Lecture 11: Basic Josephson Junctions

Outline

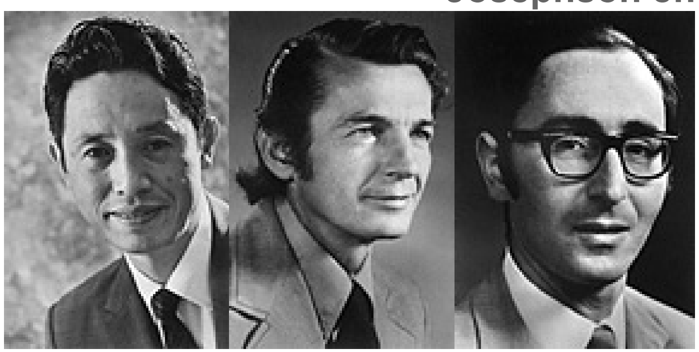
1. Quantum Tunneling
2. Josephson Tunneling
3. Josephson current-phase and current-voltage relations
4. Basic Josephson Junction lumped element
5. AC Josephson Effect
6. DC voltage standard

October 9, 2003

The Nobel Prize in Physics 1973

"for their experimental discoveries regarding tunneling phenomena in semiconductors and superconductors, respectively"

"for his theoretical predictions of the properties of a supercurrent through a tunnel barrier, in particular those phenomena which are generally known as the Josephson effects"



Leo Esaki	Ivar Giaever	Brian David Josephson
1/4 of the prize	1/4 of the prize	1/2 of the prize
Japan	USA	United Kingdom
IBM Thomas J. Watson Research Center Yorktown Heights, NY, USA	General Electric Company Schenectady, NY, USA	University of Cambridge Cambridge, United Kingdom
b. 1925	b. 1929 (in Bergen, Norway)	b. 1940

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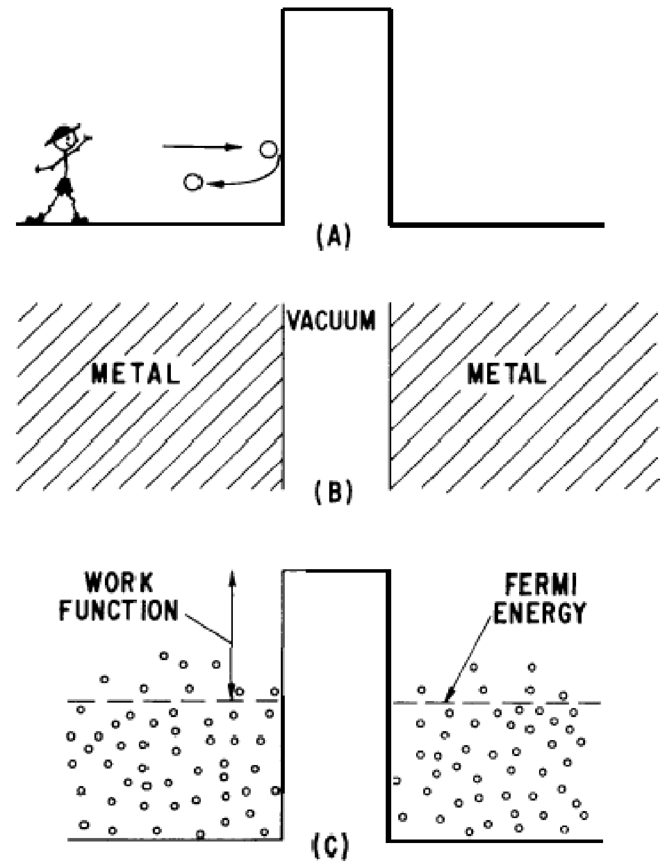


Fig. 1.

•http://www.nobel.se/physics/laureates

Tunneling between a normal metal and another normal metal or a superconductor

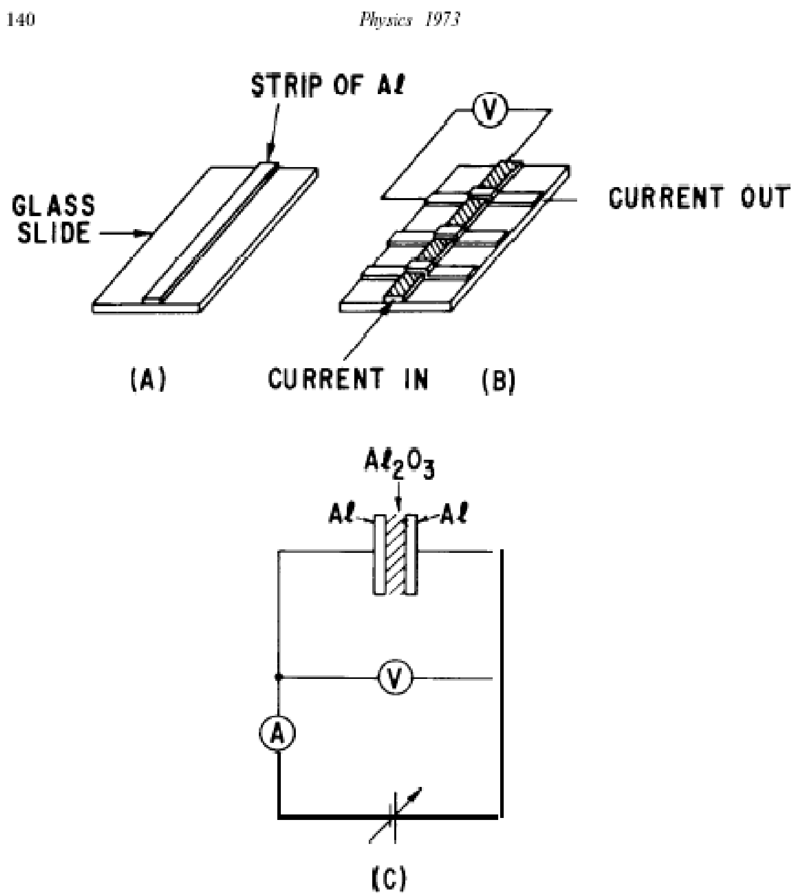


Fig. 3.

