Classical Dynamics

5ccm231

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N.B This module was previously called Intermediate Dynamics but the content is the same

N.B. These notes will be updated as the module progresses. The content is not expected to change but the numbering of the subsections will be altered to align with the video lectures.

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Contents

Chapter 1

Review

1.1 Scalars, Vectors and all That

This chapter is meant to be a review of basic concepts in the geometry of three-dimensional Euclidean space, $\mathbb{R}^3$. Therefore we will be somewhat brief and not as precise as a mathematician should be.

The simplest thing one can imagine is a scalar. Simply put a scalar is a single number; the value of which everyone agrees on. For example the temperature of a given point in this room, *e.g.*

$$T = 20^o C \ . \tag{1.1}$$

(It's a different question as to whether or not you think thats warm or cold.)

Vectors are more interesting physical quantities. In three dimensions they are given by a triplet of numbers. For example the position of point in this room is given by a vector:

$$\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \ . \tag{1.2}$$

We denote vectors by an underline. The three numerical values on the right hand side give the coordinate values of the point in some coordinate system. Why is this so different from three scalars? Because people using different coordinate systems will not use the same values of x, y and z to describe the same point.

We will often use an index notation r^a, $a = 1, 2, 3$ for the components of a vector $\underline{r}$. In particular

$$r^1 = x \qquad r^2 = y \qquad r^3 = z \ . \tag{1.3}$$

Please note that r^2 in this case does not mean r-squared. There is no meaning to the square of a vector (although below we will consider the length-squared of a vector which will be denoted by $|\underline{r}|^2$).

Vectors live in a vector space. This means that one has the following two actions that map vectors to vectors:

- multiplication by a scalar a:

$$a\underline{r} = \begin{pmatrix} ax \\ ay \\ az \end{pmatrix} \ . \tag{1.4}$$

- addition of two vectors:

$$\underline{r}_1 + \underline{r}_2 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} . \tag{1.5}$$

Note that the subscript on $\underline{r}_i$ does not denote the components r^a!.

For this course three-dimensional space is a vector space: $\mathbb{R}^3$. The three means that it is 3-dimensional. This in turn means that one can pick a basis of three vectors, $\underline{e}_1, \underline{e}_2, \underline{e}_3$ so that any other vector can be written uniquely in terms of these three:

$$\underline{r} = a\underline{e}_1 + b\underline{e}_2 + c\underline{e}_3 . \tag{1.6}$$

Of course for $\mathbb{R}^3$ the most natural choice is

$$\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \quad \underline{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \quad \underline{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \tag{1.7}$$

So that, quite simply,

$$\underline{r} = x\underline{e}_1 + y\underline{e}_2 + z\underline{e}_3 . \tag{1.8}$$

But this not the only choice.

More generally a basis for an n-dimensional vector space is a choice of n vectors $\underline{e}_1, ...\underline{e}_n$ that are linearly independent. This means that the equation

$$a_1\underline{e}_1 + ... + a_n\underline{e}_n = \underline{0} \tag{1.9}$$

only has the trivial solution $a_1 = ... = a_n = 0$

The **Scalar Product** (*a.k.a.* dot product, inner product...) is a map that takes two vectors into a number and has the following properties:

- symmetric: $\underline{r}_1 \cdot \underline{r}_2 = \underline{r}_2 \cdot \underline{r}_1$
- distributive: $\underline{r}_1 \cdot (\underline{r}_2 + \underline{r}_3) = \underline{r}_1 \cdot \underline{r}_2 + \underline{r}_1 \cdot \underline{r}_3$

Sometimes a third property is useful (but not always *e.g.* Relativity):

- positive definite: $\underline{r} \cdot \underline{r} \geq 0$ with equality iff $\underline{r} = \underline{0}$.

If this is last property is true then we can define the length of a vector to be

$$|\underline{r}| = \sqrt{\underline{r} \cdot \underline{r}} , \tag{1.10}$$

which allows us to do geometry.

Although one can consider more general possibilities we will simply take

$$\underline{r}_1 \cdot \underline{r}_2 = \sum_{a=1}^{3} r_1^a r_2^a = x_1 x_2 + y_1 y_2 + z_1 z_2 . \tag{1.11}$$

You can check yourself that this satisfies the symmetry, distributive and positive definite properties above. In this case length of a vector is

$$|\underline{r}| = \sqrt{x^2 + y^2 + z^2} , \tag{1.12}$$

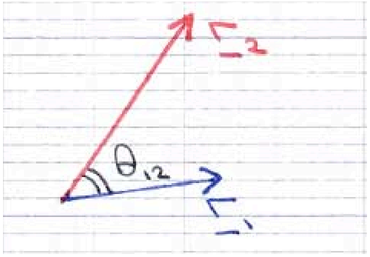

Figure 1.1.1: Angle between two vectors

which is, of course, just the Pythagorian theorem (in 3D). For example

$$\text{if} \quad \underline{r}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \underline{r}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \qquad \text{then} \quad \underline{r}_1 \cdot \underline{r}_2 = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32 \ . \tag{1.13}$$

The interpretation of the scalar product is

$$\underline{r}_1 \cdot \underline{r}_2 = |\underline{r}_1||\underline{r}_2| \cos\theta_{12} \tag{1.14}$$

where θ_{12} is the angle between the two vectors in the plane through the origin defined by $\underline{r}_1$ and $\underline{r}_2$.

This is easily seen in two dimensions. Let

$$\underline{r}_1 = \begin{pmatrix} r_1 \cos\theta_1 \\ r_1 \sin\theta_1 \end{pmatrix} \qquad \underline{r}_2 = \begin{pmatrix} r_2 \cos\theta_2 \\ r_2 \sin\theta_2 \end{pmatrix} \ . \tag{1.15}$$

Then it is easy to see that $|\underline{r}_1| = r_1$ and $|\underline{r}_2| = r_2$ and also

$$\underline{r}_1 \cdot \underline{r}_2 = r_1 r_2 (\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2) = r_1 r_2 \cos(\theta_1 - \theta_2) \tag{1.16}$$

(do you remember your trig identities?!) and so indeed we find $\theta_{12} = \theta_1 - \theta_2$. In three dimensions one can simply rotate the basis until both vectors take the form

$$\underline{r}_1 = \begin{pmatrix} r_1 \cos\theta_1 \\ r_1 \sin\theta_1 \\ 0 \end{pmatrix} \qquad \underline{r}_2 = \begin{pmatrix} r_2 \cos\theta_2 \\ r_2 \sin\theta_2 \\ 0 \end{pmatrix} \ , \tag{1.17}$$

and the result follows again.

Thus, not surprisingly, the basis chosen in (1.7) satisfies:

$$\underline{e}_a \cdot \underline{e}_b = \delta_{ab} = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases} \tag{1.18}$$

i.e. the basis elements are all unit length and orthogonal to each other. Such a basis is called **orthonormal**.

1.2 Vector Product

It is easy to see that the scalar product can be extended to any dimension, we've already used it in two and three dimensions. However in three dimensions there is another product as we will now see.

In three dimensions any two non-parallel vectors $\underline{r}_1$ and $\underline{r}_2$ define a plane through the origin. A plane in three-dimensions has a **normal vector**: that is a vector which is orthogonal to every vector in the plane. This allows us to define a the vector product which takes two vectors and gives a third that is orthogonal to the original two. Explicitly we define

$$(\underline{v} \times \underline{w})^a = \sum_{bc=1}^{3} \epsilon_{abc} v^b w^c \ . \tag{1.19}$$

Here ϵ_{abc} has the following properties:

$$\epsilon_{abc} = \begin{cases} +1 & \text{if}(a,b,c) = (1,2,3),(2,3,1),(3,1,2) \\ -1 & \text{if}(a,b,c) = (3,2,1),(2,1,3),(1,3,2) \\ 0 & \text{otherwise} \end{cases} \tag{1.20}$$

What does this mean? It means that ϵ_{abc} vanishes unless a, b, c are all distinct. In that case $\epsilon_{abc} = +1$ if (a,b,c) is in 'clockwise'[1] (or cyclic) order or $\epsilon_{abc} = -1$ if (a,b,c) is in 'anti-clockwise' (or anti-cyclic) order.

If this seems tricky then it maybe easier to see whats going on by writing out the components:

$$\begin{aligned} (\underline{v} \times \underline{w})^1 &= v^2 w^3 - v^3 w^2 \\ (\underline{v} \times \underline{w})^2 &= v^3 w^1 - v^1 w^3 \\ (\underline{v} \times \underline{w})^3 &= v^1 w^2 - v^2 w^1 \ . \end{aligned} \tag{1.21}$$

Notice again the the sign is determined by whether or not 1,2,3 appear in clockwise or anti-clockwise order.

There are two fundamental properties of the vector product:

- anti-symmetry: $(\underline{v} \times \underline{w}) = -(\underline{w} \times \underline{v})$
- orthogonality: $\underline{v} \cdot (\underline{v} \times \underline{w}) = \underline{w} \cdot (\underline{v} \times \underline{w}) = 0$

Let us check these. In fact anti-symmetry is clear from the definition. Swapping $v^a \leftrightarrow w^a$ in (1.19) or (1.21) changes the overall sign of the right hand sides.

Let us check that the vector product $(\underline{v} \times \underline{w})$ is indeed orthogonal to both $\underline{v}$ and $\underline{w}$. First the fast way:

$$\underline{v} \cdot (\underline{v} \times \underline{w}) = \sum_a v^a (\underline{v} \times \underline{w})^a = \sum_{abc=1}^{3} \epsilon_{abc} v^a v^b w^c = 0 \ . \tag{1.22}$$

Why? because the sum involves $\epsilon_{abc} v^a v^b w^c$ which contains terms of the form

$$\epsilon_{123} v^1 v^2 w^3 + \epsilon_{213} v^2 v^1 w^3 = (+1) v^1 v^2 w^3 + (-1) v^2 v^1 w^3 = 0 \ . \tag{1.23}$$

[1] Imagine a clock face corresponding to a day with just 3 hours.

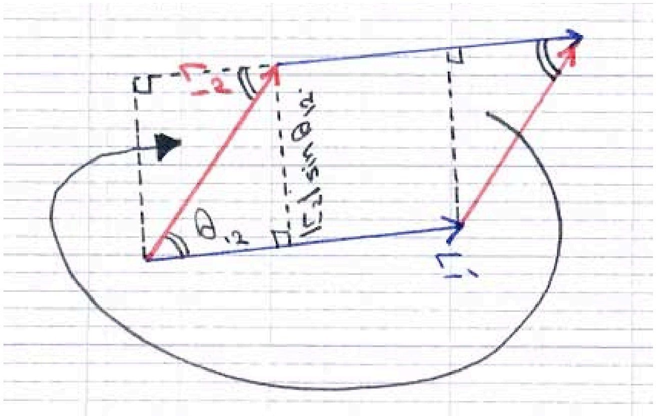

Figure 1.2.1: Area of a parallelagram

Or we can do the slow way:

$$\begin{aligned}
\underline{v} \cdot (\underline{v} \times \underline{w}) &= \sum_a v^a (\underline{v} \times \underline{w})^a \\
&= (\epsilon_{123} v^1 v^2 + \epsilon_{213} v^2 v^1) w^3 + (\epsilon_{312} v^3 v^1 + \epsilon_{132} v^1 v^3) w^2 + (\epsilon_{231} v^2 v^3 + \epsilon_{321} v^3 v^2) w^1 \\
&= (v^1 v^2 - v^2 v^1) w^3 + (v^3 v^1 - v^1 v^3) w^2 + (v^2 v^3 - v^3 v^2) w^1 \\
&= 0 + 0 + 0 \\
&= 0 \ . \qquad (1.24)
\end{aligned}$$

Lastly we see that

$$\underline{w} \cdot (\underline{v} \times \underline{w}) = -\underline{w} \cdot (\underline{w} \times \underline{v}) = 0 \ , \qquad (1.25)$$

which simply follows from anti-symmetry and switching the names $\underline{v} \leftrightarrow \underline{w}$.

Next we show that

$$|\underline{r}_1 \times \underline{r}_2| = |\underline{r}_1||\underline{r}_2| \sin\theta_{12} \ . \qquad (1.26)$$

You can prove this by writing out all the terms (it helps to observe $|\underline{r}_1|^2|\underline{r}_2|^2\sin^2\theta_{12} = |\underline{r}_1|^2|\underline{r}_2|^2(1-\cos^2\theta_{12}) = |\underline{r}_1|^2|\underline{r}_2|^2 - |\underline{r}_1 \cdot \underline{r}_2|^2$). Or we can go back to or choice before (1.17), where we adapted the basis to be convenient for the plane defined by $\underline{r}_1$ and $\underline{r}_2$, Here we see that

$$\underline{r}_1 \times \underline{r}_2 = \begin{pmatrix} r_1\cos\theta_1 \\ r_2\sin\theta_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} r_2\cos\theta_2 \\ r_2\sin\theta_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ r_1 r_2(\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2) \end{pmatrix} \qquad (1.27)$$

and so

$$|\underline{r}_1 \times \underline{r}_2|^2 = r_1^2 r_2^2(\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2)^2 = |\underline{r}_1|^2|\underline{r}_2|^2\sin^2(\theta_1-\theta_2) \ , \qquad (1.28)$$

as promised.

There is also an interpretation of $|\underline{r}_1 \times \underline{r}_2|$ as the area of the parallelogram defined by $\underline{r}_1$ and $\underline{r}_2$:

From Figure 2 one can see that the area consists of two triangles with hight $h_t = |\underline{r}_2|\sin\theta_{12}$ and base $b_t = |\underline{r}_2|\cos\theta_{12}$ as well as rectangle with base $b_r = |\underline{r}_1| - b_t$ and hight $h_r = h_t$. Thus

$$Area = 2 \cdot \frac{1}{2} b_t h_t + b_r h_r = h_t(b_t + b_r) = |\underline{r}_1||\underline{r}_2|\sin\theta_{12} \ . \qquad (1.29)$$

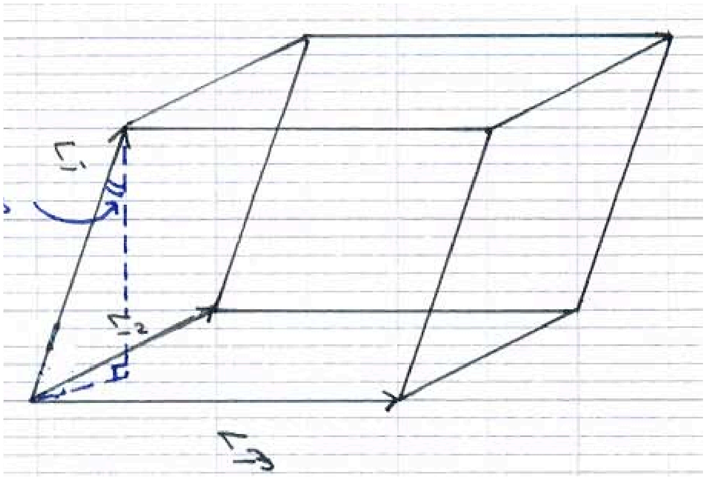

Figure 1.3.1: Volume of a Parallelepiped

1.3 Triple Products

One can construct a **scalar triple product** of three vectors:

$$\underline{r}_1 \cdot (\underline{r}_2 \times \underline{r}_3) \tag{1.30}$$

which gives a scalar. Geometrically this gives the area (or more accurately the absolute value gives the area) of a parallelepiped with sides defined by $\underline{r}_1$, $\underline{r}_2$ and $\underline{r}_3$.

One could also consider the **vector triple product**

$$\underline{r}_1 \times (\underline{r}_2 \times \underline{r}_3)\ . \tag{1.31}$$

But it is not really independent of what we have already constructed. Indeed note that $\underline{r}_2 \times \underline{r}_3$ is orthogonal to both $\underline{r}_2$ and $\underline{r}_3$. Similarly $\underline{r}_1 \times (\underline{r}_2 \times \underline{r}_3)$ is orthogonal to $(\underline{r}_2 \times \underline{r}_3)$, which is orthogonal to the plane defined by $\underline{r}_2$ and $\underline{r}_3$. So it must be that $\underline{r}_1 \times (\underline{r}_2 \times \underline{r}_3)$ lies in the plane defined by $\underline{r}_2$ and $\underline{r}_3$. Therefore we have

$$\underline{r}_1 \times (\underline{r}_2 \times \underline{r}_3) = A\underline{r}_2 + B\underline{r}_3\ . \tag{1.32}$$

For some scalars A and B.

To compute A and B we can use the definition:

$$\begin{aligned}(\underline{r}_1 \times (\underline{r}_2 \times \underline{r}_3))^a &= \sum_{b,c} \varepsilon_{abc} r_1^b (\underline{r}_2 \times \underline{r}_3)^c \\ &= \sum_{b,c,d,e} \varepsilon_{abc}\varepsilon_{cde} r_1^b r_2^d r_3^e\ .\end{aligned} \tag{1.33}$$

Next we observe that

$$\sum_c \varepsilon_{abc}\varepsilon_{cde} = \delta_{ad}\delta_{be} - \delta_{ae}\delta_{bd}\ . \tag{1.34}$$

Why? Its actually just a matter thinking it through: Clearly to be non-zero a must be different from b. Let us fix $a = 1, b = 2$. Then the left hand side is only non-zero if $c = 3$ and $(d, e) = (1, 2)$ or $(d, e) = (2, 1)$. Thus there are only two choices for a non-vanishing answer: $a = d$ and $b = e$ or $a = e$ and $b = d$. Similarly for the other choices of a, b.

This is what the right hand side says. The only issue is the minus sign but this arises as $\varepsilon_{abc}\varepsilon_{cab} = (\varepsilon_{abc})^2 = 1$ but $\varepsilon_{abc}\varepsilon_{cba} = -(\varepsilon_{abc})^2 = -1$ (assuming a, b, c are all different).

We can now compute

$$\begin{aligned}(\underline{r}_1 \times (\underline{r}_2 \times \underline{r}_3))^a &= \sum_{b,d,e}(\delta_{ad}\delta_{be} - \delta_{ae}\delta_{bd})r_1^b r_2^d r_3^e \\ &= (\underline{r}_1 \cdot \underline{r}_3)r_2^a - (\underline{r}_1 \cdot \underline{r}_2)\underline{r}_3^a \ . \end{aligned} \tag{1.35}$$

which is just

$$\underline{r}_1 \times (\underline{r}_2 \times \underline{r}_3) = (\underline{r}_1 \cdot \underline{r}_3)\underline{r}_2 - (\underline{r}_1 \cdot \underline{r}_2)\underline{r}_3 \ , \tag{1.36}$$

i.e. $A = (\underline{r}_1 \cdot \underline{r}_3)$, $B = -(\underline{r}_1 \cdot \underline{r}_2)$.

1.4 Matrices

.

Lastly we look at matrices which act on vectors as linear maps $M : \mathbb{R}^3 \to \mathbb{R}^3$. Linear here means that

$$\mathbf{M}(A\underline{r}_1 + \underline{r}_2) = A\mathbf{M}(\underline{r}_1) + \mathbf{M}(\underline{r}_2) \tag{1.37}$$

where a is a scalar. We will usually drop the parenthesis and denote matrix multiplication by $\mathbf{M}\underline{r}$.

In terms of the components notation we can write[2]

$$(\mathbf{M}\underline{r})^a = \sum_{b=1}^{3} M^a{}_b \underline{r}^b \ . \tag{1.38}$$

This can be read as follows: the a^{th} component of $\mathbf{M}\underline{r}$ is given by the scalar product of the a^{th}-row of $\mathbf{M}$ with $\underline{r}$. Thus a matrix has two indices and can be written as an array: *e.g.* the identity matrix $\mathbf{I}$, which doesn't change a vector after multiplication, is

$$(\mathbf{I})^a{}_b = \delta^a{}_b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ , \tag{1.39}$$

here $\delta^a{}_b$ is known as the Kronecker-delta.

It is clear that we can multiply a matrix by a scalar or add to matrices to get a new matrix in the obvious way:

$$(A\mathbf{M} + \mathbf{N})^a{}_b = AM^a{}_b + N^a{}_b \ . \tag{1.40}$$

In addition we can also define the product of two matrices (we will always be looking at 3×3 matrices) in what may not seem like the obvious way:

$$(\mathbf{MN})^a{}_b = \sum_{c=1}^{3} M^a{}_c N^c{}_b \ . \tag{1.41}$$

[2] In this course you need not worry about why one index is up and the other down. In more general cases, such as special relativity this is important, and there is a rule that you only ever sum over and index that appears in an expression exactly once up and once down.

This can be read as follows: the ab^{th} component of $\mathbf{MN}$ is given by the scalar product of the a^{th}-row $\mathbf{M}$ with the b^{th}-column of $\mathbf{N}$. The reason this definition is useful, as opposed to what might be the obvious way of simply multiplying the individual components together, is that this is what you'd get if you first acted on a vector by $\mathbf{N}$ and then acted again by $\mathbf{M}$.

An important set of matrices are those that leave the scalar product between two vectors invariant:

$$(\mathbf{O}\underline{r}_1)\cdot(\mathbf{O}\underline{r}_2) = \underline{r}_1\cdot\underline{r}_2 \ , \tag{1.42}$$

for any pair of vectors $\underline{r}_1$ and $\underline{r}_2$. Substituting (1.38) into the left-hand side gives

$$(\mathbf{O}\underline{r}_1)\cdot(\mathbf{O}\underline{r}_2) = \sum_{abc=1}^{3} \mathbf{O}^c{}_a r_1^a \mathbf{O}^c{}_b r_2^b \ , \tag{1.43}$$

while the right hand side is

$$\underline{r}_1\cdot\underline{r}_2 = \sum_{c=1}^{3} \underline{r}_1^c \underline{r}_2^c \ . \tag{1.44}$$

Thus we must have

$$\sum_{a,b,c=1}^{3} \mathbf{O}^c{}_a r_1^a \mathbf{O}^c{}_b r_2^b = \sum_{c=1}^{3} r_1^c r_2^c = \sum_{a,b=1}^{3} \delta^a{}_b r_1^a r_2^b \ , \tag{1.45}$$

for any $\underline{r}_1^a$, $\underline{r}_2^a$. This in turn requires that

$$\sum_{c=1}^{3} \mathbf{O}^c{}_a \mathbf{O}^c{}_b = \delta^a{}_b \ . \tag{1.46}$$

In other words

$$\mathbf{O}^T\mathbf{O} = \mathbf{I} \ , \tag{1.47}$$

where

$$(\mathbf{O}^T)^a{}_b = \mathbf{O}^b{}_a \ , \tag{1.48}$$

is called the **transpose**. Linear transformations that don't change angles are more usually referred to as **rotations**[3]. For example you can check that

$$\mathbf{O} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \ , \tag{1.49}$$

is a rotation in the x, y plane. But more complicated examples exist (although they can always be viewed as a two-dimensional rotation in some plane - meaning that there is always one non-zero vector, the normal to the plane, that is left invariant under the action of a rotation in three-dimensions).

[3]Technically, for those who know, one usually also requires that a rotation has determinant 1, which excludes reflections about one axis (but not two).

1.5 Derivatives vs Partial Derivatives

In this course there will be a lot of calculus applied to a functions of many variables. It will be important to clearly understand the difference between the following:

$$\frac{df}{dt} , \qquad \frac{dF}{dt} , \qquad \frac{\partial F}{\partial u_i} , \qquad \frac{\partial F}{\partial t} . \tag{1.50}$$

Here $f(t)$ is a function of one variable t and $F(u_1, ..., u_n, t)$ is a function of n variables labelled u_i and possibly of t as well (we will often take t to be time but it could be anything). Let us recall some definitions:

$$\begin{aligned}
\frac{df}{dt} &= \lim_{\epsilon\to 0} \frac{f(t+\epsilon) - f(t)}{\epsilon} \\
\frac{\partial F}{\partial u_i} &= \lim_{\epsilon\to 0} \frac{f(u_1, ..., u_i + \epsilon, ...u_n, t) - f(u_1, ..., u_i, ..., u_n, t)}{\epsilon} \\
\frac{\partial F}{\partial t} &= \lim_{\epsilon\to 0} \frac{f(u_1, ...u_n, t+\epsilon) - f(u_1, ..., u_n, t)}{\epsilon} .
\end{aligned} \tag{1.51}$$

The first is the ordinary derivative and measures the rate of change of f with respect to its argument t. The second is a partial derivative and measures the rate of change of F with respect to one of its arguments u_* while holding all the others and t fixed. The third is the rate of change of F with respect to its argument t holding all the u_i fixed.

In this course we will often encounter functions $F(u_1, ..., u_n, t)$ where the variables u_i themselves depend on time t. Thus we might look at a function of the form

$$f(t) = F(u_1(t), ..., u_n(t), t) . \tag{1.52}$$

If we want to know the rate of change of f with respect to t then we use the chain rule:

$$\frac{df}{dt} = \frac{dF}{dt} = \sum_{i=1}^{n} \frac{\partial F}{\partial u_i}\frac{du_i}{dt} + \frac{\partial F}{\partial t} . \tag{1.53}$$

Here the first terms give the change in f that arises from the fact that the variables u_i change with t and F depends on u_i. The last term arises if F has an explicit dependence on t.

On a practical level this means that if we consider a small but finite variation: $u_i \to u_i + \delta u_i$ and $t \to t + \delta t$ then to first order in δu_i, δt we can approximate

$$\delta F = \sum_{i=1}^{n} \frac{\partial F}{\partial u_i}\delta u_i + \frac{\partial F}{\partial t}\delta t + \dots . \tag{1.54}$$

where

$$\delta F = F(u_i + \delta u_i, t + \delta t) - f(u_i, t) , \tag{1.55}$$

where from now on we will use a shortcut to refer to the whole collection of variables u_i: $F(u_1(t), ..., u_n(t), t) = F(u_i(t), t)$. This is simply the first term in a Taylor expansion. The ellipsis denotes higher order terms. We will often write $\delta u_i = \epsilon T_i$, $\delta t = \epsilon T$ where ϵ is a small parameter, which we can take to be as small as we wish, and T_i, T are some expressions that are not small. Thus the Taylor expansion is

$$\delta F = \epsilon \left(\sum_{i=1}^{n} \frac{\partial F}{\partial u_i} T_i + \frac{\partial F}{\partial t} T \right) + \mathcal{O}(\epsilon^2) . \tag{1.56}$$

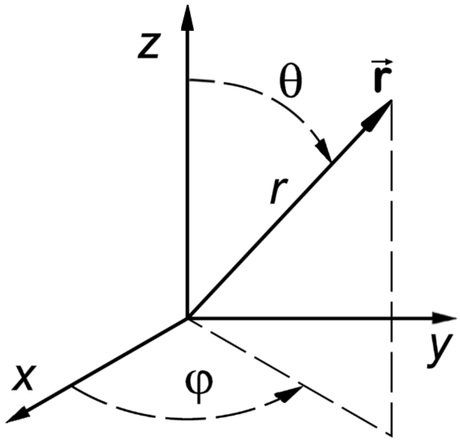

Figure 1.6.1: Spherical coordinates

By taking ϵ suitably small we can neglect the higher order terms as much as we wish.

On a more abstract level we might express this as

$$dF = \sum_{i=1}^{n} \frac{\partial F}{du_i} du_i + \frac{\partial F}{\partial t} dt \ . \tag{1.57}$$

Note that we have not defined dF, du_i or dt. One can think of them as meaning δF, δu_i and δt in the limit that $\epsilon \to 0$. Alternatively one can simply think of (1.57) as a substitute for (1.53) in the sense that if we allow the u_i to depend on any parameter, such as t, then (1.53) holds. This is just like how you are not supposed to think of df/dt as a fraction but in practice it is often helpful to do so. In a sense (1.57) is just formal expression[4] which encodes the statement that F depends on $u_1, ..., u_n$ and t.

1.6 Coordinate systems

When computing derivatives and scalar and vector products we have been working in the cartesian co-ordinate system $\{x, y, z\}$. We often use spherical co-ordinates to describe systems with angular momentum or in the case of a central potential (see next chapter). In order to compute vector products we need to be able to convert from one system to another.

In the diagram above we have coordinates $\{r,\ \theta,\ \phi\}$.

$$r \in (0,\ \infty), \quad \theta \in (0,\ \pi), \quad \phi \in (0,\ 2\pi). \tag{1.58}$$

In order to express a vector $\mathbf{r} = \{r,\ \theta,\ \phi\}$ in cartesian coordinates (so we can perform vector products) we must make the transformations,

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r\cos\theta \ . \tag{1.59}$$

Exercise: What are the opposite transformations? i.e. what are $r(x, y, z), \theta(x, y, z)$ and

[4]Formal because we are not supposed to give a numerical value to dF, du_i dt. Rather we must always understand it as applying within the context of (1.53) or (1.56).

$\phi(x, y, z)\}$?

Care should be taken with any example as different conventions are often in use. Sometimes the 'azimuthal' angle θ is measured from below the $z = 0$ plane. In this case the only transformation to change is that for the z coordinate to become negative, $z = -r\cos\theta$. Some books may even swap the names for the azimuthal angle and the longitudinal angle. It is usually easy to establish the convention in use.

Also note that when $r = 0$ then θ and ϕ are redundant and when $\theta = 0$ or π, ϕ becomes redundant.

Chapter 2

Newton and His Three Laws

So now its time to do Physics. Newtons laws are the basis of classical physics. Classical physics is the basis of almost every technological invention, including the industrial revolution, until the 20th century and is still highly relevant. But more than that they offered, for the first time, a set of universal laws that governed our every day lives as well as the motions of the heavenly bodies. So with Newton we learn that we are at one with the stars.

Newton's greatest work is his (in)famous Principia Mathematica in 1687. Although he had already developed calculus he did not use it in the Principia, forcing him to use some very elaborate reasoning. Just because he thought it might be viewed with little suspicion. Or maybe just to show off! We will simply use calculus. Amusingly the book was to be published by the Royal Society (near Piccadilly circus, not far from King's) but they had spent all their publishing budget that year on a book called "The History of Fishes", which is now only famous for not being the Principia.[1] So the cost was paid for by Edmund Halley who was their clerk at the time. If that wasn't bad enough the "History of Fishes" was a commercial failure and such a drain on the financial resources of the Royal Society that they could only pay Halley his salary by giving him the unsold copies. So that will make you think twice about accepting an FRS.

Newton's laws are incredibly powerful and they basically put Physics in pole position as the most predictive science. We still use it all the time even though they have, in a formal sense, been superseded somewhat by relativity but more so by quantum theory. Indeed perhaps the first major new ideas beyond Newton came from Maxwell's theory of electromagnetism, which he published while he was a Professor at King's in 1861. You will learn about Maxwell's laws in another course but they ushered in two new concepts: dynamical fields which permeate space and time and the Lorentz transformations of special relativity. Maxwell, who was one of the greatest physicists by any standard, came to King's because he had been down-sized in a merger at the University of Aberdeen in Scotland (a worse *faux pas* than not signing the Beatles or letting Taylor Swift leave your label to record her own songs). So there is hope for everyone except, perhaps, administrators.

[1] It's also not clear what history it talks about since it predates evolution - it must have missed that boat too.

2.1 Newton's Laws

In his Principia Newton proposed three laws that govern all motion. We will state them here for the case of a point particle, that is to say a particle small enough that any structure it has does not affect its motion. One can also discuss 'rigid bodies' which do have structure that is important for their motion but that structure itself doesn't change (hence the term rigid). Examples include spinning disney characters `<http://www.youtube.com/watch?v=qquek0c5bt4>`. Of course we think of rigid bodies as being made of particles which are subjected to Newton's Laws.

- [NI] A particle will stay at rest or move with a constant velocity along a straight line unless acted on by an external force.
- [NII] The rate of change of momentum of a particle is equal to, and in the direction of, the net force acting on it.
- [NIII] Every action has an opposite and equal reaction.

You've probably heard these before. However before we continue there are some important comments to make.

First it should be stated that [NI] effectively defines what is meant by an inertial frame. The reason being that the word 'velocity' is frame dependent. Not all frames are inertial. Picture yourself on a roller coaster. There is a frame, that is a choice of coordinates, where you are at rest and the amusement park is flying all around you. It would be very hard to convince yourself that [NI] held. Rather you'd have to come up with all sorts of fictitious forces (centrifugal force is one) to explain the motions of everything that was not strapped in traveling around with you the roller coaster (including your stomach and your lunch). To define an inertial frame we simply say it is a frame where [NI] holds true. It should be clear that if we have one inertial frame and then boost it then we obtain a new inertial frame. By boost we mean give it a constant velocity relative to what it had before. Rotations and translations also take one inertial frame to another. Mathematically this means that we can change coordinates according to

$$\begin{aligned} \text{translation}: &\quad \underline{r} \to \underline{r} + \underline{a} \\ \text{rotation}: &\quad \underline{r} \to \mathbf{R}\underline{r} \\ \text{boost}: &\quad \underline{r} \to \underline{r} + \underline{v}t \end{aligned} \tag{2.1}$$

and still be in a frame where [NI] is true. Here $\underline{a}$ and $\underline{v}$ are a fixed vectors and $\mathbf{R}$ is a rotation matrix, that is a matrix such that $\mathbf{R}^T\mathbf{R} = 1$. These transformations form the Galilean Group[2] (a group is an important mathematical topic that you might know about but if not should do soon) and this is known as Galilean relativity.

Special relativity takes these ideas further and 1) declares that physical laws must be the same in all inertial frames and 2) uses a different notion of boost which leaves the speed of light constant in all frames (which in turn requires that time changes when we go between inertial frames - leading one to replace the Galilean group by the Poincare

[2] From the line from Queen's Bohemian Rhapsody

group (which is the Lorentz group along with translations)). That was a bit of name dropping not to show off but in case you have heard the words before.

Secondly note that [NII] does not say $\underline{F} = m\ddot{\underline{r}}$, *i.e.* $\underline{F} = m\underline{a}$, where a dot denotes a time derivative. This is true in the simplest cases but not all. Rather, as we have stated it, [NII] is

$$\underline{F} = \dot{\underline{p}} \ , \tag{2.2}$$

where p is called the momentum. The more familiar $\underline{F} = m\ddot{\underline{r}}$ then arises when m is constant, where

$$\underline{p} = m\dot{\underline{r}} \tag{2.3}$$

and

$$\dot{\underline{p}} = m\ddot{\underline{r}} \ . \tag{2.4}$$

But for a simple counter example (that we will look at later) consider a rocket ship then m decreases as it burns its fuel so that

$$\underline{F} = m\ddot{\underline{r}} + \dot{m}\dot{\underline{r}} \ . \tag{2.5}$$

In full generality one also can allow for $\underline{F}$ to depend on $\underline{r}, \dot{\underline{r}}$ and t. In these cases [NII] sets up a second order differential equation for $\underline{r}$ which can, in principle if not practice, be solved for all t by knowing the values of $\underline{r}$ and $\dot{\underline{r}}$ at some initial time t.

Finally [NIII] is the famous statement that if you push against something, say the wall, then the wall pushes back on you with equal force but in the opposite direction, after all you can feel it. This is important since if it weren't the case then there would be a net force and, by [NII], the wall would move. There are by now many ways to demonstrate this. For example this is how a rocket works: it moves forward by throwing its fuel backwards. Another example is a gun where the shooter feels a recoiling force when they pull the trigger.

Note that in the absence of any Force Newton's law tell us that a particle moves in a straight line. Here one simply has

$$\underline{F} = \frac{d}{dt}\underline{p} = \underline{0} \ . \tag{2.6}$$

In other words momentum is conserved $\underline{p}$ is time independent.

2.2 Skiing: A Simple Example of Linear Motion

Skiing: Let us look at a skier who descends a slope which makes and angle θ with the horizon.

The force of gravity is constant and acts in the downward direction: $\underline{F}_g = -mg\underline{e}_y$. However it is useful to think of $\underline{e}_y$ in terms of two components

$$\underline{e}_y = \sin\theta\underline{e}_h + \cos\theta\underline{e}_R \ . \tag{2.7}$$

Here $\underline{e}_h$ is the unit vector pointing up the hill against the direction of the skier's motion and $\underline{e}_R$ points up perpendicularly from the slope. Accordingly we can write

$$\begin{aligned} \underline{F}_g &= -mg\underline{e}_y \\ &= -mg\sin\theta\underline{e}_h - mg\cos\theta\underline{e}_R \\ &= \underline{F}_h + \underline{F}_R \ , \end{aligned} \tag{2.8}$$

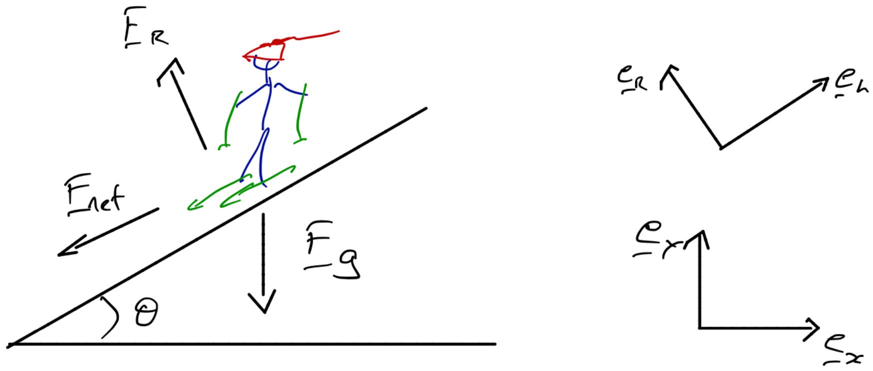

Figure 2.2.1: Skiing

corresponding to the parts which push the skier down the hill $\underline{F}_h$ and the part $\underline{F}_R$ that pulls the skier into the slope.

The component $\underline{F}_R = -mg\cos\theta\underline{e}_R$ is canceled by an opposite and equal reactive force of the hill pushing back on the skier. This is intuitively obvious but fundamentally due to the electromagnetic forces inside the atoms of snow and ski-boots. Now in this case $\underline{p} = m\dot{\underline{r}}$ where m is constant so we are simply left with

$$m\ddot{\underline{r}} = \underline{F}_h = -mg\sin\theta\underline{e}_h \ . \tag{2.9}$$

Let us write $\underline{r} = r\underline{e}_h$ then we find

$$\ddot{r} = -g\sin\theta \ . \tag{2.10}$$

To solve this we simply integrate both sides:

$$\dot{r}(t) = -g\sin\theta t + \dot{r}(0) \ , \tag{2.11}$$

where $\dot{r}(0)$ is a constant of integration. And then integrate again

$$r(t) = -\frac{1}{2}g\sin\theta t^2 + \dot{r}(0)t + r(0) \ , \tag{2.12}$$

where $r(0)$ is a another constant of integration. Now clearly the arbitrary constants $\dot{r}(0)$ and $r(0)$ correspond to the speed and position of the skier at $t = 0$. Given these as inputs we can then compute how the skier will continue to go down the hill. The answer is faster and faster as any novice skier can tell you.

