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Engineering Mechanics

Lecture Notes
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Abstract

The course "*Engineering Mechanics*" is held for students of the Master Programme "*Materials Science and Engineering*" at the Faculty of Engineering of the Christian Albrechts University in Kiel. It addresses continuum mechanics of solids as the theoretical background for establishing mathematical models of engineering problems. In the beginning, the concept of continua compared to real materials is explained. After a review of the terms motion, displacement, and deformation, measures for strains and the concepts of forces and stresses are introduced. The description allows for finite deformations. After this, the basic governing equations are presented, particularly the balance equations for mass, linear and angular momentum and energy. After a cursory introduction into the principles of material theory, the constitutive equations of linear elasticity are presented for small deformations. Finally, some practical problems in engineering like stresses and deformation of cylindrical bars under tension, bending or torsion and of pressurised tubes are presented.

A good knowledge in vector and tensor analysis is essential for a full uptake of continuum mechanics. This is not a subject of the course. Hence, the nomenclature used and some rules of tensor algebra and analysis as well as theorems on tensor properties are included in the Appendix of the present lecture notes. Generally, these notes provide significantly more background information than can be presented and discussed during the course, giving the chance of home study.

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Isaac Newton (1643-1727)
painted by Godfrey Kneller, National Portrait Gallery London, 1702

1. Introduction

In the early stages of scientific development, "*physics*" mainly consisted of *mechanics* and *astronomy*. In ancient times CLAUDIUS PTOLEMAEUS of Alexandria (*87) explained the *motions* of the sun, the moon, and the five planets known at his time. He stated that the planets and the sun orbit the Earth in the order Mercury, Venus, Sun, Mars, Jupiter, Saturn. This purely phenomenological model could predict the positions of the planets accurately enough for naked-eye observations. Researchers like NIKOLAUS KOPERNIKUS (1473-1543), TYCHO BRAHE (1546-1601) and JOHANNES KEPLER (1571-1630) described the movement of celestial bodies by mathematical expressions, which were based on *observations* and a universal *hypothesis* (model). GALILEO GALILEI (1564-1642) formulated the laws of free fall of bodies and other laws of motion. His "discorsi" on the heliocentric conception of the world encountered fierce opposition at those times.

After the renaissance a fast development started, linked among others with the names CHRISTIAAN HUYGENS (1629-1695), ISAAC NEWTON (1643-1727), ROBERT HOOKE (1635-1703) and LEONHARD EULER (1707-1783). Not only the motion of *material points* was investigated, but the observations were extended to bodies having a spatial dimension. With HOOKE's work on elastic steel springs, the first material law was formulated. A general theory of the strength of materials and structures was developed by mathematicians like JAKOB BERNOULLI (1654-1705) and engineers like CHARLES AUGUSTIN COULOMB (1736-1806) and CLAUDE LOUIS MARIE HENRI NAVIER (1785-1836), who introduced new intellectual concepts like *stress* and *strain*.

The achievements in *continuum mechanics* coincided with the fast development in mathematics: *differential calculus* has one of its major applications in mechanics, *variational principles* are used in analytical mechanics.

These days mechanics is mostly used in engineering practice. The problems to be solved are manifold:

- Is the car's suspension strong enough?
- Which material can we use for the aircraft's fuselage?
- Will the bridge carry more the 10 trucks at the same time?
- Why did the pipeline burst and who has to pay for it?
- How can we redesign the bobsleigh to win a gold medal next time?
- Shall we immediately shut down the nuclear power plant?

For the scientist or engineer, the important questions he must find answers to are:

- How shall I formulate a problem in mechanics?
- How shall I state the governing field equations and boundary conditions?
- What kind of experiments would justify, deny or improve my hypothesis?
- How exhaustive should the investigation be?
- Where might errors appear?
- How much time is required to obtain a reasonable solution?
- How much does it cost?

One of the most important aspects is the load–deformation behaviour of a structure. This question is strongly connected to the choice of the appropriate mathematical model, which is used for the investigation and the chosen material. We first have to learn something about different models as well as the terms motion, deformation, strain, stress and load and their mathematical representations, which are vectors and tensors.

Figure 1-1: Structural integrity is commonly not tested like this.

The objective of the present course is to emphasise the formulation of problems in engineering mechanics by reducing a complex "reality" to appropriate mechanical and mathematical models. In the beginning, the concept of continua is expounded in comparison to real materials.. After a review of the terms *motion*, *displacement*, and *deformation*, measures for *strains* and the concepts of *forces* and *stresses* are introduced. Next, the basic governing equations of continuum mechanics are presented, particularly the *balance equations* for *mass*, *linear* and *angular momentum* and *energy*. After a cursory introduction into the principles of material theory, the constitutive equations of *linear elasticity* are presented for small deformations. Finally, some *practical problems* in engineering, like stresses and deformation of cylindrical bars under tension, bending or torsion and of pressurised tubes are presented.

A good knowledge in vector and tensor analysis is essential for a full uptake of continuum mechanics. A respective presentation will not be provided during the course, but the nomenclature used and some rules of tensor algebra and analysis as well as theorems on properties of tensors are included in the Appendix.

2. Models in the Mechanics of Materials

2.1 Disambiguation

Models are generally used in science and engineering to reduce a complex reality for detailed investigations. The prediction of a future state of a system is the main goal, which has to be achieved. Due to the hypothetical nature of this approach, it is irrelevant whether the assumed state will be achieved or not: Safety requirements often demand for assumptions that are equivalent to a catastrophic situation, which during the lifetime of a structure probably never takes place. More important is the question, what *scenario* is going to be investigated. Depending on the needs, the physical situation can be modelled in different ways.

Modelling has become an important and fashionable issue, likewise. Every serious research project will claim modelling activities to increase the chances of being awarded grants. Modern technology and product development have detected the saving effects of modelling: "*The development and manufacture of advanced products, such as cars, trucks and aircraft require very heavy investments. Experience has shown that a large portion of the total life cycle cost – as much as 70-80 percent – is already committed in the early stages of the design. It is important to realize that the best chance to influence life cycle costs occurs during the early, conceptual phase of the design process. Improvements in efficiency and quality during this phase should enable us to obtain the right solutions and make the right decisions from the beginning. This requires good design, analysis and synthesis methods and tools, as well as good simulation techniques including computational prototyping and digital mock-ups*".[1]

Modelling, however, is an ambiguous term and needs further explanation and a more precise definition. The common understanding of a model is manifold. Collins Compact English Dictionary (1998) explains it as follows:

1. *a three-dimensional representation, usually on a smaller scale, of a device or structure: an architect's model of the proposed new housing estate*
2. *an example or pattern that people might want to follow: her success makes her an excellent role model for other young Black women*
3. *an outstanding example of its kind: the report is a model of clarity*
4. *a person who poses for a sculptor, painter, or photographer*
5. *a person who wears clothes to display them to prospective buyers; a mannequin*
6. *a design or style of a particular product: the cheapest model of this car has a 1300cc engine*
7. *a theoretical description of the way a system or process works: a computer model of the British economy*
8. *adj excellent or perfect: a model husband*
9. *being a small scale representation of: a model aeroplane*
10. *vb -elling, -elled or US -eling, -eled to make a model of: he modeled a plane out of balsa wood*
11. *to plan or create according to a model or models: it had a constitution modeled on that of the United States*
12. *to display (clothing or accessories) as a mannequin*
13. *to pose for a sculptor, painter, or photographer*

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[1] B. FREDERIKSSON and L. SJÖSTRÖM: "The role of mechanics an modelling in advanced product development" European Journal of Mechanics A/Solids, Vol <u>16</u> (1997), 83-86.

For natural and engineering sciences we shall generally adopt items 1 and 7 as definitions. In a broad sense, every scientific activity might be looked at as "modelling" since dealing with a complex reality always requires reduction and idealization of problems. Thus, modelling may be understood as novel only in the sense of "*computational simulation of reality*", which is the underlying comprehension in the quotation "*simulation techniques including computational prototyping*" given above. At least in engineering sciences, modelling has to combine and integrate computational and experimental efforts in order to proceed to an understanding of the physical phenomena which allows for realistic predictions of the performance, availability and safety of technical products and systems.

2.2 Characterisation of Materials

Materials testing has a long tradition and is based on the desire of scientists to measure the mechanical properties of materials and the need of design engineers to improve the performance and safety of buildings, bridges and machines. Mechanical sciences started with GALILEO GALILEI (1564-1642). He did not only promote COPERNICUS' concept of a heliocentric planetary system, but studied the laws of falling bodies and strength of materials both theoretically and experimentally [2]. An actual engineering problem was the dependence of the strength or a bending bar on its cross sectional dimensions for which GALILEI designed an experiment shown in Fig. 2.1.

The test configuration reduces the complex problem of structural bars, e.g. in housing, to a cantilever beam under a single load at its end. He found the "*bending resistance*" was proportional to the width, b, and the square of the height, h, of the bar's cross-section. Expressing this result in modern mathematical terms, we can derive today that the *section modulus* is $W = bh^2/6$. Neglecting the dead weight of the bar, the bending moment is $M = G\ell$, where G is the applied weight $\mathbb{E}$ at the end of the bar of length ℓ, and finally, the maximum tensile strength occurring in point $\mathbb{A}$ becomes

$$\sigma_{\max} = \frac{6G\ell}{bh^2} , \tag{2-1}$$

if a linear distribution of stresses over the cross section is assumed. But these mathematical formulas and a general theory of bending did not exist at GALILEI's times. They were developed about one and a half century later by mathematicians like JAKOB BERNOULLI (1655-1705) and engineers like CH. COULOMB (1736-1806) and L.M.H. NAVIER (1785-1836), who introduced new concepts and abstract ideas like *bending moment*, *stress* and *strain*, see section 8.4, which allow for relating *bending strength* with *tensile strength*. GALILEI did not consider the deformation of the bar, either, as the law of elasticity, later found by R. HOOKE (1635-1703), was unknown. As the section modulus, W, is a purely geometrical quantity, which is determined by the shape and dimensions of the cross section, GALILEI's structural experiment actually did not reveal material properties.

The obvious question that arises from any experiment is:

- What can we learn from it? -

or more precisely:

- How does this *test configuration* compare to the "*real*" situation?

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[2] G. GALILEI: "Discorsi e dimostrazioni matematiche, intorno à due nueve scienza attenti alla mechanica & i movimenti locali" Elsevir, 1638.

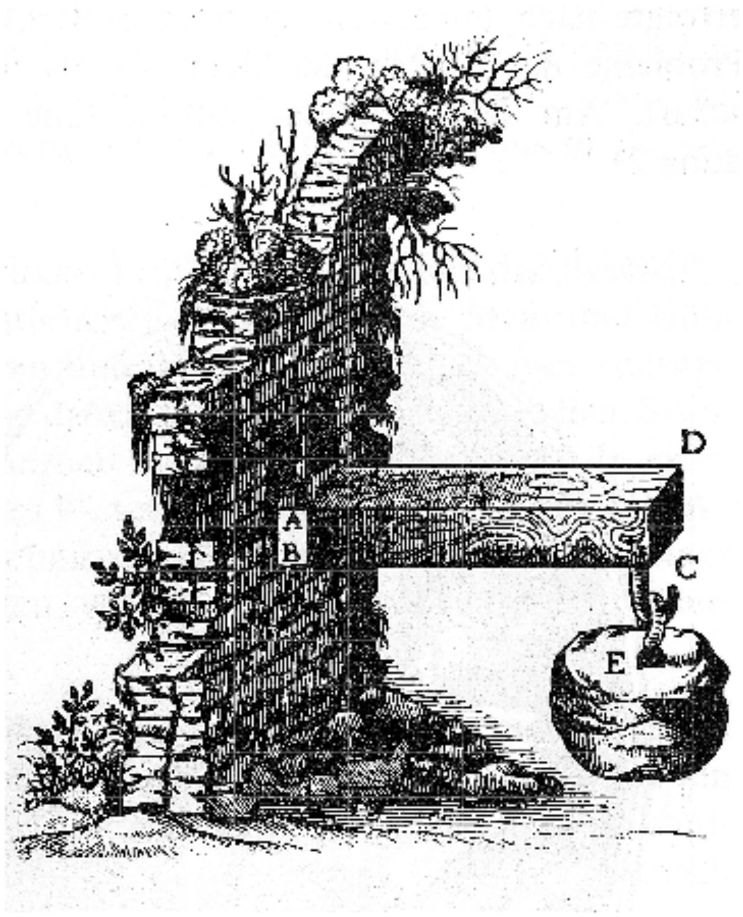

Figure 2-1: Test set-up by GALILEI (1638) for the investigation of the load carrying behaviour of a cantilever beam

For instance, can we take the fracture load obtained in the above test to design the supporting beams in a building? Finally, we reach the fundamental and still present-day problem of materials testing: are the test data measured on a specimen transferable to an actual large-scale structure? Specimens used in materials testing are models in the sense of a "*three-dimensional representation, usually on a smaller scale, of a structure*", see above. In addition, they are of a simpler geometry and under simpler loading conditions. Whether the information from a (simplified) model may or may not be transferred to (complex) reality, is still controversial in many cases and cannot be answered by experiments alone. It needs a *model* in the sense of a *theoretical description*.

A deeper understanding of GALILEI's bending problem would have required a theory, which did not exist in the 17th century. Nevertheless, engineers wanted to design structures and get information on the mechanical behaviour of different materials. Hence, they had to develop special test set-ups for various loading conditions such as tension, compression, bending, buckling, etc.. With expanding technology, other material properties became relevant, not only under static loading but also under impact or oscillating stresses. Engineers had found, that the ductility of a metallic material was an important property, which influences the safety margins of a structure or plant. In order to measure this ductility, the French metallurgist G. CHARPY designed his pendulum impact testing machine in 1901 to measure the mechanical work necessary to fracture a notched bar.

All these tests on comparatively simple specimens are performed in order to obtain information on the materials strength and toughness and to conclude to the mechanical behaviour and performance of complicated structural geometries und different kinds of loading and loading histories. The fundamental problem of materials testing, i.e. how much these tests tell us about inherent material properties, however, has still remained controversial. Separating material properties from structural properties is an intellectual process of abstracting, which is typical for modelling. It requires a theory, namely *continuum mechanics*, which has been developed in the late 19th and early 20th century and been permanently improved ever since.

3. Continuum Hypothesis

3.1 Introduction

Continuum mechanics is concerned with motion and deformation of material objects, called *bodies*, under the action of forces. If these objects are solid bodies, the respective subject area is termed *solid mechanics*, if they are fluids, it is fluid mechanics or fluid dynamics. The mathematical equations describing the fundamental physical laws for both solids and fluids are alike, so the different characteristics of solids and fluids have to be expressed by constitutive equations. Obviously, the number of different constitutive equations is huge considering the large number of materials. All of this can be written using a unified mathematical framework and common tools. In the following we concentrate on solids.

Continuum mechanics is a phenomenological field theory based on a fundamental hypothesis called continuum hypothesis. The governing equations comprise *material independent principles*, namely

- ***Kinematics***, being a purely geometrical description of motion and deformation of material bodies;
- ***Kinetics***, addressing forces as external actions and stresses as internal reactions;
- ***Balance equations*** for conservation of mass, momentum and energy;

and *material dependent laws*, the

- ***Constitutive equations***.

Altogether, these equations form an *initial boundary value problem*.

3.2 Notion and Configuration of a Continuum

It is commonly known, that matter consists of elementary particles, atoms and molecules, which are small but finite and not homogeneously distributed. The mechanical behaviour of materials is determined by the interaction of these elementary constituents. However, an engineering modelling cannot be done at this level and length scale. Even on a next higher length scale, the microstructure of materials appears as inhomogeneous and consisting of different constituents. Again, if one is interested in the macroscopic behaviour of an engineering structure, modelling on a microscopic length scale is in general not feasible. While studying the external actions on objects, it will, except for specific questions concerning the relations between micro- and macroscopic properties, not be necessary to account for the non-homogeneous microstructure. The discretely structured matter will hence be represented by a phenomenological model, the *continuum*, by averaging its properties in space and neglecting any discontinuities and gaps. By a continuum, we mean the hypothetical object in which the matter is continuously distributed over the entire object. It does not contain any intrinsic length scale.

The concept of a continuum is deduced from mathematics, namely the system of real numbers: between any two distinct real numbers there is always another distinct real number, and therefore, there are infinitely many real numbers between any two distinct numbers. The three-dimensional (EUKLIDean) space is a continuum of points, which can be represented by three real numbers, the coordinates, x_i (i = 1, 2, 3). The same holds for the physical time, t, which can also be represented by a real number, and thus time and space together form a four-dimensional continuum.

Following is a description Albert EINSTEIN gave on p. 83 of his Relativity; The Special and the General Theory: *The surface of a marble table is spread out in front of me. I can get from any one point on this table to any other point by passing continuously from one point to a “neighboring" one, and repeating this process a (large) number of times, or, in other words, by going from point to point without executing “jumps". I am sure the reader will appreciate with sufficient clearness what I mean here by “neighboring" and by “jumps" (if he is not too pedantic). We express this property of the surface by describing the latter as a continuum.*

Consider now a *material body*, $\mathcal{B}$, defined as a three-dimensional differentiable manifold, the elements of which are called *particles* (or material points), $\mathcal{X} \in \mathcal{B}$. The body is endowed with a non-negative scalar measure, *m*, which is called the *mass* of the body (see section 3.3). It occupies a region, $\mathbb{B} \subset \mathbb{E}^3$, in the three-dimensional EUKLIDean space, $\mathbb{E}^3$, at a given time, *t*.

- Every particle, $\mathcal{X} \in \mathcal{B}$, has a position $\mathbb{X} \in \mathbb{B}$, in the region $\mathbb{B}$ occupied by $\mathcal{B}$;
- Every point $\mathbb{X} \in \mathbb{B}$ is the position of a particle $\mathcal{X} \in \mathcal{B}$.

So, there exists a one-to-one correspondence between the particles of a continuum and the geometrical points of a region occupied by the continuum at any given time. The geometrical region that a body occupies at a given time is called its *configuration*. In mathematical terms, a configuration of a body, $\mathcal{B}$, is a smooth homeomorphism of $\mathcal{B}$ onto a region $\mathbb{B} \subset \mathbb{E}^3$ of the three-dimensional EUKLIDean space, $\mathbb{E}^3$. At no instant of time, a particle can have more than one distinct position or can two distinct particles have the same position.

This one-to-one correspondence allows us to speak of points, lines, surfaces and volumes in a continuum. For simplicity, material points, material surfaces and material bodies are often referred to as points, surfaces and volumes.

The above definition is commonly supplemented by three characteristics, which are introduced axiomatically.

(1) A material continuum remains a continuum under the action of forces. Hence, two particles that are neighbours at one time remain neighbours at all times. We do allow bodies to be fractured, but the surfaces of fracture must be identified as newly created external surfaces.

(2) Stresses and strains can be defined everywhere in the body, i.e. they are field quantities.

(3) The theory of "*simple materials*" postulates, that the stress at any point in the body depends only on the deformation history of that very point but not on the deformation history of other points of the body. This relation may be affected by other physical quantities, such as temperature, but these effects can be studied separately. This is of course a greatly simplifying assumption, which facilitates the establishing of constitutive relations on the one hand but restricts their generality.

A *scalar* field describes a one-to-one correspondence between a single scalar number and the value of a physical quantity. A triple of numbers can be assigned to a geometric point, representing its coordinates, but also to any other physical quantity characterised by a magnitude and an orientation, i.e. a *vector*. A 3×3 matrix of numbers can be assigned to a *tensor* of rank two, if these numbers obey certain transformation rules. Within this concept, scalar fields may be referred to as tensor fields of rank or order zero, whereas vector fields are called tensor fields of rank or order one. Generally, tensors have certain *invariant* properties, i.e. properties, which are independent of the coordinate system used to describe them. Because of this attribute, we can use tensors for representing various fundamental laws of physics, which are supposed to be independent of the coordinate systems considered.

Finally, every function of space and time considered in the following is supposed to be differentiable up to any desired order in the region of interest.

3.3 Density and Mass

It has already been stated above, that a body is endowed with a non-negative scalar measure, m, called the mass. Like time and space, mass is a primitive quantity. Let us now consider a vicinity, $\Delta\mathcal{B}$, of a particle $\mathcal{X} \in \mathcal{B}$, occupying a subregion, $\Delta\mathbb{B} \subset \mathbb{B}$, and having the volume ΔV. Because of the hypothesis of continuity, this region $\Delta\mathbb{B}$ is not empty, however small it may be. Let Δm be the mass of ΔV, then as $\Delta m > 0$ and $\Delta V > 0$, the ratio $\Delta m / \Delta V$ is a positive real number. The limit as ΔV tends to zero defines the *mass density* ρ,

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} \tag{3-1}$$

In general, the value of ρ depends on the point $\mathbb{X} \in \mathbb{B}$ and the time. While defining density using eq. (3-1), the assumption was made that the mass is a differentiable function of volume. This means that the mass is assumed to be distributed *continuously* over a region. No part of a material body is assumed to possess concentrated mass. Thus, we assume that ρ is continuously differentiable over $\mathbb{B}$. It follows that eq. (3-1) is equivalent to

$$m(\mathcal{B}) = \int_{V(\mathcal{B})} \rho \, dV \; . \tag{3-2}$$

4. Kinematics: Motion and Deformation

The following chapter provides an introduction into the description of *finite* (large) deformations and hence abandons the common assumption made engineering mechanics of "small" deformations. Though elastic deformations of metals are commonly small[3], a general description of deformation without this restriction makes sense for several reasons:

1. The general formalism of describing deformations without any restrictions is pointed out, thus allowing for a later quantification of what is small;
2. Materials other than metals, e.g. elastomers, may show large elastic deformations;
3. The representation allows also for the description of large inelastic deformations as it is essential in fracture mechanics or metal forming;
4. Commercial finite element codes like ABAQUS, ADINA, ANSYS, MARC provide options for large deformations, so that some insight into the underlying theory helps avoiding to use these programmes as "black boxes", only.

4.1 Motion

A material body, $\mathcal{B}$, has been defined as the manifold of all material points, $\mathcal{X} \in \mathcal{B}$, which is accessed to (geometrical) observation by a *placement* in the three-dimensional EUKLIDean space, $\mathbb{E}^3$. This placement is done by a mapping, $\boldsymbol{\chi}$, called *configuration***,** which assigns every particle $\mathcal{X}$ to a geometrical point $\mathbb{X} \in \mathbb{B}$, and thus represents the body $\mathcal{B}$ as a subregion $\mathbb{B} \subset \mathbb{E}^3$,

$$\boldsymbol{\chi}: \quad (\mathcal{X},t) \rightarrow \mathbb{X} = \boldsymbol{\chi}(\mathcal{X},t) \quad , \quad \mathbb{X} \in \mathbb{B} \ . \tag{4-1}$$

Let us now choose some (arbitrary) point, $\mathbb{O}$, called the *origin*, as reference point, then a *position vector*,

$$\mathbf{x} = \overrightarrow{\mathbb{O}\mathbb{X}} = \mathbb{X} - \mathbb{O} \, , \tag{4-2}$$

can be assigned to every point $\mathbb{X}$. This identifies every particle, $\mathcal{X} \in \mathcal{B}$, by its position vector, $\mathbf{x} \in \mathcal{E}^3$, with $\mathcal{E}^3$ being the three-dimensional vector space. The body can now be gauged, since a norm is defined in $\mathcal{E}^3$ by the inner product

$$|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}} \ . \tag{4-3}$$

The spatial distance, $d = \left\| \overline{\mathbb{X}_1\mathbb{X}_2} \right\| = \left\| \mathbb{X}_2 - \mathbb{X}_1 \right\|$, of two particles, $\mathbb{X}_1$, $\mathbb{X}_2$, results as

$$d = |\mathbf{x}_2 - \mathbf{x}_1| = \sqrt{(\mathbf{x}_2 - \mathbf{x}_1) \cdot (\mathbf{x}_2 - \mathbf{x}_1)} \ . \tag{4-4}$$

A *coordinate system* is now introduced, consisting of a triad of *base vectors* $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \subset \mathcal{E}^3$ and an origin $\mathbb{O} \in \mathbb{E}^3$. These base vectors are commonly assumed as unit and orthogonal vectors, i.e. $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$. In particular, *Cartesian*, i.e. straight lined *coordinates* will be assumed [4]. The position vector is represented as[5]

[3] For comprehending the meaning of "small" and "large" see section 4.4.

[4] Other than Cartesian coordinate systems, e.g. cylindrical or spherical coordinates, are not addressed here (but see Appendix A1.4 for *cylindrical coordinates*) It can be convenient to introduce them for certain geometries, however. It is also possible to introduce different curvilinear coordinate systems in different configurations, see below in section 4.5.

$$\mathbf{x} = x_i \mathbf{e}_i \tag{4-5}$$

with the (Cartesian) coordinates

$$x_i = \mathbf{x} \cdot \mathbf{e}_i \quad , \quad i = 1,2,3 . \tag{4-6}$$

The body changes its position in $\mathbb{E}^3$ under external actions, i.e. it takes up a new configuration. The sequence of configurations with time is called *motion*, ${}^t\boldsymbol{\chi}$, of the body $\mathcal{B}$. The actual configuration at time *t*, given by

$$ {}^t\mathbb{X} = \mathbb{O} + {}^t\mathbf{x} = {}^t\boldsymbol{\chi}(\mathcal{X},t) \quad , \quad {}^t\mathbb{X} \in {}^t\mathbb{B} \tag{4-7}$$

is called *current configuration*. It is compared with a reference configuration in order to describe the change of the position. Commonly, the *initial configuration* at some fixed time t_0 (for convenience $t_0 = 0$) is taken as reference,

$$ {}^0\mathbb{X} = \mathbb{O} + {}^0\mathbf{x} = \boldsymbol{\chi}(\mathcal{X},t_0) = {}^0\boldsymbol{\chi}(\mathcal{X}) \quad , \quad {}^0\mathbb{X} \in {}^0\mathbb{B} \ , \tag{4-8}$$

in which no forces are acting on the body [6]. The spatial points, ${}^0\mathbb{X}$, or the position vectors, ${}^0\mathbf{x}$ at t_0, respectively, will be used for identifying the material points, $\mathcal{X}$, of the body $\mathcal{B}$, in other words, the position vector ${}^0\mathbf{x}$ acts as a label of $\mathcal{X}$, and, for simplicity, will be used synonymously for characterising the particle, in the following.

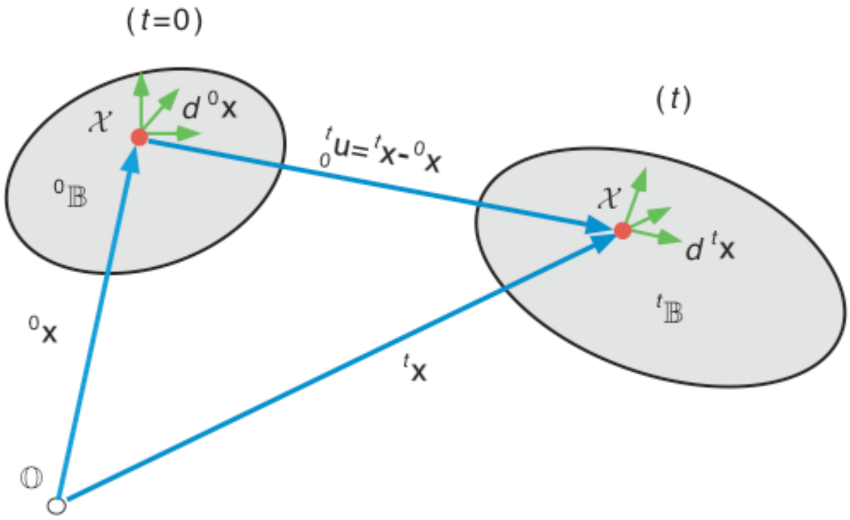

Figure 4-1: Body $\mathcal{B}$ in the initial and current configurations

As the body $\mathcal{B}$ moves from ${}^0\mathbb{B}$ to ${}^t\mathbb{B}$, each particle moves from ${}^0\mathbb{X}$ to ${}^t\mathbb{X}$. According to eqs. (4-7) and (4-8), the spatial points are identified by the position vectors, ${}^0\mathbf{x}$, ${}^t\mathbf{x}$, respectively. Thus, the motion of $\mathcal{B}$ is described by

$$ {}^t\mathbf{x} = {}^t\bar{\boldsymbol{\chi}}(\mathcal{X}) = {}^t\bar{\boldsymbol{\chi}}\left({}^0\bar{\boldsymbol{\chi}}^{-1}({}^0\mathbf{x}),t\right) = {}^t\hat{\boldsymbol{\chi}}\left({}^0\mathbf{x},t\right) . \tag{4-9}$$

The mapping, ${}^t\hat{\boldsymbol{\chi}}$, called *deformation*, links initial and current configuration. Since no two distinct particles of $\mathcal{B}$ can have the same position in any configuration and no two distinct points in a configuration can be positions of the same particle, eq. (4-9) does not only assign a

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5 The summation convention of EINSTEIN, $a_i b_i = \sum_{i=1}^{3} a_i b_i$, is assumed here and in the following, see Appendix.

6 The initial configuration is sometimes addressed as *stress-free* or *undeformed* configuration. However even if no external forces are acting on the body in this configuration, it may nevertheless be subject to residual stresses and deformations due to preceding processing of the material.

unique ${}^t\mathbf{x}$ to a given ${}^0\mathbf{x}$ and t, but also a unique ${}^0\mathbf{x}$ to a given ${}^t\mathbf{x}$ and t, i.e. the mapping must be invertible,

$$ {}^0\mathbf{x} = {}^0\overline{\boldsymbol{\chi}}(\mathcal{X}) = {}^0\boldsymbol{\chi}\left({}^t\boldsymbol{\chi}^{-1}({}^t\mathbf{x},t)\right) = {}^0\hat{\boldsymbol{\chi}}^{-1}\left({}^t\mathbf{x},t\right). \tag{4-10} $$

with ${}^0\hat{\boldsymbol{\chi}}^{-1}$ being the inverse of ${}^t\hat{\boldsymbol{\chi}}$. For any point ${}^t\mathbf{x}$ and any instant of time $t > t_0$, eq (4-10) specifies the particle of which ${}^t\mathbf{x}$ is the position at that instant.

4.2 Material and Spatial Description

If we focus attention on a specific point ${}^t\mathbf{x}$, eq. (4-10) determines all those particles, which pass through this point at different instants of time, $t > t_0$. On the other hand, at a specific instant of time, eq. (4-10) specifies all particles that are positioned at different points of the current configuration ${}^t\mathbb{B}$, and the totality of all these particles constitutes the body $\mathcal{B}$. Thus, for a given t, eq. (4-10) defines a mapping from ${}^t\mathbb{B}$ onto ${}^0\mathbb{B}$. Since the functions ${}^t\hat{\boldsymbol{\chi}}$ in eq. (4-9) and ${}^0\hat{\boldsymbol{\chi}}^{-1}$ in eq. (4-10) are inverse, it follows that both equations can be recovered as a unique solution of each other. Thus, the motion described by eq. (4-9) is described by equation eq. (4-10) also. But the ways the two equations describe the motion are not identical; they are only equivalent. While eq. (4-9) contains the particle ${}^0\mathbf{x}$ and time t as independent variables and specifies the position ${}^t\mathbf{x}$ of ${}^0\mathbf{x}$ for a given t, eq. (4-10) contains the point ${}^t\mathbf{x}$ and time t as independent variables and specifies the particle ${}^0\mathbf{x}$ that occupies ${}^t\mathbf{x}$ for a given t. Thus, in the description of motion given by eq. (4-9), attention is focused on a particle and we observe what is happening to the particle as it moves. This description is called the *material description*, and the independent variables (${}^0\mathbf{x}$, t) are referred to as *material variables*. On the other hand, in the description of motion given by eq. (4-10), attention is given to a point in space, and we study what is happening at that point as time passes. This description is called the *spatial description*, and the independent variables (${}^t\mathbf{x}$, t) present in eq. (4-10) are referred to as *spatial variables*. Traditionally, material description is referred to as LAGRANGEan[7], and the spatial description as EULERean[8].

Note, however, when using the term "spatial", that continuum mechanics of solid bodies is always based on a *material* approach in the sense, that particles of a body, $\mathcal{X} \in \mathcal{B}$, are considered and the spatial points, $\mathbb{X} \in \mathbb{B}$, are traced, which they occupy in the course of time. The volume V of $\mathcal{B}$, i.e. the domain ${}^t\mathbb{B}$ occupied by $\mathcal{B}$, will change with time, whereas the mass of $\mathcal{B}$, eq. (3-2), is constant (see conservation law in section 6.2). No particle is ever allowed to leave or enter the domain ${}^t\mathbb{B}$. In contrast, fluid mechanics applies an original spatial approach, considering state variables like pressure, density, velocity at fixed spatial points, where different particles are located at different times. Particles may (and will) leave or enter the defined spatial "control zone" which has constant volume but generally varying mass.

Material (LAGRANGEan) and spatial (EULERean) formulations affect the calculation of time derivatives (see section 4.5).