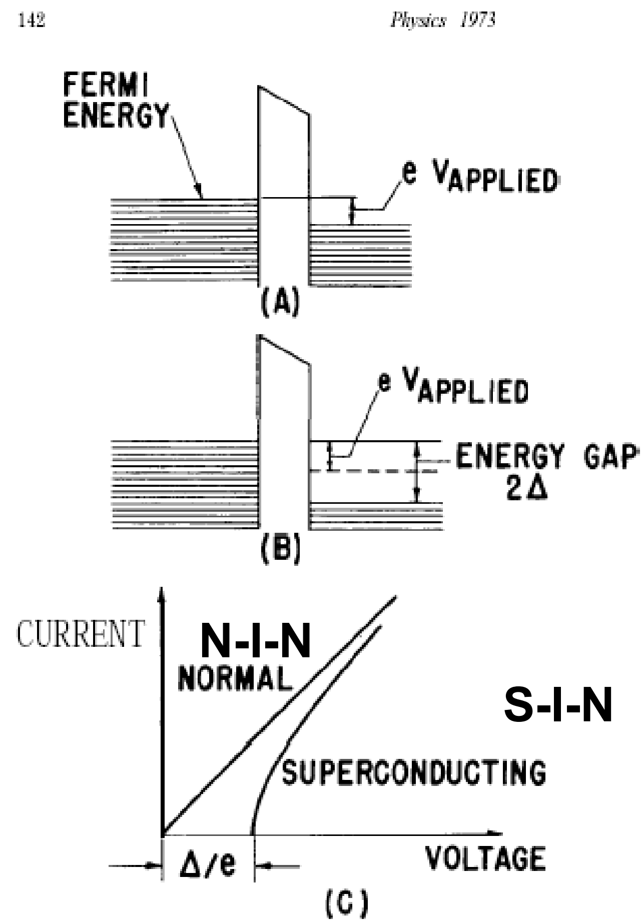


Fig. 5.