But to be more explicit an assume that at $t = 0$ the skier is at $r = 100m$ and traveling at $-1m/s$ (*i.e.* down the hill) then

$$r(t) = -5\sin\theta t^2 - t + 100 \tag{2.13}$$

where we've used $g = 10m/s^2$ (its actually more like $9.8m/s^2$). For a green run one might have $\theta = \pi/6$ so $\sin\theta = 1/2$ leading to

$$r(t)_{green} = -2.5t^2 - t + 100 \tag{2.14}$$

In the first second the skier arrives at $r(1)_{green} = 96.5m$, just a few meters, but after 5 seconds $r(10)_{green} = 67.5m$, thats already quite far: 2/3 of the way down the hill. For a red run maybe one has $\theta = \pi/4$, $\sin\theta \sim 0.7$ and hence

$$r(t)_{red} \sim -3.5t^2 - t + 100 \tag{2.15}$$

so that after 1 second $r(1)_{red} = 95.5m$ also just a few meters but now after 5 seconds $r(5)_{red} = 7.5m$ which is almost all the way down.

2.3 Friction

That's not so realistic since the skier will eventually reach a so-called terminal velocity due to the presence of friction. How do we include friction? It corresponds to adding another force

$$\underline{F} = \underline{F}_R + \underline{F}_h + \underline{F}_F \tag{2.16}$$

where

$$\underline{F}_F = -\nu\underline{\dot{r}}\ . \tag{2.17}$$

Here $\nu > 0$ is the friction coefficient. Note the minus sign which means that friction acts in the opposite direction to the velocity.

The effect of friction is therefore to change the equation to

$$m\underline{\ddot{r}} = \underline{F}_h + \underline{F}_F = -mg\sin\theta\underline{e}_h - \nu\underline{\dot{r}}. \tag{2.18}$$

Putting $\underline{r} = r\underline{e}_h$ gives us the equation

$$\ddot{r} = -g\sin\theta - \frac{\nu}{m}\dot{r}. \tag{2.19}$$

To solve this we multiply both sides by $e^{\nu t/m}$ and write it as

$$\frac{d}{dt}(\dot{r}e^{\nu t/m}) = -g\sin\theta e^{\nu t/m}\ . \tag{2.20}$$

We an now integrate both sides once:

$$\dot{r}e^{\nu t/m} = -\frac{mg}{\nu}\sin\theta e^{\nu t/m} + A\ , \tag{2.21}$$

where A is an integration constant. Multiplying both sides by $e^{-\nu t/m}$ gives

$$\dot{r} = -\frac{mg}{\nu}\sin\theta + Ae^{-\nu t/m}\ , \tag{2.22}$$

which can again be integrated to

$$r(t) = -\frac{mg}{\nu}\sin\theta t - \frac{mA}{\nu}e^{-\nu t/m} + B\ , \tag{2.23}$$

Here B is another integration constant. It is easy to see that A and B can be related to the initial position and momentum but not such a simple way as before. However we can see one universal feature that doesn't depend on these. At late times, $t \to \infty$, the exponential becomes negligible and

$$r(t)_{t\to\infty} = -\frac{mg}{\nu}\sin\theta t + B\ . \tag{2.24}$$

meaning that the skier will no longer speed up but will travel with a constant terminal velocity $v_\infty = -\frac{mg}{\nu}\sin\theta$ down the hill. Of course the fun of skiing is that ν is small so v_∞ is big! The same equation also applies if you try skiing without snow, its just that ν is very large. This assumes that $\nu > 0$. For $\nu < 0$ we find the opposite: the skier speeds up exponentially! Thats why the sign in the coefficient of friction is so important.

Friction also explains why rain drops don't hurt (usually). Imagine a 1g rain drop that falls from 1km. Without wind resistance its final velocity is

$$r(t) = -5t^2 + 1000 \tag{2.25}$$

where we have used the same equation as the skier but taken the initial value of r to be 1km = 1000m, the initial $\dot{r}$ to be zero and set $\theta = \pi/2$. So it hits your head at $t = 10\sqrt{2}s$. The speed it hits you with is

$$\dot{r} = -10t = -100\sqrt{2}m/s\ , \tag{2.26}$$

and carries momentum

$$p = m\dot{r} = -0.1\sqrt{2}kgm/s \sim -.14kgm/s\ . \tag{2.27}$$

How painful would that be? Consider a 1kg brick dropped on you from a height of 1m. Again use the same equations:

$$r(t) = -5t^2 + 1 \tag{2.28}$$

so $t = \sqrt{5}/5$, the final velocity is $\dot{r} = -10t = -2\sqrt{5}$ and the final momentum is

$$p = m\dot{r} = -2\sqrt{5}kgm/s \sim -4.47kgm/s\ . \tag{2.29}$$

So one rain drop would give about 1/40 the punch of such a brick. Maybe not too bad but you can expect more than one rain drop to fall on your head. Indeed you can expect 40 per second! Of course life isn't so cruel. The rain drops reach a modest terminal velocity and only carry a small amount of momentum when they hit hour head.

2.4 Angular Motion

A more complicated but still quite trackable example of motion arises when particles move in angular (*e.g.* circular or elliptical) orbits. For example we could be studying planets as they moved around the sun

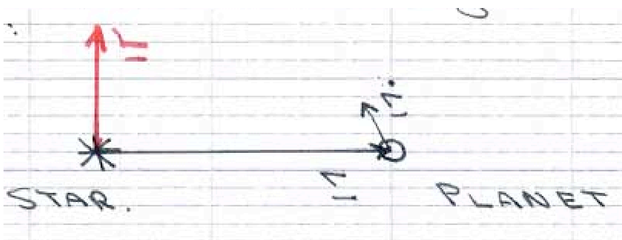

Figure 2.4.1: Planetary motion

Since they are not moving in straight lines linear momentum, $\underline{p}$ is not conserved. However in many cases an analogous quantity, angular momentum, is conserved. One can also introduce a suitable notion of force, known as torque, that is better adapted to angular motion.

We start with the definition of angular momentum about the origin:

$$\underline{L} = \underline{r} \times \underline{p} \ . \tag{2.30}$$

Note that $\underline{L}$ points in a direction orthogonal to both $\underline{r}$ and $\underline{p}$.

Next we want to define the analogue of force for angular motion: Torque $\underline{N}$. This is important in sports cars as it tell you how much the engine can turn the wheels around, which then leads to forward motion. In particular

$$\underline{N} = \underline{r} \times \underline{F} \tag{2.31}$$

We observe that, from (NII), if we have a point particle with fixed mass m and momentum $\underline{p} = m\dot{\underline{r}}$ then

$$\begin{aligned} \underline{N} &= \underline{r} \times \underline{F} \\ &= \underline{r} \times \dot{\underline{p}} \\ &= \frac{d}{dt}(\underline{r} \times \underline{p}) \\ &= \dot{\underline{L}} \end{aligned} \tag{2.32}$$

Here we have used the fact that

$$\begin{aligned} \frac{d}{dt}(\underline{r} \times \underline{p}) &= \dot{\underline{r}} \times \underline{p} + \underline{r} \times \dot{\underline{p}} \\ &= m\dot{\underline{r}} \times \dot{\underline{r}} + \underline{r} \times \dot{\underline{p}} \\ &= \underline{0} + \underline{r} \times \dot{\underline{p}} \end{aligned} \tag{2.33}$$

So indeed torque and angular momentum play analogous roles to force and momentum. Lets see how this works in a simple example.

Let us imagine a particle the moves in a circle. Thus we take

$$\underline{r} = \begin{pmatrix} r\cos\theta(t) \\ r\sin\theta(t) \\ 0 \end{pmatrix} \tag{2.34}$$

Circular motion means that $r = |\underline{r}|$ is constant but not $\theta(t)$!. In particular

$$\frac{d}{dt}|\underline{r}| = 0 \ , \qquad i.e. \qquad \dot{r} = 0 \qquad \text{but} \qquad \frac{d}{dt}\underline{r} \neq \underline{0} \qquad \text{because} \qquad \frac{d\theta}{dt} \neq 0 \ . \tag{2.35}$$

As you keep this in mind, we here drop the dependence on t everywhere to simplify the notation. As anticipated we have

$$\dot{\underline{r}} = \begin{pmatrix} -r\dot{\theta}\sin\theta \\ r\dot{\theta}\cos\theta \\ 0 \end{pmatrix} \tag{2.36}$$

and hence

$$|\dot{\underline{r}}|^2 = r^2\dot{\theta}^2(\sin^2\theta + \cos^2\theta) = r^2\dot{\theta}^2 \ . \tag{2.37}$$

Note that $\dot{\theta}$ is called the angular velocity[3] and that (2.37) will be useful in the evaluation of the kinetic energy, to be defined in the following. Moreover we have

$$\ddot{\underline{r}} = \begin{pmatrix} -r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta \\ r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta \\ 0 \end{pmatrix} \ . \tag{2.38}$$

It is important to realise that, even for constant angular velocity, this particle is accelerating. If we do restrict to $\dot{\theta} = \omega$ a constant then we see that there must be a force:

$$m\ddot{\underline{r}} = m \begin{pmatrix} -r\dot{\theta}^2\cos\theta \\ -r\dot{\theta}^2\sin\theta \\ 0 \end{pmatrix} = -m\omega^2\underline{r} \ . \tag{2.39}$$

Thus there is a force pointing inwards whose strength is linear in r:

$$\underline{F} = -m\omega^2\underline{r} \ , \tag{2.40}$$

such as a spring. This is called a centripetal force (as opposed to centrifugal force which is a fictitious force pointing outwards that one feels if one is the particle).

In the general case, let us calculate also the angular momentum. From the expression for $\dot{\underline{r}}$ above we have

$$\begin{aligned} \underline{L} &= \underline{r} \times \underline{p} \\ &= m \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 0 \end{pmatrix} \times \begin{pmatrix} -r\dot{\theta}\sin\theta \\ r\dot{\theta}\cos\theta \\ 0 \end{pmatrix} \\ &= m \begin{pmatrix} 0 \\ 0 \\ r^2\dot{\theta}(\cos^2\theta + \sin^2\theta) \end{pmatrix} \\ &= mr^2\dot{\theta} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned} \tag{2.41}$$

[3] For those either old enough or cool enough one might know of 33 and 45 rpm records (even 78 for the truly old); rpm stands for revolutions per minute and is a measure of angular velocity.

Note that this points up out of the plane of circular motion. Next we compute the torque for constant angular velocity:

$$\begin{aligned} \underline{N} &= \underline{r} \times \underline{F} \\ &= -m\omega^2 \underline{r} \times \underline{r} \\ &= \underline{0}\ . \end{aligned} \tag{2.42}$$

We see from the expression for $\ddot{\underline{r}}$ that if there is angular acceleration (but still take r constant):

$$\begin{aligned} \underline{N} &= \underline{r} \times \underline{F} \\ &= m\underline{r} \times \ddot{\underline{r}} \\ &= m \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 0 \end{pmatrix} \times \left(\begin{pmatrix} -r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta \\ r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta \\ 0 \end{pmatrix} \right) \\ &= m \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 0 \end{pmatrix} \times \begin{pmatrix} -r\ddot{\theta}\sin\theta \\ r\ddot{\theta}\cos\theta \\ 0 \end{pmatrix} \\ &= mr^2\ddot{\theta} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned} \tag{2.43}$$

Note that the second term in the second line is proportional to $\underline{r}$ and hence vanishes in the cross product with $\underline{r}$. This kind of torque is why you buy a BMW M-series and not simply a 316i. It is the ability of a car to speed up the angular velocity of its wheels.

2.5 Work, Conservative Forces and Conserved Quantities

We have seen that in the cases of linear and angular motion, if there is no force or torque then momentum $\underline{p}$ and angular $\underline{L}$ are constant in time, or conserved. In particular

$$\begin{aligned} \underline{F} = \underline{0} &\Rightarrow \underline{p} \text{ is conserved} \\ \underline{N} = \underline{0} &\Rightarrow \underline{L} \text{ is conserved} \end{aligned}$$

Conserved quantities play a critical role in the understanding of dynamical system because they can be used to solve a problem, or at least reduce its complexity.

But before we move on to conserved quantities more generally we need to introduce the notion of work, or more precisely the work done on a system (one doesn't have an absolute notion of work). In an infinitesimal step the work done on a particle by a force F is the scalar product of the force and of the consequent particle's displacement

$$dW = \underline{F} \cdot d\underline{r}\ . \tag{2.44}$$

For a complete path the work done is defined as a line integral so that as a particle moves from $\underline{r}_1$ to $\underline{r}_2$ we have

$$\Delta W = \int_{\underline{r}_1}^{\underline{r}_2} \underline{F} \cdot d\underline{r}\ . \tag{2.45}$$

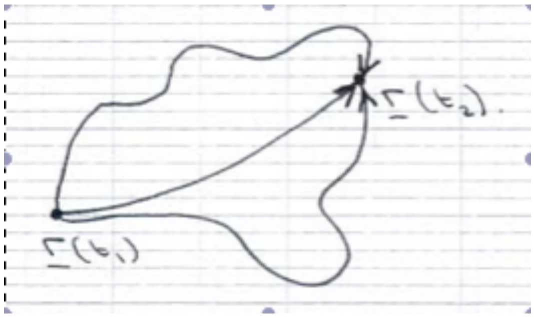

Figure 2.5.1: Paths for work

You should think of this as follows. The particle takes some path described by $\underline{r}(t)$ and goes from $\underline{r}_1 = \underline{r}(t_1)$ to $\underline{r}_2 = \underline{r}(t_2)$ as t goes from t_1 to t_2. Therefore

$$d\underline{r} = \dot{\underline{r}}dt \ , \tag{2.46}$$

and

$$\Delta W = \int_{t_1}^{t_2} \underline{F} \cdot \dot{\underline{r}}dt \ . \tag{2.47}$$

Now if we write $\underline{F} = m\ddot{\underline{r}}$ we have

$$\begin{aligned} \Delta W &= m\int_{t_1}^{t_2} \ddot{\underline{r}} \cdot \dot{\underline{r}}dt \\ &= \frac{m}{2}\int_{t_1}^{t_2} \frac{d}{dt}\left(\dot{\underline{r}} \cdot \dot{\underline{r}}\right) dt \\ &= \frac{1}{2}m|\dot{\underline{r}}(t_2)|^2 - \frac{1}{2}m|\dot{\underline{r}}(t_1)|^2 \ . \end{aligned} \tag{2.48}$$

Here we have introduced the **kinetic** energy

$$T = \frac{1}{2}m|\dot{\underline{r}}|^2 \ . \tag{2.49}$$

The work done is therefore given by the change in kinetic energy over the path of the particle:

$$\Delta W = \Delta T \ . \tag{2.50}$$

It is generally path-dependent: taking different paths will lead to different changes in kinetic energy.

For example if one has only the force of friction, with $\underline{F} = -\nu\dot{\underline{r}}$ then

$$\Delta W_{friction} = -\nu\int_{t_1}^{t_2} \dot{\underline{r}} \cdot \dot{\underline{r}}dt < 0 \ . \tag{2.51}$$

This is negative since, unless there is a counter acting force pushing the particle along, the particle will slow down. The friction will have done negative work (positive work is an achievement: such as speeding up!).

For example if you push your shopping trolley around the supermarket you must do work to keep the trolley moving at a constant speed because you are constantly fighting agains the friction. Furthermore the more you walk around the isles the more work you must do. All trips to the supermarket start at the front door and end at the check-out.

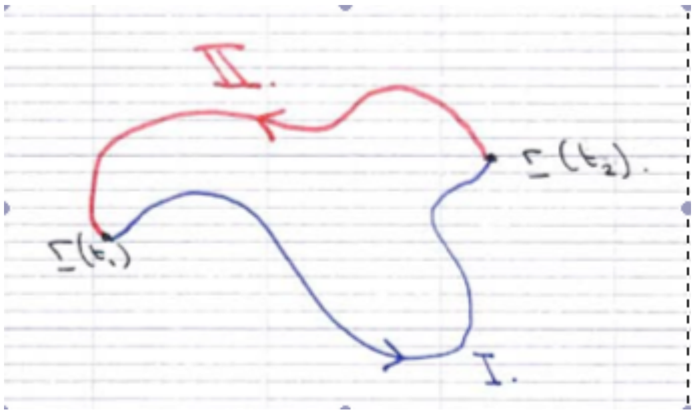

Figure 2.5.2: Closed path for work

But the amount of work you must do to overcome the negative work of friction depends on how long a path around the isles you take to find what you want.

In other words in order to fight against friction you must supply a force that does positive work. In particular to ensure that your final speed is the same as your initial velocity you must provide a force $\underline{F}_{you}$ so that

$$\Delta W = \int \underline{F}_{you} \cdot d\underline{r} + \Delta W_{friction} = 0 \ . \tag{2.52}$$

and hence you must do work:

$$\Delta W_{you} = \int \underline{F}_{you} \cdot d\underline{r} > 0 \ . \tag{2.53}$$

But there is a class of force for which the work done is independent of the path taken. Such a force is said to be **conservative**. Roughly speaking the fundamental forces we observe in Nature (gravity, electromagnetism,...) when acting in empty space ideal environment are conservative. Non-conservative forces typically arise from some kind of friction force that is due to the microscopic details of many particles bouncing around hitting each other in a disorderly way.

An important class of conservative forces arises if there exists a function $V(\underline{r})$ such that

$$\underline{F} = -\underline{\nabla} V \ . \tag{2.54}$$

Let us check that such an $\underline{F}$ does lead to a definition of work that is path independent. To do this we observe that

$$\begin{aligned} W &= \int_{\underline{r}_1}^{\underline{r}_2} \underline{F} \cdot d\underline{r} \\ &= -\int_{\underline{r}_1}^{\underline{r}_2} \underline{\nabla} V \cdot d\underline{r} \\ &= -\int_{t_1}^{t_2} \underline{\nabla} V \cdot \dot{\underline{r}} \ dt \\ &= -\int_{t_1}^{t_2} \frac{d}{dt} V(\underline{r}(t)) \ dt \\ &= V(\underline{r}(t_1)) - V(\underline{r}(t_2)) \\ &= V(\underline{r}_1) - V(\underline{r}_2) \ . \end{aligned} \tag{2.55}$$

As promised, this only depends on the end points and not the path taken.

In fact if $\underline{F}$ only depends on $\underline{r}$ and not, for example $\dot{\underline{r}}$, then such a V always exists (at least locally). We won't prove this here but to see why we note that path independence is equivalent to the statement that the total work done around a closed path vanishes.

For instance, let us consider a closed path, from $\underline{r}_1$ to $\underline{r}_2$ and then back again (along a different path) (see figure 6). In this case the work done between the first and second legs will cancel so that

$$\oint \underline{F} \cdot d\underline{r} = \int_I \underline{F} \cdot d\underline{r} + \int_{II} \underline{F} \cdot d\underline{r} = 0 \tag{2.56}$$

since path I and path II have the same endpoints but in reverse order.

Therefore we have

$$0 = \oint_\gamma \underline{F} \cdot d\underline{r} = \int_{\{B|\partial B=\gamma\}} (\underline{\nabla} \times \underline{F}) \cdot d\underline{A} \tag{2.57}$$

where we have used a corollary of the Stokes theorem applied to a region B whose boundary is the curve γ, $d\underline{r}$ is tangent to the curve and $d\underline{A}$ is perpendicular to the surface B. Since this must be true for any curve γ we deduce that

$$\underline{\nabla} \times \underline{F} = \underline{0} \tag{2.58}$$

This means (and we won't prove it here) that $\underline{F} = -\underline{\nabla} V$ for some function V.[4]

This allows us to define the most fundamental of conserved quantities: the total energy :

$$\begin{aligned} E &= T + V \\ &= \frac{1}{2} m |\dot{\underline{r}}|^2 + V(\underline{r}) \ . \end{aligned} \tag{2.59}$$

Claim: The total energy of a conserved system is conserved, *i.e.* constant. To see this we simply differentiate:

$$\begin{aligned} \frac{d}{dt} E &= m \dot{\underline{r}} \cdot \ddot{\underline{r}} + \underline{\nabla} V \cdot \dot{\underline{r}} \\ &= \dot{\underline{r}} \cdot \underline{F} + \underline{\nabla} V \cdot \dot{\underline{r}} \\ &= (\underline{F} + \underline{\nabla} V) \cdot \dot{\underline{r}} \\ &= 0 \ . \end{aligned} \tag{2.60}$$

N.B. This is a special case of a more general definition of energy. We will encouter other systems where there is a conserved energy but it takes a different form in section 3.5.4.

2.6 Solving One-dimensional Dynamics

Conservation of energy is already enough to essentially solve for the motion of a particle in one-dimension with a conservative force. In this case $\underline{r}$ is just a number $r \in \mathbb{R}$. Let the potential be $V(r)$ so that the energy is

$$E = \frac{1}{2} m \left(\frac{dr}{dt} \right)^2 + V(r) \ . \tag{2.61}$$

[4]This is an example of the Poincare lemma. More interestingly the number of solutions to this equation which are not of this form count the number of "holes" in space in a certain topological sense.

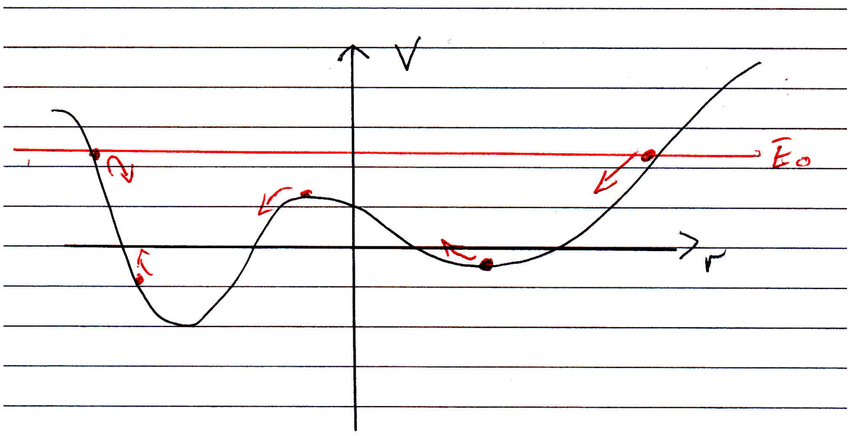

Figure 2.6.1: Motion in one dimension

Note that $E \geq V(r)$ with equality only if $\dot{r} = 0$, *i.e.* the particle is at rest. Conceptually, regardless of the origin of this system, we can think of it as a particle moving in a potential $V(x)$ and, more explicitly, imagine that $V(x)$ is the height of a hill. Then the particle's motion will simply be the same as a skier, moving without friction, along the hill. We must re-imagine our hill as a function of r distance travelled along the slope rather than of x, (this would have the effect of smoothing out the slopes).

Given that the energy is conserved along the motion, let's fix it to E_0. E_0 is a constant but one that can be changed from solution to solution. For each solution one can formally solve for $r(t)$. In fact one solves for t as a function of r and then inverts. To do this we rewrite the conservation energy equation with $E = E_0$ as

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{m}(E_0 - V(r))} \ , \tag{2.62}$$

and as

$$\frac{dr}{\sqrt{\frac{2}{m}(E_0 - V(r))}} = \pm dt \tag{2.63}$$

which we can integrate to get

$$t - t_0 = \pm \int_{r(t_0)}^{r(t)} \frac{dr'}{\sqrt{\frac{2}{m}(E_0 - V(r'))}} \ . \tag{2.64}$$

The right hand side is, for a given potential $V(r)$, just some integral that can in principle be evaluated.

Note that there is a choice of sign. For a given choice we solve the system until we reach a **turning point**. The is a point where $E_0 = V(r)$ and the expression inside the square root vanishes. Here all the energy is potential energy, *i.e.* no kinetic energy. This means that the particle is at rest at that point. The is illustrated by the left-most point in Figure 7. What typically happens is that the particle goes up the hill as far as it can and will now turn around and come back. Thus at such a point $V(r)$ was increasing, has stopped and will now start decreasing. To find the corresponding solution one must match solutions at the turning point with one choice of sign to solutions with the other choice of sign (since time always runs forwards whereas the particle can go up the hill turn around and come back).

We can illustrate this with the skier again! This is solvable for the simple case of a constant slope. Here the potential is just $V = mg\sin\theta r$ (why?). Therefore the solution we find from (2.64) is

$$\begin{aligned} t - t_0 &= \pm\int_{r(t_0)}^{r(t)} \frac{dr'}{\sqrt{\frac{2}{m}(E_0 - mg\sin\theta r')}} \\ &= \mp\frac{1}{g\sin\theta}\sqrt{\frac{2}{m}(E_0 - mg\sin\theta r')}\Bigg|_{r(t_0)}^{r(t)} \\ &= \mp\frac{1}{g\sin\theta}\sqrt{\frac{2}{m}}\left(\sqrt{E_0 - mg\sin\theta r(t)} - \sqrt{E_0 - mg\sin\theta r(t_0)}\right) \end{aligned} \tag{2.65}$$

This looks a little odd but we can square both sides

$$\begin{aligned} \frac{1}{2}mg^2\sin^2\theta(t-t_0)^2 &= (E_0 - mg\sin\theta r(t)) + (E_0 - mg\sin\theta r(t_0)) \\ &\quad - 2\sqrt{E_0 - mg\sin\theta r(t))}\sqrt{E_0 - mg\sin\theta r(t_0)} \end{aligned} \tag{2.66}$$

Next we note that

$$\sqrt{E_0 - mg\sin\theta r(t)} = \sqrt{E_0 - mg\sin\theta r(t_0)} \mp g\sin\theta\sqrt{\frac{m}{2}}(t-t_0) \tag{2.67}$$

and so

$$\begin{aligned} \frac{1}{2}mg^2\sin^2\theta(t-t_0)^2 &\mp 2g\sin\theta\sqrt{\frac{m}{2}}\sqrt{E_0 - mg\sin\theta r(t_0)}(t-t_0) \\ &= (E_0 - mg\sin\theta r(t)) - (E_0 - mg\sin\theta r(t_0)) \end{aligned} \tag{2.68}$$

The point of this mess is that it is a quadratic equation for r as a function of t. In fact we can just as well set $t_0 = 0$ by a suitable choice of time which leads to

$$r(t) = r(0) - \frac{1}{2}g\sin\theta t^2 \pm \sqrt{\frac{2}{m}}\sqrt{E_0 - mg\sin\theta r(0)}t \tag{2.69}$$

Compare this with our old result:

$$r(t) = r(0) - \frac{1}{2}g\sin\theta t^2 + v(0)t \tag{2.70}$$

From here it is clear that the choice of sign is the choice of sign of the the initial speed $v(0)$. In fact matching the coefficient of the term linear in t we identify

$$v(0) = \pm\sqrt{\frac{2}{m}}\sqrt{E_0 - mg\sin\theta r(0)} \Longleftrightarrow E_0 = \frac{1}{2}mv^2(0) + mg\sin\theta r(0) \tag{2.71}$$

which agrees with what we know. In particular the choice of sign simply corresponds to the fact that the energy is the same for a particle with velocity v or $-v$. Thus there are two branches of solutions depending on this choice.

Let us return to the case of turning points. In particular let us look at the solution we found above (again with $t_0 = 0$):

$$\begin{aligned} t &= \mp\frac{1}{g\sin\theta}\sqrt{\frac{2}{m}}\left(\sqrt{E_0 - mg\sin\theta r} - \sqrt{E_0 - mg\sin\theta r(t_0)}\right) \\ &= \mp\frac{1}{g\sin\theta}\sqrt{\frac{2}{m}}\sqrt{E_0 - mg\sin\theta r} + \frac{v(0)}{g\sin\theta}\ . \end{aligned} \tag{2.72}$$

Let us suppose that the skier starts by going up hill with some initial $v(0) > 0$ then $r(t)$ increases until she slows down to a stop. Thus both r and t are increasing (time is always increasing) which means that we must take the minus sign in (2.72) (and correspondingly the plus sign in (2.71)). When does she stop? She stopes when there is no kinetic energy so that $E = V(r(t))$, *i.e.* $E_0 - mg\sin\theta r = 0$, where the term inside the square root vanishes. It follows from (2.72) that this happens at

$$t = t_{turning} = \frac{v(0)}{g\sin\theta} \ . \tag{2.73}$$

Its called a turning point because after stopping the skier will then start to go down the hill with increasing speed. Thus after the turning point r is decreasing but t is increasing. This corresponds to the plus sign in (2.72). Thus the full solution is

$$t = \begin{cases} -\frac{1}{g\sin\theta}\sqrt{\frac{2}{m}}\sqrt{E_0 - mg\sin\theta r(t)} + \frac{v(0)}{g\sin\theta} & t \leq t_{turning} \\ +\frac{1}{g\sin\theta}\sqrt{\frac{2}{m}}\sqrt{E_0 - mg\sin\theta r(t)} + \frac{v(0)}{g\sin\theta} & t \geq t_{turning} \end{cases} \tag{2.74}$$

But of course at all times we find that

$$E_0 = \frac{1}{2}m\dot{r}^2 + mg\sin\theta r \ , \tag{2.75}$$

regardless of the choice of sign.

2.7 Angular Momentum Revisited

There are other important examples of conserved quantities. Perhaps the next most important one is angular momentum which we have already briefly seen.

A force is called central if $\underline{F} \propto \underline{r}$, *i.e.* $\underline{F} = f(\underline{r})\underline{r}$.

For a conservative force that is derived from a potential $V(\underline{r})$ then central implies that V is only a function of $|\underline{r}|$ (for some choice of the origin). Such forces include the one we used above for circular motion but also, and most importantly, gravity:

$$V_g = -\frac{G_N M m}{|\underline{r}|} \tag{2.76}$$

where G_N is Newton's constant and M,m are the masses of the two particles. To see this we note that (here we think of V as a function of $|\underline{r}|$)

$$\underline{\nabla} V = \left(\frac{dV}{d|\underline{r}|}\right)\underline{\nabla}(|\underline{r}|) \tag{2.77}$$

and

$$\underline{\nabla}(|\underline{r}|) = \frac{1}{2|\underline{r}|}\underline{\nabla}(|\underline{r}|^2) = \frac{1}{2|\underline{r}|}\underline{\nabla}(\underline{r}\cdot\underline{r}) = \frac{\underline{r}}{|\underline{r}|} \ . \tag{2.78}$$

Thus

$$\underline{F} = -\underline{\nabla} V = -\left(\frac{dV}{d|\underline{r}|}\right)\frac{\underline{r}}{|\underline{r}|} \ . \tag{2.79}$$

i.e.

$$\underline{F} = f\underline{r} \qquad f = -\frac{1}{|\underline{r}|}\frac{dV}{d|\underline{r}|} \tag{2.80}$$

Claim: The angular momentum $\underline{L} = \underline{r} \times \underline{p}$ is conserved for a central force. In other words the torque vanishes. To see this we simply note that

$$\begin{aligned}
\frac{d}{dt}\underline{L} &= \frac{d}{dt}\left(\underline{r} \times \underline{p}\right) \\
&= \dot{\underline{r}} \times \underline{p} + \underline{r} \times \dot{\underline{p}} \\
&= m\dot{\underline{r}} \times \dot{\underline{r}} + \underline{r} \times \underline{F} \\
&= \underline{0} + \underline{r} \times f\underline{r} \\
&= \underline{0}
\end{aligned} \tag{2.81}$$

2.8 Solving Three-Dimensional Motion in a Central Potential with Effective Potentials

Conservation of angular momentum, along with conservation of energy, is powerful enough to essentially solve the dynamics of a particle in $\mathbb{R}^3$, just like we did for one-dimensional motion. To this end we note that since $\underline{L} = \underline{r} \times \underline{p}$, conservation of $\underline{L}$ implies that $\underline{L}$ is orthogonal to $\underline{r}$, thus the motion is restricted to a plane: the plane orthogonal to $\underline{L}$. Let us choose coordinates where the plane is

$$\underline{r} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad \dot{\underline{r}} = \begin{pmatrix} \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \dot{r}\sin\theta + r\dot{\theta}\cos\theta \\ 0 \end{pmatrix} \tag{2.82}$$

From this we find

$$\begin{aligned}
\underline{L} &= m\underline{r} \times \dot{\underline{r}} \\
&= m \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 0 \end{pmatrix} \times \begin{pmatrix} \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \dot{r}\sin\theta + r\dot{\theta}\cos\theta \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ mr^2\dot{\theta} \end{pmatrix} ,
\end{aligned} \tag{2.83}$$

and

$$\begin{aligned}
|\dot{\underline{r}}|^2 &= (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)^2 + (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)^2 \\
&= \dot{r}^2 + r^2\dot{\theta}^2 \ .
\end{aligned} \tag{2.84}$$

Since $\underline{L}$ is conserved we can fix

$$l = mr^2\dot{\theta} \qquad \Rightarrow \qquad \dot{\theta} = \frac{l}{mr^2} \tag{2.85}$$

where l is a constant. Thus we can write the conserved energy as

$$\begin{aligned}
E_0 &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) \\
&= \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + V(r) \\
&= \frac{1}{2}m\dot{r}^2 + V_{eff}(r) \ .
\end{aligned} \tag{2.86}$$

Here

$$V_{eff}(r) = \frac{l^2}{2mr^2} + V(r) \tag{2.87}$$

is an effective potential that incorporates the effect of angular motion. We have now reduce the problem to the same as the one-dimensional case only with the effective potential $V_{eff}(r)$ in place of the original $V(r)$ (and r is now restricted to $r \geq 0$).

Note that one "effect" of angular momentum is to add a so-called angular momentum barrier that stops particles from going to $r = 0$ if $l \neq 0$. In particular since $E \geq V$ and E is constant then the particle cannot get to $r = 0$ since V_{eff} becomes arbitrarily large and hence, for small enough r, $V_{eff} > E$ which is forbidden (this assumes that there isn't a negative term in the original potential $V(r)$ that is more dominant than $l^2/2mr^2$ at small r).

2.9 Celestial Motion about the Sun

Let us now put everything we've learnt together and consider the classic case of a planet orbiting the sun with a potential

$$V = -\frac{G_N Mm}{r} \tag{2.88}$$

so that

$$V_{eff} = \frac{l^2}{2mr^2} - \frac{G_N Mm}{r} \tag{2.89}$$

Thus here there is an angular momentum barrier since the first term dominates at small r.

We can qualitatively see how the system behaves in the case of asteroid or planet orbiting the sun under the force of gravity. Looking at the plot of V_{eff} we can identify two types of trajectories. If $E_0 < 0$ then the object is a planet and cannot escape to $r \to \infty$. Rather it will oscillate around the minimum of V_{eff}. As we will see these indeed correspond to the elliptic orbits of planets. There is a also very special case of an exactly circular orbit where r is held fixed at the minimum of V_{eff}. On the other hand if $E_0 > 0$ then the object is an asteroid and will escape to $r \to \infty$. Put another way such an asteroid comes in from $r \to \infty$ and then reaches a minimal value of r and then goes back out to ∞. This is in fact a hyperbolic orbit (there is also a special case of a parabolic orbit).

In this case we can also analytically solve for the motion. To see that the solutions are ellipses and parabolas it is more helpful to think of r as a function of θ so that

$$\dot{r} = \frac{dr}{d\theta}\dot{\theta} = \frac{l}{mr^2}\frac{dr}{d\theta} \ . \tag{2.90}$$

The energy conservation equation is now

$$E_0 = \left(\frac{dr}{d\theta}\right)^2 \frac{l^2}{2mr^4} + \frac{l^2}{2mr^2} - \frac{G_N Mm}{r} \ . \tag{2.91}$$

The smart idea is to note that we can rewrite this as

$$E_0 = \frac{l^2}{2m}\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 + \frac{l^2}{2m}\frac{1}{r^2} - G_N Mm\frac{1}{r} \ . \tag{2.92}$$

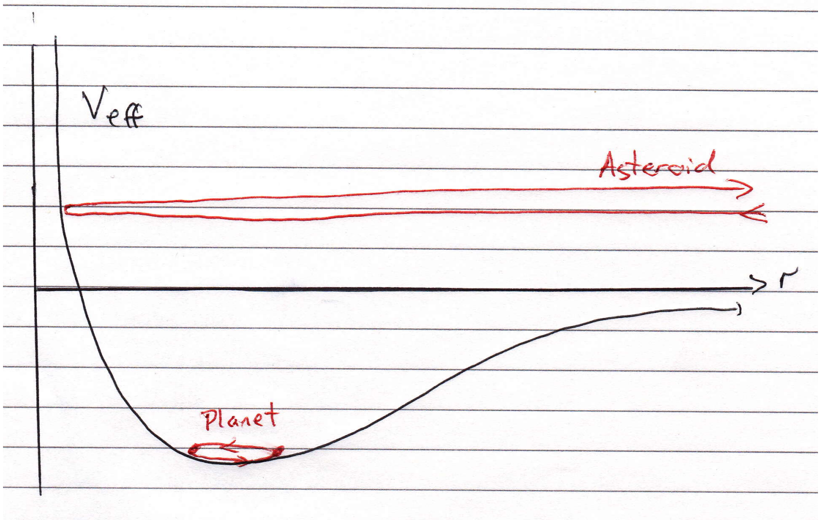

Figure 2.9.1: Planetary Effective Potential

So let us introduce $u = 1/r$ to find

$$\begin{aligned} E_0 &= \frac{l^2}{2m}\left(\frac{du}{d\theta}\right)^2 + \frac{l^2}{2m}u^2 - G_N M m u \\ &= \frac{l^2}{2m}\left(\frac{du}{d\theta}\right)^2 + \frac{l^2}{2m}\left(u - \frac{G_N M m^2}{l^2}\right)^2 - \frac{G_N^2 M^2 m^3}{2l^2} \end{aligned} \tag{2.93}$$

Next we make a shift

$$u = v + G_N M m^2/l^2 \tag{2.94}$$

so that

$$\begin{aligned} E_0 + \frac{G_N^2 M^2 m^3}{2l^2} &= \frac{l^2}{2m}\left(\frac{dv}{d\theta}\right)^2 + \frac{l^2}{2m}v^2 \\ &= \frac{l^2}{2m}\left[\left(\frac{dv}{d\theta}\right)^2 + v^2\right] \end{aligned} \tag{2.95}$$

The solution to this is

$$v = A\cos(\theta - \theta_0) \ , \tag{2.96}$$

where θ_0 is arbitrary. In fact without loss of generality we can choose coordinates such that $\theta_0 = 0$. We see that A satisfies

$$E_0 + \frac{G_N^2 M^2 m^3}{2l^2} = \frac{l^2 A^2}{2m} \qquad \Rightarrow \qquad A = \sqrt{\frac{2mE_0}{l^2} + \frac{G_N^2 M^2 m^4}{l^4}} \ . \tag{2.97}$$

Returning to the original variables we have

$$\begin{aligned} \frac{1}{r(\theta)} &= v + G_N M m^2/l^2 \\ &= A\cos(\theta - \theta_0) + B \ , \end{aligned} \tag{2.98}$$

where $B = G_N M m^2/l^2$. Or in terms of B we have

$$A^2 = B^2 + \frac{2mE_0}{l^2} \ . \tag{2.99}$$

2.10 Conic Sections

The solutions (2.98) are known as conic sections. They played a key role in dynamics since they give the motion of the planets, comet which, until the 20-th century described the motions of all heavenly bodies in the known universe (which was at that time was essentially just the solar system along with fixed stars). So they literally had cosmic signifigence and hence we should look at them more closely. The curves we find (ellipses, Parabolas and hyperbolas) are called conic sections because thats what you get by intersecting a cone with a plane in three-dimensions. But they had almost mystic significance for 300 years until the larger Universe, with a wider variety of motions, was discovered.

