Lectures in Computational Physics 3

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Contents

Preface

These are lecture notes for a class, I am giving at Ariel University on Computational Physics. These notes pertain to the third semester of the series and the topic is the solution of physics problems using Matlab/Octave.

CHAPTER 1

The pendulum

1.1. Introduction

This chapter deals with the dynamics of various pendulum problems. In order to solve these numerically, they must be expressed as a first order system

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0.$$

Here x can be a vector of any dimension, in which case, the right hand side $f(t, x)$ must also be a vector-valued function of the same dimension. The main numerical method of solution should be *Runge-Kutta* given by the following algorithm:

$$x_{n+1} = x_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \qquad t_{n+1} = t_n + h,$$

where

$$\begin{aligned} k_1 &= f(t_n, x_n) \\ k_2 &= f(t_n + h/2, x_n + k_1 h/2) \\ k_3 &= f(t_n + h/2, x_n + k_2 h/2) \\ k_4 &= f(t_n + h, x_n + k_3 h). \end{aligned}$$

1.2. The linear pendulum

We begin with the simplest oscillatory system.

$$\ddot{x} = -\omega^2 x, \quad x(0) = x_0, \quad \dot{x}(0) = x_1$$

The advantage is that it is linear and hence can be solved analytically. We can thus use it to test the Runge-Kutta code we will use for future problems in this section and the next.

The general solution is:

$$x = a\cos(\omega t) + b\sin(\omega t).$$

$$x(0) = a = x_0, \quad \dot{x}(0) = \omega b = x_1.$$

Thus $a = x_0$ and $b = x_1/b$, and we get

$$x = x_0\cos(\omega t) + \frac{x_1}{\omega}\sin(\omega t).$$

To get the conserved *energy*, multiply by $\dot{x}$ and integrate:

$$E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2. \tag{1.1}$$

Note that to solve numerically, you have to convert to a first order system:

$$\dot{x} = y, \quad \dot{y} = -\omega^2 x, \quad x(0) = x_0, \quad y(0) = x_1.$$

Written in matrix form

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}0 & y\\ -\omega^2 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}, \quad \begin{pmatrix}x(0)\\ y(0)\end{pmatrix} = \begin{pmatrix}x_0\\ x_1\end{pmatrix}$$

EXERCISE 1.1. Implement your own Runge-Kutta code in Matlab and use it to solve this system. Test your results against the built-in ODE solver and against the analytic solution.

1.3. The linear pendulum with friction

We now add friction to the system. One usually assumes that *dynamic friction* is proportional to the velocity and in the opposite direction.

$$\ddot{x} = -\omega^2 x - 2\sigma x, \quad x(0) = x_0, \quad \dot{x}(0) = x_1,$$

where $\sigma > 0$. The *characteristic polynomial* is

$$\lambda^2 + 2\sigma\lambda + \omega^2 = 0.$$

The *characteristic roots* are:

$$\lambda_{1,2} = -\sigma \pm \sqrt{\sigma^2 - \omega^2}.$$

There are two cases (+ a marginal case):

(1) Monotone decay: $\sigma > \omega$; $x = ae^{\lambda_1 t} + be^{\lambda_2 t}$. Note that both characteristic roots are negative.
(2) Oscilatory decay: $\sigma < \omega$; $x = e^{-\sigma t}(a\cos(\mu t) + b\sin(\mu t))$, where $\mu = \sqrt{\omega^2 - \sigma^2} < \omega$.

EXERCISE 1.2. Write as a first order system and in matrix form and solve using your Runge-Kutta code. Compare with the built-in solver and the analytic solution.

1.4. The nonlinear pendulum

The real pendulum is nonlinear. We now go the other way around, starting with the energy

$$\frac{E}{mL^2} = \frac{1}{2}\dot{x}^2 + \frac{g}{L}(1 - \cos x). \tag{1.2}$$

The constants on the left are so that the energy has the correct units (mass × velocity^2). Differentiating with respect to time, we get the equations of motion, to which we add the initial conditions.

$$\ddot{x} = -\frac{g}{L}\sin x, \quad x(0) = x_0, \quad \dot{x}(0) = x_1,$$

where g is the gravitational acceleration and L the length of the pendulum. This is no longer solvable analytically. However, because of the conservation of energy, we say it is *integrable*. That means there are enough *first integrals*, i.e. here, as there is only one degree of freedom, there is one first integral. As a consequence, the problem is solvable up to quadrature.

In vector form we have:

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}y\\ \frac{g}{L}\sin x\end{pmatrix}, \quad \begin{pmatrix}x(0)\\ y(0)\end{pmatrix} = \begin{pmatrix}x_0\\ x_1\end{pmatrix}$$

We can view the conservation of energy as a constraint on the trajectories to remain on the energy level curves in *phase space*; see Figure 1.1.

$$\tilde{E} = \frac{1}{2}y^2 + \frac{g}{L}(1 - \cos x).$$

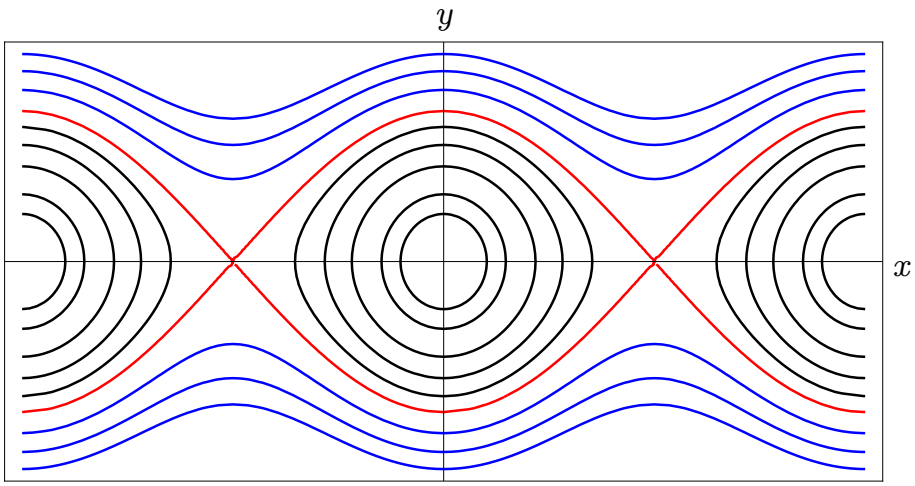

FIGURE 1.1. Phase space with some trajectories

The *separatrix* is the trajectory (red in the figure) on which $\tilde{E} = 2g/L$. On that trajectory $y = 0$ when $x = \pm\pi$. When $\tilde{E} < 2g/L$ the trajectories (black in the figure) are oscillatory. When $\tilde{E} > 2g/L$ the trajectories (blue in the figure) are transient.

Since $\sin x = x - \frac{1}{6}x^3 + \dots$ the *linearization* of the nonlinear pendulum is exactly the linear pendulum with $\omega = \sqrt{g/L}$.

Note that the period T of oscillatory trajectories is no longer constant, but instead now depends on the amplitude. The theorem that the period of is independent of the amplitude is true only of the linear pendulum. To see this, assume $E < 2g/L$ and move $\tilde{E}$ to the right hand side to write

$$\frac{1}{2}y^2 - \frac{g}{L}\left(\cos x - (1 - L\tilde{E}/g)\right) = 0$$

and denote $\cos x_0 = 1 - L\tilde{E}/g$, where $0 \le x_0 \le \pi$. Using the double angle formula we get

$$\frac{1}{2}y^2 - \frac{2g}{L}(\sin^2 x_0/2 - \sin^2 x/2) = 0.$$

Now set $\lambda = \sin x_0/2$ and $\sin x/2 = \lambda \sin\phi$ to get

$$\lambda\cos\phi\,\dot{\phi} = (\sin x/2)^{\cdot} = (\cos x/2)\frac{\dot{x}}{2} = \sqrt{1 - \sin^2 x/2}\,\frac{\dot{x}}{2} = \sqrt{1 - \lambda^2\sin^2\phi}\,\frac{\dot{x}}{2}.$$

Equivalently

$$\frac{\dot{x}}{2} = \frac{\lambda\cos\phi\,\dot{\phi}}{\sqrt{1 - \lambda^2\sin^2\phi}}$$

and

$$\frac{1}{2}\dot{x}^2 = \frac{2\lambda^2\cos^2\phi\,\dot{\phi}^2}{1 - \lambda^2\sin^2\phi} = \frac{2g}{L}(\lambda^2 - \lambda^2\sin^2\phi) = \frac{2g\lambda^2}{L}\cos^2\phi.$$

We get

$$\dot{\phi} = \sqrt{\frac{g}{L}}\sqrt{1 - \lambda^2\sin^2\phi}.$$

This can be written as

$$\frac{d\phi}{\sqrt{1 - \lambda^2\sin^2\phi}} = \sqrt{\frac{g}{L}}\,dt.$$

Since $T/4$ elapses between the lowest point $\phi = 0$ and the highest $\phi = \pi/2$, we can find

$$T = 4\int_0^{\pi/2} dt = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1 - \lambda^2\sin^2\phi}}.$$

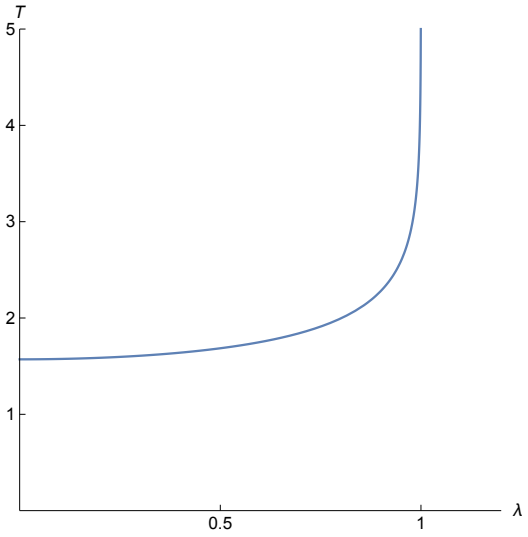

FIGURE 1.2. Plot of the period T of the nonlinear pendulum

The right hand side is called an elliptic integral of the first kind. It cannot be expressed in terms of elementary functions. The period T is plotted against λ in Figure 1.2. One observes that T is constant for small λ, and tends to infinity when $\lambda \to 1$, i.e. when approaching the separatrix.

1.5. The nonlinear pendulum with friction

We can also add dynamic friction to the nonlinear pendulum.

$$\ddot{x} = -\frac{g}{L}\sin x - \sigma\dot{x}.$$

This is no longer integrable as the nonlinear pendulum in the previous section because energy is not conserved.

We translate to a first order system:

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix} y \\ \frac{g}{L}\sin x - \sigma y\end{pmatrix}, \quad \begin{pmatrix}x(0)\\ y(0)\end{pmatrix} = \begin{pmatrix}x_0\\ x_1\end{pmatrix}$$

This can only be solved numerically.

EXERCISE 1.3. Set say $g/L = 1$. Compute various trajectories starting with $1/2\dot{x}^2 + (1-\cos x) > 2$, and vary σ to see how the bigger σ the faster the orbit enters the bounded orbit regime. Compute the crossing time and plot it against σ.

1.6. The double pendulum

Our double pendulum consists of two masses m_1 and m_2, with m_1 attached to a fixed point, say the origin, with a weightless wire of length l_1, and m_2 attached to m_1 with a weightless wire of length l_2. See Figure 1.3.