http://www.nobel.se/physics/laureates/1973/giaever-lecture.pdf

Tunneling between two superconductors

148 *Physics 1973*

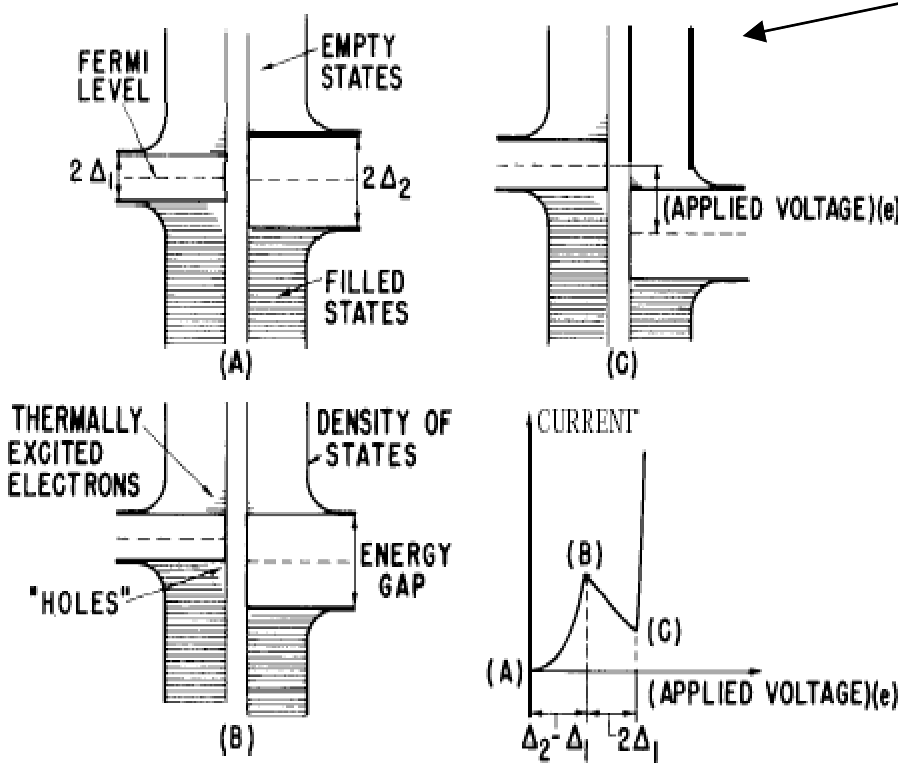
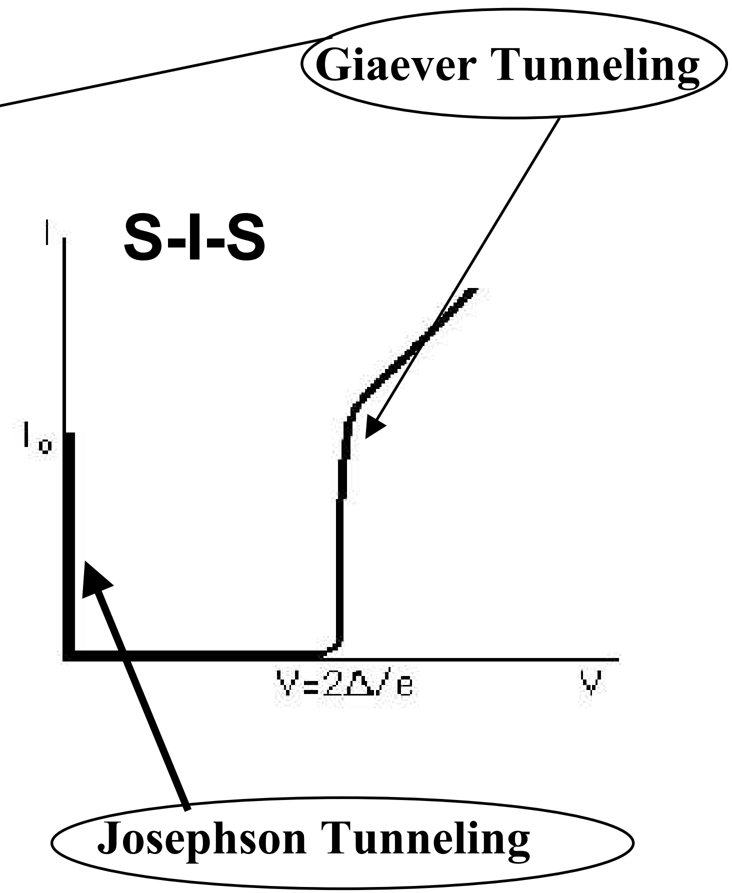
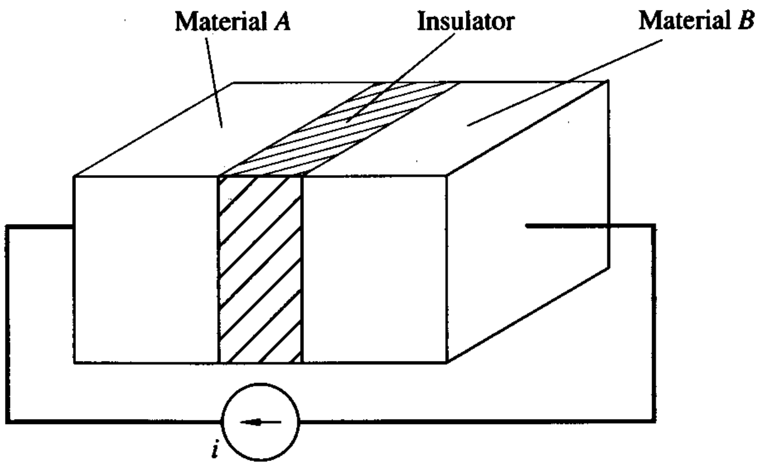


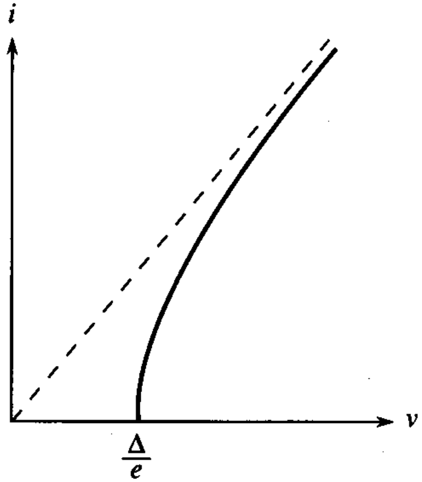
Fig. 10.
Tunneling between two superconductors with different energy gaps at a temperature larger than 0° K. A. No voltage is applied between the two conductors. B. As a voltage



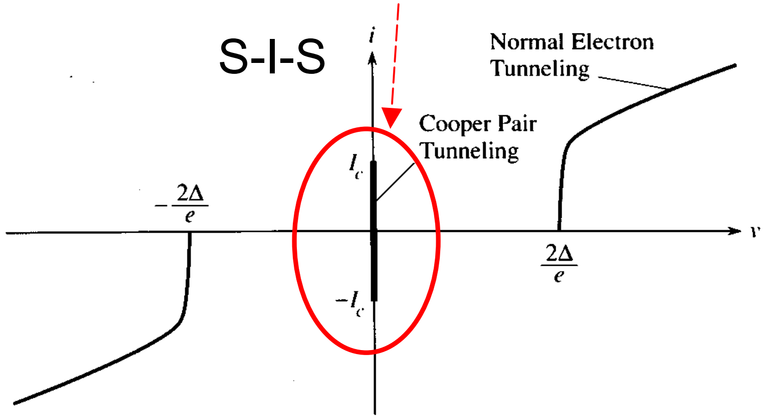
Tunneling Summary: We are only concerned with the Josephson Tunneling in a *Basic Junction*



