Elements of Linear Elasticity

A classic textbook: *Theory of Elasticity*, by S.P. Timoshenko and J.N. Goodier, McGraw-Hill, New York.

The aim of continuum mechanics. A rubber band elongates when you pull it. A spring board bends when you stand on it. A bridge vibrates when a truck runs over it. These phenomena show that solids deform. The aim of continuum mechanics is to predict the deformation of a solid in response to a load.

The method of continuum mechanics. The method of continuum mechanics is to view a solid as a continuous distribution of **material particles**, and predict the deformation of the solid by developing algorithms to calculate the motion of all the material particles.

A solid is made of atoms, each atom is made of electrons, protons and neutrons, and each proton or neutron is made of... This kind of description of matter is too detailed. We will not go very far by thinking of a bridge as a pile of atoms.

Instead, we will develop a continuum theory, in which a solid is modeled by a continuous distribution of material particles. Each material particle consists of many atoms. As time progress, the clouds of electrons deform, and the protons jiggle at a maddeningly high frequency. The material particle represents the collective behavior of many atoms.

At a given time, the material particle occupies a **place** in a three-dimensional space. The places in the space are labeled by using a system of **coordinates**. As time progresses, the material particle moves from one place to another place. We can visualize the motion of the material particle by attaching a marker to the particle. Of course we should be careful that the marker should not alter the motion of the material particle. The solid consists of many material particles. Different particles may move in different directions and at different speeds. We can visualize the motion of the entire solid by attaching many markers to the solid.

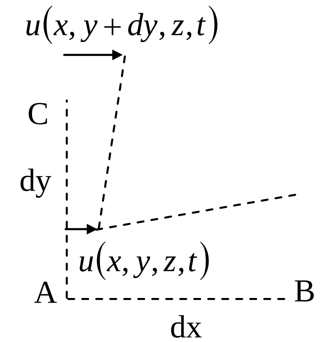
Displacement. At a given time, the positions of all the particles together describe a **configuration** of the solid. As time progresses, the particles move, and the solid changes its configuration. Any configuration of the solid can be used as a reference configuration. Say we use the configuration of the solid at time t_0 as the reference configuration. The configuration of the solid at time t is called the current configuration.

We name a material particle by the coordinates (x,y,z) of the place occupied by the material particle when the solid is in the reference configuration at time t_0. When the solid is in the current configuration at time t, the particle moves to a new place. The **displacement** of the particle is the vector by which the particle moves from its place in the reference configuration to its place in the current configuration. At time t, the particle (x,y,z) has the displacement $u(x,y,z,t)$ in the x-direction, the displacement $v(x,y,z,t)$ in the y-direction, and the displacement $w(x,y,z,t)$ in the z-direction.

A function of coordinates is known as a field. The field of displacement is a time-dependent field. At a given time, the field of displacement describes the configuration of the solid. Thus, the central aim of continuum mechanics is to develop methods to predict the field of displacement as time progresses.

It is sometimes convenient to write the coordinates of a material particle in the reference configuration as (x_1,x_2,x_3), and the field of displacement as $u_1(x_1,x_2,x_3,t)$, $u_2(x_1,x_2,x_3,t)$, $u_3(x_1,x_2,x_3,t)$.

If all the particles in the solid move by the same displacement, the solid as a whole moves by a rigid-body translation. By contrast, if different particles in the solid move by different displacement vectors, the solid deforms. For example, in a bending beam, material particles on one face of the beam move apart from one another (tension), and material particles on the other face of the beam move toward one another (compression). As another example, in a vibrating rod, the displacement varies with the material particle, and the displacement of each material particle is also a function of time.



Strain. Given a field of displacement in a solid, we can calculate the corresponding field of strain. Consider two material particles of the solid in the reference configuration: particle A at (x,y,z) and particle B at $(x+dx,y,z)$. In the reference configuration, the two particles are distance dx apart. At a given time t, the two particles move to new places. The x-component of the displacement of particle A is $u(x,y,z,t)$, and that of particle B is $u(x+dx,y,z,t)$. Consequently, the distance between the two particles elongates by $u(x+dx,y,z,t)-u(x,y,z,t)$. By definition, the **axial strain** in the x-direction is

$$\varepsilon_x = \frac{u(x+dx,y,z,t)-u(x,y,z,t)}{dx} = \frac{\partial u(x,y,z,t)}{\partial x}.$$

This is a component of strain of the material particle (x,y,z) at time t.

The **shear strain** is defined as follows. Consider two lines of material particles. In the reference configuration, the two lines are perpendicular to each other. The deformation changes the included angle by some amount. This change in the angle defines the shear strain, γ. We now translate this definition into a strain-displacement relation. Consider three material particles A, B, and C. In the reference configuration, their coordinates are A (x,y,z), B$(x+dx,y,z)$, and C$(x,y+dy,z)$. In the deformed configuration, in the x-direction, particle A moves by $u(x,y,z,t)$ and particle C by $u(x,y+dy,z,t)$. Consequently, the deformation rotates line AC about axis z by an angle

$$\frac{u(x,y+dy,z,t)-u(x,y,z,t)}{dy} = \frac{\partial u}{\partial y}.$$

Similarly, the deformation rotates line AB about axis z by an angle

$$\frac{v(x+dx,y,z,t)-v(x,y,z,t)}{dx} = \frac{\partial v}{\partial x}.$$

By definition, the shear strain in the xy plane is the net change in the included angle:

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}.$$

For a solid in a three-dimensional space, the state of strain of a material particle is described by a total of six components. The components of strain relate to the components of displacement as