The position of m_1 is $\mathbf{r}_1 = (l_1\cos(x_1 - \pi/2), l_2\sin(x_1 - \pi/2))$ and the position of m_2 is $\mathbf{r}_2 = \mathbf{r}_1 + (l_2\cos(x_2 - \pi/2), l_2\sin(x_2 - \pi/2))$. From this we can calculate the *kinetic energy*:

$$T = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1l_1^2\dot{x}_1^2 + \frac{1}{2}m_2\left(l_1^2\dot{x}_1^2 + l_2^2\dot{x}_2^2 + 2l_1l_2\dot{x}_1\dot{x}_2\cos(x_1 - x_2)\right),$$

where $v_1^2 = |\dot{\mathbf{r}}_1|^2$ and $v_2^2 = |\dot{\mathbf{r}}_2|^2$. The *potential energy* is:

$$V = -m_1gy_1 - m_2g_2y_2 = -(m_1 + m_2)gl_1\cos x_1 - m_2gl_2\cos x_2.$$

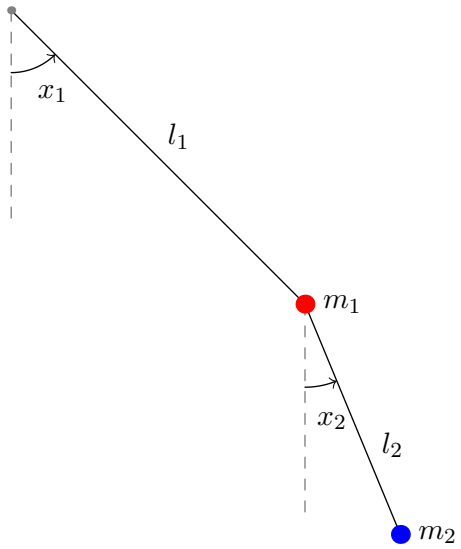

FIGURE 1.3. The double pendulum

The total energy (also called the *Hamiltonian*) is $\mathcal{H} = T + V$. The *Lagrangian* is

$$\mathcal{L} = T - V = \frac{1}{2}m_1 l_1^2 \dot{x}_1^2 + \frac{1}{2}m_2\left(l_1^2\dot{x}_1^2 + l_2^2\dot{x}_2^2 + 2l_1 l_2 \dot{x}_1\dot{x}_2\cos(x_1 - x_2)\right) + (m_1 + m_2)gl_1\cos x_1 + m_2 g l_2 \cos x_2$$

Since the system now has two degrees of freedom, it is no longer possible to derive the equations of motion simply by differentiating the energy. Instead, the equations of motion are the *Euler-Lagrange equations*:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_j}\right) - \frac{\partial \mathcal{L}}{\partial x_j} = 0, \quad i = 1, 2.$$

After some simplifications, we get:

$$(m_1 + m_2)l_1\ddot{x}_1 + m_2 l_2 \ddot{x}_2 \cos(x_1 - x_2) + m_2 l_2 \dot{x}_2^2 \sin(x_1 - x_2) + g(m_1 + m_2)\sin x_1 = 0$$
$$m_2 l_2 \ddot{x}_2 + m_2 l_1 \ddot{x}_1 \cos(x_1 - x_2) - m_2 l_1 \dot{x}_1^2 \sin(x_1 - x_2) + m_2 g \sin x_2 = 0.$$

To solve numerically, we have to convert these equations to a first order system by introducing $y_1 = \dot{x}_1$ and $y_2 = \dot{x}_2$, and then solving for $\dot{y}_1$ and $\dot{y}_2$:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{x}_2 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ -\frac{g(m_1+m_2)\sin x_1 - g m_2 \cos(x_1-x_2)\sin x_2 + m_2 l_1 \cos(x_1-x_2)\sin(x_1-x_2)y_1^2 + m_2 l_2 \sin(x_1-x_2)y_2^2}{l_1(m_1+m_2\sin^2(x_1-x_2))} \\ y_2 \\ \frac{g(m_1+m_2)(\cos(x_1-x_2)\sin x_1 - \sin x_2) + l_1(m_1+m_2)\sin(x_1-x_2)y_1^2 + l_2 m_2 \cos(x_1-x_2)\sin(x_1-x_2)y_2^2}{l_2(m_1+m_2\sin^2(x_1-x_2))} \end{pmatrix}$$

To simplify, we pick units in which $g = 1$ and define $m_1/m_2 = \lambda$. Thus we get:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ -\frac{(1+\lambda)\sin x_1 - \lambda\cos(x_1-x_2)\sin x_2 + \lambda l_1 \cos(x_1-x_2)\sin(x_1-x_2)y_1^2 + \lambda l_2 \sin(x_1-x_2)y_2^2}{l_1(1+\lambda\sin^2(x_1-x_2))} \\ \frac{(1+\lambda)(\cos(x_1-x_2)\sin x_1 - \sin x_2) + l_1(1+\lambda)\sin(x_1-x_2)y_1^2 + l_2\lambda\cos(x_1-x_2)\sin(x_1-x_2)y_2^2}{l_2(1+\lambda\sin^2(x_1-x_2))} \end{pmatrix}$$

The order of the variables in the 4-d problem has been changed to make it more convenient when coding.

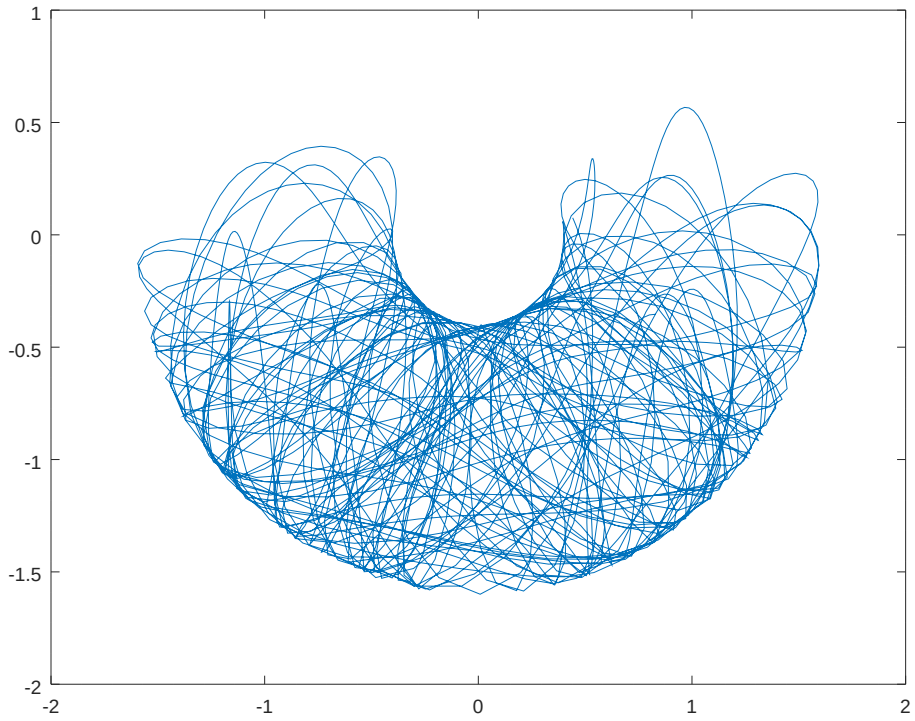

FIGURE 1.4. Trajectory of m_2 in double pendulum

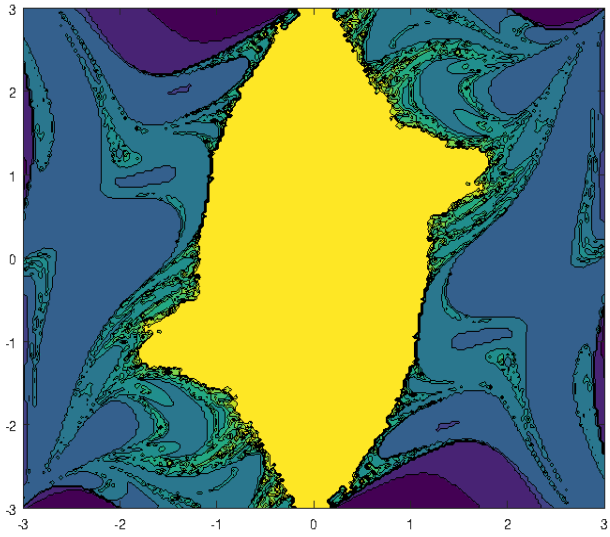

FIGURE 1.5. Contour plot of first flip time (yellow=∞)

Figure 1.4 shows the trajectory of the second mass with initial conditions $x_1 = 2\pi/3$, $x_2 = -\pi/4$ over the time interval $[0, 250]$. The rod length were chosen $l_1 = 1$, and $l_2 = 0.6$. The masses were chosen equal, i.e. $\lambda = 1$. The solution was computed using Runge-Kutta with a step size of $h = 0.1$. The *chaotic* nature of the system is already apparent.

Chaos is usually defined as a high sensitivity to initial conditions, i.e. trajectories that are initially close and eventually diverge greatly. Here this is mostly caused by flips, i.e. one of the masses reaching or coming close to the vertical position. Figure 1.5 shows a contour plot of the time of the first flip.

The chaotic features are well presented here via the idea of fractal sets and self-similarity. When zooming in on any part of the boundary of the yellow region, the features do not simplify. Instead, the zoomed-in version is qualitatively just a scaling of the un-zoomed one. Figure 1.6 is a zoom-in on the region $1 \leq x_1 \leq 2.5$, $0.5 \leq x_2 \leq 2$.

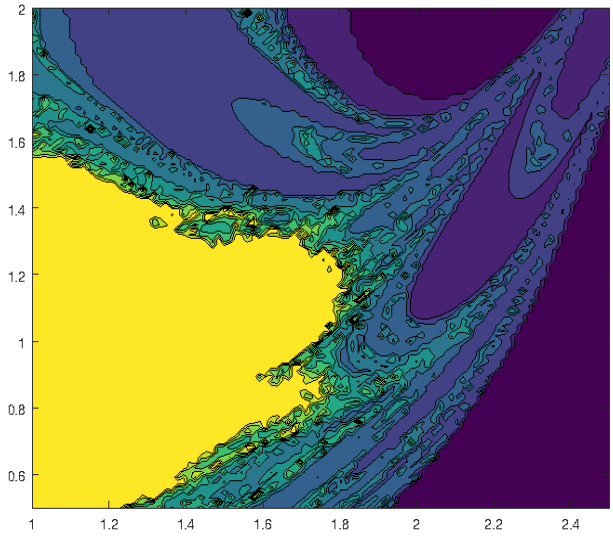

FIGURE 1.6. Contour plot of first flip time - Zoom-in to $[1, 2.5] \times [0.5, 2]$

EXERCISE 1.4. (a) Derive the equations of motion for a double compound pendulum, i.e. replacing each of the weightless wires and weights in the configuration above by an arm having a uniform mass distribution, i.e. constant mass per length.
HINT: The kinetic energy of the first rod is due entirely to the rotational motion: $\frac{1}{2} I_1 \dot{x}_1^2$ where I_1 is the moment of inertia through the end point of the rod. The kinetic energy of the second rod can be decomposed into the energy due to the motion of the center of mass and the energy due to the rotation about the center of mass $\frac{1}{2} m_2 v_c^2 + \frac{1}{2} I_2 \dot{x}_2^2$ where I_2 is the moment of inertia about the center of mass, m_2 is the mass of the second rod, and v_c is the velocity of the center of mass.

(b) Solve the equations numerically in Matlab/Octave and plot the trajectory of the second arm as in Figure 1.4.

(c) Create a contour plot for the first flip time as in Figure 1.5.

(d) Add dynamic friction, both in the first pivot and in the second pivot. You may choose the same coefficient σ. Run the contour plot code with two values of σ, one large enough so that there is no chaos, and one small enough to allow some chaos. It is recommended to run the code first with a small resolution, and increase the resolution only after having determined the two values of σ.

(e) Zoom-in a few times in the case of small σ and determine if there is a difference with the case where $\sigma = 0$.

1.7. Octave Code

Below are 4 files written in Octave which were used to produce Figure 1.5. Please note that in `bf_st_v.m` the symmetry of the window $[-3,3]\times[-3,3]$, and of the stopping time T was used.

`double_pendulum.m` - right hand side of the ODE.

```
function y=double_pendulum(~,x)
  global lambda l1 l2;
  x1=x(1);
  x2=x(2);
  y1=x(3);
  y2=x(4);
  y(1)=y1;
  y(2)=y2;
  y(3)=-((1+lambda)*sin(x1)-lambda*cos(x1-x2)*sin(x2) \
    +lambda*l1*cos(x1-x2)*sin(x1-x2)*y1^2 \
    +lambda*l2*sin(x1-x2)*y2^2)/(l1*(1+lambda*sin(x1-x2)^2));
  y(4)=((1+lambda)*(cos(x1-x2)*sin(x1)-sin(x2)) \
    +l1*(1+lambda)*sin(x1-x2)*y1^2 \
    +l2*lambda*cos(x1-x2)*sin(x1-x2)*y2^2)/(l2*(1+lambda*sin(x1-x2)^2));
end
```

`bf_st.m` - first flip time as a function of x_1 and x_2.

```
function t=bf_st(x1,x2)
   t=0;
   x=[x1,x2,0,0];
   h=0.1;
% Runge-Kutta steps
   while (t<500 && pi>abs(x(2)) && pi>abs(x(1)))
      k1 = double_pendulum(t,x);
      k2 = double_pendulum(t+h/2,x+k1*h/2);
      k3 = double_pendulum(t+h/2,x+k2*h/2);
      k4 = double_pendulum(t+h,x+h*k3);
      x=x+h/6*(k1+2*k2+2*k3+k4);
      t=t+h;
   endwhile
end
```

`bf_st_v.m` - vectorized (and parallelized) version of `bf_st.m`.

```
function T=bf_st_v(x1,x2)
s=size(x1);
n=s(1);
m=s(2);
T=zeros(m,n);
for i=1:n/2
% Comment line above and uncomment line below when not using symmetry
%  for i=1:n
   f=@(k) bf_st(x1(i,k),x2(i,k));
   j=1:m;
   T(i,:)=pararrayfun(nproc-1,f,j);
%Comment line below when not using symmetry
   T(n-i,:)=flip(T(i,:));
endfor
end
```

`dp_flip.m` - driver script.

```
clear all;
pkg load parallel;
global l1=1 l2=.6 lambda=1 sigma=.05
r1=linspace(-3.14,3.14,100);
r2=linspace(-3.14,3.14,100);
[x1,x2]=meshgrid(r1,r2);
T=bf_st_v(x1,x2);
contourf(x1,x2,log(T));
```

CHAPTER 2

Kepler orbits

2.1. Introduction

In this chapter, we will first study the dynamics of a mass under a central force, and in particular one which obeys an inverse square law as in Newtonian gravity, and electrodynamics.

