LECTURE NOTES ON DYNAMICAL SYSTEMS, CHAOS AND FRACTAL GEOMETRY

Geoffrey R. Goodson

Towson University Mathematics Department

Preface

These notes were created over a number of years of teaching senior seminar type courses to senior year students at Towson University and graduate courses in the Towson University Applied Mathematics Graduate Program. Originally I based the course on some of the existing text books currently available, such as [63], [22], [28] and [20], but gradually realized that these didn't fully suit the needs of our students, so I produced my own lecture notes. However, these notes owe a debt to the above mentioned books. The first 12 chapters cover fairly standard topics in (mostly) one-dimensional dynamics and should be accessible to most upper level undergraduate students. The requirements from real analysis and topology (metric spaces) are developed as the material progresses. Chapter 13 gives a (non-rigorous) introduction to the theory of substitutions (via examples) and we show how these can give rise to certain types of fractals. Subsequent chapters develop the rigorous mathematical theory of substitutions and Sturmian sequences. Although there are numerous expositions of this material, most either give a non-rigorous account with no depth, or assume a level of sophistication above that of the upper level undergraduate or beginning graduate student. Our study of substitutions is purely topological and avoids any measure theory. Consequently we do not touch on some important topics of current interest such as the spectral properties of substitutions (see [49]).

Some of this material owes a debt to quite recent publications in the field of dynamical systems appearing in journals such as the *American Mathematical Monthly*, the *College Mathematics Journal*, *Mathematical Intelligencer* and *Mathematics Magazine*. Various Internet resources have been used such Wolfram's MathWorld, some of these without citation because of difficulties in knowing who the author is. All of the figures were created using the computer algebra system Mathematica. Computer algebra systems are indispensable tools for studying all aspects of this subject. We often use Mathematica to simplify complicated algebraic manipulations, but avoid its use where possible. These notes are still a work in progress and will be regularly updated - many of the sections are still incomplete.

I would like to thank some of my teachers whose courses I attended at various times and from whom I learnt a lot about dynamical systems. These include William Parry, Peter Walters and Rufus Bowen at the University of Warwick, Dan Newton at the University of Sussex, Michael Sears and Harvey Keynes at the University of the Witwatersrand, and Dan Rudolph at the University of Maryland.

I welcome any comments for improvement and would like to hear about any typographical or mathematical errors.

Geoffrey Goodson, Towson University, 2014.

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Dynamical Systems and Chaos: Spring 2013

CONTENTS

Chapter 1. The Orbits of One-Dimensional Maps.

1.1 Iteration of Functions and Examples of Dynamical Systems.

Dynamical systems is the study of how things change over time. Examples include the growth of populations, the change in the weather, radioactive decay, mixing of liquids and gases such as the ocean currents, motion of the planets, the interest in a bank account. Ideally we would like to study these with continuously varying time (what are called *flows*), but in this text we will simplify matters by only considering discrete changes in time. For example, we might model a population by measuring it daily. Suppose that x_n is the number of members of a population on day n, where x_0 is the initial population. We look for a function $f : \mathbb{R} \to \mathbb{R}$, (where $\mathbb{R}$ = set of all real numbers), for which

$$x_1 = f(x_0), x_2 = f(x_1) \quad \text{and generally} \quad x_n = f(x_{n-1}), \quad n = 1, 2, \ldots.$$

This leads to the iteration of functions in the following way:

Definition 1.1.1 Given $x_0 \in \mathbb{R}$, the *orbit* of x_0 under f is the set

$$O(x_0) = \{x_0, f(x_0), f^2(x_0), \ldots\},$$

where $f^2(x_0) = f(f(x_0))$, $f^3(x_0) = f(f^2(x_0))$, and continuing indefinitely, so that

$$f^n(x) = f \circ f \circ f \circ \cdots \circ f(x); \quad (n\text{-times composition}).$$

Set $x_n = f^n(x_0)$, $x_1 = f(x_0)$, $x_2 = f^2(x_0)$, so that in general

$$x_{n+1} = f^{n+1}(x_0) = f(f^n(x_0)) = f(x_n).$$

More generally, f may be defined on some subinterval I of $\mathbb{R}$, but in order for the iterates of $x \in I$ under f to be defined, we need the range of f to be contained in I, so $f : I \to I$.

This is what we call the iteration of one-dimensional maps (as opposed to higher dimensional maps $f : \mathbb{R}^n \to \mathbb{R}^n$, $n > 1$, which will be studied briefly in a later chapter).

Definition 1.1.2 A (one dimensional) *dynamical system* is a function $f : I \to I$ where I is some subinterval of $\mathbb{R}$.

Given such a function f, equations of the form $x_{n+1} = f(x_n)$ are examples of *difference equations*. These arise in the types of examples we mentioned above. For example in biology x_n may represent the number of bacteria in a culture after n hours. There is an obvious correspondence between one-dimensional maps and these difference equations. For example, a difference equation commonly used for calculating square roots:

$$x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n}),$$

corresponds to the function $f(x) = \frac{1}{2}(x + \frac{2}{x})$. If we start by setting $x_0 = 2$ (or in fact any real number), and then find x_1, x_2 etc., we get a sequence which rapidly approaches $\sqrt{2}$ (see page 9 of Sternberg [63]). One of the issues we examine is what exactly is happening here.

Examples of Dynamical Systems 1.1.3

1. **The Trigonometric Functions** Consider the iterations of the trigonometric functions starting with $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$. Select $x_0 \in \mathbb{R}$ at random, e.g., $x_0 = 2$ and set $x_{n+1} = \sin(x_n)$, $n = 0, 1, 2, \ldots$. Can you guess what happens to x_n as n increases? One way to investigate this type of dynamical system is to enter 2 into our calculator, then repeatedly press the **2nd**, **Answer** and **Sin** keys (you will need to do this many times to get a good idea - it may be easier to use Mathematica, or some similar computer algebra system). Do the same, but replace the sine function with the cosine function. How do we explain what appears to be happening in each case? These are questions that we aim to answer quite soon.

2. **Linear Maps** Probably the simplest dynamical system (and least interesting from a chaotic dynamical point of view), for population growth arises from the iteration of *linear maps*: maps of the form $f(x) = a \cdot x$. Suppose that x_n = size of a population at time n, with the property

$$x_{n+1} = a \cdot x_n$$

for some constant $a > 0$. This is an example of a *linear model* for the growth of the population.

If the initial population is $x_0 > 0$, then $x_1 = ax_0$, $x_2 = ax_1 = a^2x_0$, and in general $x_n = a^n x_0$ for $n = 0, 1, 2, \ldots$. This is the exact solution (or closed form solution) to the difference equation $x_{n+1} = a \cdot x_n$. Clearly $f(x) = ax$ is the corresponding dynamical system. We can use the solution to determine the long term behavior of the population:

x_n is very well behaved since:

(i) if $a > 1$, then $x_n \to \infty$ as $n \to \infty$,

(ii) if $0 < a < 1$ then $x_n \to 0$ as $n \to \infty$ (i.e., the population becomes extinct),

(iii) if $a = 1$, then the population remains unchanged.

3. **Affine maps** These are functions $f : \mathbb{R} \to \mathbb{R}$ of the form $f(x) = ax + b$ $(a \neq 0)$, for constants a and b. Consider the iterates of such maps:

$$f^2(x) = f(ax + b) = a(ax + b) + b = a^2x + ab + b,$$

$$f^3(x) = a^3x + a^2b + ab + b,$$

$$f^4(x) = a^4x + a^3b + a^2b + ab + b,$$

and generally

$$f^n(x) = a^nx + a^{n-1}b + a^{n-2}b + \cdots + ab + b.$$

Let $x_0 \in \mathbb{R}$ and set $x_n = f^n(x_0)$, then we have

$$\begin{aligned} x_n &= a^nx_0 + (a^{n-1} + a^{n-2} + \cdots + a + 1)b \\ &= a^nx_0 + b\left(\frac{a^n - 1}{a - 1}\right), \quad \text{if } a \neq 1, \end{aligned}$$

or

$$x_n = \left(x_0 + \frac{b}{a-1}\right)a^n + \frac{b}{1-a}, \quad \text{if } a \neq 1,$$

is the closed form solution in this case (here we have used the formula for the sum of a finite geometric series:

$$\sum_{k=0}^{n-1} r^k = \frac{r^n - 1}{r - 1},$$

when $r \neq 1$). If $a = 1$, the solution is $x_n = x_0 + nb$.

We can use these equations to determine the long term behavior of x_n. We see that:

(i) if $|a| < 1$ then $a^n \to 0$ as $n \to \infty$, so that

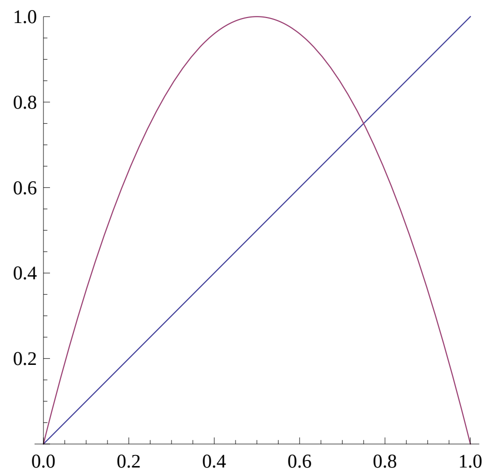
$$\lim_{n\to\infty} x_n = \frac{b}{1-a},$$

(ii) if $a > 1$, then $\lim_{n\to\infty} x_n = \infty$ for $b, x_0 > 0$,

(iii) if $a = 1$, then $\lim_{n\to\infty} x_n = \infty$ if $b > 0$.

The limit won't exist if $a \leq -1$.

4. **The Logistic Map** $L_\mu : \mathbb{R} \to \mathbb{R}$, $L_\mu(x) = \mu x(1-x)$ was introduced to model a certain type of population growth (see [41]). Here μ is a real parameter which is fixed. Note that if $0 < \mu \leq 4$ then L_μ is a dynamical system of the interval $[0,1]$, i.e. $L_\mu : [0,1] \to [0,1]$). For example, when $\mu = 4$, $L_4(x) = 4x(1-x)$, with $L_4([0,1]) = [0,1]$ with graph given below. If $\mu > 4$, then L_μ is no longer a dynamical system of $[0,1]$ as $L_\mu([0,1])$ is not a subset of $[0,1]$.



Recurrence Relations 1.1.4

Many sequences can be defined *recursively* by specifying the first term (or two), and then stating a general rule which specifies how to obtain the nth term from the $(n-1)$th term (or other additional terms), and using mathematical induction to see that the sequence is "well defined" for every $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. For example, $n! = n$-factorial can be defined in this way by specifying $0! = 1$, and $n! = n \cdot (n-1)!$, for $n \in \mathbb{Z}^+$. The Fibonacci sequence F_n, can be defined by setting

$$F_0 = 1, \quad F_1 = 1 \quad \text{and} \quad F_{n+1} = F_n + F_{n-1}, \quad \text{for} \quad n \in \mathbb{Z}^+,$$

so that $F_2 = 2$, $F_3 = 5$ etc.

The orbit of a point $x_0 \in \mathbb{R}$ under a function f is then defined recursively as follows: we are given the starting value x_0, and we set

$$x_n = f(x_{n-1}), \quad \text{for} \quad n \in \mathbb{Z}^+.$$

The principle of mathematical induction then tells that x_n is defined for every $n =\geq 0$, since it is defined for $n = 0$, and assuming it has been defined for $k = n-1$ then $x_n = f(x_{n-1})$ defines it for $k = n$.

Ideally, given a recursively defined sequence x_n, we would like to have a specific formula for x_n in terms of elementary functions (so called *closed form solution*), but this is often very difficult, or impossible to achieve. In the case of affine maps and certain logistic maps, there is a closed form solution. One can use the former to study problems of the following type:

Example 1.1.5 An amount \$ T is deposited in your bank account at the end of each month. The interest is $r\%$ per period. Find the amount $A(n)$ accumulated at the end of n months (Assume $A(0) = T$).

Answer. $A(n)$ satisfies the difference equation

$$A(n+1) = A(n) + A(n)r + T, \quad \text{where } A(0) = T,$$

or

$$A(n+1) = A(n)(1+r) + T,$$

so we set $x_0 = T$, $a = 1 + r$ and $b = T$ in the formula of Example 1.1.3, then the solution is

$$\begin{aligned} A(n) &= (1+r)^n T + T\left(\frac{(1+r)^n - 1}{1+r-1}\right) \\ &= (1+r)^n T + \frac{T}{r}((1+r)^n - 1). \end{aligned}$$

Remark 1.1.6 It is conjectured that closed form solutions for the difference equation arising from the logistic map are only possible when $\mu = \pm 2, 4$ (see Exercises 1.1 # 3 for the cases where $\mu = 2, 4$, and # 7 for the case where $\mu = -2$ and also [67] for a discussion of this conjecture).

Exercises 1.1

1. If $L_\mu(x) = \mu x(1-x)$ is the logistic map, calculate $L_\mu^2(x)$ and $L_\mu^3(x)$.

2. Use Example 1.1.3 for affine maps to find the solutions to the difference equations:

(i) $x_{n+1} - \dfrac{x_n}{3} = 2$, $x_0 = 2$,

(ii) $x_{n+1} + 3x_n = 4$, $x_0 = -1$.

3. A logistic difference equation is one of the form $x_{n+1} = \mu x_n(1 - x_n)$ for some fixed $\mu \in \mathbb{R}$. Find exact solutions to the logistic equations:

(i) $x_{n+1} = 2x_n(1 - x_n)$. Hint: use the substitution $x_n = (1 - y_n)/2$ to transform the equation into a simpler equation that is easily solved.

(ii) $x_{n+1} = 4x_n(1 - x_n)$. Hint: set $x_n = \sin^2(\theta_n)$ and simplify to get an equation that is easily solved.

4. You borrow \$ P at r % per annum, and pay off \$ M at the end of each subsequent month. Write down a difference equation for the amount owing $A(n)$ at the end of each month (so $A(0) = P$). Solve the equation to find a closed form for $A(n)$. If $P = 100,000$, $M = 1000$ and $r = 4$, after how long will the loan be paid off?

5. If $T_\mu(x) = \begin{cases} \mu x & \text{if } \ 0 \le x < 1/2 \\ \mu(1-x) & \text{if } \ 1/2 \le x < 1 \end{cases}$, show that T_μ is a dynamical system of $[0,1]$ for $\mu \in (0,2]$

6. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined below. For each of the intervals given, determine whether f can be considered as a dynamical system $f : I \to I$:

(a) $f(x) = x^3 - 3x$,

$$\text{(i)} \quad I = [-1,1], \quad \text{(ii)} \quad I = [-2,2].$$

(b) $f(x) = 2x^3 - 6x$,

$$\text{(i)} \quad I = [-1,1], \quad \text{(ii)} \quad I = [-\sqrt{\frac{7}{2}}, \sqrt{\frac{7}{2}}], \quad \text{(iii)} \quad I = [-4,4].$$

7. For the following functions, find $f^2(x)$, $f^3(x)$ and a general formula for $f^n(x)$:

$$\text{(i)} \ \ f(x) = x^2, \ \ \text{(ii)} \ \ f(x) = |x+1|, \ \ \text{(iii)} \ \ f(x) = \begin{cases} 2x & \text{if } \ 0 \le x < 1/2 \\ 2x - 1 & \text{if } \ 1/2 \le x < 1. \end{cases}$$

8. Use mathematical induction to show that if $f(x) = \dfrac{2}{x+1}$, then

$$f^n(x) = \frac{2^n(x+2) + (-1)^n(2x-2)}{2^n(x+2) - (-1)^n(x-1)}.$$

9*. Show that a closed form solution to the logistic difference equation when $\mu = -2$ is given by

$$x_n = \frac{1}{2}\left[1 - f\left[r^n f^{-1}(1-2x_0)\right]\right], \quad \text{where} \quad r = -2 \quad \text{and} \quad f(\theta) = 2\cos\left(\frac{\pi - \sqrt{3}\theta}{3}\right).$$

(Hint: Set $x_n = \dfrac{1 - f(\theta_n)}{2}$ and use steps similar to 3(ii)).

1.2 Newton's Method and Fixed Points

Given a differentiable function $f : \mathbb{R} \to \mathbb{R}$, Newton's method often allows us to find good approximations to zeros of $f(x)$, i.e., approximate solutions to the equation $f(x) = 0$. The idea is to start with a first approximation x_0 and look at the tangent line to $f(x)$ at the point $(x_0, f(x_0))$. Suppose this line intersects the x-axis at x_1, then we can check that

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, \quad \text{if} \quad f'(x_0) \neq 0.$$

For reasons that we will make clear, it is often the case that x_1 is a better approximation to the zero $x = c$ than x_0 was. In other words, Newton's method is an algorithm for finding approximations to zeros of a function $f(x)$. The algorithm gives rise to a difference equation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

where x_0 is a first approximation to a zero of $f(x)$. The corresponding real function is

$$N_f(x) = x - \frac{f(x)}{f'(x)}, \quad \text{(the \textit{Newton} function).}$$

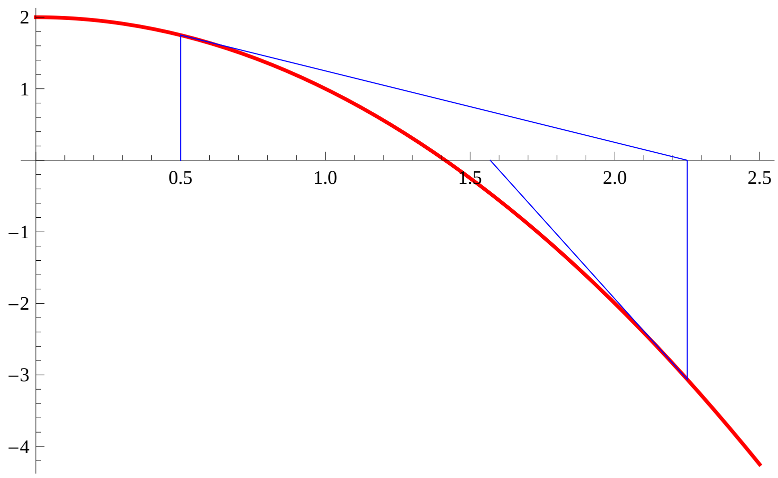
Note that if $f(x) = x^2 - a$, then $f'(x) = 2x$ and

$$N_f(x) = x - \frac{x^2 - a}{2x} = \frac{1}{2}\left(x + \frac{a}{x}\right),$$

so that

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right),$$

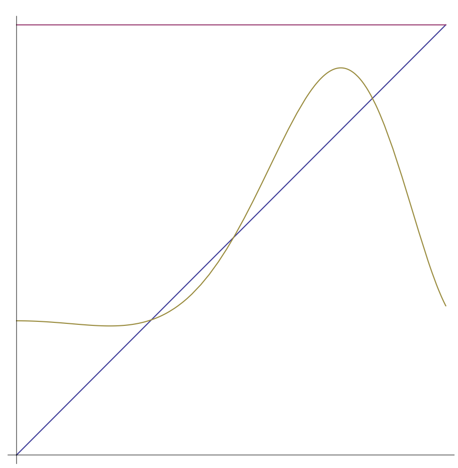
the difference equation we mentioned in Section 1.1 that is used for approximating $\sqrt{2}$ when $a = 2$.



The first two approximations for $f(x) = 2 - x^2$, starting with $x_0 = .5$.

Note that we are looking for where $f(x) = 0$, and this happens if and only if $N_f(x) = x$.

Definition 1.2.1 For a function $f : \mathbb{R} \to \mathbb{R}$, a point $c \in \mathbb{R}$ for which $f(c) = c$ is called a *fixed point* of f. It is a point where the graph of $f(x)$ intersects the line $y = x$. We denote the set of fixed points of f by $\text{Fix}(f)$.



Fixed points occur where the graph of $f(x)$ intersects the line $y = x$.

Example 1.2.2 Suppose that $f(x) = x^2$, then $x^2 = x$ gives $x(x-1) = 0$, so has fixed points $c = 0$ and $c = 1$, so $\text{Fix}(f) = \{0, 1\}$. If $f(x) = x^3 - x$, then $x^3 - x = x$ gives $x(x^2 - 2) = 0$, so the fixed points are $c = 0$ and $c = \pm\sqrt{2}$, $\text{Fix}(f) = \{0, \pm\sqrt{2}\}$.

The logistic map $f(x) = 4x(1-x) = 4x - 4x^2$, $0 \le x \le 1$ has the properties: $f(0) = 0$, (a fixed point), $f(1) = 0$, a maximum at $x = 1/2$ (with $f(1/2) = 1$). Solving $f(x) = x$ gives $4x - 4x^2 = x$, so $4x^2 = 3x$, so the fixed points are $c = 0$ and $c = 3/4$.

This map has what we call *eventual fixed points*: $f(1) = 0$ and $f(0) = 0$, so we say that $c = 1$ is eventually fixed. Also $f(1/4) = 3/4$, so $c = 1/4$ is eventually fixed, as is $c = (2+\sqrt{3})/4$.

Definition 1.2.3 $x^* \in \mathbb{R}$ is an *eventual fixed point* of $f(x)$ if there exists a fixed point c of $f(x)$ and $r \in \mathbb{Z}^+$ satisfying $f^r(x^*) = c$, but $f^s(x^*) \neq c$ for $0 < s < r$.

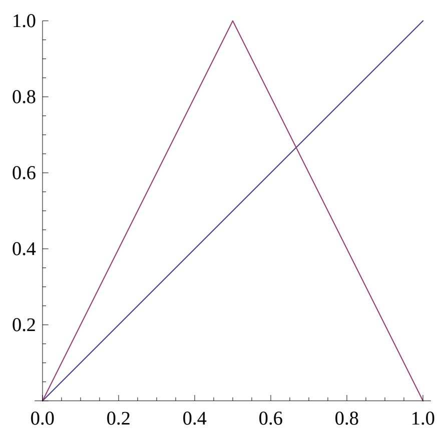
Example 1.2.4 The Tent Map

Define a function $T : [0,1] \to [0,1]$ by

$$T(x) = \begin{cases} 2x & : 0 \le x \le 1/2 \\ 2(1-x) & : 1/2 < x \le 1 \end{cases}$$

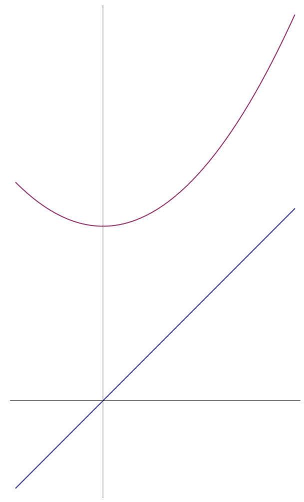
$$= 1 - 2|x - 1/2|.$$

$T(x)$ is called the *tent map*, it has fixed points: $T(0) = 0$ and $T(2/3) = 2/3$. Since $T(1/4) = 1/2$, $T(1/2) = 1$ and $T(1) = 0$, $1/4, 1/2$ and 1 are eventually fixed. It is not difficult to see that there are many other eventually fixed points (infinitely many).



Note that some maps do not have fixed points:

Example 1.2.5 $f(x) = x^2 + 1$ has no fixed points.

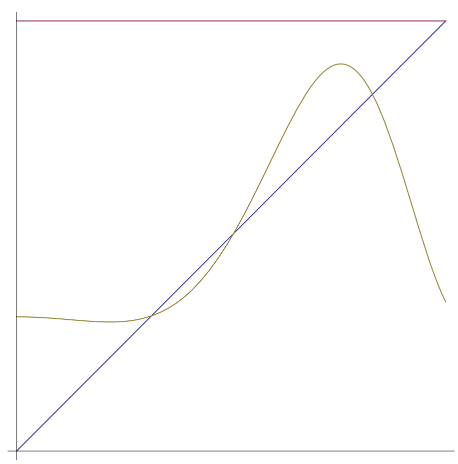


$f(x) = x^2 + 1$ does not intersect the line $y = x$.

Question 1.2.6 For what values of $a \in \mathbb{R}$ does $Q_a(x) = x^2 + a$ have fixed points? We see below that certain functions always have fixed points:

Theorem 1.2.7 *Let $f : I \to I$ be a continuous function, where $I = [a, b]$, $a < b$ is a bounded interval. Then $f(x)$ has a fixed point $c \in I$.*

Proof. Set $g(x) = f(x) - x$. We may assume that $f(a) \neq a$ and $f(b) \neq b$, so that $f(a) > a$ and $f(b) < b$.



The graph of $f(x)$ always intersects the line $y = x$.

It follows that

$$g(a) = f(a) - a > 0, \quad \text{and} \quad g(b) = f(b) - b < 0,$$

so that $g(x)$ is a continuous function which is positive at a and negative at b, so by the Intermediate Value Theorem (IVT), there exists $c \in (a, b)$ (open interval) with $g(c) = 0$, i.e., $f(c) = c$, so c is a fixed point of $f(x)$. $\square$

Remark 1.2.8 The above is an example of an existence theorem, it says nothing about how to find the fixed point, where it is or how many there are. It tells us that if $f(x)$ is a continuous function on an interval I with $f(I) \subseteq I$, then $f(x)$ has a fixed point in I. It is also true that if instead $f(I) \supseteq I$ then $f(x)$ has a fixed point in I as we see now:

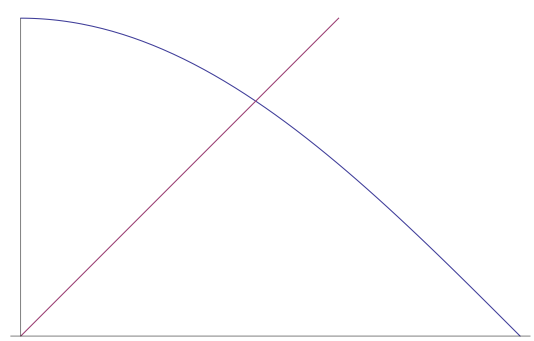
Theorem 1.2.9 *Let $f : I \to \mathbb{R}$ ($I = [a, b], a < b$) be a continuous function with $f(I) \supseteq I$, then $f(x)$ has a fixed point in I.*

Proof. As before, set $g(x) = f(x) - x$, then there exists $c_1 \in (a, b)$ with $f(c_1) < c_1$ (in fact $f(c_1) < a < c_1$). Also there is $c_2 \in (a, b)$ with $f(c_2) > c_2$.

Then $g(c_1) < 0$ and $g(c_2) > 0$ and since $g(x)$ is a continuous function, it follows by the Intermediate Value Theorem that there exists $c \in I$, ($c_1 < c < c_2$ or $c_2 < c < c_1$), with $g(c) = 0$; $f(c) = c$. $\square$

Remark 1.2.10 It is often difficult to find fixed points explicitly:

Example 1.2.11 Set $f(x) = \cos x$. If $f(c) = c$, then $\cos c = c$. It is possible to find an approximation to the fixed point $c = .739085\ldots$ using (for example) Newton's method.



$f(x) = \cos x$ has a fixed point in $[0, \pi/2]$.

Exercises 1.2

1. Find all fixed points and eventual fixed points of the map $f(x) = |x - 1|$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be such that for some $n \in \mathbb{Z}^+$, the nth iterate of f has a unique fixed point c (i.e., $f^n(c) = c$ and c is unique). Then show that c is a fixed point of f.

3. Use the `Mathematica` command `NestList` (see below) to find how the following functions behave when the given point is iterated (comment on what appears to be happening in each case):

(i) $L_1(x) = x(1 - x)$, with starting point $x_0 = .75$.

(ii) $L_2(x) = 2x(1 - x)$, with starting point $x_0 = .1$.

(iii) $L_3(x) = 3x(1 - x)$, with starting point $x_0 = .2$.

(iv) $L_{3.2}(x) = 3.2x(1 - x)$, with starting point $x_0 = .95$.

(v) $f(x) = \sin(x)$, with starting point $x_0 = 9.5$.

(vi) $g(x) = \cos(x)$, with starting point $x_0 = -15.3$.

4. Show that the logistic map $L_\mu(x) = \mu x(1 - x)$, $x \in [0, 1]$, for $0 < \mu < 4$, the fixed point $x = 0$ has no eventual fixed points other than $x = 1$.

Show that for $1 < \mu \leq 2$, the fixed point $x = 1 - 1/\mu$ only has $x = 1/\mu$ as an eventual fixed point, but this is not the case for $2 < \mu < 3$.

To use `Mathematica`, *first define your function, say* $f(x) = sin(x)$ *in an* `input cell` *using the syntax*

```
f[x_]:= Sin[x].
```

Execute the cell by doing `Shift` *and* `Enter` *simultaneously. The command*

```
NestList[f, 9.5, 100]
```

when executed in a new `input cell` *will give 100 iterations of* $\sin(x)$ *with starting value* $x_0 = 9.5$. *If you want to graph* $f(x)$ *on the interval* $[a, b]$, *use the command:*

```
Plot[f[x], { x, a, b } ].
```

5. (a) Let $f(x) = (1+x)^{-1}$. Find the fixed points of f and show that there are no points c with $f^2(c) = c$ and $f(c) \neq c$ (period 2-points). Note that $f(-1)$ is not defined, but points that get mapped to -1 belong to the interval $[-2, -1)$ and are of the form

$$\nu_n = -\frac{F_{n+1}}{F_n}, \quad n \geq 1,$$

where $\{F_n\}$, $n \geq 0$, is the Fibonacci sequence (see 1.1.4). Note that $\nu_n \to -r$ as $n \to \infty$, where $-r$ is the negative fixed point of f (see [12] for more details).

(b) If $x_0 \in (0,1]$, set $x_n = \dfrac{F_{n-1}x_0 + F_n}{F_n x_0 + F_{n+1}}$. Use mathematical induction to show that

$$x_{n+1} = f(x_n) = \frac{F_n x_0 + F_{n+1}}{F_{n+1}x_0 + F_{n+2}}.$$

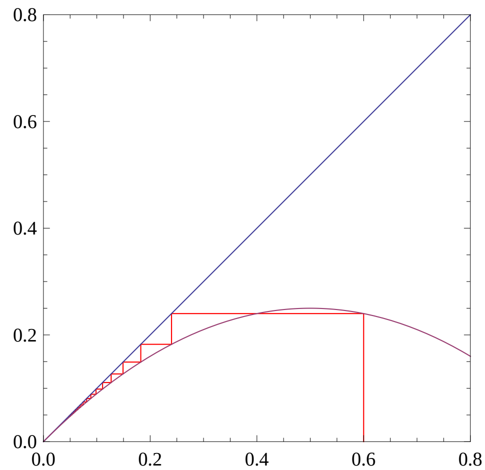
Deduce that as $n \to \infty$, $x_n \to 1/r$, the positive fixed point of f.

1.3 Graphical Iteration

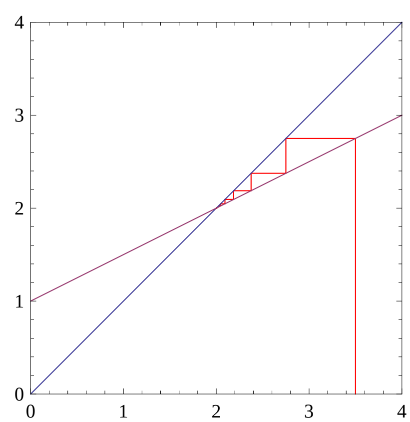
It is possible to follow the iterates of a function $f(x)$ at a point x_0 using *graphical iteration* (sometimes called *web diagrams*).

We start at x_0 on the x-axis and draw a line vertically to the function. We then move horizontally to the line $y = x$, then vertically to the function and continue in this way. Notice that in some examples the iterations converges to a fixed point. In others it goes off to ∞, whilst in others still, it oscillates between two points indefinitely.

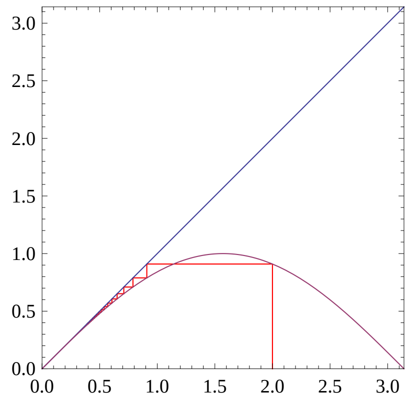
Example 1.3.1 $f(x) = x(1-x)$. In this example, an examination of graphical iteration seems to suggest that the orbits of any point in $[0,1]$ approach the fixed point $x = 0$. This is an example of an *attracting* fixed point.



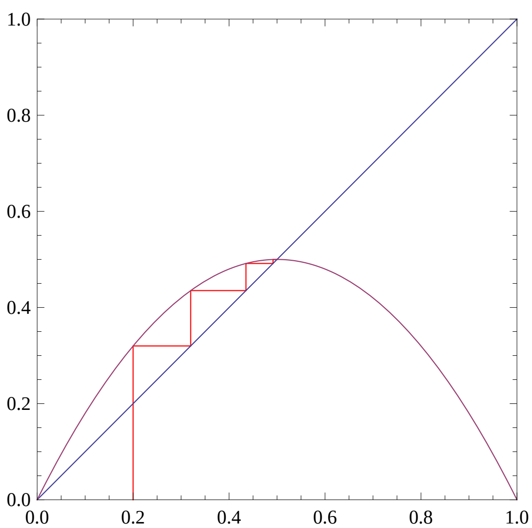
Example 1.3.2 Let $f(x) = x/2 + 1$. This is an affine transformation with $a = 1/2$, $b = 1$. According to what we saw in Example 1.1.3, the iterates should converge to $\frac{b}{1-a} = 2$ since $|a| < 1$. What is actually happening is that $c = 2$ is an attracting fixed point of $f(x)$ with the property that it attracts all members of $\mathbb{R}$ (said to be *globally attracting*).

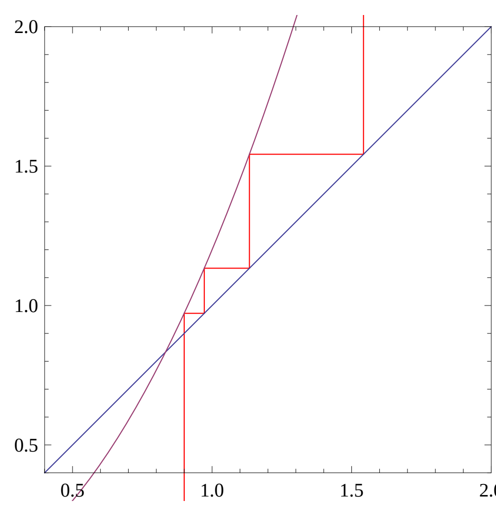


Example 1.3.3 We see from the graph of $f(x) = \sin x$ that $c = 0$ seems to be an attracting fixed point. We shall show later that it is actually globally attracting.



We see two basic situations arising (together with some variations of these). In particular, we notice that if the sequence $x_n = f^n(x_0)$ converges to some point c as $n \to \infty$, then c is a fixed point.
(i) Stable orbit, where the web diagram approaches a fixed point.
(ii) Unstable orbit, where the web diagram moves away from a fixed point.





Proposition 1.3.4 *If $y = f(x)$ is a continuous function and $\lim_{n\to\infty} f^n(x_0) = c$, then $f(c) = c$ (i.e., if the orbit converges to a point c, then c is a fixed point of f) .*

Proof. We see that

$$\lim_{n\to\infty} f^n(x_0) = c \Rightarrow f(\lim_{n\to\infty} f^n(x_0)) = f(c),$$

and since f is continuous,

$$\lim_{n\to\infty} f^{n+1}(x_0) = f(c).$$

But clearly $c = \lim_{n\to\infty} f^{n+1}(x_0)$, so that $c = f(c)$ by the uniqueness of limit. $\square$

1.4 Attractors and Repellers

Suppose that $f : X \to X$ is a function (where we assume that X is some subset of $\mathbb{R}$ such as an interval or possibly $\mathbb{R}$ itself). Let $c \in X$ be a fixed point of f. We define the notions on attracting fixed point and repelling fixed that we looked at graphically in the last section.

Definition 1.4.1 Let $f : X \to X$ with $f(c) = c$.

(i) c is a *stable fixed point* if for all $\epsilon > 0$ there exists $\delta > 0$ such that if $x \in X$ and $|x - c| < \delta$, then $|f^n(x) - c| < \epsilon$ for all $n \in \mathbb{Z}^+$.

If this does not hold, c will be called *unstable*.

(ii) c is said to be *attracting* if there is a real number $\eta > 0$ such that

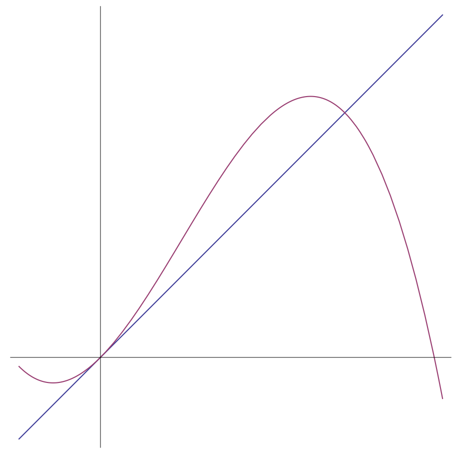
$$|x - c| < \eta \Rightarrow \lim_{n\to\infty} f^n(x) = c.$$

(iii) c is *asymptotically stable* if it is both stable and attracting.

Remark 1.4.2 We will show that a fixed point c of $f : X \to X$ (suitably differentiable) with the property $|f'(c)| < 1$ is an asymptotically stable fixed point. If c is an unstable fixed point, we can find $\epsilon > 0$ and x arbitrarily close to c such that some iterate of x, say $f^n(x)$, will be greater than distance ϵ from c. This is the case when $|f'(c)| > 1$ as the next theorem shows. We call c a *repeller* (c is a *repelling fixed point*), since the iterates move away from the fixed point (c is unstable). We will also see that a fixed point can be stable without being attracting, and that it can be attracting without being stable.

Definition 1.4.3 A fixed point c of f is *hyperbolic* if $|f'(c)| \neq 1$. If $|f'(c)| = 1$ it is *non-hyperbolic*.

Thus if c is a non-hyperbolic fixed point, then $f'(c) = 1$ or $f'(c) = -1$, so the graph of $f(x)$ either meets the line $y = x$ tangentially, or at 90°:



$f(x) = -2x^3 + 2x^2 + x$ has both types of non-hyperbolic fixed points.

Hyperbolic fixed points have the following properties:

Theorem 1.4.4 *Let $f : X \to X$ be a differentiable function with continuous first derivative (we say that f is of class C^1).*

(i) *If a is a fixed point for $f(x)$ with $|f'(a)| < 1$, then a is asymptotically stable. The iterates of points close to a, converge to a geometrically (i.e., there is a constant $0 < \lambda < 1$ for which $|f^n(x) - a| < \lambda^n |x - a|$ for all $n \in \mathbb{Z}^+$ and for all $x \in X$ sufficiently close to a).*

(ii) *If a is a fixed point for $f(x)$ for which $|f'(a)| > 1$, then a is a repelling fixed point for f.*

Proof. (i) We may assume that X is an open interval with $a \in X$. Suppose $|f'(a)| < \lambda < 1$ for some $\lambda > 0$, then using the continuity of $f'(x)$, there exists an open interval I with $|f'(x)| < \lambda < 1$ for all $x \in I$.

By the Mean Value Theorem, if $x \in I$ there exists $c \in I$, lying between x and a, satisfying

$$f'(c) = \frac{f(x) - f(a)}{x - a},$$

so that

$$|f(x) - a| = |f'(c)||x - a| < \lambda |x - a|,$$

(i.e., $f(x)$ is closer to a than x was).

Repeating this argument with $f(x)$ replacing x gives

$$|f^2(x) - a| < \lambda^2 |x - a|, \ldots, |f^n(x) - a| < \lambda^n |x - a|.$$

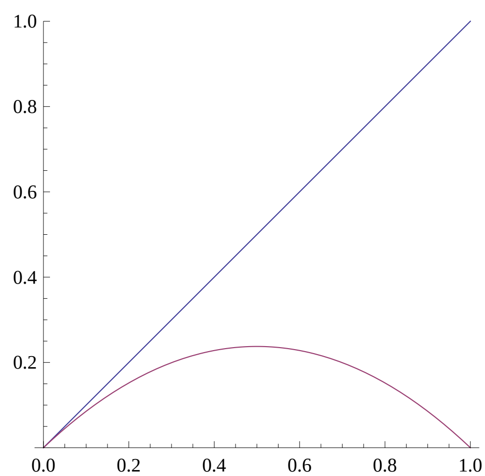
Since $\lambda^n \to 0$ as $n \to \infty$, it follows that $f^n(x) \to a$ as $n \to \infty$.

The proof of (ii) is similar. $\square$

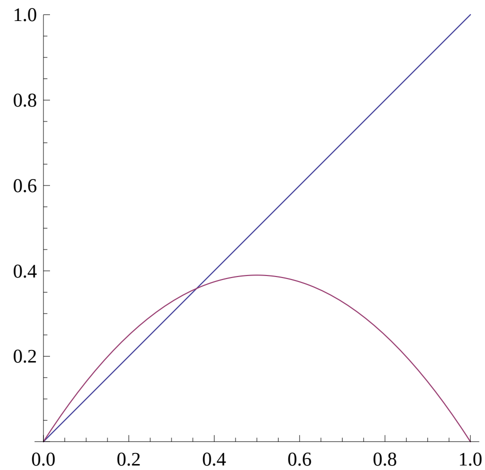
Example 1.4.5 Denote the Logistic map by $L_\mu(x) = \mu x(1-x) = \mu x - \mu x^2$. We are interested in this map for $0 \le x \le 1$ and $0 < \mu \le 4$, since for these parameter values μ, L_μ is a function which maps the interval $[0,1]$ into the interval $[0,1]$, so it necessarily has a fixed point and iterates stay inside $[0,1]$ (if $\mu > 4$, $L_\mu(x) > 1$ for some values of x in $[0,1]$ and further iterates will go to $-\infty$).

Solving $L_\mu(x) = x$ gives $x = 0$ or $x = 1 - 1/\mu$. If $0 < \mu \le 1$, then $1 - 1/\mu \le 0$, so $c = 0$ is the only fixed point in $[0,1]$ in this case.

For $0 < \mu \le 1$, $L'_\mu(x) = \mu - 2\mu x$, so $L'_\mu(0) = \mu$ and 0 is an attracting fixed point for $0 < \mu < 1$.



If $\mu > 1$, then 0 and $1 - 1/\mu$ are both fixed points in $[0,1]$, but now 0 is repelling.



Also,

$$L'_\mu(1-1/\mu) = \mu - 2\mu(1-1/\mu) = 2-\mu,$$

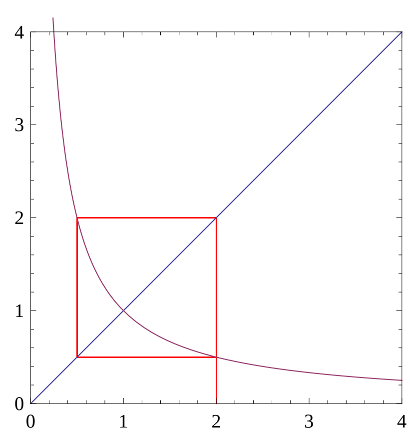
so that

$$|L'_\mu(1-1/\mu)| = |2-\mu| < 1 \quad \text{iff} \quad 1 < \mu < 3.$$

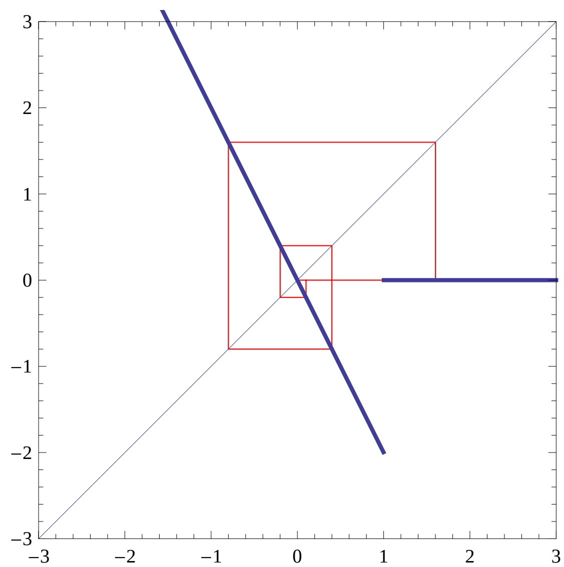
In this case $c = 1 - 1/\mu$ is an attracting fixed point, and repelling for $\mu > 3$. Note that $L'_1(0) = 1$ and $L'_3(2/3) = 1$ so we have non-hyperbolic fixed points.

We will examine in Chapter 2 what happens in the cases where $\mu = 1$ and $\mu = 3$.

Example 1.4.6 Let $f(x) = 1/x$. The two fixed points $x = \pm 1$ are hyperbolic since $f'(x) = -1/x^2$, so $f'(\pm 1) = -1$. However, they are stable but not attracting since $f^2(x) = f(1/x) = x$, and we see points close to ± 1 neither move closer nor further apart.



Example 1.4.7 Consider instead the function $f_a(x) = \begin{cases} -2x & \text{if} \quad x < a \\ 0 & \text{if} \quad x \geq a \end{cases}$, where a is any positive real number. Then $c = 0$ is an unstable (repelling) fixed point of f_a which is attracting i.e., $\lim_{n\to\infty} f^n(x) = 0$ for all $x \in \mathbb{R}$. A fixed point having this latter property is said to be *globally attracting*. Note that the function f_a is not continuous at $x = a$. It has been shown by Sedaghat [58] that a continuous mapping of the real line cannot have an unstable fixed point that is globally attracting.



The map f_1 (in bold) where $x = 0$ is an attracting but unstable fixed point.

Example 1.4.8 Newton's Method Revisited.

Suppose that $f : I \to I$ is a function whose zero c is to be approximated using Newton's method. The Newton function is $N_f(x) = x - f(x)/f'(x)$, where we are assuming $f'(c) \neq 0$. Notice that since $f(c) = 0$, $N_f(c) = c$, i.e., c is a fixed point of N_f. Consider $N_f'(c)$:

$$N_f'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2} = \frac{f(x)f''(x)}{[f'(x)]^2},$$

so that

$$N_f'(c) = \frac{f(c)f''(c)}{[f'(c)]^2} = 0,$$

since $f(c) = 0$.

It follows that $|N_f'(c)| = 0 < 1$, so that c is an attracting fixed point for N_f, and in particular

$$\lim_{n\to\infty} N_f^n(x_0) = c,$$

provided x_0, the first approximation to c, is sufficiently close to c.

Definition 1.4.9 A fixed point c for $f(x)$ is called a *super-attracting fixed-point* if $f'(c) = 0$. This gives a very fast convergence to the fixed-point for points nearby.

Remark 1.4.10 Suppose that $f'(c) = 0$, then $N_f(x)$ is not defined at $x = c$, since the quotient $f(x)/f'(x)$ is not defined there. Suppose we can write $f(x) = (x - c)^k h(x)$ where $h(c) \neq 0$ and $k \in \mathbb{Z}^+$ (for example if $f(x)$ is a polynomial with a multiple root), then we have

$$\frac{f(x)}{f'(x)} = \frac{(x-c)^k h(x)}{k(x-c)^{k-1}h(x) + (x-c)^k h'(x)} = \frac{(x-c)h(x)}{kh(x) + (x-c)h'(x)} = 0,$$

when $x = c$, so that $N_f(x) = c$ has a removable discontinuity at $x = c$ (removed by setting $N_f(c) = c$). Then we can find $N_f'(x)$ and show that $N_f'(x)$ has a removable discontinuity at $x = c$, which can again be removed by setting $N_f'(c) = (k-1)/k$, so that $|N_f'(c)| < 1$ (see the exercises).

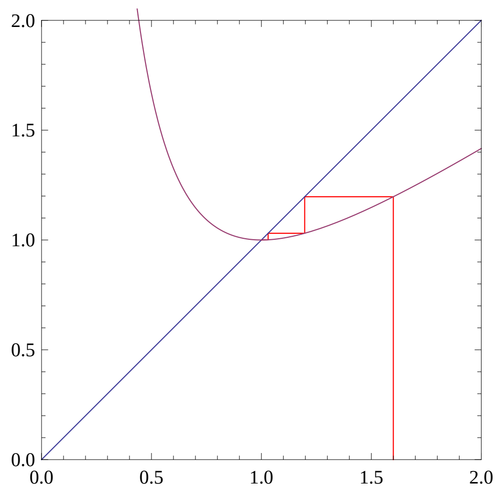
We summarize the above with a theorem:

Theorem 1.4.11 *For a differentiable function $f : I \to I$, the zero c of $f(x)$ is a super-attracting fixed point of the Newton function N_f, if and only if $f'(c) \neq 0$.*

Example 1.4.12 Suppose $f(x) = x^3 - 1$, then $f(1) = 0$ and

$$N_f(x) = x - f(x)/f'(x) = x - \left(\frac{x^3-1}{3x^2}\right) = \frac{2x}{3} + \frac{1}{3x^2},$$

so $N_f(1) = 1$ and $N_f'(x) = \frac{2}{3} - \frac{2}{3x^3}$, so that $N_f'(1) = 0$. We see from graphical iteration (next section), very fast convergence to the fixed point of N_f.



Very fast convergence to the fixed point.

Exercises 1.4

1. Find the fixed points and determine their stability for the function $f(x) = 4 - \frac{3}{x}$.

2. For the family of quadratic maps $Q_c(x) = x^2 + c$, $x \in [0, 1]$, use the Mathematica program `webPlot` below to give graphical iteration for the values shown (use 20 iterations):

(i) $c = 1/2$, starting point $x_0 = 1$, (ii) $c = 1/4$, starting point $x_0 = .1$, (iii) $c = 1/8$, starting point $x_0 = .7$.

Mathematica Program: Use as in:

```
 webPlot[Cos[x], {x, 0, Pi}, {.1, 20}, AspectRatio->Automatic]

webPlot[g_, {x_, xmin_, xmax_}, {a1_, n_}, opts_] :=
 Module[{seq, r, pts, web, graph}, r[t_] := N[g /. x -> t];
  seq = NestList[r, a1, n];     pts = Flatten[
Table[{{seq[[i]],seq[[i + 1]]},{seq[[i + 1]],seq[[i + 1]]}},{i, 1, n}], 1];
  web = Graphics[{Hue[0], Line[PrependTo[pts, {seq[[1]], 0}]]}];
  graph = Plot[{x, r[x]}, {x, xmin, xmax},
    DisplayFunction -> Identity]; Print["last iterate =", Last[seq]];
  Show[web, graph, opts, Frame -> True,
```

```
PlotRange -> {{xmin, xmax}, {xmin, xmax}}]]
```

3. Let $S_\mu(x) = \mu\sin(x)$, $0 \leq \mu \leq 2\pi$, $0 \leq \mu \leq \pi$ and $C_\mu(x) = \mu\cos(x)$, $-\pi \leq x \leq \pi$ and $0 \leq \mu \leq \pi$.

(a) Show that S_μ has a super-attracting fixed point at $x = \pi/2$, when $\mu = \pi/2$.

(b) Find the corresponding values for C_μ having a super-attracting fixed point.

4. Show that the map $f(x) = \dfrac{2}{x+1}$ has no periodic points of period $n > 1$ (Hint: Use the closed formula in Exercise 1.1.6).

5. Show that if $f(x) = x + 1/x$ and $x > 0$, then $R^n(x) \to \infty$ as $n \to \infty$. (Hint: R has no fixed points in $\mathbb{R}$ and it is continuous with $0 < R(x) < R^2(x) < \cdots$, so suppose the limit exists and obtain a contradiction).

6. Let N_f be the Newton function of the map $f(x) = x^2 + 1$. Clearly there are no fixed points of the the Newton function as there are no zeros of f. Show that there are points c where $N_f^2(c) = c$ (called *period 2-points* of N_f).

7. (a) Suppose that $f(c) = f'(c) = 0$ and $f''(c) \neq 0$. If $f''(x)$ is continuous at $x = c$, show that the Newton function $N_f(x)$ has a removable discontinuity at $x = c$ (hint: apply L'Hopital's rule to N_f at $x = c$).

(b) If in addition, $f'''(x)$ is continuous at $x = c$ with $f'''(c) \neq 0$, show that $N_f'(c) = 1/2$, so that $x = c$ is not a super-attracting fixed point in this case.

(c) Check the above for the function $f(x) = x^3 - x^2$ with $c = 0$.

8. Continue the argument in Remark 1.4.11 (generalizing the last exercise), to show that the derivative of the Newton function N_f has a removable discontinuity at $x = c$, which can be removed by setting $N_f'(c) = (k-1)/k$.

9. Let f be a twice differentiable function with $f(c) = 0$. Show that if we find the Newton function of $g(x) = f(x)/f'(x)$, then $x = c$ will be a super-attracting fixed point for N_g, even if $f'(c) = 0$ (this is called Halley's method).

1.5 Non-hyperbolic Fixed Points.

Example 1.5.1 We have seen that it $f : X \to X$ (where usually X is some subinterval of $\mathbb{R}$) and $a \in X$ with $f(a) = a$, $|f'(a)| < 1$, with suitable differentiability conditions, then a is a stable fixed point for f. This term is used because points close to a will approach a under iteration. If we look at a function like $f(x) = \sin x$, we see that $c = 0$ is a fixed point but in this case $|f'(0)| = 1$, so it is a non-hyperbolic fixed point. However, graphical iteration suggests that the basin of attraction of f is all of $\mathbb{R}$, so $c = 0$ is a stable fixed point. Before considering non-hyperbolic fixed points in more detail, let us prove the last statement analytically:

Proposition 1.5.2 *The fixed point $c = 0$ of $f(x) = \sin x$ is globally attracting .*

Proof. First notice that $c = 0$ is the only fixed point of $\sin x$. This is clear for if $\sin x = x$ then we cannot have $|x| > 1$ since $|\sin x| \leq 1$ for all x. If $0 < x \leq 1$, the Mean Value Theorem implies there exists $c \in (0, x)$ with

$$f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{\sin x}{x},$$

so that

$$\sin x = x \cos c < x,$$

since $|\cos c| < 1$ for $c \in (0, 1)$. Similarly if $-1 \leq x < 0$.

To show that $f(x)$ is globally attracting, let $x \in \mathbb{R}$. We may assume that $-1 \leq x \leq 1$, since this will be the case after the first iteration.

Suppose that $0 < x \leq 1$, then we note that $0 < f'(x) < 1$ on this interval. By the Mean Value Theorem, there exists $c \in (0, x)$ with

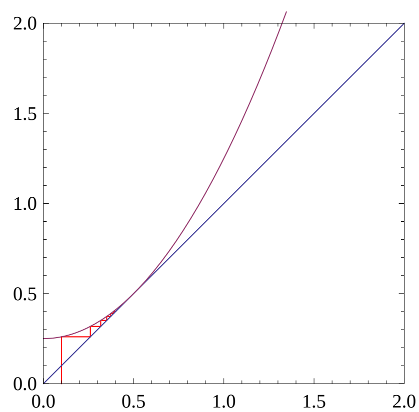
$$f'(c) = \frac{f(x) - f(0)}{x - 0}, \quad \text{or} \quad 0 < f(x) = f'(c)x < x.$$

Continuing in this way, we see that

$$0 < f^n(x) < f^{n-1}(x) < \ldots < f(x) < x,$$

so we have a decreasing sequence $x_n = f^n(x)$ bounded below by 0. It follows that this sequence converges. But Proposition 1.4.5 implies that if this sequence converges, it must converge to a fixed point. $c = 0$ being the only fixed point gives $f^n(x) \to 0$ as $n \to \infty$. A similar argument can be used for $-1 \leq x < 0$. $\square$

Example 1.5.3 It is possible for the fixed point to be unstable, but to have a one-sided stability (to be *semi-stable*). For example, consider $f(x) = x^2 + 1/4$ which has the single (non-hyperbolic) fixed point $c = 1/2$. This fixed point is stable from the left, but unstable on the right.



In the following we give some criteria for non-hyperbolic fixed points to be asymptotically stable/unstable etc. It also gives a criteria for semi-stability.

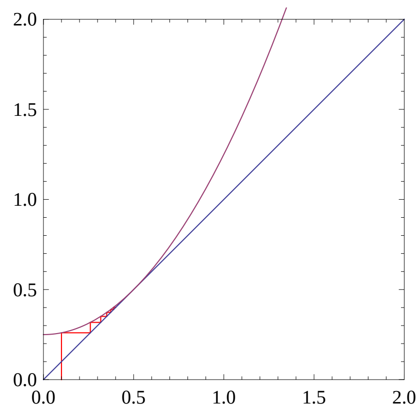
Theorem 1.5.4 *Let c be a non-hyperbolic fixed point of $f(x)$ with $f'(c) = 1$. If $f'(x)$, $f''(x)$ and $f'''(x)$ are continuous at $x = c$, then:*

(i) *if $f''(c) \neq 0$, the c is semi-stable,*

(ii) *if $f''(c) = 0$ and $f'''(c) > 0$, then c is unstable,*

(iii) *If $f''(c) = 0$ and $f'''(c) < 0$, then c is asymptotically stable.*

Proof. (i) If $f'(c) = 1$ then $f(x)$ is tangential to $y = x$ at $x = c$. Suppose that $f''(c) > 0$, then $f(x)$ is concave up at $x = c$ and the picture must look like the following:



We see this gives stability on the left and instability on the right.

More formally, since the derivatives are continuous, and $f''(c) > 0$, this will be true in some small interval $(c-\delta, c+\delta)$ surrounding c. In particular, the derivative function $f'(x)$ must be increasing on that interval, so that since $f'(c) = 1$ then

$$f'(x) < 1 \quad \text{for all} \quad x \in (c-\delta, c), \quad \text{and} \quad f'(x) > 1 \quad \text{for all} \quad x \in (c, c+\delta),$$

for some $\delta > 0$. Also, from the continuity of $f'(x)$, we can assume that $f'(x) > 0$ in this interval.

Now by the Mean Value Theorem applied to the interval $[x, c] \subset (c-\delta, c]$, there exists $q \in (x, c)$ with

$$f'(q) = \frac{f(x) - f(c)}{x - c},$$

Now since $0 < f'(q) < 1$ and $c > x$, we have

$$0 < \frac{f(x) - f(c)}{x - c} < 1,$$

or

$$x < f(x) < c.$$

Repeating this argument, we see that the sequence $f^n(x)$ is increasing and bounded above by c, so must converge to a fixed point. There can be no other fixed point (say $d \neq c$), in this interval as the Mean Value Theorem would give $f'(q_1) = 1$ for some $q_1 \in (x, c)$, a contradiction. Consequently we see that c is stable on the left.

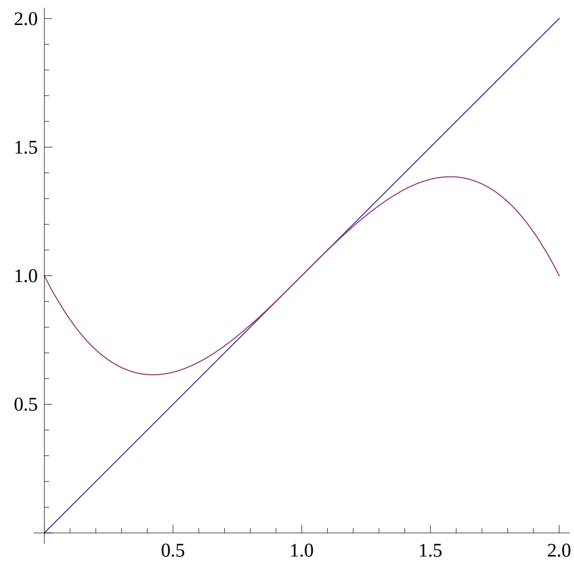
On the other hand, if $[c, x] \subset (c, c+\delta)$, then applying the Mean Value Theorem as above gives

$$f'(q) = \frac{f(x) - f(c)}{x - c} > 1, \quad \text{so} \quad f(x) > x > c,$$

since $x - c > 0$. This tells us that the point moves away from c under iteration, so the fixed point is unstable on the right. Similar considerations can be used when $f''(c) < 0$ and the graph is concave down at $x = c$.

(ii) is similar to (iii), so is omitted.

(iii) In this case $f'''(c) < 0$, $f''(c) = 0$ and $f'(c) = 1$. We will show that we have a point of inflection at $x = c$ as in the following picture:



By the second derivative test, $f'(x)$ has a local maximum at $x = c$ (the continuous function $f'(x)$ is concave down). It follows that

$$f'(x) < 1 \quad \text{for all} \quad x \in (c - \delta, c + \delta),\ x \neq c,$$

for some $\delta > 0$ ($f''(x) > 0$ for $x \in (c-\delta, c)$, so $f'(x)$ is increasing there, and $f''(x) < 0$ for $x \in (c, c+\delta)$, so $f'(x)$ is decreasing there). In particular $f'(x) \neq 1$ if $x \neq c$.

Now use an argument similar to that of (i) above to deduce the result. □

Example 1.5.5 Returning to the function $f(x) = \sin x$, we see that $f'(0) = 1$, $f''(0) = 0$ and $f'''(0) = -1$, so the conditions of Theorem 1.5.4 (iii) are satisfied and we conclude that $x = 0$ is an asymptotically stable fixed point.

If $f(x) = \tan x$, then $f'(0) = 1$ and we can check that Theorem 1.5.4 (ii) holds so that the fixed point $x = 0$ is unstable. For $f(x) = x^2 + 1/4$, with $f(1/2) = 1/2$, $f'(1/2) = 1$, we can apply Theorem 1.5.4 (i).

How do we treat the case where $f'(c) = -1$ at the fixed point? We use:

Definition 1.5.6 The *Schwarzian derivative* $Sf(x)$ of $f(x)$ is the function

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2,$$

so that

$$Sf(x) = -f'''(x) - \frac{3}{2}[f''(x)]^2, \quad \text{when} \quad f'(x) = -1.$$

Theorem 1.5.7 *Suppose that c is a fixed point for $f(x)$ and $f'(c) = -1$. If $f'(x)$, $f''(x)$ and $f'''(x)$ are continuous at $x = c$ then:*

(i) *if $Sf(c) < 0$, then $x = c$ is an asymptotically stable fixed point,*

(ii) *if $Sf(c) > 0$, then $x = c$ is an unstable fixed point.*

Proof. (i) Set $g(x) = f^2(x)$, then $g(c) = c$ and we see that if c is asymptotically stable with respect to g, then it is asymptotically stable with respect to f. Now

$$g'(x) = \frac{d}{dx}\left(f(f(x))\right) = f'(f(x)) \cdot f'(x),$$

so that $g'(c) = f'(c) \cdot f'(c) = (-1)(-1) = 1$.

The idea is to apply Theorem 1.5.4 (iii) to the function $g(x)$. Now

$$g''(x) = f'(f(x)) \cdot f''(x) + f''(f(x)) \cdot [f'(x)]^2,$$

thus

$$g''(c) = f'(c)f''(c) + f''(c)[f'(c)]^2 = 0, \quad \text{since} \quad f'(c) = -1.$$

Also

$$g'''(x) = f''(f(x))\cdot f'(x)\cdot f''(x) + f'(f(x))\cdot f'''(x) + f'''(f(x))[f'(x)]^3 + f''(f(x))\cdot 2f'(x)f''(x).$$

Therefore

$$\begin{aligned} g'''(c) &= [f''(c)]^2(-1) - f'''(c) - f'''(c) + 2f''(c)(-1)f''(c) \\ &= -2f'''(c) - 3[f''(c)]^2 \\ &= 2Sf(c) < 0, \end{aligned}$$

and the result follows from Theorem 1.5.4 (iii).

(ii) This now follows from Theorem 1.5.4 (ii) $\square$

Remark 1.5.8 The above proof shows how the Schwarzian derivative arises from differentiating $g = f \circ f = f^2$. In the case where $f'(c) = -1$, it follows that

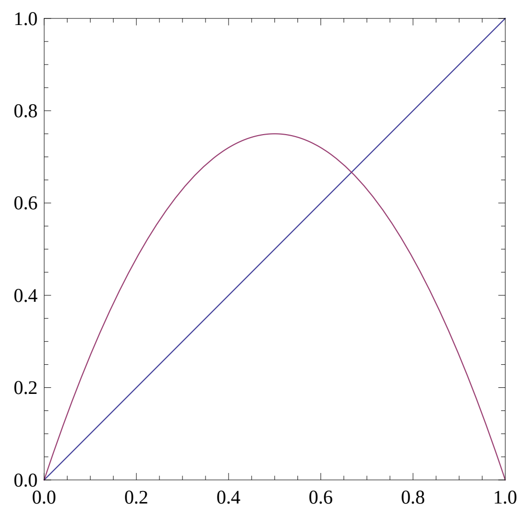
$$g''(c) = 0 \quad \text{and} \quad Sf(c) = \frac{1}{2}g'''(c).$$

Example 1.5.9 For the logistic map $L_\mu(x) = \mu x(1-x)$ we have

$$L'_\mu(x) = \mu - 2\mu x, \quad L''_\mu(x) = -2\mu \quad \text{and} \quad L'''_\mu(x) = 0.$$

When $\mu = 1$, $x = 0$ is the only fixed point and Theorem 1.5.4 (i) shows that $x = 0$ is semi-stable (attracting on the right). However, we regard this as a stable fixed point for L_μ defined on the interval $[0,1]$, since points to the left of 0 are not in the domain of L_μ.

When $\mu = 3$, $c = 2/3$ is fixed and $L'_\mu(2/3) = -1$ giving a non-hyperbolic fixed point. However, $Sf(2/3) = 0 - \frac{3}{2}[6]^2 < 0$ (negative Schwarzian derivative), so by Theorem 1.5.7 (i), $x = 2/3$ is asymptotically stable.



Exercises 1.5

1. Show that $f(x) = -2x^3+2x^2+x$ has two non-hyperbolic fixed points and determine their stability.

2. For the family of quadratic maps $Q_c(x) = x^2 + c$, $x \in \mathbb{R}$, use the Theorems of Section 1.5 to determine the stability of the fixed points for all possible values of c. Find any values of c so that Q_c has a non-hyperbolic fixed point, and determine their stability.

3. Find the fixed points of the following maps and use the appropriate theorems to determine whether they are asymptotically stable, semi-stable or unstable:

$$\text{(i) } f(x) = \frac{x^3}{2} + \frac{x}{2}, \quad \text{(ii) } f(x) = \arctan x, \text{(iii) } f(x) = x^3 + x^2 + x,$$

$$\text{(iv) } f(x) = x^3 - x^2 + x, \quad \text{(v) } f(x) = \begin{cases} 0.8x & ;\text{if} \quad x \leq 1/2 \\ 0.8(1-x) & ;\text{if} \quad x > 1/2 \end{cases}$$

4. Let $f(x) = ax^2 + bx + c$, $a \neq 0$, and p a fixed point of f. Prove the following:

(i) If $f'(p) = 1$, then p is semistable.

(ii) If $f'(p) = -1$, then p is asymptotically stable.

5. If $f(x) = \dfrac{ax+b}{cx+d}$, $a, b, c, d \in \mathbb{R}$ is the linear fractional transformation, show that its Schwarzian derivative is $Sf(x) = 0$ for all x in its domain.

6. If $Sf(x)$ is the Schwarzian derivative of $f(x)$, a C^3 function and $F(x) = \dfrac{f''(x)}{f'(x)}$, show that $Sf(x) = F'(x) - (F(x))^2/2$.

7. If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$, find the fixed points of f and show that for $x \neq 0$ they are non-hyperbolic. Show that $x = 0$ is not an isolated fixed point (i.e., there are other fixed points arbitrarily close to 0). Is $x = 0$ a stable, attracting or repelling fixed point?

8. ([50]) Let N_f be the Newton function of a four times continuously differentiable function f. If $f(\alpha) = 0$, show that $N_f'''(\alpha) = 2Sf(\alpha)$, where Sf is the Schwarzian derivative of f.

9. (a) Use the Intermediate Value Theorem to show that $f(x) = \cos(x)$ has a fixed point c in the interval $[0, \pi/2]$. We can show experimentally that this fixed point is approximately $c = .739085\ldots$, for example by iterating any $x_0 \in \mathbb{R}$.

(b) Show that the basin of attraction of c is all of $\mathbb{R}$ (Hint: You may assume that $x \in [-1, 1]$ - why? Now use the Mean Value Theorem to show that $|f(x)-c| < \lambda|x-c|$ for some $0 < \lambda < 1$).

(c) Does $f(x)$ have any eventual fixed points?

(d) Can $f(x)$ have any points p with $f^2(p) = p$ other than c?

Chapter 2, Bifurcations and the Logistic Family

2.1 The Basin of Attraction

In this chapter we examine the basin of attraction of the logistic maps, i.e., for a given μ and fixed point $x = c$ we look for the set of those $x \in [0, 1]$ which converge to c under iteration by $L_\mu(x) = \mu x(1-x)$.

Definition 2.1.1 The *basin of attraction* $B_f(c)$ of a fixed point c of $f(x)$ is the set of all x for which the sequence $x_n = f^n(x)$ converges to c:

$$B_f(c) = \{x \in X : f^n(x) \to c, \text{ as } n \to \infty\}.$$

The *immediate basin of attraction* of f is the largest interval containing c, contained in the basin of attraction of c. We first show that this is always an open interval when c is an attracting fixed point.

Proposition 2.1.2 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function having an attracting fixed point p. The immediate basin of attraction of p is an open interval.*

Proof. Since p is an attracting fixed point there is $\epsilon > 0$ such that for all $x \in I_\epsilon = (p-\epsilon, p+\epsilon)$, $f^n(x) \to p$ as $n \to \infty$. Denote by J the largest interval containing p for which $f^n(x) \to p$ for $x \in J$, as $n \to \infty$.

Suppose that $J = [a, b]$, a closed interval, then there exists $r \in \mathbb{Z}^+$ with $f^r(a) \in I_\epsilon$. Now f^r is also a continuous function, so points close to a will also get mapped into I_ϵ, leading to a contradiction.

More specifically, there exists $\delta > 0$ such that if $|x-a| < \delta$, then $|f^r(x)-f^r(a)| < \eta$, where $\eta = \min\{|f^r(a) - (p-\epsilon)|, |(p+\epsilon) - f^r(a)|\}$. Thus there are points $x < a$ close to a for which $f^r(x) \in I_\epsilon$, so $f^{rn}(x) \to p$ as $n \to \infty$, a contradiction. We conclude that $a \notin J$ (and similarly for b), so J is an open interval. $\square$

Example 2.1.3 1. If $f(x) = x^2$, the fixed points are $c = 0$ and $c = 1$, both hyperbolic, the first being attracting and the second repelling. Clearly $B_f(0) = (-1, 1)$ and $B_f(1) = \{-1, 1\}$. We sometimes regard $c = \infty$ as an attractive fixed point of f, so that $B_f(\infty) = [-\infty, -1) \cup (1, \infty]$.

2. If $f : I \to I$ is a continuous function on a closed interval $I = [a, b]$ having an attracting fixed point $p \in (a, b)$, then we cannot exclude the possibility that the basin of attraction includes either a or b, or both. On the other hand, if $x = a$ is the attracting fixed point, the basin of attraction may be a set of the form $[a, c)$ for some

$c \in (a, b]$. Such a set can be regarded as being open as a subset of $[a, b]$ (see Chapter 4).

2.2 The Logistic Family

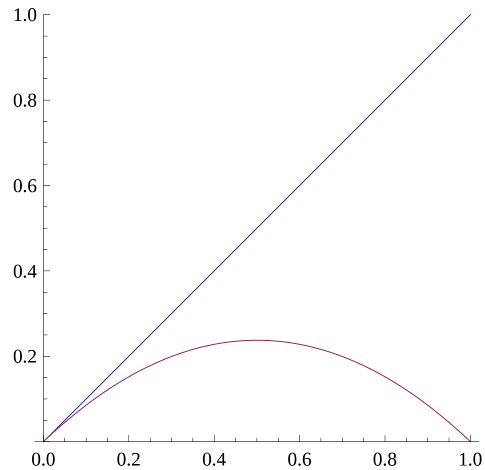
The logistic maps $L_\mu(x) = \mu x(1-x)$ are functions of two real variables μ and x. We usually restrict x to the interval $[0, 1]$ and consider $\mu \in (0, 4]$. μ is a parameter which we allow to vary, but then we study the function L_μ for specific fixed values of μ. As the parameter μ is varied, we see a corresponding change in the nature of the function L_μ. This is what is called *bifurcation*. For example, for $0 < \mu \leq 1$, L_μ has exactly one fixed point in $[0, 1]$, $c = 0$, which is attracting. As μ increases beyond 1, a new fixed point $c = 1 - 1/\mu$, is created in $[0, 1]$, so now L_μ has two fixed points. $c = 0$ is now repelling and $c = 1 - 1/\mu$ is attracting (for $1 < \mu \leq 3$). At $\mu = 3$ the nature of these fixed points changes again as we shall see. In this section we determine the basin of attraction of these fixed points as μ increases from 0 to 3. We see that the "dynamics" (long term behavior) of L_μ is quite uncomplicated for this range of values of μ.

The function $L_\mu(x) = \mu x(1-x)$, $0 \leq x \leq 1$ has a maximum value of $\mu/4$ when $x = 1/2$. Consequently, for $0 < \mu \leq 4$, L_μ maps the unit interval $[0, 1]$ into itself. We shall consider later what happens when $\mu > 4$. We start by showing that the basin of attraction of L_μ for $0 < \mu \leq 1$ is all of the domain of L_μ, namely $[0, 1]$. We say that 0 is a *global attractor* in this case.

Theorem 2.2.1 *Let* $L_\mu(x) = \mu x(1-x)$, $0 \leq x \leq 1$ *be the logistic map. For* $0 < \mu \leq 1$, $B_{L_\mu}(0) = [0, 1]$ *and for* $1 < \mu \leq 3$, $B_{L_\mu}(1 - 1/\mu) = (0, 1)$.

We split the proof into a number of different cases:

Case 2.2.2 $\boldsymbol{0 < \mu \leq 1}$.



For $0 < \mu < 1$, the only fixed point is 0.

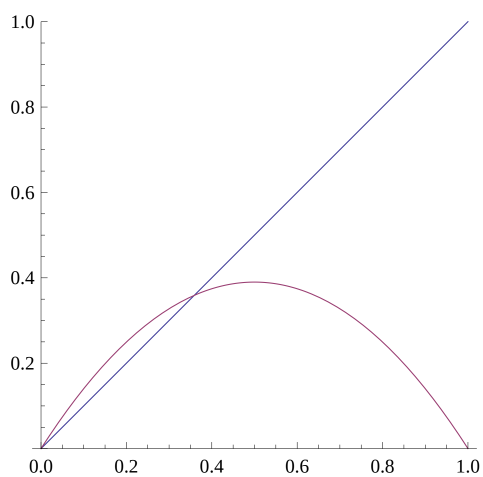
We have seen that for $\mu \in (0,1)$, L_μ has only the one fixed point $x = 0$ in $[0,1]$ (the other fixed point is $1 - 1/\mu \leq 0$). For $\mu < 1$ this fixed point is asymptotically stable, but in any case

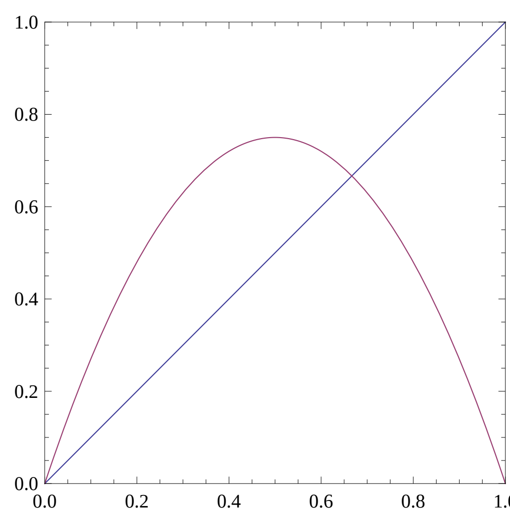
$$0 < \mu \leq 1, \;\; 0 < 1 - x < 1 \Rightarrow 0 < \mu(1-x) < 1$$

$$\Rightarrow 0 < L_\mu(x) = \mu x(1-x) < x, \;\; x \in (0,1],$$

and in a similar way, $L_\mu^2(x) < L_\mu(x)$ etc., so the sequence $L_\mu^n(x)$ is decreasing, bounded below by 0, and hence must converge to the only fixed point, namely 0. It follows that the basin of attraction is $B_{L_\mu}(0) = [0,1]$ $(L_\mu(1) = 0)$.

Case 2.2.3 $\boldsymbol{1 < \mu \leq 3}$.





$1 - 1/\mu < 1/2$ for $1 < \mu < 2$. $1 - 1/\mu \geq 1/2$ for $2 \leq \mu \leq 3$.

We have seen that for $\mu > 1$ the fixed point 0 is repelling, but a new fixed point $c = 1 - 1/\mu$ has been "born" which is attracting (for $1 < \mu \leq 3$), so by Proposition 2.1.2 there is a largest open interval $I = (a, b)$ containing the fixed point for which $x \in I$ implies $L_\mu^n(x) \to c$ as $n \to \infty$. If the basin of attraction of c is $B_\mu(c)$, then $0, 1 \notin B_\mu(c)$ because $L_\mu(0) = 0$ and $L_\mu(1) = 0$, so $B_\mu(c) \neq [0, 1]$. Furthermore, clearly $a, b \notin B_\mu(c)$.

From the Intermediate Value Theorem, $L_\mu(a, b)$ is an interval which must be contained in (a, b), for if $x \in (a, b)$, $L_\mu^n(L_\mu(x)) \to c$ as $n \to \infty$. Let x_n be a sequence in (a, b) with $\lim_{n\to\infty} x_n = a$, then by the continuity of L_μ, $\lim_{n\to\infty} L_\mu(x_n) = L_\mu(a)$. Since $x_n \in (a, b)$ for every $n \in \mathbb{Z}^+$, we have $L_\mu(x_n) \in (a, b)$ for all $n \in \mathbb{Z}^+$. Since $L_\mu(a) \notin (a, b)$, the only way this is possible is if $L_\mu(a) = a$ or $L_\mu(a) = b$, and similarly for b. This is only possible if a and b are fixed points, are eventual fixed points or $L_\mu(a) = b, L_\mu(b) = a$ (we call $\{a, b\}$ a 2-cycle - see Section 2.3). We show that the latter case cannot happen, and this leads to the conclusion that $a = 0$ and $b = 1$, since there are no other fixed or eventual fixed points in $[0, 1]$ that can satisfy these condition. Consequently, we must have $B_\mu(1 - 1/\mu) = (0, 1)$.