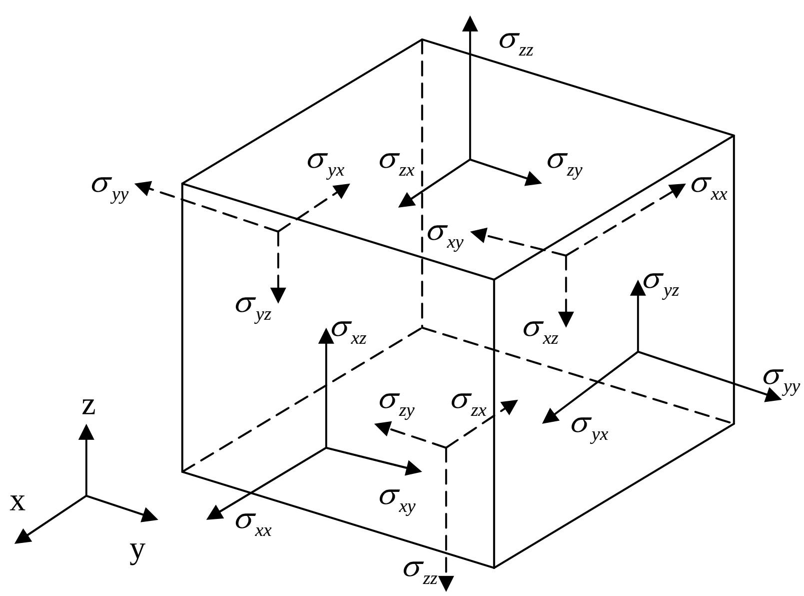
$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

Another definition of the shear strain relates to the definition above by $\varepsilon_{xy} = \gamma_{xy}/2$. With this new definition, we can write the six strain-displacement relations neatly as

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Stress. A material particle suffers a **state of stress**. To talk about stress we need to talk about internal forces. We must expose the internal forces by drawing a **free-body diagram**. Represent the material particle by a small cube, with its edges parallel to the coordinate axes. Cut the cube out from the body to expose all the internal forces on its 6 faces. Define a component of stress by a component of force per unit area. On each face of the cube, there are three **components of stress**, one normal to the face (normal stress), and the other two tangential to the face (shear stresses). Now the cube has six faces, so there are a total of 18 components of stress. A few points below get us organized.



- Notation: σ_{ij}. The first subscript signifies the direction of the component of the force. The second subscript signifies the direction of the vector normal to the face.
- Sign convention. On a face whose normal is in the positive direction of a coordinate axis, the component of stress is positive when it points to the positive direction of the axis. On a face whose normal is in the negative direction of a coordinate axis, the component of stress is positive when it points to the negative direction of the axis.
- Equilibrium of the cube. As the size of the cube shrinks, the forces that scale with the volume (gravity, inertia) are negligible. Consequently, the forces acting on the cube faces must be in static equilibrium. Normal components of stress form pairs. Shear components of stress form quadruples. Consequently, only 6 independent components of stress are needed to describe the state of stress of a material particle.

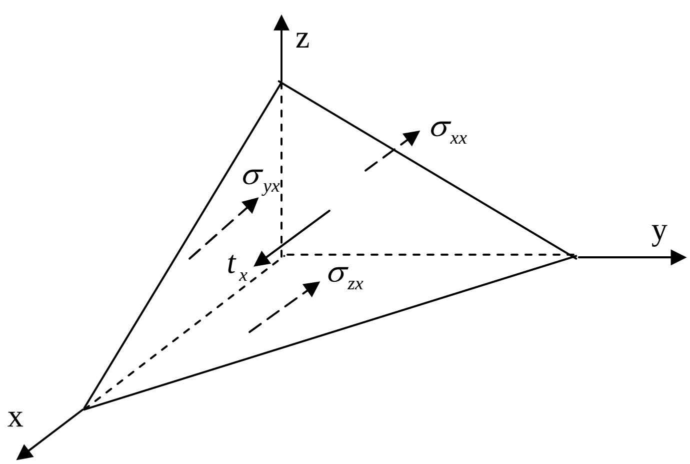
- Write the six components of stress in a 3 by 3 symmetric matrix:

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}.$$

In the above, we have been careless. We have agreed to use (x, y, z) to denote the coordinates of the place in the space occupied by a material particle when the solid is in the reference configuration. But we have then used the same coordinates to denote the place of the material particle in the current configuration when we try to balance force and moment in the current configuration. This practice might be OK when the deformation is small, but will be abandoned later when we do things more rigorously.

Traction. Imagine a plane inside a solid. The plane has the unit normal vector ***n***, with three components n_1, n_2, n_3. The force per area on the plane is called the traction. The traction is a vector, with three components:

$$\mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}.$$



Question: There are infinite many planes through a material particle. How do we determine the traction on each plane?

Answer: You can calculate the traction from

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

We now see the merit of writing the components of stress as a matrix. Also, the six components are indeed sufficient to characterize a state of stress of a material particle, because the six components allow us to calculate the traction on any plane.

Proof of the stress-traction relation. This traction-stress relation is the consequence of the equilibrium of a **tetrahedron** formed by the particular plane and the three coordinate planes. Denote the areas of the four triangles by A, A_x, A_y, A_z. Recall a relation from geometry:

$$A_x = An_x, A_y = An_y, A_z = An_z$$

Balance of the forces in the x-direction requires that

$$t_x A = \sigma_{xx} A_x + \sigma_{xy} A_y + \sigma_{xz} A_z .$$

This gives the desired relation

$$t_x = \sigma_{xx} n_x + \sigma_{xy} n_y + \sigma_{xz} n_z .$$

This relation can be rewritten using the index notation:

$$t_1 = \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3 .$$

It can be further rewritten using the summation convention:

$$t_1 = \sigma_{1j} n_j .$$

Similar relations can be obtained for the other components of the traction vector:

$$t_2 = \sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3$$

$$t_3 = \sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3$$

The three equations for the three components of the traction vector can be written collectively in the matrix form, as give in the beginning of this section. Alternatively, they can be written as

$$t_i = \sigma_{ij} n_j .$$

Here the summation is implied for the repeated index j. The above expression represents three equations. We have just described the **index notation and summation convention.**

Example: stress and traction. A material particle is in a state of stress with the following components:

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 6 \\ 5 & 6 & 4 \end{bmatrix}.$$

(a) Compute the traction vector on a plane intersecting the axes x, y and z at 1, 2 and 3, respectively.
(b) Compute the magnitude of the normal stress on the plane.
(c) Compute the *magnitude* of the shear stress on the plane.
(d) Compute the *direction* of the shear stress on the plane.

