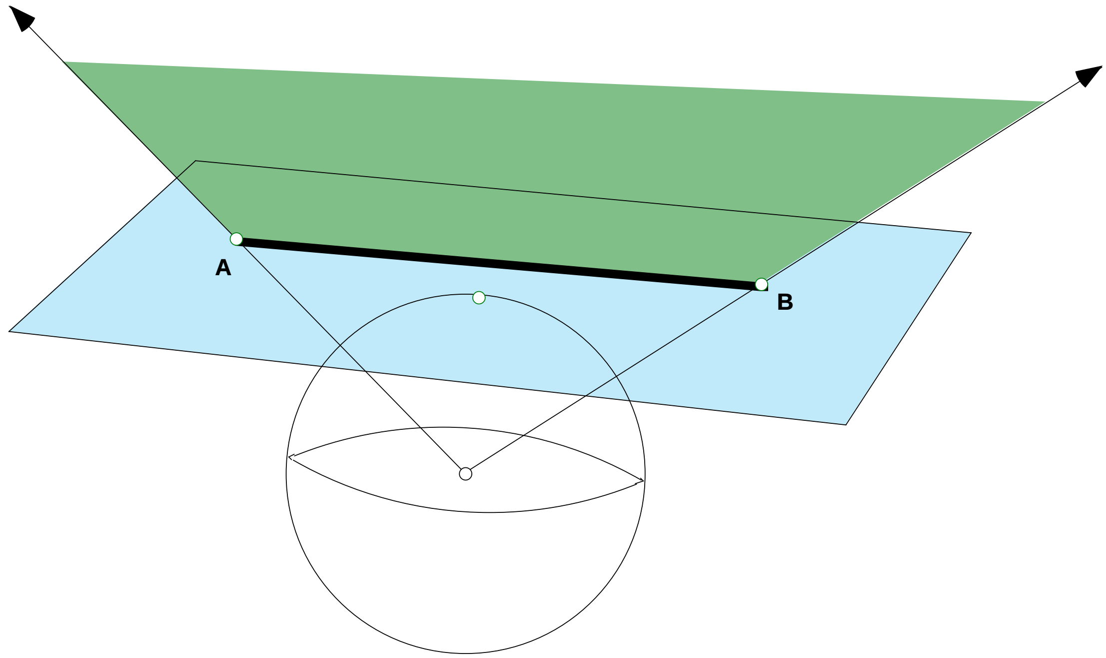
The two Euclidean lines passing through A and the origin and through B and the origin specify two P-points in $\mathbf{P^2}$, while the indicated portion of the plane containing these lines through A and B specify the ‘P-line segment’ $\overline{AB}$.

Because of Theorem 3.1.6, the geometry $\mathbf{P^2}$ cannot be a model for Euclidean plane geometry, but it comes very ‘close’. Fix a plane passing through the origin in 3-space and call it the *Equatorial Plane* by analogy with the plane through the equator on the earth.

3.1.7 Example. Denote by $\mathbf{E^2}$ the geometry in which the E-points consist of all lines through the origin in 3-space that are not contained in the equatorial plane and the E-lines consist of all planes through the origin save for the equatorial plane. In other words, $\mathbf{E^2}$ is what is left of $\mathbf{P^2}$ after one P-line and all the P-points on that P-line in $\mathbf{P^2}$ are removed.

The claim is that $\mathbf{E^2}$ can be identified with the Euclidean plane. Thus there must be parallel E-lines in this new geometry $\mathbf{E^2}$. Do you see why? Furthermore, $\mathbf{E^2}$ satisfies Euclid’s Fifth Postulate.

The figure below indicates how $\mathbf{E^2}$ can be identified with the Euclidean plane. Look at a fixed sphere in Euclidean 3-Space centered at the origin whose equator is the circle of intersection with the fixed equatorial plane. Now look at the plane which is tangent to this sphere at the North Pole of this sphere.



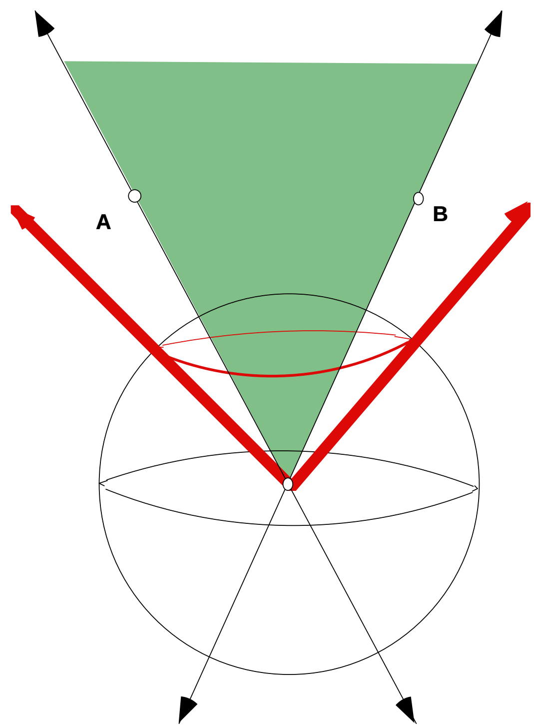
Every line through the origin in 3-space will intersect this tangent plane in exactly one point unless the line is parallel in the usual 3-dimensional Euclidean sense to the tangent plane at the North Pole. But these parallel lines are precisely the lines through the origin that lie in the equatorial plane. On the other hand, for each point A in the tangent plane at the North Pole there is exactly one line in 3-space passing through both the origin and the given point A in the tangent plane. Thus there is a 1-1 correspondence between the E-points in $\mathbf{E}^2$ and the points in the tangent plane at the North Pole. In the same way we see that there is a 1-1 correspondence between E-lines in $\mathbf{E}^2$ and the usual Euclidean lines in the tangent plane. The figure above illustrates the 1-1 correspondence between E-line segment $\overline{AB}$ in $\mathbf{E}^2$ and the line segment $\overline{AB}$ in Euclidean plane geometry.

For reasons, which will become very important later in connection with transformations, this 1-1 correspondence can be made explicit through the use of coordinate geometry and ideas from linear algebra. Let the fixed sphere centered at the origin having radius 1. Then the point (x, y) in the Euclidean plane is identified with the point $(x, y, 1)$ in the tangent plane at the North Pole, and this point is then identified with the line $\{\alpha\,(x, y, 1): -\infty < \alpha < \infty\}$ through the origin in 3-space.

Since there are no parallel lines in $\mathbf{P}^2$ it is clear that the removal from $\mathbf{P}^2$ of that one P-line and all P-points on that P-line must be very significant.

3.1.8 Exercise. What points do we need to add to the Euclidean plane so that under the identification of the Euclidean plane with $\mathbf{E}^2$ the Euclidean plane together with these additional points are in 1-1 correspondence with the points in $\mathbf{P}^2$? What line do we need to add to the Euclidean plane so that we get a 1-1 correspondence with all the lines in $\mathbf{P}^2$?

Note first that by restricting further the points and lines in $\mathbf{P}^2$ we get a model of a different geometry. The set of all lines passing through the origin in 3-space and through the 45th parallel in the Northern Hemisphere of the fixed sphere model determines a *cone* in 3-space to be denoted by **L**.

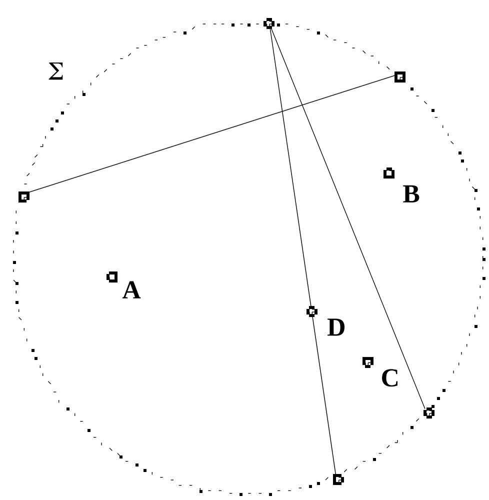


3.1.9 Definition. Denote by $\mathbf{H}^2$ the geometry whose h-points consists of Euclidean lines through the origin in 3-space that lie in the inside the cone **L** and whose h-lines consist of the intersections of the **interior** of **L** and planes through the origin in 3-space.

Again the Euclidean lines through A and B represent h-points A and B in $\mathbf{H}^2$ and the 'h-line segment' $\overline{AB}$ is (as indicated in the above figure by the shaded region) the sector of a plane containing the Euclidean lines through the origin which are passing through points on the line segment connecting A and B. $\mathbf{H}^2$ is a model of *Hyperbolic* plane geometry. The reason why it's a model of a 'plane' geometry is clear because we have only defined points and lines, but what is not at all obvious is why the name 'hyperbolic' is used. To understand that let's try to use $\mathbf{H}^2$ to create other models. For instance, our intuition about 'plane' geometries suggests that we should try to find models in which h-points really are points, not lines through the origin! One way of doing this is by looking at surfaces in 3-space, which intersect the lines inside the cone **L** exactly once. There are two natural candidates, both presented here. The second one presented realizes Hyperbolic plane geometry as the points on a hyperboloid, - hence the name 'Hyperbolic' geometry. The first one presented realizes Hyperbolic plane geometry as the points inside a disk. This first one, known as the *Klein Model,* is very useful for solving the following exercise because its h-lines are realized as open Euclidean line segments. In the next section we study a third model known as the *Poincaré Disk*.

3.1.10 Exercise. Given an h-line l in Hyperbolic plane geometry and an h-point P not on the h-line, how many h-lines parallel to l through P are there?

3.1.11 Klein Model. Consider the tangent plane M, tangent to the unit sphere at its North Pole, and let the origin in M be the point of tangency of M with the North Pole. Then M intersects the cone **L** in a circle, call it Σ, and it intersects each line inside **L** in exactly one point inside Σ. In fact, there is a 1-1 correspondence between the lines inside **L** and the points inside Σ. On the other hand, the intersection of M with planes is a Euclidean line, so the lines in $\mathbf{H}^2$ are in 1-1 correspondence with the chords of Σ, except that we must remember that points *on* circle Σ correspond to lines *on* **L**. So the lines in the Klein model of Hyperbolic plane geometry are exactly the chords of Σ, omitting the endpoints of a chord. In other words, the hyperbolic h-lines in this model are *open* line segments. The following picture contains some points and lines in the Klein model,



the dotted line on the circumference indicating that these points are omitted.

3.1.11a Exercise. Solve Exercise 3.1.10 using the Klein model.

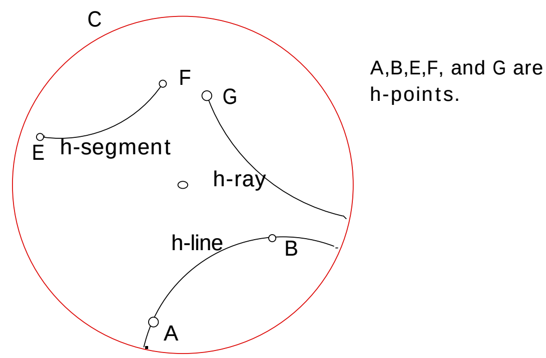
3.1.12 Hyperboloid model. Consider the hyperbola $z^2 - x^2 = 1$ in the x,z-plane. Its asymptotes are the lines $z = \pm x$. Now rotate the hyperbola and its asymptotes about the z-axis. The asymptotes generate the cone **L**, and the hyperbola generates a two-sheeted hyperboloid lying inside **L**; denote the upper hyperboloid by **B**. Then every line through the origin in 3-space intersects **B** exactly once – see **Exercise 3.1.13**; in fact, there is a 1-1 correspondence between the points on **B** and the points in $\mathbf{H}^2$. The lines in $\mathbf{H}^2$ correspond to the curves on **B** obtained by intersecting the planes though the origin in 3-space. With this model, the hyperboloid **B** is a realization of Hyperbolic plane geometry.

3.1.13 Exercise. Prove that every line through the origin in 3-space intersects **B** (in the Hyperbolic model above) exactly once.

3.2 POINCARÉ DISK. Although the line geometries of the previous section provide a very convenient, coherent, and illuminating way of introducing models of non-Euclidean geometries, they are not convenient ones in which to use Sketchpad. More to the point, they are not easy to visualize or to work with. The Klein and Hyperboloid models are more satisfactory ones that conform more closely to our intuition of what a 'plane geometry' should be, but the definition of distance between points and that of angle measure conform less so. We instead focus on the Poincaré Model **D**, introduced by Henri Poincaré in 1882, where 'h-points' are points as we usually think of them - points in the plane - while 'h-lines' are arcs of particular Euclidean circles. This too fits in with our usual experience of Euclidean plane geometry if one thinks of a straight line through point **A** as the limiting case of a circle through point **A** whose radius approaches ∞ as the center moves out along a perpendicular line through **A**. The Poincaré Disk Model allows the use of standard Euclidean geometric ideas in the development of the geometric properties of the models and hence of Hyperbolic plane geometry. We will see later that **D** is actually a model of the "same" geometry as $\mathbf{H}^2$ by constructing a 1-1 transformation from $\mathbf{H}^2$ onto **D**.

Let **C** be a circle in the Euclidean plane. Then **D** is the geometry in which the 'h-points' are the points inside **C** and the 'h-lines' are the arcs *inside* **C** of any circle intersecting **C** at right angles. This means that we omit the points of intersection of these circles with **C**. In addition, any diameter of the bounding circle will also be an h-line, since any straight line through the center of the bounding circle intersects the bounding circle at right angles and (as before) can be regarded as the limiting case of a circle whose radius approaches infinity.

As in the Klein model, points on the circle are omitted and hyperbolic h-lines are *open* --- in this case, open arcs of circles. As we are referring to points inside **C** as *h-points* and the hyperbolic lines inside **C** as *h-lines*; it will also be convenient to call **C** the *bounding circle*. The following figure illustrates these definitions:



More technically, we say that a circle intersecting **C** at right angles is *orthogonal* to **C**. Just as for Euclidean geometry, it can be shown that through each pair of h-points there passes exactly one h-line. A coordinate geometry proof of this fact is included in Exercise 3.6.2. We suggest a synthetic proof of this in Section 3.5. Thus the notion of *h-line segment* between h-points A and B makes good sense: it is the portion between A and B of the unique h-line through A and B. In view of the definition of h-lines, the h-line segment between A and B can also be described as the arc between A and B of the unique circle through A and B that is orthogonal to **C**. Similarly, an *h-ray* starting at an h-point A in **D** is either one of the two portions, between A and the bounding circle, of an h-line passing through A.

Having defined **D**, the first two things to do are to introduce the *distance*, $d_h(A, B)$, between h-points A and B as well as the *angle measure* of an *angle* between h-rays starting at some h-point A. The distance function should have the same properties as the usual Euclidean distance, namely:

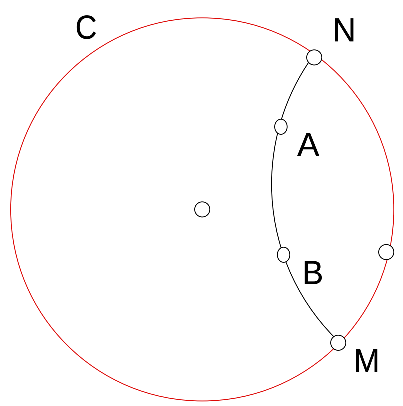
- (Positive-definiteness): For all points A and B ($A \neq B$),
$$d_h(A, B) > 0 \quad \text{and} \quad d_h(A, A) = 0;$$
- (Symmetry): For all points A and B,
$$d_h(A, B) = d_h(B, A);$$
- (Triangle inequality): For all points A, B and C,
$$d_h(A, B) \leq d_h(A,C) + d_h(C, B).$$

Furthermore the distance function should satisfy the **Ruler Postulate.**

3.2.0. Ruler Postulate: The points in each line can be placed in 1-1 correspondence to the real numbers in such a way that:

- each point on the line has been assigned a unique real number (its *coordinate*);
- each real number is assigned to a unique point on the line;
- for each pair of points A, B on the line, $d_h(A, B) = |a - b|$, where a and b are the respective coordinates of A and B.

The function we adopt for the distance looks very arbitrary and bizarre at first, but good sense will be made of it later, both from a geometric and transformational point of view. Consider two h-points A, B in **D** and let M, N be the points of intersection with the bounding circle of the h-line through A, B as in the figure:



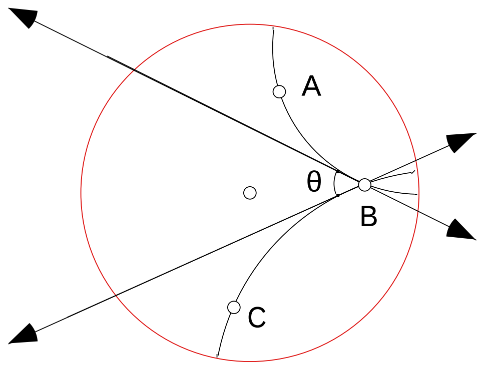
We set

$$d_h(A,B) = \left|\ln\left(\frac{d(A,M)d(B,N)}{d(A,N)d(B,M)}\right)\right|$$

where $d(A,M)$ is the usual Euclidean distance between points A and M. Using properties of logarithms, one can check that the role of M and N can be reversed in the above formula (see Exercise 3.3.7).

3.2.1 Exercise. Show that $d_h(A, B)$ satisfies the positive-definiteness and symmetry conditions above.

We now introduce angles and angle measure in **D.** Just as in the Euclidean plane, two *h*-rays starting at the same point form an angle. In the figure below we see two intersecting h-lines forming BAC.



To find the hyperbolic measure $m_h\ BAC$ of BAC we appeal to angle measure in Euclidean geometry. To do that we need the tangents to the arcs at the point *A*. The hyperbolic measure of the angle BAC is then defined to be the Euclidean measure of the Euclidean angle between these two tangents, i.e. $m_h\ BAC = m\ \theta$.

Just as the notions of points, lines, distance and angle measure are defined in Euclidean plane geometry, these notions are all defined in **D**. And, we can exploit the hyperbolic tools for Sketchpad, which correspond to the standard Euclidean tools, to discover facts and theorems about the Poincaré Disk and hyperbolic plane geometry in general.

- Load the “Poincaré” folder of scripts by moving the sketch “Poincare Disk.gsp” into the Tool Folder. To access this sketch, first open the folder “Samples”, then “Sketches”, then “Investigations”. Once Sketchpad has been restarted, the following scripts will be available:

  - **Hyperbolic Segment** - Given two points, constructs the h-segment joining them
  - **Hyperbolic Line** - Constructs an h-line through two h-points
  - **Hyperbolic P. Bisector** - Constructs the perpendicular bisector between two h-points
  - **Hyperbolic Perpendicular** - Constructs the perpendicular of an h-line through a third point not on the h-line.
  - **Hyperbolic A. Bisector –** Constructs an h-angle bisector.
  - **Hyperbolic Circle By CP** - Constructs an h-circle by center and point.

- **Hyperbolic Circle By CR –** Constructs an h-circle by center and radius.
- **Hyperbolic Angle –** Gives the hyperbolic angle measure of an h-angle.
- **Hyperbolic Distance** - Gives the measure of the hyperbolic distance between two h-points which do not both lie on a diameter of the Poincare disk.

- The sketch “Poincare Disk.gsp” contains a circle with a specially labeled center called, ‘P. Disk Center’, and point on the disk called, ‘P. Disk Radius’. The tools listed above work by using Auto-Matching to these two labels, so if you use these tools in another sketch, you must either label the center and radius of your Poincare Disk accordingly, or match the disk center and radius before matching the other givens for the tool**.** We are now ready to investigate properties of the Poincaré Disk. Use the line tool to investigate how the curvature of *h-lines* changes as the line moves from one passing close to the center of the Poincaré disk to one lying close to the bounding circle. Notice that this line tool never produces h-lines passing through the center of the bounding circle for reasons that will be brought out in the next section. In fact, if you experiment with the tools, you will find that the center of the Poincare Disk and the h-lines which pass through the center are problematic in general. Special tools need to be created to deal with these cases.

(There is another very good software simulation of the Poincaré disk available on the web at

http://math.rice.edu/~joel/NonEuclid.

You can download the program or run it online. The site also contains some background material that you may find interesting.)

3.2.2 Demonstration: Parallel Lines. As in Euclidean geometry, two *h-lines* in **D** are said to be *parallel* when they have no *h-points* in common.

- In the Poincaré disk construct an *h-line* l and an *h-point* P not on l. Use the *h-line* script to investigate if an *h-line* through P parallel to l can be drawn. Can more than one be drawn? How many can be drawn? **End of Demonstration 3.2.2.**

3.2.3 Shortest Distance. In Euclidean plane geometry the line segment joining points P and Q is the path of shortest distance; in other words, a line segment can be described both in *metric* terms and in *geometric* terms. More precisely, there are two natural definitions of a line segment $\overline{PQ}$, one as the shortest path between P and Q, a metric property, the other as all points between P, Q on the unique line l passing through P and Q - a geometric property. But what do we mean by *between*? That is easy to answer in terms of the metric: the line segment $\overline{PQ}$ consists of all points R on l such that $d_h(P,R) + d_h(R,Q) = d_h(P,Q)$. This last definition makes good sense also in **D** since there we have defined a notion of distance.

3.2.3a Demonstration: Shortest Distance.

- In the Poincaré disk select two points A and B. Use the "Hyperbolic Distance" tool to investigate which points C minimize the sum

$$d_h(C,A) + d_h(C,B).$$

What does your answer say about an *h-line segment* between A and B?

End of Demonstration 3.2.3a.

3.2.4 Demonstration: Hyperbolic Versus Euclidean Distance. Since Sketchpad can measure both Euclidean and hyperbolic distances we can investigate hyperbolic distance and compare it with Euclidean distance.

- Draw two h-line segments, one near the center of the Poincaré disk, the other near the boundary. Adjust the segments until both have the same hyperbolic length. What do you notice about the Euclidean lengths of these arcs?
- Compute the ratio

$$d_h(A,B) \big/ d(A,B)$$

of the hyperbolic and Euclidean lengths of the respective hyperbolic and Euclidean line segments between points A, B in the Poincaré disk. What is the largest value you can obtain? **End of Demonstration 3.2.4.**

3.2.5 Demonstration: Investigating d_h further.

- Does this definition of d_h depend on where the boundary circle lies in the plane?
- What is the effect on d_h if we change the center of the circle?
- What is the effect on d_h of doubling the radius of the circle?

By changing the size of the disk, but keeping the points in the same proportion we can answer these questions. Draw an h-line segment $\overline{AB}$ and measure its length.

Over on the toolbar change the select arrow to the **Dilate tool**. Select "P. Disk Center", then **Transform** "Mark Center." Under the **Edit** menu "Select All," then deselect the “Distance =”. Now, without deselecting these objects, drag the P. Disk Radius to vary the size of the P-Disk and of all the Euclidean distances between objects inside proportionally. What effect does changing the size of the P-Disk proportionally (relative to the P-Disk Center) have on the hyperbolic distance between the two endpoints of the hyperbolic segment?

Over on the toolbar change the select arrow to the **Rotate tool**. Select “P-Disk Center”, then **Transform** “Mark Center.” Under the **Edit** menu “Select All,” then deselect the “Distance =”.   Now, without deselecting these objects, drag the P-Disk Radius to rotate the orientation of the P-Disk. What effect does changing the orientation of the P-Disk uniformly have on the hyperbolic distance between the two endpoints of the hyperbolic segment?

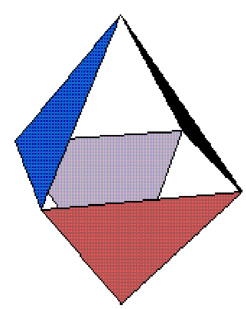
Over on the toolbar, change the Rotate tool back to the **select arrow**. Under the **Edit** menu “Select All,” then deselect the “Distance =”. Grab the P-Disk Center, and drag the Disk around the screen. What effect does changing the location of the P-Disk have on the hyperbolic distance between the two endpoints of the h-line segment?

End of Demonstration 3.2.5.

3.3 Exercises. This Exercise set contains questions related to Abstract Geometries and properties of the Poincaré Disk.

Exercise 3.3.1. Prove that in a 4-Point geometry there passes exactly 3 lines through each point.

Exercise 3.3.2. The figure to the right is an octahedron. Use this to exhibit a model of a 4-Point geometry that is very different from the tetrahedron model we used in class. Four of the faces have been picked out. Use these as the 4 points. What must the lines be if the octahedron is to be a model of a 4-Point geometry? Make sure you check that all the axioms of a 4-Point geometry are satisfied.



Exercise 3.3.3. We have stated that our definition for the hyperbolic distance between two points satisfies the ruler postulate, but it is not easy to construct very long *h-line segments*, say ones of length 10. The source of this difficulty is the rapid growth of the exponential

function. Suppose that the radius of the bounding circle is 1 and let *A* be an *h-point* that has Euclidean distance *r* from the origin ($r < 1$, of course). The diameter of the bounding circle passing through *A* is an *h-line*. Show the hyperbolic distance from the center of the bounding circle to *A* is

$$\left|\ln\frac{(1+r)}{(1-r)}\right|;$$

Find *r* when the hyperbolic distance from *A* to the center of the bounding circle is 10.

Exercise 3.3.4. Use Exercise 3.3.3 to prove that the second statement of the ruler postulate holds when the hyperbolic line is a diameter of the bounding circle and if to each point we assign the hyperbolic distance between it and the center of the bounding circle. That is, why are we guaranteed that each real number is assigned to a unique point on the line? Hint: Show your function for *r* from Exercise 3.3.3 is 1-1 and onto the interval *(-1,1)*.

Exercise 3.3.5. Explain why the ruler postulate disallows the use of the Euclidean distance formula to compute the distance between two points in the Poincaré Disk.

Exercise 3.3.6. Using Sketchpad open the Poincaré Disk Starter and find a counterexample within the Poincaré Disk to each of the following.

(a) If a line intersects one of two parallel lines, then it intersects the other.

(b) If two lines are parallel to a third line then the two lines are parallel to each other.

Exercise 3.3.7. Using properties of logarithms and properties of absolute value, show that, with the definition of hyperbolic distance,

$$d_h(A,B) = \left|\ln\left(\frac{d(A,M)d(B,N)}{d(A,N)d(B,M)}\right)\right| = \left|\ln\left(\frac{d(A,N)d(B,M)}{d(A,M)d(B,N)}\right)\right|,$$

i.e., the roles of *M* and *N* can be reversed and the same distance value results.

3.4 CLASSIFYING THEOREMS. For many years mathematicians attempted to deduce Euclid's fifth postulate from the first four postulates and five common notions. Progress came in the nineteenth century when mathematicians abandoned the effort to find a contradiction in the denial of the fifth postulate and instead worked out carefully and completely the consequences of such a denial. It was found that a coherent theory arises if one assumes the Hyperbolic Parallel Postulate instead of Euclid's fifth Postulate.

Hyperbolic Parallel Postulate: Through a point *P* not on a given line *l* there exists at least two lines parallel to *l*.

The axioms for hyperbolic plane geometry are Euclid's 5 common notions, the first four postulates and the Hyperbolic Parallel Postulate. Three professional mathematicians are credited with the discovery of Hyperbolic geometry. They were Carl Friedrich Gauss (1777-1855), Nikolai Ivanovich Lobachevskii (1793-1856) and Johann Bolyai (1802-1860). All three developed non-Euclidean geometry axiomatically or on a synthetic basis. They had neither an analytic understanding nor an analytic model of non-Euclidean geometry. Fortunately, we have a model now; the Poincaré disk **D** is a model of hyperbolic plane geometry, meaning that the five axioms, consisting of Euclid's first four postulates and the Hyperbolic Parallel Postulate, are true statements about **D**, and so any theorem that we deduce from these axioms must hold true for **D.** In particular, there are several lines though a given point parallel to a given line not containing that point.

Now, an abstract geometry (in fact, any axiomatic system) is said to be **categorical** if any two models of the system are equivalent. When a geometry is categorical, any statement which is true about one model of the geometry is true about all models of the geometry and will be true about the abstract geometry itself. Euclidean geometry and the geometries that result from replacing Euclid's fifth postulate with Alternative A or Alternative B are both categorical geometries.

In particular, Hyperbolic plane geometry is categorical and the Poincaré disk **D** is a model of hyperbolic plane geometry. So any theorem valid in **D** must be true of Hyperbolic plane geometry. To prove theorems about Hyperbolic plane geometry one can either deduce them from the axioms (*i.e.*, give a synthetic proof) or prove them from the model **D** (*i.e.*, give an analytic proof).

Since both the model **D** and Hyperbolic plane geometry satisfy Euclid's first four postulates, any theorems for Euclidean plane geometry that do not require the fifth postulate will also be true for hyperbolic geometry. For example, we noted in Section 1.5 that the proof that the angle bisectors of a triangle are concurrent is independent of the fifth postulate. By comparison, any theorem in Euclidean plane geometry whose proof used the Euclidean fifth postulate might not be valid in hyperbolic geometry, though it is not automatically ruled out, as there may be a proof that does not use the fifth postulate. For example, the proof we gave of the existence of the centroid used the fifth postulate, but other proofs, independent of the fifth postulate, do exist. On the other hand, all proofs of the existence of the circumcenter must rely in some way on the fifth postulate, as this result is false in hyperbolic geometry.

Exercise 3.4.0 After the proof of Theorem 1.5.5, which proves the existence of the circumcenter of a triangle in Euclidean geometry, you were asked to find where the fifth postulate was used in the proof. To answer this question, open a sketch containing a Poincare Disk with the center and radius appropriately labeled (P. Disk Center and P. Disk Radius). Draw a hyperbolic triangle and construct the perpendicular bisectors of two of the sides. Drag the vertices of the triangle and see what happens. Do the perpendicular bisectors always intersect? Now review the proof of Theorem 1.5.5 and identify where the Parallel postulate was needed.

We could spend a whole semester developing hyperbolic geometry axiomatically! Our approach in this chapter is going to be either analytic or visual, however, and in chapter 5 we will begin to develop some transformation techniques once the idea of Inversion has been adequately studied. For the remainder of this section, therefore, various objects in the Poincaré disk **D** will be studied and compared to their Euclidean counterparts.

3.4.1 Demonstration: Circles. A circle is the set of points equidistant from a given point (the center).

- Open a Poincaré Disk, construct two points, and label them by A and O.
- Measure the hyperbolic distance between A and O, $d_h(A, O)$, Select the point A and under the **Display** menu select Trace Points. Now drag A while keeping $d_h(A, O)$ constant.
- Can you describe what a hyperbolic circle in the Poincaré Disk should look like?
- To confirm your results, use the circle script to investigate hyperbolic circles in the Poincaré Disk. What do you notice about the center? **End of Demonstration 3.4.1.**

3.4.2 Demonstration: Triangles. A triangle is a three-sided polygon; two hyperbolic triangles are said to be congruent when they have congruent sides and congruent interior angles. Investigate hyperbolic triangles in the Poincaré Disk.

- Construct a hyperbolic triangle ΔABC and use the “Hyperbolic Angle” tool to measure the hyperbolic angles of ΔABC (keep in mind that three points are necessary to name the angle, the vertex should be the second point clicked).
- Calculate the sum of the three angle measures. Drag the vertices of the triangle around. What is a lower bound for the sum of the hyperbolic angles of a triangle? What is an

upper bound for the sum of the hyperbolic angles of a triangle? What is an appropriate conclusion about hyperbolic triangles? How does the sum of the angles change as the triangle is dragged around **D**?

The proofs of SSS, SAS, ASA, and HL as valid shortcuts for showing congruent triangles did not require the use of Euclid's Fifth postulate. Thus they are all valid shortcuts for showing triangles are congruent in hyperbolic plane geometry. Use SSS to produce two congruent hyperbolic triangles in **D.** Drag one triangle near the boundary and one triangle near the center of **D.** What happens?

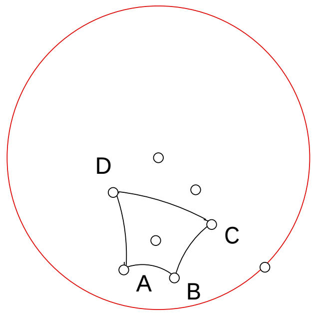
We also had AA, SSS, and SAS shortcuts for similarity in Euclidean plane geometry. Is it possible to find two hyperbolic triangles that are similar but not congruent? Your answer should convince you that it is impossible to magnify or shrink a triangle without distortion! **End of Demonstration 3.4.2.**

3.4.3 Demonstration: Special Triangles. An equilateral triangle is a triangle with 3 sides of equal length. An isosceles triangle has two sides of equal length.

- Create a tool that constructs hyperbolic equilateral triangles in the Poincaré disk. Is an equilateral triangle equiangular? Are the angles always $60°$ as in Euclidean plane geometry?
- Can you construct a hyperbolic isosceles triangle? Are angles opposite the congruent sides congruent? Does the ray bisecting the angle included by the congruent sides bisect the side opposite? Is it also perpendicular? How do your results compare to Theorem 1.4.6 and Corollary 1.4.7? **End of Demonstration 3.4.3.**

3.4.4 Demonstration: Polygons.

- A rectangle is a quadrilateral with four right angles. Is it possible to construct a rectangle in **D**?
- A regular polygon has congruent sides and congruent interior angles.
- To construct a regular quadrilateral in the Poincaré Disk start by constructing an h-circle and any diameter of the circle. Label the intersection points of the diameter and the circle as *A* and *C*. Next construct the perpendicular bisector of the diameter and label the intersection points with the circle as *B* and *D*. Construct the line segments $\overline{AB},\ \overline{BC},$ $\overline{CD},$ and $\overline{DA},$.



ABCD is a regular quadrilateral.

Then *ABCD* is a regular quadrilateral. Why does this work? Create a tool from your sketch.

