Lecture 10: Supercurrent Equation

Outline

1. Macroscopic Quantum Model
2. Supercurrent Equation and the London Equations
3. Fluxoid Quantization
4. The Normal State
5. Quantized Vortices

October 7, 2003

Macroscopic Quantum Model

1. **The wavefunction describes the whole ensemble of superelectrons such that**

$$\Psi^*(\mathbf{r},t)\Psi(\mathbf{r},t) = n^\star(\mathbf{r},t) \quad \longrightarrow \quad \text{density}$$

and $$\int d\mathbf{r}\,\Psi^*(\mathbf{r},t)\Psi(\mathbf{r},t) = \mathsf{N}^\star \quad \longrightarrow \quad \text{Total number}$$

2. **The flow of probability becomes the flow of particles, with the physical current density given by**

$$\mathbf{J}_\mathsf{S} = q^\star\, \mathsf{Re}\left\{\Psi^*\left(\frac{\hbar}{im^\star}\nabla - \frac{q^\star}{m^\star}\mathbf{A}\right)\Psi\right\}$$

3. **This macroscopic quantum wavefunction follows**

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \frac{1}{2m^\star}\left(\frac{\hbar}{i}\nabla - q^\star\mathbf{A}(\mathbf{r},t)\right)^2\Psi(\mathbf{r},t) + q^\star\phi(\mathbf{r},t)\,\Psi(\mathbf{r},t)$$

Wave function

Writing $\Psi(\mathbf{r},t) = \sqrt{n^\star(\mathbf{r},t)}\, e^{i\theta(\mathbf{r},t)}$, we find

The real part of the S-Eqn gives

$$
\begin{aligned}
-\hbar \frac{\partial}{\partial t}\theta(\mathbf{r},t) \;=\; & \frac{\hbar^2 n_s^\star}{2m^\star}\left(\nabla\theta(\mathbf{r},t) - \frac{q^\star}{\hbar}\mathbf{A}(\mathbf{r},t)\right)^2 \\
& + \frac{\hbar^2}{8m^\star n_s^\star(\mathbf{r},t)}\left(\nabla^2 n_s^\star(\mathbf{r},t)\right)^2 + q^\star\phi(\mathbf{r},t)
\end{aligned}
$$

The imaginary part of the S-Eqn gives the supercurrent equation:

$$
\mathbf{J}_\mathsf{S} = q^\star n^\star(\mathbf{r},t)\left(\frac{\hbar}{m^\star}\nabla\theta(\mathbf{r},t) - \frac{q^\star}{m^\star}\mathbf{A}(\mathbf{r},t)\right)
$$

Supercurrent Equation with n* constant

Let $n^\star(\mathbf{r},t) = n^\star$ be a constant, so that $\Psi(\mathbf{r},t) = \sqrt{n^\star}\, e^{i\theta(\mathbf{r},t)}$

we find

$$-\hbar\frac{\partial\theta}{\partial t} = \underbrace{\frac{1}{2n^\star}\Lambda\,\mathbf{J}_\mathsf{S}{}^2 + q^\star\phi}_{\text{Energy of a superelectron}} \quad \text{with} \quad \Lambda \equiv \frac{m^\star}{n^\star(q^\star)^2}$$

and

$$\Lambda\mathbf{J}_\mathsf{S} = -\left(\mathbf{A}(\mathbf{r},t) - \frac{\hbar}{q^\star}\nabla\theta(\mathbf{r},t)\right)$$

London's Equations

1.Take the curl of the supercurrent equation

$$\Lambda \mathbf{J}_s = -\left(\mathbf{A}(\mathbf{r},t) - \frac{\hbar}{q^\star}\nabla\theta(\mathbf{r},t)\right)$$

gives the Second London Equation: $\nabla \times (\Lambda \mathbf{J}_s) = -\nabla \times \mathbf{A} = -\mathbf{B}$