Solution. We need to find the unit vector normal to the plane. This is a problem in analytical geometry. The equation of a plane intersecting the axes x, y and z at 1, 2 and 3 is

$$\frac{x}{1} + \frac{y}{2} + \frac{z}{3} = 1$$

Alternatively, a plane can be defined by a given point on the plane, $\mathbf{x}_0$, and a unit vector normal to the plane, $\mathbf{n}$. For any point $\mathbf{x}$ on the plane, $\mathbf{x} - \mathbf{x}_0$ is a vector lying in the plane, so that $\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = 0$, or

$$n_x (x - x_0) + n_y (y - y_0) + n_z (z - z_0) = 0$$

A comparison of the two equations of the plane shows that the normal vector is in the direction $\left[\frac{1}{1}, \frac{1}{2}, \frac{1}{3}\right]$. Normalizing this vector, we obtain the unit vector normal to the plane:

$$\mathbf{n} = \begin{bmatrix} 6/7 \\ 3/7 \\ 2/7 \end{bmatrix}.$$

(a) The traction vector on the plane is

$$\begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 6 \\ 5 & 6 & 4 \end{bmatrix} \begin{bmatrix} 6/7 \\ 3/7 \\ 2/7 \end{bmatrix} = \begin{bmatrix} 22/7 \\ 33/7 \\ 56/7 \end{bmatrix}$$

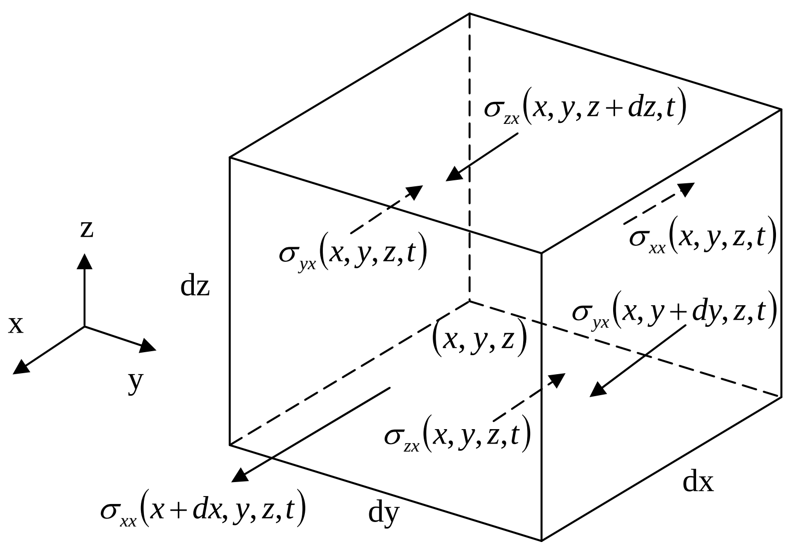
(b) Normal stress on the plane is the traction vector projected on to the normal direction of the plane

$$\sigma_n = \mathbf{t} \cdot \mathbf{n} = \frac{22}{7} \times \frac{6}{7} + \frac{33}{7} \times \frac{3}{7} + \frac{56}{7} \times \frac{2}{7} = 7$$

(c) and (d) The shear stress in the plane is a vector:

$$\tau = \mathbf{t} - \sigma_n \mathbf{n} = \frac{1}{7} \begin{bmatrix} -20 \\ 12 \\ 42 \end{bmatrix}.$$

The direction of this vector is the direction of the shear stress on the plane. The magnitude of the shear stress is 6.86.



A field of stress. Imagine again the body in the three-dimensional space. At time t, the material particle (x, y, z) is under a state of stress $\sigma_{ij}(x, y, z, t)$. Denote the distributed external force per unit volume by $\mathbf{b}(x, y, z, t)$. An example is the gravitational force, $b_z = -\rho g$. The stress and the displacement are time-dependent fields. Each material particle has the acceleration vector $\partial^2 u_i / \partial t^2$. Cut a small differential element, of edges dx, dy and dz. Let ρ be the density. The mass of the differential element is $\rho dxdydz$. Apply Newton's second law in the x-direction, and we obtain that

$$\begin{aligned}&dydz[\sigma_{xx}(x+dx,y,z,t)-\sigma_{xx}(x,y,z,t)]\\&+dxdz[\sigma_{yx}(x,y+dy,z,t)-\sigma_{yx}(x,y,z,t)]\\&+dxdy[\sigma_{zx}(x,y,z+dz,t)-\sigma_{zx}(x,y,z,t)]+b_x dxdydz=\rho dxdydz\frac{\partial^2 u_x}{\partial t^2}\end{aligned}$$

Divide both sides of the above equation by $dxdydz$, and we obtain that

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}+\frac{\partial\sigma_{zx}}{\partial z}+b_x=\rho\frac{\partial^2 u_x}{\partial t^2}.$$

This is the momentum balance equation in the x-direction.

Similarly, the momentum balance equations in the y- and z-direction are

$$\frac{\partial\sigma_{yx}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{zy}}{\partial z}+b_y=\rho\frac{\partial^2 u_y}{\partial t^2}$$

$$\frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}+b_z=\rho\frac{\partial^2 u_z}{\partial t^2}$$

When the body is in equilibrium, we drop the acceleration terms from the above equations.