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7 JOSEPH LOUIS LAGRANGE (1736-1813)

8 LEONHARD EULER (1707-1783)

4.3 Deformation

The change of position of a particle, ${}^{0}\mathbf{x}$, is described by the *displacement vector*,

$$ {}^{t}_{0}\mathbf{u} = {}^{t}\mathbf{x} - {}^{0}\mathbf{x} = \bar{\mathbf{u}}\left({}^{0}\mathbf{x}, t\right) \text{ [9]}. \tag{4-11} $$

The respective spatial description with respect to the current configuration is obtained from eq. (4-10),

$$ {}^{t}_{0}\mathbf{u} = \bar{\mathbf{u}}\left({}^{0}\hat{\boldsymbol{\chi}}^{-1}\left({}^{t}\mathbf{x}, t\right), t\right) = \hat{\mathbf{u}}\left({}^{t}\mathbf{x}, t\right). \tag{4-12} $$

For the analysis of "geometrical changes" during a motion of a body we focus on only two of its configurations: ${}^{0}\mathbb{B}$ as the initial ("undeformed") configuration and in ${}^{t}\mathbb{B}$ the current ("deformed") configuration, see figure 4.1. Applying TAYLOR's expansion to ${}^{t}\boldsymbol{\phi}$,

$$ {}^{t}\mathbf{x} + d\,{}^{t}\mathbf{x} = {}^{t}\hat{\boldsymbol{\chi}} \approx {}^{t}\hat{\boldsymbol{\chi}}({}^{0}\mathbf{x} + d\,{}^{0}\mathbf{x}, t) + \frac{\partial\,{}^{t}\hat{\boldsymbol{\chi}}}{\partial\,{}^{0}\mathbf{x}} \cdot d\,{}^{0}\mathbf{x}\,, \tag{4-13} $$

an infinitesimal line element, $d\,{}^{t}\mathbf{x}$, in the current configuration can be expressed in terms of the material line element, $d\,{}^{0}\mathbf{x}$,

$$ d\,{}^{t}\mathbf{x} = {}^{t}_{0}\mathbf{F} \cdot d\,{}^{0}\mathbf{x}\,, \tag{4-14} $$

by a non-symmetric rank 2 tensor, the *deformation gradient*

$$ {}^{t}_{0}\mathbf{F} = \frac{\partial\,{}^{t}\hat{\boldsymbol{\chi}}({}^{0}\mathbf{x}, t)}{\partial\,{}^{0}\mathbf{x}} = \frac{\partial\,{}^{t}x_i}{\partial\,{}^{0}x_j}\,{}^{0}\mathbf{e}_i\,{}^{0}\mathbf{e}_j = \left({}^{0}\nabla\,{}^{t}\mathbf{x}\right)^{\mathrm{T}} = {}^{0}\mathrm{grad}\,{}^{t}\mathbf{x}\,. \tag{4-15} $$

The inverse transformation is

$$ d\,{}^{0}\mathbf{x} = {}^{0}_{t}\mathbf{F}^{-1} \cdot d\,{}^{t}\mathbf{x} \tag{4-16} $$

with

$$ {}^{0}_{t}\mathbf{F}^{-1} = \frac{\partial\,{}^{0}\hat{\boldsymbol{\chi}}^{-1}({}^{t}\mathbf{x}, t)}{\partial\,{}^{t}\mathbf{x}} = \frac{\partial\,{}^{0}\mathbf{x}}{\partial\,{}^{t}\mathbf{x}} = \left({}^{t}\nabla\,{}^{0}\mathbf{x}\right)^{\mathrm{T}} = {}^{t}\mathrm{grad}\,{}^{0}\mathbf{x} \tag{4-17} $$

and

$$ {}^{t}_{0}\mathbf{F} \cdot {}^{0}_{t}\mathbf{F}^{-1} = \mathbf{1} \tag{4-18} $$

In the trivial case of ${}^{t}_{0}\mathbf{F} = \mathbf{1}$ the line element experiences no change of length or orientation, $d\,{}^{t}\mathbf{x} = d\,{}^{0}\mathbf{x}$. If ${}^{t}_{0}\mathbf{F}$ is orthogonal, i.e. ${}^{t}_{0}\mathbf{F} \cdot {}^{t}_{0}\mathbf{F}^{\mathrm{T}} = \mathbf{1}$, the length of the line element remains unaltered, $\left|d\,{}^{t}\mathbf{x}\right| = \left|d\,{}^{0}\mathbf{x}\right|$, but the orientation changes, which represents a *rigid-body rotation*. If two deformations differ by a *translation* only, they have the same deformation gradient.

Corresponding to the deformation gradient, the (material) *displacement gradient* is defined by

$$ {}^{t}_{0}\mathbf{H} = {}^{0}\mathrm{grad}\,{}^{t}_{0}\mathbf{u} = \left({}^{0}\nabla\,{}^{t}_{0}\mathbf{u}\right)^{\mathrm{T}} = {}^{t}_{0}\mathbf{F} - \mathbf{1} = \left({}^{t}\nabla\,{}^{t}_{0}\mathbf{u}\right)^{\mathrm{T}} \cdot {}^{t}_{0}\mathbf{F}\,. \tag{4-19} $$

---

[9] Here and in the following, the left subscript characterises the reference state, and the left superscript the acting state. In a LANGRANGEan description, the reference state is "0" and the acting state is "t", whereas in an EULERean description "t" is the reference and "0" the acting state.

As the deformation of line elements, also the deformations of area and volume elements can be expressed by the deformation gradient [10],

$$\begin{aligned} d\,{}^t\mathbf{a} &= d\,{}^t\mathbf{x}_1 \times d\,{}^t\mathbf{x}_2 = \left({}^t_0\mathbf{F}\cdot d\,{}^0\mathbf{x}_1\right)\times\left({}^t_0\mathbf{F}\cdot d\,{}^0\mathbf{x}_2\right) = \det{}^t_0\mathbf{F}\;{}^0_t\mathbf{F}^{-\mathrm{T}}\cdot d\,{}^0\mathbf{a} \\ d\,{}^tV &= d\,{}^t\mathbf{a}\cdot d\,{}^t\mathbf{x}_3 = \det{}^t_0\mathbf{F}\;{}^t_0\mathbf{F}^{-\mathrm{T}}\cdot d\,{}^0\mathbf{a}\cdot{}^t_0\mathbf{F}\cdot d\,{}^0\mathbf{x}_3 = \det{}^t_0\mathbf{F}\;d\,{}^0V \end{aligned} \quad . \tag{4-20}$$

The invertibility of the mapping ${}^t\hat{\boldsymbol{\chi}}$ requires $\det{}^t_0\mathbf{F} \neq 0$ and as the volume has to be positive, the condition $\det{}^t_0\mathbf{F} > 0$ results. For an incompressible material $\det{}^t_0\mathbf{F} = 1$. *Conservation of mass*, eq. (6-7), calls for

$${}^tm = {}^t\rho\, d\,{}^tV = {}^0m = {}^0\rho\, d\,{}^0V \tag{4-21}$$

and hence

$$\frac{{}^0\rho}{{}^t\rho} = \det{}^t_0\mathbf{F} = J\,. \tag{4-22}$$

J is called JACOBIan[11].

An alternative representation of the tensors describing deformation, which provides a descriptive interpretation, is based upon *material* or *convective coordinates*. A network of generally curvilinear and non-orthogonal coordinates, ξ^i, i = 1, 2, 3, is engraved on the body in its initial configuration and deforms together with the body. Each material point, $\mathcal{X} \in \mathcal{B}$, is identified by a single triple of numbers $\left\{\xi^1,\xi^2,\xi^3\right\}$ during the whole deformation process, and the position vectors in the initial and current configuration are

$${}^0\mathbf{x} = {}^0x_i\left(\xi^k\right){}^0\mathbf{e}_i \quad , \quad {}^t\mathbf{x} = {}^tx_i\left(\xi^k\right){}^0\mathbf{e}_i \tag{4-23}$$

respectively[12]. The base vectors of convective coordinates are defined as the local tangent vectors to the ξ^i-curves, the so-called *covariant base vectors*

$${}^0\mathbf{g}_i = \frac{\partial\,{}^0\mathbf{x}}{\partial\xi^i} \quad , \quad {}^t\mathbf{g}_i = \frac{\partial\,{}^t\mathbf{x}}{\partial\xi^i}\,. \tag{4-24}$$

As these base vectors are not orthogonal, in general, the dual *covariant base vectors* are introduced as

$${}^0\mathbf{g}^i\cdot{}^0\mathbf{g}_j = \delta^i_j \quad , \quad {}^t\mathbf{g}^i\cdot{}^t\mathbf{g}_j = \delta^i_j\;, \tag{4-25}$$

by means of which the *metric tensors*[13],

$$\mathbf{1} = \mathbf{g}^i\mathbf{g}_j = \mathbf{g}_i\mathbf{g}^j = \left(\mathbf{g}_i\cdot\mathbf{g}_j\right)\mathbf{g}^i\mathbf{g}^j = g_{ij}\mathbf{g}^i\mathbf{g}^j = g^{ij}\mathbf{g}_i\mathbf{g}_j = \delta^i_j\mathbf{g}_i\mathbf{g}^j\,, \tag{4-26}$$

can be written in initial and current configuration, ${}^0\mathbb{B}$ and ${}^t\mathbb{B}$, respectively, and also the gradients of arbitrary field quantities as

$${}^0\mathrm{grad}\,\boldsymbol{\Phi} = \frac{\partial\boldsymbol{\Phi}}{\partial\xi^i}\,{}^0\mathbf{g}^i \quad , \quad {}^t\mathrm{grad}\,\boldsymbol{\Phi} = \frac{\partial\boldsymbol{\Phi}}{\partial\xi^i}\,{}^t\mathbf{g}^i\;. \tag{4-27}$$

---

[10] Note that $\left(\mathbf{A}\cdot\mathbf{v}\right)\times\left(\mathbf{A}\cdot\mathbf{w}\right) = \mathbf{A}^*\cdot\left(\mathbf{v}\times\mathbf{w}\right)$ with $\mathbf{A}^*$ being the adjunct to $\mathbf{A}$, i.e. $\mathbf{A}^*\cdot\mathbf{A}^\mathrm{T} = \mathbf{A}^\mathrm{T}\cdot\mathbf{A}^* = \det\left(\mathbf{A}\right)\mathbf{I}$.

[11] K. G. JACOBI (1804-1851)

[12] As stated above, different coordinates can be applied in initial and current configurations. However, modified rules hold in tensor analysis for other than Cartesian coordinates, as the unit vectors are neither normalised nor orthogonal nor are the coordinates straight lined.

[13] Note that $\mathbf{e}_i\cdot\mathbf{e}_j = \delta_{ij}$ as for a Cartesian base holds only in a mixed co-contravariant base, here.

In particular, the displacement gradient takes the form

$$ {}^t_0\mathbf{H} = \frac{\partial\, {}^t_0\mathbf{u}}{\partial \xi^i}\, {}^0\mathbf{g}^i \,. \tag{4-28}$$

In this representation, the deformation gradient ${}^t_0\mathbf{F}$ connects the material base vectors of initial and current configurations,

$$ {}^t_0\mathbf{F} = {}^t\mathbf{g}_i\, {}^0\mathbf{g}^i \quad , \quad {}^0_t\mathbf{F}^{-1} = {}^0\mathbf{g}_i\, {}^t\mathbf{g}^i \quad , \quad {}^t_0\mathbf{F}^{\mathrm{T}} = {}^0\mathbf{g}^i\, {}^t\mathbf{g}_i \ , \tag{4-29}$$

which motivates its characterisation as a "*two-field*" tensor.

4.4 Strain tensors

As the deformation gradient may include (rigid) rotations, it cannot be used to describe the deformation of a material body. With the theorem of *polar decomposition* one can define appropriate measures for deformation. By this, ${}^t_0\mathbf{F}$ can be decomposed uniquely into two parts,

$$ {}^t_0\mathbf{F} = {}^t_0\mathbf{R} \cdot {}^t_0\mathbf{U} = {}^t_0\mathbf{V} \cdot {}^t_0\mathbf{R} \tag{4-30}$$

with the orthogonal *rotation tensor*

$$ {}^t_0\mathbf{R} \cdot {}^t_0\mathbf{R}^{\mathrm{T}} = \mathbf{1} \quad , \quad \det\left({}^t_0\mathbf{R} \right) = 1 \ , \tag{4-31}$$

and the symmetric and positive definite *right* and *left stretch tensors*, ${}^t_0\mathbf{U}$ and ${}^t_0\mathbf{V}$,

$$ \begin{aligned} {}^t_0\mathbf{U} &= \left({}^t_0\mathbf{F}^{\mathrm{T}} \cdot {}^t_0\mathbf{F} \right)^{1/2} = {}^t_0\mathbf{U}^{\mathrm{T}} \\ {}^t_0\mathbf{V} &= \left({}^t_0\mathbf{F} \cdot {}^t_0\mathbf{F}^{\mathrm{T}} \right)^{1/2} = {}^t_0\mathbf{V}^{\mathrm{T}} \end{aligned} \ , \tag{4-32}$$

respectively[14], the squares of which are addressed as *right* and *left* CAUCHY-GREEN[15] *tensors*, ${}^t_0\mathbf{C}$ and ${}^t_0\mathbf{B}$.

Symmetric tensors have real *eigenvalues*, λ_i, $i = \mathrm{I, II, III}$, with associated *eigenvectors* (principal axes), $\mathbf{n}_i$, which are solutions of the *eigenvalue problem*

$$ {}^t_0\mathbf{U} \cdot {}^0\mathbf{n}_i = \lambda_{(i)}\, {}^0\mathbf{n}_{(i)} \ ,^{16} \tag{4-33}$$

see Appendix A.2. Applying a rotation, ${}^t_0\mathbf{R}$, and introducing eq. (4-30),

$$ \lambda_{(i)}\, {}^t_0\mathbf{R} \cdot {}^0\mathbf{n}_{(i)} = {}^t_0\mathbf{R} \cdot {}^t_0\mathbf{U} \cdot {}^0\mathbf{n}_i = {}^t_0\mathbf{V} \cdot {}^t_0\mathbf{R} \cdot {}^0\mathbf{n}_i = {}^t_0\mathbf{V} \cdot {}^t\mathbf{n}_i = \lambda_{(i)}\, {}^t\mathbf{n}_{(i)} \ , \tag{4-34}$$

we can conclude that the right and left stretch tensor are similar, i.e. they have the same eigenvalues but rotated axes

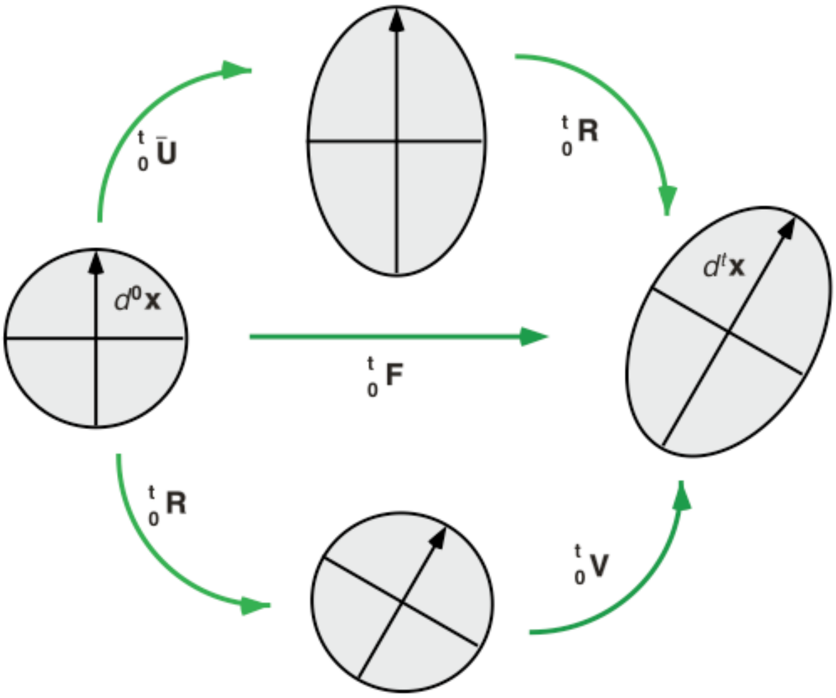
$$ {}^t\mathbf{n}_i = {}^t_0\mathbf{R} \cdot {}^0\mathbf{n}_i \quad , \quad i = \mathrm{I, II, III} \,. \tag{4-35}$$

---

[14] The square root of a tensor is defined via its spectral form as $\sqrt{\mathbf{T}} = \sum_{i=\mathrm{I}}^{\mathrm{III}} \sqrt{\lambda_{(\mathrm{i})}}\, \mathbf{n}_{(\mathrm{i})}\mathbf{n}_{(\mathrm{i})}$, see Appendix A2.5.

[15] AUGUSTIN LOUIS CAUCHY (1789-1857), GEORGE GREEN (1793-1842)

[16] No summation over (i) !



<u>Figure 4.2:</u> Polar decomposition of the deformation gradient

In the base of the eigenvectors, tensors can be represented in the spectral form,

$$\begin{aligned} {}^t_0\mathbf{U} &= \lambda_{\mathrm{I}}\, {}^0\mathbf{n}_{\mathrm{I}}\, {}^0\mathbf{n}_{\mathrm{I}} + \lambda_{\mathrm{II}}\, {}^0\mathbf{n}_{\mathrm{II}}\, {}^0\mathbf{n}_{\mathrm{II}} + \lambda_{\mathrm{III}}\, {}^0\mathbf{n}_{\mathrm{III}}\, {}^0\mathbf{n}_{\mathrm{III}} \\ {}^t_0\mathbf{V} &= \lambda_{\mathrm{I}}\, {}^t\mathbf{n}_{\mathrm{I}}\, {}^t\mathbf{n}_{\mathrm{I}} + \lambda_{\mathrm{II}}\, {}^t\mathbf{n}_{\mathrm{II}}\, {}^t\mathbf{n}_{\mathrm{II}} + \lambda_{\mathrm{III}}\, {}^t\mathbf{n}_{\mathrm{III}}\, {}^t\mathbf{n}_{\mathrm{III}} \end{aligned} \qquad (4\text{-}36)$$

The polar decomposition, eq. (4-30), thus states that a material line element $d\,{}^0\mathbf{x} = d\,{}^0x_{(i)}\, {}^0\mathbf{n}_{(i)}$ (i = I, II, III, no summation!) in one of the principal orientations of ${}^t_0\mathbf{U}$, which is subject to a deformation according to eq. (4-14), $d\,{}^t\mathbf{x} = {}^t_0\mathbf{F}\cdot d\,{}^0\mathbf{x} = {}^t_0\mathbf{R}\cdot\left({}^t_0\mathbf{U}\cdot d\,{}^0\mathbf{x}\right) = {}^t_0\mathbf{V}\cdot\left({}^t_0\mathbf{R}\cdot d\,{}^0\mathbf{x}\right)$, undergoes

- either a *stretching*, ${}^t_0\mathbf{U}$, by λ_i followed by a *rotation*, ${}^t_0\mathbf{R}$, into the current principal orientation, ${}^t\mathbf{n}_i$,
- or, vice versa, a rotation, ${}^t_0\mathbf{R}$, into the current principal orientation, ${}^t\mathbf{n}_I$, followed by a *stretching*, ${}^t_0\mathbf{V}$, by λ_i,

$$d\,{}^0\mathbf{x} = d\,{}^0x_{(\mathrm{i})}\, {}^0\mathbf{n}_{(\mathrm{i})} \qquad \Rightarrow \qquad d\,{}^t\mathbf{x} = \lambda_{(\mathrm{i})}\, d\,{}^0x_{(\mathrm{i})}\, {}^t\mathbf{n}_{(\mathrm{i})}\,, \quad i = \mathrm{I, II, III}\ (\text{no summation})\,, \qquad (4\text{-}37)$$

see Fig. 4-2. The *principal stretches*, λ_i, are related to the *relative elongations* or *linear principal strains* by

$${}^t_0\varepsilon_i = \frac{d\,{}^tx_i - d\,{}^0x_i}{d\,{}^0x_i} = \lambda_i - 1\,. \qquad (4\text{-}38)$$

By means of these principal stretches, *strain tensors* of the form

$${}^t_0\mathbf{E}^{(*)} = f(\lambda_{\mathrm{I}})\, {}^0\mathbf{n}_{\mathrm{I}}\, {}^0\mathbf{n}_{\mathrm{I}} + f(\lambda_{\mathrm{II}})\, {}^0\mathbf{n}_{\mathrm{II}}\, {}^0\mathbf{n}_{\mathrm{II}} + f(\lambda_{\mathrm{III}})\, {}^0\mathbf{n}_{\mathrm{III}}\, {}^0\mathbf{n}_{\mathrm{III}} \qquad (4\text{-}39)$$

can be defined, where $f(\lambda)$ is a continuously differentiable function with $f(1) = 0$ and $f\,'(1) = 1$. The most common strain tensors are

- **BIOT's**[17] (linear) **strain tensor**,

$$f(\lambda_i) = \lambda_i - 1 = {}^t_0\varepsilon_i$$
$${}^t_0\mathbf{E}^{(B)} = {}^t_0\mathbf{U} - \mathbf{I} = \tfrac{1}{2}\left[{}^0\nabla {}^t_0\mathbf{u} + \left({}^0\nabla {}^t_0\mathbf{u}\right)^{\mathrm{T}} \right], \tag{4-40}$$

- **GREEN-LAGRANGEan** (quadratic) **strain tensor**,

$$f(\lambda_i) = \tfrac{1}{2}\left(\lambda_i^2 - 1\right) = {}^t_0\varepsilon_i^{(G)} = {}^t_0\varepsilon_i + \tfrac{1}{2}\,{}^t_0\varepsilon_i^2$$
$${}^t_0\mathbf{E}^{(G)} = \tfrac{1}{2}\left({}^t_0\mathbf{U}^2 - \mathbf{I}\right) = \tfrac{1}{2}\left[{}^0\nabla {}^t_0\mathbf{u} + \left({}^0\nabla {}^t_0\mathbf{u}\right)^{\mathrm{T}} + {}^0\nabla {}^t_0\mathbf{u} \cdot \left({}^0\nabla {}^t_0\mathbf{u}\right)^{\mathrm{T}} \right], \tag{4-41}$$

- **HENCKY's**[18] (logarithmic) **strain tensor**,

$$f(\lambda_i) = \ln\lambda_i = {}^t_0\varepsilon_i^{(H)} = \ln\left(1 + {}^t_0\varepsilon_i\right)$$
$${}^t_0\mathbf{E}^{(H)} = \ln\left({}^t_0\mathbf{U}\right). \tag{4-42}$$

The advantage of HENCKY strains, ${}^t_0\varepsilon_i^{(H)}$, is that they are additive for two subsequent deformations steps, i.e. ${}^{t_2}_0\varepsilon_i^{(H)} = {}^{t_1}_0\varepsilon_i^{(H)} + {}^{t_2}_{t_1}\varepsilon_i^{(H)}$.

GREEN's strain tensor. ${}^t_0\mathbf{E}^{(G)}$ describes the change of the square of a line element in the current configuration compared to the initial one,

$$d\,{}^t\mathbf{x} \cdot d\,{}^t\mathbf{x} - d\,{}^0\mathbf{x} \cdot d\,{}^0\mathbf{x} = d\,{}^0\mathbf{x} \cdot \left({}^t_0\mathbf{F}^{\mathrm{T}} \cdot {}^t_0\mathbf{F} - \mathbf{I}\right) \cdot d\,{}^0\mathbf{x} = 2d\,{}^0\mathbf{x} \cdot {}^t_0\mathbf{E}^{(G)} \cdot d\,{}^0\mathbf{x}. \tag{4-43}$$

A correspondent representation with respect to the current configuration (spatial description),

$$d\,{}^0\mathbf{x} \cdot d\,{}^0\mathbf{x} - d\,{}^t\mathbf{x} \cdot d\,{}^t\mathbf{x} = d\,{}^t\mathbf{x} \cdot \left(\mathbf{I} - {}^0_t\mathbf{F}^{-1} \cdot {}^0_t\mathbf{F}^{-\mathrm{T}}\right) \cdot d\,{}^t\mathbf{x} = 2d\,{}^t\mathbf{x} \cdot {}^0_t\mathbf{E}^{(A)} \cdot d\,{}^t\mathbf{x}, \tag{4-44}$$

leads to

- **ALMANSI's**[19] **strain tensor**,

$${}^0_t\mathbf{E}^{(A)} = \tfrac{1}{2}\left(\mathbf{I} - {}^0_t\mathbf{F}^{-1} \cdot {}^0_t\mathbf{F}^{-\mathrm{T}}\right) = \tfrac{1}{2}\left(\mathbf{I} - {}^0_t\mathbf{V}^{-2}\right) = {}^0_t\mathbf{F}^{-\mathrm{T}} \cdot {}^0_t\mathbf{E}^{(G)} \cdot {}^0_t\mathbf{F}^{-1}. \tag{4-45}$$

Again, a representation in convective (material) coordinates allows for a simple and descriptive geometric interpretation,

$${}^t_0\mathbf{E}^{(G)} = \tfrac{1}{2}\left({}^t g_{ij} - {}^0 g_{ij}\right) {}^0\mathbf{g}^i\,{}^0\mathbf{g}^j$$
$${}^0_t\mathbf{E}^{(A)} = \tfrac{1}{2}\left({}^t g_{ij} - {}^0 g_{ij}\right) {}^t\mathbf{g}^i\,{}^t\mathbf{g}^j. \tag{4-46}$$

The components of GREEN's and ALMANSI's strain tensors, ${}^t_0\mathbf{E}^{(G)}$ und ${}^0_t\mathbf{E}^{(A)}$, in a material coordinate system are equal, namely the difference of the metrics in current and initial configurations. Just the system of base vectors is rotated.

All strain tensors, ${}^t_0\mathbf{E}^{(*)}$, are symmetric, ${}^t_0\mathbf{E}^{(*)} = \left({}^t_0\mathbf{E}^{(*)}\right)^{\mathrm{T}}$. They can be written with respect to a normalised and orthogonal vector base, ${}^t_0\mathbf{E}^{(*)} = \varepsilon_{ij}^{(*)}\mathbf{e}_i\mathbf{e}_j$, where their components form a 3×3 matrix,

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[17] MAURICE ANTHONY BIOT (1905-1985)
[18] HEINRICH HENCKY (1885-1951)
[19] EMILIO ALMANSI (1869-1948)

$$\left(\varepsilon_{ij}^{(*)}\right)=\begin{pmatrix}\varepsilon_{11}^{(*)} & \varepsilon_{12}^{(*)} & \varepsilon_{13}^{(*)}\\ \varepsilon_{21}^{(*)} & \varepsilon_{22}^{(*)} & \varepsilon_{23}^{(*)}\\ \varepsilon_{31}^{(*)} & \varepsilon_{32}^{(*)} & \varepsilon_{33}^{(*)}\end{pmatrix}=\left(\varepsilon_{ji}^{(*)}\right). \tag{4-47}$$

The strain components are denoted as *normal strains* for $i = j$ and *shear strains* for $i \neq j$.

For *small* principal strains, ${}^t_0\varepsilon_i \ll 1$, quadratic and logarithmic strain measures, eqs. (4-41) and (4-42), merge with the linear one, eq. (4-38), as the quadratic term in eq. (4-41) and in the series expansion of eq. (4-42) can be neglected compared to the linear term. Small deformations are characterised by small strains and small rotations. Define

$$\varepsilon \doteq \left\|{}^0\mathrm{grad}\,{}^t_0\mathbf{u}\right\| = \left\|{}^t_0\mathbf{H}\right\| \ll 1\,, \tag{4-48}$$

than

$${}^t_0\mathbf{E}^{(\mathrm{G})} = \mathbf{E} + \mathcal{O}(\varepsilon^2) \tag{4-49}$$

with the *linear strain tensor* (CAUCHY[20] 1827),

$$\mathbf{E} = \varepsilon_{ij}\mathbf{e}_i\mathbf{e}_j = \tfrac{1}{2}\left(\mathbf{H}+\mathbf{H}^{\mathrm{T}}\right) = \tfrac{1}{2}\left[\nabla\mathbf{u} + \left(\nabla\mathbf{u}\right)^{\mathrm{T}}\right] = \tfrac{1}{2}\left(u_{i,j}+u_{j,i}\right)\mathbf{e}_i\mathbf{e}_j\,, \tag{4-50}$$

being the symmetric part of the displacement gradient [21], as used in the theory of small deformations. The JACOBIan for small deformations is

$$J = \det\left(\mathbf{F}\right) = \det\left(\mathbf{1}+\mathbf{H}\right) \approx 1 + \varepsilon_{kk}\,, \tag{4-51}$$

i.e. according to eq. (4-20), ε_{kk} represents the *volume dilatation* of a material element under small deformations,

$$\varepsilon_{kk} = \mathrm{tr}\,\mathbf{E} \approx \frac{d^tV - d^0V}{d^0V}\,, \tag{4-52}$$

Eq. (4-50) allows for calculating a tensor field, **E**, from a given displacement field, **u**, uniquely. If a tensor field **E** is given, it does not automatically follow, that such a field indeed represents a strain field, that is, that there exists a displacement field, **u**, such that eq. (4-50) holds. If it does, then the strain field is called *compatible*. The necessary and sufficient condition on **E**, that ensures the existence of **u** as a solution of eq. (4-50) reads

$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{E} = \nabla\times\left(\nabla\times\mathbf{E}\right)^{\mathrm{T}} = \mathbf{0}\ . \tag{4-53}$$

4.5 Material and Local Time Derivatives

Studying the motion of a continuum, we deal with time rates of changes of quantities that vary from one particle to the other. Material time derivatives in a LAPLACEan (material) description are straightforward. Consider a real-valued function, $\psi({}^0\mathbf{x},t)$, that represents a scalar or a component of a vector or a tensor. The position vector ${}^0\mathbf{x}$ uniquely determines a continuum particle, $\mathcal{X}$, namely the one located at ${}^0\mathbb{X}$ at $t = 0$, referred to as *particle* ${}^0\mathbf{x}$. The partial derivative of ψ with respect to t, with ${}^0\mathbf{x}$ held fixed, is the time rate of change of ψ at the particle ${}^0\mathbf{x}$. This derivative is called the *material time derivative* of ψ,

[20] AUGUSTIN LOUIS CAUCHY (1789-1857)

[21] For small deformations no difference has to be made between differentiation with respect to the initial or the current coordinates

$$\dot{\psi} = \frac{d\psi}{dt} = \left.\frac{\partial\psi({}^{0}\mathbf{x},t)}{\partial \mathrm{t}}\right|_{{}^{0}\mathbf{x}} . \tag{4-54}$$

Since ${}^{t}\mathbf{x} = {}^{t}\hat{\boldsymbol{\chi}}\left({}^{0}\mathbf{x},t\right)$ in the material description of motion, the material time derivative of ${}^{t}\mathbf{x}$ represents the time rate of change of the position of the particle ${}^{0}\mathbf{x}$ at time *t*, i.e. the *velocity* of the particle ${}^{0}\mathbf{x}$ at time *t*,

$${}^{t}\mathbf{v} = {}^{t}\dot{\mathbf{x}} = \frac{d\,{}^{t}\mathbf{x}}{dt} = \left.\frac{\partial\,{}^{t}\hat{\boldsymbol{\chi}}({}^{0}\mathbf{x},t)}{\partial \mathrm{t}}\right|_{{}^{0}\mathbf{x}} . \tag{4-55}$$

As ${}_{0}^{t}\mathbf{u} = {}^{t}\mathbf{x} - {}^{0}\mathbf{x} = \bar{\mathbf{u}}({}^{0}\mathbf{x},t)$, we get ${}^{t}\mathbf{v} = {}_{0}^{t}\dot{\mathbf{u}}$.

Consider now a real-valued function, $\varphi({}^{t}\mathbf{x},t)$, that represents a scalar or a component of a vector or a tensor. Since ${}^{t}\mathbf{x}$ is a point in the current configuration, $\varphi({}^{t}\mathbf{x},t)$ is the value of φ experienced by the particle ${}^{0}\mathbf{x}$ currently located at ${}^{t}\mathbf{x}$. The partial derivative of φ with respect to *t*, with ${}^{t}\mathbf{x}$ held fixed, is the time rate of change of φ at the particle currently located at ${}^{t}\mathbf{x}$. This derivative is called the *local time derivative* of φ,

$$\frac{\partial\varphi}{\partial t} = \left.\frac{\partial\varphi({}^{t}\mathbf{x},t)}{\partial \mathrm{t}}\right|_{{}^{t}\mathbf{x}} . \tag{4-56}$$

When calculating the *material time derivative* in a EULERean (*spatial*) *description*, one has to bear in mind, that the actual particle ${}^{0}\mathbf{x}$ located in a spatial point ${}^{t}\mathbf{x}$ varies with time. Consider again $\varphi({}^{t}\mathbf{x},t)$ with ${}^{t}\mathbf{x} = {}^{t}\hat{\boldsymbol{\chi}}\left({}^{0}\mathbf{x},t\right)$, than by the chain rule of partial differentiation,

$$\dot{\varphi} = \frac{d\varphi}{dt} = \left.\frac{\partial\varphi({}^{t}\mathbf{x},t)}{\partial t}\right|_{{}^{0}\mathbf{x}} = \left.\frac{\partial\varphi}{\partial t}\right|_{{}^{t}\mathbf{x}} + \left.\left(\frac{\partial\varphi}{\partial\,{}^{t}\mathbf{x}}\right)\right|_{t} \cdot \left.\left(\frac{\partial\,{}^{t}\mathbf{x}}{\partial \mathrm{t}}\right)\right|_{{}^{0}\mathbf{x}} = \frac{\partial\varphi}{\partial \mathrm{t}} + {}^{t}\mathbf{v}\cdot{}^{t}\boldsymbol{\nabla}\varphi , \tag{4-57}$$

we obtain the *material derivative operator*,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \underline{\mathrm{v}}\cdot{}^{t}\underline{\nabla} , \tag{4-58}$$

for calculating the rate of change with time of an arbitrary field quantity in the spatial description. $(\,)^{\bullet} = d/dt$ is the *material* or *substantial derivative*, $\partial/\partial t$ is called *local* and $\underline{\mathrm{v}}\cdot{}^{t}\underline{\nabla}$ *convective derivative*, respectively.