2. Take the time derivative of the supercurrent equation:

$$\frac{\partial}{\partial t}(\Lambda \mathbf{J}_s) = -\left[\frac{\partial \mathbf{A}}{\partial t} - \frac{\hbar}{q^\star}\nabla\left(\frac{\partial \theta}{\partial t}\right)\right]$$

with $-\hbar\frac{\partial \theta}{\partial t} = \frac{1}{2n^\star}\Lambda {\mathbf{J}_s}^2 + q^\star\phi$ gives

$$\frac{\partial}{\partial t}(\Lambda \mathbf{J}_s) = \mathbf{E} - \frac{1}{n^\star q^\star}\nabla\left(\frac{1}{2}\Lambda {\mathbf{J}_s}^2\right)$$

Something more than First London Equation?

First London revisited

$$\frac{\partial}{\partial t}(\Lambda \mathbf{J}_s) = \mathbf{E} - \frac{1}{n^\star q^\star}\nabla\left(\frac{1}{2}\Lambda \mathbf{J}_s{}^2\right)$$

Full First London

With a number of vector identities, this can be shown to be equivalent to **full** Lorentz force

$$m^\star \frac{d\mathbf{v}_s}{dt} = q^\star \mathbf{E} + q^\star \mathbf{v}_s \times \mathbf{B}$$

Hence, the above is the full first London Equation. However, for MQS problems we never used the first London Equation!! So all our previous results are valid.

Our "short" form of the first London equation is valid in the limit where we ignored the magnetic field, that is ignored the Hall effect. One can show that this is true as long as

$$|\mathbf{E}| \gg \left|\frac{1}{n^\star q^\star}\nabla\left(\Lambda \mathbf{J}_s{}^2\right)\right| \quad \text{or} \quad |\mathbf{E}| \gg |\mathbf{v}_s|\,|\mathbf{B}|$$

Flux Quantization

3. Take the line integral of the supercurrent equation around a closed contour within a superconductor:

$$\oint_C \left\{ \Lambda \mathbf{J}_\mathsf{s} = -\left(\mathbf{A}(\mathbf{r},t) - \frac{\hbar}{q^\star}\nabla\theta(\mathbf{r},t)\right) \right\}$$

The line integral of each of the parts:

$$\oint_C \mathbf{A}\cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A})\cdot d\mathbf{s} = \int_S \mathbf{B}\cdot d\mathbf{s} = \Phi_C$$

flux

$$\oint_C \nabla\theta \cdot d\mathbf{l} = \lim_{\mathbf{r}_\mathsf{b}\rightarrow\mathbf{r}_\mathsf{a}} \left(\theta(\mathbf{r}_\mathsf{b},t) - \theta(\mathbf{r}_\mathsf{a},t)\right) = 2\pi \mathsf{n}'$$

integer

Therefore,

$$\oint_C (\Lambda \mathbf{J}_\mathsf{s})\cdot d\mathbf{l} + \int_S \mathbf{B}\cdot d\mathbf{s} = \frac{\hbar}{q^\star} 2\pi \mathsf{n}$$

n = - n’

Fluxoid Quantization

The **flux quantum** is defined as

$$\Phi_o \equiv \frac{2\pi\hbar}{|q^\star|} = \frac{h}{|q^\star|}$$

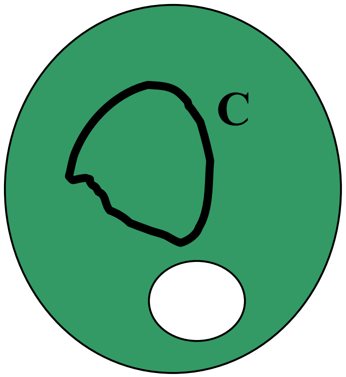
And the **Fluxoid Quantization** condition becomes

$$\underbrace{\oint_C (\Lambda \mathbf{J}_\mathsf{s}) \cdot d\mathbf{l} + \int_S \mathbf{B} \cdot d\mathbf{s}}_{\text{Fluxiod}} = \mathsf{n}\Phi_o$$