- The following theorems are true for hyperbolic plane geometry as well as Euclidean plane geometry: Any regular polygon can be inscribed in a circle. Any regular polygon can be circumscribed about a circle. Consequently, any regular *n*-gon can be divided into *n* congruent isosceles triangles just as in Euclidean plane geometry.
- Modify the construction to produce a regular octagon and regular 12-gon. Create tools from your sketches.

End of Demonstration 3.4.4.

By now you may have started to wonder how one could define area within hyperbolic geometry. In Euclidean plane geometry there are two natural ways of doing this, one geometric, the other analytic. In the geometric definition we begin with the area of a fixed shape, a square, and then build up the area of more complicated figures as sums of squares so that we could say that the area of a figure is *n* square inches, say. Since squares don't exist in hyperbolic plane geometry, however, we cannot proceed in this way.

Now any definition of area should have the following properties:

- Every polygonal region has one and only one area, (a positive real number).
- Congruent triangles have equal area.
- If a polygonal region is partitioned into a pair of sub regions, the area of the region will equal the sum of the areas of the two sub regions.

Recall, that in hyperbolic geometry we found that the sum of the measures of the angles of any triangle is less than 180. Thus we will define the defect of a triangle as the amount by which the angle sum of a triangle misses the value 180.

3.4.5 Definition. The defect of triangle ΔABC is the number

$$\delta(\Delta ABC) = 180 - m\angle A - m\angle B - m\angle C$$

More generally, the defect can be defined for polygons.

3.4.6 Definition. The defect of polygon $P_1P_2\ldots P_n$ is the number

$$\delta(\Delta P_1P_2\ldots P_n) = 180(n-2) - m\quad P_1 - m\quad P_2 - \ldots - m\quad P_n$$

It may perhaps be surprising, but this will allow us to define a perfectly legitimate area function where the area of a polygon $P_1P_2\ldots P_n$ is *k* times its defect. The value of *k* can be specified once a unit for angle measure is agreed upon. For example if our unit of angular measurement is degrees, and we wish to express angles in terms of radians then we use the constant $k = \pi/180^\circ$. It can be shown that this area function defined below will satisfy all of the desired properties listed above.

3.4.7 Definition. The area $Area_h(P_1P_2\ldots P)$ of a polygon $P_1P_2\ldots P_n$ is defined by

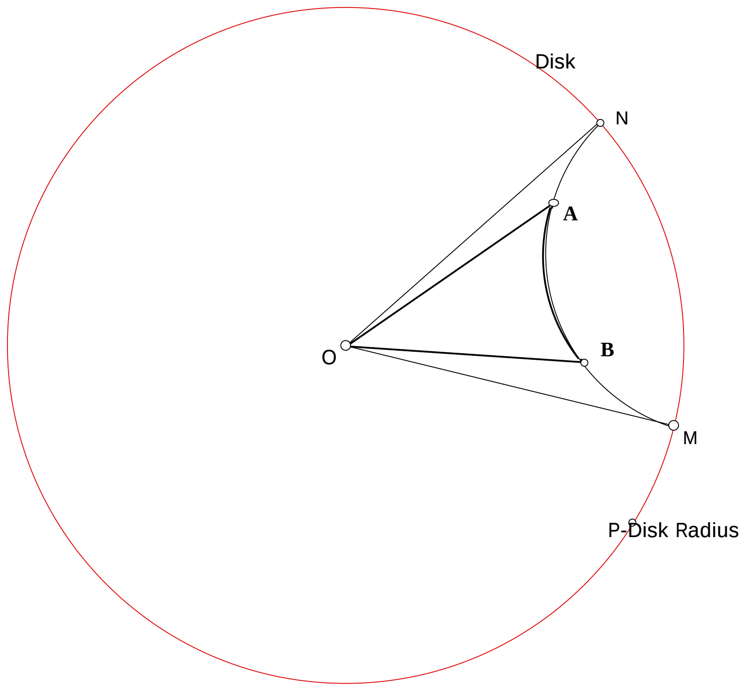
$$Area_h(P_1P_2\ldots P_n) = k\delta(P_1P_2\ldots P_n)$$

where *k* is a positive constant.

Note, that this puts an upper bound on the area of all triangles, namely $180\quad k$. (More generally, $180\quad (n-2)\quad k$ for *n*-gons.) This definition becomes even stranger when we look at particular examples.

3.4.7a Demonstration: Areas of Triangles.

- Open a Poincare disk. Construct a hyperbolic 'triangle' $\Delta_h OMN$ having one vertex *O* at the origin and the remaining two vertices *M*, *N* on the bounding circle. This is not a triangle in the strict sense because points on the bounding circle are not points in the Poincare disk. Nonetheless, it is the limit of a hyperbolic triangle $\Delta_h OAB$ as *A*, *B* approach the bounding circle.



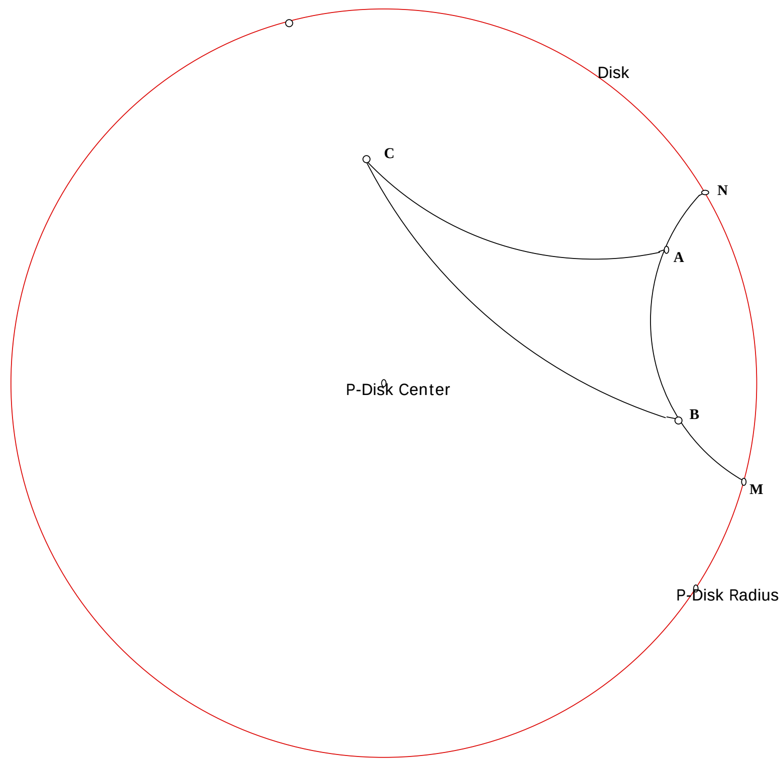
The 'triangle' $\Delta_h OMN$ is called a *Doubly-Asymptotic* triangle.

- Determine the length of the hyperbolic line segment $\overline{AB}$ using the length script. Then measure each of the interior angles of the triangle and compute the area of $\Delta_h OAB$ (use $k=1$). What happens to these values as A, B approach M, N along the hyperbolic line through A, B? Set

$$Area_h(\Delta_h OMN) = \lim Area_h(\Delta_h OAB)$$

Explain this value by relating it to properties of $\Delta_h OMN$.

- Repeat this construction, replacing the center O by any point C in the Poincaré disk.



What value do you obtain for $Area_h(\Delta_h CAB)$? Now let A, B approach M, N along the hyperbolic line through A, B and set

$$Area_h(\Delta_h CMN) = \lim Area_h(\Delta_h CAB);$$

again we say that $\Delta_h(CMN)$ is a doubly-asymptotic triangle. Relate the value of $Area_h(\Delta_h CMN)$ to properties of $\Delta_h CMN$.

- Select an arbitrary point L on the bounding circle and let C approach L. We call $\Delta_h LMN$ a *triply-asymptotic* triangle. Now set

$$Area_h(\Delta_h LMN) = \lim Area_h(\Delta_h CMN).$$

Explain your value for $Area_h(\Delta_h LMN)$ in terms of the properties of $\Delta_h LMN$.

End of Demonstration 3.4.7a.

Your investigations may lead you to conjecture the following result.

3.4.8 Theorem.

(a) The area of a hyperbolic triangle is at most 180*k* even though the lengths of its sides can be arbitrarily large.

(b) The area of a triply-asymptotic triangle is always 180*k* irrespective of the location of its vertices on the bounding circle.

By contrast, in Euclidean geometry the area of a triangle can become unboundedly large as the lengths of its sides become arbitrarily large. In fact, it can be shown that Euclid's Fifth Postulate is equivalent to the statement: *there is no upper bound for the areas of triangles.*

3.4.9 Summary. The following results are true in both Euclidean and Hyperbolic geometries:

- SAS, ASA, SSS, HL congruence conditions for triangles.
- Isosceles triangle theorem (Theorem 1.4.6 and Corollary 1.4.7)
- Any regular polygon can be inscribed in a circle.

The following results are strictly Euclidean

- Sum of the interior angles of a triangle is $180°$.
- Rectangles exist.

The following results are strictly Hyperbolic

- The sum of the interior angles of a triangle is less than $180°$.
- Parallel lines are not everywhere equidistant.
- Any two similar triangles are congruent.

Further entries to this list are discussed in Exercise set 3.6.

As calculus showed, there is also an analytic way introducing the area of a set *A* in the Euclidean plane as a double integral

$$\iint_A dxdy\,.$$

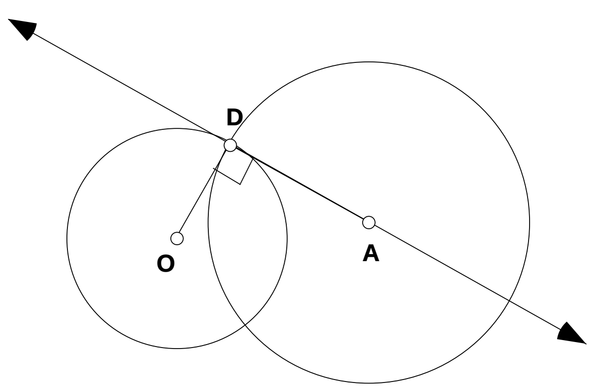
An entirely analogous analytic definition can be made for the Poincaré disk. What is needed is a substitute for *dxdy*. If we use standard polar coordinates (r,θ) for the Poincaré disk, then the hyperbolic area of a set *A* is defined by

$$Area_h(A) = \iint_A \frac{4r\,dr\,d\theta}{1-r^2}\,.$$

Of course, when *A* is an *n*-gon, it has to be shown that this integral definition of area coincides with the value defined by the defect of *A* up to a fixed constant *k* independent of *A*. Calculating areas with this integral formula often requires a high degree of algebraic ingenuity, however.

3.5 ORTHOGONAL CIRCLES. Orthogonal circles, *i.e.* circles intersecting at right angles, arise on many different occasions in plane geometry including the Poincaré disk model **D** of hyperbolic plane geometry introduced in the previous section. In fact, their study constitutes a very important part of Euclidean plane geometry known as Inversion Theory. This will be studied in some detail in Chapter 5, but here we shall develop enough of the underlying ideas to be able to explain exactly how the tools constructing h-lines and h-segments are obtained.

Note first that two circles intersect at right angles when the tangents to both circles at their point of intersection are perpendicular. Another way of expressing this is say that the tangent to one of the circles at their point of intersection *D* passes through the center of the other circle as in the figure below.

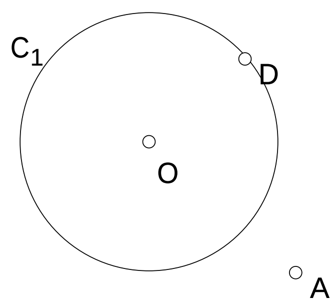


Does this suggest how orthogonal circles might be constructed?

3.5.1 Exercise. Given a circle C_1 centered at O and a point D on this circle, construct a circle C_2 intersecting C_1 orthogonally at D. How many such circles C_2 can be drawn?

It should be easily seen that there are many possibilities for circle C_2. By requiring extra properties of C_2 there will be only one possible choice of C_2. In this way we see how to construct the unique h-line through two points P, Q in **D**.

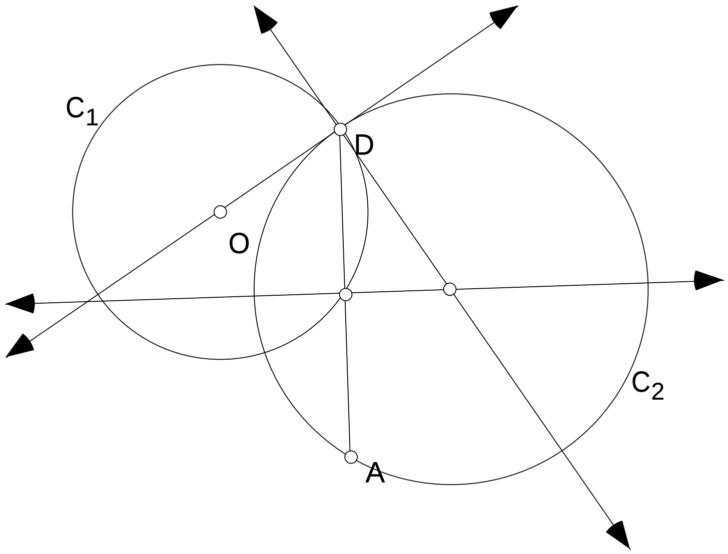
3.5.2 Demonstration. Given a circle C_1 centered at O, a point A not on C_1 , as well as a point D on C_1, construct a circle C_2 passing through A and intersecting C_1 at D orthogonally. How many such circles C_2 can be drawn?



Sketchpad provides a very illuminating solution to this problem.

- Open a new sketch. Draw circle C_1, labeling its center O, and construct point A not on the circle as well as a point D on the circle.
- Construct the tangent line to the circle C_1 at D and then the segment $\overline{AD}$.
- Construct the perpendicular bisector of $\overline{AD}$. The intersection of this perpendicular bisector with the tangent line to the circle at D will be the center of a circle passing through both A and D and intersecting the circle C_1 orthogonally at D. Why?

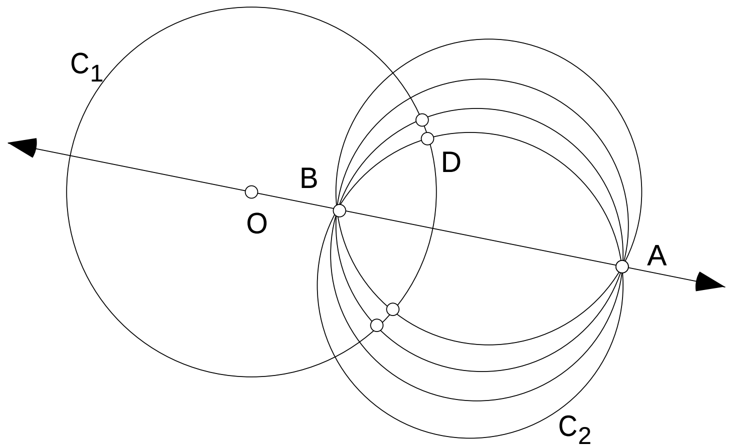
The figure below illustrates the construction when A is outside circle $C_{1.}$



What turns out to be of critical importance is the locus of circle C_2 passing through A and D and intersecting the given circle C_1 orthogonally at D, as D moves. Use Sketchpad to explore the locus.

- Select the circle C_2, and under the **Display menu** select trace circle. Drag D.
- Alternatively you can select the circle C_2, then select the point D and under the **Construct menu** select locus.

The following figure was obtained by choosing different D on the circle C_1 and using a script to construct the circle through A (outside C_1) and D orthogonal to C_2. The figure you obtain should look similar to this one, but perhaps more cluttered if you have traced the circle.



Your figure should suggest that all the circles orthogonal to the given circle C_1 that pass through A have a second common point on the line through O (the center of C_1) and A. In the figure above this second common point is labeled by B. [Does the figure remind you of anything in Physics - the lines of magnetic force in which the points A and B are the poles of the magnet. say?] Repeat the previous construction when A is inside C_1 and you should see the same result.

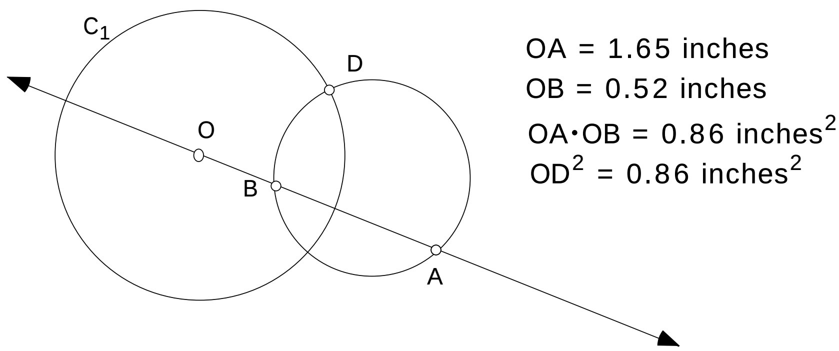
End of Demonstration 3.5.2.

At this moment, Theorem 2.9.2 and its converse 2.9.4 will come into play.

3.5.3 Theorem. Fix a circle C_1 with center O, a point A not on the circle, and point D on the circle, Now let B be the point of intersection of the line through O with the circle through A and D that is orthogonal to C_1. Then B satisfies

$$OA.OB = OD^2.$$

In particular, the point B is independent of the choice of point D. The figure below illustrates the theorem when A is outside C_1.



Proof. By construction the segment $\overline{OD}$ is tangential to the orthogonal circle. Hence $OA.OB = OD^2$ by Theorem 2.9.2. **QED**

Theorem 3.5.3 has an important converse.

3.5.4 Theorem. Let C_1 be a circle of radius r centered at O. Let A and B be points on a line through O (neither A or B on C_1). If $OA.OB = r^2$ then any circle through A and B will intersect the circle C_1 orthogonally.

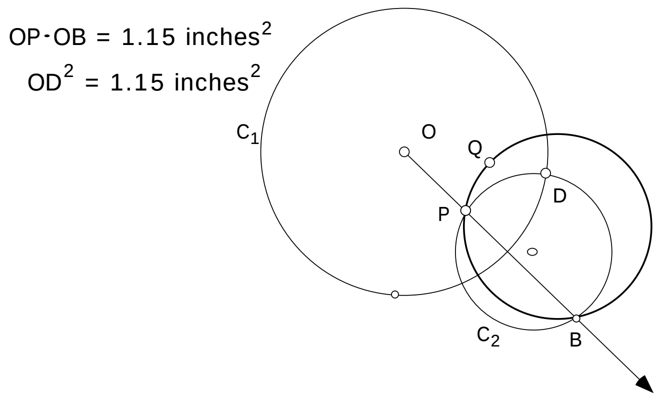
Proof. Let D denote a point of intersection of the circle C_1 with any circle passing through A and B. Then $OA.OB = OD^2$. So by Theorem 2.9.4, the line segment $\overline{OD}$ will be tangential to the circle passing through A, B, and D. Thus the circle centered at O will be orthogonal to the circle passing through A, B, and D. **QED**

Theorems 3.5.3 and 3.5.4 can be used to construct a circle orthogonal to a given circle C_1 and passing through two given points P, Q inside C_1. In other words, we can show how to construct the unique h-line through two given points P, Q in the hyperbolic plane **D.**

3.5.5 Demonstration.

- Open a new sketch and draw the circle C_1, labeling its center by O. Now select arbitrary points P and Q inside C_1.
- Choose any point D on C_1.
- Construct the circle C_2 passing through P and D that is orthogonal to C_1. Draw the ray starting at O and passing through P. Let B be the other point of intersection of this ray with C_2. By Theorem 3.5.3 $OP.OB = OD^2$. Confirm this by measuring OP, OB, and OD in your figure.

· Construct the circumcircle passing through the vertices P, Q and B of ΔPQB. By Theorem 3.3.4 this circumcircle will be orthogonal to the given circle.



If P and Q lie on a diameter of C_1 then the construction described above will fail. Why? This explains why there had to be separate scripts in Sketchpad for constructing h-lines passing through the center of the bounding circle of the Poincaré model **D** of hyperbolic plane geometry.

The points A, B described in Theorem 3.5.3 are said to be *Inverse Points*. The mapping taking A to B is said to be *Inversion*. The properties of inversion will be studied in detail in Chapter 5 in connection with tilings of the Poincaré model **D**. Before then in Chapter 4, we will study transformations. **End of Demonstration 3.5.5.**

3.6 Exercises. In this set of exercises, we'll look at orthogonal circles, as well as other results about the Poincaré Disk.

Exercise 3.6.1. To link up with what we are doing in class on orthogonal circles, recall first that the equation of a circle **C** in the Euclidean plane with radius r and center (h, k) is

$$(x - h)^2 + (y - k)^2 = r^2$$

which on expanding becomes

$$x^2 - 2hx + y^2 - 2ky + h^2 + k^2 = r^2.$$

Now consider the special case when **C** has center at the origin (0, 0) and radius 1. Show that the equation of the circle orthogonal to **C** and having center (h, k) is given by

$$x^2 - 2hx + y^2 - 2ky + 1 = 0.$$

Exercise 3.6.2. One very important use of the previous problem occurs when **C** is the bounding circle of the Poincaré disk. Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$ be two points inside the circle **C**, *i.e.*, two h-points. Show that there is one and only one choice of (h, k) for which the circle centered at (h, k) is orthogonal to **C** and passes through A, B. This gives a coordinate geometry proof of the basic Incidence Property of hyperbolic geometry saying that there is one and only one h-line through any given pair of points in the Poincaré Disk. Assume that A and B do not lie on a diagonal of C.

Exercise 3.6.3. Open a Poincaré Disk and construct a hyperbolic right triangle. (A right triangle has one $90°$ angle.) Show that the Pythagorean theorem does not hold for the Poincaré disk **D**. Where does the proof of Theorem 2.3.4 seem to go wrong?

Exercise 3.6.4. Open a Poincaré Disk. Find a triangle in which the perpendicular bisectors for the sides do not intersect. In Hyperbolic plane geometry, can any triangle be circumscribed by a circle? Can any triangle be inscribed by a circle? Why or why not?

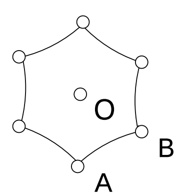
Exercise 3.6.5. Find a counterexample in the Poincaré Disk model for each of the following theorems. That is show each theorem is strictly Euclidean.

(a) The opposite sides of a parallelogram are congruent. (A parallelogram is a quadrilateral where opposite sides are parallel.)

(b) The measure of an exterior angle of a triangle is equal to the sum of the measure of the remote interior angles.

Exercise 3.6.6. Using Sketchpad open a Poincaré Disk. Construct a point P and any diameter of the disk not through P. Devise a script for producing the h-line through P perpendicular to the given diameter (also an h-line).

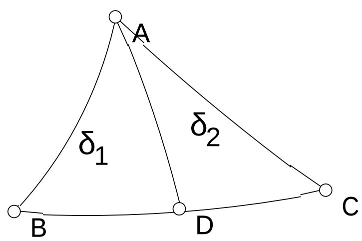
Exercise 3.6.7. The defect of a certain regular hexagon in hyperbolic geometry is 12. ($k=1$)



- Find the measure of each angle of the hexagon.
- If O is the center of the hexagon, find the measure of each interior angle of each sub-triangle making up the hexagon, such as ΔABO as shown in the figure.

- Are each of these sub-triangles equilateral triangles, as they would be if the geometry were Euclidean?

Exercise 3.6.8. Given ΔABC as shown with δ_1 and δ_2 as defects of the sub triangles ΔABD and ΔADC



prove $\delta(\Delta ABC) = \delta_1 + \delta_2$.

Chapter 4

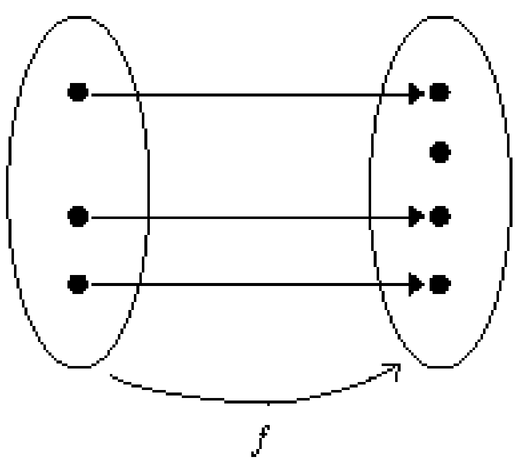
TRANSFORMATIONS

4.1 TRANSFORMATIONS, ISOMETRIES. The term *transformation* has several meanings in mathematics. It may mean any change in an equation or expression to simplify an operation such as computing a derivative or an integral. Another meaning expresses a functional relationship because the notion of a function is often introduced in terms of a mapping

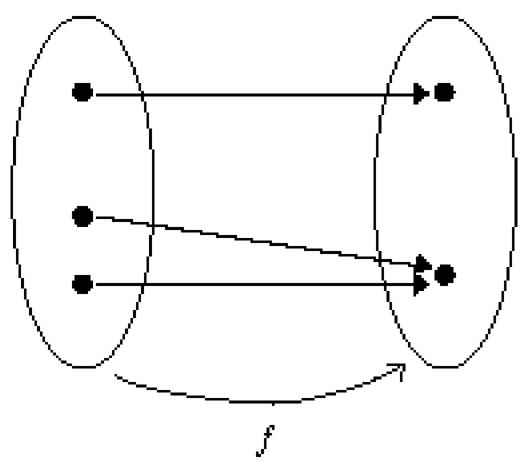
$$f : A \to B$$

between sets A and B; for instance, the function $y = x^2$ can be thought of as a mapping $f: x \to x^2$ of one number line into another. On the other hand, in linear algebra courses a *linear* transformation maps vectors to vectors and subspaces to subspaces. When we use the term transformation in geometry, however, we have all of these interpretations in mind, plus another one, namely the idea that the transformation should map a geometry to a geometry. A formal definition makes this precise.

Recall first that if $f : A \to B$ is a mapping such that every point in the range of f has a unique pre-image in A, then f is said to be *one to one* or *injective*. If the range of f is all of B, then f is said to be *onto* or *surjective*. When the function is both one to one and onto, it is called a *bijection* or is said to be *bijective*. The figures below illustrate these notions pictorially.



one to one



onto

4.1.1 Definition. Let $\mathcal{G}_1 = (\mathcal{P}_1, \mathcal{L}_1)$ and $\mathcal{G}_2 = (\mathcal{P}_2, \mathcal{L}_2)$ be two abstract geometries, and let f : $\mathcal{P}_1 \to \mathcal{P}_2$ a function that is bijective. Then we say that f is a *geometric transformation* if f also maps $\mathcal{L}_1$ onto $\mathcal{L}_2$.

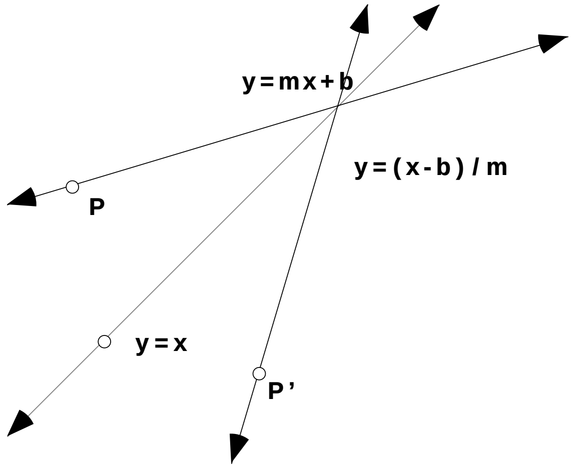
In other words, a 1-1 transformation $f : \mathcal{P}_1 \to \mathcal{P}_2$ is *geometric* if takes the set $\mathcal{P}_1$ of all points in $\mathcal{G}_1$ onto the set $\mathcal{P}_2$ of all points in $\mathcal{G}_2$, and takes the set $\mathcal{L}_1$ of all lines in $\mathcal{G}_1$ onto the set $\mathcal{L}_2$ of all lines in $\mathcal{G}_2$. It is this last property that distinguishes geometric transformations from more general transformations. A more sophisticated way of formulating definition 4.1.1 is simply to say that $f : \mathcal{G}_1 \to \mathcal{G}_2$ is bijective. Notice that the definition makes good sense for models of both Euclidean and hyperbolic geometries. For instance, we shall see later that there is geometric bijection from the model $\mathbf{H}^2$ of hyperbolic geometry in terms of lines and planes in three space and the Poincaré disk model $\mathbf{D}$ in terms of points and arcs of circles.

Some simple examples from Euclidean plane geometry make the formalism much clearer. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ both be models of Euclidean plane geometry so that $\mathcal{P}_1$ and $\mathcal{P}_2$ can be identified with all the points in the plane. For $f : \mathcal{P}_1 \to \mathcal{P}_2$ to be geometric it must map the plane onto itself, and do so in a 1-1 way, as well as map any straight line in the plane to a straight line. It will be important to see how such transformations can be described both algebraically and geometrically. It is easy to come up with functions mapping the plane onto itself, but it is much more restrictive for the function to map a straight line to a straight line. For example, $(x, y) \to (x, y^3)$ maps the plane onto itself, but it maps the straight line $y = x$ to the cubic $y = x^3$.

4.1.2 Examples. (a) Let

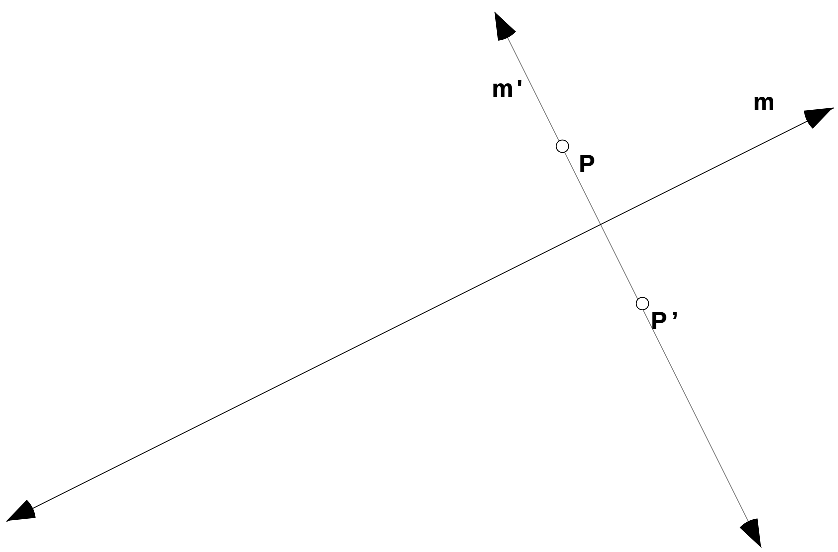
$$f : (x, y) \to (y, x)$$

be the function mapping any point $P = (x, y)$ in the plane to its reflection $P' = (y, x)$ in the line $y = x$. Since successive reflections $P \to P' \to P$ maps P back to itself, this mapping is 1-1 and maps the plane onto itself. But does it map a straight line to a straight line? Well the equation of a non-vertical straight line is $y = mx + b$. The mapping f interchanges x and y, so f maps the straight line $y = mx + b$ to the straight line $y = (x - b)/m$. Algebraically, f maps a non-vertical straight line to its *inverse*. Geometrically, f maps the graph of the straight line $y = mx + b$ to the graph of its straight line inverse $y = (x - b)/m$ as the figure below shows



One can show also that f maps any vertical straight line to a horizontal straight line, and conversely. Hence f maps the family of all lines in Euclidean plane geometry onto itself - hence f is a geometric transformation of Euclidean plane geometry.

(b) More generally than in (a), given any fixed line m, let f be the mapping defined by *reflection* in the line m. In other words, f maps any point in the plane to its 'mirror image' with respect to the *mirror line* m. For instance, when m is the x-axis, then f takes the point $P = (x, y)$ in the plane to its mirror image $P' = (x, -y)$ with respect to the x-axis. In general it is not so easy to express an arbitrary reflection in algebraic terms (see **Exercise Set 4.3**), but it is easy to do so in geometric terms. Given a point P, let m' be the straight line through P that is perpendicular to m. Then P' is the point on m' on the opposite side of m to P that is equidistant from m. Again a figure makes this much clearer



What is important to note here is that all these geometric notions make sense in hyperbolic geometry, so it makes good sense to define reflections in a hyperbolic line. This will be done in Chapter 5 where we will see that this hyperbolic reflection can be interpreted in terms of the idea of inversion as hinted at in the last section of Chapter 3.

(c) Let *f* be a *rotation* through 90˚ counter-clockwise about the some fixed point in the plane. In algebraic terms, when the fixed point is the origin, *f* is given algebraically by $f:(x,y) \to (-y,x)$. So *f* is 1-1 and maps the plane onto itself. What does *f* do to the straight line $y = mx + b$? (see **Exercise Set 4.3**)

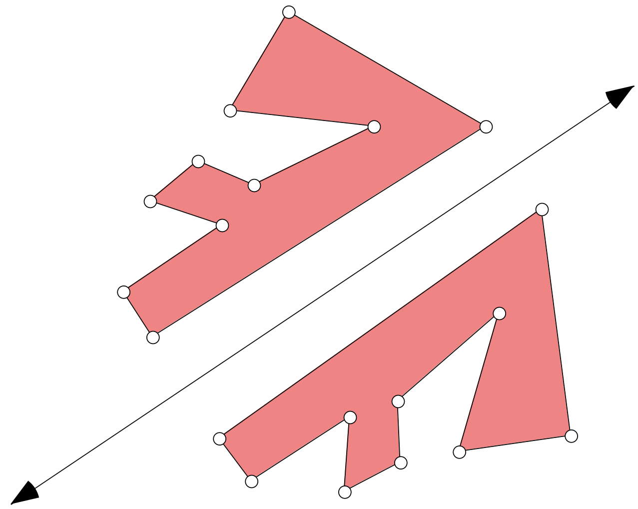
(d) Let *f* be a *translation* of the plane in some direction. Then *f* is given algebraically by $f:(x,y) \to (x+a, y+b)$ for some real numbers *a* and *b*. Again, it is clear that *f* is 1-1 and maps the plane onto itself.