4.6 Strain Rates

While the deformation gradient, ${}_{0}^{t}\mathbf{F}$, describes the change of length and orientation of a material line element and consequently the change of size and shape of a material volume element during deformation, the actual *velocity gradient*

$${}^{t}\mathbf{L} = {}^{t}\mathrm{grad}\,{}^{t}\mathbf{v} = \frac{\partial\,{}^{t}v_i}{\partial\,{}^{t}x_j}\,{}^{0}\mathbf{e}_i\,{}^{0}\mathbf{e}_j = \frac{\partial\,{}^{t}\mathbf{v}}{\partial\xi_j}\,{}^{t}\mathbf{g}^{j} = \left({}^{t}\nabla\,{}^{t}\mathbf{v}\right)^{\mathrm{T}} . \tag{4-59}$$

measures the rate of change with time of a line element in the current configuration ${}^{t}\mathbb{B}$,

$$\left(d\,{}^{t}\mathbf{x}\right)^{\bullet} = {}^{t}\mathbf{L}\cdot d\,{}^{t}\mathbf{x} \quad \text{, i.e.} \quad \left({}_{0}^{t}\mathbf{F}\right)^{\bullet} = {}^{t}\mathbf{L}\cdot{}_{0}^{t}\mathbf{F} . \tag{4-60}$$

It can be calculated from ${}_{0}^{t}\mathbf{F}$

$${}^{t}\mathbf{L} = {}^{t}\dot{\mathbf{F}}\cdot{}_{0}^{t}\mathbf{F}^{-1} = -\,{}_{0}^{t}\mathbf{F}\cdot{}^{t}\dot{\mathbf{F}}^{-1} = {}^{t}\mathbf{D} + {}^{t}\mathbf{W} \tag{4-61}$$

and additively decomposed into a symmetric part, the *deformation* (*stretching*) *rate* (EULER 1770), ${}^t\mathbf{D}$, and a skew part, the *vorticity tensor* (CAUCHY 1841), ${}^t\mathbf{W}$,

$$ {}^t\mathbf{D} = \tfrac{1}{2}\left({}^t\mathbf{L} + {}^t\mathbf{L}^{\mathrm{T}} \right) = {}^t\mathbf{D}^{\mathrm{T}} \quad , \quad {}^t\mathbf{W} = \tfrac{1}{2}\left({}^t\mathbf{L} - {}^t\mathbf{L}^{\mathrm{T}} \right) = -\, {}^t\mathbf{W}^{\mathrm{T}} . \tag{4-62} $$

The coordinates of ${}^t\mathbf{D}$ are the rates of change with time of lengths and angles of material volumes, and the coordinates of ${}^t\mathbf{W}$ are the angular velocities of line elements. Since ${}^t\mathbf{W}$ is skew, ${}^t\mathbf{W}\cdot d\,{}^t\mathbf{x} = {}^t\boldsymbol{\omega}\times d\,{}^t\mathbf{x}$, representing the velocity due to a rigid rotation about an axis through the point ${}^t\mathbf{x}$ with *angular velocity* ${}^t\boldsymbol{\omega} = \tfrac{1}{2}\mathrm{curl}\,{}^t\mathbf{v} = \tfrac{1}{2}\left({}^t\nabla\times{}^t\mathbf{v} \right)^{\mathrm{T}}$. Thus $d\,{}^t\mathbf{v} = \left(d\,{}^t\mathbf{x} \right)^{\bullet}$ is considered a superposition of the velocity caused by the stretching and determined by ${}^t\mathbf{D}$ and the velocity due to rigid rotation determined by ${}^t\mathbf{W}$. Consider a line element, $d\,{}^t\mathbf{x} = \left| d\,{}^t\mathbf{x} \right| {}^t\mathbf{n}$, in one of the principal orientations of ${}^t\mathbf{D}$, then the rate of change of its length is solely described by ${}^t\mathbf{D}$, and the rate of change of its orientation by ${}^t\mathbf{W}$,

$$ \begin{aligned} d\,{}^t\mathbf{v} &= \left(d\,{}^t\mathbf{x} \right)^{\bullet} = \left| d\,{}^t\mathbf{x} \right|^{\bullet} {}^t\mathbf{n} + \left| d\,{}^t\mathbf{x} \right| {}^t\dot{\mathbf{n}} = \left({}^t\mathbf{n}\cdot{}^t\mathbf{D}\cdot{}^t\mathbf{n} \right) d\,{}^t\mathbf{x} + {}^t\mathbf{W}\cdot d\,{}^t\mathbf{x} \\ &= \left({}^t\mathbf{n}\cdot{}^t\mathbf{D}\cdot{}^t\mathbf{n} \right) d\,{}^t\mathbf{x} + {}^t\boldsymbol{\omega}\times d\,{}^t\mathbf{x} \end{aligned} \tag{4-63} $$

${}^t\mathbf{D} = \mathbf{0}$ characterises a local rigid-body rotation.

The rates of change with time of material area and volume elements in ${}^t\mathbb{B}$ are

$$ \begin{aligned} \left(d\,{}^t\mathbf{a} \right)^{\bullet} &= \left(\mathrm{div}\,{}^t\mathbf{v}\,\mathbf{I} - {}^t\mathbf{L}^{\mathrm{T}} \right)\cdot d\,{}^t\mathbf{a} \\ \left(d\,{}^tV \right)^{\bullet} &= \mathrm{div}\,{}^t\mathbf{v}\; d\,{}^tV \end{aligned} \tag{4-64} $$

with

$$ \mathrm{div}\,{}^t\mathbf{v} = {}^t\nabla\cdot{}^t\mathbf{v} = \mathrm{tr}\,{}^t\mathbf{L} = \mathrm{tr}\,{}^t\mathbf{D} . \tag{4-65} $$

The current deformation rate tensor, ${}^t\mathbf{D}$, is related to the material time derivative of GREEN's strain tensor, ${}^t\dot{\mathbf{E}}^{(\mathrm{G})}$, by

$$ {}^t\mathbf{D} = {}^0_t\mathbf{F}^{-\mathrm{T}}\cdot{}^t\dot{\mathbf{E}}^{(\mathrm{G})}\cdot{}^0_t\mathbf{F}^{-1} , \tag{4-66} $$

A representation in convective coordinates points up this relation more clearly,

$$ {}^t\mathbf{D} = \tfrac{1}{2}\,{}^t\dot{g}_{ij}\,{}^t\mathbf{g}^i\,{}^t\mathbf{g}^j \quad , \quad {}^t\dot{\mathbf{E}}^{(\mathrm{G})} = \tfrac{1}{2}\,{}^t\dot{g}_{ij}\,{}^0\mathbf{g}^i\,{}^0\mathbf{g}^j . \tag{4-67} $$

The components of ${}^t\mathbf{D}$ and ${}^t\dot{\mathbf{E}}^{(\mathrm{G})}$ are equal, namely half the material time derivatives of the covariant time metric coefficients in ${}^t\mathbb{B}$, but the base vectors differ, i.e. the convective base vectors in ${}^t\mathbb{B}$ for ${}^t\mathbf{D}$ and the convective base vectors in ${}^0\mathbb{B}$ for ${}^t\dot{\mathbf{E}}^{(\mathrm{G})}$. ${}^t\mathbf{D}$ cannot be written and interpreted as material time derivative of a strain tensor. By means of the relation to the material time derivative of ALMANSI's strain tensor, ${}^t\mathbf{D}$ is introduced as OLDROYD's *time derivative* of ${}^0_t\mathbf{E}^{(\mathrm{A})}$,

$$ {}^t\mathbf{D} = {}^t\overset{\circ}{\mathbf{E}}{}^{(\mathrm{A})} = {}^t\dot{\mathbf{E}}^{(\mathrm{A})} + {}^t\mathbf{E}^{(\mathrm{A})}\cdot{}^t\mathbf{L} + {}^t\mathbf{L}^{\mathrm{T}}\cdot{}^t\mathbf{E}^{(\mathrm{A})} . \tag{4-68} $$

The problem which has become manifest here, raises the general question of appropriate time derivatives of material quantities. It is of particular and fundamental interest for describing the material behaviour by constitutive equations, which are often established as *rate* equations. Constitutive equations have to be *objective*, that is independent of the specific observer and his frame of reference. This condition has to be met for the involved field quantities as well as for their time derivatives. The respective conditions are addressed in the following section.

4.7 Change of Reference Frame

If two observers describe their spaces by position vectors with respect to their individual points of reference, then

- the first observer sees the position vector of a spatial point $\mathbb{X} \in \mathbb{E}^3$ at time *t* with respect to his point of reference ("origin"), $\mathbb{O} \in \mathbb{E}^3$, as

$$\mathbf{x}(\mathbb{X},t) = \overrightarrow{\mathbb{O}\mathbb{X}} = \mathbb{X} - \mathbb{O}\,, \tag{4-69}$$

- and the second observer with respect to his point of reference, $\mathbb{O}' \in \tilde{\mathbb{E}}^3$, as

$$\tilde{\mathbf{x}}(\tilde{\mathbb{X}},t) = \overrightarrow{\mathbb{O}'\tilde{\mathbb{X}}} = \overrightarrow{\mathbb{O}'\tilde{\mathbb{O}}} + \overrightarrow{\tilde{\mathbb{O}}\tilde{\mathbb{X}}}\,. \tag{4-70}$$

As the distance $\left|\overrightarrow{\mathbb{O}\mathbb{X}}\right|$ and $\left|\overrightarrow{\tilde{\mathbb{O}}\tilde{\mathbb{X}}}\right|$ is the same for both observers,

$$\overrightarrow{\tilde{\mathbb{O}}\tilde{\mathbb{X}}} = \mathbf{Q}(t)\cdot\mathbf{x}(\mathcal{X},t) \qquad \text{with} \qquad \mathbf{Q}\cdot\mathbf{Q}^{\mathrm{T}} = \mathbf{1}\,, \tag{4-71}$$

and taking $\mathbf{c}(t) = \overrightarrow{\mathbb{O}'\tilde{\mathbb{O}}}$, we obtain a EUCLID*ean transformation* of the position vectors under *change of the observer* or "*change of the reference frame*",

$$\tilde{\mathbf{x}}(\tilde{\mathbb{X}},t) = \mathbf{Q}(t)\cdot\mathbf{x}(\mathbb{X},t) + \mathbf{c}(t)\,. \tag{4-72}$$

The second observer reports his space as being shifted by $\mathbf{c}(t)$ and rotated by $\mathbf{Q}(t)$ with respect to the space of the first observer. The transformation preserves the spatial distance between simultaneous events,

$$\left|\tilde{\mathbf{x}}(\tilde{\mathbb{X}}_1) - \tilde{\mathbf{x}}(\tilde{\mathbb{X}}_2)\right| = \left|\mathbf{Q}\cdot\left(\mathbf{x}(\mathbb{X}_1) - \mathbf{x}(\mathbb{X}_2)\right)\right| = \left|\mathbf{x}(\mathbb{X}_1) - \mathbf{x}(\mathbb{X}_2)\right|. \tag{4-73}$$

A vector, $\mathbf{w}$, or a (2nd order) tensor, $\mathbf{T}$, are called *objective* under the change of frame of reference, if they are just rotated by $\mathbf{Q}$ under a EUCLIDean transformation,

$$\tilde{\mathbf{w}} = \mathbf{Q}\cdot\mathbf{w} \quad , \qquad \tilde{\mathbf{T}} = \mathbf{Q}\cdot\mathbf{T}\cdot\mathbf{Q}^{\mathrm{T}}, \tag{4-74}$$

and *invariant*, if

$$\tilde{\mathbf{w}} = \mathbf{w} \quad , \qquad \tilde{\mathbf{T}} = \mathbf{T}\,, \tag{4-75}$$

For scalars, objectivity and invariance coincide.

$$\tilde{\alpha} = \alpha\,. \tag{4-76}$$

We can now apply the EUCLIDean transformation to the kinematic quantities describing motion and deformation of a body. The motion of a body, ${}^t\mathbf{x} = {}^t\bar{\boldsymbol{\chi}}(\mathcal{X},t)$, is described by identifying its materials points, $\mathcal{X}$, by their placement in a reference configuration, ${}^0\mathbf{x} = {}^0\bar{\boldsymbol{\chi}}(\mathcal{X})$, as in eq. (4-10). If the reference configuration is observer independent[22], then a EUCLID*ean transformation* of the motion is

$${}^t\mathbf{x} = {}^t\hat{\boldsymbol{\chi}}\left({}^0\mathbf{x},t\right) \quad \Leftrightarrow \quad {}^t\tilde{\mathbf{x}} = {}^t\tilde{\boldsymbol{\chi}}\left({}^0\mathbf{x},t\right) = {}^t\mathbf{Q}\cdot{}^t\hat{\boldsymbol{\chi}}\left({}^0\mathbf{x},t\right) + {}^t\mathbf{c}\,. \tag{4-77}$$

Eq. (4-77) describes *one* motion recorded by *two* observers (in two different reference frames). It is in form identical to the description of a rigid body displacement, which describes *two* motions recorded by *one* observer.

---

[22] This is an assumption, which is part of the definition of the reference configuration. One is of course free to introduce it observer dependent, as well.

The deformation gradient transforms as

$$ {}^t_0\tilde{\mathbf{F}} = \operatorname{Grad} {}^t\tilde{\mathbf{x}} = \left({}^0\nabla\, {}^t\tilde{\mathbf{x}}\right)^{\mathrm{T}} = \frac{\partial\, {}^t\tilde{\boldsymbol{\chi}}}{\partial\, {}^0\mathbf{x}} = \frac{\partial\, {}^t\tilde{\boldsymbol{\chi}}}{\partial\, {}^t\mathbf{x}} \cdot \frac{\partial\, {}^t\mathbf{x}}{\partial\, {}^0\mathbf{x}} = {}^t\mathbf{Q}\cdot {}^t_0\mathbf{F}\,, \tag{4-78}$$

and is obviously not objective. From this, the transformations of the other kinematical quantities follow,

$$
\begin{aligned}
&{}^t_0\tilde{\mathbf{R}} = {}^t\mathbf{Q}\cdot {}^t_0\mathbf{R} && \text{rotation tensor}\\
&{}^t_0\tilde{\mathbf{U}} = {}^t_0\mathbf{U} && \text{right stretch tensor,}\\
&{}^t_0\tilde{\mathbf{E}}^{(\mathrm{B})} = {}^t_0\mathbf{E}^{(\mathrm{B})} && \text{BIOT's strain tensor,}\\
&{}^t_0\tilde{\mathbf{E}}^{(\mathrm{G})} = {}^t_0\mathbf{E}^{(\mathrm{G})} && \text{GREEN-LAGRANGE strain tensor,}\\
&{}^t_0\tilde{\mathbf{V}} = {}^t\mathbf{Q}\cdot {}^t_0\mathbf{V}\cdot {}^t\mathbf{Q}^{\mathrm{T}} && \text{left stretch tensor,}\\
&{}^t_0\tilde{\mathbf{E}}^{(\mathrm{A})} = {}^t\mathbf{Q}\cdot {}^t_0\mathbf{E}^{(\mathrm{A})}\cdot {}^t\mathbf{Q}^{\mathrm{T}} && \text{ALMANSI strain tensor.}
\end{aligned} \tag{4-79}
$$

The tensors ${}^t_0\mathbf{V}$ and ${}^t_0\mathbf{E}^{(\mathrm{A})}$ are objective, ${}^t_0\mathbf{U}$ and ${}^t_0\mathbf{E}^{(\mathrm{G})}$ are invariant. All observers measure the same volume, and hence

$$ \tilde{J} = \det\left({}^t_0\tilde{\mathbf{F}}\right) = \det\left({}^t\mathbf{Q}\cdot {}^t_0\mathbf{F}\right) = \det\left({}^t\mathbf{Q}\right)\det\left({}^t_0\mathbf{F}\right) = \det\left({}^t_0\mathbf{F}\right) = J\,, \tag{4-80}$$

and the density is ${}^t\tilde{\rho} = {}^t\rho$.

Objective time dependent quantities α, $\mathbf{w}$, and $\mathbf{T}$ in the EULERean description give objective *spatial derivatives* $\operatorname{grad}\alpha$, $\operatorname{grad}\mathbf{w}$, $\operatorname{div}\mathbf{w}$, $\operatorname{grad}\mathbf{T}$, and $\operatorname{div}\mathbf{T}$.

Let us now consider *material time derivatives* of objective quantities. Obviously $\dot{\alpha}$ is objective, but $\dot{\mathbf{w}}$ and $\dot{\mathbf{T}}$ are not. If $\mathbf{w}$ and $\mathbf{T}$ are an objective vector and an objective (2nd order) tensor, respectively, and $\dot{\mathbf{w}}$ and $\dot{\mathbf{T}}$ their time-derivatives for one observer, then the time-derivatives for the other observer are

$$
\begin{aligned}
&\dot{\tilde{\mathbf{w}}} = \left(\mathbf{Q}\cdot\mathbf{w}\right)^{\boldsymbol{\cdot}} = \dot{\mathbf{Q}}\cdot\mathbf{w} + \mathbf{Q}\cdot\dot{\mathbf{w}}\\
&\dot{\tilde{\mathbf{T}}} = \left(\mathbf{Q}\cdot\mathbf{T}\cdot\mathbf{Q}^{\mathrm{T}}\right)^{\boldsymbol{\cdot}} = \mathbf{Q}\cdot\dot{\mathbf{T}}\cdot\mathbf{Q}^{\mathrm{T}} + \dot{\mathbf{Q}}\cdot\mathbf{Q}^{\mathrm{T}}\cdot\tilde{\mathbf{T}} - \tilde{\mathbf{T}}\cdot\dot{\mathbf{Q}}\cdot\mathbf{Q}^{\mathrm{T}},
\end{aligned} \tag{4-81}
$$

The velocity, $\mathbf{v}$, of a material point is the time derivative of its current position,

$$ {}^t\mathbf{v} = {}^t\dot{\mathbf{x}} = \frac{d\,{}^t\boldsymbol{\chi}}{dt} = \left.\left(\frac{\partial\,{}^t\boldsymbol{\chi}({}^0\mathbf{x},t)}{\partial t}\right)\right|_{{}^0\mathbf{x}}\,. \tag{4-82}$$

To the second observer, it appears as

$$
\begin{aligned}
{}^t\tilde{\mathbf{v}} &= {}^t\dot{\tilde{\mathbf{x}}} = {}^t\dot{\mathbf{Q}}\cdot {}^t\mathbf{x} + {}^t\mathbf{Q}\cdot {}^t\dot{\mathbf{x}} + {}^t\dot{\mathbf{c}} = {}^t\dot{\mathbf{Q}}\cdot {}^t\mathbf{Q}^{\mathrm{T}}\cdot\left({}^t\tilde{\mathbf{x}} - {}^t\mathbf{c}\right) + {}^t\mathbf{Q}\cdot {}^t\mathbf{v} + {}^t\dot{\mathbf{c}}\\
&= {}^t\boldsymbol{\omega}\times\left({}^t\tilde{\mathbf{x}} - {}^t\mathbf{c}\right) + {}^t\mathbf{Q}\cdot {}^t\mathbf{v} + {}^t\dot{\mathbf{c}}
\end{aligned} \tag{4-83}
$$

Beside the rotated part, ${}^t\mathbf{Q}\cdot {}^t\mathbf{v}$, the relative translational velocity, ${}^t\dot{\mathbf{c}}$, and the relative angular velocity, ${}^t\boldsymbol{\omega}$, which is the dual axial vector to the skew tensor ${}^t\dot{\mathbf{Q}}\cdot {}^t\mathbf{Q}^{\mathrm{T}}$, emerge. By a second differentiation with respect to time, we obtain the acceleration

$$ {}^t\mathbf{a} = {}^t\dot{\mathbf{v}} = {}^t\ddot{\mathbf{x}} = \left.\left(\frac{\partial^2\,{}^t\boldsymbol{\chi}({}^0\mathbf{x},t)}{\partial t^2}\right)\right|_{{}^0\mathbf{x}}\,, \tag{4-84}$$

and under the change of reference frame

$$
\begin{aligned}
{}^t\mathbf{a} &= {}^t\ddot{\tilde{\mathbf{x}}} = {}^t\dot{\boldsymbol{\omega}} \times \left({}^t\tilde{\mathbf{x}} - {}^t\mathbf{c} \right) + {}^t\boldsymbol{\omega} \times \left({}^t\dot{\tilde{\mathbf{x}}} - {}^t\dot{\mathbf{c}} \right) + {}^t\dot{\mathbf{Q}} \cdot {}^t\mathbf{v} + {}^t\mathbf{Q} \cdot {}^t\dot{\mathbf{v}} + {}^t\ddot{\mathbf{c}} \\
&= {}^t\dot{\boldsymbol{\omega}} \times \left({}^t\tilde{\mathbf{x}} - {}^t\mathbf{c} \right) + {}^t\boldsymbol{\omega} \times \left[\left({}^t\tilde{\mathbf{x}} - {}^t\mathbf{c} \right) \times {}^t\boldsymbol{\omega} \right] + 2\,{}^t\boldsymbol{\omega} \times \left({}^t\tilde{\mathbf{v}} - {}^t\dot{\mathbf{c}} \right) + {}^t\mathbf{Q} \cdot {}^t\mathbf{a} + {}^t\ddot{\mathbf{c}}
\end{aligned}
, \tag{4-85}
$$

with

${}^t\mathbf{Q} \cdot {}^t\mathbf{a}$	relative acceleration,
${}^t\ddot{\mathbf{c}}$	translational acceleration,
${}^t\dot{\boldsymbol{\omega}} \times \left({}^t\tilde{\mathbf{x}} - {}^t\mathbf{c} \right)$	angular acceleration,
$2\,{}^t\boldsymbol{\omega} \times \left({}^t\tilde{\mathbf{v}} - {}^t\dot{\mathbf{c}} \right)$	CORIOLIS[23] acceleration,
${}^t\boldsymbol{\omega} \times \left[\left({}^t\tilde{\mathbf{x}} - {}^t\mathbf{c} \right) \times {}^t\boldsymbol{\omega} \right]$	centripetal acceleration.

The velocity and the acceleration are neither objective nor invariant vectors, and the same holds for the rates of linear and angular momentum. The laws of motion are only valid for an *inertial reference frame*, and consequently only for those observers for which the acceleration transforms as an objective vector, ${}^t\mathbf{a} = {}^t\mathbf{Q} \cdot {}^t\mathbf{a}$. Such special transformations are called GALILEI *transformation*, characterised by ${}^t\ddot{\mathbf{c}} = \mathbf{o}$ and ${}^t\dot{\mathbf{Q}} = \mathbf{0}$, i.e. ${}^t\boldsymbol{\omega} = \mathbf{o}$.
The velocity gradient transforms like

$$
\begin{aligned}
{}^t\mathbf{L} &= {}^t\mathrm{grad}\,{}^t\mathbf{v} = \left({}^t\nabla {}^t\mathbf{v} \right)^{\mathrm{T}} = {}^t_0\dot{\tilde{\mathbf{F}}} \cdot {}^t_0\tilde{\mathbf{F}}^{-1} \\
&= {}^t\mathbf{Q} \cdot {}^t\mathbf{L} \cdot {}^t\mathbf{Q}^{\mathrm{T}} + {}^t\dot{\mathbf{Q}} \cdot {}^t\mathbf{Q}^{\mathrm{T}}
\end{aligned}
. \tag{4-86}
$$

If it is decomposed into its symmetric and skew part, we obtain

$$
\begin{aligned}
{}^t\mathbf{D} &= {}^t\mathbf{Q} \cdot {}^t\mathbf{D} \cdot {}^t\mathbf{Q}^{\mathrm{T}} \\
{}^t\mathbf{W} &= {}^t\mathbf{Q} \cdot {}^t\mathbf{W} \cdot {}^t\mathbf{Q}^{\mathrm{T}} + {}^t\dot{\mathbf{Q}} \cdot {}^t\mathbf{Q}^{\mathrm{T}}
\end{aligned}
, \tag{4-87}
$$

that means the stretching rate, ${}^t\mathbf{D}$, is objective against a GALILEI transformation but the vorticity is not.

[23] GASPARD GUSTAVE DE CORIOLIS (1792-1843)

5. Kinetics: Forces and Stresses

In the previous chapter, the geometrical description of deformation and motion of a continuum have been addressed. Both are generally caused by external forces acting on the body and giving rise to interactions between neighbouring parts of a continuum. Such interactions are studied through the concept of stress, which is discussed in the following chapter.

5.1 Body Forces and Contact Forces

Two distinct types of forces are considered in continuum mechanics, forces acting on the volume, *body forces*, and forces acting on the surface, *contact* (*surface*) *forces*, both resulting from densities. The total force acting on a material body $\mathcal{B}$ occupying a configuration ${}^t\mathbb{B}$ of volume ${}^tV$ at time t is

$$ {}^t\mathbf{f}(\mathcal{B}) = {}^t\mathbf{f}_\mathrm{b}(\mathcal{B}) + {}^t\mathbf{f}_\mathrm{c}(\mathcal{B}) \quad . \tag{5-1} $$

All vector fields are assumed to be *objective*,

$$ {}^t\tilde{\mathbf{f}} = \mathbf{Q}\cdot{}^t\mathbf{f} \quad . \tag{5-2} $$

Body forces are forces that act on every element $\mathrm{d}\mathcal{B} \subset \mathcal{B}$ and hence on the entire volume of the body $\mathcal{B}$ or any part, $\mathcal{P} \subseteq \mathcal{B}$, of it. We postulate that the total body (or volume) force can be expressed in the form,

$$ {}^t\mathbf{f}_\mathrm{b}(\mathcal{P}) = \int\limits_{{}^tV(\mathcal{P})} {}^t\rho\, {}^t\mathbf{b}\, dV \quad , \quad \mathcal{P} \subseteq \mathcal{B} \ , \tag{5-3} $$

where ${}^t\rho = \rho({}^t\mathbf{x},t)$ is the mass density at a point $\mathcal{X} \in \mathcal{P} \subseteq \mathcal{B}$ and at time t and ${}^t\mathbf{b} = \mathbf{b}({}^t\mathbf{x},t)$ is a vector with the physical dimension force per unit mass, which is referred to as *body force density*. Gravitational force, $\mathbf{b} = -g\mathbf{e}_z$, with $g \approx 9.81\ \mathrm{ms}^{-2}$ being the gravitational constant, is an example of a body force.

Contact forces act on the surface of a material body. This surface may be either a part or the whole of the boundary surface, $\partial\mathcal{B}$, or any (imaginary) surface, $\partial\mathcal{P}$, of a part, $\mathcal{P} \subseteq \mathcal{B}$. We postulate that the total surface force can be expressed in the form

$$ {}^t\mathbf{f}_\mathrm{c}(\mathcal{P}) = \int\limits_{{}^tA(\partial\mathcal{P})} {}^t\mathbf{t}_n\, dA \quad , \quad \mathcal{P} \subseteq \mathcal{B} \ , \tag{5-4} $$

where ${}^t\mathbf{t}_\mathrm{n} = \mathbf{t}({}^t\mathbf{x},{}^t\mathbf{n},t)$ is a vector with the physical dimension force per unit area, which is referred to as *surface force density* or *stress vector* or *traction*. It depends on the locus and the orientation of the surface element, which we describe by the position vector, ${}^t\mathbf{x}$, and the exterior normal to the surface, ${}^t\mathbf{n}$, respectively. This is known as CAUCHY's stress principle.

If the surface is a part or the whole of the boundary surface, $\partial\mathcal{B}$, surface forces are *external forces* that act on the boundary surface of the body. Wind forces and forces exerted by a liquid on a solid immersed in it are examples of such surface forces. If the surface is any (imaginary internal) surface, $\partial\mathcal{P}$, of a part, $\mathcal{P} \subseteq \mathcal{B}$, contact forces are *internal forces* that arise from the action of one part, $\mathcal{P}_1$, of the body upon an adjacent part, $\mathcal{P}_2$, across the respective interface, see Fig. 5.1. For example, if we consider a heavy rod suspended vertically and visualise a horizontal cross section separating the rod into upper and lower parts, the weight of the lower part of the rod acts as a surface force on the upper part across the cross section.

Looking at the two parts, $\mathcal{P}_1$ and $\mathcal{P}_2$, generated by the imaginary cutting, the respective exterior normals of every surface element at the interface are opposite to each other, $\mathbf{n}_1 + \mathbf{n}_2 = \mathbf{0}$. According to NEWTON's third law of motion, see chapter 6, it is postulated that the corresponding surface tractions are of equal magnitude but opposite orientation,

$$ {}^t\mathbf{t}_1 = \mathbf{t}({}^t\mathbf{x}, {}^t\mathbf{n}, t) = -{}^t\mathbf{t}_2 = -\mathbf{t}({}^t\mathbf{x}, -{}^t\mathbf{n}, t) \ . \tag{5-5} $$

This relation is known as CAUCHY'*s reciprocal relation*. The internal forces across surfaces in the interior of the volume balance each other out so that their resultant is zero.

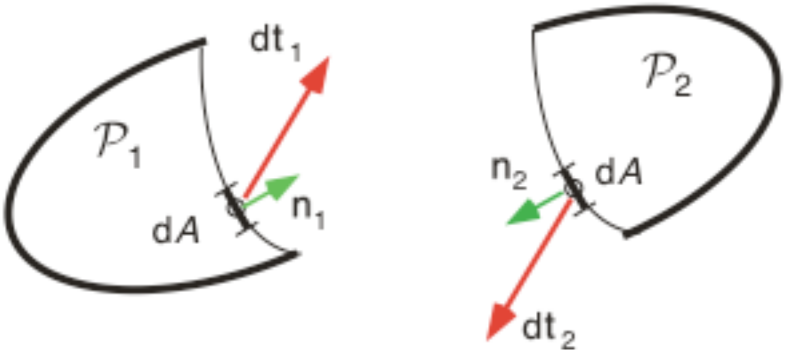

Figure 5.1: Section principle: Contact forces acting on corresponding sectional surfaces of a body, virtually cut into two parts, $\mathcal{P}_1 \cup \mathcal{P}_2 = \mathcal{B}$.

5.2 CAUCHY's Stress Tensor

In order to specify the dependence of the stress vector, $\mathbf{t}_\mathrm{n}$, on the normal, $\mathbf{n}$, we consider an infinitesimal tetrahedron with surfaces, dA_1, dA_2, dA_3, dA_n, and their respective exterior normals, $\mathbf{n}_i = -\mathbf{e}_i$ (i = 1, 2, 3), $\mathbf{n}$, see Fig. 5.2. As the tetrahedron has a closed surface,

$$ \mathbf{n}\, dA_\mathrm{n} + \sum_{i=1}^{3} \mathbf{n}_i\, dA_i = \mathbf{n}\, dA_\mathrm{n} - \sum_{i=1}^{3} \mathbf{e}_i\, dA_i = \mathbf{o} \ . \tag{5-6} $$

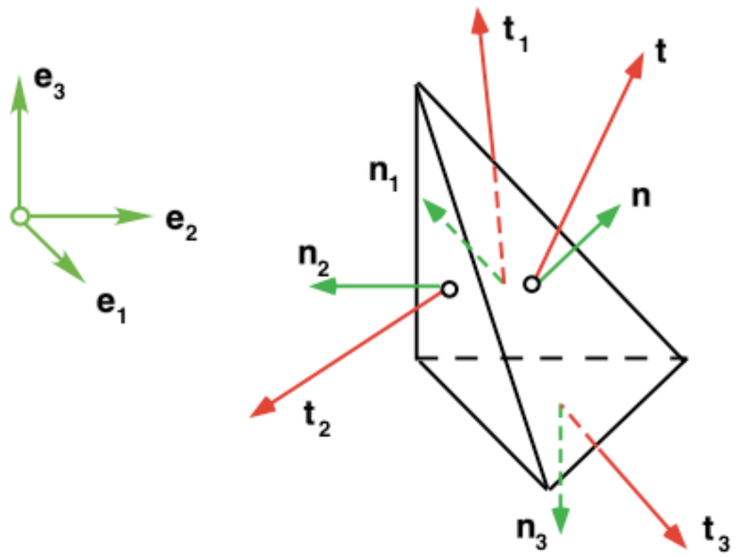

Figure 5.2: Stresses acting on the faces of an infinitesimal tetrahedron

We now apply the balance of linear momentum (or NEWTON's second law of motion, see chapter 6) in the limit of $dV \to 0$, then the volume integrals, i.e. body forces and mass accelerations, are small of higher order compared to the surface integrals, that is

$$\sum_{i=1}^{3} \mathbf{t}_i \, dA_i + \mathbf{t}(\mathbf{n}) \, dA_\mathrm{n} = \sum_{i=1}^{3} \mathbf{t}(-\mathbf{e}_i) \, dA_i + \mathbf{t}(\mathbf{n}) \, dA_\mathrm{n} = \mathbf{o} \ . \tag{5-7}$$

As by eq. (5-6), the fourth normal, $\mathbf{n}$, can be expressed by the first three, $\mathbf{n}_i = -\mathbf{e}_i$, we obtain

$$\mathbf{t}(\mathbf{n}) \, dA_\mathrm{n} = \mathbf{t}\left(\sum_{i=1}^{3} \mathbf{e}_i \frac{dA_i}{dA_\mathrm{n}} \right) = -\sum_{i=1}^{3} \mathbf{t}(-\mathbf{e}_i) \frac{dA_i}{dA_\mathrm{n}} \ . \tag{5-8}$$

Thus, $\mathbf{t}_\mathrm{n}$ is linearly dependent on $\mathbf{n}$.