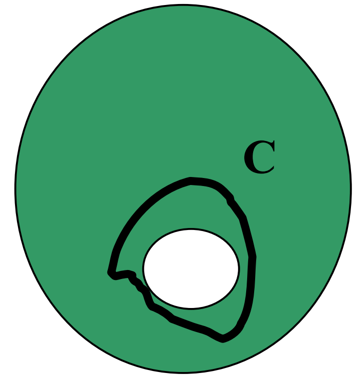
Experiments testing fluxiod quantization will determine q*= -2e, so that

$$\Phi_o = \frac{h}{2e} = 2.07 \times 10^{-15}\ \mathrm{T\text{-}m^2}$$

Simply and Multiply Connected Regions in a Superconductors



simply connected region



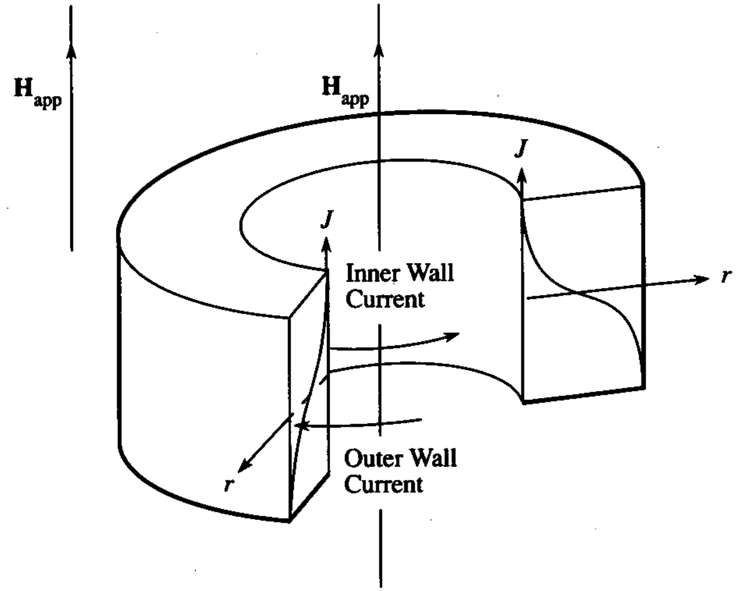
multiply connected region

$$\oint_C (\Lambda \mathbf{J}_s) \cdot d\mathbf{l} + \int_S \mathbf{B} \cdot d\mathbf{s} = n\Phi_o$$

For the simply connected region, fluxoid quantization holds for every contour C, no matter how small. As the contour shrinks to zero, both integrals vanishes and n=0.

For the mutiply connected region, the contour can only be shrink to the contour outlining the normal region. Hence, the integrals need not vanish, so n can be any integer.

Flux Quantization



Consider a hollow cylinder, in bulk limit so that the thickness of the walls is less than the penetration depth.

Let an applied field $\mathbf{H}_{app}$ be trapped in the hole as the cylinder is cooled down.

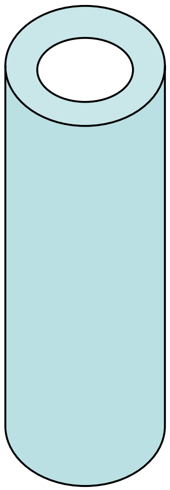
$$\oint_C (\Lambda \mathbf{J}_s) \cdot d\mathbf{l} + \int_S \mathbf{B} \cdot d\mathbf{s} = n\Phi_o$$