Sketchpad is particularly useful for working with transformations because the basic transformations are all built into the program. We can use Sketchpad to look at the properties of reflections, rotations, and translations.

4.1.2a Demonstration.

- Open a new sketch on Sketchpad and draw a line. This will be the mirror line.
  Construct a polygon in the general shape of an "F". Color its interior.
- To reflect the polygon across the mirror line, select the line and use the **Transform** menu to select "Mark Mirror". Under the **Edit** menu, select "Select All". Then under the **Transform** menu, select "Reflect".
- Try dragging some of the vertices of the polygon to investigate the properties of reflection in the mirror line. What happens when the mirror line is dragged?

Your figure should look like the following:



The *orientation* of the reflected“F” is said to be *opposite* to that of the original “F” because the clockwise order of the vertices of the image is the reverse of the clockwise order of the vertices of the pre-image. In other words, a reflection *reverses* orientation.

· Measure the area of each image polygon and its pre-image. Measure corresponding side lengths. Measure corresponding angles. Check what happens to your measurements as the vertices of the pre-image are dragged. What happens to the measurements when the mirror line is dragged? Now, complete **Conjecture 4.1.3.**

End of Demonstration 4.1.2a.

4.1.3 Conjecture. Reflections __________________ distance, angle measure and area.

4.1.4 Definition. A geometric transformation f of the Euclidean plane is said to be an *isometry* when it preserves the distance between any pair of points in the plane. In other words, f is an isometry of the Euclidean plane, when the equality $d(f(a), f(b)) = d(a,b)$ holds for every pair of points a, b in the plane.

By using triangle congruences one can prove the following.

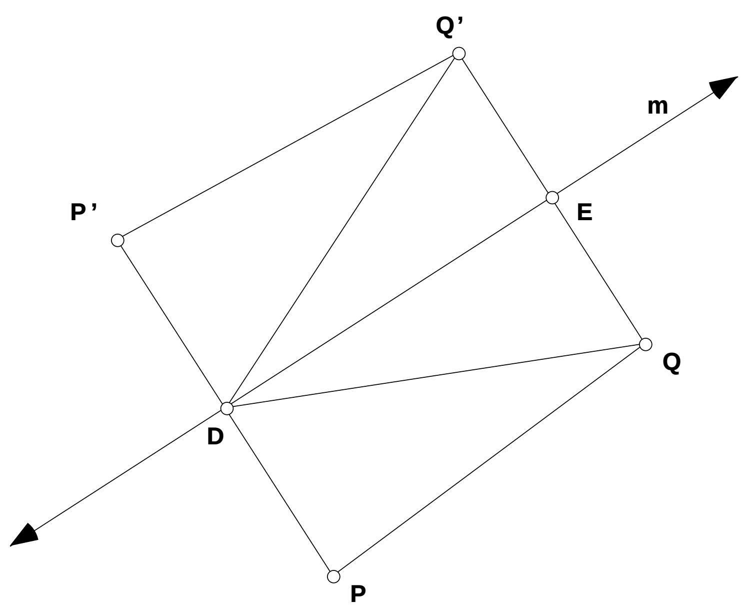
4.1.5 Lemma. Any isometry preserves angle measure.

The earlier Sketchpad activity supports the conjecture that every reflection of the Euclidean plane is an isometry. A proof of this can be given using congruence properties.

4.1.6 Theorem. Every reflection of the Euclidean plane is an isometry.

Proof. In the figure below P and Q are arbitrary points, while P' and Q' are their respective images with respect to reflection in the mirror line m. D and E are the intersection points between the mirror line and the segments $\overline{PP'}$ and $\overline{QQ'}$. For convenience we have assumed that P, Q lie on the same side of the mirror line. Use the definition of a reflection to show first that ΔEDQ is congruent to $\Delta EDQ'$, and hence that $\overline{DQ}$ is congruent to $\overline{DQ'}$. Now use this to show that ΔPDQ is congruent to $\Delta P'DQ'$. Hence $\overline{PQ}$ is congruent to $\overline{P'Q'}$.

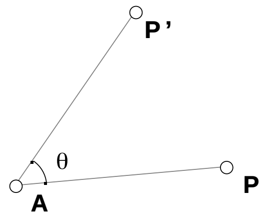
QED



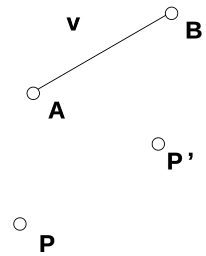
How would this proof have to be modified if P, Q lie on opposite sides of the mirror line? Notice by combining Lemma 4.1.5 with Theorem 4.1.6 we now have a proof of Conjecture 4.1.3.

Two other very familiar transformations of the Euclidean plane are rotations through a given angle about a given fixed point, and translation in a given direction by a fixed amount.

The most precise definition of these are terms of compositions of reflections (as we'll see in the next section), but direct geometric definitions can be given.



Rotation



Translation

Formally, a rotation $\rho_{A,\theta}$ about the point *A* through a directed angle θ is the transformation that fixes *A* and otherwise sends a point *P* to the point P' such that $\overline{AP}$ is congruent to $\overline{AP'}$ and θ is the directed angle measure of PAP'. A translation T_v is the transformation that sends every point *P* the same distance direction, as determined by a given vector *v*. Again, Sketchpad makes the idea clear.

4.1.6a Demonstration.

- Open a new sketch and draw an "F".
- First we'll look at rotations. Construct a point and label it *A*. This will be the 'center' of the rotation, *i.e*., the fixed point. Select the point *A* and then use the **Transform** menu to select "Mark Center A".
- Under the **Edit** menu, select "Select All". Then under the **Transform** menu select "Rotate". The rotate screen will pop up with the angle of rotation θ selected. You can change the degrees in a positive or negative direction.
- Investigate if rotation preserves distance, angle measure and area. Does rotation preserve or reverse orientation?
- Now for translations. Open a new sketch and draw an "F". Construct a line segment in a corner of your sketch and label the endpoints *A* and *B*. First select the endpoints in that order and the use the **Transform** menu to "Mark Vector A->B".

- Using the Marquee (Arrow Tool) select the “F”. Under the **Transform** menu select “Translate”. The translate screen will pop up with “By Marked Vector” selected. Click on “OK”.
- Investigate if translation preserves distance, angle measure and area. Does rotation preserve or reverse orientation? Now, complete **Conjecture 4.1.7.**

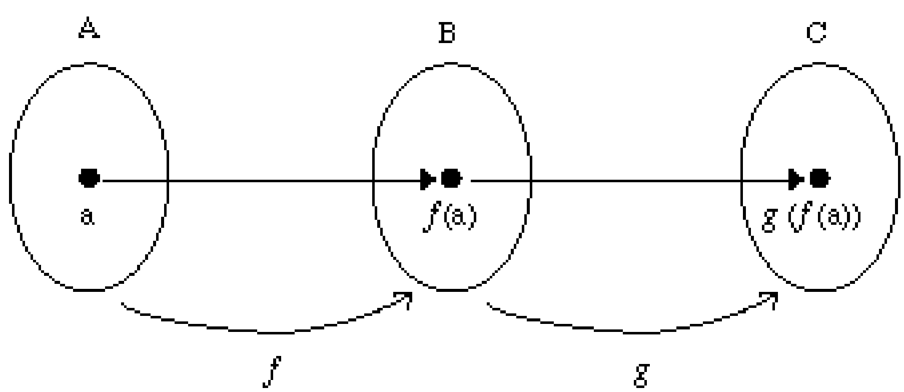
End of Demonstration 4.1.6a.

4.1.7 Conjecture. The rotation $\rho_{A,\theta}$ is ___________________ and also __________________orientation. The translation $T_{A,B}$ is ___________________ and also __________________orientation.

4.2 COMPOSITIONS. The usual composition of functions plays a very important role in the theory of transformations. Recall the general idea of composition of functions. Given functions f: $A \to B$ and g: $B \to C$, mapping a set A into a set B and B into a set C respectively, then the *composition*

$$(g \circ f)(a) = g(f(a)), \qquad (a \quad A)$$

maps A into C. Pictorially, composition can be represented by the figure below



Notice that if f: $A \to B$ and g: $B \to C$ are bijective, then the composition will also be bijective.

4.2.1 Exercise. Show that if f: $\mathcal{G}_1 \to \mathcal{G}_2$ and g: $\mathcal{G}_2 \to \mathcal{G}_3$ are bijective, then the composition $g \circ f$ is bijective from $\mathcal{G}_1$ onto $\mathcal{G}_3$. In other words, the composition of geometric transformations is again geometric.

The concept of geometric transformation is very general. What we do is impose restrictions on a transformation f: $\mathcal{G}_1 \to \mathcal{G}_2$ by imposing extra structure on $\mathcal{G}_1$ and $\mathcal{G}_2$ and then requiring that *f preserve* this extra structure. For instance, when a distance function is defined on $\mathcal{G}_1$ and $\mathcal{G}_2$, we can focus on geometric transformations $f: \mathcal{G}_1 \to \mathcal{G}_2$ that preserve the distance between points - what we called isometries in the case of Euclidean geometry. If a notion of angle measure is defined on $\mathcal{G}_1$ and $\mathcal{G}_2$, then we could focus on geometric transformations that preserve the angle between lines; such transformations are called *conformal* transformations. A complex-valued function $f : \Omega \to \Omega$ which is 1-1 and invertible on a set Ω in the complex plane is conformal whenever f is analytic. This is one reason why analytic function theory is closely connected with geometry. (There are many interesting ideas for semester projects here if one knows something about complex numbers and analytic function theory.)

4.2.2 Theorem. Let f and g be isometric transformations of the Euclidean plane. Then the composition $g \circ f$ of f and g also is an isometric transformation of the Euclidean plane.

Proof. Let P and Q be arbitrary points in the plane. Since f is an isometry,

$$dist(P,Q) = dist(f(P), f(Q)) .$$

But g also is an isometry, so

$$dist(f(P), f(Q)) = dist(g(f(P)), g(f(Q))) .$$

Combining these two results we see that

$$dist(P,Q) = dist((g \circ f)(P), (g \circ f)(Q)).$$

Hence the composition $g \circ f$ preserves lengths and so is an isometry. **QED**

This theorem shows why there are close connections between geometry and group theory. For if $f: \mathcal{G} \to \mathcal{G}$ is a geometric transformation, then f will have an inverse $f^{-1}: \mathcal{G} \to \mathcal{G}$Ā and f^{-1} will be a geometric transformation; in addition, if f is an isometry, then f^{-1} will be an isometry. Thus the set of all geometric transformations $f: \mathcal{G} \to \mathcal{G}$ is a group under composition, while the set of all isometries is a *subgroup* of this group. Now let's look more closely at the set of all isometries of the Euclidean plane - in more elaborate language, we are going to study the *Isometry Group* of the Euclidean plane. In the previous section we saw that any reflection is an isometry. Theorem 4.2.2 ensures that the composition of two reflections will be an isometry, and hence the composition of three, four or more reflections will be isometries as well. But how can we describe the composition of reflections in geometric terms? Let’s first use Sketchpad to see what happens for the composition of two reflections.

4.2.2a Demonstration. The Composition of Two Reflections.

- Open a new sketch and draw two mirror lines *l* and *l'*. Draw an “F” somewhere in the plane.
- Now reflect this “F” first in the mirror line *l* and then in the mirror line *l'*, producing a new image of “F”.
- Describe carefully the position of the final image “F” in relation to the first “F”.
- What happens if the lines *l* and *l'* are parallel. What if they are not parallel?
- You should now be able to complete **Conjecture 4.2.3.**

End of Demonstration 4.2.2a.

4.2.3 Conjecture. The composition of reflections in two mirror lines is a ____________________ when the mirror lines are parallel. The composition of reflections in two mirror lines is a ____________________ when the mirror lines intersect.

To investigate this more carefully, let's go once more to Sketchpad.

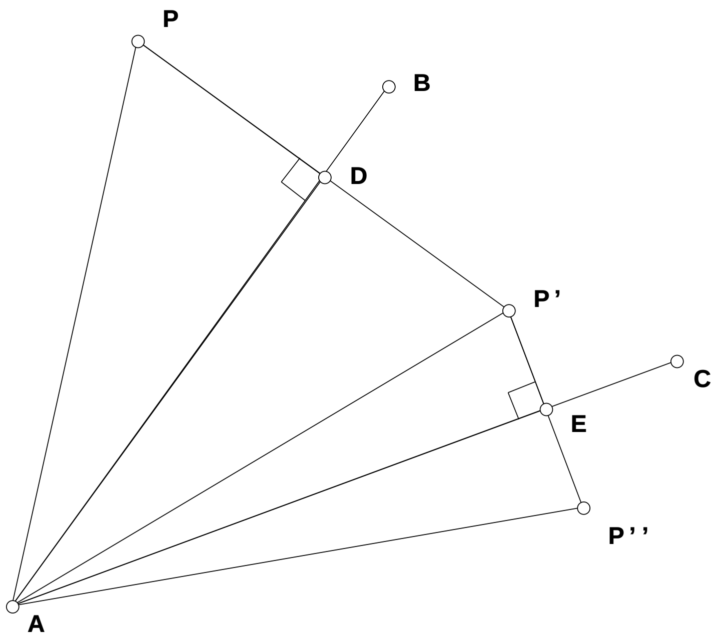
4.2.3a Demonstration.

- Open a new sketch and draw intersecting lines by first choosing three points *A*, *B*, and *C* then drawing two line segments *AB* and *AC*. The reason for constructing the mirror lines in this way is that dragging on *B* or *C* changes the angle between the mirror lines by rotating one of them about the vertex *A*.
- Now draw an “F” on one side of a mirror line and then reflect it successively in the two mirror lines, producing a new image “F” which should appear to be a rotation of the first “F”. Measure first the angle between the mirror lines and then measure the angle by line segments joining the vertex A to corresponding points on the first “F” and its image. Compare the two values. This suggests that **Theorem 4.2.4** is true.

End of Demonstration 4.2.3a.

4.2.4 Theorem. Successive reflection in two intersecting mirror lines produces a rotation about the point of intersection through twice the angle between the mirror lines.

Proof. Consider the following figure, where P is first reflected in the mirror line AB with image P'. Then P' is reflected in the mirror line AC with image P''. There are two pairs of congruent triangles. By construction $PD = DP''$, so ΔPAD is congruent to $\Delta P'AD$ by the SAS criterion. Thus $PAD = P'AD$. By a similar argument $P'AE = P''AE$. Combining these two equalities we see that $PAP'' = 2 \quad DAE$. **QED**



Now lets go back to Sketchpad and look at the case of parallel mirror lines.

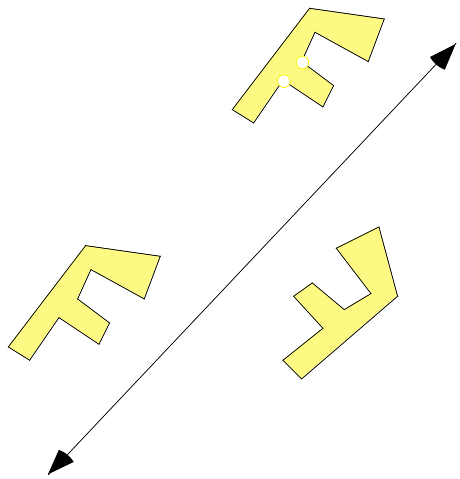
- Open a new sketch and draw two parallel lines. On one side of these lines draw an “ ” and then reflect this successively in the two mirror lines. Drag one of the mirror lines so that it remains parallel to the other mirror line - you can do this by grabbing the line and then dragging. The image “ ”should then appear to be a translate of the first one.
- Measure the distance between the parallel mirror lines and then measure the distance between corresponding points on the first “ ” and the image “ ”. Compare the two values.

4.2.5 Theorem. Successive reflection in parallel mirror lines produces a translation in a direction perpendicular to the mirrors through a distance equal to twice the distance between the mirrors.

Proof. See **Exercise Set 4.3.**

Next it makes sense to look at the composition of three reflections and see if we can describe the result in terms of rotations and translations as well. First we need to introduce one more Euclidean motion of the plane.

4.2.6 Definition. A glide reflection is the composition of a reflection with a translation parallel to the line of reflection.



We should note that sketchpad does not have the glide reflection transformation built into the program. But we could easily build our own using scripts or custom transformations. We'll see how to use custom transformations in the next section.

A transformation in the plane has **direct orientation** if it preserves the orientation of any triangle. If the transformation does not preserve the orientation but reverses it then it has **opposite orientation**. Thus if a motion is the product of an even number of reflections then it will have direct orientation. If a motion is the product of an odd number of reflections then it will have opposite orientation. Rotations and translations are examples of _____________ orientation while reflections and glide reflections show ________________ orientation. This observation will help us when trying to describe the results of composing three reflections.

There are different cases that need to be considered when looking the possible outcomes of reflecting in three mirror lines.

4.2.6a Demonstration.

- **Three Parallel Lines**: What do you get when you reflect something in three parallel lines? Draw three parallel lines and a simple polygonal figure. Reflect the figure successively about the 3 lines. (Hide intermediate figures to avoid confusion)
  What sort of transformation is this? What do the connected midpoints create?
  Draw at least 3 segments joining corresponding points on the pre-image to the final image. For each adjoining segment construct a midpoint and connect them together. Ignoring the three original lines what does this line suggest? How does your answer depend on the order of the lines? Investigate what happens when you change the order of reflection. (Drag the lines, say from #1 to #2)

- **Two Parallel Lines and One Non Parallel**: What is this a composition of?
  Draw two parallel lines and one that crosses them both. Now draw a simple figure on the outside of the parallel line and below the transverse line. Reflect it about the parallel line, then again about the other parallel line. What kind of motion is this? Now reflect it in the transversal. What is this motion called and what is the result of the two combined? Does it make any difference where the figure ends if you reflect it in another sequence, say reflecting it in the transversal first? Does it matter if the transversal is perpendicular to the parallel lines?

- **No Parallel Lines**: What sort of transformation does this case result in?
  Draw three lines that only intersect each other in one place. They should look like a triangle with its sides extended. Pick a place and draw yourself a small figure. Begin reflecting over the lines. What is the end result?

- **Three Concurrent Lines**: What is the line of reflection for this case? To construct concurrent lines make sure the lines intersect at one point. Draw such lines. Draw a small figure between two of the lines. (It will be contained in a V shaped segment)
  Begin your reflections here. What sort of transformation is this? If you reflect a point all the way around the six lines what do you get? Start with a point where you had drawn your figure. Reflect it around each of the lines until you get back to the start. Is the last point is the same place as the first?

What happens if two of the mirror lines are identical? What happens if all three are identical?

You should be able to complete the following:

Product of Two Reflections	If the 2 lines of the reflection are parallel then the motion is a ______________________________.
Product of Two Reflections	If the 2 lines of the reflection are not parallel then the motion is a ___________________________.
Product of Three Reflections	If all 3 of the lines of the reflection are parallel then the motion is a ____________________________.
Product of Three Reflections	If 2 of the lines of the reflection are parallel th the motion is a _______________________________.
Product of Three Reflections	If the 3 lines of the reflection are concurrent then the motion is a ________________________________.
Product of Three Reflections	If the 3 lines of reflection intersect each other only once then the motion is a ____________________.

End of Demonstration 4.2.6a.

With these notes in mind we can realize two of the most important theorems in the theory of isometric transformations of the Euclidean plane.

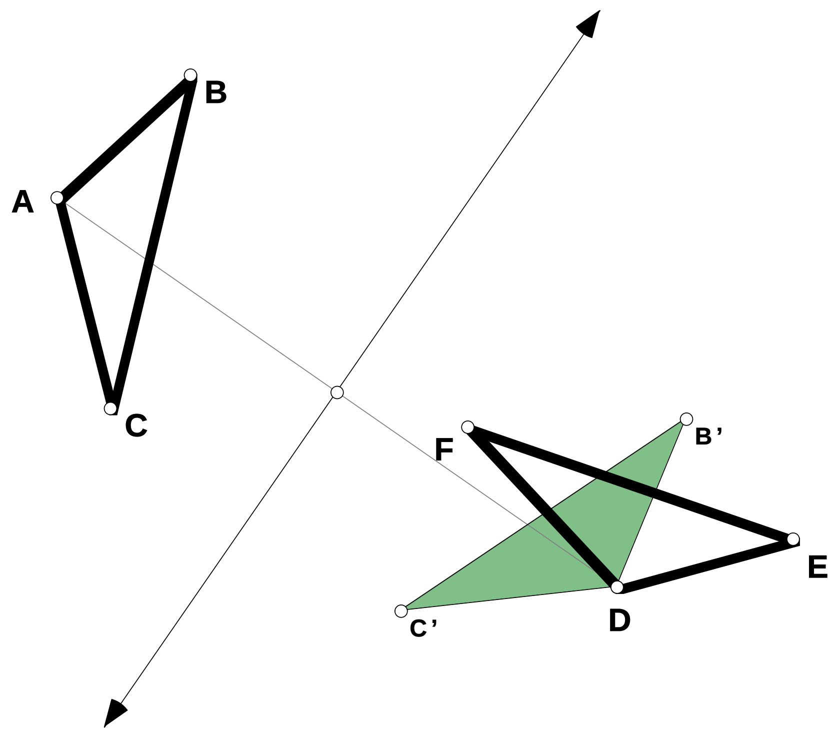
4.2.7 Theorem. Any isometry of the Euclidean plane can be written as a composition of no more than 3 reflections.

As a consequence of our exploration on composition of reflections we get the following as well.

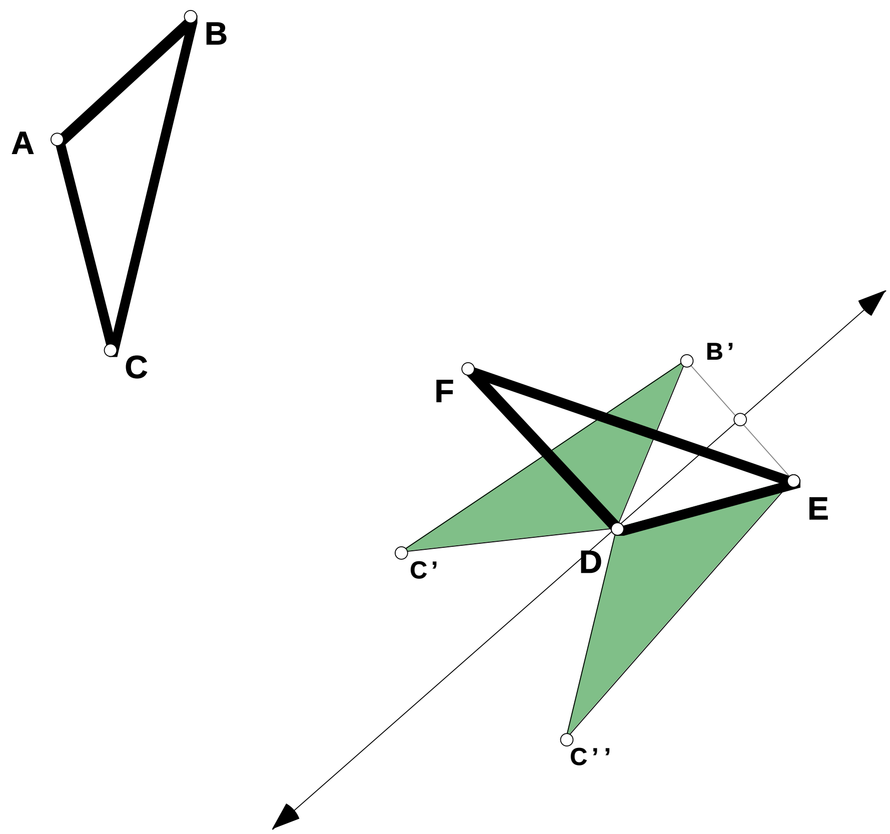
4.2.8 Theorem. Any isometry of the Euclidean plane can be written as one of the following transformations: reflection, rotation, translation or glide reflection.

Crucial to the proof of Theorem 4.2.7 will be the following. To show if we are given an isometry and three points A, B, and C with image points D, E, and F we can take the composition of (at most) three reflections and also map A, B, and C to D, E, and F respectively. If the orientation of the points is preserved it will take two reflections, and otherwise it will take three reflections.

- Open a new sketch and draw two congruent triangles, ΔABC and ΔDEF. We will find a transformation which maps ΔABC to ΔDEF.
- Draw a line segment between A and D and find the midpoint. Construct the line l perpendicular to the line segment and through the midpoint. Reflect ΔABC in l and A will be mapped onto D. So there is one point in the correct position and one reflection.



- If B and C also land on E and F then you would be done. If this is not this case, then we are to map B′ to E by reflecting through the perpendicular bisector of $B'E$ where B' is the image of B under the first reflection. This maps B' to E and keeps D fixed. Why does D stay fixed?



- This leaves you with only C'' (from the original C) to be mapped. If it falls on F after the second reflection then you would be done, but if it does not, map C'' to map to F by reflecting about the line $\overleftrightarrow{DE}$. Why is $\overleftrightarrow{DE}$ the perpendicular bisector of $\overline{FC''}$? Now you are done and it has taken 3 reflections to get from the pre-image to the final image.

Before proving Theorem 4.2.7 we need to establish another property of isometries.

4.2.9 Lemma. An isometry maps any three non-collinear points into non-collinear points.

Proof. Let A, B, and C be non-collinear points. Then by the triangle inequality the non-collinearity means that

$$dist(A,B) + dist(B,C) > dist(A,C).$$

Now let $A', B',$ and C' be the images of $A, B,$ and C. Since the isometry preserves distances,

$$dist(A',B') + dist(B',C') > dist(A',C').$$

But this ensures that $A', B',$ and C' cannot be collinear, proving the lemma. **QED**

Proof of Theorem 4.2.7. Given an isometry F, choose a set of non-collinear points A, B, and C. Let $A' = F(A), B' = F(B),$ and $C' = F(C)$ be their images. Suppose that F has preserved orientation of ΔABC. Then the Sketchpad activity on 'Composition of reflections' shows that there exist reflections S_1 and S_2 so that their composition $S_1 \circ S_2$ has the properties

$$(S_1 \circ S_2)(A) = A' ,\ (S_1 \circ S_2)(B) = B' ,\ (S_1 \circ S_2)(C) = C' .$$

We will prove that

$$(S_1 \circ S_2)(P) = F(P).$$

holds for every point P. So set

$$(S_1 \circ S_2)(P) = P' ,\ F(P) = P'' .$$

We have to show that $P' = P''$. Because $S_1 \circ S_2$ and F are isometries,

$$dist(A',P') = dist(A',P''),\ dist(B',P') = dist(B',P''),\ dist(C',P') = dist(C',P'').$$

Thus A', B', and C' will all lie on the perpendicular bisector of the segment $\overline{P'P''}$ if $P' \neq P''$. But this can happen only if A', B', and C' are collinear. But $A, B,$ and C are not collinear, so A', B', and C' are not collinear. Hence $P' = P''$ showing that $S_1 \circ S_2 = F$. If F does not preserve the orientation of ΔABC then the same proof will show that F can be written as the composition of either one reflection or three reflections. This completes the proof. **QED**

4.3 Exercises. The problems in this assignment are a combination of algebraic and geometric ones.

Exercise 4.3.1. Show that the function $f:(x,y) \to (-y,x)$ maps the straight line $y = mx + b$ to the straight line $y = -(x+b)/m$. Explain the relationship between the slopes of these two lines in terms of the transformation in 4.1.2 (c).

Exercise 4.3.2. Show that reflection in the line $y = mx$ is given by

$$f:(x,y) \to \left(\left(\frac{2m}{m^2+1}\right)y - \left(\frac{m^2-1}{m^2+1}\right)x, \left(\frac{2m}{m^2+1}\right)x + \left(\frac{m^2-1}{m^2+1}\right)y\right).$$

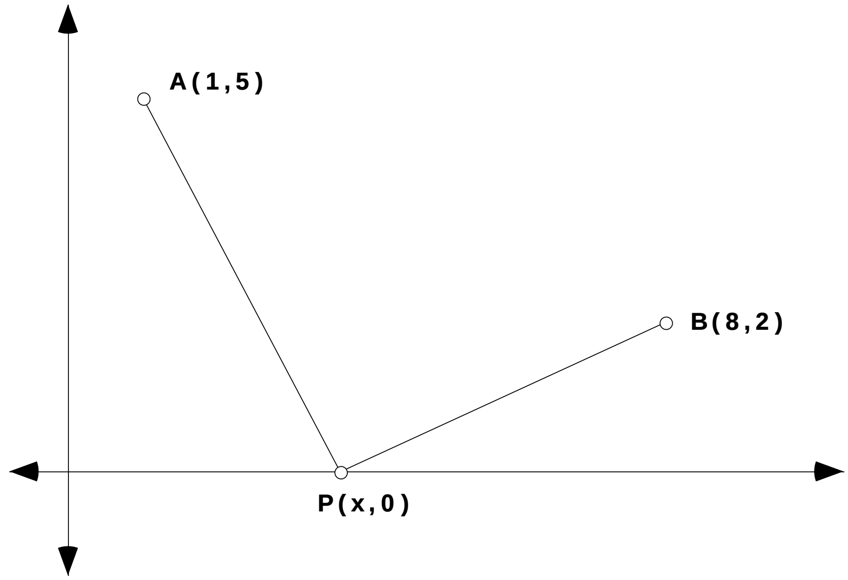
Hint: Let the reflection of the point $P = (x, y)$ be $P' = (x', y')$. You need to find two equations and then solve for x', y'. Let Q be the midpoint of PP'; so what are its coordinates? The point Q also lies on the mirror line $y = mx$; so what does this say about the coordinates of Q'? Use this to get the first equation for x', y'. The line PP' is perpendicular to the mirror line = mx. How can we use this to get a second equation for x', y'? Now solve the two equations you have obtained.

Exercise 4.3.3. Prove synthetically that every rotation $\rho_{A,\theta}$ is an isometry.

Exercise 4.3.4. Prove that successive reflections in parallel mirror lines produce a translation in a direction perpendicular to the mirrors through a distance equal to twice the distance between the mirrors.

Exercise 4.3.5. Suppose you wish to join the two towns $A(1,5)$ and $B(8,2)$ via a pipeline. A pumping station is to be placed along a straight river bank (the x-axis). Determine the location of a pumping station, $P(x,0)$, that minimizes the amount of pipe used? Solve this

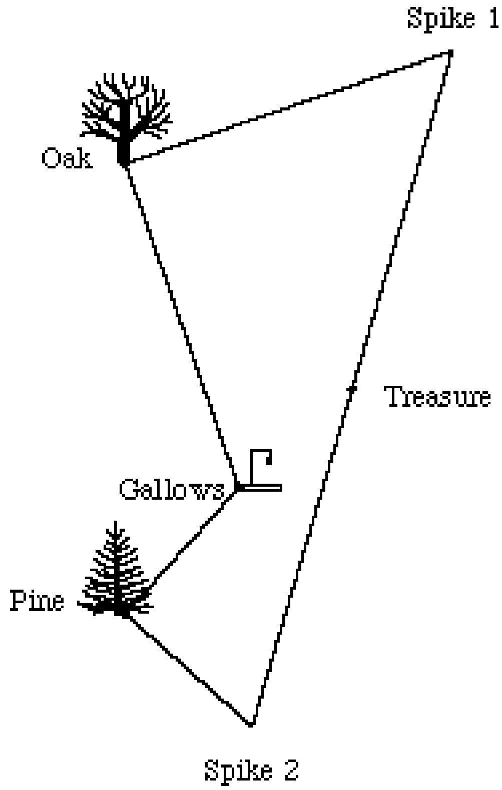
- by transformations.
- by calculus.



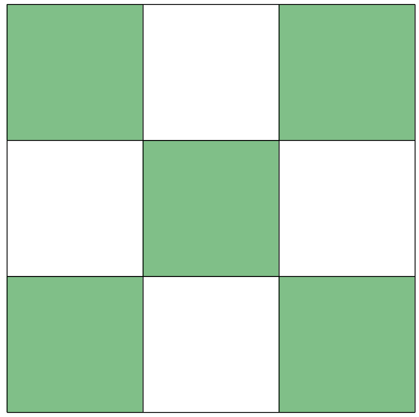
Exercise 4.3.6 Buried Treasure. Among his great-grandfather's papers, José found a parchment describing the location of a hidden treasure. The treasure was buried by a band of pirates on a deserted island which contained an oak tree, a pine tree, and a gallows where the pirates hanged traitors. The map looked like the accompanying figure and gave the following directions.

"Count the steps from the gallows to the oak tree. At the oak, turn 90° to the right. Take the same number of steps and then put a spike in the ground. Next, return to the gallows and walk to the pine tree, counting the number of steps. At the pine tree, turn 90° to the left, take the same number of steps, and then put another spike in the ground. The treasure is buried halfway between the spikes."

José found the island and the trees but could not find the gallows or the spikes, which had long since rotted. José dug all over the island, but because the island was large, he gave up. Devise a plan to help José find the treasure.



4.4 TILINGS REVISITED. To illustrate further the idea of reflections, rotations, translations, and glide reflections we want to begin the geometric analysis of 'wallpaper' designs. A wallpaper design is a tiling of the plane that admits translational symmetry in two directions. That is the design can be "moved" in two different directions and coincide with itself. The checkerboard below would produce a wallpaper design if continued indefinitely.



