Lecture notes on

Kinetic Transport Theory

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Preface

Kinetic transport equations are mathematical descriptions of the dynamics of large particle ensembles in terms of a phase space (i.e. position-velocity space) distribution function. They are sometimes called *mesoscopic models*, which places them between *microscopic models,* where the dynamics of the individual particles are described, and *macroscopic* or *continuum mechanics models*, where the material is described by a finite number of position dependent quantities such as the mass density, the mean velocity, the stress tensor, the temperature etc. Mathematically, kinetic equations typically are integro-differential equations of a particular form. Due to their importance from a modeling point of view, their mathematical theory has become a highly developed subfield of the theory of partial differential equations.

The undisputedly most important kinetic equation is the *Boltzmann equation* for hard spheres, where the underlying particle system is an idealized 3D-billiard. It is well accepted as a model for ideal gases which, on a macroscopic level, are typically described by the *Euler* or the *Navier-Stokes* equations. The latter systems can be formally derived from the Boltzmann equation by procedures called *macroscopic limits.* The main goals of this course are to present

- a *formal derivation* of the Boltzmann equation from the microscopic model,
- a proof of a variant of the *DiPerna-Lions theorem* [7] on the existence of large global solutions of the Boltzmann equation, and
- a formal derivation of the incompressible Navier-Stokes equations from the Boltzmann equation in a combined macroscopic-small-Mach-number limit. Some aspects of the rigorous justification of this limit by *Golse/Saint-Raymond* [10] are discussed.

The presented material is mostly taken from the books of C. Cercignani, R. Illner, M. Pulvirenti [5], and L. Saint-Raymond [16], referenced in the following by [CIP] and [StR], respectively.

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Historical remarks – the sixth problem of Hilbert

At the *International Congress of Mathematicians 1900* in Paris, Hilbert posed 23 outstanding mathematical problems, the 6th of which was formulated very vaguely. Under the title *'The Mathematical Treatment of the Axioms of Physics'* he asked for making Theoretical Physics mathematically rigorous, motivated by the work of several physicists, most importantly Boltzmann [3]: *'Thus Boltzmann's work on the principles of mechanics suggests the problem of developing mathematically the limiting processes, those merely indicated, which lead from the atomistic view to the laws of continua'.* One possible path for tackling this challenge in the context of gas dynamics is to take the 'detour' via the Boltzmann equation on the way from the microscopic to a continuum description. The Boltzmann equation had been derived (formally) in 1872 [2], and Hilbert himself worked on its connection with continuum theories [11].

This program requires as a first step a rigorous convergence result of solutions of the microscopic dynamics to solutions of the Boltzmann equation. The starting point is a probabilistic interpretation of the microscopic dynamics. The derivation of the Boltzmann equation then relies on a certain independence assumption (called *molecular chaos*) on the probability distribution of the particles. The main mathematical problem is to prove that molecular chaos is propagated in time by the microscopic dynamics. As Boltzmann understood already, this assumption led to the seeming contradiction (called *Loschmidt's paradox* [14]) that the time irresversible Boltzmann equation arises as a limit of the reversible microscopic dynamics. The irreversibility is a consequence of the famous *H-theorem* [2]. Rigorous mathematical results concerning the validity of the Boltzmann equation as a limit of particle dynamics came much later. Even now the status of the theory is not completely satisfactory, since smallness assumptions are needed, either smallness of the time interval [13] or smallness of the gas density [12].

The second step from the Boltzmann equation to continuum models is in somewhat better shape. This is mainly due to two results. First, global existence of solutions with initial data only satisfying 'natural' bounds without smallness assumptions has been proven in 1989 by DiPerna and Lions. Second, a program initiated by Bardos, Golse and Levermore has been completed in 2008 by Golse and Saint-Raymond [10], where one of the possible macroscopic limits of the Boltzmann equation has been justified rigorously and in great generality, namely the derivation of the *incompressible Navier-Stokes equations.* For the probably most natural macroscopic limit to the *Euler equations* of gas dynamics only partial results are available [4].