Now we can check that

$$L_\mu^2(x) - x = \mu^2 x(1 - (\mu+1)x + 2\mu x^2 - \mu x^3) - x = x(\mu^2 x^2 - \mu(\mu+1)x + \mu+1)(\mu x - \mu + 1),$$

and a and b must satisfy this equation. We can disregard the linear factors as they give the two fixed points. The discriminant of the quadratic factor is $\mu^2(\mu + 1)^2 - 4\mu^2(\mu+1) = \mu^2(\mu+1)(\mu-3) < 0$ for $1 < \mu < 3$, so there is no 2-cycle when $1 < \mu < 3$. When $\mu = 3$, the discriminant is zero, the fixed point is $c = 2/3$ and the quadratic factor gives rise to no additional roots, so again there is no 2-cycle. $\square$

Remark 2.2.4 Note that when $\mu = 2$, $x = 1/2$ is a super attracting fixed point (since $L_2'(1/2) = 0$). As we saw above the basin of attraction will be $(0, 1)$.

Exercises 2.2

1. If $L_\mu(x) = \mu x(1 - x)$ is the logistic map, show that $x = 1/2$ is the only turning point of $L^2(x)$ for $0 < \mu \leq 2$, but when $\mu > 2$, two new turning points are created.

Use this to show that for $2 < \mu < 3$, the interval $[1/\mu, 1-1/\mu]$ is mapped by L_μ^2 onto the interval $[1/2, 1-1/\mu]$.

2.3 Periodic Points

Points with finite orbits are of importance in the study of dynamical systems and their long-term behavior:

Definition 2.3.1 Let $f : X \to X$ be a function with $c \in X$.

(i) c is a *periodic point* of $f(x)$ with period $r \in \mathbb{Z}^+$ if $f^r(c) = c$ and $f^k(c) \neq c$ for $0 < k < r$ (in particular, c is a fixed point of f^r). We call r the *period* of c and the set $O(c) = \{c, f(c), f^2(c), \ldots, f^{r-1}(c)\}$ is an *r-cycle*. We write

$$\mathrm{Per}_r(f) = \{x \in X : f^r(x) = x\},$$

so that $\mathrm{Fix}(f) \subseteq \mathrm{Per}_n(f)$, $n = 1, 2, \ldots$, since the points in $\mathrm{Per}_n(f)$ may not be of period n, but of some lesser period.

(ii) c is *eventually periodic* for f if there exists $m \in \mathbb{Z}^+$ such that $f^m(c)$ is a periodic point of f (we assume that c is not a periodic point).

(iii) c is *stable* (respectively *asymptotically stable, unstable* etc.) if it is a stable fixed point of f^r.

The following criteria for stability now immediately follows from Theorem 1.3.3:

Theorem 2.3.2 *Suppose that c is a point of period r for f and that $f'(x)$ is continuous at $x = c$. If $c_i = f^i(c)$, $i = 0, 1, \ldots, r-1$ then:*

(i) *c is asymptotically stable if*

$$|f'(c_0) \cdot f'(c_1) \cdot f'(c_2) \cdots f'(c_{r-1})| < 1.$$

(ii) *c is unstable if*

$$|f'(c_0) \cdot f'(c_1) \cdot f'(c_2) \cdots f'(c_{r-1})| > 1.$$

Proof. Let us look at the case where $r = 3$ as this is typical;

$$O(c) = \{c, f(c), f^2(c)\} = \{c_0, c_1, c_2\}.$$

Then

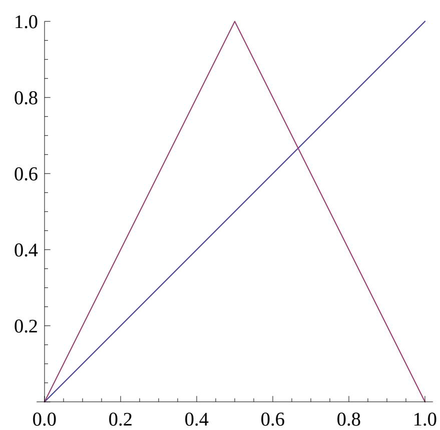
$$\frac{d}{dx}(f^3(x)) = \frac{d}{dx}(f(f^2(x))) = f'(f^2(x))(f^2(x))' = f'(f^2(x))f'(f(x))f'(x)$$

$$= f'(c_2)f'(c_1)f'(c_0), \quad \text{when} \quad x = c.$$

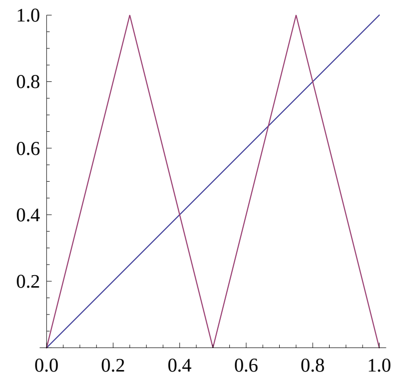
The result now follows from 1.3.3. $\square$

Example 2.3.3 If we look at the graph of the tent map $T(x)$, we see it has two fixed points, $c = 0$ and $c = 2/3$. If we graph T^3 it has eight fixed points: These arise from two 3-cycles: $\{2/7, 4/7, 6/7\}$ and $\{2/9, 4/9, 8/9\}$, together with the two fixed points, so that

$$\text{Per}_3(T) = \{0, 2/3, 2/7, 4/7, 6/7, 2/9, 4/9, 8/9\}.$$



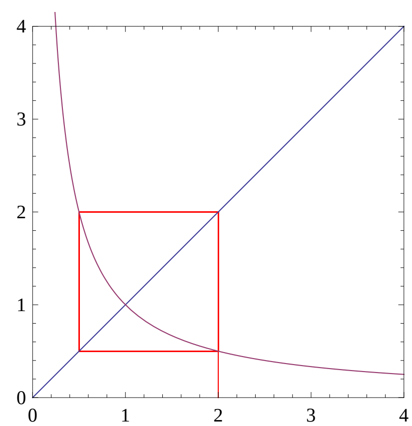
The graph of T^2 shows that T^2 has four fixed points coming from a 2-cycle $\{2/5, 4/5\}$ and the two fixed points. Since $|T'(x)| = 2$ and $|(T^2)'(x)| = |T'(x)||T'(Tx)| = 4$, etc. (except at points of non-differentiability), all these periodic points will be unstable.



Example 2.3.4 Let $f(x) = 1/x$, $x \neq 0, x \neq \pm 1$. Note that $f^2(x) = x$ and $f(x) \neq x$ for all such x, giving rise to the 2-cycle $\{x, 1/x\}$. In this case

$$|f'(x)f'(1/x)| = |-1/x^2(-x^2)| = 1,$$

so the theorem is inconclusive. However we see that the periodic points are stable but neither attracting nor repelling.



Example 2.3.5 Consider the quadratic function $f(x) = x^2 - 2$. We have seen that to find the fixed points we solve $f(x) = x$, or $x^2 - 2 = x$, $x^2 - x - 2 = (x-2)(x+1) = 0$, so $x = 2$ or $x = -1$.

To find the period 2-points we solve $f^2(x) = x$ or $f^2(x) - x = 0$. This is simplified when we realize that the fixed points must be solutions of this equation, so that $(x-2)(x+1)$ is a factor. We can then check that

$$f^2(x) - x = x^4 - 4x^2 - x + 2 = (x-2)(x+1)(x^2 + x - 1).$$

Solving the quadratic gives

$$x = \frac{-1 \pm \sqrt{5}}{2},$$

so that $\{\frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}\}$ is a 2-cycle. In general, finding periodic points of quadratics can be complicated. If $f(x)$ is a quadratic, $f^n(x) - x$ is a polynomial of degree 2^n.

To check the stability, we calculate

$$|f'((-1-\sqrt{5})/2)f'((-1+\sqrt{5})/2)| = |(-1-\sqrt{5})(-1+\sqrt{5})| = |1-5| = 4 > 1,$$

giving an unstable 2-cycle.

Remark 2.3.6 1. As before, periodic points can be stable but not attracting (as above with $f(x) = 1/x$ at $x \neq 1$). They can also be attracting but not stable as in Example 1.4.7).

2. Functions such as $f(x) = \sin x$ can have no period 2 points or points of a higher period since this would contradict the basin of attraction of $x = 0$ being all of $\mathbb{R}$. Similarly, the logistic map L_μ, $0 < \mu \leq 3$ cannot have period n-points for $n > 1$.

Exercises 2.3

1. For each of the following functions, $c = 0$ lies on a periodic cycle. Classify this cycle as attracting, repelling or neutral (non-hyperbolic). Say if it is super-attracting:

$$\text{(i)}\ \ f(x) = \frac{\pi}{2}\cos x, \quad \text{(ii)}\ \ g(x) = -\frac{1}{2}x^3 - \frac{3}{2}x^2 + 1.$$

2. (a) Show that $C_\mu(x) = \mu\cos(x)$ has a super-attracting 3-cycle $\{0, \lambda, \pi/2\}$ where $\mu = \lambda$ and λ satisfies the equation $\lambda\cos(\lambda) = \pi/2$.

(b) Give similar conditions for $S_\mu(x) = \mu\sin(x)$ to have (i) a super-attracting 2-cycle, (ii) a super-attracting 3-cycle.

(c) Explain why for families of maps, say F_μ, one member of a super-attracting n-cycle is a super-attracting fixed point (for a different value of μ).

3. Find the fixed points and the period two points of following maps (if any) and determine the stability of the 2-cycle:

(i) $f(x) = \dfrac{x^2}{2} - x + \dfrac{1}{2}$, (ii) $f(x) = a - \dfrac{b}{x}$ $(b \neq 0)$, (iii) $f(x) = \dfrac{1-x}{3x+1}$, (iv) $f(x) = |x-1|$.

4. Let $Q_c(x) = x^2 + c$. Show that for $c < -3/4$, Q_c has a 2-cycle and find it explicitly. For what values of c is the 2-cycle attracting?

5. Let $f(x) = ax^2 + bx + c$, $a \neq 0$. Let $\{x_0, x_1\}$ be a 2-cycle for $f(x)$,

(a) If $f'(x_0)f'(x_1) = -1$, prove that the 2-cycle is asymptotically stable.
(b) If $f'(x_0)f'(x_1) = 1$, prove that the 2-cycle is asymptotically stable.

6. Let $f(x) = ax^3 - bx + 1$, $a \neq 0$. If $\{0, 1\}$ is a 2-cycle for $f(x)$, give conditions on a and b under which it is asymptotically stable.

7. Let $f(x)$ be a polynomial with $f(c) = c$.

(i) If $f'(c) = 1$, show that $(x - c)^2$ is a factor of $g(x) = f(x) - x$.

(ii) If $f'(c) = 1$, or $f'(c) = -1$, show that $(x-c)^2$ is a factor of $h(x) = f^2(x) - x$ (i.e., if $f(x)$ has a non-hyperbolic fixed point c, then c is a repeated root of $f^2(x) - x$). Hint: Recall that a polynomial $p(x)$ has $(x-c)^2$ as a factor if and only if both $p(c) = 0$ and $p'(c) = 0$.

(iii) Show in the case that $f'(c) = -1$ we actually have that $(x - c)^3$ is a factor of $h(x) = f^2(x) - x$.

(iv) Check that (iii) holds for the non-hyperbolic fixed point $x = 2/3$, of the logistic map $L_3(x) = 3x(1 - x)$.

(v) Check that (i), (ii) and (iii) hold for the (non-hyperbolic) fixed points of the polynomial $f(x) = -2x^3 + 2x^2 + x$.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is differentiable everywhere. If $\{x_0, x_1, \ldots, x_{n-1}\}$ is an n-cycle for f, show that the derivative of f^n is the same at each x_i, $i = 0, 1, \ldots, n-1$.

9. Show that if x_0 is a peridic point of f with period n and $f^m(x_0) = x_0$ $m \in \mathbb{Z}^+$, then $m = kn$ for some $k \in \mathbb{Z}^+$. (Hint: Write $m = qn + r$ for some r, $0 \leq r < n$ and show that $r = 0$).

10. Show that if $f^p(x) = x$ and $f^q(x) = x$, and n is the highest common factor of p and q, then $f^n(x) = x$. (Hint: Use the previous question and the fact that every common factor of p and q is a factor of n).

11. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be an *odd function* ($f(-x) = -f(x)$ for all $x \in \mathbb{R}$). Show that $x = 0$ is a fixed point and that the intersection of the graph of f with the line $y = -x$ gives rise to points c of period 2 (when $c \neq 0$).

(b) Use part (a) to find the 2-cycles of $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - 3x/2$ and determine their stability.

(c)) Let $f : \mathbb{R} \to \mathbb{R}$ be an *even function* ($f(-x) = f(x)$ for all $x \in \mathbb{R}$). Show that the intersection of the graph of f with the line $y = -x$ gives rise to eventually fixed points c (when $c \neq 0$). What are the eventually fixed points of $f(x) = \cos(x)$?

12. (a) Let $f(x) = \begin{cases} x\sin(1/x) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$. In Exercises 1.7. # 7, we found the fixed points of f and showed that the non-zero fixed points are non-hyperbolic. Find the eventual fixed points of f (note that $f(x)$ is an even function).

(b) (Open ended question) Discuss the stability of the fixed points of $f(x)$, their basins of attractions and the existence of period 2-points. Note that $f(x) \leq x$ for all $x > 0$.

(c) Now set $g(x) = \begin{cases} x\cos(1/x) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$. Find the fixed points of g, and show that in this case there are period 2-points (Hint: note that $g(x)$ is an odd function).

2.4 Periodic Points of the Logistic Map

We try to find the 2-cycles of the logistic map $L_\mu(x) = \mu x(1-x)$, $0 \leq x \leq 1$. To do this we solve the equation

$$L_\mu^2(x) = x,$$

or

$$\mu x(1-x)[1-\mu x(1-x)] - x = 0,$$

or

$$-\mu^3 x^4 + 2\mu^3 x^3 - (\mu^3+\mu^2)x^2 + \mu^2 x - x = 0,$$

Clearly x is a factor (since $c = 0$ is a fixed point of $L_\mu(x)$) and similarly $x - (1-1/\mu)$ must be a factor so we get

$$L_\mu^2(x) - x = -x(\mu x - \mu + 1)(\mu^2 x^2 - \mu(\mu+1)x + \mu + 1),$$

giving a quadratic equation which has no roots if $\mu < 3$:

$$\mu^2 x^2 - \mu(\mu+1)x + \mu + 1 = 0.$$

Solving using the quadratic formula gives

$$\begin{aligned} c &= \frac{\mu(\mu+1) \pm \sqrt{\mu^2(\mu+1)^2 - 4\mu^2(\mu+1)}}{2\mu^2} \\ &= \frac{(1+\mu) \pm \sqrt{(\mu-3)(\mu+1)}}{2\mu}. \end{aligned}$$

This is real only for $\mu \geq 3$ (called the "birth of period two"). Let us call these two roots c_1 and c_2 (dependent on μ).

This 2-cycle is asymptotically stable if

$$|(L_\mu^2)'(c_1)| = |L_\mu'(c_1)L_\mu'(c_2)| < 1,$$

or

$$\begin{gathered} -1 < \mu^2(1-2c_1)(1-2c_2) < 1, \\ -1 < (-1-\sqrt{(\mu^2-2\mu-3)})(-1+\sqrt{(\mu^2-2\mu-3)}) < 1, \\ -1 < 1 - (\mu^2 - 2\mu - 3) < 1, \end{gathered}$$

and this gives rise to the two inequalities

$$\mu^2 - 2\mu - 3 > 0 \quad \text{and} \quad \mu^2 - 2\mu - 5 < 0,$$

and solving gives

$$3 < \mu < 1 + \sqrt{6}.$$

This is the condition for asymptotic stability of the 2-cycle $\{c_1, c_2\}$.

For $\mu = 1+\sqrt{6}$, it can be seen that

$$L'_\mu(c_1)L'_\mu(c_2) = -1, \quad \text{and} \quad SL^2_\mu(c_1) < 0,$$

so Theorem 1.5.7 (i) shows that the 2-cycle is asymptotically stable. Also, the 2-cycle is unstable for $\mu > 1+\sqrt{6}$. In summary:

Theorem 2.4.1 *For $3 < \mu \leq 1+\sqrt{6}$, the logistic map $L_\mu(x) = \mu x(1-x)$ has an asymptotically stable 2-cycle. The 2-cycle is unstable for $\mu > 1+\sqrt{6}$.*

The above shows that we have a bifurcation when $\mu = 3$ where a 2-cycle is created and which was not previously present. There is another bifurcation at $\mu = 1+\sqrt{6}$. This means that for $3 < \mu \leq 1+\sqrt{6}$, when we use graphical iteration of points close to c_1 and c_2, they will approach the period 2 orbit, and not the fixed point (which is now unstable). In fact it can be shown that for this range of values of μ, the basin of attraction of the 2-cycle consists of all of $(0,1)$, (except for the fixed point $1-1/\mu$ and eventual fixed points such as $1/\mu$). When μ exceeds $1+\sqrt{6} = 3.449499\ldots$, the period 2-points become unstable and this no longer happens, but we shall see that something different happens. We have another bifurcation when $\mu = 1+\sqrt{6}$, with the birth of an attracting period 4-cycle.

2.5 The Period Doubling Route to Chaos

We summarize what we have determined so far and talk about what happens as μ increases from zero to around 3.57.

For $\mu < b_1 := 1$, $c = 0$ is the only fixed point and it is attracting for these values of μ. There is a bifurcation at $b_1 = 1$, where a non-zero fixed point $c = 1-1/\mu$ is created. This fixed point is attracting for $\mu \leq 3$ (and $c = 0$ is no longer attracting) and super attracting when $\mu = s_1 = 2$. The second bifurcation occurs when $\mu = b_2 = 3$. The fixed point $c = 1-1/\mu$ becomes unstable, and an attracting 2-cycle is created for $3 < \mu < 1+\sqrt{6} = b_3$.

2.5.1 A Super-Attracting Period 2-Cycle

We saw that when $\mu = 2$, $c = 1/2$ is a super-attracting fixed point for L_μ. We now look for a super-attracting period 2-cycle for L_μ when $3 < \mu < 1+\sqrt{6}$, as it illustrates an important general method that can be used for finding where period three is born:

Suppose that $\{x_1, x_2\}$ is a 2-cycle for the logistic map L_μ, which is super-attracting, then

$$x_1 = \mu x_2(1 - x_2), \quad \text{and} \quad x_2 = \mu x_1(1 - x_1),$$

so multiplying these equations together gives the equation

$$\mu^2(1 - x_1)(1 - x_2) = 1.$$

In addition we must have

$$(L_\mu^2)'(x_1) = L_\mu'(x_1)L_\mu'(x_2) = 0,$$

so that

$$\mu^2(1 - 2x_1)(1 - 2x_2) = 0.$$

Thus either $x_1 = 1/2$, or $x_2 = 1/2$, so suppose the former holds, then $x_2 = \mu/4$. Substituting into the first equation gives

$$\mu^2(1 - \mu/4)(1 - 1/2) = 1,$$

or

$$\mu^3 - 4\mu^2 + 8 = 0.$$

$\mu - 2$ must is a factor of this cubic, so we have

$$(\mu - 2)(\mu^2 - 2\mu - 4) = 0,$$

and this gives $\mu = 1 + \sqrt{5}$. We have shown:

Proposition 2.5.2 *When $\mu = s_2 = 1 + \sqrt{5}$, $\{1/2, \dfrac{1+\sqrt{5}}{4}\}$ is a super-attracting 2-cycle for L_μ.*

When μ exceeds $b_3 = 1 + \sqrt{6}$, the 2-cycle ceases to be attracting and becomes repelling. In addition, a 4-cycle is created which is attracting until μ exceeds a value b_4 when it becomes repelling and an attracting 8-cycle is created. This type of period doubling continues so that when μ exceeds b_n, an attracting 2^{n-1}-cycle is created until μ reaches b_{n+1}. These cycles become super attracting at some s_n ($b_n < s_n < b_{n+1}$). This behavior continues with 2^n-cycles for all $n \in \mathbb{Z}^+$ being created, until μ reaches approximately 3.57. In other words, for $b_n < \mu < b_{n+1}$, L_μ has a stable 2^n-cycle. It can be shown that $b_\infty = \lim_{n\to\infty} b_n = 3.570$ approximately.

Although $b_{n+1} - b_n \to 0$ as $n \to \infty$, it can be shown that

$$\lim_{n\to\infty} \frac{b_n - b_{n-1}}{b_{n=1} - b_n} = \delta = 4.6692016\ldots,$$

(called Feigenbaum's number). Feigenbaum showed that you get the same constant δ in this way for any family of unimodal maps ($f_\mu(x)$ is *unimodal* if $f_\mu(0) = 0$, $f_\mu(1) = 0$, f_μ is continuous on $[0, 1]$ with a single critical point between 0 and 1).

2.6 The Bifurcation Diagram.

The behavior described above can be illustrated graphically using a *bifurcation diagram.* To create a bifurcation diagram we plot μ, $0 \leq \mu \leq 4$ along the x-axis, and values of $L_\mu^n(x)$ along the y-axis. The idea is to calculate (say) for each value of μ the first 500 iterates of some (arbitrarily chosen) point x_0. We ignore the first 450 iterates and plot the next 50. So for example, if $1 < \mu < 3$, because the fixed point is attracting, the iterates will approach the fixed point $1 - 1/\mu$, so for n large, what we see plotted will be (very close to) the value $1 - 1/\mu$. For $3 < \mu < 1 + \sqrt{6}$ the fixed point has become repelling, so this no longer shows up, but the 2-cycle has become attracting, so we see plotted the 2 points of the 2-cycle. This continues with the 4-cycle, 8-cycle etc. This is called the *period doubling route to chaos.*

We can create a bifurcation diagram for L_μ using Mathematica in the following way:

First we define the logistic map for $\mu = 1$, and then a function of two variables:

```
f[x_]:= x(1-x)

h[x_,a_]:= a*f[x]
```

For a given $a \in [0, 4]$ we pick $x \in (0, 1)$ randomly and iterate $h(x, a)$ 100 times. Since this gives rise to a single number, we can define a function as the output:

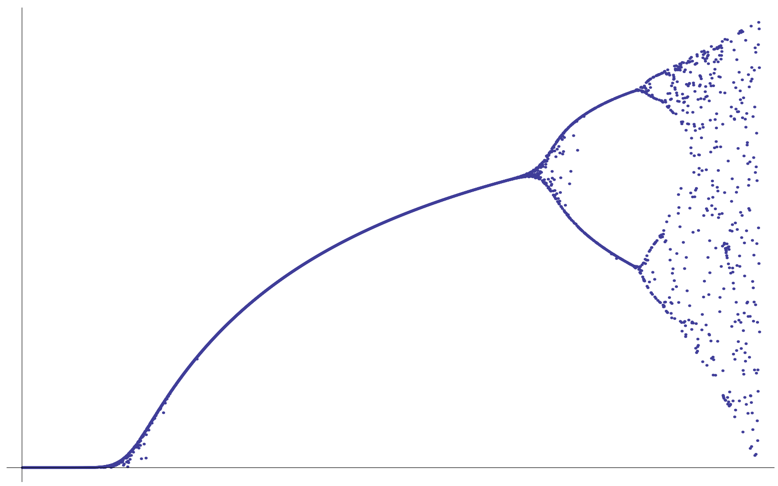
```
g[a_]:=(
  k[x_]:=h[x,a];
  A=NestList[k, Random[Real, {0,1}], 100];
  Return[A[[100]]]
  )
```

Now we generate a list using the `Table` command for $a \in [1, 4]$:

```
B=Table[g[j/1000],{j,1000,4000}];
```

and now we plot this list:

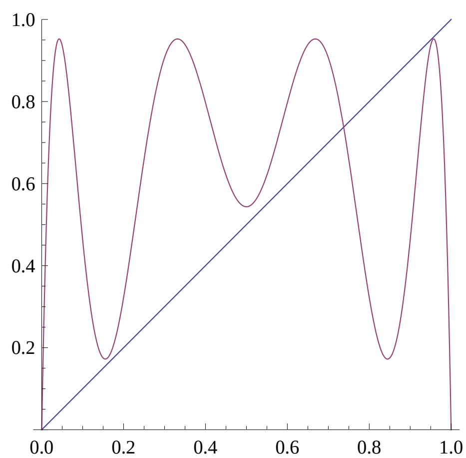
```
ListPlot[B, PlotStyle->PointSize[0.001],Ticks->False]
```



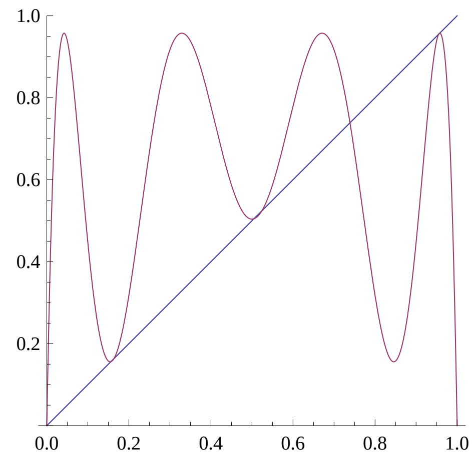
The bifurcation diagram for the logistic map.

2.6.1 Where Does Period Three Occur for the Logistic Map

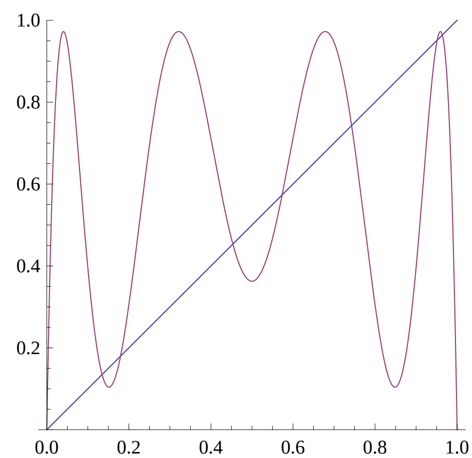
If we look at the graph of L_μ^3 for values of μ close to 3.8, we see that is where a 3-cycle is created:



The Graph of $L_\mu^3(x)$ for $\mu < 1 + \sqrt{8}$.



The Graph of $L_\mu^3(x)$ for $\mu = 1+\sqrt{8}$.



The Graph of $L_\mu^3(x)$ for $\mu > 1+\sqrt{8}$.

We aim to show that this happens when $\mu = 1+\sqrt{8}$. For values of μ slightly smaller than this we only see the two fixed points. For $\mu = 1+\sqrt{8}$ we see that a 3-cycle is born, and for larger values of μ, we see two 3-cycles. For a small range of values of μ, the 3-cycle is attracting.

To show that the bifurcation occurs when $\mu = 1+\sqrt{8}$ we follow the argument of Feng [24] (see also [7], [27] and [56]):

Period 3-points of L_μ occur where $L_\mu^3(x) = x$, so we look at where

$$L_\mu^3(x) - x = 0.$$

To disregard the fixed points of L_μ we set

$$g_\mu(x) = \frac{L_\mu^3(x) - x}{L_\mu(x) - x}.$$

Using Mathematica (for example) we can check that

$$g_\mu(x) = \mu^6 x^6 - (\mu^5 + 3\mu^6)x^5 + (\mu^4 + 4\mu^5 + 3\mu^6)x^4 - (\mu^3 + 3\mu^4 + 5\mu^5 + \mu^6)x^3$$
$$+(\mu^2 + 3\mu^3 + 3\mu^4 + 2\mu^5)x^2 - (\mu + 2\mu^2 + 2\mu^3 + \mu^4)x + 1 + \mu + \mu^2.$$

Set $\lambda = 7 + 2\mu - \mu^2$ and let

$$h_\mu(z) = g_\mu(-z/\mu),$$

then

$$h_\mu(z) = z^6 + (3\mu + 1)z^5 + (3\mu^2 + 4\mu + 1)z^4 + (\mu^3 + 5\mu^2 + 3\mu + 1)z^3$$
$$+(2\mu^3 + 3\mu^2 + 3\mu + 1)z^2 + (\mu^3 + 2\mu^2 + 2\mu + 1)z + \mu^2 + \mu + 1.$$

Then if

$$k_\mu(z) = \left\{ z^3 + z^2\frac{(3\mu+1)}{2} + z(2\mu + 3 - \frac{\lambda}{2}) + \frac{(\mu+5)}{2} - \frac{\lambda}{2} \right\}^2 + \frac{\lambda}{4}(z+1)^2(z+\mu)^2,$$

then we can check that $k_\mu(z) = h_\mu(z)$ for all z (e.g., using Mathematica).

Note that

$$\lambda > 0 \quad \text{for} \quad \mu < 1 + \sqrt{8}, \quad \lambda = 0 \quad \text{for} \quad \mu = 1 + \sqrt{8}, \quad \text{and} \quad \lambda < 0 \quad \text{for} \quad \mu > 1 + \sqrt{8}.$$

This means that for $\mu < 1+\sqrt{8}$, $h_\mu(z) > 0$ for all z (we say $h_\mu(z)$ is *positive definite*), so cannot have any roots, i.e., $g_\mu(x) = 0$ has no solution, so L_μ cannot have any 3-cycles. We summarize this as follows:

Theorem 2.6.2 [24] (i) *If $0 < \mu < 1 + \sqrt{8}$, then $h_\mu(z)$ is positive definite and the equation $h_\mu(z) = 0$ does not have any real roots. Consequently, the logistic map $L_\mu(x)$ does not have a 3-cycle.*

(ii) *If $\mu = 1 + \sqrt{8}$, then $h_\mu(z) = 0$ has three distinct roots, each of multiplicity two. These three roots constitute a 3-cycle for $L_\mu(x)$.*

(iii) *If $\mu > 1 + \sqrt{8}$ (with $\mu - (1 + \sqrt{8})$ sufficiently small), the equation $h_\mu(z) = 0$ has six simple roots which give rise to two 3-cycles for $L_\mu(x)$.*

Proof. (i) If $\mu < 1 + \sqrt{8}$, then since $h_\mu(z)$ is positive definite the result follows.

(ii) If $\mu = 1 + \sqrt{8}$, then $\lambda = 0$ and the equation becomes

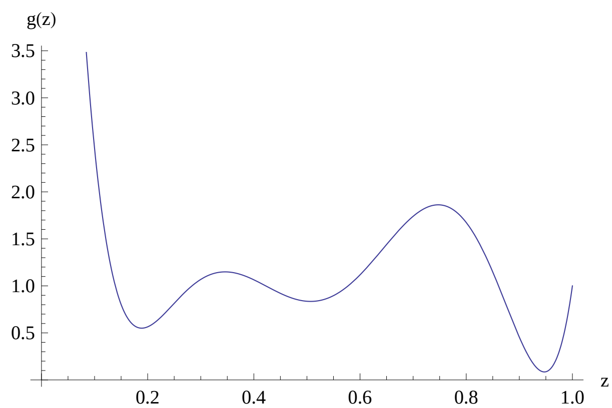
$$h_\mu(z) = \left(z^3 + (2 + 3\sqrt{2})z^2 + (5 + 4\sqrt{2})z + 3 + \sqrt{2}\right)^2.$$

The resulting cubic can be solved using the cubic formula to give three real solutions, z_1, z_2, z_3 and these can be used to give the three solutions to $g_\mu(x) = 0$, corresponding to the 3-cycle of $L_\mu(x)$:

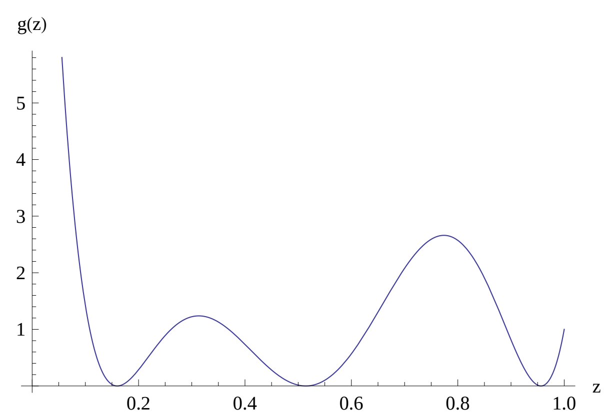
$$z_k = \frac{2\sqrt{7}}{3}\cos\left(\frac{1}{3}\arccos(-\frac{1}{2\sqrt{7}} + \frac{2k\pi}{3})\right) - \frac{2+3\sqrt{2}}{3}, \quad k = 0, 1, 2,$$

(see the graph of $g_\mu(x)$ below).

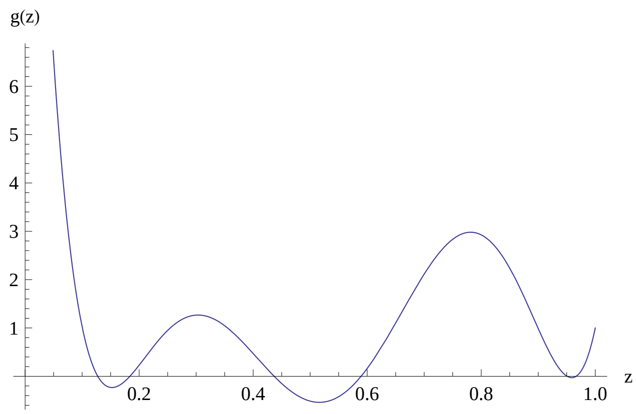
(iii) For $\lambda < 0$ we can factor $h_\mu(z) = h_1(z)h_2(z)$ using the difference of two squares, and then use the Intermediate Value Theorem on each of $h_1(z)$ and $h_2(z)$, to see that they each have three different roots corresponding to two 3-cycles, which can then be shown to be distinct (see [24] for details). $\square$



$g_\mu(x)$ for $\mu < 1 + \sqrt{8}$.



$g_\mu(x)$ for $\mu = 1 + \sqrt{8}$.



$g_\mu(x)$ for $\mu > 1 + \sqrt{8}$.

Example 2.6.3: A Super-Attracting 3-Cycle for the Logistic Map

Recall that super-attracting periodic points occur where the derivative is zero, so for a super-attracting 3-cycle $\{c_1, c_2, c_3\}$, we require

$$(L_\mu^3)'(c_1) = L'_\mu(c_1)L'_\mu(c_2)L'_\mu(c_3) = 0,$$

i.e.,

$$(1-2c_1)(1-2c_2)(1-2c_3) = 0,$$

so we may assume $c_1 = 1/2$. This means that $x = 1/2$ is a solution of the equation $L_\mu^3(x) = x$, or $L_\mu^3(1/2) = 1/2$. But if μ satisfies the equation $L_\mu(1/2) = 1/2$, then it will also satisfy the equation involving the third iterate. Consequently we solve for μ: $g_\mu(1/2) = 1/2$, where $g_\mu(x) = (L_\mu^3(x) - x)/(L_\mu(x) - x)$ as defined in the last section (this eliminates the root $\mu = 2$ which gave rise to the superstable fixed point at $x = 1/2$). We obtain

$$\frac{1}{64}(64 + 32\mu + 16\mu^2 - 24\mu^3 - 4\mu^4 + 6\mu^5 - \mu^6) = 0.$$

Set

$$p(a) = a^6 - 6a^5 + 4a^4 + 24a^3 - 16a^2 - 32a - 64,$$

then Mathematica indicates that there is a single real root μ_0 larger than $1+\sqrt{8}$ with exact value

$$\mu_0 = \frac{1}{6}\left\{6 + 2\sqrt{3\left(11 + 2\cdot 2^{2/3}(25 - 3\sqrt{69})^{1/3} + 2\cdot 2^{2/3}(25 + 3\sqrt{69})^{1/3}\right)}\right\}.$$

Mathematica is able to solve this equation exactly because it can be reduced to a cubic (and then the cubic formula may be used):

Following Lee [37], replace a by $a+1$ and check (using Mathematica) that

$$p(a+1) = a^6 - 11a^4 + 35a^2 - 89 = b^3 - 11b^2 + 35b - 89,$$

a cubic in $b = a^2$ which can now be solved exactly for b, then for a, from which the original equation can be solved. It is seen that

$$\mu_0 = 3.8318740552\ldots.$$

The other period 3-points may now be found since $c_1 = 1/2$, $c_2 = L_{\mu_0}(1/2) = \mu_0/4 = 0.95796\ldots$, and $c_3 = L_{\mu_0}^2(1/2) = \mu_0^2/4(1-\mu_0/4) = 0.15248\ldots$.

Example 2.6.4: The 3-Cycle when $\mu = 4$

When $\mu = 4$, $L_4(x) = 4x(1-x)$. In this case we have

$$g_4(x) = 4096x^6 - 13312x^5 + 16640x^4 - 10048x^3 + 3024x^2 - 420x + 21,$$

and we can check using Mathematica (see also Lee [38])

$$\begin{aligned} g_4(x/4) &= x^6 - 13x^5 + 65x^4 - 157x^3 + 189x^2 - 105x + 21 \\ &= (x^3 - 7x^2 + 14x - 7)(x^3 - 6x^2 + 9x - 3), \end{aligned}$$

the product of two cubics. The solutions give a pair of 3-cycles, which we will show are given by the following theorem:

Theorem 2.6.5 *For the logistic map $L_4(x) = 4x(1-x)$ we have:*

(i) The 2-cycle is $\{\sin^2(\pi/5), \sin^2(2\pi/5)\}$

(ii) The 3-cycles are

$$\{\sin^2(\pi/7),\ \sin^2(2\pi/7),\ \sin^2(3\pi/7)\} \quad \textit{and} \quad \{\sin^2(\pi/9),\ \sin^2(2\pi/9),\ \sin^2(4\pi/9)\ \}.$$

Proof. Recall that the difference equation $x_{n+1} = 4x_n(1-x_n)$, $n = 0, 1, 2, \dots$, has the solution

$$x_n = \sin^2\left(2^n \arcsin\sqrt{x_0}\right).$$

This was obtained by setting $x_n = \sin^2(\theta_n)$ for some $\theta_n \in (0, \pi/2]$, $(n = 1, 2 \dots)$, so that

$$\sin^2(\theta_{n+1}) = 4\sin^2(\theta_n)(1 - \sin^2(\theta_n)) = 4\sin^2(\theta_n)\cos^2(\theta_n) = \sin^2(2\theta_n).$$

We can use this to show that $\sin^2(\theta_{n+1}) = \sin^2(4\theta_{n-1})$, so in general

$$x_n = \sin^2(\theta_n) = \sin^2(2^n\theta_0), \quad \text{where} \quad \theta_0 = \arcsin(\sqrt{x_0}).$$

In particular we have

$$\theta_1 = \arcsin\left(\sqrt{\sin^2 2\theta_0}\right) = \begin{cases} 2\theta_0 & \text{for} \quad 0 \le \theta_0 \le \pi/4 \\ \pi - 2\theta_0 & \text{for} \quad \pi/4 \le \theta_0 \le \pi/2 \end{cases}$$

Applying this to the situation where we have a 2-cycle $\{c_0, c_1\}$, if $c_i = \sin^2(\theta_i)$, we get θ_0 is equal to $4\theta_0$, $\pi - 2\theta_0$, $2\pi - 4\theta_0$ or $\pi - 4\theta_0$. This gives $\theta = 0$ or $\theta = \pi/3$ (giving rise to the two fixed points) or $\theta_0 = \pi/5$, or $2\pi/5$ from which the result follows. A similar, but more complicated analysis gives the 3-cycles. □

Remark 2.6.6 Using the above ideas, if we want to find the period n-points of L_4, we solve the equation $L_4^n(x) = x$. When $x = \sin^2(\theta)$, this becomes

$$\sin^2(\theta) = \sin^2(2^n\theta).$$

This gives rise to the two equations

$$\pm\theta = 2^n\theta + 2k\pi, \quad \text{or} \quad \pm\,\theta = (2k+1)\pi - 2^n\theta, \quad \text{for some} \quad k \in \mathbb{Z}.$$

This can be summarized as a single equation:

$$\pm\theta = 2^n\theta + k\pi \Rightarrow \ \theta = \frac{k\pi}{2^n \pm 1}, \quad n = 1, 2, 3 \ldots, \ k \in \mathbb{Z}$$

so that

$$\mathrm{Per}_n(L_4) = \{\sin^2(\frac{k\pi}{2^n - 1}) : \ 0 \le k < 2^{n-1}\} \cup \{\sin^2(\frac{k\pi}{2^n + 1}) : \ 0 < k \le 2^{n-1}\}.$$

It follows that L_4 has points of all possible periods. We shall see that the set of all periodic points constitutes a "dense" set in $[0, 1]$ and each of these points is unstable.

Exercises 2.6

1. Recall the family of maps defined by $S_\mu(x) = \mu \sin(x)$ for $x \in [0, \pi]$ and $\mu \in [0, \pi]$. Use the Mathematica commands below together with the `ManipulatePlot` command to estimate the values of μ where periods two and three are born. Use

```
h[a_,x_]:= a*Sin[x]
g[a_,x_]:= h[a,h[a,x]]
k[a_,x_]:= h[a,h[a,h[a,x]]]

   Manipulate[Plot[{x,g[a,x]}, {x, 0, Pi}, PlotRange->{0,Pi},
AspectRatio->Automatic], {a,0,Pi}]
```

(Click on the + at the end of the `Manipulate` line to obtain the `Animation Controls`).

2. Modify the bifurcation diagram of the logistic maps to give a bifurcation diagram for the family S_μ, $0 \le \mu \le \pi$.

3. Do the same as in question 1 for the family $C_\mu(x) = \mu \cos(x)$, $x \in [-\pi, \pi]$ and $\mu \in [0, \pi]$.

4. Can you use the method of question 1 to estimate a value of μ for which S_μ has a superattracting 2-cycle, or 3-cycle?

5. Let $g_\mu(x) = \mu x \dfrac{(1-x)}{(1+x)}$, $\mu > 0$.

(a) Show that g_μ has a maximum at $x = \sqrt{2}-1$ and the maximum value is $\mu(3-2\sqrt{2})$.

(b) Deduce that g is a dynamical system on $[0,1]$ for $0 \leq \mu \leq 3+2\sqrt{2}$ (i.e., $g_\mu([0,1]) \subseteq [0,1]$).

(c) Find the fixed points of g_μ for $\mu \geq 1$.

(d) Give conditions on μ for the fixed points of g_μ to be attracting.

(e) Use Mathematica to graph g_μ^2 and g_μ^3, and estimate when a period 2-point is created.

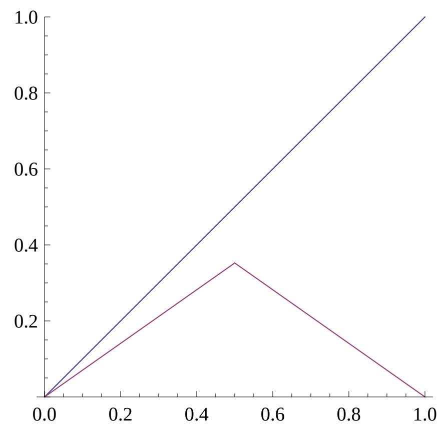
(f) Use Mathematica to give a bifurcation diagram for g_μ, for $0 \leq \mu \leq 3+2\sqrt{2}$.

2.7 The Tent Family T_μ.

We define another parameterized family in a piecewise linear manner: $T_\mu : [0,1] \to [0,1]$, $0 < \mu \leq 2$,

$$T_\mu(x) = \begin{cases} \mu x & \text{if } \ 0 \leq x \leq 1/2 \\ \mu(1-x) & \text{if } \ 1/2 < x \leq 1. \end{cases}$$

When $\mu = 2$ we get the familiar tent map $T(x)$ seen earlier. We now look at the parameter values $0 < \mu \leq 2$ (and later we shall see what happens when $\mu > 2$, and where $T_\mu : \mathbb{R} \to \mathbb{R}$). If $0 < \mu < 1$ we see that the only fixed point is $c = 0$. If $\mu = 1$, then all $c \in [0, 1/2]$ are fixed points and if $1 < \mu \leq 2$, then there are 2 fixed points. We look at each of these cases separately.



Case 2.7.1 $0 < \mu < 1$. We see that 0 is the only fixed point of T_μ and if $0 < x \leq 1/2$, then $0 \leq T_\mu(x) = \mu x < x$, and if $1/2 < x \leq 1$, then

$$0 \leq T_\mu(x) = \mu(1-x) < 1 - x < \frac{1}{2} < x,$$

so in a similar manner to what we have seen previously, the sequence $T_\mu^n(x)$ is decreasing, bounded below by 0, so must converge to the fixed point 0. Thus $B_{T_\mu}(0) = [0, 1]$.

Case 2.7.2 $\mu = 1$ Clearly if $0 < x \leq 1/2$, then $T_1(x) = x$, so is a fixed point, and if $1/2 < x \leq 1$, then x is an eventual fixed point since

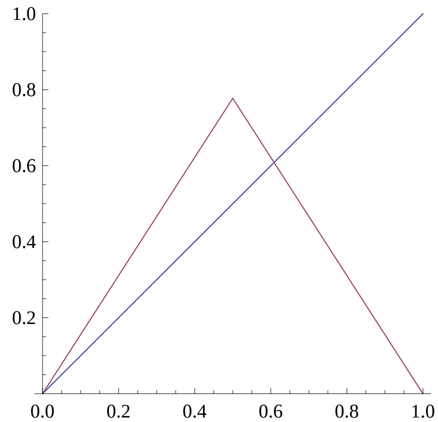
$$T_1^2(x) = T_1(1 - x) = x.$$

The fixed points are stable but not attracting.

Case 2.7.3 $1 < \mu < 2$**.** There is a bifurcation at $\mu = 1$ with a second fixed point c created, $1/2 < c < 1$. We solve $T_\mu(x) = x$ to find this point:

$$T_\mu(x) = \mu(1-x) = x, \quad \text{so that} \quad c = \frac{\mu}{1+\mu}.$$

Since $|T_\mu(x)| = \mu > 1$ in this case, the fixed point is repelling.



Case 2.7.4 $\mu = 2$**.** In this case $T_2 = T$, the familiar tent map. The fixed points 0 and 2/3 are repelling. This time the range of T is all of $[0, 1]$. T has the effect of mapping the interval $[0, 1/2]$ onto all of $[0, 1]$ and then folding the interval $[1/2, 1]$ back over the interval $[0, 1]$. It is this stretching and folding that gives rise to the chaotic nature of T that we will see later, and is typical of many transformations of this type. We will examine the periodic points of T in the next section.

2.8 The 2-Cycles and 3-Cycles of the Tent Family

At some stage for $\mu \geq 1$ a 2-cycle is created. It is interesting to use Mathematica to do a dynamic iteration of T_μ to see how this happens (use the `ManipulatePlot` command described in Exercises 2.6, but with

```
h[x_,a_]:= a*Piecewise[{{x, x < 1/2}, {(1-x), 1/2 < x}}]
```

in place of `a*Sin[x]`). It can be checked that for $\mu > 1$, T_μ^2 is given by the formula

$$T_\mu^2(x) = \begin{cases} = & \mu^2 x & \text{if} \quad 0 \leq x \leq \frac{1}{2\mu} \\ & \mu(1-\mu x) & \text{if} \quad \frac{1}{2\mu} < x \leq \frac{1}{2} \\ & \mu(1-\mu+\mu x) & \text{if} \quad \frac{1}{2} < x \leq 1 - \frac{1}{2\mu} \\ & \mu^2(1-x) & \text{if} \quad 1 - \frac{1}{2\mu} < x \leq 1 \end{cases}$$

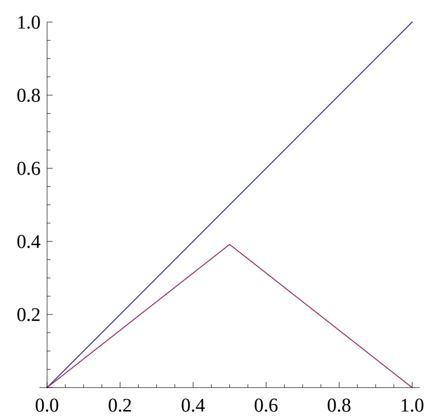
The 2-cycle is created when $T_\mu^2(1/2) = 1/2$, i.e., when

$$\mu(1-\mu/2) = 1/2, \quad \text{or} \quad (\mu-1)^2 = 0, \quad \mu = 1,$$

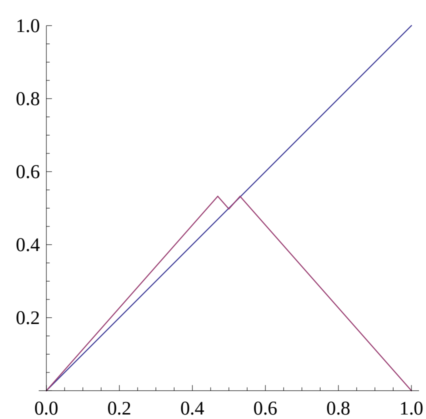
so the 2-cycle appears when $\mu > 1$. For $\mu \le 1$, there are no period 2 points. The 2-cycle $\{c_1, c_2\}$ say, is unstable since

$$|(T_\mu^2)'(c_1)| = |T_\mu'(c_1)T_\mu'(c_2)| = \mu^2 > 1.$$

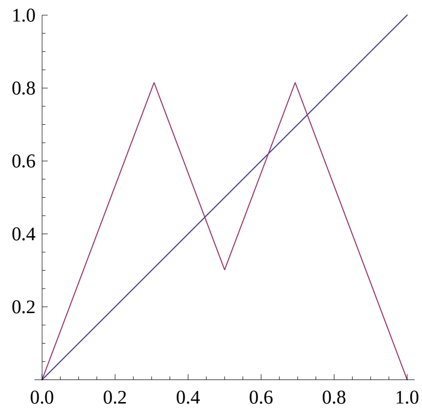
For example, if we solve $\mu(1-\mu x) = x$, we get $c_1 = \frac{\mu}{1+\mu^2}$ and solving $\mu^2(1-x) = x$ gives $c_2 = \frac{\mu^2}{1+\mu^2}$ as the period 2-points. Solving $\mu(1-\mu+\mu x) = x$ gives $c = \frac{\mu}{1+\mu}$ as the (non-zero) fixed point.



T_μ^2 for $\mu < 1$.



T_μ^2 for $\mu > 1$ (but close to 1).



T_μ^2 for $\mu > 1$.

We now look for the smallest value of μ for which there exists a 3-cycle. In a similar way to the situation for the 2-cycle, we look for where $T_\mu^3(1/2) = 1/2$ and we compute the value of the largest root of this equation. In general (see Heidel [29]) the smallest value of μ for which there exists a periodic orbit of period k is precisely the value of μ for which $1/2$ has period k. Using the formula for $T_\mu^2(1/2) = \mu(1-\mu/2)$ above, then since $\mu(1-\mu/2) \leq 1/2$ for all μ, we get

$$T_\mu^3(1/2) = \mu^2(1-\mu/2) = 1/2 \quad \text{when} \quad \mu^3 - 2\mu^2 + 1 = 0,$$

or $(\mu - 1)(\mu^2 - \mu - 1) = 0$. Disregarding $\mu = 1$ and solving the quadratic gives $\mu = (1+\sqrt{5})/2$ as the value of μ where period 3 first occurs.

In general it can be shown that for $k > 3$ odd, period k first occurs when μ is equal to the largest real root of the equation

$$\mu^k - 2\mu^{k-1} + 2\mu^{k-3} - 2\mu^{k-4} + \ldots - 2\mu + 1$$
$$= (\mu - 1)(\mu^{k-1} - \mu^{k-2} - \mu^{k-3} + \mu^{k-4} - \mu^{k-5} + \mu^{k-6} \cdots + \mu - 1) = 0,$$

so has $\mu - 1$ as a factor (see [29]). We will return to look at the tent family in Section 6.4.

Exercises 2.8

1. Find $\mu \in [0, 2]$ such that $c = 1/2$ is periodic of period 3 under the tent map T_μ (so that $\{1/2, \mu/2, \mu(1-\mu/2)\}$ is a 3-cycle).

2. For $\mu > 1$, show that T_μ has no attracting periodic points.

3. If $L_\mu(x) = \mu x(1-x)$ is such that $c = 1/2$ is n-periodic for some $n \in \mathbb{Z}^+$, prove that $1/2$ is an attracting periodic point. Is it necessarily a super-attracting periodic point?

4. Modify the bifurcation diagram of the logistic maps to give a bifurcation diagram for the tent family T_μ, $x \in [0,1]$ and $\mu \in [0,2]$.

Chapter 3. Sharkovsky's Theorem

3.1 Period 3 Implies Chaos. In 1975 in a paper entitled "Period three implies chaos", Li and Yorke proved a remarkable theorem:

Theorem 3.1.1 *Let $f : X \to X$ be a continuous function defined on an interval $X \subseteq \mathbb{R}$. If $f(x)$ has a point of period three, then for any $k = 1, 2, 3, \ldots$, there is a point having period k.*

This paper stirred considerable interest and shortly after it was pointed out that a Ukrainian mathematician by the name of Sharkovsky had in 1964 published a much more general theorem (in Russian) in a Ukrainian journal. His theorem was unknown in the west until the appearance of the Li-Yorke Theorem. To state his theorem we need to define a new ordering of the positive integers $\mathbb{Z}^+$. In the "Sharkovsky ordering", 3 is the largest number, followed by 5 then 7 (all of the odd integers), then $2\cdot 3$, $2\cdot 5$, (2 times the odd integers), then 2^2 times the odd integers etc., finishing off with powers of 2 in descending order:

$$3 \rhd 5 \rhd 7 \rhd \cdots \rhd 2\cdot 3 \rhd 2\cdot 5 \rhd \cdots \rhd 2^2\cdot 3 \rhd 2^2\cdot 5 \rhd \cdots \rhd 2^n\cdot 3 \rhd 2^n\cdot 5 \rhd \cdots \rhd 2^n \rhd 2^{n-1} \rhd \cdots 2^3 \rhd 2^2 \rhd 2 \rhd 1.$$

Sharkovsky's Theorem says that if a continuous map has a point of period k, then it has points of all periods less than k in the Sharkovsky ordering. The converse is also true in the sense that for each $k \in \mathbb{Z}^+$ there is a map having points of period k, but no points of period larger than k in the Sharkovsky ordering.

Theorem 3.1.2 (**Sharkovsky's Theorem, 1964.**) *Let $f : X \to X$ be a continuous map on an interval X (where X may be any bounded or unbounded subinterval of $\mathbb{R}$). If f has a point of period k, then it has points of period r for all $r \in \mathbb{Z}^+$ with $k \rhd r$.*

For example, this theorem tells us that if f has a 4-cycle, then it also has a 2-cycle and a 1-cycle (fixed point). If f has a 3-cycle, then f has all other possible cycles. If f has a 6-cycle, then since $6 = 2\cdot 3$, f will have $2\cdot 5, 2\cdot 7, \ldots 2^2\cdot 3, 2^2\cdot 5 \ldots$-cycles, etc.

We omit the proof of this theorem, but shall prove the case where $k = 3$ (often known as the Li-Yorke Theorem because of their independent proof of this result in 1975). James Yorke is a professor at UMD, College Park and Li was his graduate student. A few years ago, Yorke and Benoit Mandelbrot were awarded the Japan Prize (the Japanese equivalent of the Nobel Prize) for their work in Dynamical Systems and Fractals (Mandelbrot is regarded as the "father" of fractals, and we will review some of his work later). In order to prove this result we shall need some preliminary lemmas. The first was proved in Chapter 1 (Theorem 1.2.9).

Lemma 3.1.3 *Let $f : I \to \mathbb{R}$ be a continuous map, I an interval with $J = f(I) \supseteq I$, then $f(x)$ has a fixed point in I.*

Lemma 3.1.4 *Let $f : I \to \mathbb{R}$ be a continuous map. If $J \subseteq f(I)$ is a closed bounded interval, then there exists a closed bounded interval $K \subseteq I$ with $f(K) = J$.*

Proof. Write $J = [a, b]$ for some $a, b \in \mathbb{R}$, $a < b$.

There are $r, s \in I$ with $f(r) = a$ and $f(s) = b$. Set

$$\beta = y, \quad \text{where} \quad y \in I, \quad |y - r| \quad \text{is a minimum, and} \quad f(y) = b,$$

(β exists from continuity and $f(\beta) = b$: β is the point of I that is closest to r with the property that $f(\beta) = b$). Similarly, set

$$\alpha = x, \quad \text{where } x \text{ lies between } \beta \text{ and } r, \quad |x - \beta| \quad \text{is a minimum, and} \quad f(x) = a$$

(possibly equal to r), so as before $f(\alpha) = a$. In this way, there is no $c \in (\alpha, \beta)$ where $f(c) = a$, or $f(c) = b$. Then the closed interval K (either $K = [\alpha, \beta]$ or $K = [\beta, \alpha]$), has the property $f(K) = [a, b]$, since firstly, $f(\alpha) = a$, $f(\beta) = b$, so by the Intermediate Value Theorem $[a, b] \subseteq f(K)$. On the other hand, if $w \in f(K)$, then $w = f(z)$ for some z between α and β. This must give $f(z) \in [a, b]$ since otherwise, another application of the Intermediate Value Theorem would contradict the choices of α and β. □

3.1.5 Proof of Sharkovsky's Theorem for $k = 3$

We are assuming that f has a point of period 3, so there is a 3-cycle $\{a, b, c\}$ with $f(a) = b$, $f(b) = c$ and $f(c) = a$. We assume that $a < b < c$ (the other case with $a < c < b$ may be treated similarly).

We give the idea of the proof by showing why there must be points of period one, two and four. Let

$$[a, b] = L_0 \quad \text{and} \quad [b, c] = L_1.$$

Observe that

$$f(L_0) \supseteq L_1 \quad \text{and} \quad f(L_1) \supseteq L_0 \cup L_1.$$

Case 1. f has a fixed point.

Since

$$f(L_1) \supseteq L_0 \cup L_1 \supseteq L_1,$$

Lemma 3.1.3 implies that f has a fixed point in L_1.

Case 2. f has a point of period 2.

This time we use

$$f(L_1) \supseteq L_0 \cup L_1 \supseteq L_0,$$

so by Lemma 3.1.4 there is a set $B \subseteq L_1$ such that $f(B) = L_0$. We then have

$$f^2(B) = f(L_0) \supseteq L_1 \supseteq B,$$

so by Lemma 3.1.3, B contains a fixed point c of f^2. c is a period 2-point of f (and not a fixed point of f) because

$$f(c) \in L_0 \quad \text{and} \quad c \in L_1, \quad \text{so} \quad f(c) \neq c.$$

Case 3. f has a point of period 4.

The above two constructions do not illustrate the general method, but the following construction is easily generalized to any number of fixed points greater than 3. Our aim is to show that there is a subset B of L_1 which is mapped first by f into L_1, then into L_1 again, then onto L_0 and then onto L_1, so that $f^4(B) \supseteq B$. Thus f^4 has a fixed point c in B, which cannot be a point of lesser period because $f(c) \in L_1$, $f^2(c) \in L_1$, $f^3(c) \in L_0$ and $f^4(c) \in B$ (so cannot have $f(c) = c$, $f^2(c) = c$ or $f^3(c) = c$).

It is useful to think of 5 copies of $L_0 \cup L_1$ with f mapping the first to the second etc. as shown:

We find sets B_1, B_2 and B_3 as follows:

$$f(L_0) \supseteq L_1, \quad f(L_1) \supseteq L_0 \cup L_1,$$

so there exists $B_1 \subseteq L_1$ such that $f(B_1) = L_0$.

There exists $B_2 \subseteq L_1$ such that $f(B_2) = B_1$ and there exists $B_3 \subseteq L_1$ such that $f(B_3) = B_2$. Set $B = B_3$, then

$$f^2(B_3) = f(B_2) = B_1, \quad \text{and so} \quad f^3(B) = L_0, \;\; f^4(B) \supseteq L_1 \supseteq B_3.$$

In other words $f^4(B) \supseteq B$, so there exists $c \in B$, a fixed point of f^4, which is not a point of period 3 or less, so must be a point of period 4. $\square$

In general if a function has points of period 4 the most we can deduce is that there are points of period 2 and fixed points. However, the following is true:

Proposition 3.1.6 *If $f : I \to I$ is continuous on an interval I with*

$$f(a) = b, \; f(b) = c, \; f(c) = d, \; f(d) = a, \quad a < b < c < d,$$

then $f(x)$ has a point of period 3, so also has points of all other periods.

Proof. We may assume that

$$f[a,b] = [b,c], \;\; f[b,c] = [c,d], \;\; f[c,d] = [a,d].$$

In particular, there exists $B_1 \subseteq [c,d]$ with $f(B_1) = [c,d]$. There exists $B_2 \subseteq [c,d]$ with $f(B_2) = [b,c]$.

Take $K_1 \subseteq B_1$ with $f(K_1) = B_2$, then

$$f^3(K_1) = f^2(B_2) = f[b,c] = [c,d] \supseteq K_1,$$

so f^3 has a fixed point in K_1 which is not a fixed point of $f(x)$.

$\square$

The above proofs can be summarized with the following type of result:

Proposition 3.1.7 *Let $f : I \to I$ be a continuous map, and let I_1 and I_2 be two closed subintervals of I with at most one point in common. If $f(I_1) \supset I_2$ and $f(I_2) \supset I_1 \cup I_2$, then f has a 3-cycle.*

Proof. Exercise: Use the ideas of the proof of 3.1.5. $\square$

3.2 Converse of Sharkovsky's Theorem

As we mentioned, for each $m \in \mathbb{Z}^+$ in the Sharkovsky ordering of $\mathbb{Z}^+$, Sharkovsky showed that there is a continuous map $f : I \to I$ (I an interval), such that $f(x)$ has a point of period m, but no point of period k for $k \triangleright m$. The following were shown (as usual, I is either the real line or an interval):

Theorem 3.2.1 *For every $k \in \mathbb{Z}^+$, there exists a continuous map $f : I \to I$ that has a k-cycle, but has no cycles of period n for any n appearing before k in the Sharkovsky ordering.*

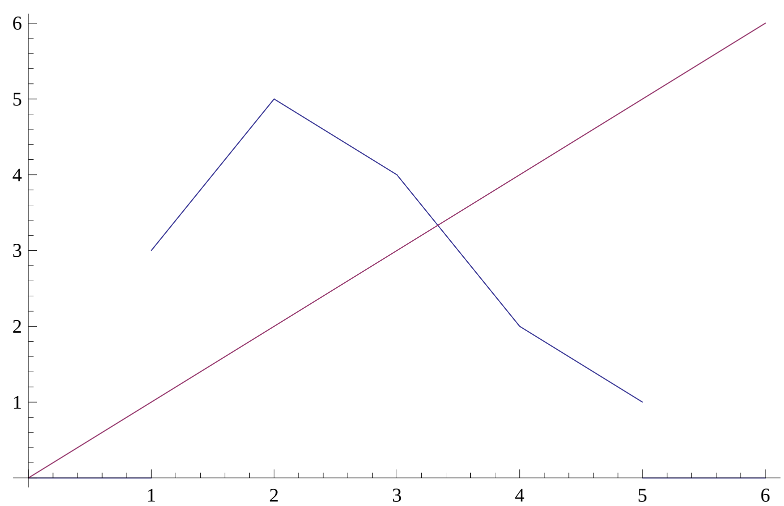
Theorem 3.2.2 *There exists a continuous map $f : I \to I$ that has a 2^n-cycle, for every $n \in \mathbb{Z}^+$ and has no other cycles of any other period.*

Strictly speaking, Sharkovsky's Theorem is really the combination of Theorem 3.1.2, Theorem 3.2.1 and Theorem 3.2.2 (see [44]), and sometimes the latter two theorems are referred to as the converse of Sharkovsky's Theorem (see [23]). We look at some particular cases of this:

Example 3.2.3 Define a function $f : [1, 5] \to [1, 5]$ as shown by the graph below (so

$$f(1) = 3, \quad f(2) = 5, \quad f(3) = 4, \quad f(4) = 2, \quad f(5) = 1,$$

with $f(x)$ piecewise linear between these points). Then f has a point of period 5, but no point of period 3.



Proof. Clearly no point from the set $\{1, 2, 3, 4, 5\}$ is of period 3, but this set is a 5-cycle. Also, Theorem 1.2.7 tells us that $f(x)$ has a fixed point c, so c is a fixed point for f^3. We shall show that f^3 has no other fixed points. Suppose to the contrary that

f^3 has another fixed point α. Now we can check that:

$$f^3[1,2] = [2,5], \quad f^3[2,3] = [3,5], \quad f^3[4,5] = [1,4],$$

so f^3 cannot have a fixed point in the intervals $[1,2]$, $[2,3]$ or $[4,5]$, so α must lie in the interval $[3,4]$. In fact $f^3[3,4] = [1,5] \supseteq [3,4]$ and we show that f^3 cannot have another fixed point in $[3,4]$.

If $\alpha \in [3,4]$, then $f(\alpha) \in [2,4]$, so either $f(\alpha) \in [2,3]$ or $f(\alpha) \in [3,4]$. If the former holds, then $f^2(\alpha) \in [4,5]$ and $f^3(\alpha) \in [1,2]$ which is impossible as we have to have $f^3(\alpha) = \alpha \in [3,4]$.

Thus we must have $f(\alpha) \in [3,4]$, so that $f^2(\alpha) \in [2,4]$. Again there are two possibilities: if $f^2(\alpha) \in [2,3]$ then $f^3(\alpha) \in [4,5]$, another contradiction, so that $f^2(\alpha) \in [3,4]$.

We have shown that the orbit of α: $\{\alpha, f(\alpha), f^2(\alpha)\}$ is contained in the interval $[3,4]$. On the interval $[3,4]$ we can check that $f(x)$ is given by the straight line formula

$$f(x) = 10 - 2x, \quad \text{and} \quad f(10/3) = 10/3,$$

so $c = 10/3$ is the unique fixed point of f. Also

$$f^2(x) = -10 + 4x, \quad f^3(x) = 30 - 8x,$$

also with the unique fixed point $x = 10/3$. It follows that f cannot have any points of period 3. □

Remark 3.2.2 The above can be directly generalized to give a map having points of period $2n+1$, but no points of period $2n-1$, $n = 2, 3, \ldots$.

Exercises 3.2

1. Use the ideas of Chapter 3 to show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and has a 2-cycle $\{a, b\}$, then f has fixed point.

2. Show that the map $f(x) = (x - 1/x)/2$, $x \neq 0$, has no fixed points but it has period 2-points. Find the 2-cycle and by looking at the graph of $f^3(x)$, check to see whether it has a 3-cycle. Why does this not contradict Sharkovsky's Theorem?

3. A map $f : [1,7] \to [1,7]$ is defined so that $f(1) = 4, f(2) = 7, f(3) = 6, f(4) = 5, f(5) = 3, f(6) = 2, f(7) = 1$, and the corresponding points are joined so the map is piecewise linear. Show that f has a 7-cycle but no 5-cycle.

4. If $F_\lambda(x) = 1 - \lambda x^2$ for $x \in \mathbb{R}$, show

(i) F_λ has fixed points for $\lambda \geq -1/4$.

(ii)F_λ has a 2-cycle for $\lambda > 3/4$.

(iii) The 2-cycle is attracting for $3/4 < \lambda < 5/4$.

5. Show that an increasing function $f : \mathbb{R} \to \mathbb{R}$ cannot have a 3-cycle. Can it have a 2-cycle? Answer the same questions when f is decreasing.

Chapter 4. Metric Spaces

In order to generalize some of the ideas we have met so far, it is convenient to introduce the idea of a *metric space* (see [60] for example). This is simply a pair (X, d) where X is a set and d is a distance defined on the set, called a *metric*. d must satisfy certain natural properties that one would expect of a distance function:

4.1 Basic Properties of Metric Spaces

Definition 4.1.1 Let X be a set with $x, y, z \in X$. A metric on X is a function $d : X \times X \to \mathbb{R}^+$ satisfying:

1. $d(x, y) \geq 0$ for all $x, y \in X$,

2. $d(x, y) = 0$ if and only if $x = y$,

3. $d(x, y) = d(y, x)$,

4. $d(x, y) \leq d(x, z) + d(z, y)$ (the *triangle inequality*).

We see that a metric on X is just a "distance function" ($d(x, y) =$ the distance from x to y) on X with certain natural properties).

Definition 4.1.2 The pair (X, d) is called a *metric space*, i.e., a metric space is just a set X with a distance function d satisfying (1)-(4) above.

Examples 4.1.3 1. $X = \mathbb{R}$ with $d(x, y) = |x - y|$ for $x, y \in \mathbb{R}$ is a metric space. In a similar way, $X = [0, 1]$ with the same distance function is a metric space.

2. $X = \mathbb{R}^2$ with the usual distance in the plane defined by

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

for $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ is a metric space. More generally $\mathbb{R}^n$ with its usual distance function and $\mathbb{C}$ the set of all complex numbers with $d(z, w) = |z - w|$, $z, w \in \mathbb{C}$ are metric spaces.

3. Let $X = S^1$ where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle in the complex plane. The natural metric on S^1 is given by the (shortest) distance around the circle between the two points in question.

4. If $X =$ any set and $d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$, for $x, y \in X$, then d is a metric on X, called the *discrete metric*.

5. Let $X = \Sigma = \{(s_1, s_2, s_3, \ldots) : s_i = 0 \ \text{ or } \ 1\}$, so that the members of Σ are infinite sequences of zeros and ones. We define a distance on Σ as follows: If

$$\omega_1 = (s_1, s_2, s_3, \ldots), \ \ \omega_2 = (t_1, t_2, t_3, \ldots),$$

are members of Σ, then set

$$d(\omega_1, \omega_2) = \sum_{n=1}^{\infty} \frac{|s_n - t_n|}{2^n},$$

then it easy to check that d is a metric on Σ, so (S, d) is a metric space.

For example, suppose that $\omega_1 = (1, 1, 1, 1, \ldots)$ and $\omega_2 = (1, 0, 1, 0, 1, \ldots)$, then

$$d(\omega_1, \omega_2) = \sum_{n=1}^{\infty} \frac{1}{2^{2n}} = \frac{1}{3}.$$

With this metric, points with coordinates which differ for small n are further apart than those with coordinates that differ for large values of n. For example, the distance between $(1, 0, 1, 1, 1, \ldots)$ and $(1, 1, 1, 1, \ldots)$ is $1/4$, whereas the distance between $(1, 1, 1, 0, 1, 1, 1, \ldots)$ and $(1, 1, 1, 1, 1, 1, \ldots)$ is $1/16$.

Definition 4.1.4 If (X, d) is a metric space and $a \in X$, $\epsilon > 0$, then

$$B_\epsilon(a) = \{x \in X : d(a, x) < \epsilon\},$$

is called the *open ball* centered on a with radius ϵ.

Examples 4.1.5 If $X = \mathbb{R}$ as in the above example, then the open balls are intervals of the form $(a - \epsilon, a + \epsilon)$. If $X = \mathbb{R}^2$, and $a = (a_1, a_2)$ is a point of $\mathbb{R}^2$, the points of $B_\epsilon(a)$ are those interior to the circle centered on (a_1, a_2) and of radius ϵ (the boundary of the circle is not included).

Definition 4.1.6 (i) A set $A \subseteq X$ is said to be *open* if for each $a \in A$ there exists $\epsilon > 0$ satisfying $B_\epsilon(a) \subseteq A$ (i.e., every point of A can be surrounded by an open ball entirely contained in A).

(ii) The set $A \subseteq X$ is *closed* if its complement $X - A$ (sometimes written A^c) is open.

(iii) A point $a \in X$ is a *limit point* of a set $A \subseteq X$ if every ball $B_\epsilon(a)$ contains a point of A other than a.

Examples 4.1.7 1. The open intervals (a, b) in $\mathbb{R}$ are open sets and any union of open intervals is open. The closed intervals such as $[a, b], [a, \infty)$ are closed sets. Sets such as $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$ are neither open nor closed.

2. Any open ball $B_\epsilon(a)$ in a metric space X is an open set. The *closed ball*:

$$\overline{B}_\epsilon(a) = \{x \in X : d(a,x) \leq \epsilon\},$$

will be a closed set. The empty set $\emptyset$ and the whole space X are both open and closed.

3. The union of open sets is open and the intersection of closed sets is closed.

Proof. If $A = \cup_{\alpha\in\Lambda} A_\alpha$ is a union of open sets, let $a \in A$, then by definition, $a \in A_\alpha$ for some $\alpha \in \Lambda$. Since A_α is open, there exists $\delta > 0$ with $B_\delta(a) \subseteq A_\alpha$. It follows that $B_\delta(a) \subseteq A$, so A is open.

If $C = \cap_{\alpha\in\Lambda} C_\alpha$ is an intersection of closed sets, then each $X - C_\alpha$ is open. But DeMorgan's laws tell us that

$$\cup_{\alpha\in\Lambda}(X - C_\alpha) = X - \cap_{\alpha\in\Lambda} C_\alpha,$$

so that $\cap_{\alpha\in\Lambda} C_\alpha$ is closed. $\square$

4. In $\mathbb{R}$, 0 is a limit point of both the interval $(0,1)$ and the set $\{1, 1/2, 1/3, \ldots, 1/n, \ldots\}$. This is because every open ball centered on 0 contains points of the set in question, other than 0.

5. If X is given the discrete metric and $0 < \epsilon < 1$ then

$$B_\epsilon(a) = \{a\}, \quad \text{for all} \quad a \in X.$$

6. If (X,d) is a metric space and A is a subset of X, then we may regard (A,d) as a metric space. For example if $A = [0,1]$ the subset of $\mathbb{R}$ with the usual metric, then in A, $B_\epsilon(0) = [0,\epsilon]$ is the open ball centered on 0.

4.2 Dense Sets

In order to define the notion of chaos for one-dimensional maps, we need various topological notions such as denseness and transitivity.

Definition 4.2.1 The *closure* of a set A in a metric space X is defined to be

$$\overline{A} = A \cup \{\text{ the limit points of } A\}.$$

Proposition 4.2.2 $\overline{A}$ *is the smallest closed set containing* A, (i.e., if B is another closed set containing A then $\overline{A} \subseteq B$).

Proof. Clearly $A \subset \overline{A}$. To see that $\overline{A}$ is a closed set, let $a \in X - \overline{A}$. Then a is not a limit point of A so there exists $\delta > 0$ with $B_\delta(a) \cap A = \emptyset$.

We claim $B_\delta(a) \cap \overline{A} = \emptyset$, for if x belongs to this set then $x \notin A$ but x is a limit point of A. Also $x \in B_\delta(a)$, an open set, so there exists $\epsilon > 0$ with $B_\epsilon(x) \subseteq B_\delta(a)$. Hence

$$B_\epsilon(x) \cap A \subseteq B_\delta(a) \cap A = \emptyset,$$

contradicting the fact that x is a limit point of A.

It now suffices to show that if B is any other closed set containing A, then $\overline{A} \subseteq B$. Let $x \in X - B$, an open set, then

$$B_\delta(x) \subset X - B \subseteq X - A \quad \text{for some} \quad \delta > 0,$$

and in particular $B_\delta(x) \cap A = \emptyset$. This says that x is not a limit point of A, so $x \in X - \overline{A}$. This shows that $X - B \subseteq X - \overline{A}$, so $\overline{A} \subseteq B$. □

Definition 4.2.3 The set $A \subseteq X$ is *dense* in X if $\overline{A} = X$.

Examples 4.2.4 1. Let $I \subseteq \mathbb{R}$ be an interval. $A \subseteq I$ is *dense* in I if for any open interval U contained in I we have $U \cap A \neq \emptyset$. This is because every $x \in I$ is a limit point of A: if $x \in I$, $B_\delta(x) \cap A \neq \emptyset$ for every $\delta > 0$.

Equivalently A is dense in I if for any $x \in I$ and any $\delta > 0$, the interval $(x-\delta, x+\delta)$ contains a point of A.

Intuitively, the points of A are spread uniformly over the interval I in such a way that every subinterval of I (no matter how small) contains some points of A.

2. The set $\mathbb{Q}$ of all rational numbers is dense in $\mathbb{R}$. The set $\mathbb{Q} \cap [0, 1]$ is dense in $[0, 1]$.

Proof. We show that the set $\mathbb{Q} \cap [0, 1]$ is dense in $[0, 1]$. Let $x \in (0, 1)$, from the definition it suffices to find a rational number y arbitrarily close to x, i.e., satisfying $|x-y| < \delta$, for some arbitrary number $\delta > 0$. Suppose that x has a decimal expansion

$$x = \sum_{n=1}^{\infty} \frac{d_n}{10^n} = \cdot d_1 d_2 d_3 \ldots, \quad \text{where} \quad d_n \in \{0, 1, 2, \ldots, 9\}.$$

Choose $m \in \mathbb{Z}^+$ so large that $10^{-m} < \delta$ and set $y = \sum_{n=0}^{m} \frac{d_n}{10^n} \in \mathbb{Q}$, then

$$|x - y| = |\cdot d_{m+1} d_{m+2} \ldots| = \sum_{n=m+1}^{\infty} \frac{d_n}{10^n} \leq \sum_{n=m+1}^{\infty} \frac{9}{10^n} \leq \frac{1}{10^m} < \delta.$$

□

3. The set $\mathbb{R} - \mathbb{Q}$ of all irrational numbers is dense in $\mathbb{R}$.

4. For intervals in $\mathbb{R}$, $\overline{(a, b)} = [a, b]$. If $A = \{1, 1/2, 1/3, \ldots\}$, then $\overline{A} = A \cup \{0\}$.

5. A proper subset of a metric space X can be both open and dense. For example, in $\mathbb{R}$, the set $E = \mathbb{R} - \{\sqrt{2}\}$ is both open and dense, with the property that $\mathbb{Q} \subset E \neq \mathbb{R}$.

Definition 4.2.5 If (X, d) is a metric space, then we say $\lim_{n\to\infty} x_n = a$ if for all $\epsilon > 0$ there exists $N \in \mathbb{Z}^+$ such that $d(a, x_n) < \epsilon$ for all $n > N$.