Theorem of CAUCHY (1823):

The stress vector ${}^t\mathbf{t}_n = \mathbf{t}({}^t\mathbf{x}, \mathbf{n}, t)$ in a point ${}^t\mathbf{x}$ of a body depends linearly on the normal ${}^t\mathbf{n}$ of the surface element, i.e., there exists a tensor field, ${}^t\mathbf{S} = \mathbf{S}({}^t\mathbf{x}, t)$, such that

$${}^t\mathbf{t}_\mathrm{n} = {}^t\mathbf{n} \cdot {}^t\mathbf{S} \ . \tag{5-9}$$

The tensor ${}^t\mathbf{S} = {}^t\sigma_{ij} \, \mathbf{e}_i \mathbf{e}_j$ is called CAUCHY 's *stress tensor*. As a particular case, we obtain the *reaction principle* (NEWTON's[24] third law of motion), eq. (5-5). The stress tensor has the components,

$${}^t\sigma_{ij} = \mathbf{e}_i \cdot {}^t\mathbf{S} \cdot \mathbf{e}_j = {}^t\mathbf{t}_i \cdot \mathbf{e}_j = {}^t t_{ij} \ , \tag{5-10}$$

which represent the j th component of the stress vector $\mathbf{t}_i = \mathbf{t}(\mathbf{e}_i)$ acting on a surface element having $\mathbf{n} = \mathbf{e}_i$ as a unit normal, see Fig. 5.3.

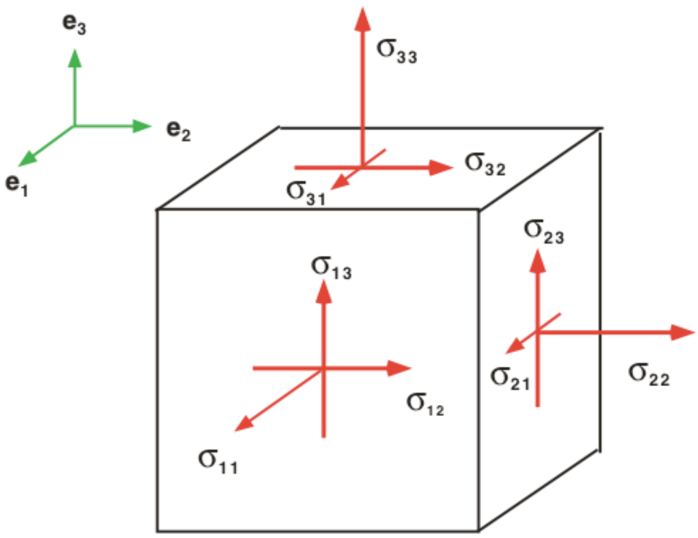

Figure 5.3: Stress state in a material point

The physical dimension of the components of the stress tensor is force per area. By definition the area is taken in the actual configuration. As a consequence, ${}^t\mathbf{S}$ is also called *true stress*

[24] ISAAC NEWTON (1643-1727)

tensor. It follows from the balance of angular momentum (see section 6.3), that ${}^t\mathbf{S}$ is *symmetric* for *non-polar media* [25],

$$ {}^t\mathbf{S} = {}^t\mathbf{S}^{\mathrm{T}} \quad , \quad {}^t\sigma_{ij} = {}^t\sigma_{ji} \ , \tag{5-11}$$

and hence

$$ {}^t\mathbf{t}_{\mathrm{n}} = {}^t\mathbf{n} \cdot {}^t\mathbf{S} = {}^t\mathbf{S} \cdot {}^t\mathbf{n} \ . \tag{5-12}$$

In a convective or material coordinate system, eq. (4-25), the CAUCHY stress tensor writes as

$$ {}^t\mathbf{S} = {}^t\sigma^{ij}\, {}^t\mathbf{g}_i\, {}^t\mathbf{g}_j \ . \tag{5-13}$$

Any stress vector on a given surface element in the current configuration, ${}^t\mathbb{B}$, can be resolved along the normal, $\mathbf{e}_{\mathrm{n}} = {}^t\mathbf{n}$, and perpendicular to it. Let $\mathbf{e}_{\mathrm{t}}$ be the unit vector perpendicular to $\mathbf{e}_{\mathrm{n}}$, i.e. $\mathbf{e}_{\mathrm{t}} \cdot \mathbf{e}_{\mathrm{n}} = 0$, then

$$ \begin{aligned} \sigma &= \sigma_{\mathrm{nn}} = {}^t\mathbf{t}_{\mathrm{n}} \cdot \mathbf{e}_{\mathrm{n}} = \mathbf{e}_{\mathrm{n}} \cdot {}^t\mathbf{S} \cdot \mathbf{e}_{\mathrm{n}} \\ \tau &= \sigma_{\mathrm{nt}} = {}^t\mathbf{t}_{\mathrm{n}} \cdot \mathbf{e}_{\mathrm{t}} = \mathbf{e}_{\mathrm{n}} \cdot {}^t\mathbf{S} \cdot \mathbf{e}_{\mathrm{t}} \end{aligned} \tag{5-14}$$

are called *normal* and *shear stresses*, respectively. Adopting this definition, all stress components having unequal subscripts in Fig. 5.2 are referred to as shear stresses. The stresses $\sigma_{11}, \sigma_{22}, \sigma_{33}$ in the diagonal of the stress tensor are called normal stresses. Normal stresses are called *tensile stresses*, if $\sigma > 0$, and *compressive stresses*, if $\sigma > 0$.

Since there is an infinite number of different tangent vectors, $\mathbf{e}_{\mathrm{t}}$, on one surface, the shear stress can be regarded as a vector and can be computed from the vector difference of stress vector, $\mathbf{t}_{\mathrm{n}}$, and its normal component, $\mathbf{t}_{\mathrm{n}} = \left(\mathbf{e}_{\mathrm{n}} \cdot \mathbf{S} \cdot \mathbf{e}_{\mathrm{n}}\right)\mathbf{e}_{\mathrm{n}}$. The shear stress τ is then given as the absolute value of the stress component in the tangential direction,

$$ \tau\, \mathbf{e}_{\mathrm{t}} = \mathbf{e}_{\mathrm{n}} \cdot {}^t\mathbf{S} - \left(\mathbf{e}_{\mathrm{n}} \cdot {}^t\mathbf{S} \cdot \mathbf{e}_{\mathrm{n}}\right)\mathbf{e}_{\mathrm{n}} \tag{5-15}$$

We shall now investigate whether there exists an orientation of the surface element at a given point, along which the stress vector is collinear with the normal of the element, i.e. the stress vector has a normal component only and no shear stresses appear [26],

$$ \mathbf{t}_{\mathrm{n}} = \sigma\, \mathbf{n} \ . \tag{5-16}$$

Using CAUCHY’s theorem (5-9) together with eq. (5-12), the above condition can be rewritten as

$$ \left(\mathbf{S} - \sigma\mathbf{1}\right) \cdot \mathbf{n} = \mathbf{o} \ , \tag{5-17}$$

which means that eq. (5-15) holds if and only if $\mathbf{n}$ and σ are eigenvectors and eigenvalues of $\mathbf{S}$, respectively. Since every symmetric tensor of rank two has exactly three, not necessarily distinct, principal values, $\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}, \sigma_{\mathrm{III}}$, there exist three orthogonal principal orientations, $\mathbf{n}_{\mathrm{I}}, \mathbf{n}_{\mathrm{II}}, \mathbf{n}_{\mathrm{III}}$, of stresses, and if the stress tensor is written with respect to the base system of principal axes, it is purely diagonal (spectral form),

$$ \mathbf{S} = \sum_{i=\mathrm{I}}^{\mathrm{III}} \sigma_{(i)}\, \mathbf{n}_{(i)} \mathbf{n}_{(i)} \ , \tag{5-18}$$

[25] Materials having no surface or volume distributed torques

[26] The left superscript indicating the current configuration is omitted in the following.

The principal stresses are the roots of the characteristic equation

$$\det\left(\mathbf{S}-\sigma\mathbf{1}\right)=-\sigma^3+I_{\mathrm{S}}\sigma^2-II_{\mathrm{S}}\sigma+III_{\mathrm{S}}=0 \qquad (5\text{-}19)$$

with the fundamental invariants of **S**

$$\begin{aligned} I_{\mathrm{S}} &= \sigma_{kk}=\operatorname{tr}\mathbf{S} \\ II_{\mathrm{S}} &= \tfrac{1}{2}\left(\sigma_{ii}\sigma_{jj}-\sigma_{ij}\sigma_{ji}\right)=\tfrac{1}{2}\left(\operatorname{tr}^2\mathbf{S}-\operatorname{tr}\mathbf{S}^2\right) \\ III_{\mathrm{S}} &= \det\mathbf{S} \end{aligned} \qquad (5\text{-}20)$$

Any tensor can be decomposed into a *spherical* part and a *deviatoric* part, $\hat{\mathbf{S}}$,

$$\mathbf{S}=\tfrac{1}{3}\left(\operatorname{tr}\mathbf{S}\right)\mathbf{1}+\hat{\mathbf{S}}=-p\mathbf{1}+\hat{\mathbf{S}}\ , \qquad (5\text{-}21)$$

where

$$p=-\tfrac{1}{3}\sigma_{kk}=-\sigma_{\mathrm{hyd}}\ , \qquad (5\text{-}22)$$

is the mean *pressure* or (negative) *hydrostatic stress*. Deviatoric stresses play an important role for plastic behaviour of metals.

As body forces and contact forces have been assumed to be objective with respect to a EUCLIDean transformation, eq. (5-2), and as ${}^t\tilde{\mathbf{n}}=\mathbf{Q}\cdot{}^t\mathbf{n}$, the CAUCHY stress tensor is *objective*,

$${}^t\tilde{\mathbf{S}}=\mathbf{Q}\cdot{}^t\mathbf{S}\cdot\mathbf{Q}^{\mathrm{T}}\ . \qquad (5\text{-}23)$$

As CAUCHY's law (5-9) is valid at every point in a continuum, $\mathbb{X}\in{}^t\mathbb{B}$, it holds also at points, $\mathbb{X}\in\partial^t\mathbb{B}$, on the boundary surface of a material body. Problems in mechanics commonly appear as *boundary value problems*, i.e. something is known about the forces and displacements at the boundary of a body, and one has to calculate what is happening inside the body. The formulation of *boundary conditions* is hence an essential part of establishing the mathematical problem. The interaction between two bodies is given by NEWTON's third law of motion (see chapter 6) or eq. (5-5). If the unit normal vector at the boundary is denoted by ${}^t\mathbf{n}(\mathbb{X})$, the imposed traction, ${}^t\mathbf{t}_{\mathrm{n}}(\mathbb{X})$, at a point on the boundary is

$${}^t\mathbf{n}(\mathbb{X})\cdot{}^t\mathbf{S}(\mathbb{X})={}^t\mathbf{t}_{\mathrm{n}}(\mathbb{X})\quad\text{for}\quad\mathbb{X}\in\partial\,{}^t\mathbb{B}\ , \qquad (5\text{-}24)$$

and if the boundary is traction free, ${}^t\mathbf{n}(\mathbb{X})\cdot{}^t\mathbf{S}(\mathbb{X})=\mathbf{o}$.

5.3 PIOLA-KIRCHHOFF Stresses

Beside CAUCHY stresses, many other stress tensors are in use. In the previous section, stress is understood as force, $d\,{}^t\mathbf{f}_{\mathrm{c}}$, per area, $d\,{}^tA$. of the current configuration, ${}^t\mathbb{B}$. As the current configuration arising under the acting forces is unknown, the idea of a (current) surface force on an element, $d\,{}^0A$, in the reference configuration may appear convenient. If a (material) surface element is considered, which has the area $d\,{}^0\mathbf{a}=\left(d\,{}^0A\right){}^0\mathbf{n}$ in the reference configuration and $d\,{}^t\mathbf{a}=\left(d\,{}^tA\right){}^t\mathbf{n}$ in the actual configuration, respectively, the force, $d\,{}^t\mathbf{f}_{\mathrm{c}}$, can be expressed by means of eq. (4-20) as

$$d\,{}^t\mathbf{f}_{\mathrm{c}}={}^t\mathbf{t}_{\mathrm{n}}\left(d\,{}^tA\right)={}^t\mathbf{S}^{\mathrm{T}}\cdot d\,{}^t\mathbf{a}=\left(\det{}^t_0\mathbf{F}\right){}^t\mathbf{S}^{\mathrm{T}}\cdot{}^0_t\mathbf{F}^{-\mathrm{T}}\cdot d\,{}^0\mathbf{a} \qquad (5\text{-}25)$$

and by defining

$${}^t_0\mathbf{T}=\left(\det{}^t_0\mathbf{F}\right){}^0_t\mathbf{F}^{-1}\cdot{}^t\mathbf{S} \qquad (5\text{-}26)$$

one gets the analogous expression to eq. (5-9),

$$d\,{}^t\mathbf{f}_\mathrm{c} = {}^t_0\mathbf{T}^\mathrm{T}\cdot d\,{}^0\mathbf{a} = d\,{}^0\mathbf{a}\cdot {}^t_0\mathbf{T} \ . \tag{5-27}$$

The unsymmetric tensor ${}^t_0\mathbf{T}$ is called *first* PIOLA-KIRCHHOFF[27] *tensor* [28]. It is also referred to as *nominal* or *engineering stress tensor*, since actual forces are related to the initial (undeformed) configuration as in standard tensile testing.

If we also pull the stress vector back into the reference configuration

$${}^0_t\mathbf{F}^{-1}\cdot d\,{}^t\mathbf{f}_\mathrm{c} = {}^0_t\mathbf{F}^{-1}\,{}^t_0\mathbf{T}^\mathrm{T}\cdot d\,{}^0\mathbf{a} = {}^0\mathbf{T}^\mathrm{T}\cdot d\,{}^0\mathbf{a} = d\,{}^0\mathbf{a}\cdot {}^0\mathbf{T} \ , \tag{5-28}$$

we obtain the symmetric *second* PIOLA-KIRCHHOFF *tensor,*

$${}^0\mathbf{T} = J\,{}^0_t\mathbf{F}^{-1}\cdot{}^t\mathbf{S}\cdot{}^0_t\mathbf{F}^{-\mathrm{T}} = {}^0\mathbf{T}^\mathrm{T} \ . \tag{5-29}$$

Inversely, CAUCHY stresses result from

$${}^t\mathbf{S} = \frac{1}{J}\,{}^t_0\mathbf{F}\cdot{}^0\mathbf{T}\cdot{}^t_0\mathbf{F}^\mathrm{T} \,. \tag{5-30}$$

First and second PIOLA-KIRCHHOFF tensors write as

$${}^t_0\mathbf{T} = {}^0\sigma^{ij}\,{}^0\mathbf{g}_i\,{}^t\mathbf{g}_j \quad , \quad {}^0\mathbf{T} = {}^0\sigma^{ij}\,{}^0\mathbf{g}_i\,{}^0\mathbf{g}_j \tag{5-31}$$

in a material coordinate system, eq. (4-25). The components of the 1st PIOLA-KIRCHHOFF tensor in the mixed base $\left({}^0\mathbf{g}_i\,{}^t\mathbf{g}_j\right)$ equal the components of the 2nd in the base of the reference configuration, $\left({}^0\mathbf{g}_i\,{}^0\mathbf{g}_j\right)$, and according to eq. (4-22), they are related to the components of the CAUCHY tensor by

$${}^0\sigma^{ij} = \frac{1}{J}\,{}^t\sigma^{ij} = \frac{{}^t\rho}{{}^0\rho}\,{}^t\sigma^{ij} \ . \tag{5-32}$$

There are many other stress tensors. All of them have different properties, and it depends on the specific application which one is preferable. If the deformation gradient is known, all of them can be uniquely determined from CAUCHY stresses. In the *theory of small deformations*, all stress tensors *coincide*, as no difference between the actual and the reference configuration has to be made. The stress tensor will be simply denoted by **S** in this case.

With respect to constitutive equations (chapter 7), the appropriate combination of stress and strain measures is a key issue. A stress tensor, $\mathbf{T}^{(*)}$, is called *work conjugate* to some strain tensor, $\mathbf{E}^{(*)}$, if the *stress power density* in the reference configuration is

$${}^0\dot{w}^\mathrm{in} = \mathbf{T}^{(*)}\cdot\cdot\,\dot{\mathbf{E}}^{(*)} = J\,{}^t\mathbf{S}\cdot\cdot\,{}^t\mathbf{D}\,, \tag{5-33}$$

see eq. (6-19). According to this definition, ${}^0\mathbf{T}$ is work conjugate to ${}^t_0\mathbf{E}^{(\mathrm{G})}$. For CAUCHY's stress tensor there is no work conjugate strain measure, see eq. (4-68), where ${}^t\mathbf{D}$ was introduced as OLDROYD's time derivative of ALMANSI's strain tensor, ${}^0_t\mathbf{E}^{(\mathrm{A})}$.

The EUCLIDean transformation of motion, eq. (4-71), can be applied to all stress tensors, and by the objectivity of CAUCHY stresses, eq. (5-23), it follows,

---

[27] GUSTAV ROBERT KIRCHHOFF (1824-1877)

[28] Sometimes, ${}^t_0\mathbf{T}^\mathrm{T}$ is termed as 1st PIOLA-KIRCHHOFF stress tensor. Due to the agreement of eq. (5-10), that the first subscript of the stress components denotes the orientation of the surface element, the present definition is preferred.

$$
\begin{aligned}
{}_0^t\tilde{\mathbf{T}} &= \mathbf{Q}\cdot{}_0^t\mathbf{T} \\
{}^0\tilde{\mathbf{T}} &= {}^0\mathbf{T}
\end{aligned}\quad , \tag{5-34}
$$

i.e. the 1st PIOLA-KIRCHHOFF tensor is neither objective nor invariant, the 2nd PIOLA-KIRCHHOFF tensor is invariant.

The observer dependence of all kinematical and dynamical quantities is now completely determined.

5.4 Plane Stress State

For some applications, in particular if analytical solutions are looked for, it may be regarded unnecessary to account for all three components of the stress vector. In sheet materials or under in-plane loading conditions, all stress vectors can be assumed to lie in one plane, and the stress tensor in a Cartesian coordinate system, $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, becomes [29],

$$
\mathbf{S} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{e}_i \mathbf{e}_j \tag{5-35}
$$

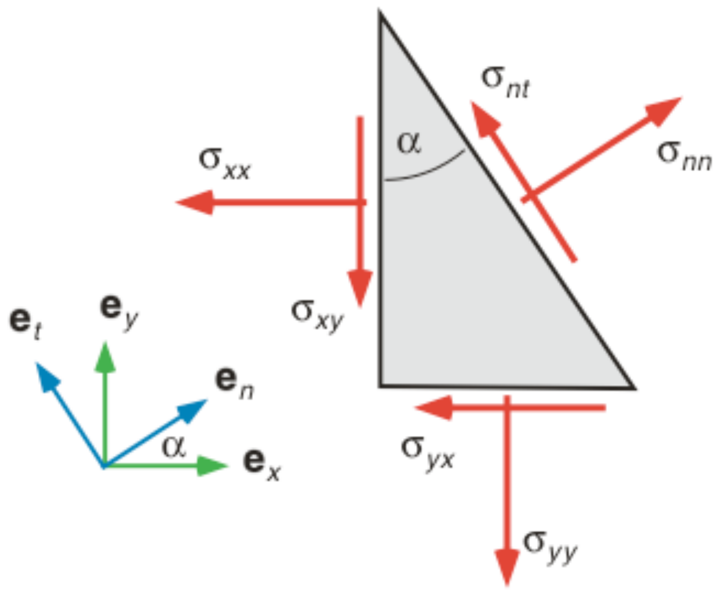

Figure 5.4: Plane stress state at a triangular volume element

From Fig. 5.4 or by applying the transformation rules for tensor components given in the Appendix, one gets the normal stress, σ, and shear stress, τ, as a function of the angle α,

$$
\begin{aligned}
\sigma &= \sigma_{\mathrm{nn}} = \tfrac{1}{2}(\sigma_{xx}+\sigma_{yy}) + \tfrac{1}{2}(\sigma_{xx}-\sigma_{yy})\cos 2\alpha + \sigma_{xy}\sin 2\alpha \\
\tau &= \sigma_{\mathrm{nt}} = \sigma_{xy}\cos 2\alpha - \tfrac{1}{2}(\sigma_{xx}-\sigma_{yy})\sin 2\alpha
\end{aligned} \tag{5-36}
$$

Squaring both equations and adding them leads to

[29] Again, the left superscript indicating the current configuration is omitted in the following.

$$\left[\sigma-\tfrac{1}{2}\left(\sigma_{xx}+\sigma_{yy}\right)\right]^2+\tau^2=\left[\tfrac{1}{2}\left(\sigma_{xx}-\sigma_{yy}\right)\right]^2+\sigma_{xy}^2\ , \tag{5-37}$$

which is the equation of a circle in a $\left(\sigma,\tau\right)$ coordinate system, having its centre located in $\left\{\tfrac{1}{2}\left(\sigma_{xx}+\sigma_{yy}\right),0\right\}$ and a radius of $\sqrt{\left[\tfrac{1}{2}\left(\sigma_{xx}-\sigma_{yy}\right)\right]^2+\sigma_{xy}^2}$, see Fig 5.5.

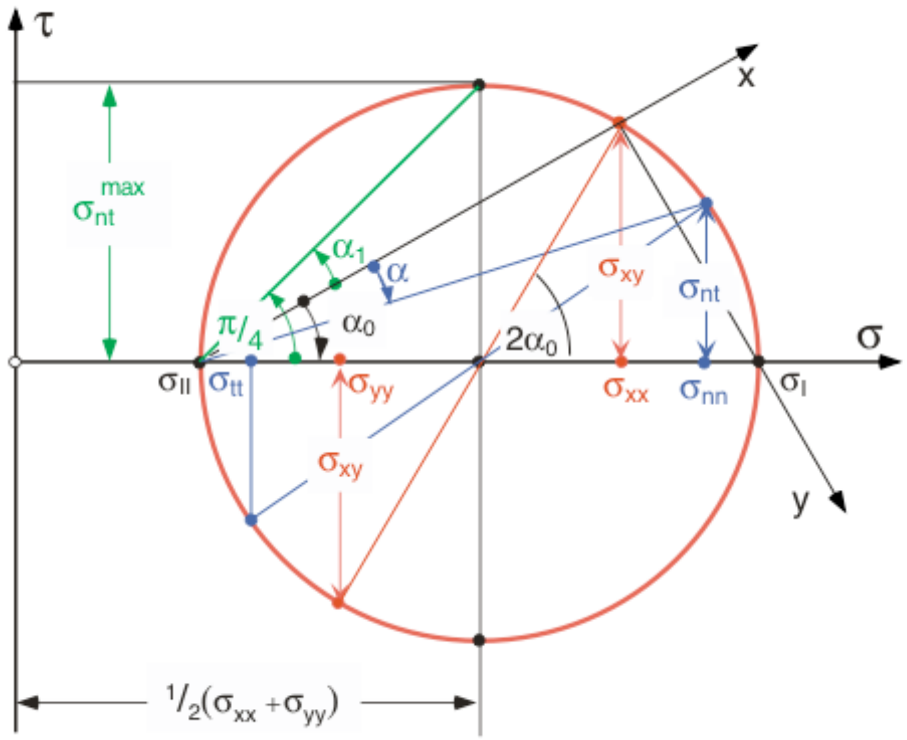

Figure 5.5: MOHR's circle in plane stress state

This circle, which is referred to as MOHR*'s circle* [30], is the locus of the components of all possible stress vectors in a material point $\mathcal{X}$, acting on area elements under varying orientation. The stress components in the actual coordinate system, $\left(\sigma_{xx},\sigma_{yy},\sigma_{xy}\right)$, define centre and radius of the circle. The components $\left(\sigma_{nn},\sigma_{tt},\sigma_{nt}\right)$ in any rotated system are obtained by a clockwise rotation of the (x, y) axes by the angle α.

Though at the present time of computers, the transformation can be made much more easily numerically instead of graphically, MOHR's circle still provides some demonstrative understanding of (plane) stress states. It shows three particular loci of interest.

➢ *Shear stresses vanish* when *normal stresses take extrema*,

$$\sigma_{\mathrm{I,II}}=\tfrac{1}{2}(\sigma_{xx}+\sigma_{yy})\pm\sqrt{\left[\tfrac{1}{2}(\sigma_{xx}-\sigma_{yy})\right]^2+\sigma_{xy}^2}\ , \tag{5-38}$$

under an angle of

$$\tan 2\alpha_0=\frac{2\sigma_{xy}}{\sigma_{xx}-\sigma_{yy}}\ . \tag{5-39}$$

[30] OTTO MOHR (1835-1918)

➢ *Shear stress becomes maximum* when the two corresponding normal stresses become equal,

$$\begin{aligned}\tau_{\max} &= \sqrt{\left[\tfrac{1}{2}\left(\sigma_{xx}-\sigma_{yy}\right)\right]^2+\sigma_{xy}^2} = \tfrac{1}{2}\left(\sigma_{\mathrm{I}}-\sigma_{\mathrm{II}}\right),\\ \sigma_{\mathrm{nn}} &= \sigma_{\mathrm{tt}} = \tfrac{1}{2}\left(\sigma_{xx}+\sigma_{yy}\right) = \tfrac{1}{2}\left(\sigma_{\mathrm{I}}+\sigma_{\mathrm{II}}\right) = \tfrac{1}{2}\operatorname{tr}\mathbf{S},\end{aligned} \tag{5-40}$$

under an angle of

$$\alpha_1 = \frac{\pi}{4} \mp \alpha_0 \ . \tag{5-41}$$

Fig. 5.6 shows the stress components in various orientations of the volume element.

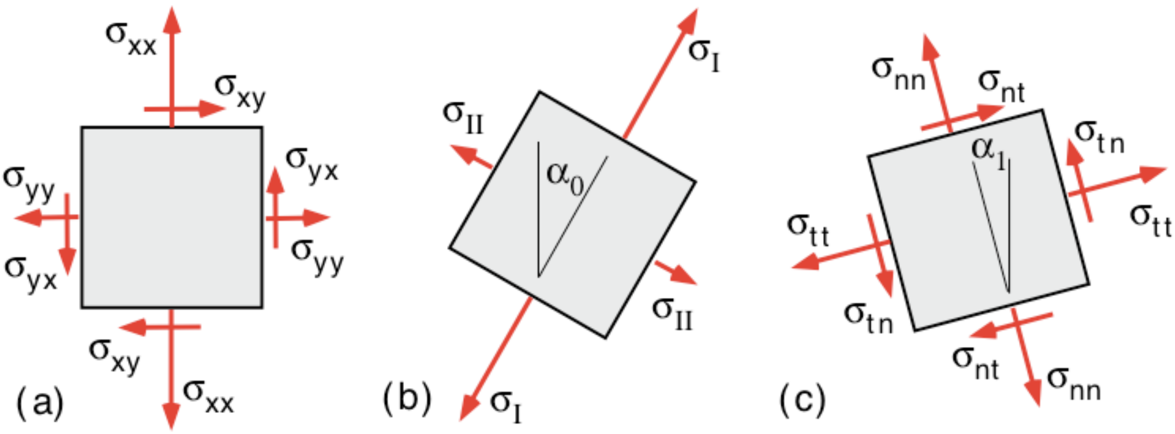

Figure 5.6: Plane stress state in different coordinate systems: (a) original (x,y), (b) principal axes, σ_{I}, σ_{II}, eqs. (5-38), (5-39), (c) direction of maximum shear stress, $\sigma_{\mathrm{nt}} = \sigma_{\mathrm{tn}} = \tau_{\max} = \tfrac{1}{2}\left(\sigma_{\mathrm{I}}-\sigma_{\mathrm{II}}\right)$, $\sigma_{\mathrm{nn}} = \sigma_{\mathrm{tt}} = \tfrac{1}{2}\left(\sigma_{\mathrm{I}}+\sigma_{\mathrm{II}}\right)$, eqs. (5-40), (5-41)

The *extrema of normal stresses*, σ_{I}, σ_{II}, are the *principal stresses* as resulting from eq. (5-19) or from either of the conditions

$$\frac{\partial\sigma(\alpha)}{\partial\alpha} = 0 \quad \text{or} \quad \tau(\alpha) = 0 \ . \tag{5-42}$$

Principal stresses, maximum shear stress as well as $\operatorname{tr}\mathbf{S} = \sigma_{xx}+\sigma_{xx} = \sigma_{\mathrm{I}}+\sigma_{\mathrm{II}}$ are invariants of the stress tensor.

5.5 *Stress Rates*

For constitutive relations which are based in incremental formulations, stress rates are required. The 2nd PIOLA-KIRCHHOFF tensor, ${}^t_0\mathbf{T}$, is a material stress tensor and hence invariant under a EUKLIDean transformation, eq. (5-29), and so is its material time derivative, ${}^t_0\dot{\mathbf{T}}$, which is simply the derivative with respect to time, see eq. (4-53). The same holds for every material stress tensor, e.g.

$${}^t_0\mathbf{S} = {}^0_t\mathbf{F}^{-1}\cdot{}^t\mathbf{S}\cdot{}^0_t\mathbf{F}^{-\mathrm{T}} = J^{-1}\,{}^t_0\mathbf{T} \ . \tag{5-43}$$

Its time rate is

$$
\begin{aligned}
{}_0^t\dot{\mathbf{S}} &= \left({}_t^0\mathbf{F}^{-1} \cdot {}^t\mathbf{S} \cdot {}_t^0\mathbf{F}^{-\mathrm{T}} \right)^{\bullet} = {}_t^0\mathbf{F}^{-1} \cdot {}^t\dot{\mathbf{S}} \cdot {}_t^0\mathbf{F}^{-\mathrm{T}} + {}_t^0\dot{\mathbf{F}}^{-1} \cdot {}^t\mathbf{S} \cdot {}_t^0\mathbf{F}^{-\mathrm{T}} + {}_t^0\mathbf{F}^{-1} \cdot {}^t\mathbf{S} \cdot {}_t^0\dot{\mathbf{F}}^{-\mathrm{T}} \\
&= {}_t^0\mathbf{F}^{-1} \cdot \left({}^t\dot{\mathbf{S}} - {}^t\mathbf{L} \cdot {}^t\mathbf{S} - {}^t\mathbf{S} \cdot {}^t\mathbf{L}^{\mathrm{T}} \right) \cdot {}_t^0\mathbf{F}^{-\mathrm{T}}
\end{aligned}
. \qquad (5\text{-}44)
$$

The term in parenthesis is OLDROYD's rate of CAUCHY's stress tensor,

$$
{}^t\overset{\circ}{\mathbf{S}} = {}^t\dot{\mathbf{S}} - {}^t\mathbf{L} \cdot {}^t\mathbf{S} - {}^t\mathbf{S} \cdot {}^t\mathbf{L}^{\mathrm{T}} . \qquad (5\text{-}45)
$$

It is an *objective derivative*, i.e. it transforms as an objective tensor under a change of observer, just as the JAUMANN rate,

$$
{}^t\mathbf{S}^{\nabla} = {}^t\dot{\mathbf{S}} + {}^t\mathbf{S} \cdot {}^t\mathbf{W} - {}^t\mathbf{W} \cdot {}^t\mathbf{S} , \qquad (5\text{-}46)
$$

which is frequently used for rate dependent formulations in finite element programmes. Because of the skew term, ${}^t\mathbf{W}$, which stands for angular velocities, it is also called *corotational rate*.

6. Fundamental Laws of Continuum Mechanics

Continuum mechanics is essentially based on fundamental principles having the character of axioms, that are laws, which cannot be proven but are commonly assumed to be true. In his "Philosophiae Naturalis Principia Mathematica" of 1687, ISAAC NEWTON established *laws of motion* und thus was the first relating kinematical and dynamic quantities. Without applying mathematical equations at that time, they read as follows:

Lex I

[Constat] corpus omne perseverare in statu suo quiscendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare.

Lex II

[Constat] mutationem motus proportionalem esse vi motrici impressae, et fieri secundum lineam rectam qua vis illa imprimitur.

Lex III

[Constat] actioni contrariam semper et aequalem esse reactionem: sive corporum duorum actiones in se mutuo semper esse aequales et in partes contrarias dirigi.