Let us choose a coordinate system where $\theta_0 = 0$ and set

$$x = r\cos\theta \qquad y = r\sin\theta \ . \tag{2.100}$$

We can rewrite (2.98) as

$$\begin{aligned} 1 &= Ar\cos\theta + Br \\ &= Ax + B\sqrt{x^2+y^2} \end{aligned} \tag{2.101}$$

Rearranging this and squaring both sides gives

$$(1-Ax)^2 = B^2x^2 + B^2y^2 \tag{2.102}$$

Some more rearranging gives

$$1 = (B^2-A^2)x^2 + 2Ax + B^2y^2 \tag{2.103}$$

which, for $A^2 \neq B^2$, can be written as

$$(B^2-A^2)\left(x + \frac{A}{B^2-A^2}\right)^2 + B^2y^2 = 1 + \frac{A^2}{B^2-A^2} \ . \tag{2.104}$$

or with a little more rearranging

$$\frac{(B^2-A^2)^2}{B^2}\left(x + \frac{A}{B^2-A^2}\right)^2 + (B^2-A^2)y^2 = 1 \ . \tag{2.105}$$

Let us first look at the $E_0 \geq 0$ possibilities. For $E_0 > 0$, $A^2 - B^2 > 0$ and we have

$$\frac{(A^2-B^2)^2}{B^2}\left(x - \frac{A}{A^2-B^2}\right)^2 - (A^2-B^2)y^2 = 1 \ . \tag{2.106}$$

which is a hyperbolic trajectory. The object, call it an asteroid, has come in from infinity with an non-zero initial velocity so that $E_0 > 0$. At some point it will reach a minimum value of r where it turns around and back goes out. The minimum value of r is easily seen from (2.98) to be $1/(B+A)$.

And for $B^2 = A^2$ we find a parabolic trajectory. Here it is easiest to back to the form (2.103) to find

$$1 - 2Bx = B^2y^2 \qquad \Leftrightarrow \qquad x = \frac{1}{2B} - \frac{1}{2}By^2 \ . \tag{2.107}$$

Here it is as if it is being dropped into the sun with a vanishing initial velocity. Again there it makes a closest approach before returning to infinity. The case $E_0 < 0$ is left as an exercise.

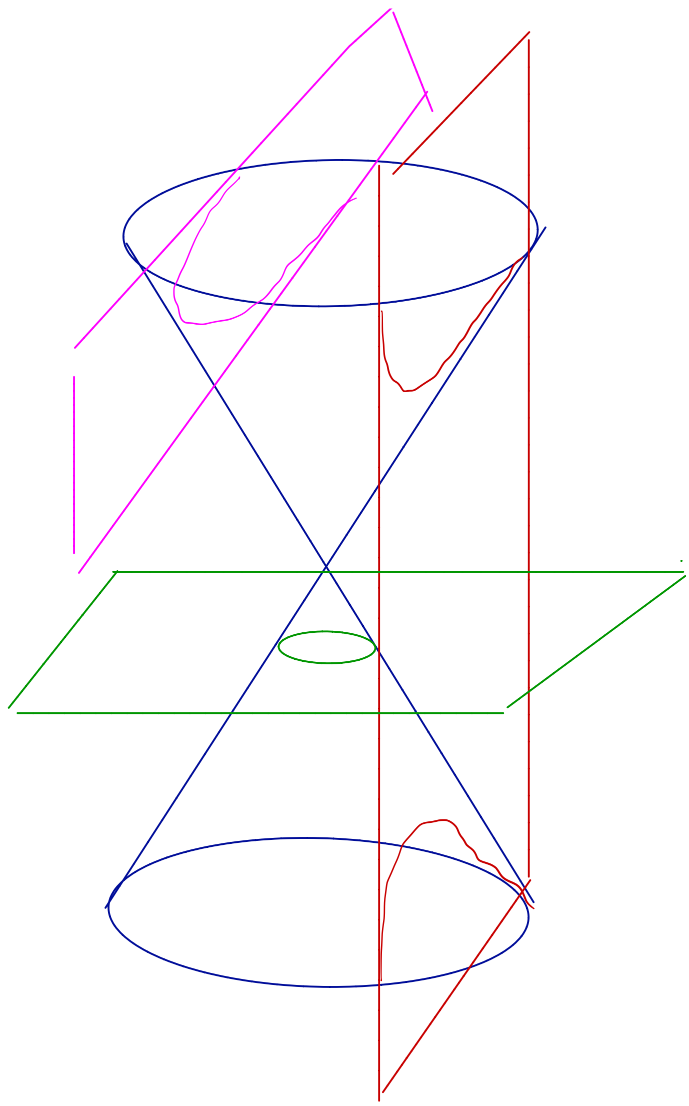

Figure 2.10.1: Conical Cross Sections

2.11 Kepler's Laws

Now we consider the case of planets. These are "bound" to the sun and cannot escape to infinity (as opposed to asteroids). Therefore they correspond to solutions with $E_0 < 0$. Before Newton Kepler was led by observations to propose three laws of planetary motion:

- [KI] The planets move in an ellipse with the sun at one focus.
- [KII] The line joining a given planet to the sun sweeps out equal areas over an equal times
- [KIII] The square of a planets orbital period is proportional to the cube of the semi-major axis.

Newton was able to derive these three laws from his universal Law of gravitation. So let's do that.

Looking at the conic sections above it is clear that they move in ellipses. It remains to see that the sun, located at $(0, 0)$ is one focus. By definition an ellipse is the set of points on the plane such the line joining them to one focus plus the line joining them to a second focus has fixed length $2d$. Let us show that what we found corresponds to an ellipse with focii at the origin and at $(-2a, 0)$ (we will have to deduce a and d). The distances to the point (x, y) are

$$d_1 = \sqrt{x^2 + y^2} \qquad d_2 = \sqrt{(x + 2a)^2 + y^2} \tag{2.108}$$

Thus the equation of an ellipse is

$$2d = d_1 + d_2 = \sqrt{x^2 + y^2} + \sqrt{(x + 2a)^2 + y^2} \tag{2.109}$$

We can rearrange this as

$$\begin{aligned}
(2d - \sqrt{x^2 + y^2})^2 &= (x + 2a)^2 + y^2 \\
\iff 4d^2 - 4d\sqrt{x^2 + y^2} &= 4ax + 4a^2 \\
\iff d^2(x^2 + y^2) &= (ax + a^2 - d^2)^2 \\
\iff (d^2 - a^2)x^2 + 2a(d^2 - a^2)x + d^2y^2 &= (d^2 - a^2)^2 \\
\iff (d^2 - a^2)(x + a)^2 + d^2y^2 &= (d^2 - a^2)^2 + (d^2 - a^2)a^2 \\
\iff (d^2 - a^2)(x + a)^2 + d^2y^2 &= (d^2 - a^2)d^2 \\
\iff \frac{1}{d^2}(x + a)^2 + \frac{1}{d^2 - a^2}y^2 &= 1
\end{aligned} \tag{2.110}$$

This agrees with (2.105) and we learn that

$$a = \frac{A}{B^2 - A^2} \qquad d = \frac{B}{B^2 - A^2} \tag{2.111}$$

Thus we have proven KI.

To prove KII we note that for an infinitesimal change in θ the infinitesimal area swept out by the line joining the planet to $(0, 0)$ is just given by a triangle with base r and height $rd\theta$ (see figure 9). Thus

$$dArea = \frac{1}{2}r^2 d\theta \tag{2.112}$$

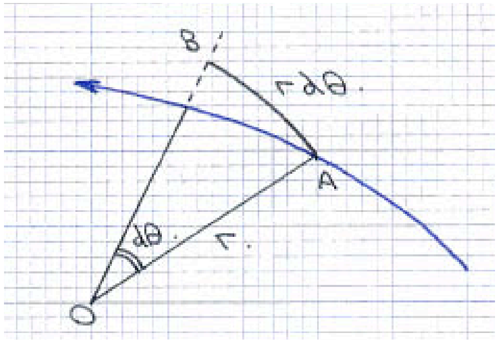

Figure 2.11.1: Area Swept-out by a Planet

for infinitesimal $d\theta$. Thus the rate of change in time is

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = l/2m \tag{2.113}$$

where l is the conserved angular momentum. This proves KII.

Finally we look at KIII). To do this we compute the area of the ellipse in two ways. First we note that since the planet sweeps out the same area at equal times we have

$$Area = \int_0^T \frac{dArea}{dt}dt = \int_0^T \frac{l}{2m}dt = \frac{lT}{2m} \tag{2.114}$$

where T is the period of the orbit, *i.e.* how long it takes to go around.

On the other hand for an ellipse of the form

$$\frac{(x+a)^2}{d^2} + \frac{y^2}{d^2-a^2} = 1 \qquad i.e. \qquad y = \pm\sqrt{d^2-a^2}\sqrt{1-\frac{(x+a)^2}{d^2}} \tag{2.115}$$

the area is simply

$$\begin{aligned} Area &= 2\sqrt{d^2-a^2}\int_{-a-d}^{-a+d}\sqrt{1-\frac{(x+a)^2}{d^2}}dx \\ &= 2d\sqrt{d^2-a^2}\int_{-1}^{1}\sqrt{1-z^2}dz \end{aligned} \tag{2.116}$$

where we introduced $z = (x+a)/d$. Continuing we find, introducing $z = \sin\varphi$,

$$\begin{aligned} Area &= 2d\sqrt{d^2-a^2}\int_{-\pi/2}^{\pi/2}\cos\varphi d\sin\varphi \\ &= 2d\sqrt{d^2-a^2}\int_{-\pi/2}^{\pi/2}\cos^2\varphi d\varphi \\ &= \pi d\sqrt{d^2-a^2}\ . \end{aligned} \tag{2.117}$$

We already saw that $d = B/(B^2-A^2)$ and $a = A/(B^2-A^2)$ so

$$d^2 - a^2 = \frac{B^2}{(B^2-A^2)^2} - \frac{A^2}{(B^2-A^2)^2} = \frac{1}{B^2-A^2} \tag{2.118}$$

and setting the areas equal we find

$$\frac{lT}{2m} = \pi\frac{B}{(B^2-A^2)^{3/2}}\ . \tag{2.119}$$

Finally we note that the semi-major axis is defined as half the widest distance across an ellipse:

$$
\begin{aligned}
R_{smaj} &= \frac{1}{2}(r(0) + r(\pi)) \\
&= \frac{1}{2}\left(\frac{1}{A+B} + \frac{1}{-A+B}\right) \\
&= \frac{B}{B^2 - A^2}
\end{aligned} \tag{2.120}
$$

Thus

$$
\frac{lT}{2m} = \pi B \left(\frac{R_{smaj}}{B}\right)^{3/2} = \pi B^{-1/2} R_{smaj}^{3/2} \tag{2.121}
$$

and hence

$$
T = \frac{2\pi m}{l} B^{-1/2} R_{smaj}^{3/2} = 2\pi (G_N M)^{-1/2} R_{smaj}^{3/2} \tag{2.122}
$$

and we have proven KIII, including a calculation of the constant of proportionality.

2.12 Weighing Planets

We see from the above that by observing the planets we could deduce $G_N M_{sun}$. Similar equations apply for the case of the moon orbiting the earth. In this case we could measure $G_N M_{earth}$. We can also deduce $g = 9.8m/s^2$ by simply expanding the force of gravity near the surface of the the earth. In particular if we lift a particle up a height h above the earth's surface then

$$
\begin{aligned}
V(h) &= -\frac{G_N M_{earth} m}{r} \\
&= -\frac{G_N M_{earth} m}{R_{earth} + h} \\
&= -\frac{G_N M_{earth} m}{R_{earth}} \left(\frac{1}{1 + h/R_{earth}}\right) \\
&= -\frac{G_N M_{earth} m}{R_{earth}} + \frac{G_N M m}{R_{earth}^2} h + \dots
\end{aligned} \tag{2.123}
$$

The first term is a constant that can be ignored. Comparing the second term to the formula $V = mgh$ we deduce that

$$
g = \frac{G_N M_{earth}}{R_{earth}^2} . \tag{2.124}
$$

Since one can measure R_{earth} and g we can again deduce $G_N M_{earth}$.

However we would really like to know G_N as a universal constant. This was first done by Lord Cavendish up near Russell Square. He was the first to perform an experiment to measure the force of gravitational attraction between two massive balls. Since he knew their mass he could deduce G_N. It is a subtle experiment based on measuring the oscillations of the two balls suspended in a so-called torsion balance configuration, but he was able to do it. In fact he was very close to the current value of

$$
G_N \sim 6.67 \times 10^{-11} m^3/kg/s^2 \tag{2.125}
$$

(he obtained something like $6.75 \times 10^{-11} m^3/kg/s^2$). Although this determined G_N it is more commonly known as "weighing the earth" since one could then deduce M_{earth} and M_{sun} and this was his primary objective.

2.13 The Runge-Lenz Vector

Lastly it is worth mentioning the famous Runge-Lenz vector. Although it is generally acknowledged that it was known to Euler, Lagrange and others before Runge and Lenz. Runge apparently wrote about it in a text book, which Lenz referenced in a paper, and ever since then its had its current name. In any case the Runge-Lenz vector is

$$\underline{A} = \underline{p} \times \underline{L} + mV(\underline{r})\underline{r} \ . \tag{2.126}$$

To see that it is conserved we compute:

$$\begin{aligned}\dot{\underline{A}} &= \dot{\underline{p}} \times \underline{L} + \underline{p} \times \dot{\underline{L}} + m(\underline{\nabla} V \cdot \dot{\underline{r}})\underline{r} + mV(|\underline{r}|)\dot{\underline{r}} \\ &= -\underline{\nabla} V \times \underline{L} + \underline{p} \times \underline{0} + m(\underline{\nabla} V \cdot \dot{\underline{r}})\underline{r} + mV(|\underline{r}|)\dot{\underline{r}}\end{aligned} \tag{2.127}$$

where we have used NI with $\underline{F} = -\underline{\nabla} V$ and $\dot{\underline{L}} = \underline{0}$ because $\underline{F}$ is central. Next we note that and $\underline{L} = m\underline{r} \times \dot{\underline{r}}$ so that

$$\begin{aligned}\dot{\underline{A}} &= -m\underline{\nabla} V \times (r \times \dot{\underline{r}}) + m(\underline{\nabla} V \cdot \dot{\underline{r}})\underline{r} + mV(|\underline{r}|)\dot{\underline{r}} \\ &= -m(\underline{\nabla} V \cdot \dot{\underline{r}})\underline{r} + m(\underline{\nabla} V \cdot \underline{r})\dot{\underline{r}} + m(\underline{\nabla} V \cdot \dot{\underline{r}})\underline{r} + mV(|\underline{r}|)\dot{\underline{r}}\end{aligned} \tag{2.128}$$

Here we used the triple product identity (1.36). Clearly the first and third terms cancel. To show that the second and fourth also cancel we note that $V \propto |\underline{r}|^{-1}$ so that

$$\begin{aligned}\underline{\nabla} V &= \frac{dV}{d|\underline{r}|}\underline{\nabla}|\underline{r}| \\ &= -\frac{V}{|\underline{r}|}\frac{1}{2|\underline{r}|}\underline{\nabla}(\underline{r} \cdot \underline{r}) \\ &= -\frac{V}{|\underline{r}|^2}\underline{r}\end{aligned} \tag{2.129}$$

Thus $\underline{\nabla} V \cdot \underline{r} = -V$ and indeed the second and fourth terms cancel. The existence of this extra conserved vector is in fact rather miraculous. It arises due to a so-called hidden symmetry that we will return to later. Note also that the only important feature is that $V \sim 1/r$ so it also works for an electric charge orbiting an opposite charge as in the Hydrogen atom.

To see how useful it is we simply evaluate

$$\begin{aligned}\underline{A} \cdot \underline{r} &= \underline{r} \cdot (\underline{p} \times \underline{L}) + mV(\underline{r})|\underline{r}|^2 \\ &= (\underline{r} \times \underline{p}) \cdot \underline{L} + mV(\underline{r})|\underline{r}|^2 \\ &= \underline{L} \cdot \underline{L} + mV(\underline{r})|\underline{r}|^2 \\ &= l^2 - G_N M m^2 r \ ,\end{aligned} \tag{2.130}$$

where $r = |\underline{r}|$. In the second line we used the fact that

$$\begin{aligned}\underline{u} \cdot (\underline{v} \times \underline{w}) &= \sum_{abc} \epsilon_{abc} u^a v^b w^c \\ &= \sum_{abc} \epsilon_{cab} u^a v^b w^c \\ &= (\underline{u} \times \underline{v}) \cdot \underline{w} \ .\end{aligned} \tag{2.131}$$

On the other hand the right hand side is just $|\underline{A}| r \cos\theta$, where θ is the angle between $\underline{r}$ and the fixed vector $\underline{A}$. Thus we have

$$|\underline{A}| r \cos\theta = l^2 - G_N M m^2 r \tag{2.132}$$

which is the same equation for $r(\theta)$ that we derived if we identify $|\underline{A}| = Al^2$. Thus we have been able to derive the equation for $r(\theta)$ without ever solving a differential equation!

2.14 Multi-Particle Systems

So far we have mainly concerned our selves with the motion of a single particle in some external force. Our next step is to consider many particles which we label by i. In this case we can distinguish between two types of forces acting on the ith particle:

- external forces $\underline{F}_i^{ext}$
- inter particle forces $\underline{F}_{ij}^{int}$ between the ith and jth particle.

Note that (NIII) implies that $\underline{F}_{ij}^{int} = -\underline{F}_{ji}^{int}$. And hence also that $\underline{F}_{ii}^{int} = \underline{0}$. Thus (NII) can be written as

$$\frac{d}{dt}\underline{p}^i = \underline{F}_i^{ext} + \sum_j \underline{F}_{ij}^{int} \tag{2.133}$$

It may not be necessary to know exactly what each particle is doing and one might just be interested in the average. To study this we can sum (2.133) over i:

$$\begin{aligned}\frac{d}{dt}\sum_i \underline{p}^i &= \sum_i \underline{F}_i^{ext} + \sum_{ij} \underline{F}_{ij}^{int} \\ &= \sum_i \underline{F}_i^{ext} - \sum_{ij} \underline{F}_{ji}^{int} \\ &= \sum_i \underline{F}_i^{ext} - \sum_{ij} \underline{F}_{ij}^{int} \\ &= \sum_i \underline{F}_i^{ext}\end{aligned} \tag{2.134}$$

Here we have used NIII: $\underline{F}_{ij}^{int} = -\underline{F}_{ji}^{int}$ so that the second sum vanishes. Let us also suppose that each particle has $\underline{p}^i = m_i \dot{\underline{r}}_i$. It is helpful to introduce the centre of mass and total mass:

$$\underline{R} = \frac{\sum_i m_i \underline{r}_i}{M}\ , \qquad M = \sum_i m_i \tag{2.135}$$

This is a weighted sum over the various positions with the weight of each particle given by its mass. From this it is easy to see that

$$\begin{aligned}\frac{d}{dt}\sum_i \underline{p}_i &= \frac{d}{dt}\sum_i m_i \dot{\underline{r}}_i \\ &= \sum_i m_i \ddot{\underline{r}}_i \\ &= M\ddot{\underline{R}}\ .\end{aligned} \tag{2.136}$$

And therefore

$$M\ddot{\underline{R}} = \sum_i \underline{F}_i^{ext} \tag{2.137}$$

Thus the centre of mass is only sensitive to the external forces. This makes sense as without it we wouldn't be able to describe a tennis ball without also thinking of all the individual atoms, or even quarks, or whatever quarks are made of, inside it.

To continue with our generalization to many bodies we define the angular momentum of the system to be the sum of the individual angular momenta:

$$\begin{aligned} \underline{L} &= \sum_i \underline{r}_i \times \underline{p}_i \\ &= \sum_i m_i \underline{r}_i \times \dot{\underline{r}}_i \end{aligned} \tag{2.138}$$

where the second line holds in the case that $\underline{p}^i = m_i \dot{\underline{r}}_i$ with m_i constant. In particular we don't expect that the individual angular momentum will be conserved for many bodies. Indeed even if the force between any pair of particles is central there will not be a common origin for all particles. We therefore have the that the torque is

$$\begin{aligned} \underline{N} &= \dot{\underline{L}} \\ &= \sum_i \dot{\underline{r}}_i \times \underline{p}_i + \underline{r}_i \times \dot{\underline{p}}_i \ . \end{aligned} \tag{2.139}$$

Let us assume that $\underline{p}^i = m_i \dot{\underline{r}}_i$ then $\dot{\underline{r}}_i \times \underline{p}_i = \underline{0}$ and hence

$$\begin{aligned} \underline{N} &= \sum_i \underline{r}_i \times \dot{\underline{p}}_i \\ &= \sum_i \underline{r}_i \times \left(\underline{F}_i^{ext} + \sum_j \underline{F}_{ij}^{int} \right) \end{aligned} \tag{2.140}$$

To proceed we need to assume something about $\underline{F}_{ij}^{int}$. A natural choice is to assume that

$$\underline{F}_{ij}^{int} \propto (\underline{r}_i - \underline{r}_j) \qquad i.e. \qquad \underline{F}_{ij}^{int} = F_{ij}^{int}(\underline{r}_i - \underline{r}_j) \tag{2.141}$$

where F_{ij}^{int} is some scalar function of the positions: $F_{ij}^{int}(\underline{r}_i, \underline{r}_j)$. This is quite reasonable for an internal force between two particles i and j. Indeed it is a straightforward generalization of our definition of a central force. Thus we have

$$\begin{aligned} \underline{N} &= \sum_i \underline{r}_i \times \underline{F}_i^{ext} + \sum_{ij} F_{ij}^{int} \underline{r}_i \times (\underline{r}_i - \underline{r}_j) \\ &= \sum_i \underline{r}_i \times \underline{F}_i^{ext} - \sum_{ij} F_{ij}^{int} \underline{r}_i \times \underline{r}_j \ . \end{aligned} \tag{2.142}$$

Next we note that by NIII $\underline{F}_{ij}^{int} = -\underline{F}_{ji}^{int}$ and hence $F_{ij}^{int} = F_{ji}^{int}$. Thus by changing the

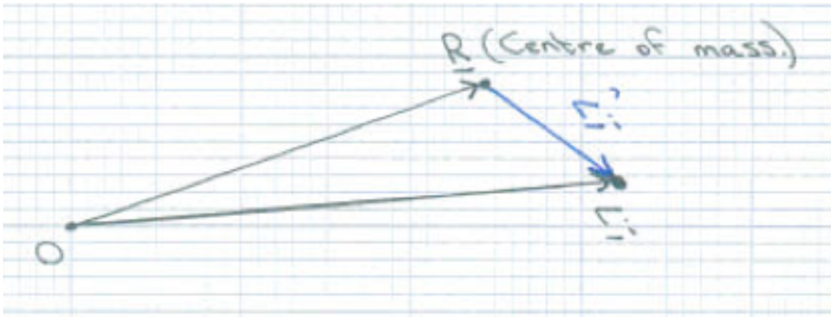

Figure 2.14.1: The Centre of Mass

summation variables

$$\begin{aligned}\sum_{ij} F^{int}_{ij}\underline{r}_i \times \underline{r}_j &= \sum_{ji} F^{int}_{ji}\underline{r}_j \times \underline{r}_i \\ &= -\sum_{ji} F^{int}_{ji}\underline{r}_i \times \underline{r}_j \\ &= -\sum_{ji} F^{int}_{ij}\underline{r}_i \times \underline{r}_j \\ &= -\sum_{ij} F^{int}_{ij}\underline{r}_i \times \underline{r}_j \ .\end{aligned} \tag{2.143}$$

Thus

$$\sum_{ij} F^{int}_{ij}\underline{r}_i \times \underline{r}_j = 0 \ , \tag{2.144}$$

and hence

$$\underline{N} = \dot{\underline{L}} = \sum_i \underline{r}_i \times \underline{F}^{ext}_i \ . \tag{2.145}$$

It is instructive to separate off this centre-of-mass motion and write

$$\underline{r}_i = \underline{r}'_i + \underline{R} \tag{2.146}$$

The total angular momentum is

$$\begin{aligned}\underline{L} &= \sum_i m_i \underline{r}_i \times \dot{\underline{r}}_i \\ &= \sum_i m_i(\underline{r}'_i + \underline{R}) \times (\dot{\underline{r}}'_i + \dot{\underline{R}}) \\ &= \sum_i m_i \underline{r}'_i \times \dot{\underline{r}}'_i + \sum_i m_i \underline{r}'_i \times \dot{\underline{R}} + \sum_i m_i \underline{R} \times \dot{\underline{r}}'_i + \sum_i m_i \underline{R} \times \dot{\underline{R}} \ .\end{aligned} \tag{2.147}$$

However we note that

$$\begin{aligned}M\underline{R} &= \sum_i m_i \underline{r}_i \\ &= \sum_i m_i(\underline{r}'_i + \underline{R}) \\ &= \sum_i m_i \underline{r}' + M\underline{R} \ ,\end{aligned} \tag{2.148}$$

and therefore

$$\sum_i m_i \underline{r}' = 0 \qquad \Rightarrow \qquad \sum_i m_i \dot{\underline{r}}' = 0 \ . \tag{2.149}$$

Thus things simplify and we have

$$\underline{L} = \sum_i m_i \underline{r}_i' \times \dot{\underline{r}}_i' + \sum_i m_i \underline{R} \times \dot{\underline{R}} \ . \tag{2.150}$$

which is simply the sum of the individual angular momenta about the centre of mass and the centre of mass angular momentum.

2.15 Work and Energy for Many Particles

It is also easy to see that the work just the sum of the individual work done by each particle:

$$\begin{aligned} \Delta W &= \sum_i \int_{t_1}^{t_2} \underline{F}_i \cdot d\underline{r}_i \\ &= \sum_i \int_{t_1}^{t_2} m_i \ddot{\underline{r}}_i \cdot \dot{\underline{r}}_i dt \\ &= \sum_i \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{1}{2} m_i |\dot{\underline{r}}_i|^2\right) dt \\ &= \sum_i (T_i(t_2) - T_i(t_2)) \end{aligned} \tag{2.151}$$

is the sum over the changes of the individual kinetic energies. As before this will be path independent if all the forces are conservative.

How is it that in a world with only fundamental forces friction arises? Where does the excess work and energy go? Well as you are pushing a shopping trolley around the supermarket you are heating up the wheels and floor. This leads to an increase in the kinetic energy of the molecules in the wheels and floor. But it is disorganized and not useful as the molecules are pushed in all sorts of directions. So from a macroscopic perspective it is just wasted. This is essentially the definition of heat: wasted energy (although you might not think the heat of a fire is wasted on a cold day or a romantic evening).

Suppose the internal and external forces can be obtained through potentials:

$$\underline{F}_i^{ext} = -\underline{\nabla}_i V^{ext} \qquad \underline{F}_{ij}^{int} = -\underline{\nabla}_i V_{ij}^{int} \ . \tag{2.152}$$

where $\underline{\nabla}_i$ is the gradient with respect to $\underline{r}_i$. Note that Newtons third law $\underline{F}_{ij}^{int} = -\underline{F}_{ji}^{int}$ implies that

$$\underline{\nabla}_i V_{ij}^{int} = -\underline{\nabla}_j V_{ji}^{int} \tag{2.153}$$

In the case where $\underline{F}_{ij}^{int} = F_{ij}^{int}(\underline{r}_i - \underline{r}_j)$ that we considered above arises when V_{ij}^{int} is only a function of the separation between two particles: $V_{ij}^{int} = V_{ij}^{int}(\underline{r}_i - \underline{r}_j)$ this means that Newton's third law is simply (up to an unimportant constant)

$$V_{ij}^{int} = V_{ji}^{int} \ . \tag{2.154}$$

It is easy to see that the work done will be path independent

$$\Delta W = \sum_i \int_{\underline{r}_1}^{\underline{r}_2} \underline{F}_i \cdot d\underline{r}_i$$
$$= -\sum_i \int_{t_1}^{t_2} \left(\underline{\nabla}_i V^{ext} + \sum_j \underline{\nabla}_i V_{ij} \right) \cdot \dot{\underline{r}}_i dt \tag{2.155}$$

As before we recognise the first term as just dV^{ext}/dt but the second term requires a bit of care. We observe that

$$\frac{d}{dt}\sum_{ij} V_{ij}^{int} = \sum_{ijk} \underline{\nabla}_k V_{ij}^{int} \cdot \dot{\underline{r}}_k$$
$$= \left(\sum_{ij} \underline{\nabla}_i V_{ij}^{int} \cdot \dot{\underline{r}}_i + \sum_{ij} \underline{\nabla}_j V_{ij}^{int} \cdot \dot{\underline{r}}_j \right)$$
$$= \left(\sum_{ij} \underline{\nabla}_i V_{ij}^{int} \cdot \dot{\underline{r}}_i + \sum_{ij} \underline{\nabla}_i V_{ji}^{int} \cdot \dot{\underline{r}}_i \right)$$
$$= 2\sum_{ij} \underline{\nabla}_i V_{ij}^{int} \cdot \dot{\underline{r}}_i \tag{2.156}$$

Here in the second line we used the fact that V_{ij}^{int} is a function of $\underline{r}_i - \underline{r}_j$ (and in particular does not depend on any third coordinate $\underline{r}_k$ with $k \neq i, j$). So we must have $k = i$ or $k = j$. In the third line we swapped the i, j indices on the sum (which does nothing). In the last line we used $V_{ji}^{int} = V_{ij}^{int}$. Thus going back to ΔW we see that

$$\Delta W = -\int_{t_1}^{t_2} \frac{d}{dt}\left(V^{ext} + \frac{1}{2}\sum_{ij} V_{ij}^{int} \right) dt$$
$$= -\left(V^{ext} + \frac{1}{2}\sum_{ij} V_{ij}^{int} \right)(t_2) + \sum_i \left(V^{ext} + \frac{1}{2}\sum_j V_{ij}^{int} \right)(t_1) \tag{2.157}$$

One way to understand this factor of $1/2$ is that takes into account the over-counting in the sum over all i, j as $V_{ij}^{int} = V_{ji}^{int}$.

Comparing (2.151) with we see that the total energy

$$E = \frac{1}{2}\sum_i m_i |\dot{\underline{r}}_i|^2 + V_{ext} + \frac{1}{2}\sum_{ij} V_{ij}^{int} \ , \tag{2.158}$$

Indeed a short calculation shows that:

$$\dot{E} = \sum_i m_i \dot{\underline{r}}_i \cdot \ddot{\underline{r}}_i + \sum_i \underline{\nabla}_i V^{ext} \cdot \dot{\underline{r}}_i + \frac{1}{2}\frac{d}{dt}\sum_{ij} V_{ij}^{int}$$
$$= \sum_i m_i \dot{\underline{r}} \cdot \ddot{\underline{r}} + \sum_i \underline{\nabla}_i V^{ext} \cdot \dot{\underline{r}}_i + \sum_{ij} \underline{\nabla}_i V_{ij}^{int} \cdot \dot{\underline{r}}_i$$
$$= \sum_i \dot{\underline{r}}_i \cdot \left(m_i \ddot{\underline{r}} + \underline{\nabla}_i V^{ext} + \sum_j \underline{\nabla}_i V_{ij}^{int} \right)$$
$$= \sum_i \dot{\underline{r}}_i \cdot \left(m_i \ddot{\underline{r}}_i - \underline{F}_i^{ext} - \sum_j \underline{F}_{ij}^{int} \right)$$
$$= 0 \ . \tag{2.159}$$

Furthermore, as we have seen above, if the external force is central, in the sense that $\underline{r}_i \times \underline{F}_i^{ext} = \underline{0}$ then we also have a conserved total angular momentum:

$$\underline{L} = \sum_i m_i \underline{r}'_i \times \dot{\underline{r}}'_i + \sum_i m_i \underline{R} \times \dot{\underline{R}} \ . \tag{2.160}$$

However unlike in the single particle case these are not enough to solve for the system in general. There are simply too many variables, too many degrees of freedom. An exception is the two-body case that we now consider.

2.16 Solving the two-body problem

The simplest example of a many-body problem is the 2-body problem. This is meant to refer to two particles under-going a mutual interaction along with a possible external force. In fact we can solve the 2 body problem for a wide class of forces.

Let us suppose that we have two particles with positions $\underline{r}_1$ and $\underline{r}_2$ which move subject to an external force $\underline{F}_{ext}$ as well as an internal force $\underline{F}_{12} = -\underline{F}_{21}$. We have seen that the center-of-mass $\underline{R}$ only sees $\underline{F}_{ext}$. So let us change variables to

$$\underline{R} = \frac{m_1\underline{r}_1 + m_2\underline{r}_2}{m_1+m_2} \qquad \underline{r}_{12} = \underline{r}_1 - \underline{r}_2 \ . \tag{2.161}$$

where $M = m_1 + m_2$ is the total mass. We need to invert this to find $\underline{r}_1$ and $\underline{r}_2$ as functions of $\underline{R}$ and $\underline{r}_{12}$

$$(m_1+m_2)\underline{R} = m_1\underline{r}_1 + m_2(\underline{r}_1 - \underline{r}_{12}) \tag{2.162}$$

rearranging gives

$$\underline{r}_1 = \underline{R} + \frac{m_2}{M}\underline{r}_{12} \tag{2.163}$$

and

$$\begin{aligned} \underline{r}_2 &= \underline{r}_1 - \underline{r}_{12} \\ &= \underline{R} - \frac{m_1}{M}\underline{r}_{12} \end{aligned} \tag{2.164}$$

The conserved energy is

$$\begin{aligned} E &= \frac{1}{2}m_1|\dot{\underline{r}}_1|^2 + \frac{1}{2}m_2|\dot{\underline{r}}_2|^2 + V_{12} + V^{ext} \\ &= \frac{1}{2}M|\dot{\underline{R}}|^2 + \frac{1}{2}\mu|\dot{\underline{r}}_{12}|^2 + V_{12} + V^{ext} \end{aligned} \tag{2.165}$$

where

$$\mu = \frac{m_1 m_2}{m_1+m_2} \ , \tag{2.166}$$

is called the reduced mass. Note that in the limit that one mass is much larger than the other, say $m_1 >> m_2$ we simply have

$$\mu = \frac{m_2}{1+m_2/m_1} = m_2\left(1 - m_2/m_1 + \ldots\right) = m_2 - .. \ , \tag{2.167}$$

corresponding to the lighter particle (but slightly reduced - hence the name) and

$$M = m_1 + m_2 = m_1\left(1 + m_2/m_1\right) = m_1 + ... \ , \tag{2.168}$$

corresponding to the heavy particle. In addition we find

$$\begin{aligned}
\underline{R} &= \frac{m_1}{m_1+m_2}\underline{r}_1 + \frac{m_2}{m_1+m_2}\underline{r}_2 \\
&= \frac{1}{1+m_2/m_1}\underline{r}_1 + \frac{m_2/m_1}{1+m_2/m_1}\underline{r}_2 \\
&= \underline{r}_1 - (m_2/m_1)\,\underline{r}_{12} + \dots \\
&= \underline{r}_1 + \dots \ ,
\end{aligned} \tag{2.169}$$

so that the centre of mass is essentially just where the heavy particle sits. On the other hand if both particles have the same mass then

$$\mu = \frac{m}{2} \qquad M = 2m \ . \tag{2.170}$$

and

$$\underline{R} = \frac{\underline{r}_1 + \underline{r}_2}{2} \ , \tag{2.171}$$

is the average position.