Theorem 4.2.6 *The following are equivalent for a metric space* (X, d)*:*

(i) *The set* A *is dense in* X*.*

(ii) *Let* $\epsilon > 0$*, then for all* $x \in X$ *there exists* $a \in A$ *such that* $d(a, x) < \epsilon$*.*

(iii) *For all* $x \in X$*, there is a sequence* $\{a_n\} \subset A$ *with* $\lim_{n\to\infty} a_n = x$*.*

Proof. (i) $\Rightarrow$ (ii). Suppose that A is dense in X and let $x \in X$, then either $x \in A$ (so (ii) holds) or x is a limit point of A:
i.e., any ball $B_\epsilon(x)$ contains points of A, so there exists $a \in A$ with $d(x, a) < \epsilon$.

(ii) $\Rightarrow$ (iii). Let $x \in X$ and $n \in \mathbb{Z}^+$, then there exists $a_n \in A$ with $d(x, a_n) < 1/n$ $(n = 1, 2, \ldots)$, so that as $n \to \infty$, $d(x, a_n) \to 0$.
i.e., given any $\epsilon > 0$ there exists $N \in \mathbb{Z}^+$ such that

$$n > N \Rightarrow d(x, a_n) < \epsilon, \quad \text{or} \quad \lim_{n\to\infty} a_n = x.$$

(iii) $\Rightarrow$ (i). We must show that $\overline{A} = X$. Let $x \in X$, then there exists a sequence $\{a_n\} \subset A$ with $\lim_{n\to\infty} a_n = x$, i.e., x is a limit point of A since $B_\delta(x) \cap A \neq \emptyset$ for each $\delta > 0$, so $\overline{A} = X$. □

Example 4.2.7 1. For a continuous function $f : \mathbb{R} \to \mathbb{R}$ the set $\mathrm{Per}_1(f) = \mathrm{Fix}(f)$ of all fixed points is a closed set. The same is true for continuous functions on a metric space (see below).

Proof. Let $x \in X$ and suppose that x_n is a sequence of points in $\mathrm{Fix}(f)$ with $x_n \to x$. The continuity of f implies $\lim_{n\to\infty} f(x_n) = f(x)$, or

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_n) = f(x),$$

so x is also fixed point and $\mathrm{Fix}(f)$ is closed.

2. If $f(x) = x\sin(1/x)$, then $f(x) = x$ when $x = 0$ and when $x = x_k = \dfrac{2}{\pi(1+4k)}$, $k \in \mathbb{Z}$. Note that $\lim\limits_{k\to\infty} x_k = 0$. Also each x_k is a non-hyperbolic fixed point and $f'(0)$ is not defined. The set $\mathrm{Fix}(f) = \{x_k : k \in \mathbb{Z}\} \cup \{0\}$ is clearly a closed set. This function has many eventually fixed points which are easy to find. The fixed point $x = 0$ is stable but not attracting since there are other fixed points arbitrarily close to it.

Exercises 4.1 - 4.2

1. (i) Prove that every open ball $B_\epsilon(a)$ in a metric space (X, d) is an open set and that every finite subset is a closed set.

(ii) Show that in general, the closed ball $\{x \in X : d(a, x) \leq \epsilon\}$ need not be equal to the closure of the open ball $B_\epsilon(a)$ (Hint: Consider the two point space $X = \{a, b\}$ with metric $d(a, b) = 1$).

2. For $x, y \in \mathbb{R}$, distances d_1, d_2, d_3, d_4, d_5 are defined by:

$$d_1(x,y) = (x-y)^2, \quad d_2(x,y) = \sqrt{x-y}, \quad d_3(x,y) = |x^2 - y^2|,$$

$$d_4(x,y) = |x - 2y|, \quad d_5(x,y) = \frac{|x-y|}{1+|x-y|}.$$

Which of these (if any) defines a metric on $\mathbb{R}$? (Hint: For d_5, consider the properties of the map $f(x) = x/(1+x)$).

3. Show that for $x = (x_1, x_2), y = (y_1, y_2)$, points in $\mathbb{R}^2$, the following define metrics on $\mathbb{R}^2$:

$$d'(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}, \quad d''(x, y) = |x_1 - y_1| + |x_2 - y_2|.$$

What do the open balls $B_1(a)$, where $a = (a_1, a_2)$, look like in each case?

4. Show that the intersection of a finite number of open sets $A_1, A_2, \ldots, A_n$, in a metric space (X, d) is an open set. Show, by considering the intervals $(-1/n, 1/n)$ $(n \in \mathbb{Z}^+)$, in $\mathbb{R}$ that the intersection of infinitely many open sets need not be open.

5. Denote by Σ the metric space of all sequences of 0's and 1's

$$\Sigma = \{\omega = (s_1, s_2, s_3, \ldots) : s_i = 0 \text{ or } 1\},$$

with metric

$$d(\omega_1, \omega_2) = \sum_{k=1}^{\infty} \frac{|s_k - t_k|}{2^k}$$

when $\omega_1 = (s_1, s_2, \ldots)$ and $\omega_2 = (t_1, t_2, \ldots)$. Find $d(\omega_1, \omega_2)$ if:

(i) $\omega_1 = (0, 1, 1, 1, 1, \ldots$ and $\omega_2 = (1, 0, 1, 1, 1, \ldots)$,

(ii) $\omega_1 = (0, 1, 0, 1, 0, \ldots$ and $\omega_2 = (1, 0, 1, 0, 1, \ldots)$.

6. Denote by $C[a, b]$ the set of all continuous functions $f : [a, b] \to \mathbb{R}$. If the distance between two such continuous functions is given by:

$$d_1(f, g) = \max_{x \in [a,b]} |f(x) - g(x)| \quad \text{or} \quad d_2(f, g) = \int_a^b |f(x) - g(x)|\, dx,$$

then show that both d_1 and d_2 dfine metrics on $C[a, b]$.

4.3 Functions Between Metric Space

Given two different metric spaces (X, d_1) and (Y, d_2), we may need to consider functions between them. Here the notion of continuity is often important:

Definition 4.3.1 A function $f : X \to Y$ between two metric spaces (X, d_1) and (Y, d_2) is said to be continuous at $a \in X$ if given any $\epsilon > 0$ there exists $\delta > 0$ such that if $x \in X$ then

$$d_1(x, a) < \delta \Rightarrow d_2(f(x), f(a)) < \epsilon.$$

Proposition 4.3.2 *The following are equivalent for $f : X \to Y$ between metric spaces (X, d_1) and (Y, d_2):*

(i) *f is continuous at $x = a$.*

(ii) *Given any ϵ-ball $B_\epsilon(f(a))$ centered on $f(a)$ there exists a δ-ball $B_\delta(a)$ centered on a such that*

$$f(B_\delta(a)) \subset B_\epsilon(f(a)).$$

(iii) *Given any open set V containing $f(a)$ there exists an open set U containing a such that*

$$f(U) \subseteq V.$$

Proof. Since the equivalence of (i) and (ii) is a fairly straightforward restatement of the definition of continuity, we will only show the equivalence of (i) and (iii).

(i) $\Rightarrow$ (iii) Let $V \subset Y$ be open with $f(a) \in V$. Then there exists $\epsilon > 0$ with $B_\epsilon(f(a)) \subseteq V$ (since V is open). Since f is continuous at a, there exists $\delta > 0$ with

$$f(B_\delta(a)) \subseteq B_\epsilon(f(a)) \subseteq V,$$

so set $U = B_\delta(a)$ and the result follows.

(iii) $\Rightarrow$ (i) $B_\epsilon(f(a))$ is open in Y and contains $f(a)$, so by hypothesis there exists U open in X, containing a such that

$$f(U) \subseteq B_\epsilon(f(a)).$$

Since U is open, we can find $\delta > 0$ such that $B_\delta(a) \subseteq U$. Thus

$$f(B_\delta(a)) \subseteq f(U) \subseteq B_\epsilon(f(a)),$$

so that f is continuous at a $\square$

Definition 4.3.3 A function $f : X \to Y$ is *continuous* if it is continuous at every $a \in X$.

Theorem 4.3.4 *The following are equivalent for a function $f : X \to Y$ between metric spaces:*

(i) $f : X \to Y$ *is continuous.*

(ii) $f^{-1}(V)$ *is open in* X *whenever* V *is open in* Y ($f^{-1}(V) = \{x \in X : f(x) \in V\}$ is defined even if f is not invertible).

Proof. (i) $\Rightarrow$ (ii) Let V be open in Y, then we may assume $f^{-1}(V) \neq \emptyset$, so let $a \in f^{-1}(V)$, then $f(a) \in V$. Since V is open there exists $B_\epsilon(f(a)) \subset V$. Now f is continuous at a, so there exists $\delta > 0$ such that

$$f(B_\delta(a)) \subseteq B_\epsilon(f(a)), \quad \text{so} \quad B_\delta(a) \subseteq f^{-1}B_\epsilon(f(a)) \subseteq f^{-1}(V),$$

so $f^{-1}(V)$ is open in X.

(ii) $\Rightarrow$ (i) If $f^{-1}(V)$ is open in X whenever V is open in Y, let $a \in X$, then $f(a) \in B_\epsilon(f(a))$, an open set in Y ($\epsilon > 0$).

By hypothesis, $f^{-1}(B_\epsilon(f(a)))$ is open in X and since $a \in f^{-1}(B_\epsilon(f(a)))$, there exists $\delta > 0$ such that

$$B_\delta(a) \subseteq f^{-1}B_\epsilon(f(a)) \quad \text{or} \quad f(B_\delta(a)) \subseteq B_\epsilon(f(a)),$$

so that f is continuous at a. $\square$

As an application of this theorem we show that the basin of attraction $B_f(p)$, of a fixed point p for a continuous map $f : X \to X$, X a metric space, is an open set. Let p be an attracting fixed point for f. In this context it means that there exists $\epsilon > 0$ such that if $x \in B_\epsilon(p)$, the open ball of radius ϵ centered on p, then $f^n(x) \to p$ as $n \to \infty$. Recall that the basin of attraction of p is the set

$$B_f(p) = \{x \in X : f^n(x) \to p \quad \text{as} \quad n \to \infty\}.$$

We saw that for $0 < \mu < 1$, 0 is an attracting fixed point of the logistic map $L_\mu(x) = \mu x(1-x)$ with basin of attraction $[0, 1]$. In this case the metric space is $X = [0, 1]$, so the basin of attraction is an open set namely the whole space (the fixed point 0 is globally attracting). For continuous maps $f : \mathbb{R} \to \mathbb{R}$ having an asymptotically stable fixed point p, the basin of attraction is an open set, which in this case is just the union of open intervals. The largest such open interval to which p belongs is called the *immediate basin of attraction of* p under f.

To say that $A \subseteq X$ is *invariant* under f means that $f(x) \in A$ for all $x \in A$.

Theorem 4.3.5 *If* $f : X \to X$ *is a continuous map of a metric space* X *and* p *is an attracting fixed point of* f*, then* $B_f(p)$*, the basin of attraction of* f*, is an invariant open set.*

Proof. If $x \in B_f(p)$, then $f^n(x) \to p$ as $n \to \infty$, so that $f^n(f(x)) \to f(p) = p$ as $n \to \infty$, so $f(x) \in B_f(p)$ and $B_f(p)$ is an invariant set.

Since p is an attracting fixed point there exists $\epsilon > 0$ such that if $x \in B_\epsilon(p)$, $f^n(x) \to p$ as $n \to \infty$. Since f is a continuous function the set $f^{-1}(B_\epsilon(p))$ is open (from Theorem 4.3.4). We claim that

$$B_f(p) = \cup_{n=1}^{\infty} f^{-n}\left(B_\epsilon(p)\right),$$

from which the result follows since the union of open sets is open.

Let $x \in \cup_{n=1}^{\infty} f^{-n}\left(B_\epsilon(p)\right)$, then $x \in f^{-n}\left(B_\epsilon(p)\right)$ for some $n \in \mathbb{Z}^+$, or $f^n(x) \in B_\epsilon(p)$, so that clearly $x \in B_f(p)$.

On the other hand, suppose that $x \in B_f(p)$, then there exists $n \in \mathbb{Z}^+$ with $f^n(x) \in B_\epsilon(p)$ (since p is attracting), so that $x \in f^{-n}\left(B_\epsilon(p)\right)$ and $x \in \cup_{n=1}^{\infty} f^{-n}\left(B_\epsilon(p)\right)$. $\square$

For metric spaces, the idea of "sameness" is given by the notion of "homeomorphism". Two metric spaces are regarded as "topologically" the same if there is a homeomorphism between them.

Definition 4.3.6 The function $h : X \to Y$ between metric spaces is a *homeomorphism* if h is one-to-one, onto and both h and h^{-1} are continuous. The spaces X and Y are said to be *homeomorphic*, when there is a homeomorphism between them.

Examples 4.3.7 1. If $f : [0,1] \to [0,1]$, $f(x) = x^2$ then f is a homeomorphism. In fact any strictly increasing continuous function $f : [0,1] \to [0,1]$ with $f(0) = 0$ and $f(1) = 1$ is a homeomorphism. Also any strictly decreasing continuous function $f : [0,1] \to [0,1]$ with $f(0) = 1$ and $f(1) = 0$ is a homeomorphism and it can be shown that any homeomorphism of $[0,1]$ to itself is of one of the above two forms. The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is a homeomorphism. The inverse function is $f^{-1}(x) = x^{1/3}$.

2. The function $f : \mathbb{R} \to (-\pi/2, \pi/2)$, $f(x) = \arctan x$ is a homeomorphism of the respective metric spaces. The logistic map $L_\mu(x) = \mu x(1-x)$, $0 < \mu \leq 4$, is not a homeomorphism of $[0,1]$ as it is not one-to-one (it is only onto when $\mu = 4$).

3. Consider the metric spaces $X = [0,1]$ and $\Sigma = \{(s_1, s_2, s_3, \ldots) : s_i = 0 \text{ or } 1\}$ with the usual metrics. We shall show that these two metric spaces are not homeomorphic.

Exercises 4.3

1. Let $f : \mathbb{R} \to (-1, 1)$ be defined by:

$$f(x) = \frac{x}{1 + |x|}.$$

(a) Show that f is a homeomorphism and find the inverse map.

(b) Extend f to a function $f : [-\infty, \infty] \to [-1, 1]$ by setting $f(-\infty) = -1$ and $f(\infty) = 1$, and define a metric d on $[-\infty, \infty]$ by

$$d(x, y) = |f(x) - f(y)| \quad \text{for all} \quad x, y \in [-\infty, \infty].$$

Show that d defines a metric on this space whose restriction to $\mathbb{R}$ is different to the usual metric defined on $\mathbb{R}$. $[-\infty, \infty]$ with this metric is called the *extended real line*.

2. Let $f : X \to X$ be a map defined on the metric space (X, d) and let $\alpha \in \mathbb{R}^+$ be fixed, with the property:

$$d(f(x), f(y)) \leq \alpha d(x, y) \quad \text{for all} \quad x, y \in X.$$

Show that f is continuous on X.

3. A map $f : X \to X$ defined on the metric space (X, d) satisfying

$$d(f(x), f(y)) = d(x, y) \quad \text{for all} \quad x, y \in X,$$

is called an *isometry*. Show that f is continuous, one-to-one, and hence a homeomorphism onto its range. What are the isometries $f : \mathbb{R} \to \mathbb{R}$ ($\mathbb{R}$ with the usual metric)?

4. Show that if $f : [a, b] \to [a, b]$ is a homeomorphism, then either a and b are fixed points or $\{a, b\}$ is a 2-cycle.

5. Let $f : X \to X$ be a continuous function on a metric space (X, d). Use the definition of continuity, limits and limit point, in this context to prove:

(a) If $\lim_{n\to\infty} x_n = a$, then show that $\lim_{n\to\infty} f(x_n) = f(a)$.

(b) If $\lim_{n\to\infty} f^n(x_0) = p$, then p is a fixed point of f.

(c) If there is exactly one limit point of the set $\mathrm{O}(x_0) = \{x_0, f(x_0), f^2(x_0), \ldots\}$, then it is a fixed point of f.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map with fixed point c and basin of attraction $B_f(c) = (a, b)$ an interval. Show that one of the following must hold:

(a) a and b are fixed points.

(b) a or b is fixed and the other is eventually fixed.

(c) $\{a, b\}$ is a 2-cycle.

7. Let $f : X \to Y$ be a continuous map of a metric spaces. Prove that if C is a closed set in Y, then $f^{-1}(C)$ is a closed set in X. (Hint: Use the fact that $f^{-1}(Y - U) = X - f^{-1}(U)$, and properties of open sets).

8. Let $f : X \to X$ be a continuous map of the metric space (X, d). $x \in X$ is a *wandering point* of f if there is an open set U containing x such that $U \cap f^n(U) = \emptyset$ for all $n \in \mathbb{Z}^+$. A point $x \in X$ is called *non-wandering* if it is not wandering. The set of non-wandering points of f is denoted by $\Omega(f)$. Show the following:

(a) The set of all wandering points is open. Deduce that $\Omega(f)$ is a closed set which contains all of the periodic points of f.

(b) $\Omega(f)$ is an invariant set.

(c) $\Omega(f^n) \subset \Omega(f)$ for all $n \in \mathbb{Z}^+$.

(d) If f is a homeomorphism, then $f(\Omega(f)) = \Omega(f)$ and $\Omega(f^{-1}) = \Omega(f)$.

9. Let $f : X \to X$ be a continuous map of the metric space (X, d). If $x \in X$, then the ω-limit (*omega-limit*) set $\omega(x)$ is defined to be

$$\omega(x) = \cap_{n=0}^{\infty} \overline{\{f^k(x) : k > n\}}.$$

It can be shown that if $X = [0,1]$, then $\omega(x) \subset \Omega(f)$ for all $x \in X$. Show the following:

(a) $y \in \omega(x)$ if and only if there is an increasing sequence $\{n_k\}$ such that $f^{n_k}(x) \to y$ as $k \to \infty$.

(b) $\omega(x)$ is a closed invariant set $(f(\omega(x)) \subset \omega(x))$.

(c) If x is a periodic point of f, then $\omega(x) = \mathrm{O}(x)$. Also if x is eventually periodic with $y \in \mathrm{O}(x)$, then $\omega(x) = \mathrm{O}(y)$.

(e) If $\omega(x)$ consists of a single point, then that point is a fixed point.

10. Let $f : X \to X$ be a continuous map of the metric space (X, d). If $x \in X$, it is said to be *recurrent* if $x \in \omega(x)$, i.e., x belongs to its ω-limit set. Recurrent points are a generalization of periodic points. Periodic points return to themselves, whereas recurrent points return closely to themselves infinitely often, becoming closer as the iteration procedure progresses. Show

(a) A periodic point is recurrent but an eventually periodic point is not recurrent.

(b) Any recurrent point x belongs to $\Omega(f)$.

(c) If $f : [0,1] \to [0,1]$ is a homeomorphism, then the only recurrent points are the periodic points.

(d) Isolated points are recurrent.

(e) $\omega(x) \subset \overline{\mathrm{O}(x)}$ and $\omega(x) = \overline{\mathrm{O}(x)}$ if and only if x is recurrent (Need X compact?).

11. Let $f : X \to X$ be a continuous map of the metric space (X, d). f is said to be *minimal* if $\overline{\mathrm{O}(x)} = X$ for every $x \in X$. Prove that f is minimal if and only if every non-empty closed invariant subset A of X satisfies $A = X$. (Hint: for the $\Leftarrow$ direction, use 8(e) above). We will show later that rotations $R_\alpha : [0,1) \to [0,1)$, $R_\alpha(x) = x + \alpha \pmod 1$, are minimal when α is irrational.

12. Let $f : X \to X$ be a map of the metric space (X, d). f is *transitive* if there exists $x \in X$ with $\overline{\mathrm{O}(x)} = X$, i.e., there is a point having a dense orbit. We will show

later that the logistic map L_4 and the tent map T_2 are transitive, and that this is an ingredient of the map being *chaotic*.

(a) Prove that a transitive map is onto.

(b) If f is an isometry which is transitive, prove that f is minimal.

13. Let $f : X \to X$ be a map of the metric space (X, d). A non-empty subset $A \subset X$ is said to be *minimal* if the restriction of f to A is a minimal map. It can be shown that any continuous map $f : X \to X$ of a metric space has at least one minimal set. Prove:

(a) Two minimal sets A and B for f are either disjoint or equal.

(b) If $f : X \to X$ is an isometry and $\omega(x) \neq \emptyset$, then x is a recurrent point and $\omega(x)$ is a minimal set.

4.4 Diffeomorphisms of $\mathbb{R}$

We study homeomorphisms that are also differentiable functions. If f is a homeomorphism, then the inverse function f^{-1} exists with $y = f(x)$ if and only if $x = f^{-1}(y)$. Denote by I and J open subintervals of $\mathbb{R}$, e.g., $I = (a, b)$ for some $a, b \in \mathbb{R}$, $(a < b)$, or $I = (a, \infty)$ etc.

Definition 4.4.1 1. A function $f : I \to J$ is said to be of class C^1 on I if $f'(x)$ exists and is continuous at all $x \in I$. Such a function is also said to be *smooth*.

2. A homeomorphism $f : I \to J$ is called a *diffeomorphism* on I if f and f^{-1} are both C^1-functions on I. If f is a diffeomorphism on $\mathbb{R}$ and $I \subset \mathbb{R}$ is some closed interval, we sometimes refer to f as a diffeomorphism on I, onto the closed interval J where $f(I) = J$.

If $y = f(x)$, then $f^{-1}(f(x)) = x$, so applying the chain rule gives

$$(f^{-1})'(f(x)) \cdot f'(x) = 1 \quad \text{or} \quad (f^{-1})'(y) = \frac{1}{f'(x)}.$$

It follows that for a diffeomorphism we must have $f'(x) \neq 0$ for all $x \in I$. Since $f'(x)$ is a continuous function, if it were allowed to be both positive and negative, then by the Intermediate Value Theorem it must also take the value 0, a contradiction. It follows that a diffeomorphism has either $f'(x) > 0$ always or $f'(x) < 0$ always. This proves:

Proposition 4.4.2 *A diffeomorphism $f : I \to J$ is either:*

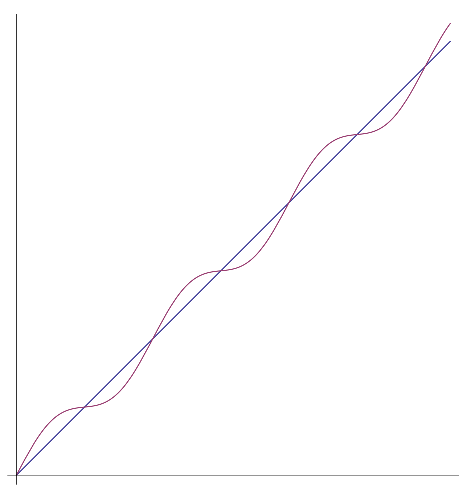
1. Order preserving: $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ (f is a strictly increasing function) or

2. Order reversing: $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ (f is a strictly decreasing function).

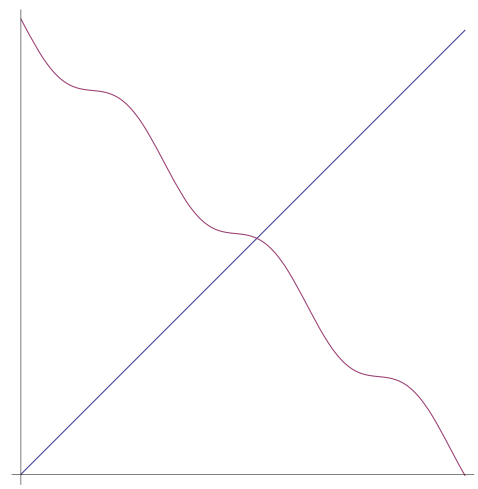
Examples 4.4.3 1. The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is a C^1 function but it is not a diffeomorphism since $f'(0) = 0$, so the inverse function $f^{-1}(x) = x^{1/3}$ is not differentiable at $x = 0$ (vertical tangent).

We can see that $f : \mathbb{R} \to (-\pi/2, \pi/2)$, $f(x) = \arctan x$ is a diffeomorphism. The inverse function is $f^{-1} : (-\pi/2, \pi/2) \to \mathbb{R}$, $f^{-1}(x) = \tan x$

2. Order preserving diffeomorphisms can have any number of fixed points, but we see that this is not true in the order reversing case:



Order preserving.



Order reversing.

Theorem 4.4.4 *Let I and J be open intervals and $f : I \to J$ an order reversing diffeomorphism with $f(I) = J \subseteq I$, then f has a unique fixed point in I.*

Proof. If $I = (a, b)$, then a continuity argument similar to that in Theorem 1.2.9 is enough to ensure the existence of a fixed point. If $I = \mathbb{R}$ let $\alpha = \lim_{x\to-\infty} f(x)$ and $\beta = \lim_{x\to\infty} f(x)$ (α and β could be $\pm\infty$). Then $\alpha > \beta$ since f is order reversing.

Let $g(x) = f(x) - x$, then

$$\lim_{x \to -\infty} g(x) = \infty, \quad \text{and} \quad \lim_{x \to \infty} g(x) = -\infty,$$

so by IVT there exists $c \in \mathbb{R}$ with $g(c) = 0$, so $f(c) = c$.

The situation for other types of intervals is similar.

Suppose now f is order reversing and has two fixed points, say $f(\alpha) = \alpha$ and $f(\beta) = \beta$ with $\alpha < \beta$, then $f(\alpha) > f(\beta)$ or $\alpha > \beta$ since they are fixed points, a contradiction. □

Examples 4.4.5 Order preserving diffeomorphisms have been seen to have any number of fixed points. We see below that they cannot have points of period greater than 1. On the other hand, order reversing diffeomorphisms can have points of period 2, e.g., consider $f(x) = -x$, but no points of greater period. The bottom line is that the dynamics of one-dimensional diffeomorphisms is not complicated. This is not the case for two-dimensional diffeomorphisms (i.e., diffeomorphisms $f : \mathbb{R}^2 \to \mathbb{R}^2$).

Theorem 4.4.6 *Let $f : I \to I$ be a diffeomorphism on an open interval I.*

(i) *If f is order preserving, then f has no periodic points of period greater than 1.*

(ii) *If f is order reversing, then f has no periodic points of period greater than 2.*

Proof. (i) Let $O(x_0) = \{x_0, x_1, x_2, \ldots\}$ be the orbit of x_0 and suppose $x_1 = f(x_0) \neq x_0$.

If $x_1 > x_0$ then $f(x_1) > f(x_0)$ or $x_2 > x_1$. Repeating this argument gives

$$x_0 < x_1 < x_2 < \cdots < x_n, \quad \text{so that} \quad f^n(x_0) = x_n \neq x_0.$$

On the other hand, if $x_1 < x_0$, then $f(x_1) < f(x_0)$ so $x_2 < x_1$ and

$$x_0 > x_1 > x_2 > \cdots x_n, \quad \text{so that} \quad f^n(x_0) = x_n \neq x_0.$$

(ii) Notice that if f is an order reversing diffeomorphism, then f^2 is an order preserving diffeomorphism. In fact if $f'(x) < 0$, then

$$(f^2)'(x) = f'(f(x)) \cdot f'(x) > 0,$$

so by (i), f^2 has no n-cycles with $n > 1$, so that f has no $2n$-cycles with $n > 1$.

If n is odd, then f^n is a diffeomorphism with $(f^n)'(x) < 0$, so that f^n has a unique fixed point which must be the fixed point of f. □

We can improve upon some of the above results in the following way:

Proposition 4.4.7 *Let $f : [a, b] \to [a, b]$ $(a < b)$ be a continuous and one-to-one function.*

(i) *Either f is strictly increasing or strictly decreasing on $[a, b]$.*

(ii) *If f is strictly increasing, then every periodic point of f is a fixed point of f.*

(iii) *If f is strictly decreasing, then f has exactly one fixed point and all other periodic points have period 2.*

Proof. (i) Suppose that $f(a) < f(b)$ (they cannot be equal as f is one-to-one), and let $x_1, x_2 \in (a, b)$ with $x_1 < x_2$ and $f(x_1) > f(x_2)$. Suppose that $f(a) < f(x_2) < f(x_1)$, then by the Intermediate Value Theorem there exists $c \in (a, x_1)$ with $f(c) = f(x_2)$, contradicting the one-to-oneness of f, so this is not possible. Similarly if $f(x_2) < f(a) < f(x_1)$ we find $c \in (x_1, x_2)$ with $f(c) = f(a)$. Other possibilities are treated in a similar way.

(ii) Suppose that c is a period point of f having period p. If $c < f(c)$, then since f is strictly increasing, $f(c) < f^2(c)$ and we get an increasing sequence whose limit exists (say L). Then

$$c < L = \lim_{n\to\infty} f^n(c) = \lim_{n\to\infty} f^{np}(c) = c,$$

a contradiction. Similarly $f(c) < c$ leads to a contradiction, so $f(c) = c$ and $p = 1$.

(iii) Since f is continuous on a compact interval into itself, f has at least one fixed point. We must have $f(a) > a$ since otherwise $a = f(a) > f(b) \geq a$ which is impossible. Similarly $f(b) < b$.

If c_1 and c_2 are different fixed points of f with $c_1 < c_2$, then $c_1 = f(c_1) > f(c_2) = c_2$, which is impossible, so f has exactly one fixed point. Now notice that since f is order reversing, f^2 is order preserving, so the only periodic points f^2 can have are fixed points, and these are period 2 points of f. $\square$

Remarks 4.4.8 1. It follows that if $f : [a, b] \to [a, b]$ is a homeomorphism, then it is either strictly increasing or strictly decreasing. If it is increasing, f is order preserving with $f(a) = a$, $f(b) = b$ and the only other periodic points are fixed points. If it is decreasing (order reversing), then it has exactly one fixed point with all other periodic points having period 2. Also, f^2 is orientation preserving and we must have $f(a) = b$, $f(b) = a$. See [46] for a discussion of one-to-oneness.

Exercises 4.4

1. Show that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is not a diffeomorphism of $\mathbb{R}$. Give an example of a diffeomorphism $f : \mathbb{R} \to \mathbb{R}$.

2^*. Show that $f(x) = 1 + x + x^2/2! + x^3/3! + \cdots + x^n/n!$ is a diffeomorphism of $\mathbb{R}$ when n is odd.

4.5 Countability, Sets of Measure Zero and the Cantor Set

Given an infinite set A, there are different ways of thinking of A as being large. For example, we might say that $\mathbb{Q} \subset \mathbb{R}$ is a large set since it is dense in $\mathbb{R}$. However, in some sense it is a small set, what we call a *countable set* and also a set of *measure zero*. Sets of measure zero are sets that are small in terms of "length". Many of the ideas in this section are due to George Cantor, a German mathematician of the 19th century. Before we examine the notion of countability, we look at sets of measure zero.

Definition 4.5.1 Let I be a bounded subinterval of $\mathbb{R}$, having end-points a, b $(a \leq b)$. The *length* of I is then $|I| = b - a$. If I is an unbounded interval, we set $|I| = \infty$.

Definition 4.5.2 We say that a set $A \subseteq \mathbb{R}$ is a *set of measure zero* if we can cover A by bounded open intervals indexed by the set $\mathbb{Z}^+$, so that the total length of the intervals can be chosen to be arbitrarily small. More precisely, given any $\epsilon > 0$ there is a collection of open intervals $\{I_n : n \in \mathbb{Z}^+\}$ with

$$A \subseteq \cup_{n \in \mathbb{Z}^+} I_n \quad \text{and} \quad \sum_{n \in \mathbb{Z}^+} |I_n| \leq \epsilon.$$

A collection of open sets O_λ, $(\lambda \in \Lambda)$, whose union contains a set A (in a metric space) is called an *open cover* of the set A.

The requirement that the intervals be open is not a serious one, they may be closed or half open. For example, if the intervals I_n are closed, replace I_n by J_n with $I_n \subset J_n$ and $|J_n| = |I_n| + \epsilon/2^n$, then

$$\sum_{n \in \mathbb{Z}^+} |J_n| = \sum_{n \in \mathbb{Z}^+} |I_n| + \sum_{n \in \mathbb{Z}^+} \epsilon/2^n \leq 2\epsilon.$$

If a property P holds everywhere except for a set of measure zero, we sometimes say that it holds *almost everywhere* (abbreviated a.e.), so for example, if $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q} \\ 0 & \quad \text{otherwise} \end{cases},$$

then we say $f(x) = 0$ a.e.

Example 4.5.3 1. A finite subset $A = \{a_1, a_2, \ldots, a_n\}$ of $\mathbb{R}$ is a set of measure zero because if $\epsilon > 0$, we can find open intervals say $I_i = (a_i - \epsilon/2n, a_i + \epsilon/2n)$, $i = 1, 2, \ldots, n$, with $A \subset \cup_{i=1}^{n} I_i$ and the total length of the intervals being ϵ.

2. The set $\mathbb{Q}$ of rationals, is a set of measure zero. To prove this we make a digression on countability. Countably infinite sets (said to be *denumerable*) are in a sense the smallest type of infinite set.

Definition 4.5.4 Any set A that is finite or can be put into a one-to-one correspondence with the set $\mathbb{Z}^+$ is said to be *countable*. This is equivalent to saying that the members of A may be listed as a sequence.

Examples 4.5.5 1. If $A = \{1, 4, 9, 16, \ldots, n^2, \ldots,\}$, then A is countable since we can define $f : \mathbb{Z}^+ \to A$ by $f(n) = n^2$, a one-to-one and onto map. Any subset of a countable set can be seen to be countable.

2. The set $\mathbb{Q}$ of rational numbers is a countable set. We list the positive rationals as a sequence (with some repetitions) by following the arrows below. The general case can be demonstrated with some minor modifications:

$$\begin{array}{ccccccccccc}
1/1 & \rightarrow & 1/2 & & 1/3 & \rightarrow & 1/4 & & 1/5 & \rightarrow & \ldots \\
 & \swarrow & & \nearrow & & \swarrow & & \nearrow & & \swarrow & \ldots \\
2/1 & & 2/2 & & 2/3 & & 2/4 & & 2/5 & \ldots & \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & \swarrow & & \ldots & \\
3/1 & & 3/2 & & 3/3 & & 3/4 & & 3/5 & \ldots & \\
 & \swarrow & & \nearrow & & \swarrow & & & & \ldots & \\
4/1 & & 4/2 & & 4/3 & & 4/4 & & 4/5 & \ldots & \\
\downarrow & \nearrow & & \swarrow & & & & & & \ldots & \\
5/1 & & 5/2 & \ldots & \ldots & \ldots & \ldots & \ldots & & & \\
\vdots & \swarrow & \vdots & \vdots & \vdots & \vdots & & & & &
\end{array}$$

3. The set $\mathbb{R}$ is not countable (it is said to be *uncountable*). In fact we show (by contradiction) that the interval $[0, 1]$ is uncountable:

Suppose that $[0,1]$ is a countable set, then we can list its members as a sequence $x_1, x_2, x_3, \ldots, x_n, \ldots$. Each member of this sequence has a binary expansion, say

$$\begin{array}{ccl} x_1 & = & \cdot a_{11}\, a_{12}\, a_{13}\, a_{14} \ldots \\ x_2 & = & \cdot a_{21}\, a_{22}\, a_{23}\, a_{24} \ldots \\ x_3 & = & \cdot a_{31}\, a_{32}\, a_{33}\, a_{34} \ldots \\ \vdots & \vdots & \qquad \vdots \\ x_n & = & \cdot a_{n1}\, a_{n2}\, a_{n3}\, a_{n4} \ldots \\ \vdots & \vdots & \qquad \vdots \end{array} .$$

Let $y = .b_1\, b_2\, b_3\, b_4 \ldots b_n \ldots$, where $b_i = \begin{cases} 0 & \text{if} \quad a_{ii} = 1 \\ 1 & \text{if} \quad a_{ii} = 0 \end{cases}$, then y differs from x_1 in the first binary digit, it differs from x_2 in the second binary digit etc., differing from x_n in the nth binary digit, so y does not appear anywhere in the sequence, contradicting the fact that we have listed all members of $[0,1]$ as a sequence.

Remarks 4.5.6 1. Each $x \in (0,1)$ has a unique *non-terminating* binary expansion, for example:

$$\frac{1}{2} = .10000\ldots, \quad \text{and} \quad \frac{1}{2} = \cdot 0111\ldots,$$

so $1/2$ has two different binary representations, but only the second is non-terminating. In the proof of the uncountability of $[0,1]$ we should assume the x_i have a non-terminating expansion. This gives $y = .b_1\, b_2\, b_3\, b_4 \ldots b_n \ldots$, but we don't know if it is terminating or not. To avoid this difficulty we should start by listing all possible binary expansions (both terminating and non-terminating), so some numbers are listed twice, but there are only countably many having terminating expansions (these are the binary rationals such as $1/2$, $3/4 = \cdot 1100\ldots$ etc.) Now proceed as previously to see that y is not one of the enumerated elements.

2. It is easy to see that there is a one-to-one correspondence between the interval $(0,1)$ and $\mathbb{R}$ (we say these sets have the same *cardinality*). For example, the map $F : \mathbb{R} \to (-\pi/2, \pi/2)$, $F(x) = \arctan(x)$ is both one-to-one and onto. If we take a linear (affine) map between $(-\pi/2, \pi/2)$ and $(0,1)$, the result is clear.

We can now prove that the rationals (and in fact any countable set) is a set of measure zero, i.e., can be covered by a collection of open intervals whose total length can be made as small as we like. It can be shown that an interval such as $[0,1]$ is not a set of measure zero and also that the irrational numbers in $[0,1]$ do not constitute a set of measure zero.

Proposition 4.5.7 *The rational numbers form a set of measure zero.*

Proof. We can enumerate $\mathbb{Q}$ as a sequence $r_1, r_2, r_3, \ldots, r_n, \ldots$, say. Let $\epsilon > 0$ and for each $n \in \mathbb{Z}^+$ define an interval I_n by

$$I_n = (r_n - \epsilon/2^{n+1}, r_n + \epsilon/2^{n+1}).$$

Clearly $r_n \in I_n$, $n = 1, 2, \ldots$, so that $\mathbb{Q} \subset \cup_{i=1}^{\infty} I_n$. The total length of these intervals is

$$\sum_{i=1}^{\infty} |I_n| = \sum_{i=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon,$$

which can be made as small as we please. The set $E = \cup_{n=1}^{\infty} I_n$ is another example of a set in $\mathbb{R}$ which is both open and dense, containing $\mathbb{Q}$ and not equal to $\mathbb{R}$. $\square$

We now show that an interval I, with end-points a and b, $a < b$, cannot be a set of measure zero. The following result is due to Borel (see Oxtoby [47]). We first need a lemma which is of interest in its own right, and is related to a result known as the *Heine-Borel Theorem*:

Lemma 4.5.8 *Let $\{O_\lambda\}_{\lambda \in \Lambda}$ be a collection of open sets which cover the interval $[a, b]$, $a \leq b$ (i.e., $[a, b] \subset \cup_{\lambda \in \Lambda} O_\lambda$). Then there is a finite subcollection of these sets that cover $[a, b]$.*

Proof. Let

$\mathcal{S} = \{x \in [a, b]$: there is a finite subcollection of intervals from $O_\lambda, \lambda \in \Lambda$ that cover $[a, x]\}$.

Clearly $a \in \mathcal{S}$ and the set $\mathcal{S}$ is bounded above by b, so $\alpha = \sup(\mathcal{S})$ exists and $\alpha \leq b$. Suppose that $\alpha < b$, then α belongs to some open set say O_μ from our collection. Since this set is open, there is an open interval $(\alpha - \epsilon, \alpha + \epsilon) \subset O_\mu$. If $\beta \in (\alpha, \alpha + \epsilon)$, then it is clear that we can cover the interval $[a, \beta]$ by a finite subcollection of the open sets since we can always include O_μ. This contradicts the fact that α is the least upper bound of the set $\mathcal{S}$, so we must have $\alpha = b$. $\square$

Proposition 4.5.9 *If a finite or infinite sequence of interval I_n, $n = 1, 2, \ldots$, covers an interval I with end points $a, b \in \mathbb{R}$, $a < b$, then $\sum_n |I_n| \geq |I|$.*

Proof We give the proof for the case where the intervals I_n are open, and $I = [a, b]$ is a closed interval. A slight modification will give the general result.

Denote by (a_1, b_1) the first interval that contains the point a. If $b_1 < b$, let (a_2, b_2) be the first interval in the sequence I_n that contains the point b_1. Continue in this way so that if $b_{n-1} < b$, (a_n, b_n) is the first interval that contains the point b_{n-1}.

This procedure must terminate with some $b_N > b$, for if not we would have an increasing sequence bounded above by b, so must converge to a limit $\alpha \le b$, where α belongs to I_k for some k. All but a finite number of the intervals (a_n, b_n) would have to precede I_k in the given sequence, namely, all those for which $b_{n-1} \in I_k$. This is impossible since no two of these intervals are equal.

It follows that

$$b - a < b_N - a_1 = \sum_{i=2}^{N}(b_i - b_{i-1}) + b_1 - a_1 \le \sum_{i=1}^{N}(b_i - a_i),$$

and the result follows. $\square$

We deduce that the interval $[a, b]$, $a < b$, is not a set of measure zero. This also gives an alternative proof of the fact that an interval is an uncountable set:

Corollary 4.5.10 *An interval I with end-points $a, b \in \mathbb{R}$, $(a < b)$, is not of measure zero, and hence cannot be a countable set.*

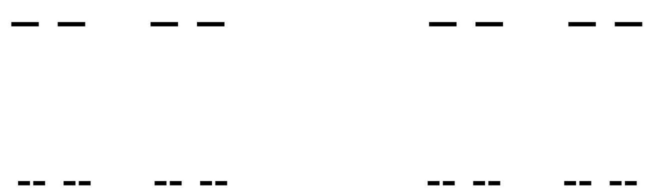
Example 4.5.11 The Cantor Set.

The Cantor set C is defined as follows: Set $S_0 = [0, 1]$, the unit interval. We remove the open interval $(1/3, 2/3)$ from S_0 to give $S_1 = [0, 1/3] \cup [2/3, 1]$. We continue removing *open middle thirds* to give

$$S_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1],$$

and continue in this way removing open middle thirds so that $S_n = [0, 1/3^n] \cup \ldots$. Denoting the total length of the of the subintervals making up S_n by $|S_n|$ (so $|S_0| = 1$), we have:

S_1 consists of 2 intervals of total length $|S_1| = 2/3$,
S_2 consists of 2^2 intervals of total length $|S_2| = 2^2/3^2$,
S_3 consists of 2^3 intervals of total length $|S_3| = 2^3/3^3$,
and generally
S_n consists of 2^n intervals of total length $|S_n| = 2^n/3^n$.



The first three steps in the construction of the Cantor Set.

Definition 4.5.12 The Cantor set C is defined by

$$C = \cap_{n=1}^{\infty} S_n.$$

Our aim now is to show that the Cantor set is a large set in the sense that it is not countable, but small in the sense that it is a set of measure zero. We also investigate the properties of C, and show a connection between C and the ternary expansion of certain numbers in $[0,1]$. The Cantor set is an example of a *fractal* - it has the property of *self-similarity*. For example $C \cap [0,1/3]$ looks exactly like the Cantor set, but on a smaller scale. $C \cap [0,1/9]$ is a replica of C but on a smaller scale still. We can continue like this indefinitely to see C on smaller and smaller scales.

Proposition 4.5.13 *The Cantor set C is a closed non-empty subset of* $[0,1]$*, having measure zero.*

Proof. C is non-empty because it contains all of the end points of each of the intervals constituting the set S_n, $n \in \mathbb{Z}^+$ (for example, $1/3 \in S_n$ for every $n \in \mathbb{Z}^+$). $C \subset S_n$ for $n = 1, 2, \ldots$ where $|S_n| = (2/3)^n \to 0$ as $n \to \infty$, so that C may be covered by a collection of intervals whose total length can be made arbitrarily small. C is a closed set because each of the sets S_n, $n \in \mathbb{Z}^+$ is closed, and C is the intersection of closed sets. $\square$

Exercises 4.5

1. *Cantor's Ternary Function* is defined as follows: Let $x \in [0,1]$ have ternary expansion $x = \cdot x_1 x_2 x_3 \ldots$ ($x_i \in \{0,1,2\}$, $i = 1, 2, 3, \ldots$). Set $N = \infty$ if $x_n \neq 1$ for

all $n \in \mathbb{Z}^+$, otherwise set $N = \min\{n \in \mathbb{Z}^+ : x_n = 1\}$. Let $y_n = x_n/2$ for $n < N$ and $y_N = 1$.

(a) Show that $\sum_{n=1}^{N} y_n/2^n$ is independent of the ternary expansion of x, if x has two such expansions.

(b) Show that the function $\kappa : [0,1] \to [0,1]$ defined by

$$\kappa(x) = \sum_{n=1}^{N} \frac{y_n}{2^n}$$

is a continuous, onto, and monotone increasing ($\kappa(x) \leq \kappa(y)$ for all $x < y$ in $[0,1]$).

(c) Show that κ is constant on each interval contained in the complement of the Cantor set (for example, $\kappa(x) = 1/2$ for all $x \in (1/3, 2/3)$).

2. Prove the following about sets of *measure zero*:

(a) A subset of a set of measure zero has measure zero.

(b) Any countable set has measure zero.

(c) The countable union of sets of measure zero has measure zero.

3. Prove that the set $\Sigma = \{(s_1, s_2, s_3, \ldots) : s_i = 0 \text{ or } s_i = 1\}$ of all sequences of 0's and 1's is uncountable.

4. Determine whether the set of numbers in $(0,1)$ whose decimal expansion contains no 1's or 5's is:

(i) closed, open or neither,

(ii) totally disconnected.

5. Lemma 4.5.8 shows that if we cover an interval $[a,b]$ with an arbitrary collection of open sets, there is a finite subcollection of these sets that also cover $[a,b]$. A set $A \subseteq \mathbb{R}$ having this property (every open cover has a finite subcover) is said to be

compact, and therefore the interval $[a, b]$ is a compact set. Clearly every finite subset of $\mathbb{R}$ is compact.

(a) Show that the sets (i) $A = [0, \infty)$, (ii) $B = \{1, 1/2, 1/3, \ldots\}$ are not compact by exhibiting an open cover that does not have a finite subcover. On the other hand, show that $B \cup \{0\}$ is compact.

(b) Show that any closed bounded subset K of $\mathbb{R}$ is compact (Hint: Use the fact that K is a subset of a closed interval of the form $[a, b]$, and extend an open cover of K to one for $[a, b]$ and then use the compactness of $[a, b]$).

6. Show that a set of measure zero cannot contain an open interval (we say that a subset $A \subset X$ of a metric space X has *empty interior* if it does not contain any open balls).

4.6 Ternary Expansions and the Cantor Set.

Each $x \in [0, 1]$ has a *ternary expansion*, i.e., can be written as

$$x = \cdot\, a_1\, a_2\, a_3\, a_4\, \ldots = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \frac{a_4}{3^4} + \ldots, \quad \text{where} \quad a_i \in \{0, 1, 2\}.$$

Note that if $a_1 = 0$ then $x \in [0, 1/3]$ since $x = \cdot 0\, a_2\, a_3 \ldots \leq \cdot 0\,2\,2\,2 \ldots = 1/3$. Similarly, if $a_1 = 1$, then $x \in [1/3, 2/3]$, and if $a_1 = 2$, then $x \in [2/3, 1]$. Conversely, if $x \in [0, 1/3]$, the x has a ternary expansion with $a_1 = 0$, and similarly for the other two intervals.

In addition, just as in the binary expansion situation, every $x \in (0, 1)$ has a unique *non-terminating* ternary expansion. For example

$$\frac{1}{3} = \cdot 1000 \ldots = \cdot\, 0222 \ldots, \quad \text{and} \quad \frac{2}{3} = \cdot\, 2000 \ldots = \cdot 1222 \ldots.$$

We shall show that $x \in C$ if and only if x has a ternary expansion (possibly terminating) consisting only of 0's and 2's. This means, for example, that $1/3 \in C$, $1 = \cdot 2222 \ldots \in C$, $2/3 \in C$ and $3/4 = \cdot 20202 \ldots \in C$.

Theorem 4.6.1 *$x \in C$ if and only if x has a ternary expansion*

$$x = \cdot\, a_1\, a_2\, a_3\, \ldots, \quad \textit{where} \quad a_i = 0 \ \textit{ or } \ 2.$$

Proof. First suppose that $x \in C$ with a ternary expansion

$$x = \cdot\, a_1\, a_2\, a_3\, \ldots,$$

then $x \in S_n$ for each $n \in \mathbb{Z}^+$. In particular $x \in S_1$, so either $x \in [0, 1/3]$ and we may take $a_1 = 0$, or $x \in [2/3, 1]$ and $a_1 = 2$. Also $x \in S_2$, so either $x \in [0, 1/9]$ and $a_2 = 0$, or $x \in [2/9, 1/3]$ and $a_2 = 2$, or $x \in [2/3, 7/9]$ and $a_2 = 0$, or $x \in [8/9, 1]$ and $a_2 = 2$. Continuing in this way we see that we may take $a_n = 0$ or $a_n = 2$ for each $n \in \mathbb{Z}^+$.

Conversely, suppose that we can choose the expansion so that $a_n = 0$ or $a_n = 2$ for each $n \in \mathbb{Z}^+$, then since this holds for $n = 1$ we must have $x \in [0, 1/3]$ or $x \in [2/3, 1]$ so that $x \in S_1$. Similarly $a_2 = 0$ or 2 implies that $x \in S_2$ and continuing in this way, $x \in S_n$ for $n = 1, 2, \ldots$, we deduce that $x \in \cap_{n=1}^{\infty} S_n = C$. □

We can now prove that C is not a countable set (in particular, it can be put into a one-to-one correspondence with $[0, 1]$).

Theorem 4.6.2 *The Cantor set C is uncountable.*

Proof. We define a one-to-one function $f : [0, 1] \to C$ as follows:
Let $x \in [0, 1]$ have a binary representation (non-terminating)

$$x = \cdot\, a_1\, a_2\, a_3\, a_4\, \ldots \quad \text{where} \quad a_i \in \{0, 1\}.$$

Define $f(x)$ by

$$f(x) = \cdot\, b_1\, b_2\, b_3\, b_4\, \ldots \quad \text{where} \quad b_i = \begin{cases} 0 & \text{if} \quad a_i = 0, \\ 2 & \text{if} \quad a_i = 1. \end{cases}$$

Then $f(x)$ is one-to-one (it is not be onto because certain numbers such as $2/3 = \cdot 1222\ldots \in C$ are not in the range of f). It follows that there is a subset of C that is uncountable, so C itself is uncountable. □

Finally we show that the Cantor set is *totally disconnected* and *perfect.* In this context, to say that C is totally disconnected means that it contains no intervals, and the fact that it is perfect means that every point of C is a limit point of C. More formally:

Definition 4.6.3 A subset $A \subset \mathbb{R}$ is said to be *totally disconnected* (or has *empty interior*), if A contains no non-empty open intervals. For example, discrete sets of points are totally disconnected, $\mathbb{Q}$ the set of rationals in $\mathbb{R}$ is totally disconnected.

Definition 4.6.4 A subset $A \subset \mathbb{R}$ is said to be *perfect* if every point of A is a limit point of A.

Theorem 4.6.5 *The Cantor set C is totally disconnected and perfect.*

Proof. If U is a non-empty open interval contained in C, then U is contained in S_n for each $n \in \mathbb{Z}^+$. But $|S_n| = (2/3)^n \to 0$ as $n \to \infty$, so this is impossible as every non-empty open interval has positive length.

To see that C is perfect, let $x \in C$ have ternary expansion

$$x = \cdot\, a_1\, a_2\, \ldots\, a_n \ldots \quad \text{where} \quad a_i \in \{0, 2\}.$$

Set

$$x_n = \cdot\, a_1\, a_2\, \ldots\, a_n * * * \ldots,$$

chosen so that $x_n \in C$ and agrees with x in the first n-places and $x_n \neq x$ and $x_n \neq x_m$ for all $n, m \in \mathbb{Z}^+$, $n \neq m$. Then

$$|x - x_n| \leq \sum_{k=n+1}^{\infty} \frac{2}{3^k} = \frac{1}{3^n} \to 0 \text{ as } n \to \infty.$$

It follows that x is a limit point of C. $\square$

There are other "Cantor Sets" besides C (see Exercises 4.6). Any subset of $\mathbb{R}$ that is closed, bounded, perfect and totally disconnected is said to be a *Cantor Set.* We shall see that these arise quite naturally in dynamical systems theory.

Exercises 4.6

1. (a) Let $\Sigma = \{(a_1, a_2, a_3, \ldots) : a_i = 0 \quad \text{or} \quad 1\}$, the sequence space of zeros and ones with the metric defined in Section 4.1. Let C be the Cantor set and define a map $f : \Sigma \to C$ by

$$f(a_1, a_2, a_3, \ldots) = \cdot b_1\, b_2\, b_3 \ldots, \quad \text{where} \quad b_i = 0 \text{ if } a_i = 0 \text{ and } b_i = 2 \text{ if } a_i = 1,$$

giving the ternary expansion of a real number in $[0, 1]$. Show that this defines a homeomorphism between Σ and the Cantor set.

(b) It can be shown that $[0, 1]$ and C are not homeomorphic (for example, C is totally disconnected whilst $[0, 1]$ is not). What goes wrong if we try to define a homeomorphism between $[0, 1]$ and C by mapping the binary expansion of $x \in [0, 1]$ to the ternary expansion?

2. We have seen that the Cantor set is an uncountable set having measure zero. Here we generalize the notion of Cantor set to give an example of a set having empty interior which does not have measure zero:

Fix $\alpha \in (0,1)$ and for each $n \in \mathbb{Z}^+$, let $a_n = 2^{n-1}\alpha/3^n$. Set $S_0(\alpha) = [0,1]$, and from its center, remove an open interval of length a_1 and denote by $S_1(\alpha)$ the resulting set, a union of two disjoint closed intervals. From the center of each of these two intervals, remove an open interval of length $a_2/2$ to obtain a set $S_2(\alpha)$ which is the union of 2^2 disjoint closed intervals of equal length. Continue in this way as in the construction of the Cantor set, so that at the nth stage we have a set $S_n(\alpha)$ obtained by removing 2^{n-1} disjoint closed intervals in $S_{n-1}(\alpha)$ an open interval of length $a_n/2^{n-1}$. Set

$$C(\alpha) = \cap_{n=1}^{\infty} S_n(\alpha).$$

$C(\alpha)$ is a *generalized Cantor set*. It is a subset of $[0,1]$ which can be shown to be uncountable and perfect. When $\alpha = 1$ we obtain the usual Cantor set. Show:

(a) $C(\alpha)$ is a closed non-empty set.

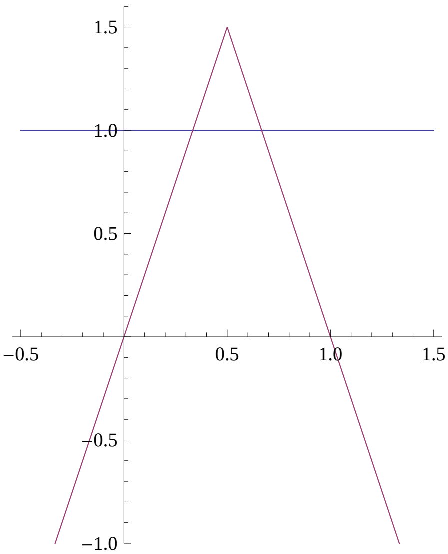
(b) $C(\alpha)$ has empty interior. (Hint: Suppose that $C(\alpha)$ contains an open interval I and show this is impossible since the lengths of the intervals in $C_n(\alpha)$ can be made arbitrarily small by making n large enough).

(c) $C(\alpha)$ is not a set of measure zero. (Hint: Find the total lengths of the intervals removed in the construction of $C(\alpha)$).

4.7 The Tent Map for $\mu = 3$.

Now consider the tent family T_μ for $\mu = 3$, where we think of T_3 as a map defined on all of $\mathbb{R}$ by

$$T_3(x) = \begin{cases} 3x & \text{if} \quad x \leq 1/2, \\ 3(1-x) & \text{if} \quad x > 1/2. \end{cases}$$



The tent map T_μ with $\mu = 3$.

Note that $\{3/13, 9/13, 12/13\}$ and $\{3/28, 9/28, 27/28\}$ are both 3-cycles for T_3 and that if $x > 1$, then $T_3^n(x) \to -\infty$ as $n \to \infty$. We shall show that the orbit of $x \in [0,1]$ is bounded if and only if $x \in C$, the Cantor set. In particular, if $x \notin C$, then $T_3^n(x) \to -\infty$ as $n \to \infty$. We also see that C is an invariant set under T_3 ($T_3(C) = C$), so we can consider T_3 as a map $T_3 : C \to C$, and this is where the interesting dynamics of T_3 takes place.

Proposition 4.7.1 *If* $\Lambda = \{x \in [0,1] : T_3^n(x) \in [0,1] \ \forall\ n \in \mathbb{Z}^+\}$, *then* $\Lambda = C$, *the Cantor set. In addition,* $T(C) \subseteq C$.

Proof.
Claim 1. If $x \in (1/3, 2/3)$ then $T_3^n(x) \to -\infty$ as $n \to \infty$.

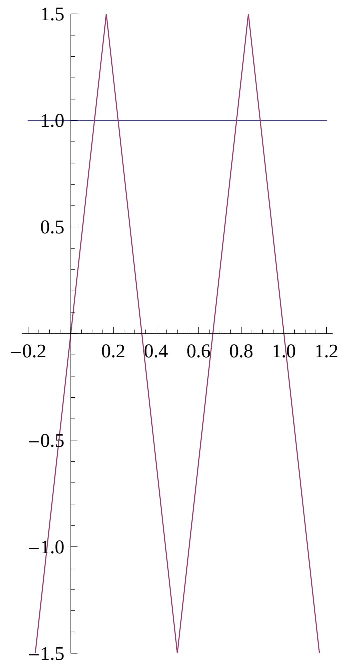
This is because T_3 is increasing on $(1/3, 1/2]$ and decreasing on $[1/2, 2/3)$ so that if $1/3 < x \le 1/2$, then $T_3(1/3) < T_3(x) \le T_3(1/2)$ so $1 < T_3(x) \le 3/2$.

Similarly if $1/2 \le x < 2/3$ then $1 < T_3(x) \le 3/2$, so $T_3^n(x) \to -\infty$ as $n \to \infty$.

Claim 2. If $x \in (1/9, 2/9) \cup (7/9, 8/9)$, then $T_3^n(x) \to -\infty$ as $n \to \infty$.

We use the fact that

$$T_3^2(x) = \begin{cases} 9x & \text{if} \quad x \le 1/6, \\ 3 - 9x & \text{if} \quad 1/6 \le x < 1/2, \\ 9x - 6 & \text{if} \quad 1/2 \le x < 5/6, \\ 9 - 9x & \text{if} \quad x \ge 5/6, \end{cases}$$



The tent map T_3^2.

T_3^2 is strictly increasing on $(1/9, 1/6]$ and strictly decreasing on $[1/6, 2/9)$, so that if $1/9 < x \leq 1/6$, then $T_3^2(1/9) < T_3^2(x) \leq T_3^2(1/6)$ and $1 < T_3^2(x) \leq 3/2$ and similarly if $1/6 < x \leq 2/9$ then $1 < T_3^2(x) \leq 3/2$. A similar argument also applies when $x \in (7/9, 8/9)$ to give $T_3^2(x) > 1$ in each case, so that $T_3^n(x) \to -\infty$ as $n \to \infty$.

Claim 3 If $x \notin C$, then $T_3^n(x) \to -\infty$ as $n \to \infty$.

We have seen that if $x \in (1/3, 2/3)$ then $T_3^n(x) \to -\infty$ as $n \to \infty$. Suppose instead $x \in [0, 1/3] \cup [2/3, 1]$ and has the ternary expansion

$$x = \cdot\, a_1\, a_2\, a_3\, a_4\, \ldots \quad \text{where} \quad a_i \in \{0, 1, 2\},$$

then

$$T_3(x) = \begin{cases} \cdot\, a_2\, a_3\, a_4\, \ldots & \text{if} \quad 0 \leq x \leq 1/3 \\ \cdot\, b_2\, b_3\, b_4\, \ldots & \text{if} \quad 2/3 \leq x \leq 1, \end{cases}$$

where $b_i = \begin{cases} 0 & \text{if} \quad a_i = 2 \\ 1 & \text{if} \quad a_i = 1 \\ 2 & \text{if} \quad a_i = 0 \end{cases}$. Consequently, if x has a 1 in its ternary expansion, there exists $k \in \mathbb{Z}^+$ with

$$T_3^k(x) = \cdot\, 1\, a_{k+1}\, \ldots, \quad \text{and so} \quad T_3^k(x) \in [1/3, 2/3).$$

We can only have $T_3^k(x) = 1/3$ when the ternary expansion of x consists of a sequence of 0's and 2's (k terms), followed by a 1, and then followed by an infinite string of 0's (in this case $x \in C$ because the terms $1000\ldots$ can be written as $0222\ldots$). Thus $T_3^{k+1}(x) > 1$ and the orbit of x will go to $-\infty$. Consequently, we deduce that if $x \notin C$ then $T_3^n(x) \to -\infty$ as $n \to \infty$.

Claim 4. If $x \in C$, then $T_3(x) \in C$.

If each $a_i = 0$ or 2, then each b_i (in Claim 3) is 0 or 2, so the result follows. □

In conclusion, we see that we may consider T_3 as a map $T_3 : C \to C$. The Cantor set C is the set on which the interesting dynamics takes place. We can see that T_3 is an onto map, but it is clearly not one-to-one. A similar analysis can be made for T_μ when $\mu > 2$ to deduce that there is some set C_μ, a type of Cantor set, on which the interesting dynamics takes place. Similar (but more complicated) reasoning also shows that the logistic map $L_\mu = \mu x(1 - x)$, $\mu > 4$, $x \in \mathbb{R}$ has the same type of behavior.

4.8 Another Cantor Set

We look at a Cantor set arising from the logistic map $L_\mu(x) = \mu x(1-x)$ for $\mu > 4$. We think of L_μ as a function defined on all of $\mathbb{R}$, but are mainly interested in its restriction to $[0, 1]$. Now $L_\mu(1/2) = \mu/4 > 1$, and $x = 1/2$ is a critical point for L_μ. From the graph of L_μ we see that there are two point a_0 and $a_1 = 1 - a_0$ where $L_\mu(a_0) = 1 = L_\mu(a_1)$, and in addition, $L_\mu(x) > 1$ for all $x \in (a_0, a_1)$. Just as for the tent map T_3, we see that if $x \in (a_0, a_1)$, then $L_\mu^n(x) \to -\infty$ as $n \to \infty$. Set $I_0 = [0, a_0]$ and $I_2 = [a_1, 1]$ and $\Lambda_1 = I_0 \cup I_1$, a disjoint union of two closed interval, so is a closed set:

$$\Lambda_1 = \{x \in [0, 1] : L_\mu(x) \in [0, 1]\}.$$

We continue in this way as we did with the tent map T_3 to find a set Λ_2, a disjoint union of 4 closed intervals:

$$\Lambda_2 = \{x \in [0, 1] : L_\mu^2(x) \in [0, 1]\}.$$

And at the nth stage

$$\Lambda_n = \{x \in [0, 1] : L_\mu^n(x) \in [0, 1]\}$$

is the disjoint union of 2^n closed intervals. We set $\Lambda = \cap_{n=1}^{\infty} \Lambda_n$, the intersection of closed sets, so is a closed set contained in $[0, 1]$ (the fact that the set Λ_n is closed also follows from the fact that it is the inverse image of a closed set - see the following paragraph).

Recall that if $f : X \to X$ is a function on a metric space, the inverse image of a set $U \subseteq X$ is the set $f^{-1}(U) = \{x \in X : f(x) \in U\}$. We see that

$$\Lambda_1 = L_\mu^{-1}\left([0,1]\right), \Lambda_2 = L_\mu^{-2}\left([0,1]\right), \ldots, \Lambda_n = L_\mu^{-n}\left([0,1]\right),$$

$$\Lambda = \cap_{n=1}^{\infty} \Lambda_n = \cap_{n=1}^{\infty} L_\mu^{-n}[0,1],$$

so that $x \in \Lambda$ if and only if $L_\mu^n(x) \in [0,1]$ for all $n \in \mathbb{Z}^+$. Also, Λ is a non-empty set because it must contain the period points of L_μ. It can be shown to be a perfect and totally disconnected set of measure zero, so it is a Cantor set. As before we can consider L_μ as a map $L_\mu : \Lambda \to \Lambda$, as this is where the interesting dynamics takes place.

The above analysis can be made easier in the case where $\mu > 2 + \sqrt{5}$. In this situation, if we find the points a_0 and a_1 by setting $L_\mu(x) = 1$, we get the equation $\mu x^2 - \mu x + 1 = 0$. Solving gives

$$a_i = \frac{\mu \pm \sqrt{\mu^2 - 4\mu}}{2\mu}, \quad i = 0 \text{ or } 1.$$

For $x \in \Lambda_1$, $|L'_\mu(x)|$ is a minimum when $x = a_0$, or $x = a_1$, and in this case

$$L'_\mu(a_0) = \mu - 2\mu a_0 = \sqrt{\mu^2 - 4\mu}.$$

Thus for $\mu > 0$ and $x \in \Lambda_1$,

$$L'_\mu(x) \geq L'_\mu(a_0) = \sqrt{\mu^2 - 4\mu} > 1,$$

when $\mu^2 - 4\mu - 1 > 0$, for which we require $\mu > 2 + \sqrt{5}$. This shows that L_μ cannot have any attracting periodic points if $\mu > 2 + \sqrt{5}$. It can now be shown that the individual intervals making up each set Λ_n has length less that $1/r^n$, for some $r > 1$ and this is used to show that Λ is a Cantor set. This is also true for any $\mu > 4$, but the analysis is more difficult. (see [20] for more details)

Chapter 5. Devaney's Definition of Chaos

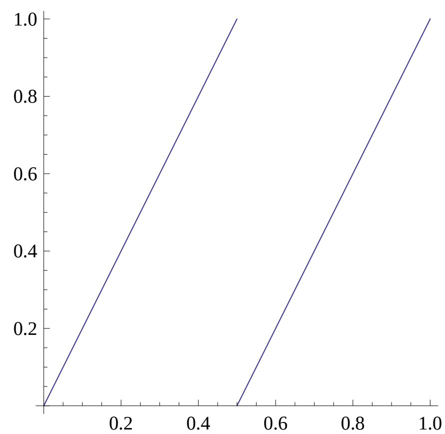
In this chapter we will give Devaney's definition of chaos for one-dimensional maps and also for more general maps defined on metric spaces. One-dimensional maps are functions $f : I \to \mathbb{R}$ for some interval $I \subseteq \mathbb{R}$. It turns out that maps whose periodic points form a dense set often have highly chaotic properties.

5.1 The Doubling Map and the Angle Doubling Map

Example 5.1.1 The Doubling Map.

The doubling map $D : [0, 1] \to [0, 1]$ is defined by

$$D(x) = 2x(\text{mod } 1) = \begin{cases} 2x\ ; & 0 \leq x < 1/2, \\ 2x - 1\ ; & 1/2 \leq x \leq 1. \end{cases}$$



The doubling map D.

It is useful to describe D in terms of the binary expansion of a real number in [0,1]. Let $x \in [0, 1]$ with binary expansion

$$x = \cdot a_1 a_2 a_3 \dots \quad \text{where} \quad a_i = 0 \text{ or } 1.$$

In other words,

$$x = \frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \cdots = \sum_{i=1}^{\infty} \frac{a_i}{2^i}.$$

Suppose that $a_1 = 0$ in the binary expansion of x, then

$$D(x) = 2x = \frac{a_2}{2} + \frac{a_3}{2^2} + \cdots = \cdot a_2 a_3 \dots .$$

On the other hand, if $a_1 = 1$, then

$$D(x) = 2x - 1 = (a_1 + \frac{a_2}{2} + \frac{a_3}{2^2} + \cdots) - 1 = \frac{a_2}{2} + \frac{a_3}{2^2} + \cdots = \cdot a_2 a_3 \ldots .$$

We see that in general

$$D(\cdot a_1 a_2 a_3 \ldots) = \cdot a_2 a_3 \ldots \quad \text{and} \quad D^n(\cdot a_1 a_2 a_3 \ldots) = \cdot a_{n+1} a_{n+2} a_{n+3} \ldots .$$

Consequently, if $x = \cdot a_1 a_2 \ldots a_n a_1 a_2 \ldots a_n a_1 \ldots$ has an expansion which repeats every n places, then $D^n(x) = x$, so that x is periodic of period n.

For example $D^2(\cdot 010101\ldots) = \cdot 010101\ldots$, so is a point of period 2. We use this to show that the set of periodic points of D are dense in $[0, 1]$. This can also be used to count the number of periodic points of period n. Notice that since $D'(x) > 1$ everywhere it is defined, all of the periodic orbits of D are unstable.

Proposition 5.1.2 *The periodic points of the doubling map are dense in* $[0, 1]$.

Proof. Let $\epsilon > 0$ and choose N so large that $1/2^N < \epsilon$. If $x \in [0, 1]$ it suffices to show that there is a periodic point y for D that is within ϵ of x. Suppose that the binary expansion of x is

$$x = \cdot a_1 a_2 a_3 \ldots = \sum_{i=1}^{\infty} \frac{a_i}{2^i},$$

then we set

$$y = \cdot a_1 a_2 \ldots a_N a_1 a_2 \ldots a_N a_1 \ldots ,$$

a point of period N. Then

$$|x - y| = |\sum_{j=N+1}^{\infty} \frac{b_j}{2^j}| \leq \sum_{j=N+1}^{\infty} \frac{1}{2^j} = \frac{1}{2^N} < \epsilon,$$

(where $b_i = 0, 1$ or -1).

Example 5.1.2 The Angle Doubling Map.

As before, we denote by $\mathbb{C} = \{z = a + ib : a, b \in \mathbb{R}\}$, the set of all complex numbers. If $z = a + ib \in \mathbb{C}$, then its *absolute value* (or *modulus*) is given by $|z| = \sqrt{a^2 + b^2}$. The conjugate of z is $\bar{z} = a - ib$ and we can check that $z\bar{z} = |z|^2$. We can represent $\mathbb{C}$ using points in the (*complex*) *plane* $\{(a, b) : a, b \in \mathbb{R}\}$. The *unit circle* S^1 in the complex plane is the set

$$S^1 = \{z \in \mathbb{C} : |z| = 1\}.$$

Points in S^1 may be represented as:

$$z = e^{i\theta} = \cos\theta + i \sin\theta, \quad \text{for some} \quad \theta \in \mathbb{R}.$$

Here θ is the *argument* of z (written Arg(z)), and it is the angle subtended by the ray from $(0,0)$ to (a,b) (when $z = a + ib$, $a, b \in \mathbb{R}$), and the *real axis.*

S^1 is a metric space if the distance between two points $z, w \in S^1$ is defined to be the shortest distance between the two points, going around the circle. We define a map $f : S^1 \to S^1$ by $f(z) = z^2$. This map is called the *angle doubling map* because of the effect it has on $\theta =$Arg(z): $f(e^{i\theta}) = e^{2i\theta}$. We see that the angle θ is doubled. It is clear that there are a lot of similarities between the doubling map and the angle doubling map, and we shall examine this in the next section when we study the notion of conjugacy for dynamical systems. First we show that the periodic points of f are dense in S^1.

Consider the periodic points of $f(z) = z^2$: Solving $z^2 = z$ gives $z = 1$ (we can disregard $z = 0$), $f^2(z) = z$ gives $z^4 = z$ or $z^3 = 1$ and continuing in this way we see that the periodic points are certain nth roots of unity.

Proposition 5.1.3 *The periodic points of the angle doubling map $f : S^1 \to S^1$ are dense in S^1.*

Proof. If $f^n(z) = z$ for some $n \in \mathbb{Z}^+$, then $z^{2^n} = z$ or $z^{2^n-1} = 1$. Write $z = e^{i\theta}$, then we want to find the $(2^n - 1)$th roots of unity. This gives:

$$e^{(2^n-1)i\theta} = e^{2k\pi i}, \quad \text{for some} \quad k \in \mathbb{Z}^+,$$

giving the $2^n - 1$ distinct roots: $z_k = e^{2k\pi i/(2^n-1)}$, $k = 0, 1, 2, \ldots, 2^n - 2$, showing that

$$\text{Per}_n(f) = \{e^{2k\pi i/(2^n-1)} : 0 \le k < 2^n - 1\}, \;\; n \in \mathbb{Z}^+.$$

These points are equally spaced around the circle, a distance $2\pi/(2^n - 1)$ apart, which can be made arbitrarily small by taking n large enough. It follows that the periodic points are dense in S^1. $\square$

5.2 Transitivity

Sometimes, given $f : X \to X$ (X a metric space), when we iterate $x_0 \in X$, the orbit $O(x_0) = \{x_0, f(x_0), \ldots\}$, spreads itself evenly over X, so that $O(x_0)$ is a dense set in X. This leads to:

Definition 5.2.1 $f : X \to X$ is said to be *(topologically) transitive* if there exists $x_0 \in X$ such that $O(x_0)$ is a dense subset of X. A *transitive point* for f is a point x_0 which has a dense orbit under f. If f is transitive, then there is a dense set of transitive points, since each point in $O(x_0)$ will be a transitive point.

Example 5.2.2 The doubling map $D : [0,1] \to [0,1]$ is transitive: To show this we explicitly construct a point $x_0 \in [0,1]$ which has a dense orbit under D. x_0 is defined using its binary expansion in the following way: first write down all possible "1-blocks", i.e., 0 followed by 1. Then write down all possible "2-blocks", i.e., 00, 01, 10, 11, then all possible '3-blocks", i.e., 000, 001, 010, 011, 100, 101, 110, 111, and then continue in this way with all possible "4-blocks" etc. (we could write them down in the order in which they appear as in the binary expansion of the integers). This gives:

$$x_0 = \cdot 01\ 00\,01\,10\,11\ 000\,001\,010\,011\,100\,101\,110\,111\ 0000\,0001\ldots,$$

a point of $[0,1]$.

To show that $O(x_0)$ is dense in $[0,1]$, let $y \in [0,1]$ with binary expansion

$$y = y_1y_2y_3\ldots = \sum_{i=1}^{\infty}\frac{y_i}{2^i}, \quad y_i = 0 \quad \text{or} \quad 1,$$

and let $\delta > 0$.

Choose N so large that $\frac{1}{2^N} < \delta$. All possible finite strings of 0's and 1's appear in the binary expansion of x_0, so the string $y_1y_2y_3\ldots y_N$ must also appear in the binary expansion of x_0.

It follows that for some $r \in \mathbb{Z}^+$ we have

$$D^r(x_0) = \cdot y_1y_2y_3\ldots y_Nb_{N+1}b_{N+2}\ldots, \quad \text{for some} \quad b_{N+1}, b_{N+2}, \ldots,$$

so that

$$\begin{aligned}|D^r(x_0) - y| &= |\cdot y_1y_2\ldots y_Nb_{N+1}b_{N+2}\ldots - \cdot y_1y_2\ldots y_Ny_{N+1}y_{N+2}\ldots| \\ &\leq \sum_{i=N+1}^{\infty}\frac{1}{2^i} = \frac{1}{2^N} < \delta.\end{aligned}$$

This shows that any point $y \in [0,1]$ is arbitrarily close to the orbit of x_0 under D, so this orbit is dense in $[0,1]$ □

Remark 5.2.3 It is quite easy to see that for a transitive map $f : X \to X$ on a metric space, given any non-empty open sets U and V in X there exists $m \in \mathbb{Z}^+$ such that

$$U \cap f^m(V) \neq \emptyset.$$

Less easy to see is the converse of this statement, which holds for complete separable metric spaces (X is *separable* if it has a countable dense subset - see Chapter 7 for the notion of completeness). This is the *Birkhoff Transitivity Theorem:*

Theorem 5.2.4 *A continuous map f of a complete separable metric space X is transitive if and only if for every pair U and V of non-empty open subsets of X there exists $m \in \mathbb{Z}^+$ with $U \cap f^m(V) \neq \emptyset$.*

Using this result it is easy to show directly that the angle doubling map is transitive, however, the proof is beyond the scope of this text.

5.3 The Definition of Chaos

Devaney was the first to define the notion of chaos, saying that a function is chaotic if it has a dense set of periodic points, it is transitive and also has what is called *sensitive dependence on initial conditions* (known as the "butterfly effect" in the popular literature). Subsequently it was shown that the first two requirements imply the third, so we define chaos as follows:

Definition 5.3.1 Let $f : X \to X$ be a map of the metric space X, then f is said to be *chaotic* if:

(i) The set of periodic points of f is dense in X.

(ii) f is transitive.

Examples 5.3.2 1. Homeomorphisms or diffeomorphisms of an interval $I \subseteq \mathbb{R}$ cannot be chaotic as they are never transitive. The same type of considerations apply to functions such as $\sin x$, $\cos x$, $\arctan x$ and the logistic map L_μ for $0 < \mu < 3$.

2. We have shown that the doubling map $D : [0, 1] \to [0, 1]$ has a dense set of periodic points and is transitive, so it is a chaotic map.