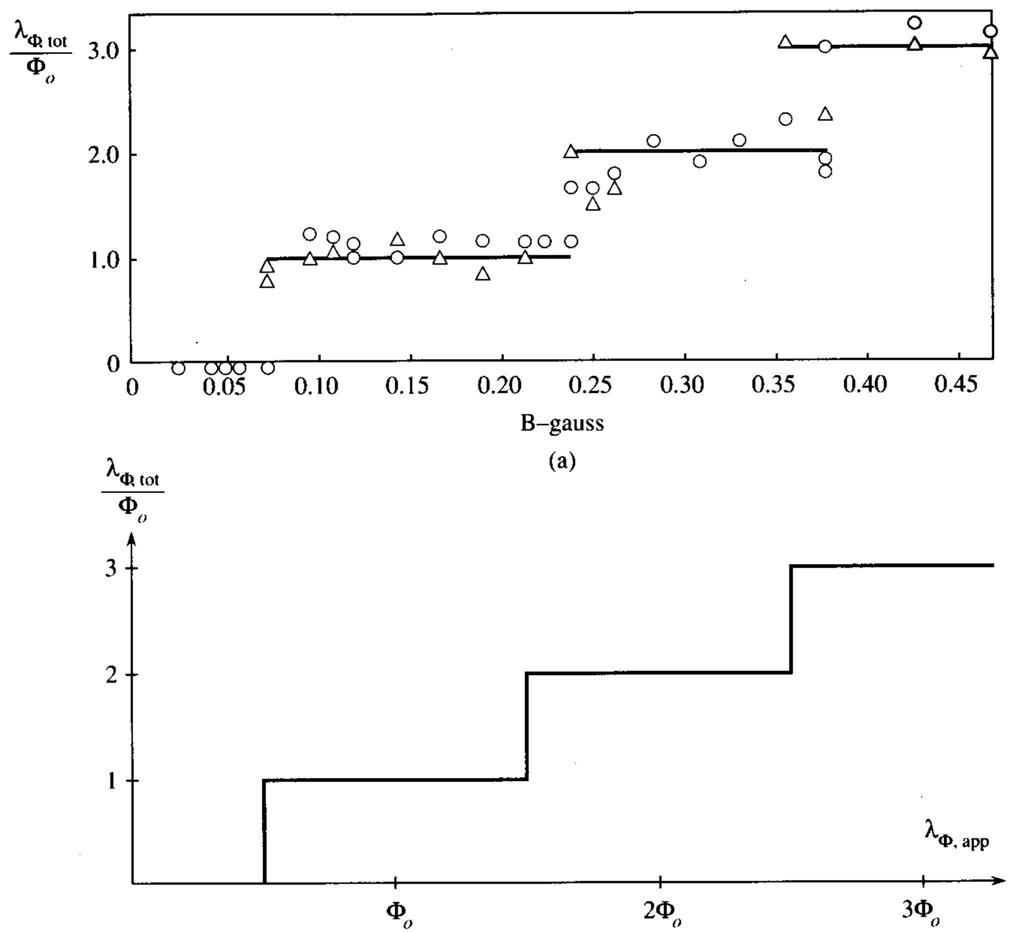
Let the contour be deep within the superconductor where **J**=0. Then

$$\int_S \mathbf{B} \cdot d\mathbf{s} = n\Phi_o$$

Flux is quantized in the bulk limit.

Flux Quantization Experiments





(a)

(b)

Flux trapped in hollow cylinder

Deaver and Fairbank, 1961, measure

$\Phi_o = 2.07 \times 10^{-15}$ T-m^2

and show that q*=2e; Cooper Pairs.