2.2. Kepler's laws

Astronomy has always been a source of scientific inspiration. Probably the most significant revolution in science was spurred by astronomy. Since antiquity, civilizations have wondered about the laws governing the motion of celestial bodies. The theory in force during the middle-ages was due to Ptolemy. In this model, the sun, the moon, and all the stars revolved about the earth, and their motion was determined by super-imposing circular motions. Some "stars" (which turned out to be the planets) were erratic, and each required many parameters to describe their motion. The theory gave very accurate predictions, but was rather cumbersome.

In the seventeenth century, Johannes Kepler, using data collected by Tycho Brahe, formulated three simple laws to explain the motion of the planets:

1st Law *Every planet moves on an ellipse with the sun at a focus.*

2nd Law *The radius vector from the sun to the planet sweeps out equal areas in equal times.*

3rd Law *The square of the orbital period of the planet is proportional to the cube of the semi-major axis of the orbit.*

Here, only two parameters are required for each planet: the eccentricity of the ellipse, and its semi-major axis. The motion is then determined from these three laws. For example, the second law gives the relative angular velocity of the planet at each point on its orbit. Note for instance that when the planet is close to the sun it is moving faster than when it is further away. The third law is then used to determine its absolute velocity.

In 1687, in his masterpiece *Philosophiae Naturalis Principia Mathematica*, Isaac Newton formulated the laws of dynamics and gravity, and derived Kepler's three laws from these using the tools of the newly developed Calculus. The most remarkable feature, is that the principles he formulated are universal, hence apply to any body moving under the influence of gravity. They are used to describe the motion of asteroids, comets, satellites, in addition to the planets.

2.3. Newton's Law of Motion

In Newtonian mechanics, the motion of a particle, such as a planet, is described by a parametric curve in 3-dimensional space $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$. The position of the particle at time t is given by the terminal point of the vector $\mathbf{r}(t)$. The velocity of such a particle is $\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}$, and the acceleration is $\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$. For future reference, note the following consequences from the product rule:

$$\frac{d}{dt}\big(\mathbf{u}\cdot\mathbf{w}\big) = \frac{d\mathbf{u}}{dt}\cdot\mathbf{w} + \mathbf{u}\cdot\frac{d\mathbf{w}}{dt}, \qquad \frac{d}{dt}\big(\mathbf{u}\times\mathbf{w}\big) = \frac{d\mathbf{u}}{dt}\times\mathbf{w} + \mathbf{u}\times\frac{d\mathbf{w}}{dt},$$

for any differentiable functions $\mathbf{u}(t)$ and $\mathbf{w}(t)$.

Now, *Newton's Law of Motion* states that the acceleration of a particle is proportional to the force exerted on the particle, the constant of proportionality being defined as the *mass* of the particle. This is written:

$$\mathbf{F} = m\mathbf{a}. \tag{2.1}$$

2.4. Newton's Law of Gravity

This law states that the gravitational force that a body of mass M exerts on a body of mass m is proportional to the product of their masses, inversely proportional to the square of the distance between the two bodies, and is directed so that it points from the second body to the first. If the first body's position is given by $\mathbf{r}_1$, and the second body's position is given by $\mathbf{r}_2$, then the force is:

$$\mathbf{F} = -\frac{GMm\left(\mathbf{r}_2 - \mathbf{r}_1\right)}{\left|\mathbf{r}_2 - \mathbf{r}_1\right|^3}. \tag{2.2}$$

We wish to consider the case where there are only two bodies. We make the assumption that $m \ll M$, i.e. m is much smaller than M. This is satisfied if, say, the first body is the sun and the second body is a planet. In that case, it is also possible to assume that the position of the first body is fixed at the origin of our coordinate system. Finally, we choose our system of units so that $GM = 1$, and $m = 1$. The only unknown is then the position of the second body which we take to be $\mathbf{r}(t)$. Combining (2.1) and (2.2), we obtain the system of differential equations:

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{\mathbf{r}}{r^3}, \tag{2.3}$$

where we have put $r = |\mathbf{r}|$. Of course, the solution will depend on the initial data $\mathbf{r}(0)$, and $\dot{\mathbf{r}}(0)$. More generally the equation $\frac{d\mathbf{r}}{dt} = f(\mathbf{r})\mathbf{r}$ is called a *central force law*. Many of the results below apply equally well to any central force law.

2.5. Conserved Quantities

For any vector-valued function $\mathbf{r}(t)$, define its *energy*:

$$E(t) = \frac{1}{2}\left|\dot{\mathbf{r}}\right|^2 - \frac{1}{r}.$$

THEOREM 2.1 (Conservation of energy). *Let* $\mathbf{r}(t)$ *be a solution of* (2.3). *Then*

$$\frac{dE}{dt} = 0.$$

PROOF. Note that $r^2 = \mathbf{r} \cdot \mathbf{r}$, so that

$$2r\dot{r} = 2\mathbf{r} \cdot \dot{\mathbf{r}},$$

and it follows that

$$\dot{r} = \frac{\mathbf{r} \cdot \dot{\mathbf{r}}}{r}. \tag{2.4}$$

Now, $\left|\dot{\mathbf{r}}\right|^2 = \dot{\mathbf{r}} \cdot \dot{\mathbf{r}}$, so that using (2.3), we obtain

$$\frac{d}{dt}\left|\dot{\mathbf{r}}\right|^2 = 2\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} = -2\frac{\dot{\mathbf{r}} \cdot \mathbf{r}}{r^3} \tag{2.5}$$

Therefore, combining (2.4) with (2.5), we conclude:

$$\dot{E} = \frac{1}{2}\frac{d}{dt}\left|\dot{\mathbf{r}}\right|^2 - \frac{d}{dt}\left(\frac{1}{r}\right) = -\frac{\dot{\mathbf{r}} \cdot \mathbf{r}}{r^3} + \frac{1}{r^2}\dot{r} = -\frac{\dot{\mathbf{r}} \cdot \mathbf{r}}{r^3} + \frac{\dot{\mathbf{r}} \cdot \mathbf{r}}{r^3} = 0.$$

□

It follows that if $\mathbf{r}(t)$ is a solution of (2.3), then $E(t) = E(0) =$ constant. A similar argument can be applied for any central force replacing the potential $\frac{1}{r}$ in the energy by a anti-derivative of the force.

For any vector-valued function $\mathbf{r}(t)$ define its *angular momentum*:

$$\mathbf{L}(t) = \mathbf{r} \times \dot{\mathbf{r}}.$$

THEOREM 2.2 (Conservation of angular momentum). *Let* $\mathbf{r}(t)$ *be a solution of* (2.3). *Then*

$$\frac{d\mathbf{L}}{dt} = 0.$$

PROOF. Recall that $\mathbf{w} \times \mathbf{w} = 0$ for any vector $\mathbf{w}$ in 3-space. Thus,

$$\dot{\mathbf{L}} = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{r} \times \left(-\frac{\mathbf{r}}{r^3}\right) = -\frac{1}{r^3}\,\mathbf{r} \times \mathbf{r} = 0.$$

□

Thus, if $\mathbf{r}(t)$ is a solution of (2.3), then $\mathbf{L}(t) = \mathbf{L}(0) =$ constant. Once again, the same holds for any central force.

2.6. Integration of the Equations of Motion

Henceforth $\mathbf{r}(t)$ will denote a solution of (2.3). we have constants E and $\mathbf{L}$ such that

$$\frac{1}{2}\,|\dot{\mathbf{r}}|^2 - \frac{1}{r} = E, \qquad \mathbf{r} \times \dot{\mathbf{r}} = \mathbf{L}. \tag{2.6}$$

Now, if $\mathbf{L} = \mathbf{0}$, then it is not difficult to check that $\mathbf{r}(t)$ will lie in a fixed line through the origin. In this case, there is always a collision at some time $t = t_0$, i.e. $\mathbf{r}(t_0) = \mathbf{0}$. This case is left for the reader to work out. From now on, we assume that $\mathbf{L} \neq \mathbf{0}$. Note that, since $\mathbf{r}(t)$ is always perpendicular to the constant vector $\mathbf{L} \neq \mathbf{0}$, we obtain that $\mathbf{r}(t)$ lies in a fixed plane through the origin. We may assume that this is the xy-plane, and we now introduce polar coordinates (r, θ) in this plane, centered at the origin. Note that $r = |\mathbf{r}|$ is the same as before. There are now only two unknowns $r(t)$, and $\theta(t)$, that is two degrees of freedom.

From the polar to Cartesian coordinates transformation $x = r\cos\theta, y = r\sin\theta$, we obtain

$$|\dot{\mathbf{r}}| = \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2.$$

and,

$$\mathbf{r} \times \dot{\mathbf{r}} = (\dot{x}y - x\dot{y})\,\mathbf{k} = -r^2\dot{\theta}\,\mathbf{k}$$

Thus, if we define $L = |\mathbf{L}|$, then we have $L = r^2\dot{\theta}$, and equations (2.6) can be written

$$\frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{1}{r} = E \tag{2.7}$$

$$r^2\dot{\theta} = L. \tag{2.8}$$

Since there are two conserved quantities and two degrees of freedom the system is integrable. From now on, when we say a solution, we mean a pair of functions $\big(r(t), \theta(t)\big)$ which satisfy the equations (2.7)–(2.8). From equation (2.8), we see that $r(t) \neq 0$, and also

$$\dot{\theta} = \frac{L}{r^2}. \tag{2.9}$$

Substituting into (2.7), we obtain

$$\frac{1}{2}\dot{r}^2 + \frac{L^2}{2r^2} - \frac{1}{r} = E. \tag{2.10}$$

There is now only one unknown r, hence the problem has been reduced to one degree of freedom. In fact, this is now very similar to (1.1) and (1.2), but with a different potential, $V(r) = L^2/2r^2 - 1/r$ also called the *effective potential*.

2.7. Derivation of Kepler's laws from Newton's laws

We start with the second law of Kepler since it is the easiest to prove. This section is included for completeness but not all of it is needed for the computational aspects, which are discussed next.

THEOREM 2.3. KEPLER'S SECOND LAW
Let $\big(r(t), \theta(t)\big)$ *be a solution. Then the radius vector* $\mathbf{r}$ *sweeps out equal areas in equal time.*

PROOF. This follows from the conservation of angular momentum:

$$\frac{d\theta}{dt} = \frac{L}{r^2}.$$

Note that the area swept by the radius vector is given by:

$$A(\theta) = \frac{1}{2}\int_0^\theta r^2 d\theta.$$

Thus,

$$\frac{dA}{d\theta} = \frac{1}{2}r^2,$$

and from the chain rule

$$\frac{dA}{dt} = \frac{dA}{d\theta}\frac{d\theta}{dt} = \frac{1}{2}r^2\frac{L}{r^2} = \frac{1}{2}L.$$

□

We now prove the first law.