Chapter 1

A formal derivation of the Boltzmann equation

Hard sphere dynamics

Without discussing the physical validity of such an assumption, a gas will be modeled as an ensemble of N *identical, elastic, hard spheres* with diameter σ moving in a container represented by a domain $\Omega' \subset \mathbb{R}^3$ (with smooth boundary $\partial\Omega'$). The state of the system is then given by the positions of the centers and by the velocities of all particles:

$$z = (x_1, v_1, \ldots, x_N, v_N) \in (\Omega \times \mathbb{R}^3)^N \,,$$

where

$$\Omega = \{x \in \Omega' : \; d(x, \partial\Omega') > \sigma/2\} \,.$$

We assume that spheres touch the boundary at at most one point or, more precisely, that $\partial\Omega$ is smooth with unit outward normal $\nu(x)$, $x \in \partial\Omega$. The set $(\Omega \times \mathbb{R}^3)^N$ is still too large, however, since it allows for overlapping spheres. It has to be replaced by the open set

$$\Lambda = \{(x_1, v_1 \ldots, x_N, v_N) \in (\Omega \times \mathbb{R}^3)^N : \; |x_i - x_j| > \sigma \text{ for } i \neq j\}$$

The first basic assumption on the dynamics is that the particles move independently and with constant velocity, as long as they do not touch each other. This means that for an initial state $z_0 \in \Lambda$ (that means that all particles are separated from each other and from the boundary of the container) there exists a time $t_1 > 0$ such that, for time $t \in [0, t_1)$, the dynamics is very simple and given by solving the ODE

$$\dot{z} = (v_1, 0, \ldots, v_N, 0) \,, \tag{1.1}$$

subject to the initial conditions

$$z(t = 0) = z_0 = (x_{10}, v_{10}, \ldots, x_{N0}, v_{N0}) \,, \tag{1.2}$$

with the *free flow* solution

$$z(t) = (x_1(t), v_1(t), \ldots, x_N(t), v_N(t)), \quad x_j(t) = x_{j0} + t\, v_{j0}, \ v_j(t) = v_{j0}.$$

Let t_1 be the smallest time, such that $z(t_1-) \in \partial\Lambda$. We claim that *generically* (what this means precisely will be discussed later) we have one of two situations

1. $\exists\, i: \quad x_i(t_1-) \in \partial\Omega, \quad v_i(t_1-) \cdot \nu(x_i(t_1-)) > 0,$
2. $\exists\, i, j: \quad |x_i(t_1-) - x_j(t_1-)| = \sigma,$

i.e., either a particle has collided with the wall of the container, or two particles have collided with each other. The second basic assumption on the dynamics is that at times like t_1 *hard, elastic collisions* take place which, in case 1, conserve the component of the particle momentum parallel to the container boundary and, in case 2, conserve the total momentum of the involved particles. Mathematically, this means that the positions of the particles are continuous in time ($x_i(t_1+) = x_i(t_1-)$), whereas the velocities of the involved particles have jump discontinuities, where the post-collisional velocities are computed as follows:

1. With the abbreviations $v_i = v_i(t_1-)$, $v_i' = v_i(t_1+)$, $\nu = \nu(x_i(t_1))$, the collision being elastic means *conservation of kinetic energy*:

   $$|v_i'|^2 = |v_i|^2,$$

   and the conservation of the parallel component of the momentum gives the requirement

   $$v_i' - (\nu \cdot v_i')\nu = v_i - (\nu \cdot v_i)\nu.$$

   This implies the rule of *specular reflection*

   $$v_i' = v_i - 2(\nu \cdot v_i)\nu. \tag{1.3}$$

2. We again introduce abbreviations:

   $$v_i = v_i(t_1-), \quad v_i' = v_i(t_1+), \quad v_j = v_j(t_1-), \quad v_j' = v_j(t_1+),$$

   $$n = \frac{x_i(t_1) - x_j(t_1)}{\sigma}.$$

   For the collision between two particles we need, besides conservation of total kinetic energy and of total momentum,

   $$|v_i'|^2 + |v_j'|^2 = |v_i|^2 + |v_j|^2, \qquad v_i' + v_j' = v_i + v_j,$$

the additional assumption (which can be supported by symmetry arguments) that the momentum changes (i.e. the forces) have to be in the direction n between the centers of the spheres:

$$v_i' = v_i + \lambda n\,, \qquad v_j' = v_j + \mu n\,.$$

These requirements lead to the collision rule

$$v_i' = v_i - (n \cdot (v_i - v_j))n\,, \qquad v_j' = v_j + (n \cdot (v_i - v_j))n\,.$$