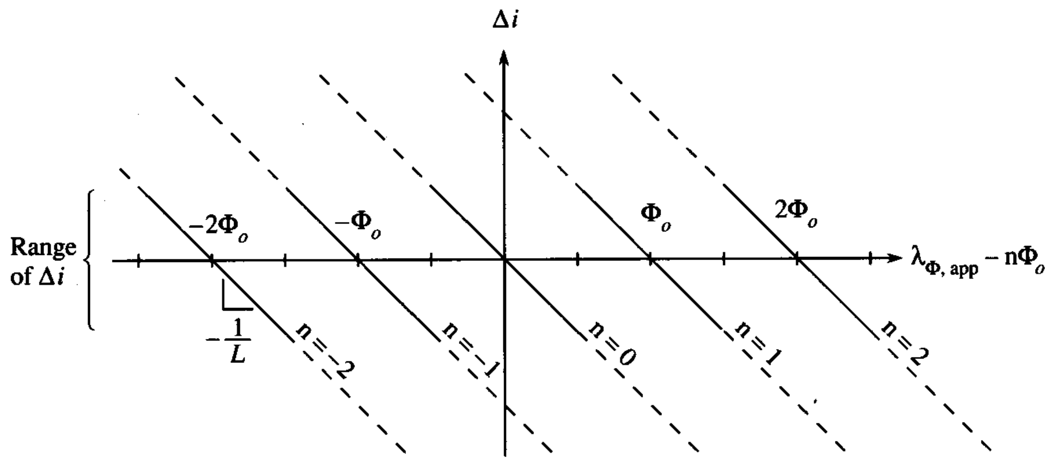
Induced Currents

To have flux quantization, currents must be induced in the cylinder to add to or oppose the applied magnetic field.

$$\lambda_{\Phi,\mathrm{tot}} = n\Phi_o = \lambda_{\Phi,\mathrm{app}} + \Delta\lambda_{\Phi}$$

Induced flux= L i

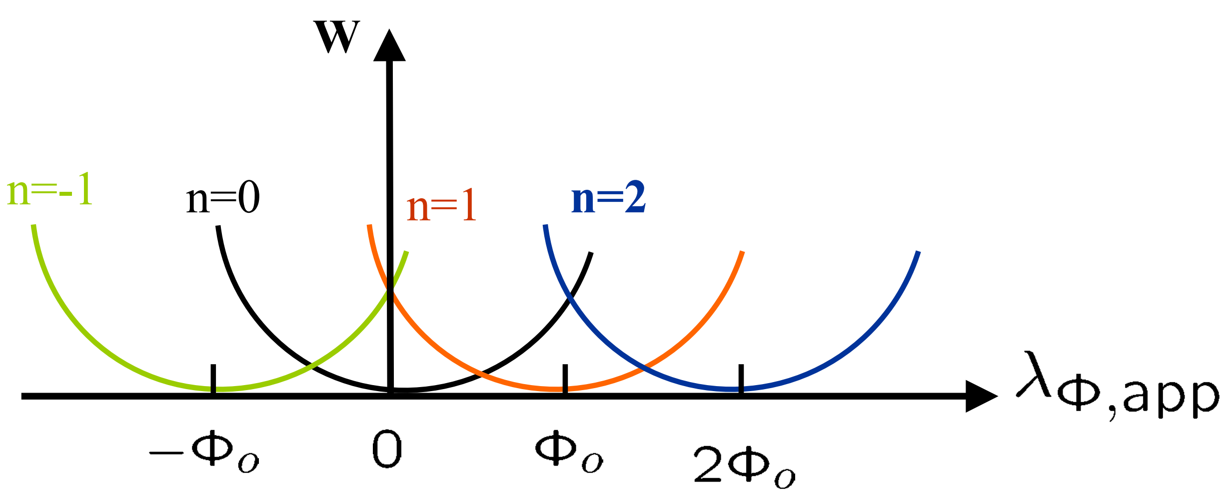
$$\Delta i = \frac{\Delta\lambda_{\Phi}}{L} = \frac{n\Phi_o - \lambda_{\Phi,\mathrm{app}}}{L}$$



Energy of Hollow Cylinder

If the walls of the cylinder are thin compared to the total area of the hole but thicker than the penetration depth, the difference in electromagnetic energy in the superconducting is

$$W \approx \frac{1}{2} L i^2 = \frac{1}{2L}\left(\lambda_{\Phi,\mathrm{app}} - \mathrm{n}\Phi_o\right)^2$$



The Normal State

If the electrons in both the superconducting and the normal state are described by quantum mechanics, what is the wavefunction for the normal state and how does it differ from the superconducting state?

Quantum Mechanics describes both states as a wavefunction that depends on the coordinates of all the electrons.