A key point of the two body problem is that, under certain conditions, namely if V^{ext} only depends on $\underline{R}$ and V_{12}^{int} only depends on $\underline{r}_{12}$, then we can reduce it to two one body problems: one for $\underline{R}$ and one for $\underline{r}_{12}$. To see this we note that the equations for $\underline{r}_1$ and $\underline{r}_2$ are

$$\begin{aligned}
m_1\ddot{\underline{r}}_1 &= -\underline{\nabla}_1 V^{ext} - \underline{\nabla}_1 V_{12} \\
m_2\ddot{\underline{r}}_2 &= -\underline{\nabla}_2 V^{ext} - \underline{\nabla}_2 V_{12} \ .
\end{aligned} \tag{2.172}$$

The equation for $\underline{R}$ is obtained by summing the two equations in (2.172)

$$\begin{aligned}
M\ddot{\underline{R}} &= -\underline{\nabla}_1 V^{ext} - \underline{\nabla}_2 V^{ext} \\
&= -\frac{m_1}{M}\underline{\nabla}_{\underline{R}} V^{ext} - \frac{m_2}{M}\underline{\nabla}_{\underline{R}} V^{ext} \\
&= -\underline{\nabla}_{\underline{R}} V^{ext} \\
&= \sum_i \underline{F}_i^{ext}
\end{aligned} \tag{2.173}$$

which agrees with our general result above. Here we used the NIII condition that $\underline{\nabla}_1 V_{12} = -\underline{\nabla}_2 V_{12}$ as well as the chain rule to write

$$\underline{\nabla}_1 V^{ext} = \frac{m_1}{M}\underline{\nabla}_{\underline{R}} V^{ext} \ , \qquad \underline{\nabla}_2 V^{ext} = \frac{m_2}{M}\underline{\nabla}_{\underline{R}} V^{ext} \tag{2.174}$$

assuming that V^{ext} only depends on $\underline{R}$. Therefore the equation for $M\ddot{\underline{R}}$ only involves $\underline{R}$ and a again leads to the conservation of

$$E_{cm} = \frac{1}{2}M|\dot{\underline{R}}|^2 + V^{ext} \ . \tag{2.175}$$

To obtain an equation for the relative position we multiply the first equation in (2.172) by m_2 and the second by m_1 and then subtract them to find

$$M\mu\ddot{\underline{r}}_{12} = -m_2\underline{\nabla}_1 V^{ext} + m_1\underline{\nabla}_2 V^{ext} - m_2\underline{\nabla}_1 V_{12} + m_1\underline{\nabla}_2 V_{12} \tag{2.176}$$

If V^{ext} is only a function of $\underline{R}$ then

$$-m_2\underline{\nabla}_1 V^{ext} + m_1\underline{\nabla}_2 V^{ext} = -\frac{m_2 m_1}{M}\underline{\nabla}_{\underline{R}} V^{ext} + \frac{m_1 m_2}{M}\underline{\nabla}_{\underline{R}} V^{ext} = \underline{0}\ . \tag{2.177}$$

Also if V_{12} is just a function of $\underline{r}_{12}$ then

$$\underline{\nabla}_2 V_{12} = -\underline{\nabla}_1 V_{12} = -\underline{\nabla}_{12} V_{12}\ . \tag{2.178}$$

Thus (2.176) becomes

$$\mu\ddot{\underline{r}}_{12} = -\underline{\nabla}_{12} V_{12}\ . \tag{2.179}$$

This equation only depends on $\underline{r}_{12}$ and not $\underline{R}$. As before this equation tells us that

$$E_{12} = \frac{1}{2}\mu|\dot{\underline{r}}_{12}|^2 + V_{12} \tag{2.180}$$

is conserved on its own.

Thus we have reduced the two-body problem to two one-body problems. In particular the total energy $E = E_{12} + E_{cm}$ is actually the sum of two conserved energies.

Finally if V_{12} only depends on $r_{12} = |\underline{r}_{12}|$ then energy and angular momentum of the relative system will be conserved leading to a single one-dimensional problem:

$$E_{12} = \frac{1}{2}\mu\dot{r}_{12}^2 + V_{eff}^{int} \qquad V_{eff}^{int} = V_{12}(r_{12}) + \frac{l_{12}^2}{2\mu r_{12}^2} \tag{2.181}$$

and similarly if V^{ext} only depends on $R = |\underline{R}|$ then

$$E_{cm} = \frac{1}{2}M\dot{R}^2 + V_{eff}^{ext} \qquad V_{eff}^{ext} = V^{ext}(R) + \frac{l_{cm}^2}{2MR^2}\ . \tag{2.182}$$

Next you might try the three-body problem, but that is unsolvable in general. But some things can be said in special cases or limits (such as when one mass is much heavier than the others).

2.17 Examples

Let us first consider a simple example. Let us consider two skiers going down the same ski slope that we had before. We only consider one-dimensional motion where both skiers move straight down the hill. They have masses m_1 and m_2 and their distance from the bottom of the slope are r_1 and r_2 respectively. Thus they are subject to an external force of gravity

$$F_1^{ext} = -m_1 g\sin\theta \qquad F_2^{ext} = -m_2 g\sin\theta \tag{2.183}$$

However they are tied together by a spring which induced an internal force

$$F_{12}^{int} = -F_{21}^{int} = -k(r_1 - r_2) \tag{2.184}$$

with some constant $k > 0$. Thus the equations of motion are

$$\begin{aligned} m_1\ddot{r}_1 &= -m_1 g\sin\theta - k(r_1 - r_2) \\ m_2\ddot{r}_2 &= -m_2 g\sin\theta - k(r_2 - r_1) \end{aligned} \tag{2.185}$$

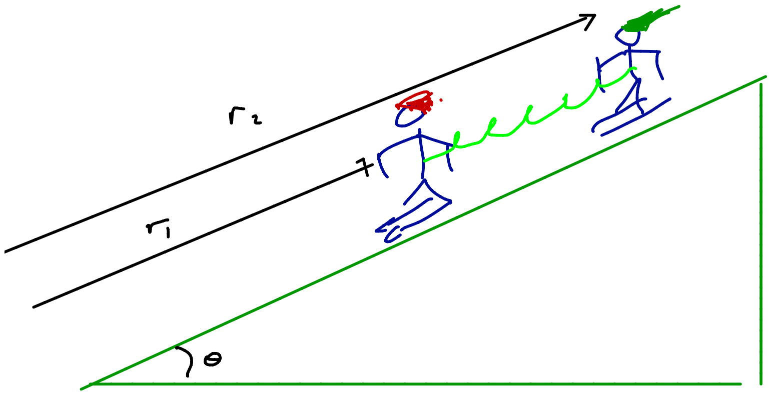

Figure 2.17.1: Two Skiers Tied Together With A Spring

If we add the two equations of motion we readily see that

$$M\ddot{R} = -Mg\sin\theta \tag{2.186}$$

where the centre of mass is $R = (m_1r_1 + m_2r_2)/(m_1 + m_2)$ and the total mass is M. As expected this equation is independent of r_{12} and can be solved by

$$R(t) = -\frac{1}{2}g\sin\theta t^2 + \dot{R}(0)t + R(0) \tag{2.187}$$

Thus the centre of mass behaves just like a single skier did.

On the other hand taking m_2 times the first equation and subtracting m_1 times the second equation we find

$$\mu\ddot{r}_{12} = -kr_{12} \tag{2.188}$$

where $\mu = m_1m_2/M$ is the reduced mass. If we let $\omega = \sqrt{k/\mu}$ then the solutions to this equation are (see Problem 3.3).

$$r_{12}(t) = \frac{1}{\omega}\dot{r}_{12}(0)\sin(\omega t) + r_{12}(0)\cos(\omega t) \tag{2.189}$$

Thus the separation between the skiers behaves as a simple harmonic oscillator. Putting things back we find

$$\begin{aligned} r_1 &= R + \frac{m_2}{M}r_{12} \\ &= -\frac{1}{2}g\sin\theta t^2 + \dot{R}(0)t + R(0) + \frac{m_2}{M\omega}\dot{r}_{12}(0)\sin(\omega t) + \frac{m_2}{M}r_{12}(0)\cos(\omega t) \\ r_2 &= R - \frac{m_1}{M}r_{12} \\ &= -\frac{1}{2}g\sin\theta t^2 + \dot{R}(0)t + R(0) - \frac{m_1}{M\omega}\dot{r}_{12}(0)\sin(\omega t) - \frac{m_1}{M}r_{12}(0)\cos(\omega t) \end{aligned} \tag{2.190}$$

We also see that we these forces come from the potentials

$$\begin{aligned} V^{ext} &= m_1g\sin\theta r_1 + m_2g\sin\theta r_2 \\ &= gM\sin\theta R \end{aligned} \tag{2.191}$$

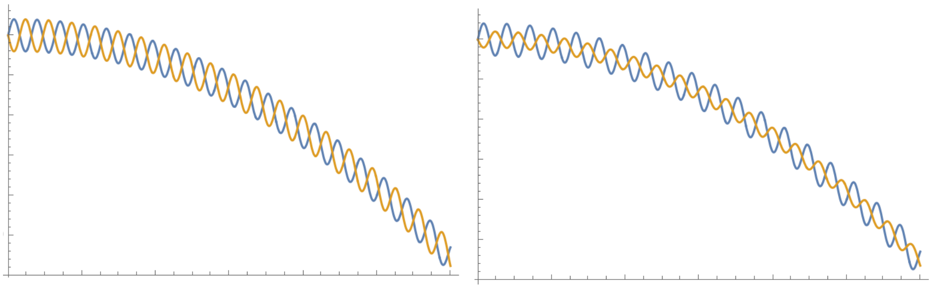

Figure 2.17.2: r_1 and r_2 for two skiers of equal masses (left) and unequal masses (right)

and

$$V_{12}^{int} = V_{21}^{int} = \frac{1}{2}k(r_1 - r_2)^2$$
$$= \frac{1}{2}kr_{12}^2 \ . \tag{2.192}$$

Thus we see that both

$$E_{cm} = \frac{1}{2}M\dot{R}^2 + Mg\sin\theta R$$
$$E_{12} = \frac{1}{2}\mu\dot{r}_{12}^2 + \frac{1}{2}kr_{12}^2 \tag{2.193}$$

are conserved.

Let us consider N electrons moving in the presence of an electric field $\underline{E}$ which we assume to be constant. This gives an external force

$$\underline{F}_i^{ext} = e\underline{E} \ . \tag{2.194}$$

that acts on the ith electron. However in addition there will be inter-electron repulsive forces due to the fact that like charges repel. These give rise to the internal forces

$$\underline{F}_{ij}^{int} = \frac{e^2}{4\pi}\frac{1}{|\underline{r}_i - \underline{r}_j|^2}\frac{\underline{r}_i - \underline{r}_j}{|\underline{r}_i - \underline{r}_j|} \ . \tag{2.195}$$

This force is similar in form to gravity and one can easily check that it arises from the potential

$$V_{ij} = \frac{e^2}{4\pi}\frac{1}{|\underline{r}_i - \underline{r}_j|} \tag{2.196}$$

Thus the equation for a single electron is

$$m\ddot{\underline{r}}_i = e\underline{E} + \sum_{j\neq i}\frac{e^2}{4\pi}\frac{1}{|\underline{r}_i - \underline{r}_j|^2}\frac{\underline{r}_i - \underline{r}_j}{|\underline{r}_i - \underline{r}_j|} \tag{2.197}$$

This is clearly a tricky thing to solve in general (impossible might be a better term). However if the electrons are well separated then the final term is small compared to the $\underline{E}$ and $\underline{B}$ terms. In this case the problem of each electron is separated and individually solvable. Roughly speaking this is true if

$$\frac{e^2}{|\underline{r}_i - \underline{r}_j|^2} << e|\underline{E}| \tag{2.198}$$

where v is the speed of the electrons. This can be achieved by making the background electric field to be large.

Alternatively, we can essentially solve the system with inter-electron force if we consider just two electrons. There isn't a potential V^{ext} for the magnetic field (though the magnetic force is conservative!) so let us set that to zero and only look at a constant electric field $\underline{E}$. For the centre of mass we find (recall $M = 2m$)

$$M\ddot{\underline{R}} = \underline{F}_1^{ext} + \underline{F}_2^{ext} = 2e\underline{E} \tag{2.199}$$

which corresponds to the potential

$$V^{ext} = -e\underline{E} \cdot (\underline{r}_1 + \underline{r}_2) = -2e\underline{E} \cdot \underline{R} \tag{2.200}$$

This is of the form we needed above in that it only depends on $\underline{R}$. The electric field is just a constant force like gravity was for the skier going down the hill (with down now being determined by the direction in which $\underline{E}$ points). So the solution will be similar. Thus we can solve for the centre-of-mass by writing

$$\underline{R} = \frac{1}{2}at^2\underline{E} + \dot{\underline{R}}(0)t + \underline{R}(0) \tag{2.201}$$

where $\dot{\underline{R}}(0)$ and $\underline{R}(0)$ are the initial centre of mass speed and position and a is a constant. The electrons are therefore accelerated in the direction $\underline{E}$. To determine a we substitute into (2.199) we find

$$Ma = 2e \qquad \Longrightarrow \qquad a = \frac{2e}{M} = \frac{e}{m} \tag{2.202}$$

Thus the background electric field causes a constant acceleration of the electrons along the direction of $\underline{E}$.

Next we look at $\underline{r}_{12}$. From the formulae above we find the relation (recall $\mu = m/2$)

$$E = \frac{1}{2}\mu\dot{r}_{12}^2 + V_{eff} \qquad V_{eff}(r_{12}) = \frac{e^2}{4\pi}\frac{1}{r_{12}} + \frac{l^2}{2\mu r_{12}^2} \ . \tag{2.203}$$

This is similar to to the gravitational case we saw before when studying planetary motion except that all terms are positive and hence V_{eff} is monotonically decreasing from infinity at $r_{12} = 0$ to zero as $r_{12} \to \infty$. Therefore there are no bound state solutions and r_{12} always ends up growing arbitrarily large. In other words if the two electrons are sufficiently far away from each other there is little attractive force but once they come close there is a repulsive force that sends them far away again from each other. Thus the pair of electrons will constantly accelerate along the direction of $\underline{E}$ and scatter off each other whenever they come too close.

Chapter 3

Lagrangian Mechanics

So when a rain drop falls do you think that it is trying to solve differential equations arising from Newtons laws to figure out what to do on the way down? Surely not. So how does it know? Here we need to introduce a new level of abstraction. And an apparent miracle.

3.1 The Principle of Least Action

You may have noticed in the treatment of the planets, that we never really used the force at all. Although we used Newtons Laws to introduce the notion of a potential which we then used to derive a conserved energy. Let us return to the case of a single particle. In particular we had kinetic energy

$$T = \frac{1}{2}m|\underline{\dot{r}}|^2 \tag{3.1}$$

and if the force was conservative, a potential energy $V(\underline{r})$. This lead to the total energy

$$E = T + V \ . \tag{3.2}$$

This is conserved: for a given path of a particle E remains constant.

There is something else that we can consider:

$$L = T - V = \frac{1}{2}m|\underline{\dot{r}}|^2 - V(\underline{r}) \ . \tag{3.3}$$

This is not conserved, it changes in time as the particle moves. But we can consider instead the functional

$$S[\underline{r}] = \int_{t_1}^{t_2} L(\underline{r}, \underline{\dot{r}})dt \ . \tag{3.4}$$

S is called the **action** and L is the **Lagrangian**. The action is a functional in the sense that it is a function of a function: given a function, namely the entire path $\underline{r}(t)$ of a particle from t_2 to t_2 then $S[\underline{r}]$ gives a number. This number depends on the whole path not just any given point on it. This is indicated by the square brackets $S = S[\underline{r}]$. Note that the Lagrangian is not a functional because it depends on $\underline{r}$ and $\underline{\dot{r}}$ at a single time. Rather one tends to think of it as a formal function of $\underline{r}$ and $\underline{\dot{r}}$ as independent variables, without thinking of the fact that $\underline{r}$ and $\underline{\dot{r}}$ are also themselves functions time.

Finally the action is somewhat analogous to the work which is also a functional (*i.e.* a function of a function):

$$W[\underline{r}] = \int_{t_1}^{t_2} \underline{F} \cdot \dot{\underline{r}} dt \tag{3.5}$$

except that whereas work is path-independent for conservative forces the action S is very much path-dependent also in this case. Although there is the following crude analogy. When you are pushing a shopping cart around the supermarket looking for the marmite, if you know what you are doing you will not wander around everywhere but rather take the shortest path to the marmite and then the shortest path to the check-out. You do this to minimize the amount of work that you must do to push the shopping trolley. As we will now see particles make a similar calculation. We now state the

Principle of Least Action: Particles move so as to extremize the action S as a functional of all possible paths between $\underline{r}(t_1)$ and $\underline{r}(t_2)$. That is to say Newton's Laws of motion are equivalent to the statement that

$$\delta S = 0 \ , \tag{3.6}$$

where δS is the first order variation of the action obtained by shifting the path $\underline{r} \to \underline{r} + \delta\underline{r}$ (and $\dot{\underline{r}} \to \dot{\underline{r}} + \delta\dot{\underline{r}}$), subject to the condition that the end points of the path are fixed: $\delta\underline{r}(t_1) = \delta\underline{r}(t_2) = \underline{0}$.

Let us prove this. To do this we compute

$$\begin{aligned} S[\underline{r} + \delta\underline{r}] &= \int_{t_1}^{t_2} \frac{1}{2} m(\dot{\underline{r}} + \delta\dot{\underline{r}}) \cdot (\dot{\underline{r}} + \delta\dot{\underline{r}}) - V(\underline{r} + \delta\underline{r}) dt \\ &= \int_{t_1}^{t_2} \frac{1}{2} m\dot{\underline{r}} \cdot \dot{\underline{r}} + \frac{1}{2} m\delta\dot{\underline{r}} \cdot \dot{\underline{r}} + \frac{1}{2} m\dot{\underline{r}} \cdot \delta\dot{\underline{r}} - V(\underline{r} + \delta\underline{r}) dt + \ldots \\ &= \int_{t_1}^{t_2} \frac{1}{2} m\dot{\underline{r}} \cdot \dot{\underline{r}} + m\dot{\underline{r}} \cdot \delta\dot{\underline{r}} - V(\underline{r} + \delta\underline{r}) dt \end{aligned} \tag{3.7}$$

where the dots denote higher order terms in $\delta\underline{r}$ which we have dropped in the last line as we will only be interested in the first order variation. Next we need to taylor expand $V(\underline{r} + \delta\underline{r})$:

$$V(\underline{r} + \delta\underline{r}) = V(\underline{r}) + \underline{\nabla} V \cdot \delta\underline{r} + \ldots \tag{3.8}$$

where again the dots denote higher order terms in $\delta\underline{r}$. Putting this back in we find

$$\begin{aligned} S[\underline{r} + \delta\underline{r}] &= \int_{t_1}^{t_2} \frac{1}{2} m\dot{\underline{r}} \cdot \dot{\underline{r}} - V(\underline{r}) + m\dot{\underline{r}} \cdot \delta\dot{\underline{r}} - \underline{\nabla} V \cdot \delta\underline{r} dt \\ &= S[\underline{r}] + \int_{t_1}^{t_2} \left[m\dot{\underline{r}} \cdot \delta\dot{\underline{r}} - \underline{\nabla} V \cdot \delta\underline{r} \right] dt + \ldots \end{aligned} \tag{3.9}$$

Our next step is to note that the third term can be manipulated using integration by parts:

$$m\dot{\underline{r}} \cdot \delta\dot{\underline{r}} = \frac{d}{dt} \left(m\dot{\underline{r}} \cdot \delta\underline{r} \right) - m\ddot{\underline{r}} \cdot \delta\underline{r} \ . \tag{3.10}$$

Thus we have

$$\begin{aligned} S[\underline{r} + \delta\underline{r}] &= S[\underline{r}] + \int_{t_1}^{t_2} \frac{d}{dt} \left(m\dot{\underline{r}} \cdot \delta\underline{r} \right) dt - \int_{t_1}^{t_2} \left[m\ddot{\underline{r}} \cdot \delta\underline{r} + \underline{\nabla} V \cdot \delta\underline{r} \right] dt + \ldots \\ &= S[\underline{r}] + m\dot{\underline{r}} \cdot \delta\underline{r} \,\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \left[m\ddot{\underline{r}} + \underline{\nabla} V \right] \cdot \delta\underline{r} dt + \ldots \end{aligned} \tag{3.11}$$

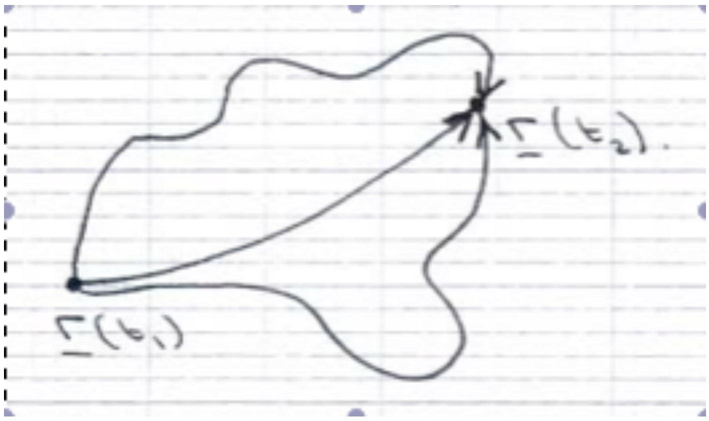

Figure 3.1.1: Paths in Space

Now we assume that we vary the action over all paths which begin at $\underline{r}(t_1)$ and end at $\underline{r}(t_2)$. Thus we impose that $\delta\underline{r}(t_1) = \delta\underline{r}(t_2) = \underline{0}$. Therefore we find

$$S[\underline{r} + \delta\underline{r}] = S[\underline{r}] - \int_{t_1}^{t_2} [m\ddot{\underline{r}} + \underline{\nabla}V] \cdot \delta\underline{r}dt + \dots \tag{3.12}$$

from which we read off the first order variation:

$$\delta S[\underline{r}] = S[\underline{r} + \delta\underline{r}] - S[\underline{r}] = -\int_{t_1}^{t_2} [m\ddot{\underline{r}} + \underline{\nabla}V] \cdot \delta\underline{r}dt \ . \tag{3.13}$$

The claim is that for arbitrary variations in the path the particle will extremise S: $\delta S = 0$. The path which extremises S against an arbitrary variation is therefore the one for which:

$$m\ddot{\underline{r}} + \underline{\nabla}V = \underline{0} \ . \tag{3.14}$$

This is just NII with $\underline{F} = -\underline{\nabla}V$ and $\underline{p} = m\dot{\underline{r}}$!

3.2 Generalized Coordinates and Lagrangians

The power of the Lagrangian method lies in the fact that we can relatively simply construct Lagrangians for different physical systems by determining the kinetic and potential energies as a function of the dynamical variables. This leads to more general Lagrangians. In particular this can happen in essentially two ways: either we wish to modify what we mean by kinetic and potential energy, for example we may allow for the mass to change. Or we impose some kind of constraint to the system which effectively eliminates some coordinates in terms of others.

This leads to the notion of **generalized coordinates**. This means that instead of thinking of the dynamical variables as the positions $\underline{r}_i$ of N particles, we simply consider a generic system with "generalized" coordinates that we denote by q_i. Here each q_i is treated as a scalar, not a vector, and we will take the index i to be rather generic and ranges over all the generalized coordinates in the problem at hand.

A related concept is that of a **degree of freedom** is a generalized coordinate that is allowed to evolve in time without restrictions on its initial conditions. For example if we do have a system of N free particles with positions $\underline{r}_1, ..., \underline{r}_N$ then the generalized coordinates q_i are just the $3N$ components of the positions so that $i = 1, ..., 3N$. This has $3N$ degrees of freedom.

However we will also want to look at constrained systems where the various generalized coordinate are related to each other leading to a reduction in the number of degrees of freedom. An example of this was the skier we considered early on. The skier is constrained to lie on a hill slope and furthermore we assume that they went straight down the hill. Thus even though a skier is described by a three-vector $\underline{r}$ in the end we only used the distance that the skier was up the slope from the bottom, which we denoted by r. Another example is a pendulum where the weight is constrained to sit at the end of rod of fixed length. In this case the generalized coordinate is simply the angle of the pendulum from the equilibrium position.

Thus we wish to know how to evaluate the principle of least action, $\delta S = 0$ for a general Lagrangian. We will assume that the Lagrangian is a function of $q_i(t)$ and $\dot{q}_i(t)$ but not higher order derivatives (although this can also be considered). It may also have an explicit dependence on t. Thus we start with

$$S[q_i] = \int_{t_1}^{t_2} L(q_i(t), \dot{q}_i(t), t)dt \ . \tag{3.15}$$

To evaluate δS we expand $q_i \to q_i + \delta q_i$, $\dot{q}_i \to \dot{q}_i + \delta\dot{q}_i$. We treat q_i and $\dot{q}_i$ as independent. Although these are functions, at a fixed value of t we could just think of them as ordinary variables. Therefore we can expand L using the familiar rules of calculus:[1]

$$\begin{aligned} S[q_i + \delta q_i] &= \int_{t_1}^{t_2} L(q_i(t) + \delta q_i(t), \dot{q}_i(t) + \delta\dot{q}_i(t), t)dt \\ &= \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) + \sum_i \frac{\partial L}{\partial q_i}\delta q_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i + \ldots dt \end{aligned} \tag{3.16}$$

where the dots denote higher order terms in δq_i and $\delta\dot{q}_i$. We do not need these terms to evaluate δS which is, by definition, the first order term in the variation:

$$\delta S = S[q_i + \delta q_i] \mid_{\mathcal{O}(\delta)} - S[q_i] \ . \tag{3.17}$$

Therefore we find

$$\delta S = \sum_i \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i \right) dt \ . \tag{3.18}$$

Next we want to write the second term in terms of δq_i, not $\delta\dot{q}_i$. To do this we rewrite the second term as a total derivative plus something else:

$$\frac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i \ . \tag{3.19}$$

Substituting this into δS we find

$$\begin{aligned} \delta S &= \sum_i \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i + \frac{\partial L}{\partial q_i}\delta q_i dt \\ &= \frac{\partial L}{\partial \dot{q}_i}\delta q_i \Big|_{t_1}^{t_2} - \sum_i \int_{t_1}^{t_2} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}\right]\delta q_i dt \ . \end{aligned} \tag{3.20}$$

[1] In the literature one sometimes sees $\delta L/\delta q_i$ instead of $\partial L/\partial q_i$.

To get deal with the boundary term we note that we want to make an arbitrary variation the path

$$q_i(t) \to q_i(t) + \delta q_i(t) \ , \tag{3.21}$$

however we only want to consider paths that start at a fixed starting point $q_i(t_1)$ and end at a fixed end point. Thus we keep $\delta r(t)$ arbitrary except that

$$\delta q_i(t_1) = \delta q_i(t_2) = 0 \ . \tag{3.22}$$

And therefore

$$\frac{\partial L}{\partial \dot{q}_i}\delta q_i\bigg|_{t_1}^{t_2} = 0 \ , \tag{3.23}$$

so that finally we find

$$\delta S = -\sum_i \int_{t_1}^{t_2} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}\right]\delta q_i dt \ . \tag{3.24}$$

Now the principle of least action asserts that the dynamical path of the system is the one for which $\delta S = 0$ for any choice of $\delta q_i(t)$ (subject to (3.22)). This will only be the case if

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \ . \tag{3.25}$$

This is known as the Euler-Lagrange equation.

Finally we note that two Lagrangians that differ by a total derivative will give the same Euler-Lagrange equation and hence correspond to the same physical system. In particular if

$$L'(q_i, \dot{q}_i, t) = L(q_i, \dot{q}_i, t) + \frac{d}{dt}\Omega(q_i, t) \ , \tag{3.26}$$

then the associated actions differ by boundary terms

$$S'[q_i] = S[q_i] + \Omega(q_i(t_2), t_2) - \Omega(q_i(t_1), t_1) \ . \tag{3.27}$$

Since we do not vary the boundary values $\delta q_i(t_1) = \delta q_i(t_2) = 0$ it follows that

$$\delta S' = \delta S \ , \tag{3.28}$$

and hence we would find the same Euler-Lagrange equation.

3.3 Simple Examples

This is all quite abstract. Let us see how it works in several examples.

Example 1: a particle in 3D: Lets go back to the single particle that we originally studied:

$$\begin{aligned} L &= \frac{1}{2}m|\dot{\underline{r}}|^2 - V(\underline{r}) \\ &= \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - V(q_1, q_2, q_3) \end{aligned} \tag{3.29}$$

where in the second line we have rewritten the Lagrangian in terms of the "generalized" coordinates q_i, $i = 1,2,3$ which are simply the components of $\underline{r} = (q^1, q^2, q^3)$, *i.e.* $q_i = r^a$. From here we see that (recall that we think of q_i and $\dot{q}_i$ as independent):

$$\begin{aligned}\frac{\partial L}{\partial q_i} &= -\frac{\partial V}{\partial q_i} \\ \frac{\partial L}{\partial \dot{q}_i} &= m\dot{q}_i\end{aligned} \tag{3.30}$$

Thus the Euler-Lagrange equation (3.25) is

$$m\ddot{q}_i + \frac{\partial V}{\partial q_i} = 0 \ , \tag{3.31}$$

which is the component version of (3.14). In particular we have simply rediscovered NII with $F_i = -\partial V/\partial q_i$.

Example 2: The skier: Lets look at our skier again. Here $\underline{r} = r\underline{e}_h$, where $\underline{e}_h$ was the constant unit vector pointing up the hill. Thus

$$\begin{aligned}T &= \frac{1}{2}m|\dot{\underline{r}}|^2 \\ &= \frac{1}{2}m(\dot{r}\underline{e}_h)\cdot(\dot{r}\underline{e}_h) \\ &= \frac{1}{2}m\dot{r}^2 \ .\end{aligned} \tag{3.32}$$

The potential energy is just proportional to the height $h = r\sin\theta$:

$$V = mgr\sin\theta \ . \tag{3.33}$$

Thus the Lagrangian is

$$L = \frac{1}{2}m\dot{r}^2 - mgr\sin\theta \ . \tag{3.34}$$

Here there is just one generalized coordinate $q = r$. We can easily evaluate

$$\begin{aligned}\frac{\partial L}{\partial r} &= -mg\sin\theta \\ \frac{\partial L}{\partial \dot{r}} &= m\dot{r}\end{aligned} \tag{3.35}$$

and hence the Euler-Lagrange equation is

$$m\ddot{r} + mg\sin\theta = 0 \ , \tag{3.36}$$

which is what we found before. Note that here we did not have to worry about resolving the forces into the part $\underline{F}_D$ along the direction of motion and the part $\underline{F}_V$ that is cancelled by the upwards force of the hill pushing back on the skier. All we needed was to identify the potential, which is the height.

In fact we can easily allow for a more interesting ski slope where the ski slope has a non-trivial profile $y = h(x)$. Here x is the direction along the horizon and y is the height above the ground.

In this case the potential is

$$V = mgh(x) \tag{3.37}$$

The kinetic energy is (we only consider motion straight up and down the hill and not side-to-side across the slope)

$$\begin{aligned} T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\ &= \frac{1}{2}m\left[\dot{x}^2 + \left(\frac{dh}{dx}\dot{x}\right)^2\right] \\ &= \frac{1}{2}m\left[1 + \left(\frac{dh}{dx}\right)^2\right]\dot{x}^2 \ . \end{aligned} \tag{3.38}$$

Thus the Lagrangian is

$$L = \frac{1}{2}m\left[1 + \left(\frac{dh}{dx}\right)^2\right]\dot{x}^2 - mgh(x) \tag{3.39}$$

From here we find

$$\begin{aligned} \frac{\partial L}{\partial x} &= m\frac{dh}{dx}\frac{d^2h}{dx^2}\dot{x}^2 - mg\frac{dh}{dx} \\ \frac{\partial L}{\partial \dot{x}} &= m\left[1 + \left(\frac{dh}{dx}\right)^2\right]\dot{x} \end{aligned} \tag{3.40}$$

and hence the Euler-Lagrange equation is

$$\begin{aligned} &\frac{d}{dt}\left(m\left[1 + \left(\frac{dh}{dx}\right)^2\right]\dot{x}\right) - m\frac{dh}{dx}\frac{d^2h}{dx^2}\dot{x}^2 + mg\frac{dh}{dx} = 0 \\ \Longleftrightarrow \qquad & m\left[1 + \left(\frac{dh}{dx}\right)^2\right]\ddot{x} + m\frac{dh}{dx}\frac{d^2h}{dx^2}\dot{x}^2 + mg\frac{dh}{dx} = 0 \end{aligned} \tag{3.41}$$

This equation would have been pretty hard to determine based directly on Newton's Laws. Although in this case it is simply equivalent to the conservation of energy

$$E = T + V = \frac{1}{2}m\left[1 + \left(\frac{dh}{dx}\right)^2\right]\dot{x}^2 + mgh \tag{3.42}$$

as can be seen by evaluating $dE/dt = 0$.

We find the simple skier we had before if we identify $x = r\cos\theta$ and $h = r\sin\theta = x\tan\theta$.

Example 3: Circular motion and centrifugal force: Let us look at a free particle $V = 0$, lying in a plane but using polar coordinates. We looked at this problem before and took:

$$\underline{r} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 0 \end{pmatrix} \ . \tag{3.43}$$

To compute the Lagrangian we note that

$$\dot{\underline{r}} = \begin{pmatrix} \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \dot{r}\sin\theta + r\dot{\theta}\cos\theta \\ 0 \end{pmatrix} \ . \tag{3.44}$$

and hence

$$
\begin{aligned}
L &= \frac{1}{2}m|\underline{\dot{r}}|^2 \\
&= \frac{1}{2}m((\dot{r}\cos\theta - r\dot{\theta}\sin\theta)^2 + (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)^2) \\
&= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \quad . \qquad (3.45)
\end{aligned}
$$

The generalized coordinates are $q_1 = r$ and $q_2 = \theta$. Lets look at the Euler-Lagrange equations, there will be one for r and one for θ. First r:

$$
\begin{aligned}
\frac{\partial L}{\partial r} &= mr\dot{\theta}^2 \\
\frac{\partial L}{\partial \dot{r}_i} &= m\dot{r} \ , \qquad (3.46)
\end{aligned}
$$

so

$$
m\ddot{r} - mr\dot{\theta}^2 = 0 \ . \qquad (3.47)
$$

The second term is the centrifugal force term. Recall this was a fictitious force and indeed it arises here because we have not used Cartesian coordinates. Indeed in the Lagrangian we have set $V = 0$ so that there is no 'real' force.

For θ we find

$$
\begin{aligned}
\frac{\partial L}{\partial \theta} &= 0 \\
\frac{\partial L}{\partial \dot{\theta}} &= mr^2\dot{\theta} \ , \qquad (3.48)
\end{aligned}
$$

so

$$
\frac{d}{dt}\left(mr^2\dot{\theta}\right) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = 0 \ . \qquad (3.49)
$$

This equation immediately gives us the conservation of angular momentum:

$$
|\underline{L}| = mr^2\dot{\theta} = constant. \qquad (3.50)
$$

We could also considering adding a potential term $V(r,\theta)$. This would then correct the Euler-Lagrange equations to

$$
\begin{aligned}
m\ddot{r} - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} &= 0 \\
mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + \frac{\partial V}{\partial \theta} &= 0 \ , \qquad (3.51)
\end{aligned}
$$

Corresponding to a force with components $F_r = -\partial V/\partial r$, $F_\theta = -\partial V/\partial\theta$. We also see that the F_θ component generates torque, corresponding to non-conservation of angular momentum.

3.4 Constraints

The last two examples were actually cases where there was a constraint on the system. In particular, rather than looking at an object that was free to move in all three-dimensions there was always some kind of restriction: The skier was required to stay on a the slope and the particle undergoing circular motion was required to lie in the plane. These

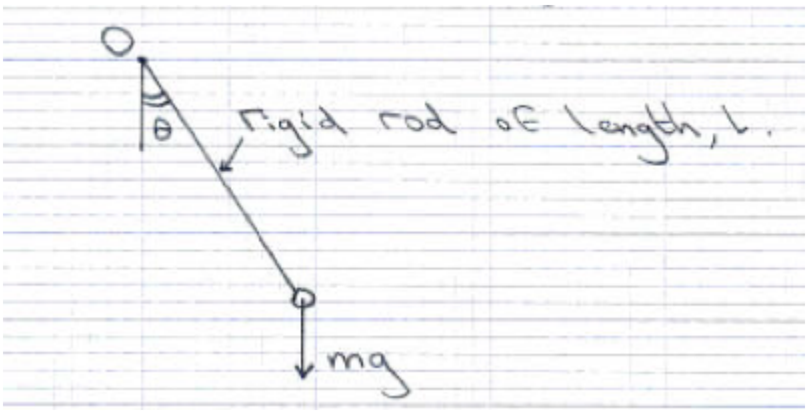

Figure 3.5.1: A Pendulum

constraints were so simple that we hardly noticed them at all. Nevertheless they are examples of **holonomic** constraints:

Definition A holonomic constraint on a system is a function of the form

$$C(q_i, t) = 0 \ . \tag{3.52}$$

that is imposed on the coordinates, but not their derivatives, at all times for some function C. There can of course be more than one such constraint on a system. Constraints that are not of this type are called, no surprise here, **non-holonomic**.

In general each holonomic constraint reduces the number degrees of freedom of the system by one. In particular one uses each constraint to solve for one of the generalized coordinates in terms of the others.

One of the great powers of the Lagrangian formulation is that it can handle holonomic constraints quite easily (in principle - one can always come up with examples that are tough to solve in practice). This is because we are only required to determine the kinetic and potential energies of the system. We do not need to analyze each of the forces and counter-forces to determine the net force on each particle. For the examples above this isn't so hard to do. We did it for some of them. Let us now look at some more difficult problems that involve constraints. Although possible, you would find it very tricky to solve them by analysing all the forces.

3.5 The Pendulum and Double Pendulum in the Plane

A Pendulum is a weight of mass m that is attached to a rigid rod of length l which is itself held fixed at the other end. There is a story that Galileo was in church and saw the chandeliers swinging and by the end of the, evidently less than riveting, sermon he had deduced the form of their motion with the classic result that the period of oscillation is independent of the mass of the chandelier or the size of its swing (its amplitude).

If we choose coordinates such that x points out of the page, y is horizontal and z points up the page (as in figure 1.6.1) then the constraints are

$$\begin{aligned} C_1 &= \underline{r} \cdot \underline{e}_x = 0 \\ C_2 &= |\underline{r}| - l = 0 \end{aligned} \tag{3.53}$$

In particular the first one asserts that the pendulum only moves in the $y-z$ plane whereas the second states that the distance to the origin, where the chandelier is attached to the ceiling, is fixed. Writing

$$\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{3.54}$$

we see that C_1 imposes $x = 0$. To solve C_2 we write $y = r\sin\theta$, $z = -r\cos\theta$ (the minus sign is because the chandelier is taken to lie below the ceiling). Then C_2 simply states that $|\underline{r}| = r = l$.