THEOREM 2.4. KEPLER'S FIRST LAW
Let $\big(r(t), \theta(t)\big)$ *be a solution. Then the orbit is a conic section.*

PROOF. From (2.7)–(2.8), we obtain:

$$\dot{r}^2 = 2E + 2/r - L^2/r^2$$
$$\dot{\theta}^2 = L^2/r^4.$$

Thus, using the chain rule, $dr/d\theta = \dot{r}/\dot{\theta}$, we have:

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{L^2}(2Er^4 + 2r^3 - L^2r^2). \tag{2.11}$$

Differentiating this equation with respect to θ, we get:

$$2\frac{dr}{d\theta}\frac{d^2r}{d\theta^2} = \frac{1}{L^2}(8Er^3 + 6r^2 - 2L^2r)\frac{dr}{d\theta}.$$

Hence

$$\frac{d^2r}{d\theta^2} = \frac{1}{L^2}(4Er^3 + 3r^2 - L^2r). \tag{2.12}$$

Now, let

$$u = \frac{1}{r},$$

then

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta},$$

and

$$\frac{d^2u}{d\theta^2} = -\frac{1}{r^2}\frac{d^2r}{d\theta^2} + \frac{2}{r^3}\left(\frac{dr}{d\theta}\right)^2.$$

Substituting $(dr/d\theta)^2$ from (2.11), and $d^2r/d\theta^2$ from (2.12), we arrive at:

$$\begin{aligned}\frac{d^2u}{d\theta^2} &= -\frac{1}{r^2L^2}(4Er^3 + 3r^3 - L^2r) + \frac{2}{r^3L^2}(2Er^4 + 2r^2 - L^2r^2)\\ &= \frac{1}{L^2} - \frac{1}{r}\\ &= \frac{1}{L^2} - u.\end{aligned}$$

This equation:

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{L^2},$$

is relatively easy to solve. All its solutions are of the form:

$$u = \frac{1}{L^2} + k\cos(\theta - \theta_0),$$

where k and θ_0 are constants. Writing $e = kL^2$, we can write

$$r = \frac{L^2}{1 + e\cos(\theta - \theta_0)}. \tag{2.13}$$

This is the equation of a conic section in polar coordinates. □

Equation (2.13) is the equation of a circle if $|e| = 0$, an ellipse if $0 < |e| < 1$, a parabola if $|e| = 1$, and a hyperbola if $1 < |e| < \infty$. An orbit is called *elliptical* if it is a circle or an ellipse, *parabolic* if it is a parabola, and *hyperbolic* if it is a hyperbola; compare with the four cases in section 2.6.

Let $\big(r(t), \theta(t)\big)$ be a solution. Let T be the time it takes for the particle to complete one full orbit. We call T the *period*.

We now prove the third law.

Theorem 2.5. Kepler's Third Law
Let $\big(r(t), \theta(t)\big)$ be a solution with an elliptical orbit. Then

$$\frac{T^2}{a^3} = 4\pi^2,$$

where T is the period, and a the semi-major axis of the ellipse.

Proof. From the conservation of angular momentum, we have:

$$\frac{d\theta}{dt} = \frac{L}{r^2}.$$

Thus, $d\theta/dt > 0$, and

$$\frac{dt}{d\theta} = \frac{r^2}{L}.$$

It follows, by substitution, that

$$T = \int_0^T dt = \int_0^{2\pi} \frac{dt}{d\theta}\, d\theta = \frac{1}{L}\int_0^{2\pi} r^2 d\theta.$$

We substitute (2.13) into this integral to obtain:

$$T = L^3 \int_0^{2\pi} \frac{d\theta}{(1+e\cos\theta)^2} = \frac{2\pi L^3}{(1-e^2)^{3/2}}.$$

We have taken $\theta_0 = 0$, which clearly does not affect the calculation. We also omitted the computation of the integral, which is rather technical. Now, we have:

$$r_{\max} + r_{\min} = 2a$$

$$r_{\max} - r_{\min} = 2c,$$

where a is the semi-major axis of the ellipse, while c is the distance of the focus from the center. Thus, the eccentricity e of the ellipse is

$$e = \frac{c}{a} = \frac{r_{\max} - r_{\min}}{r_{\max} + r_{\min}}.$$

It follows that

$$1 + e = \frac{2r_{\max}}{r_{\max} + r_{\min}}$$

$$1 - e = \frac{2r_{\min}}{r_{\max} + r_{\min}},$$

hence

$$1 - e^2 = (1+e)(1-e) = \frac{4r_{\max}r_{\min}}{r_{\max} + r_{\min}} = \frac{r_{\min}r_{\max}}{a^2}.$$

On the other hand,

$$\frac{L^2}{r_{\max}} = 1 - e \tag{2.14}$$

$$\frac{L^2}{r_{\min}} = 1 + e, \tag{2.15}$$

hence

$$L^2\left(\frac{1}{r_{\max}} + \frac{1}{r_{\min}}\right) = 2.$$

It follows that

$$L^2 = \frac{2r_{\min}r_{\max}}{r_{\max} + r_{\min}} = \frac{r_{\min}r_{\max}}{a}.$$

It follows that $L^2/(1-e^2) = a$, and therefore

$$T^2 = 4\pi^2 a^3.$$

□

2.8. Numerical integration of the equation of motion

In order to write down a first order system which can be integrated numerically using Runge-Kutta, we first differentiate (2.10) and solve for $\ddot{r}$:

$$\ddot{r} = \frac{L^2}{r^3} - \frac{1}{r^2}.$$

This is easily converted to a first order system to which we add Equation (2.9):

$$\begin{pmatrix} \dot{r} \\ \dot{v} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} v \\ \frac{L^2}{r^3} - \frac{1}{r^2} \\ \frac{L}{r^2} \end{pmatrix} \tag{2.16}$$

We now have to supply initial conditions. We pick two parameters, the angular momentum $L \neq 0$, and the eccentricity $0 \leq e < 1$. We can assume that $t = 0$ corresponds to the perihelion $r = r_0$ which can be found from (2.15):

$$r_0 = \frac{L^2}{1+e}.$$

Since r is at a minimum at $t = 0$, we have $v_0 = \dot{r}(0) = 0$. We assume the perihelion occurs when $\theta_0 = \theta(0) = 0$. Finally we can find the period T from

$$a = \frac{L^2}{1-e^2}, \qquad T = 2\pi a^{3/2}.$$

Thus we can now solve (2.16) with initial conditions

$$\begin{pmatrix} r(0) \\ v(0) \\ \theta(0) \end{pmatrix} = \begin{pmatrix} r_0 \\ 0 \\ 0 \end{pmatrix},$$

where $r_0 = L^2/(1+e)$, on $[0,T]$ where $T = 2\pi|L|^3/(1-e^2)^{3/2}$.

Exercise 2.1. Solve (2.16) numerically in Matlab/Octave with the initial conditions set in the previous paragraph using your own Runge-Kutta code. Produce an animation. Use an eccentricity large enough to observe a significant difference in speeds at perihelion versus aphelion.

Exercise 2.2. In general relativity, the problem of the motion of a test mass in the gravitational field of a Schwarzschild black hole also reduces to a central force problem. The only difference is a single term in the potential energy. Simply replace (2.10) with

$$\frac{1}{2}\dot{r}^2 - \frac{1}{r} + \frac{L^2}{2r^2} - \frac{L^2}{r^3} = E$$

and repeat the same procedure. In order to get orbits close to an ellipse, you should choose $L >> \sqrt{12}$ (but not too large, otherwise you may end up with an orbit indistinguishable from an ellipse). Pick a as the larger root of the quadratic equation $r^2 - L^2 r + 3L^2 = 0$, $r_0 = a(1-e)$ and $T = 2\pi a^{3/2}$. Solve numerically as above over a few periods and plot to observe the precession of the perihelion.

CHAPTER 3

The restricted planar three body problem

In this chapter, we consider the *restricted planar three body problem*: the planar motion of a test mass, in the gravitational field of two masses orbiting each other in a circular orbit. This problem also has two degrees of freedom and like the double pendulum in the previous chapter exhibits chaos. However, it has five equilibrium points, known as *Lagrange points*, two of which are stable.

3.1. The equations of motion

We consider two bodies of masses m_1 and m_2 orbiting each other in a circular orbit with angular velocity $\mathbf{\Omega}$. The position of the first body is $\mathbf{r}_1$ and of the second body is $\mathbf{r}_2$. In addition, there is a third *massless* body m at position $\mathbf{r}$ in the same plane, see Figure 3.1. The third body being massless here means that m is negligible compared to m_1 and m_2, or equivalently that m_1 and m_2 act on m but are unaffected by m. According to Kepler's third law

$$\Omega^2 R^3 = \frac{4\pi^2 R^3}{T^2} = G(m_1 + m_2),$$

where R is the separation between m_1 and m_2 and T the period.

By Newton's law of gravity and law of motion the equations of motion are given by

$$m\frac{d^2\mathbf{r}}{dt^2} = m\mathbf{a} = -\frac{Gmm_1(\mathbf{r}-\mathbf{r}_1)}{|\mathbf{r}-\mathbf{r}_1|^3} - \frac{Gmm_2(\mathbf{r}-\mathbf{r}_2)}{|\mathbf{r}-\mathbf{r}_2|^3}.$$

Cancelling m and choosing units in which $G(m_1 + m_2) = 1$, we get

$$\ddot{\mathbf{r}} = -\frac{p(\mathbf{r}-\mathbf{r}_1)}{|\mathbf{r}-\mathbf{r}_1|^3} - \frac{q(\mathbf{r}-\mathbf{r}_2)}{|\mathbf{r}-\mathbf{r}_2|^3}, \tag{3.1}$$

where $p = m_1/(m_1 + m_2)$, $q = m_2/(m_1 + m_2)$. The right hand side is the gravitational acceleration $m^{-1}\mathbf{F}$ due to the action of m_1 and m_2 on m.

3.2. The Lagrange points

We now set to find the five equilibrium points of this dynamical system.

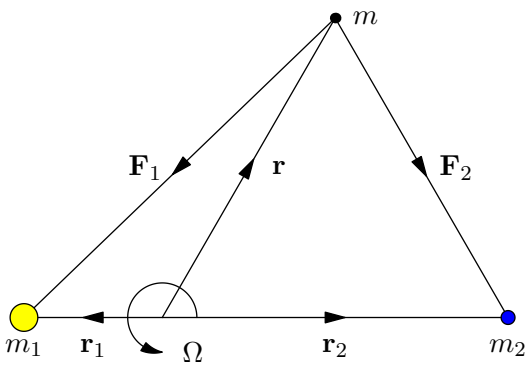

FIGURE 3.1. The restricted planar 3-body poblem

We will describe the system in a co-rotating coordinate system, i.e. one in which the two main bodies m_1 and m_2 are stationary. The rotation of the coordinate system introduces two pseudo forces, the *coriolis* and *centrifugal* forces hence the *effective force* is:

$$\mathbf{F}_\Omega(\mathbf{r},\dot{\mathbf{r}}) = \mathbf{F}(\mathbf{r}) - 2m\,\boldsymbol{\Omega}\times\dot{\mathbf{r}} - m\,\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}). \tag{3.2}$$

The first additional term $-2m\,\boldsymbol{\Omega}\times\dot{\mathbf{r}}$ is the coriolis force, and the second term $m\,\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})$ is the centrifugal force. We are looking for stationary points $\mathbf{r} = \mathbf{r}_0$ which are solutions. Thus we can set $\dot{\mathbf{r}} = 0$ in $\mathbf{F}_\Omega$ and look for points such that $\mathbf{F}_\Omega(\mathbf{r}_0,\mathbf{0}) = 0$.

We take Cartesian coordinates in the plane with the origin at the center of mass and write:

$$\boldsymbol{\Omega} = \Omega\mathbf{k}, \quad \mathbf{r}_1 = -qR\mathbf{i}, \quad \mathbf{r}_2 = pR\mathbf{i}, \quad \mathbf{r} = x\mathbf{i} + y\mathbf{j}.$$

Using $\Omega^2R^3 = 1$, we find that we have to solve

$$\begin{aligned} m^{-1}\mathbf{F}_\Omega(\mathbf{r},\mathbf{0}) = \Omega^2\left(x - \frac{p(x+qR)R^3}{((x+qR)^2+y^2)^{3/2}} - \frac{q(x-pR)R^3}{((x-pR)^2+y^2)^{3/2}}\right)\mathbf{i} \\ + \Omega^2\left(y - \frac{pyR^3}{((x+qR)^2+y^2)^{3/2}} - \frac{qyR^3}{((x-pR)^2+y^2)^{3/2}}\right)\mathbf{j} = \mathbf{0}. \end{aligned} \tag{3.3}$$

We note that $m^{-1}\mathbf{F}_\Omega(\mathbf{r},\mathbf{0}) = -\nabla U(\mathbf{r})$ where $U(\mathbf{r}) = -p/|\mathbf{r}_1| - q/|\mathbf{r}_2| - \Omega^2|\mathbf{r}|^2/2$.