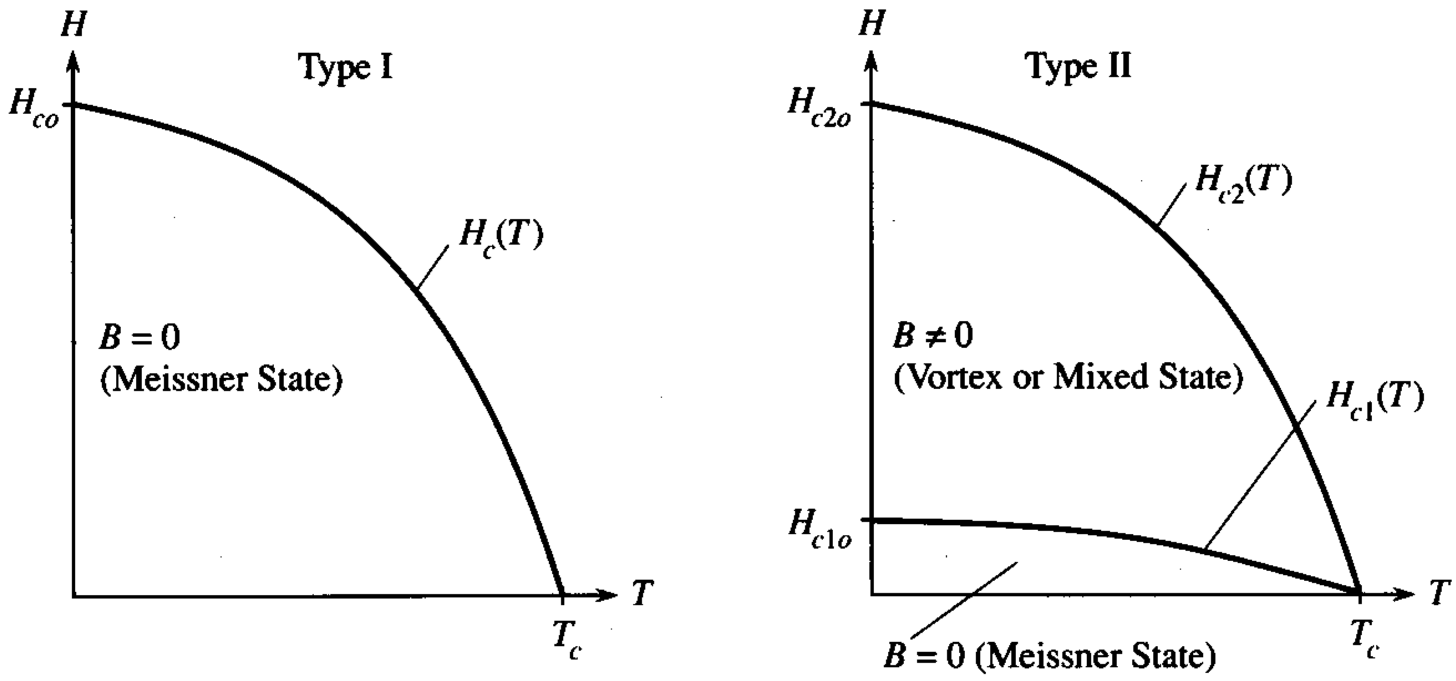
The MQM wavefunction for the superconductor is the spatial average of this phase coherent wavefunction and is preserved with an applied field. The coherence persists over the macroscopic scale of the superconductor.

In the normal state, the applied field causes dissipation; this energy loss causes the phase of the wavefunction is randomized, on a length scale which is usually much smaller than the scale of the material.

The electrons is the normal state have a background speed, the fermi velocity v_F. The electrons undergo collisions on a time scale τ_{tr}. The average distance between collisions is the mean free path,