To construct the Lagrangian we note that the position of the chandelier is

$$\underline{r} = \begin{pmatrix} 0 \\ l\sin\theta \\ -l\cos\theta \end{pmatrix} . \tag{3.55}$$

Since l is fixed we have

$$\dot{\underline{r}} = \begin{pmatrix} 0 \\ l\dot{\theta}\cos\theta \\ l\dot{\theta}\sin\theta \end{pmatrix} . \tag{3.56}$$

and hence the kinetic energy is

$$T = \frac{1}{2}m|\dot{\underline{r}}|^2 = \frac{1}{2}ml^2\dot{\theta}^2 \tag{3.57}$$

Indeed this is the same in example 3 just with $r \to l$ a constant (as well as $\theta \to \theta - \pi/2$). Unlike example 3 there is a potential due to gravity which is just the height (much like the skier):

$$V = -mgl\cos\theta . \tag{3.58}$$

Therefore the Lagrangian is

$$L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta . \tag{3.59}$$

We evaluate

$$\begin{aligned} \frac{\partial L}{\partial \theta} &= -mgl\sin\theta \\ \frac{\partial L}{\partial \dot{\theta}} &= ml^2\dot{\theta} , \end{aligned} \tag{3.60}$$

so that the Euler-Lagrange equation for θ is

$$ml^2\ddot{\theta} + mgl\sin\theta = 0 . \tag{3.61}$$

Here we see that the factors of m cancel, corresponding to Galileo's observation:

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 . \tag{3.62}$$

This is a little tricky to solve however for small oscilations, $\sin\theta = \theta + \dots$ and one can simply take

$$\ddot{\theta} + \frac{g}{l}\theta = 0 . \tag{3.63}$$

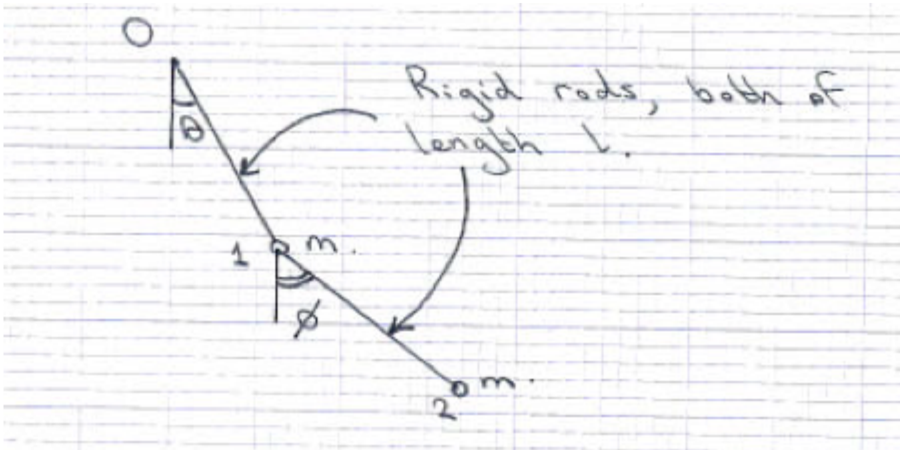

Figure 3.5.2: A Double Pendulum

The solution to this is

$$\theta = A\sin\left(\sqrt{\frac{g}{l}}t\right) + B\cos\left(\sqrt{\frac{g}{l}}t\right) \tag{3.64}$$

for arbitrary constants A and B. Therefore if you know the length of the cord holding a chandelier you can measure the gravitational constant g.

Let us next consider a pendulum that is attached to a second pendulum (see Figure (3.5.2). The rods connecting then both have length l and they both have the same mass m. The details of this are left as a problem so let us just give some sketch of what to do.

Let their positions be (x_1, y_1, z_1) and (x_2, y_2, z_2). Pendulum one has the same form as the single pendulum above in terms of θ. In particular it satisfies the constraints

$$\begin{aligned} C_1(x_1, y_1, z_1, x_2, y_2, z_2, t) &= x_1 = 0 \\ C_2(x_1, y_1, z_1, x_2, y_2, z_2, t) &= y_1^2 + z_1^2 - l^2 = 0 \end{aligned} \tag{3.65}$$

We are taking the z direction to point out of the page in Figure 13. These reduce the 3 degrees of freedom of the first pendulum to a single degree of freedom θ that was saw above.

The second pendulum satisfies the constraints

$$\begin{aligned} C_3(x_1, y_1, z_1, x_2, y_2, z_2, t) &= x_2 = 0 \\ C_4(x_1, y_1, z_1, x_2, y_2, z_2, t) &= (y_2 - y_1)^2 + (z_2 - z_1)^2 - l^2 = 0\ . \end{aligned} \tag{3.66}$$

Again this reduces the 3 degrees of freedom down to one

$$y_2 - y_1 = l\sin\phi \qquad z_2 - z_1 = -l\cos\phi\ . \tag{3.67}$$

This allows one to compute the kinetic energy T in terms of $\dot\theta$ and $\dot\phi$. The potential is the sum of two terms, one for each weight,

$$\begin{aligned} V_1 &= mgz_1 \\ V_2 &= mgz_2 \end{aligned} \tag{3.68}$$

This allows you to write down the Lagrangian as a function of the generalized coordinates θ, ϕ and their time derivatives. In particular we have

$$
\begin{aligned}
y_1 &= l\sin\theta & z_1 &= -l\cos\theta \\
y_2 &= l\sin\theta + l\sin\phi & z_2 &= -l\cos\theta - l\cos\phi
\end{aligned}
$$

So the kinetic term is

$$
\begin{aligned}
T &= \frac{1}{2}m(\dot{y}_1^2 + \dot{z}_1^2 + \dot{y}_2^2 + \dot{z}_2^2) \\
&= \frac{1}{2}ml^2(\dot{\theta}^2 + \dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}(\cos\theta\cos\phi + \sin\theta\sin\phi)) \\
&= \frac{1}{2}ml^2(2\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta-\phi))
\end{aligned}
\tag{3.69}
$$

and hence

$$
L = ml^2\dot{\theta}^2 + \frac{1}{2}ml^2\dot{\phi}^2 + ml^2\dot{\theta}\dot{\phi}\cos(\theta-\phi) + mgl(2\cos\theta + \cos\phi) \ . \tag{3.70}
$$

From here we can read off the θ equation of motion

$$
\begin{aligned}
\frac{d}{dt}\Big(2ml^2\dot{\theta} + ml^2\dot{\phi}\cos(\theta-\phi)\Big) - \Big(-ml^2\dot{\theta}\dot{\phi}\sin(\theta-\phi) - 2mgl\sin\theta\Big) &= 0 \\
&\Longleftrightarrow \\
2ml^2\ddot{\theta} + ml^2\ddot{\phi}\cos(\theta-\phi) - ml^2\dot{\phi}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi) + ml^2\dot{\theta}\dot{\phi}\sin(\theta-\phi) + 2mgl\sin\theta &= 0 \\
&\Longleftrightarrow \\
\ddot{\theta} + \frac{1}{2}\ddot{\phi}\cos(\theta-\phi) + \frac{1}{2}\dot{\phi}^2\sin(\theta-\phi) + \frac{g}{l}\sin\theta &= 0 \ ,
\end{aligned}
\tag{3.71}
$$

and the ϕ equation of motion:

$$
\begin{aligned}
\frac{d}{dt}\Big(ml^2\dot{\phi} + ml^2\dot{\theta}\cos(\theta-\phi)\Big) - \Big(ml^2\dot{\theta}\dot{\phi}\sin(\theta-\phi) - mgl\sin\phi\Big) &= 0 \\
&\Longleftrightarrow \\
ml^2\ddot{\phi} + ml^2\ddot{\theta}\cos(\theta-\phi) - ml^2\dot{\theta}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi) - ml^2\dot{\theta}\dot{\phi}\sin(\theta-\phi) + mgl\sin\phi &= 0 \\
&\Longleftrightarrow \\
\ddot{\phi} + \ddot{\theta}\cos(\theta-\phi) - \dot{\theta}^2\sin(\theta-\phi) + \frac{g}{l}\sin\phi &= 0 \ .
\end{aligned}
\tag{3.72}
$$

Of course it is altogether a different problem to solve these equations! In fact they are known to exhibit chaotic behaviour. You can take a look at some cool pictures and movies about it here:

`http://en.wikipedia.org/wiki/Double_pendulum`

3.6 Interlude: Linearized Analysis and Normal Modes

While we are looking at the double pendulum it is instructive to consider the equations in the limit where (θ, ϕ) are small, as we did for the single pendulum - although here we also need to assume that $(\dot{\theta}, \dot{\phi})$ are small to obtain linear differential equations. In this case we approximate $\sin(\theta-\phi)\sim\theta-\phi$, $\cos(\theta-\phi)\sim 1$ and $\sin\phi\sim\phi$, and neglect terms of higher order in θ and ϕ. In this case we find

$$\begin{aligned}\ddot\theta+\frac{1}{2}\ddot\phi\ +\frac{g}{l}\theta&=0\\ \ddot\phi+\ddot\theta+\frac{g}{l}\phi&=0\end{aligned}\tag{3.73}$$

To solve this we write our system in terms of matrices:

$$K\ddot\Theta+\Omega\Theta=0\tag{3.74}$$

$$\Theta=\begin{pmatrix}\theta\\ \phi\end{pmatrix}\ ,\qquad K=\begin{pmatrix}1 & 1/2\\ 1 & 1\end{pmatrix},\qquad \Omega=\frac{g}{l}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\tag{3.75}$$

We then invert K and write our equation as

$$\ddot\Theta+K^{-1}\Omega\Theta=0\tag{3.76}$$

Note that K^{-1} always exists otherwise it would mean that some linear combination of θ and ϕ did not have kinetic energy in the Lagrangian. Next we construct the eigenvalues $\omega^2_{1,2}$ of $K^{-1}\Omega$ and their eigenvectors Θ_1 and Θ_2 respectively. The solution to the equation $\ddot\Theta+K^{-1}\Omega\Theta=0$ is then

$$\Theta=\mathrm{Re}\left(A_1e^{i\omega_1t}\Theta_1+A_2e^{i\omega_2t}\Theta_2\right)\tag{3.77}$$

where A_1 and A_2 are arbitrary complex numbers. Here we use the linearity of the equation to take the real part (or we could take the imaginary part) to obtain a real solution.

The $\Theta_{1,2}$ and $\omega_{1,2}$ are called the **normal modes** and **normal frequencies** respectively. In our case the normal frequencies are real but in general they could be complex. This is okay as the equations are linear and so one just takes the real (or imaginary) part to obtain a physically acceptable solution. However an imaginary part to the frequency ω indicates an instability as the solution will have an exponential dependence of the form $e^{i\omega t}\sim e^{-\mathrm{Im}(\omega)t}$ which diverges in the past or future.

In our case we find

$$K^{-1}\Omega=\frac{1}{1-1/2}\begin{pmatrix}1 & -1/2\\ -1 & 1\end{pmatrix}\frac{g}{l}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\frac{g}{l}\begin{pmatrix}2 & -1\\ -2 & 2\end{pmatrix}\tag{3.78}$$

The eigenvalue equation is therefore

$$\begin{aligned}0&=(2g/l-\omega^2_{1,2})^2-2g^2/l^2\\ &\Longleftrightarrow\\ \omega^2_{1,2}&=\frac{g}{l}(2\mp\sqrt{2})\ .\end{aligned}\tag{3.79}$$

Next we find the eigenvectors. To this end we write

$$\Theta=\begin{pmatrix}1\\ b\end{pmatrix}\tag{3.80}$$

and substitute into $K^{-1}\Omega\Theta = \omega_{1,2}^2\Theta$ which leads to the condition

$$\frac{g}{l}(2-b) = \frac{g}{l}(2 \mp \sqrt{2}) \tag{3.81}$$

Thus $b = \pm\sqrt{2}$ and our eigenvectors are

$$\Theta_1 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} , \qquad \Theta_2 = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \tag{3.82}$$

Thus our solution is

$$\begin{pmatrix} \theta \\ \phi \end{pmatrix} = \begin{pmatrix} \mathrm{Re}\left(A_1 e^{i\omega_1 t} + A_2 e^{i\omega_2 t}\right) \\ \sqrt{2}\mathrm{Re}\left(A_1 e^{i\omega_1 t} - A_2 e^{i\omega_2 t}\right) \end{pmatrix} \tag{3.83}$$

For example taking $A_1 = A_2 = A$ a real constant gives:

$$\begin{pmatrix} \theta \\ \phi \end{pmatrix} = \begin{pmatrix} A\cos\omega_1 t + A\cos\omega_2 t \\ \sqrt{2}A\cos\omega_1 t - \sqrt{2}A\cos\omega_2 t \end{pmatrix} \tag{3.84}$$

Lastly we note that it is curious to see irrational values showing up which gives a hint of the complicated and chaotic motion of the full system. In particular the ratio

$$\frac{\omega_1}{\omega_2} = \sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}} \tag{3.85}$$

is irrational so that the motion is not periodic, *i.e.* there is no time $t \neq 0$ for which both $\omega_1 t$ and $\omega_2 t$ are integer multiples of 2π. So there is no time in the future where the pendulums return to their original positions and velocities.

3.7 A Marble in a Bowl

Let us derive the equations of motion of a marble rolling about without friction in a bowl under the force of gravity. In particular suppose that the bowl is defined by the curve, for $z \geq 0$,

$$z = x^2 + y^2 \qquad \Longleftrightarrow \qquad C(x,y,z) = z - x^2 - y^2 = 0 \ . \tag{3.86}$$

Solving this constraint reduces us from three degrees of freedom to two. In particular let us switch to polar coordinates for x and y:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \tag{3.87}$$

so that the constraint is simply solved by taking

$$z = r^2 \ . \tag{3.88}$$

The kinetic energy is then

$$\begin{aligned} T &= \frac{1}{2}m((\dot{r}\cos\theta - r\dot{\theta}\sin\theta)^2 + (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)^2 + 4r^2\dot{r}^2) \\ &= \frac{1}{2}m(1+4r^2)\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 \ . \end{aligned} \tag{3.89}$$

The potential energy is again just the height:

$$V = mgr^2 \ , \tag{3.90}$$

so that

$$L = \frac{1}{2}m(1+4r^2)\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - mgr^2 \ . \tag{3.91}$$

More generally one could consider a bowl with a shape given by a function $z = f(x^2+y^2)$ so that the constraint is satisfied by $z = f(r^2)$. Therefore $\dot{z} = 2r\dot{r}f'(r^2)$ and hence

$$L = \frac{1}{2}m(1+4r^2(f'(r^2))^2)\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - mgf(r^2) \ . \tag{3.92}$$

To find the Euler-Lagrange equations we first evaluate

$$\begin{aligned}\frac{\partial L}{\partial \dot{r}} &= m(1+4r^2)\dot{r}\\ \frac{\partial L}{\partial r} &= 4mr\dot{r}^2 + mr\dot{\theta}^2 - 2mgr \ ,\end{aligned} \tag{3.93}$$

so that the r Euler-Lagrange equation is

$$m(1+4r^2)\ddot{r} + 8mr\dot{r}^2 - 4mr\dot{r}^2 - mr\dot{\theta}^2 + 2mgr = 0 \ . \tag{3.94}$$

For the θ equation we again notice that since $\partial L/\partial\theta = 0$ there is a conservation law:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}(mr^2\dot{\theta}) = 0 \ , \tag{3.95}$$

which is equivalent to the conservation of angular momentum $l = mr^2\dot{\theta}$.

We can now use the conservation of l to obtain a reduced dynamical system that only involves r and an effective potential as we did before. To this end we note that the energy

$$\begin{aligned}E &= T + V\\ &= \frac{1}{2}m(1+4r^2)\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + mgr^2\\ &= \frac{1}{2}m(1+4r^2)\dot{r}^2 + \frac{l^2}{2mr^2} + mgr^2\end{aligned} \tag{3.96}$$

is conserved.

This can also be seen by starting from the r-equation of motion and substituting in for $\dot{\theta}$:

$$m(1+4r^2)\ddot{r} + 4mr\dot{r}^2 - \frac{l^2}{mr^3} + 2mgr = 0 \tag{3.97}$$

Next we multiply by $\dot{r}$ and integrate up with respect to time:

$$\begin{gathered}m(1+4r^2)\ddot{r}\dot{r} + 4mr\dot{r}^3 - \frac{l^2}{mr^3} + 2mgr\dot{r} = 0\\ \Longleftrightarrow\\ \frac{d}{dt}\left(\frac{1}{2}m(1+4r^2)\dot{r}^2 + \frac{l^2}{2mr^2} + mgr^2\right) = 0\end{gathered} \tag{3.98}$$

Thus E is indeed constant.

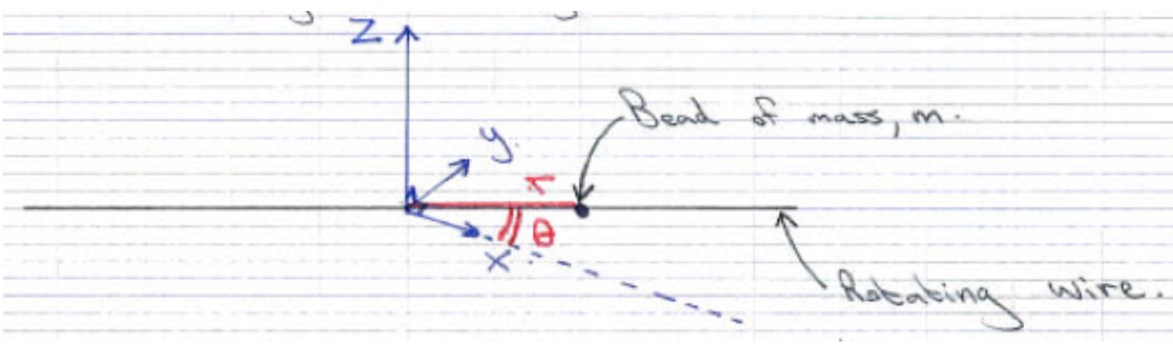

Figure 3.8.1: A Bead On A Rotating Wire

Just as before we can rewrite the conservation of energy as

$$\frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2}\frac{1}{1+4r^2} + \frac{mgr^2}{1+4r^2} - \frac{E}{1+4r^2} = 0 \tag{3.99}$$

This is of the form

$$\frac{1}{2}m\dot{r}^2 + V_{eff} = 0 \tag{3.100}$$

but with

$$V_{eff} = \frac{l^2}{2mr^2}\frac{1}{1+4r^2} + \frac{mgr^2}{1+4r^2} - \frac{E}{1+4r^2}\ . \tag{3.101}$$

This can be quantitatively and qualitatively analysed as we did before for 3D problems with conserved angular momentum. Note that E now appears as part of V_{eff} rather than as a line which the particle must stay below.

We could also have simply written the energy as

$$E = \frac{1}{2}m(1+4r^2)\dot{r}^2 + \hat{V}_{eff} \qquad \hat{V}_{eff} = \frac{l^2}{2mr^2} + mgr^2 \tag{3.102}$$

In this case there is a non-standard kinetic term but this won't have much of a qualitative effect on the dynamics since the non-constant coefficient $1+4r^2$ never vanishes. Its effect is to modify the relation between kinetic energy and velocity depending on the value of r. But qualitatively one still has that the kinetic energy is an increasing positive function of velocity. It will of course have quantitative effects.

3.8 A Bead on a Rotating Wire

Let us now look at something with a time-dependent constraint. We consider a straight wire that is lying in the $x-y$ plane and rotating about its midpoint at the origin with constant angular velocity $\omega = \dot{\theta}$. Let us imagine a bead moves on the wire without friction. We also assume that the wire is infinitely long so the bead never falls off the end.

Let us write the position of the bead in cylindrical coordinates

$$\underline{r} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix}\ . \tag{3.103}$$

The constraint that the bead is on the wire, and the wire is rotating, can be written as

$$\begin{aligned} C_1(r,\theta,z,t) &= \theta - \omega t = 0 \\ C_2(r,\theta,z,t) &= z = 0 \ , \end{aligned} \tag{3.104}$$

for a fixed ω. These are solved by taking

$$\theta = \omega t \qquad z = 0 \ . \tag{3.105}$$

Thus there are initially three degrees of freedom but there are two constraints leading to just one generalized coordinate or degree of freedom r. To continue we just compute:

$$\dot{\underline{r}} = \begin{pmatrix} \dot{r}\cos(\omega t) - \omega r \sin(\omega t) \\ \dot{r}\sin(\omega t) + \omega r\cos(\omega t) \\ 0 \end{pmatrix} \ . \tag{3.106}$$

There is no potential energy so the Lagrangian is just the kinetic energy

$$\begin{aligned} L &= \frac{1}{2}m|\dot{\underline{r}}|^2 \\ &= \frac{1}{2}m((\dot{r}\cos(\omega t) - \omega r\sin(\omega t))^2 + (\dot{r}\sin(\omega t) + \omega r\cos(\omega t))^2) \\ &= \frac{1}{2}m(\dot{r}^2 + \omega^2 r^2) \ . \end{aligned} \tag{3.107}$$

This is just like an unconstrained particle in a potential $V = -m\omega^2 r/2$ corresponding to a force $F = m\omega^2 r$ that points radially outwards. This is a **centrifugal** force and is again fictitious, in the sense that there is no force or potential term in the original Lagrangian.

Let us look at the equation of motion:

$$\ddot{r} - \omega^2 r = 0 \ . \tag{3.108}$$

Rather than finding sine and cosine as solutions the minus sign in second term indicates an instability. The solutions are given by

$$r = Ae^{\omega t} + Be^{-\omega t} \ . \tag{3.109}$$

At late times only the first term is important and the bead flies off to $r \to \infty$ getting ever faster and faster due to the centrifugal force. In particular if at $t = 0$ we assume $\dot{r} = 0$ then we require

$$A\omega - B\omega = 0 \qquad \Longrightarrow A = B \tag{3.110}$$

so that $r = 2A\cosh(\omega t)$.

3.9 The Coriolis Effect

Next we consider a more involved and famous example: the Coriolis effect which is important for the weather. This is not particularly related to constraints (although we will impose one) but rather relates to what (fictitious) forces arise when one switches between different coordinate systems where there is an explicit time dependence. In

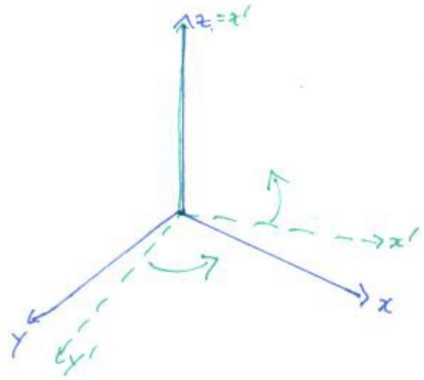

Figure 3.9.1: Rotating Coordinates

other words what happens when we are not in an inertial frame and Newton's first law is violated. This leads to so-called fictitious forces of which the most famous is the Coriolis effect. For some videos demonstrating this see

https://www.youtube.com/watch?v=mPsLanVS1Q8

https://www.youtube.com/watch?v=dt_XJp77-mk

In particular consider a mass of air above the Earth with coordinates $\underline{r}' = (x', y', z')$ where the z' coordinate runs north-south. Since the Earth is rotating $\underline{r}'$ is not an inertial frame. Therefore it makes sense to switch to a inertial coordinate system $\underline{r} = (x, y, z)$:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x' \cos(\omega t) - y' \sin(\omega t) \\ y' \cos(\omega t) + x' \sin(\omega t) \\ z' \end{pmatrix} , \tag{3.111}$$

where $\omega = 2\pi/60/60/24 \sim 0.00007$ is the angular velocity of the Earth per second. For $\omega > 0$ this means that the (x', y') plane is rotating with angular velocity ω with respect to the (x, y) plane. We can denote this as

$$\underline{r} = \mathbf{R}\underline{r}' \qquad \mathbf{R} = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{3.112}$$

We don't need a potential. The Atmosphere is of course subjected to Earth's gravity but the pressure of the lower air levels keeps the higher air from falling. So in effect the gravitational force is cancelled, at least in the approximation that we will make. In other words we will impose the constraint that the height $|\underline{r}|$ of the air molecule is fixed. Thus the Lagrangian of an air molecule is the kinetic energy

$$\begin{aligned} L &= T \\ &= \frac{1}{2} m |\dot{\underline{r}}|^2 \\ &= \frac{1}{2} m \dot{\underline{r}}^T \dot{\underline{r}} \end{aligned} \tag{3.113}$$

where we are thinking of the positions as 1×3 matrices. Now

$$\dot{\underline{r}} = \mathbf{R}\dot{\underline{r}}' + \dot{\mathbf{R}}\underline{r}' \tag{3.114}$$

so that

$$
\begin{aligned}
L &= \frac{1}{2}m(\mathbf{R}\dot{\underline{r}}' + \dot{\mathbf{R}}\underline{r}')^T(\mathbf{R}\dot{\underline{r}}' + \dot{\mathbf{R}}\underline{r}') \\
&= \frac{1}{2}m(\dot{\underline{r}}'^T\mathbf{R}^T + \underline{r}'^T\dot{\mathbf{R}}^T)(\mathbf{R}\dot{\underline{r}}' + \dot{\mathbf{R}}\underline{r}') \\
&= \frac{1}{2}m(\dot{\underline{r}}'^T\mathbf{R}^T\mathbf{R}\dot{\underline{r}}' + \underline{r}'^T\dot{\mathbf{R}}^T\mathbf{R}\dot{\underline{r}}' + \dot{\underline{r}}'^T\mathbf{R}^T\dot{\mathbf{R}}\underline{r}' + \underline{r}'^T\dot{\mathbf{R}}^T\dot{\mathbf{R}}\underline{r}') \ . \qquad (3.115)
\end{aligned}
$$

The middle two terms are actually equal:

$$
\begin{aligned}
\underline{r}'^T\dot{\mathbf{R}}^T\mathbf{R}\dot{\underline{r}}' &= \left(\underline{r}'^T\dot{\mathbf{R}}^T\mathbf{R}\dot{\underline{r}}'\right)^T \\
&= \dot{\underline{r}}'^T\mathbf{R}^T\dot{\mathbf{R}}\underline{r}' \ . \qquad (3.116)
\end{aligned}
$$

Further since $\mathbf{R}$ is a rotation we have $\mathbf{R}^T\mathbf{R} = \mathbf{I}$. Thus we see that

$$
L = \frac{1}{2}m|\dot{\underline{r}}'|^2 + m\dot{\underline{r}}'^T\mathbf{R}^T\dot{\mathbf{R}}\underline{r}' + \frac{1}{2}m\underline{r}'^T\dot{\mathbf{R}}^T\dot{\mathbf{R}}\underline{r}' \ . \qquad (3.117)
$$

Next we need to compute

$$
\dot{\mathbf{R}} = \omega\begin{pmatrix} -\sin(\omega t) & -\cos(\omega t) & 0 \\ \cos(\omega t) & -\sin(\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (3.118)
$$

so that

$$
\begin{aligned}
\mathbf{R}^T\dot{\mathbf{R}} &= \omega\begin{pmatrix} \cos(\omega t) & \sin(\omega t) & 0 \\ -\sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -\sin(\omega t) & -\cos(\omega t) & 0 \\ \cos(\omega t) & -\sin(\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \omega\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (3.119) \\
\dot{\mathbf{R}}^T\dot{\mathbf{R}} &= \omega^2\begin{pmatrix} -\sin(\omega t) & \cos(\omega t) & 0 \\ -\cos(\omega t) & -\sin(\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -\sin(\omega t) & -\cos(\omega t) & 0 \\ \cos(\omega t) & -\sin(\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \omega^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ . \qquad (3.120)
\end{aligned}
$$

Putting this all together we find

$$
L = \frac{1}{2}m((\dot{x}')^2 + (\dot{y}')^2 + (\dot{z}')^2) + m\omega(x'\dot{y}' - y'\dot{x}') + \frac{1}{2}m\omega^2((x')^2 + (y')^2) \ . \qquad (3.121)
$$

This can be rewritten as

$$
L = \frac{1}{2}m|\dot{\underline{r}}'|^2 + m\omega(\underline{r}' \times \dot{\underline{r}}') \cdot \underline{e}_z + \frac{1}{2}m\omega^2|\underline{r}' \times \underline{e}_z|^2. \qquad (3.122)
$$

where $\underline{e}_z$ is the unit vector point north.

What are these terms? The first is the familiar kinetic term and is independent of the rotation of the earth. The third is a centrifugal force term. In particular the third term can written as a potential

$$
V = -\frac{1}{2}m\omega^2|\underline{r}' \times \underline{e}_z|^2 = -\frac{1}{2}m\omega^2|\underline{r}'|^2\sin^2\theta \ , \qquad (3.123)
$$

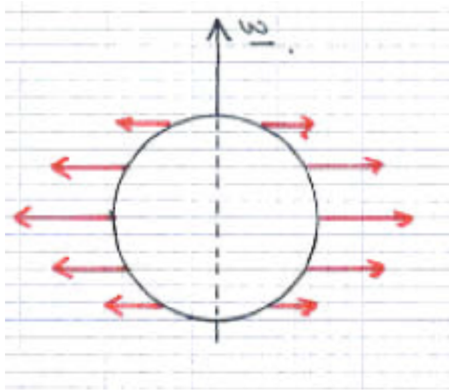

Figure 3.9.2: Centrifugal Force on Earth

where θ is the angle between $\underline{r}'$ and $\underline{e}_z$ - so the minimum at $\theta = \pi/2$ is the equator. So you are lighter, by about 0.3% at the equator.

The second term gives a velocity dependent force and is known as the Coriolis effect. Let us look at the effect of this term on the equations of motion (Euler-Lagrange equation). To do this we consider the x' equation:

$$\begin{aligned} \frac{d}{dt}\left(m\dot{x}' - m\omega y'\right) - m\omega\dot{y}' + \mathcal{O}(\omega^2) &= 0 \\ \ddot{x}' - 2\omega\dot{y}' + \mathcal{O}(\omega^2) &= 0 \ . \end{aligned} \tag{3.124}$$

Here we do not want to worry about the effects of the centrifugal force which is higher order in ω. The y' equation is:

$$\begin{aligned} \frac{d}{dt}\left(m\dot{y}' + m\omega x'\right) + m\omega\dot{x}' + \mathcal{O}(\omega^2) &= 0 \\ \ddot{y}' + 2\omega\dot{x}' + \mathcal{O}(\omega^2) &= 0 \ . \end{aligned} \tag{3.125}$$

On the other hand the z' equation is unaffected:

$$\ddot{z}' = 0 \ . \tag{3.126}$$

Thus the Coriolis term gives and extra velocity dependent force. We can integrate these equations:

$$\begin{aligned} \dot{x}' - 2\omega y' &= 2\omega A \\ \dot{y}' + 2\omega x' &= 2\omega B \\ \dot{z}' &= C \end{aligned} \tag{3.127}$$

where A, B, C are constants. Clearly we can integrate up the z' equation again to find

$$z' = Ct + D \ . \tag{3.128}$$

To solve for x' and y' we first substitute $\dot{y}' = 2\omega B - 2\omega x'$ into the $\ddot{x}'$ equation:

$$\ddot{x}' + 4\omega^2 x' = 4\omega^2 B \ . \tag{3.129}$$

Again we can solve this by writing

$$x' = B + x'_0 \ , \tag{3.130}$$

where x_0' satisfies

$$\ddot{x}_0' + 4\omega^2 x_0' = 0 \ . \tag{3.131}$$

Thus x' is oscillating about B with frequency ω:

$$x' = B + \alpha\sin(2\omega t) + \beta\cos(2\omega t) \ . \tag{3.132}$$

We can now solve for y' by writing:

$$\begin{aligned} \dot{y}' &= 2\omega B - 2\omega x' \\ &= -2\omega\alpha\sin(2\omega t) - 2\omega\beta\cos(2\omega t) \\ y' &= E + \alpha\cos(2\omega t) - \beta\sin(2\omega t) \ . \end{aligned} \tag{3.133}$$

Here E is just a constant (not related to the energy). We see that

$$(x' - B)^2 + (y' - E)^2 = \alpha^2 + \beta^2 \ . \tag{3.134}$$

Thus a particle will move in circles in the $x' - y'$ plane. Of course since ω is so small a given air mass doesn't make it very far around the circle before other weather effects become important.

3.10 Symmetries

So far we have been using the principle of least action to obtain the equations of motion of a system from the Euler-Lagrange Equations. But we haven't usually been trying to solve them. Indeed almost all systems of interest will be too complicated to solve exactly. One would must use a computer to find numerical approximations. So we now want to think more abstractly about Lagrangians.

However the action defines the dynamics and we can learn a lot about a system by thinking about the Lagrangian. An important part of the analysis involves symmetries and these are in turn deeply related to conserved quantities. In order to proceed we need to study conserved quantities again, but this time in the Lagrangian formulation.

If we define the **conjugate momentum** as

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \ , \tag{3.135}$$

and

$$F_i = \frac{\partial L}{\partial \dot{q}_i} \ , \tag{3.136}$$

then the Euler-Lagrange equation reads as

$$\frac{d}{dt}p_i = F_i \ . \tag{3.137}$$

which is in the form of NII. In the cases where

$$L = \frac{1}{2}\sum m_i \dot{q}_i^2 - V(q_i) \ , \tag{3.138}$$

we have that the conjugate momentum is $p_i = m_i\dot{q}_i$ and the generalized force is

$$F_i = \frac{\partial L}{\partial \dot{q}_i} = -\frac{\partial V}{\partial \dot{q}_i} \ , \tag{3.139}$$

which is just the usual expression in terms of a potential V. But we have also seen examples where the conjugate momentum is more complicated. For example in the marble in a bowl we find

$$L = \frac{1}{2}m(1+4r^2)\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - mgr^2 \ , \tag{3.140}$$

and so

$$\begin{aligned} p_r &= \frac{\partial L}{\partial \dot{r}} = m(1+4r^2)\dot{r} \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \ . \end{aligned} \tag{3.141}$$

This arises as the curvature of the bowl modifies what one means by momentum. The generalized force is also a little different:

$$\begin{aligned} F_r &= \frac{\partial L}{\partial r} = 4mr\dot{r}^2 + mr\dot{\theta}^2 - 2mgr \\ F_\theta &= \frac{\partial L}{\partial \theta} = 0 \ . \end{aligned} \tag{3.142}$$

In the examples above we saw several times that if the Lagrangian was independent of a particular coordinate q_* (quite often it was the angle θ), but not $\dot{q}$, then we could immediately identify a conserved quantity (when the Lagrangian is independent of the angle θ, the conserved quantity was the angular momentum):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_*}\right) = 0 \qquad \Longrightarrow \qquad Q = \frac{\partial L}{\partial \dot{q}_*} \text{ is conserved} \ . \tag{3.143}$$

Such a coordinate is said to be **ignorable**.

If L is independent of a particular coordinate, say q_*, then there is a symmetry: $L[q_i + \delta q_i, \dot{q}_i + \delta\dot{q}_i, t] = L[q_i, \dot{q}_i, t]$ where

$$\delta q_* = \epsilon \ , \qquad \delta q_i = 0 \text{ otherwise} \ , \qquad \delta\dot{q}_* = \delta\dot{q}_i = 0 \ , \tag{3.144}$$

where ϵ is a constant. This can be made more general as follows. A (continuous or infinitesimal) symmetry of the Lagrangian is a transformation

$$q_i \to q_i + \epsilon T_i \qquad \dot{q}_i \to \dot{q}_i + \epsilon\dot{T}_i \ , \tag{3.145}$$

where ϵ is an infinitesimal parameter[2] and T_i is a function of the q_i's and t, under which L is invariant:

$$L[q_i + \epsilon T_i, \dot{q}_i + \epsilon\dot{T}_i, t] = L[q_i, \dot{q}_i, t] \ . \tag{3.146}$$

i.e. $\delta L = L[q_i + \epsilon T_i, \dot{q}_i + \epsilon\dot{T}_i, t] - L[q_i, \dot{q}_i, t] = 0$ (to first order in ϵ).

3.11 Noether's Theorem

We can now state and prove the famous **Noether's Theorem**: For every continuous symmetry of the Lagrangian there is a conserved quantity:

$$Q = \sum_i \frac{\partial L}{\partial \dot{q}_i} T_i \ . \tag{3.147}$$

[2]Meaning that we are free to make ϵ as small as we like.

Note that the term continuous is important. Lagrangians can also have discrete symmetries, where $q_i \to q_i'$, which do not have a small expansion parameter ϵ. A common one might by a reflection $q_i \to -q_i$. Such symmetries are also important but they do not lead to conserved charges.

Let us prove Noether's theorem. We first note that the condition that this is a symmetry, *i.e.* $\delta L = 0$, when expanded to first order in ϵ using Taylor's theorem gives

$$\begin{aligned}\delta L &= \sum_i \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i \\ &= \epsilon\left(\sum_i \frac{\partial L}{\partial q_i}T_i + \frac{\partial L}{\partial \dot{q}_i}\dot{T}_i\right) \\ &= 0\ .\end{aligned} \tag{3.148}$$

Thus we must have

$$\sum_i \frac{\partial L}{\partial q_i}T_i + \frac{\partial L}{\partial \dot{q}_i}\dot{T}_i = 0\ . \tag{3.149}$$

It is important to emphasize that we have only considered a specific variation of the coordinates and their derivatives given by

$$\delta q_i = \epsilon T_i \qquad \delta \dot{q}_i = \epsilon \dot{T}_i\ , \tag{3.150}$$

where T_i is some specified function of the coordinates (for example, in the case of a marble in a bowl, $\delta r = 0, \delta\theta = \epsilon$ corresponding to $T_r = 0,\ T_\theta = 1$). This is quite different, in a sense opposite, to when we evaluated $\delta S = 0$. In that case we required that the action was invariant under all variations of the coordinates and their derivatives and this led to the Euler-Lagrange equation that selects a particular path q_i, $\dot{q}_i$.

To do so we simply compute

$$\begin{aligned}\frac{dQ}{dt} &= \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)T_i + \frac{\partial L}{\partial \dot{q}_i}\frac{dT_i}{dt} \\ &= \sum_i \frac{\partial L}{\partial q_i}T_i + \frac{\partial L}{\partial \dot{q}_i}\dot{T}_i \\ &= 0,\end{aligned} \tag{3.151}$$

where in the second line we used the Euler-Lagrange equation and in the last line equation (3.149).

Finally we note that actually all we require is a symmetry of the action S. Thus it is enough if the Lagrangian is invariant up to a total derivative:

$$\delta L = \epsilon\frac{d\Omega}{dt}\ . \tag{3.152}$$

Here Ω is some function of the q_i's and $\dot{q}_i$'s. Thus (3.149) is modified to

$$\sum_i \frac{\partial L}{\partial q_i}T_i + \frac{\partial L}{\partial \dot{q}_i}\dot{T}_i = \frac{d\Omega}{dt}\ . \tag{3.153}$$

In this case all we need to do is shift the definition of Q to

$$Q = \sum_i \frac{\partial L}{\partial \dot{q}_i}T_i - \Omega\ , \tag{3.154}$$

so that the second to last line in

$$\begin{aligned}\frac{dQ}{dt} &= \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) T_i + \frac{\partial L}{\partial \dot{q}_i}\frac{dT_i}{dt} - \frac{d\Omega}{dt} \\ &= \sum_i \frac{\partial L}{\partial q_i} T_i + \frac{\partial L}{\partial \dot{q}_i}\dot{T}_i - \frac{d\Omega}{dt} \\ &= 0. \end{aligned} \tag{3.155}$$

Even more generally one might also allow for t to change under the transformation. In this case one must also be careful to include the change in t when evaluating the action as in integral over time, as we will see below.