Clearly, the $\mathbf{j}$ component vanishes on the x-axis $y = 0$. Setting the $\mathbf{i}$ component to zero and defining $x = R(s+p)$, we get

$$s + p - p\frac{s+1}{|s+1|^3} - (1-p)\frac{s}{|s|^3} = 0,$$

where we have used $p + q = 1$. Note that s denotes the directed distance to m_2 in units of R. Thus $s = -1$ and $s = 0$ represent collisions of m with m_1 and m_2 respectively, hence have to be excluded. There are then three cases to consider: (i) $s < -1$; $-1 < s < 0$; and $s > 0$. We get respectively

$$(s+p)(s+1)^2s^2 + ps^2 + (1-p)(s+1)^2 = 0, \quad s < -1 \tag{3.4}$$

$$(s+p)(s+1)^2s^2 + ps^2 - (1-p)(s+1)^2 = 0, \quad -1 < s < 0 \tag{3.5}$$

$$(s+p)(s+1)^2s^2 - ps^2 - (1-p)(s+1)^2 = 0, \quad s > 0. \tag{3.6}$$

THEOREM 3.1. *Each of the three Equations* (3.4)-(3.6) *has one solution.*

EXERCISE 3.1. Check the validity of Theorem 3.1 graphically, choosing various values of $0 < p < 1$ for each of the three cases. Indicate if you see any trend.
BONUS: Prove Theorem 3.1 analytically.

The stationary point corresponding to the solution of Equation (3.4) is to the left of m_1. It is denoted L_3. Similarly the point corresponding to the solution of (3.5) lies between the two masses m_1 and m_2. It is denoted L_1. The point corresponding to the solution of (3.6) lies to the right of m_2 and is denoted L_2.

Solutions in closed form cannot be found for Equations (3.4)-(3.6). Instead we now find approximations when $q \ll 1$. Writing (3.4) in terms of q we get

$$s^2((s+1)^3+1) = q((s+1)^2s^2 + s^2 - (s+1)^2).$$

We think of this equation as defining s implicitly as a function of q. Clearly $s(0) = -2$ and differentiating implicitly we find $s'(0) = -7/12$. Thus to first order we get $s = -2 - 7q/12$ and the position of the stationary point L_3 is:

$$(x_3, y_3) \approx \left(-R\left(1 + \frac{5q}{12}\right), 0\right).$$

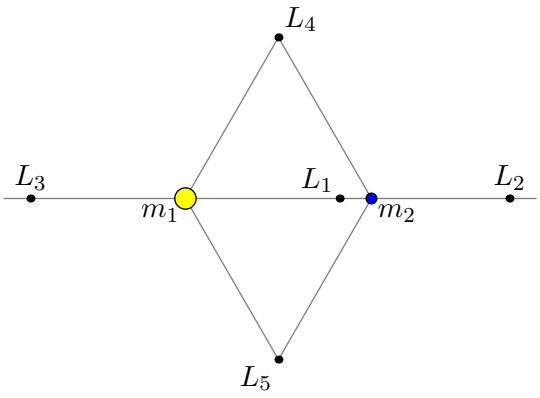

FIGURE 3.2. The Lagrange points

EXERCISE 3.2. Show that for $q \ll 1$

$$(x_1, y_1) \approx \left(R\left(1 - \left(\frac{q}{3}\right)^{1/3}\right), 0\right), \quad (x_2, y_2) \approx \left(R\left(1 + \left(\frac{q}{3}\right)^{1/3}\right), 0\right).$$

To find the remaining two Lagrange points, we decompose the force into directions parallel and perpendicular to $\mathbf{r}$, i.e. project $\mathbf{F}_\Omega$ onto $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ and $\mathbf{r}^\perp = y\mathbf{i} - x\mathbf{j}$. We compute

$$F_\Omega^\perp = \mathbf{F}_\Omega \cdot \mathbf{r}^\perp = pqyR^4 \left(-\frac{1}{((x+qR)^2 + y^2)^{3/2}} + \frac{1}{((x-pR)^2 + y^2)^{3/2}} \right) = 0.$$

This implies the distances from m to m_1 and m_2 are equal. Using this fact, we now find

$$F_\Omega^\parallel = \mathbf{F}_\Omega \cdot \mathbf{r} = R^3(x^2 + y^2) \left(\frac{1}{R^3} - \frac{1}{((x-pR)^2 + y^2)^{3/2}} \right).$$

This in turn implies that the distance from m to m_1 is equal to the distance between m_1 and m_2 hence the three mass lie on an equilateral triangle. These last two Lagrange points L_4 and L_5 are thus given by

$$(x_4, y_4) = \frac{R}{2}\left(p - q, \sqrt{3}\right), \qquad (x_5, y_5) = \frac{R}{2}\left(p - q, -\sqrt{3}\right).$$

3.3. Stability

Using (3.2), let us write the equations of motion as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{v}_x \\ \dot{v}_y \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \\ -\dfrac{\partial U}{\partial x} + 2\Omega v_y \\ -\dfrac{\partial U}{\partial y} - 2\Omega v_x \end{pmatrix}. \tag{3.7}$$

To linearize, consider a family of solutions (x, y, v_x, v_y) depending on a parameter τ. The equation satisfied by the deviation $(\xi, \eta, \nu_x, \nu_y) = (x', y', v_x', v_y')$, where $' = d/d\tau$, is obtained by differentiating (3.7)

with respect to τ:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{v}_x \\ \dot{v}_y \end{pmatrix}' = \begin{pmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{\nu}_x \\ \dot{\nu}_y \end{pmatrix} = \begin{pmatrix} \nu_x \\ \nu_y \\ -\left(\frac{\partial U}{\partial x}\right)' + 2\Omega\nu_y \\ -\left(\frac{\partial U}{\partial y}\right)' - 2\Omega\nu_x \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\partial^2 U}{\partial x^2} & -\frac{\partial^2 U}{\partial x \partial y} & 0 & 2\Omega \\ -\frac{\partial^2 U}{\partial y \partial x} & -\frac{\partial^2 U}{\partial^2 y} & -2\Omega & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \nu_x \\ \nu_y \end{pmatrix}. \tag{3.8}$$

In the case of a stationary point, when $(x, y, v_x, v_y) = (x_0, y_0, 0, 0)$ is constant at $\tau = 0$, the matrix A on the right hand side is to be evaluated at the stationary point $(x_0, y_0, 0, 0)$. The stationary point $(x_0, y_0, 0, 0)$ is *linearly stable* if all eigenvalues of $A|_{(x_0,y_0,0,0)}$ have non-positive real part. The point is *linearly unstable* if some eigenvalue has positive real part. Note that in general linear stability and instability are different from the nonlinear analogous concepts. However if a stationary point is strictly linearly stable it can be shown to be nonlinearly stable. If a stationary point is linearly unstable it is usually nonlinearly unstable [1].

EXERCISE 3.3. Show that L_1, L_2, and L_3 are linearly unstable.
HINT: Compute to lower order in q:

$$\begin{pmatrix} -\frac{\partial^2 U}{\partial x^2} & -\frac{\partial^2 U}{\partial x \partial y} \\ -\frac{\partial^2 U}{\partial y \partial x} & -\frac{\partial^2 U}{\partial^2 y} \end{pmatrix}\Bigg|_{L_{1,2}} = \begin{pmatrix} 9\Omega^2 & 0 \\ 0 & -3\Omega^2 \end{pmatrix}.$$

Similarly:

$$\begin{pmatrix} -\frac{\partial^2 U}{\partial x^2} & -\frac{\partial^2 U}{\partial x \partial y} \\ -\frac{\partial^2 U}{\partial y \partial x} & -\frac{\partial^2 U}{\partial^2 y} \end{pmatrix}\Bigg|_{L_3} = \begin{pmatrix} -3\Omega^2 & 0 \\ 0 & \frac{7}{8}q\Omega^2 \end{pmatrix},$$

BONUS: Compute the *e-folding time*, i.e. the time it takes for the deviation to grow by a factor of e, for the Earth-Sun system.

On the other hand L_4 and L_5 are stable. For example, at L_4 we find:

$$\begin{pmatrix} -\frac{\partial^2 U}{\partial x^2} & -\frac{\partial^2 U}{\partial x \partial y} \\ -\frac{\partial^2 U}{\partial y \partial x} & -\frac{\partial^2 U}{\partial^2 y} \end{pmatrix}\Bigg|_{L_4} = \begin{pmatrix} \frac{3}{4}\Omega^2 & -\frac{3\sqrt{3}}{4}\kappa\Omega^2 \\ -\frac{3\sqrt{3}}{4}\kappa\Omega^2 & \frac{9}{4}\Omega^2 \end{pmatrix},$$

where $\kappa = p - q$. The eigenvalues of A are

$$\pm i\frac{\Omega}{2}\sqrt{2 - \sqrt{27\kappa^2 - 23}}, \quad \pm i\frac{\Omega}{2}\sqrt{2 + \sqrt{27\kappa^2 - 23}}.$$

The point L_4 will be linearly stable if the four eigenvalues are purely imaginary. This is equivalent to

$$\kappa^2 \geq \frac{23}{27}, \sqrt{27\kappa^2 - 23} \leq 2.$$

The first condition is satisfied if and only if

$$m_1 \geq \frac{25m_2}{2}\left(1 + \sqrt{1 - 4/625}\right),$$

and the second is then automatically satisfied; see Figure 3.3.

[1] However that is not necessarily always so.

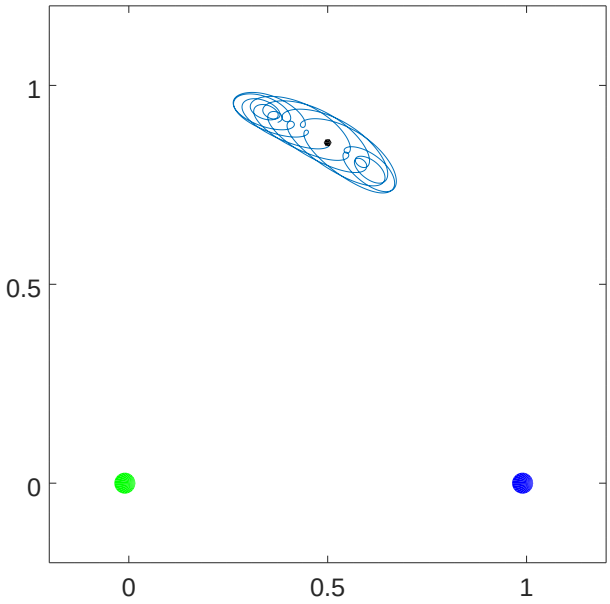

FIGURE 3.3. Motion of a test particle near L4. ($R = 1$, $\Omega = 1$, $q = 0.01$)

EXERCISE 3.4. Verify that L_5 is also stable when the mass ratio m_2/m_1 is small enough.

EXERCISE 3.5. Solve the equations of motion (3.7) near each of the stationary points over an interval large enough to illustrate instability of L_1, L_2, L_3 and stability of L_4, L_5.

EXERCISE 3.6. Check that the $z = 0$ plane is linearly stable, i.e. that deviations ζ from $z = 0$ tend to decay.

CHAPTER 4

Linear wave equations in one dimension

4.1. The string

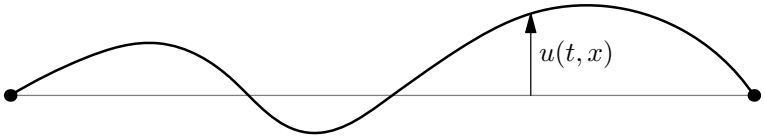

FIGURE 4.1. The string

Consider a taut string stretching on the x-axis from $x = 0$ to $x = L$. We assume the string is vibrating in the vertical direction by a small amount $u(t, x)$. If u is small enough then it satisfies the equation

$$u_{tt} - c^2 u_{xx} = 0, \tag{4.1}$$

where $c^2 = \frac{T}{\mu}$, μ is the *linear density* of the string, and T is the *tension*. In order to solve this equation we must prescribe an initial position $f(x)$ and initial velocity $g(x)$:

$$u(0, x) = f(x), \quad u_t(0, x) = g(x). \tag{4.2}$$

The problem (4.1)-(4.2) is called a Cauchy problem.