$$\ell_{tr} = v_F \tau_{tr}$$

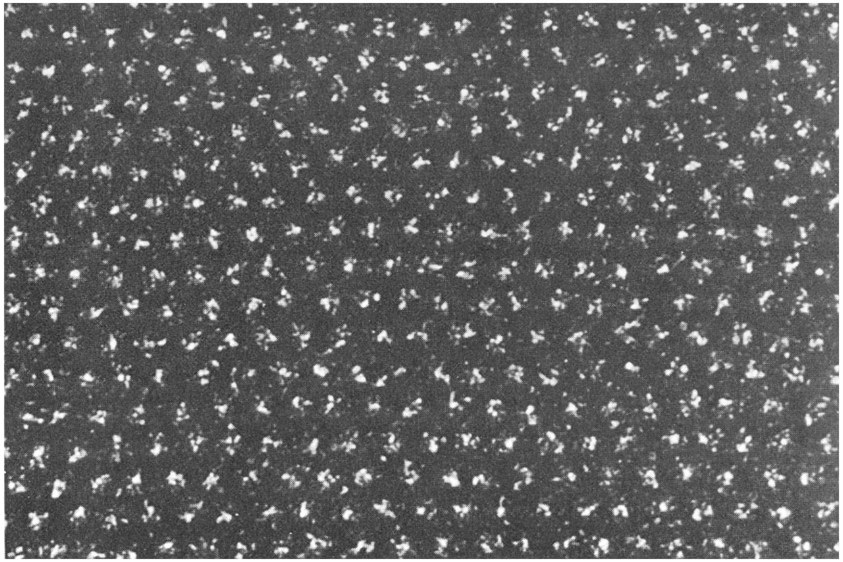
Fluxoid Quantization and Type II Superconductors



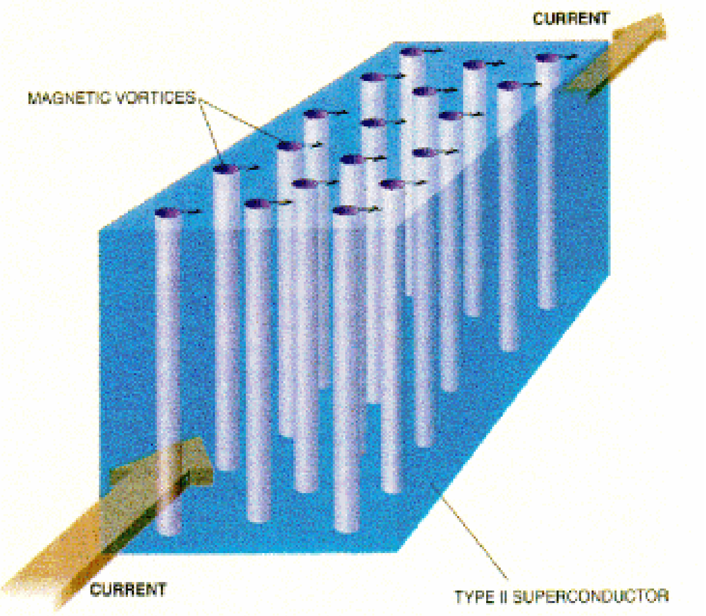
The Vortex State

$$\langle B \rangle = n_V \Phi_V$$

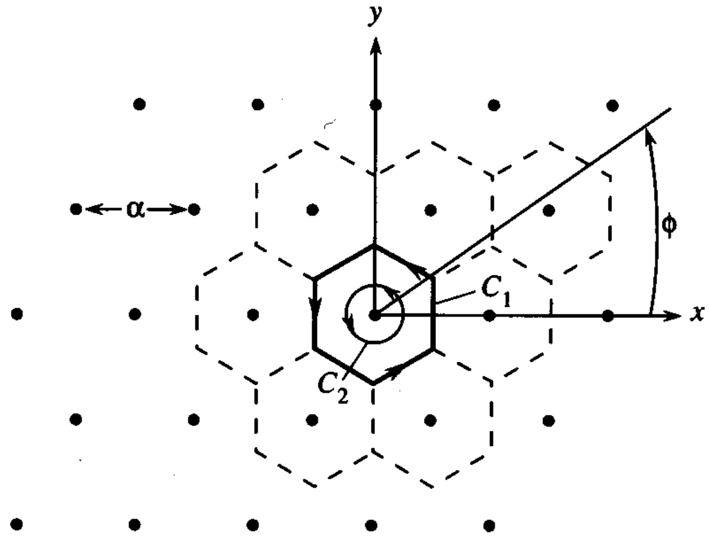
n_V is the areal density of vortices, the number per unit area.