Translated:

Law I

[It is established that] Each body remains in its state of rest or motion uniform in direction until it is made to change this state by imposed forces.

Law II

[It is established that] The change of motion is proportional to the imposed driving force and occurs along a straight line in which the force acts.

Law III

[It is established that] To every action there is always an equal reaction: or the mutual interactions of two bodies are always equal but directed contrary.

Brought into a more modern and mathematical form, they still constitute the fundament of NEWTONian mechanics. In particular, the first one states that the laws of motion are only valid for an *inertial reference frame* (see section 4.6). The second law is the balance of (linear) momentum and the third is the reaction principle of eq. (5.5).

6.1 General Balance Equation

The axioms of continuum mechanics are formulated as *balance laws* or *conservation laws* for mechanical quantities like mass, momentum, energy etc. and established as equations of integrals over a material body (global form), or as *field equations* (local form). They are material independent and applicable to all continua, regardless of their internal physical structure. These fundamental equations will be formulated in their spatial (EULERean) form, which means that all field quantities are expressed as functions of the coordinates of a material point in the actual configuration. As time derivatives of the integrals will be required, they are transformed to the reference configuration ((LAGRANGEan form), differentiated under the integral and transformed back to the EULERean description, again. time derivatives of integrals over material volumes or surfaces in the actual configuration. By this procedure of exchanging the integration and differentiation process, the local field equation is obtained from a global balance law.

Let $\Phi(\mathcal{P},t)$ be a global physical quantity (scalar, tensor- or vector-valued) defined for a material body $\mathcal{B}$ or any part $\mathcal{P} \subseteq \mathcal{B}$,, having a density (per unit volume), ${}^{t}\phi({}^{t}\mathbf{x},t)$, as a field of same order in the current configuration, and ${}^{0}\phi({}^{0}\mathbf{x})$ in the reference configuration, then by means of eq. (4-20).

$$\Phi(\mathcal{P},t) = \int_{{}^{t}V(\mathcal{P})} {}^{t}\phi \, d\,{}^{t}V = \int_{{}^{0}V(\mathcal{P})} {}^{t}\phi \, J \, d\,{}^{0}V = \int_{{}^{0}V(\mathcal{P})} {}^{0}\phi \, d\,{}^{0}V \quad , \quad \mathcal{P} \subseteq \mathcal{B} . \tag{6-1}$$

Conservation of mass, see section 6.2 and eq. (4-22), finally results in,

$$ {}^{0}\phi = J\,{}^{t}\phi = \frac{{}^{0}\rho}{{}^{t}\rho}\,{}^{t}_{0}\phi({}^{0}\mathbf{x},t) \ . \tag{6-2}$$

i.e. the densities, ${}^{t}\phi$, ${}^{0}\phi$, transform like the mass densities. The time derivative is

$$\begin{aligned} \dot{\Phi}(\mathcal{P},t) &= \frac{d}{dt} \int_{{}^{0}V(\mathcal{P})} {}^{0}\phi \, d\,{}^{0}V = \int_{{}^{0}V(\mathcal{P})} \left({}^{t}\dot{\phi}\, J + {}^{t}\phi\, \dot{J} \right) d\,{}^{0}V \\ &= \frac{d}{dt} \int_{{}^{t}V(\mathcal{P})} {}^{t}\phi \, d\,{}^{t}V = \int_{{}^{t}V(\mathcal{P})} \left[{}^{t}\dot{\phi}\, d\,{}^{t}V + {}^{t}\phi \left(d\,{}^{t}V \right)^{\cdot} \right] \end{aligned} , \tag{6-3}$$

and with eqs. (4-63) and (4-57)

$$\frac{d}{dt} \int_{{}^{t}V(\mathcal{P})} {}^{t}\phi \, d\,{}^{t}V = \int_{{}^{t}V(\mathcal{P})} \left({}^{t}\dot{\phi} + {}^{t}\phi \ {}^{t}\nabla \cdot {}^{t}\mathbf{v} \right) d\,{}^{t}V = \int_{{}^{t}V(\mathcal{P})} \left[\frac{\partial\,{}^{t}\phi}{\partial t} + {}^{t}\nabla \cdot \left({}^{t}\phi\,{}^{t}\mathbf{v} \right) \right] d\,{}^{t}V \ , \tag{6-4}$$

By means of the *divergence theorem* of GAUSS[31], REYNOLDS[32] *transport equation* is obtained,

$$\frac{d}{dt} \int_{{}^{t}V(\mathcal{P})} {}^{t}\phi \, d\,{}^{t}V = \int_{{}^{t}V(\mathcal{P})} \frac{\partial\,{}^{t}\phi}{\partial t} d\,{}^{t}V + \int_{{}^{t}A(\partial\mathcal{P})} {}^{t}\phi\,{}^{t}\mathbf{v} \cdot \mathbf{n} \, d\,{}^{t}A \ . \tag{6-5}$$

Note, that the domain of the integral is fixed to the body. If it is an arbitrarily moving control volume as in fluid mechanics, a similarly looking equation is obtained with ${}^{t}\mathbf{v}$ being the velocity of the control volume in space.

6.2 Conservation of Mass

The mass *m* of a continuum remains unchanged during the motion of a material body $\mathcal{B}$ as well as for any part, $\mathcal{P} \subseteq \mathcal{B}$, of it

$$m(\mathcal{P}) = \int_{{}^{0}V(\mathcal{P})} {}^{0}\rho({}^{0}\mathbf{x}) \, d\,{}^{0}V = \int_{{}^{t}V(\mathcal{P})} {}^{t}\rho({}^{t}\mathbf{x}) \, d\,{}^{t}V = const \quad \text{for} \quad \mathcal{P} \subseteq \mathcal{B} \ , \tag{6-6}$$

with ${}^{0}\rho$ and ${}^{t}\rho$ denoting the mass density in the reference configuration and actual configuration, respectively. The material time derivative of *m* has to vanish and by applying the eq. (6-4),

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[31] KARL FRIEDRICH GAUSS (1777-1855)
[32] OSBORNE REYNOLDS (1842-1912)

$$\dot{m}(\mathcal{P}) = \frac{d}{dt}\int\limits_{{}^tV(\mathcal{P})} {}^t\rho({}^t\mathbf{x})\, d\,{}^tV = \int\limits_{{}^tV(\mathcal{P})} \left(\frac{d\,{}^t\rho}{dt} + {}^t\rho\,{}^t\nabla \cdot {}^t\mathbf{v} \right) d\,{}^tV = 0\,, \tag{6-7}$$

we derive the local field equation as eq. (6-7) has to hold for an arbitrary volume $V(\mathcal{P})$,

$$\frac{d\,{}^t\rho}{dt} + {}^t\rho\,{}^t\nabla \cdot {}^t\mathbf{v} = \frac{\partial\,{}^t\rho}{\partial t} + {}^t\nabla \cdot \left({}^t\rho\,{}^t\mathbf{v} \right) = 0 \ . \tag{6-8}$$

In fluid mechanics, the respective equation is known as *continuity equation*.

6.3 Balance of Linear and Angular Momentum

The linear momentum of a material body $\mathcal{B}$ or any part $\mathcal{P} \subseteq \mathcal{B}$ is introduced as

$$ {}^t\mathbf{p}(\mathcal{P}) = \int\limits_{{}^tV(\mathcal{P})} {}^t\mathbf{v}\,{}^t\rho\, d\,{}^tV = \int\limits_{{}^tV(\mathcal{P})} {}^t\dot{\mathbf{x}}\,{}^t\rho\, d\,{}^tV = m(\mathcal{P})\,{}^t\dot{\mathbf{x}}_{\mathrm{M}} \qquad , \quad \mathcal{P} \subseteq \mathcal{B}\,, \tag{6-9}$$

where

$$ {}^t\mathbf{x}_{\mathrm{M}} = \frac{1}{m(\mathcal{P})} \int\limits_{{}^tV(\mathcal{P})} {}^t\mathbf{x}\,{}^t\rho\, d\,{}^tV \tag{6-10}$$

is the *centre of mass* of $\mathcal{P} \subseteq \mathcal{B}$. According to NEWTON's second law of motion, it is now postulated that the material time rate of the linear momentum ("*mutationem motus*") is equal to the total force ${}^t\mathbf{f}(\mathcal{P})$ acting on the body, see eq. (5-1):

$$ {}^t\dot{\mathbf{p}}(\mathcal{P}) = m(\mathcal{P})\,{}^t\ddot{\mathbf{x}}_{\mathrm{M}} = {}^t\mathbf{f}(\mathcal{P}) = {}^t\mathbf{f}_{\mathrm{c}}(\mathcal{P}) + {}^t\mathbf{f}_{\mathrm{b}}(\mathcal{P})\,, \tag{6-10}$$

assuming the proportionality constant without loss of generality as 1, by which the physical dimension of forces is defined as [mass·length/time^2] with the unit 1 N = 1 kg m s^{-2}, in honour of NEWTON. Introducing the density functions of body forces and contact forces, eqs. (5-3) and (5-4), it follows

$$\frac{d}{dt}\int\limits_{{}^tV(\mathcal{P})} {}^t\mathbf{v}\,{}^t\rho\, d\,{}^tV = \int\limits_{{}^tV(\mathcal{P})} {}^t\mathbf{b}\,{}^t\rho\, d\,{}^tV + \int\limits_{{}^tA(\partial\mathcal{P})} {}^t\mathbf{t}_{\mathrm{n}}\, d\,{}^tA \ . \tag{6-11}$$

Introducing CAUCHY's law (5-9), conservation of mass (6-7) and the divergence theorem as above in eq. (6-5), we obtain

$$\int\limits_{{}^tV(\mathcal{P})} \left({}^t\rho \frac{d\,{}^t\mathbf{v}}{dt} - {}^t\rho\,{}^t\mathbf{b} - {}^t\boldsymbol{\nabla} \cdot {}^t\mathbf{S} \right) d\,{}^tV = 0 \ , \tag{6-12}$$

which finally leads to CAUCHY*'s equation of motion* as the local form of the balance of linear momentum

$$ {}^t\boldsymbol{\nabla} \cdot {}^t\mathbf{S} + {}^t\rho\,{}^t\mathbf{b} = {}^t\rho\,{}^t\ddot{\mathbf{x}} \qquad \text{or} \qquad {}^t\sigma_{ij,i} + {}^t\rho\,{}^tb_j = {}^t\rho\,{}^t\ddot{x}_j \tag{6-13}$$

in tensor or index notation, respectively. In *static equilibrium*, the acceleration term on the right hand side vanishes, so that the *equilibrium equations* write as ${}^t\boldsymbol{\nabla} \cdot {}^t\mathbf{S} + {}^t\rho\,{}^t\mathbf{b} = \mathbf{o}$.

The *angular momentum* is defined as the moment of momentum with respect to the origin $\mathbb{O}$,

$$ {}^t\mathbf{d}^{(\mathbb{O})}(\mathcal{P}) = \int\limits_{{}^tV(\mathcal{P})} {}^t\mathbf{x} \times {}^t\mathbf{v}\, {}^t\rho \, d\,{}^tV \qquad , \quad \mathcal{P} \subseteq \mathcal{B}, \tag{6-14} $$

and the balance law requires, that the time rate of angular momentum equals the moment of external forces [33] acting on $\mathcal{P}$ with respect to the origin $\mathbb{O}$,

$$ {}^t\dot{\mathbf{d}}^{(\mathbb{O})}(\mathcal{P}) = \mathbf{m}^{(\mathbb{O})} = \int\limits_{{}^tV(\mathcal{P})} {}^t\mathbf{x} \times {}^t\mathbf{b}\, {}^t\rho \, d\,{}^tV + \int\limits_{{}^tA(\partial\mathcal{P})} {}^t\mathbf{x} \times {}^t\mathbf{t}_\mathrm{n} \, d\,{}^tA, \tag{6-15} $$

Conservation of mass yields

$$ {}^t\dot{\mathbf{d}}^{(\mathbb{O})}(\mathcal{P}) = \int\limits_{{}^tV(\mathcal{P})} {}^t\mathbf{x} \times {}^t\ddot{\mathbf{x}}\, {}^t\rho \, d\,{}^tV \;, \tag{6-16} $$

and CAUCHY's law (5-9) with divergence theorem,

$$ \begin{aligned} \int\limits_{{}^tA(\partial\mathcal{P})} {}^t\mathbf{x} \times {}^t\mathbf{t}_\mathrm{n} \, d\,{}^tA &= \int\limits_{{}^tA(\partial\mathcal{P})} {}^t\mathbf{x} \times \left({}^t\mathbf{n} \cdot {}^t\mathbf{S} \right) d\,{}^tA = \int\limits_{{}^tA(\partial\mathcal{P})} \left({}^t\mathbf{x} \times {}^t\mathbf{S}^\mathrm{T} \right) \cdot {}^t\mathbf{n} \, d\,{}^tA \\ &= \int\limits_{{}^tV(\partial\mathcal{P})} {}^t\nabla \cdot \left({}^t\mathbf{x} \times {}^t\mathbf{S} \right) d\,{}^tV = \int\limits_{{}^tV(\partial\mathcal{P})} \left[2\,{}^t\mathbf{q} + {}^t\mathbf{x} \times \left({}^t\nabla \cdot {}^t\mathbf{S} \right) \right] d\,{}^tV \,, \end{aligned} \tag{6-17} $$

where ${}^t\mathbf{q}$ is the axial vector to skw(${}^t\mathbf{S}$). Thus,

$$ \int\limits_{{}^tV(\mathcal{P})} \left[{}^t\mathbf{x} \times \left({}^t\nabla \cdot {}^t\mathbf{S} + {}^t\mathbf{b}\, {}^t\rho - {}^t\ddot{\mathbf{x}}\, {}^t\rho \right) + 2\,{}^t\mathbf{q} \right] d\,{}^tV = \mathbf{o}\,, \tag{6-18} $$

and because of CAUCHY's equation of motion (6-13) and as eq. (6-18) has to hold for arbitrary volumes, we have ${}^t\mathbf{q} = \mathbf{o}$ and hence $\mathrm{skw}\left({}^t\mathbf{S} \right) = \mathbf{0}$ or

$$ {}^t\mathbf{S} = {}^t\mathbf{S}^\mathrm{T} \quad \text{or} \quad {}^t\sigma_{ij} = {}^t\sigma_{ji} \;, \tag{6-19} $$

the symmetry of CAUCHY's stress tensor.

Eqs. (6-8), (6-13), (6-19) are the field equations representing the laws of conservation of mass and balance of linear and angular momentum in the local form. They provide seven partial differential equations, which hold at every point of a continuum and for all time. As the initial density, ${}^0\rho$, and the body force, ${}^t\mathbf{b}$, are known, 13 unknown field functions are contained in the field equations, namely the density ${}^t\rho$, the three velocity components, ${}^tv_i = {}^t\dot{u}_i$, and the nine stress components ${}^t\sigma_{ij}$. By means of the conservation of mass, eq. (6-8) or (4-22), and the symmetry of the stress tensor, eq. (6-19), the number of unknowns can be immediately reduced to nine, six stresses and three velocity (or displacement) components, but at the same time, the number of equations reduces to three, namely CAUCHY's equation of motion (6-13). Hence, the number of equations given so far is inadequate to determine all the unknown field functions, and we need six more basic equations. Recall that the general field equations have been developed for any continuum without focusing on a particular material. Different materials have individual characteristic properties. We should therefore have additional basic equations that reflect these properties. Equations that represent the characteristic properties of a material (or a class of materials) and distinguish one material from the other are called *constitutive equations*. They will be treated in chapter 7. But before, two other universal principles used in continuum mechanics will be addressed, which can be derived from the

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[33] Assuming non-polar media, i.e. absence of mass and surface distributed torques, $\mathbf{m}_\mathrm{b}$, $\mathbf{m}_\mathrm{c}$,

above ones and hence used alternatively, but do not change the balance between the number of unknowns and number of equations.

6.4 Balance of Energy

The *rate of work* performed by forces acting on a body is defined by

$$ {}^t\dot{W}^{\mathrm{ex}} = \int_{\partial\mathcal{B}} {}^t\mathbf{t}_{\mathrm{n}} \cdot {}^t\mathbf{v}\, d\,{}^tA + \int_{\mathcal{B}} {}^t\rho\,{}^t\mathbf{b} \cdot {}^t\mathbf{v}\, d\,{}^tV \tag{6-20}$$

With the use of the *kinetic* and *internal energies*

$$ {}^tE = \tfrac{1}{2}\int_{\mathcal{B}} {}^t\rho\,{}^t\mathbf{v} \cdot {}^t\mathbf{v}\, d\,{}^tV \qquad , \qquad {}^tU = \int_{\mathcal{B}} {}^t\rho\,{}^tu\, d\,{}^tV \tag{6-21}$$

the postulate of the balance of mechanical energy can be formulated as follows: the material time rate of the total energy of a body is equal to the sum of rate of work done by external forces acting on the volume and the boundary,

$$ \frac{d}{dt}\left({}^tE + {}^tU\right) = \int_{\partial\mathcal{B}} {}^t\mathbf{t}_{\mathrm{n}} \cdot {}^t\mathbf{v}\, d\,{}^tA + \int_{\mathcal{B}} {}^t\rho\,{}^t\mathbf{b} \cdot {}^t\mathbf{v}\, d\,{}^tV \tag{6-22}$$

With the definitions given above this postulate reads

$$ \frac{d}{dt}\int_{\mathcal{B}} {}^t\rho\left[\tfrac{1}{2}\left({}^t\mathbf{v}\cdot{}^t\mathbf{v}\right) + {}^tu\right] d\,{}^tV = \int_{\partial\mathcal{B}} \left({}^t\mathbf{S}^{\mathrm{T}} \cdot \mathbf{n}\right) \cdot {}^t\mathbf{v}\, d\,{}^tA + \int_{\mathcal{B}} {}^t\rho\,{}^t\mathbf{b} \cdot {}^t\mathbf{v}\, d\,{}^tV \tag{6-23}$$

which can be transferred in a local formulation using CAUCHY´s eq. (6-13) and the symmetry of ${}^t\mathbf{S}$:

$$ {}^t\rho \frac{d\,{}^tu}{dt} = {}^t\mathbf{S} \cdot\cdot\, {}^t\nabla\,{}^t\mathbf{v} \qquad . \tag{6-24}$$

Eq. (6-24) represents the balance of energy and is usually referred to as *energy equation*. It includes the *specific internal energy*, ${}^tu$, as a new unknown quantity and hence does not provide any additional information with respect to the governing equations of the boundary value problem. The term

$$ {}^t\dot{w}^{\mathrm{in}} = {}^t\mathbf{S} \cdot\cdot\, {}^t\nabla\,{}^t\mathbf{v} = {}^t\mathbf{S} \cdot\cdot\, {}^t\mathbf{D} = \frac{{}^t\rho}{{}^0\rho}\,{}^t_0\mathbf{T} \cdot\cdot\, {}^t_0\dot{\mathbf{F}} = \frac{{}^t\rho}{{}^0\rho}\,{}^0\mathbf{T} \cdot\cdot\, {}^t_0\dot{\mathbf{E}}^{(\mathrm{G})} \tag{6-25}$$

is called *stress power density*[34]. As stated above, the stress power determines the appropriate choice of corresponding stress and strain measures, requiring that they have to be *work conjugate*, see eq. (5-33).

6.5 Principle of Virtual Work

Alternative to the balance equations of linear and angular momentums formulated above, which lead to a boundary value problem, *variational principles* have been established for describing the motion of a body, leading to integral equations. They are particularly used in numerical methods of continuum mechanics, as there are the finite element or the boundary element method. They are extremum principles for energy type quantities, like work, kinetic

[34] Subscript "in" denotes "internal forces"

energy, potential energy, etc.. Reversely, the differential equations of motion can be established by methods of variational calculus.

In order to understand terms like *virtual* or *variation* used later, some introductory remarks appear helpful. The fundamental mathematical problem of *variational calculus* is to find a set of functions $x_i(t)$, $i = 1,...,n$, for which the integral

$$I = \int_{t_0}^{t_1} F\left(t, x_1, ..., x_n, \dot{x}_1, ..., \dot{x}_n\right) dt \tag{6-26}$$

becomes an extremum under given "boundary" conditions, i.e. at beginning and end of the time interval,

$$x_i(t_0) = x_i^{(0)} \; ; \quad x_i(t_1) = x_i^{(1)} \; . \tag{6-27}$$

The problem is solved by defining *varied* functions

$$\bar{x}_i(t) = x_i(t) + \varepsilon\xi_i(t) \;\text{ with }\; \xi_i(t_0) = \xi_i(t_1) = 0\,, \tag{6-28}$$

where $\xi_i(t)$ are arbitrary, differentiable functions (also addressed as *test functions*) meeting eq. (6-27), and ε is a real number. $\delta x_i = \xi_i(t)$ and $\delta \dot{x}_i = \dot{\xi}_i(t)$ are called variations of x_i and $\dot{x}_i$, respectively. The integral of eq. (6.26) can now be written as a function of ε,

$$I(\varepsilon) = \int_{t_0}^{t_1} F\left(t, x_1 + \varepsilon\xi_1, ..., \dot{x}_1 + \varepsilon\dot{\xi}_1, ...\right) dt\,, \tag{6-29}$$

and the condition for I becoming an extremum is

$$\delta I = \left(\frac{\partial I}{\partial \varepsilon}\right)_{\varepsilon=0} = 0 \; . \tag{6-30}$$

Eq. (6-29) is the variational problem with δI being the (first) variation of I. The variational problem leads to

$$\delta I = \left(\frac{\partial I}{\partial \varepsilon}\right)_{\varepsilon=0} = \int_{t_0}^{t_1} \left(\frac{\partial F}{\partial x_i}\xi_i + \frac{\partial F}{\partial \dot{x}_i}\dot{\xi}_i\right) dt = 0\,. \tag{6-31}$$

Partial integration of the second integrand yields

$$\int_{t_0}^{t_1} \left(\dot{\xi}_i \frac{\partial F}{\partial \dot{x}_i}\right) dt = \left[\xi_i \frac{\partial F}{\partial \dot{x}_i}\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \left(\xi_i \frac{d}{dt}\frac{\partial F}{\partial \dot{x}_i}\right) dt \;, \tag{6-32}$$

where the first bracket vanishes due to eq. (6-27), so that the variational problem finally writes as

$$\delta I = \int_{t_0}^{t_1} \xi_i \left(\frac{\partial F}{\partial x_i} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}_i}\right) dt = 0 \; . \tag{6-33}$$

As $\xi_i(t)$ are arbitrary (test functions), the term written in brackets has to vanish in order to satisfy eq. (6-33) and EULER's differential equation of the variational problem is obtained,

$$\frac{\partial F}{\partial x_i} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}_i} = 0 \ . \tag{6-34}$$

Back to continuum mechanics, we consider a velocity field $\mathbf{u} = {}^t_0\mathbf{u} = {}^t\mathbf{x} - {}^0\mathbf{x}$ and an arbitrary *virtual displacement*

$$\delta\mathbf{u} = \left.\frac{\partial(\mathbf{u} + \varepsilon\mathbf{w})}{\partial\varepsilon}\right|_{\varepsilon=0} = \mathbf{w} \tag{6-35}$$

being independent of t for $t > 0$, but ${}^0\mathbf{w} = \mathbf{w}(t_0) = \mathbf{o}$ in the reference configuration[35]. Let $Z(\mathbf{u})$ be some energy functional, then by eqs. (6-29) and (6-30), its first variation is defined via the GÂTEAUX derivative of $Z(\mathbf{u})$ at time t in $\mathbf{w}$ direction,

$$\delta Z = \lim_{\varepsilon\to 0}\left(\frac{Z(\mathbf{u} + \varepsilon\mathbf{w}) - Z(\mathbf{u})}{\varepsilon}\right) = \left.\frac{\partial Z(\mathbf{u} + \varepsilon\mathbf{w})}{\partial\varepsilon}\right|_{\varepsilon=0} \ . \tag{6-36}$$

The variation $\delta Z\left(\mathbf{u},\mathbf{w}\right)$ is linear in $\mathbf{w} = \delta\mathbf{u}$, i.e.

$$\begin{aligned} &\delta Z(\mathbf{u},\alpha\mathbf{w}) = \alpha\,\delta Z(\mathbf{u},\mathbf{w}) \\ &\delta Z(\mathbf{u},\mathbf{w}_1 + \mathbf{w}_2) = \delta Z(\mathbf{u},\mathbf{w}_1) + \delta Z(\mathbf{u},\mathbf{w}_2) \ . \end{aligned} \tag{6-37}$$

We now derive the *principle of virtual work* from CAUCHY's field equations of motion[36], eq. (6-13),

$$\frac{\partial\sigma_{ij}}{\partial x_i} + \rho b_j = \rho\ddot{u}_j \ , \tag{6-38}$$

multiplying them by the virtual displacement δu_j and integrating them over the (current) volume V,

$$\int_V \frac{\partial\sigma_{ij}}{\partial x_i}\delta u_j \, dV + \int_V \rho b_j \delta u_j \, dV = \int_V \rho\ddot{u}_j \delta u_j \, dV \ . \tag{6-39}$$

We now consider the first terms of eq, (6-39). It can be converted as

$$\int_V \frac{\partial\sigma_{ij}}{\partial x_i}\delta u_j \, dV = \int_V \frac{\partial}{\partial x_i}\left(\sigma_{ij}\,\delta u_j\right)dV - \int_V \sigma_{ij}\,\frac{\partial\left(\delta u_j\right)}{\partial x_i}dV \ , \tag{6-40}$$

and GAUß' theorem can be applied to

$$\int_V \frac{\partial}{\partial x_i}\left(\sigma_{ij}\,\delta u_j\right)dV = \int_{\partial V} n_i\,\sigma_{ij}\,\delta u_j \, dA = \int_{\partial V} t_j\,\delta u_j \, dA \ , \tag{6-41}$$

while the differentiability of the displacement field and the symmetry of CAUCHY's stress tensor yield

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35 With $\delta\mathbf{u} = \delta\mathbf{v}\,dt$, the principle can be formulated with a virtual velocity field as a *principle of virtual power*.

36 All quantities refer to the current configuration at time t; the superscript t is omitted for convenience. An analogical derivation can be performed in the reference configuration based on the second PIOLA-KIRCHHOFF stress tensor.

$$\int_V \sigma_{ij} \frac{\partial\left(\delta u_j\right)}{\partial x_i} dV = \int_V \sigma_{ij}\, \delta \frac{\partial u_j}{\partial x_i} dV = \int_V \sigma_{ij}\, \delta\varepsilon_{ij}\, dV\,. \tag{6-42}$$

Defining the *virtual work of external forces* by

$$\delta W^{\mathrm{ex}} = \int_{\partial V} t_j\, \delta u_j\, dA + \int_V \rho b_j \delta u_j\, dV\,, \tag{6-43}$$

the *virtual work of stresses* (virtual strain energy, virtual work of internal forces) by

$$\delta W^{\mathrm{in}} = \int_V \sigma_{ij}\, \delta\varepsilon_{ij}\, dV\,, \tag{6-44}$$

and the *virtual work of mass acceleration* by

$$\delta B = \int_V \rho \ddot{u}_j \delta u_j\, dV \tag{6-45}$$

we obtain the *principle of virtual work*

$$\delta W^{\mathrm{ex}} - \delta W^{\mathrm{in}} = \delta B\,. \tag{6-46}$$

It can also be written as

$$\delta\left(W^{\mathrm{ex}} - W^{\mathrm{in}} - B\right) = 0 \tag{6-47}$$

stating that the variation of the energy functional $\left(W^{\mathrm{ex}} - W^{\mathrm{in}} - B\right)$ vanishes or that the energy functional becomes an extremum (more precisely: a minimum) among all admissible states defined by the virtual displacements. Taking the mass accelerations as negative fictitious external forces, we obtain D'ALEMBERT's[37] principle, $\delta\left(W^{\mathrm{ex}} - B\right) = \delta W^{\mathrm{in}}$, that external and internal forces are balanced.

Special cases of the principle of virtual work are obtained for

- Rigid bodies, $\delta W^{\mathrm{in}} = 0$,
- Elastic bodies (see section 7.2), $\delta W^{\mathrm{in}} = \int_V C_{ijkl}\, \varepsilon_{kl} \delta\varepsilon_{ij}\, dV$, with $W^{\mathrm{in}} = \frac{1}{2}\int_V C_{ijkl}\, \varepsilon_{ij}\, \varepsilon_{kl}\, dV$ being the elastic strain energy,
- Static problems, $\delta B = 0$, resulting in the equilibrium condition, $\delta W^{\mathrm{ex}} = \delta W^{\mathrm{in}}$.

The virtual work of mass acceleration, eq. (6-45), can be decomposed into the *virtual power of linear momentum, δP,* and the *virtual kinetic energy, δE,*

$$\delta B = \int_V \rho \ddot{u}_j \delta u_j\, dV = \frac{d}{dt}\int_V \rho \dot{u}_j \delta u_j\, dV - \delta \int_V \tfrac{1}{2} \rho \dot{u}_j \dot{u}_j\, dV = \delta P - \delta E\,, \tag{6-48}$$

from which HAMILTON's principle is derived. And following the procedure above leading to EULER's differential equations (6-34) of the variational problem, LAGRANGE's equations of motion are obtained.

37 JEAN LEROND D'ALEMBERT (1717-1783)

7. Constitutive Equations

All foregoing axioms, laws, equations are supposed to be valid for all material bodies and are thus labelled *universal*. In contrast, material theory aims at describing the individual behaviour of different materials. This is done by so-called *material laws, constitutive equations* or *material models*. Before going into details of the differences in material behaviour however, we want to find common features, something like the general form of material laws. For this, we first assume certain basic *principles* of material theory, which are partly based on our experience and partly on plausibility. Further on, we want to develop criteria to classify the universe of material behaviour to obtain more concrete models for certain classes of materials. Finally we will not be able to continue without experiments. But even for the design of appropriate experiments, theory is expected to give guidance. Any material characterisation is meaningful only in the context of a constitutive model.

7.1 The Principles of Material Theory

It is generally assumed that there is a *deterministic relation* between stresses in a body and the motion of the body, In order to put this into a functional form, we have to decide which quantities to use as independent variables and which as dependent variables. It has become more or less common practice to consider the stresses as dependent variables, and motions as independent ones. The whole history of motion may affect the present stresses, but of course, we do not allow the future motion of a body to have any effect on them.

Principle of Determinism:

The stresses ${}^t\mathbf{S}$ in a material point, $\mathcal{X}$, at a certain instant of time, t, are determined by the whole history of motion (-∞, t] of all particles $\mathcal{Y}$ of the body $\mathcal{B}$,

$$ {}^t\mathbf{S}(\mathcal{X}) = \underset{\tau=-\infty}{\overset{t}{f}} \left\{ \boldsymbol{\chi}(\mathcal{Y},\tau), \mathcal{X} \right\} \quad \forall \quad \mathcal{X},\mathcal{Y} \in \mathcal{B} \ , \tag{7-1} $$

where f is a tensor valued functional.

The fundamental concepts like motions and stresses depend on observers. As was shown in section 4.6, the change of the observer (or frame of reference) is described by a EUCLIDean transformation, eq. (4-76). Some of the kinematical quantities have been classified as objective or as invariant, eq. (4-78). Assuming the objectivity of forces, CAUCHY's stress tensor has been shown to be objective. eq. (5-22). It is also necessary to establish a rule for the *observer dependence* of the constitutive equations.

Principle of Material Objectivity:

The stress power is *objective* (and thus also *invariant*) under EUKLIDean transformations,

$$ {}^t\dot{\tilde{w}}^{\text{in}} = {}^t\tilde{\mathbf{S}} \cdot\cdot\, {}^t\tilde{\mathbf{L}} = {}^t\dot{w}^{\text{in}} = {}^t\mathbf{S} \cdot\cdot\, {}^t\mathbf{L} \tag{7-2} $$

The necessary and sufficient condition for this invariance to hold for all materials is that CAUCHY's stress tensor is objective, ${}^t\tilde{\mathbf{S}} = \mathbf{Q} \cdot {}^t\mathbf{S} \cdot \mathbf{Q}^{\mathrm{T}}$, eq. (5-22).