Using the summation convention, we write the three equations of momentum balance as

$$\frac{\partial\sigma_{ij}}{\partial x_j}+b_j=\rho\frac{\partial^2 u_i}{\partial t^2}.$$

Material model. Homogeneity. When talking about homogeneity, you should think about at least two length scales: a large (macro) length scale, and a small (micro) length scale. A material is said to be *homogeneous* if the macro-scale of interest is much larger than the scale of microstructures. A fiber-reinforced material is regarded as homogeneous when used as a component of an airplane, but should be thought of as *heterogeneous* when its fracture mechanism is of interest. Steel is usually thought of as a homogeneous material, but really contains numerous voids, particles and grains.

Material model. **Isotropy.** A material is isotropic when response in one direction is the same as in any other direction. Metals and ceramics in polycrystalline form are isotropic at macro-scale, even though their constituents—grains of single crystals—are anisotropic. Woods, single crystals, uniaxially fiber reinforced composites are *anisotropic* materials.

Hooke's law. For an isotropic, homogeneous solid, only two independent constants are needed to describe its elastic property: Young's modulus E and Poisson's ratio ν. In addition, a thermal expansion coefficient α characterizes strains due to temperature change. When temperature changes by ΔT, thermal expansion causes a strain $\alpha\Delta T$ in all three directions. The combination of multi-axial stresses and a temperature change causes strains

$$\varepsilon_x=\frac{1}{E}\left[\sigma_x-\nu\left(\sigma_y+\sigma_z\right)\right]+\alpha\Delta T$$

$$\varepsilon_y=\frac{1}{E}\left[\sigma_y-\nu\left(\sigma_z+\sigma_x\right)\right]+\alpha\Delta T$$

$$\varepsilon_z=\frac{1}{E}\left[\sigma_z-\nu\left(\sigma_x+\sigma_y\right)\right]+\alpha\Delta T$$

The relations for shear are

$$\gamma_{xy} = \frac{2(1+\nu)}{E}\sigma_{xy}, \gamma_{yz} = \frac{2(1+\nu)}{E}\sigma_{yz}, \gamma_{zx} = \frac{2(1+\nu)}{E}\sigma_{zx}.$$

Recall the notation $\varepsilon_{xy} = \gamma_{xy}/2$, and we have

$$\varepsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy}, \varepsilon_{yz} = \frac{1+\nu}{E}\sigma_{yz}, \varepsilon_{zx} = \frac{1+\nu}{E}\sigma_{zx}$$

The six stress-strain relation may be written as

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}.$$

The symbol δ_{ij} stands for 0 when $i \neq j$ and for 1 when $i = j$. We adopt the convention that a repeated index implies a summation over 1, 2 and 3. Thus, $\sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33}$.

Express stress in terms of strain. In the above, the 6 components of strain are expressed in terms of the 6 components of stress. From the above relations, we can solve for the components of stress in terms of the components of strain. The resulting relations are

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij},$$

where μ and λ are known as the Lame constants, given by

$$\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}.$$

Theory of Elasticity in a Nutshell

Players: Fields (A total of 15 functions)

- **The field of displacement** is represented by 3 functions, $u(x,y,z;t), v(x,y,z;t), w(x,y,z;t)$.
- **The field of strain** is represented by 6 functions, $\varepsilon_{xx}(x,y,z;t)$, $\varepsilon_{xy}(x,y,z;t)$,...
- **The field of stress** is represented by 6 functions, $\sigma_{xx}(x,y,z;t)$, $\sigma_{xy}(x,y,z;t)$,...

Rules: 3 elements of solid mechanics

- Deformation geometry
- Momentum balance
- Material model

Complete set of equations (see next page)**.**

Boundary conditions

- Prescribe displacement.
- Prescribe traction.

Initial conditions

- Prescribe initial displacement field.
- Prescribe initial velocity field.

Goals

- **Solve boundary value problems.** ODE and PDE.
- **Relate boundary value problems to phenomena.**

Methods:

- **Analytical methods**. S.P. Timoshenko and J.N. Goodier, *Theory of Elasticity*, McGraw-Hill, New York.
- **Numerical methods**. Finite Element Methods. ABAQUS.
- **Handbooks**. R.E. Peterson, *Stress Concentration Factors*, John Wiley, New York, 1974. 2nd edition by W.D. Pilkey, 1997.

Linear Elasticity: Collected Equations

Deformation geometry: strain-displacement relation

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$
$$\varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$
$$\varepsilon_z = \frac{\partial w}{\partial z}, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

Stress-traction relation

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

Momentum balance

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + b_x = \rho \frac{\partial^2 u}{\partial t^2}$$
$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + b_y = \rho \frac{\partial^2 v}{\partial t^2}$$
$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + b_z = \rho \frac{\partial^2 w}{\partial t^2}$$

Hooke's Law

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y - \nu\sigma_z\right) + \alpha\Delta T, \quad \varepsilon_{yz} = \frac{1+\nu}{E}\sigma_{yz}$$
$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_z - \nu\sigma_x\right) + \alpha\Delta T, \quad \varepsilon_{zx} = \frac{1+\nu}{E}\sigma_{zx}$$
$$\varepsilon_z = \frac{1}{E}\left(\sigma_z - \nu\sigma_x - \nu\sigma_y\right) + \alpha\Delta T, \quad \varepsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy}$$

3D Elasticity: Equations in other coordinates

Cylindrical Coordinates $(r,\ \theta,\ z)$

u, v, w are the displacement components in the radial, circumferential and axial directions, respectively. Inertia terms are neglected.