Top view of Bitter decoration experiment on YBCO



Quantized Vortices



Fluxoid Quantization along C_1

$$\mathsf{n}\Phi_o = \oint_{C_1} \mu_o \lambda^2 \mathbf{J}_\mathsf{S} \cdot d\mathbf{l} + \int_{S_1} \mathbf{B} \cdot d\mathbf{s}$$

But along the hexagonal path C_1 **B** is a mininum, so that **J** vanishes along this path.

Therefore, $$\mathsf{n}\Phi_o = \int_{S_1} \mathbf{B} \cdot d\mathbf{s}$$

And experiments give n = 1, so each vortex has one flux quantum associated with it.

Along path C_2, $$\Phi_o = \oint_{C_2} \mu_o \lambda^2 \mathbf{J}_\mathsf{S} \cdot d\mathbf{l} + \int_{S_2} \mathbf{B} \cdot d\mathbf{s}$$

For small C_2, $$\Phi_o = \lim_{r \to 0} \oint_{C_2} \mu_o \lambda^2 \mathbf{J}_\mathsf{S} \cdot d\mathbf{l} \quad \Longrightarrow \quad \lim_{r \to 0} \mathbf{J}_\mathsf{S} = \frac{\Phi_o}{2\pi \mu_o \lambda^2} \frac{1}{r} \mathbf{i}_\phi$$

Normal Core of theVortex

The current density $\lim_{r \to 0} \mathbf{J}_{\mathsf{s}} = \frac{\Phi_o}{2\pi\mu_o\lambda^2}\frac{1}{r}\mathbf{i}_\phi$ diverges near the vortex center,

Which would mean that the kinetic energy of the superelectrons would also diverge. So to prevent this, below some core radius ξ the electrons become normal. This happens when the increase in kinetic energy is of the order of the gap energy. The maximum current density is then

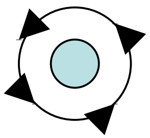
$$\mathbf{J}_{\mathsf{s}}^{\mathsf{max}} = \frac{\Phi_o}{2\pi\mu_o\lambda^2}\frac{1}{\xi}\mathbf{i}_\phi \quad \Longrightarrow \quad \mathbf{v}_{\mathsf{s}}^{\mathsf{max}} = \frac{\hbar}{m^\star}\frac{1}{\xi}\mathbf{i}_\phi$$

In the absence of any current flux, the superelectrons have zero net velocity but have a speed of the fermi velocity, v_F. Hence the kinetic energy with currents is

$$\mathcal{E}_{\mathsf{kin}}^0 = \frac{1}{2}m^\star v_F^2 = \frac{1}{2}m^\star\left(v_{F,x}^2 + v_{F,y}^2 + v_{F,z}^2\right)$$

Coherence Length ⊠

The energy of a superelectron at the core is



$$\mathcal{E}^1_{\text{kin}} = \frac{1}{2} m^\star \left[v^2_{F,x} + \left(v_{F,y} + v^{\text{max}}_{s,\phi} \right)^2 + v^2_{F,z} \right]$$

The difference in energy, is to first order in the change in velocity,

$$\delta\mathcal{E} \approx m^\star v_{F,y}\, v^{\text{max}}_{s,\phi} \approx \Delta$$

With $\mathbf{v}^{\text{max}}_{\text{s}} = \frac{\hbar}{m^\star} \frac{1}{\xi} \mathbf{i}_\phi$ this gives $\xi \approx \frac{\hbar v_F}{2\Delta}$

The full BCS theory gives the *coherence length* as $\xi_o = \frac{\hbar v_F}{\pi \Delta_o}$

Therefore the maximum current density, known as the *depairing current density*, is

$$J_{\text{depair}} \approx \frac{\Phi_o}{2\pi\mu_o\lambda^2\xi}$$