The velocities of all the particles not involved in the collision remain unchanged. It is easily seen that the post-collisional velocity vector $(v_1(t_1+), 0, \ldots, v_N(t_1+), 0)$ points into Λ, which shows that the dynamics can be continued by a free flow until, at time $t_2 > t_1$, another collision occurs.

The above procedure defines a dynamical system on Λ except for a subset, which leads to either a multiple collision (three or more particles collide at the same time or two or more particles collide with each other and at least one of them with the container wall) or to an accumulation of collision events in finite time. Fortunately, this subset can be shown to be small enough to be neglected in our further considerations (see [CIP] for details). Thus, for an initial state $z_0 \in \Lambda$, the state $T_t z_0$ at any later time $t > 0$ can be computed by a finite number of free flows interrupted by collisions.

It is an important property of this dynamical system to be *mechanically reversible.* This means that, if the dynamics is stopped, all the velocities are reversed, and then it is restarted, then the observed dynamics is exactly the previous dynamics running backwards in time.

Another important property is the *indistinguishability* of the particles. In other words, if two particles are exchanged in an initial state and then the dynamics evolves for a certain time t, then the result is the same as if the dynamics is applied to the original state and the particles are exchanged at the end. In order to express this statement in a formula, the exchange operator

$$\pi_{ij} z := (x_1, v_1, \ldots, x_j, v_j, \ldots, x_i, v_i, \ldots, x_N, v_N)\,,$$

for $z = (x_1, v_1, \ldots, x_i, v_i, \ldots, x_j, v_j, \ldots, x_N, v_N)$, $1 \leq i < j \leq N$, is introduced. The indistinguishability of particles is then equivalent to the commutation relation

$$T_t \pi_{ij} = \pi_{ij} T_t \qquad \text{for all } t > 0\,,\ 1 \leq i < j \leq N\,.$$

A probabilistic description

At typical atmospheric pressures and temperatures, $1\,\mathrm{cm}^3$ of air contains on the order of 10^{20} molecules. For several reasons it is out of question (and also not very interesting) to have exact knowledge about their positions and velocities at a certain time. This is a motivation to adopt a stochastic description with a time dependent probability density $P(z,t) \geq 0$, $(z,t) \in \Lambda \times \mathbb{R}$, where $\int_B P(z,t)dz$ is the probability to find the ensemble in $B \subset \Lambda$ at time t. Of course, $\int_\Lambda P(z,t)dz = 1$ has to hold for all t. There is a simple rule for the evolution of the probability density. It relies on the property of the dynamics to preserve volume in the state space: If a time dependent set $B(t) \subset \Lambda$ consists of the states of the particles at time t, which had states in $B(0)$ at time $t = 0$,

$$B(t) = \{T_t z : z \in B(0)\}\,, \tag{1.4}$$

then we claim that

$$\int_{B(t)} dz' = \int_{B(0)} dz\,,$$

holds for arbitrary $B(0) \subset \Lambda$ and $t > 0$. The change of variables $z' = T_t z$ carried out in the integral on the left hand side, shows that this property is equivalent to

$$\det(D_z(T_t z)) = 1 \qquad \text{for all } z \in \Lambda\,,\ t > 0\,. \tag{1.5}$$