Together with the

Principle of Form Invariance:
The material functionals are invariant under change of observer,

$$\tilde{f}\{\tilde{\boldsymbol{\chi}},\mathcal{X}\} = f\{\boldsymbol{\chi},\mathcal{X}\} \tag{7-3}$$

we obtain the

Principle of Frame Invariance:
If a constitutive equation holds for a process described by the motion ${}^t\mathbf{x} = \boldsymbol{\chi}(\mathcal{X},t) = {}^t\hat{\boldsymbol{\chi}}({}^0\mathbf{x},t)$ and a stress tensor ${}^t\mathbf{S}(\mathcal{X})$, it must also hold for a motion ${}^t\tilde{\mathbf{x}} = \tilde{\boldsymbol{\chi}}(\mathcal{X},t) = {}^t\mathbf{c} + {}^t\mathbf{Q}\cdot\boldsymbol{\chi}(\mathcal{X},t)$ and a transformed stress tensor ${}^t\tilde{\mathbf{S}}(\mathcal{X}) = {}^t\mathbf{Q}\cdot{}^t\mathbf{S}(\mathcal{X})\cdot{}^t\mathbf{Q}^{\mathrm{T}}$, where ${}^t\mathbf{Q}\cdot{}^t\mathbf{Q}^{\mathrm{T}} = \mathbf{1}$, i.e.

$${}^t\mathbf{Q}\cdot f\{\boldsymbol{\chi},\mathcal{X}\}\cdot{}^t\mathbf{Q}^{\mathrm{T}} = f\left\{\left({}^t\mathbf{c} + {}^t\mathbf{Q}\cdot\boldsymbol{\chi}\right),\mathcal{X}\right\} \tag{7-4}$$

This principle is a strong restriction on the material behaviour. We can generate accelerations of arbitrary magnitude with respect to the inertial observer by $\mathbf{c}(t)$ and $\mathbf{Q}(t)$. They do not directly effect the stresses, however, but only through CAUCHY's law of motion, eq. (6-13). Eq. (7-4) implies that eq. (7-1) takes the form

$${}^t\mathbf{S}(\mathcal{X}) = \underset{\tau=-\infty}{\overset{t}{f}}\left\{\boldsymbol{\chi}(\mathcal{Y},\tau) - \boldsymbol{\chi}(\mathcal{X},\tau)\right\} \quad \forall \quad \mathcal{X},\mathcal{Y}\in\mathcal{B}\,. \tag{7-5}$$

It is known from experience that the stresses in a material point do not depend on the motion of other points, provided that those are sufficiently remote. We therefore assume, that there is a finite neighbourhood of the point, whose motion alone affects the stresses, while the rest of the body has no direct effect.

Principle of Local Action:
The stresses in a material point depend on the motion of only a finite neighbourhood
$\Omega:\left\{\,\|\mathbf{y}-\mathbf{x}\| \le \delta \;\middle|\; \mathbf{x} = \boldsymbol{\chi}(\mathcal{X}),\, \mathbf{y} = \boldsymbol{\chi}(\mathcal{Y})\right\}$.

By a TAYLOR's expansion of $\boldsymbol{\chi}(\mathcal{Y},\tau) - \boldsymbol{\chi}(\mathcal{X},\tau)$ in eq. (7-5) we obtain

$${}^t\mathbf{S}(\mathcal{X}) = \underset{\tau=-\infty}{\overset{t}{f}}\left\{{}_{-\infty}^{\tau}\mathbf{F}(\mathcal{X}),{}^\tau\nabla\left({}_{-\infty}^{\tau}\mathbf{F}(\mathcal{X})\right),\ldots\right\}. \tag{7-4}$$

If we take the neighbourhood as arbitrarily small, $\delta \to 0$, we obtain the material functional of so-called *simple materials*

$${}^t\mathbf{S}(\mathcal{X}) = \underset{\tau=-\infty}{\overset{t}{f}}\left\{{}_{-\infty}^{\tau}\mathbf{F}(\mathcal{X})\right\}. \tag{7-5}$$

Additional principles and assumptions allow for further simplifications. The same way, as the spatial neighbourhood affecting the stress state in a material point is confined by the principle of local action, the relevant time domain can be restricted, leading to the

Principle of Fading Memory:

The memory of a material fades with time, i.e. stresses are determined by the motion history within a finite past [t-τ_0, t] only,

$$ {}^{t}\mathbf{S}(\mathcal{X}) = \underset{\tau=t-\tau_0}{\overset{t}{f}} \left\{ {}^{\tau}_{\tau_0}\mathbf{F}(\mathcal{X}) \right\}. \tag{7-6}$$

In the limit of $\tau_0 \to 0$, stresses depend only on the present deformation,

$$ {}^{t}\mathbf{S}(\mathcal{X}) = \mathbf{\Phi}\left\{ {}^{t}_{0}\mathbf{F}(\mathcal{X}) \right\}. \tag{7-7}$$

where $\mathbf{\Phi}$ is an isotropic tensor function. From the polar decomposition of the deformation gradient, eq. (4-30), and the principle of invariance under a rigid body rotation taking particularly ${}^{t}\mathbf{Q} = {}^{t}_{0}\mathbf{R}^{\mathrm{T}}$, other forms of constitutive equations for simple materials based on the right stretch tensor, ${}^{t}_{0}\mathbf{U}$, the right CAUCHY-GREEN tensor, ${}^{t}_{0}\mathbf{C} = {}^{t}_{0}\mathbf{U}^2$, or GREEN's strain tensor, ${}^{t}_{0}\mathbf{E}^{(\mathrm{G})} = \frac{1}{2}\left({}^{t}_{0}\mathbf{C} - \mathbf{I} \right)$, can be derived. In the following, we shall consider the simplest case of a linear relation between CAUCHY's stress tensor, ${}^{t}\mathbf{S}$, and the linear strain tensor, $\mathbf{E}$, known as HOOKE's law of elasticity for small deformations.

7.2 Linear Elasticity

An elastic solid is a deformable continuum that recovers its original configuration when forces causing deformation are removed. For linear elastic behaviour and small deformations, a linear relation between stresses, $\mathbf{S}$, and strains, $\mathbf{E}$, is postulated, which writes

$$ \mathbf{S} = \underline{\underline{\mathbf{C}}} \cdot\cdot \mathbf{E} \quad \text{or} \quad \sigma_{ij} = C_{ijkl}\varepsilon_{kl} . \tag{7-8}$$

In the theory of small deformations no distinction between actual and reference configuration is necessary, CAUCHY and PIOLA-KIRCHHOFF stresses tensors coincide, $\mathbf{S} = \sigma_{ij}\mathbf{e}_i\mathbf{e}_j$, and strains can be described by the infinitesimal strain tensor of eq. (4-50). $\underline{\underline{\mathbf{C}}} = C_{ijkl}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\mathbf{e}_l$ is a material tensor of rank four. It consists of 81 scalar coefficients that depend on the physical properties of the solid but not on the strains, called the elastic moduli. Because $\mathbf{S}$ and $\mathbf{E}$ are symmetric, $C_{ijkl} = C_{jikl} = C_{ijlk} = C_{jikl}$, no more than 36 components of $\underline{\underline{\mathbf{C}}}$ can be independent. From the condition, that the strain energy has to be a quadratic form of $\mathbf{E}$,

$$ w = \mathbf{S} \cdot\cdot \mathbf{E} = \mathbf{E} \cdot\cdot \mathbf{S} = \mathbf{E} \cdot\cdot \underline{\underline{\mathbf{C}}} \cdot\cdot \mathbf{E}, \tag{7-9}$$

it follows furthermore, that $C_{ijkl} = C_{klij}$ so that only 27 elastic constants may be independent in the most general case of an anisotropic material.

In the case that the material is isotropic, i.e. its deformation does not depend on the orientation with respect to the loading, the components of $\underline{\underline{\mathbf{C}}}$ can be written in the form

$$ C_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk} , \tag{7-10}$$

which allows for writing equation (7-8) as

$$ \sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} \quad \text{or} \quad \mathbf{S} = \lambda\left(\mathrm{tr}\,\mathbf{E}\right)\mathbf{1} + 2\mu\mathbf{E}, \tag{7-11}$$

with just two independent elastic constants, λ and μ, which were introduced by LAMÉ and hence are called LAMÉ`s[38] coefficients of elasticity.

In engineering practise, the HOOKE's law of elasticity is commonly written as

$$\mathbf{S} = \frac{E}{1+\nu}\left[\mathbf{E} + \frac{\nu}{1-2\nu}(\mathrm{tr}\,\mathbf{E})\,\mathbf{1}\right] \tag{7-12}$$

with E as YOUNG's[39] modulus and ν as POISSON's[40] ratio. The physical dimension of E is force per area, that is N/mm^2 = MPa, and ν is dimensionless. The inverse relation to eq. (7-12) reads

$$\mathbf{E} = \frac{1+\nu}{E}\mathbf{S} - \frac{\nu}{E}(\mathrm{tr}\,\mathbf{S})\,\mathbf{1} \tag{7-13}$$

For a uniaxial stress state, $\mathbf{S} = \sigma_{11}\mathbf{e}_1\mathbf{e}_1$, the strains are $\varepsilon_{11} = \sigma_{11}/E$, $\varepsilon_{22} = \varepsilon_{33} = -\nu\varepsilon_{11}$. In this way, YOUNG's modulus and POISSON's ratio of an elastic material can be determined from a uniaxial tensile test (see section 8.3). Other elastic constants used are

- the *shear modulus G*, relating shear stresses and strains by $\sigma_{ij} = 2G\varepsilon_{ij}\ (i \neq j)$ and
- the *bulk modulus K*, relating hydrostatic stress, eq. (5-22), and volume dilatation, eq. (4-52), by $\sigma_{\mathrm{hyd}} = \tfrac{1}{3}\sigma_{kk} = K\varepsilon_{kk}$,

resulting from a decomposition of eq. (7-13) into a spherical and a deviatoric part, see eq. (5-21)

$$\mathbf{E} = \hat{\mathbf{E}} + \tfrac{1}{3}(\mathrm{tr}\,\mathbf{E})\mathbf{1} = \frac{1+\nu}{E}\hat{\mathbf{S}} + \frac{1-2\nu}{3E}(\mathrm{tr}\,\mathbf{S})\mathbf{1} = 2G\,\hat{\mathbf{S}} + K(\mathrm{tr}\,\mathbf{S})\mathbf{1} \tag{7-14}$$

As only two elastic constants are independent, any pair of two constants can be expressed by any other, see the following table .

	$\lambda =$	$\mu =$	$E =$	$\nu =$	$K =$	$G =$
λ, μ	λ	μ	$\dfrac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$\dfrac{\lambda}{2(\lambda+\mu)}$	$\lambda + \tfrac{2}{3}\mu$	μ
G, K	$K - \tfrac{2}{3}G$	G	$\dfrac{9K\cdot G}{3K+G}$	$\dfrac{3K-2G}{6K+2G}$	K	G
E, ν	$\dfrac{E\nu}{(1+\nu)(1-2\nu)}$	$\dfrac{E}{2(1+\nu)}$	E	ν	$\dfrac{E}{3(1-2\nu)}$	$\dfrac{E}{2(1+\nu)}$

Table: Elastic constants for isotropic materials and their relations

If the change of volume (volume dilatation) during a deformation is zero, i.e. $\varepsilon_{kk} = \mathrm{tr}\,\mathbf{E} = 0$, the medium is called *incompressible*. This can be fulfilled for arbitrary hydrostatic stresses $\sigma_{\mathrm{hyd}} = \tfrac{1}{3}\sigma_{kk} = \tfrac{1}{3}\mathrm{tr}\,\mathbf{S}$ if and only if $K = 0$ or $\nu = 0.5$. In this case, hydrostatic stresses can not be determined from the strains, see eq. (7-12). The range of physically possible values of POISSON's ratio is hence $0 \leq \nu \leq 0.5$.

[38] GABRIEL LAMÉ (1795-1870)
[39] THOMAS YOUNG (1773-1829)
[40] S.D. Poisson (1781-1840)

8. Elementary Problems of Engineering Mechanics

8.1 Equations of Continuum Mechanics for Linear Elasticity

We now have the complete system of equations governing the deformation of an elastic solid. They are given in tensor notation, Cartesian coordinates $\{\mathbf{e}_i\} = \{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and cylindrical coordinates $\{\mathbf{e}_r, \mathbf{e}_\varphi, \mathbf{e}_z\}$[41] in the following.

- The relations between *strains and displacements* for small deformations are

$$\mathbf{E} = \tfrac{1}{2}\left[\nabla\mathbf{u} + \left(\nabla\mathbf{u}\right)^{\mathrm{T}}\right] . \tag{8-1a}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) . \tag{8-1b}$$

$$\left.\begin{aligned}
\varepsilon_{rr} &= \frac{\partial u_r}{\partial r} \\
\varepsilon_{\varphi\varphi} &= \frac{\partial u_\varphi}{r\partial\varphi} + \frac{u_r}{r} \\
\varepsilon_{zz} &= \frac{\partial u_z}{\partial z} \\
\varepsilon_{r\varphi} = \varepsilon_{\varphi r} &= \frac{1}{2}\left(\frac{\partial u_r}{r\partial\varphi} + \frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right) \\
\varepsilon_{rz} = \varepsilon_{zr} &= \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \\
\varepsilon_{\varphi z} = \varepsilon_{z\varphi} &= \frac{1}{2}\left(\frac{\partial u_z}{r\partial\varphi} + \frac{\partial u_\varphi}{\partial z}\right)
\end{aligned}\right\} . \tag{8-1c}$$

- For *static equilibrium*, CAUCHY's equations take the form,

$$\nabla \cdot \mathbf{S} + \rho\mathbf{b} = \mathbf{o} . \tag{8-2a}$$

$$\frac{\partial\sigma_{ij}}{\partial x_i} + \rho b_j = 0 \tag{8-2b}$$

$$\left.\begin{aligned}
&\frac{\partial\sigma_{rr}}{\partial r} + \frac{\partial\sigma_{\varphi r}}{r\partial\varphi} + \frac{\partial\sigma_{zr}}{\partial z} + \frac{1}{r}\left(\sigma_{rr} - \sigma_{\varphi\varphi}\right) + \rho b_r = 0 \\
&\frac{\partial\sigma_{r\varphi}}{\partial r} + \frac{\partial\sigma_{\varphi\varphi}}{r\partial\varphi} + \frac{\partial\sigma_{z\varphi}}{\partial z} + \frac{1}{r}\sigma_{r\varphi} \qquad\quad + \rho b_\varphi = 0 \\
&\frac{\partial\sigma_{rz}}{\partial r} + \frac{\partial\sigma_{\varphi z}}{r\partial\varphi} + \frac{\partial\sigma_{zz}}{\partial z} + \frac{1}{r}\sigma_{rz} \qquad\quad + \rho b_z = 0
\end{aligned}\right\} \tag{8-2c}$$

[41] See Appendix A 1.4

➢ And finally, the *constitutive behaviour* is described by HOOKE's law,

$$\left.\begin{aligned}\mathbf{S} &= \frac{E}{1+\nu}\left[\mathbf{E} + \frac{\nu}{1-2\nu}(\operatorname{tr}\mathbf{E})\,\mathbf{1}\right] \\ \mathbf{E} &= \frac{1+\nu}{E}\mathbf{S} - \frac{\nu}{E}(\operatorname{tr}\mathbf{S})\,\mathbf{1}\end{aligned}\right\} . \qquad (8\text{-}3a)$$

$$\left.\begin{aligned}\sigma_{ij} &= \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right) \\ \varepsilon_{ij} &= \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}\end{aligned}\right\} [42]. \qquad (8\text{-}3b,c)$$

These equations hold for all material points of an arbitrary body. They have to be completed by boundary conditions, specifying the particular geometry and loading. Together, they form the boundary value problem for the specific component.

General analytical solutions for arbitrarily shaped and loaded bodies do not exist. They require the application of numerical methods like the finite element method, which is based on the principle of virtual work (see section 6.5). In the following, the boundary value problems of simple configurations are established and solved.

8.2 Bars, Beams, Rods

Components like bars, beams and rods are frequently applied in mechanical and civil engineering. Their characteristic feature is that one dimension, the length *L*, is much greater than the two others, width *b* and height *h*. They allow for approximate analytical solutions of the boundary value problem by introducing certain assumptions and simplifications, which will be introduced and discussed in the following.

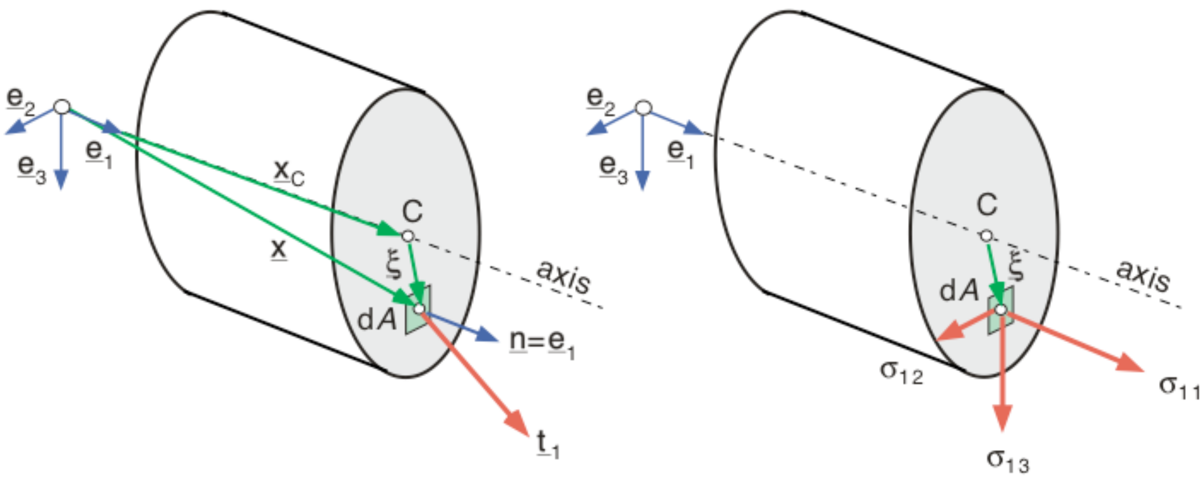


Figure 8.1: Sectioned bar with stress vector $\mathbf{t}_1(\mathbf{x}) = \sigma_{1i}(\mathbf{x})\,\mathbf{e}_i$ acting on a surface element d*A* of the cross section.

We assume that the length coordinate is $\mathbf{e}_1$. The centres, C, of all cross sections constitute the axis of the bar or beam. The bar (or beam) is notionally sectioned perpendicularly to the axis,

[42] $(i,j) = (1,2,3)$ or $(i,j) \triangleq (x,y,z)$ or $(i,j) \triangleq (r,\varphi,z)$

$\mathbf{n} = \mathbf{e}_1$, at an arbitrary point $\mathbf{x}_C = x_1\mathbf{e}_1$ of the axis, see Fig. 8.1. The dynamic interactions between the two parts have to be replaced by contact forces

$$d\mathbf{f}_1(\mathbf{x}) = \mathbf{t}_1(\mathbf{x})\,dA = \sigma_{1i}(\mathbf{x})\,\mathbf{e}_i\,dA \tag{8-4}$$

acting on each surface element of the cross section.

Adapted to the specific geometry, *sectional loads*, i.e. sectional forces and moments[43], are introduced by integrating the stress vectors, $\mathbf{t}_1(\mathbf{x})$, over the cross section,

$$\begin{aligned} \mathbf{f}_1(\mathbf{x}_C) &= \iint d\mathbf{f}_1(\mathbf{x}) = \iint_A \mathbf{t}_1(\mathbf{x})\,dA = N(x_1)\mathbf{e}_1 + Q_2(x_1)\mathbf{e}_2 + Q_3(x_1)\mathbf{e}_3 \\ \mathbf{m}_1^{(C)}(\mathbf{x}_C) &= \iint \boldsymbol{\xi}\times d\mathbf{f}_1(\mathbf{x}) = \iint_A \boldsymbol{\xi}\times\mathbf{t}_1(\mathbf{x})\,dA = M_T(x_1)\mathbf{e}_1 + M_2(x_1)\mathbf{e}_2 + M_3(x_1)\mathbf{e}_3 \end{aligned}, \tag{8-5}$$

where $\boldsymbol{\xi} = \mathbf{x} - \mathbf{x}_C = x_2\,\mathbf{e}_2 + x_3\,\mathbf{e}_3$. N is called *normal force*, Q_2, Q_3 *shear forces*, M_T *torque*, M_2, M_3 *bending moments*, see Fig 8.2. Obviously, there have to be six components of sectional loads in the general case corresponding to three translational and three rotational degrees of freedom of the cross section.

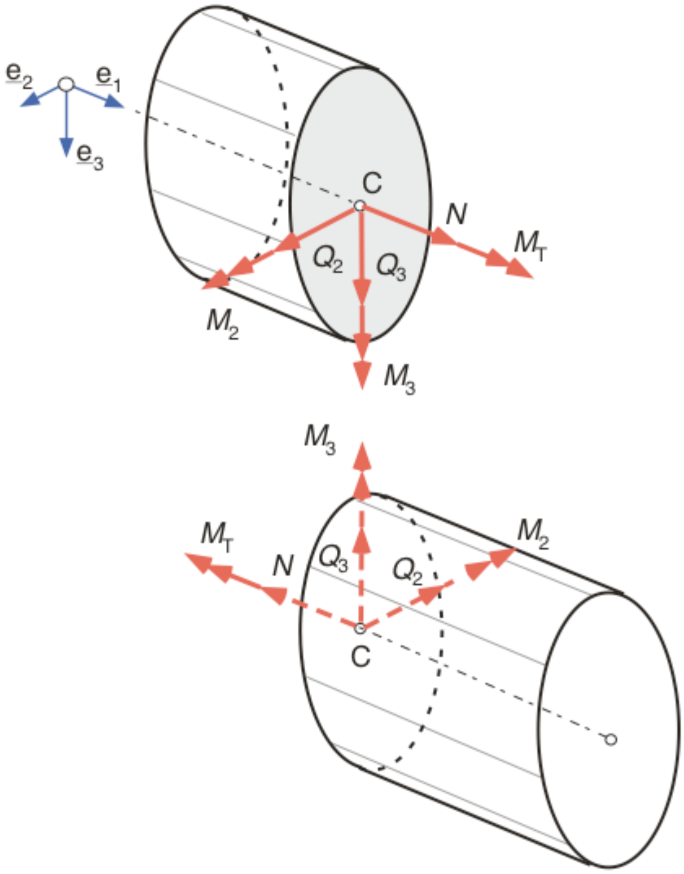


Figure 8.2: Sectioned bar with sectional forces and moments[44],

According to CAUCHY's reciprocal relation (or NEWTON's reaction principle), eq. (5-5), the corresponding forces and moments acting on the opposite cross section, $\mathbf{n} = -\mathbf{e}_1$, of the sectioned beam are

[43] Also called internal forces and moments, as they result from stresses and become observable only by sectioning.

[44] Moments are indicated by double arrows showing in the direction of the respective rotational axis.

$$\begin{aligned}\mathbf{f}_{-1}(\mathbf{x}_C) &= -N(x_1)\mathbf{e}_1 - Q_2(x_1)\mathbf{e}_2 - Q_3(x_1)\mathbf{e}_3 \\ \mathbf{m}_{-1}^{(C)}(\mathbf{x}_C) &= -M_T(x_1)\mathbf{e}_1 - M_2(x_1)\mathbf{e}_2 - M_3(x_1)\mathbf{e}_3\end{aligned} \quad . \tag{8-6}$$

Introducing $\mathbf{t}_1 = \mathbf{e}_1 \cdot \mathbf{S} = \sigma_{11}\mathbf{e}_1 + \sigma_{12}\mathbf{e}_2 + \sigma_{13}\mathbf{e}_3$, we obtain the following relations of equivalence between stresses and sectional loads

$$\begin{aligned} N(x_1) &= \iint_A \sigma_{11}\, dA & M_T(x_1) &= \iint_A \left(x_2\,\sigma_{13} - x_3\,\sigma_{12}\right) dA \\ Q_2(x_1) &= \iint_A \sigma_{12}\, dA, & M_2(x_1) &= \iint_A x_3\,\sigma_{11}\, dA \\ Q_3(x_1) &= \iint_A \sigma_{13}\, dA & M_3(x_1) &= -\iint_A x_2\,\sigma_{11}\, dA \end{aligned} \quad . \tag{8-7}$$

Assuming external forces[45] per unit length, $\mathbf{q}(x_1) = q_i(x_1)\mathbf{e}_i$ with $\mathbf{f} = \int_0^L \mathbf{q}(x_1)\,dx_1$, acting on the bar, the equilibrium results in the following relations,

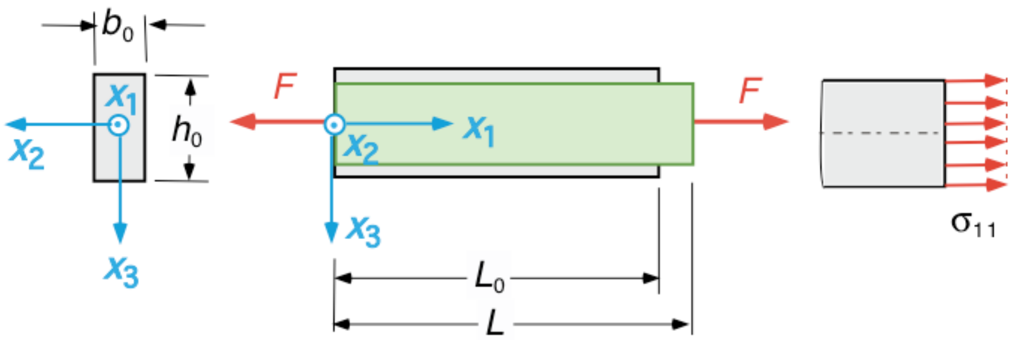
$$\begin{aligned} &\frac{dN}{dx_1} = -q_1(x_1)\,; \quad \frac{dQ_2}{dx_1} = -q_2(x_1)\,; \quad \frac{dQ_3}{dx_1} = -q_3(x_1) \\ &\frac{dM_2}{dx_1} = Q_3(x_1)\,; \quad \frac{dM_3}{dx_1} = -Q_2(x_1) \end{aligned} \tag{8-8}$$

Some elementary loading situations, namely uniaxial tension and compression, bending and torsion will be analysed in the following.

8.3 Uniaxial Tension and Compression

We consider a bar of length L_0 and rectangular cross section[46], $A_0 = b_0 h_0$, in the undeformed configuration, which is loaded by a tensile force[47] at both ends, $x_1 = 0$ and $x_1 = L_0$., see Fig. 8.3. Equilibrium of forces, eq. (8-8), in $\mathbf{e}_1$ direction states that $dN/dx_1 = 0$, and hence

$$\boxed{N(x_1) = F = \iint_{A_0} \sigma_{11}\, dA = \int_{x_2=-b_0/2}^{+b_0/2} \int_{x_3=-h_0/2}^{+h_0/2} \sigma_{11}\, dx_2\, dx_3} \ . \tag{8-9}$$



[45] Either contact forces (line loads acting on the surface) or body forces (self-weight)

[46] A rectangular cross section is assumed for simplicity but without restricting generality, as the derivation and the equations hold for arbitrary cross sections

[47] For $F < 0$, the bar is under compression

Figure 8-3: Bar under tensile force

The surfaces $\mathbf{n} = \mathbf{e}_2$ and $\mathbf{n} = \mathbf{e}_3$ of the bar are stress free,

$$\left.\begin{array}{r}
\sigma_{22}\big|_{x_2=\pm b_0/2} = 0 \quad \forall\ x_1, x_3 \\
\sigma_{33}\big|_{x_3=\pm h_0/2} = 0 \quad \forall\ x_1, x_2 \\
\sigma_{21}\big|_{x_2=\pm b_0/2} = \sigma_{23}\big|_{x_2=\pm b_0/2} = \sigma_{31}\big|_{x_3=\pm h_0/2} = \sigma_{32}\big|_{x_3=\pm h_0/2} = 0 \quad \forall\ x_1
\end{array}\right\} \tag{8-10}$$

Eqs. (8-9) and (8-10) represent the boundary conditions for external surface forces. For solving the boundary value problem, an additional assumption is made for the kinematics of deformation, namely that *all plane surfaces remain plane*:

$$\boxed{\frac{\partial u_1}{\partial x_2} = \frac{\partial u_1}{\partial x_3} = 0 \quad , \quad \frac{\partial u_2}{\partial x_1} = \frac{\partial u_2}{\partial x_3} = 0 \quad , \quad \frac{\partial u_3}{\partial x_1} = \frac{\partial u_3}{\partial x_2} = 0} \ . \tag{8-11}$$

Regarding (8-1b), this results in

$$\begin{array}{l}
\varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = 0 \\
\dfrac{\partial \varepsilon_{11}}{\partial x_2} = \dfrac{\partial \varepsilon_{11}}{\partial x_3} = 0
\end{array} \quad , \tag{8-12}$$

and from HOOKE's law, eq. (8-3b), we have

$$\sigma_{12} = \sigma_{13} = \sigma_{23} = 0 \ , \tag{8-13}$$

which satisfies the boundary conditions for shear stresses, eq. (8-10)$_3$, as well as $Q_2 = Q_3 = 0$ at $x_1 = 0, L_0$. The equilibrium conditions (8-2b), neglecting body forces, require that

$$\frac{\partial \sigma_{11}}{\partial x_1} = 0 \quad , \quad \frac{\partial \sigma_{22}}{\partial x_2} = 0 \quad , \quad \frac{\partial \sigma_{33}}{\partial x_3} = 0 \ , \tag{8-14}$$

and from the boundary conditions for σ22 and σ33, eq. (8-10)1,2, we conclude that the stress state is uniaxial,

$$\sigma_{22} = \sigma_{33} = 0 \quad \forall\ x_1, x_2, x_3 \ , \tag{8-15}$$

i.e. $\mathbf{S} = \sigma_{11}\,\mathbf{e}_1\mathbf{e}_1$. Applying HOOKE's law, eq. (8-3b),

$$\begin{array}{l}
\dfrac{\partial \varepsilon_{11}}{\partial x_1} = 0 \\
\varepsilon_{22} = \varepsilon_{33} = -\dfrac{\nu}{E}\sigma_{11} = -\nu\varepsilon_{11}
\end{array} \quad , \tag{8-16}$$

integration of the normal strains regarding eq. (8-11) finally yields

$$\begin{array}{l}
u_1(x_1, x_2, x_3) = \varepsilon_{11} x_1 + c_1 \\
u_2(x_1, x_2, x_3) = \varepsilon_{22} x_2 + c_2 = -\nu\varepsilon_{11} x_2 + c_2 \\
u_3(x_1, x_2, x_3) = \varepsilon_{33} x_3 + c_3 = -\nu\varepsilon_{11} x_3 + c_3
\end{array} \ . \tag{8-17}$$

We assume the following boundary conditions for displacements and obtain the integration constants as

$$
\begin{aligned}
&u_1(x_1 = 0, x_2, x_3) = 0 \quad \Rightarrow \quad c_1 = 0 \\
&u_2(x, x_2 = 0, x_3) = 0 \quad \Rightarrow \quad c_2 = 0 \quad . \\
&u_3(x_1, x_2, x_3 = 0) = 0 \quad \Rightarrow \quad c_3 = 0
\end{aligned}
\tag{8-18}
$$

The normal stresses, σ_{11}, are generated by the normal force, according to eq. (8-9),

$$
F = N(x_1) = \iint_{A_0} \sigma_{11}\, dA = \sigma_{11} A_0 \quad \Rightarrow \quad \sigma_{11} = \frac{F}{A_0} \, .
\tag{8-19}
$$

The *elongation* of the bar is

$$
\Delta L = L - L_0 = u_1(L_0) = \varepsilon_{11} L_0 = \frac{L_0}{EA_0} F \quad ,
\tag{8-20}
$$

and the *reduction of area*

$$
\Delta A = A - A_0 = b_0 h_0 - \left(b_0 + \Delta b\right)\left(h_0 + \Delta h\right),
\tag{8-21}
$$

with

$$
\begin{aligned}
\Delta b &= u_2\left(+\tfrac{b_0}{2}\right) - u_2\left(-\tfrac{b_0}{2}\right) = \varepsilon_{22} b_0 = -\nu \varepsilon_{11} b_0 \\
\Delta h &= u_3\left(+\tfrac{h_0}{2}\right) - u_3\left(-\tfrac{h_0}{2}\right) = \varepsilon_{33} h_0 = -\nu \varepsilon_{11} h_0
\end{aligned} \quad ,
\tag{8-22}
$$

and hence

$$
\Delta A == A_0\left(-2\nu\varepsilon_{11} + \nu^2 \varepsilon_{11}^2\right) \approx -2\nu\varepsilon_{11} A_0 = \frac{2\nu}{E} F \quad ,
\tag{8-23}
$$

assuming small deformations, i.e. $\varepsilon_{11}^2 \ll \varepsilon_{11}$. Note that these results are independent of the shape of the cross section.

By differentiating eq. (8-19), we obtain the equilibrium condition (8-8) for the normal force,

$$
\frac{dN}{dx_1} = \frac{d\sigma_{11}}{dx_1} A_0 + \frac{dA_0}{dx_1} \sigma_{11} = 0 \quad ,
\tag{8-24}
$$

and because of the equilibrium equation for the normal stress, $(8\text{-}14)_1$, this results in the condition,

$$
\frac{dA_0}{dx_1} = 0 \quad ,
\tag{8-25}
$$

that the cross section of the bar has to be constant. In engineering practice, nevertheless, the above-mentioned formulae are also applied to bars with varying cross section, if dA_0 / dx_1 is sufficiently small.

8.4 Bending of a Beam

We consider a beam of rectangular cross section[48], $A_0 = b_0 h_0$, which is subject to a bending moment, M_2, around the $\mathbf{e}_2$-axis,

$$\boxed{M_2 = \iint_{A_0} x_3 \sigma_{11}\, dA = \int_{x_2=-\frac{b_0}{2}}^{+\frac{b_0}{2}} \int_{x_3=-\frac{h_0}{2}}^{+\frac{h_0}{2}} x_3 \sigma_{11}\, dx_2\, dx_3}, \tag{8-26}$$

see Fig. 8.4. Normal forces vanish,

$$\boxed{N = \iint_{A_0} \sigma_{11}\, dA = \int_{x_2=-b_0/2}^{+b_0/2} \int_{x_3=-h_0/2}^{+h_0/2} \sigma_{11}\, dx_2\, dx_3 = 0}, \tag{8-27}$$

as well as shear forces, $Q_3 = \partial M_2 / \partial x_1 = 0$, $Q_2 = -\partial M_3 / \partial x_1 = 0$. The surfaces $\mathbf{n} = \mathbf{e}_2$ and $\mathbf{n} = \mathbf{e}_3$ of the bar are stress free as for the tensile bar above, see eq. (8-10). Since no normal force, N, and no bending moment, M_3, are acting, this loading case is denoted *pure uniaxial bending*. Due to the linearity of the problems, the case of biaxial bending with normal force is obtained by superposition of the elementary cases.