$$\frac{\partial \sigma_r}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta}+\frac{\partial \sigma_{rz}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r}=0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta}}{\partial \theta}+\frac{\partial \sigma_{\theta z}}{\partial z}+\frac{2\sigma_{r\theta}}{r}=0$$

$$\frac{\partial \sigma_{rz}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta}+\frac{\partial \sigma_{z}}{\partial z}+\frac{\sigma_{rz}}{r}=0$$

$$\varepsilon_r=\frac{\partial u}{\partial r},\ \varepsilon_{\theta z}=\frac{1}{2}\left(\frac{\partial v}{\partial z}+\frac{1}{r}\frac{\partial w}{\partial \theta}\right)$$

$$\varepsilon_\theta=\frac{1}{r}\left(\frac{\partial v}{\partial \theta}+u\right),\varepsilon_{zr}=\frac{1}{2}\left(\frac{\partial w}{\partial r}+\frac{\partial u}{\partial z}\right)$$

$$\varepsilon_z=\frac{\partial w}{\partial z},\ \varepsilon_{r\theta}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u}{\partial \theta}+\frac{\partial v}{\partial r}-\frac{v}{r}\right)$$

Spherical Coordinates (r,θ,ϕ)

θ is measured from the positive z-axis to a radius; ϕ is measured round the z-axis in a right-handed sense. u, v, w are the displacements components in the r, θ, ϕ directions, respectively. Inertia terms are neglected.

$$\frac{\partial \sigma_r}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial \sigma_{r\phi}}{\partial \phi}+\frac{1}{r}\left(2\sigma_r-\sigma_\phi-\sigma_\theta-\sigma_{r\theta}\cot\theta\right)=0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta}}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial \sigma_{\theta\phi}}{\partial \phi}+\frac{1}{r}\left[\left(\sigma_\theta-\sigma_\phi\right)\cot\theta+3\sigma_{r\theta}\right]=0$$

$$\frac{\partial \sigma_{r\phi}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta\phi}}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi}}{v\phi}+\frac{1}{r}\left(3\sigma_{r\phi}+2\sigma_{\theta\phi}\cot\theta\right)=0$$

$$\varepsilon_r=\frac{\partial u}{\partial r},\quad \varepsilon_{\theta\phi}=\frac{1}{2r\sin\theta}\left(\sin\theta\frac{\partial w}{\partial \theta}-w\cos\theta+\frac{\partial v}{\partial \phi}\right)$$

$$\varepsilon_\theta=\frac{1}{r}\left(\frac{\partial v}{\partial \theta}+u\right),\quad \varepsilon_{r\phi}=\frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u}{\partial \phi}+\frac{\partial w}{\partial r}-\frac{w}{r}\right)$$

$$\varepsilon_\phi=\frac{1}{r\sin\theta}\left(\frac{\partial w}{\partial \phi}+u\sin\theta+v\cos\theta\right),\quad \varepsilon_{r\theta}=\frac{1}{2}\left(\frac{\partial v}{\partial r}-\frac{v}{r}+\frac{1}{r}\frac{\partial u}{\partial \theta}\right)$$

Example: a rubber layer pressed between two steel plates. A very thin elastic layer, of Young's modulus E and Poisson's ratio ν, is well bonded between two perfectly rigid plates. A thin rubber layer between two thick steel plates is a good approximation of the situation. The thin layer is compressed between the plates by a known normal stress σ_z. Calculate all the components of stress and strain in the thin layer.

Solution. The stress at the edges of the elastic layer is complicated. We will neglect this edge effect, and focus on the field away from the edges, where the layer is in a state of **homogenous deformation**. This emphasis makes sense if we are interested in, for example, the displacement of one plate relative to the other. Of course, this emphasis is misplaced if we wish to study the debonding between the layer and the plates, as debonding may initiate from the edges.

To calculate the state of homogeneous deformation, we do not need to work with the differential equations, because they are automatically satisfied. We will use the algebraic equations that relate the stress and strain.

The shear stresses vanish, but all the three axial stresses, $\sigma_x, \sigma_y, \sigma_z$, are nonzero. By symmetry, we note that

$$\sigma_x = \sigma_y .$$

Because the elastic layer is bonded to the rigid plate, the two components of strain vanish:

$$\varepsilon_x = \varepsilon_y = 0 .$$

That is, the elastic layer is in a state of uniaxial strain: $\varepsilon_z \neq 0$. Using Hooke's law, we obtain that

$$0 = \varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y - \nu\sigma_z\right),$$

or

$$\sigma_x = \frac{\nu}{1-\nu}\sigma_z .$$

Using Hooke's law again, we obtain that

$$\varepsilon_z = \frac{1}{E}\left(\sigma_z - \nu\sigma_y - \nu\sigma_x\right) = \frac{(1+\nu)(1-2\nu)}{(1-\nu)E}\sigma_z .$$

This equation gives the desired result: the strain of the elastic layer is expressed in terms of the applied stress. When the elastic layer is incompressible, $\nu = 0.5$, the layer cannot be strained in just one direction, and will be in a state of hydrostatic stress: $\sigma_x = \sigma_y = \sigma_z$.

Example: **Lamé problem.** A small spherical cavity is in an elastic solid. Remote from the cavity, the solid is in a state of hydrostatic tension. Determine the field of stress in the solid.

The symmetry of the problem makes spherical coordinate system convenient.

List nonzero quantities.

- the radial displacement u,
- the radial stress σ_r, two equal hoop stresses $\sigma_\theta = \sigma_\phi$,
- the radial strain ε_r, two equal hoop strains $\varepsilon_\theta = \varepsilon_\phi$.
- A total of 5 functions of r.

List equations. Use the basic equation sheet. Simplify to the special symmetry.

Deformation geometry: $\varepsilon_r = \dfrac{du}{dr}, \varepsilon_\theta = \dfrac{u}{r}$.

