

Classical Mechanics

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Abstract

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1 Introduction

1.1 Newtonian Dynamics

Classical mechanics has not really changed, in substance, since the days of Isaac Newton. The essence of Newton's insight, encoded in his second law $\mathbf{F} = m\mathbf{a}$, is that the motion of a particle described by its trajectory, $\mathbf{r}(t)$, is completely determined once its initial position and velocity are known. His famous equation, describing the second law, relates the acceleration $d^2\mathbf{r}/dt^2$ to the force on the particle, which is implicitly assumed to depend only on the positions, and possibly the velocities of the particles in the system. Consider a system of N particles, whose trajectories are described by $3N$ coordinates $\mathbf{r}_k(t)$, $k = 1, \dots, N$. Then Newton's laws of motion take the mathematical form of $3N$ second order differential equations in time:

$$m_k \frac{d^2 \mathbf{r}_k}{dt^2} = \mathbf{F}_k(\{\mathbf{r}_i\}, \{\dot{\mathbf{r}}_i\}) \quad (1)$$

This is the general framework, but of course for each dynamical system we also need to know the force law. Newton himself specified his inverse square force law for gravitational systems:

$$\mathbf{F}_k^{\text{Grav}} = -m_k \sum_{j \neq k} \frac{Gm_j(\mathbf{r}_k - \mathbf{r}_j)}{|\mathbf{r}_k - \mathbf{r}_j|^3} \quad (2)$$

which has no dependence on the velocities of the particles. He didn't know about magnetic forces on moving charges, which were only understood much later. A particle of charge q moving in an electromagnetic field experiences a force

$$\mathbf{F}^{\text{EM}} = q[\mathbf{E}(\mathbf{r}, t) + \mathbf{v} \times \mathbf{B}(\mathbf{r}, t)] \quad (3)$$

The existence of magnetic forces means that we have to allow for velocity dependent forces to hope to describe all phenomena.

Although Newton's laws of motion were designed to describe particles and other material bodies, the fundamental insight that dynamics should be governed by differential equations that are second order in time carries over to fields such as the electric and magnetic fields, which satisfy second order wave equations, e.g.

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \mathbf{B} = \mu_0 \mathbf{J} \quad (4)$$

Everything we know about the physical world to date can be effectively understood in terms of the dynamics of particles and fields.

Newton also didn't know about Einstein's relativity, but his law of motion, appropriately modified carries over to this domain as well. The necessary modification is to replace $m\mathbf{a}$ with $d\mathbf{p}/dt$, where \mathbf{p} is the relativistic momentum of the particle:

$$\mathbf{p} \equiv \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} \quad (5)$$

We see that for particles moving slowly compared to the speed of light $v \ll c$, the momentum goes approximately to its Newtonian expression $\mathbf{p} \approx m\mathbf{v}$, so $\dot{\mathbf{p}} \approx m\mathbf{a}$. But even for fully relativistic motion the basic structure of Newtonian dynamics holds.

1.2 The Value of New Formulations

The previous subsection contains everything you need to know about Newtonian dynamics—once you solve the equations there is nothing more to say. However, there are other ways to look at the dynamics that reveal features of the motion that brute force solution of the equations might leave obscure. For example in elementary physics we learn how to exploit energy conservation when the force is the gradient of a potential $\mathbf{F} = -\nabla V$. Then instead of working with the second order differential equations we can use energy conservation

$$E = \frac{1}{2}m\mathbf{v}^2 + V(\mathbf{r}) = \text{Constant} \quad (6)$$

to immediately read off the speed of the particle in terms of its location and total energy E .

In one dimensional motion the potential energy curve tells us a lot about the types of motion that will occur. A horizontal line of height E intersects $V(x)$ at the “turning points” where the particle comes to rest. Points where $dV/dx = 0$ tell where a particle feels zero force. A particle placed there at rest will stay at rest: we spot the static solutions by looking for the maxima and minima of the potential. A minimum is stable equilibrium, whereas a maximum is unstable equilibrium.

For motion in one dimension, energy conservation implies Newton’s equations

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + \dot{x}\frac{\partial V}{\partial x} = \dot{x}\left(m\ddot{x} + \frac{\partial V}{\partial x}\right) = 0 \quad (7)$$

so as long as $\dot{x} \neq 0$, Newton’s equation holds. But in two or more dimensions energy conservation only tells us

$$\dot{\mathbf{x}} \cdot (m\ddot{\mathbf{x}} + \nabla V) = 0 \quad (8)$$

which only implies that $m\ddot{\mathbf{x}} + \nabla V$ is perpendicular to $\dot{\mathbf{x}}$. For example the magnetic force might or might not be present:

$$m\ddot{\mathbf{x}} = -\nabla V + q\dot{\mathbf{x}} \times \mathbf{B} \quad (9)$$

The ideas of Lagrange, Hamilton, and Jacobi allow us to interpret general nonstatic solutions in terms of maxima or minima of an energy-like quantity called the action. Since a nonstatic solution is a curve in space rather than simply a point, we have to study the action as a function of curves, which requires the concepts of the calculus of variations, which we will develop in the course as we go.

A great advantage of the action is that by construction it is an invariant under the symmetries of the dynamical system. It is a scalar functional of the coordinates $q_k(t)$ and velocities $\dot{q}_k(t)$. It therefore summarizes the dynamical content of a system in a compact and transparent way. As we shall see, it also greatly simplifies the problem of imposing constraints on the dynamical variables.

2 Hamilton's Principle of Least (Stationary) Action

2.1 Generalized coordinates

Cartesian coordinates are just fine for describing particles that can move unconstrained throughout space. But when the motion is constrained in some way, another choice of coordinates may be preferable. As a simple example suppose a particle is constrained to move in a circle in the xy -plane. Then we have the constraints

$$z(t) = 0, \quad x^2(t) + y^2(t) = R^2 \quad (10)$$

We could solve the constraints to eliminate the coordinate $y(t) = \pm\sqrt{R^2 - x^2(t)}$, but the sign ambiguity is a nuisance. But in passing to polar coordinates $x = \rho \cos \varphi$, $y = \rho \sin \varphi$, we see that the constraint is simply $\rho = R$, and φ gives a perfectly natural and unambiguous description of the particle's location. Thus in this situation it would be nice to use φ and $\dot{\varphi}$ as coordinate and velocity. It is standard to use q_k and \dot{q}_k to denote such generalized coordinates. There is no need to commit to a particular choice of coordinates in advance. For example a system of N particles can be described by $3N$ Cartesian coordinates. If there are k constraints, we can choose $s = 3N - k$ independent generalized coordinates in any way that is convenient.

2.2 The Action and Hamilton's Principle

The action is defined as a time integral

$$I = \int_{t_1}^{t_2} dt L(q_k(t), \dot{q}_k(t), t) \quad (11)$$

where L is called the Lagrangian of the system. For the moment we don't specify it in detail. It is a single scalar function of the generalized coordinates and their velocities, that determines the equations of motion according to Hamilton's principle: The trajectory $q_k(t)$ of the system which starts at the point q_k^1 at time t_1 and ends up at the point q_k^2 at time t_2 is that trajectory which minimizes the action I .

This means that if we evaluate I for a trajectory $q_k(t) + \delta q_k(t)$ infinitesimally different from the solution, the change in the action will be of order δq_k^2 . So calculate

$$\begin{aligned} \Delta I &= \int_{t_1}^{t_2} dt [L(q_k(t) + \delta q_k(t), \dot{q}_k(t) + \delta \dot{q}_k(t), t) - L(q_k(t), \dot{q}_k(t), t)] \\ &= \int_{t_1}^{t_2} dt \sum_l \left[\delta q_l \frac{\partial L}{\partial q_l} + \delta \dot{q}_l(t) \frac{\partial L}{\partial \dot{q}_l} \right] + O(\delta q^2) \\ &= \sum_l \delta q_l \left. \frac{\partial L}{\partial \dot{q}_l} \right|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt \sum_l \delta q_l \left[\frac{\partial L}{\partial q_l} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_l} \right] + O(\delta q^2) \end{aligned} \quad (12)$$

Now since the ends of the trajectory are fixed, $\delta q_k(t_1) = \delta q_k(t_2) = 0$, so $q_k(t)$ will satisfy Hamilton's principle if

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_l} = \frac{\partial L}{\partial q_l}, \quad \text{for all } l \quad (13)$$

Without saying so, we have just applied what is known as the calculus of variations!

If the q_k 's are Cartesian coordinates, this will be the form of Newton's equations if

$$\frac{\partial L}{\partial \dot{q}_l} = m_l \dot{q}_l(t), \quad \frac{\partial L}{\partial q_l} = -\frac{\partial V}{\partial q_l} \quad (14)$$

which tells us that the Lagrangian in that case can be taken to be

$$L = \frac{1}{2} \sum_l m_l \dot{q}_l^2 - V(q) = T(\dot{q}) - V(q) \quad (15)$$

where T is the kinetic energy of the system and V is the potential energy. Note carefully the difference of L from the total energy $T + V$! there is an all important sign difference in the second term.

The Lagrangian is not uniquely determined, because Hamilton's principle requires that $\delta q(t_1) = \delta q(t_2) = 0$. For this reason a different Lagrangian

$$L' = L + \frac{d}{dt} f(q(t), t), \quad I' = I + f(q(t_2), t_2) - f(q(t_1), t_1) \quad (16)$$

will imply the same equations of motion. In other words, two Lagrangians, that differ by the total time derivative of a function of coordinates and time, will imply the same equations of motion.

2.3 The simple pendulum

As a familiar example of a problem with constraints consider the simple frictionless pendulum with massless rod of length l , swinging in the xz -plane with the pivot at the origin of coordinates. Let φ be the angle from the vertical. Then $x = l \sin \varphi$, $z = -l \cos \varphi$, and $T = (m/2)(\dot{x}^2 + \dot{y}^2) = (ml^2/2)\dot{\varphi}^2$ and $V = -mgl \cos \varphi$. Hence

$$L = \frac{ml^2}{2} \dot{\varphi}^2 + mgl \cos \varphi \quad (17)$$

and Lagrange's equation gives the familiar $\ddot{\varphi} = -(g/l) \sin \varphi$. Here we have solved the constraint $x^2 + z^2 = l^2$ by going to polar coordinates and setting $\rho = l$.

Let's consider the same problem in terms of Cartesian coordinates. Then the unconstrained Lagrangian is

$$L = \frac{m}{2}(\dot{x}^2 + \dot{z}^2) - mgz \quad (18)$$

We have to impose the constraint $x^2 + z^2 = l^2$. The method of Lagrange multipliers adds a term $\lambda(t)(x^2 + z^2 - l^2)$ to the Lagrangian:

$$L \rightarrow \frac{m}{2}(\dot{x}^2 + \dot{z}^2) - mgz + \lambda(t)(x^2 + z^2 - l^2) \quad (19)$$

We now regard λ as a generalized coordinate. Since $\dot{\lambda}$ doesn't appear in the Lagrangian, the e.o.m. for λ is just

$$\frac{\partial L}{\partial \lambda} = x^2 + z^2 - l^2 = 0 \quad (20)$$

which is seen to be precisely the constraint we wish to impose. The e.o.m's for x, z now involve λ :

$$m\ddot{x} = 2\lambda x, \quad m\ddot{z} = -mg + 2\lambda z \quad (21)$$

From this we see that the force exerted by the constraint is $\mathbf{F}_c = 2\lambda\boldsymbol{\rho}$. Passing to polar coordinates at this point, the e.o.m's reduce to

$$\ddot{\varphi} = -\frac{g}{l} \sin \varphi, \quad 2\lambda = m(\ddot{\varphi} \cot \varphi - \dot{\varphi}^2) = -m \left(\frac{g}{l} \cos \varphi + \dot{\varphi}^2 \right) \quad (22)$$

$$\mathbf{F}_c = -m (g \cos \varphi + l\dot{\varphi}^2) (\sin \varphi \hat{x} - \cos \varphi \hat{z}) \quad (23)$$

For example, at the bottom $\varphi = 0$, $\mathbf{F}_c = m(g + l\dot{\varphi}^2)\hat{z}$ to compensate gravity and match $m \times$ the centripetal acceleration. Notice that if the rod is replaced by a rope, it can only pull on the particle, which means that it forces the constraint only when $\lambda < 0$. This is always true if $\varphi < \pi/2$. But if $\varphi > \pi/2$ the rope will only do its job if $\dot{\varphi}^2 > -(g/l) \cos \varphi$!

2.4 The energy from the Lagrangian

We are all familiar with the conservation of energy by Newton's equations when the forces are conservative. In the Lagrange formulation we can generally identify an energy conservation law when the Lagrangian has no explicit time dependence. Consider the Hamiltonian defined by

$$H \equiv \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \quad (24)$$

$$\begin{aligned} \frac{dH}{dt} &= \sum_i \ddot{q}_i \frac{\partial L}{\partial \dot{q}_i} + \sum_i \dot{q}_i \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \sum_i \dot{q}_i \frac{\partial L}{\partial q_i} - \sum_i \ddot{q}_i \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial t} \\ &= \sum_i \dot{q}_i \left[\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} \right] - \frac{\partial L}{\partial t} = -\frac{\partial L}{\partial t} \end{aligned} \quad (25)$$

by Lagrange's equations. Thus H is conserved provided $\partial L / \partial t = 0$. For standard Newtonian systems where T is quadratic in the velocities and $L = T - V$

$$\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = 2T \quad (26)$$

so $H = T + V$ as we expect.

3 Conservation Laws and Symmetries of the Lagrangian

3.1 Time translation symmetry and energy conservation

We have just seen that when the Lagrangian has no explicit time dependence, $\partial L/\partial t = 0$, the Hamiltonian H is conserved. Since the Hamiltonian is just the energy of the system, this connects energy conservation to time translation symmetry.

3.2 Space translation symmetry and momentum conservation

Consider a system of N particles with trajectories $\mathbf{r}_k(t)$, and consider an overall infinitesimal translation of the coordinate system by an amount $\boldsymbol{\epsilon}$. Then in the new system the trajectories are $\mathbf{r}_k(t) + \boldsymbol{\epsilon}$. However the velocities are unchanged. Thus the change in the Lagrangian is

$$\delta L = \sum_k \boldsymbol{\epsilon} \cdot \frac{\partial L}{\partial \mathbf{r}_k} = \boldsymbol{\epsilon} \cdot \frac{d}{dt} \sum_k \frac{\partial L}{\partial \dot{\mathbf{r}}_k} \quad (27)$$

Symmetry of the Lagrangian under space translations means that $\delta L = 0$ which implies the conservation of total momentum

$$\mathbf{P} \equiv \sum_k \frac{\partial L}{\partial \dot{\mathbf{r}}_k} \quad (28)$$

For example if the total potential energy only depends on difference of coordinate $\mathbf{r}_k - \mathbf{r}_l$, total momentum will be conserved.

The more direct consequence of translation invariance is that the total force on the system vanishes:

$$\mathbf{F} = \sum_k \frac{\partial L}{\partial \mathbf{r}_k} = 0 \quad (29)$$

For a two particle system this is just a reflection of Newton's third law.

3.3 Galilei Invariance and the center of mass.

A Galilei transformation connects two coordinate frames moving at a uniform velocity with respect to each other.

$$\mathbf{r}_k(t) \rightarrow \mathbf{r}_k(t) + \mathbf{V}t. \quad (30)$$

Newton's equations take identical form in the two frames if the force is translationally invariant. Under this transformation

$$\begin{aligned} L &= \sum_k \frac{m_k}{2} \dot{\mathbf{r}}_k^2 - V \\ &\rightarrow L + \mathbf{V} \cdot \sum_k m_k \dot{\mathbf{r}}_k + \frac{M}{2} \mathbf{V}^2 = L + \frac{d}{dt} \left[\mathbf{V} \cdot \sum_k m_k \mathbf{r}_k + \frac{M}{2} \mathbf{V}^2 t \right] \end{aligned} \quad (31)$$

Here $M = \sum_k m_k$ is the total mass in the system. The Lagrangian is not invariant but rather changes by a total time derivative of a function of coordinates and time. Note that though the quantity $\sum_k m_k \mathbf{r}_k \equiv M \mathbf{R}_{\text{CM}}$ is not conserved, $\sum_k m_k \mathbf{r}_k - \mathbf{P}t = M \mathbf{R}_{\text{CM}} - \mathbf{P}t$ is conserved. We can rewrite δL :

$$\begin{aligned}\delta L &= \mathbf{V} \cdot \frac{d}{dt} \left[\sum_k m_k \mathbf{r}_k - \mathbf{P}t \right] + \frac{(\mathbf{P} + M\mathbf{V})^2}{2M} - \frac{\mathbf{P}^2}{2M} \\ &= \mathbf{V} \cdot \frac{d}{dt} \left[\sum_k m_k \mathbf{r}_k - \mathbf{P}t \right] + \delta \frac{\mathbf{P}^2}{2M}\end{aligned}\quad (32)$$

so the conservation law follows from $\delta [L - \mathbf{P}^2/2M] = 0$. This conservation law ensures that one can always choose the center of mass frame, $\mathbf{R}_{\text{CM}} = 0$, by a Galilei transformation combined with a spatial translation.

Note on Relativity: In special relativity the relation between inertial frames is given by a Lorentz transformation instead of a Galilei transformation. For example, the relation between coordinates in a frame K' moving w.r.t. frame K parallel to the x -axis with speed V is

$$t' = \gamma(t - Vx/c^2), \quad x' = \gamma(x - Vt), \quad y' = y, \quad z' = z \quad (33)$$

in which time coordinates as well as space coordinates are changed. Here $\gamma \equiv 1/\sqrt{1 - V^2/c^2}$. Taking differentials, we calculate

$$\begin{aligned}c^2 dt'^2 - dx'^2 - dy'^2 - dz'^2 &= \gamma^2 c^2 (dt - V dx/c^2)^2 - \gamma^2 (dx - V dt)^2 - dy^2 - dz^2 \\ &= \gamma^2 (c^2 - V^2) dt^2 - \gamma^2 (1 - V^2/c^2) dx^2 - dy^2 - dz^2 \\ &= c^2 dt^2 - dx^2 - dy^2 - dz^2\end{aligned}\quad (34)$$

$$cdt' \sqrt{1 - \frac{1}{c^2} \left(\frac{d\mathbf{r}'}{dt'} \right)^2} = cdt \sqrt{1 - \frac{1}{c^2} \left(\frac{d\mathbf{r}}{dt} \right)^2} \quad (35)$$

Recalling from problem 4 that the Lagrangian for a relativistic particle is

$$L = -mc^2 \sqrt{1 - \frac{1}{c^2} \left(\frac{d\mathbf{r}}{dt} \right)^2}, \quad (36)$$

we see that

$$dt' L' = dt L \quad (37)$$

I.e. that the *action*, not the Lagrangian, is invariant under Lorentz transformations. Thus Hamilton's principle is valid in all Lorentz frames.

3.4 Rotational symmetry and angular momentum conservation

A rotation can be described by fixing an axis \mathbf{u} and specifying the angle of rotation $\delta\varphi$ about that axis. We combine these into a vector $\delta\boldsymbol{\varphi} = \mathbf{u}\delta\varphi$. We take $\delta\varphi$ infinitesimal. It is convenient to choose the origin of Cartesian coordinates on the axis of rotation. Then the infinitesimal change of each coordinate vector is given by $\delta\mathbf{r}_k(t) = \delta\boldsymbol{\varphi} \times \mathbf{r}_k(t)$. Taking time derivatives gives $\delta\dot{\mathbf{r}}_k(t) = \delta\boldsymbol{\varphi} \times \dot{\mathbf{r}}_k(t)$, and then

$$\begin{aligned}\delta L &= \sum_k (\delta\boldsymbol{\varphi} \times \mathbf{r}_k(t)) \cdot \frac{\partial L}{\partial \mathbf{r}_k} + \sum_k (\delta\boldsymbol{\varphi} \times \dot{\mathbf{r}}_k(t)) \cdot \frac{\partial L}{\partial \dot{\mathbf{r}}_k} \\ &= \delta\boldsymbol{\varphi} \cdot \sum_k \left(\mathbf{r}_k(t) \times \frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{r}}_k} + \dot{\mathbf{r}}_k(t) \times \frac{\partial L}{\partial \dot{\mathbf{r}}_k} \right) = \delta\boldsymbol{\varphi} \cdot \frac{d}{dt} \sum_k \mathbf{r}_k(t) \times \frac{\partial L}{\partial \dot{\mathbf{r}}_k}\end{aligned}\quad (38)$$

Then invariance of the Lagrangian, $\delta L = 0$ implies the conservation of angular momentum

$$\mathbf{J} \equiv \sum_k \mathbf{r}_k(t) \times \frac{\partial L}{\partial \dot{\mathbf{r}}_k} = \sum_k \mathbf{r}_k \times \mathbf{p}_k \quad (39)$$

In a general inertial frame the angular momentum depends on the choice of the origin of coordinates: under $\mathbf{r}_k \rightarrow \mathbf{r}_k + \mathbf{a}$,

$$\mathbf{J} \rightarrow \mathbf{J} + \mathbf{a} \times \mathbf{P}. \quad (40)$$

With translational invariance, \mathbf{P} is conserved, so the angular momentum computed about any point is then conserved. Note that in center of mass frame ($\mathbf{P} = 0$), \mathbf{J} is independent of the origin of coordinates. Of course under rotations \mathbf{J} rotates as any vector.

Finally, for a given system, call \mathbf{S} the angular momentum in the system's center of mass frame. Then in a frame in which the c.o.m moves with velocity \mathbf{V} the angular momentum is

$$\begin{aligned}\mathbf{J} &\equiv \sum_k (\mathbf{r}_k + \mathbf{V}t) \times (\mathbf{p}_k + m_k\mathbf{V}) \\ &= \sum_k \mathbf{r}_k \times \mathbf{p}_k + \mathbf{V}t \times \mathbf{P} + \sum_l m_l \mathbf{r}_k \times \mathbf{V} \\ &= \mathbf{S} + M\mathbf{R}_{\text{CM}} \times \mathbf{V} = \mathbf{S} + \mathbf{R}_{\text{CM}} \times \mathbf{P}\end{aligned}\quad (41)$$

Thus the total angular momentum of a system can always be decomposed into a “spin” part \mathbf{S} which is the angular momentum in the center of mass frame and an “orbital” part $\mathbf{R}_{\text{CM}} \times \mathbf{P}$. And we now appreciate that the spin part doesn't depend on the choice of origin of coordinates!

There can be situations with only partial symmetry. For instance, if one has translational invariance in say the z -coordinate, then P^z is conserved. Or isotropy about only one point implies the conservation of the angular momentum wrt that point. Or yet again If there is cylindrical symmetry about say the z -axis, J^z will be conserved.

3.5 Scaling symmetry and Virial theorems

Suppose we scale all the coordinates of a system by the same factor, $\mathbf{r}_k \rightarrow \lambda \mathbf{r}_k$. Then the Newtonian kinetic energy scales as $T \rightarrow \lambda^2 T$. The potential energy will have a simple scaling law if it is homogeneous in the coordinates. If the degree of homogeneity is k , then $V \rightarrow \lambda^k V$. We can make the Lagrangian scale by an overall factor, if we scale the time variable, $t \rightarrow \lambda^p t$, where $\lambda^{2-2p} = \lambda^k$ i.e. if $p = 1 - k/2$. Then the new L will imply the same equations of motion as the old L .

Simple examples: (1) $k = 2$ (harmonic oscillator), $p = 0$, so the frequency is independent of the amplitude; (2) $k = -1$ (Coulomb or Newtonian potential), $p = 3/2$, so the period of an orbit varies as the $3/2$ power of its size: $T^2 \propto R^3$ (Kepler's third law; (3) $k = 1$ (uniform force), $p = 1/2$, the time of fall varies as the square root of the distance fallen.

Looking a little more closely at the details of the scaling laws in situations where $L = T(\dot{q}) - V(q)$ leads us to the virial theorem. First, the quadratic dependence of the kinetic energy on velocities means $T(\lambda \dot{q}_n) = \lambda^2 T(\dot{q}_n)$. Differentiating this w.r.t. λ and setting $\lambda = 1$ implies

$$\begin{aligned} 2T &= \sum_n \dot{q}_n \frac{\partial T}{\partial \dot{q}_n} = \frac{d}{dt} \left(\sum_n q_n \frac{\partial T}{\partial \dot{q}_n} \right) - \sum_n q_n \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_n} \\ &= \frac{d}{dt} \sum_n q_n \frac{\partial T}{\partial \dot{q}_n} - \sum_n q_n \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_n} \\ 2T &= \frac{d}{dt} \left(\sum_n q_n \frac{\partial T}{\partial \dot{q}_n} \right) - \sum_n q_n \frac{\partial L}{\partial q_n} \end{aligned} \quad (42)$$

for these special Lagrangians. If we time average this equation and the motion is bounded, the average of the first term vanishes and we have the Virial theorem:

$$2\langle T \rangle = - \left\langle \sum_n q_n \frac{\partial L}{\partial q_n} \right\rangle = - \left\langle \sum_n q_n F_n \right\rangle \equiv \text{Virial} \quad (43)$$

where we have identified the generalized force $F_n = \partial L / \partial q_n$.

If V is homogeneous of degree k in the coordinates we find, by a similar argument that

$$kV = \sum_n q_n \frac{\partial V}{\partial q_n} = - \sum_n q_n \frac{\partial L}{\partial q_n} \quad (44)$$

for these special Lagrangians. Then Lagrange equations imply

$$2T - kV = \frac{d}{dt} \sum_n q_n \frac{\partial T}{\partial \dot{q}_n} = \frac{d}{dt} \sum_n q_n m_n \dot{q}_n \quad (45)$$

The final step is to average this equation over a long time interval t_0 :

$$\langle 2T - kV \rangle \equiv \frac{1}{t_0} \int_0^{t_0} dt (2T - kV) = \frac{1}{t_0} \left[\sum_n q_n(t_0) m_n \dot{q}_n(t_0) - \sum_n q_n(0) m_n \dot{q}_n(0) \right] \quad (46)$$

If the motion is bounded, e.g. planetary orbits, the right side goes to 0 when $t_0 \rightarrow \infty$. This is the virial theorem:

$$\langle T \rangle = \frac{k}{2} \langle V \rangle, \quad \text{Bounded Motion} \quad (47)$$

From energy conservation $T + V = E$ so that

$$E = \langle T \rangle + \langle V \rangle = \frac{k+2}{2} \langle V \rangle = \frac{k+2}{k} \langle T \rangle \quad (48)$$

Thus on time averaging the proportion of the total energy going into kinetic and potential energy is fixed by the scaling laws. Notice that for $k = -1$ (Keplerian motion) this relation implies that $E = -\langle T \rangle < 0$, which is indeed the condition for bounded motion in that case. Also notice that for harmonic oscillations ($k = 2$) the split between kinetic and potential is precisely 50-50.