3. The tent map $T : [0, 1] \to [0, 1]$ is chaotic. Recall that if $x \in [0, 1]$ has a binary expansion $x = \cdot\, a_1\, a_2\, a_3 \dots$, then $Tx = \begin{cases} \cdot\, a_2\, a_3\, a_4 \dots & \text{if} \quad a_1 = 0 \\ \cdot\, a_2'\, a_3'\, a_4' \dots & \text{if} \quad a_1 = 1, \end{cases}$ where $a_i' = 1$ if $a_i = 0$ and $a_i' = 0$ if $a_i = 1$. More generally we can see using an induction argument that ([61]):

$$T^n x = \begin{cases} \cdot\, a_{n+1}\, a_{n+2}\, a_{n+3} \dots & \text{if} \quad a_n = 0 \\ \cdot\, a_{n+1}'\, a_{n+2}'\, a_{n+3} \dots & \text{if} \quad a_n = 1. \end{cases}.$$

We can use this to write down the periodic points of T. For example, the fixed points are $x = 0$ and $x = \cdot\, 1\, 0\, 1\, 0 \dots = 2/3$, and the period 2-points are

$$x_1 = \cdot\, 01\, 10\, 01\, 10 \dots = 2/5 \quad \text{and} \quad x_2 = \cdot\, 11\, 00\, 11\, 00 \dots = 4/5.$$

The period 3-points are

$$\cdot\,010\,010\,010\,010\ldots = 2/7,\quad \cdot\,100\,100\,100\,100\ldots = 4/7,\quad \cdot\,110\,110\,110\,110\ldots = 6/7$$

and

$$\cdot\,001\,110\,001\,110\ldots = 2/9,\quad \cdot\,011\,100\,011\,100\ldots = 4/9,\quad \cdot\,111\,000\,111\,000\ldots = 8/9.$$

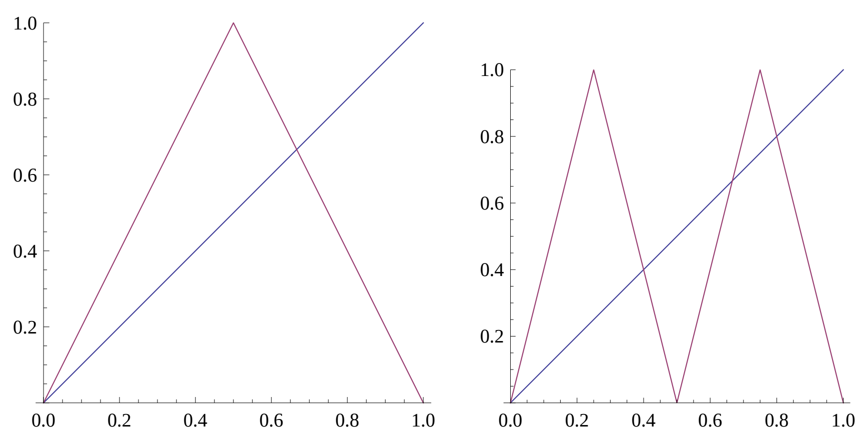
Notice that points of the form $x = k/2^n \in (0,1)$, $k \in \mathbb{Z}^+$, are almost fixed since they have a binary expansion of the form

$$x = \cdot\, a_1\, a_2\, a_3 \ldots, a_n\, 0\, 0\ \ldots,$$

where $a_n = 0$ if k is even and $a_n = 1$ if k is odd. It follows that $T^n x = 0$ if k is even and $T^n x = 1$ if k is odd. In particular

$$T^n \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right] = [0,1].$$

The Intermediate Value Theorem now implies that there is a fixed point of T^n in the interval $[\frac{k-1}{2^n}, \frac{k}{2^n}]$. Since such intervals can be made arbitrarily small and cover all of $[0,1]$, the set of periodic points must be dense in $[0,1]$.



The fixed and period 2-points of the tent map.

We use these ideas to show that the periodic points of T are numbers in $[0,1]$ of the form $x = r/s$ where r is an even integer and s is an odd integer.

Proof. If $x \in (0,1)$ is a periodic point for T, then $T^n x = x$ for some $n \in \mathbb{Z}^+$. There are two cases to consider: Suppose that x has a binary expansion

$$x = \cdot\, a_1\, a_2\, a_3 \ldots a_n\, a_{n+1}\ \ldots, \quad \text{where} \quad a_n = 0,$$

then

$$T^n x = \cdot\, a_{n+1}\, a_{n+2} \ldots a_{2n} \ldots,$$

so we must have $a_1 = a_{n+1}, a_2 = a_{n+2}, \ldots a_n = a_{2n} = 0$, and

$$x = \cdot\, a_1\, a_2 \ldots a_n\, a_1\, a_2 \ldots a_n\, a_1 \ldots, \quad \text{where} \quad a_n = 0,$$

Rewriting this gives

$$\begin{aligned} x &= \left(\frac{a_1}{2} + \frac{a_2}{2^2} + \ldots + \frac{a_{n-1}}{2^{n-1}}\right) + \frac{1}{2^n}\left(\frac{a_1}{2} + \frac{a_2}{2^2} + \ldots + \frac{a_{n-1}}{2^{n-1}}\right) + \ldots \\ &= \left(\frac{a_1}{2} + \frac{a_2}{2^2} + \ldots + \frac{a_{n-1}}{2^{n-1}}\right)\left(\frac{1}{1 - 1/2^n}\right) \\ &= \frac{a_1 2^{n-1} + a_2 2^{n-2} + \ldots + a_{n-1} 2}{2^n - 1} = \frac{r}{s}, \end{aligned}$$

where r is even and s is odd.

In the second case $x = \cdot\, a_1\, a_2\, a_3 \ldots, a_n\, a_{n+1}\ \ldots$, where $a_n = 1$, so that

$$T^n x = \cdot\, a'_{n+1}\, a'_{n+2} \ldots a'_{2n} \ldots,$$

and this gives $a_1 = a'_{n+1},\ a_2 = a'_{n+2}, \ldots, a_n = a'_{2n}, \ldots$, and

$$x = \cdot\, a_1\, a_2 \ldots a_{n-1}\, 1\, a'_1\, a'_2 \ldots a'_{n-1}\, 0\, a_1 \ldots.$$

We can now argue as before, but using the first $2n$ terms of x.

Conversely, suppose that $x = r/s \in (0, 1)$ where r is an even integer and s is an odd integer. Since s and 2 are coprime, we can apply Euler's generalization of Fermat's Theorem to give

$$2^{\phi(s)} \equiv 1 (\text{mod } s), \quad \text{or} \quad 2^p - 1 = ks \quad \text{for some} \quad p, k \in \mathbb{Z}^+,$$

$(1 < kr < 2^p - 1)$, where ϕ is Euler's function. Write the binary expansion of kr as

$$kr = a_1 2^{p-1} + a_2 2^{p-2} + \ldots a_{p-2} 2^2 + a_{p-1} 2, \quad a_i \in \{0, 1\},$$

which is even, so

$$\begin{aligned} \frac{kr}{2^p - 1} &= (a_1 2^{p-1} + a_2 2^{p-2} + \ldots a_{p-2} 2^2 + a_{p-1} 2)\left(\frac{1/2^p}{1 - 1/2^p}\right) \\ &= \left(\frac{a_1}{2} + \frac{a_2}{2^2} + \ldots + \frac{a_{p-1}}{2^{p-1}}\right)\left(\frac{1}{1 - 1/2^p}\right) \\ &= \cdot\, a_1\, a_2\ \ldots a_{p-1}\, 0\, a_1\, a_2 \ldots a_{p-1}\, 0 \ldots, \end{aligned}$$

which is a point of period p (or less). $\square$

It is now clear that T is transitive since if we define $x_0 \in (0,1)$ having a binary expansion consisting of all 1-blocks, all 2-blocks, all 3-blocks etc., as before, except that we insert a single zero between every block then it follows that if $x = \cdot x_1 x_2 \ldots x_n x_{n+1} \ldots$, then $T^p x_0 = \cdot x_1 x_2 \ldots x_n b_{n+1} \ldots$ for some $p > 0$, i.e., every block will appear in the iterates of x_0. As before we see that T is transitive, so is chaotic.

4. It is possible for a map to be transitive without being chaotic (although for continuous functions f on intervals in $\mathbb{R}$, this is not possible: see [64]). For example, consider the *irrational rotation* $R_a : S^1 \to S^1$ defined by $R_a(z) = a \cdot z$ for some (fixed) $a \in S^1$. To say that R_a is an irrational rotation means that $a^n \neq 1$ for any $n \in \mathbb{Z}^+$, i.e., a is not an nth root of unity for any $n \in \mathbb{Z}^+$. It can be shown that *every* $z_0 \in S^1$ has a dense orbit (a transformation with this property is said to be *minimal*). However, suppose that $R_a^n(z) = z$, then $a^n z = z$ or $a^n = 1$, a contradiction, so that R_a has no periodic points. R_a is an example of an *isometry*: points always stay the same distance apart:

$$|R_a(z) - R_a(w)| = |az - aw| = |a||z - w| = |z - w|.$$

Note that if instead we have $a^n = 1$ for some $n \in \mathbb{Z}^+$, then $R_a^n(z) = a^n z = z$ for all $z \in S^1$, so that R_a^n is just the *identity map* (every point of S^1 is of period n).

5.4 Some Symbolic Dynamics and the shift map

Recall that the set of all infinite sequences of 0's and 1's:

$$\Sigma = \{\omega = (s_1, s_2, s_3, \ldots) : s_i = 0 \text{ or } 1\},$$

is a metric space with metric defined by

$$d(\omega_1, \omega_2) = \sum_{k=1}^{\infty} \frac{|s_k - t_k|}{2^k}, \quad \text{where} \quad \omega_1 = (s_1, s_2, \ldots), \quad \omega_2 = (t_1, t_2, \ldots) \in \Sigma.$$

This metric has the following properties:

Property 5.4.1 If $\omega_1 = (s_1, s_2, \ldots)$, $\omega_2 = (t_1, t_2, \ldots) \in \Sigma$, with $s_i = t_i$, $i = 1, 2, \ldots, n$, then $d(\omega_1, \omega_2) \leq 1/2^n$.

Proof.

$$d(\omega_1, \omega_2) = \sum_{k=1}^{\infty} \frac{|s_k - t_k|}{2^k} = \sum_{k=n+1}^{\infty} \frac{|s_k - t_k|}{2^k} \leq \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^n}.$$

□

Property 5.4.2 If $d(\omega_1, \omega_2) < 1/2^n$, then $s_i = t_i$ for $i = 1, 2, \ldots, n$.

Proof. We give a proof by contradiction: Suppose that $s_j = t_j$ for some $1 \leq j \leq n$, then

$$d(\omega_1, \omega_2) = \sum_{k=1}^{\infty} \frac{|s_k - t_k|}{2^k} \geq \frac{1}{2^j} \geq \frac{1}{2^n},$$

a contradiction. □

The shift map σ (sometimes called the *Bernoulli shift*) is an important function defined on Σ.

Definition 5.4.3 The shift map $\sigma : \Sigma \to \Sigma$ is defined by

$$\sigma(s_1, s_2, s_3, \ldots) = (s_2, s_3, \ldots),$$

so for example $\sigma(1, 0, 1, 0, \ldots) = (0, 1, 0, 1, \ldots)$ and $\sigma^2(1, 0, 1, 0, \ldots) = (1, 0, 1, 0, \ldots)$, so that if $\omega_1 = (1, 0, 1, 0, \ldots)$ and $\omega_2 = (0, 1, 0, 1, \ldots)$, then $\{\omega_1, \omega_2\}$ is a 2-cycle for σ. In this way, it is easy to write down all of the points of period n. Any sequence which is eventually constant is clearly an eventually fixed point of Σ, and any sequence which is eventually periodic (such as $(1, 1, 1, 0, 1, 0, 1, 0, 1, \ldots)$), is an eventually periodic point.

Proposition 5.4.4 *The shift map $\sigma : \Sigma \to \Sigma$ is continuous, onto, but not one-to-one.*

Proof. Clearly σ is onto but not one-to-one. To show that σ is continuous, let $\epsilon > 0$, then we want to find $\delta > 0$ such that if

$$d(\omega_1, \omega_2) < \delta, \quad \text{then} \quad d(\sigma(\omega_1), \sigma(\omega_2)) < \epsilon.$$

We shall see that it suffices to take $\delta = 1/2^{n+1}$ if n is chosen so large that $1/2^n < \epsilon$.

In this case, if $d(\omega_1, \omega_2) < \delta = 1/2^{n+1}$, then from Property 5.4.2, $s_i = t_i$ for $i = 1, 2, \ldots, n + 1$. Clearly the first n terms of the sequences $\sigma(\omega_1)$ and $\sigma(\omega_2)$ are equal, so by Property 5.4.1, $d(\sigma(\omega_1), \sigma(\omega_2)) \leq 1/2^n < \epsilon$, so that σ is continuous. □

We can now prove:

Theorem 5.4.5 *The shift map $\sigma : \Sigma \to \Sigma$ is chaotic.*

Proof. We first show that the periodic points are dense in Σ. Let $\omega = (s_1, s_2, \ldots) \in \Sigma$, then it suffices to show that there is a sequence of periodic points $\omega_n \in \Sigma$ with $\omega_n \to \omega$ as $n \to \infty$. Set

$\omega_1 = (s_1, s_1, s_1, s_1, \ldots)$, a period 1-point for σ,

$\omega_2 = (s_1, s_2, s_1, s_2, \ldots)$, a period 2-point for σ,
$\omega_3 = (s_1, s_2, s_3, s_1, s_2, s_3, \ldots)$, a period 3-point for σ,
and continuing in this way so that
$\omega_n = (s_1, s_2, \ldots, s_n, s_1, \ldots)$, a period n-point for σ.

Since ω and ω_n agree in the first n coordinates, $d(\omega, \omega_n) \leq 1/2^n$, so $d(\omega, \omega_n) \to 0$ as $n \to \infty$, or $\omega_n \to \omega$ as $n \to \infty$.

To show that σ is transitive, we explicitly construct a point $\omega_0 \in \Sigma$ having a dense orbit under σ. Set

$$\omega_0 = (\underbrace{0\,1}_{\text{1-blocks}}\ \underbrace{0\,0\ 0\,1\ 1\,0\ 1\,1}_{\substack{\text{all possible}\\ \text{2-blocks}}}\ \underbrace{0\,0\,0\ 0\,0\,1\ 0\,1\,0\ 1\,0\,1\ 0\,1\,1\ldots}_{\text{all possible 3 blocks}}),$$

and continuing in this way so that all possible n-blocks appear in ω_0. To see that $\overline{\mathrm{O}(\omega_0)} = \Sigma$, let $\omega = (s_1, s_2, s_3, \ldots) \in \Sigma$ be arbitrary. Let $\epsilon > 0$ and choose n so large that $1/2^n < \epsilon$, then since ω_0 consists of all possible n-blocks, the sequence $(s_1, s_2, \ldots, s_n)$ must appear somewhere in ω_0, i.e., there exists $k > 0$ with

$$\sigma^k(\omega_0) = (s_1, s_2, \ldots, s_n, \ldots),$$

so that ω and $\sigma^k(\omega_0)$ agree on the first n coordinates. It follows that

$$d(\omega, \sigma^k(\omega_0)) \leq \frac{1}{2^n} < \epsilon.$$

This shows that the orbit of ω_0 comes arbitrarily close to any member of Σ, so it is dense in Σ. It follows that σ is transitive and so it is chaotic. $\square$

Remark 5.4.6 It can be shown that (topological) properties such as being totally disconnected, perfect etc. are preserved by homeomorphisms.

The shift space Σ and the Cantor set C are homeomorphic metric spaces. In addition, the half open interval $[0, 1)$ and the unit circle S^1 in the complex plane are homeomorphic. In particular, Σ and C will have identical topological properties, and so will S^1 and $[0, 1)$. It follows that Σ and $[0, 1]$ cannot be homeomorphic as C and $[0, 1]$ are not homeomorphic (C is totally disconnected, but $[0, 1]$ is not).

Proof. Σ is given its usual metric, and C has the metric induced from being a subset of $\mathbb{R}$, so that $d(x, y) = |x - y|$ for $x, y \in C$.

We define a map $h : C \to \Sigma$ by $h(\cdot\, a_1\, a_2\, a_3\, \ldots) = (s_1, s_2, s_3, \ldots)$, where $a_i = 0$ or 2 and $s_i = a_i/2$. Clearly h is both one-to-one and onto. We show that it is continuous at each $x_0 \in C$.

Let $\epsilon > 0$ and choose n so large that $1/2^n < \epsilon$. Set $\delta = 1/3^n$, then if $|x_0 - x| < \delta$, both x_0 and x must lie in the same component (sub-interval) of S_n of length $1/3^n$. It follows that x_0 and x must have an identical ternary expansions in the first n places. Correspondingly, $h(x_0)$ and $h(x)$ must have the same first n coordinates. It follows that $d(h(x_0), h(x)) \leq 1/2^n < \epsilon$, so h is continuous at x_0. In a similar way we see that h^{-1} is continuous.

To see that $[0, 1)$ and S^1 are homeomorphic, define $h : [0, 1) \to S^1$ by $h(x) = e^{2\pi i x}$, then h is clearly one-to-one and onto. The map h wraps the interval $[0, 1)$ around the circle, so that strictly speaking we are looking at $[0, 1]$ with the end points identified, so in this way h becomes continuous. □

Exercises 5.4

1. Let $f : X \to X$ be a transitive map of the metric space (X, d). Show that if U and V are non-empty open sets in X, there exists $m \in \mathbb{Z}^+$ with $U \cap f^m(V) \neq \emptyset$.

2. Let $F : [0, 1) \to [0, 1)$ be the tripling map $F(x) = 3x \bmod 1$. Follow the proof for the doubling map (but use ternary expansions) to show that F is transitive and the period points are dense (find the periodic points), and hence show that F is chaotic.

3. Let $D : [0, 1) \to [0, 1)$ be the doubling map. Show that $|\mathrm{Per}_n(D)| = 2^n$.

4. Use Proposition 4.4.7 to show that for a continuous increasing function $f : [a, b] \to [a, b]$, the periodic points cannot be dense in $[a, b]$ (so a homeomorphism of $[a, b]$ cannot be chaotic).

5.5 Sensitive Dependence on Initial Conditions

We now show that the original definition of chaos due to Devaney [20] follows from the definition we have given. This result is due to Banks et al. [6].

Definition 5.5.1 Let $f : X \to X$ be defined on a metric space (X, d). Then f has *sensitive dependence on initial conditions* if there exists $\delta > 0$ such that for any $x \in X$ and any open interval U containing x and points other than x, there is a point $y \in U$ and $n \in \mathbb{Z}^+$ with

$$d(f^n(x), f^n(y)) > \delta.$$

This is the precise definition of the idea that iterates of points close to each other may eventually be widely apart, so that a map has sensitive dependence on initial conditions if there exist points arbitrarily close to x which are eventually at least distance δ away from x. It is important to know whether we have sensitive dependence when doing computations as round-off errors may be magnified after numerous iterations. For example, suppose we iterate the doubling map, starting with $x_0 = 1/3$ and $x_1 = .333$. After 10 iterations we have $D^{10}(x_0) = 1/3$ and $D^{10}(x_1) = .92$, more than distance $1/2$ apart.

Examples 5.5.2 1. The linear map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = ax$, $|a| > 1$, has sensitive dependence since if $x \neq y$,

$$|f^n(x) - f^n(y)| = |a^n x - a^n y| = a^n |x - y| \to \infty \quad \text{as} \quad n \to \infty.$$

However, clearly the dynamics of f is not complicated (f is not chaotic).

2. The shift map $\sigma : \Sigma \to \Sigma$ has sensitive dependence on initial conditions since if $\omega_1, \omega_2 \in \Sigma$ with $\omega_1 \neq \omega_2$, then they must differ at some coordinates, say $s_i \neq t_i$. Then

$$\sigma^{i-1}(\omega_1) = (s_i, s_{i+1}, \ldots), \quad \text{and} \quad \sigma^{i-1}(\omega_2) = (t_i, t_{i+1}, \ldots),$$

so that

$$d(\sigma^{i-1}(\omega_1), \sigma^{i-1}(\omega_2)) = \sum_{k=1}^{\infty} \frac{|s_{i+k-1} - t_{i+k-1}|}{2^k} = \frac{1}{2} + \text{other terms} \geq \frac{1}{2}.$$

3. The angle doubling map $f : S^1 \to S^1$, $f(z) = z^2$ has sensitive dependence since if we iterate $z = e^{i\theta}, w = e^{i\phi} \in S^1$, their distance apart doubles after each iteration.

4. The doubling map $D : [0, 1] \to [0, 1]$ can be seen directly to have sensitive dependence on initial conditions. This also follows from the following theorem:

Theorem 5.5.3 (Banks et al. [6]) *Let $f : X \to X$ be a chaotic transformation, then f has sensitive dependence on initial conditions.*

We first prove a preliminary result:

Lemma 5.5.4 *Let $f : X \to X$ be a transformation which has at least two different periodic orbits. Then there exists $\epsilon > 0$ such that for any $x \in X$ there is a periodic point p satisfying*

$$d(x, f^k(p)) > \epsilon, \quad \textit{for all} \quad k \in \mathbb{Z}^+.$$

Proof. Let a and b be two periodic points with different orbits. Then $d(f^k(a), f^l(b)) > 0$ for all k and l (since we are dealing with finite sets).

Choose $\epsilon > 0$ small enough that $d(f^k(a), f^l(b)) > 2\epsilon$ for all k and l. Then

$$d(f^k(a), x) + d(x, f^l(b)) \geq d(f^k(a), f^l(b)) > 2\epsilon \quad \forall k, l \in \mathbb{Z}^+,$$

by the triangle inequality.

If x is within ϵ of any of the points $f^l(b)$, then it must be at a greater distance than ϵ from all of the points $f^k(a)$ and the result follows. $\square$

Proof of Theorem 5.5.3 Let $x \in X$ and U be an open set in X containing x.

Let p be a periodic point of period r for f, whose orbit is a distance greater than 4δ from x.

The periodic points of f are dense in X, so there is a periodic point q of period n say, with

$$q \in V = U \cap B_\delta(x).$$

Write

$$W_i = B_\delta(f^i(p)),$$

then

$$f^i(p) \in W_i, \forall i \Rightarrow p \in f^{-i}(W_i), \forall i,$$

so the open set

$$W = f^{-1}(W_1) \cap f^{-2}(W_2) \cap \cdots \cap f^{-n}(W_n) \neq \emptyset.$$

Since f is transitive, there is a point $z \in V$ with $f^k(z) \in W$ for some $k \in \mathbb{Z}^+$.

Let j be the smallest integer with $k < nj$, or

$$1 \leq nj - k \leq n.$$

Then

$$f^{nj}(z) = f^{nj-k}(f^k(z)) \in f^{nj-k}(W).$$

But

$$f^{nj-k}(W) = f^{nj-k}(f^{-1}(W_1) \cap f^{-2}(W_2) \cap \cdots \cap f^{-n}(W_n)) \subset f^{nj-k}(f^{-(nj-k)} W_{nj-k}) = W_{nj-k},$$

so that $d(f^{nj}(z), f^{nj-k}(p)) < \delta$. Now $f^{nj}(q) = q$, and by the triangle inequality

$$d(f^{nj-k}(p), x) \leq d(f^{nj-k}(p), f^{nj}(z)) + d(f^{nj}(z), f^{nj}(q)) + d(f^{nj}(q), x)$$

so that

$$\begin{aligned} 4\delta < d(f^{nj-k}(p), x) &\leq d(f^{nj-k}(p), f^{nj}(z)) + d(f^{nj}(z), f^{nj}(q)) + d(q, x) \\ &< \delta + d(f^{nj}(z), f^{nj}(q)) + \delta, \end{aligned}$$

so

$$d(f^{nj}(z), f^{nj}(q)) > 2\delta.$$

This inequality implies that either

$$d(f^{nj}(x), f^{nj}(z)) \geq \delta,$$

or

$$d(f^{nj}(x), f^{nj}(q)) \geq \delta,$$

for if $f^{nj}(x)$ were within distance $< \delta$ from both of these points, they would have to be within $< 2\delta$ from each other, contradicting the previous inequality above. So one of the two, z or q will serve as the y in the theorem with $m = nj$. $\square$

Chapter 6. Conjugacy of Dynamical Systems

Two metric spaces X and Y are the "same" (homeomorphic) if there is a homeomorphism from one space to the other. In this chapter we study the question of when two *dynamical systems* are the same. Given maps $f : X \to X$ and $g : Y \to Y$, we require them to have the same type of dynamical behavior, e.g., if one is chaotic, then so is the other, there is a one-to-one correspondence between their periodic points etc. One obvious requirement is that the underlying metric spaces should be homeomorphic. We have seen a lot of similarities between the logistic map $L_4(x) = 4x(1-x)$ and the tent map $T(x)$ and this will be examined in this chapter together with other examples such as the shift map and circle maps.

6.1 Conjugate Maps

This "sameness" is given by the idea of conjugacy, a notion borrowed from group theory, where two members a and b of a group G are conjugate if there exists $g \in G$ with $ag = gb$. One of the central problems of one-dimensional dynamics and dynamical systems in general is to be able to tell whether or not two dynamical systems are conjugate. We will see that if one map has a 3-cycle and another map has no 3-cycle (for example), then the maps cannot be conjugate, or if one map has 2 fixed points and the other has 3 fixed points, then they are not conjugate. These are examples of *conjugacy invariants*, which give criteria for maps to be non-conjugate. A generally harder problem is establishing a conjugacy between maps.

Definition 6.1.1 1. Let $f : X \to X$ and $g : Y \to Y$ be maps of metric spaces. Then f and g are said to be *conjugate* if there is a homeomorphism $h : X \to Y$ such that

$$h \circ f = g \circ h.$$

The map h is a *conjugacy* between f and g. Obviously conjugacy is an equivalence relation.

2. If in the above definition we only require the map $h : X \to Y$ to be continuous, then we say that g is a *factor* of f. If in addition, h is an onto map, then we say that g is a *quasi-factor* of f.

Exercises 6.1

1. Prove that if $f : X \to X$ and $g : Y \to Y$ are conjugate maps of metric spaces, then f is one-to-one if and only if g is one-to-one, and f is onto if and only if g is onto.

2. Prove that if f and g are conjugate via h and f has a local maximum at x_0, then g has a local maximum or minimum at $h(x_0)$.

3. Suppose that $h : [0,1] \to [0,1]$ is a conjugacy between $f, g : [0,1] \to [0,1]$ where $f(0) = f(1) = 0$ and $g(0) = g(1) = 0$. Show that h is increasing on $[0,1]$. Deduce that h maps the zero's of f to the zero's of g.

4. ([11]) The function $T_n(x) = \cos(n \arccos(x))$ is the nth Chebyshev polynomial. Show that T_n is conjuagte to the map $\Lambda_n : [0,1] \to [0,1]$, the piecewise linear continuous map defined by joining the points $(0,0), (1/n, 1), (2/n, 0), (3/n, 1), \ldots,$ ending with $(1,1)$ if n is odd, or $(1,0)$ if n is even. Use the conjugacy map $h : [0,1] \to [0,1]$, $h(x) = \cos(pix)$. (In [11], there is a generalization of this to maps T_λ, where $\lambda > 1$ is a real number).

6.2 Properties of Conjugate Maps

It is often easier to show that certain maps are chaotic indirectly by showing that they are conjugate to chaotic maps and using the following result:

Proposition 6.2.1 *If $f : X \to X$ and $g : Y \to Y$ are maps conjugate via a conjugacy $h : X \to Y$: $h \circ f = g \circ h$, then*

1. *$h \circ f^n = g^n \circ h$ for all $n \in \mathbb{Z}^+$, (so f^n and g^n are also conjugate).*

2. *If c is a point of period m for f, then $h(c)$ is a point of period m for g. c is attracting if and only if $h(c)$ is attracting.*

3. *f is transitive if and only if g is transitive.*

4. *f has a dense set of periodic points if and only if g has a dense set of periodic points.*

5. *f is chaotic if and only if g is chaotic.*

Proof. 1. $h \circ f^2 = h \circ f \circ f = g \circ h \circ f = g \circ g \circ h = g^2 \circ h$, and in the same way $h \circ f^3 = g^3 \circ h$, and continuing inductively the result follows.

2. Suppose that $f^i(c) \neq c$ for $0 < i < m$ and $f^m(c) = c$, then $h \circ f^i(c) \neq h(c)$ for $0 < i < m$ since h is one-to-one, and so $g^i \circ h(c) \neq h(c)$ for $0 < i < m$. In addition, $h \circ f^m(c) = g^m \circ h(c)$, or $h(c) = g^m(h(c))$, so $h(c)$ is a period m point for g.

We shall only show that if p is an attracting fixed point of f (so that there is an open ball $B_\epsilon(p)$ such that if $x \in B_\epsilon(p)$ then $f^n(x) \to p$ as $n \to \infty$), then $h(p)$ is an attracting fixed point of g.

Let $V = h(B_\epsilon(p))$, then since h is a homeomorphism, V is open in Y and contains $h(p)$. Let $y \in V$, then $h^{-1}(y) \in B_\epsilon(p)$, so that $f^n(h^{-1}(y)) \to p$ as $n \to \infty$.

Since h is continuous, $h(f^n(h^{-1}(y))) \to h(p)$ as $n \to \infty$, i.e.,

$$g^n(y) = h \circ f^n \circ h^{-1}(y) \to h(p), \quad \text{as} \quad n \to \infty,$$

so $h(p)$ is attracting.

3. Suppose that $O(z) = \{z, f(z), f^2(z), \ldots\}$ is dense in X and let $V \subset Y$ be a non-empty open set. Then since h is a homeomorphism, $h^{-1}(V)$ is open in X, so there exists $k \in \mathbb{Z}^+$ with $f^k(z) \in h^{-1}(V)$.

It follows that $h(f^k(z)) = g^k(h(z)) \in V$, so that

$$O(h(z)) = \{h(z), g(h(z)), g^2(h(z)), \ldots\}$$

is dense in Y, i.e., g is transitive. Similarly, if g is transitive, then f is transitive.

4. Suppose that f has a dense set of periodic points and let $V \subset Y$ be non-empty and open. Then $h^{-1}(V)$ is open in X, so contains periodic points of f. As in (3), we see that V contains periodic points of g. Similarly if g has a dense set of periodic points, so does f.

5. This now follows from (3) and (4). □

Example 6.2.2 We remark that sensitive dependence on initial conditions is not a conjugacy invariant. It is possible for two maps on metric spaces to be conjugate, one to have sensitive dependence, but the other not: Consider $T : (0, \infty) \to (0, \infty)$, $T(x) = 2x$ and $S : \mathbb{R} \to \mathbb{R}$ defined by $S(x) = x + \ln 2$. If $H : (0, \infty) \to \mathbb{R}$ is defined by $H(x) = \ln x$, then H is a homeomorphism and we can check that $H \circ T = S \circ H$ so T and S are conjugate, T has sensitive dependence, but S does not.

It can be shown however, that if $T : X \to X$ is a map on a compact metric space X (for example $X = [0, 1]$) having sensitive dependence, then any map conjugate to T also has sensitive dependence.

It can also be shown that the property of having negative Schwarzian derivative is not a conjugacy invariant.

Example 6.2.3 1. The logistic map $L_4 : [0,1] \to [0,1]$, $L_4(x) = 4x(1-x)$ is conjugate to the tent map $T : [0,1] \to [0,1]$, $T(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1/2 \\ 2(1-x) & \text{if } 1/2 < x \le 1 \end{cases}$.

Proof. Define $h : [0,1] \to [0,1]$ by $h(x) = \sin^2(\pi x/2)$. We can see that h is one-to-one, onto and both h and h^{-1} are continuous, so it is a homeomorphism (it is not a diffeomorphism as $h'(1) = 0$). Also

$$L_4 \circ h(x) = L_4\left(\sin^2(\frac{\pi x}{2})\right) = 4\sin^2(\frac{\pi x}{2})\left(1 - \sin^2(\frac{\pi x}{2})\right) = \sin^2(\pi x),$$

and

$$h \circ T(x) = h(Tx) = \begin{cases} h(2x) & \text{if } 0 \le x \le 1/2 \\ h(2-2x) & \text{if } 1/2 < x \le 1 \end{cases} = \sin^2(\pi x),$$

so $L_4 \circ h = h \circ T$ and L_4 and T are conjugate. □

2. The doubling map $D : [0,1] \to [0,1]$ is a quasi-factor of the shift map $\sigma : \Sigma \to \Sigma$.

Proof. D is a factor of σ since

$$h \circ \sigma = D \circ h,$$

where $h : \Sigma \to [0,1]$ is defined by

$$h(x_1, x_2, x_3, \ldots) = \cdot x_1\, x_2\, x_3\, \ldots = \sum_{i=1}^{\infty} \frac{x_i}{2^i}$$

is a continuous function (note that h is not a homeomorphism since it is not one-to-one: for example $h(1,0,0,\ldots) = 1/2 = h(0,1,1,1,\ldots)$, but $(1,0,0,\ldots) \neq (0,1,1,1,\ldots)$ in Σ). In addition $h(\Sigma) = [0,1]$, so that h is onto and D is a quasi-factor of σ □

4. The logistic map L_4 is a quasi-factor of the angle doubling map $f : S^1 \to S^1$, $f(z) = z^2$.

Proof. Define $h : S^1 \to [0,1]$ by $h(e^{ix}) = \sin^2 x$, then

$$L_4 \circ h(e^{ix}) = L_4(\sin^2 x) = 4\sin^2(1 - \sin^2 x) = \sin^2(2x),$$

and

$$h \circ f(e^{ix}) = h(e^{2ix}) = \sin^2(2x).$$

h is clearly onto and continuous, but it is not one-to-one: $h(e^{ix}) = h(e^{-ix})$, so L_4 is a quasi-factor of f, but h is not a conjugacy. □

We can now show that many of the above maps are chaotic. In order to do this we need to weaken the conditions of Proposition 6.2.1. If we drop the requirement that h is a homeomorphism, but just require it to be continuous and onto, then we can show that if f is chaotic then so is g. In other words, if g is a quasi-factor of f where f is chaotic, then g is also chaotic. This result will be useful in showing that a number of well known examples are chaotic.

Proposition 6.2.4 *Let $h : X \to Y$ be surjective (onto) and continuous. If $f : X \to X$ and $g : Y \to Y$ satisfy $h \circ f = g \circ h$ and f is chaotic, then g is chaotic.*

Before proving this proposition we need a lemma concerning continuous functions on metric spaces:

Lemma 6.2.5 *Let $h : X \to Y$ be a continuous function of metric spaces and A a subset of X, then $h(\overline{A}) \subseteq \overline{h(A)}$.*

Proof. Let $y \in h(\overline{A})$, then there exists $x \in \overline{A}$ with $y = h(x)$. We can find a sequence $x_n \in A$ with $\lim_{n\to\infty} x_n = x$.

Then $h(x_n) \in h(A)$ and since h is continuous

$$\lim_{n\to\infty} h(x_n) = h(x) = y \quad \text{so that} \quad y \in \overline{h(A)}.$$

$\square$

Proof of Proposition 6.2.4 Use $\mathrm{Per}(f)$ and $\mathrm{Per}(g)$ to denote the periodic points of f and g respectively, then we saw earlier that $h(\mathrm{Per}(f)) \subseteq \mathrm{Per}(g)$. Since f is chaotic, $\overline{\mathrm{Per}(f)} = X$, and since h is onto, $h(X) = Y$. Then using the lemma we have

$$Y = h(X) = h(\overline{\mathrm{Per}(f)}) \subseteq \overline{h(\mathrm{Per}(f))} \subseteq \overline{\mathrm{Per}(g)},$$

so that $\overline{\mathrm{Per}(g)} = Y$. In other words, the periodic points of g are dense in Y.

f is transitive so there exists $x_0 \in X$ with $\overline{\mathrm{O}_f(x_0)} = X$ (where we use the subscript to distinguish the orbits with respect to f and g). Now

$$h\left(\mathrm{O}_f(x_0)\right) = h\{f^n(x_0) : n \in \mathbb{Z}^+\} = \{h \circ f^n(x_0) : n \in \mathbb{Z}^+\}$$

$$= \{g^n \circ h(x_0) : n \in \mathbb{Z}^+\} = \mathrm{O}_g(h(x_0)),$$

so that

$$Y = h(X) = h(\overline{\mathrm{O}_f(x_0)}) \subseteq \overline{h\left(\mathrm{O}_f(x_0)\right)} = \overline{\mathrm{O}_g(h(x_0))},$$

so that $h(x_0)$ is a transitive point for g. $\square$

It is easily seen that Proposition 6.3.4 remains true if we replace the requirement that h be onto by requiring that $h(X)$ be dense in Y.

Theorem 6.2.6 *The tent map $T : [0,1] \to [0,1]$, the logistic map $L_4(x) = 4x(1-x)$, the angle doubling map $f : S^1 \to S^1$, $f(z) = z^2$, and the Doubling map $D : [0,1] \to [0,1]$ are all chaotic.*

Proof. Suppose that we can show that the angle-doubling map f is a quasi-factor of the doubling map D, then we have:

The tent map T is conjugate to the logistic map L_4, which is a quasi-factor of $f(z)$, which is quasi-factor of D, which is a quasi-factor of σ, the shift map. It has been shown that the shift map is chaotic. The result now follows.

It therefore suffices to show that for $f : S^1 \to S^1$, $f(z) = z^2$ and $D : [0,1] \to [0,1]$, $D(x) = 2x \pmod 1$, f is a quasi-factor of D. Define $h : [0,1] \to S^1$ by $h(x) = e^{2\pi i x}$, then h is continuous, onto and one-to-one everywhere except that $h(0) = 1 = h(1)$. Now

$$f \circ h(x) = f(e^{2\pi i x}) = e^{4\pi i x},$$

and

$$h \circ D(x) = h(2x \pmod 1) = e^{2\pi i (2x \pmod 1)} = e^{4\pi i x},$$

or $h \circ D = f \circ h$, and f is a quasi-factor of D. $\square$

Exercises 6.2

1. If $g(z) = z^3$ on S^1, show that g is the angle tripling map. Find the period points and show that they are dense in S^1. Show that g is conjugate to the map $F : [0,1) \to [0,1)$, $F(x) = 3x$ mod 1 of Exercises 5.4.

1. If $T_3 : C \to C$ is the tent map with $\mu = 3$, but restricted to the Cantor set. Show that T_3 is conjugate to the shift map $\sigma : \Sigma \to \Sigma$.

3. Prove that the doubling map $D : [0,1) \to [0,1)$, $D(x) = 2x \pmod 1$, and the angle doubling map $f : S^1 \to S^1$, $f(z) = z^2$, are conjugate.

4. Is the shift map $\sigma : \Sigma \to \Sigma$ conjugate to the doubling map D? (It can be shown that the shift map σ is conjugate to T_3 restricted to the Cantor set).

5. Let $U : [-1,1] \to [-1,1]$ be defined by $U(x) = 1 - 2x^2$ and $T_2 : [0,1] \to [0,1]$ be the tent map

$$T_2(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1/2, \\ 2(1-x) & \text{if } 1/2 < x \le 1. \end{cases}$$

Prove that $H : [0,1] \to [-1,1]$, $H(x) = -\cos(\pi x)$ defines a conjugacy between these maps.

6. Prove that the map $F : [-1,1] \to [-1,1]$, $F(x) = 4x^3 - 3x$ is conjugate to $T : [0,1] \to [0,1]$, $T(x) = 3x \bmod(1)$, via the conjugacy $h : [0,1] \to [-1,1]$, $h(x) = \cos(\pi x)$. (Hint: $\cos(3x) = 4\cos^3(x) - 3\cos(x)$). Deduce that F is chaotic. This question is related to Exercise 6.1, # 4 concerning the Chebyshev polynomials.

6.3 Linear Conjugacy

It is sometimes the case that the conjugacy between two real (or complex) functions is given by a map with a straight line graph (an affine map). This is called a linear conjugacy, and is stronger than the usual notion of conjugacy.

Definition 6.3.1 For functions $f : I \to I$ and $g : J \to J$ defined on subintervals of $\mathbb{R}$, we say that f and g are *linearly conjugate* and that h is a *linear conjugacy* if h maps I onto J where $h(x) = ax + b$ for some $a, b \in \mathbb{R}$, $a \neq 0$ and $h \circ f = g \circ h$.

The following example gives a criterion for two quadratic functions to be linearly conjugate:

Example 6.3.2 Let

$$F(x) = ax^2 + bx + c \quad \text{and} \quad G(x) = rx^2 + sx + t,$$

where $a \neq 0$ and $r \neq 0$. If

$$c = \frac{b^2 - s^2 + 2s - 2b + 4rt}{4a},$$

then F and G are linearly conjugate via the linear conjugacy

$$h(x) = \frac{a}{r}x + \frac{b-s}{2r}.$$

Proof.

$$\begin{aligned} h \circ F(x) = h(ax^2+bx+c) &= \frac{a(ax^2+bx+c)}{r} + \frac{b-s}{2r} \\ &= \frac{a^2}{r}x^2 + \frac{ab}{r}x + \frac{2ac+b-s}{2r}, \end{aligned}$$

and

$$\begin{aligned} G \circ h(x) = G(\frac{a}{r}x + \frac{b-s}{2r}) &= r(\frac{a}{r}x + \frac{b-s}{2r})^2 + s(\frac{a}{r}x + \frac{b-s}{2r}) + t \\ &= r\left(\frac{a^2}{r^2}x^2 + 2\frac{a(b-s)}{2r^2}x + \frac{(b-s)^2}{4r^2}\right) + \frac{sa}{r}x + \frac{bs-s^2}{2r} + t \\ &= \frac{a^2}{r}x^2 + \frac{ab}{r}x + \frac{(b-s)^2+2bs-2s^2+4rt}{4r}, \end{aligned}$$

and we see that these are equal if

$$c = \frac{b^2-s^2+2s-2b+4rt}{4a}.$$

□

For example, if F is defined on the interval $[0,1]$, then

$$h(0) = \frac{b-s}{2r} \quad \text{and} \quad h(1) = \frac{2a+b-s}{2r},$$

so if $a/r > 0$, then F is conjugate to G on the interval $[\frac{b-s}{2r}, \frac{2a+b-s}{2r}]$.

Example 6.3.3 1. If $L_\mu(x) = \mu x(1-x)$, and $Q_c(x) = x^2 + c$ and $c = \dfrac{2\mu-\mu^2}{4}$, then L_μ on the interval $[0,1]$ is linearly conjugate to Q_c on the interval $[-\mu/2, \mu/2]$. If $\mu = 4$, this shows that $L_4(x) = 4x(1-x)$ on $[0,1]$ is conjugate to $Q_c(x) = x^2 + c$ on the interval $[-2,2]$ when $c = -2$. In particular, Q_{-2} on $[-2,2]$ is chaotic.

Proof. We apply Example 6.3.2 with

$$a = -\mu,\ b = \mu,\ c = 0,\ r = 1,\ s = 0,\ t = c.$$

In this case $h(0) = \mu/2$ and $h(1) = -\mu/2$ and we can check that the conditions of the example hold when $c = \dfrac{2\mu-\mu^2}{4}$. □

2. On the other hand, if $\mu = 2$, we see that $L_2(x) = 2x(1-x)$ on $[0,1]$ is conjugate to $Q_0(x) = x^2$ on $[-1,1]$. Recall in Exercises 1.1 the difference equation $x_{n+1} = 2x_n(1-x_n)$ transforms to $y_{n+1} = y_n^2$ on setting $x_n = (1-y_n)/2$. This is just the fact that L_2 and Q_0 are conjugate via $h(x) = -2x+1$.

3. We can check that the logistic map L_4 is conjugate to $F : [-1,1] \to [-1,1]$, $F(x) = 2x^2 - 1$.

Exercises 6.3

1. Show that every quadratic polynomial $p(x) = ax^2 + bx + d$ is conjugate to a unique polynomial of the form $f_c(x) = x^2 + c$.

2. Prove that the logistic map L_4 is conjugate to $F : [-1,1] \to [-1,1]$, $F(x) = 2x^2 - 1$.

3. Show that the logistic map $L_\mu(x) = \mu x(1-x)$, $x \in [0,1]$ is conjugate to the logistic type map $F_\mu(x) = (2-\mu)x(1-x)$ $(\mu \neq 2)$, via the linear conjugacy (which is defined on the interval with end points $\frac{1-\mu}{2-\mu}$ and $\frac{1}{2-\mu}$):

$$h(x) = \frac{\mu}{2-\mu}x + \frac{1-\mu}{2-\mu}.$$

4. Let $f_c(x) = x^2 + c$. Show that if $c > -1/4$, there is a unique $\mu > 0$ such that f_c is conjugate to $L_\mu(x) = \mu x(1-x)$ via a map of the form $h(x) = ax + b$.

5. (a) If $f_a(x) = ax$, $f_b(x) = bx$, $a, b \in \mathbb{R}$, defined on $\mathbb{R}$, when are they linearly conjugate?

(b) Show that $f_{1/2}$ and $f_{1/4}$ are conjugate via the map $h(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ -\sqrt{-x} & \text{if } x < 0 \end{cases}$.

6.4 Conjugacy and the Tent Family

We saw in Section 2.7 that for $\mu \geq (1+\sqrt{5})/2$, the tent map T_μ has a point of period three, so by Sharkovsky's Theorem, it will have points of all possible periods. In this section we use a certain conjugacy to show that for $\mu > 1$, T_μ will have points of period 2^n for each $n \geq 1$. Our argument is based on that in [29]. We first show that the interval $[1/(1+\mu), \mu/(1+\mu)]$ is invariant under T_μ^2 when $1 < \mu \leq \sqrt{2}$.

The formula for T_μ^2 in Section 2.7 gives

$$T_\mu^2(x) = \begin{cases} \mu^2 x & \text{if} \quad 0 \leq x \leq \frac{1}{2\mu} \\ \mu - \mu^2 x & \text{if} \quad \frac{1}{2\mu} < x \leq \frac{1}{2} \\ \mu^2 x + \mu - \mu^2 & \text{if} \quad \frac{1}{2} < x \leq 1 - \frac{1}{2\mu} \\ \mu^2 - \mu^2 x & \text{if} \quad 1 - \frac{1}{2\mu} < x \leq 1 \end{cases}$$

Proposition 6.4.1 *For $1 < \mu \leq \sqrt{2}$, The restriction*

$$T_\mu^2 : [\frac{1}{1+\mu}, \frac{\mu}{1+\mu}] \to [\frac{1}{1+\mu}, \frac{\mu}{1+\mu}],$$

is well defined.

Proof. Note that for this range of values of μ, $1/(1+\mu) < 1/2$ and $\mu/(1+\mu) > 1/2$, so that $T_\mu(1/(1+\mu)) = \mu/(1+\mu)$ is an eventual fixed points since $T_\mu(\mu/(1+\mu)) = \mu/(1+\mu)$.

Let $x \in [1/(1+\mu), \mu/(1+\mu)]$, then from the formula for T_μ^2 and the fact that

$$\frac{1}{2\mu} < \frac{1}{1+\mu} < \frac{\mu}{1+\mu} < 1 - \frac{1}{2\mu},$$

we see that on this interval the minimum value of T_μ^2 occurs at $x = 1/2$ and this gives

$$T_\mu^2(x) \geq T_\mu^2(1/2) = \mu(1-\mu/2) \geq \frac{1}{1+\mu},$$

since this is equivalent to

$$\mu^3 - \mu^2 - 2\mu + 2 \leq 0, \quad \text{or} \quad (\mu - 1)(\mu^2 - 2) \leq 0,$$

where $1 < \mu \leq \sqrt{2}$.

Again assuming that $x \in [1/(1+\mu), \mu/(1+\mu)]$, we see that on the other hand, if $x \leq 1/2$, then $T_\mu(x) = \mu x > \mu/(1+\mu) > 1/2$, so $T_\mu^2(x) = \mu(1-\mu x) < \mu(1-\mu/(1+\mu)) = \mu/(1+\mu)$, so $T_\mu^2(x) \in [1/(1+\mu), \mu/(1+\mu)]$.

If instead $x > 1/2$, then

$$T_\mu(x) = \mu(1-x) > \mu(1 - \frac{\mu}{1+\mu}) = \frac{\mu}{1+\mu} > \frac{1}{2},$$

so

$$T_\mu^2(x) = \mu(1-\mu(1-x)) = \mu(1-\mu+\mu x) \le \mu(1-\mu+\frac{\mu^2}{1+\mu}) = \frac{\mu}{1+\mu},$$

so again $T_\mu^2(x) \in [1/(1+\mu), \mu/(1+\mu)]$. $\square$

We use the above proposition to show that T_μ and $T_{\sqrt{\mu}}^2$ are conjugate when $T_{\sqrt{\mu}}^2$ is restricted to a suitable invariant subinterval.

Proposition 6.4.2 *For $1 < \mu \le \sqrt{2}$, T_μ^2 restricted to the interval*

$$\left[\frac{1}{1+\mu}, \frac{\mu}{1+\mu}\right],$$

is conjugate to T_{μ^2} on $[0,1]$.

Proof. From Proposition 6.4.1, we see that the given interval is invariant under T_μ^2. Now we show that we actually have a linear conjugacy $h(x) = ax + b$:

$$h \circ T_\mu^2 = T_{\mu^2} \circ h,,$$

where

$$h : \left[\frac{1}{1+\mu}, \frac{\mu}{1+\mu}\right] \to [0,1],$$

and

$$a = \frac{1+\mu}{1-\mu}, \quad b = \frac{\mu}{\mu-1},$$

where we can check that

$$h(\frac{\mu}{1+\mu}) = 0, \quad \text{and} \quad h(\frac{1}{1+\mu}) = 1.$$

If $0 \le x \le 1/2$, then since $h^{-1}(x) = x/a - b/a$, we can check that $1/2 \le h^{-1}(x) \le \mu/(1+\mu) < 1 - 1/2\mu$, so that

$$h \circ T_\mu^2 \circ h^{-1}(x) = h \circ T_\mu^2 \left(\frac{x}{a} - \frac{b}{a}\right)$$

$$= h\left(\mu^2(\frac{x}{a} - \frac{b}{a}) + \mu - \mu^2\right) = \mu^2 x - \mu^2 b + a(\mu - \mu^2) + b = \mu^2 x = T_{\mu^2}(x).$$

Similarly we can check that if $1/2 < x \leq 1$, then $1/2\mu < 1/(1+\mu) \leq h^{-1}(x) \leq 1/2$, so that

$$h \circ T_\mu^2 \circ h^{-1}(x) = h \circ T_\mu^2(\frac{x}{a} - \frac{b}{a}) = h\left(\mu - \mu^2(\frac{x}{a} - \frac{b}{a})\right)$$

$$= a\mu - \mu^2 x + \mu^2 b + b = \mu^2(1-x) = T_{\mu^2}(x),$$

i.e., in both cases we have $h \circ T_\mu^2 \circ h^{-1}(x) = T_{\mu^2}(x)$, giving the desired conjugacy. □

Corollary 6.4.3 *For $1 < \mu \leq 2$, $T_{\sqrt{\mu}}^2$ restricted to the interval*

$$\left[\frac{\sqrt{\mu}-1}{\mu-1}, \frac{\mu-\sqrt{\mu}}{\mu-1}\right],$$

is conjugate to T_μ on $[0,1]$.

We apply these results to give us information about the period points of T_μ:

Proof. Replace μ by $\sqrt{\mu}$ in the previous result. □

Theorem 6.4.4 *For $1 < \mu \leq 2$, T_μ has a 2^n-cycle for each $n \in \mathbb{Z}^+$.*

Proof. We have seen that for each $\mu > 1$, T_μ has a period 2-point distinct from the fixed point of T_μ. In particular, as $\mu^2 > 1$, T_{μ^2} has a period 2-point distinct from the fixed point of T_{μ^2}. But by the last result, T_{μ^2} and T_μ^2 are conjugate, so T_μ^2 has a period 2-point distinct from the fixed point of T_μ^2. This must be a period 4-point for T_μ, for if not it would be a period 2-point, giving a fixed point for T_μ^2.

Continuing this argument, starting with a period 2-point for T_{μ^4} and the conjugacy between $T_{\mu^2}^2$ and T_{μ^4} we deduce that T_μ has a period 8-point. In this way, for each $n \in \mathbb{Z}^+$ we see that T_μ has a period 2^n-point. □

Example 6.4.5 Consider the case where $\mu = 2$, then we see that T_2, the standard tent map, is conjugate to $T_{\sqrt{2}}^2$ when it is restricted to the interval $\left[\sqrt{2}-1, 2-\sqrt{2}\right]$. This implies that $T_{\sqrt{2}}^2$ has the same dynamics as T_2 on this subinterval. For example, it must have a 3-cycle, say $\{c_1, c_2, c_3\}$, where the c_i's are distinct and $T_{\sqrt{2}}^6(c_1) = c_1$. It follows that c_1 is a point of period 6 for $T_{\sqrt{2}}$, and in this way we deduce that $T_{\sqrt{2}}$ has $2k$-cycles for each $k \in \mathbb{Z}^+$. We saw earlier that $\mu = (1+\sqrt{5})/2$ is where period 3 is born for the tent family. In particular $T_{\sqrt{2}}$ has no 3-cycle, but if $\alpha = (1+\sqrt{5})/2$ and using the fact that $T_{\sqrt{\alpha}}^2$ (suitably restricted) is conjugate to T_α, it follows that $T_{\sqrt{\alpha}}$ must have points of period 6.

Remark 6.4.6 1. Suppose that $\mu > 1$ and $\dfrac{\mu^2}{1+\mu^3} \leq \dfrac{1}{2}$, then $\dfrac{\mu^3}{1+\mu^3} = 1 - \dfrac{1}{1+\mu^3} \geq \dfrac{1}{2}$, so that

$$T_\mu\left(\frac{\mu}{1+\mu^3}\right) = \frac{\mu^2}{1+\mu^3},\quad T_\mu\left(\frac{\mu^2}{1+\mu^3}\right) = \frac{\mu^3}{1+\mu^3} \quad \text{and} \quad T_\mu\left(\frac{\mu^3}{1+\mu^3}\right) = \frac{\mu}{1+\mu^3}.$$

We see that we have a 3-cycle:

$$\left\{\frac{\mu}{1+\mu^3}, \frac{\mu^2}{1+\mu^3}, \frac{\mu^3}{1+\mu^3}\right\}.$$

This 3-cycle appears when $\mu > 1$ and $\dfrac{\mu^2}{1+\mu^3} \leq \dfrac{1}{2}$, or equivalently

$$\mu^3 - 2\mu^2 + 1 \geq 0, \quad \text{or} \quad (\mu - 1)(\mu^2 - \mu - 1) \geq 0.$$

We see that this will happen when $\mu \geq (1 + \sqrt{5})/2$. A similar analysis can be done for other periodic orbits. For example, if $\mu > 1$ and $\dfrac{\mu^3}{1+\mu^4} \leq \dfrac{1}{2}$ we get a 4-cycle, and this happens when $\mu^3 - \mu^2 - \mu - 1 \geq 0$.

2. Suppose that $1 < \mu < 2$, then if $x \in [\mu - \mu^2/2, \mu/2]$, we can check that $T_\mu(x) \in [\mu - \mu^2/2, \mu/2]$, so that this interval is an invariant set. For $1 < \mu < \sqrt{2}$, the smallest set invariant under T_μ is a collection of subintervals of $[\mu - \mu^2/2, \mu/2]$. If $\mu > \sqrt{2}$ this becomes all of the interval $[\mu - \mu^2/2, \mu/2]$, called the *Julia set* of T_μ (named after one of the early pioneers of chaotic dynamics, Gaston Julia, who worked on complex dynamics in particular in the early 1900's). For $\mu = 2$, the Julia set is all of $[0, 1]$. The bifurcation diagram for T_μ, $\mu > 1$ gives us some insight into what is happening here.

3. The conjugacy between T_2 and L_4 can be constructed by consideration of the periodic points of these maps. Since the period points are dense for each of these maps, by carefully ordering them according to their ordering in $[0, 1]$, we can define a map h by defining it on the periodic points. h is then defined on a dense subset of $[0, 1]$, into a dense subset. This map can be continuously extended to a homeomorphism of $[0, 1]$ with $h(0) = 0, h(1) = 1$. In this way it can be shown that the conjugation between T_2 and L_4 is unique. See the exercises for a proof that the conjugation between T_2 and L_4 is unique.

6.5 Renormalization

The previous example shows that T_μ^2 restricted to the interval

$$\left[\frac{\mu-1}{\mu^2-1}, \frac{\mu^2-\mu}{\mu^2-1}\right] = \left[\frac{1}{\mu+1}, \frac{\mu}{\mu+1}\right],$$

is conjugate to T_{μ^2} on $[0,1]$ (where we are replacing μ by μ^2 in 6.4.2). How do we arrive at this conjugacy? Notice that for $\mu > 1$, T_μ has a fixed point $p_\mu = \mu/(\mu+1)$ and another point $\hat{p}_\mu = 1/(\mu+1)$ with $T_\mu(\hat{p}_\mu) = T_\mu(p_\mu)$, so it is eventually fixed. Let us look at the graph of T_μ^2 restricted to the interval $[\hat{p}_\mu, p_\mu]$. Inside the square shown we see that the graph obtained looks like an "upside-down" version of T_μ, and we consider the possibility that T_μ^2 restricted to the interval $[\hat{p}_\mu, p_\mu]$ is actually conjugate to T_μ (or in fact T_{μ^2}).

Define a linear map $h_\mu : [\hat{p}_\mu, p_\mu] \to [0,1]$ of the form $h_\mu(x) = ax + b$ in such a way that $h_\mu(p_\mu) = 0$ and $h_\mu(\hat{p}_\mu) = 1$. We can check that

$$h_\mu(x) = \frac{1}{\hat{p}_\mu - p_\mu}(x - p_\mu), \quad \text{and} \quad h_\mu^{-1}(x) = (\hat{p}_\mu - p_\mu)x + p_\mu.$$

h_μ expands the interval $[\hat{p}_\mu, p_\mu]$ onto the interval $[0,1]$ and changes the orientation. This is exactly the conjugacy defined in Proposition 6.4.2.

We define a *renormalization operator* of T_μ by

$$(RT_\mu)(x) = h_\mu \circ T_\mu^2 \circ h_\mu^{-1}(x).$$

What we actually showed in the previous section is that $(RT_\mu)(x) = T_{\mu^2}(x)$, giving us the conjugacy claimed. This procedure can be continued for T_μ^4, T_μ^8 etc, and similar considerations can be made with the logistic map L_μ (see [20] for more details).

6.6 Conjugacy and Fundamental Domains

We have seen that two dynamical systems f and g with different dynamical properties cannot be conjugate. On the other hand, sometimes we have dynamical systems having seemingly very similar dynamical properties and which we would like to show are conjugate. This is sometimes possible using the notion of *fundamental domain*, a set on which we construct a map h in an arbitrary manner and show that it extends to a conjugacy on the whole space. We first illustrate this idea with homeomorphisms $f, g : \mathbb{R} \to \mathbb{R}$. We look at a fairly straightforward case where both homeomorphisms are order preserving and have no fixed points (in fact lie strictly above the line $y = x$).

Proposition 6.6.1 *Let $f, g : \mathbb{R} \to \mathbb{R}$ be homeomorphisms satisfying $f(x) > x$ and $g(x) > x$ for all $x \in \mathbb{R}$. Then f and g are conjugate.*

Proof. The idea for the proof is a follows: Select $x_0 \in \mathbb{R}$ arbitrarily and consider the *2-sided orbit*

$$\mathrm{O}_f(x_0) = \{f^n(x_0) : n \in \mathbb{Z}\} = \{\dots, x_{-1}, x_0, x_1, x_2, \dots\}.$$

Since $f(x) > x$ for all x, this is an increasing sequence: $\dots x_{-1} < x_0 < x_1 < x_2 < \dots$, so that the sets

$$\dots, [x_{-1}, x_0), [x_0, x_1), [x_1, x_2), \dots,$$

are disjoint and their union is all of $\mathbb{R}$. We must have $\lim_{n\to\infty} x_n = \infty$ since otherwise the limit would exist and would have to be a fixed point. There are no fixed points since $f(x) > x$ always.

The set $I = [x_0, f(x_0)) = [x_0, x_1)$ is called a *fundamental domain* for f. Set $J = [x_0, g(x_0))$ and define a map $h : I \to J$ arbitrarily as a continuous bijection (e.g., we can set $h(x_0) = x_0$ and $h(f(x_0)) = g(x_0)$ and then linearly from I to J).

Now every other orbit of f intertwines with $\mathrm{O}_f(x_0)$: if $y_0 \in (x_0, x_1)$, then $y_n = f^n(y_0) \in f^n(I)$, so lies between x_i and x_{i+1}. It follows that every orbit has a unique member in the interval $[x_0, x_1)$ and we use this to extend the definition of h to all of $\mathbb{R}$.

If $x \in f^n(I)$ we define $h(x)$ by mapping x back to I via f^{-n}, then using $h(f^{-n}(x))$ which is well defined, and then mapping back to $g^n(J)$ using g^n. I.e., if $x \in f^n(I)$, $n \in \mathbb{Z}$, define

$$h(x) = g^n \circ h \circ f^{-n}(x).$$

In this way, h is defined on all of $\mathbb{R}$. We can check that h is one-to-one. It is onto because $h(f^n(I)) = g^n(J))$ for each n, and we can check that it is continuous. Finally, because of the definition of h, if $x \in \mathbb{R}$, then $x \in f^n(I)$ for some $n \in \mathbb{Z}$, so $x = f^n(y)$ for some $y \in I$. Then

$$g \circ h(x) = g(g^n \circ h \circ f^{-n}(x)) = g^{n+1} \circ h \circ f^{-(n+1)}(f(x)) = h \circ f(x),$$

so that f and g are conjugate. $\square$

Examples 6.6.2 1. The above argument can be generalized to the situation where $f(x)$ and $g(x)$ are homeomorphisms with corresponding fixed points. To be more precise, consider a homeomorphism $f : [0, 1] \to [0, 1]$ which is orientation preserving, so that $f(0) = 0$, $f(1) = 1$ and f is increasing. Suppose that f has fixed points (in addition to 0 and 1), at $c_1, c_2, \dots, c_n$, then f^2 has the same collection of fixed points (no additional fixed points as f cannot have points of period 2 or higher). If $f(x) > x$ for $c_k < x < c_{k+1}$, then we can use the argument of the proposition to construct a homeomorphism between f and f^2, and do the same for each interval

$[c_i, c_{i+1}]$ (treating the case where $f(x) < x$ in an analogous way). In this way we see that f and f^2 are conjugate maps.

2. Consider the logistic maps $L_\mu(x) = \mu x(1-x)$ for various values of $\mu \in (0, 4]$ and $x \in [0, 1]$. We first show that for $0 < \mu < \lambda \leq 1$, L_μ and L_λ are conjugate. There is a slight complication here as these maps are not increasing, but they do have an attracting fixed point at 0, and we saw earlier that the basin of attraction is all of $[0, 1]$. We first deal with the interval on which the maps are increasing, $[0, 1/2]$, and look at the restriction of the functions to this interval.

Our aim is to construct a homeomorphism $h : [0, 1] \to [0, 1]$ with the property $L_\lambda \circ h = h \circ L_\mu$. Take $(L_\mu(1/2), 1/2] = (\mu/4, 1/2]$ as a fundamental domain for L_μ and $(L_\lambda(1/2), 1/2] = (\lambda/4, 1/2]$ as a fundamental domain for L_λ. Define $h : (\mu/4, 1/2] \to (\lambda/4, 1/2]$ by $h(1/2) = 1/2$ and $h(\mu/4) = \lambda/4$ and then linearly on the remainder of the interval.

Set $I = (\mu/4, 1/2]$ and $J = (\lambda/4, 1/2]$, then since 0 is an attracting fixed point, the intervals $L_\mu^n(I)$ and $L_\lambda^n(J)$ are disjoint for $n \in \mathbb{Z}^+$, and their union is all of $(0, 1/2]$. Extend the definition of h so that it is defined on $(0, 1/2]$ by;

$$h(x) = L_\lambda^n \circ h \circ L_\mu^{-n}(x), \quad \text{for} \quad x \in L_\mu^n(I).$$

We can now check that h is continuous and increasing on $[0, 1/2]$ when we set $h(0) = 0$.

Now define h on $(1/2, 1]$ by setting $h(1-x) = 1 - h(x)$ for $x \in [0, 1/2)$, clearly giving a homeomorphism on $[0, 1]$. Then

$$L_\lambda(h(1-x)) = L_\lambda(1 - h(x)) = L_\lambda(h(x)) = h(L_\mu(x)) = h(L_\mu(1-x)),$$

so that h is the required conjugation. $\square$

3. A similar proof shows that L_μ and L_λ are conjugate whenever $1 < \mu < \lambda < 2$. Look at the intervals $[0, 1-1/\mu]$ and $[1-1/\mu, 1/2]$ separately and the fact that $1-1/\mu$ is an attracting fixed point, then use the symmetry about the point $x = 1/2$.

However, these maps cannot be conjugate to L_2 since any conjugating map $h : [0, 1] \to [0, 1]$ must have the property that $h(1/2) = 1/2$ (see the exercises). This leads to a contradiction.

4. The maps L_4 and L_μ, $\mu \in (0, 4)$ cannot be conjugate since $L_4 : [0, 1] \to [0, 1]$ is an onto map, but L_μ is not (see the exercises).

Exercises 6.6

1. (a) Let $a, b \in (0,1)$ and $f_a(x) = ax, f_b(x) = bx$ be dynjamical systems on $[0,1]$. We saw in Exercises 6.3 that these maps need not be linearly conjugate. Prove that f_a and f_b are conjugate (Hint: Use the method of examples in this section).

(b) Let $g : [0,1] \to [0,1]$ be continuous, strictly increasing with $g(0) = 0$ and $g(x) < x$ for all $x \in (0,1]$. Prove that g is conjugate to f_a for any $a \in (0,1)$.

2. Consider the tent map $T_{\sqrt{2}}$.

(i) Show that $x = 1/2$ is an eventual fixed point for $T_{\sqrt{2}}$.

(ii) Use Section 6.4 to show that there is a subinterval of $[0,1]$ on which $T_{\sqrt{2}}$ is conjugate to T_2.

(iii) Deduce that $T_{\sqrt{2}}$ has periodic points of period $2k$ for any $k > 1$, but no points of odd period greater than 1.

(iv) Prove that if $\mu > \sqrt{2}$, then T_μ has a 3-cycle.

(v) Prove that there is an interval on which $T_{\sqrt{2}}$ is chaotic.

3. Let $0 < \lambda, \mu < 1$. If $h : [0,1] \to [0,1]$ is an orientation preserving homeomorphism with $h \circ L_\mu(x) = L_\lambda \circ h(x)$ for all $x \in [0,1]$, show that $h(1/2) = 1/2$ (Hint: h is a conjugation between two different logistic maps with $h \circ L_\mu(x) = L_\lambda \circ h(x)$. Note that this equation also holds if we replace x by $1 - x$. Use this to deduce that $h(x) + h(1-x) = 1$ for all $x \in [0,1]$).

4. Use exercise 2 above to deduce that $h(\mu/4) = \lambda/4$. It follows that the orientation preserving homeomorphism of Example 6.6.2 can be extended in only one way from $[\mu/4, 1/2]$ to $[\lambda/4, 1/2]$.

5*. Prove that for the logistic map L_μ, if $0 < \mu \leq 2$, then L_μ is conjugate to L_μ^2, the composition of L_μ with itself. Show that this is not true for $\mu > 2$. What is the corresponding result for the tent family?

6. Let $S_\mu(x) = \mu \sin(x)$. Prove that if $0 < \mu < \lambda < 1$, then S_μ and S_λ are conjugate.

7. Prove that the rotation $R_a : S^1 \to S^1$, $R_a(z) = az$ is conjugate to the map $T_\alpha : [0,1) \to [0,1)$, $T_\alpha(x) = x + \alpha \pmod 1$, when $a = e^{2\pi i\alpha}$. Can R_a be conjugate to R_b for $a \neq b$?

8. Prove that $T_\alpha : [0,1) \to [0,1)$, $T_\alpha(x) = x + \alpha \pmod 1$ is conjugate to its inverse map $T_\alpha^{-1}(x) = x - \alpha \pmod 1$. Can T_α be conjugate to T_α^2?

9. The aim of this exercise is to show the uniqueness of the conjugacy between the tent map T_2 and the logistic map L_4.

(i) Check that this conjugacy $k : [0,1] \to [0,1]$ is given by,

$$k(x) = \frac{2}{\pi}\arcsin(\sqrt{x}); \quad T_2 \circ k = k \circ L_4.$$

(ii) Suppose that $h : [0,1] \to [0,1]$ is another conjugacy between T_2 and L_4, then $h(0) = 0$, $h(1) = 1$ and h is a strictly increasing continuous function (why?), i.e., h is an orientation preserving homeomorphism of $[0,1]$.

(iii) Show that h maps the local maxima (respectively minima) of T_2^n to local maxima (respectively minima) of L_4^n.

(iv) Use the fact that any such conjugation is order preserving to show that $h(x) = k(x)$ at all local maxima and local minima.

(v) Use the continuity of h and k to deduce that $h(x) = k(x)$ for all $x \in [0,1]$.

(vi) Deduce that there is no C^1 conjugacy between T_2 and L_4.

10. Use the above exercise to show that if $L_4(x) = 4x(1-x)$, then L_4^n has turning points at $\sin^2(k\pi/2^{n+1})$, for $k = 1, 2, \ldots, 2^{n+1} - 1$.

11. Use the fact that the conjugation between T_2 and L_4 is unique to show that if $\phi : [0,1] \to [0,1]$ is a homeomorphism satisfying $L_4 \circ \phi = \phi \circ L_4$, then $\phi(x) = x$ for all $x \in [0,1]$, i.e., ϕ is the identity map (hint: first show that $k \circ \phi$ is also a conjugation between T_2 and L_4).

12^*. Let T_μ be the tent map. Show that if $\mu > \sqrt{2}$, then for each open interval $U \subset [0,1]$, there exists $n > 0$ such that

$$\left[T_\mu^2(1/2), T_\mu(1/2)\right] \subseteq T_\mu^n(U).$$

(Hint: Use the fact that $|T_\mu(U)| \geq \mu|U|$ if U does not contain $1/2$, so that the length keeps increasing. We claim there exists $m > 0$ such that $T^m(U)$ and $T^{m+1}(U)$ both contain $1/2$, for if not, $|T^{m+2}(U)| \geq \mu^2|U|/2$ for all $m \in \mathbb{Z}^+$, a contradiction, since this eventually exceeds 1).

Chapter 7. Singer's Theorem

We shall now show that the logistic map $L_\mu : [0,1] \to [0,1]$, $L_\mu(x) = \mu x(1-x)$ with $0 < \mu < 4$ has at most one attracting cycle. We use a result due to Singer (1978) which is applicable to maps having a negative Schwarzian derivative. Recall that a map $f : \mathbb{R} \to \mathbb{R}$ is C^3 if $f'''(x)$ exists and is continuous.

7.1 The Schwarzian Derivative Revisited

Recall the Schwarzian derivative of $f(x)$ is:

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x))}\right]^2 = F'(x) - \frac{1}{2}\left[F(x)\right]^2,$$

where $F(x) = \dfrac{f''(x)}{f'(x)}$.

Our first goal is to show that many polynomials have negative Schwarzian derivatives.

Lemma 7.1.1 *Let f(x) be a polynomial of degree n for which all the roots of its derivative $f'(x)$ are distinct and real. Then $Sf(x) < 0$ for all x.*

Proof. Suppose that the derivative of $f(x)$ is given by

$$f'(x) = a(x - r_1)(x - r_2)\cdots(x - r_{n-1}),$$

where $a \in \mathbb{R}$, then

$$F(x) = \frac{f''(x)}{f'(x)} = (\ln f'(x))' = \sum_{i=1}^{n-1} \frac{1}{x - r_i},$$

and so

$$F'(x) = -\sum_{i=1}^{n-1} \frac{1}{(x - r_i)^2}.$$

Now put this into the Schwarzian derivative:

$$Sf(x) = F'(x) - \frac{1}{2}[F(x)]^2 < 0.$$

$\square$

Lemma 7.1.2 *Assume that f is a C^3 map on $\mathbb{R}$, then*

(i) $S(f \circ g)(x) = Sf(g(x)) \cdot (g'(x))^2 + Sg(x)$.

(ii) *If $Sf < 0$ and $Sg < 0$, then $S(f \circ g) < 0$.*

(iii) *If* $Sf < 0$, *then* $Sf^k < 0$ *for all* $k \in \mathbb{Z}^+$.

Proof. (i) As above we have $F(x) = \dfrac{f''(x)}{f'(x)}$, so set $G(x) = \dfrac{g''(x)}{g'(x)}$ and $H(x) = \dfrac{h''(x)}{h'(x)}$, where $h = f \circ g$. Then

$$h'(x) = f'(g(x)) \cdot g'(x), \quad h''(x) = (f''(g(x))g'(x)^2 + (f'(g(x))g''(x),$$

so that

$$H(x) = \frac{f''(g(x))g'(x)^2 + f'(g(x))g''(x))}{f'(g(x))g'(x)}$$

$$= \frac{f''(g(x))g'(x)}{f'(g(x))} + \frac{g''(x)}{g'(x)} = F(g(x))g'(x) + G(x).$$

This gives

$$H'(x) = F'(g(x))g'(x)^2 + F(g(x))g''(x) + G'(x),$$

so that

$$S(f \circ g)(x) = H'(x) - \frac{1}{2}H^2(x)$$

$$= \left[F'(g(x)) - \frac{1}{2}F^2(g(x))\right] g'(x)^2 + F(g(x))g''(x) - F(g(x))g'(x)G(x) + G'(x) - \frac{1}{2}G^2(x)$$

$$= Sf(g(x)) \cdot g'(x)^2 + Sg(x),$$

since $G(x) = g''(x)/g'(x)$.

(ii) is now immediate, and (iii) follows using induction. □

Example 7.1.3 Let $g(x) = \dfrac{ax+b}{cx+d}$, $a, b, c, d \in \mathbb{R}$, a linear fractional transformation. A direct calculation shows that $Sg(x) = 0$ everywhere in its domain. It now follows from Lemma 1 that if $h(x) = g(f(x)) = \dfrac{af(x)+b}{cf(x)+d}$. then $Sh(x) = Sf(x)$.

We now prove a version of Singer's Theorem. Recall that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $c \in \mathbb{R}$ is an attracting fixed point or attracting periodic point, then the basin of attraction $B_f(c)$ is an open set. Denote by W the maximal open interval contained in $B_f(c)$ which contains c (called the *immediate basin of attraction of* c).

Theorem 7.1.4 *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a* C^3 *map with negative Schwarzian derivative. If* c *is an attracting periodic point for* f, *then either:*

(i) *the immediate basin of attraction of* c *extends to* ∞ *or* $-\infty$, *or*

(ii) *there is a critical point of* f *(i.e., a root of* $f'(x) = 0$*), whose orbit is attracted to the orbit of* c *under* f*.*

Proof. We first look at the case where c is a fixed point of f. Suppose that its immediate basin of attraction is the open interval W and that (i) does not hold, then W is a bounded set.

Thus $W = (a, b)$ for some $a, b \in \mathbb{R}$. Because of the continuity of f and the fact that $f(a), f(b) \notin (a, b)$, there are three possibilities for $f(a)$ and $f(b)$.

Case 1: $f(a) = f(b)$ - this will happen for example if a is a fixed point and b is an eventual fixed point. It follows from the Mean Value Theorem that (a, b) contains a critical point of f.

Case 2: $f(a) = a$ and $f(b) = b$, then by the Mean Value Theorem, there are points $x_1 \in (a, c)$ and $x_2 \in (c, b)$ such that $f'(x_1) = f'(x_2) = 1$ (see picture). But since c is an attracting fixed point, $|f'(c)| \leq 1$. It follows that either $f'(x_0) = 0$ for some $x_0 \in (x_1, x_2)$, or $f'(x)$ has a minimum value $f'(x_0) > 0$. In the latter case we have $f'(x_0) > 0$, $f''(x_0) = 0$ and $f'''(x_0) > 0$, so that $Sf(x_0) > 0$, contradicting the Schwarzian derivative being everywhere negative.

Case 3: $f(a) = b$ and $f(b) = a$. Here c is fixed by f^2 and so are a and b, so that $(f^2)'$ has a zero x_0 in (a, b) (as in Case 1). But

$$(f^2)'(x_0) = (f'(f(x_0))f'(x_0) = 0,$$

so either x_0 or $f(x_0)$ is a root of f', but both lie in (a, b).

Now suppose that c is a point of period k, then $f^k(c) = c$, an attracting fixed point for f^k, then from our earlier arguments, the immediate basin of attraction of c (for f^k) contains a critical point of f^k, say x_0:

$$(f^k)'(x_0) = f'((x_0)f'(f(x_0)) \cdots f'(f^{k-1}(x_0)) = 0,$$

so that $f'(f^m(x_0)) = 0$ for some $0 \leq m < k$. In this case $f^m(x_0) \in f^m(W) \subset W$, the basin of attraction of c. $\square$

Example 7.1.5 Consider the map $f(x) = x - x^5$. We see that $f'(x)$ has two real roots: $\pm(1/5)^{1/4}$, and $f''(x)$ and $f'''(x)$ are both continuous. We have $f''(x) = -20x^3$, and $f'''(x) = -60x^2$.

Substituting these into the Schwarzian derivative gives:

$$Sf(x) = (-60x^2)/(1-5x^4) - 3/2[(-20x^3)/(1-5x^4)]^2 = \frac{-60x^2}{(1-5x^4)^2}(1+5x^4),$$

and this is always negative. We can check that the critical points are in the basin of attraction of the fixed point $x = 0$.

Example 7.1.6 The map $f(x) = 3x/4 + x^3$ cannot have a negative Schwarzian derivative. It has fixed points $x = 0$ and $x = \pm 1/2$, 0 being attracting with bounded basin of attraction, but f has no critical points.

7.2 Singer's Theorem

Corollary 7.2.1 *Let $f : [0, 1] \to [0, 1]$ be a C^3 map with $Sf(x) < 0$ for all x. The basin of attraction of an attracting cycle contains $0, 1$ or a critical point of $f(x)$.*

Proof. If $J = (a, b)$, $0 < a < b < 1$, is the basin of attraction of an attracting cycle, then we have seen above that it must contain a critical point of f. Any other attracting cycles will be of the form $[0, a)$ or $(b, 1]$, so will contain 0 or 1. $\square$

Example 7.2.2 We now see that the logistic map $L_\mu(x) = \mu x(1 - x)$, $0 < \mu < 4$, $x \in [0, 1]$, has at most one attracting periodic cycle. If $0 < \mu \leq 1$, 0 is the only attracting fixed point, having basin of attractions $[0, 1]$. For $1 < \mu < 4$, L_μ has exactly one critical point $x_0 = 1/2$.