This is a deep connection between symmetry and conservation laws. It is widely viewed as one of the most fundamental cornerstones of physics. There are a few important symmetries that many physical systems have and these lead to well known conserved charges. Let us look at some.

3.12 Elementary examples of symmetries: A free particle on plane

Let us look at a single particle in a plane with Cartesian coordinates (x, y):

$$L_{cartesian} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \ . \tag{3.156}$$

Here we see that both x and y are ignorable. This leads to two symmetries:

$$\begin{aligned} x &\to x + \epsilon_1 \qquad y \to y \\ y &\to y + \epsilon_2 \qquad x \to x \end{aligned} \tag{3.157}$$

parameterized by ϵ_1 and ϵ_2. These are simply translations in space as there is no preferred point in space in the absence of an external force.

Let us look at the same system but in polar coordinates:

$$L_{polar} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \tag{3.158}$$

where

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \theta &= \arctan(y/x) \ . \end{aligned} \tag{3.159}$$

Here θ is an ignorable coordinate and hence one has the symmetry

$$\theta \to \theta + \epsilon_3 \qquad r \to r \tag{3.160}$$

This is simply a rotation about the origin which is a symmetry as there is no preferred direction.

However these represent the same system. So both Lagrangians must have all three symmetries. So we must show that: **(A)** rotations are a symmetry of $L_{cartesian}$ and **(B)** translations in x and y are a symmetries of L_{polar}. (In fact there is a fourth symmetry due to translations in time but we will see that a little later.)

(A): To see that this is indeed the case we can first compute the ϵ_3 symmetry in cartesian coordinates

$$\begin{aligned} \delta x &= \frac{\partial x}{\partial r}\delta r + \frac{\partial x}{\partial \theta}\delta\theta \\ &= -r\sin\theta\epsilon_3 \\ &= -y\epsilon_3 \end{aligned} \tag{3.161}$$

and

$$\begin{aligned} \delta y &= \frac{\partial y}{\partial r}\delta r + \frac{\partial y}{\partial \theta}\delta\theta \\ &= r\cos\theta\epsilon_3 \\ &= x\epsilon_3 \ . \end{aligned} \tag{3.162}$$

Therefore $\delta\dot{x} = -\dot{y}\epsilon_3$ and $\delta\dot{y} = \dot{x}\epsilon_3$.

$$\begin{aligned} \delta L_{cartesian} &= \frac{\partial L_{cartesian}}{\partial x}\delta x + \frac{\partial L_{cartesian}}{\partial y}\delta y + \frac{\partial L_{cartesian}}{\partial \dot{x}}\delta\dot{x} + \frac{\partial L_{cartesian}}{\partial \dot{y}}\delta\dot{y} \\ &= m\dot{x}\delta\dot{x} + m\dot{y}\delta\dot{y} \\ &= -m\dot{x}\dot{y}\epsilon + m\dot{y}\dot{x}\epsilon \\ &= 0 \end{aligned} \tag{3.163}$$

Again we can compute compute the conserved charge due to the symmetry generated by ϵ_3:

$$\begin{aligned} Q_3 &= \frac{1}{\epsilon_3}\frac{\partial L}{\partial \dot{x}}\delta x + \frac{1}{\epsilon_3}\frac{\partial L}{\partial \dot{y}}\delta y \\ &= m\dot{x}(-y) + m\dot{y}(x) \\ &= m(\dot{y}x - y\dot{x}). \end{aligned} \tag{3.164}$$

If we substitute $x = r\cos\theta$ and $y = r\sin\theta$ then

$$\begin{aligned} Q_3 &= m(\dot{r}\sin\theta + r\dot{\theta}\cos\theta)r\cos\theta - mr\cos\theta(\dot{r}\cos\theta - r\dot{\theta}\sin\theta) \\ &= mr^2\dot{\theta}(\cos^2\theta + \sin^2\theta) \\ &= mr^2\dot{\theta} \\ &= l \ . \end{aligned} \tag{3.165}$$

(B): On the other hand we can also compute the change in r, θ coming from ϵ_1 and ϵ_2:

$$\begin{aligned} \delta r &= \frac{\partial r}{\partial x}\delta x + \frac{\partial r}{\partial y}\delta y \\ &= \frac{x}{r}\epsilon_1 + \frac{y}{r}\epsilon_2 \\ &= \cos\theta\epsilon_1 + \sin\theta\epsilon_2 \end{aligned} \tag{3.166}$$

and

$$\begin{aligned}
\delta\theta &= \frac{\partial\theta}{\partial x}\delta x + \frac{\partial\theta}{\partial y}\delta y \\
&= \frac{1}{1+y^2/x^2}\frac{-y}{x^2}\epsilon_1 + \frac{1}{1+y^2/x^2}\frac{1}{x}\epsilon_2 \\
&= -\frac{y}{x^2+y^2}\epsilon_1 + \frac{x}{x^2+y^2}\epsilon_2 \\
&= -\frac{\sin\theta}{r}\epsilon_1 + \frac{\cos\theta}{r}\epsilon_2 \ .
\end{aligned} \tag{3.167}$$

From these we can compute

$$\begin{aligned}
\delta\dot r &= -\dot\theta\sin\theta\epsilon_1 + \dot\theta\cos\theta\epsilon_2 \\
\delta\dot\theta &= -\dot\theta\frac{\cos\theta}{r}\epsilon_1 - \dot\theta\frac{\sin\theta}{r}\epsilon_2 + \dot r\frac{\sin\theta}{r^2}\epsilon_1 - \dot r\frac{\cos\theta}{r^2}\epsilon_2
\end{aligned} \tag{3.168}$$

From these we can compute

$$\begin{aligned}
\delta L_{polar} &= \frac{\partial L_{polar}}{\partial r}\delta r + \frac{\partial L_{polar}}{\partial\theta}\delta\theta + \frac{\partial L_{polar}}{\partial\dot r}\delta\dot r + \frac{\partial L_{polar}}{\partial\dot\theta}\delta\dot\theta \\
&= m\dot r\delta\dot r + mr\delta r\dot\theta^2 + mr^2\dot\theta\delta\dot\theta \\
&= m\dot r(-\dot\theta\sin\theta\epsilon_1 + \dot\theta\cos\theta\epsilon_2) + mr(-\dot\theta\cos\theta\epsilon_1 + \dot\theta\sin\theta\epsilon_2)\dot\theta^2 \\
&\quad + mr^2\dot\theta\left(-\dot\theta\frac{\cos\theta}{r}\epsilon_1 - \dot\theta\frac{\sin\theta}{r}\epsilon_2 + \dot r\frac{\sin\theta}{r^2}\epsilon_1 - \dot r\frac{\cos\theta}{r^2}\epsilon_2\right) \\
&= 0
\end{aligned} \tag{3.169}$$

Thus the symmetries generated by ϵ_1 and ϵ_2 also extend to symmetries of L_{polar}. We can compute the Noether charges. For the symmetry generated by ϵ_1 we set $\epsilon_2 = 0$ and find

$$\begin{aligned}
Q_1 &= \frac{1}{\epsilon_1}\frac{\partial L}{\partial\dot r}\delta r + \frac{1}{\epsilon_1}\frac{\partial L}{\partial\dot\theta}\delta\theta \\
&= m\dot r(\cos\theta) + mr^2\dot\theta\left(-\frac{\sin\theta}{r}\right) \\
&= m\frac{d}{dt}(r\cos\theta) \\
&= m\dot x \ .
\end{aligned} \tag{3.170}$$

While for the symmetry generated by ϵ_2 we set $\epsilon_1 = 0$ and find

$$\begin{aligned}
Q_2 &= \frac{1}{\epsilon_2}\frac{\partial L}{\partial\dot r}\delta r + \frac{1}{\epsilon_2}\frac{\partial L}{\partial\dot\theta}\delta\theta \\
&= m\dot r(\sin\theta) + mr^2\dot\theta\left(\frac{\cos\theta}{r}\right) \\
&= m\frac{d}{dt}(r\sin\theta) \\
&= m\dot y \ .
\end{aligned} \tag{3.171}$$

The point of this is to show that although some symmetries may be realised rather trivially, such as those generated by ϵ_1 and ϵ_2 in cartesian coordinates, there may still be other symmetries which have a non-trivial realization. In this case the there is a third symmetry generated by ϵ_3. Similarly in polar coordinates where ϵ_3 is rather simple there are in fact still two more symmetries generated by ϵ_1 and ϵ_2. Later in the course we will see that there can be still more symmetries that are not al all apparent in the Lagrangian formulation.

3.13 Invariance under spatial translations gives conserved momentum

This is the simplest example of a symmetry. Let us suppose that the generalized coordinates are positions in space and that the potential and kinetic terms only depend on the separation between any two pairs of particles, *e.g.* we might have

$$L = \sum_i \frac{1}{2} m_i |\dot{\underline{r}}_i|^2 - V(\underline{r}_i - \underline{r}_j) \tag{3.172}$$

where V only depends on $\underline{r}_i - \underline{r}_j$. Then we have an overall translational symmetry:

$$\underline{r}_i \to \underline{r}_i + \epsilon \underline{a} \qquad \dot{\underline{r}}_i \to \dot{\underline{r}}_i \ , \tag{3.173}$$

where $\underline{a}$ is a fixed vector. Therefore $\underline{r}_i - \underline{r}_j$ is invariant. This corresponds to picking up every particle in your system and moving it over a tiny bit in the direction of $\underline{a}$. Since we assume $\dot{\underline{a}} = \underline{0}$ the Lagrangian will be invariant and we find

$$Q = \sum_i \frac{\partial L}{\partial \dot{\underline{r}}_i} \cdot \underline{a} = \sum_i \underline{p}_i \cdot \underline{a} \ . \tag{3.174}$$

This is just the total momentum along the direction $\underline{a}$. This symmetry reflects the **homogeneity** of space, namely that there is no preferred location.

3.14 Invariance under rotations gives conserved angular momentum

Let us make the same assumption as for spatial translations but then also assume that the potential and kinetic terms only depend on the distance $|\underline{r}_i - \underline{r}_j|$ between any pair of particles (and not the direction). This is the case for all known fundamental forces. Then we can consider a rotation of all the particles:

$$\underline{r}_i \to \underline{r}_i + \epsilon \mathbf{T} \underline{r}_i \qquad \dot{\underline{r}}_i \to \dot{\underline{r}}_i + \epsilon \mathbf{T} \dot{\underline{r}}_i \ , \tag{3.175}$$

where $\mathbf{T}$ is a constant anti-symmetric matrix: $\mathbf{T}^T = -\mathbf{T}$. To show that the Lagrangian is invariant we must show that $|\underline{r}_i - \underline{r}_j|$ is invariant:

$$\begin{aligned} \delta(|\underline{r}_i - \underline{r}_j|^2) &= 2(\underline{r}_i - \underline{r}_j) \cdot \delta(\underline{r}_i - \underline{r}_j) \\ &= 2\epsilon(\underline{r}_i - \underline{r}_j) \cdot \mathbf{T}(\underline{r}_i - \underline{r}_j) \ , \end{aligned} \tag{3.176}$$

Now $\mathbf{T}$ is an anti-symmetric 3×3 matrix so we can write is as

$$\mathbf{T} = \begin{pmatrix} 0 & -T^3 & T^2 \\ T^3 & 0 & -T^1 \\ -T^2 & T^1 & 0 \end{pmatrix} \ . \tag{3.177}$$

If we think in terms of components we have

$$\mathbf{T}^a{}_b = -\sum_{c=1}^{3} \epsilon_{abc} T^c \ . \tag{3.178}$$

and therefore

$$\begin{aligned}\delta(|\underline{r}_i - \underline{r}_j|^2) &= 2\epsilon(\underline{r}_i - \underline{r}_j) \cdot \mathbf{T}(\underline{r}_i - \underline{r}_j) \\ &= -2\epsilon \sum_{abc} \epsilon_{abc}(r_i^a - r_j^a)(r_i^b - r_j^b)T^c \\ &= 0 \ . \end{aligned} \tag{3.179}$$

The expression vanishes because $\epsilon_{abc} = -\epsilon_{bac}$ but $(r_i^a - r_j^a)(r_i^b - r_j^b)$ is symmetric in $a \leftrightarrow b$. Thus the Lagrangian will be invariant.

The Noether charge is

$$\begin{aligned} Q &= \sum_i \frac{\partial L}{\partial \dot{\underline{r}}_i} \cdot \mathbf{T}\underline{r}_i \\ &= \sum_i \underline{p}_i \cdot \mathbf{T}\underline{r}_i \\ &= \sum_i \sum_{ab} \underline{p}_i^a \mathbf{T}^a{}_b \underline{r}_i^b \\ &= -\sum_i \sum_{abc} p_i^a \epsilon_{abc} r_i^b T^c \\ &= \underline{T} \cdot \sum_i \underline{r}_i \times \underline{p}_i \ . \end{aligned} \tag{3.180}$$

This is just the component of the total angular momentum along the direction $\underline{T} = (T^1, T^2, T^3)$.

This symmetry reflects the **isotropy** of space, namely that it looks the same in all directions.

Note that putting in a massive body such as the sun and holding it fixed will break homogeneity but not isotropy about the sun. Of course in reality the sun isn't fixed, just heavy compared to the planets, and so space is really homogeneous as the sun is free to move.

3.15 Invariance Under Time Translations Gives Conservation of Energy

Lastly, but most importantly, let us show that conservation of energy arises from invariance under time translations. So let us assume that the Lagrangian does not have any explicit time dependence:

$$\frac{\partial L}{\partial t} = 0 \ . \tag{3.181}$$

First note that the action S depends only on t_1 and t_2 (and not on the integration variable t which has been integrated over). Thus a translation in time means a shift $t_1 \to t_1 + \epsilon$, $t_2 \to t_2 + \epsilon$. In this case we must be a little more subtle with the variation of the action which is now

$$\begin{aligned}\delta S &= \int_{t_1+\epsilon}^{t_2+\epsilon} L(q_i(t), \dot{q}_i(t))dt - \int_{t_1}^{t_2} L(q_i(t), \dot{q}_i(t))dt \\ &= \int_{t_1}^{t_2} L(q_i(t+\epsilon), \dot{q}_i(t+\epsilon)) - L(q_i(t), \dot{q}_i(t))dt \ , \end{aligned} \tag{3.182}$$

where in the first term of the second line we use a change of variables $t' = t + \epsilon$ in the integral. In this case the coordinates will transform as

$$q_i(t) \to q_i(t+\epsilon) = q_i(t) + \epsilon \dot{q}_i(t) \qquad \dot{q}_i(t) \to \dot{q}(t+\epsilon) = \dot{q}_i(t) + \epsilon \ddot{q}_i(t) \ . \tag{3.183}$$

In other words

$$\delta q_i = \epsilon \dot{q}_i \qquad \delta \dot{q}_i = \epsilon \ddot{q}_i \tag{3.184}$$

Since the Lagrangian has no explicit t dependence it too simply transforms as

$$L(q_i(t+\epsilon), \dot{q}_i(t+\epsilon)) = L(q_i(t), \dot{q}_i(t)) + \epsilon \frac{dL}{dt} \ , \tag{3.185}$$

where

$$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \ . \tag{3.186}$$

Thus

$$\delta S = \epsilon \int_{t_1}^{t_2} \frac{dL}{dt} dt \ , \tag{3.187}$$

and we need to use the modified form for the Noether charge (3.154):

$$\begin{aligned} Q &= \frac{1}{\epsilon} \sum_i \frac{\partial L}{\partial \dot{q}_i} \delta q_i - L \\ &= \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L \\ &= \sum_i p^i \dot{q}_i - L \end{aligned}$$

If we evaluate this for a simple Lagrangian of the form

$$L = \frac{1}{2} m |\dot{\underline{r}}|^2 - V(\underline{r}_1) \ , \tag{3.188}$$

then

$$\underline{p} = m \dot{\underline{r}} \ , \tag{3.189}$$

and hence the conserved charge is indeed the energy E:

$$\begin{aligned} Q &= \underline{p} \cdot \dot{\underline{r}} - \frac{1}{2} m |\dot{\underline{r}}|^2 + V(\underline{r}) \\ &= \frac{1}{2} m |\dot{\underline{r}}|^2 + V(\underline{r}) \\ &= E \end{aligned} \tag{3.190}$$

as previously defined. Here we see how to extend it to a general Lagrangian. Note that for a general Lagrangian, one that isn't of the form (3.188), this definition of energy is not simply of the form $E = \frac{1}{2} m |\dot{\underline{r}}|^2 + V$. This can happen for example if the Lagrangian contains terms which are linear, or which are higher than second order, in $\dot{\underline{r}}_i$.

Let us check that this is indeed conserved. To do so we first compute:

$$
\begin{aligned}
\frac{dL}{dt} &= \frac{\partial L}{\partial t} + \sum_i \left(\frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \right) \\
&= \frac{\partial L}{\partial t} + \sum_i \left(\frac{\partial L}{\partial q_i}\dot{q}_i + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right] - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\right]\dot{q}_i \right) \\
&= \frac{\partial L}{\partial t} + \frac{d}{dt}\sum_i \left[\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right] + \sum_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\right]\right)\dot{q}_i \\
&= \frac{\partial L}{\partial t} + \frac{d}{dt}\sum_i p_i\dot{q}_i \ ,
\end{aligned}
\tag{3.191}
$$

where we have used the Euler-Lagrange equation. Combining this with

$$
E = \sum_i p^i\dot{q}_i - L \tag{3.192}
$$

tells us that

$$
\frac{dE}{dt} = -\frac{\partial L}{\partial t} \ . \tag{3.193}
$$

Thus, provided that L does not explicitly depend on time, then E will be conserved.

In particular if we return to the free particle in two dimensions that we studied before we can identify a fourth symmetry corresponding to time translations. The associated conserved quantity is then just

$$
\begin{aligned}
E &= p_x\dot{x} + p_y\dot{y} - L \\
&= m\dot{x}^2 + m\dot{y}^2 - \frac{1}{2}m(\dot{x}^2+\dot{y}^2) && (3.194) \\
&= \frac{1}{2}m(\dot{x}^2+\dot{y}^2) \ . && (3.195)
\end{aligned}
$$

Or in terms of polar coordinates

$$
\begin{aligned}
E &= p_r\dot{r} + p_\theta\dot{\theta} - L \\
&= m\dot{r}^2 + mr^2\dot{\theta}^2 - \frac{1}{2}m(\dot{r}^2+r\dot{\theta}^2) && (3.196) \\
&= \frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2) \ . && (3.197)
\end{aligned}
$$

It is easy to check that these two expression for E agree. Thus there are four conserved quantities (p_x,p_y, $p_\theta = l$ and E).

3.16 Example: A spherical pendulum

Let us return to the simple pendulum but now we not longer constrained it to lie in a plane but can move in all three dimensions. However it still has one constraint: it must be a fixed distance l from the origin:

$$
C_1(\underline{r},t) = |\underline{r}| - l = 0 \ . \tag{3.198}
$$

To solve this constraint we introduce spherical coordinates, $\underline{r} = (x,y,z)$ with

$$
\begin{aligned}
x &= r\sin\theta\cos\phi \\
y &= r\sin\theta\sin\phi \\
z &= -r\cos\theta
\end{aligned}
\tag{3.199}
$$

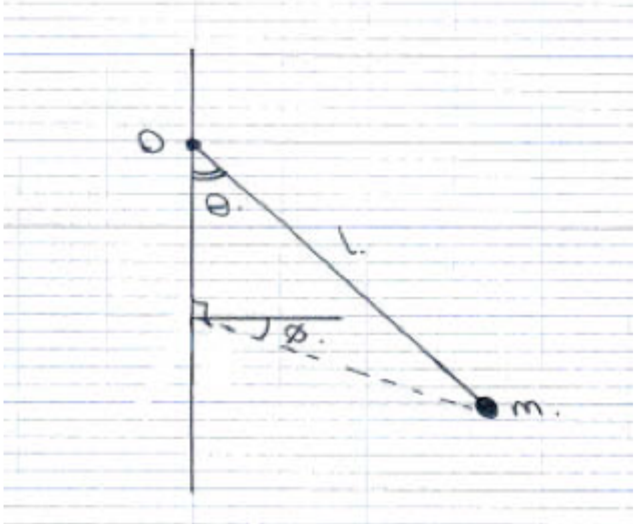

Figure 3.16.1: A Spherical Pendulum

Note that the range of θ and ϕ are $[0,\pi]$ and $[0,2\pi)$ respectively, therefore the constraint is

$$x^2+y^2+z^2-l^2=r^2-l^2=0 \tag{3.200}$$

solved by taking $r=l$.

To continue we compute the kinetic energy

$$\dot{\underline{r}}=l\begin{pmatrix}\dot\theta\cos\theta\cos\phi-\dot\phi\sin\theta\sin\phi\\ \dot\theta\cos\theta\sin\phi+\dot\phi\sin\theta\cos\phi\\ \dot\theta\sin\theta\end{pmatrix} \tag{3.201}$$

and hence

$$\begin{aligned}T&=\frac{1}{2}m|\dot{\underline{r}}|^2\\&=\frac{1}{2}ml^2\left((\dot\theta\cos\theta\cos\phi-\dot\phi\sin\theta\sin\phi)^2+(\dot\theta\cos\theta\sin\phi+\dot\phi\sin\theta\cos\phi)^2+\dot\theta^2\sin^2\theta\right)\\&=\frac{1}{2}ml^2\left(\dot\theta^2\cos^2\theta\cos^2\phi+\dot\phi^2\sin^2\theta\sin^2\phi+\dot\theta^2\cos^2\theta\sin^2\phi+\dot\phi^2\sin^2\theta\cos^2\phi+\dot\theta^2\sin^2\theta\right)\\&=\frac{1}{2}ml^2\left(\dot\theta^2+\sin^2\theta\dot\phi^2\right)\ .\end{aligned} \tag{3.202}$$

The potential is just $V=mgz$:

$$V=-mgl\cos\theta\ . \tag{3.203}$$

Hence we arrive at the Lagrangian

$$L=\frac{1}{2}ml^2\left(\dot\theta^2+\sin^2\theta\dot\phi^2\right)+mgl\cos\theta\ . \tag{3.204}$$

Here we see that ϕ is an ignorable coordinate.

What are the conserved charges? Since L does not depend explicitly on t we have a conserved energy:

$$\begin{aligned}
E &= \frac{\partial L}{\partial \dot{\theta}}\dot{\theta} + \frac{\partial L}{\partial \dot{\phi}}\dot{\phi} - L \\
&= ml^2\dot{\theta}^2 + ml^2\sin^2\theta\dot{\phi}^2 - \frac{1}{2}ml^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) - mgl\cos\theta \\
&= \frac{1}{2}ml^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) - mgl\cos\theta \ . \qquad (3.205)
\end{aligned}$$

From the symmetry $\phi \to \phi + \epsilon$ we have conserved angular momentum about the z-axis

$$\begin{aligned}
L_z &= \frac{\partial L}{\partial \dot{\phi}} \\
&= ml^2\sin^2\theta\dot{\phi} \ . \qquad (3.206)
\end{aligned}$$

As before this is enough to reduce the system to two a single first order differential equation. In particular we write

$$\dot{\phi} = \frac{L_z}{ml^2\sin^2\theta} \ , \qquad (3.207)$$

so that the energy is

$$E = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{L_z^2}{2ml^2\sin^2\theta} - mgl\cos\theta \ . \qquad (3.208)$$

This gives the effective potential

$$V_{eff} = \frac{L_z^2}{2ml^2\sin^2\theta} - mgl\cos\theta \ , \qquad (3.209)$$

so that the solution for $\theta(t)$ comes from the integral

$$t - t_0 = \sqrt{\frac{m}{2}}l\int_{\theta_0}^{\theta}\frac{d\theta'}{\sqrt{(E - V_{eff}(\theta'))}} \ . \qquad (3.210)$$

Once one knows $\theta(t)$ one can integrate (3.207) to obtain $\phi(t)$. Clearly finding $\theta(t)$ is not easy but it can be done numerically with computers.

However qualitatively one can understand the dynamics by looking at a sketch of V_{eff} for $L_z \neq 0$ (see figure 17). We see that there is a minimum of V_{eff} at

$$-\frac{L_z^2}{ml^2\sin^3\theta}\cos\theta + mgl\sin\theta = 0 \qquad \Longleftrightarrow \qquad \frac{\sin^4\theta}{\cos\theta} = \frac{L_z^2}{m^2gl^3} \qquad (3.211)$$

At this value of, $\theta = \theta_0$, we obtain circular orbits where $\theta = \theta_0$ is constant and

$$\phi = \frac{L_z t}{ml^2\sin^2\theta_0} \qquad (3.212)$$

Notice that small values of L_z lead to small values of θ_0 but large values lead to $\theta_0 \sim \pi/2$.

More typically θ will oscillate around this minimum, while ϕ winds around, leading to paths depicted in figure 18. Of course we could also choose initial conditions where $\dot{\phi} = 0$, *i.e.* $L_z = 0$. In which case we recover the simple pendulum in a plane.

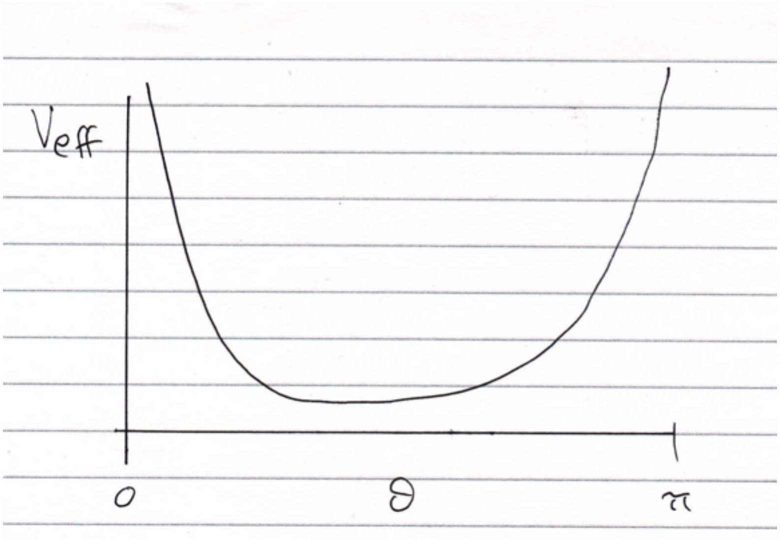

Figure 3.16.2: Effective Potential for the Spherical Pendulum

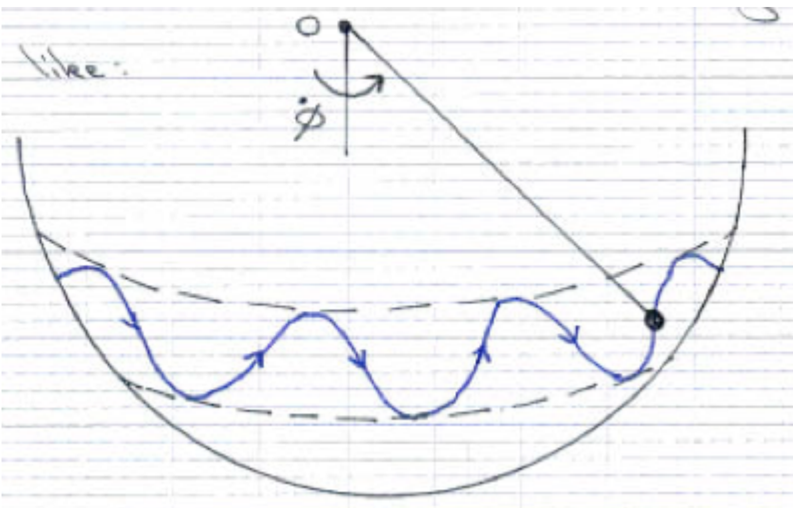

Figure 3.16.3: Typical Motion Of A Spherical Pendulum

3.17 The Brachistochrone

The principle of least action and resulting Euler-Lagrange equations are an example of the calculus of variations. This is where one performs 'calculus' on functionals, *i.e.* where one needs to differentiate with respect to a function. One of the first examples of this technique was to solve the so-called brachistochrone problem:

`https://en.wikipedia.org/wiki/Brachistochrone_curve`

`https://mathcurve.com/courbes2d.gb/brachistochrone/brachistochrone.shtml`

The short version of the story is that Bernouli posed the problem as a challenge:

"*I, Johann Bernoulli, address the most brilliant mathematicians in the world. Nothing is more attractive to intelligent people than an honest, challenging problem, whose possible solution will bestow fame and remain as a lasting monument. Following the example set by Pascal, Fermat, etc., I hope to gain the gratitude of the whole scientific community by placing before the finest mathematicians of our time a problem which will test their methods and the strength of their intellect. If someone communicates to me the solution of the proposed problem, I shall publicly declare him worthy of praise.*"

Newton received the challenge at 4pm on 29 January 1697 and had solved it by 4am the following day.[3] He mailed his solution anonymously to Bernoulli who upon reading it declared that he "*recognizes a lion from his claw mark*".

So before we finish with our discussion of Lagrangians lets us discuss this problem using terms our skier would understand. Our skier is by now quite experienced and wants to know the shape of a ski slope that gets her down the hill as quickly as possible. Clearly the shortest ski slope would be a straight line but this will not be the fastest as it is beneficial and more fun to speed up as much as possible early on, even if that means having to go further.

Let us consider a skier on a slope given by $y = h(x)$. We saw before that the Lagrangian is

$$L = \frac{1}{2}m((1+(h')^2)\dot{x}^2) - mgh(x) \ , \tag{3.213}$$

where $h' = dh/dx$. This means that the energy is

$$\begin{aligned} E &= \frac{\partial L}{\partial \dot{x}}\dot{x} - L \\ &= \frac{1}{2}m((1+(h')^2)\dot{x}^2) + mgh(x) \ . \end{aligned} \tag{3.214}$$

What the skier wants to minimize is the time

$$\begin{aligned} t &= \int dt \\ &= \int_{x_1}^{x_2} \frac{dx}{\dot{x}} \end{aligned} \tag{3.215}$$

where x_1 and x_2 are the starting and finishing positions (along the x-axis). If we write $E = mgh_0$ for some h_0 then using the equation for the energy we have

$$\dot{x} = \sqrt{2g}\sqrt{\frac{h_0 - h(x)}{1+(h'(x))^2}} \ . \tag{3.216}$$

[3] It's reassuring to observe that Physicists working patterns haven't changed much.

Note that if the skier starts from rest, *i.e.* with $\dot{x} = 0$, then $h_0 = h(x(0))$. Thus we have

$$\begin{aligned} t &= \sqrt{\frac{1}{2g}} \int_{x_1}^{x_2} \sqrt{\frac{1 + (h'(x))^2}{h_0 - h(x)}} dx \\ &= \int_{x_1}^{x_2} \mathcal{L}(h, h') dx \ . \end{aligned} \tag{3.217}$$

This is our functional that we want to minimise. We know how to do that! We just derive the Euler-Lagrange equation for $\mathcal{L}$. Only now $h(x)$ is the function we want to vary. Thus we need to compute (note that the role of t is now replaced by x)

$$\begin{aligned} 0 = \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial h'}\right) - \frac{\partial \mathcal{L}}{\partial h} &= \sqrt{\frac{1}{2g}} \frac{d}{dx}\left(\frac{h'}{\sqrt{h_0 - h}\sqrt{1 + h'^2}}\right) - \frac{1}{2}\sqrt{\frac{1}{2g}} \frac{\sqrt{1 + h'^2}}{(h_0 - h)^{3/2}} \\ &= \sqrt{\frac{1}{2g}} \frac{h''}{\sqrt{h_0 - h}\sqrt{1 + h'^2}} + \frac{1}{2}\sqrt{\frac{1}{2g}} \frac{h'^2}{(h_0 - h)^{3/2}\sqrt{1 + h'^2}} \\ &\quad - \sqrt{\frac{1}{2g}} \frac{h'^2 h''}{\sqrt{h_0 - h}(1 + h'^2)^{3/2}} - \frac{1}{2}\sqrt{\frac{1}{2g}} \frac{\sqrt{1 + h'^2}}{(h_0 - h)^{3/2}} \\ &= \sqrt{\frac{1}{2g}} \frac{h''}{\sqrt{h_0 - h}(1 + h'^2)^{3/2}} + \frac{1}{2}\sqrt{\frac{1}{2g}} \frac{1}{(h_0 - h)^{3/2}\sqrt{1 + h'^2}} \ . \end{aligned} \tag{3.218}$$

So finally we can write this as

$$(h_0 - h)h'' + \frac{1}{2}(1 + h'^2) = 0 \ . \tag{3.219}$$

But there is an easier way to proceed. Although there is no 'time' in our new problem x behaves like time and furthermore the Lagrangian $\mathcal{L}$ is independent of x. So the analogue of 'energy', $\mathcal{E}$, is still 'conserved' (meaning independent of x):

$$\mathcal{E} = \frac{\partial \mathcal{L}}{\partial h'} h' - \mathcal{L} \ . \tag{3.220}$$

Check that $d\mathcal{E}/dx = 0$ if you don't believe me! This gives us a first order equation

$$\begin{aligned} \sqrt{2g}\mathcal{E} &= \frac{h'^2}{\sqrt{h_0 - h}\sqrt{1 + h'^2}} - \sqrt{\frac{1 + h'^2}{h_0 - h}} \\ &= \frac{1}{\sqrt{h_0 - h}\sqrt{1 + h'^2}} \ . \end{aligned} \tag{3.221}$$

If we square both sides and rearrange we find

$$(h_0 - h)(1 + h'^2) = C^2 \tag{3.222}$$

where $C^2 = 1/2g\mathcal{E}^2$. We can solve for h':

$$h' = \pm\sqrt{\frac{C^2 - h_0 + h}{h_0 - h}} \ , \tag{3.223}$$

and rewrite this

$$\sqrt{\frac{h_0 - h}{C^2 - h_0 + h}} dh = \pm dx \ . \tag{3.224}$$

A computer (or hard work) will tell you that

$$\pm(x - x_0) = C^2 \arcsin\left(\sqrt{\frac{C^2 - h_0 + h}{C^2}}\right) + \sqrt{(h_0 - h)(C^2 - h_0 + h)} \tag{3.225}$$

Another way to solve this we write

$$\begin{aligned} x &= \frac{1}{2}C^2(\theta - \sin\theta) \\ h &= h_0 - \frac{1}{2}C^2(1 - \cos\theta) \ . \end{aligned} \tag{3.226}$$

To check this we note that $h' = -\frac{1}{2}C^2 \sin\theta d\theta/dx$ and $dx/d\theta = \frac{1}{2}C^2(1 - \cos\theta) = h_0 - h$ so

$$\begin{aligned} h' &= -\frac{1}{2}\frac{C^2 \sin\theta}{dx/d\theta} = \frac{\sin\theta}{1 - \cos\theta} \\ 1 + h'^2 &= 1 + \frac{sin^2\theta}{(1 - \cos\theta)^2} = \frac{\sin^2\theta + 1 - 2\cos\theta + \cos^2\theta}{(1 - \cos\theta)^2} = \frac{2}{1 - \cos\theta} \end{aligned} \tag{3.227}$$

From these you can see that $(h_0 - h)(1 + h'^2) = C^2$.

If you think about the curve $(x(\theta), -h(\theta))$ then you realise that it describes the motion of a point on wheel of radius $C^2/2$ as it rolls along a flat surface, corresponding to $h = h_0 - C^2$ (think of θ as depending linearly on time) and is known as a cycloid:

`https://mathworld.wolfram.com/Cycloid.html`

Chapter 4

Hamiltonian Mechanics

In our discussion of Lagrangians we already introduced the notion of the conjugate momentum:

$$p^i = \frac{\partial L}{\partial \dot{q}_i} \ . \tag{4.1}$$

Once all the (holonomic only) constraints have been solved for, leading to a reduced number of degrees of freedom there is one conjugate momentum for each generalized coordinate q_i. The Euler-Lagrange equation then gives a second order differential equation for the time evolution of the system. Since it is second order one must specify the initial values of q_i and $\dot{q}_i$. That is the initial positions and velocities of the particles.

We now consider an equivalent description of dynamical systems known as the Hamiltonian formulation. Here one essentially swaps $\dot{q}_i$ for the conjugate momentum p^i and doubles the number of variables. Since both positions and velocities are needed to describe a system the Hamiltonian formulation puts both of q_i and p^i on an equal footing. The upside of this is that the second order Euler-Lagrange equations are replaced by first order evolution equations known as Hamilton's equations. In effect we wish to go from thinking in terms of the Lagrangian $L(q_i, \dot{q}_i, t)$ to a new function $H(q_i, p^i, t)$ known as the Hamiltonian which encodes the same information. In particular in the Hamiltonian view $\dot{q}_i$ never appears, only q_i and p^i. To emphasize this point we will not use a dot as a short-hand for a time derivative once we are in the Hamiltonian formulation (although we will use it when we discuss Lagrangians). Thus, once in the Hamiltonian formulation $\dot{q}$ never appears.