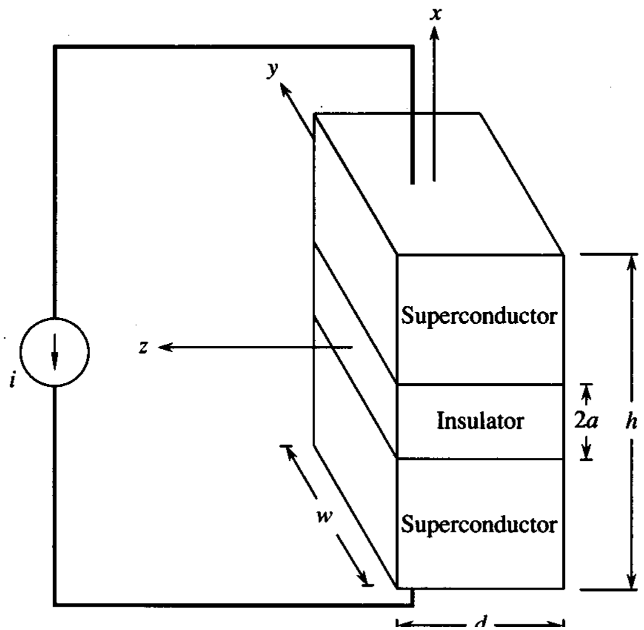
S-I-N



S-I-S



In the superconducting electrodes:



The Supercurrent Equations govern the electrodes,

$$\mathbf{J}_s(\mathbf{r},t) = -\frac{1}{\Lambda}\left(\mathbf{A}(\mathbf{r},t) + \frac{\Phi_o}{2\pi}\nabla\theta(\mathbf{r},t)\right)$$

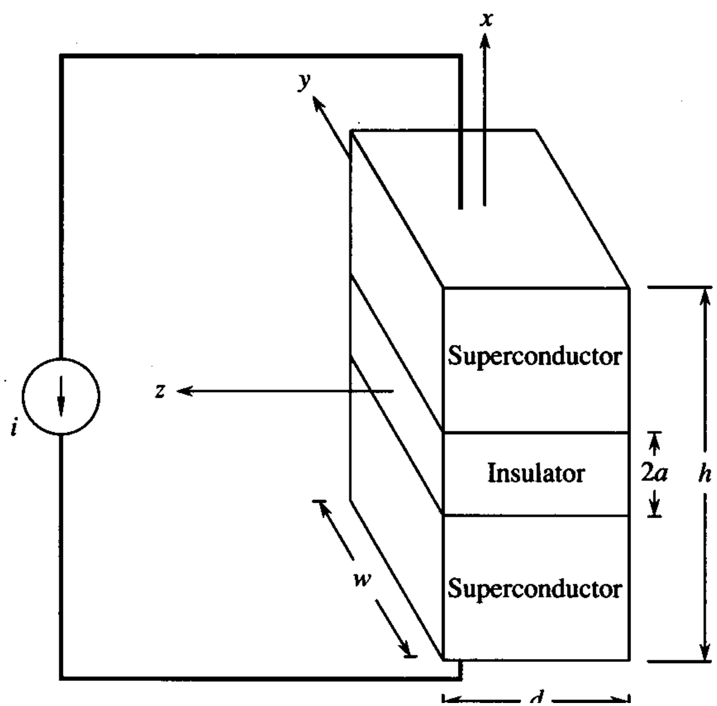
$$\frac{\partial}{\partial t}\theta(\mathbf{r},t) = -\frac{1}{\hbar}\left(\frac{\Lambda \mathbf{J}_s^2}{2n^\star} + q^\star\phi(\mathbf{r},t)\right)$$

Even in the absence of E&M fields, a gradient of the phase can cause a current and the time change of that phase can cause a voltage. For example, for a constant current $\mathbf{J}_o$, at the boundaries we find

$$\mathbf{J}_s(\pm a,t) = -\frac{\Phi_o}{2\pi\Lambda}\nabla\theta(\pm a,t) = \mathbf{J}_o \quad \& \quad \frac{\partial}{\partial t}\theta(\pm a,t) = -\frac{1}{\hbar}\left(\frac{\Lambda \mathbf{J}_o^2}{2n^\star}\right) = -\frac{\mathcal{E}_o}{\hbar}$$

So that the wavefunction in the electrode is $\Psi(\mathbf{r},t) = \Psi(\mathbf{r})e^{-i(\mathcal{E}_o t/\hbar)}$