4 Solving the Equations of Motion

4.1 Motion in One Dimension

The Lagrangian for a Newtonian particle moving in one dimension is simply

$$L = \frac{1}{2} m \dot{x}^2 - V(x, t) \quad (49)$$

If $V(x)$ is independent of time, energy is conserved and we can write immediately

$$\dot{x}^2 = \frac{2(E - V(x))}{m} \geq 0 \quad (50)$$

We see that motion is only possible in regions where $V(x) \leq E$. since this is a first order equation we can directly integrate it to obtain

$$t = \pm \sqrt{\frac{m}{2}} \int_{x(0)}^{x(t)} \frac{dx'}{\sqrt{E - V(x')}} \quad (51)$$

The particle comes to rest at points where $V(x) = E$, called turning points of the motion where the velocity reverses. Oscillatory motion occurs when there are two turning points, say $x_1 < x_2$. Then the period of an oscillation is just twice the travel time from x_1 to x_2 :

$$T = \sqrt{2m} \int_{x_1}^{x_2} \frac{dx'}{\sqrt{E - V(x')}} \quad (52)$$

As an example, take the simple harmonic oscillator potential $V(x) = kx^2/2$. The turning points are $x_{\pm} = \pm \sqrt{2E/k}$, and the period is

$$T = \sqrt{\frac{4m}{k}} \int_{-x_+}^{x_+} \frac{dx'}{\sqrt{x_+^2 - x'^2}} = \frac{4}{\omega} \int_0^1 \frac{du}{\sqrt{1 - u^2}} = \frac{2\pi}{\omega} \quad (53)$$

Independent of the amplitude x_+ . Letting the upper endpoint be variable we solve the equations of motion for this case directly:

$$\begin{aligned}
t &= \frac{1}{\omega} \int_{x(0)/x_+}^{x(t)/x_+} \frac{du}{\sqrt{1-u^2}} \\
\omega t &= \sin^{-1} \frac{x(t)}{x_+} - \sin^{-1} \frac{x(0)}{x_+} \\
x(t) &= x_+ \sin \left(\omega t + \sin^{-1} \frac{x(0)}{x_+} \right)
\end{aligned} \tag{54}$$

a familiar result!

The pendulum provides a less trivial example.

$$L = \frac{1}{2}ml^2\dot{\varphi}^2 + mgl \cos \varphi, \quad E = \frac{1}{2}ml^2\dot{\varphi}^2 - mgl \cos \varphi$$

Necessarily $E \geq -mgl$. To simplify the writing, define $\omega = \sqrt{g/l}$, $\epsilon = E/ml^2 \geq -g/l$. so that

$$\begin{aligned}
\epsilon &= \frac{1}{2}\dot{\varphi}^2 - \omega^2 \cos \varphi \\
\dot{\varphi}^2 &= 2(\epsilon + \omega^2 \cos \varphi) \\
t &= \int_0^\varphi \frac{d\varphi'}{\sqrt{2(\epsilon + \omega^2 \cos \varphi')}} = \int_0^\varphi \frac{d\varphi'}{\sqrt{2(\epsilon + \omega^2 - 2\omega^2 \sin^2(\varphi'/2))}}
\end{aligned} \tag{55}$$

There are two qualitatively different motions. If $\epsilon > \omega^2$ the mass goes “over the top” and keeps circling indefinitely (assuming the absence of friction!). If $\epsilon < \omega^2$ the pendulum reaches a maximum angle φ_0 given by $\cos \varphi_0 = -\epsilon/g$. Note that if $0 < \epsilon < \omega^2$ $\varphi_0 > \pi/2$; and if $-\omega^2 < \epsilon < 0$, $\varphi_0 < \pi/2$.

In the first case $\epsilon > \omega^2$, we can rearrange and change integration variables $u = \sin(\varphi'/2)$:

$$\begin{aligned}
t\sqrt{2(\epsilon + \omega^2)} &= \int_0^\varphi \frac{d\varphi'}{\sqrt{1 - k^2 \sin^2(\varphi'/2)}} = 2 \int_0^{\sin(\varphi/2)} \frac{du}{\sqrt{1 - u^2}\sqrt{1 - k^2u^2}} \\
k^2 &\equiv \frac{2\omega^2}{\epsilon + \omega^2} < 1
\end{aligned} \tag{56}$$

The integral here cannot be expressed in terms of elementary functions. But it turns out that we can use it to define elliptic functions which are higher transcendental functions which generalize trigonometric. Jacobi introduced a version of them which has gained wider acceptance than other and earlier versions. The Jacobi elliptic function $\text{sn}(z, k^2)$ can be defined by the integral

$$z = \int_0^{\text{sn}(z, k^2)} \frac{du}{\sqrt{1 - u^2}\sqrt{1 - k^2u^2}} \tag{57}$$

With this new function we can write the solution of the pendulum motion with $\epsilon > \omega^2$ as follows:

$$\sin \frac{\varphi(t)}{2} = \operatorname{sn} \left(t \sqrt{\frac{\epsilon + \omega^2}{2}}, \frac{2\omega^2}{\epsilon + \omega^2} \right), \quad \epsilon > \omega^2 \quad (58)$$

The period of this motion is twice the time it takes for the bob to go from bottom to top:

$$T = \frac{4}{\sqrt{2(\epsilon + \omega^2)}} \int_0^1 \frac{du}{\sqrt{1-u^2}\sqrt{1-k^2u^2}} \equiv \frac{4K}{\sqrt{2(\epsilon + \omega^2)}} \quad (59)$$

When $\epsilon < \omega^2$, the pendulum oscillates between $\varphi = \pm\varphi_0$. We can write $\epsilon = -\omega^2 \cos \varphi_0$, so that $k^2 = 1/\sin^2(\varphi_0/2) > 1$. Then it is convenient to change variables to $v = ku = u/\sin(\varphi_0/2)$. leading to

$$\omega t = \int_0^{\sin(\varphi/2)/\sin(\varphi_0/2)} \frac{dv}{\sqrt{1-\sin^2(\varphi_0/2)v^2}\sqrt{1-v^2}} \quad (60)$$

$$\sin \frac{\varphi}{2} = \sin \frac{\varphi_0}{2} \operatorname{sn} \left(\omega t, \sin^2 \frac{\varphi_0}{2} \right) \quad (61)$$

The motion is oscillatory between $-\varphi_0$ and φ_0 . The period of oscillation is 4 times the time it takes to rise from $\varphi = 0$ to φ_0 :

$$\omega T = 4 \int_0^1 \frac{dv}{\sqrt{1-\sin^2(\varphi_0/2)v^2}\sqrt{1-v^2}} = 4K \left(\sin \frac{\varphi_0}{2} \right) \quad (62)$$

Mathematical properties of elliptic functions. Compare the definition of $\operatorname{sn}(z, k^2)$ to a similar definition of $\sin z$:

$$z = \int_0^{\sin z} \frac{du}{\sqrt{1-u^2}} \quad (63)$$

We see that $\operatorname{sn}(z, k^2) \rightarrow \sin z$ for $k \rightarrow 0$. Recall that the sine function is periodic under $z \rightarrow z + 2\pi$: it is singly periodic. It turns out that $\operatorname{sn}(z, k^2)$ has *two* periods. The ratio of the two periods in this case is actually imaginary, that is the periodicities are in different directions in the complex z -plane. Fig. 1 compares the function $\operatorname{sn}(xK, k^2)$ for two values of $k = 0.05, 0.999$. As $k \rightarrow 1$ the maxima flatten out more and more.

Differentiating both sides of the integral defining sn with respect to z implies the differential equation

$$\operatorname{sn}' = \sqrt{1-\operatorname{sn}^2}\sqrt{1-k^2\operatorname{sn}^2} \quad (64)$$

$$\operatorname{sn}'' = -(1+k^2)\operatorname{sn} + 2k^2\operatorname{sn}^3 \quad (65)$$

which can be used in exploring its properties.

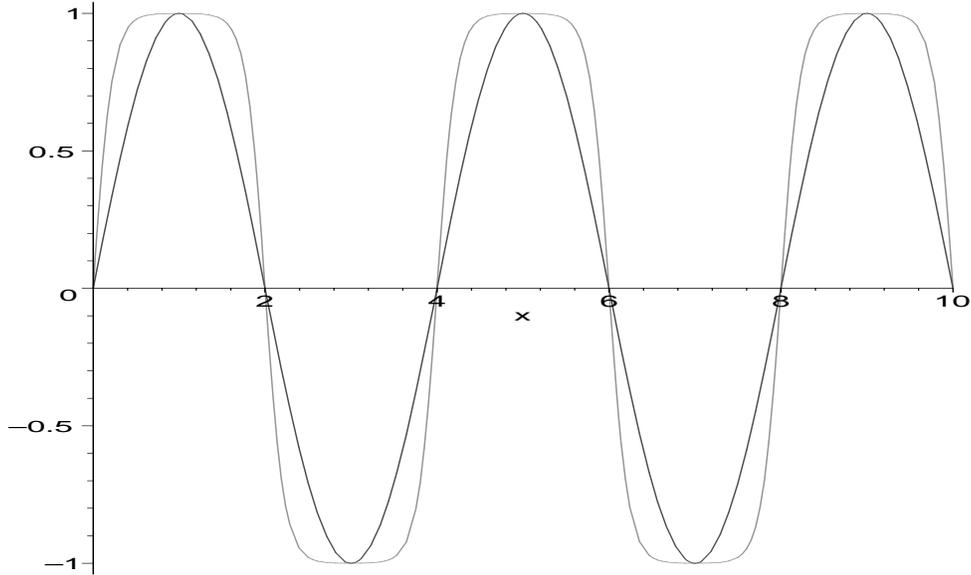


Figure 1: Plots of $\text{sn}(xK(k), k)$ versus x for $k = 0.05, 0.999$

The parameter k is known as the modulus of the elliptic function. The periods of sn can be expressed using k and the conjugate modulus $k' \equiv \sqrt{1 - k^2}$. Define the Complete elliptic integrals

$$K(k) \equiv \int_0^1 \frac{du}{\sqrt{1 - u^2}\sqrt{1 - k^2u^2}}, \quad K'(k) \equiv \int_0^1 \frac{du}{\sqrt{1 - u^2}\sqrt{1 - k'^2u^2}} \quad (66)$$

Then the two periods of $\text{sn}(z, k)$ are $4K$ and $2iK'$. Notice that, in the limit $k \rightarrow 0$, $K \rightarrow \pi/2$ and $K' \rightarrow \infty$, in accord with the single periodicity of $\sin z$. The proof of these periodicities is difficult.

In a sense elliptic functions are to the complex plane what trig functions are to the real line. One can tile the complex plane with rectangles of dimensions $4K \times 2K'$ and the values of sn assumed at corresponding points in each rectangle are the same.

Just as there are several useful trig functions, so are there several useful elliptic functions: Three are sn , cn , dn related by

$$\text{sn}^2 + \text{cn}^2 = 1, \quad k^2\text{sn}^2 + \text{dn}^2 = 1 \quad (67)$$

These definitions allow us to write, for instance, $\text{sn}' = \text{cn dn}$. Evaluating these functions at K , we have

$$\text{sn}(K, k) = 1, \quad \text{cn}(K, k) = 0, \quad \text{dn}(K, k) = \sqrt{1 - k^2} = k'. \quad (68)$$

From a fundamental mathematical point of view the elliptic functions are defined as doubly periodic functions analytic everywhere with the exception of a finite number of poles in each period rectangle.

Using the integral definition of sn we can immediately see that $\text{sn} \rightarrow \infty$ at finite values of z because the integral converges as $u \rightarrow \infty$. (This is in contrast to the integral definition of \sin because the integral diverges as $u \rightarrow \infty$.)

4.2 Motion in a General Central Potential $V(r)$

We now consider motion in a potential that depends only on the radial distance from the central point of attraction. More generally we can start with two particles under the influence of a mutual interaction potential that depends only on the distance between the two particles,

$$L = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 - V(\mathbf{r}_1 - \mathbf{r}_2) \quad (69)$$

But as we already have discussed, we can remove the motion of the system as a whole by going to the center of mass frame. So we change coordinates

$$\begin{aligned} \mathbf{R} &= \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}, & \mathbf{r} &= \mathbf{r}_1 - \mathbf{r}_2 \\ \mathbf{r}_1 &= \mathbf{R} + \frac{m_2}{M}\mathbf{r}, & \mathbf{r}_2 &= \mathbf{R} - \frac{m_1}{M}\mathbf{r} \end{aligned} \quad (70)$$

$$L = \frac{M}{2}\dot{\mathbf{R}}^2 + \frac{m_1m_2}{2M}\dot{\mathbf{r}}^2 - V(\mathbf{r}) \quad (71)$$

where $M = m_1 + m_2$. The dynamics of \mathbf{R} is completely independent of the dynamics of \mathbf{r} , which is just the dynamics of a particle with reduced mass $m = m_1m_2/M$ moving in the potential $V(\mathbf{r})$.

From now on we restrict V to be central i.e. to depend only on $r = |\mathbf{r}|$. Thus we consider the Lagrangian

$$L = \frac{1}{2}m\dot{\mathbf{r}}^2 - V(r) \quad (72)$$

Because the system is rotationally invariant about $\mathbf{r} = 0$, Both energy E and angular momentum $\mathbf{J} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \dot{\mathbf{r}}$ are conserved. Since both \mathbf{r} and $\dot{\mathbf{r}}$ are perpendicular to the constant direction \mathbf{J} , the trajectory of the particles stays within a fixed plane. Finally since the cross product of two vectors has magnitude equal to the area of the parallelogram spanned by the two vectors, the constancy of \mathbf{J} implies that the trajectory sweeps out equal areas in equal times (Kepler's Second Law).

To go further let's use spherical polar coordinates with z -axis chosen parallel to \mathbf{J} . Then the orbit lies in the xy -plane, i.e. the angle $\theta = \pi/2$. The position vector of the particle is then $\mathbf{r} = r(t)(\cos \varphi(t), \sin \varphi(t), 0)$,

$$\mathbf{J} = mr^2\dot{\varphi}\hat{z}, \quad \dot{\varphi} = \frac{J}{mr^2}, \quad \dot{\mathbf{r}}^2 = \dot{r}^2 + r^2\dot{\varphi}^2 = \dot{r}^2 + \frac{J^2}{m^2r^2} \quad (73)$$

The Lagrangian in polar coordinates is

$$L = \frac{m}{2}\dot{r}^2 + \frac{m}{2}r^2\dot{\varphi}^2 - V(r) \quad (74)$$

Note that the e.o.m for φ is simply that $r^2\dot{\varphi} = \text{constant}$ which is nothing but angular momentum conservation. The dynamics of the radial coordinate is then completely given by energy conservation

$$E = \frac{m}{2}\dot{r}^2 + \frac{J^2}{2mr^2} + V(r) = \text{Constant} \quad (75)$$

We can think of the radial motion as a particle moving in one dimension in the effective potential

$$V_{\text{eff}} \equiv \frac{J^2}{2mr^2} + V(r) \quad (76)$$

with solution

$$t = \sqrt{\frac{m}{2}} \int_{r_{\min}}^{r(t)} \frac{dr'}{\sqrt{E - V_{\text{eff}}(r')}} \quad (77)$$

If the input potential $V(r)$ is negative approaching 0 less rapidly than $1/r^2$ at $r = \infty$, and $J \neq 0$ the effective potential goes to $+\infty$ at $r = 0$, goes negative at a finite r reaches a negative minimum V_{\min} and thence rises to 0. For $E = V_{\min}$, r is a constant and we have a circular orbit. Then $\varphi(t) = (J/mr^2)t$ and the period of the orbit is $2\pi mr^2/J$.

When $E > V_{\min}$ there are two turning points for $r(t)$, r_{\min} and r_{\max} . If E stays very close to V_{\min} , we can approximate V_{eff} by a quadratic

$$V_{\text{eff}}(r) \approx V_{\text{eff}}(r_0) + \frac{1}{2}(r - r_0)^2 V_{\text{eff}}''(r_0) \quad (78)$$

So we see that r will undergo simple harmonic motion of angular frequency $\omega = \sqrt{V_{\text{eff}}''(r_0)/m}$.

We can also find the orbit $r(\varphi)$ by noticing

$$\frac{d\varphi}{dr} = \frac{\dot{\varphi}}{\dot{r}} = \frac{J}{r^2 \sqrt{2m} \sqrt{E - V_{\text{eff}}(r')}} \quad (79)$$

$$\varphi = \frac{J}{\sqrt{2m}} \int_{r_{\min}}^{r(\varphi)} \frac{dr'}{r'^2 \sqrt{E - V_{\text{eff}}(r')}} \quad (80)$$

notice that there is no mathematical reason that a bound orbit should close on itself. That simply means that $r(\pi)$ need not be r_{\max} . This will only happen for very special potentials, in fact only for $V(r) \propto 1/r$ or r .

So for a generic potential which allows bounded orbits we can calculate the total change in φ when after r goes from r_{\min} to r_{\max} and back to r_{\min} :

$$\Delta\varphi = 2 \frac{J}{\sqrt{2m}} \int_{r_{\min}}^{r_{\max}} \frac{dr'}{r'^2 \sqrt{E - V_{\text{eff}}(r')}} = 2 \frac{J}{\sqrt{2m}} \int_{1/r_{\max}}^{1/r_{\min}} \frac{du}{\sqrt{E - V_{\text{eff}}(1/u)}} \quad (81)$$

where the second form obtained, by the change of variables $u = 1/r'$, can be very useful especially if $V(r)$ is a negative power of r . If by good fortune $\Delta\varphi = 2\pi$, the orbit is a simple

closed curve encircling the origin. More generally if $\Delta\varphi = 2\pi p/q$ for integers p, q , the orbit will close after q revolutions.

In the case of a nearly circular orbit of radius $r_0 = 1/u_0$, it is convenient to use the approximation

$$V_{\text{eff}}(1/u) = V_{\text{eff}}(1/u_0) + \frac{1}{2}(u - u_0)^2 \frac{d^2 V_{\text{eff}}(1/u)}{du^2} \Big|_{u=u_0} \quad (82)$$

Then the turning points in u are given by

$$u_{\pm} = u_0 \pm \sqrt{2(E - V_{\text{eff}}(1/u_0)) \left(\frac{d^2 V_{\text{eff}}(1/u)}{du^2} \Big|_{u=u_0} \right)^{-1}}$$

Then we write

$$\begin{aligned} \Delta\varphi &\approx 2 \frac{J}{\sqrt{m}} \left(\frac{d^2 V_{\text{eff}}(1/u)}{du^2} \Big|_{u=u_0} \right)^{-1/2} \int_{-u_+}^{u_+} \frac{du}{\sqrt{u_+^2 - u^2}} = 2\pi \frac{J}{\sqrt{m}} \left(\frac{d^2 V_{\text{eff}}(1/u)}{du^2} \Big|_{u=u_0} \right)^{-1/2} \\ &\approx 2\pi \left(1 + \frac{m}{J^2} \frac{d^2 V(1/u)}{du^2} \Big|_{u=u_0} \right)^{-1/2} \end{aligned} \quad (83)$$

which we see immediately is 2π when $V \propto 1/r = u$. When $V = kr^2/2 = k/2u^2$ on the other hand $d^2V/du^2 = 3k/u^4$. But then $u_0^4 = km/J^2$, so $\Delta\varphi = \pi$ in this case, so the orbit closes after r goes from min to max to min twice.

4.3 Motion in the $1/r$ potential

Now we turn our attention to the important special case of a $1/r$ potential, the famous Kepler problem. We put $V(r) = -k/r = -ku$, with $k > 0$. Plugging this into the formula for φ :

$$\varphi = \frac{J}{\sqrt{2m}} \int_{1/r(\varphi)}^{1/r_{\min}} \frac{du}{\sqrt{E + ku - J^2 u^2 / 2m}} \quad (84)$$

This is an elementary integral which we identify by completing the square

$$\begin{aligned} E + ku - J^2 u^2 / 2m &= E + \frac{mk^2}{2J^2} - \frac{J^2}{2m} (u - mk/J^2)^2 \\ &= \frac{J^2}{2m} [(1/r_{\min} - mk/J^2)^2 - (u - mk/J^2)^2] \end{aligned} \quad (85)$$

Thus

$$\begin{aligned}\varphi &= \int_{1/r(\varphi)}^{1/r_{\min}} \frac{du}{\sqrt{(1/r_{\min} - mk/J^2)^2 - (u - mk/J^2)^2}} \\ &= \frac{\pi}{2} - \sin^{-1} \frac{1/r - mk/J^2}{1/r_{\min} - mk/J^2}\end{aligned}\quad (86)$$

$$\frac{1}{r} = \frac{mk}{J^2} + \left(\frac{1}{r_{\min}} - \frac{mk}{J^2} \right) \cos \varphi \quad (87)$$

This is the generic equation for a conic section

$$r(\varphi) = \frac{p}{1 + e \cos \varphi} \quad (88)$$

where the latus rectum p and the eccentricity are given by

$$p = \frac{J^2}{mk}, \quad e = \sqrt{1 + \frac{2EJ^2}{mk^2}} \quad (89)$$

When $e < 1$ ($E < 0$) the motion stays bounded and we can expect it to be an ellipse. In this case we see that for fixed $E < 0$ there is a maximum possible angular momentum $J_{\max} = k\sqrt{-m/2E}$, which occurs when $e = 0$, i.e. for circular motion. If $e \geq 1$ ($E \geq 0$) the motion is unbounded and will be a hyperbola or if $e = 1$ ($E = 0$) a parabola.

To see this equation in Cartesian coordinates, we use $x = r(\varphi) \cos \varphi$ and the polar equation to get $r = p - ex$. Then

$$y^2 = r^2 \sin^2 \varphi = r^2 - x^2 = (p - ex)^2 - x^2 = p^2 - 2pex - (1 - e^2)x^2 \quad (90)$$

which clearly shows three cases $e < 1, e = 1, e > 1$. By completing the square, in the case $e \neq 1$,

$$\begin{aligned}y^2 &= p^2 - (1 - e^2) \left(x - \frac{ep}{e^2 - 1} \right)^2 + \frac{e^2 p^2}{1 - e^2} \\ \frac{p^2}{1 - e^2} &= y^2 + (1 - e^2) \left(x - \frac{ep}{e^2 - 1} \right)^2,\end{aligned}\quad (91)$$

we see that the center of the conic is at the point $(-ep/(1 - e^2), 0)$, which is on the negative (positive) x -axis for $e < 1, e > 1$. We can put this in the standard form

$$\frac{(x - x_0)^2}{a^2} \pm \frac{y^2}{b^2} = 1, \quad a = \frac{p}{|1 - e^2|}, \quad b = \frac{p}{\sqrt{|1 - e^2|}}, \quad x_0 = \frac{ep}{e^2 - 1} \quad (92)$$

where the sign is defined by $1 - e^2 = \pm|1 - e^2|$.

Ellipse, $e < 1$: An ellipse is defined as the set of points such that the the sum of the distances of each point to two fixed points, called the *foci*, is a constant. With the representation $r(\varphi)$

given above for $e < 1$, one focus is at the origin. The nearest point (perihelion) on the curve to this focus is the point $\varphi = 0$, $r = p/(1 + e)$. The furthest point (aphelion) is $\varphi = \pi$, $r = p/(1 - e)$ (in Cartesian coordinates the aphelion is $(x, y) = (-p/(1 - e), 0)$ and the perihelion is at $(p/(1 + e), 0)$). The major axis of the ellipse is the line from perihelion to aphelion. Its length $2a$ is given by

$$2a = \frac{p}{1 + e} + \frac{p}{1 - e} = \frac{2p}{1 - e^2} \quad (93)$$

The semi-major axis length is a . By symmetry the second focus is a distance $p/(1 + e)$ to the right of the aphelion: the Cartesian point $(-2ep/(1 - e^2), 0)$. The sum of the distances of the perihelion, and of the aphelion from the two foci is $2a$. Consider a general point on the curve $r(\varphi)$. Its distance from the focus at the origin is clearly $r(\varphi)$. Its Cartesian location is $(r(\varphi) \cos \varphi, r(\varphi) \sin \varphi)$. The square of its distance from the other focus is then

$$\begin{aligned} d^2 &= [r(\varphi) \cos \varphi + 2ep/(1 - e^2)]^2 + r^2(\varphi) \sin^2 \varphi \\ &= r^2(\varphi) + 4 \frac{epr(\varphi) \cos \varphi}{1 - e^2} + 4 \frac{e^2 p^2}{(1 - e^2)^2} \\ &= r^2(\varphi) + 4 \frac{p(p - r)}{1 - e^2} + 4 \frac{e^2 p^2}{(1 - e^2)^2} \\ &= r^2 - \frac{4pr}{1 - e^2} + \frac{4p^2}{(1 - e^2)^2} \\ &= \left(\frac{2p}{1 - e^2} - r \right)^2 = (2a - r)^2 \end{aligned} \quad (94)$$

Thus $d = 2a - r$ so that $r + d = 2a$ in accordance with the geometrical definition of an ellipse. Finally the minor axis of the ellipse is the diameter perpendicular to the major axis at its midpoint, which has Cartesian coordinates $(-ep/(1 - e^2), 0) = (-ea, 0)$. its length is $2b$, where $b = \sqrt{a^2 - e^2 a^2} = a\sqrt{1 - e^2} = p/\sqrt{1 - e^2}$

Hyperbola, $e > 1$: A hyperbola is the set of points such that the difference of the distances of each point from two foci is constant. This defines two disjoint curves in the xy -plane. Again, using the polar coordinate description, $r = p/(1 + e \cos \varphi)$, the origin is at one focus, the center is at $(ep/(e^2 - 1), 0)$, and the second focus is at the point $(2ep/(e^2 - 1), 0)$. The distance of a point on the curve from the first focus is $r(\varphi)$. The squared distance to the other focus is

$$\begin{aligned} d^2 &= [r(\varphi) \cos \varphi - 2ep/(e^2 - 1)]^2 + r^2(\varphi) \sin^2 \varphi \\ &= \left(\frac{2p}{1 - e^2} - r \right)^2 = \left(\frac{2p}{e^2 - 1} + r \right)^2 \end{aligned} \quad (95)$$

by identical algebra to the ellipse case. Thus $d = r + 2p/(e^2 - 1)$ and $d - r = 2p/(e^2 - 1)$, confirming that the curve $r(\varphi)$ for $e > 1$ traces out one of the two branches of a hyperbola.

The other branch is traced out by the formula

$$r(\varphi) = \frac{p}{-1 + e \cos \varphi}, \quad \cos \varphi \geq \frac{1}{e} \quad (96)$$

It solves the equation of motion for the potential $V(r) = +k/r$ (we always assume $k > 0$), for which the energy is always positive $E > 0$, and the quantity under the square root is

$$\begin{aligned} E - ku - J^2 u^2 / 2m &= E + \frac{mk^2}{2J^2} - \frac{J^2}{2m} (u + mk/J^2)^2 \\ &= \frac{J^2}{2m} [(1/r_{\min} + mk/J^2)^2 - (u + mk/J^2)^2] \end{aligned} \quad (97)$$

so the change of variables that enables the integration is $u = (-1 + e \cos \varphi)/p$ with $e = \sqrt{1 + 2EJ^2/mk^2}$ and $p = J^2/mk$ given by the same formulas as before.