Since $L'_\mu(0) = \mu > 1$, the fixed point 0 is unstable; therefore $[0, a)$ cannot be a basin of attraction. Furthermore, $L_\mu(1) = 0$ and hence $(b, 1]$ is not a basin of attraction either. Since $SL_\mu(x) < 0$ everywhere (at $x = 1/2$, $\lim_{x\to 1/2} L_\mu(x) = -\infty$), we conclude that there is at most one attracting periodic cycle in $(0, 1)$ and the result follows.

If we look at the bifurcation diagram of the logistic map, it follows, for example that where we see six horizontal lines, we have an attracting 6-cycle, and not two attracting 3-cycles.

Remark 7.2.3 In a similar way we get Singers Theorem [59], proved by David Singer in 1978. This theorem is actually a real version of a theorem about complex polynomials proved by the French mathematician Gaston Julia in 1918 [32]:

Suppose that f is a C^3 map on a closed interval I such that $Sf(x) < 0$, for all $x \in I$. If f has n critical points in I, then f has at most $n + 2$ attracting cycles.

Example 7.2.4 Let $G(x) = \lambda \arctan(x)$, $\lambda \neq 0$. Then $G'(x) = \lambda/(1 + x^2)$. Clearly $G(x)$ has no critical points. Now, if $|\lambda| < 1$, then $c = 0$ is an asymptotically stable fixed point with basin of attraction $(-\infty, \infty)$. If $\lambda > 1$, then G has two attracting fixed points x_1 and x_2 with basins of attraction $(-\infty, 0)$ and $(0, \infty)$, respectively. Finally, if $\lambda < -1$, then G has an attracting 2-cycle $\{\bar{x}_1, \bar{x}_2\}$ with basin of attraction $(-\infty, 0) \cup (0, \infty)$.

Exercises 7.2

1. If $G(x) = \lambda \arctan(x)$, $(\lambda \neq 0)$, show that the Schwarzian derivative is

$$SG(x) = \frac{-2}{(1 + x^2)^2}.$$

Use this to verify the conclusion from Example 7.2.4.

2. If $f(x) = \dfrac{ax + b}{cx + d}$, we have seen that the Schwarzian derivative satisfies $Sf(x) = 0$. Now suppose that f is a function for which $Sf(x) = 0$. Show

(i) $(f''(x))^2/(f'(x))^3 = \text{constant}$,

(ii) $f(x)$ is of the form $f(x) = \dfrac{ax + b}{cx + d}$ (Hint: Set $y = f'$ to obtain a separable differential equation of the form $y' = cy^{3/2}$).

Chapter 8. Fractals

Many objects in nature have been modeled mathematically by approximating the curves or surfaces involved by 'smooth' curves or surfaces, in order that the calculus be applied to their study. However, in recent years it has been realized that for certain of these objects, the calculus is not the best tool for their study.

For example, the motion of a particle suspended in a fluid (Brownian motion), the length of the coastline of an island or the surface area of the human lung. The length of a coast line is dependent on how carefully it is measured; for such an object the more closely you look at it, the more irregular it appears and the greater the length appears to be. The study of such objects has resulted in a new area of mathematics called Fractal Geometry. Fractal geometry was popularized by the mathematician Benoit Mandelbrot and it was him who coined the term *fractal* in 1977 ([42]). Mandelbrot was originally from Poland, educated in France and later moved to the United States is particularly famous for the so called "Mandelbrot set".

Much of the current interest in fractals is a consequence of Mandelbrot's work. His computer simulations of maps of the complex plane have resulted in extremely complicated and beautiful fractals. The mathematical work was initiated by Cayley, Fatou and Julia in the late 19th and early 20th centuries, but progress ceased until the development of the electronic computer. Later we shall see how fractals arise from the study of complex analytic maps.

8.1 Examples of Fractals

Probably the first published example of a fractal was given by Karl Weierstrass in 1872. He constructed an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous everywhere, but nowhere differentiable. Until this time mathematicians believed that continuous functions had to be differentiable at "most" points, and this was often tacitly assumed. We saw earlier how to construct the Cantor set (published in 1883), probably the simplest example of a fractal.

Another early example is due to Helge von Koch (1904), who constructed the famous "Koch Snowflake". Starting with an equilateral triangle of side length 1 unit, three equilateral triangle are constructed (one on each side as shown), each having side length 1/3. This construction is then continued so that at each stage, an equilateral triangle of side length 1/3 that of the previous ones is added to each exposed line. The Koch Snowflake itself is a limiting curve in a sense to be described shortly.

The Koch Snowflake curve is constructed inductively as follows:

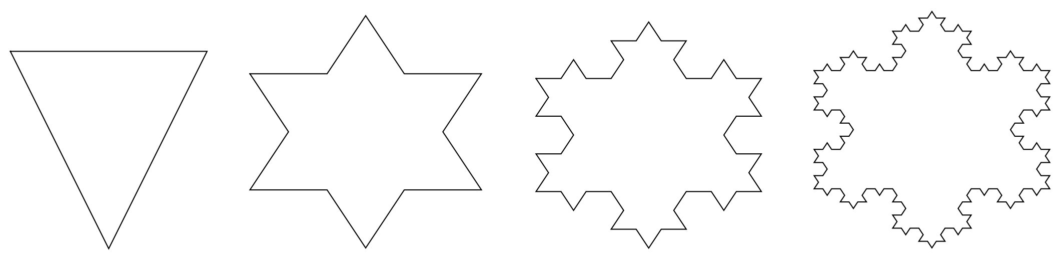
(i) Start with an equilateral triangle with side length = 1, the perimeter of the resulting curve is $L(0) = 3$.

(ii) Now add equilateral triangles of side length 1/3. The perimeter is now $L(1) = 3 \times 4 \times 1/3 = 4$.

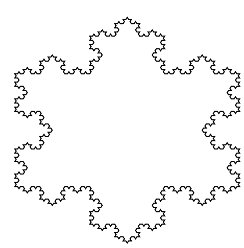
(iii) Add new equilateral triangles as seen below by taking $n = 2$. The perimeter is now $L(2) = 3 \times 4^2 \times (1/3^2) = 4^2/3$.

(iv) Continue in this way so that at the nth stage the perimeter is $L(n) = 4^n/3^{n-1}$.

Take in turn $n = 3, 4, \ldots$ to get a good idea of what the limiting curve, (the Koch snowflake) looks like. As $n \to \infty$, $L(n) \to \infty$, so that the Koch Snowflake is a curve having infinite length. It can be seen to be continuous, and it can be shown to be nowhere differentiable. The area enclosed by the curve is easily found and is clearly finite.



The first 4 iterations of the Koch Snowflake.



The Koch Snowflake

The Koch snowflake is an example of a fractal in the sense that it has the self-similarity property typical of fractals - as we zoom in on any part of the curve (no matter how far), it resembles (gives an exact copy) of what we saw previously. It is a fractal curve in $\mathbb{R}^2$, which typically are continuous curves, differentiable nowhere and consequently have infinite length.

8.2 An Intuitive Introduction to the Idea of Fractal Dimension

The idea of (topological) dimension is intuitively clear. For example, a point or a number of points has 0-dimensions, a curve is 1-dimensional, a surface 2-dimensional, etc. Thus the topological dimension of the snowflake curve is 1. More precisely:

Definition 8.2.1 A set $K \subseteq \mathbb{R}^n$ has *topological dimension* 0 if for every point $x \in K$ there is an open ball $B_\delta(x)$ in $\mathbb{R}^n$ having arbitrarily small radius, whose boundary does not intersect K.

K has topological dimension $k \in \mathbb{Z}^+$ if every point $x \in K$ is surrounded by an open ball $B_\delta(x)$ having arbitrarily small radius whose boundary intersects K in a set of topological dimension $k-1$, and k is the least positive integer with this property.

Any discrete set such as the Cantor set or the set of rationals will have topological dimension 0. The topological dimension of a line, a circle or the Koch curve is 1. A filled in circle or square will have topological dimension 2 and for a solid cube or sphere it will be 3.

It turns out that for sets such as the snowflake curve and the Cantor set, there is another very useful idea of dimension, originally called *Hausdorff-Besicovitch dimension*, now often called *fractal dimension* (following B. Mandelbrot). We motivate its definition as follows:

(i) Given a piece of string, two copies of it result in a string "twice the size".

(ii) For a square we need 4 copies.

(iii) For a cube we need 8 copies.

(iv) For a 4-dimensional cube we need 16 copies. i.e. to double the (side length) of a d-dimensional cube we need $c = 2^d$ copies, so that $d = (\log c)/(\log 2)$,

Thus for example, if we have an object whose size doubles if three copies are stuck together, then it would have fractal dimension $\log 3/\log 2$.

Returning to the snowflake curve, notice that (one side of it) is made up of 4 copies of itself; each 1/3 of the size, so $a = 3$, $c = 4$ and

$$d = \frac{\log c}{\log a} = \frac{\log 4}{\log 3} = 1.2616\ldots.$$

Definition 8.2.2 This number d is defined to be the *fractal dimension* of the curve.

The Cantor set is seen to be made up of two copies of itself, each reduced in size by a factor of three, so that $c = 2$ and $a = 3$ and $d = \log 2/\log 3$, again a fractal since the topological dimension of C is zero.

Roughly speaking a fractal is defined to be a geometrical object whose fractal dimension is strictly greater than its (topological) dimension (this was the original definition due to Mandelbrot). The coastline of Britain has $d = 1.25$ (approximately), so it is very rough. That of the U.S. is closer to 1 (fairly smooth).

Generally, fractals have the property of being self-similar, i.e. they do not change their appearance significantly when viewed under a microscope of arbitrary magnifying power. This self similarity can take a linear form as in the snow flake curve, or a non-linear form as in the Mandelbrot set (see later).

8.3 Box Counting Dimension

We try to make the discussion of the last section more precise. A *box* in $\mathbb{R}^n$ is a set of the form

$$[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] = \{(x_1, x_2, \ldots, x_n) : a_i \leq x_i \leq b_i, \quad i = 1, 2, \ldots, n\}.$$

Let $K \subseteq \mathbb{R}^n$ and set $N_\delta(K)$ = the smallest number of boxes of equal side length δ needed to cover the set K. As $\delta \to 0$, we need more such boxes, so that $N_\delta(K)$ will increase. The idea is that as δ decreases, the number of sets required to cover K increases as some power of the length of the boxes:

$$N_\delta \sim \delta^{-d}.$$

This power d is the *box counting dimension*, which is defined more formally as follows:

Definition 8.3.1 The *box counting dimension* of $K \subseteq \mathbb{R}^n$ is given by

$$\dim(K) = \lim_{\delta \to 0} \frac{\log N_\delta(K)}{-\log \delta},$$

if this limit exists (it is independent of the base of the logarithm).

Examples 8.3.2 1. Let $K = [0, 1]$, the unit interval in $\mathbb{R}$. We can cover K with one interval of length $\delta = 1$, two intervals of length $\delta = 1/2$. In general we need 2^n intervals of length $\delta = 2^{-n}$ so that

$$\dim([0, 1]) = \lim_{n \to \infty} \frac{\log 2^n}{-\log 2^{-n}} = 1,$$

which is what we would expect (strictly speaking we have only shown that $\dim([0, 1]) \leq 1$ since we have not shown that the covering gives the minimum N).

It can be shown that if $K \subseteq \mathbb{R}^n$ contains an open subset of $\mathbb{R}^n$, then $\dim(K) = n$ (what is called the *topological dimension*).

2. The Cantor set C has $N_1(C) = 1$ (using the single interval $[0,1]$ to cover C). Dividing $[0,1]$ into three equal subintervals, we can cover C with two intervals of length 1/3.

Repeating this process, we can cover C with 2^n subinterval of length 3^{-n}, so that $N_{3^{-n}}(C) = 2^n$ and

$$\dim(C) = \lim_{n\to\infty} \frac{\log 2^n}{-\log 3^{-n}} = \frac{\log 2}{\log 3}.$$

3. The Menger sponge M can be shown to have box counting dimension equal to $\dim(M) = \lim_{n\to\infty} \frac{\log 20^n}{\log 3^{-n}} = \frac{\log 20}{\log 3} \sim 2.73$, since when $\delta = 1$, $N_1(M) = 1$, when $\delta = 1/3$, $N_\delta(M) = 20$ and continuing in this way gives the result.

8.4. The Mathematical Theory of Fractals

In the remainder of this chapter we attempt to give a rigorous definition of the idea of a limiting set. For example, the Koch Snowflake is the limiting curve of a sequence of curves in $\mathbb{R}^2$. We intend to make rigorous this notion. In order to do so, we need to define a metric on "sets of sets", known as the *Hausdorff metric*, and then we can look at the convergence of sequences in the resulting metric spaces.

8.4.1 Contraction Mappings and Complete Metric Spaces

Given a sequence x_n in a metric space X, it is possible that the sequence converges, but not to a point of X. For example, consider the following sequence in $\mathbb{Q}$ (the set of rationals with its usual metric $d(x,y) = |x-y|$ for $x, y \in \mathbb{Q}$):

$x_0 = 1$, $x_1 = 1.1$, $x_2 = 1.14$, $x_3 = 1.141$, and x_n being the member of $\mathbb{Q}$ equal to $1.141\ldots$, with a decimal expansion consisting of 1 followed by the first n terms in the decimal expansion of $\sqrt{2}$, followed by 0's. It is clear that x_n is a sequence in $\mathbb{Q}$, but $\lim_{n\to\infty} x_n = \sqrt{2} \notin \mathbb{Q}$. The reals $\mathbb{R}$ and complex numbers $\mathbb{C}$ do not have this type of difficulty, they are what we call *complete metric spaces*.

Definition 8.4.2 Let $\{x_n\}$ be a sequence in a metric space (X,d), then $\{x_n\}$ is a *Cauchy sequence* if given any $\epsilon > 0$ there exists $N \in \mathbb{Z}^+$ for which

$$m, n > N \Rightarrow d(x_n, x_m) < \epsilon.$$

Definition 8.4.3 A metric space (X, d) is said to be *complete* if every Cauchy sequence $\{x_n\}$ in X converges to a member of X.

Proposition 8.4.4 *In a metric space (X, d), any convergent sequence is a Cauchy sequence.*

Proof. Let $\{x_n\}$ be a convergent sequence, converging to $x \in X$, then given $\epsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that

$$n > N \Rightarrow d(x, x_n) < \epsilon/2.$$

By the triangle inequality, if $m, n > N$, then

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \epsilon/2 + \epsilon/2 = \epsilon,$$

so that $\{x_n\}$ is a Cauchy sequence. $\square$

Example 8.4.5 The sequence $x_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n}$, is not a Cauchy sequence in $\mathbb{R}$.

Proof. A sequence $\{x_n\}$ in a metric space (X, d) fails to be Cauchy if we can find $\epsilon > 0$ such that for all $N \in \mathbb{Z}^+$, we can find x_n, x_m with $n, m \geq N$ and $d(x_n, x_m) \geq \epsilon$. We shall see that it suffices to take $\epsilon = 1/2$. In this case, $d(x_n, x_m) = |x_n - x_m|$.

Set $n = 2N$ and $m = N$, then

$$|x_{2N} - x_N| = \frac{1}{N+1} + \frac{1}{N+2} + \frac{1}{N+3} \cdots + \frac{1}{2N} > \frac{N}{2N} = \frac{1}{2},$$

so that x_n is not a Cauchy sequence. $\square$

The problem is that the converse of Proposition 8.4.4 is not true. It can be seen that the rational numbers do not constitute a complete metric space since for example, the sequence mentioned above, where x_n is the first n terms of the decimal expansion of $\sqrt{2}$, followed by 0's, can be seen to be a Cauchy sequence in $\mathbb{Q}$ (with respect to the usual metric), but does not converge in $\mathbb{Q}$. The reals $\mathbb{R}$, $\mathbb{R}^n$ and the complex numbers $\mathbb{C}$ are examples of complete metric spaces. The completeness of $\mathbb{R}$ is a consequence of the *Completeness Axiom* (See the Appendix), which says that any non-empty, bounded subset of $\mathbb{R}$ has a *least upper bound* (called the *supremum*), and the completeness of $\mathbb{R}^n$ and $\mathbb{C}$ can be deduced from this.

8.5 The Contraction Mapping Theorem and Self-Similar Sets

The main property of complete metric spaces we shall study is concerned with functions $f : X \to X$ having the effect of bringing points closer together.

Definition 8.5.1 A function $f : X \to X$ of the metric space (X, d) is called a *contraction mapping* if there exists a real number α, $0 < \alpha < 1$ satisfying

$$d(f(x), f(y)) \leq \alpha \cdot d(x, y), \quad \text{for all} \quad x, y \in X.$$

The constant α is called the *contraction constant* of f.

We shall see that contraction mappings always have a fixed point. The following theorem is called the Contraction Mapping Theorem, or the Banach Fixed-Point Theorem, after the Polish mathematician Stefan Banach who stated and proved the theorem in 1922. The proof we shall give is due to R. Palais [48].

Theorem 8.5.2 *Let $f : X \to X$ be a contraction mapping on a non-empty complete metric space X, then f has a unique fixed point. In addition, if p is the fixed point, then $p = \lim_{n\to\infty} f^n(x)$ for every $x \in X$.*

Proof. Choose $x_0 \in X$ arbitrarily and set $x_n = f^n(x_0)$, then we shall show that $\{x_n\}$ is a Cauchy sequence and hence is convergent. Suppose that $0 < \alpha < 1$ has the property that $d(f(x), f(y)) \leq \alpha \cdot d(x, y)$ for $x, y \in X$. Then inductively we see that for any $k \in \mathbb{Z}^+$

$$d(f^k(x), f^k(y)) \leq \alpha^k \cdot d(x, y).$$

By the triangle inequality we have for $x, y \in X$:

$$d(x, y) \leq d(x, f(x)) + d(f(x), f(y)) + d(f(y), y)$$

$$\leq d(x, f(x)) + \alpha d(x, y) + d(f(y), y).$$

Solving for $d(x, y)$ gives the "Fundamental Contraction Inequality":

$$d(x, y) \leq \frac{d(f(x), x) + d(f(y), y)}{1 - \alpha}.$$

Note that if x and y are both fixed points, then $d(x, y) = 0$, so any fixed point is unique. Now, in the Fundamental Inequality, replace x and y by $x_n = f^n(x_0)$ and $x_m = f^m(x_0)$, then

$$d(x_n, x_m) \leq \frac{d(f(x_n), x_n) + d(f(x_m), x_m)}{1 - \alpha}$$

$$= \frac{d(f^n(f(x_0)), f^n(x_0)) + d(f^m(f(x_0)), f^m(x_0))}{1 - \alpha}$$

$$\leq \frac{\alpha^n d(f(x_0), x_0) + \alpha^m d(f(x_0), x_0)}{1-\alpha}$$

$$= \frac{\alpha^n + \alpha^m}{1-\alpha} d(f(x_0), x_0),$$

Since α^k becomes arbitrarily small for k large enough, the sequence $\{x_n\}$ is Cauchy and hence $\lim_{n\to\infty} x_n = \lim_{n\to\infty} f^n(x_0)$ exists. We showed earlier that if such a limit exists, it must converge to a fixed point, say p, which must be unique by our earlier remark. □

If we let $m \to \infty$ in the last inequality, we get the rate at which $f^n(x_0)$ converges to p:

Corollary 8.5.3 $d(f^n(x_0), p) \leq \dfrac{\alpha^n}{1-\alpha} d(f(x_0), x_0)$.

Example 8.5.4 Let $f : I \to I$ where $I \subseteq R$ is an interval. If $|f(x) - f(y)| \leq \alpha|x - y|$ for all $x, y \in I$, where $0 < \alpha < 1$, then f is a contraction mapping, and f has a unique fixed point. The basin of attraction of f is all of I. This is no longer true if $\alpha = 1$ (see the exercises).

The next theorem gives us the notion of *self-similar set*. Here we follow Huchinson [31] (see also [34]). We use the notion of *compact set*, which we will not define for general metric spaces, but in the case of our main examples, $X = \mathbb{R}^n$ (usually with $n = 2$), a compact set is one that is closed and bounded. Compactness for metric spaces is treated in Chapter 14.

Theorem 8.5.5 (Huchinson's Theorem) *Let (X, d) be a metric space, and for each $1 \leq i \leq N$, let $f_i : X \to X$ be a contraction. Then there exists a unique non-empty compact subset K of X that satisfies*

$$K = f_1(K) \cup \cdots \cup f_N(K).$$

K is called the self-similar set with respect to $\{f_1, \ldots, f_N\}$.

Before we can prove Theorem 8.5.5 we need some preliminary notions. Define a function on the subsets of X by

$$F(A) = \cup_{i=1}^N f_i(A), \quad \text{for} \quad A \subseteq X,$$

then the idea of the proof is to show the existence of a fixed point of F using the Contraction Mapping Theorem. In order to make this work, we need to turn the set of non-empty subsets of X into a complete metric space. We use the notion of

Hausdorff metric. Set

$$\mathcal{C}(X) = \{A : A \ \text{ is a non-empty compact subset of } X\},$$

so that $F : \mathcal{C}(X) \to \mathcal{C}(X)$. We turn this into a metric space $(\mathcal{C}(X), D)$ as follows:

Suppose $A \in \mathcal{C}(X)$ and $\bar{B}_\delta(x) = \{y \in X : d(x,y) \leq \delta\}$ is the closed ball of radius δ centered on x. Then set

$$U_\delta(A) = \cup_{y\in A}\bar{B}_\delta(y) = \{x \in X : d(x,y) \leq \delta \ \text{ for some } \ y \in A\}.$$

Think of $U_\delta(A)$ as a closed set that contains A whose boundary lies within δ of A.

Definition 8.5.6 The *Hausdorff metric* D is defined on $\mathcal{C}(X)$ for $A, B \in \mathcal{C}(X)$ by

$$D(A,B) = \inf\{\delta > 0 : A \subseteq U_\delta(B) \ \text{ and } \ B \subseteq U_\delta(A)\}.$$

We now show that $D(A,B)$ defines a metric on $\mathcal{C}(X)$, but we omit the proof of completeness:

Theorem 8.5.7 *Let (X,d) be a metric space. Then the Hausdorff metric $D(A,B)$ defines a metric on $\mathcal{C}(X)$. If the metric space (X,d) is complete, then so is the metric space $(\mathcal{C}(X), D)$.*

Proof. From the symmetry of the definition of D we see that $D(A,B) = D(B,A) \geq 0$, and also that $D(A,A) = 0$.

Suppose that $D(A,B) = 0$, then for any $n \in \mathbb{Z}^+$, $A \subseteq U_{1/n}(B)$, so that for any $x \in A$ we can find a sequence $x_n \in B$ with $d(x,x_n) \leq 1/n$. Since B is a closed set (being compact) we have $x \in B$ so that $A \subseteq B$. Similarly $B \subseteq A$, so $A = B$.

To prove the triangle inequality let A, B and C belong to $\mathcal{C}(X)$ and suppose that $D(A,B) < \delta$ and $D(B,C) < \epsilon$, then $U_{\delta+\epsilon}(A) \supseteq C$ and $U_{\delta+\epsilon}(C) \supseteq A$, so that

$$D(A,C) < \delta + \epsilon = D(A,B) + \delta' + D(B,C) + \epsilon',$$

where $\delta' = \delta - D(A,B) > 0$ and $\epsilon' = \epsilon - D(B,C) > 0$ are arbitrary, so that

$$D(A,C) \leq D(A,B) + D(B,C).$$

$\square$

We now show that a contraction on the metric space (X,d) gives rise to a contraction on $(\mathcal{C}(X), D)$.

Proposition 8.5.8 *Let $f : X \to X$ be a contraction mapping on (X,d) with*

$$d(f(x), f(y)) \leq \alpha \cdot d(x,y), \ (0 < \alpha < 1), \quad \textit{for all} \ \ x, y \in X.$$

Then

$$D(f(A), f(B)) \leq \alpha \cdot D(A, B) \quad \textit{for all} \quad A, B \in \mathcal{C}(X).$$

Proof. If $\delta > D(A, B)$ then $U_\delta(A) \supseteq B$, so if $x \in B$ there exists $y \in A$ with $d(x, y) \leq \delta$. Since f is a contraction, this implies $d(f(x), f(y)) \leq \alpha d(x, y) \leq \alpha\delta$. Consequently

$$U_{\alpha\delta}(f(A)) \supseteq f(B).$$

In a similar way $U_{\alpha\delta}(f(B)) \supseteq f(A)$, so that $D(f(A), f(B)) \leq \alpha\delta$ and the result follows. □

Proof of Theorem 8.5.5 We first show that if $A_1, A_2, B_1, B_2 \in \mathcal{C}(X)$, then

$$D(A_1 \cup A_2, B_1 \cup B_2) \leq \max\{D(A_1, B_1), D(A_2, B_2)\}.$$

To see this, suppose that $\delta > \max\{D(A_1, B_1), D(A_2, B_2)\}$, then

$$\delta > D(A_1, B_1) \geq \inf\{\delta' : U_{\delta'}(A_1) \supseteq B_1\},$$

so in particular

$$U_\delta(A_1) \supseteq B_1 \quad \text{and} \quad U_\delta(A_2) \supseteq B_2.$$

This clearly implies $U_\delta(A_1 \cup A_2) \supseteq B_1 \cup B_2$. A similar argument implies that $U_\delta(B_1 \cup B_2) \supseteq A_1 \cup A_2$, so that $\delta > D(A_1 \cup A_2, B_1 \cup B_2)$ and it follows that $D(A_1 \cup A_2, B_1 \cup B_2) \leq \max\{D(A_1, B_1), D(A_2, B_2)\}$.

Now using this result repeatedly we have

$$D(F(A), F(B)) = D(\cup_{j=1}^N f_j(A), \cup_{j=1}^N f_j(B)) \leq \max_{1 \leq j \leq N} D(f_j(A), f_j(B)).$$

By Proposition 8.5.5, $D(f_i(A), f_i(B)) \leq \alpha_i D(A, B)$, where α_i are the contraction constants for the maps f_i. If $\alpha = \max_{1 \leq i \leq N} \alpha_i$, then

$$D(F(A), F(B)) \leq \alpha D(A, B),$$

so that F is a contraction on $(\mathcal{C}(X), D)$, a complete metric space. The result follows by the contraction mapping theorem. □

Examples 8.5.9 Take $X = \mathbb{R}$ and denote by C the Cantor middle-thirds set. If we define $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, by

$$f_1(x) = \frac{x}{3}, \quad f_2(x) = \frac{x}{3} + \frac{2}{3},$$

then clearly these are contractions on $\mathbb{R}$, and then we define F on $\mathcal{C}(\mathbb{R})$ by $F(A) = f_1(A) \cup f_2(A)$ for all compact subsets $A \subset \mathbb{R}$. According to Huchinson's Theorem, if J is any compact subset of $\mathbb{R}$, $\lim_{n\to\infty} F^n(J) = K$ is a unique fixed point of F in $\mathcal{C}(\mathbb{R})$. It is easy to check that C is a fixed point of F, so we must have $K = C$, the Cantor set.

2. Set $X = \mathbb{R}^2$ and let (a_j, b_j), $j = 1, 2, 3$ be the vertices of an equilateral triangle. Define contractions $f_j : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x+a_1}{2} \\ \frac{y+b_1}{2} \end{pmatrix}, \quad f_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x+a_2}{2} \\ \frac{y+b_2}{2} \end{pmatrix}, \quad f_3\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x+a_3}{2} \\ \frac{y+b_3}{2} \end{pmatrix}.$$

Then we shall see that $F = f_1 \cup f_2 \cup f_3$ gives rise to the Sierpinski triangle as its unique fixed point.

Exercises 8.5

1. We have used the Intermediate Value Theorem throughout these notes in the following form as one of our main tools: *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. If $f(a) < 0$ and $f(b) > 0$ then there exists $c \in (a, b)$ with $f(c) = 0$.* Prove the Intermediate Value Theorem using the following steps (our proof depends on the Monotone Convergence Theorem for real sequences - see the Appendix):

(a) Use the Bisection Method to obtain a sequence of nested intervals $[a_n, b_n]$ of length $(b-a)2^{-n}$ where $f(a_n) < 0$ and $f(b_n) \geq 0$. (I.e., subdivide $[a, b]$ and set $a_1 = (a+b)/2$ if $f(a + b)/2) < 0$, otherwise set it equal to b_1, and continue in this way.)

(b) Show that $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist and are equal.

(c) Use the continuity of f to conclude that

$$f\left(\lim_{n\to\infty} a_n\right) \leq 0 \quad \text{and} \quad f\left(\lim_{n\to\infty} b_n\right) \geq 0.$$

(d) Conclude that $c = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ satisfies $f(c) = 0$.

2. Let $f : X \to X$ be a contraction mapping on a metric space (X, d). Prove that f is continuous.

3. Let $f : I \to I$ where I is a non-empty subinterval of $\mathbb{R}$. If f has the property

$$|f(x) - f(y)| < |x - y| \quad \text{for all} \quad x, y \in I.$$

Note that $f(x)$ is necessarily continuous. Show that:

(a) f has at most one fixed point.

(b) The function $f(x) = x + 1/x$ has this property on $[1, \infty)$, but has no fixed points.

(c) The function $g(x) = (x + \sin(x))/2$ has this property on $\mathbb{R}$. Does g have a fixed point?

(d) Show that if f has this property on $I = [a, b]$, $(a < b)$, then $f(x)$ has a unique fixed point. (Hint: Consider $\inf\{|f(x) - x| : x \in [a, b]\}$).

4. Let $f(x) = x^2 - a$ with $1 < a < 3$, and let N_f be the corresponding Newton function. Show that N_f satisfies the hypothesis of the Contraction Mapping Theorem on $(1, \infty)$. What is the fixed point?

5. Brouwer's Fixed Point Theorem says that if we have a continuous map $f : D \to D$, where D is a closed ball in $\mathbb{R}^n$, then f has a fixed point in D (e.g. $D = [0, 1] \subseteq \mathbb{R}$, or $D = \{(x, y); x^2 + y^2 \leq 1\} \subseteq \mathbb{R}^2$). Give an example of a continuous onto map $f : (0, 1) \to (0, 1)$ which does not have a fixed point.

Chapter 9, Newton's Method

Newton's method applied to quadratic polynomials is fairly straightforward when the polynomial has two distinct roots. We leave it to the exercises to show that for the quadratic polynomials

$$f(x) = ax^2 + bx + c, \quad \text{and} \quad g(x) = x^2 - \alpha,\ a \neq 0,$$

(where a, b, c and α are real), N_f and N_g are conjugate when $\alpha = b^2 - 4ac$. The conjugacy is given by the map $h(x) = 2ax + b$. It is then easy to see that for $\alpha > 0$, the basin of attraction of the fixed points $\sqrt{\alpha}$ and $-\sqrt{\alpha}$ of N_g are $(0, \infty)$ and $(-\infty, 0)$ respectively. When $\alpha = 0$, the only fixed point is 0 with basin of attraction for N_g being $\mathbb{R}$. The conjugacy then shows that N_f has similar properties.

For this reason we now examine Newton's method for quadratics of the form $f_c(x) = x^2 + c$ where $c < 0$, so that f_c does not have any zeros in $\mathbb{R}$. We start with a detailed look at the binary representation of real numbers. Subsequently we look at the situation for certain cubic polynomials. For the first section we follow [14] and [51] and for the second we follow [65] (see also [30]).

9.1 Binary Representation of Real Numbers

Let $b \in \mathbb{Z}^+$ be a given base. Any $x \in \mathbb{R}$ may be represented in base b in the form

$$\begin{aligned} x &= a_N\, a_{N-1} \dots a_1\, a_0 \cdot c_1\, c_2\, c_3 \dots c_k\, c_{k+1} \dots \\ &= a_N\, b^N + a_{N-1}\, b^{N-1} + \cdots + a_1\, b + a_0 + \frac{c_1}{b} + \frac{c_2}{b^2} + \cdots + \frac{c_k}{b^k} + \cdots\ , \end{aligned}$$

where $a_i, c_j \in \{0, 1, \dots, b-1\}$ for all i and j.

We will restrict our attention to $x \in [0, 1]$ and the case $b = 2$ (binary representation), but the situation is much the same for any base b. In this case $a_i, c_j \in \{0, 1\}$. The following properties of binary expansions are well known (see for example [52]). The *pre-period* is the number of terms after the "decimal point" prior to the start of the periodic part of the expansion. The *period* is the minimal length of a repeating part of the binary expansion.

9.1.1 Properties of Binary Expansions

Rational numbers either have finite binary expansions (dyadic rationals) or infinite periodic representations, in general not unique. Irrationals have unique infinite non-periodic representations. More detail for the case of rationals is given by the following properties:

(a) *A rational* $r \in (0,1)$ *has a finite binary representation if and only if it is dyadic, i.e., it can be written as* $r = k/2^n$ *for some* $k, n \in \mathbb{Z}^+$ *where* k *is odd.*

(b) *A rational* $r \in (0,1)$ *has an infinite repeating binary representation if and only if it can be written as* $r = t/q$ *where* q *is odd. In this case the pre-period is zero and the period of the repeating sequence does not exceed* $\phi(q)$ *(when* $(t,q) = 1$*), where* ϕ *is Euler's function.*

(c) *A rational* $r \in (0,1)$ *is of the form* $r = t/2^n q$ *irreducible where* q *is odd and* $n > 0$ *if and only if the binary expansion of* r *is eventually repeating, with repeating part having period* $\phi(q)$ *and pre-period* n.

In cases (b) and (c) we get additional information:

(b′) *If* $r \in (0,1)$ *is of the form* $r = t/q$ *(irreducible,* q *odd), we may write* $r = s/(2^p - 1)$ *for some (minimal) positive integers* s *and* p*. In this case* p *gives the period of the binary representation.*

(c′) *If* $r \in (0,1)$ *is of the form* $r = t/2^n q$ *(irreducible* q *odd), we may write* $r = s/2^n(2^p - 1)$ *for some (minimal) positive integers* n*,* s *and* p*. Again,* p *gives the length of the period of the binary representation and* n *is the pre-period.*

Examples 1. $r = 11/16 = 1/2 + 1/2^3 + 1/2^4 = \cdot 1011$ is a dyadic rational.

2. If $r = 1/5 = 0 \cdot \overline{0011}$ (where the overline indicates that this part is repeated indefinitely), then the period is $\phi(5) = 4$, whereas $r' = 1/13 = 0 \cdot \overline{000100111011}$ has period $\phi(13) = 12$.

3. We have $r = 1/(2 \cdot 5) = 0.0\overline{00011}$ with period 4 (same as that of 1/5), and pre-period 1 since $2^1 = 2$.

4. $1/5 = 3/(2^4 - 1) = 0 \cdot \overline{0011}$, $1/13 = 5 \cdot 63/(2^{12} - 1) = 0 \cdot \overline{000100111011}$.

5. $1/12 = 1/(2^2(2^2 - 1)) = 0 \cdot 00\overline{01}$, $1/14 = 1/2(2^3 - 1) = 0 \cdot \overline{001}$.

In summary we have:

(1) *If* $r \notin \mathbb{Q} \cap [0,1]$*, then* r *has a unique binary representation which is infinite and non-periodic.*

(2) *If* $r \in \mathbb{Q} \cap [0,1]$*, then either*

(i) *there exists* $k, n \in \mathbb{Z}^+$ *such that* $r = k/2^n$ *so that* r *has a binary representation that terminates after* n *digits, or*

(ii) *there exists* $k, n, p \in \mathbb{Z}^+$ *such that* $r = k/(2^p(2^n - 1))$ *so that* r *has a unique binary representation with pre-period* p *and period* n*.*

9.2 Newton's Method for Quadratic Polynomials

Returning to Newton's method, for the function $f_1(x) = x^2 + 1$, so that Newton's function is

$$N_1(x) = x - \frac{f_1(x)}{f_1'(x)} = \frac{x^2 - 1}{2x}.$$

The sequence $N_1^n(x)$ cannot converge since it would have to converge to a fixed point, and there are no fixed points. We set $N_1(0) = 0$ for convenience and say that the orbit *terminates* if it lands on 0. For example, $N_1(1) = 0$ and $N_1^2(1 + \sqrt{2}) = 0$.

On the other hand, $N_1(1/\sqrt{3}) = -1/\sqrt{3}$ and $N_1(-1/\sqrt{3}) = 1/\sqrt{3}$, giving a point of period 2 and since $N_1(\sqrt{3}) = 1/\sqrt{3}$ we have an eventually periodic point.

Notice that the recurrence formula

$$x_{n+1} = \frac{x_n^2 - 1}{2x_n}$$

is reminiscent of the trigonometric identity:

$$\cot(2\theta) = \frac{\cot^2(\theta) - 1}{2\cot(\theta)}, \quad \theta \in (0, \pi), \quad \theta \neq \pi/2.$$

The map $h(\theta) = \cot(\pi\theta)$ is a homeomorphism $h : (0, 1) \to \mathbb{R}$, so any conjugation involving h will preserve orbits of N_1 in the usual way. Thus if $x_0 = \cot(\pi r_0)$, then $N_1^n(x_0) = \cot(2^n \pi r_0)$ for each n provided $2^n \pi r_0$ is not an integer multiple of π, i.e., r_0 is not of the form $k/2^n$ for some $k, n \in \mathbb{Z}^+$. In other words, $N_1 \circ h(r) = h \circ T(r)$ if $r \neq 1/2$, where $T : (0, 1) \to (0, 1)$ is $T(x) = 2x \pmod 1$, and it can be seen that N_1 (on the set of numbers x_0 whose orbits do not terminate at 0), is conjugate to the map

$$T' : (0, 1) - \{\text{dyadic numbers}\} \to (0, 1) - \{\text{dyadic numbers}\}, \quad T'(x) = 2x \pmod 1.$$

This leads to:

Proposition 9.2.1 (i) *If* $r_0 \in (0, 1)$*, then* $r_0 = k/2^n$ *for some* $k, n \in \mathbb{Z}^+$*, with* k *odd, if and only if the orbit of* $x_0 = \cot(\pi r_0)$ *under* N_1 *terminates at 0 after* n *iterations.*

(ii) *The orbit* $O(x_0) = \{N_1^n(x_0) : n \in \mathbb{Z}^+\}$ *is either finite or infinite (periodic/eventually-periodic) if and only if* r_0 *is rational.*

(iii) *If $r_0 \notin \mathbb{Q} \cap (0,1)$, then the orbit of $x_0 = \cot(\pi r_0)$ under N_1 is infinite and there are points x_0 having a dense orbit (so that $N_1 : \mathbb{R} \to \mathbb{R}$ is topologically transitive).*

Proof. (i) If $r_0 = k/2^n$ with k odd, then

$$x_{n-1} = \cot(2^{n-1}\pi r_0) = \cot(\pi k/2) = 0,$$

so that x_m is not defined for $m \geq n$, and the orbit of $x_0 = \cot(\pi r_0)$ terminates at the fixed point 0 after n iterations.

Conversely, if the orbit of x_0 terminates at 0 after n iterations, then

$$N_1^n(x_0) = \cot(2^n \pi r_0) = 0,$$

so there exists $m \in \mathbb{Z}$ with $2^n \pi r_0 = m\pi + \pi/2$, so $r_0 = (2m+1)/2^{n+1}$.

(ii) If r_0 gives rise to a periodic or eventually periodic orbit, then it cannot by dyadic: there must exist m and p with

$$N_1^{m+p}(x_0) = N^m(x_0), \quad \text{or} \quad \cot(2^{m+p}\pi r_0) = \cot(2^m \pi r_0),$$

and solving gives $r_0 = k/2^m(2^p - 1)$, $k, p, m \in \mathbb{Z}^+$, $p \geq 2$.

(iii) We have a conjugacy between N_1 and T, the doubling map (at least if we exclude the dyadic rationals), so that since the orbit of any irrational r_0 under T will be infinite, the corresponding orbit will be infinite under N_1. In addition, if r_0 has a dense orbit in $(0,1)$ under T (such orbits exist since T is transitive), then $x_0 = \cot(\pi r_0)$ will have a dense orbit under N_1. □

Examples 9.2.2 1. If $r_0 = 1/3 = 1/(2^2 - 1) = 0 \cdot \overline{01}$, then $m = 0, p = 2$ and $x_0 = \cot(\pi/3) = 1/\sqrt{3}$, $x_1 = \cot(2\pi/3) = -1/\sqrt{3}$. We see that the orbit of x_0 is periodic with period $p = 2$.

2. If $r_0 = 1/7 = 1/(2^3 - 1) = 0 \cdot \overline{001}$, then $m = 0$, $p = 3$ and $x_0 = \cot(\pi/7)$, giving rise to the 3-cycle $\{\cot(\pi/7), \cot(2\pi/7), \cot(4\pi/7)\}$.

Remark 9.2.3 Returning to $f_c(x) = x^2 + c$ for $c \in \mathbb{R}$ with $c < 0$, we see that f_c has two roots $\pm\sqrt{c}$. The corresponding dynamics of N_c is trivial since the respective basins of attraction are $(-\infty, 0)$ and $(0, \infty)$ with $N_c(0)$ being undefined.

If $c > 0$, then a change of variable can be shown to reduce the situation to the case where $c = 1$. When $c = 0$, $N_0(x) = x/2$, so the dynamics is trivial.

9.3 Newton's Method for Cubic Polynomials

In this section we restrict ourselves to cubic polynomials. We follow the development in Walsh [65]. Let $f(x)$ be a cubic polynomial and $N_f = x - f(x)/f'(x)$ the corresponding Newton's function. We first show that it suffices to consider only monic polynomials. We then show that shifting the polynomial (for example, replacing x by $x-h$) does not change the dynamics of Newton's method. In this way, the study of the dynamics of Newton's method for any cubic polynomial can be reduced to the study of a few special cases.

Lemma 9.3.1 *Let $f(x)$ be a cubic polynomial with corresponding Newton's function $N_f(x)$. Denote by $A(x) = ax+b$, $a, b \in \mathbb{R}, a \neq 0$, an affine transformation. Then*

(i) If $k \in \mathbb{R}$, and $g(x) = kf(x)$, then $N_f(x) = N_g(x)$.

(ii) If $g(x) = f(A(x))$, then $AN_gA^{-1}(x) = N_f(x)$, i.e., N_f and N_g are conjugate via an affine transformation.

(iii) Let $f_{a,c}(x) = (x-a)(x^2+c)$. There exists $a, c \in \mathbb{R}$ such that $N_f(x)$ is conjugate to $N_{f_{a,c}}(x)$.

Proof. (i) This is straightforward since $g'(x) = kf'(x)$.

(ii) We show that $AN_g(x) = N_fA(x)$. The right-hand side is

$$N_fA(x) = A(x) - \frac{f(A(x))}{f'(A(x))} = ax + b - \frac{f(A(x))}{f'(A(x))}.$$

The left-hand side is

$$AN_g(x) = A\left(x - \frac{g(x)}{g'(x)}\right) = A\left(x - \frac{f(A(x))}{af'(A(x))}\right) = a\left(x - \frac{f(A(x))}{af'(A(x))}\right) + b,$$

and these are equal.

(iii) We may assume that the polynomial is monic, say

$$f(x) = x^3 + kx^2 + mx + n,$$

then replacing x by $x-h$ gives

$$g(x) = f(x-h) = (x-h)^3 + k(x-h)^2 + m(x-h) + n$$

$$= x^3 + (k-3h)x^2 + (3h^2 - 2hk + m)x - h^3 + kh^2 - mh + n.$$

It can be shown (for example, using Mathematica) that h can be chosen so that the other coefficients give the form:

$$g(x) = x^3 - ax^2 + cx - ac = (x-a)(x^2+c).$$

Now use (ii) to see that N_f and N_g are conjugate, where $g = f_{a,c}$. $\square$

9.4 The Cubic Polynomials $f_c(x) = (x+2)(x^2+c)$.

As in [65], we now investigate Newton's method for cubic polynomials of the form $f(x) = (x+2)(x^2+c)$, first for the case where $c < 0$ (so that $f(x) = 0$ when $x = -2$ and at $x = \pm\sqrt{-c}$), and then when $c > 0$, so that $f(x) = 0$ has a single root at $x = -2$. The Newton's function is

$$N_c(x) = x - \frac{f_c(x)}{f_c'(x)} = \frac{2x^3 + 2x^2 - 2c}{3x^2 + 4x + c},$$

a one-parameter family of rational functions with fixed points the zeros of $f_c(x)$ and since $N_c'(x) = f_c''(x)f_c(x)/(f_c'(x))^2$, it has critical points the zeros of $f_c(x)$, together with $x = -2/3$, the root of $f_c''(x) = 0$.

Case 1: $c < 0$. Here $f_c(x) = 0$ when $x = -2$ or $x = \pm\sqrt{-c}$, three distinct roots. Following [65], let us take $c = -1$ and write $f(x) = f_{-1}(x)$ and $N(x) = N_{-1}(x)$, so that the fixed points of N are -2 and ± 1. Denote the immediate basin of attraction of a fixed point p by $W(p)$ (the largest interval containing p, contained in the basin of attraction of p). If e_1 and e_2 are the critical points of f, then it can be seen that

$$W(1) = (e_2, \infty), \quad \text{and} \quad W(-2) = (-\infty, e_1), \quad \text{where} \quad e_1 < e_2.$$

Since $x = -1$ is a (super)attracting fixed point for N ($N'(-1) = 0$) and N is continuous on (e_1, e_2), $W(-1) = (a, b)$ is an open interval for some $a, b \in \mathbb{R}$ with $e_1 < a < b < e_2$.

As we have seen previously, we must have $N(a) = b$ and $N(b) = a$ (so $\{a, b\}$ is a 2-cycle) since neither a nor b is a fixed point of N. We shall see that there are no other periodic points, and that this is a repelling 2-cycle.

Following the steps in [65], we can find a sequence $\{e_n\}_{n=1}^{\infty}$ with

$$e_1 < e_3 < e_5 < \cdots < a, \quad \text{and} \quad e_2 > e_4 > e_6 > \cdots > b,$$

with

$$N(e_{2n+1}) = e_{2n} \quad \text{and} \quad N(e_{2n+2}) = e_{2n-1}, \quad n \geq 1,$$

so that $N(e_1, a) = (b, \infty)$, $N(e_3, a) = (b, e_2)$ etc., with

$$e_{2n-1} \to a \quad \text{and} \quad e_{2n} \to b \quad \text{as} \quad n \to \infty.$$

In addition, the basin of attraction of $x = 1$ contains the set $\cup_{n=1}^{\infty}(e_{2n-1}, e_{2n+1})$, and the basin of attraction of $x = -2$ contains the set $\cup_{n=1}^{\infty}(e_{2n}, e_{2n+2})$.

Using these results, it can be shown that the set

$$E = \{x \in \mathbb{R} : \text{the sequence } N^n(x) \text{ does not converge to a root of } f(x) \text{ as } n \to \infty\}$$

is just the set consisting of the 2-cycle $\{a, b\}$ and the sequence $\{e_n\}_{n=1}^{\infty}$. Of course the critical points of f and "eventual critical points" (which is where the sequence $\{e_n\}_{n=1}^{\infty}$ arises), will always lie in E. Consequently, Newton's method gives convergence to a root of $f(x)$ everywhere except on a countable set.

Case 2: $c > 0$. The situation is now more complicated, for example we will see that as c decreases from 1 to 0, a 3-cycle is born at around $c = .466$. Also, when $c = 1/5$, $\{1/10, -3/5\}$ (and also $\{0.093561\ldots, -0.634223\ldots\}$) is a 2-cycle. Sarkovsky's Theorem does not apply in the usual way to $N_f(x)$, since in general it is not a continuous function, but other results can be used to show that when we have a 3-cycle, we have cycles of all other possible orders.

The situation becomes increasingly more complicated for polynomials of degree 4 or more. The following result is due to Barna [5]:

Theorem 9.4.1 *Let f be an nth degree polynomial with $n > 3$ distinct real roots. The set E is a Cantor set. In particular it has measure zero, is closed, totally disconnected and each point of E is a limit point of E.*

Exercises 9.1 - 6.4

1. Prove that if $r \in (0, 1)$ has a finite binary representation, then $r = k/2^n$ for some $k, n \in \mathbb{Z}^+$, where k is odd. Now prove the converse.

2. Show that $1/13 = 5 \cdot 63/(2^12 - 1) = 0 \cdot \overline{000100111011}$.

3. Prove that $r \in (0, 1)$ has an infinite repeating binary representation if and only if $r = t/q$ for some $t, q \in \mathbb{Z}^+$, where q is odd.

4. Show that for the quadratic polynomials

$$f(x) = ax^2 + bx + c, \quad \text{and} \quad g(x) = x^2 - \alpha, \ a \neq 0,$$

the Newton's functions N_f and N_g are conjugate when $\alpha = b^2 - 4ac$. The conjugacy is given by $h(x) = 2ax + b$ so that $h \circ N_f(x) = N_g \circ h(x)$ for all x in the domain of N_f. Are f and g conjugate in this situation? (No, for example if $f(x) = (x+1)^2 = x^2 + 2x + 1$ and $g(x) = x^2$, then $\alpha = 0 = 4 - 4 = b^2 - 4ac$, but since g has 2 fixed points and f has none, f and g cannot be conjugate). Note that if $f(x) = (x+c)^2$ where $c < 1/4$, then $f(x)$ has two fixed points. Are f and $g(x) = x^2$ conjugate? We know conditions for f and g to be linearly conjugate - when are they conjugate?

5. If $g(x) = x^2 - 1$, show that the basins of attraction of the fixed points 1 and -1 of N_g are $(0, \infty)$ and $(-\infty, 0)$ respectively.

Hint: Look at a web plot for N_g. Show that if $x_0 \in (1, \infty)$, then $1 < N_g(x_0) < x_0$, so that $N_g^n(x_0)$ is a decreasing sequence bounded below, so has to converge to the fixed point $c = 1$ (since N_g is continuous on $(0, \infty)$). The other cases are treated similarly.

6. If $g(x) = x^2 - \alpha$ for $\alpha > 0$, generalize the above argument to show that the basins of attraction of the fixed points $\sqrt{\alpha}$ and $-\sqrt{\alpha}$ of N_g are $(0, \infty)$ and $(-\infty, 0)$ respectively. When $\alpha = 0$, show that the only fixed point of N_g is 0 with basin of attraction $\mathbb{R}$.

Chapter 10, Iterations of Continuous Functions

10.1 Coppel's Theorem

Our aim is to prove a result due to Coppel (1955, [16]), that was a precursor of Sharkovsky's Theorem although it seems that Sharkovsky was unaware of Coppel's work. Prior to establishing his major results, in the early 1960's, Sharkovsky reproved Coppel's result and went on to prove his famous theorem on the periodic points of one-dimensional maps:

Theorem 10.1.1 *Let $f : [a, b] \to [a, b]$ be a continuous map. Then $\lim_{n\to\infty} f^n(x)$ exists for every $x \in [a, b]$ if and only if $f(x)$ has no points of period 2.*

Corollary 10.1.2 *A continuous map $f : [a, b] \to [a, b]$ must have a 2-cycle if it has any periodic points that are not fixed.*

Proof. If f has an m-cycle, $m > 1$, it has points in $[a, b]$ for which the sequence $\{f^n(x)\}$ does not converge. It follows by Theorem 1 that f has a point of period 2. $\square$

Before proving Coppel's Theorem, we prove some preliminary results.

Lemma 10.1.3 *Let $f : [a, b] \to [a, b]$ be a continuous map. If there exists $c \in [a, b]$ with*

$$f^2(c) < c < f(c), \quad \textit{or} \quad f(c) < c < f^2(c),$$

then $f(x)$ has a period 2-point.

Proof. Suppose that $c \in [a, b]$ with $f^2(c) < c < f(c)$. Set $g(x) = f^2(x) - x$, then $g(a) = f^2(a)-a \geq 0$ and $g(c) = f^2(c)-c < 0$, so by the Intermediate Value Theorem, there exists $p \in [a, c]$ with $g(p) = 0$, or $f^2(p) = p$.

If $f(p) \neq p$, we are done, so suppose that $f(p) = p$. Since $c \in f(p, c)$, there exists $q \in (p, c)$ with $f(q) = c$. We may assume without loss of generality that there are no other fixed points in $[p, c]$ (e.g., by taking p to be the largest such fixed point).

As before, $g(c) = f^2(c) - c < 0$, and

$$g(q) = f^2(q) - q = f(c) - q > f(c) - c > 0,$$

so $f^2(r) = r$ for some r in $[q, c]$, which cannot be a fixed point of f since there are no fixed points of f in (q, c). This completes the proof in this case.

The case where $f(c) < c < f^2(c)$ is similar. $\square$

The next Lemma generalizes Lemma 1 in the sense that it tells us that if there exists $c \in [a, b]$ and $n \in \mathbb{Z}^+$ with $f^n(c) < c < f(c)$, then f must have a 2-cycle. It is easy to give examples of continuous functions on bounded intervals for which there exist c in the interval with $c < f^2(c) < f(c)$, but f having no point of period 2. Roughly speaking, Lemma 1 shows that you cannot have oscillation about a repelling fixed point unless there is a 2-cycle. If p is a fixed point, which is attracting, we may have a scenario like:

$$f(c) < p < f^2(c) < c.$$

Conjecture: If $f'(x)$ is continuous near p, where $f(p) = p$ and $f'(p) > -1$, then f has a 2-cycle because if $c > p$ is close to p, then $f(c) < p < c < f^2(c)$.

Lemma 10.1.4 *Let $f : [a, b] \to [a, b]$ be a continuous map with no points of period 2. Then*

(i) *If $f(c) > c$, then $f^n(c) > c$ for all $n \in \mathbb{Z}^+$.*

(ii) *If $f(c) < c$, then $f^n(c) < c$ for all $n \in \mathbb{Z}^+$.*

Proof. We use a proof by induction: Let $x \in [a, b]$ and $m \in \mathbb{Z}^+$ be fixed.

As our induction hypothesis we take:

$$f(x) < x \Rightarrow f^n(x) < x,\ n = 1, 2, \ldots, m, \ \text{ and } \ f(x) > x \Rightarrow f^n(x) > x,\ n = 1, 2, \ldots, m.$$

Step 1 We first show that a consequence of the induction hypothesis is:

$$f^{m+1}(x) = x \Rightarrow f(x) = x \quad \text{for} \quad x \in [a, b],$$

(this is clearly true for $m = 1$). Suppose that $f^{m+1}(c) = c$ where $f(c) > c$, then $d = f^m(c) > c$ (by the induction hypothesis).

Suppose that

$$d = f^{m-1}(f(c)) \geq f(c),$$

then $f(f(c)) \geq f(c)$ (since if not, $f(f(c)) < f(c) \Rightarrow f^{m-1}(f(c)) < f(c)$ by the induction hypothesis).

Then $f^m(f(c)) \geq f(c)$ for the same reason, and this says $f^{m+1}(c) = c \geq f(c)$, a contradiction. Hence $d = f^m(c) < f(c)$. Consequently

$$f(d) = f^{m+1}(c) = c.$$

It follows that there exists $q \in [c, d]$ where $f(q) = d$. We may assume that q is the nearest point to c at which this happens.

Then

$$c < q < d, \ \ f(q) = d, \quad \text{and} \quad f(x) > d > q \quad \text{for} \quad c \leq x < q.$$

Hence $f^2(q) = f(d) = c < q$. But $f^2(c) > c$ because $f(c) > c$.

It follows (using $g(x) = f^2(x) - x$ as before) that at some point x between c and q we have $f^2(x) = x$ and hence $f(x) = x$. But this is impossible because $f(x) > q$ for $c \leq x < q$.

Similarly the assumption that $f(c) < c$ leads to a contradiction, so we must have $f(c) = c$

Step 2 We use the ideas of the proof of Lemma 1. The induction hypothesis clearly holds for $m = 1$ (and also for $m = 2$ by Lemma 1), so suppose that $f(x) > x \Rightarrow f^n(x) > x$, $n = 1, 2, \ldots, m$, $x \in [a, b]$ and let $c \in [a, b]$ with $f(c) > c$ and $f^{m+1}(c) < c$, we will show that this leads to a contradiction.

As in Lemma 1 (using $g(x) = f^{m+1}(x) - x$ in place of $f(x) - x$) there is a point $p \in (a, c)$ with $f^{m+1}(p) = p$, so by the induction hypothesis and Step 1, $f(p) = p$. As before, we may assume that there is no other such fixed point in (p, c).

Since $f(x) \neq x$ for $x \in (p, c)$ and $f(c) > c$, it follows that

$$f(x) > x \quad \text{for all} \quad x \in (p, c). \tag{1}$$

Again the induction hypothesis gives

$$f^m(x) > x > p \quad \text{for all} \quad x \in (p, c). \tag{2}$$

Choose $q \in (p, c)$ very close to p so that $p < f^m(q) < c$ (again using continuity). Then from (1) and (2):

$$f^{m+1}(q) = f(f^m(q)) > f^m(q) > q,$$

contradicting, the definition of p since it gives rise to another fixed point in (q, c). This contradiction gives us $f^{m+1}(c) > c$.

In a similar way we show that $f(c) < c$ and $f^{m+1}(c) > c$ leads to a contradiction. and the Lemma follows. $\square$

Proof of Coppel's Theorem

If f has a 2-cycle $\{x_1, x_2\}$, $x_1 \neq x_2$, $f(x_1) = x_2$, $f(x_2) = x_1$, then $\lim_{n\to\infty} f^n(x_1)$ does not exist as it oscillates between x_1 and x_2.

To prove the converse we use Lemma 2. Let $x \in [a, b]$ and $x_n = f^n(x)$. If $x_{m+1} = x_m$ for some m, then $x_n = x_m$ for all $n > m$, so the sequence converges. Thus we can assume that $x_{n+1} > x_n$ infinitely often and $x_{n+1} < x_n$ infinitely often.

Fix $x \in [a, b]$ and set

$$A = \{x_n : f(x_n) > x_n\} \quad \text{and} \quad B = \{x_n : f(x_n) < x_n\}.$$

Then A and B are disjoint sets and their union is the orbit of x.

Suppose that

$$A = \{x_{n_1}, x_{n_2}, x_{n_3}, \dots x_{n_p}, \dots\} \quad \text{where} \quad n_1 < n_2 < \dots < n_p < \dots,$$

so that $x_{n_p} = f^{n_p}(x)$, then by the definition of A, $f(x_{n_p}) > x_{n_p}$ for each $p = 1, 2, \dots$.

It follows that

$$x_{n_2} = f^{n_2}(x) = f^{n_2 - n_1}(f^{n_1}(x)) = f^{n_2 - n_1}(x_{n_1}) > x_{n_1}$$

by Lemma 2, since $f(x_{n_1}) > x_{n_1}$. In a similar way

$$x_{n_{p+1}} = f^{n_{p+1}}(x) = f^{n_{p+1} - n_p}(f^{n_p}(x)) = f^{n_{p+1} - n_p}(x_{n_p}) > x_{n_p},$$

so gives rise to an increasing sequence which is contained in $[a, b]$ (a subsequence of $\{x_n\}$).

It follows that $r = \lim_{p\to\infty} x_{n_p}$ exists, and in a similar manner the terms of the B give a decreasing sequence, with limit q say, $q \leq r$.

Now infinitely often there exists $x_n \in A$ with $f(x_n) \in B$ (since we are assuming A and B are infinite sets whose union is all of the orbit of x). Take a subsequence of $\{x_{n_p}\}$ (also denoted by $\{x_{n_p}\}$), with the property that $f(x_{n_p}) \in B$ for all p. Then

$$\lim_{p\to\infty} x_{n_p} = r \quad \text{and} \quad \lim_{p\to\infty} f(x_{n_p}) = f(r) = q,$$

(by continuity). A similar argument shows that $f(q) = r$, so that $\{q, r\}$ is a 2-cycle, which contradicts the hypothesis, so we must have $q = r$, i.e., the sequence converges for every $x \in [a, b]$. $\square$

Example 10.1.5 The logistic map $L_\mu : [0, 1] \to [0, 1]$, $L_\mu(x) = \mu x(1 - x)$, is continuous and for $0 < \mu \leq 3$, has no points of period 2, so that $\lim_{n\to\infty} L^n_\mu(x)$ exists for all $x \in [0, 1]$. It follows that this sequence must converge to a fixed point. For $0 < \mu \leq 1$ the only fixed point is $x = 0$, and so the basin of attraction of 0 is all of $[0, 1]$. For $1 < \mu \leq 3$, $x = 0$ is a repelling fixed point and $x = 1 - 1/\mu$ is attracting. The basin of attraction of $1 - 1/\mu$ is all of $[0, 1]$ except for the fixed point 0 and its only eventual fixed point $x = 1$. If $3 < \mu \leq 4$, then L_μ has period 2-points and Coppel's Theorem is not applicable. We can now see that web plots can oscillate around a fixed point?

10.2 Results Related to Sharkovsky's Theorem

The following result is a principal tool in proving Sharkovsky's Theorem:

Theorem 10.2.1 *Let $f : I \to I$ be continuous on an interval I. Suppose that I_1 and I_2 are two subintervals of I with at most one point in common.*

If $f(I_1) \supset I_2$ and $f(I_2) \supset I_1 \cup I_2$, then f has a point of period 3.

Proof. There is a set $B_1 \subseteq I_1$ with $f(B_1) = I_2$.
There is a set $B_2 \subseteq I_2$ with $f(B_2) = B_1$.
There is a set $B_3 \subseteq I_2$ with $f(B_3) = B_2$.
Then $f^3(B_3) = f^2(B_2) = f(B_1) = I_2 \supseteq B_3$, so there exists $c \in B_3$, a fixed point for f^3, which must be a period 3-point. $\square$

Theorem 10.2.2 *If $f : [0,1] \to [0,1]$ satisfies: $\forall I \subseteq [0,1]$ an interval, $\exists n \in \mathbb{Z}^+$ with $f^n(I) = [0,1]$, then f is chaotic on $[0,1]$. In addition, there is no attracting periodic orbit.*

Proof. Above says that f is transitive (using the alternate definition). Suppose $n \in \mathbb{Z}^+$ with

$$f^n(I) = [0,1] \supseteq I, \exists x \in I : f^n(x) = x,$$

It follows that the periodic points of f are dense in $[0,1]$. Also f has sensitive dependence since if $x \in I$, there exists $y \in I$ with

$$|f^n(x) - f^n(y)| > \frac{1}{2}.$$

If there is an attracting periodic orbit, each point in the orbit would be contained in an interval that was attracted to the points of the orbit by repeated application of f - contradiction. $\square$

Question Does L_4 have this property (cf. *strong mixing*: $f^n(U) \cap V \neq \emptyset$ for all n large enough, U, V open).

Lemma 10.2.3 *Let $f : I \to I$ be a continuous interval map and J a subinterval containing no periodic points. If $x, y, f^m(x), f^n(y)$ belong to J and*

$$y < x < f^m(x), \quad \textit{then} \quad y < f^n(y).$$

Lemma 10.2.4 *Let $f : I \to I$ be a continuous interval map. If there exists $x, y \in I$ and $n \geq 2$ such that*

$$f(y) \leq y < x < f(x), \quad \textit{and} \quad f^n(x) \leq y,$$

then f has a 3-cycle.

Chapter 11, Linear Transformations and their Induced Maps on the Circle and Torus

11.1 Linear Transformations

We saw in the first few chapters how continuous linear transformations $f : \mathbb{R} \to \mathbb{R}$ behave. They are necessarily of the form $f(x) = ax$ for some $a \in \mathbb{R}$, so that $c = 0$ is the only fixed point (when $a \neq 1$). This fixed point is attracting if $|a| < 1$, repelling if $|a| > 1$ and neutral when $|a| = 1$ (of course when $a = 1$, every point is fixed). Consequently, the dynamics of these maps is trivial in each case.

Definition 11.1.1 A linear transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ is a function with the property:

$$f(ax + by) = af(x) + bf(y), \quad \text{for all} \quad x, y \in \mathbb{R}^n \quad \text{and} \quad a, b \in \mathbb{R}.$$

Suppose now that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation with $n > 1$. It is well known that such a map can be represented by matrix multiplication: there is a matrix $A \in M_n(\mathbb{R})$ (the vector space of all n-by-n matrices having real entries) for which $f(x) = A \cdot x$, where we are writing $x \in \mathbb{R}^n$ as an n-by-1 vector (or matrix). Let us examine the case where $n = 2$ in more detail. In this case

$$\mathbb{R}^2 = \{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1, x_2 \in \mathbb{R}\} = \{(x_1, x_2)^t : x_1, x_2 \in \mathbb{R}\},$$

where A^t represents the transpose of the matrix A.

Suppose that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and write $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, then

$$f(x) = A \cdot x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}.$$

The *eigenvalues* of f are those $\lambda \in \mathbb{C}$ for which there exists $x \in \mathbb{R}^n$, $x \neq 0$, with $f(x) = \lambda \cdot x$. These can be found by solving the polynomial equation $\det(A - \lambda I) = 0$, where $\det(A)$ is the determinant of A ($\det(A) = ad - bc$ when $n = 2$), and I is the n-by-n identity matrix. The corresponding x-values are the *eigenvectors* of f, and the set

$$E_\lambda = \{x \in \mathbb{R}^n : f(x) = \lambda x\}$$

is a subspace of $\mathbb{R}^n$ called the *eigenspace* of f corresponding to the eigenvalue λ.

Again $x = 0$ is always a fixed point of f, and the dynamics of f is quite easily seen to be trivial, but we shall examine it in the case of $n = 2$ in more detail.

Theorem 11.1.2 *Given a linear transformation $f : \mathbb{R}^n \to \mathbb{R}^n$, there is an invertible matrix $P \in M_n(\mathbb{R})$ such that if $f(x) = A \cdot x$, then*

$$A = P \begin{pmatrix} A_s & 0 \\ 0 & A_u \end{pmatrix} P^{-1},$$

where A_s is a matrix having all of its eigenvalues less than one, A_u is a matrix having all of its eigenvalues greater than or equal to one.

Definition 11.1.3 The linear map $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *hyperbolic*, if its corresponding matrix A has no eigenvalues of absolute value equal to one.

Proposition 11.1.4 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be hyperbolic with corresponding matrix A having s eigenvalues with absolute value less than one, and u eigenvalues of absolute value greater than one. Then there are subspaces E^s and E^u having dimensions s and u respectively, where $s + u = n$ such that*

$$\mathbb{R}^n = E^s \oplus E^u,$$

where $|f^n(x)| \to 0$ as $n \to \infty$ if $x \in E^s$, and $|f^n(x)| \to \infty$ as $n \to \infty$ if $x \in E^u$.

11.2 Circle maps induced by linear transformations on $\mathbb{R}$

Consider the linear transformation $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x$. This map gives rise to a mapping defined on the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ which is determined as follows. First notice that we can write

$$S^1 = \{e^{2\pi i x} : x \in [0, 1)\} = \{e^{i\theta} : \theta \in [0, 2\pi)\},$$

so we identify S^1 with the interval $[0, 1)$ via the mapping $H : [0, 1) \to S^1$, $H(x) = e^{2\pi i x}$ (think of the circle as the interval $[0, 1)$ bent around so that 0 and 1 become adjacent, with the resulting figure appropriately scaled). If we define a mapping $D : [0, 1) \to [0, 1)$ by

$$D(x) = 2x \pmod 1,$$

then D is the transformation defined on $[0, 1)$ induced by the linear transformation f. We think of D as a circle map as it is conjugate to the map $T : S^1 \to S^1$, $T(e^{ix}) = e^{2ix}$ via the transformation H. This is just the squaring function (angle doubling map) $T(z) = z^2$ defined on S^1.

We ask the question: when does a linear transformation $f(x) = ax$ on $\mathbb{R}$ give rise to a well defined circle map?

In order to be well defined, we require the induced map $T(e^{ix}) = e^{aix}$ to "wrap around" the circle in the appropriate manner. This means that we require $T(e^{i(x+2\pi)}) = T(e^{ix})$ for $x \in [0, 1)$, or $e^{ai(x+2\pi)} = e^{aix}$, so that $e^{2\pi ia} = 1$ and so $a \in \mathbb{Z}$ is the required condition. It is now readily seen that this is a necessary and sufficient condition for $f(x) = ax$ to give rise to a circle map in this way.

We have examined the dynamical properties of $D(x) = 2x$ (mod 1), (the doubling map), and maps such as $D_3(x) = 3x$ (mod 1), can be analyzed in a similar fashion. Such maps, when regarded as maps of the circle, are continuous and onto, but not one-to-one and have very strong chaotic properties (for $a \neq 1$).

11.3 Endomorphisms of the torus

Given a linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}$, we can use it to define a map from $X = [0, 1) \times [0, 1)$ to itself in an analogous way as in the case of $n = 1$. The corresponding map T' is now defined on the *torus* $S^1 \times S^1$ by

$$T'\begin{pmatrix} e^{ix_1} \\ e^{ix_2} \end{pmatrix} = \begin{pmatrix} e^{i(ax_1+bx_2)} \\ e^{i(cx_1+dx_2)} \end{pmatrix}.$$

The torus can be thought of as being obtained from the unit square $[0, 1) \times [0, 1)$, by identifying opposite edges of the square to form a cylinder, and bending around to join the remaining edges of the cylinder to form the doughnut shape called a torus. In this way, if we define $T : [0, 1) \times [0, 1) \to [0, 1) \times [0, 1)$ by

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 & \text{mod } 1 \\ cx_1 + dx_2 & \text{mod } 1 \end{pmatrix},$$

then we see (by examining the wrap around properties of T'), that T is well defined if and only if $a, b, c, d \in \mathbb{Z}$ (see the exercises). As before, it can be seen that T is a continuous map of the torus which is onto when the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is non-singular ($\det(A) \neq 0$), but is not in general one-to-one.

Example 11.3.1 Consider $T : [0, 1) \times [0, 1) \to [0, 1) \times [0, 1)$ defined by

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 & (\text{mod } 1) \\ 2x_2 & (\text{mod } 1) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (\text{mod } 1).$$

In other words,

$$T\left(x_1, x_2\right)^t = \begin{cases} \left(2x_1, 2x_2\right)^t, & \text{if } 0 \leq x_1 < 1/2,\ 0 \leq x_2 < 1/2, \\ \left(2x_1 - 1, 2x_2\right)^t, & \text{if } 1/2 \leq x_1 < 1,\ 0 \leq x_2 < 1/2, \\ \left(2x_1, 2x_2 - 1\right)^t, & \text{if } 0 \leq x_1 < 1/2,\ 1/2 \leq x_2 < 1, \\ \left(2x_1 - 1, 2x_2 - 1\right)^t, & \text{if } 1/2 \leq x_1 < 1,\ 1/2 \leq x_2 < 1, \end{cases}$$

where $(x_1, x_2)^t$ denotes the transpose of (x_1, x_2).

This mapping T is not one-to-one, it is in fact everywhere four-to-one - if we partition $[0, 1) \times [0, 1)$ into four equal sub-squares, we see that each gets mapped onto $[0, 1) \times [0, 1)$. Consequently, it is clearly onto and it can be shown to be area preserving. We will see that it is chaotic.

Our aim in this section is to give precise conditions under which maps of this type are chaotic. Let us denote the n-dimensional torus by $\mathbb{T}^n = [0, 1)^n = [0, 1) \times [0, 1) \times \cdots \times [0, 1)$ (the direct product of n copies of $[0, 1)$). Notice that $\mathbb{T}^n$ is a group when given the group operation of addition modulo one on each coordinate. Recall that an *endomorphism* of a group is a homomorphism that is also onto. If it is also one-to-one, then it is a *group automorphism*. As above, use A^t to denote the transpose of the matrix A. We then have:

Proposition 11.3.2 (i) *Every endomorphism $T : \mathbb{T}^n \to \mathbb{T}^n$ is of the form*

$$T(x_1, x_2, \ldots, x_n)^t = A \cdot (x_1, x_2, \ldots, x_n)^t \pmod 1,$$

where $A = [a_{ij}]$ and $a_{ij} \in \mathbb{Z}$ for all $1 \leq i, j \leq n$.

(ii) *T maps $\mathbb{T}^n$ onto $\mathbb{T}^n$ if and only if* $\det(A) \neq 0$.

(iii) *T is an automorphism of $\mathbb{T}^n$ if and only if* $\det(A) = \pm 1$.

Proof. (i) We omit the proof of this result (see for example Walters [66]).

11.4 Chaos for Endomorphisms of $\mathbb{T}^2$

Suppose now that T is an endomorphism of the torus $\mathbb{T}^2$ defined by:

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \mod 1 \\ cx_1 + dx_2 \mod 1 \end{pmatrix},$$

where $a, b, c, d \in \mathbb{Z}$. Let $x_1, x_2 \in \mathbb{T}^2$ be of the form $x_1 = p_1/q, x_2 = p_2/q$, $p_1, p_2, q \in \mathbb{Z}$, then

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (ap_1 + bp_2)/q \mod 1 \\ (cp_1 + dp_2)/q \mod 1 \end{pmatrix}.$$

The number of points in $\mathbb{T}^2$ of the form $(s/q, t/q)$ with $0 \leq |s|, |t| < q$ is finite, so there exists integers $m, n \geq 0$ with $m < n$ such that $T^n(x_1, x_2)^t = T^m(x_1, x_2)^t$, so that $T^{n-m}(x_1, x_2)^t = (x_1, x_2)^t$. In other words, every point of this form is periodic. As a consequence, we see that the periodic points of T are dense in $\mathbb{T}^2$. Our aim is to show that T is chaotic.

Exercises 11.1 -11.3

1. Prove that the maps $T(x) = ax \pmod 1$ on $[0, 1)$ and $T'(e^{ix}) = e^{iax}$ on S^1 are conjugate for $a \in \mathbb{Z}$.

2. Prove that the map $T : [0, 1) \times [0, 1) \longrightarrow [0, 1) \times [0, 1)$

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \mod 1 \\ cx_1 + dx_2 \mod 1 \end{pmatrix},$$

is well defined if and only if $a, b, c, d \in \mathbb{Z}$.

3. Prove that the maps $T : [0, 1) \times [0, 1) \longrightarrow [0, 1) \times [0, 1)$ and $T' : S^1 \times S^1 \longrightarrow S^1 \times S^1$, defined by

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \mod 1 \\ cx_1 + dx_2 \mod 1 \end{pmatrix},$$

and

$$T'\begin{pmatrix} e^{ix_1} \\ e^{ix_2} \end{pmatrix} = \begin{pmatrix} e^{i(ax_1+bx_2)} \\ e^{i(cx_1+dx_2)} \end{pmatrix},$$

are conjugate when $a, b, c, d \in \mathbb{Z}$.

4. Prove that the map T of the last exercise is onto when $\det(A) \neq 0$, and one-to-one if and only if $\det(A) = \pm 1$. In this case T is an invertible area-preserving mapping of the torus. The torus can be given a group structure in the obvious way (with addition modulo one), and we now see that T will be an automorphism of this group. When

the determinant is not ± 1, T is still a homomorphism (actually an endomorphism of the torus since it is onto).

Chapter 12. Some Elementary Complex Dynamics

We start this section by generalizing some of the results from the early chapters. Much of the modern theory of dynamical systems arose from the study of Newton's method in the complex plane. The work of Schröder, Cayley, Fatou and Julia is particularly noteworthy in this regard (see "A History of Complex Dynamics" by Daniel Alexander [1] for a detailed treatment of this). For this reason we start with a review of some basic complex analysis.

12.1 The Complex Numbers

We briefly remind the reader of some of the basic properties of the complex numbers

$$\mathbb{C} = \{a + ib : a, b \in \mathbb{R}, i^2 = -1\}.$$

The complex numbers are often defined as the set of points in the plane $(a, b) \in \mathbb{R}^2$, together with the operations of addition and multiplication defined by

$$(a, b) + (c, d) = (a + c, b + d) \ ; \quad (a, b) \cdot (c, d) = (ac - bd, ad + bc).$$

It is then easily verified that with this addition and multiplication, $\mathbb{R}^2$ is a field with identity $(1, 0)$ and an element $i = (0, 1)$ having the property $i^2 = (-1, 0) = -(1, 0)$. We identify $(1, 0)$ with $1 \in \mathbb{R}$ and write $(a, b) = a(1, 0) + b(0, 1) = a + ib$ and denote the space obtained by $\mathbb{C}$. Because the complex numbers form a field, all the usual laws of arithmetic and algebra hold, for example, if $(a, b) \neq (0, 0)$, then the multiplicative inverse of (a, b) is $(a, b)^{-1} = (a/(a^2 + b^2), -b/(a^2 + b^2))$, so that if $z = a + ib$, $1/z = (a + ib)^{-1} = (a - ib)/(a^2 + b^2) = \bar{z}/|z|^2$.

The *absolute value* or *modulus* of $z \in \mathbb{C}$ is $|z| = \sqrt{a^2 + b^2}$ where $z = a + ib$ and $a, b \in \mathbb{R}$ are the *real* and *imaginary* parts of z respectively, written $a = \mathrm{Rl}(z)$ and $b = \mathrm{Im}(z)$. We can also write $|z|^2 = z \cdot \bar{z}$, where $\bar{z} = a - ib$ is the *complex conjugate* of z.

Points in $\mathbb{C}$ may also be represented in polar form, or what is called *modulus-argument* form, for if $z = a + ib$, $a, b \in \mathbb{R}$, set $r = |z|$ and let θ be the angle subtended from the ray going from $(0, 0)$ to (a, b) (written $\arg(z)$, called the *argument* of z and measured in the anti-clockwise direction), so that $a = r\cos(\theta)$ and $b = r\sin(\theta)$ and

$$z = r(\cos(\theta) + i\sin(\theta)).$$

We do not define the argument when $z = 0$ as this leads to ambiguities. The argument of $z \in \mathbb{C}$ is not unique since arguments differing by a multiple of 2π give rise to the same complex number. The argument θ of z for which $-\pi \leq \theta < \pi$ is called the *principal argument* of z and denoted by $\mathrm{Arg}(z)$.

Notice that if we formally manipulate the power series for sine and cosine we obtain the identity

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

This equation, which we take as the definition of $e^{i\theta}$ for $\theta \in \mathbb{R}$, is *Euler's formula* and may be justified by showing that $e^{i\theta}$ really behaves like an exponential. For example, $(e^{i\theta})^n = e^{in\theta}$, for $n \in \mathbb{Z}$ is a consequence of DeMoivre's Theorem:

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta), \quad n \in \mathbb{Z},$$

(clearly true when $n = 1$ and can be checked for $n = -1$, then proved by induction for $n > 1$ and $n < -1$). Care needs to be taken when n is a fraction such as $n = 1/2$, as the expression becomes multi-valued.