In the insulator

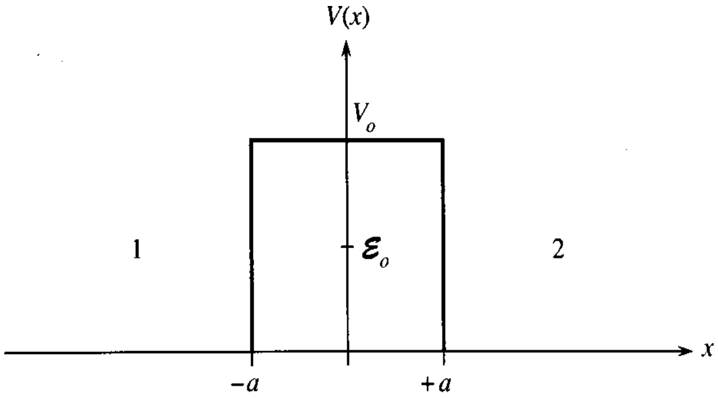


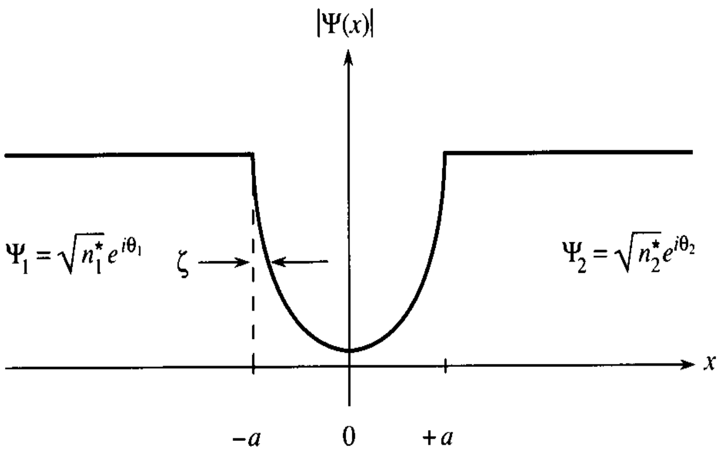
The current must be continuous, so it must flux through the insulating barrier; a process which is not allowed classically. But quantum mechanically the superelectrons can tunnel through the insulating barrier as a supecurrent with zero voltage. This is the Josephson current.

Because the supercurrent equation does not hold in the insulating region, the full macroscopic wave equation must be used to find Y in the insulating region, with the boundary conditions given by the wavefunction at the electrodes.

$$i\hbar \frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \frac{1}{2m^\star}\left(\frac{\hbar}{i}\nabla - q^\star \mathbf{A}(\mathbf{r},t)\right)^2 \Psi(\mathbf{r},t) + q^\star \phi(\mathbf{r},t)\,\Psi(\mathbf{r},t) + \underbrace{V(x)}_{\text{Tunneling Potential Barrier}}\Psi(\mathbf{r},t)$$

Tunneling through the Barrier





The energy of the superelectron is less than the barrier height, so that no classical particles flow.

$$-\frac{\hbar^2}{2m^\star}\nabla^2\Psi(\mathbf{r}) = \underbrace{(\mathcal{E}_o - V_o)}_{\text{constant}}\,\Psi(\mathbf{r}) \qquad \text{for } |x| \le a$$

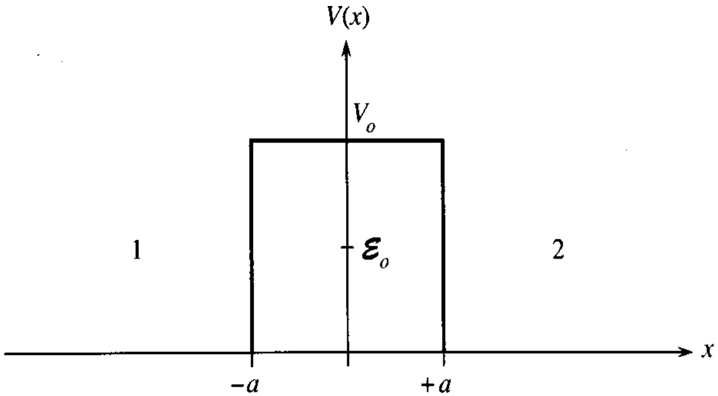
Therefore, in the insulating region

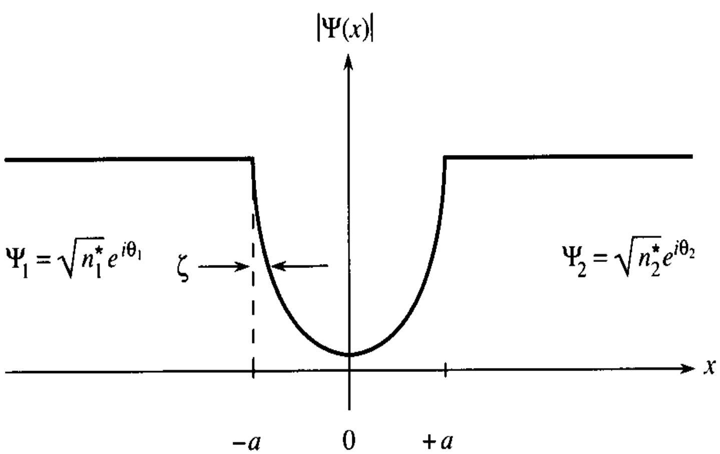
$$\Psi(x) = C_1 \cosh x/\zeta + C_2 \sinh x/\zeta$$

Where $\zeta \equiv \sqrt{\dfrac{\hbar^2}{2m^\star(V_o - \mathcal{E}_o)}}$ so that

$$\mathbf{J}_\mathrm{s} = \frac{2q^\star}{m^\star}\,\mathrm{Re}\left\{\Psi^* \frac{\hbar}{i}\nabla\Psi\right\} = \frac{q^\star\hbar}{m^\star\zeta}\,\mathrm{Im}\left\{C_1{}^* C_2\right\}$$

Tunneling through the Barrier





$$\mathrm{J_S} = \frac{2q^\star}{m^\star} \,\mathrm{Re}\left\{ \Psi^* \frac{\hbar}{i} \nabla \Psi \right\} = \frac{q^\star \hbar}{m^\star \zeta} \,\mathrm{Im}\left\{ C_1{}^* C_2 \right\}$$

At the boundaries.