To understand the hyperbolic trajectory far from the center of force, examine the Cartesian equation for large $x - x_0, y$. We find that $y \sim \pm(b/a)(x - x_0)$ which describes two straight lines of slopes $\pm b/a$ intersecting at the center of the hyperbola. The two branches of the hyperbola approach these straight lines asymptotically. By considering similar triangles one can see that the distance of closest approach of these straight lines to one of the foci is precisely b . If we interpret the trajectory as that followed by a particle with momentum $\sqrt{2mE}$ launched from a great distance toward the center of force along one of these asymptotic lines, we call this distance of closest approach the *impact parameter*. The conserved angular momentum of this initial state is clearly $b\sqrt{2mE}$. For the $1/r$ potential of either sign, this quantity reduces to

$$b\sqrt{2mE} = \frac{p}{\sqrt{e^2 - 1}} \sqrt{2mE} = \frac{J^2/mk}{\sqrt{2EJ^2/mk^2}} \sqrt{2mE} = J \quad (98)$$

confirming this interpretation. Specifying E, b is a way of characterizing the initial state in a general potential. Whatever the potential, the particle of definite energy and impact parameter will follow a definite trajectory eventually arriving at a great distance traveling in a new direction, making an angle $\theta(b, E)$ with the initial momentum direction. Consulting the geometry of the hyperbola, we can see that

$$\tan \frac{\pi - \theta}{2} = \frac{b}{a} = b \frac{e^2 - 1}{p} = b \frac{2E}{k}, \quad b(\theta, E) = \frac{k}{2E} \cot \frac{\theta}{2} \quad (99)$$

To link the properties of these trajectories to the physics of the motion, we recall the relations of p, e to J, E . We immediately see that $e < 1$ for $E < 0$, $e > 1$ for $E > 0$, and $e = 1$ for $E = 0$. The semi major axis length for an elliptical orbit is

$$a = \frac{p}{1 - e^2} = \frac{J^2/mk}{-2EJ^2/mk^2} = \frac{k}{-2E} \quad (100)$$

which is seen to depend only on E not on J . The semi minor axis length is $b = a\sqrt{-2EJ^2/mk^2}$.

Time Dependence: Finally we turn to the time dependence of the trajectories in Keplerian motion. We have by direct integration

$$t = \int_{r_{\min}}^{r(t)} \frac{m dr'}{\sqrt{2m}\sqrt{E + k/r' - J^2/2mr'^2}} = \int_{r_{\min}}^{r(t)} \frac{mr' dr'}{\sqrt{2mEr'^2 + 2mkr' - J^2}} \quad (101)$$

This is an elementary integral that we can identify by completing the square

$$\begin{aligned} 2mEr'^2 + 2mkr' - J^2 &= 2mE(r' + k/2E)^2 - mk^2/2E + (1 - e^2)mk^2/2E \\ &= 2mE(r' - a)^2 - 2mEe^2a^2 \end{aligned} \quad (102)$$

If $e < 1$ ($E < 0$) we make the substitution $r' = a + ea \sin \theta$ in the integral:

$$\begin{aligned} t &= m \int d\theta \frac{a + ea \sin \theta}{\sqrt{-2mE}} = m \frac{(\theta + \pi/2)a - ea \cos \theta}{\sqrt{-2mE}} = \frac{a^{3/2}}{\sqrt{k/m}} (\theta + \pi/2 - e \cos \theta) \\ r(t) &= a + ea \sin \theta \end{aligned} \quad (103)$$

The minimum r value occurs when $\theta = -\pi/2$ and the maximum when $\theta = +\pi/2$, and we have defined the integration constants so that r is at its minimum at $t = 0$. In particular, the period of the orbit is twice the value of t at $\theta = \pi/2$:

$$T = \frac{-2\pi km}{2E\sqrt{-2mE}} = \frac{2\pi a^{3/2}}{\sqrt{k/m}} \quad (104)$$

When $E > 0$ corresponding to a hyperbolic trajectory, the distance between the two branches of the hyperbola at their closest approach is $2d = -2a = k/E$, and the quantity under the square root in the integrand is now

$$2mEr'^2 + 2mkr' - J^2 = 2mE(r' + d)^2 - 2mEe^2d^2 \quad (105)$$

and the integral is done with the hypertrig substitution $r' = -d + ed \cosh \lambda$:

$$\begin{aligned} t &= m \int d\lambda \frac{-d + ed \cosh \lambda}{\sqrt{2mE}} = m \frac{-\lambda d + ea \sinh \lambda}{\sqrt{2mE}} = \frac{d^{3/2}}{\sqrt{k/m}} (e \sinh \lambda - \lambda) \\ r(t) &= -d + ed \cosh \lambda \end{aligned} \quad (106)$$

Clearly $t = 0$ corresponds to $\lambda = 0$ for which $r = r_{\min}$. Late time is achieved at large λ from which we can infer that $r(t) \sim t\sqrt{k/md} = t\sqrt{2E/m} = vt$ as $t \rightarrow \infty$.

Runge-Lenz Vector: The $1/r$ potential is very special in that bound orbits close. You may recall that in quantum mechanics the energy levels in an attractive $1/r$ potential are degenerate are degenerate, with states of differing angular momentum having the same

energy. Such unexpected degeneracies frequently point to a new symmetry and associated conservation law. In fact there is a new quantity, the Runge-Lenz vector:

$$\mathbf{A} = \mathbf{p} \times \mathbf{J} - mk \frac{\mathbf{r}}{r}. \quad (107)$$

Let's check that it is conserved:

$$\begin{aligned} \dot{\mathbf{A}} &= (\dot{\mathbf{p}}) \times \mathbf{J} - mk \frac{\dot{\mathbf{r}}}{r} + mk \dot{r} \frac{\mathbf{r}}{r^2} = -\frac{k}{r^3} (\mathbf{r} \times (\mathbf{r} \times \mathbf{p})) - mk \frac{\dot{\mathbf{r}}}{r} + mk \dot{r} \frac{\mathbf{r}}{r^2} \\ &= -\frac{k}{r^3} (m(\mathbf{r} \cdot \dot{\mathbf{r}})\mathbf{r} - mr^2 \dot{\mathbf{r}}) - mk \frac{\dot{\mathbf{r}}}{r} + mk \dot{r} \frac{\mathbf{r}}{r^2} = -\frac{mk\mathbf{r}}{r^3} \frac{d}{dt} \frac{1}{2} r^2 + mk \dot{r} \frac{\mathbf{r}}{r^2} \\ &= -\frac{mk\mathbf{r}}{r^3} r \dot{r} + mk \dot{r} \frac{\mathbf{r}}{r^2} = 0 \end{aligned} \quad (108)$$

We can relate the magnitude of \mathbf{A} to J, E :

$$\begin{aligned} \mathbf{A}^2 &= (\mathbf{p} \times \mathbf{J}) \cdot (\mathbf{p} \times \mathbf{J}) - \frac{2mk}{r} \mathbf{r} \cdot (\mathbf{p} \times \mathbf{J}) + m^2 k^2 \\ &= \mathbf{J}^2 \left(\mathbf{p}^2 - \frac{2mk}{r} \right) + m^2 k^2 = 2mEJ^2 + m^2 k^2 = m^2 k^2 \left(1 + \frac{2EJ^2}{mk^2} \right) \end{aligned} \quad (109)$$

To see what \mathbf{A} is with respect to the elliptical orbit, calculate

$$\begin{aligned} \mathbf{r} \cdot \mathbf{A} &= rA \cos \varphi = \mathbf{r} \cdot (\mathbf{p} \times \mathbf{J}) - mkr = (\mathbf{r} \times (\mathbf{p}) \cdot \mathbf{J}) - mkr = J^2 - mkr \\ r &= \frac{J^2}{mk + A \cos \varphi} = \frac{p}{1 + (A/mk) \cos \varphi} \end{aligned} \quad (110)$$

which is the familiar polar equation for the elliptical orbit with $e = A/mk$!

4.4 From Center of Mass to a general inertial frame

We started our discussion with the motion of two particles under a mutual interaction governed by a potential $V(\mathbf{r}_1 - \mathbf{r}_2)$. We then noticed that the change of variables to $\mathbf{R} = (m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2)/M$, $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ reduced the nontrivial dynamics to that of a single particle of the reduced mass $m = m_1 m_2 / M$, and position \mathbf{r} . Then we chose the center of mass system of coordinates with $\mathbf{R} = 0$ and solved for \mathbf{r} . If we ask what is the motion of the individual particles in the center of mass we note that $\mathbf{r}_2 = -m_1 \mathbf{r}_1 / m_2$, so $\mathbf{r} = (1 + m_1 / m_2) \mathbf{r}_1$ or

$$\mathbf{r}_1 = \frac{m_2}{M} \mathbf{r}, \quad \mathbf{r}_2 = -\frac{m_1}{M} \mathbf{r} \quad (111)$$

Thus for $E < 0$ each particle executes an elliptic orbit about the center of mass as focus, with the size of each orbit scaled by a factor m_2/M for particle 1 and m_1/M for particle 2. If the system is moving as a whole at uniform velocity \mathbf{V} , we simply add $\mathbf{V}t$ to each right side

$$\mathbf{r}_1 = \frac{m_2}{M} \mathbf{r} + \mathbf{V}t, \quad \mathbf{r}_2 = -\frac{m_1}{M} \mathbf{r} + \mathbf{V}t \quad (112)$$

From these we can get the momenta:

$$\begin{aligned}\mathbf{p}_1 &= m\dot{\mathbf{r}} + m_1\mathbf{V} = m\dot{\mathbf{r}} + \frac{m_1}{M}(\mathbf{p}_1 + \mathbf{p}_2) \\ \mathbf{p}_2 &= -m\dot{\mathbf{r}} + m_2\mathbf{V} = -m\dot{\mathbf{r}} + \frac{m_2}{M}(\mathbf{p}_1 + \mathbf{p}_2)\end{aligned}\quad (113)$$

5 Scattering Processes

5.1 Scattering from a fixed central potential

Classical trajectories that are unbounded are interpreted physically in terms of scattering processes. One aims a particle toward a target a great distance away and then observes the resulting situation at a much later time when the incident particle is again at great distances from the target particle, which generally recoils and ends up with altered momentum.

In the center of mass system the momentum of the incident particle is at any time $\mathbf{p}_1(t) = m\dot{\mathbf{r}}(t)$ and that of the target particle is $\mathbf{p}_2(t) = -m\dot{\mathbf{r}}(t)$ where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and $m = m_1m_2/(m_1 + m_2)$ is the reduced mass. The initial momenta are $\mathbf{p}_1 = m\dot{\mathbf{r}}(-\infty) = -\mathbf{p}_2$ and the final momenta are $\mathbf{p}'_1 = m\dot{\mathbf{r}}(+\infty) = -\mathbf{p}'_2$.

In a scattering experiment one does not scatter one particle at a time but rather prepares a beam of many similar particles all having as nearly as possible the same initial momentum, and spread out uniformly over its cross section. Let us first consider the target to be a fixed center of a central potential $V(r)$. Then the classical trajectory is determined by

$$\varphi = J \int_0^{1/r} \frac{du}{\sqrt{2m(E - V(1/u)) - J^2u^2}} \quad (114)$$

It will be symmetric about its point of closest approach r_{\min} , which is a zero of the argument of the square root. Let us define Φ as φ at $r = r_{\min}$

$$\Phi = J \int_0^{1/r_{\min}} \frac{du}{\sqrt{2m(E - V(1/u)) - J^2u^2}} \quad (115)$$

Then the angle between the final and initial momentum of the incident particle is just $\theta = \pi - 2\Phi$.

Definition of Scattering Cross Section An incident beam of particles can be characterized by the flux F : number of particles per unit area per unit time crossing a given plane perpendicular to the beam. The number of particles per unit time passing through an element of area at impact parameter b is

$$Fb db d\varphi = Fb(\theta, E) \frac{db}{d\theta} d\theta d\varphi = F \frac{1}{\sin\theta} b(\theta, E) \frac{db}{d\theta} d\Omega \quad (116)$$

All these particles will end up at angle θ, φ so the counting rate at a detector at subtending solid angle $d\Omega$ will be

$$F \frac{1}{\sin\theta} b(\theta, E) \frac{db}{d\theta} d\Omega \equiv F d\Omega \frac{d\sigma}{d\Omega} \quad (117)$$

which defines the differential scattering cross section $d\sigma/d\Omega$. Multiplied by the incident flux, it gives the number of particles scattered per unit solid angle, in direction (θ, φ) .

$$\frac{d\sigma}{d\Omega} = \frac{1}{\sin\theta} b(\theta, E) \left| \frac{db}{d\theta} \right| \quad (118)$$

Recall that for the potential $V(r) = \pm k/r$ we determined for a hyperbolic trajectory that

$$b(\theta, E) = \frac{k}{2E} \cot \frac{\theta}{2}, \quad \frac{db}{d\theta} = -\frac{k}{4E \sin^2(\theta/2)} \quad (119)$$

so

$$\frac{d\sigma}{d\Omega} = \frac{k^2}{16E^2 \sin^4(\theta/2)} \quad (120)$$

which is the famous formula for Rutherford scattering. Although we obtained it in classical mechanics, the formula also applies unmodified for quantum mechanics.

5.2 The scattering cross section for elastic two body scattering

The result of the previous section applies unaltered to two body scattering in the center of mass system $\mathbf{R} = 0$, provided we use the reduced mass $m = m_1 m_2 / (m_1 + m_2)$ for m . This is because in this frame the momentum of the incident particle is $m\dot{\mathbf{r}}(t)$ at any time t . The target particle has momentum $-m\dot{\mathbf{r}}(t)$ at the same time. Furthermore the energy and angular momentum of the two body system in this frame are the same as used in the previous section:

$$\frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} + V(|\mathbf{r}_1 - \mathbf{r}_2|) = \frac{\mathbf{p}_1^2}{2m} + V(r) = \frac{1}{2} m \dot{\mathbf{r}}^2 + V(r) = E \quad (121)$$

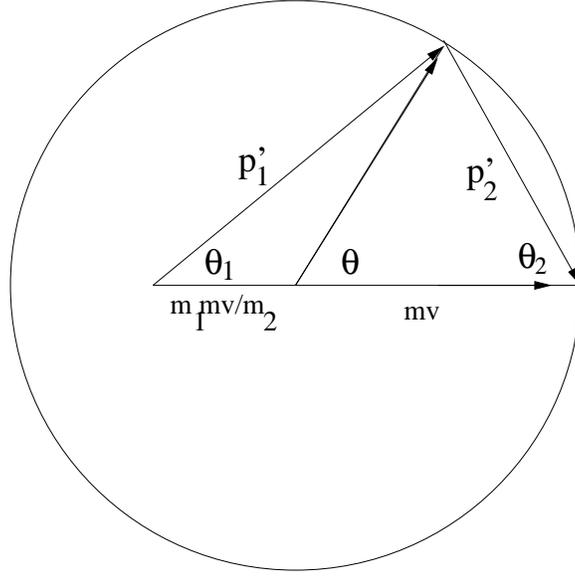
$$\mathbf{r}_1 \times \mathbf{p}_1 + \mathbf{r}_2 \times \mathbf{p}_2 = \mathbf{r} \times \mathbf{p}_1 = m\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{J} \quad (122)$$

In the center of mass the target is moving toward the incident particle so the flux of incident particles on the target is determined by the relative velocity $\mathbf{v}_1 - \mathbf{v}_2 = \dot{\mathbf{r}}$. In short every aspect of the two body scattering in the center of mass is correctly described by the effective one body problem treated in the previous section.

However, in other inertial frames some reinterpretation of scattering angle is necessary. The most important other such frame is the Lab frame, in which the target particle is initially at rest. The scattering angle in the Lab system can be worked out in terms of that in the center of mass by considering the relation of the corresponding momentum vectors.

$$\begin{aligned} \mathbf{p}'_1 &= m\dot{\mathbf{r}}(+\infty) + \frac{m_1}{M} \mathbf{p}_1 \\ \mathbf{p}'_2 &= -m\dot{\mathbf{r}}(+\infty) + \frac{m_2}{M} \mathbf{p}_1 \end{aligned} \quad (123)$$

Initially $\dot{\mathbf{r}}(-\infty) = \dot{\mathbf{r}}_1(-\infty) - \dot{\mathbf{r}}_2(-\infty) = \dot{\mathbf{r}}_1(-\infty) = \mathbf{p}_1/m_1$ since particle 2 is initially at rest. Conservation of energy in the center of mass system guarantees that the initial and final kinetic energies are equal (since $V = 0$ at $r = \infty$) and therefore that $|\dot{\mathbf{r}}(-\infty)| = |\dot{\mathbf{r}}(+\infty)| \equiv v$. Then the vectors are related as in the figure:



From the figure we see that the scattering angle θ_1 of the incident particle in the lab is related to the scattering angle θ in the center of mass by

$$\tan \theta_1 = \frac{m_2 \sin \theta}{m_1 + m_2 \cos \theta}, \quad \theta_2 = \frac{\pi - \theta}{2} \quad (124)$$

The trig identity $1 + \tan^2 = \sec^2$ leads to the alternative relation

$$\cos \theta_1 = \frac{m_1 + m_2 \cos \theta}{\sqrt{m_1^2 + m_2^2 + 2m_1 m_2 \cos \theta}} \quad (125)$$

The figure makes clear some general features of the scattering. For $m_1 < m_2$, as θ sweeps through 2π θ_1 also sweeps through 2π . However if $m_1 > m_2$ the left vertex of the triangle lies outside the circle so that θ_1 reaches a maximum when \mathbf{p}'_1 becomes tangent to the circle, and subsequently decreases as θ increases. This maximum occurs when $\sin \theta_1^{\max} = mv / (m_1 mv / m_2) = m_2 / m_1$. From the figure we see that in the equal mass case the maximum actually corresponds to $p'_1 = 0$: that is the incident particle comes to rest and the target particle exits with the momentum of the incident particle. For equal mass elastic scattering, $\theta_1^{\max} = \pi/2$, occurring when $\theta = \pi$, and further the angle between the directions of the final particles is always $\pi/2$.

The formula for the scattering cross section in the Lab system is obtained by writing

$$F b db d\varphi = F \frac{1}{\sin \theta_1} b(\theta, E) \frac{db}{d\theta_1} d\Omega_1 \equiv F \frac{d\sigma_{\text{Lab}}}{d\Omega_1} d\Omega_1 \quad (126)$$

so that

$$\frac{d\sigma_{\text{Lab}}}{d\Omega_1} = \frac{1}{\sin \theta_1} b(\theta(\theta_1), E) \frac{db}{d\theta} \frac{d\theta}{d\theta_1} = \frac{d\sigma_{\text{CM}}}{d\Omega} \frac{\sin \theta}{\sin \theta_1} \frac{d\theta}{d\theta_1} = \frac{d\sigma_{\text{CM}}}{d\Omega} \frac{d \cos \theta}{d \cos \theta_1} \quad (127)$$

If we like we can evaluate the relative factor explicitly

$$\begin{aligned} \frac{d \cos \theta_1}{d \cos \theta} &= \frac{m_2}{\sqrt{m_1^2 + m_2^2 + 2m_1m_2 \cos \theta}} - m_1m_2 \frac{m_1 + m_2 \cos \theta}{(m_1^2 + m_2^2 + 2m_1m_2 \cos \theta)^{3/2}} \\ &= \frac{m_2^3 + m_1m_2^2 \cos \theta}{(m_1^2 + m_2^2 + 2m_1m_2 \cos \theta)^{3/2}} \end{aligned} \quad (128)$$

5.3 Total Cross Section

The total cross section is defined as the integral of the differential cross section over all angles

$$\sigma \equiv \int d\Omega \frac{d\sigma}{d\Omega} = \int_0^{2\pi} d\varphi \int_0^{b_{\max}} b db = \pi b_{\max}^2. \quad (129)$$

If $V(r) \neq 0$ for all r , *no matter how rapidly it falls to zero as $r \rightarrow \infty$* , $b_{\max} = \infty$. The total cross section is finite only when the potential is strictly zero outside of some radius.

5.4 Inelastic scattering

In classical particle scattering there is no explicit mechanism for loss of energy in a scattering process, though we know that such physical mechanisms abound: friction, radiation, and chemical changes of state, to name a few. One can take these possibilities into account by allowing some kinetic energy to be transformed into heat or some other kind of energy. In the center of mass, the upshot is that the relative momenta initially and finally have different magnitudes $|\mathbf{p}'| \neq |\mathbf{p}|$. This translates to a modified formula for the lab scattering angle:

$$\tan \theta_1 = \frac{m_2 p' \sin \theta}{m_1 p + m_2 p' \cos \theta} = \frac{\sin \theta}{m_1 p / (m_2 p') + \cos \theta} \quad (130)$$

which reduces to the elastic scattering formula when $p' = p$.

In relativistic scattering processes, on the other hand, it is the total energy $\sum \gamma_i m_i c^2$ that is conserved, not the kinetic energy $K = \sum (\gamma_i - 1) m_i c^2$. Only in processes where the total rest mass $\sum m_i$ is conserved will K be conserved. In practice this only happens if the particles in the final state are the same as those in the initial state. As a simple example of a relativistic inelastic process, consider e^+e^- pair production by the scattering of two photons in the center of mass system. the photon momenta are back to back with total energy $2p_\gamma c$. conservation of energy gives $p_\gamma c = \sqrt{p_e^2 c^2 + m_e^2 c^4}$.

6 Small Oscillations

For generic systems with several degrees of freedom the equations of motion are intractable and approximations must be made to make progress. One important approximation method is to linearize the equations of motion. In general linear equations cannot be approximately

valid for a long time, but an exception occurs when there is a stable equilibrium state of the system. Such a state exists for configurations for which the potential energy has a minimum. Then a departure from equilibrium will induce a restoring force tending to return the system to equilibrium. For small departures from equilibrium the equations of motion can be linearized (equivalently the Lagrangian is expanded to quadratic order in the small displacements and velocities), and the system oscillates harmonically about its equilibrium. Corrections to the approximation remain small for all time and may be calculated in perturbation theory.

6.1 Oscillations in one dimension

We start with a one dimensional system

$$L = \frac{1}{2}m\dot{x}^2 - V(x) \quad (131)$$

A possible equilibrium point x_0 satisfies $V'(x_0) = 0$. The equilibrium is stable if $V''(x_0) > 0$. Then expanding V about x_0 to second order gives

$$V(x) = V(x_0) + \frac{1}{2}(x - x_0)^2 V''(x_0) + \dots \equiv V(x_0) + \frac{k}{2}(x - x_0)^2 + \dots \quad (132)$$

where we have defined the spring constant by $k = V''(x_0) > 0$. Writing $x = x_0 + q(t)$ we have

$$L = -V(x_0) + \frac{m}{2}\dot{q}^2 - \frac{k}{2}q^2 + O(q^3) \quad (133)$$

Small oscillations are controlled by the approximate Lagrangian

$$L_0 = \frac{m}{2}\dot{q}^2 - \frac{k}{2}q^2 = \frac{m}{2}(\dot{q}^2 - \omega_0^2 q^2). \quad (134)$$

The equations of motion are simply $m\ddot{q} = -kq = -m\omega_0^2 q$ and the general solution can be written

$$q(t) = |A| \cos(\omega_0 t - \delta) = \text{Re } |A| e^{i\delta} e^{-i\omega_0 t} \quad (135)$$

The complex number $A = |A|e^{i\delta}$ contains all the initial data information. Representing the oscillations by the complex exponential $e^{-i\omega_0 t}$ will turn out to be extremely convenient when we include damping and periodic forcing terms. Indeed one can introduce the complex conjugate combinations

$$a_{\pm} = \dot{q} \pm i\omega_0 q, \quad \dot{q} = \text{Re} a_{\pm}, \quad q = \pm \frac{1}{\omega_0} \text{Im} a_{\pm} \quad (136)$$

Then using the equations of motion we find

$$\dot{a}_{\pm} = \ddot{q} \pm i\omega_0 \dot{q} = -\omega_0^2 q \pm i\omega_0 \dot{q} = \pm i\omega_0 (\dot{q} \pm i\omega_0 q) = \pm i\omega_0 a_{\pm} \quad (137)$$

So the a_{\pm} satisfy first order differential equations in time, with solutions $e^{\pm i\omega_0 t}$. Notice also that the energy or Hamiltonian is

$$E = H = \frac{m}{2}(\dot{q}^2 + \omega_0^2 q^2) = \frac{m}{2}|a_+|^2 \quad (138)$$

Forced oscillations without damping A driving force enters the Lagrangian as a term $qF(t)$ or in the equation of motion as

$$\ddot{q} + \omega_0^2 q = \frac{1}{m} F(t) \quad (139)$$

The effect of the forcing term on the a_{\pm} is simple to work out

$$\begin{aligned} \dot{a}_{\pm} &= \dot{q} \pm i\omega_0 \dot{q} = \pm i\omega_0 a_{\pm} + \frac{F(t)}{m} \\ \frac{d}{dt}(a_{\pm} e^{\mp i\omega_0 t}) &= \frac{F(t)}{m} e^{\mp i\omega_0 t} \end{aligned} \quad (140)$$

which can be directly integrated

$$\begin{aligned} a_{\pm}(t) e^{\mp i\omega_0 t} - a_{\pm}(0) &= \int_0^t dt' \frac{F(t')}{m} e^{\mp i\omega_0 t'} \\ a_{\pm}(t) &= a_{\pm}(0) e^{\pm i\omega_0 t} + \int_0^t dt' \frac{F(t')}{m} e^{\pm i\omega_0(t-t')} \end{aligned} \quad (141)$$

Of course energy is not conserved in the presence of $F(t)$. In fact we have

$$H(t) = \frac{m}{2}|a_+|^2 = \frac{m}{2} \left| a_{\pm}(0) + \int_0^t dt' \frac{F(t')}{m} e^{-i\omega_0 t'} \right|^2 \quad (142)$$

Suppose $F(t) = 0$ for early and late times, and suppose also that the oscillator is unexcited at early times. Then the total energy delivered to the system after F has turned off is simply

$$E = \frac{1}{2m} \left| \int_{-\infty}^{\infty} dt' F(t') e^{-i\omega_0 t'} \right|^2 \quad (143)$$

expressed in terms of the Fourier transform of the force. If the force is active over a time much shorter than $1/\omega_0$, this expression reduces to the square of the impulse $\int dt F(t)$ divided by $2m$. Of course by Newton's law the impulse is equal to the change in momentum of a particle.

Going to complex notation we can represent a periodic force as $F_0 e^{-i\omega t}$. Making the complex ansatz $q = q_0 e^{-i\omega t}$ we find a particular solution

$$q(t) = \frac{F_0/m}{\omega_0^2 - \omega^2} e^{-i\omega t} \quad (144)$$

and the general solution

$$q(t) = Ae^{-i\omega_0 t} + \frac{F_0/m}{\omega_0^2 - \omega^2} e^{-i\omega t} \quad (145)$$

We see that $q(t)$ generally possesses two frequencies, the driving frequency and the natural frequency, the latter being absent if $A = 0$. To get the actual motion from the complex solution, we simply take the real part, putting $A = |A|e^{i\delta_0}$, $F_0 = |F_0|e^{i\delta}$ we have

$$q(t) = \text{Re } q(t) = |A| \cos(\omega_0 t - \delta_0) + \frac{|F_0|/m}{\omega_0^2 - \omega^2} \cos(\omega t - \delta) \quad (146)$$

Clearly we have to rethink what happens when we try to drive the system at its natural frequency, when the second term blows up.

Resonance When $\omega = \omega_0$ we get a particular solution in the form $q(t) = te^{-i\omega_0 t}$:

$$q(t) = t \frac{F_0}{-2im\omega_0} e^{-i\omega_0 t} = t \frac{F_0}{2m\omega_0} e^{-i\omega_0 t + \pi i/2} \quad (147)$$

and the general solution can have $Ae^{-i\omega_0 t}$ added on the right side. We can describe this solution as a oscillation at the natural frequency with an amplitude increasing linearly with time. Notice the phase shift of the response compared to the force: with F_0 real the force behaves as $\cos \omega_0 t$ whereas the response behaves as $\sin \omega_0 t$.