4.2. D'Alembert's solution

We first consider the problem on the whole real line $-\infty < x < \infty$. It is not difficult[1] to check that any solution is of the form

$$u(t, x) = \varphi(x + ct) + \psi(x - ct). \tag{4.3}$$

Given the initial conditions (4.2), we can find the functions φ and ψ from the system

$$\varphi(x) + \psi(x) = f(x), \quad \varphi(x) - \psi(x) = \frac{1}{c} \int_0^x g(x)\, dx.$$

The first equation is obtained by substituting $t = 0$ in (4.3) and using the initial condition for u. The second equation is obtained by differentiating (4.3), substituting $t = 0$, using the initial condition fo u_t, and integrating. Solving this sytem, we obtain

$$\varphi(x) = \frac{1}{2}\left(f(x) + \frac{1}{c} \int_0^x g(s)\, ds\right), \quad \psi(x) = \frac{1}{2}\left(f(x) - \frac{1}{c} \int_0^x g(s)\, ds\right).$$

Hence the solution of the 1-d infinite string problem

$$u(t, x) = \frac{1}{2}(f(x + ct) + f(x - ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\, ds. \tag{4.4}$$

From this solution, one immediately obtains *Huygen's principle*: The infinite string returns to an undisturbed state after passage of the wave, i.e. $u(t, x) = 0$ for all t large enough. In fact, if

[1] By substituting $\xi = x + ct$, $\eta = x - ct$, the wave equation (4.1) transforms into $u_{\xi\eta=0}$.

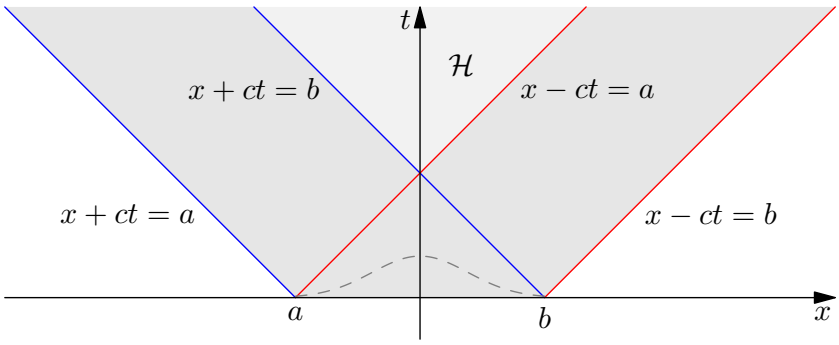

FIGURE 4.2. Huygen's Principle. The dashed curve represents an initial disturbance (whose amplitude should really be in a third dimension) compactly supported in the interval $[a, b]$. The disturbance propagates in the two dark gray regions, one bounded by the blue lines (the $\varphi(x+t)$ part of the d'Alembert solution), the other by the red lines (the $\psi(x-t)$ part of the d'Alembert solution). The Huygens principle states that the solution is identically zero in the light gray region denoted $\mathcal{H}$.

$f(x) = g(x) = 0$ outside the interval (a, b), then $u(t, x) = 0$ for x outside $(a + ct, b + ct) \cup (a - ct, b - ct)$, see Figure 4.2.

EXERCISE 4.1. Verify directly that (4.4) satisfies the Cauchy problem (4.1)-(4.2).

EXERCISE 4.2. Write Matlab/Octave code to solve the Cauchy problem for the wave equation on the infinite line given initial conditions f and g of compact support. Write code which works for functions f and g given analytically, as well as given numerically. Solve on a time interval large enough to observe Huygen's principle. Create an animation.

THEOREM 4.1. *Suppose* $\int_{\infty}^{\infty}(g^2 + f^2)\,dx < \infty$ *then for any solution* u *of the Cauchy problem* (4.1)-(4.2)

$$E(u) = \frac{1}{2}\int_{-\infty}^{\infty}(u_t^2 + c^2u_x^2)\,dx$$

is constant in time.

PROOF. Differentiate $E(u)$ with respect to t to find

$$\frac{dE}{dt} = \int_{-\infty}^{\infty}(u_t u_{tt} + c^2 u_x u_{xt})\,dx = \int_{-\infty}^{\infty} u_t(u_{tt} - c^2 u_{xx})\,dx = 0,$$

where we integrated by parts. The boundary terms in the integration by parts vanish for functions of compact support. For functions of finite energy, as assumed in the statement, it can be shown that the boundary terms tend to zero. We omit the details. □

COROLLARY 4.2. *Given* $\int_{\infty}^{\infty}(g^2 + f^2)\,dx < \infty$, *the Cauchy problem* (4.1)-(4.2) *has a unique finite energy solution.*

EXERCISE 4.3. Prove Corollary 4.2.

4.3. The bounded string

We now consider the case of a bounded string $0 \le x \le L$. This yields a initial boundary value problem:

$$\begin{cases} u_{tt} - c^2u_{xx} = 0, & 0 \le x \le L,\ 0 \le t, \\ u(t,0) = u(t,L) = 0, & 0 \le t, \\ u(0,x) = f(x), \quad u_t(0,x) = g(x), & 0 \le x \le L. \end{cases} \tag{4.5}$$

First note that the energy

$$E(u) = \frac{1}{2}\int_0^L (u_t^2 + u_x^2)\,dx$$

is conserved. Indeed, one can use $u_t(t,0) = u_t(t,L) = 0$ to show the boundary terms vanish in the integration by parts in the proof of Theorem 4.1. In particular uniqueness holds as above. Next, we note that d'Alembert's solution can be used to write a solution here as well, by extending f and g to be odd on $[-L,L]$ and then periodic on $\mathbb{R}$ with period $2L$. Clearly the d'Alembert solution satisfies the equation and the initial conditions on $[0,L]$. It remains to check that it verifies the boundary conditions.

EXERCISE 4.4. Show that the d'Alembert solution with the odd $2L$ periodic extensions of the initial conditions satisfies the boundary conditions.

EXERCISE 4.5. Write Matlab/Octave code to solve (4.5) using the code written in exercise 4.2. HINT: If one wishes to solve for $0 \le t \le T$, then it is only necessary to extend f and g to the interval $[-T, L+T]$.

4.4. Separation of variables

Although an explicit solution to the problem was obtained via the d'Alembert solution, it is worthwhile to examine a second method of solution: *separation of variables.* The motivation is twofold: this second method generalizes to higher dimensions easily, and it introduces the concept of *modes of vibration* which have applications in numerous situations.

Consider the *ansatz*: $u(t,x) = T(t)X(x)$ then substituting in (4.1) and dividing by u we get:

$$\frac{\ddot{T}}{T} - c^2\frac{X''}{X} = 0,$$

from which it follows immediately that $c^{-2}\ddot{T}/T = X''/X = \lambda$ a constant. Also from the boundary conditions we get $X(0) = X(L) = 0$. Since if $\lambda > 0$, $X(x) = X(0)\cosh(x\sqrt{\lambda}) + X'(0)\sinh(x\sqrt{\lambda})$, the only solution satisfying both boundary conditions is $X \equiv 0$. We conclude that $\lambda < 0$, and from the boundary condition $X(L) = 0$, in fact $X(x) = \sin(n\pi x/L)$, hence $\lambda = -(n\pi/L)^2$, for $n = 1, 2, \ldots$. It follows that $T = a_n\cos(n\pi ct/L) + b_n\sin(n\pi ct/L)$ and we get the solutions

$$u_n(t,x) = \left(a_n\cos\left(\frac{n\pi ct}{L}\right) + b_n\sin\left(\frac{n\pi ct}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right)$$

which satisfies the equation and the boundary conditions. We then look for a solution which satisfies also the initial conditions as a series $\sum_{n=1}^{\infty} u_n$. We get

$$f(x) = \sum_{n=1}^{\infty} a_n\sin\left(\frac{n\pi x}{L}\right), \tag{4.6}$$

$$g(x) = \frac{c\pi}{L}\sum_{n=1}^{\infty} n b_n\sin\left(\frac{n\pi x}{L}\right). \tag{4.7}$$

The next section is devoted to find the *mode amplitudes*, i.e. coefficients a_n and b_n.

4.5. Fourier series

We want to express a real function $f(x)$ on $[0,L]$ which vanishes at the end points as a *trigonometric series*:

$$f(x) = \sum_{n=1}^{\infty} a_n\sin\left(\frac{n\pi x}{L}\right). \tag{4.8}$$

In order to determine the coefficients, it is more convenient to first consider a more general problem: express an arbitrary complex valued function $f(x)$ on $[-L, L]$ as a *Fourier series*

$$f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\pi x/L}. \tag{4.9}$$

Clearly, if $n \neq m$ then

$$\int_{-L}^{L} e^{in\pi x/L} e^{-im\pi x/L}\, dx = \int_{-L}^{L} e^{i(n-m)\pi x/L}\, dx = \frac{-i\pi}{(n-m)L}\, e^{i(n-m)\pi x/L}\Big|_{x=-L}^{L} = 0. \tag{4.10}$$

If the series (4.9) converges uniformly[2] on $[-L, L]$, then multiplying by $e^{-im\pi x/L}$ and integrating term by term we get

$$\int_{-L}^{L} f(x) e^{-im\pi x/L}\, dx = \sum_{n=1}^{\infty} \alpha_n \int_{-L}^{L} e^{in\pi x/L} e^{-im\pi x/L}\, dx = \alpha_m \int_{-L}^{L} dx = \alpha_m 2L.$$

It follows that the *Fourier coefficients* are given by:

$$\alpha_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i\pi nx/L}\, dx. \tag{4.11}$$

We now wish to go back to our series (4.8) over the reals. Note that if $f(x)$ is real valued, as is the case in our situation, and odd on $[-L, L]$, which we can assume thanks to the boundary conditions, we have $\operatorname{Re} \alpha_n = 0$, i.e. $\alpha_n = -\frac{i}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$, and thus $\alpha_{-n} = -\alpha_n$. Setting $a_n = 2i\alpha_n$ we get

$$a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \tag{4.12}$$

and

$$\alpha_n e^{in\pi x/L} + \alpha_{-n} e^{-in\pi x/L} = \alpha_n \left(e^{in\pi x/L} - e^{-in\pi x/L}\right) = a_n \sin\left(\frac{n\pi x}{L}\right),$$

so that we recover series (4.8).

Exercise 4.6. (a) Write your own Matlab/Octave code to compute the coefficients (4.12) of a function $f(x)$ on $[0, L]$ which vanishes at the end points. Recompose the function using the series (4.8) and compare with the original function.

(b) Use (a) to solve the initial value problem for various initial conditions. In particular study the wave solution for the *plucked string*: $u(0, x) = L/2 - |x - L/2|$, $u_t(0, x) = 0$; and the *hammered string*: $u(0, x) = 0$, $u_t(0, x) = \delta_{L/2}$.

4.6. The Fast Fourier Transform

Definition 4.1. The *Discrete Fourier Transform* (DFT) of a vector $y = (y_0, \ldots, y_{N-1})$ is the vector $Y = (Y_0, \ldots, Y_{N-1})$ with

$$Y_n = \sum_{j=0}^{N-1} y_j e^{-2\pi ijn/N}, \quad n = 0, \ldots, N-1. \tag{4.13}$$

We denote $\mathcal{F}(y) = Y$. We note that if $f(x)$ is a periodic complex valued function on $[-L, L]$, this approximates the Fourier coefficients (4.11) except for a frequency shift and division by N. Indeed, noting first that the function $f(x)e^{-i\pi nx/L}$ is periodic of period $2L$, and taking the step size $h = 2L/N$

[2]This hypothesis can be significantly weakened.

and using the trapezoid rule, taking into account that $f(-L) = f(L)$, we find

$$\alpha_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\pi nx/L}\,dx = \frac{1}{2L}\left(\int_{-L}^{0} + \int_{0}^{L}\right) f(x)e^{-i\pi nx/L}\,dx$$
$$= \frac{1}{2L}\left(\int_{L}^{2L} + \int_{0}^{L}\right) f(x)e^{-i\pi nx/L}\,dx \approx \frac{1}{N}\sum_{j=0}^{N-1} f(x_j)e^{-i\pi nx_j/L} = \frac{1}{N}\sum_{j=0}^{N-1} y_j e^{-2\pi ijn/N} = \frac{1}{N}Y_n,$$

where we have set $x_j = jh$, $j = 0, \ldots, N-1$, and $y_j = f(x_j)$. Furthermore, the coefficients Y_n are periodic in n with period N. It is often customary to shift the last $N/2$ coefficients to the beginning of the sequence so the sequence is $\tilde{Y} = (Y_{-N/2}, \ldots, Y_{N/2-1})$ when N is even, and $\tilde{Y} = (Y_{-(N-1)/2}, \ldots, Y_{(N-1)/2})$ when N is odd. This is more in line with the coefficients α_n which are distributed over positive and negative frequencies.