$$\Psi(-a) = \sqrt{n_1^\star}\, e^{i\theta_1} \quad \& \quad \Psi(+a) = \sqrt{n_2^\star}\, e^{i\theta_2}$$

So that

$$C_1 = \frac{\sqrt{n_1^\star}\, e^{i\theta_1} + \sqrt{n_2^\star}\, e^{i\theta_2}}{2\cosh(a/\zeta)} \quad \& C_2 = -\frac{\sqrt{n_1^\star}\, e^{i\theta_1} - \sqrt{n_2^\star}\, e^{i\theta_2}}{2\sinh(a/\zeta)}$$

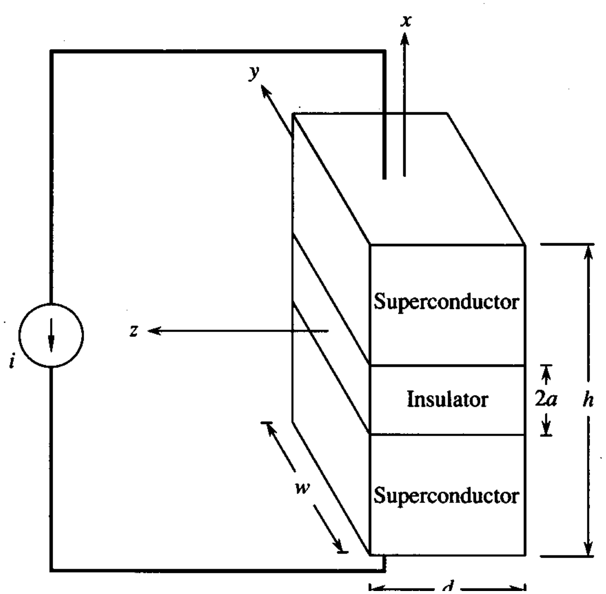
Therefore,

$$\mathrm{J_S} = \mathrm{J_C} \sin(\theta_1 - \theta_2)$$

with $$\mathrm{J_C} = \frac{e\hbar \sqrt{n_1 n_2}}{m\zeta \sinh(2a/\zeta)}$$

Josephson Current-Phase relation

$$\mathbf{J}_\mathsf{S} = \mathbf{J}_\mathsf{C} \sin\left(\theta_1 - \theta_2\right)$$



In the presence of and electromagnetic field, the Josephson current-phase relation generalizes to

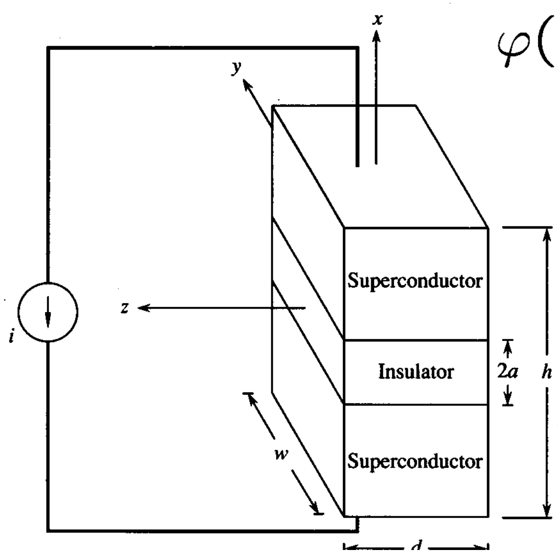
$$\boxed{\mathbf{J}_\mathsf{S}(\mathbf{r},t) = \mathbf{J}_\mathsf{C}(y,z,t) \sin\varphi(y,z,t)}$$

where the *gauge-invariant phase* is defined as

$$\varphi(y,z,t) = \theta_1(y,z,t) - \theta_2(y,z,t) - \frac{2\pi}{\Phi_o}\int_1^2 \mathbf{A}(\mathbf{r},t)\cdot d\mathbf{l}$$

Which is invariant under $\mathbf{A}' = \mathbf{A} + \nabla\chi$, $\theta' = \theta + \frac{q^\star}{\hbar}\chi$, $\phi' \equiv \phi - \frac{\partial\chi}{\partial t}$

Josephson Voltage-Phase relation



The *gauge-invariant phase* is

$$\varphi(y,z,t) = \theta_1(y,z,t) - \theta_2(y,z,t) - \frac{2\pi}{\Phi_o}\int_1^2 \mathbf{A}(\mathbf{r},t)\cdot d\mathbf{l}$$

The rate of change of the *gauge-invariant phase* is

$$\frac{\partial \varphi}{\partial t} = \frac{\partial \theta_1}{\partial t} - \frac{\partial \theta_2}{\partial t} - \frac{2\pi}{\Phi_o}\frac{\partial}{\partial t}\int_1^2 \mathbf{A}(\mathbf{r},t)\cdot d\mathbf{l}$$