Damping: The indefinitely growing amplitude we have just seen at resonance is only valid if there is no friction or dissipative mechanism in which energy is lost to the environment: the driving force is delivering energy to the system and it has nowhere to go except into the amplitude of oscillation. Friction and dissipation take us outside the framework of pure classical mechanics unless we greatly complicate the system to include the environment. However, by adding drag terms to the equation of motion we can take friction into account in a phenomenological way. In the context of small oscillations the simplest drag force is linear in the velocity: $F_{\text{Drag}} = -m\gamma\dot{q}$.

$$\ddot{q} + \gamma\dot{q} + \omega_0^2 q = \frac{F(t)}{m} \quad (148)$$

Note that we can't add a term to a Lagrangian with no explicit time dependence beyond that in $F(t)^2$ that will generate this term in Lagrange's equation

²However the time dependent Lagrangian

$$L_\gamma = e^{\gamma t} \left(\frac{m}{2} \dot{q}^2 - \frac{k}{2} q^2 + qF(t) \right) \quad (149)$$

does imply this equation of motion. Its time dependence signifies that energy will *not* be conserved when $F(t) \neq 0$: $dH/dt = -\partial L/\partial t = -\gamma L - q\dot{F}e^{\gamma t} \rightarrow -\gamma L$ for $F = 0$.

To explore the consequences of this term, we first seek a (complex) solution of the homogeneous equation ($F = 0$), with time dependence $e^{\alpha t}$. Then α satisfies the quadratic equation

$$\alpha^2 + \alpha\gamma + \omega_0^2 = 0, \quad \alpha = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2} = -\frac{\gamma}{2} \pm i\sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \quad (150)$$

We can identify two damping regimes, under and over damping:

$$q(t) = Ae^{-\gamma/2}e^{\pm i\omega'_0 t}, \quad \gamma < 2\omega_0, \quad \omega'_0 = \sqrt{\omega_0^2 - \gamma^2/4} < \omega_0 \quad (151)$$

$$q(t) = A_+e^{-\gamma_+/2} + A_-e^{-\gamma_-/2}, \quad \gamma > 2\omega_0, \quad \gamma_{\pm} = \gamma \pm \sqrt{\gamma^2 - 4\omega_0^2} \quad (152)$$

In the first case we identify an oscillatory factor with reduced frequency with exponentially damped amplitude. In the second case there is no oscillatory behavior, just two possible exponentially damped terms, the slowest damping corresponding to γ_- . In the critical case $\gamma = 2\omega_0$, $\gamma_- = \gamma_+ = \gamma$ and the two damping behaviors are $e^{-\gamma t/2}$ and $te^{-\gamma t/2}$.

Forced oscillation with damping: When damping is present all solutions of the homogeneous equation of motion (without a forcing term) are exponentially damped. This means that when we construct the general solution of the forced oscillator with damping those terms, necessary to set up the initial conditions die away after a long enough time $t \gg 1/\gamma$. That is why they are called transients: no matter what initial conditions are applied, the system eventually settles down to the unique terms oscillating with the driving frequency ω . For periodic forcing, $F = F_0e^{-i\omega t}$, we find that (after a long time) the solution only displays the driving frequency with an amplitude proportional to F_0 which we assume is real and positive, so $\text{Re } F(t) = F_0 \cos \omega t$.

$$q(t) = \frac{F_0/m}{\omega_0^2 - \omega^2 - i\omega\gamma} e^{-i\omega t} = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}} e^{i\delta} e^{-i\omega t},$$

$$\tan \delta = \frac{\omega\gamma}{\omega_0^2 - \omega^2} \quad (153)$$

Damping has produced a small negative imaginary part in the denominator, so that at resonance the amplitude oscillating at frequency ω stays finite as $\omega \rightarrow \omega_0$. Of course if damping is small the amplitude becomes quite large. Going back to real q , we have

$$\text{Re } q(t) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}} \cos(\omega t - \delta) \quad (154)$$

The phase δ measures the lag of the response relative to the driving force. Notice that at resonance $\delta \rightarrow \pi/2$ and $q(t) \rightarrow F_0 \sin \omega t / m\omega_0\gamma$. That is, at resonance the steady state response is out of phase with the driving force by 90° .

6.2 Systems with several degrees of freedom

We turn now to small oscillations in a general dynamical system with any number of degrees of freedom described by generalized coordinates $q_i(t)$, and a Lagrangian $L(q_i, \dot{q}_i)$. We assume that there is a state of stable equilibrium, and choose the q_i so that this state is described by $q_i = 0$ for all i . Then the small oscillation approximation starts by expanding L up to quadratic order in q_i, \dot{q}_i :

$$L = L_0 + \frac{1}{2} \sum_{ij} M_{ij} \dot{q}_i \dot{q}_j + \sum_{ij} A_{ij} \dot{q}_i q_j - \frac{1}{2} \sum_{ij} K_{ij} q_i q_j + O(q^3) \quad (155)$$

There are no linear terms in this expansion because $q_i = 0$ is assumed to be a static equilibrium solution:

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0, \quad \text{for } q_i = 0. \quad (156)$$

Of course a term linear in \dot{q}_i would be a total derivative and hence can be dropped from the Lagrangian without altering the Lagrange equations of motion.

We see that the dynamics is parameterized by the three matrices M, A, K each of size $n \times n$ where n is the number of degrees of freedom. Because we can write

$$\dot{q}_i q_j = \frac{1}{2} (\dot{q}_i q_j - \dot{q}_j q_i) + \frac{1}{2} \frac{d}{dt} (q_i q_j) \quad (157)$$

and the second term, which is a total time derivative, can be dropped from the Lagrangian, we can assume that A is an antisymmetric matrix: $A_{ij} = -A_{ji}$. Furthermore, since M and K both multiply quantities symmetric in ij , we may assume they are both symmetric: $M_{ij} = M_{ji}$ and $K_{ij} = K_{ji}$.

The next step is to derive the equations of motion

$$\begin{aligned} \frac{d}{dt} (M_{kj} \dot{q}_j + A_{kj} q_j) &= \dot{q}_j A_{jk} - K_{kj} q_j \\ M_{kj} \ddot{q}_j + 2A_{kj} \dot{q}_j + K_{kj} q_j &= 0 \end{aligned} \quad (158)$$

where we have used the symmetry properties of the matrices, and we have adopted the summation convention that repeated indices are always summed over $j = 1 \cdots n$. Finally we look for the normal modes by putting $q_j(t) = a_j e^{-i\omega t}$ and plugging into the equations of motion:

$$(-\omega^2 M_{ij} - 2i\omega A_{ij} + K_{ij}) a_j \equiv H_{ij} a_j = 0 \quad (159)$$

Regarding the a_j as the components of an n vector a and H_{ij} as the components of an $n \times n$ matrix H , this equation can be written $Ha = 0$. If $\det H \neq 0$ this equation would imply that $a = 0$ i.e. the equations of motion would have no solution. In other words, the normal mode frequencies ω must satisfy the $2n$ -order polynomial equation

$$\det H = \det(-\omega^2 M - 2i\omega A + K) = 0, \quad \text{for normal modes} \quad (160)$$

So far we have imposed no physical requirements on the real matrices M, A, K . All we have noticed is that M, K are symmetric and A is antisymmetric: in particular M, iA, K are all hermitian matrices. By now you have certainly encountered hermitian matrices in quantum mechanics, and know that their eigenvalues are real and the eigenstates form a basis.

One physical consequence of the A terms is that they are odd under time reversal, which is a very good symmetry in nature, broken very weakly. Certainly for molecular and atomic systems assuming time reversal invariance in the absence of external fields is essentially exact. Magnetic fields are odd under time reversal, so the presence of external magnetic fields will break the symmetry and allow the A terms. Indeed these A terms are exactly of the form coming from a magnetic term $q\dot{\mathbf{r}} \cdot \mathbf{A}$ in the Lagrangian. But we can control the magnetic field and in particular turn it off, in which case we would have $A = 0$.

Thus the M and K terms must satisfy the assumption we made at the beginning that we were expanding about a point of stable equilibrium. Consider first the kinetic terms. We can always rotate coordinates to bring the matrix M into diagonal form

$$M = \begin{pmatrix} m_1 & 0 & 0 & \cdots & 0 \\ 0 & m_2 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & m_n \end{pmatrix} \quad (161)$$

Stability requires that all these mass eigenvalues $m_k > 0$. They must all be nonzero and positive. A matrix M that satisfies this condition is said to be positive definite. If one of the mass eigenvalues were zero, it would mean that there are fewer than n degrees of freedom since at least one equation of motion would be a constraint. In our discussion we assume that all constraints have already been implemented.

A similar requirement is made for spring constant matrix K , however we only require that the eigenvalues of K be non-negative. A vanishing eigenvalue of K would signify that there is a direction in which $V'' = 0$. Either there is no restoring force at all or it is higher than linear as $x \rightarrow 0$. In summary we can say that M must be positive definite and K merely positive.

Since M is positive definite its square root \sqrt{M} and inverse are well defined. This allows us to redefine generalized coordinates

$$q = \frac{1}{\sqrt{M}} q' \quad (162)$$

so that

$$\begin{aligned} L &= \frac{1}{2} \sum_k \dot{q}'_k{}^2 + \sum_{kl} A'_{kl} \dot{q}'_k q'_l - \frac{1}{2} \sum_{kl} \Omega_{ij}^2 q'_k q'_l \\ A' &= \frac{1}{\sqrt{M}} A \frac{1}{\sqrt{M}}, \quad \Omega^2 = \frac{1}{\sqrt{M}} K \frac{1}{\sqrt{M}} \end{aligned} \quad (163)$$

Another way of saying this is that we lose no generality in assuming that the kinetic term of the Lagrangian is diagonal, that each degree of freedom has unit mass. Then the spring constant matrix is simply the frequency squared matrix.

For the rest of this section we assume that $A = 0$ (no magnetic fields), and that the mass matrix is the identity matrix:

$$L = \frac{1}{2} \sum_k \dot{q}_k^2 - \frac{1}{2} \sum_{kl} \Omega_{kl}^2 q_k q_l \quad (164)$$

Since Ω^2 is a real symmetric matrix, we can always rotate the coordinates in such a way that it is diagonal, with diagonal entries ω_k^2 . This problem of finding the normal modes of oscillation is mathematically the eigenvalue problem

$$\Omega^2 V_l = \omega_l^2 V_l \quad (165)$$

familiar from quantum mechanics. The eigenvalues must satisfy the characteristic equation

$$\det(\Omega^2 - \omega_l^2 I) = 0 \quad (166)$$

The left side is a polynomial of order n in the variable ω_l^2 . Generically there will be precisely n roots, though some of them may coincide (degeneracy). Our stability assumption guarantees that all the roots are positive.

For each root ω_l^2 of the characteristic equation, we plug it back into the eigenvalue equation to determine the eigenvector V_l . The simplest situation is nondegeneracy: there are n distinct eigenvalues ω_l^2 and to each one a unique eigenvector V_l . We recall the proof that eigenvectors belonging to distinct eigenvalues are orthogonal:

$$\begin{aligned} V_{l_2}^i \Omega_{ij}^2 V_{l_1}^j &= \omega_{l_1}^2 V_{l_2}^i V_{l_1}^i, & V_{l_1}^i \Omega_{ij}^2 V_{l_2}^j &= \omega_{l_2}^2 V_{l_1}^i V_{l_2}^i \\ 0 = V_{l_2}^i \Omega_{ij}^2 V_{l_1}^j - V_{l_1}^i \Omega_{ij}^2 V_{l_2}^j &= (\omega_{l_1}^2 - \omega_{l_2}^2) V_{l_2}^i V_{l_1}^i \\ V_{l_2}^i V_{l_1}^i &= 0, & \text{if } \omega_{l_1}^2 &\neq \omega_{l_2}^2 \end{aligned} \quad (167)$$

where the second line uses the symmetry of Ω . When there is no degeneracy the n mutually orthogonal eigenvectors V_l form a basis which spans the n dimensional coordinate space. It is convenient to normalize these vectors to one so that

$$V_l^T V_{l'} \equiv \sum_i V_l^i V_{l'}^i = \delta_{ll'} \quad (168)$$

Then we can expand the coordinates $q_i(t)$ in this basis:

$$q_i(t) = \sum_l Q_l(t) V_l^i, \quad Q_l(t) = \sum_i V_l^i q_i(t) \quad (169)$$

The $Q_l(t)$ are called the normal coordinates. If we choose them as our generalized coordinates, the Lagrangian simplifies to a sum of independent one dimensional Lagrangians:

$$L = \frac{1}{2} \sum_{l=1}^n (\dot{Q}_l^2 - \omega_l^2 Q_l^2) \quad (170)$$

When some of the eigenfrequencies are degenerate, there is ambiguity in the selection of an eigenbasis. Suppose ω^2 is d -fold degenerate. This means that the eigenvalue equation $\Omega^2 V = \omega^2 V$ has d independent solutions. These solutions are not automatically mutually orthogonal, but it is always possible to choose (in many different ways) linear combinations of them that are. Making such a choice one comes back to the Lagrangian (170) with the understanding that the ω_i need not all be distinct.

A simple example As a concrete example consider the motion of three particles moving in a line with harmonic potential interactions:

$$\begin{aligned} L &= \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - \frac{k}{2}((x_1 - x_2)^2 + (x_2 - x_3)^2) \\ &= \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - \frac{k}{2}(x_1^2 + x_2^2 + 2x_2^2 - 2x_1x_2) \end{aligned} \quad (171)$$

The frequency squared matrix is

$$\Omega^2 = \frac{k}{m} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \quad (172)$$

In this system it is easy to guess the normal modes: (1) Motion of the center of mass with no oscillations $(x_1, x_2, x_3) = Q_0(1, 1, 1)/\sqrt{3}$. It is easy to check

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0, \quad \omega_0^2 = 0. \quad (173)$$

(2) Particle 2 at rest and particles 1 and 2 opposite $(x_1, x_2, x_3) = Q_1(1, 0, -1)/\sqrt{2}$:

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \omega_1^2 = \frac{k}{m}. \quad (174)$$

(3) Particles 1 and 3 move together and opposite to particle 2, $(x_1, x_2, x_3) = Q_2(1, -2, 1)/\sqrt{6}$:

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \quad \omega_2^2 = 3\frac{k}{m}. \quad (175)$$

Then

$$\begin{aligned} L &= \frac{m}{2} \left(\dot{Q}_0^2 + \dot{Q}_1^2 - \omega_1^2 Q_1^2 + \dot{Q}_2^2 - \omega_2^2 Q_2^2 \right) \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= Q_0 V_0 + Q_1 V_1 + Q_2 V_2, \quad x_{\text{CM}} = \frac{1}{3}(x_1 + x_2 + x_3) = \frac{Q_0}{\sqrt{3}} \end{aligned} \quad (176)$$

6.3 M particle long chain

A tractable example of a many body system that has interesting applications in many areas of physics is a chain of M particles \mathbf{r}_k , of equal mass m , organized in a long chain interacting with nearest neighbor harmonic potentials. This system generalizes the simple example we just discussed to an arbitrary number of particles all moving in 3 dimensions:

$$L = \frac{m}{2} \sum_{l=1}^M \dot{\mathbf{r}}_l^2 - \frac{k}{2} \sum_{l=1}^{M-1} (\mathbf{r}_{l+1} - \mathbf{r}_l)^2 \quad (177)$$

Note that the particles at the end of the chain, particles $l = 1$ and $l = M$ only interact with one particle. The chain is free to move about in space as a whole. To find the normal modes we derive the equations of motion

$$\begin{aligned} \ddot{\mathbf{r}}_l &= -\frac{k}{m}(2\mathbf{r}_l - \mathbf{r}_{l+1} - \mathbf{r}_{l-1}), & l = 2, \dots, M-1 \\ \ddot{\mathbf{r}}_1 &= -\frac{k}{m}(\mathbf{r}_1 - \mathbf{r}_2), & \ddot{\mathbf{r}}_M = -\frac{k}{m}(\mathbf{r}_M - \mathbf{r}_{M-1}) \end{aligned} \quad (178)$$

Assuming harmonic time dependence $\mathbf{r}_k = \mathbf{a}_k e^{-i\omega t}$, the normal modes are determined by

$$\begin{aligned} \omega^2 \mathbf{a}_l &= \frac{k}{m}(2\mathbf{a}_l - \mathbf{a}_{l+1} - \mathbf{a}_{l-1}), & l = 2, \dots, M-1 \\ \omega^2 \mathbf{a}_1 &= \frac{k}{m}(\mathbf{a}_1 - \mathbf{a}_2), & \omega^2 \mathbf{a}_M = \frac{k}{m}(\mathbf{a}_M - \mathbf{a}_{M-1}) \end{aligned} \quad (179)$$

The first equation can be diagonalized with the ansatz $\mathbf{a} e^{i\lambda l}$ which solves the equation provided

$$\omega^2 = (k/m)(2 - e^{i\lambda} - e^{-i\lambda}) = 4(k/m) \sin^2(\lambda/2) \quad (180)$$

Since this is even in λ it follows that another solution of the first equation with the same ω^2 is $\mathbf{b} e^{-i\lambda}$. We will need to use a linear combination of these two solutions to meet the $l = 1, M$ equations. Put $\mathbf{a}_l = \mathbf{a} e^{i\lambda l} + \mathbf{b} e^{-i\lambda l}$ and find

$$\begin{aligned} (2 - e^{i\lambda} - e^{-i\lambda})(\mathbf{a} e^{i\lambda} + \mathbf{b} e^{-i\lambda}) &= \mathbf{a} e^{i\lambda}(1 - e^{i\lambda}) + \mathbf{b} e^{-i\lambda}(1 - e^{-i\lambda}) \\ (2 - e^{i\lambda} - e^{-i\lambda})(\mathbf{a} e^{iM\lambda} + \mathbf{b} e^{-iM\lambda}) &= (\mathbf{a} e^{iM\lambda}(1 - e^{-i\lambda}) + \mathbf{b} e^{-iM\lambda}(1 - e^{i\lambda})) \end{aligned} \quad (181)$$

The first equation gives $\mathbf{b} = -\mathbf{a}(e^{i\lambda} - 1)/(e^{-i\lambda} - 1) = e^{i\lambda} \mathbf{a}$, after which the second equation gives

$$0 = (1 - e^{i\lambda}) \mathbf{a} e^{iM\lambda} + (1 - e^{-i\lambda}) \mathbf{b} e^{-iM\lambda} = (1 - e^{i\lambda}) (e^{iM\lambda} - e^{-iM\lambda}) \quad (182)$$

Or $e^{2iM\lambda} = 1$. This last condition is met if $\lambda = n\pi/M$, which gives $\omega^2 = 4(k/m) \sin^2(n\pi/2M)$. Choosing $n = 0, \dots, M-1$ gives M distinct values of ω^2 which shows that these are all the normal modes. The normal mode eigenvectors are

$$\mathbf{V}^l = \mathbf{a}(e^{i\lambda l} + e^{-i(l-1)\lambda}) = 2\mathbf{a} e^{i\lambda/2} \cos \lambda \left(l - \frac{1}{2} \right) \quad (183)$$

The constant is arbitrary, so we choose the normalized vectors

$$\begin{aligned} V_n^l &= \sqrt{\frac{2}{M}} \cos \frac{n\pi}{M} \left(l - \frac{1}{2} \right), & n = 1, \dots, M-1 \\ V_0^l &= \frac{1}{\sqrt{M}} \\ \sum_{l=1}^M V_n^l V_{n'}^l &= \delta_{nn'}, & \sum_{n=0}^{M-1} V_n^l V_{n'}^l = \delta_{ll'} \end{aligned} \quad (184)$$

Then we can expand the coordinates of the particles in the chain in normal modes as follows:

$$\mathbf{r}_l(t) = \sum_{n=0}^{M-1} \mathbf{Q}_n(t) V_n^l, \quad \mathbf{Q}_n(t) = \sum_{l=1}^M \mathbf{r}_l(t) V_n^l \quad (185)$$

and of course \mathbf{Q}_n oscillates with frequency $\omega_n = 2\omega_0 \sin(n\pi/2M)$. The center of mass coordinate is $\mathbf{R}_{\text{CM}} = \sum_l \mathbf{r}_l/M$:

$$\mathbf{R}_{\text{CM}} = \frac{1}{M} \sum_l \mathbf{r}_l V_0^l \sqrt{M} = \frac{\mathbf{Q}_0}{\sqrt{M}} \quad (186)$$

when M is very large, the normal modes with low n have very small frequency $\omega_n \sim \omega_0 n\pi/M$ for large M . (Here $\omega_0 \equiv \sqrt{k/m}$.) If we consider that a macroscopic sample of matter has Avogadro's number of particles in it, we realize that using this chain as a model of a string, M could be of order 10^8 , and these low frequencies would be that factor smaller than the molecular scale frequencies of the constituent atoms. This gives a qualitative explanation of why macroscopic frequencies are so much smaller than the fundamental frequencies of the underlying dynamics.

To summarize this subsection, we express the Lagrangian in terms of normal mode coordinates

$$L = \frac{m}{2} \sum_{n=0}^{M-1} (\dot{Q}_n^2 - \omega_n^2 Q_n^2), \quad \omega_n = 2 \sin \frac{n\pi}{2M} \quad (187)$$

It is also easy to give the energy and angular momentum in terms of normal mode coordinates:

$$\begin{aligned} H &= \frac{m}{2} \sum_{n=0}^{M-1} (\dot{Q}_n^2 + \omega_n^2 Q_n^2) \\ \mathbf{J} &= m \sum_l \mathbf{r}_l \times \dot{\mathbf{r}}_l = m \sum_{n=0}^{M-1} \mathbf{Q}_n \times \dot{\mathbf{Q}}_n \end{aligned} \quad (188)$$

An interesting special motion is the $n = 1$ mode with $\mathbf{Q}_1 = Q(\cos \omega_1 t, \sin \omega_1 t, 0)$. Then $\dot{\mathbf{Q}}_1 = \omega_1 Q(-\sin \omega_1 t, \cos \omega_1 t, 0)$. Then $H = m\omega_1^2 Q^2$ and $\mathbf{J} = \hat{z} m\omega_1 Q^2$ so that $E = H = \omega_1 J$. For this motion

$$\mathbf{r}_l(t) = \mathbf{Q}_1(t) \sqrt{\frac{2}{M}} \cos \frac{\pi}{M} \left(l - \frac{1}{2} \right) \quad (189)$$

At $t = 0$ the particles are lined up on the x -axis between $x = -Q\sqrt{2/M}$ and $x = +Q\sqrt{2/M}$ and this line rotates about the z -axis at angular frequency ω_1 , pinwheel fashion.

6.4 Forcing and damping with several degrees of freedom

We first turn to the application of driving terms to coupled oscillator problems, first without damping. Let q_l be the original generalized coordinates of the system, before finding the normal modes. Then a general forcing term in the Lagrangian has the form $\sum_k F_k(t)q_k$. We can immediately apply our knowledge of normal modes to rewrite this term as

$$\sum_k F_k(t)q_k = \sum_k F_k(t) \sum_l Q_l(t)V_l^k = \sum_l Q_l(t)f_l(t) \quad (190)$$

$$f_l(t) = \sum_k F_k(t)V_l^k \quad (191)$$

In other words the effect of external driving forces on the system can be evaluated independently on each normal mode.

$$\ddot{Q}_l + \omega_l^2 Q_l = f_l(t) \quad (192)$$

and the discussion reverts to our discussion of oscillations in one degree of freedom! For a harmonic driving term $f_l^0 e^{-i\omega t}$ the general solution for each normal mode coordinate is

$$Q_l(t) = A_l e^{-i\omega t} + \frac{f_l^0}{\omega_l^2 - \omega^2} e^{-i\omega t} \quad (193)$$

it is now a simple matter to return to the original coordinates

$$q_k(t) = \sum_l A_l V_l^k e^{-i\omega t} + \sum_l \frac{f_l^0}{\omega_l^2 - \omega^2} V_l^k e^{-i\omega t} \quad (194)$$

The new feature here is that the amplitude of the response of one of the original coordinates to the driving term shows resonance at all of the normal mode frequencies coupling to that coordinate. Because we have neglected damping the response amplitude blows up at each resonant frequency. But just as in the case of one degree of freedom, we know that the solution for $\omega = \omega_l$ doesn't literally blow up but acquires a factor of t

$$q_k(t) \rightarrow \sum_l A_l' V_l^k e^{-i\omega t} + it \frac{f_l^0}{2\omega_l} V_l^k e^{-i\omega t}, \quad \omega = \omega_l \quad (195)$$

We again note the $\pi/2$ phase shift between response and driving force at resonance.

Damping with several degrees of freedom For the case of one degree of freedom, we introduced a damping force $-m\gamma\dot{q}$ linear in the single velocity of the degree of freedom.

With several degrees of freedom the obvious generalization of the damping of coordinate k is

$$F_k = - \sum_j \sqrt{m_k} \Gamma_{kj} \sqrt{m_j} \dot{q}_j \quad (196)$$

We recall that a similar force term appeared in small oscillations when time reversal was violated by an external magnetic field:

$$m_k \ddot{q}_k + 2 \sum_j A_{kj} \dot{q}_j + \sum_j \sqrt{m_k m_j} \Omega_{kj}^2 q_j = 0 \quad (197)$$

But in this case A is an antisymmetric matrix. For this reason that force term does no work:

$$\sum_{kj} \dot{q}_k \dot{q}_j A_{kj} = - \sum_{kj} \dot{q}_k \dot{q}_j A_{kj} = 0 \quad (198)$$

Similarly the antisymmetric part of Γ_{kj} is non-dissipative. Thus no generality is lost in assuming Γ is a symmetric matrix. The most general small oscillation equation of motion including damping and external forcing is

$$m_k \ddot{q}_k + 2 \sum_j A_{kj} \dot{q}_j + \sum_j \sqrt{m_k m_j} \Omega_{kj}^2 q_j + \sum_j \sqrt{m_k m_j} \Gamma_{kj} \dot{q}_j = F_k(t) \quad (199)$$

As always with homogeneous linear differential equations with constant coefficients the $F_k = 0$ case can be converted to algebra by assuming the time dependence e^{rt} leading to the characteristic equation

$$\det \{ r^2 m I + r(2A + \sqrt{m} \Gamma \sqrt{m}) + \sqrt{m} \Omega^2 \sqrt{m} \} = 0 \quad (200)$$

With general A, Γ this equation quickly becomes unwieldy: even for only 2 degrees of freedom it is a general quartic equation in r . If $A = 0$ and Γ commutes with Ω^2 , we can bring Γ and Ω^2 simultaneously into diagonal form, in which case each normal mode simply has its own damping constant γ . Then driving the system at frequency ω leads to the solution

$$q_k(t) = \sum_l A_l V_l^k e^{-\gamma/2} e^{-i\omega_l t} + \sum_l \frac{f_l^0}{\omega_l^2 - \omega^2 - i\omega\gamma} V_l^k e^{-i\omega t} \quad (201)$$

showing damped resonances at potentially all normal mode frequencies.