First we notice that certain rotations are admissible. For the checkerboard we can rotate by $90°$ (quarter-turn) about the center of any green or white square and repeat the same figure. Also we can rotate by $180°$ (half-turn) about the vertex of any square and repeat the same figure. There are wallpaper designs that admit $60°$ (sixth-turn) rotations and $120°$ (third-turn) rotations. What is more remarkable is that these are the only rotations allowed in any wallpaper design! A simple argument shows why. (See Crowe) To get you started on the fifth-turn case, try the following. Choose one center of rotation P and then choose another center of rotation that is closest to Q. Next argue why this cannot happen. The n-th turn case is even easier.

This restriction on rotations provides a convenient way to analyze wallpaper patterns. In fact, it can be shown that there are only 17 different types of wallpaper designs!

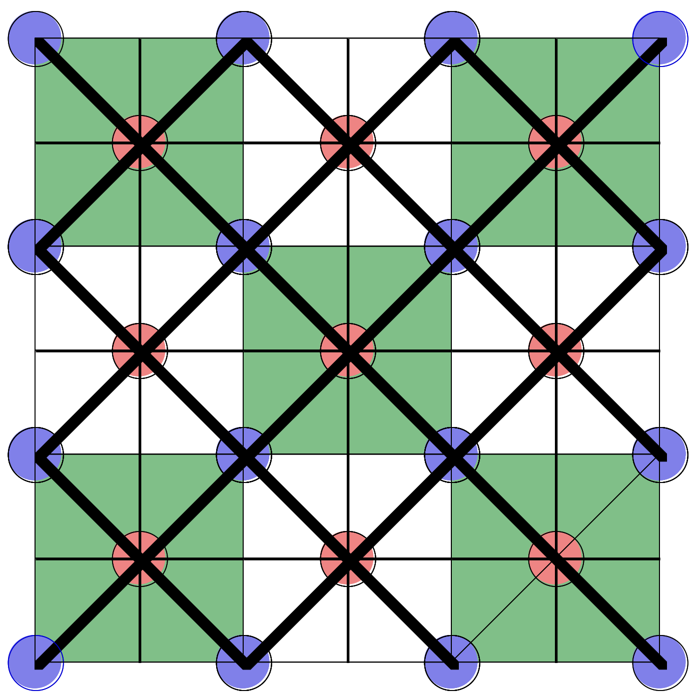
- Four which have no rotations at all;
- Five whose smallest rotation is $180°$;
- Three whose smallest rotation is $120°$;
- Three whose smallest rotation is $90°$;
- Two whose smallest rotation is $60°$;

There is a simple flowchart one can use to classify any wallpaper design. The symbols for the patterns have special meaning: *m* means mirror, *g* means glide, and a number like 2 or 4 means half-turn or quarter-turn.

Let's go back to our checkerboard design - we shall think of it as extending over the whole plane to form a tiling by congruent copies of a single square. An alternating coloring has been added for extra effect. This tiling will be left unchanged by various reflections and rotations about various points.

- Go back to the checkerboard figure and mark in the mirror lines with respect to which a reflection leaves the design unchanged. Mark the mirror lines in bold. Mark in red the centers of rotation through 90˚ that leave the design unchanged. Mark also in blue the centers of rotation through 180˚ that leave the design unchanged.

Your pattern should look like the one below.



- This tiling is classified as "p4m". The smallest rotations allowed are quarter -turns and there are reflections in four directions.
- Successive use of the reflections and rotations fixing the design would replicate the whole tiling from just one white square and one colored tile. Can the whole tiling be generated from any part smaller than theses two squares? Find the smallest piece from which the whole tiling could be generated by successive reflections and rotations. This smallest piece is called a *Fundamental Domain.*

How would the pattern of reflections and rotations differ if the tiling consisted of all white squares? What is a Fundamental domain of the new monochromatic tiling?

You can continue to examine wallpaper designs in the next set of exercises. Now we will assemble all the results and ideas developed about transformations and tilings to show how to use Sketchpad to construct figures with a prescribed symmetry. First let's see how to use Custom Tools to define our own transformations.

4.4.1 Demonstration. Custom transformations.

A custom transformation is a sequence of one or more transformations. The basic steps are given below.

- Transform an object one or more times.
- Hide any intermediate objects or format them as you wish them to appear when you apply your transformation.
- Select the pre-image and image, and select and show the labels of all marked transformation parameters.
- Create a new tool. The pre-image and transformation parameters will become given objects in the custom tool.
- In the custom tool's Script View, set each of the given transformation parameters—mirrors, centers, and so forth—to automatically match objects with the same label..

For example, let's define a rotation $\rho_{A,\theta}$ through a given angle θ about a given point *A*.

- Open a new sketch and construct a point *A* and any point *P*. Mark *A* as a center of rotation. Then construct the point P' which is the rotation of *P* about *A* through an angle θ (choose any θ).
- Next select *P* and P' and *A*. Choose "Create New Tool from the **Tools** menu. Type a name that describes the transformational sequence. In the Script View window, double click on Point A in the "Given" section and check the box "Automatically Match Sketch Object".
- You can now apply your custom transformation to any figure in your sketch. Draw any polygonal figure in your sketch and construct its interior. Select the polygon interior and apply the tool.

Repeat this process to define a reflection S_m about a given mirror line m and a translation T_v in a given direction.

End of Demonstration 4.4.1.

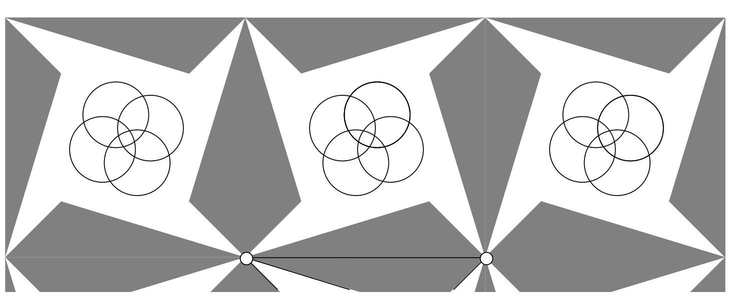
When you define a multi-step transformation, Sketchpad remembers the formatting you've applied to each step's image—whether you've colored it, or hidden it, and so forth. When you apply the transformation to new objects, Sketchpad creates intermediate images with exactly the same formatting. If you are interested only in the final image of the sequence of transformational steps, and not in the intermediate images, hide each intermediate image between your two selected objects before defining the transformation. If you want your transformed images to have a certain color, then be sure your image has the appropriate color when you define the transformation.

4.4.2 Demonstration. Producing a picture with *p4g* symmetry.

To utilize these ideas and generate the symmetries necessary for producing a picture having *p4g* symmetry:

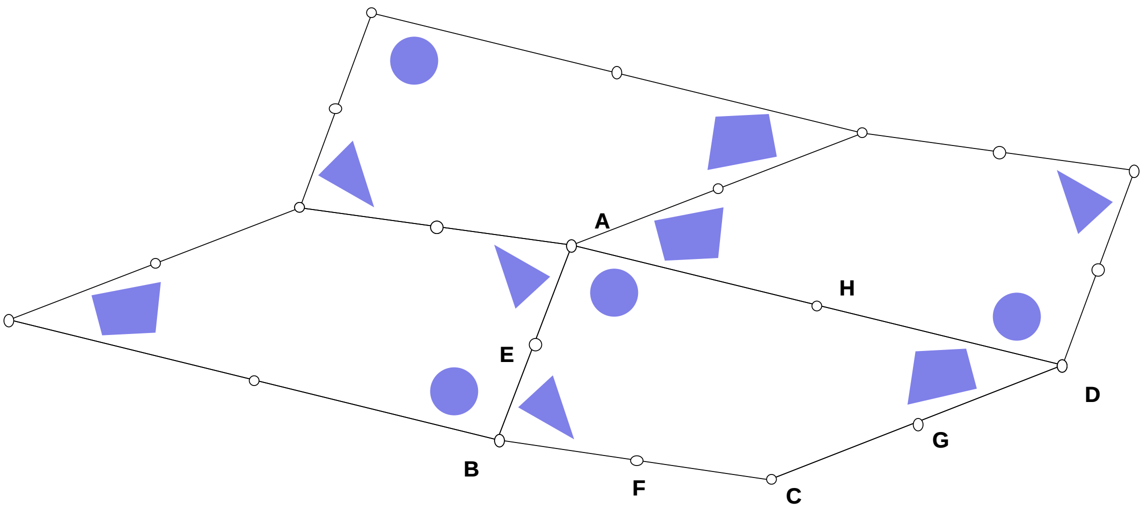
- Create a tool which performs a 4-fold rotation about *A*; call it *4-foldrot.* Construct a 2-fold rotation about *B*; call it *2-foldrot.* Finally construct a reflection about the side $\overline{BC}$ of ΔABC.
- Construct a right-angled isosceles triangle ΔABC having a right angle at *A*; this will be the fundamental domain of the figure.

Now you are free to draw any figure having p4g symmetry. Below is one example. The original Δ has been left in. The picture was constructed from one triangle inside the fundamental domain and one circle. The most interesting designs usually occur when the initial figure 'pokes' outside the fundamental domain. The vertices of the original triangle can be dragged to change the appearance of the design; the original design can be dragged too. This often results in a radical change in the design.



End of Demonstration 4.4.2

Earlier, as a consequence of the Euclidean parallel postulate, we saw that the sum of the angles of a triangle is always 180˚ no matter the shape of the triangle; similarly the sum of the angles of a quadrilateral is always 360˚ no matter the shape of the quadrilateral. Somewhat later we gave a more careful proof of this fact by determining the sum of the angles of any polygon - in fact we saw that the value depends only on the number sides of the polygon. This value was then used to show that equilateral triangles, squares and regular hexagons are the only regular polygons that tile the Euclidean plane. But nothing was said about the possibility of non-regular polygons tiling the plane. In fact, any triangle or quadrilateral can tile the plane. The figure below illustrates the case of a convex quadrilateral. *ABCD* was the original quadrilateral and *E, F, G, H* are the respective midpoints. One can obtain the figure below by rotating by 180˚ about the midpoint of each side of the quadrilateral. (You can tile the plane with any triangle by the same method – try it!)

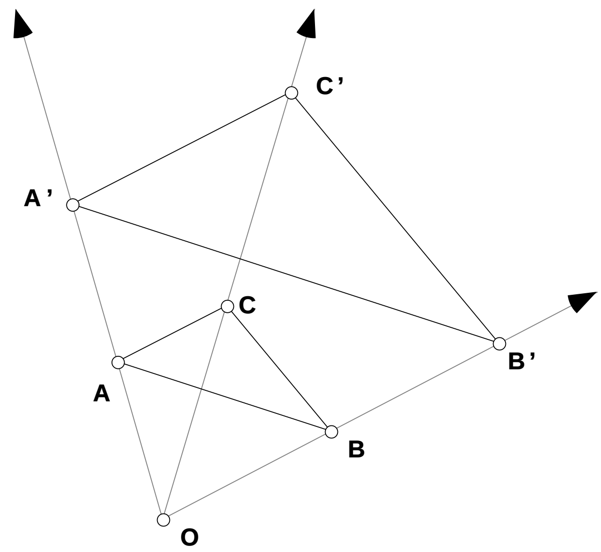


To do this for yourself, you can use custom transformations. Define a transformation for each midpoint. I've drawn a different figure in each of the corners of the chosen quadrilateral to help me distinguish among the corners. Use your four rotations to produce a tiling of the plane by congruent copies of the original quadrilateral with one copy of each of the four corners occurring at every vertex. Join neighboring images of the midpoints by line segments. What resulting repeating diagram emerges? You should see an overlay of parallelograms. Can you find a parallelogram and points so that successive rotations of the parallelogram through 180˚ about the points would produce the same tiling?

4.5 DILATIONS. In this section we would like look at another type of mapping, dilation, that is frequently used in geometry. Dilation will not be an isometry but it will have another useful property, namely that it preserves angle measure.

4.5.1 Definition. A geometric transformation of the Euclidean Plane is said to be conformal when it preserves angle measure. That is, if A', B', and C' are the images of A, B, and C then $m \quad A'B'C' = m \quad ABC$.

4.5.2 Definition. A dilation with center O and dilation constant $k \neq 0$ is a transformation that leaves O *fixed* and maps any other point P to the point P' on the ray OP such that $OP' = k \quad OP$.



4.5.2a Demonstration. Dilation with Sketchpad.

Sketchpad has the dilation transformation built into the program.

· Open a new sketch and construct a point O and ΔABC.

· Select O and then "Mark Center O." under the **Transform** menu.

· Select ΔABC and then select "dilate" from the **Transform** menu.

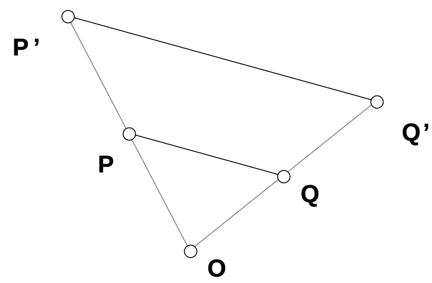
· Enter the desired scale factor (dilation constant). (In the figure above the dilation constant is equal to 2. Notice that in the dialogue box, the scale factor is given as a fraction. In this case, we would either enter $\frac{2}{1}$ or $\frac{1}{0.5}$.)

· What is the image of a segment under dilation? Is the dilation transformation is conformal?

· Next construct a circle and dilate about the center O by the same constant. What is the image of a circle?

End of Demonstration 4.5.2a.

4.5.3 Theorem. The image of $\overline{PQ}$ under dilation is a parallel segment, $\overline{P'Q'}$ such that $P'Q' = |k|\, PQ$



Proof. From SAS similarity it follows that $\Delta POQ \sim \Delta P'OQ'$ and thus $P'Q' = |k|\ PQ$. The proof needs to be modified when *O,P,* and *Q* are collinear.

4.5.4 Theorem. The dilation transformation is conformal.

Proof. See **Exercise Set 4.6**.

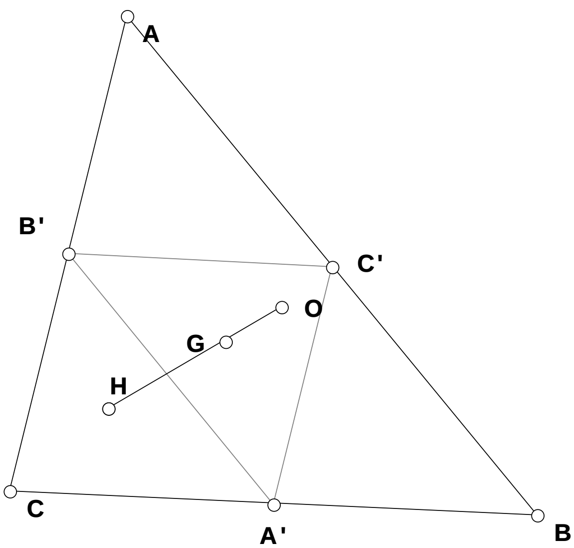
One can easily see that the following theorem is also true. The idea for the proof is to show that all points are a fixed distance from the center.

4.5.5 Theorem. The image of a circle under dilation is another circle.

Proof. Let O be the center of dilation, Q be the center of the circle, and P be a point on the circle. *Q'* will be the center of the image circle. By Theorem 4.5.3, $\dfrac{P'Q'}{PQ} = \dfrac{OQ'}{OQ}$ or

$P'Q' = \dfrac{PQ\ \ OQ'}{OQ}$. Now each segment in the right-hand expression has a fixed length so *P'Q'* is a constant. Thus for any position of *P, P'* lies on a circle with center *Q'*.

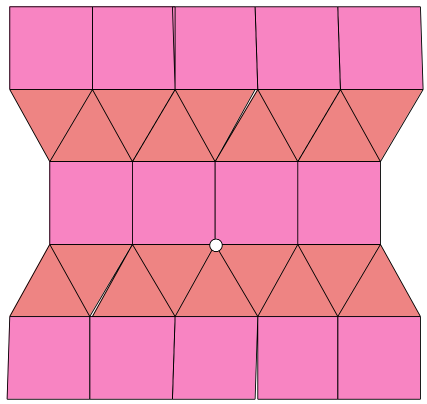
Using dilations we can provide an alternate proof for the fact that the centroid of a triangle trisects the segment joining the circumcenter and the orthocenter (The Euler Line).

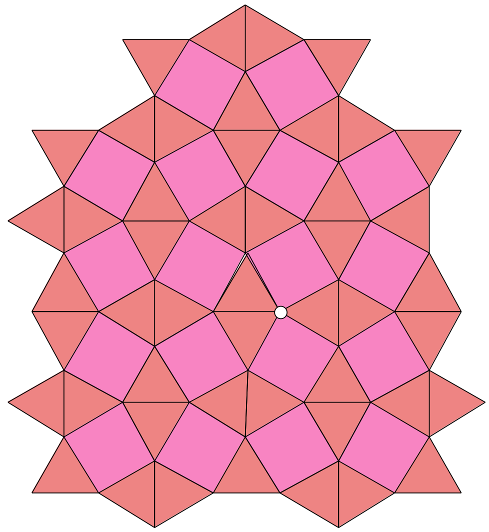


Given ΔABC with centroid G, orthocenter H, and circumcenter O. Let A', B', and C' be the midpoints of the sides. First note that O is the orthocenter of $\Delta A'B'C'$ and that G divides each median into a 2:3 ratio. Thus if we dilate ΔABC about G with a dilation constant of $-\frac{1}{2}$, ΔABC will get mapped to $\Delta A'B'C'$ and H will get mapped to O (their orthocenters must correspond). Hence O, G, and H must be collinear by the definition of a dilation and $OG = \frac{1}{2}HG$. **QED.**

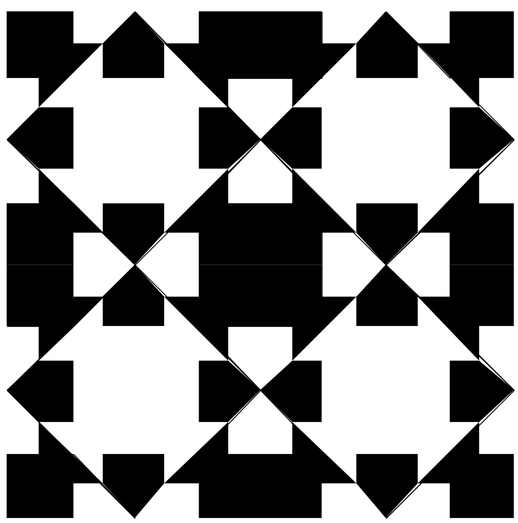
4.6 Exercises.

Exercise 4.6.1. Recall the two regular tilings of order 2 produced with squares and triangles. Classify each as a wallpaper design.

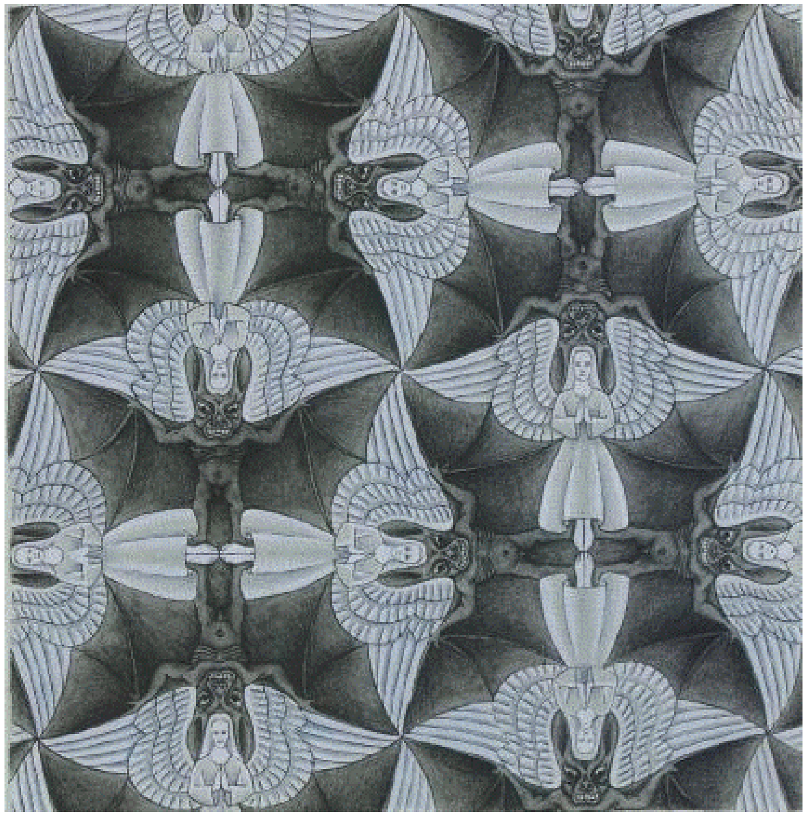




Exercise 4.6.2. Classify the following wallpaper design. Is there any relation to the checkerboard tiling?

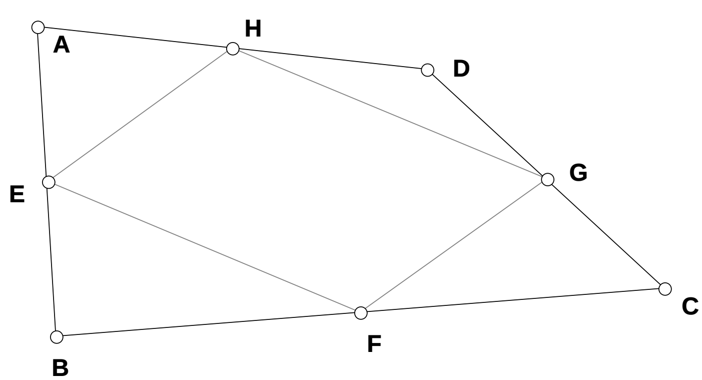


Exercise 4.6.3. What type of wallpaper design is Escher's version of 'Devils and Angels' for Euclidean geometry?



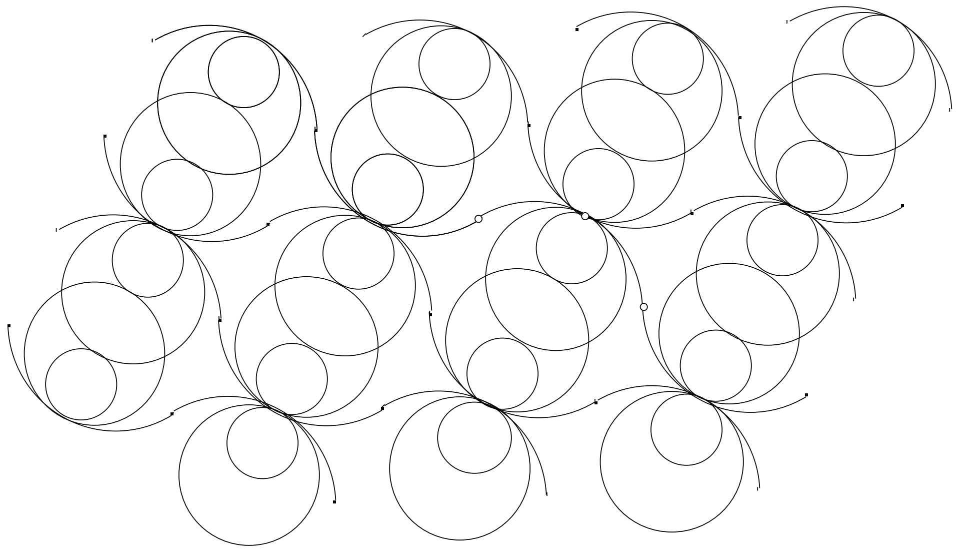
Exercise 4.6.4. On sketchpad use custom transformations to create a wallpaper design other than a p4g.

Exercise 4.6.5. Let *ABCD* be a quadrilateral. In the figure below *E*,*F*,*G*, and *H* are the midpoints of the sides. Prove that *EFGH* is a parallelogram. Hint: Similar triangles.



Exercise 4.6.6. Escher's lizard graphic is shown below. Mark all the points in the picture about which there are rotations by 180˚. What do you notice about these points? Exhibit a parallelogram and three points about which successive rotations through 180˚ would produce Escher's design. What is the wallpaper classification for the lizard design?

Exercise 4.6.7. Now pretend that you are Escher. Start with a parallelogram *PQRS.* Draw some geometric design inside this parallelogram - a combination of circles and polygons, say. Choose three points and define rotations through 180˚ about these points so that successive rotations about these three point tiles the plane with congruent copies of your design. Try making a second design allowing some of the circles and polygons to fall outside the initial parallelogram - this usually produces a more interesting picture. Here's one based on two circles and an arc of a circle

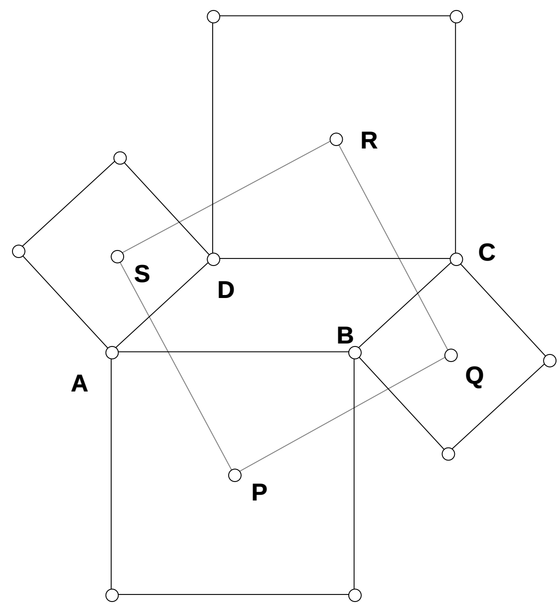


Exercise 4.6.8. Prove Theorem 4.5.4. The dilation transformation is conformal.

4.7 USING TRANSFORMATIONS IN PROOFS

Transformations can also be useful in proving certain theorems, sometimes providing a more illuminating proof than those accomplished by synthetic or analytic methods. We "discovered" Yaglom's Theorem in the second assignment and re-visited it while looking at tilings. There is an easy proof that uses transformations.

4.7.1 Theorem. Let *ABCD* be any parallelogram and suppose we construct squares externally on each side of the parallelogram. Then centers of these squares also form a square.



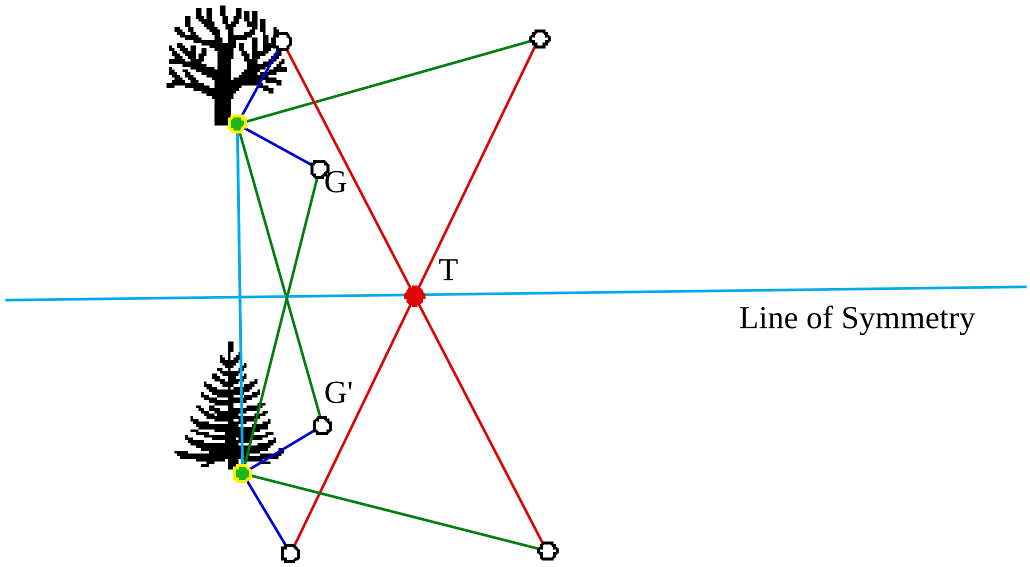
Proof. Consider the rotation about P by 90˚. (Try it on sketchpad.) The square centered at P will rotate onto its original position and $\overline{AB}$ must rotate to $\overline{A'A}$, so the square centered at Q will rotate to onto the square centered at S. Thus their centers will coincide. This tells us that the segment $\overline{PQ}$ rotates 90˚ onto the segment $\overline{PS}$, and therefore $PQ=PS$ and $m\ \ QPS=90$. Do the same for the other centers Q, R, and S. Thus $PQRS$ is a square.

QED

Earlier in this chapter we looked at the Buried Treasure problem (**Exercise 4.3.6**). After working with the Treasure sketch one notices that the location of the treasure is likely to be independent of the position of the gallows. If we use this observation as an assumption, then perhaps we can gain an understanding as to where the treasure is buried with respect to the trees.

The map's instructions are very symmetrical. Since the only reference points are the two trees, a symmetry argument will be used with objects reflected across the perpendicular bisector of the segment joining the trees. Choose a position for the gallows (G) near the Oak tree, and its reflection (G') near the Pine tree (Figure 1).

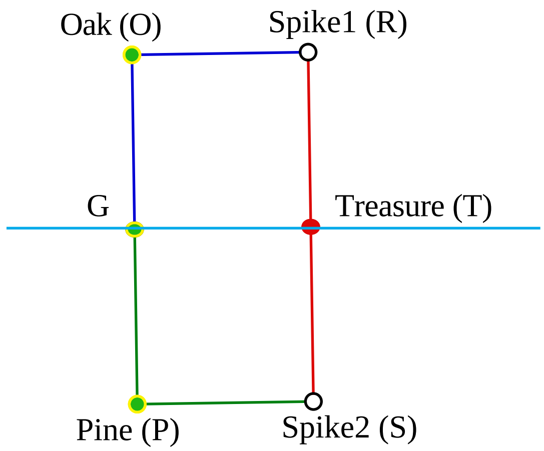
Figure 1



The treasure must lie upon the line of symmetry; or else it is in two different places. Therefore, the treasure lies upon the perpendicular bisector of the Pine Oak segment.

To calculate where upon the perpendicular bisector the treasure lies, we next choose G to be a point on the line of symmetry, specifically the midpoint between the Pine (P) and the Oak (O) trees (Figure 2). We will need to find GT. Since G is the midpoint of $\overline{OP}$, we see that $GO = GP$; in addition, by following the treasure map directions, we see that $GP = PS$ and $GO = OR$.

Figure 2



OR = PS by transitivity. $\overline{OR} \parallel \overline{PS}$ since they are both perpendicular to the same line, therefore ORSP is a parallelogram, specifically a rectangle. *OP* = *RS*, and since *G* is the midpoint of $\overline{OP}$ and *T* is the midpoint of $\overline{RS}$ it follows that *GP* = *TS*. Therefore *GTSP* is a parallelogram, more specifically a square. So one solution to help José is the following: he needs to find and mark the midpoint between the Pine and the Oak. Then starting at the pine tree he should walk toward the marker while counting his steps, then make a 90° turn to the right and pace off the same number of paces. The treasure is at this point.

We can provide a proof of our result by coordinate geometry or by transformations.

1. Solution by coordinate geometry:

José should be happy now with his treasure, but in the preceding argument we made a fairly big assumption, so our conclusion is only as strong as our assumptions. Using coordinate geometry we can develop a proof of the treasure's location without making such assumptions.

- Pick convenient coordinate axes. The pine and oak trees are the only clear references. Let the pine tree be the origin and the oak tree some point on the y-axis $(0, a)$. The gallows are in an unknown position, say (x, y).
- Calculate the position of Spike 2 (S). Rotating the gallows position -90° about the pine tree gives the coordinate of S as $(y, -x)$.

- Calculate the position of Spike1 (R). Rotating the gallows position 90° about the oak tree will take a little more effort. If the oak tree were the origin then the rotation of 90° would be simple. So lets reduce our task to a more simple task. Translate the entire picture, $T_{(0,\,-a)}$. This will place the oak tree on the origin. Rotate the translated gallows $(x, y - a)$ 90° about the origin to $(-y + a, x)$. Now translate the picture $T_{(0,\,a)}$ and the picture is back where it began. The position of R is now $(-y + a, x + a)$.
- Our last task is to calculate where the treasure is located.

Use the midpoint formula to calculate the position of the treasure halfway between the spikes.

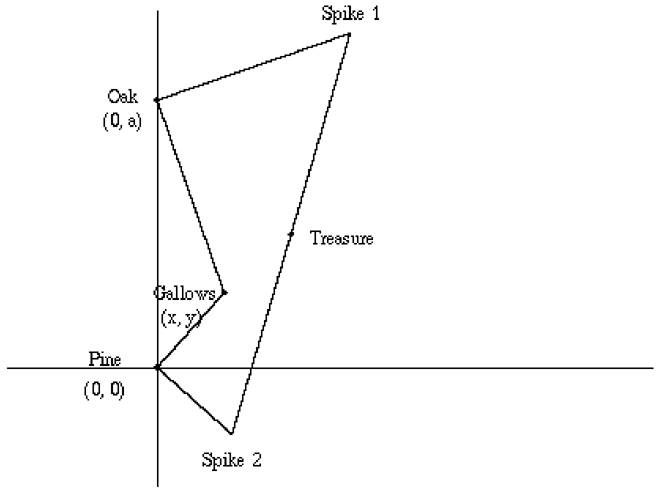
Spike 1: R $(-y + a, x + a)$

Spike 2: S $(y, -x)$

Treasure: T $(a/2, a/2)$

Coordinate geometry proves that the position of the treasure is invariant with respect to the gallows.