At the boundary in the electrodes,

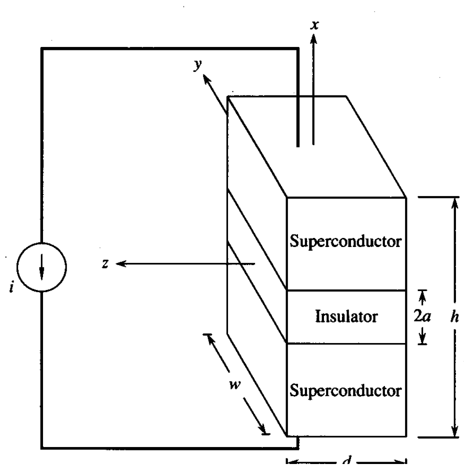
$$\frac{\partial}{\partial t}\theta(\mathbf{r},t) = -\frac{1}{\hbar}\left(\frac{\Lambda \mathbf{J}_s^2}{2n^\star} + q^\star \phi(\mathbf{r},t)\right) \quad \text{so that}$$

$$\frac{\partial \varphi}{\partial t} = -\frac{1}{\hbar}\left(\frac{\Lambda}{2n^\star}\left[\underbrace{\mathbf{J}_s^2(-a) - \mathbf{J}_s^2(a)}_{0}\right] + q^\star \underbrace{\left[\phi(-a) - \phi(a)\right]}_{\int_1^2 -\nabla\phi\cdot d\mathbf{l}}\right) - \frac{2\pi}{\Phi_o}\frac{\partial}{\partial t}\int_1^2 \mathbf{A}(\mathbf{r},t)\cdot d\mathbf{l}$$

Therefore, $\displaystyle \frac{\partial \varphi}{\partial t} = \frac{2\pi}{\Phi_o}\int_1^2\left(-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}\right)\cdot d\mathbf{l}$ or $\displaystyle \boxed{\frac{\partial \varphi(y,z,t)}{\partial t} = \frac{2\pi}{\Phi_o}\int_1^2 \mathbf{E}(\mathbf{r},t)\cdot d\mathbf{l}}$

Basic Lumped Junctions

Distributed element

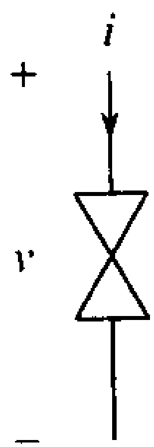


$$\mathbf{J}_\mathsf{S}(\mathbf{r},t) = \mathbf{J}_\mathsf{C}(y,z,t)\,\sin\varphi(y,z,t)$$

$$\varphi(y,z,t) = \theta_1(y,z,t) - \theta_2(y,z,t) - \frac{2\pi}{\Phi_o}\int_1^2 \mathbf{A}(\mathbf{r},t)\cdot d\mathbf{l}$$

$$\frac{\partial\varphi(y,z,t)}{\partial t} = \frac{2\pi}{\Phi_o}\int_1^2 \mathbf{E}(\mathbf{r},t)\cdot d\mathbf{l}$$

Lumped element



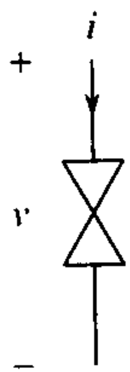
$$i = I_c \sin\varphi$$

$$v = \frac{\Phi_o}{2\pi}\frac{d\varphi}{dt}$$

$$\varphi(t) = \theta_1(t) - \theta_2(t) - \frac{2\pi}{\Phi_o}\int_1^2 \mathbf{A}(\mathbf{r},t)\cdot d\mathbf{l}$$

I_c is the *critical current*

Energy in a Basic Josephson Junction



$$i = I_c \sin\varphi$$

$$v = \frac{\Phi_o}{2\pi}\frac{d\varphi}{dt}$$

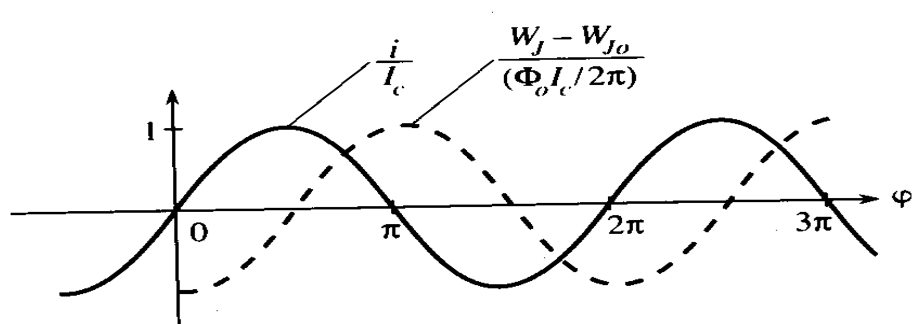
The energy W_J in the basic junction is

$$W_J = \int_0^{t_o} iv\,dt$$

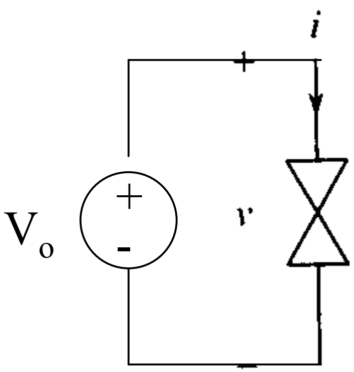
Use the Josephson relations to write as

$$W_J = \int_0^{t_o} \left(I_c \sin\varphi'\right)\left(\frac{\Phi_o}{2\pi}\frac{d\varphi'}{dt}\right) dt = \frac{\Phi_o I_c}{2\pi}\int_0^{\varphi} \sin\varphi'\,d\varphi'$$