Equilibrium equation: $$\frac{d\sigma_r}{dr} + 2\frac{\sigma_r - \sigma_\theta}{r} = 0 .$$

Material model (elasticity, Hooke's law):

$$\varepsilon_r = \frac{1}{E}(\sigma_r - 2\nu\sigma_\theta),\ \varepsilon_\theta = \frac{1}{E}[(1-\nu)\sigma_\theta - \nu\sigma_r].$$

Reduce to a single ODE. The above are a set of 5 equations for 5 functions of r. You can use a number of approaches to solve them. I'll follow an approach that obtain a single equation for the radial stress, σ_r. From the equilibrium equation, I express σ_θ in terms of σ_r:

$$\sigma_\theta = \sigma_r + \frac{r}{2}\frac{d\sigma_r}{dr}.$$

Then I use the material model to express both strains in terms of σ_r:

$$\varepsilon_r = \frac{1}{E}\left[(1-2\nu)\sigma_r - \nu r\frac{d\sigma_r}{dr}\right]$$

$$\varepsilon_\theta = \frac{1}{E}\left[(1-2\nu)\sigma_r + (1-\nu)\frac{r}{2}\frac{d\sigma_r}{dr}\right]$$

I can eliminate u from the two equations of deformation geometry, and the resulting equation is in terms of the two strains,

$$\varepsilon_r = d(r\varepsilon_\theta)/dr .$$

Express this equation in terms of the radial stress, and I have

$$\frac{d^2\sigma_r}{dr^2} + \frac{4}{r}\frac{d\sigma_r}{dr} = 0 .$$

Solve the ODE. This is an **equidimensional** equation. The solution is of form $\sigma_r = r^m$. Substitute $\sigma_r = r^m$ into the ODE, and we find two roots: $m = 0$ and $m = -3$. Consequently, the full solution is

$$\sigma_r = A + \frac{B}{r^3},$$

where A and B are constants to be determined by the boundary conditions. The hoop stress is given by

$$\sigma_\theta = A - \frac{B}{2r^3}.$$

Apply boundary conditions

- Prescribed remote stress: $\sigma_r = S$ as $r = \infty$.
- Traction-free at the surface of the cavity: $\sigma_r = 0$ as $r = a$

Upon determining the two constants A and B, we obtain the stress distribution

$$\sigma_r = S\left[1 - \left(\frac{a}{r}\right)^3\right],\ \sigma_\theta = S\left[1 + \frac{1}{2}\left(\frac{a}{r}\right)^3\right].$$

- Plot each component of stress as a function of r.
- Verify the boundary conditions.
- At a distance several times the radius of the cavity, the state of stress nearly recovers the applied hydrostatic tension.
- Near the surface of the cavity, the hoop stress is higher than the applied stress.

Stress concentration factor. Note that the hoop stress is nonzero near the cavity surface, where the hoop stress reaches the maximum. The stress concentration factor is the ratio of the maximum stress over the applied stress. In this case, the stress concentration factor is 3/2.

Stress concentration factors are used in practice to predict failure, and they are listed in handbooks for bodies of many shapes and subject to many types of loads.

Elastic energy. Consider a rod, initial length L_0 and cross-sectional area A_0. When a machine applies a force f to the rod, the length of the rod becomes L and the cross-sectional area becomes A. The experimental record gives us the function $f(L)$, which need not be linear. When the length of the rod changes from L to $L + dL$, the machine does work fdL to the rod.

We model an elastic solid with an elastic energy, $F(L)$. This function obeys the following rule. When the length of the rod increases by dL, the increase in the elastic energy of the rod equals the work done by the machine:

$$dF = fdL.$$

Define the stress and strain as

$$\sigma = \frac{f}{A_0}, \qquad \varepsilon = \frac{L - L_0}{L_0}.$$

Define the **elastic-energy density**, w, as the elastic energy per unit volume, namely

$$w = \frac{F}{A_0 L_0}.$$

Here we have used the initial area and initial length to define the stress, the strain, and the elastic-energy density.

With these definitions, we can rewrite $dF = fdL$ as

$$dw = \sigma d\varepsilon.$$

The free energy density is a function of the strain:

$$w = w(\varepsilon).$$

Once we know this function, we can obtain the stress-strain relation by taking the differentiation:

$$\sigma = \frac{\partial w(\varepsilon)}{\partial \varepsilon}.$$

We next restrict ourselves to small strains, so that we can expand the function $w(\varepsilon)$ into a Taylor series in the strain:

$$w(\varepsilon) = \frac{1}{2} E \varepsilon^2.$$

We will only go up to the quadratic term in strain.

The stress is obtained by taking partial differentiation:

$$\sigma = E\varepsilon.$$

Elastic energy density of a block under shear. Consider a block, height H_0 and cross-sectional area A_0. When a machine applies a shear force f to the block, the block deforms by an angle θ. When the angle changes from θ to $\theta + d\theta$, the machine does work $fH_0 d\theta$ to the block.

The elastic energy of the block is a function $F(\theta)$. In equilibrium, the change in the elastic energy of the block equals the work done by the machine:

$$dF = fH_0 d\theta$$

Define the shear stress and the shear strain as

$$\tau = \frac{f}{A_0}, \gamma = \theta .$$

Define the **elastic-energy density**, w, as the elastic energy per unit volume, namely

$$w = \frac{F}{A_0 H_0} .$$

From the above, we have

$$dw = \tau d\gamma .$$

The energy per unit volume is a function of the shear strain, $w(\gamma)$. Once we know this function, we can obtain the stress-strain relation by taking the differentiation:

$$\tau = \frac{\partial w(\gamma)}{\partial \gamma} .$$

When the block is made of a linearly elastic solid, under shear load, the stress-strain relation is $\tau = G\gamma$. Consequently, the energy density function is

$$w(\gamma) = \frac{1}{2} G\gamma^2$$

This result holds only for linear elastic solid in pure shear condition.