Figure 3



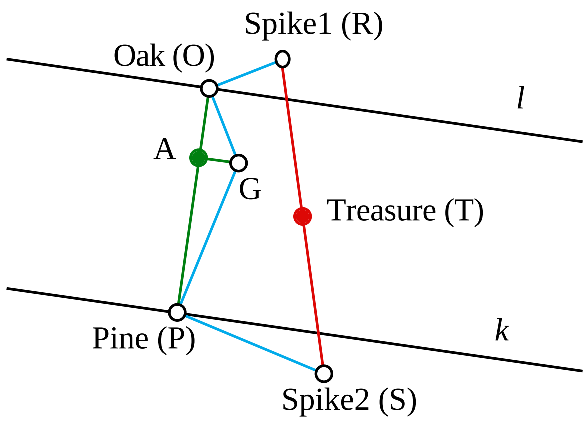
2. Explanation by Isometries:

So far the explanations have given a solution, but they haven't given us much insight as to why the location of the treasure is independent of the position of the gallows. *Sketchpad* can assist in the explanation using transformations.

4.7.2 Demonstration. The Buried Treasure Problem using Sketchpad.

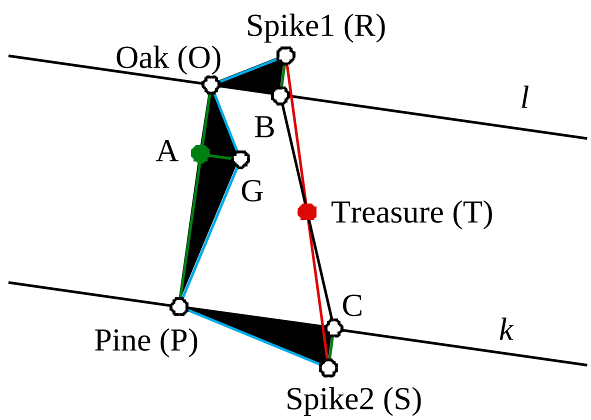
The exact position of the gallows is unknown, therefore we indicate the position of the Gallows by the letter *G* and make no more assumptions about its position. Construct the segment joining the Oak tree (*O*) and Pine tree (*P*). Construct lines *l* and *k* perpendicular to $\overline{OP}$ passing through *O* and *P* respectively. Lines *l* and *k* are parallel to each other. Construct $\overline{GA}$ as the altitude of the ΔPOG. By the instructions given in the map, construct the positions of the spikes (*R* and *S*), and the treasure (*T*). Hide all unnecessary lines and points. (Figure 4)

Figure 4



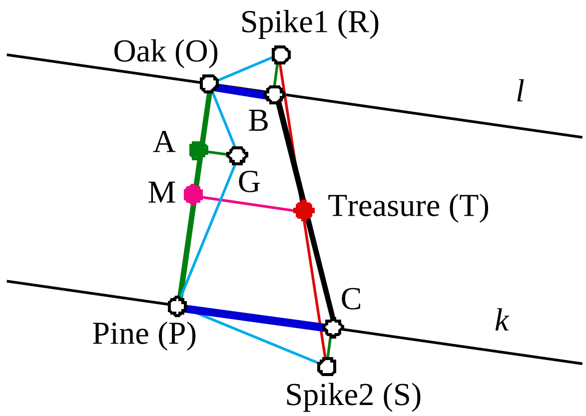
In the coordinate proof the spike positions were found by rotating the position of the gallows about the trees. We will use this technique again in this proof. Rotate ΔOAG 90° about O, forming ΔOBR. Rotate ΔPAG -90° about P, forming ΔPCS. It is simple to show B lies on l and C lies on k. Since isometries preserve distance the following congruencies hold: $\overline{GA}$ $\overline{RB}$; $\overline{GA}$ $\overline{SC}$, and by transitivity $\overline{RB}$ $\overline{SC}$. Since $\overline{RB} \| \overline{SC}$, BRT CST. By SAS ΔRBT ΔSCT. From this we can conclude *B, T, C* are collinear, *T* is the midpoint of $\overline{BC}$ and therefore equidistant from *l* and *k*. (See **Figure 5**).

Figure 5 →



With *T* established as the midpoint of $\overline{BC}$, we will change our focus to the trapezoid *OBCP* (See Figure 6). Naming *M* the midpoint of $\overline{OP}$, yields the median $\overline{MT}$. The length of the median is the average of the two bases, thus $MT = \frac{1}{2}(OB + PC)$. But by the original rotation we know that $OB + PC = OA + AP = OP$; thus $MT = \frac{1}{2}OP$. From this we can conclude that ΔPMT is an isosceles right triangle.

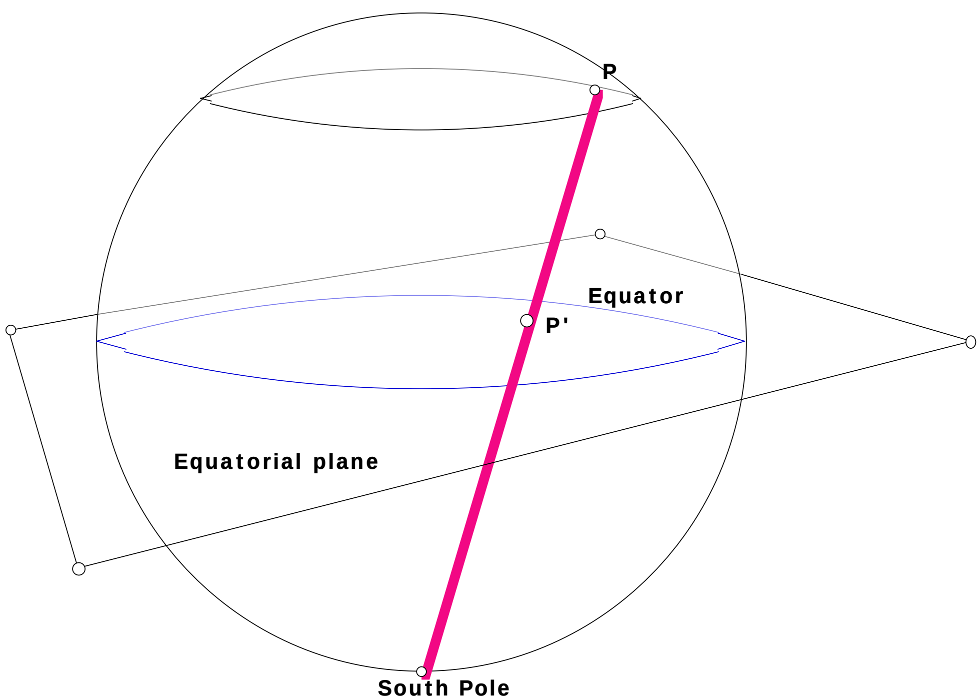
Figure 6



End of Demonstration 4.7.2.

4.8 STEREOGRAPHIC PROJECTION. In all the previous discussions the geometric transformation has mapped one model of a geometry onto the *same* model. But in map-making, for instance, the problem is to map the sphere model to a different model, in fact to a model realized as some geometry realized in the plane. One very important example of this is the transformation known as *Stereographic Projection.* We shall see this plays also a crucial role in describing the geometric transformation taking the line model of hyperbolic geometry in terms of lines and planes inside a cone in 3-space to the Poincaré model **D**.

To construct the stereographic projection of the sphere onto the plane, first draw the equatorial plane - this will serve as the plane onto which the sphere is mapped. Now take any point P on the sphere other than the South Pole and draw the ray starting at the South Pole and passing through P. Label by P' the point of intersection of this ray with the equatorial plane. For clarity in the figure below the ray has been drawn as the line segment joining the South Pole and P.



Stereographic projection is the mapping $P \to P'$ from the sphere to the equatorial plane. It has a number of important **properties**:

1. When P lies on the equator, then $P = P'$ so the image of the equator is itself. More precisely, the equator is left fixed by the transformation $P \to P'$. For convenience, let's agree to call this circle the *equatorial circle*.
2. When P lies in the Northern hemisphere then P' lies inside the equatorial circle, while if P lies in the Southern hemisphere, P' lies outside the equatorial circle.

3. Since the ray passing through the South Pole and *P* approaches the tangent line to the sphere at the South Pole, and so becomes parallel to the equatorial plane, as *P* approaches the South Pole, the image of the South Pole under stereographic projection is identified with infinity in the equatorial plane.

4. There is a 1-1 correspondence between the equatorial plane and the set of all points on the sphere excluding the South Pole.

5. The image of any line of longitude, *i.e.*, any great circle passing through the North and South Poles, is a straight line passing through the center of the equatorial circle. Conversely, the pre-image of any straight line through the center of the equatorial circle is a line of longitude on the sphere.

6. The image of any line of latitude on the sphere is a circle in the equatorial plane concentric to the equatorial circle.

7. The image of any great circle on the sphere is a circle in the equatorial plane. Now every great circle intersects the equator at diametrically opposite points on the equator. On the other hand, the points on the equator are fixed by stereographic projection, so we see that the image of any great circle on the sphere is a circle in the equatorial plane passing through diametrically opposite points on the equatorial circle

8. Stereographic projection is *conformal* in the sense that it preserves angle measure. In other words, if the angle between the tangents at the point of intersection of two great circles is θ, then the angle between the tangents at the points of intersection of the images of these great circles is again θ.

Many books develop the properties of stereographic projection listed above by using the idea of inversion in 3-space. These same properties can, however, be established algebraically. This is what we'll do at this juncture because it brings in results learned earlier in calculus courses. Let Σ be the sphere in 3-space centered at the origin having radius 1. The points on Σ can described by

$$(\xi,\eta,\zeta), \quad \xi^2+\eta^2+\zeta^2=1,$$

so in the figure above, let $P = P(\xi,\eta,\zeta)$ and let $P' = P'(x,y)$ be its image in the equatorial plane under stereographic transformation where the center of the equatorial circle is taken as the origin. In particular, the equation of the equatorial circle is $x^2 + y^2 = 1$. To determine the relation between (ξ,η,ζ) and (x, y) we use similar triangles to show that

$$\text{(A)} \qquad \Pi : P(\xi,\eta,\zeta) \to P'(x, y), \qquad x = \frac{\xi}{1+\zeta}, \qquad y = \frac{\eta}{1+\zeta}.$$

This is the algebraic formulation of stereographic projection. Since $\xi^2 + \eta^2 + \zeta^2 = 1$, the coordinates of $P'(x, y)$ satisfy the relation

$$\text{(B)} \qquad x^2 + y^2 = \frac{1-\zeta^2}{(1+\zeta)^2} = \frac{1-\zeta}{1+\zeta}.$$

As illustration, consider the case first of the North Pole $P = (0,0,1)$. Under stereographic projection Π maps $P = (0,0,1)$ maps to $P' = (0,0)$ in the equatorial plane, *i.e*., to the origin in the equatorial plane. By contrast, the South Pole is the point $P = (0,0,-1)$ and it is the only point of the sphere with $\zeta = -1$. Thus the South Pole is the only point on Σ for which the denominator $1 + \zeta = 0$. Thus the south Pole maps to infinity in the equatorial plane, and it is the only point on Σ which does so. That $P(\xi,\eta,\zeta) \to P'(x, y)$ is a 1-1 mapping from $\Sigma \setminus (0,0,-1)$ onto the equatorial plane can also be shown solving the equations

$$x = \frac{\xi}{1+\zeta}, \qquad y = \frac{\eta}{1+\zeta}$$

given a point (ξ,η,ζ) in $\Sigma \setminus (0,0,-1)$ or a point (x, y) in the equatorial plane.

Now let's turn to the important question of what Π does to circles on Σ. Every such circle is the intersection with Σ of a plane; for instance, a great circle is the intersection of Σ and a plane through the origin. In calculus you learned that a plane is given by the equation

$$\text{(C)} \qquad A\xi + B\eta + C\zeta = D$$

where the vector (A, B, C) is the normal to the plane and $\dfrac{D}{\sqrt{A^2+B^2+C^2}}$ is the distance of the plane from the origin. The simplest case is that of a line of longitude. Algebraically, this is the intersection of Σ with a *vertical* plane through the origin, so the normal lies in the (ξ,η)-plane meaning that $C = D = 0$ in the equation above. Thus a line of longitude is the set of points (ξ,η,ζ) such that

$$A\xi + B\eta = 0, \quad \xi^2 + \eta^2 + \zeta^2 = 1.$$

The image of any such point under Π is the set of points (x, y) in the equatorial plane such that $Ax + By = 0$, which is the general equation of a straight line passing through the origin. Conversely, given any straight line l in the equatorial plane, it will be given by $Ax + By = 0$ for some choice of constants A, B. So l will be the image of the great circle defined by the plane $A\xi + B\eta = 0$. This shows that there is a 1-1 correspondence between lines of longitude and straight lines through the center of the equatorial circle, proving property **5** above.

The image of a line of latitude is easily determined also since a line of latitude is the intersection of Σ with a *horizontal* plane, *i.e.*, a plane $\zeta = D$ with $-1 < D < 1$. But then, by the general relation (B) the image of the line of latitude determined by the plane $\zeta = D$ consists of all points (x, y) in the equatorial plane such that

$$x^2 + y^2 = \frac{1-D}{1+D}.$$

This is the equation of a circle centered at the origin and radius $\sqrt{(1-D)/(1+D)}$; as D varies over the range $-1 < D < 1$, this describes the family of all circles centered at the origin. So Π defines a 1-1 mapping of the lines of latitude onto the family of all circles concentric with the equatorial circle.

The proof of property **7** is a little more tricky. Consider first the case of a plane passing through the points $(0,\pm 1,0)$ on Σ; we could think of these as being the East and West 'Poles'. Also, the plane need not be vertical because otherwise its intersection with Σ

would be a line of longitude dealt with earlier in property **6**. Thus we are led to considering a great circle determined by the plane

$$\zeta = \xi \tan\theta ,$$

where θ is fixed, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$; in fact, θ is the angle between the plane and the (ξ, η)-plane. By relations (A) and (B), the points (x, y) in the image of the intersection with Σ of the plane $\zeta = \xi \tan\theta$ will satisfy the equations

$$x = \frac{\xi}{1 + \xi \tan\theta} \quad , \qquad x^2 + y^2 = \frac{1 - \zeta}{1 + \zeta} = \frac{1 - \xi \tan\theta}{1 + \xi \tan\theta} \; .$$

After eliminating ξ from these equations we see that the image point (x, y) satisfies the equation

$$x^2 + y^2 = 1 - 2x\tan\theta .$$

In other words, the image of the great circle determined by the plane $\zeta = \xi \tan\theta$ is the circle

$$(x + \tan\theta)^2 + y^2 = 1 + (\tan\theta)^2 = (\sec\theta)^2$$

which is the circle centered at $(-\tan\theta, 0)$ having radius $1/\cos\theta$. As problem 7 in Assignment 6 shows, this is a circle passing though diametrically opposite points of the circle $x^2 + y^2 = 1$; in fact, it passes through the points $y = \pm 1$ which are the image of the points of intersection of the great circles determined by the plane $\zeta = \xi \tan\theta$ and the equator in Σ.

But how do we deal with a more general great circle that is not a line of longitude and does not pass through the East and West Poles? The fundamental idea we'll use is that a rotation of the sphere about the ζ-axis through an angle ϕ will fix the ζ-coordinate of a point $P(\xi, \eta, \zeta)$ on Σ while rotating the ξ, η-coordinates, but it will also rotate the x, y-coordinates of the image $P'(x, y)$ by the same angle ϕ. So the effect of rotating a great circle is to rotate its image under stereographic projection. Since a rotation is an isometry, it maps a circle to a circle. Hence the image of any great circle is a circle. Let's do the details.

4.8.1 Theorem. Under the rotation $\rho_{O,\phi}$ about the origin the point (ξ,η) is mapped to the point $(\xi',\eta') = \rho_{O,\phi}(\xi,\eta)$ where

$$\xi' = \xi\cos\phi - \eta\sin\phi, \quad \eta' = \xi\sin\phi + \eta\cos\phi .$$

More generally, the point (ξ,η,ζ) is mapped to the point (ξ',η',ζ).

Under $\rho_{O,\phi}$ the plane $\zeta = \xi\tan\theta$ is mapped to the plane $\zeta = (\xi\cos\phi + \eta\sin\phi)\tan\theta$. The angle between this plane and the (ξ,η)-plane is again θ and the intersection of the plane with Σ is a great circle passing through the equator at the points

$$(-\sin\phi, \cos\phi, 0), \quad (\sin\phi, -\cos\phi, 0).$$

Now by (A), the point (ξ',η',ζ) is mapped to (x', y') where

$$x' = x\cos\phi - y\sin\phi, \quad y' = x\sin\phi + y\cos\phi .$$

Consequently, stereographic projection commutes with the rotation $\rho_{O,\phi}$ in the sense that

(D) $$\Pi \circ \rho_{O,\phi} = \rho_{O,\phi} \circ \Pi .$$

Since the isometry $\rho_{O,\phi}$ will map circles to circles, we obtain the following result, completing the proof of property **7** listed above.

4.8.2 Theorem. Stereographic projection maps the great circle determined by the rotated plane $\zeta = (\xi\cos\phi + \eta\sin\phi)\tan\theta$ to the circle in the equatorial plane obtained after rotation by $\rho_{O,\phi}$ of the image of the great circle determined by the plane $\zeta = \xi\tan\theta$.

The general result of property 8 can be established using similar transformation ideas to those in the proof of Theorem 4.8.2.

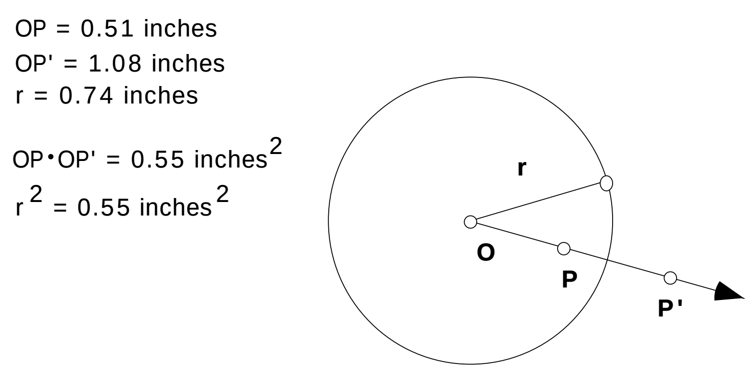
Chapter 5
INVERSION

The notion of inversion has occurred several times already, especially in connection with Hyperbolic Geometry. Inversion is a transformation different from those of Euclidean Geometry that also has some useful applications. Also, we can delve further into hyperbolic geometry once we have developed some of the theory of inversion. This will lead us to the description of isometries of the Poincaré Disk and to constructions via Sketchpad of tilings of the Poincaré disk just like the famous 'Devils and Angels' picture of Escher.

5.1 DYNAMIC INVESTIGATION. One very instructive way to investigate the basic properties of inversion is to construct inversion via a custom tool in Sketchpad. One way of doing this was described following Theorems 3.5.3 and 3.5.4 in Chapter 3, but in this section we'll describe an alternative construction based more closely on the definition of inversion. Recall the definition of inversion given in section 5 of chapter 3.

5.1.1 Definition. Fix a point O and a circle C centered at O of radius r. For a point P, $P \neq O$, the *inverse* of P is the unique point P' on the ray starting from O and passing through P such that $OP\ \ OP' = r^2$.

The point O is called the *center of inversion* and circle C is called the *circle of inversion*, while r is called the *radius of inversion.*



To create a tool that constructs the inverse of a point P given the circle of inversion and its center, we can proceed as follows using the dilation transformation.

- Open a new sketch and draw a circle by center and point. Label the center by *O* and label the point on the circle by *R*. Construct a point *P* not on the circle. Construct the ray from the center of the circle, passing through P. Construct the point of intersection between the circle and the ray, label it *D*.
- Mark the center of the circle - this will be the center of dilation. Then select the center of the circle, the point P, and then the point of intersection of the ray and the circle.Go to "Mark Ratio" under the **Transform** menu. This defines the ratio of the dilation.
- Now select the point of intersection of the ray and the circle, and dilate by the marked ratio. The dilated point is the inverse point to *P*. Label the dilated point P'.
- Select *O*, *R*, *P*, and *P'*.
- Under the **Custom Tools** Menu, choose "Create New Tool" and check "Show Script View" . You may wish to use Auto-Matching for *O* and *R* as we are about to use our inversion script to explore many examples. Under the **Givens** List for your script, double click on *O* and *R* and check the box "Automatically Match Sketch Object". To make use of the Auto-Matching you need to start with a circle that has center labeled by *O* and a point on the circle labeled by *R*.
- Save your script.

Use your tool to investigate the following.

5.1.2 Exercise. Where is the inverse of *P* if

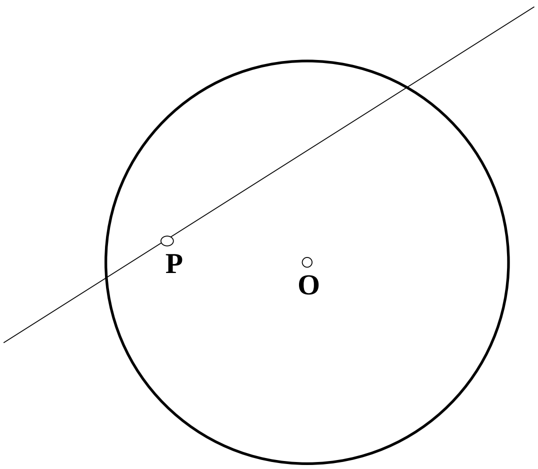
- *P* is outside the circle of inversion? ________________________
- *P* is inside the circle of inversion? ________________________
- *P* is on the circle of inversion? ________________________
- *P* is the center of the circle of inversion? ________________________

Using our tool we can investigate how inversion transforms various figures in the plane by using the construct "Locus" property in the **Construct** menu. Or by using the "trace" feature. For instance, let's investigate what inversion does to a straight line.

- Construct a circle of inversion. Draw a straight line and construct a free point on the line. Label this free point by *P*.
- Use your tool to construct the inverse point P' to *P*.

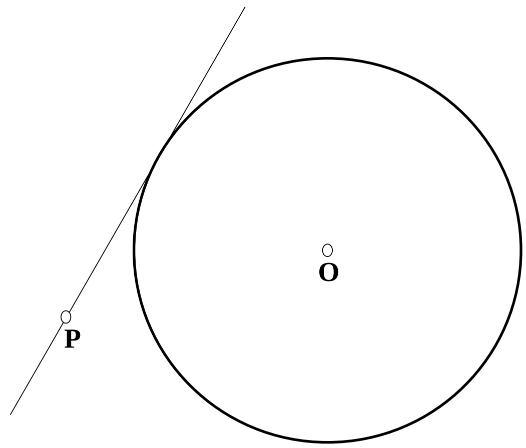
- Select the points P' and P. Then select “Locus” in the **Construct** menu. (Alternatively, one could trace the point P' while dragging the point P.)

5.1.3 Exercise. What is the image of a straight line under inversion? By considering the various possibilities for the line describe the locus of the inversion points. Be as detailed as you can.



A line, which
passes through the circle of inversion

Image:______________________________



A line, which
is tangent to the circle of inversion

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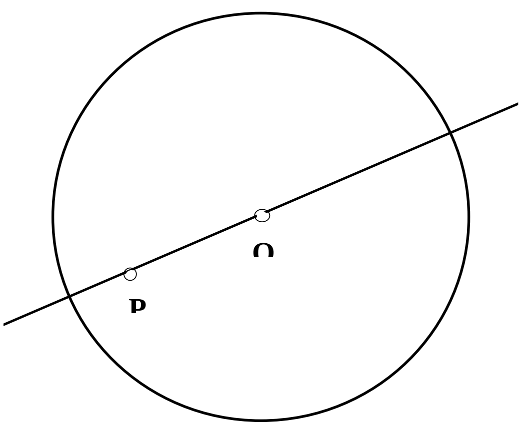
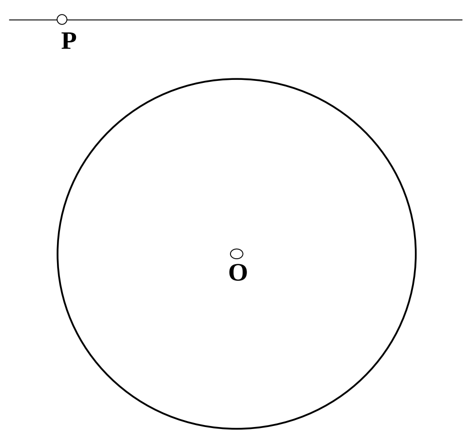


A line, which passes
through the center of the circle of inversion

Image:______________________________



A line, which doesn’t
intersect the circle of inversion

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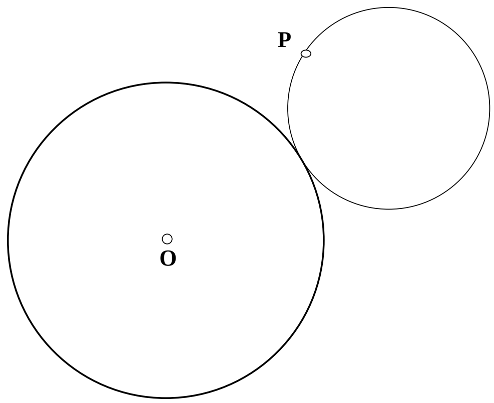
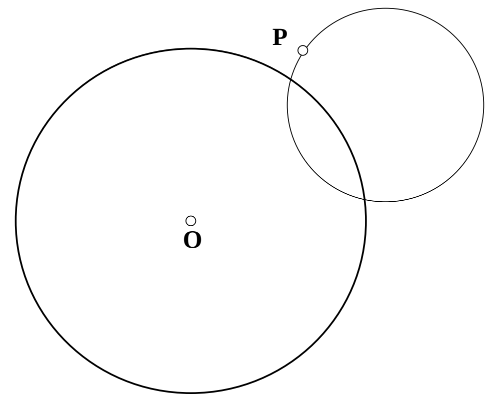
5.1.4 Exercise. What is the image of a circle under inversion? By considering the various possibilities for the line describe the locus of the inversion points. Be as detailed as you can.



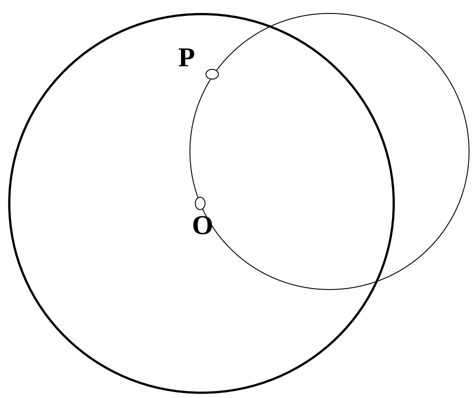
A circle, which
is tangent to the circle of inversion

Image:_______________________________



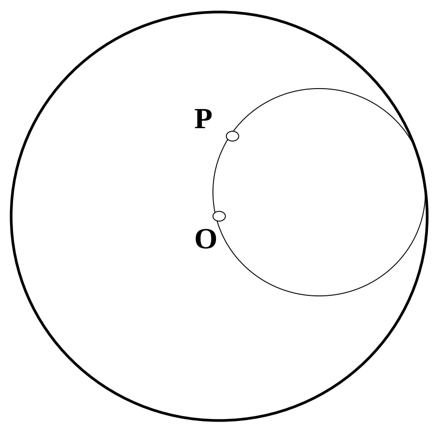
A circle, which intersects the
circle of inversion in two points.

Image:_______________________________



A circle, which passes through
the center of the circle of inversion

Image:_______________________________



A circle passing through the center of the
circle of inversion, also internally tangent

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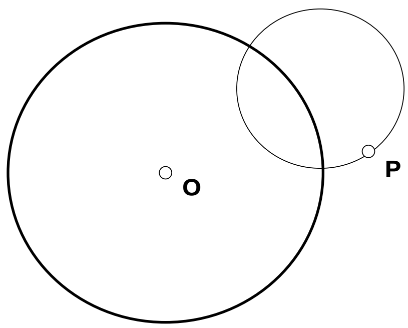

A circle which is orthogonal to the circle of inversion.

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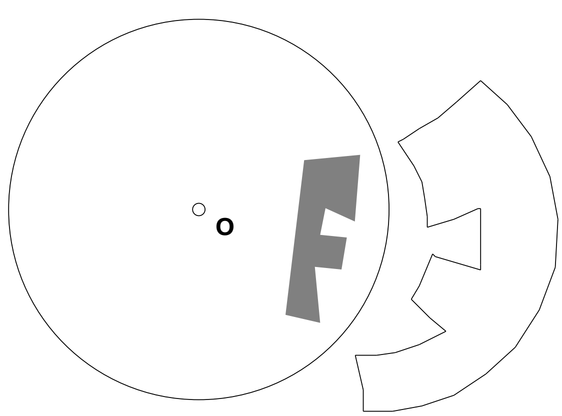
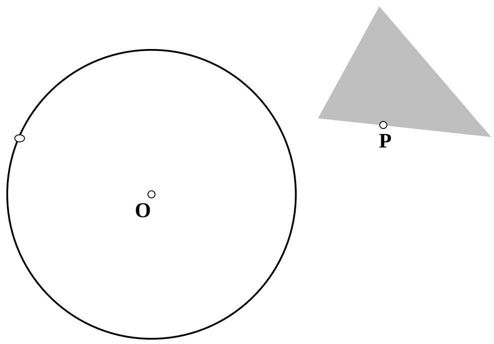
You should have noticed that some circles are transformed into another circle under the inversion transformation. Did you notice what happens to the center of the circle under inversion in these cases? Try it now.

End of Exercise 5.1.4.

You can easily construct the inverse image of polygonal figures by doing the following. Construct your figure and its interior. Next hide the boundary lines and points of your figure so that only the interior is visible. Next select the interior and choose “Point on Object” from the **Construct** Menu. Now construct the inverse of that point and then apply the locus construction. Here is an example.

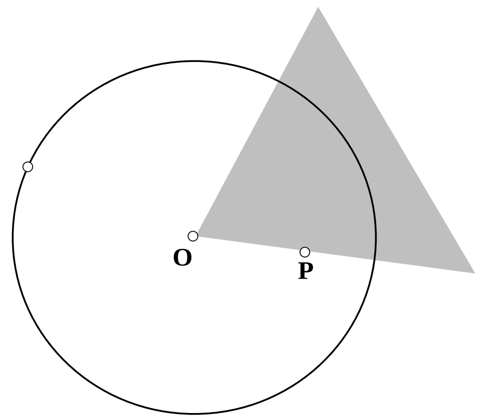


5.1.5 Exercise. What is the image of other figures under inversion? By considering the various possibilities for the line describe the locus of the inversion points. Be as detailed as you can.



A triangle, external to the circle of inversion

Image:______________________________



A triangle, with one vertex as the center of the circle of inversion

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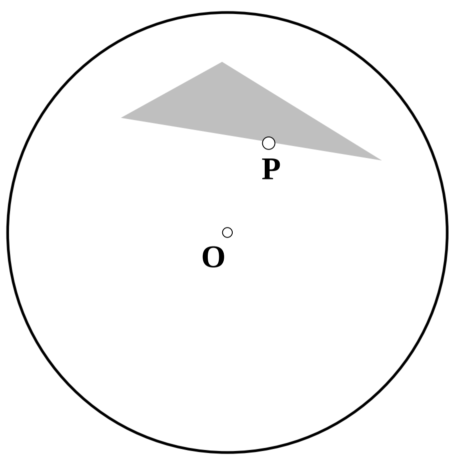

A triangle internal to the circle of inversion

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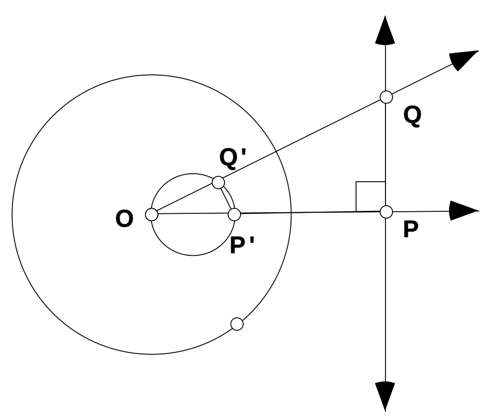
5.2 PROPERTIES OF INVERSION. Circular inversion is not a transformation of the Euclidean plane since the center of inversion does not get mapped to a point in the plane. However if we include the point at infinity, we would have a transformation of the Euclidean Plane and this point at infinity. Also worth noting is that if we apply inversion twice we obtain the identity transformation. With these observations in mind we are now ready to work through some of the basic properties of inversion. Let *C* be the circle of inversion with center *O* and radius r. Also, when we say "line", we mean the line including the point at infinity. The first theorem is easily verified by observation.

5.2.1 Theorem. Points inside *C* map to points outside of *C*, points outside map to points inside, and each point on *C* maps to itself. The center *O* of inversion maps to $\{\infty\}$

5.2.2 Theorem. The inverse of a line through *O* is the line itself.

Again, this should be immediate from the definition of inversion, however note that the line is not pointwise invariant with the exception of the points on the circle of inversion. Perhaps more surprising is the next theorem.

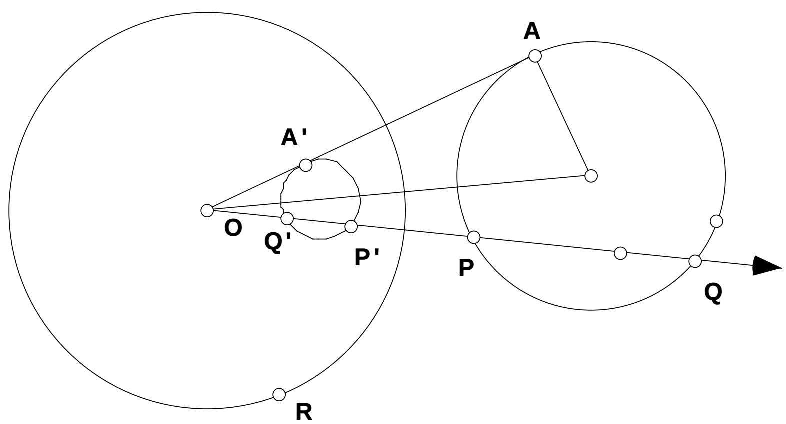
5.2.3 Theorem. The inverse image of a line not passing through *O* is a circle passing through O.