In general damping represents the loss of energy by a subsystem into its environment. These losses can be identified as heat and/or electromagnetic radiation. In any case we should expect that the damping terms should be responsible for a decrease in the energy of the subsystem. Including damping, the Lagrange equations are modified to

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = F_k^{\text{damping}} \quad (202)$$

which we use to calculate the time derivative of the Hamiltonian

$$\begin{aligned}\frac{dH}{dt} &= \sum_k \ddot{q}_k \frac{\partial L}{\partial \dot{q}_k} + \sum_k \dot{q}_k \left(\frac{\partial L}{\partial q_k} + F_k^{\text{damping}} \right) - \sum_k \ddot{q}_k \frac{\partial L}{\partial \dot{q}_k} - \sum_k \dot{q}_k \frac{\partial L}{\partial q_k} \\ &= \sum_k \dot{q}_k F_k^{\text{damping}} \rightarrow - \sum_{kj} \dot{q}_k \dot{q}_j \sqrt{m_k m_j} \Gamma_{kj} \equiv -2\mathcal{F}\end{aligned}\quad (203)$$

where the last form specializes to our expression for the damping forces we assumed for small oscillations. The function $\mathcal{F}(\dot{q}) = (1/2) \sum_{kj} \dot{q}_k \dot{q}_j \sqrt{m_k m_j} \Gamma_{kj}$ is Rayleigh's dissipation function. Its physical interpretation makes clear that it should be a positive definite bilinear form (meaning the eigenvalues of the coefficient matrix are all positive). In the current context of small oscillations we see that

$$F_k^{\text{damping}} = - \frac{\partial \mathcal{F}}{\partial \dot{q}_k} \quad (204)$$

but it can be used to model damping forces more generally.

6.5 Parametric resonance

There is one other phenomenon in driven oscillations that is important. The driving forces we have studied so far have been independent of the coordinates. When they are allowed to be linear in the coordinates, their effect can be thought of as giving time dependence to the parameters of the oscillation. We consider the simplest situation of one degree of freedom in which the frequency is given time dependence $\omega(t)$. Then resonance effects can occur when this time dependence is itself periodic, for example $\omega^2(t) = \omega_0^2(1 + \delta \cos \omega t)$, with $\delta \ll 1$.

$$\ddot{q} + \omega_0^2(1 + \delta \cos \omega t)q = 0 \quad (205)$$

Resonance can be expected when the frequency displayed in the driving terms matches those in the other terms in the equation. Since the coefficient of $\cos \omega t$ is q which in the limit $\delta = 0$ oscillates at frequency ω_0 , as does the \ddot{q} term, we want $\cos \omega t \cos \omega_0 t$ to have an oscillatory contribution close to ω_0 . this will be the case if $\omega \approx 2\omega_0$. We then try a solution of the form

$$q(t) = a(t) \cos \frac{\omega t}{2} + b(t) \sin \frac{\omega t}{2} \quad (206)$$

where a, b are slowly varying in t . Computing time derivatives

$$\begin{aligned}\dot{q} &= \left(\dot{a} + \frac{\omega}{2} b \right) \cos \frac{\omega t}{2} + \left(\dot{b} - \frac{\omega}{2} a \right) \sin \frac{\omega t}{2} \\ \ddot{q} &= \left(\ddot{a} + \omega \dot{b} - \frac{\omega^2}{4} a \right) \cos \frac{\omega t}{2} + \left(\ddot{b} - \omega \dot{a} - \frac{\omega^2}{4} b \right) \sin \frac{\omega t}{2}\end{aligned}\quad (207)$$

Writing out the equations of motion

$$\begin{aligned}0 &= \left(\ddot{a} + \omega \dot{b} - \frac{\omega^2 - 4\omega_0^2}{4} a + \frac{a\delta\omega_0^2}{2} \right) \cos \frac{\omega t}{2} + \left(\ddot{b} - \omega \dot{a} - \frac{\omega^2 - 4\omega_0^2}{4} b - \frac{b\delta\omega_0^2}{2} \right) \sin \frac{\omega t}{2} \\ &\quad + \frac{a\delta\omega_0^2}{2} \cos \frac{3\omega t}{2} + \frac{b\delta\omega_0^2}{2} \sin \frac{3\omega t}{2}\end{aligned}\quad (208)$$

The terms on the last line can be cancelled by including higher frequency terms, with amplitude of order δ compared to the terms we kept, in the ansatz for $q(t)$. The terms on the the first line will be zero if the coefficients are both zero. Putting $a(t) = ae^{rt}$, $b(t) = be^{rt}$ leads to the algebraic equations

$$\begin{aligned} r^2 a + r\omega b - \frac{\omega^2 - 4\omega_0^2}{4} a + \frac{a\delta\omega_0^2}{2} &= 0 \\ r^2 b - r\omega a - \frac{\omega^2 - 4\omega_0^2}{4} b - \frac{b\delta\omega_0^2}{2} &= 0 \end{aligned} \quad (209)$$

For $\omega^2 - 4\omega_0^2, \delta \ll 1$ we can neglect r^2 and find

$$r^2 \omega^2 = \left(-\frac{\delta\omega_0^2}{2} - \frac{\omega^2 - 4\omega_0^2}{4} \right) \left(-\frac{\delta\omega_0^2}{2} + \frac{\omega^2 - 4\omega_0^2}{4} \right) = \frac{\delta^2\omega_0^4}{4} - \left(\frac{\omega^2 - 4\omega_0^2}{4} \right)^2 \quad (210)$$

Exponentially growing behavior, signifying parametric resonance will occur if r is real which means

$$-\frac{\delta}{2} < \frac{\omega^2 - 4\omega_0^2}{4\omega_0^2} < \frac{\delta}{2} \quad (211)$$

This is just one of several parametric resonance frequencies for this case, and the simplest to analyze. More generally parametric resonance occurs for $\omega \approx 2\omega_0/n$ for integer n , but $n > 1$ are much more complicated to analyze and also turn out to be weaker than the $n = 1$ case.

7 Rigid body motion

By a rigid body we mean an extended physical structure which makes no change in shape or size throughout its motion. If \mathbf{r}_1 and \mathbf{r}_2 are any two points in the rigid body, defined in a coordinate system fixed in the body, $\mathbf{r}_1 - \mathbf{r}_2$ is strictly constant. Rigid bodies are a fiction. Their existence would contradict relativity³. But even in the nonrelativistic domain there is inevitably some elasticity or compressibility in matter. But there are clearly many structures for which the distortions under sufficiently mild stress are negligible, so it is reasonable to devote a few classes to understanding their motion in some detail.

The first step in the description is to uniquely specify the coordinate configuration of the rigid body. We can specify the location of the center of mass by the three center of mass coordinates \mathbf{R} with respect to some coordinate system fixed in space. It remains to specify the orientation of the body. For this purpose we attach an orthonormal coordinate system to the body which moves and rotates with the body. It is usually convenient, but not mandatory, to choose the center of mass as the origin of this body-fixed system. Then we can specify the body's orientation by the orientation of the body axes, relative to the fixed axes, by the some rotation about the body origin. Every rotation can be described by specifying a rotation axis \hat{u} (say in the direction θ, φ) and a rotation angle ϕ . All together the configuration is completely specified by 6 coordinates, $\mathbf{R}, \hat{u}, \theta$.

³Consider for example the pole vaulter paradox.

7.1 Angular velocity, moment of inertia, and angular momentum

The dynamical state of a rigid body also requires the specification of 6 velocities. Three of these are given by the velocity of the center of mass $\mathbf{V} = \dot{\mathbf{R}}$. But we also need 3 angular velocities. To get these, think of an infinitesimal displacement in the fixed system of a general point, with coordinates \mathbf{r} in the body system, of the rigid body. It is the vector sum of a displacement of the center of mass $d\mathbf{R}$ and the vector displacement $\hat{u}d\phi \times \mathbf{r}$ due to a rotation about the center of mass.

$$\mathbf{v}dt = d\mathbf{R} + d\phi\hat{u} \times \mathbf{r} \equiv dt(\mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}) \quad (212)$$

In this formula \mathbf{v} is the velocity of the considered point in the fixed (inertial) system, \mathbf{V} is the velocity of the center of mass and **the angular velocity** $\boldsymbol{\omega}$ is implicitly defined by this equation.

Suppose we had chosen a different origin for our body axes, say the point \mathbf{a} . Then the velocity of this new origin is $\mathbf{V}' = \mathbf{V} + \boldsymbol{\omega} \times \mathbf{a}$ and the coordinates of the point \mathbf{r} in the new system are $\mathbf{r}' = \mathbf{r} - \mathbf{a}$. Then we could seemingly define a different angular velocity $\boldsymbol{\omega}'$ by

$$\mathbf{v} = \mathbf{V}' + \boldsymbol{\omega}' \times \mathbf{r}' = \mathbf{V} + \boldsymbol{\omega} \times \mathbf{a} - \boldsymbol{\omega}' \times \mathbf{a} + \boldsymbol{\omega}' \times \mathbf{r} \quad (213)$$

But this has to equal $\mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}$ which implies that $\boldsymbol{\omega}' = \boldsymbol{\omega}$. This confirms that the angular velocity defined this way doesn't depend on the choice of body coordinates, so it is a meaningful physical property of the rigid body. From now on we will take the body origin at the center of mass.

We are now in a position to write down the kinetic energy of a rigid body in terms of \mathbf{V} and $\boldsymbol{\omega}$. It is clearest to imagine that the rigid body is made up of a large number of point particles of masses m_k and positions \mathbf{r}_k referred to the body axes. Then the velocity of the k th particle is given by

$$\mathbf{v}_k = \mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}_k \quad (214)$$

Then the kinetic energy is given by

$$\begin{aligned} T &= \frac{1}{2} \sum_k m_k \mathbf{v}_k^2 = \frac{1}{2} \sum_k m_k (\mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}_k)^2 \\ &= \frac{1}{2} \mathbf{V}^2 \sum_k m_k + \frac{1}{2} \sum_k m_k (\boldsymbol{\omega} \times \mathbf{r}_k)^2 + (\mathbf{V} \times \boldsymbol{\omega}) \cdot \sum_k m_k \mathbf{r}_k \\ &= \frac{M}{2} \mathbf{V}^2 + \frac{1}{2} \sum_k m_k (\boldsymbol{\omega} \times \mathbf{r}_k)^2 + M(\mathbf{V} \times \boldsymbol{\omega}) \cdot \mathbf{R} \end{aligned} \quad (215)$$

where $M = \sum_k m_k$ is the total mass of the system and $\mathbf{R} = (1/M) \sum_k m_k \mathbf{r}_k$ is the center of mass relative to the body axes. This shows the convenience of choosing the origin of the

body axes to be the center of mass. In that case $\mathbf{R} = 0$ and the kinetic energy reduces to

$$\begin{aligned} T &= \frac{M}{2} \mathbf{V}^2 + \frac{1}{2} \sum_k m_k (\boldsymbol{\omega} \times \mathbf{r}_k)^2 = \frac{M}{2} \mathbf{V}^2 + \frac{1}{2} \sum_k m_k [\boldsymbol{\omega}^2 r_k^2 - (\boldsymbol{\omega} \cdot \mathbf{r}_k)^2] \\ &= T_{\text{CM}} + \frac{1}{2} \sum_{ab} \omega^a \omega^b I_{ab}, \quad I_{ab} \equiv \sum_k m_k (\delta_{ab} r_k^2 - r_k^a r_k^b) \end{aligned} \quad (216)$$

Here I_{ab} is called **the moment of inertia tensor**, and T_{CM} is the kinetic energy of a mass M concentrated at the center of mass. Unless explicitly stated otherwise the moment of inertia tensor will always be referred to the center of mass.

To calculate the kinetic energy in a given application, we obviously need an expression for the angular velocity in terms of the natural coordinates for the problem. The needed information is contained in the formula

$$\mathbf{v} = \mathbf{V} + \boldsymbol{\omega} \times \mathbf{r} \quad (217)$$

Giving the velocity, \mathbf{v} , **as measured in the inertial frame**, of any point \mathbf{r} in the body, **measured relative to the body-fixed axes**. But sometimes it is much easier to use different body axes to infer $\boldsymbol{\omega}$, particularly axes whose origin is instantaneously at rest $\mathbf{V}' = 0$. In that case

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}' \quad (218)$$

where we exploited the fact that $\boldsymbol{\omega}' = \boldsymbol{\omega}$! A common example is an object rolling on a surface without slipping. Then the points of contact are always instantaneously at rest, and can be easily identified.

The moment of inertia is calculated in the body-fixed system. Since it is symmetric in its indices $I_{ab} = I_{ba}$, one can always rotate the body axes to a frame where it is diagonal $I_{ab} = I_a \delta_{ab}$. These are the principal axes and are clearly a convenient choice for body axes:

$$T = T_{\text{CM}} + \frac{1}{2} \sum_a \omega^{a2} I_a = T_{\text{CM}} + \frac{1}{2} (I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2), \quad \text{principal axes} \quad (219)$$

where I remind you that this formula only holds if origin of the body system is the center of mass, and the axes are the principal axes.

If all I_a are distinct the body can be called an asymmetrical top: this is dynamically the most complex situation. If one pair are equal, say $I_1 = I_2$, the rigid body is called a symmetrical top. Finally if $I_1 = I_2 = I_3 \equiv I$ we have the spherical top. In the last case any set of orthonormal axes are principal axes, and the internal kinetic energy is simply $I\boldsymbol{\omega}^2/2$. In the case of the symmetrical top, $I_1 = I_2$, the direction of the x_3 principal axis is unique but the $x_1 x_2$ principal axes are arbitrary. Please note that if the mass distribution within the rigid body is not uniform these symmetries may not be evident from the geometrical shape of the object. On the other hand, if the mass distribution is uniform the symmetry of a geometrical shape will imply the corresponding symmetries of I_a . So a uniform sphere will be a spherical top, and one with an axis of symmetry will be a symmetrical top.

A collinear rigid system of masses is called a rotor or rotator. Choosing the 3-axis to pass through the masses, one finds that $I_1 = I_2$ and $I_3 = 0$. There are only two rotational degrees of freedom since rotation about the 3-axis is meaningless. Another special configuration is a coplanar one. Obviously the center of mass is in the plane and the line through the center of mass perpendicular to the plane is a principal axis, say the 3-axis. The other two principal axes are in the plane. Then $I_1 = \sum_k m_k x_2^{k2}$, $I_2 = \sum_k m_k x_1^{k2}$, and $I_3 = I_1 + I_2$.

It is also possible (and sometimes easier) to calculate I_{ab} in a system translated by a displacement \mathbf{d} from the center of mass. Calling the moments in this system I'_{ab} we have

$$I'_{ab} = \sum_k m_k [(\mathbf{r}_k - \mathbf{d})^2 \delta_{ab} - (r_k^a - d^a)(r_k^b - d^b)] = I_{ab} + M(\mathbf{d}^2 \delta_{ab} - d^a d^b) \quad (220)$$

which we can interpret as the sum of the moment of inertia tensor in the center of mass plus the moment of inertia of a point mass M located at the center of mass.

Angular Momentum: Let's call the angular momentum of the rigid body about its center of mass the spin \mathbf{S} . Then by definition

$$\begin{aligned} \mathbf{S} &= \mathbf{J} - \mathbf{R} \times \mathbf{P} = \sum_k m_k \mathbf{r}_k \times \mathbf{v}_k \\ &= \sum_k m_k \mathbf{r}_k \times (\mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}_k) = \sum_k m_k \mathbf{r}_k \times (\boldsymbol{\omega} \times \mathbf{r}_k) \end{aligned} \quad (221)$$

where the last form used the fact that \mathbf{r}_k is the position of particle k relative to the center of mass, i.e. $\sum_k m_k \mathbf{r}_k = 0$. Expanding out the triple vector product and writing out the components of the equation,

$$S^a = \sum_k m_k (\omega^a r_k^2 - r_k^a \boldsymbol{\omega} \cdot \mathbf{r}_k) = \sum_b \omega^b \sum_k m_k (r_k^2 \delta_{ab} - r_k^a r_k^b) = \sum_b I_{ab} \omega^b \quad (222)$$

If we resolve $\mathbf{S}, \boldsymbol{\omega}$ into components along body fixed axes which are also principal axes, the relation between the components is simply

$$S^1 = I_1 \omega^1, \quad S^2 = I_2 \omega^2, \quad S^3 = I_3 \omega^3 \quad (223)$$

Be warned that these components will be time dependent even if the spin is conserved, because they are referred to time dependent axes!

7.2 Equations of motion

Since a rigid body has 6 degrees of freedom, we obviously need 6 equations of motion. We can take three of them to be Newton's law for the center of mass motion, which specifies the rate of change of the total momentum of the system:

$$\frac{d\mathbf{P}}{dt} = \sum_k \frac{d\mathbf{p}_k}{dt} = \sum_k \mathbf{f}_k \equiv \mathbf{F} \quad (224)$$

The \mathbf{f}_k include both external forces as well as the forces of interaction between the parts of the rigid body. But the latter all cancel out, because when the external forces are zero the total momentum of the system is conserved. This cancellation can also be ascribed to Newton's third law.

It is natural to take the remaining 3 equations to specify the rate of change of the angular momentum of the rigid body. This is also a consequence of Newton's equations for the individual constituents of the rigid body. Let us begin by calculating the time derivative of the total angular momentum in the space-fixed system of coordinates. For this purpose we call \mathbf{r}_k^0 the position vector in this space-fixed system. Then

$$\frac{d\mathbf{J}}{dt} = \frac{d}{dt} \sum_k \mathbf{r}_k^0 \times \mathbf{p}_k = \sum_k \mathbf{r}_k^0 \times \dot{\mathbf{p}}_k = \sum_k \mathbf{r}_k^0 \times \mathbf{f}_k \equiv \mathbf{N}_0 \quad (225)$$

where we used the fact that $\dot{\mathbf{r}}_k^0 \times \mathbf{p}_k = 0$ since \mathbf{p}_k is parallel to $\dot{\mathbf{r}}_k^0$. The right side is the total torque \mathbf{N}_0 on the body about the origin of the space-fixed system. In many rigid body problems there is a constraint that one point of the body is fixed. For example a compound pendulum or a top anchored to a fixed point. In this case the configuration of the body is specified by three rotation angles about the fixed point, so we only require three equations of motion. In these cases we can dispense with the equation of motion for the center of mass and take the 3 equations to be the ones just given, with the origin of the space-fixed system coinciding with the fixed point: both \mathbf{J} and \mathbf{N}_0 are calculated about the fixed point which is not necessarily the center of mass.

In the general situation, when the body as a whole participates in the motion, it is convenient to write the rotational equations of motion in terms of the spin \mathbf{S} , the angular momentum about the center of mass. We have, defining $\mathbf{r}_k = \mathbf{r}_k^0 - \mathbf{R}$ as the position vector from the center of mass:

$$\begin{aligned} \mathbf{J} &= \sum_k (\mathbf{R} + \mathbf{r}_k) \times \mathbf{p}_k = \mathbf{R} \times \mathbf{P} + \sum_k m_k \mathbf{r}_k \times (\mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}_k) \\ &= \mathbf{R} \times \mathbf{P} + \sum_k m_k \mathbf{r}_k \times (\boldsymbol{\omega} \times \mathbf{r}_k) = \mathbf{R} \times \mathbf{P} + \mathbf{S} \end{aligned} \quad (226)$$

where we used $\sum_k m_k \mathbf{r}_k = 0$. Then

$$\frac{d\mathbf{S}}{dt} = \frac{d\mathbf{J}}{dt} - \dot{\mathbf{R}} \times \mathbf{P} - \mathbf{R} \times \dot{\mathbf{P}} = \mathbf{N}_0 - \mathbf{R} \times \mathbf{F} \quad (227)$$

But

$$\mathbf{N}_0 = \sum_k (\mathbf{R} + \mathbf{r}_k) \times \mathbf{f}_k = \mathbf{R} \times \mathbf{F} + \sum_k \mathbf{r}_k \times \mathbf{f}_k \quad (228)$$

Then we have finally the 6 equations of motion

$$\frac{d\mathbf{S}}{dt} = \sum_k \mathbf{r}_k \times \mathbf{f}_k \equiv \mathbf{N}, \quad \frac{d\mathbf{P}}{dt} = \mathbf{F} \quad (229)$$

where \mathbf{N} is the torque about the center of mass.

7.3 Free motion of Rigid bodies

When no external forces or torques act on a rigid body, the equation of motion simply say that momentum and spin are conserved. Momentum conservation just means that the center of mass moves freely $\mathbf{R} = \mathbf{R}_0 + \mathbf{V}t$, with $\mathbf{V} = \mathbf{P}/M$. Thus no generality is lost by assuming that the center of mass is at rest $\mathbf{r} = 0$.

Depending on the principal moments of inertia, the conservation of spin allows some intricate motions. For a spherical top, $I_1 = I_2 = I_3 = I$, $\boldsymbol{\omega} = \mathbf{S}/I$, and the motion is simple rotation about any axis through the center of mass. A rotor $I_3 = 0, I_1 = I_2$ is similarly treated because the only meaningful motions are rotations about an axis perpendicular to the rotor, so again we have $\mathbf{S} = I_1 \boldsymbol{\omega}_\perp$.

A symmetrical top $I_1 = I_2 \neq I_3$ allows for more interesting possibilities. Of course \mathbf{S} is still a constant, but $\boldsymbol{\omega}$ need not be parallel to \mathbf{S} . If we resolve these vectors into their components along the principal body fixed axes, we have $\omega_1 = S_1/I_1, \omega_2 = S_2/I_1, \omega_3 = S_3/I_3$. Because the principal axes are rotating, both the ω_a and S_a can depend on time. However, there are two conservation laws: \mathbf{S}^2 and the kinetic energy T are independent of time:

$$S_1^2 + S_2^2 + S_3^2 = S^2, \quad \frac{S_1^2 + S_2^2}{I_1} + \frac{S_3^2}{I_3} = 2T \quad (230)$$

Together these two conservation laws imply that S_3 is a constant. Since $S_3 = S \cos \theta$ where θ is the angle between the principal 3-axis and \mathbf{S} , we conclude that θ is fixed throughout the motion. The only possible motion is a precession of the 3-axis about the fixed direction \mathbf{S} , as the top rotates about its 3-axis with angular velocity $\omega_3 = S_3/I_3 = (S/I_3) \cos \theta$. To go further notice that the projection into the 12-plane, \mathbf{S}_\perp is parallel to $\boldsymbol{\omega}_\perp$. This means that $\mathbf{S}, \boldsymbol{\omega}$ and the 3-axis all lie in the same plane which therefore rotates about \mathbf{S} with the 3-axis. The velocity of a point \mathbf{r} on the 3-axis is of course $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, and it will always be perpendicular to the rotating plane. Thus the point describes a circle of radius $r \sin \theta$ about \mathbf{S} . The precession frequency is thus

$$\omega_{\text{pre}} = \frac{v}{r \sin \theta} = \frac{\omega \sin \phi}{\sin \theta} = \frac{\omega_3 \tan \phi}{\sin \theta} = \frac{S_3 \tan \phi}{I_3 \sin \theta} = \frac{S \tan \phi}{I_3 \tan \theta} \quad (231)$$

where ϕ is the angle between $\boldsymbol{\omega}$ and the 3-axis, $\omega \cos \phi = \omega_3 = S_3/I_3$. Now $\tan \phi = \omega_\perp/\omega_3 = (I_3/I_1)(S_\perp/S_3) = (I_3/I_1) \tan \theta$ so we conclude

$$\omega_{\text{pre}} = \frac{S}{I_1}, \quad \omega_3 = \frac{S}{I_3} \cos \theta \quad (232)$$

Notice that the precession frequency is independent of the angle θ . In this formula we see that in the limit $I_3 \rightarrow 0$ we must have $\theta = \pi/2$, i.e. the rotor rotates about an axis perpendicular to its axis, in accord with our previous conclusion.

The most complex free motion is that of an asymmetrical top, with all three principal moments of inertia distinct. For definiteness let us assume that $I_1 < I_2 < I_3$. Now the two conservation laws read

$$S_1^2 + S_2^2 + S_3^2 = S^2, \quad \frac{S_1^2}{2TI_1} + \frac{S_2^2}{2TI_2} + \frac{S_3^2}{2TI_3} = 1 \quad (233)$$

In the space described by coordinates (S_1, S_2, S_3) these two equations imply that this point lies on the intersection of a sphere of radius S with an ellipsoid with semi-axes $a = \sqrt{2TI_3}$, $b = \sqrt{2TI_2}$, $c = \sqrt{2TI_1}$, $c < b < a$. This intersection will be non-empty provided $c \leq S \leq a$. When $S \approx c$ the intersection is a small closed curve encircling the x_1 axis. Similarly when $S \approx a$ is a small closed curve encircling the x_3 axis. In these two cases the motion stays near these respective axes throughout. In contrast when $S \approx b$, the intersection is a closed curve that travels far from the x_2 -axis. In this sense rotation about the 1 or 3 axes is stable whereas that about the 3-axis is unstable. We have to defer more details about the motion in this case till after a more detailed analysis of the general equations of motion.

7.4 Eulerian angles: specifying the top's configuration in space

The Eulerian angles are a fairly standard choice. Start with the fixed body axes $x_1x_2x_3$ coincident with the space axes XYZ . (1) Rotate the body axes an angle φ about the z, x_3 -axis. (2) Rotate the body axes an angle θ about the new x_1 axis (called the **line of nodes**). (3) Rotate the body axes an angle ψ about the new x_3 axis. The final x_3 axis is now in the spherical polar direction θ, φ , and ψ completes the specification of the body fixed axes with respect to rotations about this direction. The range of these angles is $0 < \varphi, \psi < 2\pi$ and $0 < \theta < \pi$. These definitions involve arbitrary choices of axes for each step, and represent one among several conventions in use. In quantum mechanics the second step is usually a rotation about the new x_2 -axis instead of the new x_1 axis.

To use the Eulerian angles to describe the motion of rigid bodies, we need to relate the angular velocity $\boldsymbol{\omega}$ to θ, φ, ψ . We begin by constructing the angular velocity vectors corresponding to only one of these time derivatives non-zero:

$$\boldsymbol{\omega}_\theta = \dot{\theta}(\cos \psi, -\sin \psi, 0), \quad \boldsymbol{\omega}_\psi = \dot{\psi}(0, 0, 1), \quad \boldsymbol{\omega}_\varphi = \dot{\varphi}(\sin \theta \sin \psi, \sin \theta \cos \psi, \cos \theta)$$

Adding these up we find the total angular velocity and kinetic energy

$$\begin{aligned} \boldsymbol{\omega} &= (\dot{\theta} \cos \psi + \dot{\varphi} \sin \theta \sin \psi, -\dot{\theta} \sin \psi + \dot{\varphi} \sin \theta \cos \psi, \dot{\psi} + \dot{\varphi} \cos \theta) \\ T &= \frac{I_1}{2}(\dot{\theta} \cos \psi + \dot{\varphi} \sin \theta \sin \psi)^2 + \frac{I_2}{2}(-\dot{\theta} \sin \psi + \dot{\varphi} \sin \theta \cos \psi)^2 + \frac{I_3}{2}(\dot{\psi} + \dot{\varphi} \cos \theta)^2 \end{aligned} \quad (234)$$

where we have taken the body axes to be principal axes.