Here $D_z(T_t z)$ is the Jacobian of $T_t z$ with respect to z. The property (1.5) is easily seen, as long as the dynamics is a free flow. For $T_t z = (x_1 + tv_1, v_1, \ldots, x_N + tv_N, v_N)$, the Jacobian is given by the block diagonal matrix

$$D_z(T_t z) = \mathrm{diag}\{A, \ldots, A\}\,, \qquad \text{with } A = \begin{pmatrix} I_{3\times3} & tI_{3\times3} \\ 0_{3\times3} & I_{3\times3} \end{pmatrix}\,,$$

whose determinant is obviously equal to one. Since the free flow is a very simple example of an *Hamiltonian flow,* this can be seen as an illustration of the *Liouville theorem.*

The relation (1.5) remains true after collisions of a particle with the container wall and after collisions between two particles. Instead of a complete proof, this will be shown for the example of the collision of a particle with a plain piece of container wall, determined by the equation $\nu \cdot x = 0$ with normal vector ν. If the pre-collisional path of the particle is given by the free flight

$$x(t) = x_0 + tv_0\,,\ v(t) = v_0\,, \qquad t < T\,,$$

then the collision time is given by $T = -(\nu \cdot x_0)/(\nu \cdot v_0)$ and, using the specular reflection formula (1.3), the post-collisional free flight by

$$x(t) = x_0 - 2(\nu \cdot x_0)\nu + t\Big(v_0 - 2(\nu \cdot v_0)\nu\Big)\,,\ v(t) = v_0 - 2(\nu \cdot v_0)\nu\,, \quad t > T\,.$$

The Jacobian $D_{x_0,v_0}(x(t),v(t))$ for $t > T$ is given by

$$\begin{pmatrix} B & tB \\ 0 & B \end{pmatrix}, \qquad \text{with } B = I_{3\times 3} - 2\nu \otimes \nu\,, \tag{1.6}$$

where $\otimes$ denotes the tensor product. A straightforward computation using $|\nu| = 1$ shows $\det(B) = -1$ and, thus, that the determinant of the Jacobian is equal to one.

Since the dynamics is assumed as deterministic, randomness enters only through the initial distribution $P(z,0)$. Therefore, for a time dependent set $B(t)$ given by (1.4),

$$\int_{B(t)} P(z',t)dz' = \int_{B(0)} P(z,0)dz\,, \qquad \text{for all } t > 0\,,$$

has to hold. The change of coordinates $z' = T_t z$, the property (1.5), and the arbitrariness of $B(0)$ imply

$$P(T_t z, t) = P(z,0)\,, \tag{1.7}$$

i.e. the probability density has to be constant along trajectories of the dynamical system. For $T_t z \in \Lambda$, i.e. away from collision events, this equation can be differentiated with respect to time, giving the partial differential equation

$$\partial_t P(z,t) + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} P(z,t) = 0\,, \qquad \text{for } z \in \Lambda \tag{1.8}$$

the *Liouville equation* in the context of the above mentioned Hamiltonian flow. Since the left hand side of (1.7) has to be constant in time, it must not change across collision events. This leads to the boundary conditions

$$\begin{aligned} &P(x_1, v_1, \ldots, x_i, v_i', \ldots, x_N, v_N, t) = P(z,t)\,, \\ &\text{with } v_i' = v_i - 2(\nu(x_i)\cdot v_i)\nu(x_i)\,, \qquad \text{for } x_i \in \partial\Omega\,, \end{aligned} \tag{1.9}$$

for the boundary part corresponding to collisions with the container wall, and

$$\begin{aligned} &P(x_1, v_1, \ldots, x_i, v_i', \ldots, x_j, v_j', \ldots, x_N, v_N, t) = P(z,t)\,, \\ &\text{with } v_i' = v_i - (n_{ij}\cdot(v_i - v_j))n_{ij}\,, \qquad v_j' = v_j + (n_{ij}\cdot(v_i - v_j))n_{ij}\,, \\ &n_{ij} = \frac{x_i - x_j}{\sigma}\,, \qquad \text{for } |x_i - x_j| = \sigma\,, \end{aligned} \tag{1.10}$$

for the boundary part corresponding to two-particle collisions. Together with initial conditions

$$P(z,0) = P_0(z)\,, \qquad \text{for } z \in \Lambda\,, \tag{1.11}$$

(1.8)–(1.10) should constitute a well posed problem for the determination of $P(z,t)$. However, for computational approaches it is completely useless due to the fact that the state space is $6N$-dimensional.