Proof. Let P be the foot of the perpendicular from O to the line. Let Q be any other point on the line. Then P' and Q' are the respective inverse points. By the definition of inverse points, $OP\ \ OP' = OQ\ \ OQ'$. We can use this to show that ΔOPQ is similar to $\Delta OQ'P'$. Thus the image of any Q on the line is the vertex of a right angle inscribed in a circle with diameter OP'.

The proof of the converse to the previous theorem just involves reversing the steps. The converse states, the inverse image of a circle passing through O is a line not passing through O. Notice that inversion is different from the previous transformations that we have studied in that lines do not necessarily get mapped to lines. We have seen that there is a connection between lines and circles.

5.2.4 Theorem. The inverse image of a circle not passing through O is a circle not passing through O.

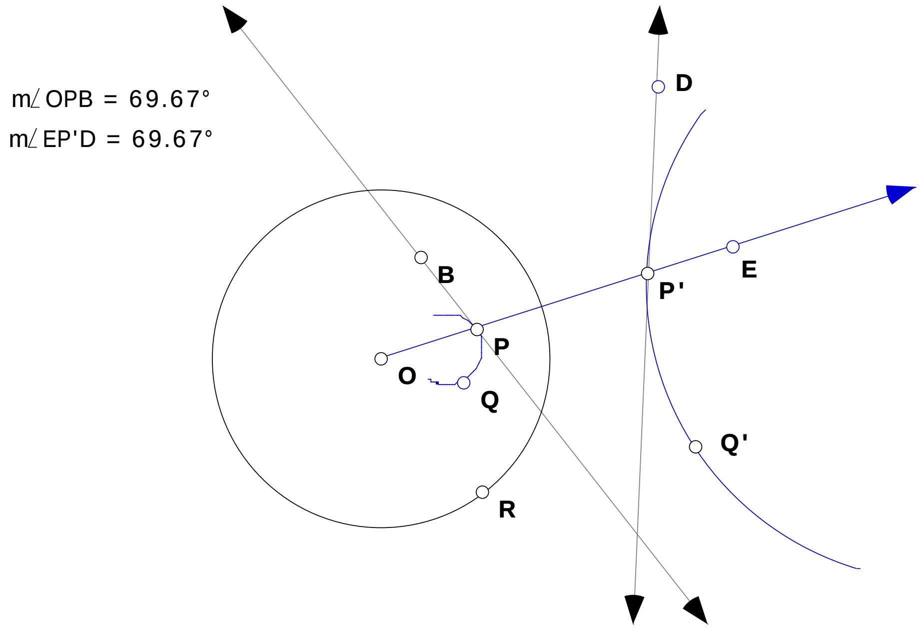


Proof. Construct any line through the center of inversion which intersects the circle in two points P and Q. Let P' and Q' be the inverse points to P and Q. We know that $OP\ \ OP' = OQ\ \ OQ' = r^2$. Also by Theorem 2.9.2 (Power of a Point), $OP\ \ OQ = OA^2 = k$.

Thus $\dfrac{OP\ \ OP'}{OP\ \ OQ} = \dfrac{OQ\ \ OQ'}{OP\ \ OQ} = \dfrac{r^2}{k}$ or $\dfrac{OP'}{OQ} = \dfrac{OQ'}{OP} = \dfrac{r^2}{k}$. In other words, everything reduces to a dilation.

5.2.5 Theorem. Inversion preserves the angle measure between any two curves in the plane. That is, inversion is conformal.

Proof. It suffices to look at the case of an angle between a line through the center of inversion and a curve. In the figure below, P and Q are two points on the given curve and P' and Q' are the corresponding points on the inverse curve. We need to show that $m\ \ OPB = m\ \ EP'D$. The sketchpad activity below will lead us to the desired result.



- Open a new sketch and construct the circle of inversion with center O and radius r. Construct an arc by 3 points inside the circle and label two of the points as P and Q. Next construct the inverse of the arc by using the locus construction and label the points P' and Q'. Finally construct the line OP (it will be its own inverse).
- Next construct tangents to each curve through P and P' respectively.
- Notice that P, Q, P', and Q' all lie on a circle. Why? Thus QPP' and $P'Q'Q$ are supplementary (Inscribed Angle Theorem).
- Thus $m \ OPQ = m \ P'Q'O$. Check this by measuring the angles.
- Next drag Q towards the point P. What are the limiting position of the angles OPQ and $P'Q'O$?
- What result does this suggest?

5.2.6 Theorem. Under inversion, the image of a circle orthogonal to C is the same circle (setwise, not pointwise).

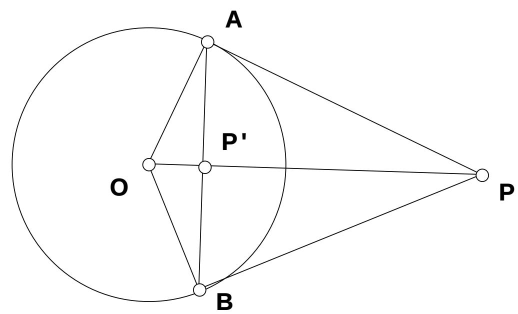
Proof. See **Exercise 5.3.1.**

5.2.7 Theorem. Inversion preserves the generalized cross ratio $\left(\frac{d(P,M)}{d(P,N)}\frac{d(Q,N)}{d(Q,M)}\right)$ of any four distinct points P,Q,M, and N in the plane.

Proof. See **Exercise 5.3.3.**

Recall our script for constructing the inverse of a point relied on the dilation transformation. A compass and straightedge construction is suggested by the next result.

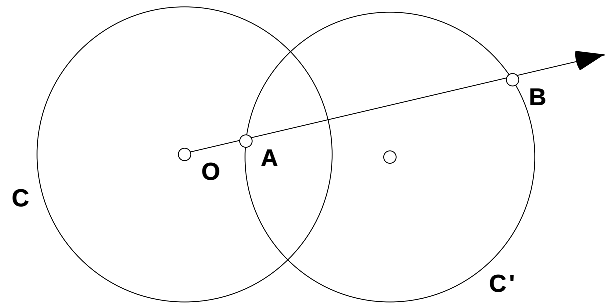
5.2.8 Theorem. The inverse of a point outside the circle of inversion lies on the line segment joining the points of intersection of the tangents from the point to the circle of inversion.



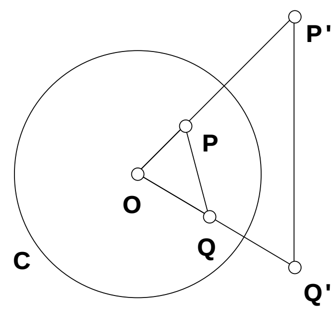
Proof. By similar triangles OAP and $OP'A$, $\dfrac{OA}{OP'} = \dfrac{OP}{OA}$. Use this to conclude that P and P' are inverse points.

5.3 Exercises. These exercises are all related to the properties of inversion.

Exercise 5.3.1. Prove Theorem 5.2.6. That is if C is the circle of inversion and C' is orthogonal to it, draw any line through O which intersects C' in A and B and show that A and B must be inverse to each other.



Exercise 5.3.2. Let C be the circle of inversion with center O. Show that if P' and Q' are the inverse images of P and Q then $\Delta OPQ \sim \Delta OQ'P'$.



Exercise 5.3.3. Do the following to prove Theorem 5.2.7. Let P, Q, N, and M be any four distinct points in the plane. Use Exercise 5.3.2 to show that $\dfrac{PM}{P'M'} = \dfrac{OP}{OM'}$ and $\dfrac{PN}{P'N'} = \dfrac{OP}{ON'}$.

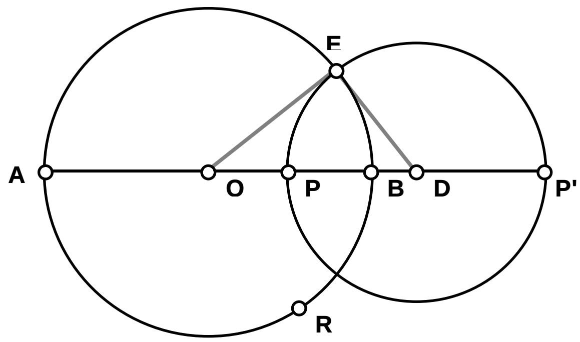
Show that these imply $\dfrac{PM}{PN} = \dfrac{PM'}{PN'} \dfrac{ON'}{OM'}$.

Complete the proof that $\dfrac{PM}{PN} \dfrac{QN}{QM} = \dfrac{P'M'}{P'N'} \dfrac{Q'N'}{Q'M'}$.

Exercise 5.3.4. Use Theorem 5.2.8 and Sketchpad to give compass and straightedge constructions for the inverse point of P when P is inside the circle of inversion and when P is outside the circle of inversion.

Exercise 5.3.5. Let C be the circle having the line segment $\overline{AB}$ as a diameter, and let P and P' be inverse points with respect to C. Now let E be a point of intersection of C with the circle

having the line segment $\overline{PP'}$ as diameter. See the figure below. Prove that $OED = 90^\circ$.
(Hint: Recall the theorems about the Power of a Point in Chapter 2.)



Exercise 5.3.6. Again, let C be the circle having the line segment $\overline{AB}$ as a diameter, and let P and P' be inverse points with respect to C. Now let C' be the circle having the line segment $\overline{PP'}$ as diameter. See the figure above. Prove that A and B are inverse points with respect to C'.
(Hint: Recall the theorems about the Power of a Point in Chapter 2.)

5.4 APPLICATIONS OF INVERSION There are many interesting applications of inversion. In particular there is a surprising connection to the Circle of Apollonius. There are also interesting connections to the mechanical linkages, which are devices that convert circular motion to linear motion. Finally, as suggested by the properties of inversion that we discovered there is a connection between inversion and isometries of the Poincaré Disk. In particular, inversion will give us a way to construct “hyperbolic reflections” in h-lines. We will use this in the next section to construct tilings of the Poincaré Disk.

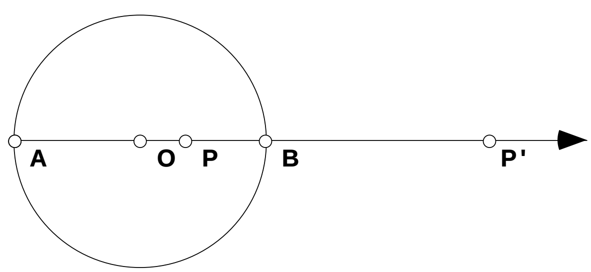
First let’s look at the Circle of Apollonius and inversion in the context of a magnet. A common experiment is to place a magnet under a sheet of paper and then sprinkle iron filings on top of the paper. The iron filings line up along circles passing through two points, the North and South poles, near the end of the magnets. These are the Magnetic lines of force. The theory of magnetism then studies equipotential lines. These turn out to be circles each of which is orthogonal to all the magnetic lines of force. The theory of inversion was created to deal with the theory of magnetism. We can interpret these magnetic lines of force and equipotential lines within the geometry of circles.

- Open a new sketch and construct a circle having center O and a point on the circle labeled R.
- Next construct any point P inside the circle and the inverse point P'. Construct the diameter $\overline{AB}$ of the circle of inversion that passes through the point P.
- Finally construct the circle with diameter $\overline{PP'}$ and construct any point Q on this circle.
- Construct the segments $\overline{AQ}$ and $\overline{BQ}$. Select them using the arrow tool in that order (while holding down the shift key) and choose "Ratio" from the **Measure** menu. You should be computing the ratio $\dfrac{AQ}{BQ}$.
- Drag the point Q. What do you notice? What does this tell you about the circle with diameter $\overline{PP'}$?

5.4.1 Conjecture. If P and P' are inverse points with respect to circle C and lie on the diameter $\overline{AB}$ of C then the circle with diameter $\overline{PP'}$ is ____________________________.

Towards the proof of the conjecture we'll need the following.

5.4.2 Theorem. Given P and P' which are inverse points with respect to a circle C and lie on the diameter $\overline{AB}$ of C, then $\dfrac{AP}{BP} = \dfrac{AP'}{BP'}$.

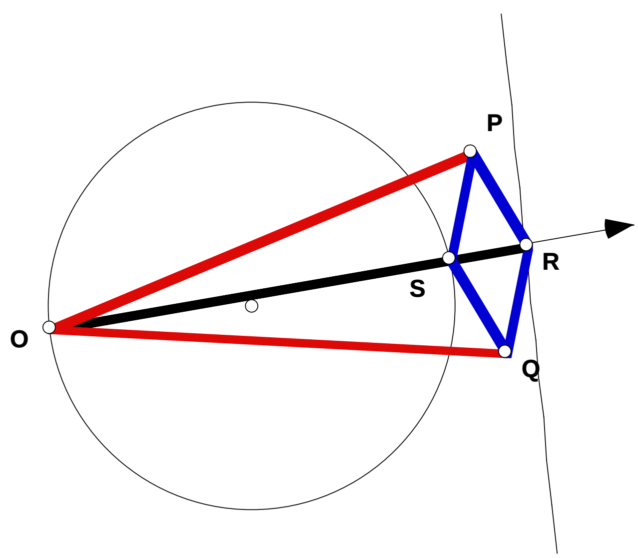


Proof. Since P and P' are inverse points $OP\ \ OP' = OB^2$ or $\dfrac{OP}{OB} = \dfrac{OB}{OP'}$. Now one can check that if $a/b = c/d$ then $\dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}$ so that $\dfrac{OP+OB}{OP-OB} = \dfrac{OB+OP'}{OB-OP'}$ or $\dfrac{AP}{BP} = \dfrac{AP'}{BP'}$. **QED**

The completion of the proof can be found in **Exercise Set 5.6.**

Another interesting application of inversion underlies one possible mechanical linkage that converts circular motion to linear motion. Such a change of motion from circular to linear occurs in many different mechanical settings from the action of rolling down the window of your car to the pistons moving within the cylinders in the engine of the car. The Peaucellier linkage figure below shows the components. The boldface line segments represent rigid rods such that $PR = PS = QR = QS$ and $OP = OQ$. There are hinges at the join of these rods at *O*, *P*, *Q*, *R*, and *S*. Points *P*, *Q*, and *R* can move freely while *S* is free to move on a circle *C* and *O* is fixed on that circle. Surprisingly, as *S* moves around the circle the point *R* traces out a straight line. It is an interesting exercise to try to construct this linkage on Sketchpad. Try it! In case you get stuck, one such construction is given below. The "proof" that *R* should trace out a straight line is part of the next assignment.

5.4.2a Demonstration. Constructing a Peaucellier Linkage.

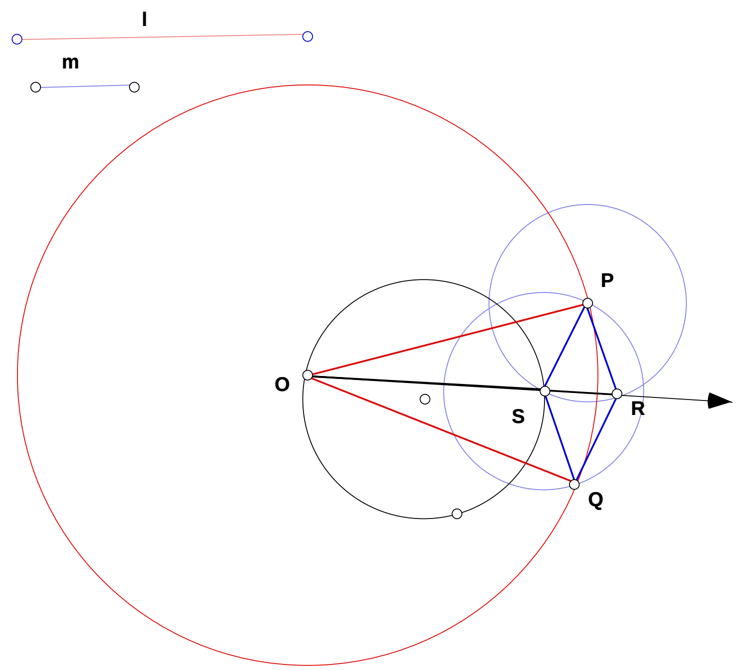


- Open a new sketch and construct a circle. Draw the ray $\overrightarrow{OS}$ where *O* and *S* are points on the circle.
- In the corner of your sketch construct two line segments *l* and *m*. (See below. Segment *l* *will* determine the length of $\overline{OP}$ and segment *m* will determine the length of $\overline{PS}$.) Color *l* red, and color *m* blue.
- Construct a circle with center *O* and the same length as segment *l* and another circle with center *S* and the same length as segment *m*. Color the circles appropriately. Adjust *l* and *m*

if necessary so that the circles intersect outside of *C*. Next construct the intersection points of the circles and label them *P* and *Q*, respectively.

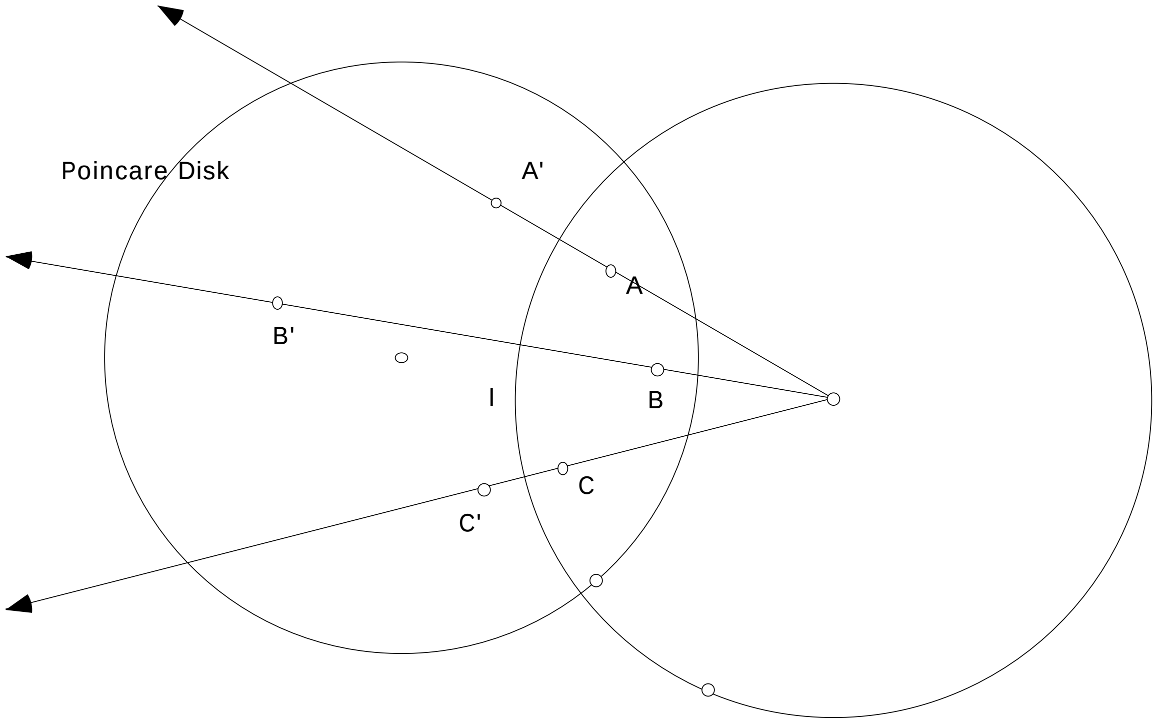
- Construct a circle with center *P* and radius the same as segment *m*. Label the intersection point with the ray $\overrightarrow{OS}$ by *R*. Join the points to construct the rhombus *PRQS* and color the segments blue.
- Construct the segments $\overline{OP}$ and $\overline{OQ}$, then color them red.
- Finally select the point R and choose "Trace Points" from the **Display** Menu and then drag *S* making sure that *O* is staying fixed. (Or alternatively, select the point *R* and then the point *S* and then choose "Locus" from the **Construct** Menu.)
- Do you notice anything special about the line that is traced out? Can you describe it in another way?

  Try various positions for *O*.



End of Demonstration 5.4.2a.

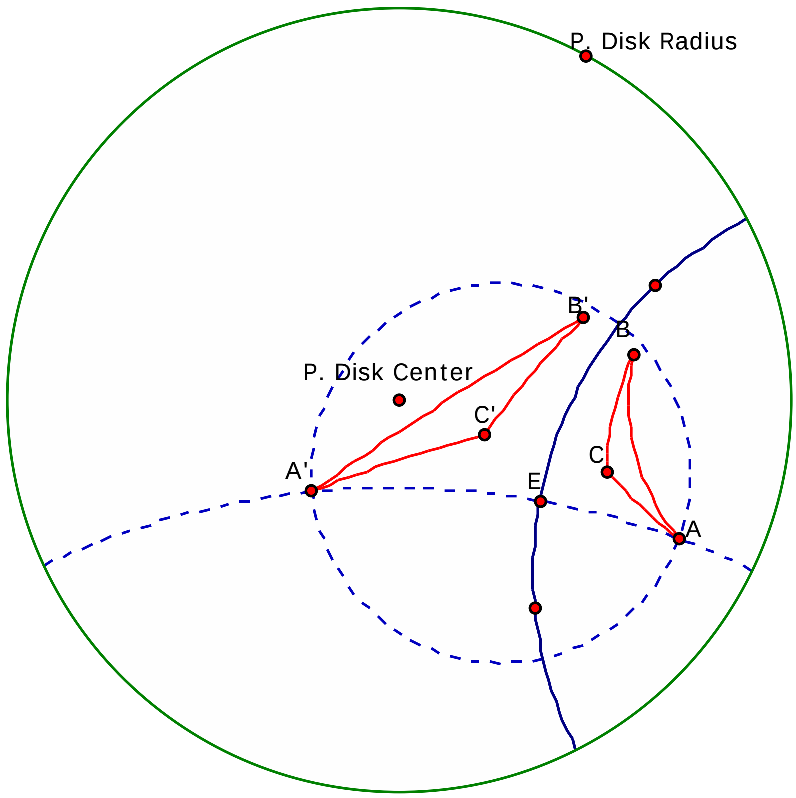
Finally, let's return to the Poincaré disk and Hyperbolic Geometry. We only need to put a few things together to realize that inversion gives us a way to construct h-reflection in an h-line *l*. If *l* is a diameter of C, take just the Euclidean reflection in the Euclidean line containing *l*. Since this is a Euclidean isometry, cross ratios, h-distance, and h-angle measure are preserved. If *l* is the arc of a circle C orthogonal to the Poincaré Disk, consider inversion with C as the circle of inversion. This provides the desired h-reflection since *l* maps to itself, the half planes of *l* map to each other and an inversion is h-distance preserving and h-conformal.



Putting this together our knowledge of inversion we can actually construct specific isometric transformations of the Poincaré Disk. We'll see that there are several useful reasons for doing so. First, let's check this out on Sketchpad.

5.4.2b Demonstration. Investigating constructions on the Poincaré Disk.

We will consider two ways to reflect a triangle in a Poincaré disk. The first way uses the definition of a reflection.



- Open a Poincaré Disk.
- Construct any h-line l and then an h-triangle ABC.
- First construct the h-line through the vertex A perpendicular to l. Then construct the intersection point of l and the perpendicular line, label it E. Next construct an h-circle by center E and point A. The image point A' will be the intersection of the circle and the perpendicular line.
- Repeat for B, and C. Connect A', B', and C' with h-segments.

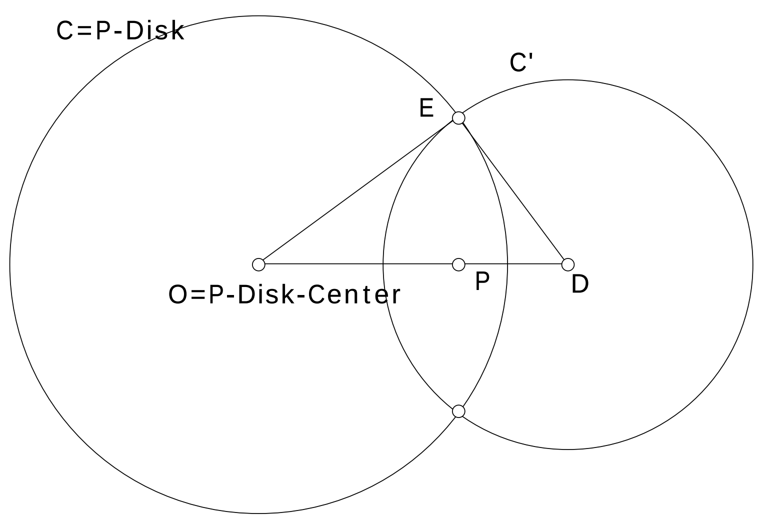
Next, try this again but now using the notion of inversion. First we need a tool that allows you to construct the inverse of a point, by only clicking on two points on the hyperbolic segment and on the point to reflect.

- Open Poincaré Disk and construct a hyperbolic segment.
- Select 3 points on the arc and construct the circle through the 3 points by any method. Label the center of your circle *O* and one point on the circle *R*.
- Construct a point *P*. Now, construct the inverse of *P* as we did before. You could even run your inverse point script.
- Hide everything except the Poincaré Disk, the hyperbolic segment, and the points *P* and *P'*.
- Create a new tool and automatically match the Poincaré Disk center and radius.
- Now use the script to construct the inverse point for each of *A*, *B*, and *C*. What do you notice?

End of Demonstration 5.4.2b.

5.4.3 Demonstration. Mapping a point P to the Origin. Given a point *P* in the Poincaré Disk, describe and then construct the hyperbolic isometry mapping *P* onto the origin.

Using Sketchpad we were able to perform an h-reflection, but the question here is to construct a specific h-reflection. What this boils down to is describing the circle *C* 'with respect to which inversion maps *P* onto the origin. To ensure that C' is an h-line we also require that C' be orthogonal to the bounding circle *C* for the Poincaré Disk.



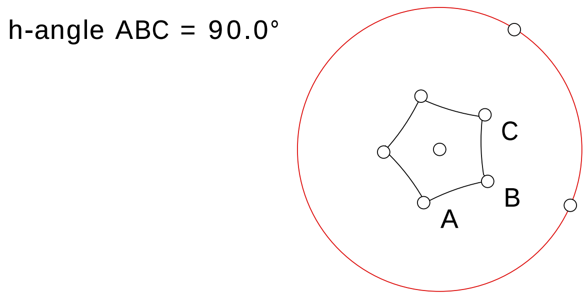
We have to construct the circle C' so that C' is orthogonal to C and $DP \;\; DO = DE^2$. Surprising the solution is easy. Let D be the inverse point to P with respect to the circle C. Then $DO \;\; PO = OE^2$. Consequently, $DP \;\; DO = DO(DO - PO) = DO^2 - OE^2 = DE^2$. Thus, D is the center of the desired circle as O and P will be inverse points. To determine the radius we need to describe E. The condition $DP \;\; DO = DE^2$ ensures that $\Delta OED \sim \Delta EPD$. Hence the line segment EP is perpendicular to the line segment OD. Thus to determine E we just need to draw the perpendicular to OD and find the intersection point with C. The point D is the intersection of this last perpendicular with the ray from O passing through P.

End of Demonstration 5.4.3.

Suppose now that we are given **any** two points P and Q in the Poincaré Disk. **We can, in fact, construct a hyperbolic isometry of the Disk that maps *P* onto *Q*.** *A*ll we have to do is first construct an isometry mapping P to the origin, and then construct an isometry mapping the origin to Q. We can also use this result to prove some results about Hyperbolic geometry. We discovered that the sum of the interior angles of an h-triangle is less than 180 degrees. This is easily seen when the origin is one of the vertices of the triangle for then two of the sides of the h-triangle will be Euclidean Line segments. Given an arbitrary h-triangle we can always map one vertex to the origin using the result above and since inversion is a hyperbolic isometry we can see the result is also true for any triangle.

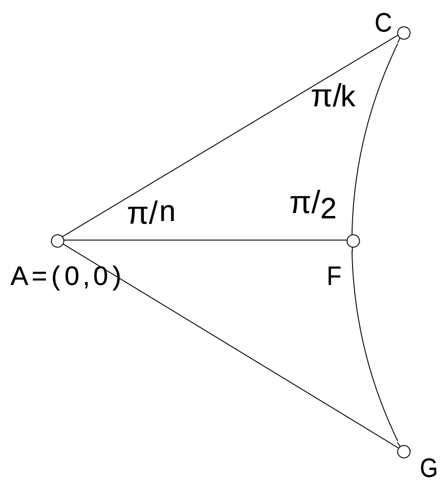
5.5 TILINGS OF THE HYPERBOLIC PLANE. Let's pull together many of the ideas developed in this course by investigating tilings of the hyperbolic plane – in its Poincaré disk model – and then use this to explain the geometry underlying the most sophisticated of Escher's repeating graphic designs. Earlier in Chapter 2 we saw that very few regular polygons could be used to provide edge–to-edge tilings of the Euclidean plane. In fact, only equilateral triangles meeting six at a vertex, squares meeting four times at a vertex, and finally regular hexagons, meeting three times at a vertex. As we have extended to the hyperbolic plane the notion of distance between points and the angle between lines, we can now formulate the notion of a regular h-polygon in exactly the same was as before. A regular h-polygon is a figure in the hyperbolic plane whose edges are h-line segments that have the same length and the same interior angles. What should be noted is that the interior angles of a regular h-polygon can have arbitrary values so long as those values are less than their Euclidean values. Thus for any n, any regular h-polygon with n sides will tile the Hyperbolic plane, so long as the interior angle evenly divides 360! The first question we face is the following:

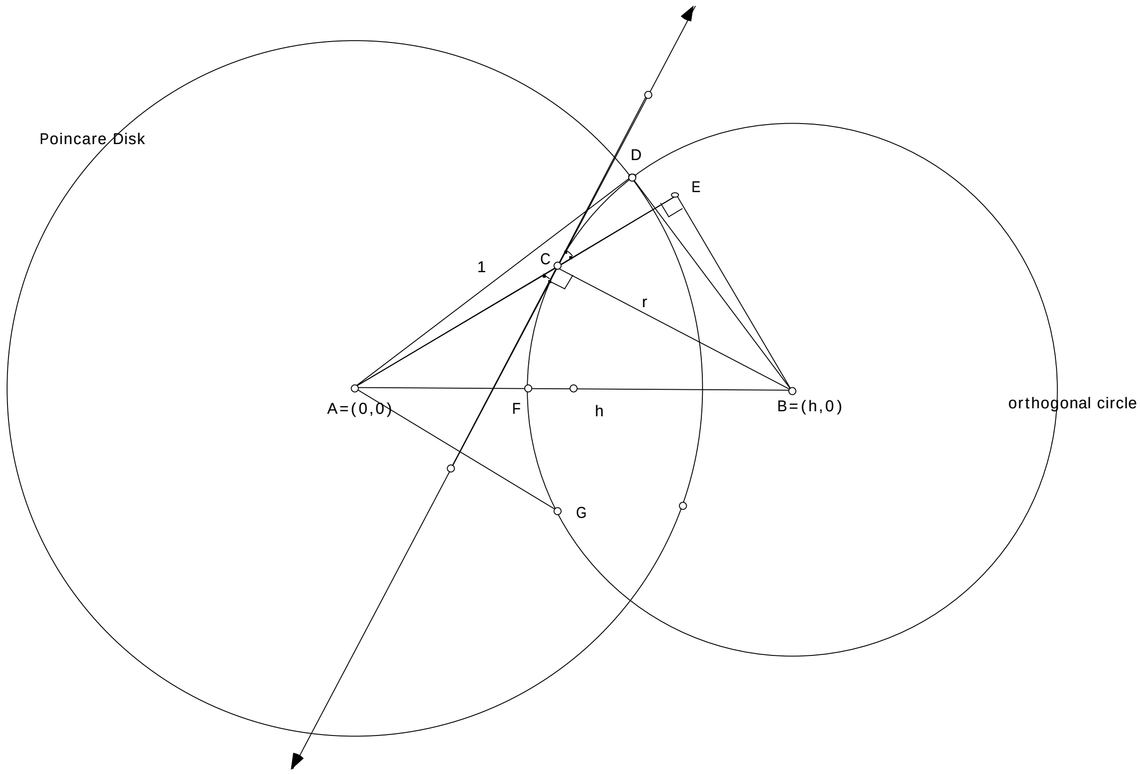
5.5.1 Demonstration. How do we construct a regular *n*-gon that will tile the hyperbolic plane?



We will construct our regular *n*-gon centered at the center of the Poincaré Disk. The edges of the regular *n*-gon are arcs of circles that are orthogonal to the Poincaré Disk. We can find the center of one of those circles by some basic trigonometry. The central h-angles of a regular *n*-gon are all equal to $2\pi/n$. For our *n*-gon to tile the plane the interior h-angles must all be equal to $2\pi/k$ where k is an appropriate positive integer. Any regular *n*-gon is comprised of n congruent isosceles triangles. ΔAGC is one of those isosceles triangles. We will focus our attention on ΔAFC, where $\overline{AF}$ is the perpendicular bisector of $\overline{GC}$.

Assume the Poincaré Disk has center *(0,0)* and radius 1 and that the desired orthogonal circle has center *(h,0)* and radius r. The key step is to extend $\overline{AC}$ to $\overline{AE}$ which gives the right ΔABE.





We are given that $m(h - angle \quad FAC) = {}^{\pi}\!/_{n}$ and $m(h - angle \quad ACF) = {}^{\pi}\!/_{k}$.

Using trigonometry,

$$\sin(\quad ECB) = EB / r \text{ and } \sin(\quad CAB) = EB / h .$$

Thus,

$$r \ \sin(\quad ECB) = h \ \sin(\quad CAB)$$

Now,

- $1 + r^2 = h^2$
- $\sin(\quad ECB) = \pi / 2 - \pi / k$
- $\sin(\quad CAB) = \pi / n$

yielding $\sqrt{h^2 - 1} \ \sin(\pi / 2 - \pi / k) = h \ \sin(\pi / n)$.