In the case of a symmetrical top, the ψ dependence cancels in T

$$T = \frac{I_1}{2}(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta) + \frac{I_3}{2}(\dot{\psi} + \dot{\varphi} \cos \theta)^2, \quad I_1 = I_2 \quad (235)$$

Although we have already solved for the free motion of a symmetrical top based on the conservation laws, it is instructive to confirm that the same results follow from the free

Lagrangian $L = T$. The Lagrange equations are

$$\begin{aligned}
\frac{d}{dt} \frac{\partial L}{\partial \dot{\psi}} &= I_3 \frac{d}{dt} (\dot{\psi} + \dot{\varphi} \cos \theta) = \frac{\partial L}{\partial \psi} = 0 \\
\frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}} &= \frac{d}{dt} (I_1 \dot{\varphi} \sin^2 \theta + I_3 \cos \theta (\dot{\psi} + \dot{\varphi} \cos \theta)) = \frac{\partial L}{\partial \varphi} = 0 \\
\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} &= I_1 \ddot{\theta} = \frac{\partial L}{\partial \theta} = \left(I_1 \dot{\varphi} \cos \theta - I_3 (\dot{\psi} + \dot{\varphi} \cos \theta) \right) \dot{\varphi} \sin \theta
\end{aligned} \tag{236}$$

The first equation simply states that $S_3 = I_3 \omega_3 = I_3 (\dot{\psi} + \dot{\varphi} \cos \theta) = \text{constant}$. That $\partial L / \partial \dot{\psi} = S^3$ is evident because ψ measures the angle of rotation about the 3-axis. Similarly, the fact that φ measures the angle of rotation about the z -axis of the fixed coordinate system implies that $\partial L / \partial \dot{\varphi} = \mathbf{S} \cdot \hat{z}$ so the second equation says simply that $\mathbf{S} \cdot \hat{z}$ is a constant. Of course we know that for free rotation \mathbf{S} is conserved as a vector. If we choose the fixed system z -axis parallel to \mathbf{S} , $\mathbf{S} \cdot \hat{z} = S$ is just the magnitude of the spin vector, which is conserved by the second equation. With that choice $S_3 = S \cos \theta$, and the first equation implies that θ is a constant, i.e. that the 3-axis precesses about \mathbf{S} . The first two equations read as conservation laws

$$\begin{aligned}
S_3 &= S \cos \theta = I_3 (\dot{\psi} + \dot{\varphi} \cos \theta) = I_3 \omega_3, & \omega_3 &= \frac{S}{I_3} \cos \theta \\
S &= I_1 \dot{\varphi} \sin^2 \theta + S_3 \cos \theta, & \omega_{\text{pre}} &= \dot{\varphi} = \frac{S(1 - \cos^2 \theta)}{I_1 \sin^2 \theta} = \frac{S}{I_1}
\end{aligned} \tag{237}$$

in agreement with our previous calculation. Of course plugging these results into the right side of the third equation shows that it vanishes, as constant θ would require.

7.5 Symmetrical top with fixed point moving under gravity

Using Eulerian angles we can set up the Lagrangian for a top with its lowest point fixed in space. For simplicity we assume a symmetrical top, $I_1 = I_2$. To obtain the kinetic energy for this application we need the moment of inertia relative to the fixed point. Let $I_1 = I_2, I_3$ be the principal moments of inertia about the center of mass. The center of mass lies on the axis of symmetry, say a distance a from the fixed point. Then the moments of inertia about the fixed point are $I'_1 = I'_2 = I_1 + Ma^2$, and $I'_3 = I_3$. Thus the Lagrangian of this top is

$$L' = \frac{I'_1}{2} (\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta) + \frac{I_3}{2} (\dot{\psi} + \dot{\varphi} \cos \theta)^2 - Mga \cos \theta \tag{238}$$

The absence of ψ, φ from the Lagrangian implies the two conservation laws

$$J_3 = \frac{\partial L'}{\partial \dot{\psi}} = I_3 (\dot{\psi} + \dot{\varphi} \cos \theta), \quad J_z = I_1 \dot{\varphi} \sin^2 \theta + J_3 \cos \theta \tag{239}$$

In addition of course energy is conserved

$$\begin{aligned}
E &= \frac{I_1'}{2}(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta) + \frac{I_3}{2}(\dot{\psi} + \dot{\varphi} \cos \theta)^2 + Mga \cos \theta \\
&= \frac{I_1'}{2}\dot{\theta}^2 + \frac{J_3^2}{2I_3} + \frac{(J_z - J_3 \cos \theta)^2}{2I_1' \sin^2 \theta} + Mga \cos \theta \equiv \frac{I_1'}{2}\dot{\theta}^2 + \frac{J_3^2}{2I_3} + mga + V_{\text{eff}}(\theta) \\
V_{\text{eff}}(\theta) &= \frac{(J_z - J_3 \cos \theta)^2}{2I_1' \sin^2 \theta} - Mga(1 - \cos \theta)
\end{aligned} \tag{240}$$

The range of motion in θ is limited to those values for which

$$E \geq \frac{J_3^2}{2I_3} + mga + V_{\text{eff}}(\theta) \tag{241}$$

When $J_3 \neq \pm J^z$ this inequality excludes θ near $0, \pi$ because $V_{\text{eff}} \rightarrow \infty$ at those points. The equality is attained for $\cos \theta$ a root of a cubic polynomial. There is either one real root and a complex conjugate pair or three real roots. More simply, notice that the effective potential is a rational function $f(z)$ of the variable $z = \cos \theta$, which is restricted to the range $-1 < z < +1$.

$$f(z) = \frac{(J_z - J_3 z)^2}{2I_1'(1 - z^2)} - Mga(1 - z) = \frac{(J_z - J_3)^2}{4I_1'(1 - z)} + \frac{(J_z + J_3)^2}{4I_1'(1 + z)} - \frac{J_3^2}{2I_1'} - Mga(1 - z) \tag{242}$$

By direct calculation we find

$$f'' = \frac{(J_z - J_3)^2}{2I_1'(1 - z)^3} + \frac{(J_z + J_3)^2}{2I_1'(1 + z)^3} \tag{243}$$

which is manifestly positive for all $-1 < z < +1$. Thus its graph is concave upward in this interval going to $+\infty$ at both ends. Thus there is a unique minimum of f and hence of V_{eff} for some $0 < \theta_{\text{min}} < \pi$. When $E = J_3^2/2I_3 + mga + V_{\text{eff}}(\theta_{\text{min}})$, θ will be fixed at θ_{min} throughout the motion: this is simple precession about the vertical. However when E is greater than this, there will be precisely two turning points $\theta_{1,2}$ and θ will oscillate between them. This oscillation is called nutation and the motion is called precession with nutation.

The way the nutation appears in space depends on whether or not

$$\dot{\varphi} = \frac{J_z - J_3 \cos \theta}{I_1 \sin^2 \theta} \tag{244}$$

changes sign as θ varies between θ_1 and θ_2 . If it does not the nutation involves a monotonic advance. If it does change sign there is a looping effect.

7.6 Euler's Equations

For a general rigid body, it is desirable to exploit the simplification achieved by choosing principal axes, which requires us to formulate the equations of motion in a body fixed frame, which is non-inertial. We formulated the equations of motion in an inertial frame

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}, \quad \frac{d\mathbf{S}}{dt} = \mathbf{N} \tag{245}$$

Let us consider the time dependence of a vector resolved into components along the body fixed axes $\mathbf{A} = \sum_{a=1}^3 A_a(t)\mathbf{e}_a(t)$:

$$\frac{d\mathbf{A}}{dt} = \sum_{a=1}^3 \dot{A}_a(t)\mathbf{e}_a(t) + \sum_{a=1}^3 A_a(t)\frac{d\mathbf{e}_a(t)}{dt} \quad (246)$$

The contribution of the second term arises from the rotating body-fixed axes. Recall that the angular velocity of a rigid body was defined implicitly by $\mathbf{v}_k = \mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}_k$ where the second term is due to the rotation of the body. This equation holds with the same $\boldsymbol{\omega}$ for every constituent of the rigid body. In particular the displacement $\boldsymbol{\Delta} = \mathbf{r}_1 - \mathbf{r}_2$ between any points fixed in the rigid body has time derivative

$$\frac{d\boldsymbol{\Delta}}{dt} = \boldsymbol{\omega} \times \boldsymbol{\Delta} \quad (247)$$

A little thought shows that any vector fixed in the rigid body will satisfy the same equation, in particular

$$\frac{d\mathbf{e}_a}{dt} = \boldsymbol{\omega} \times \mathbf{e}_a \quad (248)$$

thus for **any** vector we have

$$\frac{d\mathbf{A}}{dt} = \sum_{a=1}^3 \dot{A}_a(t)\mathbf{e}_a(t) + \boldsymbol{\omega} \times \mathbf{A} \equiv \frac{d^*\mathbf{A}}{dt} + \boldsymbol{\omega} \times \mathbf{A} \quad (249)$$

where the * on the derivative signifies that the time derivative only acts on the components of the vector along the body-fixed axes. So we can cast the equations of motion to involve only quantities specified in the body-fixed system:

$$\begin{aligned} \frac{d^*\mathbf{P}}{dt} + \boldsymbol{\omega} \times \mathbf{P} &= \mathbf{F}, & \frac{d^*\mathbf{S}}{dt} + \boldsymbol{\omega} \times \mathbf{S} &= \mathbf{N} & (250) \\ \frac{dP_1}{dt} + \omega_2 P_3 - \omega_3 P_2 &= F_1, & \frac{dP_2}{dt} + \omega_3 P_1 - \omega_1 P_3 &= F_2, & \frac{dP_3}{dt} + \omega_1 P_2 - \omega_2 P_1 &= F_3 \\ \frac{d\omega_1}{dt} + \frac{I_3 - I_2}{I_1} \omega_2 \omega_3 &= \frac{N_1}{I_1}, & \frac{d\omega_2}{dt} + \frac{I_1 - I_3}{I_2} \omega_1 \omega_3 &= \frac{N_2}{I_2}, & \frac{d\omega_3}{dt} + \frac{I_2 - I_1}{I_3} \omega_1 \omega_2 &= \frac{N_3}{I_3} \end{aligned}$$

The equations on the last line are Euler's equations for the ω_a . Once ω_a are obtained, they can be plugged into the equations on the second line to determine the motion of the center of mass.

Once we have the $\omega_a(t)$, we can find the Euler angles of the top orientation as a function of time by solving the differential equations:

$$\begin{aligned} \dot{\theta} \cos \psi + \dot{\varphi} \sin \theta \sin \psi &= \omega_1(t) \\ -\dot{\theta} \sin \psi + \dot{\varphi} \sin \theta \cos \psi &= \omega_2(t) \\ \dot{\psi} + \dot{\varphi} \cos \theta &= \omega_3(t) & (251) \end{aligned}$$

Let us first reproduce the free motion of a symmetrical top, $I_2 = I_1$ and $\mathbf{N} = 0$. the third Euler equation says simply that $\omega_3 = \text{constant}$. Then defining $\Omega = (I_3 - I_1)\omega_3/I_1$, the first two Euler equations are

$$\begin{aligned} \frac{d\omega_1}{dt} &= -\Omega\omega_2, & \frac{d\omega_2}{dt} &= \Omega\omega_1 \\ \frac{d}{dt}(\omega_1 - i\omega_2) &= -i\Omega(\omega_1 - i\omega_2), & \omega_1 - i\omega_2 &= Ae^{-i\Omega t} \end{aligned} \quad (252)$$

Choosing A real, we have $\omega_1 = A \cos \Omega t$ and $\omega_2 = A \sin \Omega t$.

To compare with our earlier solution using Euler angles, recall that we found that $\dot{\theta} = 0$, $\dot{\phi} = S/I_1$ and $\dot{\psi} = \omega_3 - \dot{\phi} \cos \theta = \omega_3(1 - I_3/I_1) = -\Omega$. Thus $\psi = -\Omega t$ and

$$\omega_1 = \frac{S \sin \theta}{I_1} \sin \psi = -\omega_{\perp} \sin(\Omega t), \quad \omega_2 = \frac{S \sin \theta}{I_1} \cos \psi = \omega_{\perp} \cos(\Omega t) \quad (253)$$

in agreement with the analysis of Euler's equations.

7.7 Free rotation of an asymmetrical top

For definiteness take $I_1 < I_2 < I_3$. The Euler equations with $N_a = 0$ can be quickly used to confirm that

$$\begin{aligned} S^2 &= I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 = S_1^2 + S_2^2 + S_3^2 \\ 2T &= I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 = \frac{S_1^2}{I_1} + \frac{S_2^2}{I_2} + \frac{S_3^2}{I_3} \end{aligned} \quad (254)$$

are both independent of time (i.e. conserved). As mentioned earlier the solution (S_1, S_2, S_3) of this pair of constraints lies on the intersection of a sphere of radius S with an ellipsoid of semi axes $a = \sqrt{2TI_3}$, $b = \sqrt{2TI_2}$, and $c = \sqrt{2TI_1}$. A non-empty intersection requires $\sqrt{2TI_1} < S < \sqrt{2TI_3}$.

We can now use these conservation laws to express, ω_1, ω_3 in terms of ω_2 :

$$\begin{aligned} I_1^2\omega_1^2 &= S^2 - I_2^2\omega_2^2 - I_3^2\omega_3^2 \\ 2TI_1 &= S^2 + I_2(I_1 - I_2)\omega_2^2 + I_3(I_1 - I_3)\omega_3^2 \\ \omega_3 &= \sqrt{\frac{S^2 - 2TI_1 - I_2(I_2 - I_1)\omega_2^2}{I_3(I_3 - I_1)}} \\ \omega_1 &= \sqrt{\frac{S^2 - I_2^2\omega_2^2}{I_1^2} - \frac{I_3}{I_1^2(I_1 - I_3)}(2TI_1 - S^2 - I_2(I_1 - I_2)\omega_2^2)} \\ &= \sqrt{\frac{2TI_3}{I_1(I_3 - I_1)} - \frac{S^2}{I_1(I_3 - I_1)} - \frac{I_2^2\omega_2^2}{I_1^2} \left(1 - \frac{I_3}{I_2(I_1 - I_3)}((I_1 - I_2))\right)} \\ &= \sqrt{\frac{2TI_3 - S^2 - I_2(I_3 - I_2)\omega_2^2}{I_1(I_3 - I_1)}} \end{aligned} \quad (255)$$

Notice that we have arranged the argument of each square root to be manifestly positive in the allowed range of S .

Finally ω_2 satisfies the zero torque Euler equation

$$\begin{aligned}\frac{d\omega_2}{dt} &= \frac{I_3 - I_1}{I_2} \omega_1 \omega_3 \\ dt &= I_2 \sqrt{I_1 I_3} \frac{d\omega_2}{\sqrt{2TI_3 - S^2 - I_2(I_3 - I_2)\omega_2^2} \sqrt{S^2 - 2TI_1 - I_2(I_2 - I_1)\omega_2^2}}\end{aligned}\quad (256)$$

We recognize the right side as the integrand of an elliptic integral. To put it in standard form we first compare the two ratios

$$R_1 = \frac{I_2(I_2 - I_1)}{S^2 - 2TI_1}, \quad R_3 = \frac{I_2(I_3 - I_2)}{2TI_3 - S^2}\quad (257)$$

Then define $k^2 = R_</math>, and we change variables to $u = \sqrt{R_>}\omega_2$, after which$

$$\begin{aligned}t &= \sqrt{\frac{I_1 I_3 I_2^2}{(S^2 - 2TI_1)(2TI_3 - S^2)R_>}} \int_0^{\omega_2 \sqrt{R_>}} \frac{du}{\sqrt{1 - u^2} \sqrt{1 - k^2 u^2}} \\ &= \sqrt{\frac{I_1 I_3 R_<}{(I_2 - I_1)(I_3 - I_2)}} \int_0^{\omega_2 \sqrt{R_>}} \frac{du}{\sqrt{1 - u^2} \sqrt{1 - k^2 u^2}} \\ \omega_2(t) &= \frac{1}{\sqrt{R_>}} \text{sn}(\Omega t), \quad \Omega = \sqrt{\frac{(I_2 - I_1)(I_3 - I_2)}{I_1 I_3 R_<}}\end{aligned}\quad (258)$$

The remaining angular velocities are

$$\omega_3 = \sqrt{\frac{S^2 - 2TI_1}{I_3(I_3 - I_1)}} \sqrt{1 - \frac{R_1}{R_>} \text{sn}^2(\Omega t)}\quad (259)$$

$$\omega_1 = \sqrt{\frac{2TI_3 - S^2}{I_1(I_3 - I_1)}} \sqrt{1 - \frac{R_3}{R_>} \text{sn}^2(\Omega t)}\quad (260)$$

The elliptic function $\text{sn}(z)$ is periodic with period $4K$ where

$$K = \int_0^1 \frac{du}{\sqrt{1 - u^2} \sqrt{1 - k^2 u^2}}\quad (261)$$

So all the ω_a have the period $4K/\Omega$.

To infer the motion of the top in space, we set up Euler angles with the space-fixed z -axis in the direction of the conserved spin $\mathbf{S} = S\hat{z}$. Then recall that the components of \hat{z} on the body fixed axes are

$$\hat{z} = (\sin \theta \sin \psi, \sin \theta \cos \psi, \cos \theta)\quad (262)$$

So we have

$$\begin{aligned} S_1 &= I_1\omega_1 = S \sin \theta \sin \psi, & S_2 &= I_2\omega_2 = S \sin \theta \cos \psi, & S_3 &= I_3\omega_3 = S \cos \theta \\ \cos \theta &= \frac{I_3\omega_3}{S}, & \tan \psi &= \frac{I_1\omega_1}{I_2\omega_2} \end{aligned} \quad (263)$$

it remains to determine $\varphi(t)$. For that we solve the two equations

$$\begin{aligned} \dot{\theta} \cos \psi + \dot{\varphi} \sin \theta \sin \psi &= \omega_1(t) \\ -\dot{\theta} \sin \psi + \dot{\varphi} \sin \theta \cos \psi &= \omega_2(t) \\ \dot{\varphi} \sin \theta &= \omega_1 \sin \psi + \omega_2 \cos \psi = \frac{I_1}{S \sin \theta} \omega_1^2 + \frac{I_2}{S \sin \theta} \omega_2^2 \\ \dot{\varphi} &= S \frac{I_1\omega_1^2 + I_2\omega_2^2}{S^2 - S_3^2} = S \frac{I_1\omega_1^2 + I_2\omega_2^2}{I_1^2\omega_1^2 + I_2^2\omega_2^2} \end{aligned} \quad (264)$$

which determines $\varphi(t)$ as an integral over a rational function of elliptic functions.

7.8 The tippy top

As a preliminary to understanding the behavior of the tippy top consider Fig. 2, which shows two configurations of a spinning sphere whose center of mass is displaced from the

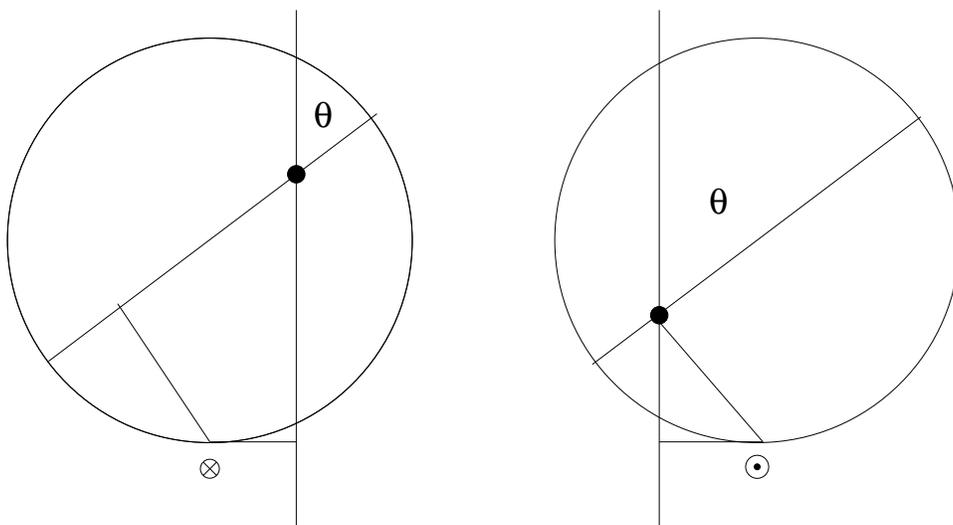


Figure 2: Spinning balls with displaced center of mass. The axis of rotation is through the center of mass and the rotation is counterclockwise viewed from the top. The force of friction is into the page in the left figure and out of the page in the right figure.

geometrical center. The torque due to friction at the point of contact differs when computed

about the vertical axis through the center of mass and about the body's symmetry axis. When the center of mass is in the upper hemisphere, as shown in the left figure, the torque about the vertical axis is downward, in accord with the fact that friction should slow down the rotation. The torque about the body symmetry axis is larger and directed from the geometrical center to the center of mass. This would increase the angular velocity about that axis which means the angle θ between that axis and the vertical would decrease. To see this let us assume the angular velocity starts out in the vertical direction so its body fixed components are

$$\boldsymbol{\omega} = \omega(\sin \theta \sin \psi, \sin \theta \cos \psi, \cos \theta) \quad (265)$$

If we assume $\omega \gg \sqrt{g/R}$, it is reasonable to assume that the x and y components of the torque average to zero. thus since we set the top spinning with $\omega_x = \omega_y = 0$ we can assume that $\omega_{x,y}$ remain negligible, and hence $\omega_3 \approx \omega \cos \theta$ throughout the motion. Thus increasing ω_3 relative to ω implies that θ decreases.

In the configuration on the right the torque about the vertical axis is still down. The torque about the body axis is from the geometrical center to the center of mass, but now this is directed downward. The tippy top is set in motion with the body axis vertical and the center of mass in the lower hemisphere. With a small perturbation the configuration on the right is assumed and the torque will decrease the initial angular velocity about the body symmetry axis more than the decrease of the angular velocity about the vertical axis, causing the angle θ to increase. In both configurations the effect of friction is to increase the height of the center of mass. The closer the center of mass is to the geometrical center the more pronounced is the effect on the body axis compared to the vertical axis. In the approximation used here, the angular velocity in the space-fixed frame remains parallel to \hat{z} throughout the motion, slowly decreasing in magnitude due to friction. Meanwhile ω_3 starts out near ω decreases through zero, changes sign and ends up near $-\omega$.

In the final phase of the tippy top's behavior consider now Fig. 3. This will cause a dramatic reduction in the angular velocity about the vertical with practically no effect on the angular velocity about the symmetry axis. Thus θ will decrease more dramatically tipping the top onto its post.

7.9 Dynamics in non-inertial frames of reference.

When we consider the motion of rigid bodies from the point of view of body-fixed axes, we are actually viewing the dynamics from non-inertial frames, This is something we can do with any physical system: it is merely a change of coordinates.

$$\mathbf{r}_0(t) = \mathbf{r}(t) + \mathbf{R}(t), \quad \dot{\mathbf{r}}_0 = \frac{d\mathbf{r}}{dt} + \dot{\mathbf{R}}(t) \equiv \frac{d\mathbf{r}}{dt} + \mathbf{V}(t) \quad (266)$$

if there is in addition to the translation $\mathbf{R}(t)$ also a rotation with angular velocity $\boldsymbol{\omega}$ we will further express

$$\frac{d\mathbf{r}}{dt} = \frac{d^*\mathbf{r}}{dt} + \boldsymbol{\omega} \times \mathbf{r} \quad (267)$$

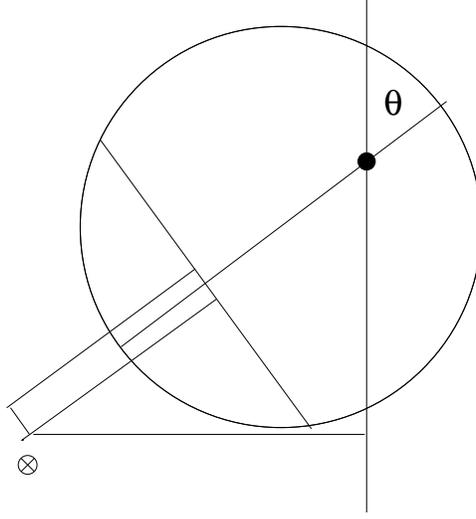


Figure 3: Spinning tippy top. Now the moment arm of friction about the vertical axis is huge, and that about the symmetry axis is tiny.

where the * signifies that only the components of \mathbf{r} on the rotating axes are differentiated. Then the kinetic energy of a particle is

$$\begin{aligned} \frac{m}{2} \dot{\mathbf{r}}_0^2 &= \frac{m}{2} \left(\frac{d^* \mathbf{r}}{dt} + \boldsymbol{\omega} \times \mathbf{r} \right)^2 + m \mathbf{V} \cdot \frac{d\mathbf{r}}{dt} + \frac{m}{2} \mathbf{V}^2 \\ &= \frac{m}{2} \left(\frac{d^* \mathbf{r}}{dt} + \boldsymbol{\omega} \times \mathbf{r} \right)^2 - m \mathbf{r} \cdot \frac{d\mathbf{V}}{dt} + \frac{d}{dt} (m \mathbf{r} \cdot \mathbf{V}) + \frac{m}{2} \mathbf{V}^2 \end{aligned} \quad (268)$$

The last two terms can be deleted from the Lagrangian, so we can take the Lagrangian in the non-inertial frame to be

$$\begin{aligned} L &= \frac{m}{2} \dot{\mathbf{r}}^2 + m \dot{\mathbf{r}} \cdot (\boldsymbol{\omega} \times \mathbf{r}) + \frac{m}{2} (\boldsymbol{\omega} \times \mathbf{r})^2 - m \mathbf{r} \cdot \mathbf{A} - V \\ &= \frac{m}{2} \dot{\mathbf{r}}^2 + m \dot{\mathbf{r}} \cdot (\boldsymbol{\omega} \times \mathbf{r}) + \frac{m}{2} (\boldsymbol{\omega}^2 \mathbf{r}^2 - (\boldsymbol{\omega} \cdot \mathbf{r})^2) - m \mathbf{r} \cdot \mathbf{A} - V \end{aligned} \quad (269)$$

where it is understood in this formula that $\dot{\mathbf{r}} \equiv d^* \mathbf{r} / dt$. Thus all of the vectors in the Lagrangian can be resolved along the rotating axes:

$$\mathbf{r}(t) = \sum_a r_a(t) \mathbf{e}_a(t), \quad \dot{\mathbf{r}}(t) = \sum_a \dot{r}_a(t) \mathbf{e}_a(t), \quad \boldsymbol{\omega} = \sum_a \omega_a(t) \mathbf{e}_a(t) \quad (270)$$

and the Lagrangian is a function of these components. Next we work out Lagrange's equations:

$$\begin{aligned} \frac{\partial L}{\partial \dot{r}_a} &= m \dot{r}_a + m \epsilon_{abc} \omega_b r_c \\ m \ddot{r}_a + m \epsilon_{abc} \dot{\omega}_b r_c + m \epsilon_{abc} \omega_b \dot{r}_c &= - \frac{\partial V}{\partial r_a} - m A_a + m \epsilon_{abc} \omega_c \dot{r}_b + m (\boldsymbol{\omega}^2 r_a - \mathbf{r} \cdot \boldsymbol{\omega} \omega_a) \end{aligned} \quad (271)$$

or in vector notation

$$m\ddot{\mathbf{r}} = -\frac{\partial V}{\partial \mathbf{r}} - m\mathbf{A} - m\dot{\boldsymbol{\omega}} \times \dot{\mathbf{r}} + 2m\dot{\mathbf{r}} \times \boldsymbol{\omega} + m\boldsymbol{\omega} \times (\mathbf{r} \times \boldsymbol{\omega}) \quad (272)$$

The fourth term on the right is called the Coriolis force and the last term is called the centrifugal force, which is directed radially outward. These are fictitious forces due to the rotation of the coordinate frame relative to an inertial frame. If the frame is uniformly rotating without translational acceleration, the second and third terms are absent.