Mechanical reversibility is still present in the probabilistic interpretation: If, for given $P(z,t)$, $0 \le t \le T$, solving (1.8)–(1.11), we define $\tilde{P}_0(z) := P(x_1, -v_1, \ldots, x_N, -v_N, T)$ as initial data for the computation of a new distribution $\tilde{P}$, then

$$\tilde{P}(z,t) = P(x_1, -v_1, \ldots, x_N, -v_N, T-t)$$

holds.

Indistinguishability of the particles is required at $t=0$: $P_0(\pi_{ij}z) = P_0(z)$ for all $z \in \Lambda$, $1 \le i < j \le N$. Then the indistinguishability property of the flow implies

$$P(\pi_{ij}z, t) = P(z,t)\,, \qquad \text{for all } z \in \Lambda\,,\ 1 \le i < j \le N\,,\ t > 0\,.$$

The one-particle probability density

In view of the huge dimension of the state space, Boltzmann (and Maxwell) considered the question, if something intelligent could be said about the evolution of the one-particle probability density, i.e. the marginal

$$P_1(x_1, v_1, t) = \int_{\Lambda_1(x_1,v_1)} P(x_1, v_1, \ldots, x_N, v_N, t) \prod_{i=2}^N dx_i dv_i\,,$$

where

$$\Lambda_1(x_1, v_1) = \{(x_2, v_2, \ldots, x_N, v_N):\ z \in \Lambda\} \subset (\Omega \times \mathbb{R}^3)^{N-1}\,.$$

Note that, by the indistinguishability, P_1 is the probability density for any single particle. An equation for the evolution of P_1 can be obtained by integrating the Liouville equation (1.8) over $\Lambda_1(x_1, v_1)$:

$$\partial_t P_1 + \int_{\Lambda_1} v_1 \cdot \nabla_{x_1} P \prod_{i=2}^N dx_i dv_i + \sum_{j=2}^N \int_{\Lambda_1} \nabla_{x_j} \cdot (v_j P) \prod_{i=2}^N dx_i dv_i = 0\,. \quad (1.12)$$

In the second term, the order of integration and differentiation cannot simply be changed, since the integration domain depends on x_1. As a correction, boundary terms have to be added, corresponding to those parts of the boundary of Λ_1, whose definitions involve x_1, i.e. $|x_1 - x_i| = \sigma$, $i = 2, \ldots, N$. Denoting the corresponding outward unit normals and surface elements by $n_{1i} = (x_1 - x_i)/\sigma$ and, respectively, σ_{1j}, we obtain

$$\begin{aligned}\int_{\Lambda_1} v_1 \cdot \nabla_{x_1} P \prod_{i=2}^N dx_i dv_i &= v_1 \cdot \nabla_{x_1} P_1 \\ &- \sum_{j=2}^N \int_{\mathbb{R}^3} \int_{|x_1 - x_j| = \sigma} v_1 \cdot n_{1j} \left(\int_{\Lambda_2} P \prod_{2 \le i \ne j} dx_i\, dv_i \right) d\sigma_{1j} dv_j\,, \quad (1.13)\end{aligned}$$

where

$$\Lambda_2 = \Lambda_2(x_1, v_1, x_j, v_j)$$
$$= \{(x_2, v_2, \dots, x_{j-1}, v_{j-1}, x_{j+1}, v_{j+1}, \dots x_N, v_N) : \ z \in \Lambda\} \subset (\Omega \times \mathbb{R}^3)^{N-2} .$$

By the indistinguishability property, the term in parantheses is equal to $P_2(x_1, v_1, x_j, v_j, t)$, where the *two-particle probability density* is given by

$$P_2(x_1, v_1, x_2, v_2, t) = \int_{\Lambda_2(x_1,v_1,x_2,v_2)} P(x_1, v_1, \dots, x_N, v_N) \prod_{i=3}^{N} dx_i\, dv_i \,.$$