Solving for h we get,

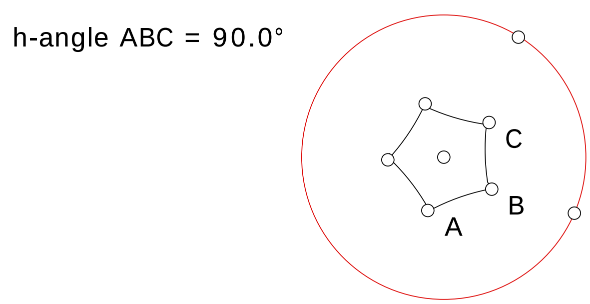
$$h = \sqrt{\frac{\sin^2(\pi / 2 - \pi / k)}{\sin^2(\pi / 2 - \pi / k) - \sin^2(\pi / n)}}$$

where h is the center of the orthogonal circle which determines the edge of a regular n-gon with interior angles equal to $2\pi/k$.

Examples:

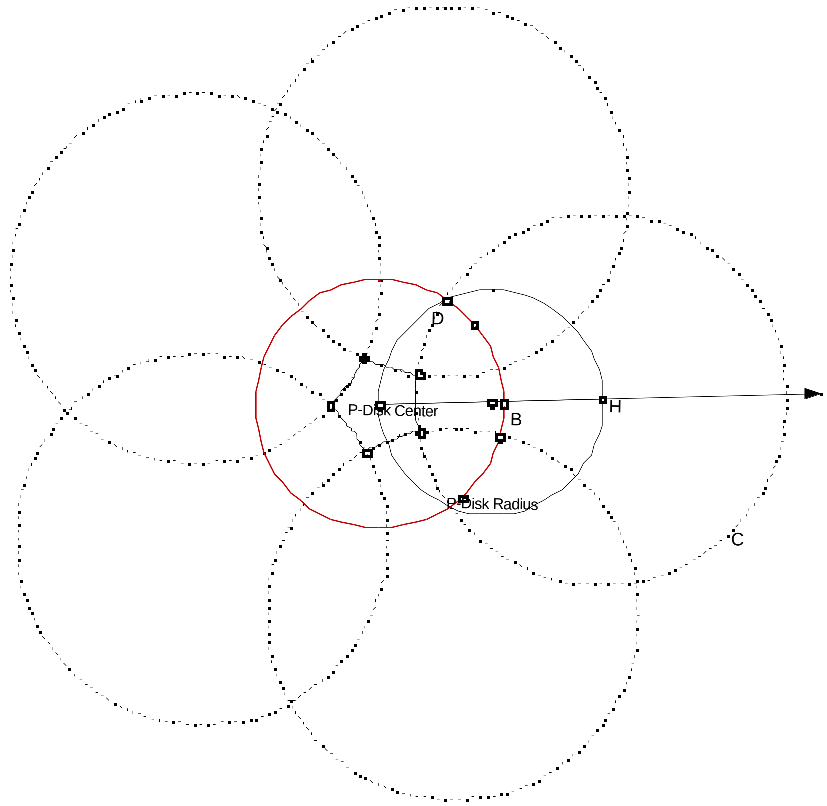
- Regular hexagon meeting 4 at vertex (i.e. interior angles are equal to $2\pi/4$): $k=4$, $n=6$ thus $h=\sqrt{2}\approx 1.414$
- Regular quadrilateral meeting 6 at vertex (i.e. interior angles are equal to $2\pi/6$): $k=6$. $n=4$, thus $h=\sqrt{3}\approx 1.732$
- Regular pentagon meeting 4 at vertex (i.e. interior angles are equal to $2\pi/4$): $k=4$, $n=5$ thus $h=\sqrt{\sqrt{5}+1}\approx 1.798$

To construct the n-gon, once we know h, we can do the following. We'll do the specific case of a regular pentagon,



- Open the Poincaré Disk Starter.
- Draw a ray through the disk center. Construct the point of intersection with the Poincare Disk. Label it *B*.
- Select the P. Disk Center and "Mark Center" under the **Transform** menu. Now dilate B by the scale factor = h =1.798. This new point is the center of the desired circle, label it *H*.
- Let O denote the P. Disk Center (do not change the label in your sketch since any script that uses auto-matching will not work). Construct a circle with diameter $\overline{OH}$. Then construct one of the points of intersection with the Poincaré Disk, label it *D*.

- Construct the circle *C* by center *H* and point *D*.
- Rotate *C* about the P-disk Center by 72 degrees. Do this 5 times.
- Construct the *5* points of intersection that are closest to the P-Disk Center. These points are the vertices of the pentagon. Connect them with h-segments. Hide anything that is unwanted.

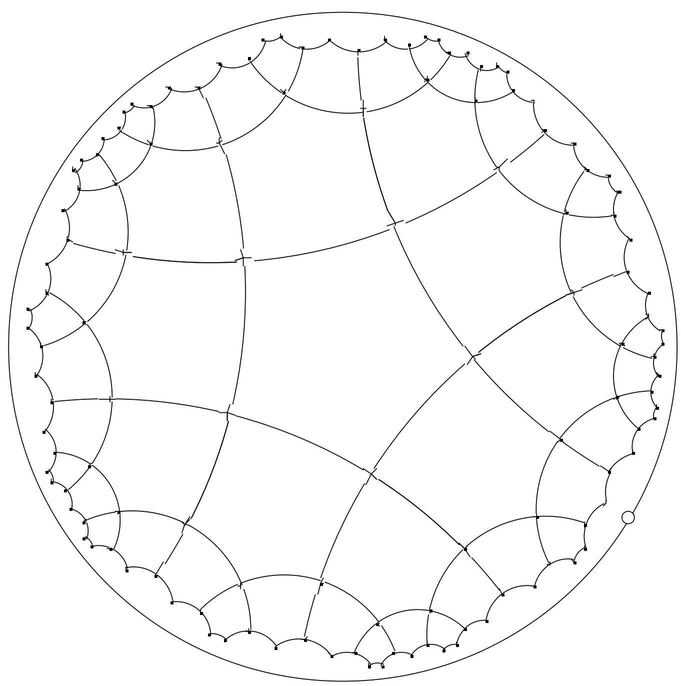


End of Demonstration 5.5.1.

5.5.1a Demonstration. Tiling the hyperbolic plane.

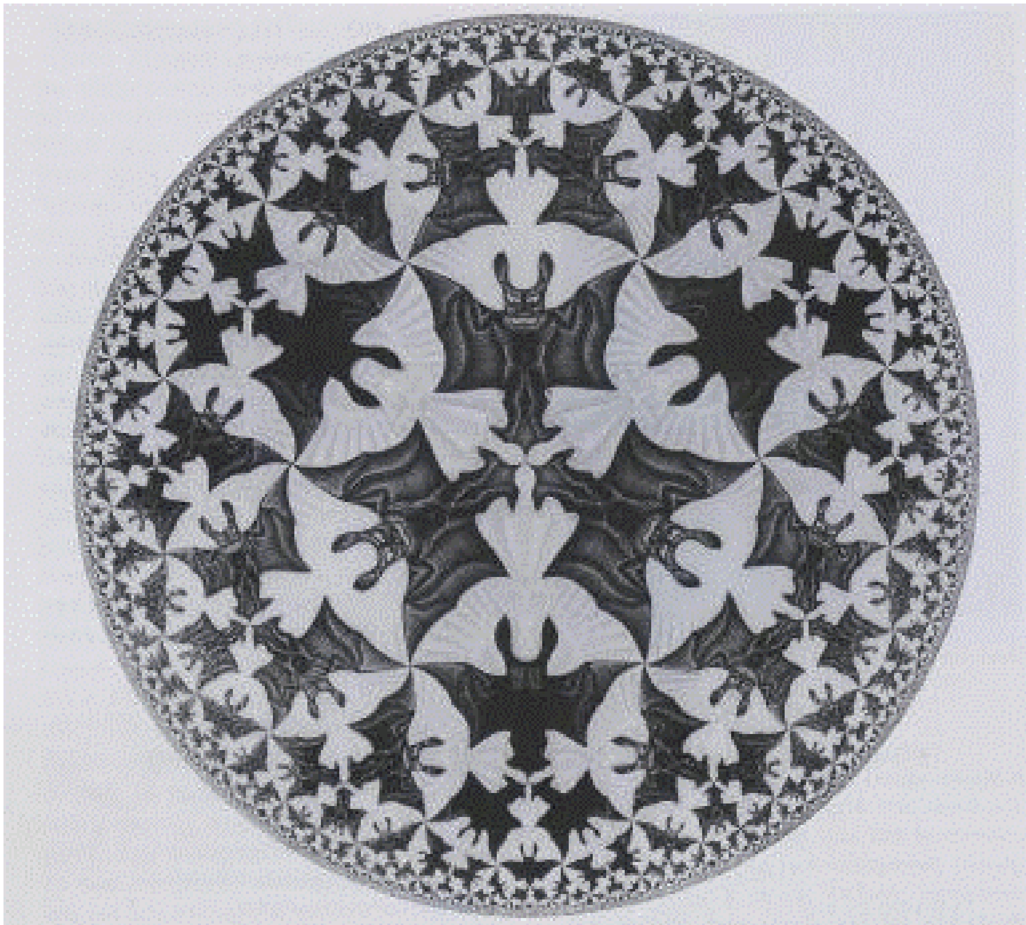
Once we have an appropriate starter *n-gon* that will tile the hyperbolic plane by meeting *k* at a vertex (i.e. the interior angles equal $2\pi/k$) we can tile plane successively h-reflecting the figure. Things will go a little quicker if we also allow ourselves rotations as well. Choose one side of the regular *n-gon* and reflect the vertices of the *n-gon* across this h-segment (we can accomplish this with an appropriate tool since this is equivalent to inverting the vertices with respect to the circle). Then connect the images of these vertices by h-segments. One could continue this process producing a tiling of the plane (up to the memory limitations of SketchPad). To make the process go faster one could also use (Euclidean) rotations about the P-Disk Center of $2\pi/n$ degrees.

For example, starting with our regular hexagon, we can create a hyperbolic as in the figure below!



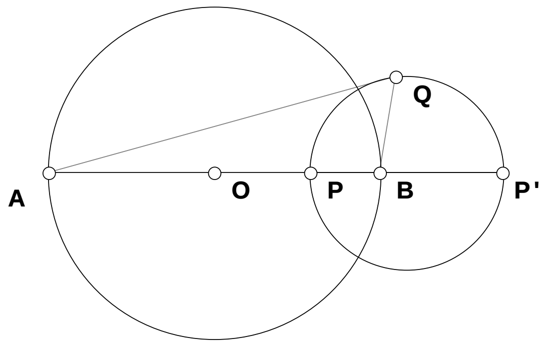
End of Demonstration 5.5.1a.

Now go back to Escher's Devils and Angels graphic in the hyperbolic plane (see below). Escher is using "colored" tiles to tile the hyperbolic plane. Can you determine what regular polygon is underlying the tiling? How many are meeting at each vertex?



5.6 Exercises. These exercises follow up on the connection between inversion and Apollonius' Circle and between inversion and linkages.

Exercise 5.6.1. Complete the proof of Conjecture 5.4.2. That is if P and P' are inverse points with respect to circle C and lie on the diameter $\overline{AB}$ of C and Q any point on the circle with diameter $\overline{PP'}$ then $\frac{AQ}{BQ} = \frac{AP}{BP}$. Follow the steps below to give a coordinate geometry proof.



· Let $A=(-1,0)$, $B=(1,0)$, and the P be the point $(a,0)$. What are the coordinates of P' ?

· What are the coordinates of the midpoint of the line segment $\overline{PP'}$?

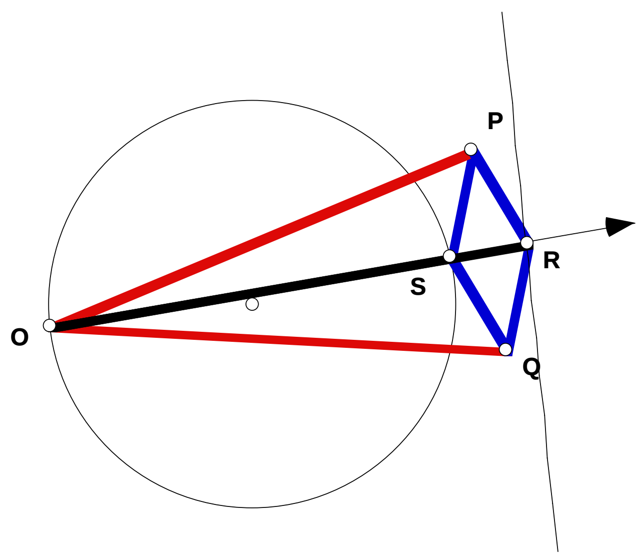
· What is the equation of the circle C' ’?

· Determine the ratio PA/PB .

· Determine the ratio QA/QB.

· Complete the solution by showing $\frac{AQ}{BQ} = \frac{AP}{BP}$.

The remaining exercises refer to the Peaucellier linkage and the figure below.



Exercise 5.6.2. Using the fact that *PRQS* is a rhombus, prove that its diagonals are perpendicular and bisect each other.

Exercise 5.6.3. Prove that *OS.OR* is a constant by proving that $OS.OR = OP^2 - PR^2$. When do *S*, *R* lie on the circle centered at *O* having radius $\sqrt{OP^2 - PR^2}$?

Exercise 5.6.4. Deduce from Exercise 5.6.3 that the locus of *R* is a straight line *l* as *S* varies over circle *C*.

Exercise 5.6.5. Prove that *l* is perpendicular to the line passing through *O* and the center of the circle *C*.

Exercise 5.6.6. As *S* varies over the circle *C* does *R* vary over all of the (infinite) line *l*? If not, give a precise description of the line segment that *R* describes. Can *S* go around all of circle *C*? If not, give a precise description of the arc of *C* that *S* traces.

M333L - BIBLIOGRAPHY FOR TOPICS – SPRING 2003

Abstract symmetry in mathematics and physics

- Bunch, Bryan, Reality's Mirror QC 174.17 S9 86 PMA
- Guillen, Michael, Bridges to Infinity,1983, pp.73-80
- Hauptmann, Herbert A., "The Role of Symmetry in Mathematical Discovery", Symmetry v.1, 1990 27-28
- Polya, How to Solve It
- Rosen, Joe, "Symmetry, Analogy, Science", Symmetry, v.1, 1990
- Zee, A., Fearful Symmetry

Applications of geometric transformations in computer graphics

- Smart, Modern Geometry

The arbelos

- Bankoff, Leon, "Are the twin circles of Archimedes really twins?", Mathematics magazine, v. 47, no. 4, September 1974, pp. 214-218
- Bankoff, Leon, "The golden arbelos", Scripta Mathematica, v. 21, no1, March 1955, pp.70-76
- Gaba, M.G., "On a generalization of the arbelos,", the American mathematical Monthly, v. 47, January 1940, pp.19-24
- Gardner, Martin, "Mathematical Games", Scientific American, Jan 1979, vol 240, 18-28
- Posamentier, Alfred and Gordon Sheridan, Investigations in Geometry, Addison-Wesley, 1982

Billiards

- Bender, et al, Billiard Ball Problems on Unusual Tables, COMAP
- Carroll, Lewis, Circular Billiards
- DeTemple, Jack and Duanene Robertson, The Calc Handbook, pp. 197-201 (billiards on conics)
- Mizerak, Steve, Inside Billiards
- Olivastro, Dominic, Hall of Mirrors, Science, v. 32 no. 1, Jan-Feb 1992, pp.54-56
- Pardon, George Frederick, Billiards; Its Theory and Practice
- Schoenberg, Isaac J., Mathematical Time Exposures
- Wells, David, Hidden Connections, Double Meanings, pp. 126 – 128

Cavalieri's Principle

- Eves, Howard, Two Surprising theorems on Cavalieri Congruence, the College Mathematics Journal, march, 1991

Chinese Geometry

- Chinese mathematics : a concise history. / Li, Yen, 1892-1963. / Oxford Oxfords# 1987 QA 27 C5 L4713 1987 PCL Stacks
- Swetz, Frank, The Volume of a Sphere: A Chinese Derivation, Mathematics Teacher, February, 1995, 142 - 145

Collinearity of points and Concurrency of lines

- Posamentier, Alfred and Gordon Sheridan, Investigations in Geometry, Addison-Wesley, 1982

Congruence conditions for triangles

- Beran, Davis, SSA and the Steiner-Lehmus theorem, Mathematics teacher, May 1992, 380-383

Conic Sections

- Broman, Arne and Lars Broman, Museum Exhibits for the Conics, Mathematics Magazine, vol 67, June, 1994, pp. 206 – 209
- Dawson, John W. Jr., Displaying the Conics: Three alternatives to computer graphics, Primus, vol.1, no.1, March, 1991
- Downs, J. W., Practical Conic Sections, Dale Seymour, 1993
- Drunker, Daniel, Reflection Properties of Curves and Surfaces, Mathematics Magazine, June 1992
- Land, Frank, The Language of Mathematics, Doubleday, 1963, pp. 147 – 159 ,206 – 214
- Schupp, H, Applications and Geometry, in M. Niss et al, ed., Teaching of Mathematical Modeling and Applications, Ellis Hoorwood, 1991, pp. 220-229 QA401 T433 PMA
- F. Van Maanen, Seventeenth century instruments for drawing conic sections, Mathematical Gazette 76:476 (1992) 222-230
- Worksheet from the San Francisco Exploratorium, Sourcebook, Urban Mathematics Collaboratives, 1990, SB-12
- De Temple, Duane, A Direct Derivation of the Equations of the Conic Sections, Mathematics Teacher, March 1990

Constructing geometric figures from string

- Bolt, Brian, More Mathematical Activities for Teachers, Cambridge University Press, 1985, pp. 63-66, 70 -71, 156 - 157
- Cundy, Henry M. and A.P. Rollett, Mathematical Models, 1961
- Dixon, Robert, Mathographics, Dover, 1987, pp.75-78
- Gardner, Martin, Penrose Tiles to Trapdoor Ciphers
- Lyng, Merwyn J., Dancing Curves: A Dynamic Demonstration of Geometric Principles, NCTM 1978
- Pedoe, Dan, Geometry and the Visual Arts
- Pohl, Victoria, How to Enrich Geometry using String Designs, Dale Seymour
- Somervell, Edith L., A Rhythmic Approach to Mathematics, NCTM, 1975
- Steinhaus, Hugo, Mathematical Snapshots
- Winter, John, String Sculpture, Creative Publications
- NCTM, Multisensory Aids in the Teaching of Mathematics, Eighteenth Yearbook, 1945
- Seymour, Dale, Geometric Design, Dale Seymour Publications, 1988

Constructing regular polygons

- Ball and Coxeter
- Bold, Famous Problems of Geometry

- DeTemple, Duane W., Simple Constructions for the Regular Pentagon and Heptadecagon, Math. Teacher, May 1989 (contains other refs) Mathematics Teacher, March 1990, p. 228
- DeTemple, Duane W., Carlyle Circles and the Lemoine Simplicity of Polygon Constructions, American Mathematical Monthly,v.98, n0. 2, February 1991, pp.97-108
- Dixon, Robert, Mathographics, Dover, 1987, pp. 34-44, 52-53

Constructing triangles

- Posamentier, Alfred and Gordon Sheridan, Investigations in Geometry, Addison-Wesley, 1982

Cross Sections of Solids

- Hunt, Richard A., Calculus with Analytic Geometry
- Wells, David, The Penguin Dictionary of Curious and Interesting Geometry, p. 195
- Wenniger, Magnus, Dual Models

Crystals

- Baeyer, Hans C., Impossible Crystals, Discover 11, February, 1990, 69 – 78
- Tromba, Anthony. / Mathematics and optimal form. / New York 1985 BH 301 N3 H55 1985 PCL Stacks (Article by Wulff)

Curves of Constant Width

- Caldwell, J.H., Topics in recreational mathematics, QA 95 C28 UGL
- Rademacher, Hans and Otto Toeplitz, The Enjoyment of Mathematics, Dover Selected Papers in Geometry, pp. 292-294
- Smith, Scott, Drilling Square Holes, Mathematics Teacher, vol 86, no.7, October, 1993, pp. 579 - 583 (Contains more references.)
- Smart, Modern Geometries, section 3.6

Cycloids, Epicycloids, and Hypocycloids

- Bolt, Brian, Mathematics Meets Technology, Cambridge Univ press, 1991, pp.112-113, 118

Dimensions

- Abbot, E., Flatland
- Banchoff, Thomas F., Discovering the Fourth Dimension, Natick, MA: PrimComputer Inc., 1987
- Bragdon, Claude, Projective Ornament, Dover.
- Burger, Dionys, Sphereland: A Fantasy About Curved Spaces and an Expanding Universe, New York: Harper and Row 1965
- Critchlow, Keith, Order in Space, New York, Thames and Hudson, 1969
- Dewdney, A. K., Planiverse
- Eckhart, Ludwig, Four-dimensional Space
- Henderson, Linda, The Fourth Dimension and Non'Euclidean Geometry in Modern Art
- Hess, Adrien L., and Carl Diekhans, The Number of Segments Needed to Extend a Cube to N Dimensions, Mathematics Magazine 43 (September1970), 187-92
- Hess, Adrien L., Viewing Diagrams in Four Dimensions, Mathematics Teacher 64 (March 1971), 247-48
- Hess, Adrien L., Four-Dimensional Geometry -- An Introduction, NCTM, 1977

- Hilbert and Cohn-Vossen, Geometry and the imagination
- Loeb, Space Structures: Their Harmony and Counterpoint, Birkhauser, 1976, QA 491 L63
- Manning, Henry Parker, The Fourth Dimension Simply Explained, New York,macmillan, 1911
- Manning, Henry Parker, Geometry of Four Dimensions, New York, Dover, 1956
- Marr, Richard, Four Dimensional Geometry, Houghton Mifflin, 1970
- Miyazaki, K., An Adventure in Multidimensional Space: The Art and Geometry of Polygons
- Peterson, Ivers, The Mathematical Tourist, New York, Freeman, 1988
- Rucker, Rudy, geometry, Relativity, and the Fourth Dimension
- Rucker, Rudy, Miindtools, The Five Levels of Mathematical Reality, Boston, Houghton Mifflin, 1987
- Rucker, Rudy, ed., Speculations on the Fourth Dimension, Selected Writings of Charles H. Hinton, New York, Dover, 1980
- Rucker, Rudy, The Fourth Dimension, Toward a Geometry of Higher Reality, Boston, Houghton Mifflin, 1984
- Senechal, Marjorie and G. Fleck, Shaping Space: A Polyhedral Approach, Birkhauser, 1988
- Steen, Lynn, A. On the Shoulders of Giants: New Approaches to Numeracy Arithmetic Teacher, vol 37, no 6, February 1990
- Weeks, Jeffrey R., The Shape of Space, Marcel Dekker, 1985, QA 612.2, W44, 1985, PMA

<u>Dissecting geometric figures</u>

- Anglin, R.H. and R. Robinson Rowe, J.R. M. 8(20), 1975,149-50
- Coffin, Stewart T. The Puzzling World of Polyhedral Dissections, Oxford, U. Press, 1990
- Galvin, Fred Letter to the Editor, Amer Math Monthly, Feb 1990, p.131
- Gardner, Martin, Penrose Tiles and Trapdoor Ciphers QA 95 G 298 PMA
- Kierstead, Friend H., Jr. and R.S. Johnson, J.R.M. 9(1), 61-63, 1976 (?)
- Harry Lindgren, Recreational Problems in Geometric Dissections and How to Solve Them . Revised and enlarged by Greg Frederickson. New York: Dover Publications, 1972.
- Frederickson, Greg N, Dissections: Plane & Fancy. New York: Cambridge University Press, 1997.
- Wills, Herbert, III, Leonardo's Dessert: No Pi, NCTM, 1985

<u>Dissections (2 and 3 D)</u>

- Bolt, Brian, The Amazing Mathematical Amusement Arcade, Cambridge, 1984
- Buchman, Edwin, The impossibility of Tiling a convex region with unequal equilateral triangles, American Mathematical Monthly, Dec 1981, pp. 748-753
- Caldwell, J.H., Topics in recreational Mathematics, QA 95 C28 UGL
- Coffin, The Puzzling World of Polyhedral Dissections, Oxford U. Press, 1991

- Domoryad, A.P. tr. Moss, Halina, Mathematical Games and Pastimes, Macmillan, New York, 1964, pp. 158-65, 193-201)
- Eves, Howard, a Survey of Geometry
- Fry, Erin K. and Peter L. Glidden, Illustrating Mathematical Connections: A Geometric Proof of Euler's Theorem, Mathematics Teacher, January, 1996, pp. 62 – 65
- Hunter, J. A.. H. and Madachy, Joseph S., Mathematical Diversions, Van Nostrand, New Jersey, 1962, pp. 56-67
- Langman, Harry, Play Mathematics, Hafner, New York, 1962, pp. 119-55
- Langman, Harry, Geometric Dissections, D. Van Nostrand, Princeton, New Jersey, 1964
- Musser, Gary L. and Burger, William F. Mathematics for Elementary Teachers, Macmillan, New York, 1988, p.516
- Soifer, Alexander, How Does One Cut a Triangle?, Center for Excellence in Mathematical Education (885 Red Mesa Dr., Colorado Springs, CO 80906); xiii + 139 pp, $18.95 (P). ISBN 0-940263-01-7
- Yates, Robert C., Geometrical Tools, Educational Publishers, 1949. 744 Y 27 (UGLI)

Duplicating the cube

- Ball and Coxeter Bold, Famous Problems of Geometry

Erroneous proofs in geometry

- Ball and Coxeter
- Carroll, Lewis, The Lewis Carroll Picture Book
- Fetisov, A.I., Proof in Geometry, Mir
- Gardner, Martin, Wheels, Life and Other Mathematical Amusements, pp. 51-59
- D.A. Maxwell, Fallacies in Mathematics (Cambridge, 1961)
- Ya. S. Dubros (Dubnov?), Mistakes in Geometric Proofs (Heath, 1963)
- Bradis, Lapses in Mathematical Reasoning, QA 43 B6953
- Lakatos, I., Proofs and Refutations: the Logic of Mathematical Discovery
- Northrop, E.P., Riddles in Mathematics, Van Nostrand, 1944
- Posamentier, Alfred and Gordon Sheridan, Investigations in Geometry, Addison-Wesley, 1982
- Posamentier, Alfred J.H. banks, and R.L. Bannister, Geometry, Its Elements and Structure, McGraw-Hill, 1977, pp.242-244, 270-271

Euler's formula

- Coxeter, H.S.M., Introduction to Geometry
- Mortenson, Michael, Geometric Modeling
- Smart, Modern Geometry
- Pedoe, Dan, Geometry and the Visual Arts
- Barr, Stephen, Experiments in Topology, 1964, Dover
- Courant and Robbins, What is Mathematics
- Hilbert and Cohn-Vossen, Geometry and the Imagination

- Loeb, Space Structures: Their Harmony and Counterpoint, Birkhauser, 1976, QA 491 L63
- Lord, E.A.. and Wilson, C.B., The Mathematical Description of Shape and Form, 1984
- Lyusternik, L.A., Convex Figures and Polyhedron, 1963, Dover
- Newman, James, The World of Mathematics
- V.N.R. Concise Encyclopedia of Mathematics, Van Nostrand

Fibonacci Sequence

- Daniel C. Litchfield and David A. Goldenheim with support from Charles H.Detrich, Euclid, Fibonacci, Sketchpad, Mathematics Teacher, January 1997. p 8 - 12

Fivefold symmetry

- Hatigittai, Istvan, Fivefold Symmetry, 1992

Fractal geometry

- Barnsley, Michael, Fractals Everywhere
- Barcellos, Anthony, The Fractal Geometry of Mandelbrot, College Mathematics Journal 15, March 1984, 98-114
- B/W Fractals, Macintosh and Hypercard programs, $8.99 from Somak Software, 535 Encinitas Blvd, Encinitas, CA 92024, (619) 942- 2556
- Devlin, Keith, Mathematics: the New Golden Age Encyclopedia of Physical Science and Technology, Vol 5
- Garcia, Linda, The Fractal Explorer, 1990
- Gardner, Martin, Penrose Tiles and Trapdoor Ciphers QA 95 G 298 PMA
- Falconer, K.J., The Geometry of Fractal Sets, Cambridge, 1985
- Feder, Jens, Fractals, 1988, Plenum Press
- Kern, Jane F. and Cherry C. Mauk, Exploring Fractals - A Problem-Solving Approach using Mathematics and Logo, Mathematics eacher, March 1990
- Lam, Lui, ed., Nonlinear Physics for Beginners, World Scientific Publishers, 199
- Long, 1981, Pascal's triangle and p self-similarity (ICM 90 ref)
- Mandlebrot, Benoit B., The Fractal Geometry of Nature
- MandelZot v2.0, Macintosh program, $8.99 from Somak Software, 535 Encinitas Blvd, Encinitas, CA 92024, (619) 942-2556
- McDermott, Jeanne, Geometrical Forms Known as Fractals Find Sense in Chaos, Smithsonian 14, December 1983, 110-117
- McWhorter, William, A., Jr, and Jane Morrill Tazeleer, Creating Fractals, Byte 12, August 1987, 123-128

- Peitgen, Heinz-Otto, The Beauty of Fractals
- Peitgen, et al, Fractal Cosmos 1990 Calendar (ref Math Teacher Dec 89)
- Peterson, Ivars, "Ants in Labyrinths" and "The Dragons of Chaos", in The Mathematical Tourist
- Schroeder, Manfred, Fractals, Chaos, Power Laws, 1992
- Six Worlds, IBM PC and compatibles 5-1/4" diskette, Turing Omnibus, Box 1456, London, ON N6A 8M2 (Review Math Teacher, May 1989)
- Newman, Rochelle and Martha Boles, The Golden Relationship: Art, Math, Nature
- Various materials available from Media Magic, Box 507B, Nicasio, CA 94946
- Wolfe, 1984, Fractal dimension (ICM 90 ref)
- Bannon, Thomas J., Fractals and Transformations, Mathematics teacher, March, 1991, 178ff
- Barton, Ray, Chaos and Fractals, Ma thematics Teacher 83, October 1990, 524-29
- Bolt, Brian, More Mathematical Activities, Cambridge University Press 1985, pp. 72-74, 157-158
- Camp, Dane R., A Fractal Excursion, Mathematics Teacher, April, 1991, pp. 265 ff
- Devaney, Robert Chaos, Fractals and Dynamics: Computer Experiments in Mathematics, Addison Wesley, 1990, Ch 9
- Lauwerier, Hans, Fractals: Endlessly Repeated Geometrical Figures, QA 614.86 L3813 1991 PMA
- Martin, Tami, Fracturing Our Ideas about Dimension, NCTM Student Math Notes, insert in NCTM News Bulletin, november, 1991
- Nievergelt, Yves, Fractals on Hewlett-Packard Supercalculators, New Directions in Math series, Wadsworth, 1991, $8
- Norton, Alec, Review of Measure, Topology, and Fractal Geometry, by Gerald A. Edgar, American Mathematical Monthly, April 1992, p. 378 [Parts of the review will be too technical, but other parts are relevant.]