In this case the energy (Hamiltonian) of the particle in the rotating frame is

$$E = H = m\dot{\mathbf{r}} \cdot (\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}) - L = \frac{m}{2}\dot{\mathbf{r}}^2 - \frac{m}{2}(\boldsymbol{\omega} \times \mathbf{r})^2 + V \quad (273)$$

For example the effect of earth's rotation on motion near its surface can be calculated using this simplification:

$$m\ddot{\mathbf{r}} = -\nabla V + 2m\dot{\mathbf{r}} \times \boldsymbol{\omega} + m\boldsymbol{\omega} \times (\mathbf{r} \times \boldsymbol{\omega}) \quad (274)$$

We could set up an earth fixed coordinate system with the z axis vertically up, the x -axis to the east and the y axis to the north. Then if the the angle between z and the earth's axis is θ (the latitude is $\pi/2 - \theta$), then $\omega_z = \omega \cos \theta$, $\omega_y = \omega \sin \theta$, and $\omega_x = 0$.

8 Hamiltonian Formulation of Mechanics.

By now we have seen the application of Lagrangian mechanics to many situations. The fact that all of the dynamics of a system is encoded in a single scalar Lagrangian has been particularly helpful in situations where non-Cartesian coordinates are the best way to attack the equations of motion, and especially in situations where nontrivial constraints are imposed on the system. We are now going to develop a third formulation of mechanics which is not so much useful in solving practical problems, but rather exposes in an insightful way the underlying fundamental structure of classical dynamics. In particular this new formulation reveals the clearest analogies between quantum and classical dynamics.

8.1 Hamilton's Equations

The Lagrange equations are second order in time, which means that two initial conditions, say $q_k(0), \dot{q}_k(0)$, for each degree of freedom are required to uniquely determine each solution. We can say that knowing these two quantities for each degree of freedom completely determines the state of the system. Mathematically one can always convert a second order equation to a pair of first order ones, for instance by giving a name $y_k = \dot{q}_k$. Then $\ddot{q}_k = \dot{y}_k$ so we double the number of variables to y_k, q_k , the original equation of motion becomes first order and the definition $\dot{q}_k = y_k$ is the second first order equation. But this is not the most insightful approach. From the structure of Lagrange's equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial L}{\partial q_k} \quad (275)$$

we see that it is better to choose $p_k = \partial L / \partial \dot{q}_k$ instead of y_k as the new variable. This equation can be implicitly solved for $\dot{q}(q, p, t)$. Then the Lagrange equation reads

$$\dot{p}_k = \left. \frac{\partial L}{\partial q_k} \right|_{\dot{q}} (q, \dot{q}(q, p, t), t) \quad (276)$$

here the subscript indicates that the \dot{q}_l are all fixed when the partial w.r.t. q_k is taken. But if we take q, p as variables, holding p fixed is more natural. So calculate

$$\begin{aligned} \left. \frac{\partial L}{\partial q_k} \right|_p &= \left. \frac{\partial L}{\partial q_k} \right|_{\dot{q}} + \sum_l \frac{\partial \dot{q}_l}{\partial q_k} \frac{\partial L}{\partial \dot{q}_l} = \left. \frac{\partial L}{\partial q_k} \right|_{\dot{q}} + \sum_l \frac{\partial \dot{q}_l}{\partial q_k} p_l = \left. \frac{\partial L}{\partial q_k} \right|_{\dot{q}} + \left. \frac{\partial}{\partial q_k} \right|_p \sum_l \dot{q}_l p_l \\ \left. \frac{\partial L}{\partial q_k} \right|_{\dot{q}} &= \left. \frac{\partial}{\partial q_k} \right|_p (L - \sum_l \dot{q}_l p_l) = - \left. \frac{\partial H}{\partial q_k} \right|_p \end{aligned} \quad (277)$$

This is the first of Hamilton's equations: $\dot{p}_k = -\partial H / \partial q_k$. The second Hamilton's equation comes from considering the derivative of H wrt p_k :

$$\left. \frac{\partial H}{\partial p_k} \right|_q = \dot{q}_k + \sum_l p_l \frac{\partial \dot{q}_l}{\partial p_k} - \sum_l \frac{\partial L}{\partial \dot{q}_l} \frac{\partial \dot{q}_l}{\partial p_k} = \dot{q}_k \quad (278)$$

We have arrived at Hamilton's equations:

$$p_k \equiv \frac{\partial L}{\partial \dot{q}_k}, \quad H(q, p) \equiv \sum_l \dot{q}_l p_l - L \quad (279)$$

$$\dot{p}_k = -\frac{\partial H}{\partial q_k}, \quad \dot{q}_k = \frac{\partial H}{\partial p_k} \quad (280)$$

The definitions on the top line constitute what is known mathematically as a Legendre transformation $L \rightarrow H$. This transformation may be executed more transparently as follows. Write out the differential dL in terms of its variables:

$$\begin{aligned} dL &= \sum_k dq_k \frac{\partial L}{\partial q_k} + \sum_k d\dot{q}_k \frac{\partial L}{\partial \dot{q}_k} \\ &= \sum_k dq_k \dot{p}_k + \sum_k p_k d\dot{q}_k = \sum_k dq_k \dot{p}_k - \sum_k dp_k \dot{q}_k + d \sum_k p_k \dot{q}_k \end{aligned} \quad (281)$$

$$d(L - \sum_k p_k \dot{q}_k) = -dH = \sum_k dq_k \dot{p}_k - \sum_k dp_k \dot{q}_k \quad (282)$$

You may recognize these manipulations as analogous to the relationship, in thermodynamics, of different thermodynamic potentials to each other.

The *phase space variables* $p_k(t), q_k(t)$ characterize the state of the system. p_k and q_k are canonically conjugate variables. We speak of p_k as the momentum canonically conjugate to q_k or more simply as the momentum conjugate to q_k . Hamilton's equations relate the phase space variables at time $t + dt$ to those at time t . Solving the equations then gives the state of the system at time t_2 in terms of the state of the system at time t_1 .

We have already encountered the Hamiltonian H as the energy of a dynamical system. We also recall that energy is conserved if the Lagrangian does not depend explicitly upon the time: $\dot{H} = -\partial L / \partial t$. Let us calculate \dot{H} using Hamilton's equations:

$$\frac{dH}{dt} = \sum_k \dot{p}_k \frac{\partial H}{\partial p_k} + \sum_k \dot{q}_k \frac{\partial H}{\partial q_k} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} \quad (283)$$

So H is conserved if H does not depend explicitly on the time. Comparing the two conclusions we see that

$$\left. \frac{\partial H}{\partial t} \right|_{p,q} = - \left. \frac{\partial L}{\partial t} \right|_{\dot{q},q} \quad (284)$$

Notice that the partial time derivatives on either side hold different sets of variables fixed!

In fact a relation of the last type is true of the derivative wrt any parameter appearing in the dynamics by the nature of the Legendre transformation:

$$\left. \frac{\partial H}{\partial \lambda} \right|_{p,q} = \sum_k p_k \frac{\partial \dot{q}_k}{\partial \lambda} - \sum_k \frac{\partial L}{\partial \dot{q}_k} \frac{\partial \dot{q}_k}{\partial \lambda} - \left. \frac{\partial L}{\partial \lambda} \right|_{\dot{q},q} = - \left. \frac{\partial L}{\partial \lambda} \right|_{\dot{q},q} \quad (285)$$

As an example let us take the Lagrangian for a particle moving in an electromagnetic field

$$L = \frac{m}{2}\dot{\mathbf{r}}^2 + Q\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t) - Q\phi(\mathbf{r}, t) \quad (286)$$

$$\mathbf{p} = m\dot{\mathbf{r}} + Q\mathbf{A}, \quad H = \frac{(\mathbf{p} - Q\mathbf{A})^2}{2m} + Q\phi \quad (287)$$

Now let's directly calculate

$$\begin{aligned} \left. \frac{\partial L}{\partial Q} \right|_{\mathbf{r}, \dot{\mathbf{r}}} &= \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t) - \phi(\mathbf{r}, t) = \mathbf{A} \cdot \frac{\mathbf{p} - Q\mathbf{A}}{m} - \phi \\ \left. \frac{\partial H}{\partial Q} \right|_{\mathbf{r}, \mathbf{p}} &= -\mathbf{A} \cdot \frac{\mathbf{p} - Q\mathbf{A}}{m} + \phi \end{aligned} \quad (288)$$

Which confirms the equality. Notice however that because there is q dependence in the relation of \mathbf{p} to $\dot{\mathbf{r}}$, H depends quadratically on q whereas L depends linearly on q . This means that the second derivatives differ:

$$\left. \frac{\partial^2 L}{\partial Q^2} \right|_{\mathbf{r}, \dot{\mathbf{r}}} = 0, \quad \left. \frac{\partial^2 H}{\partial Q^2} \right|_{\mathbf{r}, \mathbf{p}} = \frac{\mathbf{A}^2}{m} \quad (289)$$

As another example consider the Lagrangian of a relativistic particle

$$L = -mc^2\sqrt{1 - \dot{\mathbf{r}}^2/c^2}, \quad \mathbf{p} = m\frac{\dot{\mathbf{r}}}{\sqrt{1 - \dot{\mathbf{r}}^2/c^2}}, \quad H = \sqrt{\mathbf{p}^2c^2 + m^2c^4} \quad (290)$$

$$\left. \frac{\partial L}{\partial m} \right|_{\mathbf{r}, \dot{\mathbf{r}}} = -c^2\sqrt{1 - \dot{\mathbf{r}}^2/c^2} = -\frac{c^2}{\gamma} = -\frac{mc^4}{H}, \quad \left. \frac{\partial H}{\partial m} \right|_{\mathbf{r}, \mathbf{p}} = \frac{mc^4}{\sqrt{\mathbf{p}^2c^2 + m^2c^4}} = \frac{mc^4}{H} \quad (291)$$

8.2 Cyclic variables in the Hamiltonian formulation

With in the Lagrangian description we have learned that if the Lagrangian does not depend on a coordinate q , then the momentum conjugate to q is conserved. The Hamiltonian derived from such a Lagrangian is also independent of q , so the first Hamilton equation immediately says $\dot{p} = 0$. In this situation we can eliminate the degree of freedom q by simply substituting the desired value of p in the Hamiltonian and treating it as simply a parameter in the dynamics of the remaining degrees of freedom.

This is an improvement over the Lagrangian approach, where errors would be introduced by substituting the information from the values of a conserved quantity in the Lagrangian. Recall the example of motion of a particle in a central potential, where we restrict the motion to the xy -plane:

$$\begin{aligned} L &= \frac{m}{2}\dot{r}^2 + r^2\dot{\varphi}^2 - V(r) \\ p_r &= m\dot{r}, \quad p_\varphi = mr^2\dot{\varphi} \equiv J, \quad H = \frac{p_r^2}{2m} + \frac{J^2}{2mr^2} + V(r) = \frac{p_r^2}{2m} + V_{\text{eff}}(r) \end{aligned} \quad (292)$$

Treating J as a fixed parameter, we can now safely use the Hamiltonian to correctly give the dynamics of the single coordinate $r(t)$.

Routh has introduced a procedure that performs the Legendre transformation only with respect to the cyclic coordinates, leaving the remaining coordinates in Lagrange formulation. In the case just discussed the Routhian would be

$$R = \dot{\varphi}p_\varphi - L = V_{\text{eff}}(r) - \frac{m}{2}\dot{r}^2 \quad (293)$$

This is a correct way to eliminate cyclic variables from the Lagrangian without going to the full Hamiltonian formalism, but the advantages of doing only this are limited, and we will not dwell on it. We lose relatively little by simply going to the complete Hamiltonian formulation.

8.3 Hamilton's principle in Hamilton's formulation of mechanics

We have started with the Lagrangian and transformed to the Hamiltonian. But we can reverse this procedure, defining the Lagrangian by

$$L = \sum_k \dot{q}_k p_k - H(p, q, t) \quad (294)$$

and defining the action in phase space:

$$I = \int_{t_1}^{t_2} dt \left(\sum_k \dot{q}_k p_k - H(p, q, t) \right) \quad (295)$$

We can now state Hamilton's principle in phase space: The equations of motion are obtained by finding $q_k(t), p_k(t)$ which make the action stationary under infinitesimal changes $\delta q_k(t), \delta p_k(t)$ which satisfy $\delta q_k(t_1) = \delta q_k(t_2) = 0$. Notice that there need be no such conditions put on the δp_k .

$$\begin{aligned} \delta I &= \int_{t_1}^{t_2} dt \sum_k \left(\dot{q}_k - \frac{\partial H}{\partial p_k} \right) \delta p_k + \frac{d}{dt} \left(\sum_k \delta q_k p_k \right) - \sum_k \delta q_k \left(\dot{p}_k - \frac{\partial H}{\partial q_k} \right) \\ &= \left(\sum_k \delta q_k p_k \right) \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt \sum_k \left(\dot{q}_k - \frac{\partial H}{\partial p_k} \right) \delta p_k - \sum_k \delta q_k \left(\dot{p}_k + \frac{\partial H}{\partial q_k} \right) \\ &= \int_{t_1}^{t_2} dt \sum_k \left(\dot{q}_k - \frac{\partial H}{\partial p_k} \right) \delta p_k - \sum_k \delta q_k \left(\dot{p}_k + \frac{\partial H}{\partial q_k} \right) \end{aligned} \quad (296)$$

Clearly $\delta I = 0$ for arbitrary variations only if Hamilton's equations hold.

Because \dot{p}_k do not appear in I it would be legitimate to use the equation from the δp_k variation

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \quad (297)$$

to determine the p_k 's as functions of the q_k 's and the \dot{q}_k 's and eliminate them from the action principle—this returns us to the original Lagrange form of Hamilton's principle.

8.4 Poisson Brackets

Hamilton's equations instruct us how to calculate the time derivatives of the canonical phase space variables q, p . But any physical quantity is a function of the phase space variables $f(q, p, t)$ and we can easily evaluate its time derivative as well

$$\begin{aligned}\frac{df}{dt} &= \sum_k \dot{q}_k \frac{\partial f}{\partial q_k} + \sum_k \dot{p}_k \frac{\partial f}{\partial p_k} + \frac{\partial f}{\partial t} \\ &= \sum_k \frac{\partial H}{\partial p_k} \frac{\partial f}{\partial q_k} - \sum_k \frac{\partial H}{\partial q_k} \frac{\partial f}{\partial p_k} + \frac{\partial f}{\partial t} \equiv \{f, H\} + \frac{\partial f}{\partial t}\end{aligned}\quad (298)$$

after using Hamilton's equations. The terms involving f and H on the right of the last equation are a fundamental new quantity in the canonical formalism called the Poisson bracket $\{f, H\}$. It is defined for any two functions $f(q, p, t)$, $g(q, p, t)$ as follows

$$\{f, g\} \equiv \sum_k \left(\frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k} \right) \quad (299)$$

If a conserved quantity has no explicit time dependence its Poisson bracket with the Hamiltonian is zero. For the case of the canonical variables themselves they reduce to

$$\{q_k, q_l\} = \{p_k, p_l\} = 0, \quad \{q_k, p_l\} = \delta_{kl} \quad (300)$$

Poisson brackets satisfy a number of important properties, most of which are immediate consequences of their definitions:

$$\{f, g\} = -\{g, f\}, \quad \{f, gh\} = \{f, g\}h + g\{f, h\} \quad (301)$$

the second equation is sort of a Leibnitz rule. The Poisson bracket is linear in either of its entries $\{c_1 f_1 + c_2 f_2, g\} = c_1 \{f_1, g\} + c_2 \{f_2, g\}$.

A more subtle property is the Jacobi identity involving double Poisson brackets:

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0. \quad (302)$$

It can be proved by a straightforward but tedious slog. To appreciate its importance notice that the Poisson bracket of two functions of phase space is itself a function of phase space. So we can consider its time derivative

$$\begin{aligned}\frac{d}{dt}\{f, g\} &= \{\{f, g\}, H\} + \frac{\partial}{\partial t}\{f, g\} \\ &= -\{\{H, f\}, g\} - \{\{g, H\}, f\} + \left\{ \frac{\partial f}{\partial t}, g \right\} + \left\{ f, \frac{\partial g}{\partial t} \right\} \\ &= \left\{ \frac{\partial f}{\partial t} + \{f, H\}, g \right\} + \left\{ f, \frac{\partial g}{\partial t} + \{g, H\} \right\} = \left\{ \frac{df}{dt}, g \right\} + \left\{ f, \frac{dg}{dt} \right\}\end{aligned}\quad (303)$$

The Jacobi identity was used to get from the first line to the second line! This is a powerful statement: the time derivative of the Poisson bracket of two quantities is related by a Leibnitz rule to the Poisson brackets of each quantity with the time derivative of the other. As a particular case, suppose that f and g are two conserved quantities. Then it follows that $\{f, g\}$ is also a conserved quantity. (Poisson's theorem).

8.5 Canonical Transformations

The Lagrangian formulation of mechanics takes the same form for all choices of generalized coordinates: redefining $q_k \rightarrow Q_k(q, t)$ leaves Lagrange's equations invariant in form. In the Hamiltonian formulation we can ask a similar question: If we change canonical variables to $P_k(q, p, t), Q_k(q, p, t)$, are Hamilton's equations invariant in form? The answer is yes if the transformation is canonical.

We define *canonical transformations* to be those that leave Poisson brackets invariant:

$$\{f, g\}_{Q,P} = \{f, g\}_{q,p}, \quad \text{Canonical Transform.} \quad (304)$$

Note that the definition of canonical transformations does not refer in any way to the Hamiltonian of the system. We shall see that they nonetheless leave Hamilton's equations invariant. Working in q, p coordinates we can calculate

$$\dot{Q}_k = \{Q_k, H\}_{p,q} + \frac{\partial Q_k}{\partial t} = \{Q_k, H\}_{P,Q} + \frac{\partial Q_k}{\partial t} = \frac{\partial H}{\partial P_k} + \frac{\partial Q_k}{\partial t} \quad (305)$$

$$\dot{P}_k = \{P_k, H\}_{p,q} + \frac{\partial P_k}{\partial t} = \{P_k, H\}_{P,Q} + \frac{\partial P_k}{\partial t} = -\frac{\partial H}{\partial Q_k} + \frac{\partial P_k}{\partial t} \quad (306)$$

Evidently if the canonical transformations are time independent, it is immediate that the form of Hamilton's equations is invariant, with the same Hamiltonian (expressed in different coordinates) for both sets of coordinates. We shall shortly see that if the canonical transformations are time dependent Hamilton's equations will be preserved provided a change is allowed in the Hamiltonian. At least for time independent transforms the converse is true: If Hamilton's equations are invariant in form under a time independent transformation for a generic Hamiltonian, the transformation is canonical. Indeed invariance requires that the particular Poisson brackets $\{f, H\}$ be invariant. But if this is required for any Hamiltonian H , all Poisson brackets must be invariant.

To investigate canonical transformations further we appeal to Hamilton's principle, which implies Hamilton's equations. For the two coordinates systems to yield the same equations of motion, the two Lagrangians should differ by a total time derivative:

$$\begin{aligned} \sum_k \dot{q}_k p_k - H &= \sum_k \dot{Q}_k P_k - \bar{H} + \frac{dF}{dt} \\ dF &= \sum_k p_k dq_k - \sum_k P_k dQ_k + (\bar{H} - H)dt \end{aligned} \quad (307)$$

This condition will hold if the transformation is such that $F(q, Q, t)$ and

$$p_k = \frac{\partial F}{\partial q_k}, \quad P_k = -\frac{\partial F}{\partial Q_k}, \quad \bar{H} = H + \frac{\partial F}{\partial t} \quad (308)$$

Now if F does not depend explicitly on time, the first equation determines $Q_k(q, p)$ and then the second determines $P_k(q, p)$. Since Hamilton's equations will be invariant under this

transformation for any H it follows that any such transformation preserves Poisson brackets and hence is canonical.

But since the Poisson brackets are defined in terms of derivatives with respect to q, p but not with respect to t , the transformation generated by a time dependent $F(q, Q, t)$ will also be canonical. In that case the above argument shows that Hamilton's equations will still be valid if one takes a new Hamiltonian $\bar{H} = H + \partial F/\partial t$. The function $F(q, Q, t)$ is called the *generating function* of the canonical transformation.

In general for a system with s degrees of freedom a generating function depends on s old variables s new variables and time. Of the s old or new variables only one is selected from each conjugate pair. In the case just discussed the variables are chosen to be the old and new coordinates. We can just as well choose the old and new momenta, the old coordinates and new momenta, of the old momenta and new coordinates, or indeed any hybrid mixture we wish. We illustrate the four main choices obtained from $F(q, Q, t)$ via Legendre transformations.

$$F_1(q, Q, t) = F : \quad p_k = \frac{\partial F_1}{\partial q_k}, \quad P_k = -\frac{\partial F_1}{\partial Q_k}, \quad \bar{H} = H + \frac{\partial F_1}{\partial t} \quad (309)$$

$$F_2(q, P, t) = F + QP : \quad p_k = \frac{\partial F_2}{\partial q_k}, \quad Q_k = \frac{\partial F_2}{\partial P_k}, \quad \bar{H} = H + \frac{\partial F_2}{\partial t} \quad (310)$$

$$F_3(p, Q, t) = F - qp : \quad q_k = -\frac{\partial F_3}{\partial p_k}, \quad P_k = -\frac{\partial F_3}{\partial Q_k}, \quad \bar{H} = H + \frac{\partial F_3}{\partial t} \quad (311)$$

$$F_4(p, P, t) = F_2 - qp : \quad q_k = -\frac{\partial F_4}{\partial p_k}, \quad Q_k = \frac{\partial F_4}{\partial P_k}, \quad \bar{H} = H + \frac{\partial F_4}{\partial t} \quad (312)$$

Let's consider some examples. First the generating function for the identity is $F_2 = \sum_k q_k P_k$. More generally consider

$$F_2 = \sum_{kl} q_k M_{kl} P_l : \quad p_k = M_{kl} P_l, \quad Q_l = q_k M_{kl} \quad (313)$$

$$Q_k = M_{kl}^T q_l, \quad P_k = M_{kl}^{-1} p_l \quad (314)$$

If $M = R^T$ is an orthogonal matrix, $RR^T = I$, this canonical transformation is just a rotation $Q = Rq, P = Rp$.

As another important example, consider an infinitesimal canonical transformation $F_2 = \sum_k q_k P_k + \epsilon G(q, P, t)$:

$$Q_k = q_k + \epsilon \frac{\partial G}{\partial P_k}, \quad p_k = P_k + \epsilon \frac{\partial G}{\partial q_k} \quad (315)$$

$$\delta p_k = P_k - p_k = -\epsilon \frac{\partial G}{\partial q_k} \rightarrow \epsilon \{p_k, G\}$$

$$\delta q_k = Q_k - q_k = \epsilon \frac{\partial G}{\partial P_k} \rightarrow \epsilon \{q_k, G\} \quad (316)$$

Where on the right of the last two equations, we have set $P_k = p_k$ in G and identified the derivatives with Poisson brackets. In this way we see that the infinitesimal generator

ϵG induces infinitesimal canonical transformations via Poisson brackets. We can reach a finite canonical transformation by a sequence of infinitesimal transformations by solving the differential equation

$$\frac{d\Omega}{d\epsilon} = \{\Omega, G\} \quad (317)$$

for any function $\Omega(q, p)$. We recognize this as a Hamilton-type equation where the parameter ϵ plays the role of time. Indeed, we can interpret Hamilton's equations themselves as a canonical transformation obtained by integrating infinitesimal canonical transformations whose infinitesimal generator is the Hamiltonian!

In quantum mechanics the analog of a canonical transformation is a unitary transformation $U = e^{-i\epsilon G/\hbar}$ where G is the infinitesimal generator. Quantum operators transform as follows

$$\Omega(\epsilon) = U^\dagger \Omega U, \quad \frac{d\Omega}{d\epsilon} = \frac{1}{i\hbar} U^\dagger [\Omega, G] U = \frac{1}{i\hbar} [\Omega(\epsilon), G] \quad (318)$$

Here we see clearly the parallel between canonical transformations in classical mechanics and unitary transformations in quantum mechanics in which *P.B.* \leftrightarrow $(-i/\hbar)$ commutator,

8.6 Hamilton-Jacobi Theory

We have learned that a finite canonical transformation can be built up as a concatenation of infinitesimal canonical transformations by solving a differential equation. This diff eq is identical in form to Hamilton's equations for q_k, p_k or indeed any function $f(q, p)$ without explicit time dependence

$$\frac{df}{dt} = \{f, H\}. \quad (319)$$

In other words the solution of Hamilton's equations can be regarded as an evolving canonical transformation. Specifically the transformation $q_k(0), p_k(0) \rightarrow q_k(t), p_k(t)$ is a canonical transformation. If we can find the generating function of this canonical transformation, we have another route to the solution of the equations of motion. Let us regard the initial coordinates as the new canonical variables $q_k(0) = Q_k, p_k(0) = P_k$ and $q_k(t) = q_k, p_k(t) = p_k$ as the old variables. Since the initial conditions are constants of the motion, the Hamiltonian governing the new variables should be independent of the new variables, so we seek a canonical transformation such that the new Hamiltonian $\bar{H} = 0$. From $\bar{H} = H + \partial F/\partial t$ this means that

$$\frac{\partial F}{\partial t} + H(p, q, t) = 0 \quad (320)$$

Let us take an F_2 style generating function $F_2(q, P, t) \equiv S(q, P, t)$. then to interpret this equation we must eliminate $p_k = \partial F_2/\partial q_k$ from the Hamiltonian:

$$\frac{\partial F_2}{\partial t} + H\left(\frac{\partial F_2}{\partial q}, q, t\right) = 0 \quad (321)$$

This is the Hamilton-Jacobi equation that determines the generating function for the canonical transformation that maps the variables at time t to their values at $t = 0$. The solution of this equation is called Hamilton's principal function and is customarily denoted by the letter $S(q, t)$.

$$\frac{\partial S(q, t)}{\partial t} = -H\left(\frac{\partial S}{\partial q}, q, t\right) \quad \text{Hamilton - Jacobi} \quad (322)$$

To completely determine S we need an initial condition. For example we might want the canonical transformation to be simply the identity at $t = 0$. Then in the $F_2(q, P, t)$ form of generating function we would specify $F_2(q, 0) = \sum_k q_k P_k$. If we solve don't specify the initial condition then the solution would generate a canonical transformation which includes an initial redefinition of variables.