The DFT requires $O(N^2)$ operations. A *Fast Fourier Transform* (FFT) algorithm is an algorithm which computes Y_n in $O(N \log N)$ operations at most. All FFT algorithms use $O(N \log N)$ operations, but it is not known whether or not there exists an algorithm which uses fewer operations. For example the *Cooley-Tukey* algorithm saves on the order of 1,000 fold operations for $N = 4096$, from about 30 millions to 30 thousand. The Matlab/Octave function `fft` computes Y using a FFT. The Matlab/Octave function `fftshift` shift Y to obtain $\tilde{Y}$.

If we think of (4.11) as the Fourier transform on $[-L, L]$, sending $f(x) \mapsto (\alpha_n)$, then we might think of (4.9) as the inverse Fourier transform which recovers the function $f(x)$ from the coefficients (α_n). This suggests that

$$Y_n \mapsto y_j = \frac{1}{N}\sum_{n=0}^{N-1} Y_n e^{2\pi inj/N}, \quad j = 0, \ldots, N-1 \tag{4.14}$$

should be the inverse of $\mathcal{F}$.

LEMMA 4.3. *Let $0 < |k| < N$ be an integer, then*

$$\sum_{n=0}^{N-1} e^{2\pi ink/N} = 0.$$

THEOREM 4.4. *Let $y = \mathcal{G}(Y)$ be defined by* (4.14)*. Then $\mathcal{G}$ is the inverse of $\mathcal{F}$.*

EXERCISE 4.7. Prove Theorem (4.4) using Lemma 4.3. BONUS: Prove the Lemma.

The Matlab/Octave function `ifft` computes the inverse Fourier transform using an FFT.

EXERCISE 4.8. Use `fft`, `fftshift`, and `fft`, to redo the decomposition of given functions to frequency modes as in Exercise 4.6. Compare the results and the computer time used.

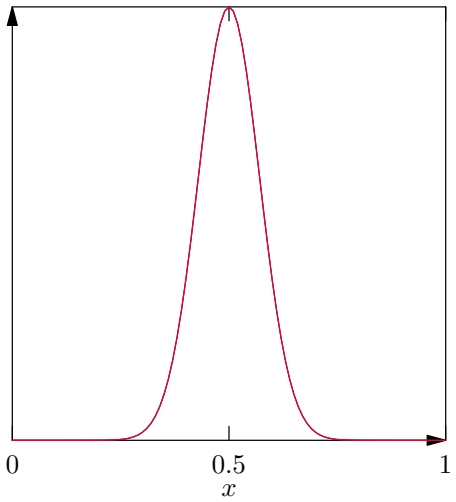

FIGURE 4.3. Comparison between original function and the function reconstructed from Fourier coefficients. The error is not visible at this resolution. See `fourier.m` in Section 4.7. See also the logarithmic plot of the error in Figure 4.4.

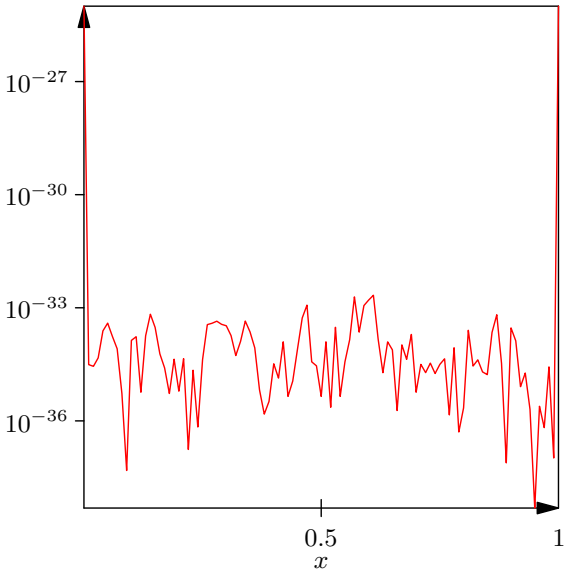

FIGURE 4.4. Logarithmic plot of the error between the original function and the function reconstructed from Fourier coefficients.

4.7. Octave Code

Code to simulate some solutions of the wave equation in 1 dimension. Solutions to some of the exercises.

`dalembert.m` - computes the d'Alembert solution at given time t.

```
function u=dalambert(f,g,x,t)
    c=1;
    u=0.5*(f(x+c*t)+f(x-c*t));
end
```

`run_dalembert.m` - driver to compute the d'Alembert solution with $f(x) = e^{-x^2}$ a Gaussian, and $g(x) = 0$.

```
f=@(x) exp(-x.^2);
c=1;
g=@(x) 2*c*x*exp(-(x).^2).*0;
x=-10:0.1:10;
dt=0.1;
for t=0:dt:12.5;
  u=dalambert(f,g,x,t);
  plot(x,u);
  axis([-10 10 -.2 1]);
  pause(dt/2);
end
```

`tc.m` - Computes trigonometric series coefficients in (4.12).

```
function a=tc(y)
  h=1/(length(y)-1);
  x=[0:h:1];
  n=[0:length(y)-1];
  a=2*h*y*sin(pi*x'*n);
end
```

`fourier.m` - Script which computes trigonometric coefficients `a`, then trigonometric series (4.8) from the coefficients and compares the two graphically. The example used is Gaussian. See Figure 4.3.

```
clear;
x=[0:.01:1];
f=@(s) exp(-100*(s-.5).^2);
y=f(x);
plot(x,y);
hold on;
a=tc(y);
n=[0:length(y)-1];
w=a*sin(pi*x'*n);
plot(x,w);
hold off;
```

CHAPTER 5

Wave equations in higher dimensions

5.1. The fundamental solution of the wave equation in dimensions 2 and 3

There is no simple analog of the D'Alembert solution in higher dimensions. Instead there is an integral representation of the solution of the initial value problem. It turns out to be easier to do in odd dimensions and then obtain the solution in even dimensions by the *method of descent*. For simplicity we will concentrate on dimensions 3 and then 2 but similar ideas can be applied in odd and even dimensions respectively. We skip the derivation and only give the final result.

Consider the Cauchy problem for the wave equation in the whole space:

$$\begin{cases} u_{tt} - c^2\Delta u = 0, & \text{on } \mathbb{R}^3 \\ u(0,p) = f(p), \quad u_t(0,p) = g(p), & p \in \mathbb{R}^3. \end{cases} \tag{5.1}$$

We assume that f is differentiable and g is continuous.

Let $\omega = \frac{4}{3}\pi$ be the volume of the unit ball in $B^3 \subset \mathbb{R}^3$. Then the surface area of its boundary, the unit sphere S^2 is $3\omega = 4\pi$. Furthermore the volume of any ball of radius r is ωr^3 and the surface area of its boundary is $3\omega r^2$. Interestingly, these formulas generalize. The volume of the unit n-ball in n dimensions is $\omega_n = \pi^{n/2}/\Gamma\left(\frac{n}{2}+1\right)$, and consequently the volume of n-ball of radius r is $\omega_n r^n$ and the 'surface' area of its boundary is $n\omega_n r^{n-1}$. This is useful to generalize what we do next to higher dimensions.

Define the spherical average of any function f on the sphere $\partial B_r(p)$ by

$$\bar{f}(p,r) = \fint_{\partial B_r(p)} f(q)\, dA(q) = \frac{1}{n\omega r^{n-1}} \int_{\partial B_r(p)} f(q)\, dA(q),$$

where $B_r(p)$ is the ball of radius r centered at p, and $\partial B_r(p)$ is its boundary. The main idea is to take spherical averages of the solution of (5.1) and to show that for fixed p, $\bar{u}(t,p;r)$ satisfies the problem:

$$\begin{cases} \bar{u}_{tt} - c^2\left(\bar{u}_{rr} + \frac{2}{r}\,\bar{u}_r\right) = 0, \quad r \geq 0 \\ \bar{u}(0,p;r) = \bar{f}(p;r), \quad \bar{u}_t(0,p;r) = \bar{g}(p;r), \quad r \geq 0. \end{cases}$$

This is the spherically symmetric wave equation in 3-d. It turns out that $v = r\bar{u}$ satisfies the usual wave equation in 1-d. Now one can use the d'Alembert solution (using an even extension for $r < 0$) to find v, and take the limit of v/r as $r \to 0+$ to get back u, thanks to continuity. When all is said and done we get

$$u(t,p) = \frac{1}{4\pi c^2 t^2} \int_{\partial B_{ct}(p)} \Big(f(q) + \nabla f(q)\cdot(q-p) + tg(q)\Big)\, dA(q). \tag{5.2}$$

This is *Kirchoff's formula*. Several things can be viewed from this immediately. First, the Huygen's principle holds in dimension 3, i.e. signals propagate along the characteristic cone, and not in its interior. Also, for compactly supported initial data, the solution decays like t^{-2} which is the area of the *wave front*, the intersection of the characteristic cone with the time slice at t. Finally, the solution is at least as smooth as g and ∇f.[1]

[1] In fact, one can do slightly better.

Now consider problem (5.1) in dimension 2. By setting $\tilde{u}(t,x,y,z) = u(x,y)$, $\tilde{f}(x,y,z) = f(x,y)$, $\tilde{g}(x,y,z) = g(x,y)$, we can view the problem as a problem in dimension 3, with the data and the solution independent of z. We can then use Kirchoff's solution (5.2) and using the z independence, translate back to dimension 2. One gets:

$$u(t,p) = \frac{1}{2\pi c^2 t^2} \int_{B_{ct}(p)} \frac{ctf(q) + ct^2 g(q) + ct\nabla f(q)\cdot(q-p)}{\left(c^2t^2 - |q-p|^2\right)^{1/2}} \, dA(q)$$

This is the so-called method of descent. We note that Huygen's principle no longer holds since signals now propagates along in the interior of the characteristic cone, not just on its boundary. The decay for compactly supported data is now t^{-1}. Finally note that some analysis is required to determine when the integral converges since the denominator vanishes on the boundary of the characteristic cone.

5.2. Separation of variables

In contrast to the above, the method of separation of variables generalizes in a straighforward manner to higher dimensions. Consider the wave equation on a bounded domain $\Omega \subset \mathbb{R}^n$:

$$\begin{cases} u_{tt} - c^2\Delta u = 0, & \text{on } \Omega \\ u(t,p) = 0, & t > 0, \quad p \in \partial\Omega \\ u(0,p) = f(p), \quad u_t(0,p) = g(p), & p \in \Omega. \end{cases} \tag{5.3}$$

For simplicity, we have chosen vanishing *Dirichlet* boundary conditions $u(t,p) = 0$. As in Section 4.4, assume that the function $u(t,p) = \tau(t)v(p)$, then as before we get

$$\frac{\ddot{\tau}}{\tau} - c^2\frac{\Delta v}{v} = 0,$$

and as before, this implies $\Delta v = -\lambda v$, where $\lambda > 0$, and hence $\tau(t) = a\cos(\sqrt{\lambda}\,ct) + b\sin(\sqrt{\lambda}\,ct)$. Furthermore, it can be proved that the possible values of λ are discrete λ_n with $\lambda_n \to \infty$ as $n \to \infty$. We thus look for a solution as a series

$$u_n(t,p) = \sum_{n=1}^{\infty} \left(\alpha_n \cos(\sqrt{\lambda_n}\,ct) + \beta_n \sin(\sqrt{\lambda_n}\,ct)\right) v_n(p) \tag{5.4}$$

where v_n is a solution of

$$\begin{cases} \Delta v_n + \lambda_n v_n = 0, \\ v_n(p) = 0, \qquad p \in \partial\Omega, \end{cases} \tag{5.5}$$

and where

$$f(p) = \sum_{n=1}^{\infty} \alpha_n v_n(p), \quad g(p) = c\sum_{n=1}^{\infty} \lambda_n \beta_n v_n(p). \tag{5.6}$$

The problem (5.5) is referred to as an *eigenvalue problem*. In this particular case, it is called the *Helmholtz equation*. The *eigenvalues* are $-\lambda_n$ and the *eigenfunctions* are v_n. On a rectangle (or its generalizations in higher dimensions), it is quite straightforward to address this type of problem by separation of variables just as above. However it is much more interesting to consider other domains. The wave equation on a circular disk in two dimensions leads to the study of Bessel functions. These can then also be used for the wave equations on a cylinder in $\mathbb{R}^3$. The wave equation on a spherical ball in $\mathbb{R}^3$ leads to spherical Bessel functions, Legendre polynomials, and spherical harmonics.