Geodesic structures

- Edmondson, Amy C. A Fuller Explanation: The Synergetic Geometry of R. Buckminster Fuller, Birkhauser, 1987
- Fuller, Buckminster, Utopia or Oblivion: the Prospects for Humanity
- Kenner, Hugh, Geodesic Math and How to Use It
- Laycock, Mary, Bucky for Beginners, Dale Seymour
- Mottel, Syeus, Charas: The Improbable Dome Builders
- Prenis, John, The Dome Builder's Handbook

- Wenniger, Magnus, Spherical Models
- Yarnall, William, Dome Builder's Handbook no.2
- Bolt, Brian, More Mathematical Activities for Teachers, Cambridge University Press, 1985, pp. 45 - 50 (rigid structures)
- Kenner, Hugh. / Geodesic math and how to use it. / Berkeley 1976 TH 2170 K46 Architecture Library
- Pugh, Anthony, Polyhedra, a Visual Approach, Univ.of California Press, 1976
- Domebook 2, Pacific Domes, 1971

Geometric constructions by paper folding

- Cundy, Henry M. and A.P. Rollett, Mathematical Models, 1961
- Fukuta, Jiro, Problem E 3369, Amer Math Monthly, Feb 1990, p. 150
- Johnson, Donovan, Paper folding for the Mathematics Class NCTM, 1957
- Martin, George E., Transformatio Geometry, Springer, section 5.2
- Olson, Alton T., Mathematics through Paper Folding, NCTM, 1975 (?)
- Pappas, Theoni, Mathematics Appreciation, Dale Seymour
- Pedoe, Dan, Geometry and the Visual Arts
- Row, T. Sundara, Geometric Excursions in Paper Folding
- Smart
- Yates, Robert, Carl, Geometrical Tools, QA 464 Y3 1949

Geometrical Puzzles

- Bolt, Brian, Mathematical Cavalcade
- Bolt, Brian, More Mathematical Activities
- Conway, J. and Guy, R., Winning Ways
- Domoryad, A. P., Mathematical games and pastimes
- Dunn, Angela, Mathematical Bafflers, Dover
- Klarner, D., Thr Mathematical Gardener
- Moscovich, Mind Benders, Games of Shape
- Mott-Smith, Geoffrey, Mathematical Puzzles, Dover
- Slocum, J. and Botermasn. J., Puzzles Old and New

Geometric quilt designs

- Labelle, Judith and Carol Ann Waugh, Patchworking
- Washburn and Crowe

Geometry in Animal Form and Function

- Haldane, J. B. S. (John Burdon Sanderson), 1892-1964. / On being the right size and other essays. / Oxford 1985 QH 311 H318 1985 Life Science Library

- Stewart, Ian and Martin Golubitsky, Fearful Symmetry, Penguin, 1992, chapters 7 and 8
- Thompson, D'Arcy Wentworth, On Growth and Form (2 vols), Cambridge U. Press, 1972

Geometry in Ancient China

- Knorr, Wilbur, textual Studies in Ancient and Medieval Geometry, Birkhauser, 1989

Geometry in Architecture

- Baglivo, Jenny, and Jack A. Graver: Incidence and Symmetry in Design and Architecture, Cambridge Universtiy Press, Cambridge, 1983
- Hay, D. R., The Orthographic Beauty of the Parthenon, 1853
- Ghyka, Matila, The Geometry of Art and Life, Dover
- Pedoe, Dan, Geometry and the Visual Arts
- Mathematics Teacher, January, 1990
- Salvadori, Mario, Why Buildings Stand Up, W.W. Norton, 1980

Geometry in Astronomy

- McGraw-Hill Encyclopedia of Science and Technology
- Milnor, John, On the geometry of the Kepler problem, Amer Math Monthly 90(1983)353-364
- Rosenfeld, B.A. and M.D. Sergeeva, Stereographic Projection, Mir
- Saari, Donald g., A Visit to the Newtonian N-body Problem via Complex Variables, Amer. Math Monthly 97, Feb 1990

Geometry in Design

- Baglivo, Jenny, and Jack A. Graver: Incidence and Symmetry in Design and Architecture, Cambridge Universtiy Press, Cambridge, 1983
- Barratt, Krome, Logic and Design, the Syntax of Art, Science and Mathematics, Eastview, Westfield, NJ, 1980
- Edwards, Edward B., Pattern and Design with Dynamic Symmetry: How to Create Art Deco Geometric Designs, Dover
- Hambridge, Jay, The Elements of Dynamic Symmetry, Dover
- Loeb, Arthur L., Concepts and Images: Visual Mathematics, Birkhauser, 1993, QA 36, L64, 1993, PMA

Geometry of Crystals and Molecular Structure

- American Scientist, January-February 1993 "a variety of insights that geometric methods give into fullerenes."
- Buerger, Martin Julian, Introduction to Crystal Geometry
- Gallian, Joseph A., 1986, Contemorary Abstract Algebra, Heath, Chapter 30
- Galiulin, R.V., Diamond Latticework: The fearful symmetry of crystalline structures, Quantum, Jan/Feb 1991, pp. 6-11

- Gallian, Joseph A., 1986, Contemorary Abstract Algebra, Heath, Chapter 30
- Hilbert and Cohn-=Vossen, Geometry and the Imagination
- Holden, Alan. / Crystals and crystal growing. / 1st ed. Garden City, NY 1960, QD 921 H58 Undergraduate Library
- Holden, Alan. / The nature of solids. / New York 1965 531 H711N
- Klein, Herbert, Manula of Minerology
- Mercer, Ian F., Crystals, Harvard University Press, Cambridge, MA, 1990 (Elementary, good pictures)
- Montesinos, Jose Maria, Classical tessellations and three-manifolds 1944- / Berlin 1987, QA 166.8 M66 1987 Phys-Math-Astron Library (the mathematics may be too advanced, but it has some good pictures)
- Newman, The World of Mathematics
- Smith, Deane, Bibliography on molecular and crystal structure models
- Steinhardt, Paul, American Scientist, v.74, 1986, p. 586 (quasicrystals)
- Senechal, Marjorie, Crystalline Symmetries: An Informal Mathematical Introduction
- Senechal, Marjorie and G. Fleck, Shaping Space: A Polyhedral Approach, Birkhauser, 1988
- Steinhardt, Paul, Endeavor, v. 14, 1990, p. 112 (quasicrystals)
- Sterling, Bruce, Buckymania, Fantasy and Science Fiction, July, 1992, 85-90
- http://www.nyu.edu/pages/mathmol/K_12.html

Geometry of Escher Prints

- Coxeter, H.S. M., M. C. Escher Art and Science, North Holland, 1986
- Ernst, The Magic Miirror of M.C. Escher, Random HOuse, 1984
- Escher, M. C., Escher on Escher, exploring the infinite, Harry N. Abrams, 1989
- Escher, M. C., Graphic Work of Escher, M. C., Hauthor Books, 1960
- Escher, M. C., Universe of Mind Play
- Schattschneider, Doris, Visions of Symmetry, W.H. Freeman, 1990
- WORLD OF ESCHER, INC., http://www.texas.net/escher/
- Art and Man, Dec 1985/Jan 1986, "How Escher Created", pp.4-5
- Britton, Jill, and Dale Seymour, Introduction to Tessellations
- Gardner, Martin, Mathematical Carnival
- MacGillavry, Caroline, Fantasy and Symmetry: The Periodic Drawings of M.C. Esher
- MacGillavry, Symmetry Aspects of Escher's Periodic Drawings, Scheltman and Hokema, Bohn, 1976
- M.C. Escher: Art and Science
- Smart, Modern Geometry

Geometry of Galileo

- Galileo, Galilei, Operations of the Geometric and Military Compass

Geometry of Golf

- Beasley, John D., The Mathematics of Games, Oxford Univ Press, 1989
- Boomer, Perry, On Learrning Golf, Knopf, 1846, pp. 187-196

- Clemence, Willian James, A cinematographical study of the variation in momentum when swinging varying clubhead weights
- Jones, Robert Tyre, Bobby Jones on The Basic Golf Swing, Doubleday, 1969, pp. 54,57, 62-3
- Thomas, Alvin, The use of a visual perceptual device in teaching the downswing in golf
- Wiren, Gary, Golf, pp.16-17,26-27, 34-35, 42-45
- Wrigglesworth, Frank L., A cinematographical analysis of the short chip in golf

Geometry in Higher Dimensions

- Banchoff, Thomas F., Beyond the Third Dimension, (Book 1990, video 1992)
- Francis, George K, A Topological Picture Book
- Gardner, Martin, Mathematical Carnival, MAA, 1984, QA 95 G286
- Leonardo (International Society for the Arts, Sciences, and technology), The Fourth Dimension in Art, Science, and Mathematics, 1992
- Robbins, Fourfield: Computers, Art and the Fourth Dimension, book and video, both 1992. Software for Macintosh available for $10 from Media Magic, 1-800-882-8284
- Rucker, Rudy, The Fourth Dimension, 1984
- Steen, ed., On the Shoulders of Giants

Geometry of billiards

- Gardner, Martin, Scientific American, v. 212, May 1965
- Gardner, Martin, Scientific American, v. 236, April 1977
- Griffel, D.H., Mathematical Gazette, June, 1989
- Jepsen, Charles, Two Year College Mathematics Journal, November, 1979
- Schultz, Harris and Ray C. Shifless, Mathematical Gazette, June, 1988

Geometry of crystals

- Anderson, Hyde, Inorganic Crystal Stuuctures
- Brown and Forsyth, The Crystal Structure of Solids
- Jaffe, Howard, Introduction to Crystal Chemistry
- Lockwood, E.H., and R.H. MacMillan, Geometric Symmetr
- Wells, David, Hidden Connections, Double Meanings, Cambridge, 1988

Geometry of genetics

- Mathematics Untwists the Double Helix, Science, 23 February 1990

Geometry of Islamic art

- Berggren, J.L., Episodes in the Mathematics of Medieval Islam, QA 27 A67 B46 1988

- Dury, Art of Islam
- Ettinghausen and Graber, The Art and Architecture of Islam, 650-1250
- Faruqui, Islam and Art
- James, Islamic Art
- Knorr, Wilbur, textual Studies in Ancient and Medieval Geometry, Birkhauser, 1989
- Kuhnel, Islamic Arts
- El-said and Parman, Geometric Concepts in Islamic Art, Dale Seymour Publications, 1976
- Rice, Islamic Art
- Prussin, Labelle, Hatumere: Islamic Design in West Africa
- Soucek, Content and Context of Visual Arts in the Islamic World
- Van der Waerden, Geometry and Algebra in Ancient Civilizations, Springer, 1983
- Washburn and Crowe

<u>Geometry of knots</u>

- Bain, G. Celtic Art, Dover, 1973
- Brown, Ronnie, and John Robinson, Symbolism: Sculptures and Tapestries by John Robinson, Mathematics and Knots, (School of Mathematics, U. of Wales, Bangor, Gwynedd LL57 1UT, Wales, UK), 1989, 36 pp., £6 + £1 surface, £250 air) ISBN 0-9514947-08. Also, 27 pp, ISBN 0-9514947-16
- Cipra, Bary A., "To Have and Have Not: When Are Knots Alike?", Science, v.242, Sept 9, 1988
- Devlin, Keith, Mathematics: the New Golden Age
- Geometry Center, Ujniversity of Minnesota, Not Knot (video), 1991
- Holden, Orderly Tangles, QA 491 H626 UGLI
- Moran, Siegfried, The Mathematical Theory of Knots, Elsevier, 1989
- Peterson, Ivars, The Mathematical Tourist
- Peterson, Ivars, "Unknotting a Knotty Tale, Science News, v. 133, May 21, 1988 "Tying up a Knotty Loose End," Science News, v.134, Oct 29, 1988 Geometry of Molecular Structure
- Gillespie, R.J., Molecular Geometry, Van Nostrand, 1972
- Hout, R.F. Jr., A Pictorial Approach to Molecular Structure and Reactivity, WIley, 1984
- Richards, W.G., Structure and Spectra of molecules, Wiley, 1985, pp.27-43
- Quasicrystals: Rules of the Game, Science, 2 March 1990

<u>Geometry of optics</u>

- Bowden and Schiffer, The role of Mathematics in Science, Chapter 3 (MAA?) (Not in UTCat?)Consortium, November, 1986
- Herzberger, Modern Geometrical Optics, pp. 71-148, 383-420

- Leathem, The Elementary Theory of the Symmetrical Optical Instrument, pp.1-25, 39-42, 44-57
- Martin, Geometrical Optics, pp.1-43,107-130,173-196
- Pitchford, Studies in Geometrical Optics, pp.1-12, 50-60, 72-86, 180-209
- Southall, The principles and methods of Geometrical Optics, pp.1-32, 35, 198-262
- Pedoe, Dan, Geometry and the Visual Arts

The geometry of perspective drawing

- Beskin, N.M., Images of Geometric Solids, Mir
- Leonardo (International Society for the Arts, Sciences, and technology), Art and Technology, 1992
- Moore, Charles G., To View an Ellipse in Perspective, College Mathematics Journal, 1989, 134 – 136
- Penna, Michael A. and Richard Patterson, Projective Geometry and its Application to Comuter Graphics

The geometry of relativity

- Penrose, Roger, The Geometry of the Universe, in Steen, Lynn A., Mathematics Today, QA7 M3447 PMA

Geometry of textiles

- New York Times Educational Supplement, April 3, 1988 Washburn and Crowe
- Grunbaum, B., Periodic ornamentation of the fabric plane: lessons from Peruvian fabrics, Symmetry 1(1990), 45 -68.
- Grunbaum, Branko, and G.C. Shepard, Satins and Twills: An Introduction to the Geometry of Fabrics, Mathematics Magazine, 53, 1980, 131-161 and 313

Geometry of the kaleidoscope

- Arithmetic Teacher, February, 1970
- Ball and Coxeter
- Brewster, Sir David, article on images in mirrors, EncyclopediaBritannica, c. 1818 (Brewster invented the kaleidoscope)
- Coxeter, Regular Polytopes
- Baker, Cozy, Through the Kaleidoscope ... and Beyond, Beechcliff Books, Annapolis, 1987 pp. 13-22,170-178, 188-195
- Mathieu, Jean Paul, Optics, parts 1 and 2, vol. 62, 1075

- Weiting, Thomas W., The Mathematical Theory of Chromatic Plane Ornaments, Marcel Dekker, NY, 1982, pp 31-36,346-362
- Williamson, Samuel J. and Cummins, Herman Z. Light and Color in Nature and Art, Wiley, NY, 1983, pp.223-270

<u>The Golden Ratio in geometry</u>

- Baravelle, H.V., The Geometry of the Pentagon and the Golden Section, Math. Teacher, Jan. 1948
- Dantzig, Tobias, The Bequest of the Greeks
- Ghyka, Matila, The Geometry of Art and Life, Dover
- Huntley, H.E. The Divine Proportion, Dover
- Musser, Gary and William F. Burger, Mathematics for Elementary Teachers
- Newman, Rochelle and Martha Boles, The Golden Relationship: Art, Math, Nature
- Ogilvy, Charles, Excursions in Geometry
- Pappas, Theoni, Mathematics Appreciation, Dale Seymour
- Pedoe, Dan, Geometry and the Visual Arts
- Posamentier, Alfred and Gordon Sheridan, Investigations in Geometry, Addison-Wesley, 1982
- Runion, Garth E. The Golden Section and Related Curiosa, Scott Foresman, 1972
- Smart, Modern Geometry
- Vajda, S., Fibonacci and Lucas Numbers, and the Golden Section:Theory and Applications, QA 241, V24, 1989 PMA

<u>Indian Geometry</u>

- Amma, T.A. Saraavati, Geometry in Ancient and Medieval India, PMA QA 444 S26 1979
- Knorr, Wilbur, textual Studies in Ancient and Medieval Geometry, Birkhauser, 1989
- Sarasvati, Svami Satya Prakesh, PMA QA 443.5 P72 1987
- Van der Waerden, Geometry and Algebra in Ancient Civilizations, Springer, 1983
- Baeyer, Hans C., Impossible Crystals, Discover 11, February, 1990, 69 – 78
- Boles, Martha and Rochelle Newman, Universal Patterns Book 1: The Golden Relationship: Art, MAth and Nature
- Cook, Theodore A., The Curves of Life, Dover
- Hargittai, I and C.A. Pickover, Spiral Symmetry, World Scientific, 1992
- Kessler, James E., Goldern Triangle, Mathematics Teacher, April, 1994, p. 324

· Loeb, Arthur L., Concepts and Images: Visual Mathematics, Birkhauser, 1993, QA 36, L64, 1993, PMA
· Markowski, George, Misconceptions about the Golden ratio, College Mathematics Journal, January, 1992, 2 – 19
· Mathematics Teacher, February 1991, 133-138, The artist's View of Points and Lines
· National Council of Teachers of Mathematics, Historical Topics for the Mathematics Classroom
· Willard, Claude-Jacques, Le nombre d'or: utilisation en mathematiques et dans les beaux-arts, Paris, 1987

Honeycombs
· Peressini, Anthony L., The Design of Honeycombs, Umap Module Unit 502
· Thompson, D'Arcy, On Growth and Form, Cambridge U Press, 1917
· Toth, L. Fejes, What the bees know and what they do not know, Bull. Amer. Math. Soc. vol 70 (1964), pp. 468 - 481

Indian Geometry

· Datta, Bibhutibhusan, The Science of the Sulba, 1932
· Taylor, Charles, Ancient and Modern geometry of Conics, pp. xvii-xxxiii

Inversion
· Pedoe, Dan, Acourse of Geometry, QA 445 P43 PMA
· Rosenfeld, B.A. and M.D. Sergeeva, Stereographic Projection, Mir
· Smart, Modern Geometry

Japanese Geometry
· Fukagawa, Hedetosi and Dan Pedoe, Japanese Temple Geometry Problems, The Charles Babbage Research Centre, P.O. Box 272, St. Norbert Postal Station, Winnipeg, Canade, R3V 1L6, 1989
· Sokolowsky, Dan, Review of "Japanese Temple Problems", American Mathematical Monthly, April, 1991

Knots
· Adams, Colin, The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots, 1994, Freeman
· Gialamas. Stafanos and Gialamos, Panagiotis, Knots Everywhere, Consortium, Spring, 1993, pp. 4 - 5 (Contains other references)
· Gialamas, S. Knots Everywhere, Chicago: Aristotle Press, 1993
· Livingston, Charles, KNot Theory, MAA 1993 (Fairly advanced?)
· Prasolov, V.V., Intuitive Topology, AMS, 1994

Line of Sight Problems
· Stewart, Ian, MAthematical Recreations: Shedding litle Darkness, Scientific American 272:2 (August 1996) 100-103
· Scientific American 270:5 (May 1994) 118 - 120 (Number of guards needed for an art gallery)-

Linkages and geometry

- Bolt, Brian, More Mathematical Activities for Teachers, Cambridge University Press, 1985, pp.12-14, 114
- Bolt, Brian, Mathematics Meets Technology, Cambridge Univ press, 1991
- Choate, Jonathan, Linkages, Consortium, Summer 1995, pp. 6 – 7
- Cundy, Henry M. and A.P. Rollett, Mathematical Models, 1961
- Kempe, A. B., How to draw a straight line: a lecture on linkages, Macmillan
- Shyers, Joan, COMAP Module 594
- Solovyov, Yury, Making the Crooked Straight: Inversors and Watt's Steam Engine, Quantum, November/December 1990, pp. 20 – 23
- Kempe, A.B., How to Draw a Straight Line, NCTM, 1977
- Smart, Modern Geometry
- Meserve, Bruce E., Linkages as visual aids, Mathematics Teacher 39m 1946, 372-79
- Multi-sensory Aids in the Teaching of Mathematics, NCTM Eighteenth Yearbook (?
- Yates, Robert, Carl, Geometrical Tools, QA 464 Y3 1949
- Yates, A Mathematical Sketch and Model Book, Educat.Pub, 1949 (?)

Map coloring

- K. Appel, and W. Haken, The four color proof suffices, Mathematical Intelligencer,8(1986), 10-20
- K. Appel, and W. Haken, th esolution of the four color problem, Scientific American, October 1977, pp.108 – 121
- Barnette, David, Map Coloring, Polyhedra, and the Four-Color Problem, MAA, 1984
- Keeports, David, A Map-coloring algorithm, Mathematics teacher, December, 1991, pp. 759-76
- Appel, Kenneth and Wolfgang Haken, The Four Color Problem, in Steen, Lynn A., Mathematics Today, QA7 M3447 PMA
- Ball and Coxeter
- Brink, Raymond W., ed, Selected Papers on Geometry QA 446 S44, PMA
- Coxeter, H.M.S., The Four Color Problem, 1890-1940, Math. Teacher, April, 1959
- Devlin, Keith, Mathematics: the New Golden Age
- Gardiner, A., Mathematical Puzzling
- Hilbert and Cohn-Vossen, Geometry and the Imagination
- Ringel G., Map Color Problem
- Saaty, Thomas, The Four Color Problem
- Smart, Modern Geometry
- Spitznagel, Edward, Selected Topics in Mathematics

Map Projections

- Consortium, Summer, 1989 (References therein)

Mayan Geometry

- Scientific American, August 1989

Minimal Surfaces

- Boys, Charles, Soap Bubbles, Dover, 1959

- Hildebrandt, Stefan and Anthony J. Tromba, Mathematics and Optimal Form, 1985 (Available from Dale Seymour)
- Leonardo (International Society for the Arts, Sciences, and technology), Art and Technology, 1992
- Taylor, Jean E., Crystals, in Equilibrium and Otherwise, video, 1990 American Mathematical Monthly, April 1992, p. 376
- Courant and Robbins, What is Mathematics?
- Hildebrandt, Stefan and Anthony J. Tromba, Mathematics and Optimal Form
- senberg, Cyril, The science of soap films and soap bubbles, Dover
- NCTM News Bulletin, November, 1995, p. 2 ("Double Bubbles Give Mathematicians No Trouble")
- Scientific American, 1976
- Selected Papers in Geometry, pp. 40-47

Optics

- Conrady, A. E., Applied Optics and Optical Design, vol.1, Dover
- Drucker, Daniel, Reflection Properties of Curves and Surfaces, Mathematics Magazine, June 1992
- House, Peggy, Interactions of Science and Mathematics: A Set of Activities, Eric Clearinghouse for Science, Mathematics and Environmental Education, Ohio State University, College of Education, 1200 Chambers Road, Columbus, Ohio 43212
- Maesumi, Mohsen, Parabolic Mirrors, Elliptic and Hyperbolic Lenses, American Mathematical Monthly, June-July 1992, pp. 558-560
- Riddle, Analytic Geometry, 3rd edition, pp. 138-143 (reflection properties of conics)

Paper Folding

- Envelopes. / Boltianskii, V. G. (Vladimir Grigor'evich) / New York 1964 QA 621 B65 Physics-Math-Astronomy Library
- "Excursions in Geometry" by C. Stanley Ogilvy
- Dutch, Stephen, Folding n-pointed stars and snowflakes, Mathematics Teacher, November, 1994, pp. 630 - 637

Perspective Drawing

- Dixon, Robert, Mathographics, Dover, 1987, pp.79-86
- Edgerton, S. Y., Jr., The Renaissance Rediscovery of Linear Perspective, Basic Books, NY, 1975

Pick's Theorem(Overlap's with Euler's Theorem?)

- Grunbaum, Branko and G. C Shephard, Pick's Theorem, American Mathematical Monthly, February, 1993, pp 150 - 161 (Additionlal references p. 150)

Polyhedra

- Ball and Coxeter
- Bassetti, Fred, at al, Math Projects: Polyhedral Shapes, 1968
- Critchlow, Keith, Order in Space, Dale Seymour

· Cundy, H. Martyn and A.P. Rollett, Mathematical Models, Clarendon Press, 1954
· Federico, Descartes on Polyhedra
· Ghyka, Matila, The Geometry of Art and Life, Dover
· Hilbert and Cohn-Vossen, Geometry and the Imagination, pp.89-93, 290-295, 143-157
· Hilton, Peter, and Jean Pedersen, Build Your Own Polyhedra
· Holden, Allan, Shapes, Spaces, and Symmetry, 1971
· Jimenez,et al, The construction of Platonic bodies from constant width continuous strips, Int J. Math. Educ. Sci. Technol, 1990, vol 21, no.1, 37-50
· Laycock, Mary, Straw Polyhedra, Creative Publications, 1970
· Lockwood, E.H., and R.H. MacMillan, Geometric Symmetry
· Loeb, et al, On the Icosahedron, the Pentagonal Dodecahedron, and the Rhombic Triacontahedron, Symmetry, v.1, no.1, 1990, 29-36.
· Lyusternik, Convex Figures and Polyhedra\
· Ogilvy, Excursions in Geometry, pp. 129-134
· Pearce, Peter, and Susan Pearce, Polyhedra Primer, Dale Seymour
· Pedoe, Dan, Geometry and the Visual Arts
· Posamentier, Alfred and Gordon Sheridan, Investigations in Geometry, Addison-Wesley, 1982
· Senechal, Marjorie and George Fleck, Shaping Space, Birkhauser, 1988
· Toth, L. Fejes, Regular Figures, Pergamon Press, 1964
· Wenninger, Magnus J., Polydron Models, Cambridge U. Press,1970
· Wenninger, Magnus J., Polyhedron Models for the Classroom, NCTM, 1966
· Williams, Robert, The Geometrical Foundation of Natural Structure, Dale Seymour

<u>Polyominoes</u>
· Ball and Coxeter
· Dudeney, Canturbury Puzzles and Others, p.119-120
· Gardner, Scientific American v. CCXIII, p.96-104
· Canadian Journal of Mathematics, vol. XXV, pp585-602

<u>Projective Geometry</u>
· Brill and Steuben, Demystifying the Projective Plane, Math. Magazine 63, no.1, Feb 1990
· Posamentier, Alfred and Gordon Sheridan, Investigations in Geometry, Addison-Wesley, 1982
· Smart, Modern Geometry
· Pedoe, Dan, Geometry and the Visual Arts

<u>Proofs of the Pythagorean theorem</u>

- Fredrichs, Kurt Otto, From Pythagoras to Einstein, QA 460 P99 F7 PMA
- Beamer, James E, Using puzzles to teach the Pythagorean Theorem, Mathematics Teacher, May 1989
- Euclid, Elements
- Heath, Thomas, Manual of Greek Mathematics
- Loomis, Elisha Scott, The Pythagorean Proposition
- Newman, The World of Mathematics (Vol 1)
- Posamentier, Alfred and Gordon Sheridan, Investigations in Geometry, Addison-Wesley, 1982
- Prentice-Hall Encyclopedia of Mathematics, 1982
- Swetz, Frank J. and T.I. Kao, Was Pythagoras Chinese?, NCTM 1977

<u>Proofs without words</u>

- Various journals (Amer Math Monthly, Math Magazine)

<u>Space Tessellations</u>

- Ball and Coxeter
- Critchlow, Keith, Order in Space, Dale Seymour
- Ghyka, Matila, The Geometry of Art and Life, Dover
- Loeb, et al, On the Icosahedron, the Pentagonal Dodecahedron, and the Rhombic Triacontahedron, Symmetry, v.1, no.1, 1990, 29-36.
- Holden, Allan, Shapes, Space, and Symmetry, 1971
- Williams, Robert, The Geometrical Foundation of Natural Structure, Dale Seymour

<u>Quilts</u>

- Nephew, Sara, Equilateral triangle patchwork quilts, Dover
- Whitman, Nancy, Line and Rotational Symmetry, Mathematics Teacher, April, 1991
- Zaslavsky, Claudia, "Symmetry in American Art", Arithmetic teacher, September, 1990

<u>Sphere Packing</u>

- Packing your n-Dimensional Marbles, Science, 2 March 1990
- The nth Dimension, Forbes, v.138, Dec 29, 1986
- Curves for a tighter fit, Science News, v. 138, May 19, 1990, pp316-317
- Loosely packed spheres, Science News, v. 137, June 16, 1990, p. 382
- Thompson, Thomas M., From Error-correcting Codes through Sphere Packings to Simple Groups, MAA, 1984

Williams, Robert, The Geometrical Foundations of Natural Structure

Spirals

- Davis, Philip J., Spirals: From Theodorus to Chaos, 1993
- Lawlor, Robert, Sacred Geometry, 1982
- Leonardo (International Society for the Arts, Sciences, and technology), Art and Technology, 1992
- Pickover, Clifford and I. Hargittai, Spiral Symmetry, 1992
- Edwards, Edward B., Pattern and Design with Dynamic Symmetry, Dover, 1967, pp.13-24, 45-46, 68 - 69, 92-95, 104-105, 111-112, 118-122
- Hargittai, Istvan, and Clifford A. Pickover, eds, Spiral Symmetry, QA 483 S68 1992
- Wells, David, Hidden Connections, DOuble Meanings, Cambridge, pp. 83 - 86

Squaring the circle

- Ball and Coxeter
- Bold, Famous Problems of Geometry
- Dixon, Robert, Mathographics, Dover, 1987, pp.44

String designs

- Envelopes. / Boltianskii, V. G. (Vladimir Grigor'evich) / New York 1964 QA 621 B65 Physics-Math-Astronomy Library
- Irvine, Max, Cable Structures, Dover

Symmetry in physics

- Altmann, Simon L., Icons and Symmetries, Oxford Univ. Press, 1992
- Stewart, Ian and Martin Golubitsky, Fearful Symmetry, Penguin, 1992

Strip (frieze) patterns

- Crowe, Donald, Symmetry, Rigid Motions, and Patterns, Himap (bib)
- Hargittai, Istavan, ed., Symmetry: Unifying Human Understanding
- Lockwood, E.H., and R.H. MacMillan, Geometric Symmetry
- Washburn and Crowe
- Newman, Rochelle and Martha Boles, The Golden Relationship: Art, Math, Nature

Symmetry in Everyday Objects

- Gallian, Joseph A., Symmetry in Logos and Hubcaps

Taxicab geometry

- Krause, Eugene F., Taxicab Geometry, Addison-Wesley

- Posamentier, Alfred and Gordon Sheridan, Investigations in Geometry, Addison Wesley, 1982

Tilings (Tessellations of the plane)

- Ball and Coxeter
- Boorman, Phil, More About Tessellating Hexagons, Math Teacher, Summer, 1971
- Clemens, Stanley, Tessellations of Pentagons, Math Teaching, June, 74
- Coxeter, H.M.S., Twelve Geometric Essays, pp.76,115,168
- Dubrovm, Fomenko, and Novikov, Modern Geometry - Methods and Applications, p.169
- Dunn, James, A. More About Tessellating Hexagons, Mathematics Teaching, Summer, 1971
- Gardner, Martin, Penrose Tiles and Trapdoor Ciphers QA 95 G 298 PMA
- Ghyka, Matila, The Geometry of Art and Life, Dover
- Grunbaum, Branko and G.C. Shepard, Tilings and Patterns, 1987
- Hilbert and Cohn-Vossen, Geometry and the Imagination
- Klarner, D.A. (ed.), The Mathematical Gardner
- Lockwood, E.H., and R.H. MacMillan, Geometric Symmetry Mathematics Teacher, April,1974
- Mold, Josephine, Tessellations, Cambridge U. Press
- Nikulin, V.V., Geometries and Groups Scientific American, July, 1975
- Serra, Michael, Discovering Geometry, Key Curriculum Press, 1989
- Smart, Modern Geometry
- Baeyer, Hans C., Impossible Crystals, Discover 11, February, 1990, 69 – 78
- Fulton, Chandler, Tessellations, American Mathematical Monthly, vol. 99, May 1992, pp. 442 - 445.
- Classical tessellations and three-manifolds. / Montesinos, Jose Maria, 1944- / Berlin 1987, QA 166.8 M66 1987 Phys-Math-Astron Library (the mathematics may be too advanced, but it has some good pictures
- Spatial tessellations : concepts and applications of Voronoi diagrams. / Okabe, Atsuyuki, 1945- / Chichester, En# 1992 QA 278.2 O36 1992 Physics-Math-Astronomy Library (See also review by Marjorie Senechal, College MAth Journal, v 26, January 1995
- Orton, A. and S.Flors(?), Analysis of an ancient Tessellation, Mathematical Gazette, vol. 73, 1989, pp 297 - 301

Tessellatons of Space

- Baeyer, Hans C., Impossible Crystals, Discover 11, February, 1990, 69 – 78
- Ellott, H.A., Maclean, and Jordan, Geometry in the Classroom, Holt, Rinehart and Winston of Canada, 1968
- Gray, Jeremy, Parsimonious Polyhedra, Nature, 367 (10 February 1994) 513 – 514
- Loeb, Space Structures: Their Harmony and Counterpoint, Birkhauser, 1976, QA 491 L63
- Classical tessellations and three-manifolds. / Montesinos, Jose Maria, 1944- / Berli 1987, QA 166.8 M66 1987 Phys-Math-Astron Libraryn (the mathematics may be too advanced, but it has some good pictures)
- Orton, A. and S.M. Flower, Mathematical Gazette, v. 73, 1989, pp. 297-301
- Wheeler, Dan and David Sklar, Two Year Colloege MAthematics Journal, September, 1981, p. 246

Topological surfaces

- Arnold, B.H., Intuitive Concepts in Elementary Topology
- Barr, Stephen, Experiments in Topology, 1964
- Brill and Steuben, Demystifying the Projective Plane, Math Mag 63, no1, Feb.1990 pp.48-51
- Courant and Robins, What is Mathematics, p.235-264
- Francis, George K., A Topological Picture Book
- Guillen, Michael, Bridges to Infinity,1983, pp. 153-174
- Johnson, Donovan, Topology, the Rubber Sheet Geometry, 1960
- Liepzen, W., Visual Topology, 1969
- Khurgin, Ya. Did You Say Mathematics?, Mir
- Bergamini, David, Mathematics, Life Science Library, Time-Life, 1973
- Pappas, Theoni, Mathematics Appreciation, Dale Seymour
- Smart, Modern Geometry
- Weeks, Jeffrey, The Shape of Space, Dekker, 1985
- Pappas, Theoni, The Joy of Mathematics
- Prasolov, V.V., Intuitive Topology, AMS, 1994

Trisecting the Angle

- Courant and Robbins, What is Mathematics, pp.137-138
- Gardner, Martin, Mathematical Carnival, Chapter 19 (and bibliography)
- Yates, Robert C., Geometrical Tools, Educational Publishers, 1949. 744 Y 27 (UGLI)
- Ball and Coxeter
- Bold, Famous Problems of Geometry
- Dudley, Underwood, A Budget of Trisections, Speinger, 1987
- Miller, Leslie, College Geometry, p. 48, 194-197, 1957
- Ogilvy, Excursions in Geometry, ch 10, 1969
- Posamentier, Alfred and Gordon Sheridan, Investigations in Geometry, Addison-Wesley, 1982
- Yates, Robert C., The Trisection Problem, NCTM, 1971

Topology of highway intersections

- Grazis, D., ed., Traffic Science, Wiley, 1974Consortium, Winter, 198

Wallpaper patterns

- Audsley, W. D. , Designs and Patterns from Historical Ornament, Dover
- Crowe, Donald, Symmetry, Rigid Motions, and Patterns, Himap (bib)
- Lockwood, E.H., and R.H. MacMillan, Geometric Symmetry
- Lynn, Catherine, Wallpaper in America: fron the 17th century to World War I
- Stevens, Peter, Handbook of Regular Patterns: an Introduction to Symmetry in Two Dimensions
- Washburn and Crowe
- Waterman, V. Ann, Design Your Own Repeat Patterns, Dover
- Newman, Rochelle and Martha Boles, The Golden Relationship: Art, Math, Nature
- Christie, Archibald, Pattern Design, Dover
- Dye, Daniel Shets, Chinese Lattice Designs, Dover
- Gallian, Joseph, Contemporary Abstract Algebra, Heath, Chapter 30
- Grunbaum, Branko and G.C. Shepard, Tilings and Patterns, 1986, Freeman
- Huinker, DeAnn M. and Eugene M. Krause, Investigations in Mathematics for Elementary Teachers, Heath, 1991, pp. 271-274
- Schattschneider, Doris, The Plane Symmetry Groups: Their Recognition and Notation, American Mathematical Monthly 85(6) 439-450
- Stevens, Peter, Handbook of Regular Patterns, MIT Press, 2980
- Waterman, V. Ann, Design your Own Repeat Patterns, Dover
- http://www.geom.umn.edu:80/~math5337/Wallpaper/