As a simple example, consider a free particle moving in one dimension $H = p^2/2m$. Then

$$\frac{\partial S(q, t)}{\partial t} = -\frac{1}{2m} \left(\frac{\partial S}{\partial x}\right)^2 \quad (323)$$

We can solve this equation by putting $S = S_0(x) - Et$ where $\partial S/\partial x = \sqrt{2mE}$, so $S(x, t) = x\sqrt{2mE} - Et = xP - \frac{P^2}{2m}t$. Here we have identified the initial momentum $P = \sqrt{2mE}$ where E is the conserved energy. We can find another solution in the form $S = m(x - X)^2/2t$ because then

$$\frac{\partial S}{\partial t} = -m\frac{(x - X)^2}{2t^2}, \quad \frac{\partial S}{\partial x} = m\frac{x - X}{t} \quad (324)$$

The two solutions we have found are

$$S_1 = m\frac{(x - X)^2}{2t}, \quad S_2 = xP - \frac{P^2}{2m}t \quad (325)$$

They in fact generate the same canonical transformations

$$\begin{aligned} p &= \frac{\partial S_1}{\partial x} = m\frac{x - X}{t}, & P &= -\frac{\partial S_1}{\partial X} = m\frac{x - X}{t}, & x &= X + \frac{P}{m}t, & p &= P \\ p &= \frac{\partial S_2}{\partial x} = P, & X &= \frac{\partial S_2}{\partial P} = x - \frac{P}{m}t, & x &= X + \frac{P}{m}t \end{aligned} \quad (326)$$

In fact S_2 is just the Legendre transform of S_1 :

$$P = -\frac{\partial S_1}{\partial X} = m\frac{x - X}{t}, \quad S_2 = XP + S_1 = P\left(-\frac{Pt}{m} + x\right) + \frac{P^2}{2m}t = xP - \frac{P^2}{2m}t$$

As a less trivial example let us apply Hamilton-Jacobi to the harmonic oscillator $H = p^2/2m + m\omega^2 x^2/2$

$$\frac{\partial S}{\partial t} = -\frac{1}{2m} \left(\frac{\partial S}{\partial x}\right)^2 - m\omega^2 x^2/2 \quad (327)$$

Again since there is no explicit time dependence we put $S = S_0 - Et$ and solve

$$\begin{aligned}\frac{\partial S_0}{\partial x} &= \sqrt{2mE - m^2\omega^2x^2} \\ S(x, t) &= \int_0^x dx' \sqrt{2mE - m^2\omega^2x'^2} - Et\end{aligned}\quad (328)$$

The integral may be done with the change of variables $x' = \sqrt{2E/m\omega^2} \sin \theta$

$$S(x, t) = \frac{2E}{\omega} \int d\theta \cos^2 \theta = \frac{E}{\omega} \sin^{-1} \frac{\omega x}{\sqrt{2E/m}} + \frac{x}{2} \sqrt{2mE - m^2\omega^2x^2} - Et \quad (329)$$

to interpret S as the generating function of the canonical transformation $x(t), p(t) \rightarrow Q, P$ with $\bar{H} = 0$, we identify $P = E$, so

$$\begin{aligned}Q &= \frac{\partial S}{\partial E} = \frac{1}{\omega} \sin^{-1} \frac{\omega x}{\sqrt{2E/m}} - t \\ x(t) &= \frac{1}{\omega} \sqrt{\frac{2E}{m}} \sin \omega(t + Q), \quad p(t) = \sqrt{2mE - m^2\omega^2x^2} = \sqrt{2mE} \cos \omega(t + Q)\end{aligned}\quad (330)$$

Notice that the canonical transformation at $t = 0$ is *not* the identity:

$$x(0) = \frac{1}{\omega} \sqrt{\frac{2P}{m}} \sin \omega Q, \quad p(0) = \sqrt{2mP} \cos \omega Q \quad (331)$$

Indeed, with hindsight, we could have done this latter canonical transformation at each time to find

$$x(t) = \frac{1}{\omega} \sqrt{\frac{2P(t)}{m}} \sin \omega Q(t), \quad p(t) = \sqrt{2mP(t)} \cos \omega Q(t), \quad \bar{H} = H = P(t) \quad (332)$$

Then Hamilton's equations would tell us $\dot{P} = 0$, $\dot{Q} = 1$.

Hamilton's principal function from the action. Consider the action $\int_{t_1}^{t_2} dt L$ evaluated for the solution of Lagrange's equations satisfying the conditions $q_k(t_1) = q_{k1}$ and $q_2(t_2) = q_{k2}$. Call the result $S(q_2, t_2; q_1, t_1)$ which is a function of the initial and final coordinates and times. Now calculate the change in S that ensues if we change $q_2 \rightarrow q_2 + \delta q_2$. this means we will need to find a slightly different solution of Lagrange's equations $q_k(t) + \delta q_k(t)$ satisfying $\delta q_k(t_1) = 0$ and $\delta q_k(t_2) = \delta q_{k2}$. Then

$$\delta S = \int_{t_1}^{t_2} dt \sum_k \left(\delta \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} + \delta q_k \frac{\partial L}{\partial q_k} \right) = \sum_k \delta q_{k2} \left. \frac{\partial L}{\partial \dot{q}_k} \right|_{t=t_2} \quad (333)$$

after an integration by parts and use of Lagrange's equations. Thus $\partial S / \partial q_{k2} = p_k(t_2)$. Similarly considering a change in q_{k1} shows that $\partial S / \partial q_{k1} = -p_k(t_1)$. These equations confirm

that S is the generating function for the canonical transformation that maps $q_k(t_2), p_k(t_2)$ to $q_k(t_1), p_k(t_1)$. Furthermore we can calculate the change in S under $t_2 \rightarrow t_2 + \delta t_2$:

$$\delta S = \delta t_2 L + \sum_k \delta q_k(t_2) p_k(t_2) = \delta t_2 \left(L - \sum_k \dot{q}_k(t_2) p_k(t_2) \right) = -\delta t_2 H(t_2) \quad (334)$$

where we used the requirement that $q_k(t_2) = q_k(t_2 + \delta t_2) + \delta q_k(t_2)$ to infer that $\delta q_k(t_2) = -\dot{q}_k(t_2) \delta t_2$. Thus $\partial S / \partial t_2 = -H(q(t_2), p(t_2))$, i.e. H satisfies the Hamilton-Jacobi equation! In summary,

$$\frac{\partial S}{\partial q_2} = p_2, \quad \frac{\partial S}{\partial q_1} = -p_1, \quad \frac{\partial S}{\partial t_2} = -H(t_2), \quad \frac{\partial S}{\partial t_1} = H(t_1) \quad (335)$$

As an example lets take a free particle for which the solution is $x(t) = X + (x - X)t / (t_2 - t_1)$. Then

$$S = \frac{m}{2} \int_{t_1}^{t_2} dt \dot{x}^2 = \frac{m}{2} \frac{(x - X)^2}{t_2 - t_1} \quad (336)$$

which, with $t = t_2 - t_1$ is the second form of Hamilton's principal function discussed above.

8.7 Separation of Variables

When we simply consider the Hamilton-Jacobi equation, it is not necessary to impose a specific initial condition, but it is necessary to find a solution that depends on a number of independent constants equal to the number of degrees of freedom. If the Hamiltonian does not depend explicitly on the time, One can always separate the time dependence of $S(q, t) = S_0(q) - Et$ so that the H-J equation reduces to

$$E = H(q, \nabla_q S_0) \rightarrow \sum_k \frac{1}{2m_k} \left(\frac{\partial S_0}{\partial q_k} \right)^2 + V(q) \quad (337)$$

when the dynamics is described by a potential. If there are any cyclic coordinates—those that don't appear in the Hamiltonian, they can be immediately separated in the same way time was separated: by writing $S_0 = S'_0 + \sum_{k=\text{cyclic}} q_k p_k$ where p_k is the constant conjugate momentum and S'_0 is independent of all the cyclic q_k 's. Then we have

$$E - \sum_{k=\text{cyclic}} \frac{p_k^2}{2m_k} = \sum_l \frac{1}{2m_l} \left(\frac{\partial S_0}{\partial q_l} \right)^2 + V(q) \quad (338)$$

The simplest example of this is a particle moving in one dimension under a potential energy with no explicit time dependence. Then we can seek a solution in the form $S(q, t) = S_0(q) - Et$ so that

$$E = \frac{1}{2m} \left(\frac{\partial S_0}{\partial q} \right)^2 + V(q), \quad S_0(q) = \int_0^q dq' \sqrt{2m(E - V(q'))} + S_0(0) \quad (339)$$

In this case E is the necessary parameter. ($S_0(0)$ does not count because it has no influence on the generated canonical transformation. Then

$$S = S_0(q, E) - Et \quad (340)$$

can be interpreted as the generating function of the canonical transformation from $q(t), p(t)$ to $Q, P = E$, which are time independent. E is the total energy and

$$Q = \frac{\partial S}{\partial E} = \frac{\partial S_0}{\partial E} - t \quad (341)$$

The statement that Q is independent of time then becomes

$$\sqrt{\frac{m}{2}} \int_0^q dq' \frac{1}{\sqrt{E - V(q')}} = t + \text{Constant} \quad (342)$$

which is seen to be the standard solution by quadratures.

Notice that $S_0(q, E \equiv P)$ may be regarded in its turn as the generating function of a time independent canonical transformation

$$p = \sqrt{2m(P - V(q))}, \quad Q = \frac{\partial S_0}{\partial P} = \sqrt{\frac{m}{2}} \int_0^q \frac{dq'}{\sqrt{P - V(q')}} \quad (343)$$

In this case the new Hamiltonian is $\bar{H} = H = P$. And $Q(t)$ is no longer a constant but satisfies $\dot{Q} = 1$ so that $Q = t + \text{constant}$.

With more degrees of freedom it becomes increasingly more difficult to find the sufficiently general solution, except in the case where separation of variables is possible. This happens, for example, when the potential is the sum of terms that each depend on one variable.

An important example is a 3 dimensional central potential $V(r)$. Then in spherical coordinates

$$2mE = \left(\frac{\partial S_0}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial S_0}{\partial \theta}\right)^2 + \frac{1}{r^2 \sin^2 \theta} \left(\frac{\partial S_0}{\partial \varphi}\right)^2 + 2mV(r) \quad (344)$$

Then we can try a solution $S_0 = R(r) + \Theta(\theta) + \varphi J_\varphi$. Then we get a solution if

$$\left(\frac{d\Theta}{d\theta}\right)^2 + \frac{J_\varphi^2}{\sin^2 \theta} = J^2 \quad (345)$$

$$\left(\frac{dR}{dr}\right)^2 + \frac{J^2}{r^2} = 2m(E - V(r)) \quad (346)$$

This is sufficiently general because the three constants E, J_φ, J can be taken as the 3 new

momenta. Solving these equations

$$\begin{aligned}
R(r) &= \int_0^r dr' \sqrt{2mE - 2mV(r') - J^2/r'^2} \\
\Theta(\theta) &= \int_{\sin^{-1}(J_\varphi/J)}^\theta d\theta' \sqrt{J^2 - J_\varphi^2/\sin^2 \theta'} \\
&= \left(J \tan^{-1} \frac{\sqrt{J^2 \sin^2 \theta - J_\varphi^2}}{J \cos \theta} - J_\varphi \tan^{-1} \frac{\sqrt{J^2 \sin^2 \theta - J_\varphi^2}}{J_\varphi \cos \theta} \right) \\
Q_E &= \sqrt{\frac{m}{2}} \int_0^r \frac{dr'}{\sqrt{E - V(r') - J^2/2mr'^2}} - t \\
Q_J &= \int_0^r \frac{-J dr'}{r'^2 \sqrt{2mE - 2mV(r') - J^2/r'^2}} + \int_{\sin^{-1}(J_\varphi/J)}^\theta d\theta' \frac{J}{\sqrt{J^2 - J_\varphi^2/\sin^2 \theta'}} \\
Q_{J_\varphi} &= \varphi - \int_{\sin^{-1}(J_\varphi/J)}^\theta d\theta' \frac{J_\varphi/\sin^2 \theta'}{\sqrt{J^2 - J_\varphi^2/\sin^2 \theta'}} \tag{347}
\end{aligned}$$

Note that from the explicit formula for Θ we can evaluate the limit $\theta \rightarrow \pi/2$ to get $\Theta(\pi/2) = \pi(J - J_\varphi)/2$. This will be useful later when we discuss action-angle variables.

To put the plane of the orbit at $\theta = \pi/2$, requires $J_\varphi \rightarrow J$. In this limit the last equation shows that the second term in the fourth equation can be replaced by $\varphi - Q_{J_\varphi}$, so that equation just reproduces our old result for the orbit. The third equation then gives the old result for t as a function of r .

Finally let's consider the free symmetrical top

$$\begin{aligned}
L &= \frac{I_1}{2}(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta) + \frac{I_3}{2}(\dot{\psi} + \dot{\varphi} \cos \theta)^2 \\
p_\theta &= I_1 \dot{\theta}, \quad p_\varphi = I_1 \dot{\varphi} \sin^2 \theta + I_3 \cos \theta (\dot{\psi} + \dot{\varphi} \cos \theta), \quad p_\psi = I_3 (\dot{\psi} + \dot{\varphi} \cos \theta) \\
H &= \frac{p_\theta^2}{2I_1} + \frac{p_\psi^2}{2I_3} + \frac{(p_\varphi - p_\psi \cos \theta)^2}{2I_1 \sin^2 \theta} \tag{348}
\end{aligned}$$

Since φ and ψ are cyclic variables the H-J equation will separate with the ansatz $S_0 = \varphi p_\varphi + \psi p_\psi + \Theta(\theta)$

$$\begin{aligned}
\Theta'^2 &= 2I_1 E - \frac{I_1 p_\psi^2}{I_3} - \frac{(p_\varphi - p_\psi \cos \theta)^2}{\sin^2 \theta} \\
\Theta(\theta) &= \int^\theta d\theta' \sqrt{2I_1 E - \frac{I_1 p_\psi^2}{I_3} - \frac{(p_\varphi - p_\psi \cos \theta')^2}{\sin^2 \theta'}} \tag{349}
\end{aligned}$$

8.8 The Jacobian of a Canonical Transform: Liouville's Theorem

The volume of phase space is invariant under a canonical transformation. for a single degree of freedom this can be seen by a direct calculation of the Jacobian

$$\det \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix} = \frac{\partial Q}{\partial q} \frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p} \frac{\partial P}{\partial q} = \{Q, P\}_{p,q} = 1 \quad (350)$$

With many degrees of freedom one can change variables in two steps so that the Jacobian is a product:

$$\frac{\partial(Q_1 \cdots Q_s, P_1 \cdots P_s)}{\partial(q_1 \cdots q_s, p_1 \cdots p_s)} = \frac{\partial(Q_1 \cdots Q_s, P_1 \cdots P_s)}{\partial(q_1 \cdots q_s, P_1 \cdots P_s)} \frac{\partial(q_1 \cdots q_s, P_1 \cdots P_s)}{\partial(q_1 \cdots q_s, p_1 \cdots p_s)} \quad (351)$$

The second factor on the right describes the transformation of the old p 's to the new P 's holding the old q 's fixed, and the first factor describes the subsequent transformation of the old q 's to the new Q 's holding the new P 's fixed. In each of these two factors the variables held fixed can simply be deleted from the Jacobian since they are unaltered:

$$\frac{\partial(Q_1 \cdots Q_s, P_1 \cdots P_s)}{\partial(q_1 \cdots q_s, p_1 \cdots p_s)} = \frac{\partial(Q_1 \cdots Q_s)}{\partial(q_1 \cdots q_s)} \frac{\partial(P_1 \cdots P_s)}{\partial(p_1 \cdots p_s)} \quad (352)$$

Now describe the canonical transformation with a generating function $F_2(q, P)$. Then

$$\frac{\partial(Q_1 \cdots Q_s)}{\partial(q_1 \cdots q_s)} = \det \left(\frac{\partial^2 F_2}{\partial q \partial P} \right), \quad \frac{\partial(p_1 \cdots p_s)}{\partial(P_1 \cdots P_s)} = \det \left(\frac{\partial^2 F_2}{\partial P \partial q} \right) \quad (353)$$

The two determinants are of matrices that are simply transposes of each other and are therefore equal to each other. Thus

$$\frac{\partial(Q_1 \cdots Q_s, P_1 \cdots P_s)}{\partial(q_1 \cdots q_s, p_1 \cdots p_s)} = \frac{\partial(Q_1 \cdots Q_s)}{\partial(q_1 \cdots q_s)} \left(\frac{\partial(p_1 \cdots p_s)}{\partial(P_1 \cdots P_s)} \right)^{-1} = 1 \quad (354)$$

What is known as Liouville's theorem is that if you follow the time evolution of a region of phase space, each point of which moves according to Hamilton's equations, the region can move and change its shape, but always in such a way that its volume is constant. This follows from the invariance of the volume of phase space under canonical transformations and the fact that the time evolution of a point of phase space is a canonical transformation.

8.9 Action-Angle Variables

When the motion of a system stays within a finite region of phase space, the separation of variables in the Hamilton-Jacobi equation, can be put into a convenient and canonical form.

Assuming complete separability, coordinates can be chosen so that Hamilton's principal function can be written

$$S(q, t) = S_0 - Et = \sum_k S_k(q_k) - Et, \quad p_k = \frac{\partial S}{\partial q_k} = \frac{dS_k}{dq_k} \quad (355)$$

where each term $S_k = \int^{q_k} p_k(q'_k) dq'_k$ depends on only one variable. If the motion stays finite, each q_k will go through repeated cycles, either returning to its original value, or if q_k is an angular variable increasing by a fixed amount each cycle, e.g. $\varphi \rightarrow \varphi + 2\pi$. In this case we can define canonical action variables I_k

$$I_k \equiv \oint \frac{p_k dq_k}{2\pi} \quad (356)$$

where \oint means an integration over one cycle. Notice that from the definition of I_k , the change in S when q_k goes through one cycle, with the other variables held fixed is $\Delta S = \Delta S_k = 2\pi I_k$.

One reason the action variables are important and interesting physical quantities is that when a system undergoes very slow time evolution, *adiabatic change*, the action variables are distinguished as variables that are constant under such change: they *are adiabatic invariants*. If the change is sufficiently slow they don't change even after a long enough time to cause an order 1 change in other quantities.

To see how the action variables are calculated, we return to the central potential problem, which we separated in spherical coordinates, specialized to the Coulomb potential $V(r) = -k/r$. Since we must have finite motion we assume $E < 0$. We have already separated variables in spherical coordinates. We have an action variable for each coordinate.

$$I_\varphi = \int_0^{2\pi} J_\varphi \frac{d\varphi'}{2\pi} = J_\varphi \quad (357)$$

$$I_\theta = \oint \frac{d\theta}{2\pi} \sqrt{J^2 - \frac{J_\varphi^2}{\sin^2 \theta}} = \frac{4}{2\pi} \Theta\left(\frac{\pi}{2}\right) = J - J_\varphi \quad (358)$$

$$I_r = \oint \frac{dr}{2\pi} \sqrt{2mE + \frac{2mk}{r} - \frac{J^2}{r^2}} = -J + \frac{k\sqrt{m}}{\sqrt{-2E}} \quad (359)$$

An efficient technique for evaluating I_r is to extend r to a complex variable z and interpret the integral as a closed contour integral in the complex z -plane. Consider the integrand

$$\frac{\sqrt{-2mEz^2 - 2mkz + J^2}}{z} = \frac{\sqrt{-2mE}}{2\pi z} \sqrt{(z - r_+)(z - r_-)} \quad (360)$$

where the roots $r_- < r_+$ are the turning points of the radial motion. For real $z > r_+$ we specify the right side as positive. Clearly it must also be positive for large real negative z , so there we should write the integrand as

$$\frac{\sqrt{-2mE}}{-2\pi z} \sqrt{(-z + r_+)(-z + r_-)} \quad (361)$$

and this form is valid for all $\text{Re}z < r_-$. Cut the complex z -plane from r_- to r_+ . Then in the region $r_- < z < r_+$ the integrand is $i \times$ positive above the cut and $-i \times$ positive below the cut. Thus a closed contour integral of this integrand with the loop enclosing the cut in a clockwise sense becomes $+iI_r$ when the contour collapses around the cut. On the other hand the contour can be deformed to a large circular contour of radius R plus a counterclockwise contour integral about the pole at $z = 0$. The latter integral evaluates to $-i\sqrt{-2mEr_+r_-}$. Putting $z = Re^{i\phi}$, the large circular integral becomes

$$-\frac{i}{2\pi}\sqrt{-2mE} \int_0^{2\pi} d\phi Re^{i\phi} \left(1 - \frac{r_+ + r_-}{2R} e^{-i\phi} + \mathcal{O}\left(\frac{r_{\pm}^2}{R^2}\right) \right) \rightarrow \frac{i}{2}(r_+ + r_-) \quad (362)$$

Thus $I_r = (r_+ + r_-)\sqrt{-2mE}/2 - \sqrt{-2mEr_+r_-}$. Now r_{\pm} are roots of the polynomial

$$z^2 + \frac{k}{E}z - \frac{J^2}{2me} = z^2 - (r_+ + r_-)z + r_+r_- \quad (363)$$

$$r_+ + r_- = -\frac{k}{E}, \quad r_+r_- = -\frac{J^2}{2mE} \quad (364)$$

$$I_r = (r_+ + r_-)\sqrt{-2mE}/2 - \sqrt{-2mEr_+r_-} = \frac{k\sqrt{m}}{\sqrt{-2E}} - J \quad (365)$$

The first two equations, which would apply to any central potential, determine $J = I_{\varphi} + I_{\theta}$. The last equation determines $E = -mk^2/2(I_r + I_{\theta} + I_{\varphi})^2$. This last equation shows that the frequencies $\partial E/\partial I_r, \partial E/\partial I_{\theta}, \partial E/\partial I_{\varphi}$ associated with motion in r, θ, φ are all the same, reflecting the fact that finite motion in the Kepler problem is strictly periodic.

Let's next turn to a system with just one degree of freedom, a particle moving in a one dimensional potential $V(q)$:

$$S_0(q) = \int_0^q dq' \sqrt{2m(E - V(q'))} \quad (366)$$

The requisite finite motion occurs between two turning points $V(q_{1,2}) = E$. Then the action variable is $1/2\pi$ times the integral defining S_0 about a complete cycle from q_1 to q_2 and back to q_1 :

$$I = 2 \int_{q_1}^{q_2} \frac{dq}{2\pi} \sqrt{2m(E - V(q))} \quad (367)$$

This formula gives $I(E)$ as a function of E , but we can imagine inverting it to give $E(I)$ as a function of I . Since the energy depends only on I , the coordinate w conjugate to I is a cyclic variable. Since S_0 generates a time independent canonical transformation, the new Hamiltonian is just the energy $E(I)$. Hamilton's equations then give $\dot{I} = 0$ and

$$\dot{w} = \frac{\partial E}{\partial I} = \frac{dE}{dI} = \text{constant} \quad (368)$$

Thus the w 's depend linearly on the time $w(t) = (dE/dI)t + w(0)$. Now recall that as q goes through one cycle S_0 changes by an amount $\Delta S_0 = 2\pi I$. Since $w \equiv \partial S_0/\partial I$, we conclude

that under this cycle w changes by $\Delta w = \partial(\Delta S_0/\partial I) = 2\pi$. This justifies calling w an angle variable, since it advances by a fixed amount 2π with each cycle of the motion. When we express single valued physical quantities in terms of action angle variables, they must be periodic functions of w with period 2π .

Notice that the definition of the action variable as a closed loop integral in phase space (here the q, p plane) gives the interpretation of the action variable as the area of the region in phase space enclosed by the loop. As an example, notice that the phase space motion for a harmonic oscillator is determined by the equation

$$\frac{p^2}{2mE} + \frac{m\omega^2 q^2}{2E} = 1. \quad (369)$$

This is an ellipse with semi axes $\sqrt{2mE}$, $\sqrt{2E/m\omega^2}$. The area πab of this ellipse is

$$\pi\sqrt{2mE}\sqrt{2E/m\omega^2} = 2\pi E/\omega.$$

Thus the action variable for the harmonic oscillator is $I = E/\omega$, so $E(I) = \omega I$. We learn immediately that $\dot{w} = dE/dI = \omega!$. Sometimes we can figure out how E depends on the action variables without obtaining a complete solution of the problem. Then we are in a position to simply read off the angular frequencies of the motion by taking derivatives of E with respect to the action variables.

We can define an action variable for each q_k, p_k pair of the system of coordinates for which the H-J equation is completely separable. We have one for each degree of freedom and it is natural to choose them as the new momenta in a Hamilton-Jacobi approach. The canonical transformation generated by $S_0(q, I)$ where all of the constants of integration are expressed as functions of the I_k then gives the transformation from the original variables to the action-angle variables. As we shall see, this language stems from the fact that the coordinates conjugate to the I_k , $w_k = \partial S_0/\partial I_k$ have the character of angle variables.

With the Kepler example in mind, we return to the completely separable case $S_0 = \sum_k S_k(q_k)$, assuming we have found a solution of the H-J equation with one integration constant α_k for each degree of freedom. Then the formulas for the action variables will give $I_k(\alpha)$ which we can invert to obtain $\alpha_k(I_k)$. Thus we can regard the I_k as the new momenta in the canonical transformation generated by Hamilton's principle function $S(q, I, t) = S_0(q, I) - E(I)t$. We can also implement a time independent canonical transformation $q_k, p_k \rightarrow w_k, I_k$ using $S_0 = \sum_k S_k(q_k, I_1, \dots, I_s)$ as the generating function. Note that each S_k can in general depend on all of the I 's even though it depends on one q . Then the angle variables

$$w_k = \frac{\partial S_0}{\partial I_k} = \sum_l \frac{\partial S_l(q_l)}{\partial I_k} \quad (370)$$

can each get contributions from several S_l . However, when q_k undergoes one complete cycle, we still have the condition that S_0 changes by $2\pi I_k$, so that w_k changes by 2π while w_l for $l \neq k$ does not change. This justifies calling the w_k angle variables. Since the canonical

transformation generated by S_0 is time independent, the new Hamiltonian is just the old one expressed in terms of action angle variables, namely $\bar{H} = E(I_1, \dots, I_s)$ and it is independent of the angle variables! The Hamilton equations for the action angle variables are thus $\dot{I}_k = 0$ and $\dot{w}_k = \partial E / \partial I_k = \text{constant}$. This confirms that the I_k are indeed constants of the motion and $w_k(t) = (\partial E / \partial I_k)t = \text{constant}$.

Since the w_k are multi-valued, i.e. $w_k + 2\pi$ corresponds to the same configuration as w_k , single valued physical quantities must be periodic in the w_k with period 2π . In other words they can be expanded in a multiple Fourier series as a linear combination of complex exponentials

$$\exp \left\{ i \sum_k n_k w_k \right\}, \quad n_k = \text{integers} \quad (371)$$

Each term in the linear combination will oscillate with the angular frequency

$$\omega = \sum_k n_k \frac{\partial E}{\partial I_k} \quad (372)$$

but since these possible frequencies are not commensurable, physical property will not necessarily be strictly periodic. Of course it is possible that very special *degenerate* systems will display commensurable frequencies, and may show some periodic properties. Completely degenerate systems will display strictly periodic motion. In this regard, recall that for a central potential the energy only depended on I_θ and I_φ in the combination $I_\theta + I_\varphi$, meaning that the frequencies of motion in θ and φ are identical. Further for the coulomb potential all three action variables only entered the energy in the combination $I_r + I_\theta + I_\varphi$, i.e. all three frequencies are identical: complete degeneracy.

A particular advantage of the change to action-angle variables is that one can quickly identify the fundamental frequencies of the system. In the early days of quantum physics, before Heisenberg, Schroedinger, and Dirac, quantization rules were assigned to adiabatic invariants such as the action variables: they were required to be integer multiples of \hbar . These *ad hoc* rules met with a certain amount of success, and we can see them coming out of specific approximation schemes to true quantum mechanics such as the WKB approximation. In the case of the Coulomb potential, assigning integer values to $I_{r,\theta,\varphi}/\hbar$ leads to Bohr's energy quantization rules

$$E_{n_1 n_2 n_3} \rightarrow -\frac{mk^2}{2\hbar^2(n_1 + n_2 + n_3)^2} \quad (373)$$

Recalling that $k = e^2/4\pi\epsilon = \alpha\hbar c$, this gives the ground state energy of hydrogen $E_G = -m\alpha^2 c^2/2$, fortuitously identical to that predicted by the Schroedinger equation.