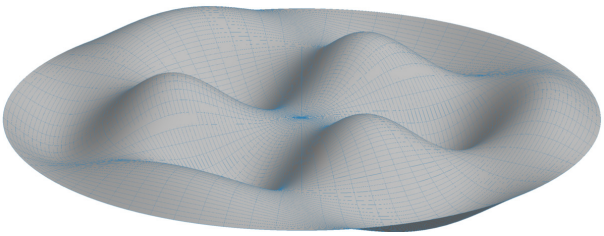

FIGURE 5.1. Circular drum with mode $J_3(\sqrt{\lambda_{3,2}}\, r)\sin(3\theta)$ shown.

5.3. The vibrating circular drum

Because of the boundary condition, it is advantageous to express problem (5.5) in polar coordinates $v = v(r,\theta)$. We drop the subscript n for now to simplify the notation.

$$\begin{cases} v_{rr} + \dfrac{1}{r}v_r + \dfrac{1}{r^2}v_{\theta\theta} + \lambda v = 0, & r < 1 \\ v(1,\theta) = 0, & 0 \le \theta \le 2\pi \end{cases} \tag{5.7}$$

Now again, suppose that $v(r,\theta) = R(r)\Psi(\theta)$, substitute in the equation and multiply by r^2/V to get

$$\frac{1}{R}\left(r^2R'' + rR' + r^2\lambda R\right) + \frac{1}{\Psi}\Psi'' = 0. \tag{5.8}$$

We have abused the notation here. In the first parenthesis, $'$ represents differentiation by r, while in the last term it represents differentiation by θ. The first term (in parenthesis) is a function of r only and the second a function of θ only hence, both are constant. Since Ψ must be periodic, we get $\Psi = a_n\cos(n\theta) + a_n'\sin(n\theta)$. The equation for R

$$r^2R'' + rR' + r^2\lambda R - n^2R = 0, \tag{5.9}$$

is transformed by a change of variable $x = \sqrt{\lambda}\, r$, and $R(r) = y(\sqrt{\lambda}\, r)$ to

$$x^2y''(x) + xy'(x) + (x^2 - n^2)y(x) = 0.$$

This equation is called the *Bessel differential equation*. Being an ordinary differential equation of second order, it has, for each $n = 0, 1, \ldots$, two linearly independent solutions $J_n(x)$, the *Bessel function of the first kind*, which is regular at the origin, and $Y_n(x)$ the *Bessel function of the second kind*, which is not regular at the origin, hence will not be of interest to us. It is not difficult to see that as $x \to \infty$, the solution $J_n(x)$ approaches a sinusoidal function[2], hence it has infinitely many roots $x_{n,k}$, see Figure 5.2. Now for each root $R(r) = J_n(x_{n,k}r)$ satisfies (5.9) with $\lambda = x_{n,k}^2$, and $R(1) = 0$. Setting $\lambda_{n,k} = x_{n,k}^2$, we get that $J_n\left(\sqrt{\lambda_{n,k}}\, r\right)\left(a_{n,k}\cos(n\theta) + a_{n,k}'\sin(n\theta)\right)$ is an eigenfunction with eigenvalue $-\lambda_{n,k}$; see Figure 5.1. Thus we can seek to write the solution of our problem as a series:

$$\begin{aligned} u(t,r,\theta) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} J_n\left(\sqrt{\lambda_{n,k}}\, r\right)\cos(\lambda_{n,k}\, ct)\left(a_{n,k}\cos(n\theta) + a_{n,k}'\sin(n\theta)\right) \\ + \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} J_n\left(\sqrt{\lambda_{n,k}}\, r\right)\sin(\lambda_{n,k}\, ct)\left(b_{n,k}\cos(n\theta) + b_{n,k}'\sin(n\theta)\right) \end{aligned} \tag{5.10}$$

For simplicity, assume for now that $g(p) = 0$, hence the second series above vanishes, and from the initial conditions (5.6), we get

$$f(r,\theta) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} J_n\left(\sqrt{\lambda_{n,k}}\, r\right)\left(a_{n,k}\cos(n\theta) + a_{n,k}'\sin(n\theta)\right)$$

[2]That is because the equation approaches $y'' + y = 0$.

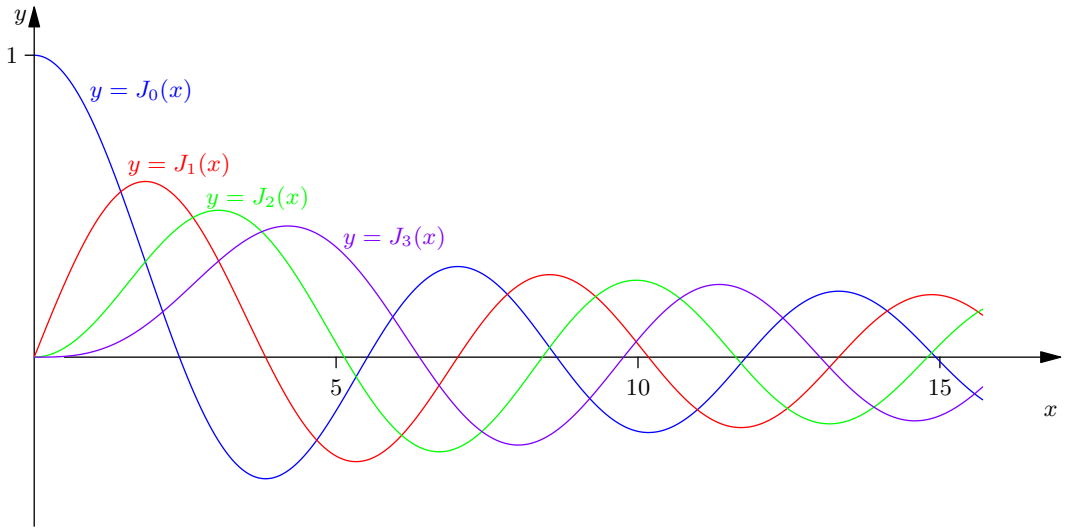

FIGURE 5.2. The Bessel functions $J_n(x)$ for $n = 0, 1, 2, 3$.

In a way similar to the way (4.12) was obtained we can, fixing r, obtain

$$\sum_{k=1}^{\infty} a_{0,k} J_0\left(\sqrt{\lambda_{0,k}}\, r\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(r,\theta)\, d\theta, \tag{5.11}$$

$$\sum_{k=1}^{\infty} a_{n,k} J_n\left(\sqrt{\lambda_{n,k}}\, r\right) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(r,\theta)\cos(n\theta)\, d\theta, \quad n \geq 1, \tag{5.12}$$

$$\sum_{k=1}^{\infty} a'_{n,k} J_n\left(\sqrt{\lambda_{n,k}}\, r\right) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(r,\theta)\sin(n\theta)\, d\theta, \quad n \geq 1. \tag{5.13}$$

On the right of these, we have functions of r and on the left, we have series expansion of these in terms of Bessel functions, much like the expansions (4.8) and (4.9). Indeed, these are called *Bessel-Fourier* series expansions. Following the same strategy as for the string, in order to determine the coefficients, we need to be able to integrate term by term. We will not demonstrate this here. In addition, we need orthogonality conditions as in (4.10). Here these are:

$$\int_0^1 J_n(\sqrt{\lambda_{n,k}}\, r)\, J_n(\sqrt{\lambda_{n,l}}\, r)\, r\, dr = 0, \quad n \geq 0, \quad k, l \geq 1, \quad k \neq l. \tag{5.14}$$

These orthogonality relations are actually a consequence of a general fact concerning so-called *self-adjoint* operators. The proof we present is appropriate for 1-d self-adjoint operator, but it can be easily adapted to any self-adjoint operator in any dimension. Note that (5.9) equation can be rewritten as

$$\frac{1}{w}(wz')' + Vz = -\lambda z, \tag{5.15}$$

where $z(r) = J_n(\sqrt{\lambda_{n,k}}\, r)$, $w(r) = r$, $\lambda = \lambda_{n,k}$ and $V(r) = -n^2/r^2$. Now suppose that z_1, z_2 satisfy (5.15) with λ_1 and λ_2 respectively, and suppose that $\lambda_1 \neq \lambda_2$. Multiplying the equation for z_1 by z_2 and vice-versa, then subtracting we get

$$\big(w(z_2 z_1' - z_1 z_2')\big)' = -(\lambda_1 - \lambda_2) w z_1 z_2.$$

Integrating over $[0, 1]$ we get:

$$w(r)(z_2(r)z_1'(r) - z_1(r)z_2'(r))\big|_{r=0}^{r=1} = -(\lambda_1 - \lambda_2) \int_0^1 z_1(r)\, z_2(r)\, w(r)\, dr.$$

Provided the boundary terms vanish, we immediately get the required orthogonality relation. In our case, $w(0) = 0$ and $z_1(1) = z_2(1) = 0$, hence we have proved (5.14). All we are missing now is the case

$k = l$ in (5.14). Here it is:

$$\int_0^1 J_n(\sqrt{\lambda_{n,k}}\, r)^2 r\, dr = \frac{1}{2} J_{n+1}(\sqrt{\lambda_{n,k}})^2.$$

The proof is relegated to a bonus exercise.

Equipped with these relations, we can now compute the coefficients $a_{n,k}$ and $a'_{n,k}$ in the Bessel-Fourier expansions (5.11)–(5.13), using the same approach as before. For example, multiply (5.11) by $rJ_0(\sqrt{\lambda_{0,l}}\, r)$ and integrate over $[0,1]$. All the terms in the series vanish except for the term $k = l$ so get

$$\frac{1}{2}\, a_{0,l}\, J_1(\sqrt{\lambda_{0,l}})^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^1 f(r,\theta)\, J_0(\sqrt{\lambda_{0,l}}\, r)\, r\, dr\, d\theta, \tag{5.16}$$

We can handle (5.12) and (5.13) in the same way, and we finally obtain:

$$a_{0,k} = \frac{1}{\pi\, J_1(\sqrt{\lambda_{0,k}})^2} \int_{-\pi}^{\pi} \int_0^1 f(r,\theta)\, J_0(\sqrt{\lambda_{0,k}}\, r)\, r\, dr\, d\theta, \tag{5.17}$$

$$a_{n,k} = \frac{1}{2\pi\, J_{n+1}(\sqrt{\lambda_{n,k}})^2} \int_{-\pi}^{\pi} \int_0^1 f(r,\theta)\, J_n(\sqrt{\lambda_{n,k}}\, r)\, \cos(n\theta)\, r\, dr\, d\theta, \quad n \geq 1 \tag{5.18}$$

$$a'_{n,k} = \frac{1}{2\pi\, J_{n+1}(\sqrt{\lambda_{n,k}})^2} \int_{-\pi}^{\pi} \int_0^1 f(r,\theta)\, J_n(\sqrt{\lambda_{n,k}}\, r)\, \sin(n\theta)\, r\, dr\, d\theta, \quad n \geq 1 \tag{5.19}$$

Exercise 5.1. Derive equations similar to (5.17)–(5.19) for the coefficients $b_{0,l}$, $b_{n,k}$ and $b'_{n,k}$.

Exercise 5.2. Write Matlab/Octave code to solve the wave equation with vanishing Dirichlet boundary conditions on the unit disk $\Omega = \{(r,\theta) : r < 1\}$ with initial conditions

$$g(r,\theta) = \begin{cases} 1, & r \leq \frac{1}{4} \\ 0, & \text{otherwise.} \end{cases}$$

HINT: First prove that all the coefficients for $n \geq 1$ are zero! Compute the coefficients $a_{0,k}$ up to k sufficiently large for the desired accuracy and substitute back into the series (5.10). Create an animation.

Exercise 5.3. BONUS: Solve when the hammered region is off-center:

$$g(p) = \begin{cases} 1, & |p - q| \leq \frac{1}{4} \\ 0 & \text{otherwise,} \end{cases}$$

for some point $q \in \Omega$ with $0 < |q| < \frac{3}{4}$.

Exercise 5.4. BONUS: Prove (5.16).
HINT: Multiply the Bessel equation (5.8) by $2y'$ to get

$$(x^2(y')^2) + ((x^2 - n^2)y^2)' - 2xy^2 = 0.$$

Integrate over $[0, \lambda_{n,k}]$ to get

$$2 \int_0^{\lambda_{n.k}} J_n(x)^2\, x\, dx = \lambda_{n,k}^2\, J'_n(\sqrt{\lambda_{n,k}})^2.$$

Finally substitute $x = \sqrt{\lambda_{n,k}}\, r$. You'll also need to know the following:

(i) $J_n(0) = 0$ for $n \geq 1$.
(ii) $J'_n(\sqrt{\lambda_{n,k}}) = J_{n+1}(\sqrt{\lambda_{n,k}})$

These in turn follow from the power series solution of the Bessel equation:

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \left(\frac{x}{2}\right)^{(n+2k)}$$