If we represent $z, w \in \mathbb{C}$ as $z = re^{i\theta}$ and $w = se^{i\phi}$, then we see that $z \cdot w = rse^{i(\theta+\phi)}$ and $z/w = r/se^{i(\theta-\phi)}$ (when $s \neq 0$).

12.2 Analytic functions in the complex plane.

In proving theorems about complex functions and limits of sequences, the various forms of the triangle inequality are often useful:

$$|z + w| \leq |z| + |w|, \quad \text{and} \quad ||z| - |w|| \leq |z - w|, \quad \text{for all} \quad z, w \in \mathbb{C}.$$

As we saw in Chapter 4, this tells us that the set $\mathbb{C}$ together with the distance $d(z, w) = |z - w|$ is a metric space. Using this metric, we can define limits and continuity as in an arbitrary metric space: open balls centered on $z_0 \in \mathbb{C}$, are then sets of the form

$$B_\delta(z_0) = \{z \in \mathbb{C} : |z - z_0| < \delta\},$$

for $\delta > 0$.

In particular, for a function $f(z)$ defined in an open ball centered on $z_0 \in \mathbb{C}$ (but not necessarily defined at z_0 itself), $\lim_{z\to z_0} f(z) = L$ if given any real number $\epsilon > 0$ there exists a real number $\delta > 0$ such that if $|z - z_0| < \delta$, then $|f(z) - L| < \epsilon$. The usual rules for limits are easily verified. Continuity can then be defined as follows:

Definition 12.2.1 (i) Let f be defined on an open ball centered on $x_0 \in \mathbb{C}$. Then f is *continuous* at z_0 if $\lim_{z\to z_0} f(z) = f(z_0)$.

(ii) If the domain of f is an open set $D \subseteq \mathbb{C}$, then f is continuous on D if it is continuous at every point of D.

All the results about continuity on a metric space are applicable. For example if f is continuous on $\mathbb{C}$ and the orbit of z_0 under f, $\mathrm{O}(z_0) = \{f^n(z_0) : n \in \mathbb{Z}^+\}$, has a unique limit point α, then the continuity f implies that α is a fixed point of f.

On the other hand, if the orbit has a finite set of limit points $\{z_1, z_2, \ldots, z_p\}$, all distinct, then a similar argument shows that this is a p-cycle (see Exercise 12.3).

Proposition 12.2.2 *Let $f, g : \mathbb{C} \to \mathbb{C}$ be functions continuous at $z = a$. Then*

(i) *$\overline{f}$, $f + g$, αf, $f \cdot g$ and f/g are continuous at a (we require $g(a) \neq 0$ for the continuity of f/g), $\alpha \in \mathbb{C}$.*

(ii) $|f|$, $\mathrm{Rl}(f)$ *and* $\mathrm{Im}(f)$ *are continuous as functions from* $\mathbb{C}$ *to* $\mathbb{R}$ *(where* $\mathrm{Rl}(f)$ *and* $\mathrm{Im}(f)$ *denote the real part and imaginary part of f respectively).*

(iii) *The composite of continuous functions is continuous (where defined).*

Proof of (ii). Let $\epsilon > 0$. Since f is continuous at $z = a$, there exists $\delta > 0$ such that if $0 < |z - a| < \delta$, then $|f(z) - f(a)| < \epsilon$. By the triangle inequality we then have

$$||f(z)| - |f(a)|| \leq |f(z) - f(a)| < \epsilon,$$

so that continuity of $|f|$ is immediate.

The continuity of $\mathrm{Rl}(f)$ follows in a similar way, but using the inequality: for $z = c + id$, $c, d \in \mathbb{R}$, $|\mathrm{Rl}(z)| = |c| \leq \sqrt{c^2 + d^2} = |z|$ and similarly for $\mathrm{Im}(f)$. $\square$

Now the derivative of a complex function can be defined in a natural way and the usual rules of differentiation will hold:

Definition 12.2.3 (i) The *derivative* $f'(z_0)$, of a complex valued function of a complex variable $f : \mathbb{C} \to \mathbb{C}$ is defined by

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

provided that this limit exists.

(ii) The function $f(z)$ is said to be *analytic* in an open ball $B_\delta(z_0)$ if the derivative exists at every point in the ball. We talk about $f(z)$ being analytic in a *neighborhood* of the point z_0 if there is an open ball containing z_0 throughout which $f(z)$ is analytic. f is analytic in an open set D if it is analytic inside every open ball contained in D.

The same limit rules and rules of algebra as in the real case show that for $z \in \mathbb{C}$, the derivative of $f(z) = z^n$ is $f'(z) = nz^{n-1}$ for $n \in \mathbb{Z}^+$, so that this function is analytic on $\mathbb{C}$. The usual differentiation rules apply: the sum, product, quotient and chain rules.

It can be shown that functions analytic on some domain D (for our purposes we may assume that D is an open ball, or possibly all of $\mathbb{C}$), are differentiable of all orders throughout that domain, so in particular, $f'(z)$ will be continuous. In fact, the property of being analytic on $B_\delta(z_0)$ is equivalent to $f(z)$ having a power series expansion about z_0, valid throughout $B_\delta(z_0)$:

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n, \quad a_n \in \mathbb{C}.$$

One of the first results of iteration theory was the fixed point theorem of Schröder (a variation of the real version from Chapter 1, the first rigorous proof was given by Koenigs: see [1] and [35]). We use the following property of limits:

Lemma 12.2.4 (i) *Suppose that* $\lim_{z\to a} f(z) = L$ *with* $|L| > 0$*, then there is a ball* $B_\delta(a)$ *for which* $|f(z)| > 0$ *for all* $z \in B_\delta(a)$*,* $z \neq a$*.*

(ii) *If* $\lim_{z\to a} f(z) = L$*, then* $\lim_{z\to a} |f(z)| = |L|$*. The converse is true when* $L = 0$*.*

Proof. (i) Choose $\epsilon = |L|/2$, then by the definition of limit, there exists $\delta > 0$ such that if $0 < |z - a| < \delta$, then $|f(z) - L| < \epsilon$. But then by the triangle inequality

$$||f(z)| - |L|| \leq |f(z) - L| < |L|/2$$

so

$$-|L|/2 < |f(z)| - |L| < |L|/2, \quad \text{or} \quad |f(z)| > |L|/2 > 0.$$

(ii) This follows in a straightforward manner from the definition of limit and the triangle inequality.

□

Theorem 12.2.5 *Let* $f(z)$ *be analytic in a neighborhood of* $\alpha \in \mathbb{C}$ *with* $f(\alpha) = \alpha$ *and* $|f'(\alpha)| < 1$*. Then for all* z *in some open ball centered on* α*,*

$$\lim_{n\to\infty} f^n(z) = \alpha.$$

Proof. We give a slightly different proof to that given in the real case, but this proof is valid there as well. By the definition of the derivative

$$|f'(\alpha)| = \left|\lim_{z\to\alpha} \frac{f(z)-f(\alpha)}{z-\alpha}\right| = \lim_{z\to\alpha} \left|\frac{f(z)-f(\alpha)}{z-\alpha}\right| < 1.$$

It follows from Lemma 12.1.3, there is an open ball D centered on α, and a constant $\lambda < 1$ for which

$$\frac{|f(z)-\alpha|}{|z-\alpha|} < \lambda$$

on D. In other words

$$|f(z)-\alpha| < \lambda|z-\alpha|.$$

Substituting $f(z)$ for z in the last inequality gives

$$|f^2(z)-\alpha| < \lambda|f(z)-\alpha| < \lambda^2|z-\alpha|,$$

and continuing in this way gives

$$|f^n(z)-\alpha| < \lambda^n|z-\alpha|.$$

Since $\lambda^n \to 0$ as $n \to \infty$, the result follows. □

Remark 12.2.6 When dealing with analytic functions, it is convenient to define an *attracting fixed point* α to be one for which $|f'(\alpha)| < 1$, and we say α is *repelling* if $|f'(\alpha)| > 1$. In a similar way to the real case, if p is a point of period r, then p is an *attracting periodic point* if

$$|(f^r)'(p)| = |f'(p)f'(f(p))f'(f^2(p))\dots f'(f^{r-1}(p))| < 1,$$

and similarly it is a *repelling periodic point* if

$$|(f^r)'(p)| = |f'(p)f'(f(p))f'(f^2(p))\dots f'(f^{r-1}(p))| > 1.$$

Such periodic points are said to be *hyperbolic* and as before, we have the *non-hyperbolic* case when the $|(f^r)'(p)| = 1$. Theorem 12.1.4 is now seen to generalize to the case of an attracting periodic point.

12.3 Some Important Complex Functions

We define some of the elementary complex functions $f : \mathbb{C} \to \mathbb{C}$ that we will be dealing with in the chapter. We are mainly concerned with *polynomial functions*:

$$f(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0, \quad a_i \in \mathbb{C}, \quad i = 0, 1, \dots, n.$$

Special cases, such as *translations*: $f(x) = z + \alpha$, *rotations*: $g(z) = az$, *quadratic functions*: $f_c(z) = z^2 + c$ will be seen to be of importance. We also take a look at the exponential, trigonometric and rational functions, and we derive some of their properties.

12.3.1 The Exponential Function: e^z. Let $z \in \mathbb{C}, z = x + iy$, where x and y are real. The *complex exponential function* is defined by:

$$e^z = e^x(\cos(y) + i\sin(y)).$$

Exercise 12.3.2 Show that the complex exponential function has the following properties for $z, w \in \mathbb{C}$:

(i) $e^{z+w} = e^z e^w$,

(ii) $(e^z)^n = e^{nz}$ if n is a positive integer,

(iii) $e^{2k\pi i} = 1$ for all $k \in \mathbb{Z}$.

Care needs to be taken in defining z^w: equations such as $(e^z)^w = e^{zw}$ are not generally true for arbitrary $z, w \in \mathbb{C}$.

Definition 12.3.3 The complex sine and cosine functions $\sin(z)$ and $\cos(z)$ are defined for $z \in \mathbb{C}$ by:

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}.$$

The tangent function $\tan(z)$ is defined to be

$$\tan(z) = \frac{\sin(z)}{\cos(z)}.$$

It is not defined for $z = \pi/2 + k\pi$, $k \in \mathbb{Z}$.

The exponential function and the functions sine and cosine are said to be *entire functions*, because it is not hard to show that they are analytic throughout the complex plane $\mathbb{C}$.

Exercise 12.3.4 Show that the trigonometric functions have the following properties for $z, w \in \mathbb{C}$:

(i) $\sin(z + w) = \sin(z)\cos(w) + \cos(z)\sin(w)$,

(ii) $\cos(z + w) = \cos(z)\cos(w) - \sin(z)\sin(w)$

(iii) $\tan(z+w) = \dfrac{\tan(z)+\tan(w)}{1-\tan(z)\tan(w)}$, $\tan(2z) = \dfrac{2\tan(z)}{1-\tan^2(z)}$.

(iv) $\sin(z)$ and $\cos(z)$ are periodic of period 2π and $\tan(z)$ is periodic of period π (so for example $\tan(z+\pi) = \tan(z)$ for all $z \neq \pi/2 + k\pi$, $k \in \mathbb{Z}$).

12.3.5 The Complex Logarithm Function: Log(z). The complex logarithm function is often defined as being multi-valued, but for our purposes we look at the logarithm function as a bijective function when the domain and range are restricted suitably. This is what is often called the *principal branch of the logarithm function*. We start with the exponential function $f : \mathbb{C} \to \mathbb{C}$, $f(z) = e^z$ and we see that it is never zero, so that its range is $\mathbb{C} \setminus \{0\}$. f is also not one-to-one since $e^{z+2k\pi i} = e^z$ for all $k \in \mathbb{Z}$ and all $z \in \mathbb{C}$, so the inverse of f is not defined. However, if we restrict f to the infinite horizontal strip, where $-\pi < \mathrm{Im}(z) \leq \pi$, then f will be one-to-one and the range remains unchanged. In particular, the function (also denoted by f):

$$f : S \to \mathbb{C} \setminus \{0\}, \quad f(z) = e^z,$$

is both one-to-one and onto, so has an inverse $g : \mathbb{C} \setminus \{0\} \to S$, where

$$S = \{z \in \mathbb{C} : z = x + iy, x, y \in \mathbb{R}, -\pi < y \leq \pi\}.$$

We write $g(z) = \mathrm{Log}(z)$, and call it the (*principal branch of the*) *logarithm function*. Since $g(f(z)) = z$ for all $z \in S$ and $f(g(z)) = z$ for all $z \in \mathbb{C} \setminus \{0\}$, we have:

$$\mathrm{Log}(e^z) = z, \ \forall\, z \in S, \quad \text{and} \quad e^{\mathrm{Log}(z)} = z, \ \forall\, z \in \mathbb{C} \setminus \{0\}.$$

Suppose now that $w = e^z$, where $z = x + iy \in S$ and $w = \rho e^{i\phi} \neq 0$, so $\rho > 0$ and $-\pi < \phi \leq \pi$, then

$$\rho e^{i\phi} = e^x e^{iy}, \quad \text{so} \quad \rho = e^x, \quad \text{and} \quad \phi = y.$$

It follows that

$$z = x + iy = \ln(\rho) + i\phi = \ln|w| + i\mathrm{Arg}(w),$$

so we have shown:

Theorem 12.3.6 *The logarithm function $g : \mathbb{C} \setminus \{0\} \to S$, $g(z) = \mathrm{Log}(z)$ satisfies*

$$\mathrm{Log}(z) = \ln|z| + i\mathrm{Arg}(z), \quad z \in \mathbb{C} \setminus \{0\},$$

where $\mathrm{Arg}(z)$ is the principal value of the argument of z ($-\pi < \mathrm{Arg}(z) \leq \pi$).

Remarks 12.3.7 1. $g(z) = \mathrm{Log}(z)$ is a bijective function, continuous and analytic everywhere except on the negative real axis.

2. In general $\text{Log}(z_1 z_2) \neq \text{Log}(z_1) + \text{Log}(z_2)$ because the sum of the arguments may not lie in $(-\pi, \pi]$. This equation holds when it is treated as a set identity for the multivalued function $\log(z)$, where $\log(z) = \{\text{Log}(z) + 2n\pi i : n \in \mathbb{Z}\}$. For example, $\text{Log}(i) = \ln|i| + i\pi/2 = i\pi/2$, and $\text{Log}(-1) = \ln|-1| + i\pi = i\pi$, and $\text{Log}(-i) = -i\pi/2$, whereas $\text{Log}(-1) + \text{Log}(i) = 3i\pi/2$.

12.3.8 The Complex Arctangent Function: Arctan(z). Starting with the tangent function $f(z) = \tan(z)$, we give an argument analogous to the one we gave for the logarithm function, to define the arctangent function. $f(z)$ is not defined where $\cos(z) = 0$, or $e^{2iz} + 1 = 0$, $z = \pi/2 + k\pi$, $k \in \mathbb{Z}$. Also, the equation $w = \tan(z)$ has no solution z when $w = \pm i$, since solving $w = \tan(z)$ gives

$$\frac{e^{2iz} - 1}{e^{2iz} + 1} = iw, \quad \text{or} \quad e^{2iz} = \frac{1 + iw}{1 - iw},$$

having no solution when $w = \pm i$.

In addition, $f(z) = \tan(z)$ is not one-to-one, for if $\tan(z_1) = \tan(z_2)$, then $z_1 - z_2 = k\pi$ for some $k \in \mathbb{Z}$. If we restrict $f(z)$ to the infinite vertical strip

$$T = \{z \in \mathbb{C} : z = x + iy, x, y \in \mathbb{R}, -\pi/2 < x < \pi/2\},$$

then we see that $f : T \to \mathbb{C} \setminus \{\pm i\}$ is both one-to-one and onto, so has an inverse $g : \mathbb{C} \setminus \{\pm i\} \to T$ which is written $g(z) = \arctan(z)$. Again we see that g is a bijective function with

$$\arctan(\tan(z)) = z, \ \forall\, z \in T, \quad \text{and} \quad \tan(\arctan(z)) = z, \ \forall\, z \in \mathbb{C} \setminus \{\pm i\}.$$

(We will need later the fact that even if $z \notin T$, $\tan(2\arctan(\tan(z))) = \tan(2z)$.)

From the equation $e^{2iz} = \dfrac{1 + iw}{1 - iw}$, $w \neq \pm i$, derived above, we get $w = \dfrac{1}{2i}\text{Log}\left(\dfrac{1 + iz}{1 - iz}\right)$, which gives:

Theorem 12.3.9 *If $g : \mathbb{C} \setminus \{\pm i\} \to T$, $g(z) = \arctan(z)$ is the principal branch of the arctangent function, then*

$$\arctan(z) = \frac{1}{2i}\text{Log}\left(\frac{1 + iz}{1 - iz}\right), \quad z \in \mathbb{C} \setminus \{\pm i\},$$

where we have used the principal branch of the logarithm function.

12.4 Newton's Method in the Complex Plane for Quadratic Functions

In this section we show that for a quadratic function having two distinct roots, the basins of attraction of the corresponding fixed points of the Newton function are given by two open half planes, divided by the perpendicular bisector of the line joining the two roots. We follow the method of Schröder (as outlined by Alexander [1]), whose study of Newton's Method for quadratic functions preceded that of Cayley by 10 years. Schröder's proof is a little more involved than the standard proof given in most texts (see the exercises), but it is interesting both from an historical point of view, and from the the study of properties of certain trigonometric functions. We first prove a result about the linear conjugacy of Newton's functions for quadratic maps having distinct roots.

Theorem 12.4.1 *Let $f(z) = az^2 + bz + c$ be a quadratic polynomial having two distinct roots. Then the Newton functions N_f and N_q are linearly conjugate, where $q(z) = z^2 - \alpha$, and $\alpha = b^2 - 4ac$.*

Proof. Since $f(z)$ has two distinct roots, $b^2 - 4ac \neq 0$. A calculation shows that

$$N_f(z) = \frac{az^2 - c}{2az + b} \quad \text{and} \quad N_q(z) = \frac{z}{2} + \frac{\alpha}{2z}.$$

Set $h(z) = 2az + b$, then we can check that $h \circ N_f(z) = N_q \circ h(z)$, using $\alpha = b^2 - 4ac$.
$\square$

Consequently, to prove the Schröder-Cayley Theorem concerning the basin of attraction of the fixed points of the Newton function arising from a quadratic polynomial, it suffices to prove it for polynomials of the form $q(z) = z^2 - \alpha$, where $\alpha \neq 0$.

Using the method of Theorem 1.4.10, it is straightforward to see that if $N_f(z) = z - f(z)/f'(z)$ is the Newton function of some complex function $f(z)$ for which $f(\alpha) = 0$, then $f'(\alpha) \neq 0$ if and only if α is a super-attracting fixed point of N_f. A generalization of this result (called Halley's method), was given by Schröder by applying Newton's method to the function $g(z) = f(z)/f'(z)$. The Newton function for g is

$$N_g(z) = z - \frac{f(z)f'(z)}{[f'(z)]^2 - f(z)f''(z)},$$

so that $f(z) = 0$ if and only if $N_g(z) = z$. Differentiating gives

$$N_g'(z) = 1 - \frac{[(f'(z))^4 - (f(z)f''(z))^2] - [f'(z)f''(z) - f(z)f'''(z)]\, f(z)f'(z)}{[(f'(z))^2 - (f(z)f''(z)]^2},$$

and substituting $z = \alpha$ gives $N_g'(\alpha) = 0$, so that α is a super-attracting fixed point for N_g (even when $f'(\alpha) = 0$).

We will use this in the proof of the following theorem:

Theorem 12.4.2 (Schröder [57], 1872, Cayley [15], 1882) *Let $f(z) = az^2 + bz + c$ be a complex quadratic function having two distinct roots α_1 and α_2. Join these points in $\mathbb{C}$ by a straight line and denote the perpendicular bisector of this line by L. The basin of attraction of the fixed points α_1 and α_2 of the Newton function N_f consists of all those points in the same open half plane determined by the line L. N_f is chaotic on L.*

Proof. It suffices to prove the theorem for $f(z) = z^2 - \alpha$, $\alpha \neq 0$ and we do this for $\alpha = 1$ as the general case is similar. In this case, if $g(z) = f(z)/f'(z)$, then $g(z) = (z^2 - 1)/2z$, and this gives the generalized Newton function as

$$M(z) = N_g(z) = z - \frac{g(z)}{g'(z)} = z - \frac{(z^2-1)}{2z}\frac{4z^2}{((z^2-1)2-(2z)^2)} = \frac{2z}{1+z^2}.$$

Now we know that

$$\frac{2\tan(z)}{1-\tan^2(z)} = \tan(2z),$$

and we set $w = \tan(z)$, then

$$\frac{2w}{1-w^2} = \tan(2\arctan(w)),$$

and now replace w by iz to give

$$M(z) = \frac{2z}{1+z^2} = -i\tan(2\arctan(iz)),$$

and iterating:

$$M^2(z) = -i\tan(2\arctan(i(-i\tan(2\arctan(iz))) = -i\tan(2^2\arctan(iz)),$$

where we have used some of the properties of the arctangent function derived earlier.

Now it is easily seen by induction that

$$M^n(z) = -i\tan(2^n\arctan(iz)),$$

where we show that $\lim_{n\to\infty} M^n(z) \to \pm 1$ (see the lemma below), depending on which side of the line L we have z lying on.

Suppose that $N(z) = N_f(z) = z/2 + 1/(2z)$, then it is clear that both $N(1/z) = N(z)$ and $M(1/z) = M(z)$. Set $h(z) = 1/z$, then it follows that

$$N \circ h(z) = N(1/z) = N(z) \quad \text{and} \quad h \circ M(z) = 1/M(z) = (z^2+1)/2z = N(z),$$

so that M and N are conjugate via h (a homeomorphism on the extended complex plane $\mathbb{C} \cup \{\infty\}$ - see the next section). In particular we see that $N^n \circ h = h \circ M^n$ for all $n \in \mathbb{Z}^+$.

We deduce that $\lim_{n\to\infty} N_f^n(z) = \pm 1$ as required. $\square$

Lemma 12.4.3 *If* $M(z) = -i\tan(2\arctan(iz))$, *then*

$$\lim_{n\to\infty} M^n(z) = \begin{cases} 1 & \textit{if} \;\; \mathrm{Rl}(z) > 0 \\ -1 & \textit{if} \;\; \mathrm{Rl}(z) < 0 \end{cases}$$

Proof. Consider

$$i\tan(2^n w) = \frac{e^{2^{n+1}iw} - 1}{e^{2^{n+1}iw} + 1} \quad \text{as} \quad n \to \infty.$$

Set $w = w_1 + iw_2$ where w_1 and w_2 are real., then

$$i\tan(2^n w) = \frac{e^{2^{n+1}iw_1}e^{-2^{n+1}w_2} - 1}{e^{2^{n+1}iw_1}e^{-2^{n+1}w_2} + 1},$$

where $|e^{2^{n+1}iw_1}| = 1$.

There are two cases to consider:

Case 1. $w_2 > 0$, then $e^{-2^{n+1}w_2} \to 0$ and $i\tan(2^n w) \to -1$ as $n \to \infty$.

Case 2. $w_2 < 0$, then $e^{2^{n+1}w_2} \to 0$ and

$$i\tan(2^n w) = \frac{e^{2^{n+1}iw_1} - e^{2^{n+1}w_2}}{e^{2^{n+1}iw_1} + e^{2^{n+1}w_2}} \to 1 \;\; \text{as} \;\; n \to \infty.$$

Now suppose that $w = \arctan(iz)$, then by Theorems 12.2.9 and 12.2.6

$$w = \frac{1}{2i}\log\left(\frac{1-z}{1+z}\right) = \frac{1}{2i}\left\{\ln\left|\frac{z-1}{z+1}\right| + i\theta\right\},$$

where $\theta = \arg\left(\dfrac{1-z}{1+z}\right)$ is the principal argument.

Thus

$$w = \frac{\theta}{2} - \frac{i}{2}\ln\left|\frac{z-1}{z+1}\right| = w_1 + iw_2,$$

where $w_2 = -\dfrac{1}{2}\ln\left|\dfrac{z-1}{z+1}\right|$. Now $w_2 > 0$ if $\left|\dfrac{z-1}{z+1}\right| > 1$, i.e., if $|z-1| > |z+1|$. This is the case when $\mathrm{Rl}(z) > 0$. Similarly $w_2 < 0$ when $\left|\dfrac{z-1}{z+1}\right| < 1$, i.e., when $\mathrm{Rl}(z) < 0$, and the result follows. □

To complete the proof of the theorem, we need to show that the Newton function is chaotic on the imaginary axis. This is left as an exercise (see Exercises 12.4). □

Exercises 12.4

1. Here we outline an alternative proof of the Schröder-Cayley Theorem.

(a) Let $f(z) = z^2 - 1$ and denote by $N_f(z)$ the Newton function of f. Show that N_f is conjugate to the map $f_0(z) = z^2$ on the extended complex plane $\widehat{\mathbb{C}}$ via a conjugacy T:

$$T \circ N_f = f_0 \circ T,$$

where T is the linear fractional transformation $T(z) = \frac{z-1}{z+1}$, which maps the attracting fixed points $\{-1, 1\}$ of N_f to the attracting fixed points $\{\infty, 0\}$ of f_0, and the repelling fixed point ∞ of N_f to the repelling fixed point 1 of f_0.

(b) Deduce that the basin of attraction of the attracting fixed point $z = 1$ of N_f is the open half-plane $\{z \in \mathbb{C} : \mathrm{Rl}(z) > 0\}$, and similarly, the basin of attraction of $z = -1$ is the open half-plane $\{z \in \mathbb{C} : \mathrm{Rl}(z) < 0\}$.

(c) Show that N_f is chaotic on the imaginary axis (Hint: Use (a) above, and the fact that $f_0(z) = z^2$ is chaotic on S^1).

2. Let $p(z)$ be a polynomial of degree $d > 1$ with Newton function

$$N_p(z) = z - \frac{p(z)}{p'(z)}.$$

(a) Show that $N_p'(z) = \dfrac{p(z)p''(z)}{(p'(z))^2}$.

(b) Suppose that $p(z) = z^m q(z)$ where $q(0) \neq 0$. Check that $N_p(0) = 0$, so that $z = 0$ is a fixed point. Show that $N_p'(0) = (m-1)/m$, so $z = 0$ is an attracting fixed point which is super-attracting only when $m = 1$.

(c) Check that the above holds for $f(z) = z^3 - z^2$.

3. (a) Show that the maps $f_0(z) = z^2$ and the Newton function N_{f_1} where $f_1(z) = z^2 + 1$ are conjugate (Hint: use the linear fractional transformation $T(z) = \frac{z-i}{z+i}$).

(b) Deduce that the Julia set of N_{f_1} is the real axis, and the basin of attraction of the fixed point i is the upper half plane, and that of $-i$ is the lower half plane.

12.5 The Dynamics of Polynomials and the Riemann Sphere

We briefly introduce the reader to the dynamics of complex polynomials. A detailed treatment is beyond the scope of this book as it requires a more advanced knowledge of complex function theory (see for example Devaney [20] or Milnor [43]). The recent survey by Stankewitz and Rolf [62], is a useful resource.

In the previous section we essentially looked at the dynamics of the Newton function $N_f(z) = (z + 1/z)/2$ for $f(z) = z^2 - 1$. We saw that if we iterate z_0 under N_f for points with $\mathrm{Rl}(z_0) > 0$, the iterates go to 1, whilst those with $\mathrm{Rl}(z_0) < 0$ go to -1. The point is, the dynamics of N_f are trivial for these values of z_0. The interesting dynamics of the map occurs on the imaginary axis, where the map is chaotic. This set is called the *Julia set* of the map, which can be defined as the closure of the repelling fixed points of the function, although we won't use this definition. Before we formally define the notion of Julia set (named after Gaston Julia whose prize winning work was published in 1918 [32]), let us consider some polynomials giving rise to Julia sets:

Examples 12.5.1 1. The quadratic map $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z^2$ has two fixed points $z = 0$ and $z = 1$. Now $f'(z) = 2z$, so $z = 0$ is a super-attracting fixed point. In fact, since $f^n(z) = z^{2^n}$, if $|z_0| < 1$, then $f^n(z_0) \to 0$ as $n \to \infty$, so the basin of attraction of $z = 0$ is the interior of the unit circle, since f maps the circle to itself. To be more specific, if $z = re^{i\theta}$, then $z^2 = r^2e^{2i\theta}$, f acts as a combination of a contraction (if $r < 1$) and a rotation. If $|z_0| > 1$, then $r > 1$ and we see that $|f^n(z_0)| \to \infty$ as $n \to \infty$. We saw in Section 5.1 that f has a countable dense collection of repelling periodic points on the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, and in fact, f restricted to S^1 is chaotic. Consequently, in this example the interesting dynamics of f occurs on the unit circle, and this is the Julia set of f. Notice that it is natural to regard ∞ as an attracting fixed point of f ($f(\infty) = \infty$), since points having modulus greater than one move towards ∞ under iteration. The basin of attraction of ∞ is therefore $B_f(\infty) = \{z \in \mathbb{C} : |z| > 1\}$.

2. Let $f(z) = z^2 - 1$, which has fixed points at $z = (-1 \pm \sqrt{5})/2$, and we can again think of $z = \infty$ as a fixed point of f. Set $h(z) = 1/z$, an analytic function on its domain, and write

$$g(z) = h^{-1} \circ f \circ h(z) = h^{-1}(f(1/z)) = h^{-1}(1/z^2 - 1) = \frac{1}{1/z^2 - 1} = \frac{z^2}{1 - z^2}.$$

Then since $h \circ g = f \circ h$, g and f are conjugate via h where h maps fixed points to fixed points ($h((1 \pm \sqrt{5})/2) = (-1 \pm \sqrt{5})/2$ and $h(0) = \infty$, $h(\infty) = 0$). (We can think of h as a homeomorphism on the set $\mathbb{C} \cup \{\infty\}$, in a manner to made more precise shortly).

This conjugation tells us that the behavior of f near ∞ is the same as the behavior of g near 0. Consequently we study the map f defined on the extended complex plane:

$$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\},$$

by requiring $f(\infty) = \infty$ and treating ∞ as just another attracting fixed point. This point of view was introduced by Koenigs [35] (see Alexander [1]). Using the function $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, the derivative of $f(z)$ at infinity is defined to be the derivative of $g(z)$ at zero. Now

$$g'(z) = \frac{2z}{(1 - z^2)^2}, \quad \text{so} \quad g'(0) = 0,$$

so $z = 0$ is a super-attracting fixed point for $g(z)$, and we conclude that $z = \infty$ is a super-attracting fixed point for $f(z)$. In general, we define the derivative of $f(z)$ at $z = \infty$ to be the derivative of $g(z)$ at $z = 0$.

3. We saw that the Newton function $N_f(z) = (z+1/z)/2$ for $f(z) = z^2-1$ is conjugate to its Halley's function $M(z) = 2z/(z^2 + 1)$ via $h(z) = 1/z$, so the fixed point $z = \infty$ of N_f has the same behavior as the fixed point $z = 0$ of M. Now $M'(0) = 2$, so it is a repelling fixed point.

4. If $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ ($a_n \neq 0$, $\deg(p(z)) \geq 2$), is a polynomial, then we can extend $p(z)$ to $\widehat{\mathbb{C}}$ by setting $p(\infty) = \infty$, a fixed point. Set $q(z) = h^{-1} \circ p \circ h(z) = 1/p(1/z)$, then

$$q(z) = \frac{z^n}{a_0 z^n + a_1 z^{n-1} + \cdots + a_n}.$$

As before, ∞ is an attracting fixed point for $p(z)$ when

$$|p'(\infty)| = |q'(0)| < 1,$$

and a calculation shows that this is always the case for the polynomial $p(z)$.

A metric can be defined on $\widehat{\mathbb{C}}$ (see Exercises 12.5) which makes the set $D = \{z \in \widehat{\mathbb{C}} : |z| > r\}$ an open ball centered on ∞, so that $h(D) = D^*$ where $D^* = \{z \in \mathbb{C} : |z| < 1/r\}$ is the corresponding open ball centered on 0. We can then check that p is continuous functions on $\widehat{\mathbb{C}}$, and that h is a homeomorphism.

12.5.2 The Riemann Sphere $\widehat{\mathbb{C}}$

The set $\widehat{\mathbb{C}}$ can be identified with the *Riemann sphere* $\mathbb{S}$: take a sphere sitting on the complex plane $\mathbb{C}$ with its south pole at the origin. From the north pole N, draw a line to the point z in $\mathbb{C}$ and denote by $S(z)$ the unique point where the line cuts the sphere. The map $S : \mathbb{C} \to \mathbb{S} \setminus \{N\}$ from the complex plane onto the sphere with the north pole omitted, sending z to $S(z)$ is a homeomorphism which extends, on setting $S(\infty) = N$ to a homeomorphism of $\widehat{\mathbb{C}}$ onto all of $\mathbb{S}$. This sphere is called the *Riemann sphere* and we can think of a function like $f(z) = z^2$ as being defined on $\mathbb{S}$. The induced map $F : \mathbb{S} \to \mathbb{S}$ is defined by $F(w) = S \circ f \circ S^{-1}(w)$. However, we usually think of F as being defined on $\widehat{\mathbb{C}}$ and identify this set with the Riemann sphere.

To make the above more precise, we need to define a metric on both $\widehat{\mathbb{C}}$ and $\mathbb{S}$ which makes S defined above a homeomorphism. We leave this to the exercises.

Examples 12.5.3 1. We studied the affine maps $f(z) = az+b$, $a \neq 0$, in the real case in Chapter 1. Setting $f(\infty) = \infty$, these are homeomorphisms of $\widehat{\mathbb{C}}$ with fixed points at $z_0 = 1/(1-a)$ (when $a \neq 1$) and $z_1 = \infty$. As before, the fixed point z_0 is attracting when $|a| < 1$ and repelling when $|a| > 1$ (in the latter case ∞ is attracting). When $a = 1$, every $z \in \widehat{\mathbb{C}}$ is a fixed point. The dynamics is trivial in all cases.

2. Consider the *linear fractional transformation*

$$f(z) = \frac{az+b}{cz+d}, \quad \text{where} \quad ad - bc \neq 0.$$

Such maps are also called *Mobius transformations*. We can extend f to all of $\widehat{\mathbb{C}}$ by setting $f(\infty) = a/c$ and $f(-d/c) = \infty$, (when $c \neq 0$). f has an inverse

$$f^{-1}(z) = \frac{-dz+b}{cz-a},$$

again a linear fractional transformation (use the fact that $ad-bc \neq 0$ to show that f is one-to-one). In this way it can be seen that f is a homeomorphism of $\widehat{\mathbb{C}}$. The induced map $F : \mathbb{S} \to \mathbb{S}$ on the Riemann sphere is given by $F(w) = S \circ f \circ S^{-1}(w)$, so in particular $F(N) = S(f(S^{-1}(N))) = S(f(\infty)) = S(a/c)$, and if $S(-d/c) = w_0$, then $F(w_0) = S(f(S^{-1}(w_0))) = S(f(-d/c)) = S(\infty) = N$. Again F is a homeomorphism of the Riemann sphere. These maps have an interesting but uncomplicated dynamics which we shall study in the exercises.

Exercises 12.5

1. Show that if $p(z)$ is a polynomial having degree at least 2, then $p(\infty) = \infty$ and $|p'(\infty)| < 1$, so that ∞ is an attracting fixed point for p. What happens if $p(z) = az+b$ for some $a, b \in \mathbb{C}$?

2. Let $p(z) = z^2 - z$. Show that p has no points of period 2. (It can be shown that if a polynomial q of degree at least 2 has no periodic points of period n, then $n = 2$, and q is conjugate to $p(z) = z^2 - z$ - see [10]).

3. Let $p(z)$ be a polynomial of degree $d \geq 2$. Show that $z = \infty$ is a repelling fixed point for the Newton function N_p, with $N_p'(\infty) = d/(d-1) > 1$. What happens if $p(z) = az + b$ for some $a, b \in \mathbb{C}$?

4. Prove that if $f(z) = \dfrac{az+b}{cz+d}$ is a linear fractional transformation with $(a-d)^2+4bc = 0$, then $f(z)$ has a unique fixed point. Prove that in this case $f(z)$ is conjugate to a translation of the form $g(z) = z + \alpha$.

5. Prove that if $f(z) = \dfrac{az+b}{cz+d}$ has two fixed points, then $f(z)$ is conjugate (via a linear fractional transformation) to a linear transformation of the form $g(z) = \alpha z$.

6. Show that the linear fractional transformation

$$T(z) = \frac{z-1}{z+1},$$

maps the imaginary axis in the complex plane onto the unit circle S^1.

7. Use the above to determine a closed form for $f^n(z)$ in each of the above two cases.

8. The Riemann sphere $\mathbb{S}$ is a sphere of radius $1/2$ sitting on the complex plane $\mathbb{C}$ so that its south pole S at $(0, 0, 0)$ is in contact with the origin $z = 0$ in $\mathbb{C}$, and the North pole N is at $(0, 0, 1)$. If P is a point on the complex plane with coordinates

(x, y) (so $z = x + iy$), we obtain the corresponding point on the sphere by joining N to P with a straight line and letting the point of intersection with the sphere be $S(z)$. Show that the coordinates of $S(z)$ are

$$S(z) = \left(\frac{\mathrm{Rl}(z)}{1+|z|^2}, \frac{\mathrm{Im}(z)}{1+|z|^2}, \frac{|z|^2}{1+|z|^2} \right).$$

9. Define a distance ρ on the Riemann sphere by $\rho(w_1, w_2) =$ the shortest distance between the points w_1 and w_2 on the sphere. Now define a distance d on the extended complex plane $\widehat{\mathbb{C}}$ by $d(z_1, z_2) = \rho(S(z_1), S(z_2))$. Show that d is a metric on $\widehat{\mathbb{C}}$ and the set $D_r = \{z \in \mathbb{C} : |z| > r\}$ is open in $\widehat{\mathbb{C}}$ for $r > 0$. It can be shown that

$$d(z_1, z_2) = \int_{z_1}^{z_2} \frac{1}{1+|z|^2}\, dz.$$

10. With respect to the metric defined in the previous problem, show that $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, $h(z) = 1/z$, $h(0) = \infty$ and $h(\infty) = 0$ is a homoemorphism. Show that any polynomial p is continuous on $\widehat{\mathbb{C}}$.

11. Let $f(x) = x^2 + a^2$ with Newton function $N_f(x) = (x - a^2/x)/2$. Show that $N_f(x)$ is conjugate to the doubling map

$$D(x) = \begin{cases} 2x & \text{if } \ 0 \le x < 1/2 \\ 2x - 1 & \text{if } \ 1/2 \le x < 1 \end{cases}$$

(Hint: Set $h(x) = a\tan(\pi x/2)$ and show that $N_f \circ h(x) = h \circ C(x)$, where

$$C(x) = \begin{cases} 2x + 1 & \text{if } \ -1 \le x < 0 \\ 2x - 1 & \text{if } \ \ \ 0 \le x < 1 \end{cases}.$$

Now show that C is conjugate to D via $k(x) = (x + 1)/2$).

Use this result to complete the proof of the Schroder-Cayley Theorem: Show that the Newton function from that theorem, restricted to the imaginary axis can be represented as N_f above, and hence it is chaotic on the imaginary axis.

12.6 The Julia Set

We have mentioned that the Julia set of a complex mapping $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is the set on which the interesting dynamics is concentrated. It will be denoted by $J(f)$, and in this section we define the set $J(f)$ and determine some of its properties.

For $f(z) = z^2$ it is the unit circle: $J(f) = S^1$, and for the map $N_f(z) = (z+1/z)/2$ it is the imaginary axis. It can be shown that if $E(z) = e^z$, then $J(E) = \mathbb{C}$ (this was shown in 1981 by Misiurewicz [45], answering an open question of Julia). We are mainly concerned with quadratic maps: $f_c(z) = z^2 + c$ for different values of the parameter $c \in \mathbb{C}$ since these are easiest to deal with, but are still interesting with many difficult questions. The Julia set of f_0 is $J(f_0) = S^1$, a closed set invariant under f_0, containing all the repelling periodic points of f_0 - a set on which all of the interesting dynamics of f_0 takes place. We shall see that these properties are typical of Julia sets.

Although the map $f_0(z) = z^2$ exhibits highly chaotic behavior on its Julia set $J(f_0)$, its study may lead one to believe that Julia sets are generally nice smooth curves. Nothing could be farther from the truth, as a variation of the parameter c gives rise to quadratic maps whose Julia sets are fractals. In fact D. Ruelle [54], has shown that for quadratic maps $f_c(z) = z^2+c$ with c small, the fractal dimension is approximately

$$d_c = 1 + \frac{|c|^2}{4\log(2)},$$

and so is indeed a fractal in these cases. The question of what value the fractal dimension can take for other values of c is still an open and difficult question.

The Julia set of f is often defined as the closure of the repelling periodic points of f. We restrict our attention to polynomials and give a definition that is easier to work with for our purposes. We saw that for polynomials of degree at least 2, ∞ is always an attracting fixed point. We use this to define the Julia set of a polynomial:

Definition 12.6.1 The *basin of attraction* of ∞ for the polynomial $p(z)$ having degree at least 2, is the set

$$B_p(\infty) = \{z \in \mathbb{C} : p^n(z) \to \infty \ \text{ as } \ n \to \infty\}.$$

Definition 12.6.2 (i) The *Julia set* $J(p)$ of the polynomial $p(z)$ is the boundary of the set $B_p(\infty)$, i.e., the set $\overline{B}_p(\infty) \setminus B_p(\infty)$.

(ii) The *filled-in Julia set* $K(p)$ is the set $K(p) = \mathbb{C} \setminus B_p(\infty)$, of all those points that do not converge to ∞ under iteration by p. Therefore, the Julia set is also the boundary of the set $K(p)$.

(iii) The *Fatou set* $F(p)$ is the complement of the Julia set: $F(p) = \mathbb{C} \setminus J(p)$.

12.6.3 Properties of the Julia Set of a Polynomial

The following are the main properties of Julia sets of polynomials (these also hold for certain more general functions). We shall prove some of these properties.

1. $J(p)$ is a non-empty, closed, bounded and uncountable set.

2. The Julia sets of p and p^r for $r \in \mathbb{Z}^+$ are identical.

3. $J(p)$ is *completely invariant* under p, i.e., and $p^{-1}(J(p)) = J(p)$.

4. The repelling periodic points of p are dense in $J(p)$.

5. The Julia set is either *path connected* (there is a curve joining any two points of $J(p)$), or $J(p)$ is totally disconnected (a Cantor like set called *fractal dust*).

We shall show in the next section that both $J(p)$ and $K(p)$ are non-empty, closed, bounded sets which are invariant under p, when p is a quadratic polynomial. It is not hard to show that conjugate maps have homeomorphic Julia sets. We shall not prove it, but it can be shown that homeomorphisms map closed bounded subsets of $\mathbb{C}$ (*compact sets*) to closed bounded subsets. Notice that if a set C is completely invariant under f (i.e., $f^{-1}(C) = C$), then $f(C) = f(f^{-1}(C)) = C$, so that $f(C) = C$.

12.6.4 The Quadratic Maps $f_c(z) = z^2 + c$

We study the quadratic maps $f_c(z) = z^2 + c$, since these are the easiest to deal with, and any quadratic map is conjugate to a map of this form. We saw in Section 6.3 that if c is real, then there is a linear conjugacy between f_c and the logistic map $L_\mu(z) = \mu z(1 - z)$ when $c = (2\mu - \mu^2)/4$. Specifically we have:

Proposition 12.6.5 *$f_c : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ defined by $f_c(z) = z^2 + c$, is conjugate to the map $L_\mu : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, $L_\mu(z) = \mu z(1 - z)$ via the conjugacy $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, $h(z) = -z/\mu + 1/2$, when $c = (2\mu - \mu^2)/4$.*

Proof. It is straight forward to check that $h \circ f_c = L_\mu \circ h$, when $c = (2\mu - \mu^2)/4$. $\square$

If $c < -2$ we see that f_c is conjugate to a logistic map L_μ with $\mu > 4$. It follows from Section 4.8 that for $x \in \mathbb{R}$, $L^n(x) \to -\infty$ for all x except for those values of x in some Cantor set contained in $[0, 1]$. The same conjugacy holds when these functions are treated as functions of a complex variable, and the above discussion indicates that we have complicated behavior of f_c for $|c|$ large. In particular, when $c = -2$, $\mu = 4$, so that f_{-2} is conjugate to L_4, a map which is chaotic as a real function on $[0, 1]$. We shall see that the Julia set is fairly simple in this case - the interval $[-2, 2]$ (the image of the interval $[0, 1]$ under h^{-1}).

Note that the above result tells us that the map $L_2(z) = 2z(1 - z)$ is conjugate to $f_0(z) = z^2$, the latter map being chaotic on the unit circle, whereas we saw that the dynamics of the real function L_2 are relatively tame. In the real case the conjugacy is between the dynamical systems L_2 on $[0, 1]$ and f_0 on $[-1, 1]$. The complicated dynamics of L_2 is on the image of the unit circle S^1 under h (an ellipse), and this will be the Julia set of L_2. We start with some more general results about polynomials of the form $F_c(z) = z^d + c$, where $c \in \mathbb{C}$ and $d \in \{2, 3, 4, \ldots\}$ is fixed. It can be shown that any polynomial of degree $d > 2$ is linearly conjugate to one of the form

$$p(z) = z^d + a_{d-2}z^{d-2} + a_{d-3}z^{d-3} + \cdots + a_1 z + a_0,$$

so polynomials of the form $F_c(z) = z^d + c$ do not represent the most general polynomials of degree d for $d > 2$.

Proposition 12.6.6 *Let $z \in \mathbb{C}$ with $|z| > 2$ and $|z| > |c|$. Then if $F_c(z) = z^d + c$, we have $F_c^n(z) \to \infty$ as $n \to \infty$.*

Proof. Suppose the hypothesis of the proposition holds for $z \in \mathbb{C}$, then by the triangle inequality and since $|z| > |c|$,

$$|F_c(z)| = |z^d + c| \geq |z^d| - |c| = |z|^d - |c| > |z|^d - |z| = |z|(|z|^{d-1} - 1).$$

Since $|z| > 2$, if $|z|^{d-1} - 1 = 1 + \alpha$, then $\alpha > 0$ and

$$|F_c(z)| > (1 + \alpha)|z|.$$

If we set $w = F_c(z)$ then $|w| > |z|$ and using the same argument, $|F_c(w)| > (1+\alpha')|w|$, where $\alpha' = |w|^{d-1}-2$, so that $|F_c^2(z)| > (1+\alpha')(1+\alpha)|z| > (1+\alpha)^2|z|$ (where it is easily checked that $\alpha' > \alpha$). An induction argument now shows that $|F_c^n(z)| > (1 + \alpha)^n|z|$, for all $n \in \mathbb{Z}^+$, so that $F_c^n(z) \to \infty$ as $n \to \infty$. $\square$

Theorem 12.6.7 *The filled-in Julia set $K(F_c)$ of $F_c(z) = z^d + c$ is a closed, bounded and F_c-invariant set.*

Proof. Set $A = \{z \in \mathbb{C} : |z| \leq r_c\}$, where $r_c = \max\{|c|, 2\}$, then A is a closed bounded set (the ball $B_{r_c}(0)$ of radius r_c centered on the origin). Then we can check that $F_c^{-1}A \subseteq A$, for if not there exists $w \in F_c^{-1}A \setminus A$, so $F_c(w) \in A$ with $w \notin A$. Thus $|w| > r_c$, so by the proof of Proposition 12.6.4,

$$|F_c(w)| > |w| > r_c,$$

contradicting the fact that $F_c(w) \in A$. Continuing this line of argument inductively, we see that

$$A \supseteq F_c^{-1}A \supseteq F_c^{-2}A \supseteq \cdots \supseteq F_c^{-n}A \cdots .$$

We now claim that

$$K(F_c) = \cap_{n=0}^{\infty} F_c^{-n}A.$$

Let $z \in K(F_c)$, then if there exists $n \in \mathbb{Z}^+$ with $z \notin F_c^{-n}A$, then $F_c^n(z) \notin A$, so that $F_c^k(F_c^n(z)) \to \infty$ as $k \to \infty$, contradicting z being in $K(F_c)$.

On the other hand, suppose that $z \in \cap_{n=0}^{\infty} F_c^{-n}A$, then $F_c^n(z) \in A$ for all $n \in \mathbb{Z}^+$, so $|F_c^n(z)| < r_c$ for all $n \in \mathbb{Z}^+$, and $z \in K(F_c)$.

Now K_c is non-empty since it contains the fixed points of F_c (these are the solutions of the equation $z^d + c = z$, which by the Fundamental Theorem of Algebra always has solutions). The iterates of these points under F_c never change, so they must lie in K_c.

The continuity of the map F_c implies that each of the sets $F_c^{-n}A$ is closed and non-empty, so their intersection is closed. Clearly $K_c \subseteq A$, so it is bounded. Finally

$$K(F_c) = A \cap F_c^{-1}A \cap F_c^{-2} \cap \cdots \quad \text{so} \quad F_c^{-1}K(F_c) = F_c^{-1}A \cap F_c^{-2} \cap \cdots \supseteq K(F_c),$$

but since $F_c^{-1}A \subseteq A$, we must have equality, i.e., $F_c^{-1}(K(F_c)) = K(F_c)$. □

Corollary 12.6.8 *The Julia set $J(F_c)$ of $F_c(z) = z^d + c$ is a closed, bounded and F_c-invariant set.*

Proof. Since $K(F_c)$ is a closed set, the basin of attraction of ∞ is open and clearly invariant under F_c. It follows that $J(F_c) = K(F_c) \cap \overline{B}_{F_c}(\infty)$ is a closed set, being the intersection of closed sets. Clearly it is bounded, and it is non-empty since any non-empty open set in $\mathbb{C}$ will have non-empty boundary. Furthermore,

$$F_c^{-1}J(F_c) = F_c^{-1}K(F_c) \cap F_c^{-1}(\overline{B}_{F_c}(\infty)) = J(F_c),$$

because clearly the basin of attraction of ∞ is an invariant set. □

Example 12.6.9 Consider the quadratic map $f_{-2}(z) = z^2 - 2$. The orbit of 0 is the set $O(0) = \{0, -2, 2\}$ (since $z = 2$ is a fixed point which is repelling). This is a bounded set, so 0, -2 and 2 are in the filled-in Julia set. Besides f_0, this map is the only f_c whose Julia set can be explicitly determined in a simple way. We will show that

$$K(f_{-2}) = J(f_{-2}) = [-2, 2].$$

Set $D^* = \{z \in \mathbb{C} : |z| > 1\} = B_{f_0}(\infty)$, the basin of attraction of infinity for f_0. Define

$$h : D^* \to \mathbb{C}, \quad \text{by} \quad h(z) = z + \frac{1}{z},$$

then we claim that h is a homeomorphism onto the set $\mathbb{C} \setminus [-2, 2]$.

Claim 1. h is one-to-one, for suppose that $h(z) = h(w)$, then

$$z + \frac{1}{z} = w + \frac{1}{w} \Rightarrow zw(z - w) = z - w,$$

so if $z \neq w$, then $zw = 1$. If $|z| > 1$, then $|w| = 1/|z| < 1$, so that h is one-to-one on D^*.

Claim 2. $h(D^*) = \mathbb{C} \setminus [-2, 2]$, for if $w \in \mathbb{C}$ with $h(z) = w$, then

$$z^2 - zw + 1 = 0, \quad \text{so that} \quad z = \frac{w \pm \sqrt{w^2 - 4}}{2}.$$

If z_1 and z_2 are the two solutions, $z_1 z_2 = 1$, so either z_1 lies in D^* or z_2 lies in D^*, or both belong to S^1 (the unit circle).

In the latter case, $h(z_1) = h(z_2) \in [-2, 2]$ (since $z + 1/z = z + \bar{z} = 2\mathrm{Rl}(z)$ for $z \in S^1$). In the former case there exists $z \in D^*$ with $f(z) = w$. It follows that h is onto.

It is easy to see that $h : D^* \to \mathbb{C} \setminus [-2, 2]$ is also continuous and its inverse is continuous, so it is a homeomorphism.

We can now check that $h \circ f_0(z) = f_{-2} \circ h(z)$ for all $z \in \widehat{\mathbb{C}}$, so that f_0 and f_{-2} are conjugate on their respective domains.

Note that the map

$$h : S^1 \to [-2, 2], \;\; h(z) = z + \frac{1}{z},$$

is a two-to-one onto map (except at $\pm 1 \in S^1$), so that f_{-2} restricted to $[-2, 2]$ is a quasi-factor of f_0 restricted to S^1 (see Section 6.1), so that f_0 being chaotic on S^1 implies that f_{-2} is chaotic on $[-2, 2]$, with $f_{-2}^n(z)$ bounded for $z \in [-2, 2]$.

Claim 3. $K(f_{-2}) = J(f_{-2}) = [-2, 2]$. We know that the basin of attraction of ∞ for f_0 is D^*, so putting the above information together, the basin of attraction of ∞ for f_{-2} must be $\mathbb{C} \setminus [-2, 2]$. It follows that the Julia set of f_{-2} is $[-2, 2]$.

Remarks 12.6.10 1. It can be shown that the restriction of the map F_c to its Julia set is chaotic.

2. If $0 \in K(F_c)$, then the Julia set of F_c is pathwise connected.

3. If $0 \notin K(F_c)$, then the Julia set is totally disconnected (a type of Cantor set called *fractal dust*).

We will examine the importance of the orbit of the critical point $z = 0$ being bounded, for the quadratic maps $f_c(z) = z^2 + c$ in the next section. First we show that the Julia set contains all of the repelling periodic points of f_c. We will need the following theorem which is a special case of a standard result in a first course in complex analysis:

Theorem 12.6.11 *Let $p(z)$ be a complex polynomial with $|p(z)| \leq M$ for all z in the closed ball $\overline{B}_r(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq r\}$. Then*

$$|p'(z_0)| < \frac{M}{r}.$$

We use this to prove:

Theorem 12.6.12 *Let z_0 be a repelling periodic point for $f_c(z) = z^2 + c$. Then $z_0 \in J(f_c)$.*

Proof. Suppose that $z_0 \notin J(f_c)$, where z_0 is a repelling periodic point having period n. Then we have

$$|(f_c^n)'(z_0)| = \lambda > 1.$$

Since z_0 is a periodic point, the orbit of z_0 is finite, so $z_0 \in K(f_c) \setminus J(f_c)$, i.e., it belongs to the interior of $K(f_c)$. It follows that there is an open ball

$$B_r(z_0) \subseteq K(f_c),$$

and no point within this ball goes to ∞ under iteration by f_c.

For each $k \in \mathbb{Z}^+$, f_c^{kn} is a polynomial and for each $z \in \overline{B}_r(z_0)$, we have

$$|f_c^{kn}(z)| \leq \max\{|c|, 2\}.$$

Let $M = \max\{|c|, 2\}$, then by the last theorem, we must have for each $k \in \mathbb{Z}^+$

$$|(f_c^{kn})'(z_0)| < \frac{M}{r}.$$

But (see the exercises)

$$|(f_c^{kn})'(z_0)| = \lambda^k \to \infty \quad \text{as} \quad k \to \infty,$$

a contradiction. $\square$

Exercises 12.6

1. Show that $z = 0$ is eventually periodic for $f_i(z) = z^2 + i$. Show that $z = -i$ is a repelling periodic point for f_i and hence belongs to the Julia set $J(f_i)$.

2. If z_0 is a period n-point for f, with $z_l = f^l(z_0)$, $l = 0, 1, \ldots, n-1$, show that $(f^n)'(z_l)$ is independent of l. Deduce that if $|(f^n)'(z_0)| = \lambda$, then $|(f^{kn})'(z_0)| = \lambda^k$ for $k \in \mathbb{Z}$.

3. Show that f_0 and f_{-2} are semi-conjugate on their Julia sets via $h(z) = z + 1/z$, i.e., $h : S^1 \to [-2, 2]$ is onto and $h \circ f_0 = f_{-2} \circ h$. (Assume the Julia sets of f_0 and f_{-2} are S^1 and $[-2, 2]$ respectively).

4. Show that $K(f_c) \neq J(f_c)$ whenever f_c has a bounded attracting periodic orbit. Is the converse true true? (Hint: For the converse, consider an n-periodic point p with $|(f^n)'(p)| = 1$).

5. Prove that if both $f(C) \subseteq C$ and $f^{-1}(C) \subseteq C$, then $f^{-1}(C) = C$, i.e., C is completely invariant under f.

6. (a) Prove that any quadratic polynomial $p(z)$ is linearly conjugate to one of the form $q(z) = z^2 + c$. Use this to show that $p(z) = z^2 - z$ is conjugate to $q(z) = z^2 - 3/4$.

(b) Let $p(z) = z^2 + c$. Explain why $p(z) - z$ divides $p^2(z) - z$. Use this to show that if p has no points of period 2, then $p(z) = z^2 - 3/4$.

(c) Prove that any cubic polynomial $p(z)$ is linearly conjugate to one of the form $q(z) = z^3 + az + b$.

12.7 The Mandelbrot Set M

Quadratic Julia sets and the Mandelbrot set arise simply from the sequence of complex numbers defined inductively by $z_{n+1} = z_n^2 + c$, where c is a constant.

Julia sets are defined by fixing c and letting z_0 vary over $\mathbb{C}$, whilst the Mandelbrot set is obtained by fixing $z_0 = 0$ and letting the parameter c vary. Let c be fixed, then if z_0 (or z_n) has large absolute value (for some $n \in \mathbb{Z}^+$) then $z_n \to \infty$ as $n \to \infty$. But there are some starting values of z_0 for which z_n remains bounded in absolute value for all n. The set of all such z_0 is called the *filled in Julia set*, $K(f_c)$ if $f_c(z) = z^2 + c$. (The Julia set being the boundary of $K(f_c)$). By varying c the types of Julia sets arising vary considerably, some being in one piece (connected), others consisting of many disjoint (totally disconnected) sets - a type of Cantor set referred to as *fractal dust*.

The Mandelbrot set M is defined to be the set of all complex numbers c for which $K(f_c)$ is a connected set. Although the Mandelbrot set does not have the same type of linear self similarity as the Koch snowflake, it is a fractal, and its fractal dimension has been calculated. It is a set with many incredible properties. It is a connected set which contains an infinite number of small copies of itself. If c lies in the interior of the main body of M, the corresponding Julia set $J(f_c)$ is a fractally deformed circle surrounding one attractive fixed point. If c lies in the interior of one of the buds, the Julia set consists of infinitely many fractally deformed circles connected to each other, each surrounding a periodic attractor. Other possibilities arise by taking c on the boundary. In particular, as c passes through the boundary to the outside of the Mandelbrot set there is a most dramatic change in the corresponding Julia sets. They decompose into a cloud of infinitely many points (the fractal dust).

The Fundamental Dichotomy 12.7.1 *Let $f_c(z) = z^2 + c$. Then either:*

(i) *The orbit of the point $z = 0$ goes to ∞ under iteration by f_c. In this case $K(f_c)$ consists of infinitely many disjoint components, or*

(ii) *The orbit of* 0 *remains bounded* $(0 \in K(f_c))$. *In this case* $K(f_c)$ *is a connected set.*

Definition 12.7.2 The *Mandelbrot set* $\mathcal{M}$ is defined to be the set of those $c \in \mathbb{C}$ for which the filled-in Julia set of f_c is a connected set.

In other words

$$\mathcal{M} = \{c \in \mathbb{C} : \text{ the orbit of } 0 \text{ is bounded under iteration by } f_c\}$$

Exercises 12.7

1. We have seen that if $|z| > 2$ and $|z| \geq |c|$, then $f_c^n(z) \to \infty$ as $n \to \infty$, where $f_c(z) = z^2 + c$.

(a) Suppose $|c| > 2$, then show that $f_c^n(0) \to \infty$ as $n \to \infty$.

(b) Deduce that the Mandelbrot set $\mathcal{M}$ has the property that

$$\mathcal{M} \subseteq \{z \in \mathbb{C} : |z| \leq 2\}.$$

(Note: We have seen that $-2 \in M$ because the map $f_{-2}(z) = z^2 - 2$ has Julia set $J(f_{-2}) = [-2, 2]$, so that $f_{-2}^n(0)$ stays inside this set).

(c) Let $F_c(z) = z^3 + c$. Prove that if $|z| > \max\{|c|, \sqrt{2}\}$, then $F_c^n(z) \to \infty$ as $n \to \infty$ (Hint: Modify the proof of Theorem 12.6.6)

Chapter 13. Examples of Substitutions

In this chapter we give an intuitive appoach to the theory of substitutions and substitution dynamical systems. We show how these give rise to various types of fractals and examine certain examples such as the Thue-Morse sequence and the Fibonacci sequence. Later in Chapters 14 and 15 we shall give a rigorous introduction to the mathematical theory of substitutions. This requires a study of compactness in metric spaces and certain sequence spaces.

13.1 One-dimensional substitutions and the Thue-Morse substitution

Before giving the definition, it is instructive to start with a simple example of a (one-dimensional) substitution.

13.1.1 The Thue-Morse sequence. Originally discovered by E. Prouhet in 1851, it was rediscovered by Axel Thue in 1912 as an example of a non-periodic sequence with some other special properties (in his study of formal languages), and the sequence generated by the Morse substitution was rediscovered by Marston Morse in 1917 in his study of the dynamics of geodesics.

Consider the integers 0, 1, 2, ..., written in binary form:

0, 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111, ...

The sequence obtained by adding the digits of these numbers (reduced modulo 2) is called the *Thue-Morse sequence*. It is:

$$u = 0\,1\,1\,0\,1\,0\,0\,1\,1\,0\,0\,1\,0\,1\,1\,0\,\ldots.$$

If n is a positive integer, we can write $n = a_0 + a_1 2 + a_2 2^2 + \ldots + a_k 2^k$ for some integer k, where $a_i \in \{0,1\}$ and $2^k \leq n < 2^{k+1}$. Define a sequence $s(n)$ by $s(n) = a_0 + a_1 + \ldots + a_k \pmod 2$, the sum of the digits, reduced modulo 2, in the binary expansion of n.

It is remarkable that this sequence may be obtained using a *substitution*, called the *Morse-substitution*. Set

$$\mathcal{A} = \{0,1\},$$

a set with 2 distinct members.

Define a map $\theta : \mathcal{A} \to \mathcal{A} \times \mathcal{A}$ in the following way (we shall make some abuse of notation):

$$\begin{aligned} \theta(0) &= 0\,1 \\ \theta(1) &= 1\,0 \end{aligned}$$

Now think of applying θ so that it has a homomorphism type property:

$$\theta^2(0) = \theta(0\,1) = \theta(0)\theta(1) = 0\,1\,1\,0,$$
$$\theta^3(0) = \theta(0\,1\,1\,0) = \theta(0)\theta(1)\theta(1)\theta(0) = 0\,1\,1\,0\,1\,0\,0\,1,$$
$$\theta^4(0) = 0\,1\,1\,0\,1\,0\,0\,1\,1\,0\,0\,1\,0\,1\,1\,0,$$

and we continue in this way to create a one-sided infinite sequence. We refer to these sequences of finite length as *words*, so if ω is such a word, we denote its length by $|\omega|$ and its *opposite* (or *reflection*) by $R(\omega)$ (so $R(0\,1\,1\,0) = 1\,0\,0\,1$)), so we see that the Morse-sequence can be defined recursively by setting

$$\theta(0) = 0\,1, \quad \theta(1) = 1\,0 \quad \text{and} \quad \theta^{n+1}(0) = \theta^n(0)R(\theta^n(0)).$$

Clearly $|\theta^n(0)| = 2^n$ is the length of the sequence at the nth stage of its construction.

Proposition 13.1.2 *Let u_n be the nth term in the Thue-Morse sequence. Then $u_n = s(n)$.*

Proof. We use induction: it is clearly true for $n = 0$, so suppose it is true for all $m < n$. We may choose k so that $2^k \le n < 2^{k+1}$.

Then u_n is the nth term in $\theta^{k+1}(0) = \theta^k(0)\theta^k(1) = X_k R(X_k)$ (where $X_k = \theta^k(0)$). Thus it is the $(n - 2^k)$th term in $R(X_k)$. This says that

$$u_n = (u_{n-2^k} + 1) \bmod 2.$$

By the inductive hypothesis, $u_{n-2^k} = s(n - 2^k)$.

Since $2^k \le n < 2^{k+1}$, we have $s(n) = s(n - 2^k) + 1 \pmod 2$ (for in this case we must have $a_k = 1$), so that

$$u_n = (s(n - 2^k) + 1) \bmod 2 = s(n).$$

$\square$

As we let $n \to \infty$, we see that the sequence $\theta^n(0)$ converges to a sequence (the *Thue-Morse sequence*) $u = 0\,1\,1\,0\,1\,0\,0\,1\,1\,0\,0\,1\,0\,1\,1\,0\,\ldots$, which has the property that $\theta(u) = u$, i.e., u is a fixed point of the substitution map θ. Clearly there is nothing special about the symbols 0 and 1, we can equally use a and b to obtain

$$u = a\,b\,b\,a\,b\,a\,a\,b\,b\,a\,a\,b\,a\,b\,b\,a\,\ldots.$$

Later we shall use the alphabet $\mathcal{S} = \{L, F\}$ to indicate the commands for a turtle in the plane (in the sense of turtle geometry as in the programming language LOGO, developed by Seymour Papert in the 1980's). The symbol F represents a forward motion in the plane by one unit and L represents a counter clockwise rotation of the turtle by the fixed angle $\phi = \pi/3$.

13.1.3 Some properties of the Thue-Morse sequence u:

1. u is non-periodic (not even eventually periodic). It can be defined recursively by:

$$u_0 = 0, \quad u_{2n} = u_n, \quad \text{and} \quad u_{2n+1} = R(u_n),$$

where $R(\omega)$ is the reflection of ω.

2. u is *recurrent*: this means that every word that occurs in u occurs infinitely many times.

3. If we write the sequence as a power series:

$$F(x) = 0 + 1 \cdot x + 1 \cdot x^2 + 0 \cdot x^3 + 1 \cdot x^4 + \cdots,$$

then $F(x)$ satisfies the quadratic equation

$$(1+x)F^2 + F = \frac{x}{1+x^2} \pmod 2.$$

This equation has two solutions, F and F', the complement of F which satisfies

$$F + F' = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1+x} \pmod 2.$$

In addition,

$$\Pi_{i\geq 0}(1 - x^{2^i}) = (1-x)(1-x^2)(1-x^4)\ldots = 1 - x - x^2 + x^3 + \cdots = \sum_{j\geq o}(-1)^{u_j}x^j.$$

4. It has been shown that if we think of $u = \cdot 0\,1\,1\,0\,1\,0\,0\,1\,1\,0\,0\,1 \ldots$ as representing the binary expansion of a real number, then u is transcendental (is not the solution to any algebraic equation, so in particular it is irrational).

5. Product formulas such as

$$\prod_{n=0}^{\infty}\left(\frac{2n+1}{2n+2}\right)^{2u_n}\left(\frac{2n+3}{2n+2}\right) = \frac{\sqrt{2}}{\pi},$$

have been established (where u_n is the nth term in the Thue-Morse sequence). As an example of this we prove the following result, whose elementary proof is due to J. -P. Allouche (see [2]).

Proposition 13.1.4 *Let* $\epsilon_n = (-1)^{u_n}$, *where* $(u_n)_{n\geq 0}$ *is the Thue-Morse sequence. Then*

$$\prod_{n=0}^{\infty}\left(\frac{2n+1}{2n+2}\right)^{\epsilon_n} = \frac{1}{\sqrt{2}}.$$

Proof. (J. -P. Allouche) Set $P = \prod_{n=0}^{\infty}\left(\frac{2n+1}{2n+2}\right)^{\epsilon_n}$, $Q = \prod_{n=1}^{\infty}\left(\frac{2n}{2n+1}\right)^{\epsilon_n}$, then

$$PQ = \frac{1}{2}\prod_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{\epsilon_n} = \frac{1}{2}\prod_{n=1}^{\infty}\left(\frac{2n}{2n+1}\right)^{\epsilon_{2n}}\prod_{n=0}^{\infty}\left(\frac{2n+1}{2n+2}\right)^{\epsilon_{2n+1}},$$

(These can be seen to converge using Abel's Theorem).

Since $\epsilon_{2n} = \epsilon_n$ and $\epsilon_{2n+1} = -\epsilon_n$, we get

$$PQ = \frac{1}{2}\prod_{n=1}^{\infty}\left(\frac{2n}{2n+1}\right)^{\epsilon_n}\left(\prod_{n=0}^{\infty}\left(\frac{2n+1}{2n+2}\right)^{\epsilon_n}\right)^{-1} = \frac{Q}{2P}.$$

Since $Q \neq 0$, this gives $P^2 = 1/2$, and the result follows as P is positive. $\square$

13.1.5 The Fibonacci substitution

The Morse substitution is an example of a substitution of *constant length*. This means that for each $x \in \mathcal{A} = \{0, 1\}$, $\theta(x)$ is always the same length. The Fibonacci substitution is a substitution of *non-constant length*, also defined on $\mathcal{A} = \{0, 1\}$ as follows:

$$\theta(0) = 0\,1$$
$$\theta(1) = 0.$$

We can check that

$$\theta^2(0) = 0\,1\,0, \quad \theta^3(0) = 0\,1\,0\,0\,1, \quad \theta^4(0) = 0\,1\,0\,0\,1\,0\,1\,0$$
$$\theta^5(0) = 0\,1\,0\,0\,1\,0\,1\,0\,0\,1\,0\,0\,1,$$

etc. The reason for the name is that it is easy to show (see exercises) that $|\theta^{n-1}(0)| = F_n$, $n = 2, 3, \ldots$, is the nth term in the Fibonacci sequence (where $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$, $n \geq 0$, is the Fibonacci sequence). This is because

$$\theta^{n+2}(0) = \theta^{n+1}(0)\theta^n(0), \quad n \geq 1.$$

13.1.6 Definition of a Substitution

To understand how substitutions work, it is best to look at examples such as those above, since the formal definition is a little more complicated. We start with a finite set $\mathcal{A}$ and denote by $\mathcal{A}^*$ the set of all *words* on $\mathcal{A}$. This means all possible finite strings of letters using our *alphabet* $\mathcal{A}$. Thus if $w \in \mathcal{A}^*$, then $w = w_1 w_2 \dots w_n$ for some $w_1, w_2, \dots, w_n \in \mathcal{A}$ and we call n the length of w and write $|w| = n$. Two words w and w' are joined by *concatenation*, so if $w = w_1 \dots w_n$ and $w' = w'_1 \dots w'_m$, then

$$ww' = w_1 \dots w_n w'_1 \dots w'_m \in \mathcal{A}^*.$$

The *empty word* (denoted ϵ), is a word of zero length with the property that $w\epsilon = w = \epsilon w$. A *substitution* is then a mapping θ from $\mathcal{A}$ into the set of words $\mathcal{A}^*$ and then extended from $\mathcal{A}^*$ to $\mathcal{A}^*$ by concatenation, with the homomorphism type property:

$$\theta(w_1 w_2) = \theta(w_1)\theta(w_2) \quad \text{for any} \quad w_1, w_2 \in \mathcal{A}^*.$$

If we take

$$\mathcal{A} = \{0, 1, 2, \dots, d-1\}, \quad \text{for some} \quad d > 1,$$

then

$$\mathcal{A}^* = \bigcup_{i=0}^{\infty} \mathcal{A}^i$$

is the set of all words of finite length (where $\mathcal{A}^i$ is the i-fold cartesian product and $\mathcal{A}^0$ contains only the empty word). If $w \in \mathcal{A}^n$, strictly speaking $w = (w_1, \dots, w_n)$ is an n-tuple, but we write it as $w = w_1 \dots w_n$, a word of length n. The substitution is a map

$$\theta : \mathcal{A} \to \mathcal{A}^*$$

which is extended to a map $\theta : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ and to $\theta : \mathcal{A}^* \to \mathcal{A}^*$ by concatenation, using the homomorphism property ($\mathcal{A}^{\mathbb{N}}$ is the set of all infinite sequences indexed by $\mathbb{N} = \{0, 1, 2, \dots\}$: $\mathcal{A}^{\mathbb{N}}$=$\{(a_0, a_1, \dots) : a_i \in \mathcal{A}\}$, but we think of these as infinite words).

Now if $\theta(0)$ begins with a 0 and $|\theta(0)| > 1$ then the sequence

$$u = \lim_{n \to \infty} \theta^n(0) \in \mathcal{A}^{\mathbb{N}}$$

satisfying $\theta(u) = u$ is called a *substitution sequence* for θ.

Later we shall see how substitutions give rise to dynamical systems having some of the properties of chaotic maps.

13.2 The Toeplitz Substitution

In this section we give another important example of a substitution. As before, set $\mathcal{A} = \{0, 1\}$ and define a substitution ϕ on $\mathcal{A}$ by

$$\phi(0) = 1\,1, \quad \phi(1) = 1\,0,$$

so that

$$1 \to 1\,0 \to 1\,0\,1\,1 \to 1\,0\,1\,1\,1\,0\,1\,0 \to 1\,0\,1\,1\,1\,0\,1\,0\,1\,0\,1\,1\,1\,0\,1\,1 \ldots$$

The fixed point of this substitution is an example of a *Toeplitz sequence* (sometimes called the 2^∞*-sequence*). We shall call the corresponding substitution the *Toeplitz substitution*. Generally, Toeplitz sequences are constructed in the following way:

$u = u_0 u_1 u_2 \ldots \in \mathcal{A}^{\mathbb{N}}$ is defined using the following steps: Start with a semi-infinite sequence of blank spaces and place a '1' in the 0'th place and a '1' in every other blank space thereafter (so $u_{2n} = 1$ for all $n \in \mathbb{N}$):

$$1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot \ldots.$$

Now place a '0' in the first blank space and every other blank space thereafter (so $u_{4n+1} = 0$ for all $n \in \mathbb{N}$):

$$1\,0\,1 \cdot 1\,0\,1 \cdot 1\,0\,1 \cdot 1\,0\,1 \cdot 1\,0\,1 \cdot 1\,0\,1 \cdot \ldots.$$

Next place a '1' in the first remaining blank space and a '1' in every other remaining blank space thereafter (so $u_{8n+3} = 1$ for all $n \in \mathbb{N}$):

$$1\,0\,1\,1\,1\,0\,1 \cdot 1\,0\,1\,1\,1\,0\,1 \cdot 1\,0\,1\,1\,1\,0\,1 \cdot \ldots.$$

Continue in this way indefinitely, alternately placing 0's in in every other blank space, then 1's in every other blank space. The construction can be varied with different alphabets - this construction gives one of the simplest examples of a *Toeplitz sequence*:

$$1\,0\,1\,1\,1\,0\,1\,0\,1\,0\,1\,1\,1\,0\,1\,1\,1\,0\,1\,1\,1\,0\,1\,0 \ldots.$$

Comparing this sequence with the fixed point of the Toeplitz substitution, we see they are identical. In Chapter 17, we shall see a connection between the Toeplitz sequence and the Morse sequence.

Definition 13.2.1 A point $u = u_0 u_1 u_2 \ldots \in \mathcal{A}^{\mathbb{N}}$ is a *Toeplitz sequence* if for all $k \in \mathbb{N}$ there exists $p_k \in \mathbb{Z}^+$ such that

$$u_k = u_{k+n \cdot p_k}, \quad \text{for all} \quad n \in \mathbb{N}.$$

The next proposition tells us that the Toeplitz sequence generated in the above way is exactly the sequence generated by the Toeplitz substitution (the 2^∞-sequence).

Proposition 13.2.2 *The fixed point of the Toeplitz substitution ϕ defined on $\mathcal{A} = \{0,1\}$ by $\phi(0) = 1\,1$, $\phi(1) = 1\,0$ is exactly the Toeplitz sequence generated above.*

Proof. Define a word A_n of length 2^n recursively by setting $A_0 = 1$, $A_0' = 0$ and then $A_n = A_{n-1}A_{n-1}'$ where $A_{n-1}' = A_{n-1}$ except they differ in the last letter. We can show by induction that $A_n = \phi^n(1)$ for $n \in \mathbb{N}$ (see Exercise 7 below).

Thus we see that $\phi^n(0)$ and $\phi^n(1)$ are identical except that they differ in the very last letter. Now we show using induction that $\phi^n(1)$ coincides with the first 2^n terms of the Toeplitz sequence u. $\phi(1) = 1\,0$, so this is clearly true when $n = 1$. Suppose that $\phi^n(1)$ coincides with the first 2^n terms of u. Then we have

$$\phi^{n+1}(1) = \phi^n(\phi(1)) = \phi^n(1\,0) = \phi^n(1)\phi^n(0) = A_nA_n'.$$

Denote by B_n the word A_n with the last letter omitted, then we must have the nth stage of the construction of u as,

$$B_n \;\cdot\; B_n \;\cdot\; B_n \;\cdot\; B_n \;\cdot \ldots,$$

where we still have a space between the different versions of the word B_n. The first space gets filled with a 0 if n is odd and the second space with a 1, and vice-versa if n is even. Thus $u = A_n\,A_n'\ldots$, which is what is required. $\square$

13.3 The Rudin-Shapiro Sequence

Another famous substitution is the *Rudin-Shapiro substitution.* This was introduced by H. S. Shapiro (1951) and W. Rudin (1959) to answer a question of R. Salem (1950) in Harmonic Analysis. Specifically they constructed a sequence $\varepsilon = \varepsilon_0\varepsilon_1\ldots$, where $\varepsilon_n \in \{-1,1\}$ has the property that for all integers $N \geq 0$:

$$\sup_{\theta\in[0,1]} \left| \sum_{n=0}^{N-1} \varepsilon_n e^{2\pi i n\theta} \right| \leq (2+\sqrt{2})\sqrt{N}.$$

It was subsequently used to solve an open question in Ergodic Theory concerning the nature to the spectrum of ergodic measure preserving transformations (see [49]). These applications are beyond the scope of this text, but we will see that it has some interesting topological and combinatorial properties.