This shows that the terms in the sum in (1.13) are all equal and we obtain

$$\int_{\Lambda_1} v_1 \cdot \nabla_{x_1} P \prod_{i=2}^{N} dx_i dv_i$$
$$= v_1 \cdot \nabla_{x_1} P_1 - (N-1) \int_{\mathbb{R}^3} \int_{|x_1-x_2|=\sigma} v_1 \cdot n_{12} P_2 \, d\sigma_{12} dv_2 \quad (1.14)$$

In the third term of (1.12), x_j is one of the integration variables and we can therefore use the divergence theorem, where the boundary integrals correspond to collisions of particle number j with a) particle number 1, b) any other particle, and c) the container wall:

$$\int_{\Lambda_1} \nabla_{x_j} \cdot (v_j P) \prod_{i=2}^{N} dx_i dv_i = \int_{\mathbb{R}^3} \int_{|x_1-x_j|=\sigma} v_j \cdot n_{1j} P_2 \, d\sigma_{1j} dv_j$$
$$+ \sum_{2 \le i \ne j} \int_{\Gamma_{ij}} v_j \cdot n_{ij} P_3 \, d\sigma_{ij} dv_j dx_i dv_i + \int_{\mathbb{R}^3} \int_{\partial\Omega} v_j \cdot \nu P_2 \, dS_j dv_j \,, \quad (1.15)$$

where dS_j is the surface element for integration with respect to $x_j \in \partial\Omega$ and

$$\Gamma_{ij} = \{(x_i, v_i, x_j, v_j) \in (\Omega \times \mathbb{R}^3)^2 : \ |x_i - x_1|, |x_j - x_1| > \sigma, \, |x_i - x_j| = \sigma\} \,.$$

Although not necessary at this point, in the second line the three-particle probability density (defined in the obvious way) has been used, which is convenient in the following. When substituted in (1.12), the terms in (1.15) have to be summed up with respect to j. For the first term in the second line of (1.15), this leads to a double sum over all $2 \le i, j \le N$, $i \ne j$. When i and j are exchanged, only $v_j \cdot n_{ij}$ has to be replaced by $v_i \cdot n_{ji} = -v_i \cdot n_{ij}$, since Γ_{ij} and P_3 are symmetric with respect to i and j, the latter because of indistinguishability. Therefore the integral in the double sum can be replaced by

$$I_{ij} := \frac{1}{2} \int_{\Gamma_{ij}} (v_j - v_i) \cdot n_{ij} P_3 \, d\sigma_{ij} dv_j dx_i dv_i$$

In this integral we introduce a coordinate transformation by the collision rule

$$v_i' = v_i - (n_{ij} \cdot (v_i - v_j))n_{ij} , \qquad v_j' = v_j + (n_{ij} \cdot (v_i - v_j))n_{ij}$$

As stated above, this transformation is measure preserving ($dv_i dv_j = dv_i' dv_j'$) and satisfies $(v_i - v_j) \cdot n_{ij} = -(v_i' - v_j') \cdot n_{ij}$. Finally, by the boundary condition (1.10),

$$P_3(x_1, v_1, x_i, v_i, x_j, v_j, t) = P_3(x_1, v_1, x_i, v_i', x_j, v_j', t)$$

holds. These observations imply $I_{ij} = -I_{ij} \Rightarrow I_{ij} = 0$. By a similar argument, the last term in (1.15) vanishes: The measure preserving collision transformation $v_j' = v_j - 2(\nu \cdot v_j)\nu$ implies $v_j \cdot \nu = -v_j' \cdot \nu$, and in P_2, the argument v_j can be replaced by v_j' because of the boundary condition (1.9). Finally, the first term on the right hand side of (1.15) is obviously independent from j, which can therefore be replaced by 2.