Therefore, $W_J = \dfrac{\Phi_o I_c}{2\pi}\left(1 - \cos\varphi\right)$



AC Josephson Effect



$$i = I_c \sin\varphi$$

$$v = \frac{\Phi_o}{2\pi}\frac{d\varphi}{dt}$$

The voltage source is DC with $v=V_0$, so that

$$\varphi(t) = \varphi(0) + \frac{2\pi}{\Phi_o} V_o t$$

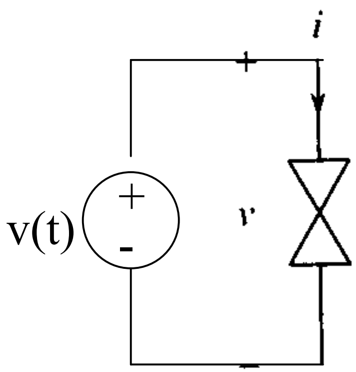
The resulting current is ac!

$$\begin{aligned} i &= I_c \sin\left(\frac{2\pi}{\Phi_o} V_o t + \varphi(0)\right) \\ &= I_c \sin\left(2\pi f_J t + \varphi(0)\right) \end{aligned}$$

The *Josephson frequency* is $f_J = \frac{V_o}{\Phi_o} = \frac{2e}{h} V_o = 483.6 \times 10^{12}\, V_o \text{ (Hz)}$

A dc voltage of 10 μV causes an oscillation frequency of about 5 GHz, a Josephson microwave oscillator. But with a typical I_c of 1 mA, this oscillator delivers a very small power of the order of 10 nW. Therefore need many synchronous oscillators.

AC and DC voltage drives



$i = I_c \sin\varphi$

$v = \dfrac{\Phi_o}{2\pi}\dfrac{d\varphi}{dt}$

Let $v(t) = V_o + V_s \cos\omega_s t$

Then the guage-invariant phase is

$$\varphi(t) = \varphi(0) + \frac{2\pi}{\Phi_o} V_o t + \frac{2\pi V_s}{\Phi_o \omega_s} \sin\omega_s t$$

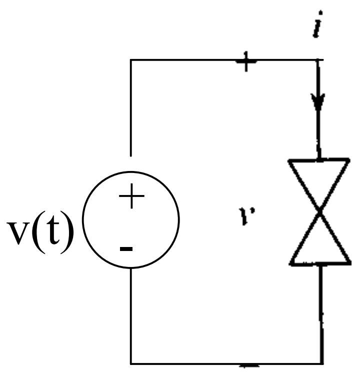
The current is FM-like: $i = I_c \sin\left(\varphi(0) + \dfrac{2\pi}{\Phi_o} V_o t + \dfrac{2\pi V_s}{\Phi_o \omega_s} \sin\omega_s t\right)$

Use the Fourier-Bessel series to express the current as a Fourier series

$$i = I_c \sum_{n=-\infty}^{\infty} (-1)^n \left[J_n\left(\frac{2\pi V_s}{\Phi_o \omega_s}\right)\right] \sin\left[(2\pi f_J - n\omega_s)t + \varphi(0)\right]$$

A dc current will occur when $2\pi f_J = n\,\omega_s$, that is, $V_o = n\left(\dfrac{\Phi_o}{2\pi}\right)\omega_s$

Principle of the DC Voltage Standard



$i = I_c \sin \varphi$

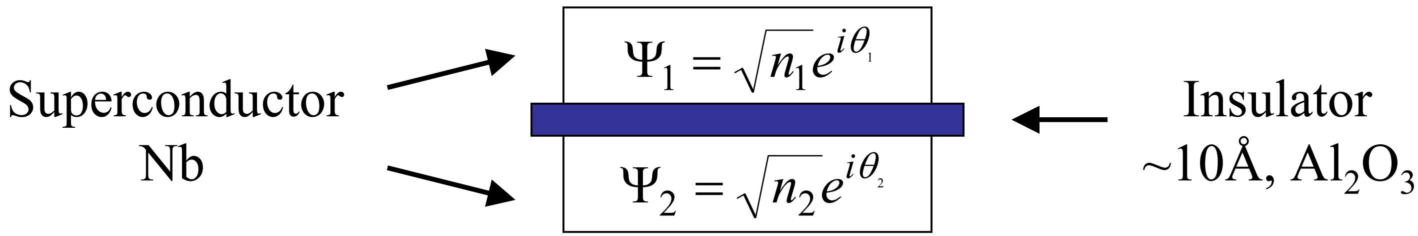
$v = \frac{\Phi_o}{2\pi} \frac{d\varphi}{dt}$

$$V_o = n \left(\frac{\Phi_o}{2\pi} \right) \omega_s$$

An ac voltage of 1 GHz applied across the junction will give a dc current, at $V_0 = 0$ and at dc voltages of integral multiples of 2μV.

The principle of the dc Volt: Put 5000 Josephson junctions in series, and apply a fixed frequency, which can be done very accurately, and measure the interval of the resulting dc voltages that occur at precise voltage intervals.

Summary: Basic Josephson Junction ($I<I_c$)



- Josephson relations:

$$I = I_c \sin\varphi$$

$$V = \frac{\Phi_0}{2\pi}\frac{d\varphi}{dt}$$

$$\varphi = \theta_2 - \theta_1 - \frac{2\pi}{\Phi_0}\int_2^1 \vec{A}(r,t)\cdot d\vec{l}$$

- Behaves as a nonlinear inductor:

$$V = L_J \frac{dI}{dt},$$

where $$L_J = \frac{\Phi_0}{2\pi\, I_c \cos\varphi}$$

Φ_0 = flux quantum

483.6 GHz / mV