Just as the Morse sequence gives the *parity* of the number of 1's in the binary expansion of $n \in \mathbb{N}$ (i.e., the number of 1's reduced modulo 2), we shall see that the Rudin-Shapiro sequence gives the parity of the number of words '1 1' in the binary expansion of n. More explicitly, it gives the number of (possibly overlapping) occurences of the word 1 1 in the base-2 expansion of n. This can be proved using induction and the relations $\varepsilon_{2n} = \varepsilon_n$ and $\varepsilon_{2n+1} = (-1)^n \varepsilon_n$, the parity being $r_n = 0$ when $\varepsilon_n = 1$ and $r_n = 1$ when $\varepsilon_n = -1$, so that $\varepsilon_n = (-1)^{r_n}$.

Definition 13.3.1 The Rudin-Shapiro sequence $\varepsilon = \varepsilon_0\varepsilon_1 \ldots$, where $\varepsilon_n \in \{-1, 1\}$ is defined recursively by: $\varepsilon_0 = 1$ and if $n \geq 1$ then

$$\varepsilon_{2n} = \varepsilon_n \quad \text{and} \quad \varepsilon_{2n+1} = (-1)^n \varepsilon_n.$$

Thus we see that if we list n, the base-2 expansion of n, r_n and ε_n we have

$n =$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	...
$n_2 =$	0	1	10	11	100	101	110	111	1000	1001	1010	1011	1100	1101	...
$r_n =$	0	0	0	1	0	0	1	0	0	0	0	1	1	1	...
$\varepsilon_n =$	1	1	1	-1	1	1	-1	1	1	1	1	-1	-1	-1	...

The Rudin-Shapiro Substitution 13.3.2

The Rudin-Shapiro sequence is closely related to a substitution ζ, called the *Rudin-Shapiro substitution*, defined on the alphabet $\mathcal{A} = \{0, 1, 2, 3\}$ in the following way:

$$\zeta(0) = 0\,2, \quad \zeta(1) = 3\,2, \quad \zeta(2) = 0\,1, \quad \zeta(3) = 3\,1.$$

If $u = u_0u_1 \ldots = \zeta^\infty(0)$ denotes the fixed point of ζ, then $\varepsilon = \tau(u)$, where τ is the map $\tau : \mathcal{A} \to \{-1, 1\}$, $\tau(0) = 1 = \tau(2)$ and $\tau(1) = -1 = \tau(3)$, extended in the usual way. The fixed point is generated as:

$$0 \to 0\,2 \to 0\,2\,0\,1 \to 0\,2\,0\,1\,0\,2\,3\,2 \to 0\,2\,0\,1\,0\,2\,3\,2\,0\,2\,0\,1\,3\,1\,0\,1 \ldots,$$

Exercises 13.1

1. If (u_n) is the Thue-Morse sequence, prove that $u_{2n} = u_n$ and $u_{2n+1} = R(u_n)$ for $n \geq 0$. (Hint: It is easier to show this for the sequence $s(n)$: If $n = a_0 + a_1 2 + a_2 2^2 + \ldots + a_k 2^k$, for $2^k \leq n < 2^{k+1}$, then $s(n) = a_0 + a_1 \ldots + a_k \pmod 2$. Now think about what $2n$ and $s(2n)$ are equal to).

2. Denote by $s_3(n)$ the sum of the digits (modulo 3) in the ternary expansion of $n \in \mathbb{N}$. What substitution on $\mathcal{A} = \{0, 1, 2\}$ gives rise to the sequence $s_3(n)$?

3. Show that for the Fibonacci substitution θ, $F_n = |\theta^{n-1}(0)|$ for each $n \in \mathbb{Z}^+$, where F_n is the nth term in the Fibonacci sequence.

4. A substitution θ is defined on $S = \{0, 1, 2\}$ by

$$\theta(0) = 0\,1\,0\ , \quad \theta(1) = 1\,2\,1\ , \quad \theta(2) = 2\,0\,2\ .$$

In this case the sequence $\theta^n(0)$ converges to a sequence

$$x = 0\,1\,0\,1\,2\,1\,0\,1\,0\,1\,2\,1\,2\,0\,2\,1\,2\,1\ \ldots$$

If (x_n) is the nth term of the resulting sequence, prove that

$$x_{3n} = x_n, \quad x_{3n+1} = x_n + 1, \quad x_{3n+2} = x_n, \;\; n \geq 0,$$

(addition being modulo three).

5*. Given a sequence $\alpha(n)$, $n \geq 0$, of complex numbers having absolute value equal to 1, the *correlation function* $\sigma(n)$ of the sequence $\alpha(n)$ is defined by:

$$\sigma(n) = \lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} \alpha(n+k)\overline{\alpha(k)}, \quad n > 0,$$

$$\sigma(-n) = \overline{\sigma(n)}, \;\; n < 0, \;\; ; \;\; \sigma(0) = 1.$$

This function was introduced by N. Wiener, the *Wiener class* $\mathcal{S}$ being those sequences for which this limit exists. This sequence gives information about what is called the *spectrum* of the sequence, and related operators (see [26] for example).

(a) Show that when this limit exists, $\sigma(n)$ is a positive definite sequence.

(b) If $\alpha(n) = \epsilon_n = (-1)^{u_n}$ where u_n is the Thue-Morse sequence, show that the correlation function σ satisfies $\sigma(2n) = \sigma(n)$, $\sigma(2n+1) = (\sigma(n+1) - \sigma(n))/2$.

(c) If $\alpha(n) = \omega^{x_n}$ where x_n is the sequence defined in question 2 above and $\omega = e^{2\pi i/3}$, show that

$$\sigma(3n) = \sigma(n)\ ; \quad \sigma(3n+1) = \frac{1}{3}(\sigma(n+1) - \sigma(n))\ ; \quad \sigma(3n+2) = \frac{1}{3}(\sigma(n) - \sigma(n+1)).$$

6. A sequence A_n is defined recursively as follows:

$$A_0 = 1, \quad A_0' = 1, \quad A_n = A_{n-1}A_{n-1}',$$

where A_{n-1}' is the word A_{n-1} except that the last letter (say a) is replaced by $1-a = R(a)$. Write down A_2 and A_3 and use induction to show that $\lim_{n\to\infty} A_n$ is the sequence obtained from the Toeplitz substitution.
7. A substitution ζ is defined on $\mathcal{A} = \{0, 1, 2, 3\}$ by

$$\zeta(0) = 0\,2\;; \quad \zeta(1) = 3\,2\;; \quad \zeta(2) = 0\,1\;; \quad \zeta(3) = 3\,1.$$

Let ε_n be the nth term in Rudin-Shapiro sequence. If $\tau : \mathcal{A} \to \{-1, 1\}$ is defined by $\tau(0) = 1 = \tau(2)$ and $\tau(1) = -1 = \tau(3)$, show that $\tau(\zeta^\infty(0)) = \varepsilon$.

13.4 Fractals Arising from Substitutions

In this section we look at the fractal nature of substitutions and study a surprising connection between the Thue-Morse sequence and the Koch curve. In particular, we see that if we look at the Morse substitution in the appropriate way, it gives rise to a fractal curve: the Koch curve.

Example 13.4.1 Consider the Morse sequence written using the alphabet $\mathcal{A} = \{a, b\}$:

$$\theta(a) = a\,b, \quad \theta(b) = b\,a.$$

This gives rise to the sequence

$$a\,b\,b\,a\,b\,a\,a\,b\,b\,a\,a\,b\,a\,b\,b\,a\dots$$

Replace $a\,b$ by A and $b\,a$ by B, then we obtain:

$$A\,B\,B\,A\,B\,A\,A\,B\dots,$$

essentially the same sequence using a different symbol, i.e., the Morse substitution has a type of self-similarity property. Similar conclusions can be seen to hold for any substitution of constant length.

13.4.2 A connection between the Thue-Morse sequence and the Koch curve

Various authors have shown (independently) that there is a connection between the Morse substitution and the Koch curve (see [17], [21], [2] and [40]). Here we present results from the latter two papers to show how the Morse substitution can be used to construct the Koch curve.

As usual, $u = (u_n) = 0\,1\,1\,0\,1\,0\,0\,1\,1\,0\,0\,1\,0\,1\,1\,0\,\ldots$, is the Thue-Morse sequence, so that $u_0 = 0, u_1 = 1$ etc. Set

$$\alpha(n) = \sum_{k=0}^{n-1} (-1)^{u_k} e^{2\pi i k/3},$$

and plot these points in the complex plane for $n = 1, 2, \ldots$. We start at the origin $(0, 0)$. Since $\alpha(1) = 1$ we represent this geometrically by drawing a line from $(0, 0)$ to $(1, 0)$. Now $\alpha(2) = 1 - e^{2\pi i/3}$ so we move 1 unit down to the right, at an angle of $\pi/3$ to the x-axis, joining $\alpha(1)$ to $\alpha(2)$ with a straight line. We continue in this way with $\alpha(3) = 1 - e^{2\pi i/3} - e^{4\pi i/3}$, $\alpha(4) = 1 - e^{2\pi i/3} - e^{4\pi i/3} + 1$. This gives the first step in the construction of one-side of the Koch snow-flake. We continue in this way so that after the 18th stage our curve looks like:

If we scale by a factor of 1/3 at each stage of the construction, then we see that we get convergence to the Koch curve. This is proved in a formal manner as follows:

Let $\Sigma = \{F, L\}$ be the alphabet where F denotes a move of one unit forward, and L is an anticlockwise rotation through an angle $\phi = \pi/3$. Denote by θ the Morse substitution.

Definition 13.4.3 The *Thue-Morse turtle programs of degree* n, denoted by TM_n and $\overline{TM}_n$ are defined to be the following words in Σ^*:

$$TM_n = \theta^n(F) \quad \text{and} \quad \overline{TM}_n = \theta^n(L),$$

so that

$$TM_n = F\,L\,L\,F\,L\,F\,F\,L\,L\,F\,F\,L\,F\,L\,L\,F\ \ldots\ F\,L\,L\,F,$$

the first 2^n terms of the Thue-Morse sequence using the alphabet Σ.

If we interpret this sequence as a polygonal path in the plane, then when it is plotted, we shall see that we get the nth step in the construction of the Koch curve.

13.4.4 Properties of the sequence $\{TM_n\}$

(a) $TM_{2n+2} = TM_{2n}\overline{TM}_{2n}\overline{TM}_{2n}TM_{2n}$ and $\overline{TM}_{2n+2} = \overline{TM}_{2n}TM_{2n}TM_{2n}\overline{TM}_{2n}$.

This implies that the even terms of $\{TM_n\}$ are palindromes, which gives the bilateral symmetry of the resulting polygonal curves.

Chapter 14. Compactness in Metric Spaces and the Metric Properties of Substitutions

In this chapter we continue our study of substitutions but in a more rigorous manner, in particular we develop their topological properties and we look at the dynamical systems that they generate.

14.1 Compactness in Metric Spaces

In earlier chapters, we have kept the use of compactness to a minimum. However, it is required when discussing properties of certain sequence spaces that arise in the mathematical theory of substitutions. In the case of the real line, the Heine-Borel Theorem is important, and was used in Section 4.5:

14.1.1 The Heine-Borel Theorem. *Every cover of a closed interval* $[a, b]$ *by a collection of open sets, has a finite subcover.*

In the case of metric spaces (X, d) we generalize this as follows:

Definition 14.1.2 A *cover* of X is a collection of sets whose union is X. An *open cover* of X is a collection of open sets whose union is X.

Definition 14.1.3 The metric space (X, d) is said to be compact if every open cover of X has a finite subcover, i.e., whenever $X = \cup_{\lambda\in I}\mathcal{O}_\lambda$, for some index set I, and open sets $\mathcal{O}_\lambda$, there is a finite subset $J \subset I$ with $X = \cup_{\lambda\in J}\mathcal{O}_\lambda$.

The Heine-Borel theorem says that any closed interval $[a, b]$ in $\mathbb{R}$ is compact. More generally it will be shown that a subset of $\mathbb{R}^n$ is compact if and only if it is both closed and bounded.

Definition 14.1.4 The metric space (X, d) is *sequentially compact* if every infinite sequence (x_n) in X has a limit point.

For A a finite subset of the metric space (X, d), $d(x, A)$ is the distance between x and the nearest point of A. For more general sets A, we set $d(x, A) = \inf_{y\in A} d(x, y)$.

Definition 14.1.5 The metric space (X, d) is *totally bounded* if for every $\epsilon > 0$, X can be covered by a finite family of open balls of radius ϵ. If A is a finite subset of X

with the property that $d(x, A) < \epsilon$ for all $x \in X$, then we call A an *ϵ-net* of X. The existence of an ϵ-net for each $\epsilon > 0$ is clearly equivalent to X being totally bounded.

This leads to the following important theorem characterizing the compactness of metric spaces.

Compactness of Metric Spaces Theorem 14.1.6 *The following are equivalent for a metric space (X, d):*

(a) *X is compact.*

(b) *If $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$ is a decreasing sequence of non-empty closed sets in X, then $\cap_{n=1}^{\infty} F_n$ is non-empty.*

(c) *X is sequentially compact.*

(d) *X is totally bounded and complete.*

Proof. (a) $\Rightarrow$ (b) Let $\{F_n\}_{n=1}^{\infty}$ be a decreasing sequence of non-empty closed sets in X and suppose that $\cap_{n=1}^{\infty} F_n = \emptyset$. Then $X - F_n$ is an increasing sequence of open sets with

$$\bigcup_{n=1}^{\infty}(X - F_n) = X - \bigcap_{n=1}^{\infty} F_n = X,$$

so that the collection of sets $\{X - F_n : n \in \mathbb{N}\}$ is an open cover of X. Since X is compact, there is a finite subcover: there is a finite index set $I \subset \mathbb{N}$ with

$$X - \bigcap_{n \in I} F_n = \bigcup_{n \in I}(X - F_n) = X.$$

A contradiction since $\cap_{n \in I} F_n = F_m \neq \emptyset$, where $m = \max(I)$.

(b) $\Rightarrow$ (c) Let (x_n) be an infinite sequence in X and let F_n be the closure of the set $\{x_n, x_{n+1}, x_{n+2}, \ldots\}$. This gives a decreasing sequence of non-empty closed sets, so by (b), $\cap_{n=1}^{\infty} F_n$ is non-empty. Let $\alpha \in \cap_{n=1}^{\infty} F_n$, then we show that there is a subsequence (x_{n_k}) which converges to α.

Since $\alpha \in F_1$, there exists x_{n_1} with $d(\alpha, x_{n_1}) < 1$. Let $n > n_1$, then $\alpha \in F_n$, so there exists $n_2 > n_1$ with $d(\alpha, x_{n_2}) < 1/2$. Continue in this way to find x_{n_k} with $n_k > n_{k-1}$ and $d(\alpha, x_{n_k}) < 1/k$. Clearly $\alpha = \lim_{k \to \infty} x_{n_k}$, so that (c) holds.

(c) $\Rightarrow$ (d) First we show that X is complete. Let (x_n) be a Cauchy sequence in X, then from above, there is a subsequence (x_{n_k}) that converges to $\alpha \in X$ (say). Let $\epsilon > 0$, then there exists $N \in \mathbb{N}$ such that if $m, n \geq N$ we have $d(x_n, x_m) < \epsilon/2$ and such that if $k \geq N$ then $d(\alpha, x_{n_k}) < \epsilon/2$. Clearly $n_k \geq k \geq N$, so that

$$d(\alpha, x_n) \leq d(\alpha, x_{n_k}) + d(x_{n_k}, x_n) < \epsilon/2 + \epsilon/2 = \epsilon,$$

so the sequence (x_n) is convergent. We have shown that every Cauchy sequence is convergent, so X is complete.

If X is not totally bounded, then there exists $\epsilon > 0$ for which X has no finite covering by balls of radius ϵ. The idea is to construct an infinite sequence of points (x_n) in X having distance apart at least ϵ ($d(x_i, x_j) \geq \epsilon$ for all $i \neq j$), contradicting the sequential compactness of X. We proceed inductively taking $x_1 \in X$ arbitrarily. Suppose that $x_1, x_2, \ldots, x_{n-1}$ have been chosen, then the open balls $B_\epsilon(x_i)$, ($1 \leq i \leq n-1$), have a union which cannot be the whole space. Consequently, if $x_n \in X - \cup_{i=1}^{n-1} B_\epsilon(x_i)$, then x_n satisfies $d(x_n, x_i) \geq \epsilon$ for all $i = 1, \ldots n-1$. It follows by the Principle of Induction that the required infinite sequence exists. This sequence cannot have a limit point, for if $\alpha \in X$ were a limit point, say within $\epsilon/2$ of some x_{n_0}, then the distance of α from every other member of the sequence will be at least $\epsilon/2$ since

$$\epsilon \leq d(x_{n_0}, x_m) \leq d(x_{n_0}, \alpha) + d(\alpha, x_m) < \epsilon/2 + d(\alpha, x_m).$$

This is clearly impossible for a limit point.

(d) $\Rightarrow$ (a) If X is not compact, there is an open cover $\mathcal{O} = \{O_\lambda : \lambda \in I\}$ of X that does not have a finite subcover. To get a contradiction, we first construct a convergent sequence (x_n) in X.

Using the fact that X is totally bounded, choose an ϵ-net A_1 with $\epsilon = 1/2$. Let $x_1 \in A_1$ with the property that no finite sub-collection of $\{O_\lambda : \lambda \in I\}$ covers $B_{1/2}(x_1)$, (x_1 exists because if every ball $B_{1/2}(a)$ of radius $1/2$ with $a \in A_1$ had a finite subcover from $\mathcal{O}$, then the whole space would have a finite subcover from $\mathcal{O}$ because A_1 is a finite set).

Now choose an ϵ-net A_2 with $\epsilon = 1/2^2 = 1/4$, and let $x_2 \in A_2$ be chosen so that $B_{1/2}(x_1) \cap B_{1/4}(x_2) \neq \emptyset$ and with the property that no finite sub-collection of $\{O_\lambda : \lambda \in I\}$ covers $B_{1/4}(x_2)$, (x_2 exists because we know that $B_{1/2}(x_1)$ has no finite subcover from $\mathcal{O}$ so we can apply the previous argument to $B_{1/2}(x_1)$).

We continue in this way so that at the nth stage we choose an ϵ-net A_n with $\epsilon = 1/2^n$, satisfying $B_{1/2^{n-1}}(x_{n-1}) \cap B_{1/2^n}(x_n) \neq \emptyset$ with the property that no finite sub-cover of $\mathcal{O}$ is a cover of $B_{1/2^n}(x_n)$.

The construction ensures that

$$d(x_{n-1}, x_n) \leq \frac{1}{2^{n-1}} + \frac{1}{2^n} \leq \frac{1}{2^{n-2}},$$

so that for $m < n$

$$d(x_m, x_n) \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n)$$

$$\leq \frac{1}{2^{m-1}} + \frac{1}{2^m} + \cdots + \frac{1}{2^{n-2}} \leq \frac{1}{2^{m-2}},$$

so that (x_n) is a Cauchy sequence in X. The completeness of X implies that this sequence converges to some point $\alpha \in X$.

Suppose that $\alpha \in O_{\lambda_0}$ for some $\lambda_0 \in I$. Since O_{λ_0} is an open set, we can choose $\epsilon > 0$ so that $B_\epsilon(\alpha) \subseteq O_{\lambda_0}$. Since $\lim_{n\to\infty} x_n = \alpha$, there exists $n \in \mathbb{N}$ with $d(x_n, \alpha) < \epsilon/2$ and $1/2^n < \epsilon$. If $x \in B_{1/2^n}(x_n)$, then

$$d(x, \alpha) \leq d(x, x_n) + d(x_n, \alpha) < 1/2^n + \epsilon/2 < \epsilon,$$

so that

$$B_{1/2^n}(x_n) \subseteq B_\epsilon(a) \subseteq O_{\lambda_0}.$$

This contradicts the fact that no finite sub-collection of $\mathcal{O}$ covers $B_{1/2^n}(x_n)$. $\square$

Examples 14.1.7 Any non-empty closed and bounded subset of $\mathbb{R}$ or $\mathbb{R}^n$ for $n \geq 1$ is a compact set (see the exercises). In particular, any non-empty closed interval $[a, b]$, the Cantor set C, the unit circle S^1 and the closed unit disc in $\mathbb{R}^2$ are compact. We shall show that the set of all one-sided sequences of 0's and 1's (with the metric given previously) is a compact metrric space.

Remarks 14.1.8 1. We see that any compact metric space (X, d) is necessarily complete. In addition, it is straightforward to show that such a space is *separable*, i.e., has a countable dense subset: From the total boundedness, for each $n \in \mathbb{N}$ there is a finite set $A_n \subseteq X$, such that if $x \in X$, then $d(x, A_n) < 1/n$. Set $A = \cup_{n=1}^\infty A_n$, a countable set with the property that for each $x \in X$, $d(x, A) \leq d(x, A_n) < 1/n$, so that $x \in \bar{A}$, i.e., $\bar{A} = X$, so X is separable (see the exercises). $\square$

2. We say that a subset A of the metric space X is compact if A, considered as a metric space in its own right (we say that A is a *subspace* of X) is compact. Then it is clear that any closed subset of a metric space is compact and any compact subset of a metric space is closed.

Exercises 14.1

1. Let $X = \{0\} \cup \{1, 1/2, 1/3, \ldots\}$. Show directly that any open cover has a finite subcover, so that X is a compact metric space (with the induced metric as a subset of $\mathbb{R}$). Why does this not contradict the Baire Category Theorem and corollary? - see Appendix A for its statement).

2. Prove that a compact subset of a compact metric space is closed and that a closed subset of a compact metric space is compact.

3. If (X, d) is a compact metric space and A is a subset of X with $d(x, A) < 1/n$ for all $n \in \mathbb{N}$, prove that $x \in \bar{A}$.

4. Prove that a subset of $\mathbb{R}^n$ is compact if and only if it is both closed and bounded.

14.2 Continuous Functions on Compact Metric Spaces

In Section 4.3 we studied continuous functions on metric spaces and we saw that for such functions the inverse images of open sets are open and the inverse images of closed sets are closed. Here we see that the images of compact sets are necessarily compact.

Theorem 14.2.1 *Let $f : X \to Y$ be a continuous map of metric spaces (X, d_X) and (Y, d_Y). If X is compact, so is $f(X) = \{f(x) : x \in X\}$, the image of X in Y.*

Proof. Let $\{O_\lambda : \lambda \in I\}$ be an open cover of $f(X)$, then each of the sets $f^{-1}(O_\lambda)$ is open in X and $X = \cup_{\lambda \in I} f^{-1}(O_\lambda)$. As X is compact there is a finite subcover: a finite set $J \subset I$ with $X = \cup_{\lambda \in J} f^{-1}(O_\lambda)$. We can now check that $\{O_\lambda : \lambda \in J\}$ is a cover of $f(X)$. $\square$

An immediate consequence is the following result, which will be needed shortly. It says that a continuous map on a compact metric space into $\mathbb{R}$, attains both its maximum and minimum values:

Corollary 14.2.2 *Let (X, d) be a compact metric space and $f : X \to \mathbb{R}$ a continuous function. Then there are points $\alpha, \beta \in X$ such that $f(\alpha) = \inf_{x \in X} f(x)$ and $f(\beta) = \sup_{x \in X} f(x)$.*

Proof. By Theorem 14.2.1, $f(X)$ is a compact subset of $\mathbb{R}$, so it is both closed and bounded. In particular, if $h = \sup_{x \in X} f(X)$ and $k = \inf_{x \in X} f(X)$ then $h, k \in f(X)$, for suppose $h \notin f(X)$, then we can find an open ball $B_\delta(h)$, surrounding h which does not intersect $f(X)$. $h - \delta/2$ would then be an upper bound for $f(X)$ smaller than h giving a contradiction. We deduce that $h = f(\beta)$ and similarly $k = f(\alpha)$ for some $\alpha, \beta \in X$. $\square$

14.3 Contraction Mapping Theorem for Compact Metric Spaces

In Section 8.5 we proved a fixed point theorem for contraction mappings on complete metric spaces. The requirement was that $d(f(x), f(y)) \leq \alpha \cdot d(x, y)$ for all $x, y \in X$, where $0 < \alpha < 1$. Compact metric spaces are complete and we can weaken the conditions of the contraction mapping theorem slightly in this case.

Theorem 14.3.1 *Let $f : X \to X$ be a function defined on the compact metric space (X, d) having the property that*

$$d(f(x), f(y)) < d(x, y), \quad \textit{for all} \quad x, y \in X, x \neq y.$$

Then f has a unique fixed point in X.

We first prove a lemma:

Lemma 14.3.2 *If $f : X \to X$ is continuous, then the function $k : X \to [0, \infty)$, $k(x) = d(x, f(x))$ is continuous.*

Proof. From the triangle inequality we obtain $|d(a, c) - d(c, b)| \leq d(a, b)$ for all $a, b, c \in X$.

Let $\epsilon > 0$ and fix $x \in X$, then we find $\delta > 0$ such that if $y \in X$ with $d(x, y) < \delta$, then

$$|d(x, f(x)) - d(y, f(y))| < \epsilon.$$

We may assume that for $y \in X$ with $d(x, y) < \delta_1$ we have $d(f(x), f(y)) < \epsilon/2$ since f is continuous at x.

The above inequality and the usual triangle inequality gives:

$$\begin{aligned} |d(x, f(x)) - d(y, f(y))| &= |d(x, f(x)) - d(f(x), y) + d(f(x), y) - d(y, f(y))| \\ &\leq |d(x, f(x)) - d(f(x), y)| + |d(f(x), y) - d(y, f(y))| \end{aligned}$$

$$\leq d(x,y) + d(f(x), f(y)) < \epsilon/2 + \epsilon/2 = \epsilon,$$

if $d(x,y) < \delta = \min\{\delta_1, \epsilon/2\}$. $\square$

Proof of the Theorem 14.3.3 Clearly f is continuous on X, so by the lemma, $k(x) = d(x, f(x))$ is a continuous function defined on a compact space. Consequently, its range is a closed and bounded subset of $\mathbb{R}^+$. It follows that $\alpha = \inf_{x\in X} k(x)$ is attained, and there is an $x_0 \in X$ with $\alpha = k(x_0)$.

Suppose that $\alpha > 0$, then since $\alpha = d(x_0, f(x_0) \neq 0$, we have $f(x_0) \neq x_0$. In addition

$$d(f^2(x_0), f(x_0)) < d(f(x_0), x_0) = \alpha,$$

a contradiction since $\alpha = \inf_{x\in X} d(x, f(x))$. Therefore we must have $\alpha = d(x_0, f(x_0)) = 0$, so $f(x_0) = x_0$, and x_0 is a fixed point of f. As usual, if y_0 is another fixed point then the inequality $d(f(x_0), f(y_0)) < d(x_0, y_0)$ leads to a contradiction unless $x_0 = y_0$. $\square$

14.4 Basic Topological Dynamics

In this section we develop the topological dynamics needed to study substitutions. For more detail, see the excellent book by P. Walters [66]. The main object of study is the (topological) dynamical system (X, T) where throughout this section we assume that (X, d) is a compact metric space and $T : X \to X$ is a continuous map (often a homeomorphism). As usual, $O(x) = \{T^n(x) : n \in \mathbb{N}\}$.

Examples 14.4.1 1. Any continuous map $f : I \to I$ on an interval $I \subseteq \mathbb{R}$ is a topological dynamical system. In particular, the logistic maps, the tent maps etc. are examples that we have studied in detail.

2. The rotation $T : S^1 \to S^1$, $T(z) = az$, where S^1 is the unit circle in the complex plane and $a \in S^1$ is fixed. This is closely related to the map $T_\alpha : [0,1) \to [0,1)$ defined by $T_\alpha(x) = x + \alpha \pmod 1$.

3. The shift map σ on a sequence space $\mathcal{S}$, was seen to be continuous, and it defines a topological dynamical system.

14.4.2 Minimality The topological dynamical system (X, T) is said to be *minimal* if for every $x \in X$, $\overline{O(x)} = X$, i.e., the orbit closure of every point is dense in X. This seems to be a very strong condition (recall that T is *transitive* if there exists $x \in X$ with $\overline{O(x)} = X$). However, we will see that there is a wide range of examples which are minimal. It can be seen that if T is transitive, it has to be onto (see the exercises).

Definition 14.4.3 Let (X, T) be a topological dynamical system. A non-empty subset $A \subset X$ is said to be *minimal* if the restriction of T to A is a minimal map. It can be shown that any continuous map $T : X \to X$ of a compact metric space has at least one minimal set.

Theorem 14.4.4 *The following are equivalent for a topological dynamical system* (X, T).

(a) (X, T) *is minimal.*

(b) *If* $E \subseteq X$ *is a closed set with* $T(E) \subset E$, *then* $E = X$ *or* $E = \emptyset$.

(c) *If* U *is a non-empty open subset of* X, *then* $\cup_{n=0}^{\infty} T^{-n}U = X$.

Proof. (a) $\Rightarrow$ (b) Suppose T is minimal and $x \in E$ where E is non-empty and closed, then if $T(E) \subset E$, we must have $O(x) \subset E$, so $X = \overline{O(x)} = E$ since E is closed.

(b) $\Rightarrow$ (c) If U is non-empty, then $E = X - \cup_{n=0}^{\infty} T^{-n}U \neq X$ is closed and $T(E) \subset E$, so we must have $E = \emptyset$.

(c) $\Rightarrow$ (a) If $x \in X$ and U is a non-empty open subset of X, then from $\cup_{n=0}^{\infty} T^{-n}U = X$, $T^n x \in U$ for some $n > 0$. It follows that $\overline{O(x)}$ is dense in X since the orbit of x enters any open set. $\square$

Propositon 14.4.5 *If* $T : X \to X$ *is transitive and* $f : X \to \mathbb{C}$ *is a continuous function with* $f(Tx) = f(x)$ *for all* $x \in X$, *then* f *is a constant function.*

Proof. If $\overline{O(x_0)} = X$, then $f(T^n x_0) = f(x_0)$ for all $n \in \mathbb{N}$. This says that f is constant on a dense subset of X, so using the continuity of f, f must be constant on X. $\square$

Examples 14.4.6 1. Let $T : S^1 \to S^1$ be the rotation $T(z) = az$, where S^1 is the unit circle in the complex plane and $a \in S^1$ is not a root of unity (i.e., $a^n \neq 1$ for all $n \in \mathbb{Z}^+$). Then T is minimal.

Proof. The set $\{a^n : n \in \mathbb{Z}^+\}$ is an infinite sequence in S^1, so by the compactness of S^1, it has a limit point, say $\alpha \in S^1$. Consequently, given $\epsilon > 0$ there are $p > q \in \mathbb{Z}^+$ with $d(a^p, a^q) < \epsilon$ (d is the usual metric on S^1, giving the shortest distance between two points around the circle). Since a rotation is an isometry on S^1, it follows that $d(a^{p-q}, 1) = d(a^p, a^q) < \epsilon$, and $d(a^{n(p-q)}, a^{(n-1)(p-q)}) < \epsilon$ for $n > 1$. Thus the set $\{a^{n(p-q)} : n \in \mathbb{N}\}$ is an ϵ-net for S^1, and it is now clear that the set $\{a^n : n \in \mathbb{Z}^+\}$ is dense in S^1. In particular, the sequence $T^n(1) = a^n$ is dense in S^1 and for any $z, w \in S^1$ we can find a sequence of integers k_n with $a^{k_n} \to wz^{-1}$. It follows that $T^{k_n}(z) = a^{k_n}z \to wz^{-1}z = w$ as $n \to \infty$, so $O(z)$ is dense in S^1, and T is minimal. □

2. Clearly $T : S^1 \to S^1$, $T(z) = z^2$ is not minimal since $T(1) = 1$, but it is transitive (we saw this previously), since given any intervals $U, V \subset S^1$, we must have $U \cap T^n V \neq \emptyset$ for some $n > 0$, and we can apply Theorem 14.4.7 (proved below).

3. Recall from Chapter 6 that the dynamical system (Y, S) is a *factor* of the dynamical system (X, T) if there is a continuous map ϕ from X onto Y with $\phi \circ T = S \circ \phi$. (X, T) is called an *extension* of (Y, S) and ϕ is called a *factor map*. The two dynamical systems are *conjugate* (with *conjugacy* ϕ), if $\phi : X \to Y$ is a homeomorphism. We saw earlier that if the two maps are conjugate, then T is transitive if and only if S is transitive. In a similar way, T is minimal if and only if S is minimal.

For example, if $T : S^1 \to S^1$ is $T(z) = az$ as in Example 1, and if $\phi(z) = z^2$, then $\phi \circ T = T^2 \circ \phi$, so T^2 is a factor of T (but they are not conjugate since ϕ is continuous and onto, but not one-to-one). On the other hand $T^{-1}(z) = a^{-1}z$, and if $\psi(z) = z^{-1}$, then $T \circ \psi = \psi \circ T^{-1}$, so that T and T^{-1} are conjugate (since ψ is continuous, one-to-one and onto). It can be shown that every conjugacy between T and T^{-1} is of the form $\psi(z) = cz^{-1}$ for some $c \in S^1$, and so is an *involution* ($\psi^2 = I$, the *identity map*).

We saw earlier that T above is conjugate to the rotation $T_\alpha : [0, 1) \to [0, 1)$ defined by $T_\alpha(x) = x + \alpha \pmod 1$ (via $\phi : [0, 1) \to S^1$, $\phi(x) = e^{2\pi i x}$), when $a = e^{2\pi i\alpha}$. If α is irrational, a won't be a root of unity, so T_α is minimal. Here is a direct proof that T_α is minimal, using the Pigeonhole Principle:

Denote by $\{x\}$ the fractional part of $x \in \mathbb{R}$, i.e., x reduced modulo one. We show that for any $0 < a < b < 1$ there exists $n \in \mathbb{N}$ with $\{n\alpha\} \in (a, b)$. Choose $N \in \mathbb{N}$ so large that $1/N < b - a$. Divide $[0, 1)$ into N segments of length $1/N$. Let $m > N$ and consider the set $\{\{\alpha\}, \{2\alpha\}, \dots, \{m\alpha\}\}$, all distinct since α is irrational. By the Pigeonhole Principle, there are j and k with $\{j\alpha\}$ and $\{k\alpha\}$ belonging to the same segment of length $1/N$. Set $r = j - k$, then we have $0 < \{r\alpha\} < 1/N$. Set $\beta = \{r\alpha\}$, then the set $\{\beta, 2\beta, \ldots, n\beta, \ldots\}$ will subdivide $[0,1)$ into intervals of length less than $1/N$, so one of them must fall into the interval (a,b). This is the required $\{n\alpha\}$. $\square$

4. It has been shown by Auslander and Yorke [4], that if $f : S^1 \to S^1$ is a continuous minimal map, then f is conjugate to an irrational rotation.

We finish this section with a result similar to Theorem 14.4.4, but requiring only that T be transitive. The proof requires the use of the Baire Category Theorem (see Appendix A), which will not be proved. Recall that a *nowhere dense set* is one whose closure does not contain any non-empty open sets. The equivalence of (a) and (c) below was mentioned in Section 5.2. This result can be given in greater generality (see [36]), but the following is sufficient for our needs:

Theorem 14.4.7 *Let (X,d) be a compact metric space with $T : X \to X$ a continuous onto map. Then the following are equivalent:*

(a) *T is transitive.*

(b) *If E is a closed set with $T(E) \subset E$, $E \neq X$, then E is nowhere dense.*

(c) *If U and V are non-empty open sets in X, there exists $n \geq 1$ with $T^n U \cap V \neq \emptyset$.*

(d) *The set $\{x \in X : \overline{O(x)} = X\}$ is dense in X.*

Proof. (a) $\Rightarrow$ (b) Suppose that $O(x_0)$ is dense in X and U is a non-empty open set with $U \subset E$. There exists $m \in \mathbb{N}$ with $T^m(x_0) \in U$. It follows that $\{T^n(x_0) : n \geq m\} \subset E$, so

$$\{x_0, Tx_0, \ldots, T^{m-1}x_0\} \cup E = X.$$

Applying T to both sides gives $\{Tx_0, \ldots, T^m x_0\} \cup E = X$, and by doing this repeatedly, we see that $E = X$.

(b) $\Rightarrow$ (c) Let U and V be non-empty and open, then $\cup_{n=0}^{\infty} T^{-n}V$ is open and

$$E = X - \bigcup_{n=0}^{\infty} T^{-n}V = \bigcap_{n=0}^{\infty} (X - T^{-n}V)$$

is closed and $T(E) \subset E$. But $E \neq X$ as V is non-empty, so contains no non-empty open sets. It follows that $\cup_{n=0}^{\infty} T^{-n}V$ is dense in X so intersects U. In particular, $T^m U$ must intersect V for some $m \in \mathbb{N}$.

(c) $\Rightarrow$ (d) X is separable, so there is a countable dense set $A = \{x_n : n \in \mathbb{N}\} \subset X$. Enumerate the positive rationals $\mathbb{Q}^+ = \{r_n : n \in \mathbb{N}\}$, then the set

$$\mathcal{U} = \{B_{r_n}(x_m) : m, n \in \mathbb{N}\} = \{U_n : n \in \mathbb{N}\} \quad \text{say,}$$

is a countable collection of open balls and any open set contains members of $\mathcal{U}$. We can check that

$$\{x \in X : \overline{O(x)} = X\} = \bigcap_{n=0}^{\infty}\left(\bigcup_{m=0}^{\infty} T^{-m}U_n\right).$$

For each $n \in \mathbb{N}$, it follows as in the last proof that $\cup_{m=0}^{\infty}T^{-m}U_n$ is an open dense subset of X, so $\{x \in X : \overline{O(x)} = X\}$ is the countable intersection of open dense sets. The Baire Category Theorem now implies that this intersection is dense in X, and (d) follows.

(d) $\Rightarrow$ (a) This is now clear. $\square$

Exercises 14.4

1. (a) Prove that a transitive map is onto.

(b) If T is an isometry ($d(Tx, Ty) = d(x, y)$ for all $x, y \in X$), which is transitive, prove that T is minimal.

(c) Show that two minimal sets A and B for the topological dynamical system (X, T) are either disjoint or equal.

2. Let $f : X \to X$ be a continuous map of the metric space (X, d). $x \in X$ is a *wandering point* of f if there is an open set U containing x such that $U \cap f^n(U) = \emptyset$ for all $n \in \mathbb{Z}^+$. A point $x \in X$ is called *non-wandering* if it is not wandering. The set of non-wandering points of f is denoted by $\Omega(f)$. Show the following:

(a) The set of all wandering points is open. Deduce that $\Omega(f)$ is a closed set which contains all of the periodic points of f.

(b) $\Omega(f)$ is an invariant set.

(c) $\Omega(f^n) \subset \Omega(f)$ for all $n \in \mathbb{Z}^+$.

(d) If f is a homeomorphism, then $f(\Omega(f)) = \Omega(f)$ and $\Omega(f^{-1}) = \Omega(f)$.

3. Let $f : X \to X$ be a continuous map of the metric space (X, d). If $x \in X$, then the ω-limit (*omega-limit*) set $\omega(x)$ is defined to be

$$\omega(x) = \cap_{n=0}^{\infty} \overline{\{f^k(x) : k > n\}}.$$

It can be shown that if $X = [0, 1]$, then $\omega(x) \subset \Omega(f)$ for all $x \in X$. Show the following:

(a) $y \in \omega(x)$ if and only if there is an increasing sequence $\{n_k\}$ such that $f^{n_k}(x) \to y$ as $k \to \infty$.

(b) $\omega(x)$ is a closed invariant set $(f(\omega(x)) \subset \omega(x))$.

(c) If x is a periodic point of f, then $\omega(x) = \mathrm{O}(x)$. Also if x is eventually periodic with $y \in \mathrm{O}(x)$, then $\omega(x) = \mathrm{O}(y)$.

(e) If $\omega(x)$ consists of a single point, then that point is a fixed point.

4. Let $f : X \to X$ be a continuous map of the metric space (X, d). If $x \in X$, it is said to be *recurrent* if $x \in \omega(x)$, i.e., x belongs to its ω-limit set. Recurrent points are a generalization of periodic points. Periodic points return to themselves, whereas recurrent points return closely to themselves infinitely often, becoming closer as the iteration procedure progresses. Show

(a) A periodic point is recurrent but an eventually periodic point is not recurrent.

(b) Any recurrent point x belongs to $\Omega(f)$.

(c) If $f : [0, 1] \to [0, 1]$ is a homeomorphism, then the only recurrent points are the periodic points.

(d) Isolated points are recurrent.

(e) $\omega(x) \subset \overline{\mathrm{O}(x)}$ and $\omega(x) = \overline{\mathrm{O}(x)}$ if and only if x is recurrent (Need X compact?).

5. Prove that if $T : X \to X$ is an isometry and $\omega(x) \neq \emptyset$, then x is a recurrent point and $\omega(x)$ is a minimal set.

Chapter 15. Substitution Dynamical Systems

15.1 Sequence Spaces

Our main purpose in studying compactness in metric spaces is to apply these results to substitution dynamical systems. First we show that with a suitable metric, the space of all sequences on a finite alphabet is a compact metric space. This is often proved using the Tychonoff Theorem: *Cartesian products of compact spaces are compact.* A proof of this result will take us too far afield, so we give a direct proof:

Let $\mathcal{A} = \{0, 1, \ldots, s-1\}$, $s \geq 2$, be a finite set (the *alphabet*) and $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, then $\mathcal{A}^{\mathbb{N}}$ is the set of all one-sided infinite sequences beginning with the index 0, so if $\omega \in \mathcal{A}^{\mathbb{N}}$,

$$\omega = \omega_0\omega_1\omega_2\omega_3\omega_4\omega_5 \ldots, \quad (\omega)_i = \omega_i,$$

(Strictly speaking we should write $\omega = (\omega_0, \omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \ldots)$, but we often use the above abuse of notation).

Define a metric on $\mathcal{A}^{\mathbb{N}}$ by

$$d(\omega, \omega') = 2^{-\min\{n \in \mathbb{N} :\, \omega_n \neq \omega'_n\}} \quad \text{if } \omega \neq \omega' \quad \text{and} \quad d(\omega, \omega) = 0,$$

so that two points are close when their first few terms are equal.

Theorem 15.1.1 *The space $\mathcal{A}^{\mathbb{N}}$ is a compact metric space. In particular, it is complete, totally bounded and sequentially compact.*

Proof. The fact that $\mathcal{A}^{\mathbb{N}}$ is a metric space is similar to the proof given in Chapter 4, and is left as an exercise. We show that $\mathcal{A}^{\mathbb{N}}$ is sequentially compact and hence is compact.

Let $\omega^n = (a_0^n, a_1^n, a_2^n, \ldots)$ be an infinite sequence in $\mathcal{A}^{\mathbb{N}}$, then we show that (ω^n) has a limit point.

Since $\mathcal{A}$ is a finite set, there exists $a_0 \in \mathcal{A}$ such that

$$A_0 = \{n \in \mathbb{N} : a_0^n = a_0\}$$

is an infinite set. We construct a sequence $(a_0, a_1, a_2, \ldots)$ inductively as follows:

(i) We have found a_0.

(ii) Suppose that for $m > 0$ we have found $a_k \in \mathcal{A}$, $k = 0, 1, \ldots, m-1$, and infinite sets $A_0, A_1, \ldots, A_{m-1}$ contained in $\mathbb{N}$ satisfying

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_{m-1},$$

and such that $a_k^n = a_k$ for all $k \in A_k$, $k = 0, \ldots, m-1$.

We can find a set A_m and $a_m \in \mathcal{A}$ such that

$$A_m = \{n \in A_{m-1} : a_m^n = a_m\}$$

is an infinite set. Then by the principle of induction, we have constructed a sequence

$$\omega = (a_0, a_1, a_2, \ldots),$$

which we show is a limit point of our given sequence.

Let $\delta > 0$ and choose $n \in \mathbb{N}$ so large that $1/2^n < \delta$, then we need to show that $\{\omega^n : n \in \mathbb{N}\} \cap B_\delta(\omega)$ contains points besides ω. It suffices to show that a member of $\{\omega^n : n \in \mathbb{N}\}$ coincides with ω in its first $n+1$ coordinates.

Let $p \in \cap_{k=0}^n A_k$, then $p \in A_0$, so $a_0^p = a_0$, $p \in A_1$, so $a_1^p = a_1$, $\ldots, p \in A_n$, so $a_n^p = a_n$, i.e.,

$$\omega^p = (a_0^p, a_1^p, a_2^p, \ldots, a_n^p, \ldots),$$

coincides with ω in the first $n+1$ coordinates, so $d(\omega^p, \omega) \leq 1/2^{n+1} < 1/2^n < \delta$ and the result follows. $\square$

The following result says that $\mathcal{A}^{\mathbb{N}}$ is a type of Cantor set. Recall that we defined a subset of $\mathbb{R}$ to be totally disconnected if it contains no open subsets. We give a more general version of this notion for metric spaces. A metric space X is said to be *disconnected* if there exists a *clopen* subset of X (i.e., a set which is both open and closed), other than X or $\emptyset$. In other words, there exist A and B open, with $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$ and $A \cup B = X$. If X is not disconnected, it is *connected*. A subset Z of X is disconnected if there are open sets A and B that have non-empty intersection with Z and satisfy $Z \subseteq A \cup B$ and $A \cap B \cap Z = \emptyset$. A subset of X is connected if it is not a disconnected subset.

For example, $\mathbb{R}$ with its usual metric is connected, but the rationals, the Cantor set C and the set of irrationals (each with the metric coming from $\mathbb{R}$) are not connected. In fact they are what we call *totally disconnected*.

A subset $Z \subseteq X$ is a *component* of X if it is maximally connected, i.e., Z is a connected set, and if $Z \subseteq Y$ with Y connected, then $Z = Y$. It can be shown that the components of X partition X into clopen sets.

Definition 15.1.2 The metric space (X, d) is *totally disconnected* if the only components of X are singletons (i.e., sets of the form $\{x\}$ for $x \in X$).

Theorem 15.1.3 *The metric space $\mathcal{A}^{\mathbb{N}}$ is totally disconnected, and every member is a limit point.*

Proof. Let $\omega \in \mathcal{A}^{\mathbb{N}}$, then ω is a limit point since we can construct a sequence (ω^n) in $\mathcal{A}^{\mathbb{N}}$ so that ω^n is equal to ω in the first n-coordinates, but with $\omega^n \neq \omega$ and $\omega^n \neq \omega^m$ for every $m, n \in \mathbb{N}$, $m \neq n$. Then it is clear that $d(\omega^n, \omega) \to 0$ as $n \to \infty$.

To show that $\mathcal{A}^{\mathbb{N}}$ is totally disconnected, let $Z \subseteq \mathcal{A}^{\mathbb{N}}$ be a component containing at least two points, say $\omega, \omega' \in Z$ with $\omega \neq \omega'$. Suppose that $\omega_k = \omega'_k$ for $k = 0, 1, \ldots, n-1$, but $\omega_n \neq \omega'_n$. Set

$$U = \{x \in \mathcal{A}^{\mathbb{N}} : d(x, \omega) < 1/2^n\} = \{x \in \mathcal{A}^{\mathbb{N}} : d(x, \omega) \leq 1/2^{n+1}\},$$

then U is both open and closed, $\omega \in U$, $\omega' \notin U$ (see Exercises 4.4). This contradicts Z being a connected component. □

Here is an application of results from this section to the existence of fixed points of substitutions:

Theorem 15.1.4 *Let θ be a substitution on $\mathcal{A} = \{0, 1, \ldots, s-1\}$, ($s > 1$), extended to be a function $\theta : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$. Then*

(a) *θ is a continuous function.*

(b) *Corresponding to each $a \in \mathcal{A}$ with $|\theta(a)| > 1$, and for which $\theta(a)$ starts with a, θ has a unique fixed point $u_a \in \mathcal{A}^{\mathbb{N}}$.*

Proof. We may assume that $\theta(0)$ starts with 0. Set $X_0 = \{\omega \in \mathcal{A}^{\mathbb{N}} : (\omega)_0 = 0\}$, a closed compact subspace of $\mathcal{A}^{\mathbb{N}}$. This follows for suppose $\omega^n \to \omega$ as $n \to \infty$, with $\omega^n \in X_0$ and $(\omega)_0 \neq 0$, then clearly $d(\omega^n, \omega) = 1$, and this is impossible.

Now $d(\theta(\omega), \theta(\omega')) \leq d(\omega, \omega')$ for all $\omega, \omega' \in \mathcal{A}^{\mathbb{N}}$, so θ is a continuous function. Clearly X_0 is invariant under θ, so we can think of θ as a continuous map on a compact space $\theta : X_0 \to X_0$. Let $\omega, \omega' \in X_0$, and suppose that $d(\omega, \omega') = 1/2^k$ for some $k > 0$. Then clearly $d(\theta(\omega), \theta(\omega')) < d(\omega, \omega')$ since $(\theta(\omega))_i = (\theta(\omega'))_i$ for $i = 0, 1, \ldots, kr$, where $r > 1$.

Thus we can apply Theorem 14.3.1 to deduce that θ has a unique fixed point in X_0, and the result follows as we can do this for each $a \in \mathcal{A}$ where $\theta(a)$ starts with a. □

Examples 15.1.5 1. In the last chapter, we saw that the fixed points of substitutions arise from iteration. If the first letter of $\theta(a)$ is a and $\theta(a)$ is a word having length at least two, then $u = \lim_{n\to\infty} \theta^n(a)$ is a fixed point. The Thue-Morse sequence arises from the substitution on $\mathcal{A} = \{0, 1\}$:

$$\theta(0) = 0\,1, \quad \theta(1) = 1\,0,$$

giving two fixed points in $\{0,1\}^{\mathbb{N}}$. These are

$$u = 0\,1\,1\,0\,1\,0\,0\,1\,1\,0\,0\,1\ldots, \quad \text{and} \quad \mathrm{R}(u) = 1\,0\,0\,1\,0\,1\,1\,0\,0\,1\,1\,0\,\ldots.$$

2. Again let $\mathcal{A} = \{0, 1\}$ and $\theta(0) = 0\,1\,0, \quad \theta(1) = 1\,0\,1$, then we have two fixed points which are both periodic sequences. This substitution is not of interest to us as it does not give rise to non-trivial dynamical behavior. For $\theta(0) = 1\,0\,1 \quad \theta(1) = 0\,1\,0$, we have no fixed points (but we do have period 2-points).

3. The Fibonacci substitution is defined on $\mathcal{A} = \{0, 1\}$ by $\theta(0) = 0\,1$, $\theta(1) = 0$, and it can be seen that θ has a unique fixed point:

$$u = 0\,1\,0\,0\,1\,0\,1\,0\,0\,1\,0\,0\,1\,\ldots$$

Exercises 15.1

1. Suppose $\mathcal{A} = \{0, 1, 2, \ldots, s-1\}$, $s \geq 2$, is an alphabet and $\mathcal{A}^{\mathbb{N}}$ is the set of all sequences with values in $\mathcal{A}$ ($\mathcal{A}^{\mathbb{N}} = \{\omega = (a_0, a_1, a_2, \ldots) : a_i \in \mathcal{A}\}$). Prove that if we define a distance d on $\mathcal{A}^{\mathbb{N}}$ by

$$d(\omega_1, \omega_2) = 2^{-\min\{k \geq 0 : a_k \neq b_k\}}, \quad \omega_1 \neq \omega_2, \quad d(\omega_1, \omega_1) = 0$$

where $\omega_1 = (a_0, a_1, a_2, \ldots)$ and $\omega_2 = (b_0, b_1, b_2, \ldots)$, then

(i) d defines a metric on $\mathcal{A}^{\mathbb{N}}$,

(ii) if $d(\omega_1, \omega_2) < 1/2^n$, then $a_0 = b_0, a_1 = b_1, \ldots, a_{n-1} = b_{n-1}$, and if $a_0 = b_0, a_1 = b_1, \ldots, a_{n-1} = b_{n-1}$, then $d(\omega_1, \omega_2) \leq 1/2^n$,

(iii) if $\alpha \in \mathcal{A}^{\mathbb{N}}$, then

$$\{\omega \in \mathcal{A}^{\mathbb{N}} : d(\alpha, \omega) < 1/2^{n-1}\} = \{\omega \in \mathcal{A}^{\mathbb{N}} : d(\alpha, \omega) \leq 1/2^n\}.$$

2. (i) Given $\alpha_1, \alpha_2, \ldots, \alpha_{n-1} \in \mathcal{A}$, the set

$$[\alpha_1, \alpha_2, \ldots, \alpha_{n-1}] = \{\omega \in \mathcal{A}^{\mathbb{N}} : a_j = \alpha_j, 0 \leq j \leq n-1, \quad \text{where} \quad \omega = (a_0, a_1, a_2, \ldots)\},$$

is called a *cylinder set*. Prove that cylinder sets are both open and closed (called *clopen*), and in fact if $\alpha \in C$

$$C = \{\omega \in \mathcal{A}^{\mathbb{N}} : d(\alpha, \omega) < 1/2^{n-1}\} = \{\omega \in \mathcal{A}^{\mathbb{N}} : d(\alpha, \omega) \leq 1/2^n\}.$$

(ii) Show that any clopen set is the finite union of cylinder sets.

3. Recall from Chapter 6 that the dynamical system (Y, S) is a *factor* of the dynamical system (X, T) if there is a continuous map ϕ from X onto Y with $\phi \circ T = S \circ \phi$. (X, T) is called an *extension* of (Y, S) and ϕ is called a *factor map*. The two dynamical systems are conjugate (with conjugacy ϕ), if $\phi : X \to Y$ is a homeomorphism.

4. Suppose that u and v are two sequences taking values in finite alphabets $\mathcal{A}$ and $\mathcal{B}$ respectively, and σ is the shift map. If the dynamical system $(\overline{O(v)}, \sigma)$ is a factor of the dynamical system $(\overline{O(u)}, \sigma)$, via a factor map ϕ, prove that there is a positive integer n such that for every i, the coordinate of index i of $\phi(x)$ depends only on $(x_i, \ldots, x_{i+n})$ (Hint: Define $\phi_0 : \overline{O(u)} \to \mathcal{B}$ by $\phi(x) = x_0$, and show that $\phi^{-1}[a])$ is a clopen set for every $a \in \mathcal{B}$).

5. The substitution θ on $\mathcal{A}$ has constant length p if $|\theta(a)| = p$ for all $a \in \mathcal{A}$. For example, $\theta(0) = 0\,1\,0$, $\theta(1) = 1\,2\,1$, $\theta(2) = 2\,0\,2$ on $\{0, 1, 2\}$ has constant length 3. Show that for such a substitution, $\sigma^p \circ \theta(\omega) = \theta \circ \sigma(\omega)$, for all $\omega \in \overline{O(u)}$, where σ is the shift map and u is a fixed point of θ.

15.2 Languages

We give some of the basic properties of sequences arising from substitutions. Much of this material can be found in the book "Substitutions in Dynamics, Arithmetics and Combinatorics", by N. Pytheas Fogg [25] (see also [49]).

15.2.1 Languages and words.

Let $\mathcal{A} = \{0, 1, 2, \ldots, s-1\}$, $s \geq 2$, be a finite set, called the *alphabet* and whose members are the *letters* of the alphabet. A *word* or *finite block* of members of $\mathcal{A}$ is a finite string of letters whose *length* is the number of letters in the string. Denote the set of all finite words by $\mathcal{A}^*$. This includes the *empty word* (the word having 0 length), and is denoted by ϵ.

If $\omega \in \mathcal{A}^*$ with $\omega = \omega_1\omega_2\ldots\omega_n$, $\omega_i \in \mathcal{A}$ then we write its length as $|\omega| = n$. For example if $\mathcal{A} = \{0, 1\}$, then $0\,1\,1\,0\,0$ and $1\,1\,1\,0$ are words of length 5 and 4 respectively.

15.2.2 The Complexity Function

Definition 15.2.3 Suppose that $u = u_0u_1u_2\ldots = (u_n)_{n\in\mathbb{N}}$ is a one-sided infinite sequence of letters from $\mathcal{A}$, then we think of u as being a member of $\mathcal{A}^{\mathbb{N}}$. The *language* of the sequence u is the set $\mathcal{L}(u)$ of all finite words that appear in the sequence u. We use $\mathcal{L}_n(u)$ to denote the subset of $\mathcal{L}(u)$ consisting of those words of length n. We call a finite word appearing in the sequence u a *factor* of u.

We write $p_u(n) = |\mathcal{L}_n(u)|$, the number of different factors of length n. $p_u(n)$ is called the *complexity function* of the sequence as it can be thought of as measuring the "randomness" of the sequence. The greater the variety of different factors appearing, the more complex is the sequence. For example, a sequence which is periodic has little complexity (see [25]).

Clearly $p_u(n) \leq p_u(n+1)$ for $n \in \mathbb{N}$ and $1 \leq p_u(n) \leq s^n$ where s is the number of members of the alphabet.

Definition 15.2.4 The sequence $u = (u_n)_{n\in\mathbb{N}}$ is *periodic*, if there exists $N \geq 1$ such that $u_n = u_{n+N}$ for all $n \in \mathbb{N}$. The minimum such N is the *period* of the sequence. It is *ultimately periodic* if it is periodic from some index on. If the sequence is neither periodic, nor ultimately periodic, we say it is *aperiodic*.

Examples 15.2.5 It can be shown that sequence u arising from the Fibonacci substitution, $\theta(0) = 0\,1$, $\theta(1) = 0$ has complexity function $p_u(n) = n+1$ for all $n \in \mathbb{N}$. A sequence having this property is said to be a *Sturmian sequence*. The Morse sequence can be shown to have complexity function satisfying $p_u(n) \leq C \cdot n$ for some constant $C > 0$, for all $n \in \mathbb{N}$.

The next result tells us that sequences with bounded complexity function have to be periodic. In particular, Sturmian sequences have the minimal complexity of all aperiodic sequences. Also, neither the Fibonacci nor the Morse sequences is periodic or eventually periodic. This is an early result due to Coven and Hedlund [18].

Proposition 15.2.6 *If $u = (u_n)_{n\in\mathbb{N}}$ is periodic or ultimately periodic, then $p_u(n)$ is a bounded sequence. If there exists $n \in \mathbb{N}$ such that $p_u(n) \leq n$, then u is an ultimately periodic sequence.*

Proof. If u is a periodic sequence with period n and $|\mathcal{L}_n(u)| = m$, then $|\mathcal{L}_{n+1}(u)| = m$ also, because a letter can be added to a member of $\mathcal{L}_n(u)$ in only one way, and we see that $|\mathcal{L}_k(u)| = m$ for all $k \geq n$. Similar considerations apply to ultimately periodic sequences.

On the other hand, assume that $p_u(n) \leq n$ for some $n \in \mathbb{N}$. We may assume that $p_u(1) \geq 2$, for otherwise u is a constant sequence. Then we have

$$2 \leq p_u(1) \leq p_u(2) \leq \cdots \leq p_u(n) \leq n.$$

We have here an increasing sequnce of n integers, the smallest being at least 2 and the largest less than or equal to n. By the pigeon-hole principle we must have $p_u(k+1) = p_u(k)$ for some k. If $\omega = \omega_0, \omega_1 \dots \omega_{k-1} \in \mathcal{L}_k(u)$, then there is at least one factor of u of the form $\omega\, a$ occuring in $\mathcal{L}_{k+1}(u)$ for some letter $a \in \mathcal{A}$. Since $p_u(k+1) = p_u(k)$ there can be at most one such factor. It follows that if $u_i u_{i+1} \dots u_{i+k-1}$ and $u_j u_{j+1} \dots u_{j+k-1}$ are two factors of length k in u with $u_i u_{i+1} \dots u_{i+k-1} = u_j u_{j+1} \dots u_{j+k-1}$, then $u_{i+k} = u_{j+k}$ since the additional letter can be added in only one way. Since there are only finitely many factors of length k, we must have $u_i u_{i+1} \dots u_{i+k-1} = u_j u_{j+1} \dots u_{j+k-1}$ for some $j > i$. If we repeatedly add a letter to the right in a unique way, then delete a letter from the left to go back to a factor of length k, we see that $u_{i+p} = u_{j+p}$ for every $p \geq 0$, and hence we have a periodic sequence. $\square$

Exercises 15.2

1. Give an example of a sequence on $\mathcal{A} = \{0, 1\}$ having complexity function $p_u(n) = 2^n$ for all $n \geq 1$.

15.3 Dynamical Systems Arising from Sequences

Our aim is to define the dynamical system arising from a substitution θ defined on a finite alphabet $\mathcal{A} = \{0, 1, \dots, s-1\}$, $s \geq 2$, but first we look at the more general situation of a dynamical system that arises from some arbitrary sequence $u \in \mathcal{A}^{\mathbb{N}}$.

Denote by $\sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ the shift map defined in the usual way:

$$\sigma(\omega_0, \omega_1, \omega_2, \ldots) = (\omega_1, \omega_2, \ldots) \quad \text{or} \quad \sigma((\omega_n)_{n\in\mathbb{N}} = (\omega_{n+1})_{n\in\mathbb{N}}.$$

We have seen that the space $\mathcal{A}^{\mathbb{N}}$ with the metric d given earlier, is a compact metric space, which is therefore complete and is also totally disconnected. Clearly the shift map is onto, but not one-to-one on $\mathcal{A}^{\mathbb{N}}$. Since $d(\sigma(\omega_1), \sigma(\omega_2)) \leq 2d(\omega_1, \omega_2)$ for all $\omega_1, \omega_2 \in \mathcal{A}^{\mathbb{N}}$, σ is continuous.

Let $u = u_0u_1u_2u_3\ldots \in \mathcal{A}^{\mathbb{N}}$ be a sequence, the orbit of u is the set $O(u) = \{u, \sigma(u), \sigma^2(u), \ldots, \sigma^n(u), \ldots\}$. The orbit closure $\overline{O(u)}$ is a closed subset of $\mathcal{A}^{\mathbb{N}}$ and hence it is compact as a subset of $\mathcal{A}^{\mathbb{N}}$, so is a compact metric space in its own right. Note that $\overline{O(u)}$ is a finite set if and only if u is a shift periodic sequence (i.e., $\sigma^k(u) = u$ for some $k > 0$).

Definition 15.3.1 The dynamical system associated with a sequence $u = u_0u_1u_2\ldots \in \mathcal{A}^{\mathbb{N}}$ is $(\overline{O(u)}, \sigma,)$, the shift map restricted to the orbit closure of u. This map is continuous (since it is continuous on $\mathcal{A}^{\mathbb{N}}$), well defined ($\overline{O(u)}$ is an invariant set, for if $\omega \in \overline{O(u)}$, then $\omega = \lim_{n\to\infty} \sigma^{k_n}(u)$ for some sequence k_n, so that by the continuity of σ, $\sigma(\omega) = \lim_{n\to\infty} \sigma^{k_n+1}(u) \in \overline{O(u)}$).

Such a dynamical system is often called a *sub-shift*, the shift map σ on $\mathcal{A}^{\mathbb{N}}$, being the *full shift*.

We can give an alternative description of the set $\overline{O(u)}$ using the language of u, $\mathcal{L}(u)$ as follows:

Proposition 15.3.2 *For every sequence $\omega \in \overline{O(u)}$, the following are equivalent:*

(i) $\omega \in \overline{O(u)}$.

(ii) *there exists a sequence k_n such that $\omega_0\omega_1\ldots\omega_n = u_{k_n}u_{k_n+1}\ldots u_{k_n+n}$.*

(iii) $\mathcal{L}_n(\omega) \subset \mathcal{L}_n(u)$ *for every $n \in \mathbb{N}$.*

Proof. Suppose that $\omega \in \overline{O(u)}$, then for each $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}$ with $d(\omega, \sigma^{k_n}(u)) < 1/2^n$. This implies that $\omega_0\omega_1\ldots\omega_n = u_{k_n}u_{k_n+1}\ldots u_{k_n+n}$, so (ii) holds. Conversely, if (ii) holds, then (i) follows in a similar way.

If (ii) holds then every word of length n appearing in ω appears in u, so (iii) holds. Conversely, if (iii) holds then (ii) must hold. $\square$

Corollary 15.3.3 $\overline{O(u)} = \{\omega \in \mathcal{A}^{\mathbb{N}} : \mathcal{L}(\omega) \subset \mathcal{L}(u)\}$.

Proof. If $\omega \in \overline{O(u)}$, then from Proposition 14.6.2, every word appearing in ω appears in u, so that $\mathcal{L}(\omega) \subset \mathcal{L}(u)$. On the other hand, if $\omega \in \mathcal{A}^{\mathbb{N}}$ with $\mathcal{L}(\omega) \subset \mathcal{L}(u)$, then clearly $\mathcal{L}_n(\omega) \subset \mathcal{L}_n(u)$ for every n and the result follows by the same proposition. $\square$

Following we define the fundamental notion of *almost periodicity* for a sequence $u \in \mathcal{A}^{\mathbb{N}}$. It will be clear that a periodic sequence is almost periodic, but the converse is not true. Almost periodic sequences are often said to be *minimal*.

Definition 15.3.4 The sequence u is *recurrent* if every factor of u appears infinitely often. It is *uniformly recurrent* or *almost periodic*, if every factor appears infinitely often with bounded gaps. More precisely, if $\omega = \omega_0\omega_1 \dots \omega_\ell$ is a factor of u, there exists $K > 0$ and a sequence of integers k_n with $(n-1)K \leq k_n < nK$, $n = 1, 2, \dots$, such that $u_{k_n}u_{k_n+1} \dots u_{k_n+\ell} = \omega$.

Recall that in Chapter 13 we showed that the Morse sequence has the property of being almost periodic. Now we are able to show that certain dynamical systems arising from sequences have some of the properties that chaotic maps have. The dynamical system (X, T) is *transitive* if there is a point $x_0 \in X$ having a dense orbit, and it is *minimal* if every point has a dense orbit. We have:

Theorem 15.3.5 *The dynamical system $(\overline{O(u)}, \sigma)$ is minimal if and only if u is an almost periodic sequence.*

Proof. Suppose that u is almost periodic, but the dynamical system is not minimal, then there exists $\alpha \in \overline{O(u)}$ for which $u \notin \overline{O(\alpha)}$. If this were not the case, then $\mathcal{L}(u) \subseteq \mathcal{L}(\alpha)$ (from the previous corollary), and since $\mathcal{L}(\alpha) \subseteq \mathcal{L}(u)$, these must be equal, so $\overline{O(u)} = \overline{O(\alpha)}$, a contradiction.

By the last proposition, there exists j such that $u_0u_1 \dots u_j \notin \mathcal{L}(\alpha)$. Now there is a sequence k_n for which $\alpha = \lim_{n\to\infty} \sigma^{k_n}(u)$ and using the fact that u is almost periodic

$$u_0u_1 \dots u_j = u_{k_n+s}u_{k_n+s+1} \dots u_{k_n+s+j},$$

for some s and infinitely many k_n. By the continuity of σ, $\sigma^{k_n+s}(u)$ approaches $\sigma^s(\alpha)$, so

$$\alpha_s\alpha_{s+1} \dots \alpha_{s+j} = u_{k_n+s}u_{k_n+s+1} \dots u_{k_n+s+j},$$

for all n large enough.. It follows that $u_0u_1 \dots u_j = \alpha_s\alpha_{s+1} \dots \alpha_{s+j}$, which is a contradiction.

Conversely suppose that $(\overline{O(u)}, \sigma)$ is minimal and let $\alpha \in \overline{O(u)}$, so that $\overline{O(\alpha)} = \overline{O(u)}$. Let $V = B_\epsilon(u)$ be some open ball containing u, where $\epsilon > 0$, then $V \cap \overline{O(\alpha)} \neq \emptyset$.

In particular, $\sigma^k(\alpha) \in V$ for some $k \in \mathbb{N}$ and so $\overline{O(u)} = \cup_{k \geq 0} \sigma^{-k} V$. This is an open cover for $\overline{O(u)}$, so by the compactness of $\overline{O(u)}$ there is a finite subcover, say

$$\overline{O(u)} = \sigma^{-k_1} V \cup \sigma^{-k_2} V \cup \cdots \cup \sigma^{-k_n} V.$$

If $K = \max_{1 \leq i \leq n} k_i$, iterations of α under σ must enter V after at most K steps.

If $\alpha = \sigma^j u$, then one of the points $\sigma^j u, \ldots \sigma^{j+K} u$ must lie in V. If we start with

$$V = \{\omega \in \mathcal{A}^{\mathbb{N}} : \omega_0 = u_0, \omega_1 = u_1, \ldots, \omega_\ell = u_\ell\},$$

(the *cylinder set* $[u_0 u_1 \ldots u_\ell]$ - see the exercises) for every $j > 0$, then $u_0 u_1 \ldots u_\ell$ is one of the words

$$u_j \ldots u_{j+\ell}, \quad u_{j+1} \ldots u_{j+\ell+1}, \quad u_{j+K} \ldots u_{j+k+\ell},$$

so u is almost periodic. $\square$

Exercises 15.3

1. Show that if u is shift periodic ($\sigma^p(u) = u$ for some $p \in \mathbb{N}$, where σ is the shift map), then $\overline{O(u)}$ is finite. Conversely, show that if $\overline{O(u)}$ is finite, then u is shift periodic. Can $\overline{O(u)}$ be countable?

15.4 Some Substitution Dynamics

In order for a substitution θ on a finite alphabet $\mathcal{A}$ to give rise to non-trivial dynamics, there are certain requirements that are often needed. These are:

1. There exists an $a \in \mathcal{A}$ such that $\theta(a)$ begins with a (this is necessary in order for θ to have a fixed point u).

2. $\lim_{n \to \infty} |\theta^n(b)| = \infty$ for all $b \in \mathcal{A}$.

3. All letters of $\mathcal{A}$ actually occur in the fixed point u.

A related property that is often useful is the following:

Definition 15.4.1 A substitution θ over $\mathcal{A}$ is said to be *primitive* if there exists $k \in \mathbb{N}$ such that for all $a, b \in \mathcal{A}$, the letter a occurs in the word $\sigma^k(b)$.

We shall show that primitive substitutions give rise to minimal dynamical systems. It is clear that the Morse and Fibonacci substitutions are primitive, but the *Chacon* *substitution*:

$$\theta(0) = 0\,0\,1\,0, \quad \theta(1) = 1,$$

is clearly not primitive and does not satisfy (2) above (nor does θ^n for any n). However, the Chacon substitution can be shown to give rise to a minimal dynamical system. Note that "non-trivial looking" substitutions such as $\theta(0) = 0\,1\,0$, $\theta(1) = 1\,0\,1\,0\,1$ on $\mathcal{A} = \{0, 1\}$, may give rise to a periodic fixed point.

The following proposition, together with the observation that if θ is a primitive substitution, then some power θ^n will satisfy properties 1, 2 and 3 above, can be used to show that certain substitutions have non-trivial dynamics. As usual, u is a periodic point of the substitution θ if there is some $p \in \mathbb{N}$ with $\theta^p(u) = u$. Note that if u is a periodic point for the substitution θ, then the dynamical system $(\overline{O(u)}, \sigma)$ is finite (i.e., $\overline{O(u)}$ is a finite set), if and only if u is also periodic under the shift map σ.

Proposition 15.4.2 *If θ is a primitive substitution, then any periodic point u is an almost periodic sequence. It follows that the shift dynamical system $(\overline{O(u)}, \sigma)$ is minimal.*

Proof. Suppose that $u = \theta^p(u)$ is a periodic point of the substitution θ, $u = u_0u_1u_2\dots$ say. Let $k \in \mathbb{N}$ be chosen so that for any $b \in \mathcal{A}$, a appears in $(\theta^p)^k(b)$. Now

$$u = (\theta^p)^k(u) = (\theta^p)^k(u_0)(\theta^p)^k(u_1)\dots,$$

where a occurs in each $(\theta^p)^k(u_i)$, so a occurs infinitely often with bounded gaps. It follows that $(\theta^p)^n(a)$ appears in $u = (\theta^p)^n(u)$ infinitely often with bounded gaps, and hence so does every factor (word) occuring in u. $\square$

Another consequence of u being recurrent, is that the shift map is onto.

Proposition 15.4.3 *If u is a recurrent sequence, then the shift map $\sigma : \overline{O(u)} \to \overline{O(u)}$ is onto.*

Proof. Let $\omega = \omega_0\omega_1\dots \in \overline{O(u)}$, then since $\mathcal{L}(\omega) \subseteq \mathcal{L}(u)$ and u is recurrent, $\omega_0\omega_1\dots\omega_n$ appears in u infinitely often. It follows that there exists $a_n \in \mathcal{A}$ such that

$$a_n\omega_0\omega_1\dots\omega_n$$

appears in u. For each $n \in \mathbb{N}$, we can construct $\omega^n \in \overline{O(u)}$ of the form $\omega^n = a_n\omega_0\omega_1\dots\omega_n * * * \dots$. Using the compactness of $\overline{O(u)}$, the infinite sequence $(\omega^n)_{n\in\mathbb{N}}$ must have a limit point $\omega' \in \overline{O(u)}$. We must have $\omega' = a\,\omega$ for some $a \in \mathcal{A}$, for otherwise $d(\omega', \omega^n)$ would be bounded below by some fixed $\epsilon > 0$. It is now clear that $\sigma(a\,\omega) = \omega$, so that σ is onto. $\square$

Exercises 15.4

1. (a) Show that if θ is any substitution on $\mathcal{A} = \{a_1, a_2, \dots, a_p\}$, then there exists $a \in \mathcal{A}$ and $n \in \mathbb{N}$ with $[\theta^n(a)]_0 = a$ (Hint: Consider a_1, $[\theta(a_1)]_0$, $[\theta^2(a_1)]_0$, $\dots$, $[\theta^n(a_1)]_0$ for some $n > p$. These cannot all be different).

(b) Deduce that if $|\theta^n(a)| \to \infty$ for all $a \in \mathcal{A}$, then θ has a periodic point.

2. A substitution θ on $\mathcal{A}$ is *one-to-one* (or *injective*), if $\theta(a) = \theta(b)$, implies $a = b$, for any $a, b \in \mathcal{A}$. For example the substitution $\theta(0) = 0\,1$, $\theta(1) = 0\,1$ is not one-to-one on $\{0, 1\}$, but $\theta(0) = 0\,1$, $\theta(1) = 1$ is one-to-one. The purpose of this exercise is to show how a substitution that is not one-to-one can be reduced to one that is, by elimating superfluous members of the alphabet $\mathcal{A}$.

Suppose θ is not one-to-one on $\mathcal{A}$. Let $\mathcal{B} \subseteq \mathcal{A}$ be defined so that for all $a \in \mathcal{A}$ there exists a unique $b \in \mathcal{B}$ with $\theta(a) = \theta(b)$. Let $\phi : \mathcal{A} \to \mathcal{B}$ be the map defined by $\phi(a) = b$ if $\theta(a) = \theta(b)$.

Denote by τ the unique substitution on $\mathcal{B}$ satisfying

$$\tau \circ \phi = \phi \circ \theta,$$

(where ϕ is extended in the obvious way).

(a) Find τ for the substitutions (i) $\theta(0) = 0\,1$, $\theta(1) = 0\,1$, $\theta(2) = 2\,0$, (ii) $\theta(0) = 021$, $\theta(1) = 12$, $\theta(2) = 021$. (Although the substitution τ may not be one-to-one, continuing this reduction, a one-to-one substitution is obtained in a finite number of steps).

(b) If θ is primitive, show that τ is also primitive.

(c) If $X_\theta = \overline{O(u)}$, where u is a fixed point of θ, then $\phi(u)$ is a fixed point of τ and the dynamical system (X_τ, σ) is a factor of the dynamical system (X_θ, σ) (where σ is the shift map, and where $\phi : X_\theta \to X_\tau$ is extended as before).

(d) Show that X_θ is a finite set if and only if X_τ is a finite set.

(e) If X_θ is not finite, show that (X_τ, σ) and (X_θ, σ) are conjugate dynamical systems.

Chapter 16, Sturmian Sequences and Irrational Rotations

Let $u = u_0u_1u_2\ldots$ be an infinite sequence. Recall that $p_u(n) = |\mathcal{L}_n(u)|$ is the number of different factors (sub-words) of length n, and is called the *complexity function* of the sequence. In this section we investigate sequences whose complexity function satisfies $p_u(n) = n+1$ for all $n \in \mathbb{N}$. These are the aperiodic sequences of minimal complexity. We will show that the sequence u arising as the fixed point of the Fibonacci substitution, $\theta(0) = 0\,1$, $\theta(1) = 0$ has this property and so also do sequences arising from a "coding" of irrational rotations.

16.1 Sturmian Sequences

Definition 16.1.1 A sequence u having the property that $p_u(n) = n+1$ for all $n \in \mathbb{N}$ is said to be a *Sturmian sequence*.

If u is a Sturmian sequence, then it has to be aperiodic (neither periodic, nor ultimately periodic), for otherwise $p_u(n)$ would be bounded. In addition, u has to be recurrent, for suppose the factor $w = w_1\ldots w_n$ only occurs a finite number of times in $u = u_1u_2\ldots$, and does not occur after u_N. Let

$$v = u_{N+1}u_{N+2}\ldots,$$

a new sequence whose language $\mathcal{L}(v)$ does not contain w. It follows that $p_v(n) \leq n$, so that v is eventually periodic, and hence so is u, a contradiction.

Also, since $p_u(1) = 2$, the sequence must use only two letters, so we write the alphabet as $\mathcal{A} = \{0, 1\}$. We will assume that this is our alphabet throughout this chapter. In addition, $p_u(2) = 3$, so one of the pairs $0\,0$, $1\,1$ does not appear in u ($0\,1$ and $1\,0$ have to appear for otherwise the sequence would be constant).

Sturmian sequences have a long history involving Jean Bernoulli III, Christoffel, A. A. Markov, M. Morse, G. Hedlund, E. Coven and many others. Our treatment is mostly based on that in Allouche and Shallit [2], Fogg [25], Rauzy [53] and Lothaire [39]. Our aim is to show that sequences that arise as "codings" of irrational rotations are Sturmian, and some (but not all) Sturmian sequences can be represented as substitutions. We mention without proof that all Sturmian sequences arise as codings of irrational rotations. We start by showing that Sturmian sequences do exist, and in fact the Fibonacci sequence is Sturmian.