When a material particle is in a state of multiaxial stress, the elastic-energy density is a quadratic form of all components of strain. The advantage of using the elastic-energy function becomes clear when the body is in a state of multiaxial stress. We model the elastic solid by stating that the elastic-energy density is a function of all components of strain:

$$w = w(\varepsilon_{11}, \varepsilon_{12}, ...).$$

The components of stress are differential coefficients:

$$dw = \sigma_{pq} d\varepsilon_{pq} .$$

That is

$$\sigma_{pq} = \frac{\partial w(\varepsilon_{11}, \varepsilon_{12}, ...)}{\partial \varepsilon_{pq}}$$

In linear elasticity, we assume that the components of stress are linear in the components of strain. Thus, the energy density is a quadratic form of the components of stain, written as

$$w = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} .$$

Here C_{ijkl} are the components of a fourth-rank tensor called the stiffness tensor. Without losing any generality, we can assume the following symmetries:

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij} .$$

If we count carefully, we should have 21 independent components for a generally anisotropic elastic solid.

The components of stress are linear in the components of strain:

$$\sigma_{pq} = C_{pqij} \varepsilon_{ij} .$$

We can also invert this relation to express the strain in terms of the stress:

$$\varepsilon_{pq} = S_{pqij} \sigma_{ij}$$

Here S_{pqij} are the components of a fourth-rank tensor called the compliance tensor. They have the same symmetry properties.

Stress-strain relation in a matrix form. We can also write the above equations in another form. The state of strain is specified by the six components:

$$\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{yz}, \gamma_{zx}, \gamma_{xy}$$

In this order we will label them as

$$\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6 .$$

The six components of strain can vary independently. The elastic energy per unit volume is a function of all the six components, $w(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6)$. This is the **energy density function**. When each strain component changes by a small amount, $d\varepsilon_i$, the energy density changes by

$$dw = \sigma_1 d\varepsilon_1 + \sigma_2 d\varepsilon_2 + \sigma_3 d\varepsilon_3 + \sigma_4 d\varepsilon_4 + \sigma_5 d\varepsilon_5 + \sigma_6 d\varepsilon_6 .$$

Here we use the engineering strains for the shear, rather than the tensor components. We do so to avoid the factor 2 in the above expression. Each stress component is the differential coefficient of the energy density function:

$$\sigma_i = \frac{\partial w(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6)}{\partial \varepsilon_i} .$$

If the function $w(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6)$ is known, we can determine the six stress-strain relations by the differentiations. Consequently, by introducing the elastic-energy function, we only need to specify one function, rather than six functions, to determine the stress-strain relations.

The above considerations apply to solids with linear or nonlinear stress-strain relations. We now examine linear elastic solids. For the stress components to be linear in the strain components, the energy density function must be a *quadratic form* of the strain components:

$$w = \frac{1}{2}\sum_{i,j} c_{ij}\varepsilon_i\varepsilon_j = \frac{1}{2}\left(c_{11}\varepsilon_1\varepsilon_1 + c_{12}\varepsilon_1\varepsilon_2 + c_{21}\varepsilon_2\varepsilon_1 ...\right).$$

Here c_{ij} are 36 constants. The cross terms come in pairs, e.g., $(c_{12} + c_{21})\varepsilon_1\varepsilon_2$. Only the combination $c_{12} + c_{21}$ will enter into the stress-strain relation, not c_{12} and c_{21} individually. We can call $c_{12} + c_{21}$ by a different name. A convenient way to say that there is only one independent constant is to just let $c_{12} = c_{21}$. We can do the same for other pairs, namely,

$$c_{ij} = c_{ji} .$$

The matrix c_{ij} is symmetric, with 21 independent elements. Consequently, 21 constants are needed to specify the elasticity of a linear anisotropic elastic solid. Because the elastic energy is positive for any nonzero strain state, the matrix c_{ij} is positive-definite.

Recall that each stress component is the differential coefficient of the energy density function, $\sigma_i = \partial w / \partial \varepsilon_i$. The stress relation becomes

$$\sigma_i = \sum_j c_{ij}\varepsilon_j .$$

We list the six components of stress as a column, and list the six components of strain as another column, so that the six stress-strain relations take the form

$$\begin{bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\sigma_4\\\sigma_5\\\sigma_6\end{bmatrix}=\begin{bmatrix}c_{11}&c_{12}&c_{13}&c_{14}&c_{15}&c_{16}\\c_{21}&c_{22}&c_{23}&c_{24}&c_{25}&c_{26}\\c_{31}&c_{32}&c_{33}&c_{34}&c_{35}&c_{36}\\c_{41}&c_{42}&c_{43}&c_{44}&c_{45}&c_{46}\\c_{51}&c_{52}&c_{53}&c_{54}&c_{55}&c_{56}\\c_{61}&c_{62}&c_{63}&c_{64}&c_{65}&c_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\\\varepsilon_4\\\varepsilon_5\\\varepsilon_6\end{bmatrix}$$

The physical significance of the constants c_{ij} is now evident. For example, when the solid is under a *uniaxial strain state*, $\varepsilon_1 \neq 0$, $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = \varepsilon_6 = 0$, the six stress components on the solid are $\sigma_1 = c_{11}\varepsilon_1$, $\sigma_2 = c_{21}\varepsilon_1$,... The matrix c_{ij} is known as the **stiffness matrix**.