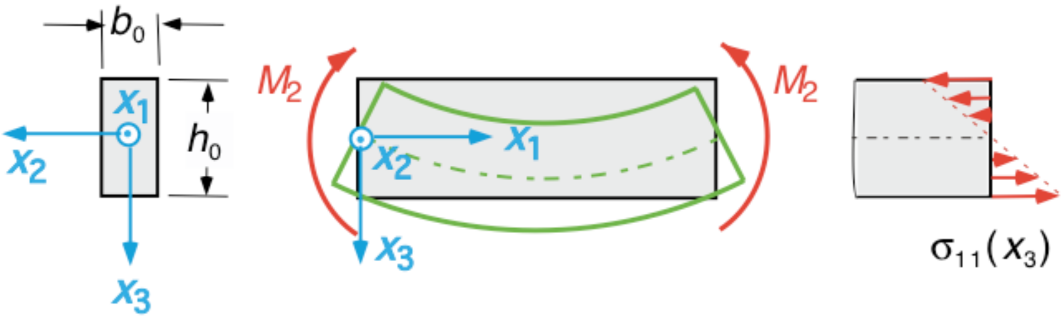


Figure 8-4: Beam subject to a bending moment

For solving the boundary value problem, assumptions for the kinematics of deformation have to be made again. The first one is, that we consider a *plane problem*, where the deformation occurs only in the ($\mathbf{e}_1$, $\mathbf{e}_3$)-plane, i.e. normal to $\mathbf{e}_2$,

$$\boxed{\frac{\partial u_1}{\partial x_2} = \frac{\partial u_2}{\partial x_1} = \frac{\partial u_3}{\partial x_2} = \frac{\partial u_2}{\partial x_3} = 0}. \tag{8-28}$$

For this, the $\mathbf{e}_3$-axis has to be a principal axis of the cross section, which is realised if the cross section is symmetric with respect to the $\mathbf{e}_3$-axis. It is an approximation, however, as HOOKE's law yields a non-zero strain $\varepsilon_{22}(x_3) = -\nu\varepsilon_{11}(x_3)$ and hence $\partial\varepsilon_{22}/\partial x_3 = \partial^2 u_2 / \partial x_3 \partial x_2 \neq 0$. In addition to eq. (8-27), BERNOULLI[49] introduced the *hypothesis*, that all *cross sections remain plane and perpendicular to the beam axis* during deformation,

$$\boxed{\varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = 0}. \tag{8-29}$$

[48] The following considerations hold for all cross sections which are symmetric with respect to the $\mathbf{e}_3$-axis.
[49] JACOB BERNOULLI (1654.1705)

From eq. (8-27), we can conclude as above, that all shear stresses have to vanish, and hence the boundary conditions eq. $(8\text{-}10)_3$ and $Q_2 = Q_3 = 0$ at $x_1 = 0, L_0$. are fulfilled. Likewise, the equilibrium conditions (8-1), neglecting body forces, and the boundary conditions for σ_{22} and σ_{33}, eq. $(8\text{-}10)_{1,2}$, yield $\sigma_{22} = \sigma_{33} = 0 \quad \forall\, x, y, z$, i.e. a uniaxial stress state, $\mathbf{S} = \sigma_{11}\,\mathbf{e}_1\mathbf{e}_1$. The equilibrium condition for σ_{11} and HOOKE's law result in $\partial\varepsilon_{11}/\partial x_1 = 0$, again, but unlike as in tension, integration permits that $\varepsilon_{11} = f(x_3)$, as $\partial u_1/\partial x_3 \neq 0$ according to BERNOULLI's hypothesis. Instead, we can differentiate eq. (8-28) twice and obtain

$$\frac{\partial^2}{\partial x_3 \partial x_1}\left(\frac{\partial u_1}{\partial x_3}\right) = \frac{\partial^2 \varepsilon_{11}}{\partial x_3^2} = -\frac{\partial^2}{\partial x_3 \partial x_1}\left(\frac{\partial u_3}{\partial x_1}\right) = -\frac{\partial^2 \varepsilon_{33}}{\partial x_1^2} \tag{8-30}$$

Applying HOOKE's law, $\varepsilon_{33} = -\nu\varepsilon_{11}$, this results 30

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_3^2} = \nu \frac{\partial^2 \varepsilon_{11}}{\partial x_1^2} = 0\,. \tag{8-31}$$

Integration shows that strains and hence stresses are linearly distributed along x_3,

$$\varepsilon_{11}(x_3) = c_0 + c_1 x_3 \quad \Rightarrow \quad \sigma_{11}(x_3) = E\left(c_0 + c_1 x_3\right). \tag{8-32}$$

The latter is also known as NAVIER´s[50] hypothesis, which is equivalent to BERNOULLI's hypothesis for linear elastic materials. For nonlinear material behaviour, as in the theory of plasticity, BERNOULLI's hypothesis is still kept, whereas NAVIER´s hypothesis does not hold any more. The integration constants can be determined from eqs. (8-25) and (8-26),

$$\begin{aligned} N &= \iint_{A_0} \sigma_{11}\, dA = E \int_{x_2=-b_0/2}^{+b_0/2} \int_{x_3=-h_0/2}^{+h_0/2} \left(c_0 + c_1 x_3\right) dx_2\, dx_3 = E c_0 A_0 = 0 \\ M_2 &= \iint_{A_0} \sigma_{11}\, dA = E \int_{x_2=-b_0/2}^{+b_0/2} \int_{x_3=-h_0/2}^{+h_0/2} z\left(c_0 + c_1 x_3\right) dx_2\, dx_3 = E\frac{b_0 h_0^3}{12} c_1 \end{aligned} \quad , \tag{8-33}$$

as $\quad c_0 = 0 \quad , \quad c_1 = \dfrac{12}{E b_0 h_0^3} M_2 \quad .$

The *stress distribution* in the cross section of a beam subject to uniaxial pure bending is hence

$$\sigma_{11}(x_3) = \frac{12}{b_0 h_0^3} M_2 x_3 = \frac{M_2}{I_{22}} x_3 \;, \tag{8-34}$$

and the *maximum stress* develops at the outer fibres of the cross section,

$$\sigma_{11}^{\max} = \left|\sigma_{11}\right|_{x_3 = \pm h_0/2} = \frac{M_2}{I_2}\frac{h_0}{2} = \frac{6 M_2}{b_0 h_0^2} = \frac{M_2}{W_2} \quad . \tag{8-35}$$

This is actually the mathematical formulation of GALILEI's experimental result, that the "bending resistance" is proportional to the width, *b*, and the square of the height, *h*, of the cross section, see section 2.2 and Fig. 2-1. Eqs. (8-34), (8-35) hold for any arbitrary simply symmetric cross section (with x_3 as axis of symmetry) as the *sectional moment of inertia*,

[50] CLAUDE LOUIS MARIE HENRI NAVIER (1785-1836)

$$I_{22} = \iint_{A_0} x_3^2 \, dA \ , \tag{8-36}$$

and the *section modulus*,

$$W_2 = \frac{I_{22}}{x_3^{\max}} \ , \tag{8-37}$$

are introduced and calculated for the respective cross section. From the assumption on the deformation kinetics, namely BERNOULLI's hypothesis, we can calculate the deformation of the beam. We differentiate eq. (8-29) once,

$$\frac{\partial^2 u_3}{\partial x_1^2} = -\frac{\partial^2 u_1}{\partial x_1 \partial x_3} = -\frac{\partial \varepsilon_{11}}{\partial x_3} = -c_1 \ , \tag{8-38}$$

introduce

$$\bar{u}_3(x_1) = u_3\big|_{x_3=0} \tag{8-39}$$

as displacement of the *beam axis,* $x_3 = 0$, and thus obtain with the help of eq. (8-33)

$$\frac{\mathrm{d}^2\bar{u}_3}{\mathrm{d}x_1^2} = -\frac{M_2}{EI_{22}} \ , \tag{8-40}$$

which is the differential equation of the *deflexion line* (elastic line) of a beam subject to bending. The deflection is calculated by integrating (8-39) twice and introducing boundary conditions for the displacement $\bar{u}_3(x_1)$ or its derivative $d\bar{u}_3 / dx_1$.

Superposition of a normal force N and a second bending moment M_3 yields the stress distribution in a beam with double symmetric cross section under both, general bending and tension,

$$\sigma_{11} = \frac{M_2}{I_{22}} x_3 - \frac{M_3}{I_{33}} x_2 + \frac{N}{A_0} \ . \tag{8-41}$$

with

$$I_{33} = \iint_{A_0} x_2^2 \, dA \tag{8-42}$$

As for the bar under tensile load, neither the cross section, A, nor the bending moments, M_2, M_3, may depend on x_1, as otherwise the stress state will not be uniaxial any more. It follows from eq. (8-8), that no shear forces are allowed. In engineering practice however, the formulae are also applied for beams with shear forces, provided that the cross section dimensions are small compared to the length dimension, $L_0^2 \gg A_0$.

8.5 Simple Torsion

We consider a rod with circular cross section[51], $A_0 = \pi R_0^2$, which is subject to a torque, M_T, around the $\mathbf{e}_1$-axis.

$$\boxed{M_T = \iint\limits_A \left(x_2 \sigma_{13} - x_3 \sigma_{12} \right) dA} \; , \tag{8-43}$$

see Fig. 8.5. Shear forces vanish throughout the rod,

$$Q_2 = \iint\limits_A \sigma_{12}\, dA = Q_3 = \iint\limits_A \sigma_{13}\, dA = 0 \;\; , \tag{8-44}$$

and as the direction of $\mathbf{e}_2$, $\mathbf{e}_3$, is obviously arbitrary, the stress and strain state has to be *axisymmetric*, i.e. $\partial / r\partial\varphi = 0$.

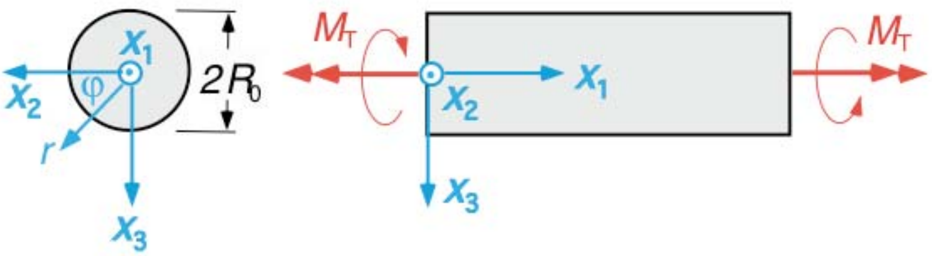


Figure 8-5: Rod subject to torque

The cylindrical surface, $\mathbf{n} = \mathbf{e}_r$, of the bar is stress free, which can be expressed in a cylindrical coordinate system, $\left\{ \mathbf{e}_1, \mathbf{e}_r, \mathbf{e}_\varphi \right\}$, most conveniently

$$\sigma_{rr}\big|_{r=R_0} = \sigma_{r1}\big|_{r=R_0} = \sigma_{r\varphi}\big|_{r=R_0} = 0 \quad \forall\; x_1, \varphi \tag{8-45}$$

As $\sigma_{1r} = \sigma_{r1} = 0 \quad \forall\; x_1, \varphi$, the stress vector can be written as

$$\mathbf{t}_1 = \sigma_{11}\mathbf{e}_1 + \sigma_{12}\mathbf{e}_2 + \sigma_{13}\mathbf{e}_3 = \sigma_{11}\mathbf{e}_1 + \sigma_{1\varphi}\mathbf{e}_\varphi \;\; , \tag{8-46}$$

with

$$\sigma_{12} = -\sigma_{1\varphi} \sin\varphi \quad , \quad \sigma_{12} = \sigma_{1\varphi} \cos\varphi \, , \tag{8-47}$$

which satisfies eq. (8-43). Employing $x_2 = r\cos\varphi$, $x_3 = r\sin\varphi$, eq. (8-42) can be written as

$$M_T = \iint\limits_{A_0} r\,\sigma_{1\varphi}\, dA = \int\limits_{r=0}^{R_0} \int\limits_{\varphi=0}^{2\pi} r\,\sigma_{1\varphi}\, r\, dr\, d\varphi = 2\pi \int\limits_{r=0}^{R_0} r^2 \sigma_{1\varphi}\; dr \;\; , \tag{8-48}$$

regarding that the stress state is axisymmetric.

Again, an assumption for the deformation kinematics is made, namely that the rod is neither elongated nor its cross section constricted, all cross sections remain plane and the cross-section radii and the generatrices of the cylinder remain straight, which is expressed by the following ansatz for the displacements

[51] A circular cross section is postulated to ensure a warp free torsion.

$$\begin{aligned} u_1(x_1,r,\varphi) &= 0 \\ u_r(x_1,r,\varphi) &= 0 \\ u_\varphi(x_1,r,\varphi) &= c_1 x_1 r \end{aligned} \quad . \tag{8-49}$$

We can immediately conclude from eq. (8-1c) that $\varepsilon_{11} = \varepsilon_{rr} = \varepsilon_{\varphi\varphi} = 0$ and hence according to HOOKE's law $\sigma_{11} = \sigma_{rr} = \sigma_{\varphi\varphi} = 0$, which satisfies $N = 0$ and the boundary conditions (8-44) for σ_{rr}. Furthermore, as

$$\begin{aligned} \sigma_{1r} &= 2G\varepsilon_{1r} = G\left(\frac{\partial u_1}{\partial r} + \frac{\partial u_r}{\partial x_1}\right) = 0 \\ \sigma_{1\varphi} &= 2G\varepsilon_{1\varphi} = G\left(\frac{\partial u_1}{r\partial \varphi} + \frac{\partial u_\varphi}{\partial x_1}\right) = c_1 r \\ \sigma_{r\varphi} &= 2G\varepsilon_{r\varphi} = G\left(\frac{\partial u_r}{r\partial \varphi} + \frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right) = 0 \end{aligned} \quad , \tag{8-50}$$

we obtain the stress tensor $\mathbf{S} = \sigma_{1\varphi}\,\mathbf{e}_1\mathbf{e}_\varphi + \sigma_{\varphi 1}\,\mathbf{e}_\varphi\mathbf{e}_1$ with $\sigma_{1\varphi} = \sigma_{\varphi 1}$ which satisfies the boundary conditions (8-44). From eq. (8-47) the constant c_1 can be determined,

$$M_\mathrm{T} = 2\pi c_1 \int_{r=0}^{R_0} r^3\,dr = \frac{\pi}{2} c_1 R_0^4 . \tag{8-51}$$

The *distribution of shear stresses* due to torsion of a circular rod is

$$\sigma_{1\varphi} = \sigma_{\varphi 1} = \frac{2}{\pi}\frac{M_\mathrm{T}}{R_0^4} r = \frac{M_\mathrm{T}}{I_p} r , \tag{8-52}$$

and the *shear angle*

$$\gamma_{\varphi 1} = 2\varepsilon_{\varphi 1} = \frac{\partial u_\varphi}{\partial x_1} = \frac{\sigma_{\varphi 1}}{G} = \frac{M_\mathrm{T}}{G I_p} r . \tag{8-53}$$

I_p is the polar *moment of inertia*,

$$I_p = \iint_{A_0} r^2 dA . \tag{8-54}$$

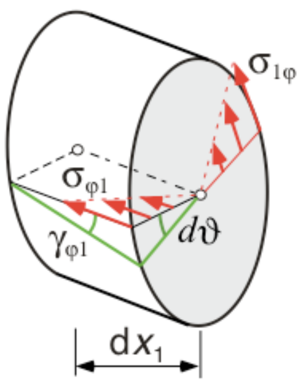


Figure 8-6: Deformation of cylindrical rod under torsion

The deformation of a rod under torsion is described by the twisting angle, $\vartheta(x_1)$, Fig. 8-6, which depends on $\gamma_{\varphi 1}$ by $\gamma_{\varphi 1}\, dx_1 = r\, d\vartheta$, and is hence calculated from the differential equation

$$\frac{d\vartheta}{dx_1} = \frac{M_T}{GI_p} . \tag{8-55}$$

Despite the assumption made for deriving the above equations, they are also applied for varying $M_T(x_1)$ and $A_0(x_1)$ in engineering practice.

8.6 Cylinder Under Internal Pressure

Consider the cylindrical part of a pressure vessel with wall thickness, t, length, L, and (mean) radius, R, which is subject to internal excess pressure p, Fig 8-7. The geometry and loading are axisymmetric and thus the stress and strain fields are, $\partial / r\partial\varphi = 0$. If L is sufficiently large, at least $L > 2R$, all shear stresses may be neglected and $\{\mathbf{e}_1, \mathbf{e}_r, \mathbf{e}_\varphi\}$ are principal axes of the stress tensor, $\mathbf{S} = \sigma_{11}\,\mathbf{e}_1\mathbf{e}_1 + \sigma_{rr}\,\mathbf{e}_r\mathbf{e}_r + \sigma_{\varphi\varphi}\,\mathbf{e}_\varphi\mathbf{e}_\varphi$.

***Thin-walled vessel*:**

We first assume, that the wall is thin, $t \ll R$, and hence, the normal stresses in radial direction, σ_{rr}, are much smaller than the circumferential (tangential) stresses, $\sigma_{rr} \ll \sigma_{\varphi\varphi}$, and can hence be neglected. With these assumptions, the boundary value problem has become statically determinate and can hence be solved by applying the equations of static equilibrium,

$$\begin{aligned} &\frac{\partial\sigma_{11}}{\partial x_1} = 0 \quad \Rightarrow \quad \sigma_1 = c_1 \\ &\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\left(\sigma_{rr} - \sigma_{\varphi\varphi}\right) \approx \frac{p}{t} - \frac{1}{R}\sigma_{\varphi\varphi} = 0 \end{aligned} . \tag{8-56}$$

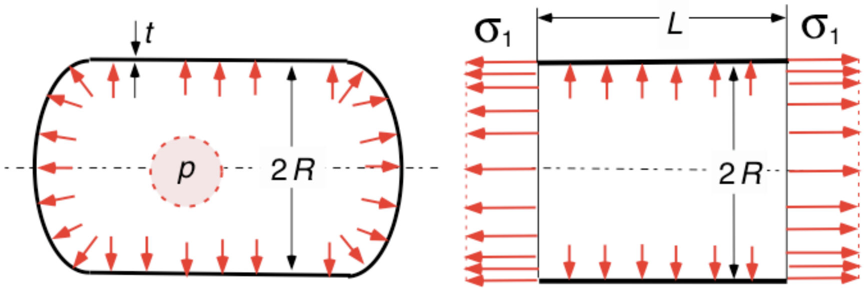


Figure 8.7: Thin-walled cylinder under internal pressure

Regarding the boundary conditions for the radial and axial stresses in the cylinder barrel,

$$\begin{aligned} &\sigma_{rr}\big|_{R-t/2} = -p \quad , \quad \sigma_{rr}\big|_{R+t/2} = 0 \\ &N = \int_{r=R-t/2}^{R+t/2} \int_{\varphi=0}^{2\pi} \sigma_{11}\, r\, d\varphi\, dr = 2\pi R t\, c_1 = p\pi R^2 \end{aligned} , \tag{8-57}$$

we obtain

$$\sigma_{11} = \frac{1}{2}\frac{R}{t}p \quad , \quad \sigma_{\varphi\varphi} = \frac{R}{t}p = 2\sigma_1 . \tag{8-58}$$

The circumferential stresses are twice the axial stresses, and $\left|\sigma_{rr}/\sigma_{\varphi\varphi}\right| \le t/R \ll 1$, as assumed. Failure of a pressure vessel will thus always occur by axial ripping of the cylinder barrel.

***Thick-walled vessel*:**

The assumption that radial stresses can be neglected, $\sigma_{rr} \ll \sigma_{\varphi\varphi}$, is not valid any more, and equilibrium results in a differential equation

$$\frac{d\sigma_{rr}}{dr} + \frac{1}{r}\left(\sigma_{rr} - \sigma_{\varphi\varphi}\right) = 0 . \tag{8-59}$$

A second differential equation is obtained from the strain relations (8-1c)

$$\varepsilon_{rr} = \frac{du_r}{dr} \quad , \quad \varepsilon_{\varphi\varphi} = \frac{u_r}{r} \ , \tag{8-60}$$

resulting in the compatibility condition

$$\frac{d\varepsilon_{\varphi\varphi}}{dr} = \frac{1}{r}\left(\varepsilon_{rr} - \varepsilon_{\varphi\varphi}\right) \ . \tag{8-61}$$

Introducing HOOKE's law, this equation takes the form

$$r\left(\nu\frac{d\sigma_{rr}}{dr} - \frac{d\sigma_{\varphi\varphi}}{dr}\right) + \left(1-\nu\right)\left(\sigma_{rr} - \sigma_{\varphi\varphi}\right) = 0 . \tag{8-62}$$

Eqs. (8-59) and (8-62) represent two first order differential equations for the two unknowns $\sigma_{rr}(r)$ and $\sigma_{\varphi\varphi}(r)$. Alternatively, HOOKE's law and eq. (8.60) can be introduced in eq. (8-59), resulting in a second order EULERian differential equation for $u_r(r)$.

$$\frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2} = 0 \ . \tag{8-63}$$

The two integration constants can be determined from the boundary conditions $(8\text{-}57)_1$. Finally, LAMÉ's solution is obtained

$$\begin{aligned} \sigma_{rr} &= \frac{R_\mathrm{i}^2}{R_\mathrm{o}^2 - R_\mathrm{i}^2}\left[1 - \left(\frac{R_\mathrm{o}}{r}\right)^2\right]p \\ \sigma_{\varphi\varphi} &= \frac{R_\mathrm{i}^2}{R_\mathrm{o}^2 - R_\mathrm{i}^2}\left[1 + \left(\frac{R_\mathrm{o}}{r}\right)^2\right]p \end{aligned} \ , \tag{8-64}$$

with $R_\mathrm{i} = R - t/2$, $R_\mathrm{o} = R + t/2$. Axial stresses have to obey eq. $(8\text{-}56)_1$ and $(8\text{-}57)_2$. As c_1 may now depend on r, the integration of $(8\text{-}57)_2$ requires an assumption on $\sigma_1(r)$. Commonly, $\partial\sigma_1/\partial r = 0$ is assumed as above, which results in

$$\sigma_{11} = \frac{R_\mathrm{i}^2}{R_\mathrm{o}^2 - R_\mathrm{i}^2}\,p = \frac{1}{2}\left(\sigma_{rr} + \sigma_{\varphi\varphi}\right) \ . \tag{8-65}$$

For both cases, the stresses in the domed ends are less simple to determine and depend on the shape of the vessel heads. They are hence not considered here.

8.7 Plane Stress State in a Disc

We consider a plane circular disc of constant thickness *h*, which is subject to in-plane loading only. No normal stresses, σ_{11}, are supposed to occur in x_1-direction, so the stress tensor is $\mathbf{S} = \sigma_{rr}\,\mathbf{e}_r\mathbf{e}_r + \sigma_{\varphi\varphi}\,\mathbf{e}_\varphi\mathbf{e}_\varphi + \sigma_{r\varphi}\,\mathbf{e}_r\mathbf{e}_r + \sigma_{\varphi r}\,\mathbf{e}_\varphi\mathbf{e}_r$. Two examples are given in the following.

Shrink fit assembly

Shrink fit assemblies are frictional shaft to collar connexions transmitting torques. The outer radius of the shaft, $R_\mathrm{o}^{(\mathrm{s})}$, is greater than the inner radius of the collar, $R_\mathrm{i}^{(\mathrm{c})}$, with $\Delta R = R_\mathrm{o}^{(\mathrm{s})} - R_\mathrm{i}^{(\mathrm{c})} > 0$ being the misfit. During the joining process, the collar is either heated or the shaft cooled down. In the assembled state at room temperature, the radial displacements of shaft and collar have to meet the condition

$$u_r^{(\mathrm{c})}(R_0) - u_r^{(s)}(R_0) = \Delta R\,, \tag{8-66}$$

where $R_\mathrm{i}^{(\mathrm{c})} \approx R_\mathrm{o}^{(\mathrm{s})} = R_0$ has been assumed for simplicity, since ΔR is small. Depending on ΔR, the stresses in the joint cause elastic or elasto-plastic deformations. It is assumed, that the deformations are elastic. The stress state is axisymmetric, $\partial / r\partial\varphi = 0$, so that the static problem results in the two first order differential equations (8-59) and (8-62) for $\sigma_{rr}(r)$ and $\sigma_{\varphi\varphi}(r)$ or the second order differential equation (8-63) for $u_\mathrm{r}(r)$ again, which have to be complemented by boundary conditions. Additionally to eq. (8-66), we have the conditions for the radial displacement in the shaft and the radial stresses in the collar,

$$u_r^{(\mathrm{s})}(0) = 0 \quad , \quad \sigma_{rr}^{(\mathrm{c})}\left(R_\mathrm{o}^{(\mathrm{c})}\right) = 0\,, \tag{8-67}$$

and the transition condition for radial stresses,

$$\sigma_{rr}^{(\mathrm{s})}(R_0) = \sigma_{rr}^{(\mathrm{c})}(R_0)\,. \tag{8-68}$$

The solution of eq. (8-63) is

$$u_r^{(\mathrm{s,c})}(r) = \frac{C_1^{(\mathrm{s,c})}}{r} + C_2^{(\mathrm{s,c})} r\,, \tag{8-69}$$

yielding the stresses

$$\begin{aligned}\sigma_{rr}^{(\mathrm{s,c})}(r) &= E^{(\mathrm{s,c})}\left[-\frac{C_1^{(\mathrm{s,c})}}{\left(1+\nu^{(\mathrm{s,c})}\right)r^2} + \frac{C_2^{(\mathrm{s,c})}}{1-\nu^{(\mathrm{s,c})}}\right]\\ \sigma_{\varphi\varphi}^{(\mathrm{s,c})}(r) &= E^{(\mathrm{s,c})}\left[\frac{C_1^{(\mathrm{s,c})}}{\left(1+\nu^{(\mathrm{s,c})}\right)r^2} + \frac{C_2^{(\mathrm{s,c})}}{1-\nu^{(\mathrm{s,c})}}\right]\end{aligned}. \tag{8-70}$$

From $u_r^{(\mathrm{s})}(0) = 0$, we obtain $C_1^{(\mathrm{s})} = 0$, which ensures that the stresses in the shaft are finite for $r \to 0$. The remaining three constants can be determined from eqs. (8-66) to (8-68)

$$\begin{aligned}&C_2^{(\mathrm{s})} = -\left(1-\nu^{(\mathrm{s})}\right)\frac{\left(\lambda^2-1\right)\kappa}{K}\\ &C_1^{(\mathrm{c})} = \left(1+\nu^{(\mathrm{c})}\right)\frac{\mu\kappa\lambda^2 R_0^2}{K} \quad , \quad C_2^{(\mathrm{c})} = \left(1-\nu^{(\mathrm{c})}\right)\frac{\mu\kappa}{K}\end{aligned}, \tag{8-71}$$

with the abbreviations

$$\kappa = \frac{\Delta R}{R_0} \quad , \quad \lambda = \frac{R_o^{(c)}}{R_i^{(c)}} = \frac{R_o^{(c)}}{R_0} \quad , \quad \mu = \frac{E^{(s)}}{E^{(c)}} \quad ,$$
$$K = \left[\left(1+\nu^{(c)}\right)\lambda^2 + \left(1-\nu^{(c)}\right)\right]\mu + \left(1-\nu^{(s)}\right)\left(\lambda^2 - 1\right) \tag{8-72}$$

As $C_1^{(s)} = 0$, we obtain from eq, (8-69)

$$\sigma_{rr}^{(s)}(r) = \sigma_{\varphi\varphi}^{(s)}(r) = -p = -E^{(s)}\left(\lambda^2 - 1\right)\kappa \big/ K \quad , \quad 0 \le r \le R_0 , \tag{8-73}$$

for the stresses in the shaft, where p is the interference pressure. The stress distribution is plotted in Fig. 8.8 for $\kappa = 10^{-3}$, $\lambda = 2$, $\mu = 1$, and $\nu^{(c)} = \nu^{(s)} = 0.3$. Circumferential stresses are highest at the inner diameter.

The assumption of a plane stress state is applicable particularly for the collar. The stress state in the shaft varies with x_1 outside the range of contact to the collar and is hence non-homogeneous and three-dimensional. The analytical solution presented here is an estimate. A full 3D solution will require a numerical calculation by finite elements, for instance.

Spinning disk

Another loading case of practical importance is the spinning disk. Instead of the equilibrium equations, CAUCHY's field equations (6-13) have to be applied,

$$\frac{d\sigma_{rr}}{dr} + \frac{1}{r}\left(\sigma_{rr} - \sigma_{\varphi\varphi}\right) = \rho\ddot{u}_r = -\rho r\dot{\varphi}^2$$
$$\frac{d\sigma_{r\varphi}}{dr} + \frac{1}{r}\sigma_{r\varphi} = \rho\ddot{u}_\varphi = \rho r\ddot{\varphi} \quad . \tag{8-74}$$

The two differential equations (8-74) for normal and shear stresses are independent. Shear stresses, $\sigma_{r\varphi} = \sigma_{\varphi r}$, vanish, if and only if $\ddot{u}_\varphi = r\ddot{\varphi} = 0$, i.e. the angular velocity $\dot{\varphi} = \omega$ is constant. Radial and circumferential stresses are

$$\sigma_{rr}(r) = -\frac{\rho\omega^2}{8}\left[(3+\nu)r^2 + (1+\nu)C_1 - (1-\nu)\frac{C_2}{r^2}\right]$$
$$\sigma_{\varphi\varphi}(r) = -\frac{\rho\omega^2}{8}\left[(3+\nu)r^2 + (1+\nu)C_1 + (1-\nu)\frac{C_2}{r^2}\right] . \tag{8-75}$$

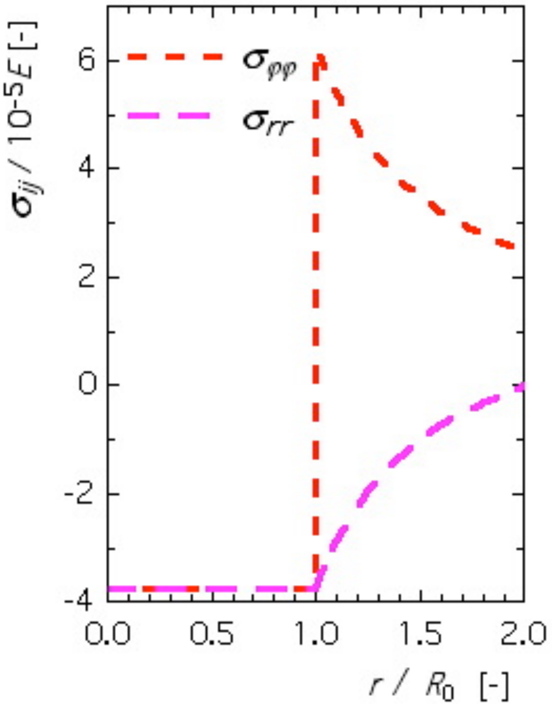
We assume the boundary conditions for a cylindrical ring with inner and outer radius, R_i, R_o, respectively as

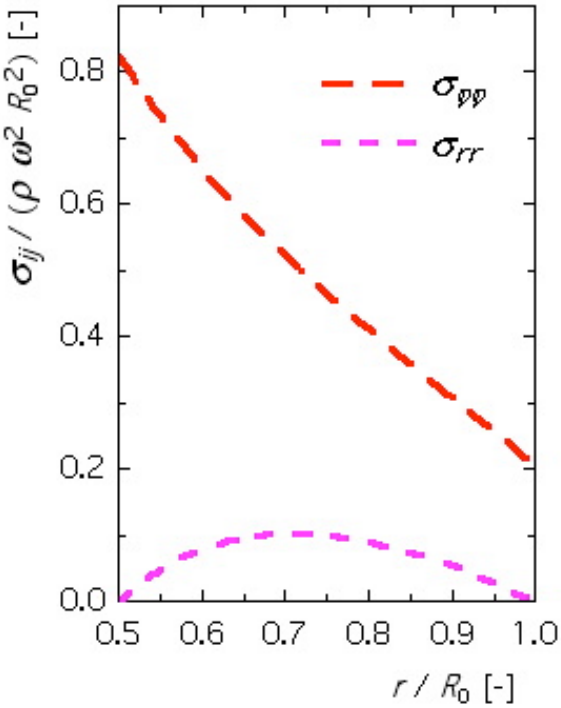
$$\sigma_{rr}(R_i) = \sigma_{rr}(R_o) = 0 , \tag{8-76}$$

and obtain the constants as

$$C_1 = -\frac{3+\nu}{1+\nu}\frac{R_o^4 - R_i^4}{R_o^2 - R_i^2}$$
$$C_2 = -\frac{3+\nu}{1-\nu}R_i^2 R_o^2 \quad . \tag{8-77}$$

The normalised stresses for $R_i / R_o = 0.5$ are plotted in Fig. 8.9. Circumferential stresses are highest at the inner diameter again.