4.1 Legendre transformation

The general procedure is called a **Legendre transformation**. Let us consider a function $F(x, y)$ which we want to swap for a new function $\tilde{F}(x, u)$ without losing any information. To do this we note that the total differential of F is

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy \ . \tag{4.2}$$

Let us introduce a new function

$$\tilde{F}(x, y, u) = uy - F(x, y) \ , \tag{4.3}$$

which is initially a function of (x, y, u) so that

$$d\tilde{F} = ydu + udy - \frac{\partial F}{\partial x}dx - \frac{\partial F}{\partial y}dy \ . \tag{4.4}$$

However we see that the dy term in $d\tilde{F}$ drops out if we take

$$u = \frac{\partial F}{\partial y} \ . \tag{4.5}$$

In this case $\tilde{F}$ is only a function of (x, u):

$$\tilde{F}(x, u) = uy(x, u) - F(x, y(x, u)) \tag{4.6}$$

where we use (4.5) to find $y(u, x)$. This is the **Legendre transformation** and it preserves all the information of the system since we can undo it by a further Legendre transformation. To see this consider

$$\tilde{\tilde{F}}(x, z) = \frac{\partial \tilde{F}}{\partial u}u(x, z) - \tilde{F}(x, u(x, z)) \ , \tag{4.7}$$

where $z = \partial\tilde{F}/\partial u$ so that $\tilde{\tilde{F}}$ is only a function of (x, z) . But from (4.6)

$$\begin{aligned} \frac{\partial \tilde{F}}{\partial u} &= y + u\frac{\partial y}{\partial u} - \frac{\partial F}{\partial y}\frac{\partial y}{\partial u} \\ &= y + u\frac{\partial y}{\partial u} - u\frac{\partial y}{\partial u} \\ &= y \ , \end{aligned} \tag{4.8}$$

where we have used (4.5) in the second line. Therefore

$$\begin{aligned} \tilde{\tilde{F}}(x, z) &= yu(x, z) - \tilde{F}(x, u(x, z)) \\ &= F(x, z) \ , \end{aligned} \tag{4.9}$$

and we have the original function back.

N.B. This isn't quite true. In computing the Legendre transform we have assumed that we can invert the expression $u = \partial F/\partial y$ to find y as a function of u (and x). If F is only linear in y then u is y-independent and hence we can not invert to find y as a function of u. We will largely ignore this special case since y will be taken to be the velocity and Lagrangians are typically quadratic in velocity.

So what? We have already encountered the conjugate momentum:

$$p^i = \frac{\partial L}{\partial \dot{q}_i} \ . \tag{4.10}$$

This can be viewed as part of a Legendre transform which produces the Hamiltonian from the Lagrangian:

$$H(q_i, p^i, t) = \sum_i p^i\dot{q}_i - L(q_i, \dot{q}_i, t) \ . \tag{4.11}$$

We have seen this before, where H was the conserved energy E (when there is no explicit time dependence). Thus the physical significance of the Hamiltonian is that it is the energy of the system.

Example: To make things concrete let us look at a Lagrangian of the form

$$L(q_i, \dot{q}_i) = \sum_i \frac{1}{2} m_i \dot{q}_i^2 - V(q_i) \ . \tag{4.12}$$

For example the q_i's could be the $3N$ position variables of N particles in $\mathbb{R}^3$. Here we see that the conjugate momenta are

$$p^i = m_i \dot{q}_i \qquad \Longleftrightarrow \qquad \dot{q}_i = p^i/m_i \ . \tag{4.13}$$

The Hamiltonian is then

$$\begin{aligned} H(q_i, p^i) &= \sum_i p^i \dot{q}_i - L(q_i, \dot{q}_i(q_i, p^i)) \\ &= \sum_i p^i \dot{q}_i - \sum \frac{1}{2} m_i \dot{q}_i^2 + V(q_i) \\ &= \sum_i \frac{(p^i)^2}{2m_i} + V(q_i) \ . \end{aligned} \tag{4.14}$$

4.2 Hamilton's Equations

As $H \equiv H(q_i, p^i; t)$ then,

$$dH = \sum_i \left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p^i} dp^i + \frac{\partial H}{\partial t} dt \right). \tag{4.15}$$

While as $H = \sum_i \dot{q}_i p^i - L$ we also have

$$\begin{aligned} dH &= \sum_i \left(d\dot{q}_i p^i + \dot{q}_i dp^i - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i - \frac{\partial L}{\partial t} dt \right) \\ &= \sum_i \left(\dot{q}_i dp^i - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial t} dt \right) \end{aligned} \tag{4.16}$$

where we have used the definition of the conjugate momentum $p^i = \frac{\partial L}{\partial \dot{q}_i}$ to eliminate two terms in the final line. By comparing the coefficients of dq_i, $d\dot{q}_i$ and dt in the two expressions for dH we find

$$\dot{q}_i = \frac{\partial H}{\partial p^i}, \qquad -\frac{\partial L}{\partial q_i} = \frac{\partial H}{\partial q_i}, \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \tag{4.17}$$

Next we use the Euler-Lagrange equation to observe that $\dot{p}_i = \frac{\partial L}{\partial q_i}$ so that the first two equations give

$$\frac{d}{dt} q_i = \frac{\partial H}{\partial p^i}, \qquad \frac{d}{dt} p^i = -\frac{\partial H}{\partial q_i} \tag{4.18}$$

These are referred to as Hamilton's equations of motion. Notice that these are $2n$ first order differential equations compared to Lagrange's equations which are n second-order differential equations. The space of (q_i, p^i) is known as **phase space** and typically for N unconstrained particles it is $\mathbb{R}^{6N}$ since each particle has three position variables and three momentum variables. In classical mechanics the state of a system is given by a point in phase space.

Example 1: Let us look at a free particle on $\mathbb{R}^3$ The Lagrangian is

$$L = \frac{1}{2}m|\dot{\underline{r}}|^2 \tag{4.19}$$

Clearly the Euler-Lagrange equations are just

$$m\ddot{\underline{r}} = 0 \tag{4.20}$$

which have linear solutions $\underline{r}(t) = \underline{v}(0)t + \underline{r}(0)$. To construct the Hamiltonian we note that

$$\underline{p} = \frac{\partial L}{\partial \dot{\underline{r}}} = m\dot{\underline{r}} \tag{4.21}$$

and hence

$$\begin{aligned} H &= \underline{p} \cdot \dot{\underline{r}} - L \\ &= \frac{1}{2m}|\underline{p}|^2 \ . \end{aligned} \tag{4.22}$$

What are Hamiltons equations? Just:

$$\begin{aligned} \frac{d}{dt}\underline{r} &= \frac{\partial H}{\partial \underline{p}} = \frac{\underline{p}}{m} \\ \frac{d}{dt}\underline{p} &= -\frac{\partial H}{\partial \underline{r}} = \underline{0} \end{aligned} \tag{4.23}$$

Thus the solution is

$$\begin{aligned} \underline{p}(t) &= \underline{p}(0) \\ \underline{r}(t) &= \frac{\underline{p}(0)}{m}t + \underline{r}(0) \end{aligned} \tag{4.24}$$

In this case the Hamiltonian flow simply consists of straight lines in phases space with a constant value of $\underline{p} \neq \underline{0}$. In particular the flow goes to the left for lines above the $\underline{r}$ axis but to the right for lines below the $\underline{r}$ axis. The $\underline{p} = \underline{0}$ axis itself is special as points that start on it remain on it. Therefore it simply consists of an infinite collection of disjoint points (this case is a bit singular).

Example 2: Let us look at the simplest next possible system in detail. Its called the harmonic oscillator and consists of a single degree of freedom q with mass m moving in a potential $V(q) = \frac{1}{2}kq^2$. Thus the force is linear $F = -kq$. Such a system could be a spring displaced by an amount q (using Hook's law with spring constant k) or the small angle approximation to a pendulum with $q = \theta << 1$, since in that case we had $V = -mgl\cos\theta \sim -mgl + \frac{1}{2}mgl\theta^2$ so that $k^2 = mgl$ and the constant term $-mgl$ is irrelevant.

First let us solve this problem using the Lagrangian approach. Here we construct

$$L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2 \tag{4.25}$$

from which we obtain the Euler-Lagrange equation

$$\begin{aligned} 0 &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} \\ &= \frac{d}{dt}(m\dot{q}) + kq \\ &= m\ddot{q} + kq \ . \end{aligned} \tag{4.26}$$

This can be readily solved by taking

$$q(t) = A\sin(\omega t) + B\cos(\omega t) \qquad \omega = \sqrt{k/m}\ . \tag{4.27}$$

Here we identify $B = q(0)$ and $\omega A = \dot{q}(0)$ so that we could write

$$q(t) = \omega^{-1}\dot{q}(0)\sin(\omega t) + q(0)\cos(\omega t)\ . \tag{4.28}$$

Let us look at this in the Hamiltonian formalism. Here the phase space is $\mathbb{R}^2$ parameterized by (q, p). First we note that

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} \tag{4.29}$$

so that

$$\dot{q} = p/m\ . \tag{4.30}$$

Thus the Hamiltonian is

$$\begin{aligned} H &= p\dot{q} - L \\ &= \frac{q^2}{m} + \frac{1}{2}\left(\frac{p}{m}\right)^2 + \frac{1}{2}kq^2 \\ &= \frac{1}{2m}p^2 + \frac{1}{2}kq^2\ . \end{aligned} \tag{4.31}$$

From here we read off Hamilton's equations:

$$\begin{aligned} \frac{d}{dt}q &= \frac{\partial H}{\partial p} = p/m \\ \frac{d}{dt}p &= -\frac{\partial H}{\partial q} = -kq\ . \end{aligned} \tag{4.32}$$

Note that we can substitute the first equation into the second equation to find

$$\frac{d}{dt}p = m\frac{d^2}{dt^2}q = -qk \tag{4.33}$$

which is just the same equation as in the Lagrangian formulation. However this is not how we want to think of the problem. Rather we want to solve for $q(t)$ and $p(t)$. The simplest way to do this is take the previous solution and reinterpret it:

$$\begin{aligned} q(t) &= A\sin(\omega t) + B\cos(\omega t) \\ p(t) &= m\dot{q} \\ &= mA\omega\cos(\omega t) - mB\omega\sin(\omega t) \end{aligned} \tag{4.34}$$

From here we see that

$$B = q(0) \qquad A = \frac{p(0)}{m\omega} \tag{4.35}$$

so we have

$$\begin{aligned} q(t) &= \frac{p(0)}{m\omega}\sin(\omega t) + q(0)\cos(\omega t) \\ p(t) &= p(0)\cos(\omega t) - mq(0)\omega\sin(\omega t)\ . \end{aligned} \tag{4.36}$$

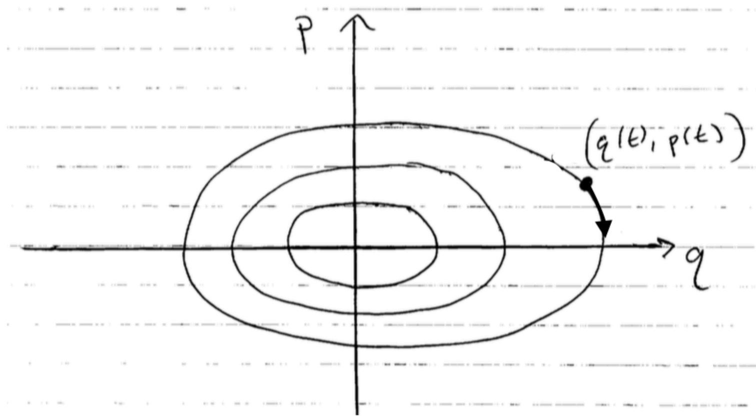

Figure 4.2.1: Phase Space Flows for the Harmonic Oscilator

Notice that these parameterize an ellipse:

$$(q(t))^2 + \frac{1}{m^2\omega^2}(p(t))^2 = +(q(0))^2 + \frac{1}{m^2\omega^2}(p(0))^2 \ . \tag{4.37}$$

In particular the right hand side is simply proportional to the Hamiltonian which is the energy of the system:

$$\begin{aligned} H &= \frac{1}{2m}p^2 + \frac{1}{2}kq^2 \\ &= \frac{k}{2}\left(q^2 + \frac{p^2}{mk}\right) \\ &= \frac{m\omega^2}{2}\left(q^2 + \frac{p^2}{mk}\right) \end{aligned} \tag{4.38}$$

so that

$$(q(t))^2 + \frac{1}{m^2\omega^2}(p(t))^2 = \frac{2E}{m\omega^2} \ . \tag{4.39}$$

Thus from the Hamiltonian point of view the dynamical motion consists of concentric ellipses in phase space. Note that any given point in phase space lies on just one ellipse.

This is a general feature: the ellipses are known as **Hamiltonian flows** and as a consequence of the first order dynamical differential equations a given point in phase space lies on just one curve of the Hamiltonian flow since the solution to Hamilton's equations only depends on the initial value of (q, p).

4.3 Poisson Brackets and Canonical Transformations

Phase space is always even-dimensional (at least as we've constructed it here). As a result there is a useful skew-symmetric structure known as a **symplectic structure**. It is determined by the **Poisson bracket** which is defined by

$$\{f, g\} \equiv \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p^i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p^i}\right) \ , \tag{4.40}$$

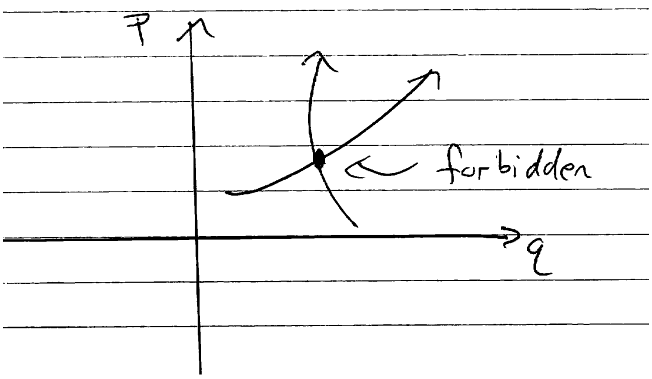

Figure 4.2.2: Forbidden Phase Space Flows

where $f = f(q_i, p^i)$ and $g = g(q_i, p^i)$ are arbitrary functions on phase space.

The Poisson bracket has several properties:

- $\{f, g\} = -\{g, f\}$.
- $\{f, g + \lambda h\} = \{f, g\} + \lambda\{f, h\}$, for $\lambda \in \mathbb{R}$
- $\{f, gh\} = \{f, g\}h + g\{f, h\}$

The first two should be obvious from the definition. The third requires a little calculation (and is given as a problem).

One can write the equations of motion using the Poisson bracket as

$$\frac{d}{dt}q = \{q_i, H\} = \frac{\partial H}{\partial p^i} \qquad \text{and} \qquad \frac{d}{dt}p = \{p^i, H\} = -\frac{\partial H}{\partial q_i}. \tag{4.41}$$

In fact for any function $f(q_i, p^i)$ on phase space we have that $\dot{f} = \{f, H\}$. To prove this we note that

$$\begin{aligned}
\{f, H\} &= \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p^i} - \frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p^i} \right) \\
&= \sum_i \left(\frac{\partial f}{\partial q_i}\frac{dq_i}{dt} + \frac{dp^i}{dt}\frac{\partial f}{\partial p^i} \right) \\
&= \frac{df}{dt} \ ,
\end{aligned} \tag{4.42}$$

if $f = f(q_i, p^i)$.

The set of Poisson brackets acting on simply q_i and p^j are known as the fundamental or canonical Poisson brackets. They have a simple form:

$$\begin{aligned}
\{q_i, p^j\} &= \delta_{ij} \\
\{q_i, q_j\} &= 0 \\
\{p^i, p^j\} &= 0 \ ,
\end{aligned} \tag{4.43}$$

which one may easily confirm by direct computation.

Definition: **Canonical transformations** $q_i \to q_i'$, $p^i \to p_i'$ are transformations which preserve fundamental Poisson brackets, *i.e.*

$$\{q_i', p_j'\} = \delta_{ij}, \quad \{q_i', q_j'\} = 0 \quad \text{and} \qquad \{p_i', p_j'\} = 0. \tag{4.44}$$

Note that q'_i and p'_i are both allowed to be functions of q_i and p^i. From the Lagrangian point of view this would be weird: it would mean redefining q'_i as a function of q_i and $\dot{q}_i$. But in the Hamiltonian formulation q_i and p^i are on equal footings. We will see that this leads to a larger class of symmetries than in the Lagrangian formulation.

The key thing about canonical transformations is that Hamilton's equations remain valid

$$\frac{d}{dt}q'_i = \frac{\partial H(q'_i, p'_i)}{\partial p'_i} \qquad \text{and} \qquad \frac{d}{dt}p'_i = -\frac{\partial H(q'_i, p'_i)}{\partial q'_i}. \tag{4.45}$$

To see this we write (4.42) for q' instead of f and we note that

$$\begin{aligned}
\frac{d}{dt}q'_i &= \{q'_i, H\} \\
&= \sum_j \left(\frac{\partial q'_i}{\partial q_j}\frac{\partial H}{\partial p^j} - \frac{\partial q'_i}{\partial p^j}\frac{\partial H}{\partial q_j} \right) \\
&= \sum_{j,k} \left(\frac{\partial q'_i}{\partial q_j}\left(\frac{\partial p'_k}{\partial p^j}\frac{\partial H}{\partial p'_k} + \frac{\partial q'_k}{\partial p^j}\frac{\partial H}{\partial q'_k} \right) - \frac{\partial q'_i}{\partial p^j}\left(\frac{\partial p'_k}{\partial q_j}\frac{\partial H}{\partial p'_k} + \frac{\partial q'_k}{\partial q_j}\frac{\partial H}{\partial q'_k} \right) \right) \\
&= \sum_{j,k} \left(\left(\frac{\partial q'_i}{\partial q_j}\frac{\partial p'_k}{\partial p^j} - \frac{\partial q'_i}{\partial p^j}\frac{\partial p'_k}{\partial q_j} \right)\frac{\partial H}{\partial p'_k} + \left(\frac{\partial q'_i}{\partial q_j}\frac{\partial q'_k}{\partial p^j} - \frac{\partial q'_i}{\partial p^j}\frac{\partial q'_k}{\partial q_j} \right)\frac{\partial H}{\partial q'_k} \right) \\
&= \sum_k \left(\{q'_i, p'^k\}\frac{\partial H}{\partial p'_k} + \{q'_i, q'_k\}\frac{\partial H}{\partial q'_k} \right) \\
&= \frac{\partial H}{\partial p'^i} \ .
\end{aligned} \tag{4.46}$$

Similarly

$$\begin{aligned}
\frac{d}{dt}p'^i &= \{p'^i, H\} \\
&= \sum_j \left(\frac{\partial p'^i}{\partial q_j}\frac{\partial H}{\partial p^j} - \frac{\partial p'^i}{\partial p^j}\frac{\partial H}{\partial q_j} \right) \\
&= \sum_{j,k} \left(\frac{\partial p'^i}{\partial q_j}\left(\frac{\partial p'^k}{\partial p^j}\frac{\partial H}{\partial p'^k} + \frac{\partial q'_k}{\partial p^j}\frac{\partial H}{\partial q'_k} \right) - \frac{\partial p'_i}{\partial p^j}\left(\frac{\partial p'_k}{\partial q_j}\frac{\partial H}{\partial p'_k} + \frac{\partial q'_k}{\partial q_j}\frac{\partial H}{\partial q'_k} \right) \right) \\
&= \sum_{j,k} \left(\left(\frac{\partial p'^i}{\partial q_j}\frac{\partial p'^k}{\partial p^j} - \frac{\partial p'^i}{\partial p^j}\frac{\partial p'_k}{\partial q_j} \right)\frac{\partial H}{\partial p'_k} + \left(\frac{\partial p'_i}{\partial q_j}\frac{\partial q'_k}{\partial p^j} - \frac{\partial p'_i}{\partial p^j}\frac{\partial q'_k}{\partial q_j} \right)\frac{\partial H}{\partial q'_k} \right) \\
&= \sum_k \left(\{p'^i, p'^k\}\frac{\partial H}{\partial p'_k} + \{p'^i, q'_k\}\frac{\partial H}{\partial q'_k} \right) \\
&= -\frac{\partial H}{\partial q'_i} \ .
\end{aligned} \tag{4.47}$$

4.4 Examples of Canonical Transformations

This might seem rather abstract and you might be thinking how in the world does one construct a canonical transformation. So let us look at this. First lets discuss one example before constructing a large class of canonical transformations.

Example: Harmonic Oscilator If we return to the Harmonic oscilator we see that

$$q' = \frac{1}{\sqrt{mk}}p \qquad p' = -\sqrt{mk}q \tag{4.48}$$

is indeed a canonical transformation:

$$\{q', p'\} = \left\{\frac{1}{\sqrt{mk}}p, -\sqrt{mk}q\right\} = -\{p, q\} = \{q, p\} = 1 \tag{4.49}$$

Furthermore $\{q', q'\} = \{p', p'\} = 0$ automatically as the Poisson bracket is anti-symmetric. Furthermore one has

$$\begin{aligned} H(q,p) &= \frac{1}{2m}\left(\sqrt{mk}q'\right)^2 + \frac{k}{2}\left(-\frac{p'}{\sqrt{mk}}\right)^2 \\ &= \frac{1}{2m}p'^2 + \frac{1}{2}kq'^2 \\ &= H(q', p') \ . \end{aligned} \tag{4.50}$$

From the point of view of the phase space diagrams this corresponds to swapping m with $1/k$ (which effectively swaps the longer and shorter radii of the ellipses), and then rotating (p, q) by 90°. This maps the ellipses of phase space back to themselves and, as we have just seen, this is a symmetry of the Hamiltonian.

However from the Lagrangian point of view this is weird: how can we swap positions with momenta? Furthermore the Lagrangian has no such symmetry to swap q with $\dot{q}$ and m with $1/k$.

Infinitesimal canonical transformation may be generated by any arbitrary function $f(q_i, p^i)$ (called the generating function) on phase space *via*

$$\begin{aligned} q_i \to q_i' &= q_i + \epsilon\{q_i, f\} \equiv q_i + \delta q_i \\ p^i \to p_i' &= p^i + \epsilon\{p^i, f\} \equiv p^i + \delta p^i \ , \end{aligned} \tag{4.51}$$

where

$$\begin{aligned} \delta q_i &= \epsilon\{q_i, f\} = \epsilon\frac{\partial f}{\partial p^i} \\ \delta p^i &= \epsilon\{p^i, f\} = -\epsilon\frac{\partial f}{\partial q_i} \ . \end{aligned} \tag{4.52}$$

Next we show that, expanding to first order in $\epsilon \ll 1$, the transformation is an infinitesimal canonical transformation. It is easy to check that this preserves the fundamental Poisson brackets up to terms of order $\mathcal{O}(\epsilon^2)$, *e.g.*

$$\begin{aligned} \{q_i', p_j'\} &= \{q_i + \epsilon\{q_i, f\}, p^j + \epsilon\{p^j, f\}\} \\ &= \{q_i, p^j\} + \epsilon\{\{q_i, f\}, p^j\} + \epsilon\{q_i, \{p^j, f\}\} + \mathcal{O}(\epsilon^2) \\ &= \{q_i, p^j\} + \epsilon(\{\frac{\partial f}{\partial p^i}, p^j\} + \{q_i, -\frac{\partial f}{\partial q_j}\}) + O(\epsilon^2) \\ &= \delta_{ij} + \epsilon\left(\frac{\partial^2 f}{\partial q_j \partial p^i} - \frac{\partial^2 f}{\partial p^i \partial q_j}\right) + \mathcal{O}(\epsilon^2) \\ &= \delta_{ij} + \mathcal{O}(\epsilon^2). \end{aligned} \tag{4.53}$$

We must also check that

$$\begin{aligned}\{q_i', q_j'\} &= \{q_i + \epsilon\{q_i, f\}, q_j + \epsilon\{q_j, f\}\} \\ &= \{q_i, q_j\} + \epsilon\{\{q_i, f\}, q_j\} + \epsilon\{q_i, \{q_j, f\}\} + \mathcal{O}(\epsilon^2) \\ &= \{q_i, q_j\} + \epsilon(\{\frac{\partial f}{\partial p^i}, q_j\} + \{q_i, \frac{\partial f}{\partial p^j}\}) + O(\epsilon^2) \\ &= 0 + \epsilon\left(-\frac{\partial^2 f}{\partial p^j \partial p^i} + \frac{\partial^2 f}{\partial p^i \partial p^j}\right) + \mathcal{O}(\epsilon^2) \\ &= 0 + \mathcal{O}(\epsilon^2).\end{aligned} \tag{4.54}$$

and

$$\begin{aligned}\{p_i', p_j'\} &= \{p^i + \epsilon\{p^i, f\}, p^j + \epsilon\{p^j, f\}\} \\ &= \{p^i, p^j\} + \epsilon\{\{p^i, f\}, p^j\} + \epsilon\{p^i, \{p^j, f\}\} + \mathcal{O}(\epsilon^2) \\ &= \{p^i, p^j\} + \epsilon(-\{\frac{\partial f}{\partial q_i}, p^j\} + \{p^i, -\frac{\partial f}{\partial q_j}\}) + O(\epsilon^2) \\ &= 0 + \epsilon\left(-\frac{\partial^2 f}{\partial q_j \partial q_i} + \frac{\partial^2 f}{\partial q_i \partial q_j}\right) + \mathcal{O}(\epsilon^2) \\ &= 0 + \mathcal{O}(\epsilon^2).\end{aligned} \tag{4.55}$$

Canonical transformations therefore generalise simple coordinate transformations of the form $q_i' = q_i'(q_i)$ to also allow for transformations that define q_i' in terms of q_i and p^i. In particular when we looked at Noether's theorem we considered infinitesimal change of variables of the form

$$q_i' = q_i + \epsilon T_i(q) \qquad i.e. \qquad \delta q_i = \epsilon T_i \tag{4.56}$$

where $\epsilon << 1$. In the Lagrangian formulation we can use this to compute the change in p^i:

$$\begin{aligned}p^{i\prime} &= \frac{\partial L}{\partial \dot{q}_i'} \\ &= \sum_j \frac{\partial \dot{q}_j}{\partial \dot{q}_i'}\frac{\partial L}{\partial \dot{q}_j} + \frac{\partial q_j}{\partial \dot{q}_i'}\frac{\partial L}{\partial q_j} \\ &= \sum_j \frac{\partial \dot{q}_j}{\partial \dot{q}_i'}\frac{\partial L}{\partial \dot{q}_j}\end{aligned} \tag{4.57}$$

where we have used the fact that q_i and $\dot{q}_i$ are independent in the Lagrangian formulation. To proceed we note that, to first order in ϵ, we can write

$$\begin{aligned}\dot{q}_j &= \dot{q}_j' - \epsilon \dot{T}_j(q') \\ &= \dot{q}_j' - \epsilon\sum_k \frac{\partial T_j(q')}{\partial q_k}\dot{q}_k\end{aligned} \tag{4.58}$$

so that

$$\begin{aligned}\frac{\partial \dot{q}_j}{\partial \dot{q}_i'} &= \delta_{ij} - \epsilon\sum_k \frac{\partial T_j}{\partial q_k'}\delta_{ki} - \epsilon\sum_k \frac{\partial^2 T_j}{\partial q_k \partial \dot{q}_i'}\dot{q}_k \\ &= \delta_{ij} - \epsilon\frac{\partial T_j}{\partial q_i'}\ ,\end{aligned} \tag{4.59}$$

where we used the fact that T does not depend on $\dot{q}_i$ so that the second derivative term vanishes. Therefore

$$\begin{aligned} p'^i &= \frac{\partial L}{\partial \dot{q}'_i} \\ &= \sum_j \frac{\partial \dot{q}_j}{\partial \dot{q}'_i}\frac{\partial L}{\partial \dot{q}_j} \\ &= \sum_j \left(\delta_{ij} - \epsilon\frac{\partial T_j}{\partial q_i}\right)\frac{\partial L}{\partial \dot{q}_j} \\ &= p^i - \epsilon\sum_j \frac{\partial T_j(q)}{\partial q_i} p^j \ , \end{aligned} \tag{4.60}$$

where we have dropped all terms that are higher order in ϵ.

To see that these transformations are canonical we can just take the generating function f to be

$$f = \sum_j p^j T_j(q) \tag{4.61}$$

(note that this is just the Noether charge (3.147) written in terms of p^j) so that

$$\begin{aligned} \delta q_i &= \epsilon\sum_j \{q_i, p^j T_j(q)\} \\ &= \epsilon\sum_j \{q_i, p^j\} T_j(q) + \epsilon\sum_j \{q_i, T_j(q)\} p^j \\ &= \epsilon\sum_j \delta_{ij} T_j(q) \\ &= \epsilon T_i(q) \ , \end{aligned} \tag{4.62}$$

whereas for δp^i we find

$$\begin{aligned} \delta p^i &= \epsilon\sum_j \{p^i, p^j T_j(q)\} \\ &= \epsilon\sum_j \{p^i, p^j\} T_j(q) + \epsilon\sum_j p^j\{p^i, T_j(q)\} \\ &= -\epsilon\sum_j p^j\frac{\partial T_j(q)}{\partial q_i} \ . \end{aligned} \tag{4.63}$$

Which agrees with what we found using the Lagrangian formulation. Thus an infinitesimal coordinate transformation in the Lagrangian formulation leads to an infinitesimal canonical transformation in the Hamiltonian formulation where the generating function f is linear in momenta.

However clearly there are many more types of canonical transformations in the Hamiltonian formulation since f can be an arbitrary function of both q_i and p^i. However the Lagrangian formulation only sees canonical transformations that are generated by a function f that is linear in the momentum variables.

4.5 Symmetries and Noether's Theorem

Let us now define what we mean by a symmetry in the Hamiltonian formulation. A **symmetry** is a canonical transformation $q \to q'_i(q_i, p^i)$, $p^i \to p'^i(q_i, p^i)$ such that

$$
\begin{aligned}
&H(q'_i, p'^i) = H(q_i, p^i) \\
&\{q'_i, p'^j\} = \delta_i^j \ , \qquad \{q'_i, q'_j\} = \{p'^i, p'^j\} = 0
\end{aligned}
$$

Example: Harmonic Oscilator Let us return again to the harmonic oscilator. In fact any rotation (along with an appropriate rescaling of coordinates) is a symmetry. In particular if

$$
\begin{aligned}
q' &= \cos\alpha q + \frac{1}{\sqrt{mk}} p \sin\alpha \\
p' &= \cos\alpha p - \sqrt{mk} q \sin\alpha
\end{aligned} \tag{4.64}
$$

then

$$
\begin{aligned}
H(q', p') &= \frac{1}{2m}(\cos\alpha p - \sqrt{mk}\sin\alpha q)^2 + \frac{k}{2}(\cos\alpha q + \frac{1}{\sqrt{mk}} p \sin\alpha)^2 \\
&= \frac{1}{2m}(\cos^2\alpha + \sin^2\alpha)p^2 + \frac{k}{2}(\cos^2\alpha + \sin^2\alpha)q^2 \\
&= H(q, p)
\end{aligned} \tag{4.65}
$$

To see whether or not it is canonical we again compute ($\{q', q'\} = \{p', p'\} = 0$ automatically)

$$
\begin{aligned}
\{q', p'\} &= \{\cos\alpha q + \frac{1}{\sqrt{mk}} p \sin\alpha, \cos\alpha p - \sqrt{mk} q \sin\alpha\} \\
&= \cos^2\alpha \{q, p\} - \sin^2\alpha \{p, q\} \\
&= \{q, p\}
\end{aligned} \tag{4.66}
$$

So this is indeed a canonical transformation. In fact if we start with a the solutions we found before:

$$
\begin{aligned}
q(t) &= \frac{p(0)}{m\omega}\sin(\omega t) + q(0)\cos(\omega t) \\
p(t) &= p(0)\cos(\omega t) - m q(0)\omega \sin(\omega t) \ .
\end{aligned} \tag{4.67}
$$

then the new solutions are (recall $\omega = \sqrt{k/m}$ and some trig identities)

$$
\begin{aligned}
q'(t) &= \frac{p(0)}{m\omega}\sin(\omega t)\cos\alpha + q(0)\cos(\omega t)\cos\alpha + \frac{p(0)}{\sqrt{mk}}\cos(\omega t)\sin\alpha - \frac{mq(0)}{\sqrt{mk}}\omega\sin(\omega t)\sin\alpha \\
&= \frac{p(0)}{m\omega}\left(\sin(\omega t)\cos\alpha + \cos(\omega t)\sin\alpha\right) + q(0)\left(\cos(\omega t)\cos\alpha - \sin(\omega t)\sin\alpha\right) \\
&= \frac{p(0)}{m\omega}\sin(\omega t + \alpha) + q(0)\cos(\omega t + \alpha) \\
p'(t) &= p(0)\cos(\omega t)\cos\alpha - m q(0)\omega\sin(\omega t)\sin\alpha - \sqrt{mk}\frac{p(0)}{m\omega}\sin(\omega t)\sin\alpha - \sqrt{mk} q(0)\cos(\omega t)\sin\alpha \\
&= p(0)\left(\cos(\omega t)\cos\alpha - \sin(\omega t)\sin\alpha\right) - m\omega q(0)\left(\sin(\omega t)\sin\alpha + \cos(\omega t)\sin\alpha\right) \\
&= p(0)\cos(\omega t + \alpha) - m q(0)\omega\sin(\omega t + \alpha) \ .
\end{aligned} \tag{4.68}
$$

Thus we see that the canonical transformations corresponding to rotations are simply time translations. Furthermore the Hamiltonian itself is the function that generates these transformations infinitesimally (up to a factor of ω that can be absorbed into α):

$$\begin{aligned}
\delta q &= \frac{\alpha}{\omega}\{q, H\} = \frac{\alpha}{2m\omega}\{q, p^2\} = \frac{\alpha}{\sqrt{mk}}p \\
\delta p &= \frac{\alpha}{\omega}\{p, H\} = \frac{k\alpha}{2\omega}\{p, q^2\} = -\alpha\sqrt{mk}q
\end{aligned}$$

which agrees with (4.64) to first order in α.

In fact this is a general result. That is to say if at time t a Hamiltonian system is at $q_i(t), p^i(t)$ then at time $t+\epsilon$ the system is in

$$\begin{aligned}
q_i(t+\epsilon) &= q_i(t) + \epsilon\frac{d}{dt}q_i(t) & p^i(t+\epsilon) &= p^i(t) + \epsilon\frac{d}{dt}p^i(t) \\
&= q_i(t) + \epsilon\{q_i, H\} & &= p^i(t) + \epsilon\{p^i, H\}
\end{aligned} \tag{4.69}$$

Thus time evolution is just a series of infinitesimal canonical transformations on phase space.

In particular if the infinitesimal canonical transformation generated by a function f on phase space is a symmetry of the Hamiltonian then $\delta H = 0$ under the transformation. Now,

$$\begin{aligned}
\delta H &= \sum_i \left(\frac{\partial H}{\partial q_i}\delta q_i + \frac{\partial H}{\partial p^i}\delta p^i\right) \\
&= \epsilon\sum_i \left(\frac{\partial H}{\partial q_i}\{q_i, f\} + \frac{\partial H}{\partial p^i}\{p^i, f\}\right) \\
&= \epsilon\sum_i \left(\frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p^i} - \frac{\partial H}{\partial p^i}\frac{\partial f}{\partial q_i}\right) \\
&= \epsilon\{H, f\} \\
&= -\epsilon\frac{df}{dt} \ ,
\end{aligned} \tag{4.70}$$

where we have assumed that f is an explicit function of the phase space variables and not time, *i.e.* $\frac{\partial f}{\partial t} = 0$. Hence if the transformation is a symmetry $\delta H = 0$ then $f(q_i, p^i)$ is a conserved quantity. Thus Noether's theorem is if and only in the Hamiltonian formulation: every conserved quantity generates a canonical transformation that is a symmetry and vice-versa: every (infinitesimal) canonical transformation is generated by a function f and if this is a symmetry then f is conserved. And indeed we will see that there are more symmetries that are manifest in the Hamiltonian formulation.

If the Hamiltonian has no explicit time dependence then clearly it generates a canonical transformation that is a symmetry since

$$\delta H = \epsilon\{H, H\} = 0 \ . \tag{4.71}$$

So that H itself is the conserved quantity: the total energy $E = H$.[1]

[1]Note that by this equation we mean that H is a function of q_i and p^i and can be evaluated for any path $(q_i(t), p^i(t))$. If it is evaluated on a specific flow that solves Hamilton's equations then it is a constant that we identify with the energy E of that flow.

4.6 Kepler Revisited: Phase Diagram

Let us construct the Hamiltonian for the Kepler problem. It is simply taken from the previous examples with a potential $V(\underline{r}) = -G_N Mm/|\underline{r}|$:

$$H = \frac{1}{2m}|\underline{p}|^2 - \frac{G_N Mm}{|\underline{r}|} \ , \tag{4.72}$$

where $\underline{q} = \underline{r}$ is the position of the planet. The phase space of this system is six-dimensional (3 from $\underline{p}$ and 3 $\underline{r}$). So its not easy to draw a phase diagram. However we can consider the effective one-dimensional theory for the radius $r = |\underline{r}|$. We saw before that that was (switching to $p = m\dot{r}$)

$$E = \frac{1}{2m}p^2 + \frac{l^2}{2mr^2} - \frac{G_N Mm}{r} \tag{4.73}$$

So in this reduced theory we have the Hamiltonian

$$\begin{aligned} H(r,p) &= \frac{1}{2m}\left(p^2 + \frac{l^2}{r^2} - \frac{2G_N Mm^2}{r}\right) \\ &= \frac{1}{2m}\left(p^2 + \left(\frac{l}{r} - \frac{G_N Mm^2}{l}\right)^2\right) - \frac{G_N^2 M^2 m^3}{2l^2} \ . \end{aligned} \tag{4.74}$$

We want to draw the phase diagram for this system. Recall that the phase flow consists of curves with constant $E = H$. At large r we have $E = p^2/2m$ which is independent of r, so lines of constant E have constant p. At small r (note that $r > 0$) we have $E = p^2/2m + l^2/2mr^2$ so $p = \pm\sqrt{2mE - l^2/r^2}$ which implies there is a minimum value of r: $r > l\sqrt{1/2mE}$ and then $|p|$ increases as r increases. On the other hand near $r = l^2/G_N Mm^2$ we can expand $r = l^2/G_N Mm^2 + \rho$ for small ρ to find

$$E = \frac{p^2}{2m} + \frac{G_N^4 M^4 m^7}{2l^6}\rho^2 - \frac{G_N^2 M^2 m^3}{2l^2} \tag{4.75}$$

which is like the harmonic oscillator (since we are near a minimum of the potential). A little bit of thought shows that the phase diagram looks like the figure below. In particular the closed orbits are the planets moving in ellipses with $E < 0$ and the open orbits that extend to $r \to \infty$ are asteroids with $E \geq 0$.