Definition 16.1.2 A *right special factor* of u is a factor w that appears in u such that $w\,0$ and $w\,1$ also appear in u. *Left special factors* are defined in a similar way.

Proposition 16.1.3 *Let $\mathcal{A} = \{0, 1\}$ and $u \in \mathcal{A}^{\mathbb{N}}$. The sequence u is Sturmian if and only if it has exactly one right special factor of each length.*

Proof. If u is Sturmian and $w_1, w_2, \dots w_{n+1}$ are the factors of length n, then all but one of them can be extended in a unique way to form a factor of length $n+1$, and exactly one must be extendable in two ways.

Conversely, given a factor w of length n, since it appears in u, either $w\,0$ or $w\,1$ appears in u. Clearly, since $p_u(n) = n+1$, exactly one such factor w can have both 0 and 1 as a suffix to form a factor of length $n+1$. □

Example 16.1.4 The Fibonacci substitution is defined on $\mathcal{A} = \{0, 1\}$ by $\theta(0) = 0\,1$, $\theta(1) = 0$, and we have seen that θ has a unique fixed point:

$$u = 0\,1\,0\,0\,1\,0\,1\,0\,0\,1\,0\,0\,1\,\dots.$$

The sequence u is a Sturmian, i.e, $p_u(n) = n+1$ for all $n \geq 0$.

Proof. We will show that for each n there is exactly one right special factor of length n.

Recall that ϵ is the empty word. Now $u = \theta(u)$ is a concatentation of the words 0 and $0\,1$, so that the word $1\,1$ does not appear as a factor of u, so $p_u(2) = 3$, and we have

$$\mathcal{L}_0(u) = \{\epsilon\}, \quad \mathcal{L}_1(u) = \{0, 1\}, \quad \mathcal{L}_2(u) = \{0\,0, 0\,1, 1\,0\}.$$

We now show that

$$\mathcal{L}_3(u) = \{0\,0\,1, 0\,1\,0, 1\,0\,0, 1\,0\,1\}.$$

Since 11 cannot appear in u, the words $0\,1\,1$, $1\,1\,0$ and $1\,1\,1$ do not appear. The factor $0\,0$ can only result from $\theta(1)\theta(0) = \theta(1\,0)$, then $0\,0\,0$ would have to result from $\theta(1)\theta(1)\theta(0) = \theta(1\,1\,0)$, and this is impossible as $1\,1$ does not appear.

Now note that for all factors w of u, either $0\,w\,0$ or $1\,w\,1$ is not a factor of u. This is clear if $w = \epsilon$, the empty word, and also if $w = 0$, or $w = 1$. We use induction on the length to prove this generally.

Consequently, assume that both $0\,w\,0$ and $1\,w\,1$ are factors of u. It follows that w must start and end with a 0 since 11 cannot appear: $w = 0\,y\,0$ for some factor y, so that $0\,0\,y\,0\,0$ and $1\,0\,y\,0\,1$ are factors of u (one can continue this analysis to see that y cannot start or end with a "0"). Since these are factors of $\theta(u)$, and examining the possibilities, we see that there exists a factor z of u with $\theta(z) = 0\,y$.

In a similar way, $\theta(1\,z\,1) = 0\,0\,y\,0$ and $\theta(0\,z\,0) = 0\,1\,0\,y\,0\,1$, so that $1\,z\,1$ and $0\,z\,0$ are both factors of u. Since the length of z is less than the length of w, this contradicts our assumption that there can be at most ones factor of this type having a lesser length.

We can now see that u has at most one right special factor of each length, for suppose that w and v are right special factors of the same length. Let x be the longest sub-word of both w and v (possibly $x = \epsilon$), with $w = z\,x$, $v = y\,x$ for some words y and z. Then the words $0\,x\,0$, $0\,x\,1$, $1\,x\,0$ and $1\,x\,1$ are all factors of u (because the last letters of y and z must be different), and this contradicts the last observation.

Finally we show that there is at least one right special factor of each length. Set $f_n = \theta^n(0)$, so $f_0 = 0$, and let $f_{-1} = 1$, then we use the identity

$$f_{n+2} = v_n \tilde{f}_n \tilde{f}_n t_n, \ \ n \geq 2,$$

where $v_2 = \epsilon$, and for $n \geq 3$

$$v_n = f_{n-3} \cdots f_1 f_0, \quad \text{and} \quad t_n = \begin{cases} 0\,1\,, & \text{if } n \text{ is odd} \\ 1\,0\,, & \text{otherwise} \end{cases},$$

(where $\tilde{w}$ denotes the reflection of the word w, so if $w = a_0 a_1 \ldots a_n$, then $\tilde{w} = a_n a_{n-1} \ldots a_1$). This equation can be proved by induction (see the Exercises).

Now we see that the word $\tilde{f}_n$ appears in two different places and since the first letter of $\tilde{f}_n$ is different to the first letter of t_n, $\tilde{f}_n$ is a right special factor (it is easy to see that the last letter of f_n alternates - 0 when n is even, 1 when n is odd). Now any word of the form $w\tilde{f}_n$ is clearly also a right special factor of u, so there are right special factors of any length, and the result follows. $\square$

16.2 Sequences Arising From Irrational Rotations

If $x \in \mathbb{R}$, we denote the integer part of x by $\lfloor x \rfloor$ and its fractional part by $\{x\}$, so that $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$ (the *floor* function), and $\{x\} = x - [x]$. With this notation, if $0 < \alpha < 1$, the rotation $T_\alpha : [0, 1) \to [0, 1)$ is defined by $T_\alpha x = \{x + \alpha\}$. We can give an explicit description of T_α as:

$$T_\alpha(x) = \begin{cases} x + \alpha\,; & \text{if } x \in [0, 1-\alpha) \\ x + \alpha - 1\,; & \text{if } x \in [1-\alpha, 1) \end{cases}.$$

Insert: Graph of T_α

Now define a sequence $f_\alpha = f_\alpha(1)f_\alpha(2)\ldots$ in $\{0,1\}^{\mathbb{Z}^+}$ by

$$f_\alpha(n) = \begin{cases} 1, & \text{if } \{n\alpha\} \in [1-\alpha, 1) \\ 0, & \text{otherwise} \end{cases},$$

(in this section it is more convenient to start our sequences at $n = 1$).

The sequence f_α is obtained from following the orbit $\{T_\alpha^n(0) : n \geq 1\}$ of 0 under the rotation T_α, giving the value 1 if the orbit falls into the interval $[1-\alpha, 1)$ and 0 otherwise. Our aim is to show that such sequences are Sturmian, and in some cases may be represented by a substitution (see Allouche and Shallit [2]). Brown [13], gives a characterization of sequences f_α, α irrational, which can be represented as substitutions, when α is the root of a quadratic equation over the rationals.

We first remark that for arbitrary $\alpha > 0$

$$f_\alpha(n) = \lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor - \lfloor \alpha \rfloor,$$

so that $f_{\alpha+1} = f_\alpha$. Consequently, without loss of generality we may assume that $0 < \alpha < 1$, and in this case

$$f_\alpha(n) = \lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor.$$

Example 16.2.1 Let $\alpha = (\sqrt{5}-1)/2 = \cdot 61803\ldots$, then we will show that

$$f_\alpha = 1\,0\,1\,1\,0\,1\,0\,1\,1\,0\ldots,$$

which is just the Fibonacci sequence $u = 0\,1\,0\,0\,1\,0\,1\,0\,0\,1\ldots$, with 0 and 1 interchanged. Thus f_α is a fixed point of the substitution $\theta(0) = 1$, $\theta(1) = 1\,0$.

Proposition 16.2.2 (a) *For $n \geq 1$ and $0 < \alpha < 1$,*

$$\sum_{i=1}^{n} f_\alpha(i) = \lfloor (n+1)\alpha \rfloor.$$

(b) *If α is irrational, then $f_{1-\alpha} = R(f_\alpha)$ (where $R(0) = 1, R(1) = 0$).*

(c) *Set $g_\alpha = g_\alpha(1)g_\alpha(2)\ldots$, where*

$$g_\alpha(n) = \begin{cases} 1, & \text{if } n = \lfloor k\alpha \rfloor \text{ for some integer } k \\ 0, & \text{otherwise} \end{cases},$$

then if $\alpha > 1$ is irrational, $g_\alpha = f_{1/\alpha}$.

Proof. (a) The series is telescoping since

$$\sum_{i=1}^{n} f_\alpha(n) = (\lfloor 2\alpha \rfloor - \lfloor \alpha \rfloor) + (\lfloor 3\alpha \rfloor - \lfloor 2\alpha \rfloor) + \cdots (\lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor) = \lfloor (n+1)\alpha \rfloor.$$

(b) $f_{1-\alpha}(n) = \lfloor (n+1)(1-\alpha) \rfloor - \lfloor n(1-\alpha) \rfloor = \lfloor -(n+1)\alpha \rfloor - \lfloor -n\alpha \rfloor + 1$, so that

$$f_\alpha(n) + f_{1-\alpha}(n) = \lfloor x \rfloor + \lfloor -x \rfloor - \lfloor y \rfloor - \lfloor -y \rfloor + 1$$

where $x = (n+1)\alpha$ and $y = n\alpha$. Now it is easy to see that $\lfloor x \rfloor + \lfloor -x \rfloor = -1$ if x is not an integer (otherwise it is 0), so $f_\alpha(n) + f_{1-\alpha}(n) = 1$, and the result follows.

(c)

$$\begin{aligned}
g_\alpha(n) = 1 &\iff \exists\, k \quad \text{such that} \quad n = \lfloor k\alpha \rfloor \\
&\iff \exists\, k \quad \text{such that} \quad n \le k\alpha < n+1 \\
&\iff \exists\, k \quad \text{such that} \quad \frac{n}{\alpha} \le k < \frac{n+1}{\alpha} \\
&\iff \exists\, k \quad \text{such that} \quad \left\lfloor \frac{n}{\alpha} \right\rfloor = k-1 \quad \text{and} \quad \left\lfloor \frac{n+1}{\alpha} \right\rfloor = k \\
&\iff \left\lfloor \frac{n+1}{\alpha} \right\rfloor - \left\lfloor \frac{n}{\alpha} \right\rfloor = 1 \iff f_{1/\alpha}(n) = 1.
\end{aligned}$$

□

Given an alphabet $\mathcal{A}$, if $a \in \mathcal{A}$ and $n \in \mathbb{N}$, we write $a^n = a\,a \ldots a$, the word of length n consisting of a repeated n times ($a^0 = \epsilon$, the empty word).

Definition 16.2.3 Let $k \ge 1$ and denote by θ_k the substitution defined on $\mathcal{A} = \{0, 1\}$ by

$$\theta_k(0) = 0^{k-1}\,1 \quad \theta_k(1) = 0^{k-1}\,1\,0.$$

When $k = 1$ the substitution is $\theta_1(0) = 1$, $\theta_1(1) = 1\,0$, essentially the Fibonacci substitution. When $k = 2$ we have $\theta_2(0) = 0\,1$, $\theta_2(1) = 0\,1\,0$.

Definition 16.2.4: Continued Fractions It is usual to denote the *continued fraction*

$$a_0 + 1/(a_1 + 1/(a_2 + \cdots)) = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}},$$

by $[a_0, a_1, a_2, \ldots]$, where $a_0, a_1, \ldots$ are positive integers. By $\alpha = [0, \overline{a_1, a_2, \ldots, a_n}]$, we mean that the sequence $a_1, a_2, \ldots, a_n$ is repeated periodically. For example, if

$$\alpha = [0, 1, 1, \ldots] = \frac{1}{1 + \frac{1}{1+\cdots}} = \frac{1}{1+\alpha},$$

then α is a root of the equation $\alpha^2 + \alpha - 1 = 0$, and it can be seen that $\alpha = \frac{\sqrt{5}-1}{2}$. Similarly, if $\beta = [0, \overline{1, 2}] = 1/(1 + 1/(2 + \beta))$, then $\beta = \sqrt{3} - 1$. These are examples of *quadratic irrationals.*

Given two substitutions of the form θ_k, θ_ℓ, we can compose them in the usual way to form a new substitution $\theta_k \circ \theta_\ell$. Using this we show that sequence f_α arises as the fixed points of a substitution when α is a certain type of quadratic irrational.

Theorem 16.2.5 *Let α be irrational with $\alpha = [0, \overline{a_1, a_2, \ldots, a_n}]$. Then f_α is a fixed point of the substitution*

$$\theta_{a_1} \circ \theta_{a_2} \circ \cdots \circ \theta_{a_n}.$$

In particular, if $\alpha = (\sqrt{5} - 1)/2$, then f_α is a fixed point of the substitution $\theta(0) = 1$, $\theta(1) = 1\,0$.

In order to prove this theorem, we need a lemma, a special case of which tells us that if $\alpha = (\sqrt{5} - 1)/2$, then $\theta_1(f_\alpha) = f_{1/(1+\alpha)} = f_\alpha$ (since $\alpha = 1/(1 + \alpha)$), so that f_α is a fixed point of the Fibonacci substitution:

Lemma 16.2.6 *Let $k \geq 1$ and $\alpha \in (0, 1)$ be irrational. Then*

$$\theta_k(f_\alpha) = g_{k+\alpha} = f_{1/(k+\alpha)}.$$

Proof. We have $f_\alpha = f_\alpha(1)f_\alpha(2)\ldots f_\alpha(j)\ldots$, where $f_\alpha(j) = \lfloor (j+1)\alpha \rfloor - \lfloor j\alpha \rfloor$, $j \geq 1$.

It follows that the sequence $\theta_k(f_\alpha)$ is a sequence of 0's and 1's with

$$\theta_k(f_\alpha) = D_1 D_2 \ldots D_q D_{q+1} \ldots,$$

where $D_j = \theta_k(f_\alpha(j))$, for $j \geq 1$.

Because of the way the substitution θ_k is defined, each block D_j contains exactly one "1", in the kth position. The length of D_j, $|D_j|$ is either k or $k + 1$.

Suppose that the $(q + 1)$st "1" appears in position n, then 1 appears in the block D_{q+1}, so that

$$n = |D_1 D_2 \ldots D_q| + k.$$

Since

$$|D_j| = k \quad \text{if} \quad f_\alpha(j) = 0 \quad \text{and} \quad |D_j| = k+1 \quad \text{if} \quad f_\alpha(j) = 1,$$

(by definition of θ_k), it follows that

$$|D_1 D_2 \ldots D_q| = qk + f_\alpha(1) + f_\alpha(2) + \cdots + f_\alpha(q) = qk + \lfloor (q+1)\alpha \rfloor,$$

(using Proposition 16.2.2 (a)).

This says that if n is the position of the $(q+1)$st "1" in the sequence $\theta_k(f_\alpha)$, then

$$n = qk + \lfloor (q+1)\alpha \rfloor + k = \lfloor (q+1)(k+\alpha) \rfloor.$$

This shows that

$$\theta(f_\alpha)(n) = 1 \iff n = \lfloor (q+1)(k+\alpha) \rfloor, \text{ for some } q \geq 0 \iff g_{k+\alpha}(n) = 1.$$

We have therefore shown that $\theta_k(f_\alpha) = g_{k+\alpha}$ and Proposition 16.2.2 (c) gives $\theta_k(f_\alpha) = g_{k+\alpha} = f_{1/(k+\alpha)}$. □

Proof of Theorem 16.2.5 From the lemma we have

$$\theta_{a_n}(f_\alpha) = f_{1/(a_n+\alpha)} = f_{[0,a_n+\alpha]},$$

$$\theta_{a_{n-1}} \circ \theta_{a_n}(f_\alpha) = \theta_{a_{n-1}}(f_{1/(a_n+\alpha)}) = f_{\frac{1}{a_{n-1}+1/(a_n+\alpha)}} = f_{1/(a_{n-1}+[0,a_n+\alpha])} = f_{[0,a_{n-1},a_n+\alpha]},$$

and continuing in this way

$$\theta_{a_1} \circ \theta_{a_2} \circ \cdots \circ \theta_{a_m}(f_\alpha) = f_\beta,$$

where $\beta = [0, a_1, a_2, \ldots, a_m + \alpha] = \alpha$. □

16.3 Cutting Sequences

There is a geometric interpretation of the sequence f_α. Let $\beta > 0$ be irrational, then the line $\mathcal{L} : y = \beta x$ in the xy-plane passes through the origin and has slope β. Subdivide the first quadrant in the plane into squares using the *grid-lines* $x = n$, $y = m$ for $n, m = 0, 1, 2, \ldots$. Since β is irrational, the line $\mathcal{L}$ cannot intersect any point of the form (m, n) for $m, n \in \mathbb{N}$, for otherwise we would have $\beta = n/m$, a rational. We examine what happens when the line $\mathcal{L}$ intersects the grid-lines. Consider a point on the line $\mathcal{L}$ moving away from the origin and traveling upwards in the positive x-direction. Define an infinite sequence $S_\beta = S_\beta(0)S_\beta(1)S_\beta(2)\ldots$ by

$$S_\beta(n) = \begin{cases} 0 & \text{if } \mathcal{L} \text{ intersects a vertical grid-line,} \\ 1 & \text{if } \mathcal{L} \text{ intersects a horizontal grid-line.} \end{cases},$$

where $S_\beta(0) = 0$ since the line starts at the origin.

Definition 16.3.1 The sequence S_β is called a *cutting sequence* for β.

We will now show that these sequences are of the form f_α for some α, and as we mentioned previously, the sequences f_α will be shown to be Sturmian. In fact, all Sturmian sequences arise from an f_α for some irrational $0 < \alpha < 1$. Since reflection in $y = x$ interchanges the lines $y = \beta x$ and $y = (1/\beta)x$ and also interchanges the horizontal and vertical grid-lines, it can be seen that

$$R(S_\beta) = S_{1/\beta}, \ \beta > 0$$

where R is the usual reflection. Consequently, if $\beta > 1$, we can obtain the sequence S_β from that of $S_{1/\beta}$, so we may assume $0 < \beta < 1$ without any loss of generality. In this case the sequence $1\,1$ cannot occur.

The connection between f_α and S_β is given by the next theorem due to Crisp, Moran, Pollington and Shiue [19], who also completely characterized those α's for which f_α is a substitution sequence:

First, we ask the question: given a sequence f_α, what does the corresponding sequence S_α look like? Suppose for example that $f_\alpha = 0\,1\,0\,0\,1\,0\,1\,0\ldots$ (the Fibonacci sequence). This sequence has the property that when $f_\alpha(n) = 0$, we have $\lfloor (n+1)\alpha \rfloor = \lfloor n\alpha \rfloor$, so that consecutive 0's appear in the sequence S_α.

If $f_\alpha(n) = 1$, then $\lfloor (n+1)\alpha \rfloor > \lfloor n\alpha \rfloor$, and we see that if

$$f_\alpha = 0\,1\,0\,0\,1\,0\,1\,0\ldots, \quad \text{then} \quad S_\alpha = 0\,0\,1\,0\,0\,0\,1\,0\,0\,1\,0\ldots,$$

which can be written as

$$S_\alpha = h(f_\alpha), \quad \text{where} \quad h(0) = 0, h(1) = 0\,1,$$

(the extra "0" at the start of S_α comes from the fact that the sequence starts at the origin). This leads to:

Theorem 16.3.2 *Let $0 < \alpha < 1$ be irrational. Then*

$$S_\alpha = f_{\frac{\alpha}{1+\alpha}}.$$

Proof. Consider the segment of the line $\mathcal{L}$ from P_n to P_{n+1}, where $P_n = (n, n\alpha)$ (where we include the point P_{n+1}, but exclude P_n). The block in S_α corresponding to P_nP_{n+1} is $1^i\,0$ where

$$i = \lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor = f_\alpha(n).$$

This block is 0 if $f_\alpha(n) = 0$ and $1\,0$ if $f_\alpha(n) = 1$. If $h'(0) = 0, h'(1) = 1\,0$, then $h'(f_\alpha) = S_\alpha$ gives the cutting sequence of $y = \alpha x$ except for the block due to the line segment L_0. The missing block is 0 and applying instead the substitution h: $h(0) = 0$, $h(1) = 0\,1$ gives entire cutting sequence, i.e., $h(f_\alpha) = S_\alpha$.

Also, recall that $\theta_1(0) = 1$ and $\theta_1(1) = 1\,0$, then clearly $h = R \circ \theta_1$, where $R(0) = 1$ and $R(1) = 0$, so that

$$S_\alpha = h(f_\alpha) = R \circ \theta_1(f_\alpha) = R(f_{\frac{1}{1+\alpha}}),$$

(using Lemma 16.2.6.). This implies

$$S_{\frac{1}{\alpha}} = R(S_\alpha) = f_{\frac{1}{1+\alpha}}, \quad \text{or} \quad S_\alpha = f_{\frac{\alpha}{1+\alpha}}.$$

$\square$

Billiards in a Square 16.3.3 Imagine shooting a ball from the origin at an irrational slope $\alpha > 0$ inside the unit square $[0, 1] \times [0, 1]$. We assume we have the usual elastic reflection when the ball strikes the sides of the square (angle of incidence equals the angle of reflection). If we write down 0 when the ball strikes a vertical side and 1 when it strikes a horizontal side, we get a sequence which is Sturmian. This can be seen by extending the trajectory of the ball in a straight line from the origin, giving rise to a cutting sequence which is the same as the original sequence.

16.4 Sequences Arising from Irrational Rotations are Sturmian

As usual, $T_\alpha : [0, 1) \to [0, 1)$ is the rotation $T_\alpha(x) = \{x + \alpha\}$ where $0 < \alpha < 1$ is irrational. We know from Section 14.2 that in this case T_α is minimal, so that the sequence $\{x + k\alpha\}_{k=0}^{\infty}$ is dense in $[0, 1)$ for every $x \in [0, 1)$.

Define partitions $\mathcal{P}$ and $\mathcal{P}'$ of $[0, 1)$ by

$$\mathcal{P} = \{[0, 1 - \alpha), [1 - \alpha, 1)\}, \quad \mathcal{P}' = \{[0, 1 - \alpha], (1 - \alpha, 1)\}.$$

A "coding" of the orbit $O(x)$ of T_α, with respect to the partition $\mathcal{P}$ is obtained in a similar way to the last section, by defining a sequence $u = u_1u_2\ldots$, of 0's and 1's with

$$u_n = 1 \iff \{x + n\alpha\} \in [1 - \alpha, 1).$$

Since T_α is minimal, the sequence can take the value $1 - \alpha$ at most once, so we get essentially the same coding using $\mathcal{P}$ or $\mathcal{P}'$. If we set $x = 0$, we get the sequence f_α defined previously. Set

$$I_0 = [0, 1 - \alpha), \quad I_1 = [1 - \alpha, 1).$$

The following proposition tells us that the coding obtained is independent of $x \in [0,1)$ in the sense that the set of factors $\mathcal{L}(u)$ (words appearing in u) is independent of x.

Proposition 16.4.1 *The finite word $w = w_1 w_2 \dots w_n$ is a factor of $u = u_0 u_1 \dots$, if and only if there exists $k \in \mathbb{N}$ such that*

$$\{x + k\alpha\} \in I(w_1, w_2, \dots, w_n) = \bigcap_{j=0}^{n-1} T_\alpha^{-j}(I_{w_{j+1}}).$$

Proof. The word $w = w_1 w_2 \dots w_n$ is a factor of u if and only if there exists $k \in \mathbb{N}$ with

$$w_1 = u_k, \quad w_2 = u_{k+1}, \quad \dots, \quad w_n = u_{k+n},$$

or if and only if

$$\{x + k\alpha\} \in I_{w_1}, \ \{x + (k+1)\alpha\} \in I_{w_2}, \ \dots \ , \{x + (k+n-1)\alpha\} \in I_{w_n},$$

if and only if

$$\{x + k\alpha\} \in I_{w_1}, \ T_\alpha(\{x + k\alpha\}) \in I_{w_2}, \ \dots \ , T_\alpha^{n-1}(\{x + k\alpha\}) \in I_{w_n}.$$

This is equivalent to

$$\{x + k\alpha\} \in \bigcap_{j=0}^{n-1} T_\alpha^{-j}(I_{w_{j+1}}).$$

□

Since the set $I(w_1, \dots, w_n)$ is independent of x, the set of factors does not depend on the initial point x. Clearly there are 2^n sets of the form $I(w_1, \dots, w_n)$ where $w_i = 0$ or $w_i = 1$, being pairwise disjoint with union equal to all of $[0,1)$. We shall show that all but $n+1$ of these sets are empty, and the remaining $n+1$ are (connected) intervals. It follows that there at exactly $n+1$ factors of length n, so that for each $x \in [0,1)$, the sequence is Sturmian. In particular, for $0 < \alpha < 1$ irrational, f_α is a Sturmian sequence.

Since T_α is a one-to-one and onto map, T_α^{-1} is defined and given by

$$T_\alpha^{-1}(x) = \begin{cases} x - \alpha + 1 & \text{if } x \in [0, \alpha) \\ x - \alpha & \text{if } x \in [\alpha, 1) \end{cases}$$

Notice that

$$\{T_\alpha^{-k}(0) : 0 \leq k \leq n-1\} = \{0, \{1 - \alpha\}, \{1 - 2\alpha\}, \dots, \{1 - (n-1)\alpha\}\},$$

and these points are spread around the unit interval (not necessarily in this order). Set

$$E_n = \{\{1 - k\alpha)\} : k = 0, 1, 2, \dots, n-1\} \cup \{0\},$$

and suppose that $E_n = \{\beta_0^n, \beta_1^n, \dots, \beta_n^n, \beta_{n+1}^n\}$, where the superscript denotes the dependence on n and

$$0 = \beta_0^n < \beta_1^n < \cdots < \beta_n^n < \beta_{n+1}^n = 1,$$

then E_n is a partition of $[0, 1)$ into $n + 1$ subintervals. Suppose that $T_\alpha^{-n}(0) = \{1 - n\alpha\} = \gamma$, then E_{n+1} is the same finite set with the addition of γ, $\beta_i < \gamma < \beta_{i+1}$ for some i. We can write

$$E_{n+1} = \{\beta_0^{n+1}, \beta_1^{n+1}, \dots, \beta_{n+1}^{n+1}, \beta_{n+2}^{n+1}\} = E_n \cup \{\gamma\}$$

We will use induction to prove that for each $n \geq 1$, there are $n + 1$ non-empty intervals of the form $I(w_1, \dots, w_n)$. For example, when $n = 1$, we have $E_1 = \{0, 1 - \alpha, 1\} = \{\beta_0^1, \beta_1^1, \beta_2^1\}$, and the two intervals are $I(0) = I_0 = [0, 1 - \alpha)$ and $I(1) = I_1 = [1 - \alpha, 1)$.

There are two possibilities when $n = 2$, depending on whether or not the discontinuity α of T_α^{-1} lies in the interval $[0, \beta_1)$ or not. Suppose that $\beta_1 < \alpha$, then we can check that if $\gamma = \{1 - 2\alpha\} = 2 - 2\alpha$, then $E_2 = \{0, \beta_1^1, \gamma, 1\}$ and

$$I(0,0) = \emptyset, \quad I(0,1) = [0, \beta_1^1), \quad I(1,0) = [\beta_1^1, \gamma), \quad I(1,1) = [\gamma, 1),$$

and if $\beta_1^1 > \alpha$, then $E_2 = \{0, \gamma, \beta_1^1, 1\}$ and

$$I(0,0) = [0, \gamma), \quad I(0,1) = [\gamma, \beta_1^1), \quad I(1,0) = [\beta_1^1, 1), \quad I(1,1) = \emptyset,$$

where $\gamma = 1 - 2\alpha = \{1 - 2\alpha\}$. Notice that in each case we have a partition of $[0, 1)$ into three intervals. We use these ideas to prove the general case, and deduce that the resulting sequences are Sturmian:

Proposition 16.4.2 *Let $x \in [0, 1)$, $0 < \alpha < 1$ irrational and u be the sequence of 0's and 1's with*

$$u_n = 1 \iff \{x + n\alpha\} \in [1 - \alpha, 1).$$

If $p_u(n)$ is the number of factors of length n of the sequence u, then

$$p_u(n) = n + 1, \quad \textit{for all} \quad n \in \mathbb{N}.$$

Proof. We need the following lemma:

Lemma 16.4.3 *Set* $E_n = \{\{1-k\alpha\} : k = 0, 1, 2, \ldots, n-1\} \cup \{0\}$ *and suppose that* $E_n = \{\beta_0^n, \beta_1^n, \beta_2^n, \ldots, \beta_{n+1}^n\}$*, where*

$$0 = \beta_0^n < \beta_1^n < \cdots < \beta_n^n < \beta_{n+1}^n = 1.$$

If $I(w_1, \ldots, w_n) \neq \emptyset$*, then* $I(w_1, \ldots, w_n)$ *is an interval of the form* $[\beta_k^n, \beta_{k+1}^n)$*, for some* $0 \le k \le n$*. Conversely, any interval of the form* $[\beta_k^n, \beta_{k+1}^n)$ *is equal to* $I(w_1, \ldots, w_n)$ *for some factor* $w_1 w_2 \ldots w_n$ *of* u*.*

Proof. We prove this result by induction. We have seen that it is true for $n = 1$ and $n = 2$, so suppose it is true for some fixed n, then any non-empty set of the form $I(w_1, \ldots, w_n)$ is an interval equal to $[\beta_k^n, \beta_{k+1}^n)$, for some $0 \le k \le n$. Consider now

$$I(w_1, w_2, \ldots, w_n, w_{n+1}) = \bigcap_{j=0}^{n} T_\alpha^{-j}(I_{w_{j+1}}) = I_{w_1} \cap T_\alpha^{-1}\left[I(w_2, w_3, \ldots w_{n+1})\right],$$

where we know by our induction hypothesis, that $I(w_2, w_3, \ldots w_{n+1})$ is an interval of the form $[\beta_k^n, \beta_{k+1}^n)$. There are two cases to consider.

Case 1 The discontinuity α of T_α^{-1} does not lie in the interval $[\beta_k^n, \beta_{k+1}^n)$:

Then $T_\alpha^{-1}\left[\beta_k^n, \beta_{k+1}^n\right) = [\beta_\ell^{n+1}, \beta_{\ell+1}^{n+1})$ for some $0 \le \ell \le n+1$ (the integers ℓ and $\ell+1$ have to be consecutive, for otherwise there would be some $\gamma \in E_{n+1}$, belonging to the interval $T_\alpha^{-1}\left[\beta_k^n, \beta_{k+1}^n\right)$, with corresponding $\beta \in E_n$ in the interval $(\beta_k^n, \beta_{k+1}^n)$, giving a contradiction).

The intersection of this interval with the interval I_{w_1} (where I_{w_1} is either I_0 or I_1), is clearly an interval of the correct form.

Case 2 The discontinuity α lies in the interval $[\beta_k^n, \beta_{k+1}^n)$:

Then

$$T_\alpha^{-1}\left[\beta_k^n, \beta_{k+1}^n\right) = [0, \beta_1^{n+1}) \cup [\beta_{n+1}^{n+1}, 1),$$

(β_1^{n+1} and β_{n+1}^{n+1} appear here since other values from E_{n+1} would lead to a contradiction as in Case 1). Since $\beta_1^{n+1} < 1-\alpha$ and $\beta_{n+1}^{n+1} > 1-\alpha$ (because $\beta_k^n \in [0, \alpha)$ and $\beta_{k+1}^n \in (\alpha, 1)$), the intersection of this set with either I_0 or I_1 is again an interval of the required form.

It now follows by induction that for each n, the sets $I(w_1, \ldots, w_n)$ are intervals of the form $[\beta_k^n, \beta_{k+1}^n)$.

Since sets of the form $I(w_1, \ldots, w_n)$ give rise to a partition of $[0, 1)$, these sets must account for all sets $[\beta_k^n, \beta_{k+1}^n)$ for $0 \le k \le n$ and the lemma follows directly. $\square$

Proof of Proposition 16.4.2 We saw from Proposition 16.4.1 that a word $w_1 w_2 \dots w_n$ appears in the sequence u if and only of $I(w_1, \dots, w_n) \neq \emptyset$. From the lemma, there are at most $n+1$ such sets (all being intervals), and the result follows. $\square$

16.5 The Three Distance Theorem

Starting with $n = 1$ we defined the partition $E_1 = \{0, 1-\alpha, 1\} = \{\beta_0^1, \beta_1^1, \beta_2^1\}$, giving rise to subintervals of $[0,1)$ having lengths $1-\alpha$ and α respectively. The sets E_2, E_3 etc. can be defined inductively by adding a single member of the orbit of 0 under T_α^{-1} at each stage, giving rise to new subintervals of $[0,1)$, of decreasing lengths. It is surprising that for each fixed $n \geq 1$ the lengths of the intervals $[\beta_k^n, \beta_{k+1}^n)$ created by the partition E_n take at most three values. This is the celebrated "Three Distance Theorem", which was conjectured by Steinhaus and proved by V. T. Vos. We digress somewhat from our studies of dynamical systems to give a proof which uses the properties of Sturmian sequences and also uses results from the theory of equidistribution (see Appendix D on Weyl's Equidistribution Theorem). This proof is due to V. Berthé, [8], [9]. The Three Distance Theorem can be stated in terms of the orbit of 0 under T_α as:

16.5.1 The Three Distance Theorem

Let $0 < \alpha < 1$ be irrational and $n \in \mathbb{Z}^+$. The points $\{k\alpha\}$, for $0 \leq k \leq n$, partition the interval $[0,1)$ into $n+1$ intervals, the lengths of which take at most three values, one being the sum of the other two.

There are many proofs of the Three Distance Theorem, usually of a combinatorial nature. We shall give a dynamical proof due to V. Berthé, which uses the theory developed so far concerning Sturmian sequences. In order to prove this theorem we need to define the *Rauzy graph* of a sequence.

Definition 16.5.2 Let $u \in \mathcal{A}^{\mathbb{N}}$ be a sequence over a finite alaphabet $\mathcal{A}$. The *Rauzy graph* Γ_n of u is an oriented graph whose vertices and edges are defined as follows:

(i) The vertices U, $V, \dots$ of Γ_n are the factors (words) of length n appearing in u (i.e., the members of $\mathcal{L}_n(u)$).

(ii) There is an edge from vertex U to vertex V if V *follows* U in the sequence u.
More precisely: V follows U if there is a factor W of u and $x, y \in \mathcal{A}$ such that

$$U = xW, \quad V = Wy \quad \text{and} \quad xWy \quad \text{is a factor of } u.$$

Such an edge is labelled xWy.

Recall that the complexity function $p_u(n) = |\mathcal{L}_n(u)|$, is the number of distinct factors of length n. We have the following straightforward facts:

Proposition 16.5.3 *$p_u(n) =$ the number of vertices of Γ_n, $p_u(n+1) =$ the number of edges of Γ_n.*

Proof. The first statement is clear. If $w = u_1u_2 \dots u_{n+1}$ is a factor of length $n+1$, set $W = u_2 \dots u_n$. Then if $U = u_1W$ and $V = Wu_{n+1}$, then U and V are factors of length n, so are vertices of the graph and $w = u_1Wu_{n+1}$ is a factor of length $n+1$, which is an edge joing them. In other words, every factor of length $n+1$ appears as an edge, and it cannot appear in more than one place.

On the other hand, by definition, every edge corresponds to a factor of length $n+1$.
□

Recall that a *right extension* of a factor $w = w_1 \dots w_n$ of a sequence u, is a factor of u of the form $w_2 \dots w_n x$. *Left extensions* are defined similarly. A factor having more than one right extension is called a *right special factor* (repectively *left special factor*).

Definition 16.5.4 Let U be a vertex of Γ_n. Set

$$U^+ = \# \quad \text{of edges of } \Gamma_n \text{ originating at } U,$$

$$U^- = \# \quad \text{of edges of } \Gamma_n \text{ ending at } U.$$

Proposition 16.5.5

$$p_u(n+1) - p_u(n) = \sum_{|U|=n} (U^+ - 1) = \sum_{|U|=n} (U^- - 1).$$

Proof. Here we are summing over all the vertices (factors of length n), so that $\sum_{|U|=n} 1 = p_u(n)$. Since U^+ gives the total number of edges (factors of length $n+1$) originating at the vertex U, $\sum_{|U|=n} U^+$ will give the total number of factors of length $n+1$. □

Definition 16.5.6 A *branch* of the Rauzy graph Γ_n is a sequence of adjacent edges $(U_1, U_2, \dots, U_m)$ of maximal length (possibly empty), with the property:

$$U_i^+ = 1 \quad \text{for} \quad i < m, \quad U_i^- = 1, \quad \text{for} \quad i > 1.$$

In order to prove the Three Distance Theorem, we need to relate the frequency of how often a factor of some length n appears in u with the lengths of the intervals $I(w_1, w_2, \ldots, w_n)$, which are intervals of the form $[\beta_k^n, \beta_{k+1}^n)$.

Definition 16.5.7 Let w be a factor of the sequence $u = u_1 u_2 \ldots$. The frequency $f(w)$ of w in u is the limit (if it exists):

$$f(w) = \lim_{k \to \infty} \frac{\#\{\text{ of occurences of } w \text{ in } u_1 u_2 \ldots u_k\}}{k}.$$

It follows from the following results that frequencies always exist for Sturmain sequences. In addition, for each fixed n, the frequencies of factors of length n can take at most three different values. Certain words necessarily have the same frequencies:

Proposition 16.5.8 *If U and V are vertices of Γ_n linked by an edge xWy with $U^+ = 1$, $V^- = 1$, then $f(U) = f(V)$. In addition, the vertices of a branch have the same frequencies.*

Proof. $U = xW$ and $V = Wy$ and since $U^+ = 1$, U has the unique right extension Wy, and since $V^- = 1$, V has the unique left extension xW. It follows for example that the frequency of the appearance of U in u and Uy in u are identical, so that

$$f(U) = f(Uy) = f(xWy) = f(xV) = f(V).$$

Continuing this argument, we see that the vertices of any branch must have the same frequencies. □

The following is now fundamental in proving the Three Distance Theorem (see [**?**]):

Proposition 16.5.9 *Let u be a recurrent sequence having complexity function $p_u(n)$. The frequencies of factors of length n take at most*

$$3\,(p_u(n+1) - p_u(n))$$

different values.

Proof. Let V_1 be the set of all factors of length n of the sequence u, having more than one extension. In other words, V_1 is the set of those vertices U in Γ_n for which

$U^+ \geq 2$. The cardinality of V_1 must satisfy:

$$|V_1| = \sum_{|U|=n,\, U^+\geq 2} 1 \leq \sum_{|U|=n} (U^+ - 1) = p_u(n+1) - p_u(n).$$

Let V_2 be the subset of the set of vertices of Γ_n defined in the following way:

$U \in V_2$ if and only if $U^+ = 1$ and if V is the unique vertex such that there is an edge from U to V in Γ_n, then $V^- \geq 2$.

In other words, $U \in V_2$ if and only if $U = xW$ where $x \in \mathcal{A}$ and where the factor W of the sequence u has a unique right extension, but at least two left extensions. The cardinality of V_2 satisfies:

$$|V_2| \leq \sum_{V^-\geq 2} V^- = \sum_{V^-\geq 2} (V^- - 1) + \sum_{V^-\geq 2} 1 \leq 2(p_u(n+1) - p_u(n)).$$

It follows that there are at most $3(p_u(n+1) - p_u(n))$ factors in $V_1 \cup V_2$. Now let U be a factor of length n that does not belong to either of V_1 or V_2. Then $U^+ = 1$, and the unique word V for which there is an edge from U to V in Γ_n satisfies $V^- = 1$. It follows from the previous proposition that U and V have the same frequencies: $f(U) = f(V)$. Now consider the path in Γ_n, starting at U and consisting of vertices that do not belong to V_1 or V_2. The last vertex of this path belongs to either V_1 or V_2 and has the same frequency as U. □

Proof of the Three Distance Theorem. We are now assuming that the sequence u arises as the coding of the sequence $\{\{x + k\alpha\} : k \in \mathbb{N}\}$, so that it is Sturmian and $p_u(n) = n + 1$ and $p_u(n+1) - p_u(n) = 1$ for each n. The last result tells us that there are at most three values the frequencies $f(w)$ can take for any factor of length n. Suppose $w = w_1 \dots w_n$, then we showed earlier that w appears in u if and only if $I(w_1, \dots, w_n) \neq \emptyset$, and in this case $I(w_1, \dots, w_n)$ is an interval formed by the partitioning of $[0, 1)$ using the points $\{T_\alpha^{-k}(0) : 0 \leq k \leq n\}$. Now using Weyl's Equidistribution Theorem (see Appendix D), the frequency of the sequence $\{x + k\alpha\}$ in a subinterval of $[0, 1)$ is equal to the length of that interval. It now follows that each of these subintervals can have at most three different lengths. □

16.6 Semi-Toplogical Conjugacy Between $([0, 1), T_\alpha)$ and $(\overline{O(u)}, \sigma)$.

Let u be the Sturmian sequence arising as the coding of some irrational rotation T_α. In other words, if $\mathcal{P} = \{[0, 1-\alpha), [1-\alpha, 1)\}$ is the partition of $[0, 1)$ defined in Section 16.4, then $u = u_0 u_1 \dots$, where $u_n = 0$ if $T_\alpha^n(0) \in [0, 1-\alpha)$ and $u_n = 1$ if $T_\alpha^n(0) \in [1-\alpha, 1)$. It is reasonable to ask whether there is a topological conjugacy between the dynamical systems $([0,1), T_\alpha)$ and $(\overline{O(u)}, \sigma)$. This is clearly not the case as there would have to be a homeomorphism between the underlying metric spaces $[0,1)$ and $\overline{O(u)}$, the first space being connected (an interval) and the second being totally disconnected (a type of Cantor set). In a similar way, the dynamical systems $([0,1), T_\alpha)$ and $(\mathbb{S}^1, R_a)$, where $R_a(z) = az$, $a = e^{2\pi i\alpha}$, are not conjugate as the underlying spaces are not homeomorphic. However, all these maps are what we call *semi-topologically conjugate*.

Definition 16.6.1 Two dynamical systems (X, f) and (Y, g) are *semi-topologically conjugate* if there exist countable sets $X_1 \subset X$, $Y_1 \subset Y$, and a one-to-one onto function $\phi : X_1 \to Y_1$ which is continuous with continuous inverse and satisfies $\phi \circ f(x) = g \circ \phi(x)$, for all $x \in X_1$.

In the case that (X, f) is a *symbolic system* (for example, a substitution dynamical system), we say that it is a *coding* of (Y, g).

It is easy to see that $([0,1), T_\alpha)$ and $(\mathbb{S}^1, R_a)$ above are semi-topologically conjugate.

Theorem 16.6.2 *Let $0 < \alpha < 1$ be irrational and let $u = u_0u_1\ldots$ be the Sturmian sequence given by $u_n = 0$ if $T_\alpha^n(0) \in [0, 1-\alpha)$, and $u_n = 1$ if $T_\alpha^n(0) \in [1-\alpha, 1)$. Denote by $(\overline{O(u)}, \sigma)$ the corresponding shift dynamical system. Then the dynamical systems $([0,1), T_\alpha)$ and $(\overline{O(u)}, \sigma)$ are semi-topologically conjugate.*

Proof. Write $\mathcal{P} = \{[0, 1-\alpha), [1-\alpha, 1)\}$, the partition of $[0,1)$ given above. Let $x \in [0,1)$, then we define a function $\phi : [0,1) \to \overline{O(u)}$ by

$$\phi(x) = \mathcal{P} - \text{name of } x,$$

where $\phi(x)$ is the sequence of 0's and 1's whose nth term is 0 if $T_\alpha^n(x) \in [0, 1-\alpha)$, and is 1 if $T_\alpha^n(x) \in [1-\alpha, 1)$. In particular, $\phi(0) = u$, and $\phi(T_\alpha^n(0)) = \phi(\{n\alpha\}) = \sigma^n(u)$. Clearly ϕ is well defined, and if x and y have the same $\mathcal{P}$-name, they are equal, since they are not seperated by arbitrarily small intervals, so ϕ is one-to-one. ϕ is onto, for if $\omega \in \overline{O(u)}$, then every factor of ω is a factor of u, so if $\omega = w_1w_2\ldots w_n\ldots$, then we can find a sequence $x_n \in I(w_1, w_2, \ldots w_n)$, $n = 1, 2, \ldots$, with $x_n \to x$ and $\phi(x) = \omega$.

Let $D = \{\{n\alpha\} : n \in \mathbb{N}\}$, a countable set dense in $[0,1)$, and $\phi(D) = \{\sigma^n(u) : n \in \mathbb{N}\}$. Clearly $\phi : [0,1) - D \to \overline{O(u)} - \phi(D)$ is also a bijection, and we show that it is a continuous map.

Let $\epsilon > 0$ and $x \in [0,1) - D$, and choose n so large that $1/2^n < \epsilon$, and also choose $\delta > 0$ so that if $y \in (x - \delta, x + \delta)$, then $(x - \delta, x + \delta) \subset I(w_1, w_2, \ldots, w_n)$ for some factor $w_1 w_2 \dots w_n$ of u. Then $\phi(x)$ and $\phi(y)$ have the same first n coordinates, so that $d(\phi(x), \phi(y)) \leq 1/2^n < \epsilon$, so ϕ is continuous at x. This argument does not work if $x \in D$, since then we cannot ensure that $(x-\delta, x+\delta) \subset I(w_1, w_2, \dots, w_n)$ for any factor $w_1 \dots w_n$. A similar argument will show that ϕ^{-1} is a continuous map.

Finally, note that the nth term of $\phi(x+\alpha)$: $[\phi(x+\alpha)]_n = 0$ if and only if $T_\alpha^n(x+\alpha) \in [0, 1-\alpha)$, if and only if $T_\alpha^{n+1}(x) \in [0, 1-\alpha)$, if and only if $[\sigma\phi(x)]_n = 0$. It follows that $\phi \circ T_\alpha(x) = \sigma \circ \phi(x)$ for $x \in [0, 1)$, and the result follows. $\square$

Remark 16.6.3 We showed earlier that if u is a rercurrent sequence, then the dynamical system $(\overline{O(u)}, \sigma)$ has the property of being onto. Since a Sturmain sequence is recurrent, the shift map $\sigma : \overline{O(u)} \to \overline{O(u)}$, is necessarily onto. The next result tells us that it is essentially one-to-one.

Proposition 16.6.4 *The shift map $\sigma : \overline{O(u)} \to \overline{O(u)}$, for u a Sturmian sequence, is one-to-one except at one point.*

Proof. Since u is recurrent, every factor in $\mathcal{L}_n(u)$ (the words in u of length n), appears infinitely often in u, so can be extended on the left. Since $|\mathcal{L}_n(u)| = n + 1$, there is exactly one factor of length n that can be extended on the left in two different ways (called a *left special factor*).

Let L_n be the unique word in $\mathcal{L}_n(u)$ that can be extended in 2 different ways. In particular,

$$0\, L_n, \; 1\, L_n \in \mathcal{L}_n(u).$$

Consider the word L_{n+1}, which can be extended in two ways to give $0\, L_{n+1}$ and $1\, L_{n+1}$. It is clear that

$$L_{n+1} = L_n\, a, \quad \text{for some} \quad a \in \{0, 1\},$$

since an extension of L_{n+1} gives rise to an extension of L_n, and L_n is unique.

Suppose now that $w \in \overline{O(u)}$ is a sequence with two preimages: say there are $v_1, v_2 \in \overline{O(u)}$ with

$$\sigma(v_1) = w = \sigma(v_2),$$

then every initial factor of w of length n has two extension: i.e., $w = L_n\, v$ for some sequence v, for each $n \in \mathbb{N}$. Such a w is clearly unique. $\square$

Exercises 16.1

1. Show that the Morse sequence is not Sturmian (Hint: look at the number of factors of length 2).

2. Let $f_n = \theta^n(0)$, $n \geq 0$ and $f_{-1} = 1$, where $u = u_0 u_1 \ldots$ is the Fibonacci sequence generated by the substitution $\theta(0) = 0\,1$, $\theta(1) = 0$.

(a) Show that $f_4 = \epsilon(0\,1\,0)\,(0\,1\,0)\,1\,0$, and $f_5 = 0\,(1\,0\,0\,1\,0)\,(1\,0\,0\,1\,0)\,0\,1$ and now use induction on the length of a word to show that for any word w

$$\theta(\tilde{w})\,0 = 0\,\theta(\tilde{w}).$$

(where $\tilde{w}$ denotes the reflection of the word w, so if $w = a_0 a_1 \ldots a_n$, then $\tilde{w} = a_n a_{n-1} \ldots a_1$).

(b) Use (a) to prove the identity

$$f_{n+2} = v_n \tilde{f}_n \tilde{f}_n t_n, \quad n \geq 2,$$

where $v_2 = \epsilon$, and for $n \geq 3$

$$v_n = f_{n-3} \cdots f_1 f_0, \quad \text{and} \quad t_n = \begin{cases} 0\,1\,, & \text{if } n \text{ is odd} \\ 1\,0\,, & \text{otherwise} \end{cases}.$$

(Hint: Show that $\theta(\tilde{f}_n t_n) = 0\,\tilde{f}_{n+1} t_{n+1}$, then use $\theta(v_n)\,0 = v_{n+1}$).

3. Let T_α be an irrational rotation on $[0, 1)$, $0 < \alpha < 1$. Show that $T_\alpha^n(0) \in [1-\alpha, 1)$ if and only if $\lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor = 1$, i.e., $\{n\alpha\} \in [1-\alpha, 1)$ if and only if $f_\alpha(n) = 1$.

4. Show that if β and γ are irrational with cutting sequences S_β and S_γ respectively, then $S_\beta = S_\gamma$ if and only if $\beta = \gamma$. It follows that there are uncountably many Sturmian sequences. Deduce from the fact that there are at most countably many substitutions on $\mathcal{A} = \{0, 1\}$, that not all Sturmian sequences can be represented as substitutions.

5. Show that if $\beta > 0$ has cutting sequence S_β, then $R(S_\beta) = S_{1/\beta}$ (where $R(0) = 1$ and $R(1) = 0$).

6. Show that the Rauzy graph Γ_n is always connected, i.e., given any two vertices U and V, there are edges that connect U and V.

7. Prove that a sequence u is recurrent if and only if every factor appears at least twice. Deduce that this is equivalent to the the graph Γ being strongly connected.

8. An infinite sequence of 0's and 1's is defined in the following way.
(a) 0 is placed in every other position (i.e., every even position).
(b) 1 is placed in every other of the unfilled positions.
(c) 0 is placed in every other of the unfilled positions.
(d) Continue in this way indefinitely. The first few terms are

$$0\,1\,0\,0\,0\,1\,0\,1\,0\,1\,0\,0\,0\,1\,0\,0\,0\,1\,0 \ldots.$$

Show that the resulting sequence is aperiodic and almost periodic. Compare this sequence with that generated by the substitution

$$\theta(0) = 1\,1, \quad \theta(1) = 1\,0.$$

9. The topological entropy of an infinite sequence $u = u_0 u_1 \ldots$, is defined to be:

$$H(u) = \lim_{n \to \infty} \frac{\log_d p_u(n)}{n},$$

where $d = |\mathcal{A}|$ and $p_u(n)$ is the complexity function of u.

(i) Find $H(u)$ for u a Sturmian sequence.

(ii) Find $H(u)$ for a sequence whose complexity function is $p_u(n) = 2^d$.

10. Use the following steps to show that $\lim_{n \to \infty} \log_d p_u(n)/n$ exists.

11. Construct a sequence u for which the frequencies of letters do not exist.

12. Prove that every prefix of a Sturmian sequence u appears at least twice in the sequence. Deduce that the factors of every Sturmian sequence appear infinitely often (u is recurrent).

Appendix A, Some Theorems from Real Analysis

The real numbers $\mathbb{R}$ together with the metric $d(x, y) = |x - y|$ are a complete metric space, i.e., every Cauchy sequence in $\mathbb{R}$ is convergent. This completeness is a consequence of the Completeness Axiom for $\mathbb{R}$: *Let $S \subset \mathbb{R}$ be a non-empty set which is bounded above, then there exists $U \in \mathbb{R}$ with the properties:*

(i) $U \geq x$ for all $x \in S$.

(ii) If $M \geq x$ for all $x \in S$, then $U \leq M$.

We call this number M the *supremum* or *least upper bound* of S. The *infimum* or *greatest lower bound* of a set which is non-empty and bounded below is defined similarly.

A1 The Intermediate Value Theorem. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Suppose that w lies between $f(a)$ and $f(b)$, then there exists $c \in (a, b)$ with $f(c) = w$.*

A2 The Monotone Sequence Theorem. (i) *Let x_n be a sequence of real numbers that is increasing and bounded above, then* $\lim_{n\to\infty} x_n$ *exists and is equal to* $\sup\{x_n : n \in \mathbb{Z}^+\}$.

(ii) *If the sequence is decreasing and bounded below, then* $\lim_{n\to\infty} x_n$ *exists and is equal to* $\inf\{x_n : n \in \mathbb{Z}^+\}$.

A3 The Closed Bounded Interval Theorem. *If $f : [a, b] \to \mathbb{R}$ is continuous, then $f([a, b]) = [c, d]$, a closed bounded interval, for some $c, d \in \mathbb{R}$.*

A4 The Mean Value Theorem. *Let $f : [a, b] \to \mathbb{R}$ be a function continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ with*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

A5 The Bolzano-Weierstrass Theorem. *If (x_n) is an infinite sequence in the closed interval $[a, b]$, then it has a limit point in $[a, b]$.*

A6 The Heine-Borel Theorem. *Every covering of a closed interval $[a, b]$ by a collection of of open sets, has a finite subcovering.*

A7 The Baire Category Theorem for Complete Metric Spaces *Every complete metric space X is a Baire space, i.e., for each collection of sets $\{U_n : n \in \mathbb{N}\}$ which are open and dense in X, their intersection $\cap_{n=0}^{\infty} U_n$ is dense in X.*

Equivalently, a non-empty complete metric space X is not the countable union of nowhere dense closed sets (i.e., closed sets that contain no open sets).

A8 Corollary *Every complete metric space with no isolated points is uncountable.*

Proof. Suppose that X is a countable complete metric space having no isolated points. If $x \in X$, then the singleton set $\{x\}$ is nowhere dense in X and so X is the countable union of nowhere dense sets, contradicting the Baire Category Theorem. □

Appendix B, The Complex Numbers

The complex numbers $\mathbb{C} = \{a + ib : a, b \in \mathbb{R}, i^2 = -1\}$ is a field with respect to the usual laws of addition and multiplication. it is a complete metric space if a distance d is defined on $\mathbb{C}$ by

$$d(z, w) = |z - w|,$$

where $|z| = \sqrt{a^2 + b^2} = \sqrt{z\bar{z}}$ when $z = a + ib$, $a, b \in \mathbb{R}$ and $\bar{z} = a - ib$.

Definition B1 Let $V \subseteq \mathbb{C}$ be open. A function $f : V \to \mathbb{C}$ is *analytic* if the derivative $f'(z)$ is defined and continuous throughout V.

This is equivalent to f having a power series expansion about any point z_0 in V which converges to f in some ball surrounding z_0.

Examples B2 The exponential function e^z and the trigonometric functions $\sin(z)$ and $\cos(z)$ are analytic functions (analytic on the whole complex plane - what are called *entire functions*). Their power series representations, valid at any point in $\mathbb{C}$, are given by:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots ,$$

$$\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz}) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots ,$$

$$\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz}) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots ,$$

We remind the reader of the following standard theorems from a first course in complex analysis:

The Maximum Modulus Principle B3 *A non-constant analytic function cannot attain its maximum absolute value at any interior point of its region of definition.*

Cauchy's Estimate B4 *If an analytic function f maps $B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$ inside some ball of radius s, then*

$$|f'(z_0)| \leq s/r.$$

Liouville's Theorem B5 *A bounded entire function is constant.*

Proof. If $M = \max\{|f(z)| : z \in \mathbb{C}\}$, then by Cauchy's Estimate, $|f'(z)| \leq M/r$, where r can be made arbitrarily large. It follows that $f'(z) = 0$ for all $z \in \mathbb{C}$, so f is constant. $\square$

Write $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$, the closed unit ball in the complex plane.

The Schwarz Lemma B6 *Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function with $f(0) = 0$. Then*

(i) $|f'(0)| \leq 1$.

(ii) *If $|f'(0)| = 1$, then $f(z) = cz$ for some $c \in S^1$ on the unit circle.*

(iii) *If $|f'(0)| < 1$, then $|f(z)| < |z|$ for all $z \neq 0$.*

Proof. (i) Since $f : \mathbb{D} \to \mathbb{D}$, $|f(z)| \leq 1$ always, so the maximum value $|f(z)|$ can take is 1. It follows from the Maximum Modulus Principle, that if $|z| < 1$, then $|f(z)| < 1$.

Note that since $f(0) = 0$, $g(z) = f(z)/z$ is defined and analytic throughout $\mathbb{D}$ (we can see this from the power series representation of $g(z)$). If $|z| = r < 1$, then $|g(z)| < 1/r$. Since this is true for all $0 < r < 1$, it follows that $|g(z)| \leq 1$ for all $z \in \mathbb{D}$. Now

$$|f'(0)| = \lim_{z \to 0} \left| \frac{f(z) - f(0)}{z - 0} \right| = \lim_{z \to 0} |g(z)| \leq 1,$$

so that (i) follows.

(ii) If $|f'(0)| = 1$, then $\lim_{z \to 0} |g(z)| = 1$, so by the continuity of g, $g(0) = 1$. But again, the Maximum Modulus Principle tells us that it is not possible for g to take its maximum possible value at an interior point, so we must have $g(z) = c \in S^1$ a constant on $\mathbb{D}$. (ii) follows.

(iii) If $|f'(0)| < 1$, then $|g(0) < 1$, and we must have $|g(z)| < 1$ for all $z \in \mathbb{D}$, for if not, then $|g(z)| = 1$ for some interior point z, so $f(z) = cz$, contradicting $|f'(0)| < 1$. $\square$

Appendix D, Weyl's Equidistribution Theorem

Let $x \in \mathbb{R}$, then the integer part of x is $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$, and the fractional part of x is $\{x\} = x - \lfloor x \rfloor$.

Definition D1 The sequence $(a_n)_{n \in \mathbb{N}}$ is *equidistributed* if for every $a, b \in [0,1]$ with $a < b$ we have

$$\lim_{N \to \infty} \frac{|\{n \leq N : \{a_n\} \in (a,b)\}|}{N} = b - a.$$

We have seen that for α irrational, the sequence $\{n\alpha\}$, $n = 1, 2, \ldots$, is dense in $[0,1)$. Our aim here is to show the much stronger fact that it is equidistributed in $[0,1)$. To say that the sequence is equidistributed means that the terms of the sequence enter any interval in proportion to the length of the interval (in the long term). In order to show this we use Weyl's criterion for equidistribution, which we state without proof.

Theorem D2: Weyl's Equidistribution Theorem *A sequence a_n is equidistributed if and only if for each $k \in \mathbb{N}$,*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k a_n} = 0.$$

We now show that for α irrational, the sequence $n\alpha$ is equidistributed in $[0,1)$. Denote by $||\alpha||$ the distance from α to the nearest integer.

Theorem D3 *Let α be irrational, then the sequence $(n\alpha)$ is equidistributed.*

Proof. We need a lemma:

Lemma D4 *Let $\alpha \in \mathbb{R}$, then for $N \in \mathbb{N}$,*

$$\left| \sum_{n=1}^{N} e^{2\pi i n\alpha} \right| \leq \min\{N, \frac{1}{2||\alpha||}\}.$$

Proof. If $\alpha = 0$, the sum is N. If $\alpha \neq 0$ we have a geometric series whose sum is

$$\frac{e^{2\pi i\alpha}(1 - e^{2\pi i n\alpha})}{1 - e^{2\pi i\alpha}}.$$

Using $\sin z = (e^{iz} - e^{-iz})/2i$, the sum is bounded by $|\sin \pi\alpha|^{-1}$, and the result follows from $|\sin \pi\alpha| \geq 2||\alpha||$. $\square$

Proof of Theorem D3 If we can show that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k n\alpha} = 0,$$

for every $k \in \mathbb{N}$, then Weyl's Theorem will give us the result. From the lemma, since $k\alpha$ is irrational, the sum is at most $1/2||k\alpha||$. Therefore

$$\left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k n\alpha} \right| < \frac{C}{N},$$

where C is the constant $C = 1/2||k\alpha||$, and the result follows. $\square$

Appendix E, Some Mathematica Codes

E1. The Koch Snowflake

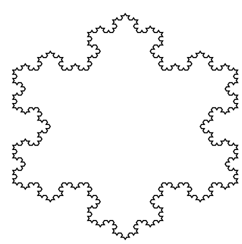
The first code we give produces various approximations to the snowflake curve. We then give some Mathematica codes that will produce the fractal curve resulting from a substitution. This enables us to give the substitution that results in the snowflake curve that we saw in Chapter 12. The code for these substitutions was written by my master's student Christopher Jones as part of his work towards an M.S. degree at Towson University.

```
Clear[initialize]
initialize[start_List : {0.‘, 0.‘},
  dim_Integer : 1000] := (path = Table[Null, {dim}]; X = N[start];
  U = {1.‘, 0.‘}; path[[1]] = X; count = 1)
```

```
right[a_] := U = {{t1 = Cos[aa = N[a \[Degree]]], t2 = Sin[aa]}, {-t2, t1}}.U
left[a_] := right[-a]
forward[s_] := path[[++count]] = X += s U
back[s_] := forward[-s]
finished := path = Take[path, count]
showturtlepath := (finished;
  Show[Graphics[Line[path]], PlotRange -> All, AspectRatio -> 1])

flakeside[length_, 0] := forward[length]
flakeside[length_, depth_] := (flakeside[length/3, depth - 1]; left[60];
  flakeside[length/3, depth - 1]; right[120]; flakeside[length/3, depth - 1];
  left[60]; flakeside[length/3, depth - 1])
KochSnowflake[n_] := (initialize[];
  Do[flakeside[1., n]; right[120], {3}] showturtlepath)
```

If we execute the following command for $n = 0, 2, 3, 4$, we get varying approximations to the snowflake curve.

```
KochSnowflake[4]
```



The Koch Snowflake

E2. The Morse Sequence

The following code can be used to generate the Morse sequence. The initial output asks how many iterations of the sequence are required. With an input of 9 iterations, $512 = 2^9$ terms of the sequence are given.

```
initialstring = "0";
numiterations = Input["Enter Number of iterations of Morse Sequence"];
```

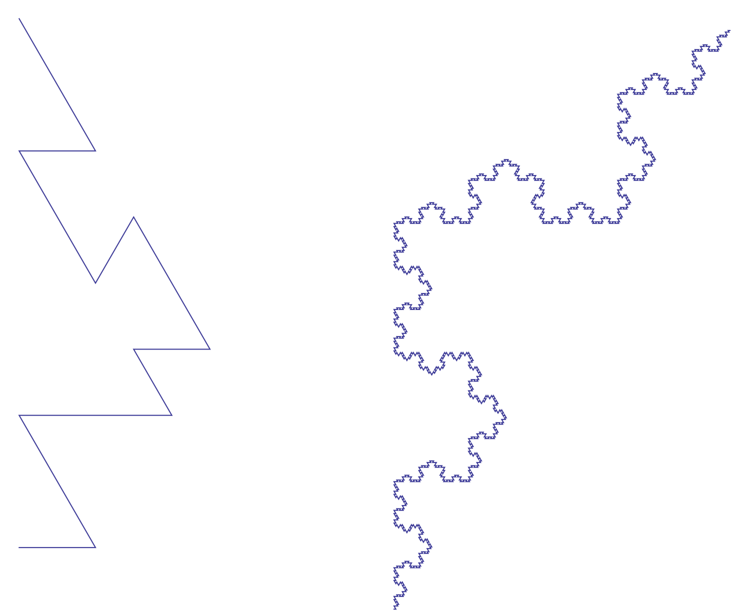
```
morse = StringReplace[initialstring, {"0" -> "01", "1" -> "10"}];
Do[morse = StringReplace[morse, {"0" -> "01", "1" -> "10"}], {i,
  numiterations - 1}]
stringlen = StringLength[morse];
Print["Morse Sequence with ", numiterations, " iterations has ", \
stringlen, " characters: "]
Print[morse]


Morse Sequence with 9 iterations has 512 characters:

01101001100101101001011001101001100101100110100101101001100101101
00101100110100101101001100101100110100110010110100101100110100110
01011001101001011010011001011001101001100101101001011001101001011
01001100101101001011001101001100101100110100101101001100101101001
01100110100101101001100101100110100110010110100101100110100101101
00110010110100101100110100110010110011010010110100110010110011010
01100101101001011001101001100101100110100101101001100101101001011
001101001011010011001011001101001100101101001011001101001
```

E3. Code for Koch Snowflake Curve produced by Thue-Morse substitution

Here is a code that gives rise to the Thue-Morse sequence, despite the fact that the initial approximations do not look promising:

```
initialstring = "0";
numiterations = Input["Enter Number of iterations of Morse Sequence"];
morse = StringReplace[initialstring, {"0" -> "01", "1" -> "10"}];
Do[morse = StringReplace[morse, {"0" -> "01", "1" -> "10"}], {i,
  numiterations - 1}]

stringlen = StringLength[morse];


xtab = Normal[SparseArray[{i_ -> 0}, stringlen]];
```

```
ytab = Normal[SparseArray[{i_ -> 0}, stringlen]];
y = 0;
x = 0;

Do[kth = StringTake[morse, {i}];
     kth =
  ToExpression[
   kth];                                                                                     \
            (*Finds element i in sequence*)
     w = (2 Pi*kth/3) + (2 Pi*i/3);
     x = x + Cos[w];
     y = y + Sin[w];
     xtab = ReplacePart[xtab, x, i];
     ytab = ReplacePart[ytab, y, i], {i, stringlen}]

Table[{xtab[[i]], ytab[[i]]}, {i, stringlen}];
ListPlot[%, Joined -> True, AspectRatio -> Automatic, Axes -> False]
Print["\n Iterations: ", numiterations, "     Root of Unity: ", \
rootunity, "      Quotient: ", quot]
```



Various approximations to the snowflake using the Morse substitution.

E4. Code to Create Julia Sets

1.

```
f[z_] := z^2 - 1;
bailOut = 50;
test[z_] := Abs[z] <= 2;
JuliaCount[z0_] := Length[NestWhileList[f, N[z0], test, 1, bailOut]];
DensityPlot[JuliaCount[x + I*y], {x, -1.4, 1.4}, {y, -1.4, 1.4},
 PlotPoints -> 300, Mesh -> False, AspectRatio -> Automatic,
 Frame -> False,
 ColorFunction -> (If[#  1, RGBColor[0, 0, 0], Hue[#]] &)]
```

2.

```
DensityPlot[
 Length[FixedPointList[#^2 + (-0.5 + I*.5) &, x + I*y, 20,
   SameTest -> (Abs[#2] > 3.0 &)]], {x, -1.5, 1.5}, {y, -1.5, 1.5},
 PlotPoints -> 150, Mesh -> False, Frame -> False,
 AspectRatio -> Automatic, ColorFunction -> Hue]
```

3.

```
Coordinates={Complex[x_,y_]->{x,y}};
```

```
JuliaSet[c_,z0_,Npoints_]:=Module[{Nc=N[c],Nz0
    =N[z0]},NestList[(1-2 Random[Integer]) Sqrt[#-Nc] &,
    Nz0,Npoints] /. Coordinates];
```

```
JuliaData=Drop[Evaluate[JuliaSet[-1+ 0 I, 0+5 I, 2^18]], 100];
```

```
ListPlot[JuliaData, Axes->False, AspectRatio->Automatic,
     PlotStyle->PointSize[0.001]]
```

E5. Code to Create the Mandelbrot Set

1.

```
DensityPlot[
 Length[FixedPointList[#^2 + (x + I*y) &, x + I*y, 20,
   SameTest -> (Abs[#2] > 2.0 &)]], {x, -2.0, 1.0}, {y, -1.0, 1.0},
 PlotPoints -> 175, Mesh -> False, Frame -> False,
 AspectRatio -> Automatic, ColorFunction -> Hue]
```

1.

```
xmin = -2.2; xmax = 0.6; ymin = -1.2; ymax = 1.2;

Itermax = 50; stepsize = 0.005;

Counter =
  Compile[{x, y, {Itermax, _Integer}},
   Module[{w, z, count = 0}, w = x + I y; z = w;
    While[(Abs[z] < 4.0) && (count <= Itermax), ++count;
     z = z^2 + w;]; Return[count]]];

MandelbrotData =
  Table[Counter[x, y, Itermax], {y, ymin, ymax, stepsize}, {x, xmin,
```

```
    xmax, stepsize}];

ListDensityPlot[MandelbrotData, Frame -> False, Mesh -> False,
 MeshRange -> {{xmin, xmax}, {ymin, ymax}}, AspectRatio -> Automatic,
 ColorFunction -> (Hue[Log[1 + #]] &)]
```

E6. Code to Create the Sierpinski Triangle

1.

```
:=V1={0,0};V2={4,0};V3={2,2 Sqrt[3]};

:=Itermax=100000;Half=1/2;P[1]={0,0};

:=P[n_]=For[n=1,n<=Itermax,n++,r=Random[ ];
  If[0<=r<=1/3,P[n+1]=N[P[n]+(V1-P[n]) Half],
  If[1/3<r<=2/3,P[n+1]=N[P[n]+(V2-P[n]) Half],
  P[n+1]=N[P[n]+(V3-P[n]) Half]]]];

  :=Siepinski=Table[P[i],{i,Itermax}];

  :=ListPlot[Sierpinski,Axes->False, AspectRatio-Automatic,
  PlotStyle->{PointSize[0.001],
  RgbColor[1,0,0]}]
```

References

[1] D. S. Alexander, A History of Complex Dynamics: From Schröder to Fatou and Julia, Vieweg, 1994.

[2] J-P. Allouche and J. Shallit, The ubiquitous Prouet-Thue-Morse sequence,

[3] J-P. Allouche and J. Shallit. Automatic Sequences: Theory, Applications and Generalizations, Cambridge University Press, 2003.

[4] J. Auslander and J. Yorke, Interval maps, factor maps and chaos, *Tohoku Math. J.* **32** (1980), 177-188.

[5] B. Barna, Uber die Divergenzpunkte des Newtonschen Verfahrens zur Bestimmung von Wurzeln algebraischer Gleichungen, I, *Publ. Math. Debrecen,* **3** (1953), 109-118.

[6] J. Banks, J. Brooks, G. Cairns, G. Davis and P. Stacey, On Devaney's definition of chaos, *American Math. Monthly,* **99**, 1992, 332-334.

[7] J. Bechhoefer, The birth of period 3, revisited. *Mathematics Magazine,* **69** (1996), 115-118.

[8] V. Berthé, Fréquences des facteurs des suites sturmian, *Theor. Comp. Sci.,* **165** (1996), 295-309.

[9] V. Berthé, Sequences of low complexity: automatic and Sturmian sequences, in Topics in Symbolic Dynamics and Applications. Ed: F. Blanchard, A. Maass and A. Nogueira. LMS Lecture Notes Ser. **279**, CUP, 1-34.

[10] P. Blanchard, Complex analytic dynamics on the Riemann sphere, *Bull. Amer. Math. Soc.*, **11** (1984), 85-141.

[11] A. Boyarsky and P. Gora, Invariant measures for Chebyshev maps, *J. Applied Math. and Stochastic Analysis*, **14** (2001), 257-264.

[12] P. Bracken, Dynamics of the mapping $f(x) = (x+1)^{-1}$, *Fibonacci Quarterly,* **33** (1995), 357-358.

[13] T. C. Brown, A characterization of the quadratic rationals, *Canad. Math. Bull.*, **34** (1991), 36-41.

[14] M. P. de Carvalho, Chaotic Newton's sequences, *Mathematical Intelligencer,* **24** (2002), 31-35.

[15] A. Cayley, Applications of the Newton-Fourier method to an imaginary root of an equation, *Quarterly J. Pure and Applied Math.,* **16** (1879), 179-185.

[16] W. A. Coppel, The solution of equations by iteration, *Proc. Cambridge Philos. Soc.* **51** (1955) 41-43.

[17] Coquet,

[18] E. M. Coven and G. A. Hedlund, Sequences with minimal block growth, *Math. Systems Theory,* **7** (1973), 138-153.

[19] D. Crisp, W. Moran, A. Pollington and P. Shiue, Substitution invariant cutting sequences, *Journal de Théorie des Nombres de Bordeaux,* **5** (1993), 123-137.

[20] R. L. Devaney, An Introduction to Chaotic Dynamical Systems, Benjamin Cummings, 1986.

[21] Dekking,

[22] S. Elaydi, Discrete Chaos, Chapman and Hall/CRC, 2000.

[23] S. Elaydi, A converse to Sharkovsky's Theorem, *American Math. Monthly,* **99**, 1992, 332-334.

[24] B. Y. Feng, A trick formula to illustrate the period three bifurcation diagram of the logistic map. *Journal of Mathematical Research and Exposition,* **30**, 2010, 286-290.

[25] N. Pytheas Fogg, Substitutions in Dynamics, Arithmetics and Combinatorics. Lecture Notes in Mathematics, **1794**, Springer, 2002.

[26] G. R. Goodson, M. Lemańczyk, On the rank of a class of substitution dynamical systems, *Studia Math.,* **96** (1990), 219-230.

[27] W. B. Gordon, Period three trajectories of the logistic map. *Mathematics Magazine,* **69** (1996), 118-120.

[28] D. Gulick, Encounters with Chaos, McGraw Hill, 1992.
[29] J. Heidel, The existence of periodic orbits of the tent map, *Physics Letters A,* **143** (1990), 195-201.
[30] R. A. Holmgren, A First Course in Discrete Dynamical Systems, Springer-Verlag, 1994.
[31] J. E. Huchinson, Fractals and self-similarity, *Indiana Univ. Math. J.* **30** (1981), 713-747.
[32] G. Julia, Memoire sur l'iteration des fonctions rationelle. *Journel des mathematiques pures et appliquées,* **4** (1918), 47-245.
[33] A. Katok and B. Hasselblatt, Introduction to the Modern Theory of Dynamical Systems, Encyclopedia of Mathematics and its Applications **54**, Cambridge University Press, 1995.
[34] J. Kigami, Analysis on Fractals. Cambridge University Press, 2001.
[35] G. Koenigs, Recherches sur le substitutions uniforms, *Bull. des Sciences math. et astron.,* Series 2, **7** (1983), 340-357.
[36]
[37] M. H. Lee, Analytical study of the superstable 3-cycle in the logistic map. *Journal of Mathematical Physics,* **50** (2009), 122702, 1-6.
[38] M. H. Lee, Three-cycle problem in the logistic map and Sharkovskii's Theorem. *Acta Physica Polonica B,* **42** (2011), 1071-1080.
[39] M. Lothaire, Algebraic Combinatorics on Words. Cambridge University Press, 2002.
[40] Ma,
[41] R. M. May, Simple mathematical models with very complicated dynamics. *Nature,* **261** (1976), 459-467.
[42] B. Mandelbrot, The Fractal Geometry of Nature. W. H. Freeman, New York, 1982.
[43] J. Milnor. Dynamics in One Complex Variable: Introductory Lectures, Vieweg, 2000.
[44] M. Misiurewicz, Remarks on Sharkovsky's Theorem. *American Mathematical Monthly.* **104** (1997), 846-847.
[45] M. Misiurewicz, On the iterates of e^z. *Ergodic Theory and Dynamical Systems.* **1** (1981), 103-106.
[46] R. Nillsen, Chaos and one-to-oneness. *Mathematics Magazine,* **72** (1999), 14-21.
[47] J. C. Oxtoby, Measure and Category. Springer-Verlag, 1971.
[48] R. Palais, A simple proof of the Banach contraction principle. *J. Fixed Point Theory and Applications,* **2** (2007), 221-223.
[49] M. Queffelec, Substitution Dynamical Systems - Spectral Analysis. Lecture Notes in Mathematics 1294, 2nd Edition, Springer-Verlag, 2010.
[50] J. Palmore, Newton's method and Schwarzian derivatives. *J. Dynamics and Differential Equations,* **6** (1994), 507-511.
[51] P. Petek, A nonconverging Newton sequence. *Mathematical Magazine,* **56** (1983), 43-45.
[52] Rademacher,
[53] G. Rauzy, Low complexity and geometry. Dynamics of complex interacting systems (Santiago, 1994) 147-177, Nonlinear Phenom, Complex Systems **2**, Kluwer Acad. Publ. Dordrecht, 1996.
[54] D. Ruelle,
[55] D. G. Saari and J. B. Urenko, Newton's method, circle maps and chaotic motion. *Amer. Math. Monthly,* **91** (1984), 3-17.
[56] P. Saha and S. H. Strogatz. The birth of period three. *Mathematics Magazine,* **68** (1995), 42-47.
[57] E. Schröder, Ueber iterite functionen, *Mathematische Annalen,* **3** (1871), 296-322.
[58] H. Sedaghat, The impossibility of unstable, globally attracting fixed points for continuous mappings of the line. *Mathematical Notes,* 1997, 356-358.
[59] D. Singer, Stable orbits and bifurcation of maps of the interval. *SIAM J. Appl. Math.* **35** (1978), 260-267.

[60] H. Sohrab, Basic Real Analysis, Birkhauser, Boston, 2003.
[61] D. Sprows, Digitally determined periodic points. *Mathematics Magazine,* **71** (1998), 304-305.
[62] R. Stankewitz and J. Rolf, Chaos, fractals, the Mandelbrot set, and more.
[63] S. Sternberg, Dynamical Systems, Dover, 2009.
[64] M. Vellekoop and R. Berglund, On intervals: transitivity → chaos. *American Math. Monthly.* **101** (1994), 353-355.
[65] J. A. Walsh, The dynamics of Newton's method for cubic polynomials. *The College Mathematics Journal,* **26** (1995), 22-28.
[66] P. Walters, An Introduction to Ergodic Theory. Springer Verlag, 1981.
[67] E. W. Weisstein, "Logistic Map". From MathWorld – A Wolfram Web Resource: http://mathworld.wolfram.com/LogisticMap.html.