<u>Figure 8.8</u>:

Stress distribution in a shrink fit assembly, $\kappa = 10^{-3}$, $\lambda = 2$, $\mu = 1$, $\nu^{(c)} = \nu^{(s)} = 0.3$

<u>Figure 8.9</u>:

Stress distribution in a spinning disc, $R_i / R_o = 0.5$

Appendix

A1. Notation and Operations

A1.1 Scalars, Vectors, Tensors - General Notation

Scalar

Latin or Greek, lowercase or capital, *italics* — b , H, α ,

Vector

Latin or Greek, lowercase, **bold** (or underscored) — $\mathbf{x}$, $\boldsymbol{\alpha}$ ($\underline{x}$, $\underline{\alpha}$)

components — Latin or Greek, lowercase, *italics* — x_i , α_i

with respect to orthogonal unit base vectors — $\mathbf{e}_i$ (i = 1, 2, 3)

suffix (index) notation — $\mathbf{x} = x_i \, \mathbf{e}_i$

2nd Order Tensor

Latin or Greek, capital, **bold** (or underscored) — $\mathbf{T}$, $\boldsymbol{\Theta}$ ($\underline{T}$, $\underline{\Theta}$)

components — Latin or Greek, capital or lowercase, *italics* — T_{ij} , σ_{ij} , θ_{ij}

suffix (index) notation — $\mathbf{T} = T_{ij} \; \mathbf{e}_i \mathbf{e}_j$

transposed tensor — $\mathbf{T}^{\mathrm{T}} = T_{ji} \; \mathbf{e}_i \mathbf{e}_j$

inverse tensor — $\mathbf{T}^{-1}$

unit tensor — $\mathbf{1} = \delta_{ij} \; \mathbf{e}_i \mathbf{e}_j$

deviatoric tensor — $\hat{\mathbf{T}} = \mathbf{T} - \tfrac{1}{3}(\mathrm{tr}\,\mathbf{T})\mathbf{1}$

4th Order Tensor

Latin or Greek, capital, double underscored — $\underline{\underline{\mathbf{C}}}$

components — Latin or Greek, capital, *italics* — C_{ijkl}

suffix (index) notation — $\underline{\underline{\mathbf{C}}} = C_{ijkl} \; \mathbf{e}_i \mathbf{e}_j \mathbf{e}_k \mathbf{e}_l$

Summation Convention of EINSTEIN

$$A_{ij} B_{jk} = \sum_{j=1}^{3} A_{ij} B_{jk} \;, \qquad C_{kk} = \sum_{k=1}^{3} C_{kk} \;, \qquad A_{ij} B_{ji} = \sum_{i=1}^{3} \sum_{j=1}^{3} A_{ij} B_{ji}$$

A1.2 Vector- und Tensor Algebra

Scalar Product (Dot Product)

two vectors: $\alpha = \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} = u_i v_i$ [52]

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \sin \sphericalangle(\mathbf{u},\mathbf{v})$$

$$\mathbf{u} \cdot \mathbf{u} = u_i u_i = \mathbf{u}^2 = |\mathbf{u}|^2$$

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$ KRONECKER symbol

vector and tensor: $\mathbf{a} = \mathbf{v} \cdot \mathbf{A} = v_i A_{ij}\, \mathbf{e}_j = a_j\, \mathbf{e}_j$, $\mathbf{b} = \mathbf{A} \cdot \mathbf{v} = v_j A_{ij}\, \mathbf{e}_i = b_i\, \mathbf{e}_i$

$\mathbf{v} \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{v}$ if and only if $\mathbf{A}$ is symmetric, $\mathbf{A} = \mathbf{A}^{\mathrm{T}}$, $A_{ij} = A_{ji}$

two (2nd order) tensors: $\mathbf{A} = \mathbf{B} \cdot \mathbf{C} = B_{ij} C_{jk}\, \mathbf{e}_i \mathbf{e}_k = A_{ik}\, \mathbf{e}_i \mathbf{e}_k$

$\mathbf{B} \cdot \mathbf{C} = \mathbf{C} \cdot \mathbf{B}$ if and only if $\mathbf{B}$ and $\mathbf{C}$ are symmetric

$$(\mathbf{B} \cdot \mathbf{C})^{\mathrm{T}} = \mathbf{C}^{\mathrm{T}} \cdot \mathbf{B}^{\mathrm{T}} = B_{ji} C_{kj}\, \mathbf{e}_i \mathbf{e}_k$$

$$\mathbf{A} \cdot \mathbf{A} = \mathbf{A}^2 = A_{ij} A_{jk}\, \mathbf{e}_i \mathbf{e}_k$$

$$\mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{1} = \delta_{ij}\, \mathbf{e}_i \mathbf{e}_j$$

Vector Product (Cross Product)

two vectors: $\mathbf{w} = \mathbf{u} \times \mathbf{v} = u_i v_j\, \mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{ijk} u_i v_j\, \mathbf{e}_k = -\mathbf{v} \times \mathbf{u}$

LEVI-CIVITA symbol (permutation symbol):

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } i, j, k \text{ take values in cyclic order} \\ -1 & \text{if } i, j, k \text{ take values in acyclic order} \\ 0 & \text{if two of } i, j, k \text{ take the same value} \end{cases}$$

$$\mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{ijk} \mathbf{e}_k$$

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \cos \sphericalangle(\mathbf{u},\mathbf{v}) \quad , \quad (\mathbf{u} \times \mathbf{v}) \perp (\mathbf{u},\mathbf{v})$$

Double Scalar Product

two (2nd order) tensors: $\alpha = \mathbf{B} \cdot\cdot \mathbf{C} = B_{ij} C_{ji} = B_{ij} C_{ij}$

4th and 2nd order tensor: $\mathbf{A} = \underline{\underline{\mathbf{C}}} \cdot\cdot \mathbf{B} = C_{ijkl} B_{kl}\, \mathbf{e}_i \mathbf{e}_j$

$$\alpha = \mathbf{A} \cdot\cdot \underline{\underline{\mathbf{C}}} \cdot\cdot \mathbf{B} = C_{ijkl} A_{ij} B_{kl}$$

[52] mind summation convention

Tensor Product

two vectors:	$\mathbf{A} = \mathbf{ab} = a_i b_j\, \mathbf{e}_i \mathbf{e}_j$
two 2nd order tensors	$\underline{\underline{\mathbf{C}}} = \mathbf{AB} = A_{ij} B_{kl}\, \mathbf{e}_i \mathbf{e}_j \mathbf{e}_k \mathbf{e}_l$

$$(\mathbf{ab})^{\mathrm{T}} = \mathbf{ba}$$

$$(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}} \mathbf{A}^{\mathrm{T}}$$

$$\mathbf{e}_i \mathbf{e}_j \cdot \mathbf{e}_k = \delta_{jk}\, \mathbf{e}_i$$

$$\mathbf{ab} \cdot \mathbf{c} = (\mathbf{ab}) \cdot \mathbf{c} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) = a_i b_j c_j\, \mathbf{e}_i$$

Vector Triple Product

$$\boxed{\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{v}(\mathbf{u} \cdot \mathbf{w}) - \mathbf{w}(\mathbf{u} \cdot \mathbf{v}) = u_j w_j v_i \mathbf{e}_i - u_j v_j w_i \mathbf{e}_i}$$

Scalar Triple Product

$$\boxed{\alpha = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{uvw}) = (\mathbf{vwu}) = (\mathbf{wuv}) = -(\mathbf{vuw}) = ... = \varepsilon_{ijk} u_i v_j w_k}$$

α is the volume of the parallelepiped determined by the three vectors **u**, **v**, **w**

$$(\mathbf{uvw}) = \det(\mathbf{uvw}) = \varepsilon_{ijk} u_i v_j w_k = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

$$= u_1 (v_2 w_3 - v_3 w_2) - u_2 (v_1 w_3 - v_3 w_1) + u_3 (v_1 w_2 - v_2 w_1)$$

$$\mathbf{e}_i \cdot (\mathbf{e}_j \times \mathbf{e}_k) = (\mathbf{e}_i \times \mathbf{e}_j) \cdot \mathbf{e}_k = \varepsilon_{ijk}$$

A1.3 Transformation of Vector und Tensor Components

An orthogonal and normalised system of base vectors $\mathbf{e}_i$ is rotated by a transformation

$$\mathbf{e}_i \to \hat{\mathbf{e}}_j : \quad \hat{\mathbf{e}}_j = a_{ij}\mathbf{e}_i \ .$$

Note that the 3×3=9 transformation coefficients, a_{ij}, do not represent the components of a tensor, though the operation looks similar to the rotation of a vector **v** or tensor **T** by an orthogonal tensor, **Q**, and the matrix $\left(a_{ij}\right)$ is also orthogonal. However, the coefficients a_{ij} do not refer to any tensor base, $\mathbf{e}_i\mathbf{e}_j$, but switch between two base-vector systems, and we look how the coordinates of one and the same vector or tensor change, if the respective system of base vectors is changed.

The inverse transformation is

$$\hat{\mathbf{e}}_j \to \mathbf{e}_i : \quad \mathbf{e}_j = \hat{a}_{ij}\hat{\mathbf{e}}_i = \hat{a}_{ij}a_{ki}\mathbf{e}_k \ .$$

and hence

$$\hat{a}_{ij}a_{ki} = a_{ji}\hat{a}_{ik} = \delta_{jk} \quad \Leftrightarrow \quad \hat{a}_{ij} = \left(a_{ij}\right)^{-1} = a_{ji} \ .$$

The transformation coefficients a_{ij} or $\hat{a}_{ij}$ are the direction cosines of the rotations $\mathbf{e}_i \to \hat{\mathbf{e}}_j$ or $\hat{\mathbf{e}}_j \to \mathbf{e}_i$, respectively.

$$\begin{aligned} a_{ij} &= \mathbf{e}_i \cdot \hat{\mathbf{e}}_j = \cos\left(\mathbf{e}_i, \hat{\mathbf{e}}_j\right) \\ \hat{a}_{ij} &= \hat{\mathbf{e}}_i \cdot \mathbf{e}_j = \cos\left(\hat{\mathbf{e}}_i, \mathbf{e}_j\right) \end{aligned} \ .$$

For a rotation in the $\left(\mathbf{e}_1, \mathbf{e}_2\right)$-plane by an angle α, the transformation matrix becomes

$$\left(a_{ij}\right) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \ ,$$

resulting in

$$\begin{aligned} \hat{\mathbf{e}}_1 &= \cos\alpha\,\mathbf{e}_1 + \sin\alpha\,\mathbf{e}_2 \\ \hat{\mathbf{e}}_2 &= -\sin\alpha\,\mathbf{e}_1 + \cos\alpha\,\mathbf{e}_2 \\ \hat{\mathbf{e}}_3 &= \mathbf{e}_3 \end{aligned} \ .$$

The components of a vector and a tensor

$$\mathbf{v} = v_i\,\mathbf{e}_i = \hat{v}_i\,\hat{\mathbf{e}}_i \quad , \quad \mathbf{T} = T_{ij}\,\mathbf{e}_i\mathbf{e}_j = \hat{T}_{ij}\,\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j \ .$$

transform like

$$\boxed{\begin{aligned} v_i &= a_{ij}\hat{v}_j \quad \text{or} \quad \hat{v}_i = \hat{a}_{ij}v_j = a_{ji}v_j \\ T_{ij} &= a_{ik}a_{jl}\hat{T}_{kl} \quad \text{or} \quad \hat{T}_{ij} = \hat{a}_{ik}\hat{a}_{jl}T_{kl} = a_{ki}a_{lj}T_{kl} \end{aligned}} \ .$$

A1.4 Vector und Tensor Analysis

Derivatives in Cartesian (rectilinear) coordinates

$$\mathbf{x} = x_i\,\mathbf{e}_i = x\,\mathbf{e}_x + y\,\mathbf{e}_y + z\,\mathbf{e}_z \quad , \qquad d\mathbf{x} = dx_i\,\mathbf{e}_i = dx\,\mathbf{e}_x + dy\,\mathbf{e}_y + dz\,\mathbf{e}_z$$

NABLA operator $$\nabla = \frac{\partial}{\partial \mathbf{x}} = \mathbf{e}_i\frac{\partial}{\partial x_i} = \mathbf{e}_x\frac{\partial}{\partial x} + \mathbf{e}_y\frac{\partial}{\partial y} + \mathbf{e}_z\frac{\partial}{\partial z}$$

Gradient $$\mathbf{v} = \nabla\phi = \operatorname{grad}\phi = \frac{\partial\phi}{\partial\mathbf{x}} = \frac{\partial\phi}{\partial x_i}\mathbf{e}_i = \phi_{,i}\,\mathbf{e}_i$$

$$\mathbf{T} = \operatorname{grad}\mathbf{v} = \left(\nabla\mathbf{v}\right)^{\mathrm{T}} = \frac{\partial\mathbf{v}}{\partial\mathbf{x}} = \frac{\partial\left(v_i\mathbf{e}_i\right)}{\partial x_j}\mathbf{e}_j = v_{i,j}\,\mathbf{e}_i\mathbf{e}_j$$

Divergence $$\alpha = \operatorname{div}\mathbf{v} = \nabla\cdot\mathbf{v} = \frac{\partial v_i}{\partial x_i} = v_{i,i}$$

$$\mathbf{w} = \operatorname{div}\mathbf{T} = \nabla\cdot\mathbf{T}^{\mathrm{T}} = \frac{\partial T_{kj}}{\partial x_i}\mathbf{e}_i\cdot\mathbf{e}_j\,\mathbf{e}_k = \frac{\partial T_{kj}}{\partial x_i}\delta_{ij}\,\mathbf{e}_k = \frac{\partial T_{ki}}{\partial x_i}\mathbf{e}_k = T_{ki,i}\,\mathbf{e}_k$$

Curl $$\mathbf{w} = \operatorname{curl}\mathbf{v} = \left(\nabla\times\mathbf{v}\right)^{\mathrm{T}} = \frac{\partial v_i}{\partial x_j}\mathbf{e}_i\times\underline{\mathbf{e}}_j = \frac{\partial v_i}{\partial x_j}\varepsilon_{ijk}\,\mathbf{e}_k = v_{i,j}\varepsilon_{ijk}\,\mathbf{e}_k$$

LAPLACEan $$\nabla\cdot\nabla = \nabla^2 = \frac{\partial^2}{\partial x_i\partial x_i} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Derivatives in cylindrical (curvilinear) coordinates

$$\mathbf{x} = r\,\mathbf{e}_r + z\,\mathbf{e}_z \quad , \qquad d\mathbf{x} = dr\,\mathbf{e}_r + r\,d\mathbf{e}_r + dz\,\mathbf{e}_z = dr\,\mathbf{e}_r + r\,d\varphi\,\mathbf{e}_\varphi + dz\,\mathbf{e}_z$$

$$\frac{\partial\mathbf{e}_r}{\partial\varphi} = \mathbf{e}_\varphi \quad , \quad \frac{\partial\mathbf{e}_\varphi}{\partial\varphi} = -\mathbf{e}_r \quad , \quad \frac{\partial\mathbf{e}_i}{\partial r} = \frac{\partial\mathbf{e}_i}{\partial z} = 0 \quad (i = r,\varphi,z)$$

NABLA operator $$\nabla = \frac{\partial}{\partial\mathbf{x}} = \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\varphi\frac{\partial}{r\partial\varphi} + \mathbf{e}_z\frac{\partial}{\partial z}$$

Divergence $$\alpha = \operatorname{div}\mathbf{v} = \nabla\cdot\mathbf{v} = \left(\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\varphi\frac{\partial}{r\partial\varphi} + \mathbf{e}_z\frac{\partial}{\partial z}\right)\cdot\left(v_r\mathbf{e}_r + v_\varphi\mathbf{e}_\varphi + v_z\mathbf{e}_z\right)$$

$$= \frac{\partial v_r}{\partial r} + \mathbf{e}_\varphi\cdot\left(\frac{v_r}{r}\frac{\partial\mathbf{e}_r}{\partial\varphi}\right) + \frac{1}{r}\frac{\partial v_\varphi}{\partial\varphi} + \frac{\partial v_z}{\partial z} = \frac{\partial v_r}{\partial r} - \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\varphi}{\partial\varphi} + \frac{\partial v_z}{\partial z}$$

and correspondingly for other operations.

Theorem of GAUSS:

Let $\mathbb{B}$ be a regular domain in the EUCLIDean space $\mathbb{E}^3$ with surface $\partial\mathbb{B}$, $\mathbf{n}$ the outer normal on $\partial\mathbb{B}$, $\mathbf{\Phi}$ a vector or (rank 2) tensor field being continuous on $\mathbb{B}$ and continuously differentiable in the interior of $\mathbb{B}$, then

$$\int_{\partial\mathbb{B}} \mathbf{\Phi} \cdot \mathbf{n} \, dA = \int_{\mathbb{B}} \operatorname{div} \mathbf{\Phi} \, dA$$

A2. 2nd Order Tensors and their Properties

A2.1 Inverse and Orthogonal Tensors

A tensor **T** is said to be *invertible* if there exists a tensor $\mathbf{T}^{-1}$, called the *inverse* of **T**, such that

$$\mathbf{T}\cdot\mathbf{T}^{-1}=\mathbf{T}^{-1}\cdot\mathbf{T}=\mathbf{1}\ .$$

A tensor **T** is said to be *orthogonal*, if **T** is invertible and $\mathbf{T}^{-1}=\mathbf{T}^{\mathrm{T}}$. Thus **T** is orthogonal, if and only if

$$\mathbf{T}\cdot\mathbf{T}^{\mathrm{T}}=\mathbf{T}^{\mathrm{T}}\cdot\mathbf{T}=\mathbf{1}\ .$$

The inverse of the transposed tensor $\mathbf{T}^{\mathrm{T}}$ is denoted by $\mathbf{T}^{-\mathrm{T}}$.

A2.2 Symmetric and Skew Tensors

Every tensor **T** can be decomposed into a symmetric and a skew part

$$\mathbf{T}=\tfrac{1}{2}\left(\mathbf{T}+\mathbf{T}^{\mathrm{T}}\right)+\tfrac{1}{2}\left(\mathbf{T}-\mathbf{T}^{\mathrm{T}}\right)=\operatorname{sym}\mathbf{T}+\operatorname{skw}\mathbf{T}$$

$$\operatorname{sym}\mathbf{T}=\tfrac{1}{2}\left(\mathbf{T}+\mathbf{T}^{\mathrm{T}}\right)=\left(\operatorname{sym}\mathbf{T}\right)^{\mathrm{T}}=\begin{pmatrix} T_{11} & \tfrac{1}{2}\left(T_{12}+T_{21}\right) & \tfrac{1}{2}\left(T_{13}+T_{31}\right) \\ \tfrac{1}{2}\left(T_{21}+T_{12}\right) & T_{22} & \tfrac{1}{2}\left(T_{23}+T_{32}\right) \\ \tfrac{1}{2}\left(T_{31}+T_{13}\right) & \tfrac{1}{2}\left(T_{32}+T_{23}\right) & T_{33} \end{pmatrix}$$

$$\operatorname{skw}\mathbf{T}=\tfrac{1}{2}\left(\mathbf{T}-\mathbf{T}^{\mathrm{T}}\right)=-\left(\operatorname{skw}\mathbf{T}\right)^{\mathrm{T}}=\begin{pmatrix} 0 & \tfrac{1}{2}\left(T_{12}-T_{21}\right) & \tfrac{1}{2}\left(T_{13}-T_{31}\right) \\ \tfrac{1}{2}\left(T_{21}-T_{12}\right) & 0 & \tfrac{1}{2}\left(T_{23}-T_{32}\right) \\ \tfrac{1}{2}\left(T_{31}-T_{13}\right) & \tfrac{1}{2}\left(T_{32}-T_{23}\right) & 0 \end{pmatrix}$$

Theorem

Given a skew tensor, $\mathbf{T}=-\mathbf{T}^{\mathrm{T}}$, there exists a unique vector **w** such that

$$\mathbf{T}\cdot\mathbf{u}=\mathbf{w}\times\mathbf{u} \qquad \text{(A-1)}$$

for every vector **u**. Conversely, given a vector **w**, there exists a unique skew tensor **T** such that (A-1) holds for every vector **u**.

The vector **w** is called the *dual vector* of the tensor **T.**

$$\mathbf{T}=\begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}\left(\mathbf{e}_i\mathbf{e}_j\right)$$

A2.3 Fundamental Invariants of a Tensor

Every (2nd order) tensor, **T**, has three (scalar valued) invariants, which do not change their value during transformation of the coordinate (base vector) system. The *fundamental* (principal) *invariants* are defined as

$$
\begin{aligned}
I_{\mathbf{T}} &= \operatorname{tr}\mathbf{T} = T_{ii} \\
II_{\mathbf{T}} &= \tfrac{1}{2}\left[\left(\operatorname{tr}\mathbf{T}\right)^2 - \operatorname{tr}\mathbf{T}^2\right] = \tfrac{1}{2}\left[T_{ii}T_{kk} - T_{ik}T_{ki}\right] \ . \\
III_{\mathbf{T}} &= \det\mathbf{T} = \varepsilon_{ijk}T_{i1}T_{j2}T_{k3} = \varepsilon_{ijk}T_{1i}T_{2j}T_{3k}
\end{aligned}
$$

Every linear combination of the three principal invariants is again invariant against a coordinate transformation. **T** and $\mathbf{T}^{\mathrm{T}}$ have the same invariants.

A2.4 Eigenvalues and Eigenvectors

Given a (2nd order) tensor, **T**, then vectors, **v**, are searched, for which

$$
\mathbf{T}\cdot\mathbf{v} = \lambda\mathbf{v} \qquad \text{or} \qquad \left(\mathbf{T} - \lambda\mathbf{1}\right)\cdot\mathbf{v} = \mathbf{0}\,. \tag{A-2}
$$

This is called the "*eigenvalue problem*". As the algebraic set of equations for determining the components of **v** is homogeneous, the solution condition is

$$
\det\left(\mathbf{T} - \lambda\mathbf{1}\right) = 0\,.
$$

It leads to the "*characteristic equation*"

$$
\lambda^3 - I_{\mathbf{T}}\lambda^2 + II_{\mathbf{T}}\lambda - III_{\mathbf{T}} = 0\,.
$$

If **T** is a symmetric tensor, $\mathbf{T} = \mathbf{T}^{\mathrm{T}}$, then all three roots of the characteristic equation are real, that means that **T** has exactly three (not necessarily distinct) real eigenvalues (or: principal values), λ_i (i = 1, 2, 3). The eigenvalues are, of course, invariants of the tensor. Eigenvectors (or: principal vectors) corresponding to two distinct eigenvalues of a symmetric tensor are orthogonal, $\mathbf{v}_i\cdot\mathbf{v}_j = 0$ for $i \neq j$, or: the principal directions corresponding to distinct principal values of **T** are orthogonal. As the algebraic set of equations (A-2) for determining the components of **v** is homogeneous, the eigenvectors are determined except for a scalar factor, i.e. their length remains undetermined. They can hence be normalised

$$
\mathbf{v}_i\cdot\mathbf{v}_j = \delta_{ij} = \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases}
$$

Eigenvectors form an orthogonal and normalised system of base vectors with respect to which the matrix of **T** is diagonal,

$$
\mathbf{T} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}\left(\mathbf{v}_i\mathbf{v}_j\right) = \lambda_1\,\mathbf{v}_1\mathbf{v}_1 + \lambda_2\,\mathbf{v}_2\mathbf{v}_2 + \lambda_3\,\mathbf{v}_3\mathbf{v}_3\,. \tag{A-3}
$$

Or: Given a symmetric tensor **T**, then with respect to its principal axes, $\mathbf{v}_i$, the matrix of **T** is diagonal. (A-3) is called the *spectral form* of **T**.

A2.5 Isotropic Tensor Functions

Real Valued Isotropic Tensor Function $\varphi(\mathbf{T})$

$$\varphi(\mathbf{T}) = \varphi(\mathbf{Q}\cdot\mathbf{T}\cdot\mathbf{Q}^{\mathrm{T}}) \quad \text{with} \quad \mathbf{Q}\cdot\mathbf{Q}^{\mathrm{T}} = \mathbf{1}$$

Theorem: Representation of real isotropic tensor functions

A real isotropic tensor function $\varphi(\mathbf{T})$ can be represented

- either as a function of the principal invariants of the tensor **T**

  $$\varphi(\mathbf{T}) = \phi(I_{\mathbf{T}}, II_{\mathbf{T}}, III_{\mathbf{T}})$$

- or as a symmetric function of the eigenvalues of **T**

  $$\varphi(\mathbf{T}) = \tilde{\phi}(\lambda_i) \quad (i = 1,2,3)$$

Tensor Valued Isotropic Tensor Function $\mathbf{\Phi}(\mathbf{T})$

$$\mathbf{\Phi}(\mathbf{T}) = \mathbf{\Phi}(\mathbf{Q}\cdot\mathbf{T}\cdot\mathbf{Q}^{\mathrm{T}}) = \mathbf{Q}\cdot\mathbf{\Phi}(\mathbf{T})\cdot\mathbf{Q}^{\mathrm{T}} \qquad \text{with} \quad \mathbf{Q}\cdot\mathbf{Q}^{\mathrm{T}} = \mathbf{1}$$

Theorem: Representation of tensorial isotropic tensor functions

A tensorial isotropic tensor function $\mathbf{\Phi}(\mathbf{T})$ can be represented

- either as a 2nd order polynomial of **T**

  $$\mathbf{\Phi}(\mathbf{T}) = \eta_0\mathbf{1} + \eta_1\mathbf{T} + \eta_2\mathbf{T}^2 \quad \text{with} \quad \eta_i = \eta_i(I_{\mathbf{T}}, II_{\mathbf{T}}, III_{\mathbf{T}})$$

- or by the spectral form

  $$\mathbf{\Phi}(\mathbf{T}) = \phi(\lambda_1,\lambda_2,\lambda_3)\mathbf{v}_1\mathbf{v}_1 + \phi(\lambda_2,\lambda_3,\lambda_1)\mathbf{v}_2\mathbf{v}_2 + \phi(\lambda_3,\lambda_1,\lambda_2)\mathbf{v}_3\mathbf{v}_3$$

  with one real function $\phi(\mu_i)$ of three arguments, μ_i, being symmetric in the second and the third, $\phi(\mu_1,\mu_2,\mu_3) = \phi(\mu_1,\mu_3,\mu_2)$, if **T** has the eigenvalues λ_i and the unit eigenvectors $\mathbf{v}_i$.

The eigenvectors of **T** are also the eigenvectors of $\mathbf{\Phi}(\mathbf{T})$, that is:

eigenvalue problem: $\mathbf{T}\cdot\mathbf{v}_i = \lambda_{(i)}\mathbf{v}_{(i)}$ [53], spectral form: $\mathbf{T} = \sum_{i=1}^{3}\lambda_i\,\mathbf{v}_i\mathbf{v}_i$,

eigenvalue problem: $\mathbf{\Phi}(\mathbf{T})\cdot\mathbf{v}_i = \phi_{(i)}\mathbf{v}_{(i)}$, spectral form: $\mathbf{\Phi}(\mathbf{T}) = \sum_{i=1}^{3}\phi_i\,\mathbf{v}_i\mathbf{v}_i$.

From the condition of isotropy, it follows that

$$\phi_1 = \phi_2 = \phi_3 = \phi(\lambda_i,\lambda_{i+1},\lambda_{i+2}) = \phi(\lambda_i,\lambda_{i+2},\lambda_{i+1}) \;, \qquad \text{i = 1, 2, 3 modulo 3.}$$

[53] no summation

A3 Physical Quantities and Units

A3.1 Definitions

Physical *quantities* are used for the qualitative and quantitative description of physical phenomena. They represent measurable properties of physical items, e.g. length, time, mass, density, force, energy, temperature, etc.

Every specific *value* of a quantity can be written as the product of a *number* and a *unit*. This product is invariant against a change of the unit.

Examples: 1 m^3 = 1000 cm^3, 1 m/s = 3,6 km/h

Physical quantities are denoted by *symbols*.

Examples: Length l, Area A, velocity v, force F

l = 1 km, A = 100 m^2, v = 5 m/s F = 10 kN

Base quantities are quantities, which are defined independently from each other in a way that all other quantities can be derived by multiplication or division.

Examples: Length, l, time, t, and mass, m, can be chosen as base quantities in *dynamics*.

The *unit* of a physical quantity is the value of a chosen and defined quantity out of all quantities of equal *dimension*.

Example: 1 meter is the unit of all quantities having the dimension of a length (height, width, diameter, ...)

The numerical value of a quantity G is denoted as $\{G\}$, its unit as $[G]$.

The invariance relation is hence $G = \{G\}\,[G]$

Base units are the units of *base quantities* The number of base units is hence always equal the number of base units.

Example: Meter, second and kilogramm can be chosen as base units in dynamics

$[l]$ = m, $[t]$ = s, $[m]$ = kg

For discriminating *symbols* (for quantities) from units, the former are printed in *italics*, the latter plain.

A3.2 SI Units

SI = Système International d'Unités - ISO 1000 (1973)

SI units are the seven *base units* of the seven *base quantities* of physics, namely

Base Quantity	SI Base Unit	
	Name	Character
Length	meter	m
Mass	kilogramm	kg
Time	second	s
Thermodynamic. Temperature	Kelvin	K
Electrical Curent	Ampere	A
Amount of Substance	mol	mol
Luminous Intensity	candela	cd

Derived units are composed by products or ratios of base units. The same holds for the unit characters. Some derived SI units have a special name and a special unit character

Quantity	SI Unit		Relation
	Name	Char.	
angle	radiant	rad	1 rad = 1 m / m
frequency	Hertz	Hz	1 Hz = 1 s^{-1}
force	Newton	N	1 N = 1 kg m s^{-2}
pressure, stress	Pascal	Pa	1 Pa = 1 N / m^2
energy, work, heat	Joule	J	1 J = 1 N m
power, heat flux	Watt	W	1 W = 1 J / s
electric charge	Coulomb	C	1 C = 1 A s
electric potential, voltage	Volt	V	1 V = 1 J / C
electric capacity	Farad	F	1 F = 1 C / V
electric resistance	Ohm	Ω	1 Ω = 1 V / A
electric conductivity	Siemens	S	1 S = 1 Ω^{-1}
magnetic flux	Weber	Wb	1 Wb = 1 V s
magnetic. flux density	Tesla	T	1 T = 1 Wb / m^2
inductivity	Henry	H	1 H = 1 Wb / A

A3.3 Decimal Fractions and Multiples of SI Units

Fractions and multiples of SI units obtained by multiplication with factors $10^{\pm 3n}$ (n = 1, 2, ...) have specific names and characters, which are generated by putting special prefixes before the names and the characters of the SI units.

Factor	**Prefix**	**Character**
10^{-15}	femto	f
10^{-12}	pico	p
10^{-9}	nano	n
10^{-6}	micro	µ
10^{-3}	milli	m
10^{3}	kilo	k
10^{6}	Mega	M
10^{9}	Giga	G
10^{12}	Tera	T

A3.4 Conversion between US and SI Units

Length:	1 m = 39.37 in	1 in = 0.0254 m
	1 m = 3.28 ft	1 ft = 0.3048 m
Force:	1 N = 0.2248 lb	1 lb = 4.448 N
	1 kN = 0.2248 kip	1 kip = 4.448 kN
Stress:	1 kPa = 0.145 lb/in^2	1 lb/in^2 = 6.895 kPa
	1 MPa = 0.145 ksi	1 ksi = 6.895 MPa

A4. MURPHY's Laws

1. Nothing is as easy as it looks.
2. Everything takes longer than you think.
3. Anything that can go wrong will go wrong.
4. If there is a possibility of several things going wrong, the one that will cause the most damage will be the one to go wrong. Corollary: If there is a worse time for something to go wrong, it will happen then.
5. If anything simply cannot go wrong, it will anyway.
6. If you perceive that there are four possible ways in which a procedure can go wrong, and circumvent these, then a fifth way, unprepared for, will promptly develop.
7. Left to themselves, things tend to go from bad to worse.
8. If everything seems to be going well, you have obviously overlooked something.
9. Nature always sides with the hidden flaw.
10. Mother nature is a bitch.
11. It is impossible to make anything foolproof because fools are so ingenious.
12. Whenever you set out to do something, something else must be done first.
13. Every solution breeds new problems.

MURPHY's Law of Research

Enough research will tend to support your theory.

MURPHY's Law of Copiers

The legibility of a copy is inversely proportional to its importance.

MURPHY's Law of the Open Road

When there is a very long road upon which there is a one-way bridge placed at random, and there are only two cars on that road, it follows that: (1) the two cars are going in opposite directions, and (2) they will always meet at the bridge.

MURPHY's Law of Thermodynamics

Things get worse under pressure.

The MURPHY Philosophy

Smile ... tomorrow will be worse.

Quantisation Revision of MURPHY's Laws

Everything goes wrong all at once.

MURPHY's Constant

Matter will be damaged in direct proportion to its value