Inverting the matrix, we express the strain components in terms of the stress components:

$$\begin{bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\\\varepsilon_4\\\varepsilon_5\\\varepsilon_6\end{bmatrix}=\begin{bmatrix}s_{11}&s_{12}&s_{13}&s_{14}&s_{15}&s_{16}\\s_{21}&s_{22}&s_{23}&s_{24}&s_{25}&s_{26}\\s_{31}&s_{32}&s_{33}&s_{34}&s_{35}&s_{36}\\s_{41}&s_{42}&s_{43}&s_{44}&s_{45}&s_{46}\\s_{51}&s_{52}&s_{53}&s_{54}&s_{55}&s_{56}\\s_{61}&s_{62}&s_{63}&s_{64}&s_{65}&s_{66}\end{bmatrix}\begin{bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\sigma_4\\\sigma_5\\\sigma_6\end{bmatrix}$$

The matrix s_{ij} is known as the **compliance matrix**. The compliance matrix is also symmetric and positive definite.

The components of the stiffness tensor relate to the corresponding components of the stiffness matrix as

$$c_{11} = C_{1111},\quad c_{14} = C_{1123},\quad c_{44} = C_{2323}.$$

However, the corresponding relations for compliance are

$$s_{11} = S_{1111},\quad s_{14} = 2S_{1123},\quad s_{44} = 4S_{2323}.$$

An isotropic, linear elastic solid is characterized by two constants (e.g., Young's modulus and Poisson's ratio) to fully specify the stress-strain relation. Some solids are anisotropic, e.g., fiber reinforced composites, single crystals. Each stress component is a function of all six strain components. Consequently, 21 constants are needed to specify the elasticity of a linear anisotropic elastic solid.

A crystal of cubic symmetry. For a crystal of cubic symmetry, such as silicon and germanium, when the coordinates are along the cube edges, the stress-strain relations are

$$\begin{bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\sigma_4\\\sigma_5\\\sigma_6\end{bmatrix}=\begin{bmatrix}c_{11}&c_{12}&c_{12}&0&0&0\\c_{12}&c_{11}&c_{12}&0&0&0\\c_{12}&c_{12}&c_{11}&0&0&0\\0&0&0&c_{44}&0&0\\0&0&0&0&c_{44}&0\\0&0&0&0&0&c_{44}\end{bmatrix}\begin{bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\\\varepsilon_4\\\varepsilon_5\\\varepsilon_6\end{bmatrix}$$

The three constants c_{11}, c_{12} and c_{44} are independent for a cubic crystal. Isotropic solid is a special case, in which the three constants are related, $c_{44} = (c_{11} - c_{12})/2$.

A fiber-reinforced composite. For a fiber reinforced composite, with fibers in the x_3 direction, the material is isotropic in the x_1 and x_2 directions. The material is said to be

transversely isotropic. Five independent elastic constants are needed. The stress-strain relations are

$$\begin{bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\sigma_4\\\sigma_5\\\sigma_6\end{bmatrix}=\begin{bmatrix}c_{11} & c_{12} & c_{13} & 0 & 0 & 0\\c_{12} & c_{11} & c_{13} & 0 & 0 & 0\\c_{13} & c_{13} & c_{33} & 0 & 0 & 0\\0 & 0 & 0 & c_{44} & 0 & 0\\0 & 0 & 0 & 0 & c_{44} & 0\\0 & 0 & 0 & 0 & 0 & (c_{11}-c_{12})/2\end{bmatrix}\begin{bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\\\varepsilon_4\\\varepsilon_5\\\varepsilon_6\end{bmatrix}$$

Example: stress in an epitaxial film. Both silicon (Si) and germanium (Ge) are crystals of cubic unit cell. The edge length of the unit cell of Si is $a_{\text{Si}}=5.428\text{Å}$, and that of Ge is $a_{\text{Ge}}=5.658\text{Å}$. A Ge film 10 nm thick is grown epitaxially (i.e. with matching atomic positions) on the [100] surface of a 100 μm thick Si substrate. Calculate the stress and strain components in the Ge film. The respective elastic constants are (in GPa)

Si: c_{11} = 165.8, c_{12} = 63.9, c_{44} = 79.6.

Ge: c_{11} = 128.5, c_{12} = 48.2, c_{44} = 66.7.

Solution. Because the Si substrate is much thicker than the Ge film, the strains in the substrate are much smaller than those in the film. We will neglect these small strains, and assume that the substrate is undeformed. Let axis 3 be normal to the film surface, and axes 1 and 2 be in the plane of the film, parallel to the cube edges of the crystal cell. To register one atom on another, Ge must be compressed in directions 1 and 2 to conform to the undeformed atomic unit cell size of Si. The two in-plane strains in the Ge film are

$$\varepsilon_{11}=\varepsilon_{22}=\frac{a_{\text{Si}}-a_{\text{Ge}}}{a_{\text{Ge}}}=-4\% \ .$$

There will be an elongation normal to the film, $\varepsilon_{33}>0$. All shear strains vanish. According to the generalized Hooke's law, the stress normal to the film surface relates to the strains as

$$\sigma_{33}=c_{11}\varepsilon_{33}+c_{12}\varepsilon_{11}+c_{12}\varepsilon_{22} \ .$$

Physically it is evident that there is no stress normal to the surface of the film, $\sigma_{33}=0$. Inserting into the above expression, we obtain that

$$\varepsilon_{33}=-\frac{2c_{12}}{c_{11}}\varepsilon_{11}=+3\% \ .$$

The two in-plane stress components are equal, given by

$$\sigma_{11}=\sigma_{22}=c_{11}\varepsilon_{11}+c_{12}\varepsilon_{22}+c_{12}\varepsilon_{33} \ ,$$

or

$$\sigma_{11}=\sigma_{22}=\left[c_{11}+c_{12}-\frac{2c_{12}^2}{c_{11}}\right]\varepsilon_{11} \ .$$

Inserting the numerical values, we obtain that $\sigma_{11}=\sigma_{22}=-5.6\text{GPa}$. This is a huge stress, it may generate dislocations in the film.