Although we won't necessarily need them we should, for completeness, compute Hamilton's equations:

$$\begin{aligned} \frac{d\underline{r}}{dt} &= \frac{\partial H}{\partial \underline{p}} = \frac{1}{m}\underline{p} \\ \frac{dr^a}{dt} &= \frac{1}{m}p^a \end{aligned} \tag{4.76}$$

and

$$\begin{aligned} \frac{d\underline{p}}{dt} &= -\frac{\partial H}{\partial \underline{r}} = -\frac{G_N Mm}{|\underline{r}|^3}\underline{r} \\ \frac{dp^a}{dt} &= -\frac{G_N Mm}{r^3}r^a \end{aligned} \tag{4.77}$$

where in the second lines we have written the equations in terms of the components r^a, p^a of $\underline{r}, \underline{p}$ with $r = \sqrt{(r^1)^2 + (r^2)^2 + (r^3)^2}$. If we substitute the first equation into

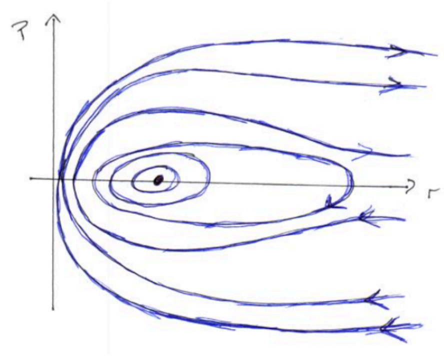

Figure 4.6.1: Phase Space For the Kepler Problem

the second then we obtain the equation of motion that we would get from NII:

$$
\begin{aligned}
m\frac{d^2\underline{r}}{dt^2} &= -\frac{G_N M m}{|\underline{r}|^3}\underline{r} \\
m\frac{d^2 r^a}{dt^2} &= -\frac{G_N M m}{r^3} r^a \ .
\end{aligned}
\tag{4.78}
$$

4.7 Kepler Revisited: Conserved Quantities

Let us look at the conserved charges. Since the Hamiltonian doesn't depend explicitly on time, $H = E$ will be conserved:

$$
\dot{E} = \{H, H\} = 0 \ . \tag{4.79}
$$

As we have seen the canonical transformation generated by $H = E$ is just time-translation: $t \to t + \epsilon$.

We also saw above that there was conservation of angular momentum $\underline{L} = \underline{r} \times \underline{p}$, or in components:

$$
L_b = \sum_{c,d} \epsilon_{bcd} r_c p_d \ . \tag{4.80}
$$

First we should check that $\{H, L_a\} = 0$. Indeed this will be true for any H of the form:

$$
H = \frac{1}{2m}\underline{p}\cdot\underline{p} + V(|\underline{r}|^2) \ . \tag{4.81}
$$

We need to evaluate

$$
\begin{aligned}
\{H, L_a\} &= \frac{1}{2m}\sum_{c,d}\epsilon_{bcd}\{|\underline{p}|^2, r_c p_d\} + \sum_{c,d}\epsilon_{bcd}\{V(|\underline{r}|^2), r_c p_d\} \\
&= \frac{1}{2m}\sum_{c,d}\epsilon_{bcd}\{|\underline{p}|^2, r_c\}p_d + \sum_{c,d}\epsilon_{bcd} r_c\{V(|\underline{r}|^2), p_d\} \\
&= \frac{1}{2m}\sum_{c,d}\epsilon_{bcd}\{|\underline{p}|^2, r_c\}p_d + \frac{dV}{d|\underline{r}|^2}\sum_{c,d}\epsilon_{bcd} r_c\{|\underline{r}|^2, p_d\}
\end{aligned}
$$

Next we simply note that

$$\{|\underline{p}|^2, r_c\} = -\sum_d \{r_c, p_d p_d\} = -2p_c \tag{4.82}$$

and similarly $\{|\underline{r}|^2, p_d\} = 2r_d$. Thus

$$\{H, L_a\} = -\frac{1}{m}\sum_{c,d} \epsilon_{bcd} p_c p_d + 2\frac{dV}{d|\underline{r}|^2}\sum_{c,d} \epsilon_{bcd} r_c r_d = 0 \ . \tag{4.83}$$

So this is indeed a conserved quantity.

Let us look at the canonical transformation generated by L_b:

$$\begin{aligned}
\delta_b r_a &= \epsilon\{r_a, L_b\} \\
&= \epsilon \sum_{c,d} \epsilon_{bcd}\{r_a, r_c p_d\} \\
&= \epsilon \sum_{c,d} \epsilon_{bcd}(\{r_a, r_c\}p_d + r_c\{r_a, p_d\}) \\
&= \epsilon \sum_{c} \epsilon_{cba} r_c \\
&= \epsilon \sum_{c} \epsilon_{abc} r_c \ ,
\end{aligned} \tag{4.84}$$

and

$$\begin{aligned}
\delta_b p_a &= \epsilon\{p_a, L_b\} \\
&= \epsilon \sum_{c,d} \epsilon_{bcd}\{p_a, r_c p_d\} \\
&= \epsilon \sum_{c,d} \epsilon_{bcd}(\{p_a, r_c\}p_d + r_c\{p_a, p_d\}) \\
&= -\epsilon \sum_{d} \epsilon_{bad} p_d \\
&= \epsilon \sum_{c} \epsilon_{abc} p_c
\end{aligned} \tag{4.85}$$

Hence for every choice of T_b a general canonical transformation generated by $L = \sum_b L_b T^b$ is just a rotation

$$\begin{aligned}
\underline{r} \to \underline{r} + \epsilon\underline{T} \times \underline{r} \qquad & \underline{p} \to \underline{p} + \epsilon\underline{T} \times \underline{p} \\
= \underline{r} + \epsilon\mathbf{T}\underline{r} \qquad & \quad = \underline{p} + \epsilon\mathbf{T}\underline{p}
\end{aligned} \tag{4.86}$$

where the matrix $\mathbf{T}$ has components $\mathbf{T}_{ac} = \sum_b \epsilon_{abc}T^b = -\sum_b \epsilon_{acb}T^b$. It is easy to see that as before these generate a symmetry of H since in particular both $|\underline{p}|^2$ and $|\underline{r}|^2$ are invariant under rotations:

$$\begin{aligned}
\delta|\underline{r}|^2 &= 2\underline{r}\cdot\delta\underline{r} \\
&= 2\sum_a r_a \delta r_a \\
&= 2\epsilon\sum_{ac} r_a \epsilon_{abc} T^b r_c \\
&= 2\epsilon\ \underline{r}\cdot(\underline{T}\times\underline{r}) \\
&= 0 \ ,
\end{aligned} \tag{4.87}$$

since $a_a r_c = r_c r_a$ but $\epsilon_{abc} = -\epsilon_{cba}$. Similarly one sees that $\delta|\underline{p}|^2 = 0$.

4.8 Kepler Revisited: Runge-Lenz

Let us also recall the Runge-Lenz vector

$$\begin{aligned}\underline{A} &= \underline{p} \times \underline{L} - \frac{G_N M m^2}{|\underline{r}|}\underline{r} \\ &= \underline{p} \times (\underline{r} \times \underline{p}) - \frac{G_N M m^2}{|\underline{r}|}\underline{r} \\ &= (\underline{p} \cdot \underline{p})\underline{r} - (\underline{r} \cdot \underline{p})\underline{p} - \frac{G_N M m^2}{|\underline{r}|}\underline{r} \end{aligned} \tag{4.88}$$

This was conserved but it does not arise from a symmetry of the Lagrangian. But by construction it generates a canonical transformation that is a symmetry of the Hamiltonian. What is it? First we write $\underline{A}$ in components:

$$A_b = r^b \sum_c p_c p_c - p^b \sum_c p_c r_c - \frac{G_N M m^2}{|\underline{r}|} r_b \ . \tag{4.89}$$

Note that this is not linear in p_a and so the associated canonical transformation is not simply a coordinate transformation of r_a. Next we evaluate

$$\begin{aligned}\delta_b r_a &= \epsilon\{r_a, A_b\} \\ &= \epsilon\{r_a, r^b \sum_c p_c p_c\} - \epsilon\{r_a, p^b \sum_c p_c r_c\} \\ &= \epsilon\left(\sum_c r^b p_c \{r_a, p_c\} + \sum_c \{r_a, p_c\} r^b p_c - \sum_c \{r_a, p^b\} p_c r_c - \sum_c p^b \{r_a, p_c\} r_c\right) \\ &= 2\epsilon r_b p_a - \epsilon\delta_{ab}(\underline{p} \cdot \underline{r}) - p^b r_a \ , \end{aligned} \tag{4.90}$$

where we have dropped terms which involve $\{r_a, r_c\}$.

For the momenta we find

$$\begin{aligned}\delta_b p_a &= \epsilon\{p_a, A_b\} \\ &= \epsilon\{p_a, r^b\} \sum_c p_c p_c - \epsilon p^b \sum_c \{p_a, r_c\} p_c - \epsilon G_N M m \{p_a, r_b/|\underline{r}|\} \\ &= -\epsilon\delta_{ab}|\underline{p}|^2 + \epsilon p_a p_b - \epsilon \frac{G_N M m}{|\underline{r}|}\{p_a, r_b\} - \epsilon G_N M m^2 r_b \{p_a, |\underline{r}|^{-1}\} \ . \end{aligned} \tag{4.91}$$

To evaluate the last term we can use the chain rule

$$\begin{aligned}\{p_a, |\underline{r}|^{-1}\} &= \sum_d \left(\frac{\partial p_a}{\partial r_d}\frac{\partial |\underline{r}|^{-1}}{\partial p_d} - \frac{\partial p_a}{\partial p_d}\frac{\partial |\underline{r}|^{-1}}{\partial r_d}\right) \\ &= -\frac{\partial |\underline{r}|^{-1}}{\partial r_a} \\ &= \frac{r_a}{|\underline{r}|^3} \ , \end{aligned} \tag{4.92}$$

so that

$$\delta_b p_a = -\epsilon\delta_{ab}|\underline{p}|^2 + \epsilon p_a p_b + \epsilon\frac{G_N M m^2}{|\underline{r}|}\delta_{ab} - \epsilon\frac{G_N M m^2}{|\underline{r}|^3} r_b r_a \ . \tag{4.93}$$

If we consider a general canonical transformation generated by $A = \sum_b A_b U^b$ we have

$$\begin{aligned}\delta\underline{r} &= 2\epsilon(\underline{U} \cdot \underline{r})\underline{p} - \epsilon(\underline{p} \cdot \underline{r})\underline{U} - \epsilon(\underline{U} \cdot \underline{p})\underline{r} \\ \delta\underline{p} &= \epsilon\underline{U}\left(-|\underline{p}|^2 + \frac{G_M m^2}{|\underline{r}|}\right) + \epsilon(\underline{U} \cdot \underline{p})\underline{p} - \epsilon\frac{G_N M m^2}{|\underline{r}|^3}(\underline{U} \cdot \underline{r})\underline{r} \ . \end{aligned} \tag{4.94}$$

To see that this is indeed a symmetry of H we first note that

$$\delta H = \frac{1}{m}\underline{p}\cdot\delta\underline{p} + \frac{G_N Mm}{|\underline{r}|^3}\underline{r}\cdot\delta\underline{r} \ . \tag{4.95}$$

Next we compute

$$\begin{aligned}
\underline{p}\cdot\delta\underline{p} &= \epsilon(\underline{U}\cdot\underline{p})\left(-|\underline{p}|^2 + \frac{G_N Mm^2}{|\underline{r}|}\right) + \epsilon(\underline{U}\cdot\underline{p})|\underline{p}|^2 - \epsilon\frac{G_N Mm^2}{|\underline{r}|^3}(\underline{U}\cdot\underline{r})(\underline{r}\cdot\underline{p}) \\
&= \epsilon\frac{G_N Mm^2}{|\underline{r}|}(\underline{U}\cdot\underline{p}) - \epsilon\frac{G_N Mm^2}{|\underline{r}|^3}(\underline{U}\cdot\underline{r})(\underline{r}\cdot\underline{p}) \ ,
\end{aligned} \tag{4.96}$$

so

$$\frac{1}{m}\underline{p}\cdot\delta\underline{p} = \epsilon\frac{G_N Mm}{|\underline{r}|}(\underline{U}\cdot\underline{p}) - \epsilon\frac{G_N Mm}{|\underline{r}|^3}(\underline{U}\cdot\underline{r})(\underline{r}\cdot\underline{p}) \ , \tag{4.97}$$

On the other hand

$$\begin{aligned}
\frac{G_N Mm}{|\underline{r}|^3}\underline{r}\cdot\delta\underline{r} &= 2\epsilon\frac{G_N Mm}{|\underline{r}|^3}(\underline{U}\cdot\underline{r})(\underline{p}\cdot\underline{r}) - \epsilon\frac{G_N Mm}{|\underline{r}|^3}(\underline{U}\cdot\underline{r})(\underline{p}\cdot\underline{r}) \\
&\quad - \epsilon\frac{G_N Mm}{|\underline{r}|^3}(\underline{U}\cdot\underline{p})(\underline{r}\cdot\underline{r}) \\
&= \epsilon\frac{G_N Mm}{|\underline{r}|^3}(\underline{U}\cdot\underline{r})(\underline{p}\cdot\underline{r}) - \epsilon\frac{G_N Mm}{|\underline{r}|}(\underline{U}\cdot\underline{p}) \\
&= -\frac{1}{m}\underline{p}\cdot\delta\underline{p}
\end{aligned} \tag{4.98}$$

Therefore these two cancel so that $\delta H = 0$ as required. You would be hard-pressed to guess this symmetry! Note that it mixes $\underline{p}$ into $\underline{r}$ and vice-versa. Therefore one would not see it as a symmetry of the Lagrangian. But it explains why we find the conserved Runge-Lenz vector.

For completeness we should also show that $\{H, A_b\} = 0$. In fact this follows from the fact that $\delta H = 0$ since

$$\begin{aligned}
\delta H &= \sum_a \frac{\partial H}{\partial r_a}\delta r_a + \sum_a \frac{\partial H}{\partial p_a}\delta p_a \\
&= \epsilon\sum_a \frac{\partial H}{\partial r_a}\{r_a, \underline{U}\cdot\underline{A}\} + \sum_a \frac{\partial H}{\partial p_a}\{p_a, \underline{U}\cdot\underline{A}\} \\
&= \epsilon\sum_a\left(\frac{\partial H}{\partial r_a}\frac{\partial(\underline{U}\cdot\underline{A})}{\partial p_a} - \frac{\partial H}{\partial p_a}\frac{\partial(\underline{U}\cdot\underline{A})}{\partial q_a}\right) \\
&= \epsilon\{H, \underline{U}\cdot\underline{A}\} \\
&= \epsilon\underline{U}\cdot\{H, \underline{A}\} \ .
\end{aligned} \tag{4.99}$$

Since $\delta H = 0$ for arbitrary $\underline{U}$ we see that $\{H, \underline{A}\} = \underline{0}$.

However for those interested we can also do the full computation:

$$\{A_b, H\} = \{r^b\sum_c p_c p_c, H\} - \{p^b\sum_c p_c r_c, H\} - \{\frac{G_N Mm^2}{|\underline{r}|}r_b, H\} \ . \tag{4.100}$$

Expanding this out will give six terms, so this is going to be complicated. To break up our calculation into smaller pieces we note that this must be true without imposing any equations of motion. Therefore it must be separately true for the terms that are proportional to various different powers of G_N: G_N^0, G_N^1 and G_N^2. The terms that are independent of G_N arise from taking $H = \frac{1}{2m}|\underline{p}|^2$ as well as dropping the last term:

$$\begin{aligned}
\{A_b, H\}_{G_N^0} &= \frac{1}{2m}\{r^b \sum_c p_c p_c, \sum_d p_d p_d\} - \frac{1}{2m}\{p^b \sum_c p_c r_c, \sum_d p_d p_d\} \\
&= \frac{1}{2m}\sum_{c,d} p_c p_c \{r^b, p_d p_d\} - \frac{1}{2m} p^b \sum_{c,d} p_c \{r_c, p_d p_d\} \\
&= \frac{1}{m}\sum_{c,d} p_c p_c \{r^b, p_d\} p_d - \frac{1}{m} p^b \sum_{c,d} p_c \{r_c, p_d\} p_d \\
&= \frac{1}{m}\sum_{c,d} p_c p_c \delta_{bd} p_d - \frac{1}{m} p^b \sum_{c,d} p_c \delta_{cd} p_d \\
&= \frac{1}{m}\sum_{c} p_c p_c p_b - \frac{1}{m} p^b \sum_{c} p_c p_c \\
&= 0 \ .
\end{aligned} \tag{4.101}$$

Next we look at the terms that are proportional to G_N (but not G_N^2). In the first two terms these come from taking $H = -G_N Mm/|\underline{r}|$ but we must take $H = \frac{1}{2m}|\underline{p}|^2$ in the third term:

$$\begin{aligned}
\{A_b, H\}_{G_N^1} &= -\{r^b \sum_c p_c p_c, \frac{G_N Mm}{|\underline{r}|}\} + \{p^b \sum_c p_c r_c, \frac{G_N Mm}{|\underline{r}|}\} - \sum_c \{\frac{G_N Mm^2}{|\underline{r}|} r_b, \frac{1}{2m} p_c p_c\} \\
&= G_N Mm \left(-r^b \sum_c \{p_c p_c, |\underline{r}|^{-1}\} + \sum_c r_c \{p^b p_c, |\underline{r}|^{-1}\} - \frac{1}{2}\sum_c \{|\underline{r}|^{-1} r_b, p_c p_c\} \right) \\
&= G_N Mm \left(-2r^b \sum_c p_c \{p_c, |\underline{r}|^{-1}\} + \sum_c r_c \{p^b p_c, |\underline{r}|^{-1}\} - \sum_c \{|\underline{r}|^{-1} r_b, p_c\} p_c \right) \\
&= G_N Mm \Big(-2r^b \sum_c p_c \{p_c, |\underline{r}|^{-1}\} + \sum_c r_c p_c \{p^b, |\underline{r}|^{-1}\} + \sum_c r_c p^b \{p_c, |\underline{r}|^{-1}\} \\
&\qquad - \sum_c r_b \{|\underline{r}|^{-1}, p_c\} p_c - |\underline{r}|^{-1} \sum_c \{r_b, p_c\} p_c \Big)
\end{aligned} \tag{4.102}$$

Next we recall that

$$\{p_d, |\underline{r}|^{-1}\} = -\frac{\partial |\underline{r}|^{-1}}{\partial r_d} = |\underline{r}|^{-3} r_d \tag{4.103}$$

so that

$$\begin{aligned}
\{A_b, H\}_{G_N^1} &= \frac{G_N Mm}{|\underline{r}|^3}\left(-2r^b \sum_c p_c r_c + \sum_c r_c p_c r^b + \sum_c r_c p^b r_c + \sum_c r_b r_c p_c - |\underline{r}|^2 \sum_c \delta_{bc} p_c \right) \\
&= \frac{G_N Mm}{|\underline{r}|^3}\left(-2r^b \sum_c p_c r_c + 2\sum_c r_c p_c r^b + |\underline{r}|^2 p_b - |\underline{r}|^2 p_b \right) \\
&= 0
\end{aligned} \tag{4.104}$$

Finally there is a term that is proportional to G_N^2 that comes from taking $H = -G_N Mm/|\underline{r}|$ in the last term:

$$\begin{aligned}
\{A_b, H\}_{G_N^2} &= \{\frac{G_N Mm^2}{|\underline{r}|} r_b, \frac{G_N Mm}{|\underline{r}|}\} \\
&= 0
\end{aligned} \tag{4.105}$$

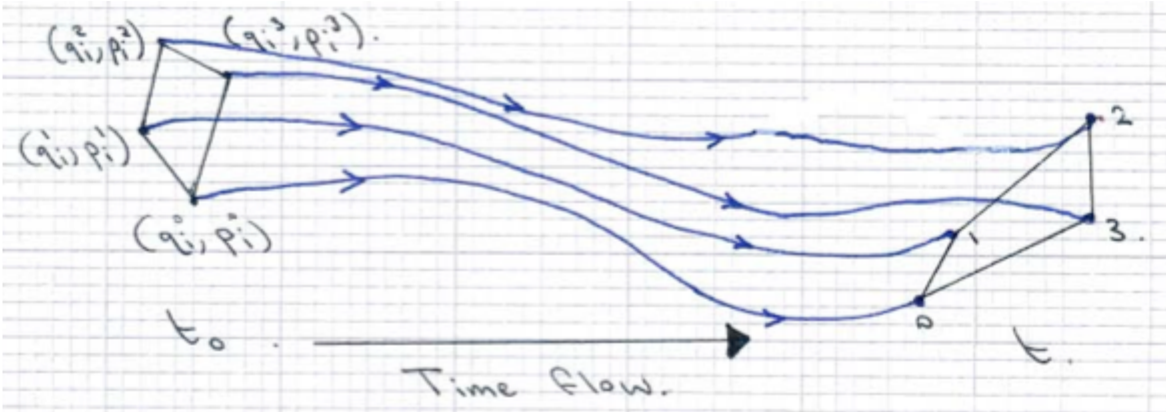

Figure 4.9.1: Time Evolution Of A Region Of Phase Space

But this is clearly zero as no p_a appears in the Poisson bracket. Thus indeed $\{A_b, H\} = 0$.

It is worth commenting that conservation of angular momentum arises from rotational symmetry which is generated by 3×3 special orthogonal matrices known as $SO(3)$. The conservation of the Runge-Lenz also gives rise to a separate "rotational" symmetry. Here we use quotes as the canonical transformations generated by the Runge-Lenz vector are not simply rotations that preserve the lengths of $\underline{r}$ and $\underline{p}$ however they are still associated with an $SO(3)$ (viewed as a group - if you know what that means). So the Kepler Hamiltonian (or the Hamiltonian for a Hydrogen atom which is structurally the same just with different constants) has an $SO(3) \times SO(3)$ symmetry. It is amusing to note that (ignoring some subtlties) $SO(3) \times SO(3) \cong SO(4)$ so it is as if there were an a hidden extra dimension of space.

4.9 Liouville's Theorem

Hamiltonian's equations define time evolution as a flow in phase space, known as the Hamiltonian flow. One of the most important features is that since Hamilton's equations are first order in time, given an initial point (p^i, q_i) there is a unique flow that passes through that point at time $t = 0$. If we start at a nearby point at $t = 0$ then we will flow along a different path. If two paths ever intersect at a point then they must be the same path everywhere (here we are assuming that there is no explicit time dependence in the Hamiltonian). To see this we stop the motion at the point were the two paths meet. Then restart the time evolution. Since it is first order the subsequent time evolution of each path must be the same. In addition we can run the Hamiltonian flow backwards to deduce that the flows must have been the same in the past.

On the other hand we could instead consider region of phase space, not just a single point. This region will then also evolve smoothly in time since no two points in it will ever intersect. Why might we be interested in this? Well since one can never measure anything with 100% precision one never really quite knows which point in phase space a system is in. By looking at regions of phase space we allow for a certain amount of experimental error. It also touches on the subject of chaos which mathematically means sensitivity to initial conditions. In particular one can ask what happens to points in phase space that start off near each other, will they stay nearby?

Here we encounter a theorem:

Liouville's Theorem: The volume of a region of phase space is constant along a Hamiltonian flow. Here volume means

$$Volume(R) = \int_R dq_1...dq_N dp_1...dp_N = \int_R dV \ , \tag{4.106}$$

where R is the region under consideration. So in general it is a high-dimensional integral in $2N$ dimensions and not simply a 'volume' in as we encounter in three-dimensions that is say measured in litres.

Proof: We wish to show that dV is time-independent. Under time evolution

$$\begin{aligned} q_i(t) &\to q_i(t+\epsilon) = q_i + \epsilon\frac{dq_i}{dt} = q_i + \epsilon\frac{\partial H}{\partial p^i} = q_i' \\ p^i(t) &\to p^i(t+\epsilon) = p^i + \epsilon\frac{dp_i}{dt} = p^i - \epsilon\frac{\partial H}{\partial q_i} = p_i' \ . \end{aligned} \tag{4.107}$$

Thus the volume at time $t+\epsilon$ is

$$\begin{aligned} dV' &= dq_1'...dq_N' dp_1'...dp_N' \\ &= \det(\mathbf{J}) dq_1...dq_N dp_1...dp_N \ , \end{aligned} \tag{4.108}$$

where

$$\begin{aligned} \mathbf{J} &= \begin{pmatrix} \frac{\partial q_i'}{\partial q_j} & \frac{\partial q_i'}{\partial p^j} \\ \frac{\partial p_i'}{\partial q_j} & \frac{\partial p_i'}{\partial p^j} \end{pmatrix} \\ &= \begin{pmatrix} \delta^i_j + \epsilon\frac{\partial^2 H}{\partial p^i \partial q_j} & \epsilon\frac{\partial^2 H}{\partial p^i \partial p^j} \\ -\epsilon\frac{\partial^2 H}{\partial q_i \partial q_j} & \delta^i_j - \epsilon\frac{\partial^2 H}{\partial q_i \partial p^j} \end{pmatrix} \ . \end{aligned} \tag{4.109}$$

Let us write this as

$$\mathbf{J} = \mathbf{1} + \epsilon\mathbf{X} \qquad \mathbf{X} = \begin{pmatrix} \frac{\partial^2 H}{\partial p^i \partial q_j} & \frac{\partial^2 H}{\partial p^i \partial p^j} \\ -\frac{\partial^2 H}{\partial q_i \partial q_j} & -\frac{\partial^2 H}{\partial q_i \partial p^j} \end{pmatrix} \ . \tag{4.110}$$

To continue we need to use the relation, valid to first order in ϵ,

$$\det(\mathbf{J}) = \mathbf{1} + \epsilon \mathrm{tr}(\mathbf{X}) \ . \tag{4.111}$$

In fact this is the first order in ϵ form of the relation:

$$\det e^{\mathbf{A}} = e^{\mathrm{tr}\mathbf{A}} \ , \tag{4.112}$$

with $A = \epsilon\mathbf{X}$ One quick way to see this relation, or at least convince oneself it's true, is to note that it is a basis independent. Let us assume that there is a basis where $\mathbf{A}$ is diagonal with eigenvalues $a_1, ..., a_n$ then

$$e^{\mathbf{A}} = \begin{pmatrix} e^{a_1} & 0 & & \\ 0 & e^{a_2} & & \\ & & \ddots & \\ & & & e^{a_n} \end{pmatrix} \tag{4.113}$$

Thus

$$\det e^{\mathbf{A}} = e^{a_1} e^{a_2} \dots e^{a_n} = e^{a_1+a_2+\dots+a_n} = e^{\mathrm{tr}\mathbf{A}} \ . \tag{4.114}$$

Let us prove this relation at order ϵ for a generic matrix $\mathbf{A} = \epsilon\mathbf{X}$ by induction. It is clearly true for a 1×1 matrix and let us assume it is true for an $n\times n$ matrix. Consider an $(n+1)\times(n+1)$ matrix then

$$\begin{aligned}\det \mathbf{J} &= J_{11}\det\hat{\mathbf{J}}_{11} - J_{12}\det\hat{\mathbf{J}}_{12} + \dots + (-1)^n J_{1n+1}\det\hat{\mathbf{J}}_{1n+1} \\ &= (1+\epsilon X_{11})\det\hat{\mathbf{J}}_{11} - \epsilon X_{12}\det\hat{\mathbf{J}}_{12} + \dots + (-1)^n\epsilon X_{1n+1}\det\hat{\mathbf{J}}_{1n+1} \ ,\end{aligned} \tag{4.115}$$

where $\hat{\mathbf{J}}_{ij}$ is the reduced matrix with the ith row and jth column deleted whereas J_{ij} is the ij entry of $\mathbf{J}$. Next we note that we only need to go to first order in ϵ so that in all but the first term we can replace $\hat{\mathbf{J}}_{ij}$ with $\hat{\mathbf{1}}_{ij}$:

$$\begin{aligned}\det \mathbf{J} &= (1+\epsilon X_{11})\det\hat{\mathbf{J}}_{11} - \epsilon X_{12}\det\hat{\mathbf{1}}_{12} + \dots(-1)^n\epsilon X_{1n+1}\det\hat{\mathbf{1}}_{1n+1} \\ &= (1+\epsilon X_{11})\det\hat{\mathbf{J}}_{11} \ ,\end{aligned} \tag{4.116}$$

the second line follows because $\det\hat{\mathbf{1}}_{ij} = 0$ if $i\neq j$ (there will always be a row of zeros somewhere in $\hat{\mathbf{1}}_{ij}$ if $i\neq j$). Thus using the induction hypothesis on $\hat{\mathbf{J}}_{11} = \hat{\mathbf{1}}_{11} + \epsilon\hat{\mathbf{X}}_{11}$ and dropping terms that are higher order in ϵ, we find

$$\begin{aligned}\det \mathbf{J} &= (1+\epsilon X_{11})(1+\epsilon\mathrm{tr}(\hat{\mathbf{X}}_{11})) \\ &= (1+\epsilon X_{11})(1+\epsilon(X_{22}+X_{33}+\dots+X_{n+1n+1})) \\ &= (1+\epsilon X_{11}) + \epsilon(X_{22}+X_{33}+\dots+X_{n+1n+1}) \\ &= 1+\epsilon\mathrm{tr}\mathbf{X} \ .\end{aligned} \tag{4.117}$$

Evaluating our expressions we obtain

$$\begin{aligned}\det \mathbf{J} &= 1 + \epsilon\sum_i \frac{\partial^2 H}{\partial q_i \partial p^i} - \epsilon\sum_i \frac{\partial^2 H}{\partial p^i \partial q_i} \\ &= 1 \ .\end{aligned} \tag{4.118}$$

Thus $dV' = dV$ so the volume is preserved by the flow. Note that the Hamiltonian appeared here because it generates time translations but was otherwise arbitrary. Thus we see that more generally any infinitesimal canonical transformation preserves the phase space volume: $dV' = dV$. As an exercise you should convince yourself of this.

4.10 Poincare Recurrence

Our last topic is enigmatic **Poincare Recurrence Theorem**: If the phase space of a system is bounded then given any open neighbourhood B of a point (q_i, p^i) then a system that starts at (q_i, p^i) will return to B in a finite time.

Proof. Let $V(t)$ be the volume of phase space that is swept out by the initial region B in time t. Since the volume is preserved we have

$$\frac{dV}{dt} = C_B \ , \tag{4.119}$$

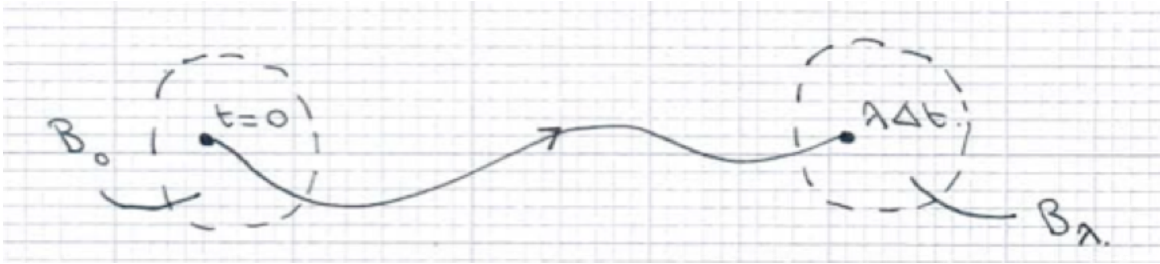

Figure 4.10.1: Poincare Recurrance

where $C_B \geq 0$ is constant and hence the total volume swept out at time t is of the form

$$V(t) = V(0) + C_B t \ . \tag{4.120}$$

If no point in B ever returns to B then the volume swept out by B will grow linearly in time. However since the total phase space volume is finite, $V(t)$ cannot become arbitrarily large. This is a contradiction which means that at least a finite volume part of B must return to B in finite time.

Note that this refers to late times. Given a finite region B of phase space at $t = 0$ then in a small enough time step $B(t)$ will naturally still intersect $B(0)$ as all the points in B will only have shifted a small amount and hence some must still remain in B. However after a finite time, call it t_1 $B(t_1)$ will generically[2] no longer intersect $B(0)$. The important part of this claim is that there must be a later time $t_{recurrence} >> t_1$ where $B(t_{recurrence})$ again intersects $B(0)$.

Next we have to ask about any remaining part of B that does not return, call it B'. If B' has finite volume then we repeat the argument above applied to B' and reach another contradiction: at least a finite volume part of B' must return to B', otherwise it would go on to sweep out an infinite volume of phase space. Thus there is at most a zero-volume set in B that does not return to B eventually.

Put another way given even the smallest region of phase space, so long as it has non-zero volume, it will sweep out an arbitrarily large region of phase space unless it eventually intersects itself again. Since there isn't an infinite amount of phase space to cover, it follows that it must intersect itself again. Thus a system either repeats its motion in phase space or evolves to fill up a subregion of the phase space, or all of it. In the last case the system is said to be **ergodic**.

Note that it is sufficient that the subsets of phase space corresponding to constant H are bounded as the system is constrained (assuming no explicit time dependence) to evolve such that H is fixed.

We can illustrate this with the simple pendulum. In the case of a single pendulum we saw that the motion was periodic (at least for small oscillations) with frequency $\omega = \sqrt{g/l}$. For example one solution is

$$\theta = A \sin \omega t \qquad\qquad p_\theta = mA\omega \cos \omega t \tag{4.121}$$

[2]Generically because one could chose B so large that it always self-intersects and then the theorem is trivially true.

which has $\theta = 0$ and $p_\theta = Aml^2\omega$ at $t = 0$. Clearly every single point in phase space returns to itself after a time $2\pi/\omega$. Note that in phase space we also require that the momenta return to themselves which means $t = 2\pi n/\omega$ and not just $t = \pi n/\omega$.

For the double pendulum we saw that a single point never returns to itself as the ratio of the periods of the normal modes was irrational. For example one solution is

$$\begin{pmatrix} \theta \\ \phi \end{pmatrix} = \begin{pmatrix} A\cos\omega_1 t + A\cos\omega_2 t \\ \sqrt{2}A\cos\omega_1 t - \sqrt{2}A\cos\omega_2 t \end{pmatrix}$$
$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} -ml^2A\omega_1\sin\omega_1 t - ml^2A\omega_2\sin\omega_2 t \\ \sqrt{2}ml^2A\omega_1\sin\omega_1 t + \sqrt{2}ml^2A\omega_2\sin\omega_2 t \end{pmatrix} \tag{4.122}$$

At $t = 0$ we have

$$\begin{pmatrix} \theta \\ \phi \end{pmatrix} = \begin{pmatrix} 2A \\ 0 \end{pmatrix} \qquad \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.123}$$

But this never happens again since this requires $\cos\omega_1 t = \cos\omega_2 t = 1$. However this means that

$$t = 2\pi n_1/\omega_1 = 2\pi n_2/\omega_2 \tag{4.124}$$

for integers n_1, n_2 which implies that $\omega_1/\omega_2 = n_1/n_2$ is rational, but it isn't. However given any rational approximation N_1/N_2 to ω_1/ω_2 we can take $t = 2\pi N_1/\omega_1$ so that $\cos\omega_1 t = 1$ and

$$\cos\omega_2 t = \cos\left(2\pi\frac{\omega_2}{\omega_1}\frac{N_1}{N_2}N_2\right) \ . \tag{4.125}$$

By taking better and better rational approximations to ω_1/ω_2 the argument can be made arbitrarily close to $2\pi N_2$. Hence $\cos\omega_2 t$ will be arbitrarily close to, but always less than, 1. Thus $\theta \leq 2A$ with equality only at $t = 0$ but it can get as close to $2A$ as one wants by waiting a sufficiently long enough time. Thus given any small region around the initial starting point the system will eventually come back to it, even if it never exactly comes back to where it started. Clearly the smaller we make the region, the better rational approximation to ω_1/ω_2 we must use and hence the value of N_1, N_2 must be large, corresponding to a long recurrence time.

This is an amazing theorem and the cause for much pub-chatter[3]. Why? One example is the long-run stability of the solar system. Let us consider a system of (say) 8 planets in orbit around the sun. Planets can't escape to infinity because they have negative energy. And they can't go faster that the speed of light. So they effectively live in a bounded phase space. Therefore the solar system is either completely stable and periodic or it will eventually explore all available phase space. It seems unlikely that the planets orbits are so "fine-tuned" that the motion will be periodic (in particular why should the orbital periods of the planets be rational multiples of an earth year). So we are led to suspect that the solar system will eventually fill up all of its phase space, which could include having at least one planet go very far away (or perhaps move very very fast). Thus one is led to expect that the solar system is, ultimately, unstable (meaning that it will eventually look very different to how it looks now).

[3]If you are lucky(...) enough to be in the pub with a physicist.

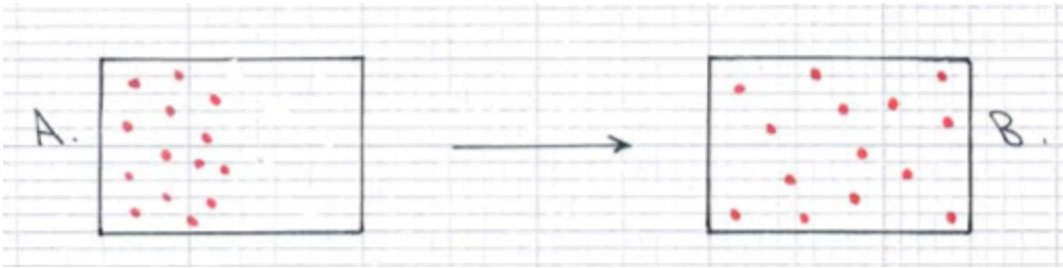

Figure 4.10.2: Dispersement Of Gas Molecules

Another example is to consider a finite sized room that is empty of air. Suppose you then place some air into a corner of the room in such a way that each air molecules have vanishing or small initial velocity. The air will subsequently naturally disperse and fill the whole room. In this case the phase space is bounded (for constant energy) because the room is bounded and the momentum of a given air molecule cannot be so large that its kinetic energy is greater that than the total energy. So the Poincare Recurrence Theorem applies. That means that in a finite time all the air will be back in the corner of the room with small velocities and the people who subsequently entered will spontaneously suffocate.

This seems to contract the notion that entropy, the amount of disorder, never decreases since the initial and final configurations are highly ordered whereas a generic room of air is a highly disordered state. So whats wrong? The cheap answer is that a finite time can still be an incredibly long time. In this case the time scale for the recurrence is longer than the age of the Universe.

But this answer is cheap because a theorem is a theorem and it can't depend on how long is a long time. Which theorem is wrong? This is much debated[4]. Arguably the theorem of the increase in entropy is more suspect. Its proof is far less simple and any known proof of it includes an assumption such as "repeated interactions between particles can be treated as independent". Whereas in essence the Poincare recurrence theorem shows that they are not truly independent.

[4]Typically also in the aforementioned pub.