Collecting our results, we substitute (1.14) and (1.15) in (1.12), use the above arguments, and obtain the evolution equation for the one-particle probability density:

$$\partial_t P_1 + v_1 \cdot \nabla_{x_1} P_1 = (N-1) \int_{\mathbb{R}^3} \int_{|x_1 - x_2| = \sigma} (v_1 - v_2) \cdot n_{12} P_2 \, d\sigma_{12} dv_2 \, . \tag{1.16}$$

Of course, one could not have expected a closed problem without any influence of correlations. Actually, it might be surprising that only the two-particle probability distribution occurs in this equation. An effort to close the problem by deriving an evolution equation for P_2 fails, since in this equation P_3 occurs in the right hand side. Continuing the process results in an infinite coupled system for $P_1, P_2, \ldots$, called the BBGKY-hierarchy (Bogoliubov-Born-Green-Kirkwood-Yvon, see [CIP]).

The next step is a simple change of variables:

$$x_1 = x \, , \quad v_1 = v \, , \quad x_2 = x - \sigma n \, , \quad v_2 = v_* \, ,$$

changing (1.16) into

$$\begin{aligned} &\partial_t P_1 + v \cdot \nabla_x P_1 \\ &= \sigma^2 (N-1) \int_{\mathbb{R}^3} \int_{S^2} (v - v_*) \cdot n \, P_2(x, v, x - \sigma n, v_*, t) \, dn \, dv_* \, , \end{aligned} \tag{1.17}$$

where dn denotes the surface measure on the unit sphere $S^2 = \{n \in \mathbb{R}^3 : |n| = 1\}$. A distinction between pre- and post-collisional situations can be achieved by the splitting

$$S^2 = S_+^2 \cup S_-^2 \, , \qquad S_\pm^2 = \{n \in S^2 : \pm(v - v_*) \cdot n > 0\} \, ,$$

where S_-^2 and S_+^2 correspond to pre- and post-collisional situations, respectively. The right hand side of (1.17) can then be written as $G - L$ with the *gain term*

$$G = \sigma^2(N-1)\int_{\mathbb{R}^3}\int_{S_+^2} |(v-v_*)\cdot n|\, P_2(x,v,x-\sigma n,v_*,t)\, dn\, dv_*\,,$$

and the *loss term*

$$L = \sigma^2(N-1)\int_{\mathbb{R}^3}\int_{S_-^2} |(v-v_*)\cdot n|\, P_2(x,v,x-\sigma n,v_*,t)\, dn\, dv_*\,, \quad (1.18)$$

Our main assumption will be concerned with P_2 in pre-collisional situations. Therefore we use the boundary condition (1.10) and rewrite the gain term as

$$G = \sigma^2(N-1)\int_{\mathbb{R}^3}\int_{S_+^2} |(v-v_*)\cdot n|\, P_2(x,v',x-\sigma n,v_*',t)\, dn\, dv_*\,, \quad (1.19)$$

with

$$v' = v - (n\cdot(v-v_*))n\,, \qquad v_*' = v_* + (n\cdot(v-v_*))n\,.$$

With the transformation $n \to -n$ in the loss term, (1.17) can be written as

$$\partial_t P_1 + v\cdot\nabla_x P_1 = \sigma^2(N-1)\int_{\mathbb{R}^3}\int_{S_+^2} |(v-v_*)\cdot n|$$
$$[P_2(x,v',x-\sigma n,v_*',t) - P_2(x,v,x+\sigma n,v_*,t)]\, dn\, dv_*\,. \quad (1.20)$$

The next step is an asymptotic limit. This requires an appropriate nondimensionalization. We assume a typical speed c_0 to be given (by the experimental situation we want to describe). A reasonable choice for a reference number density is $\varrho_0 = N/\mu(\Omega)$. The choice of the reference length l_0 will be discussed below. The nondimensionalization

$$P_1 \to \frac{\varrho_0}{c_0^3 N}P_1\,,\quad P_2 \to \frac{\varrho_0^2}{c_0^6 N^2}P_2\,,\quad x \to l_0 x\,,\quad v \to c_0 v\,,\quad t \to \frac{l_0}{c_0}t\,,$$

leads to the scaled version of (1.20):

$$\partial_t P_1 + v\cdot\nabla_x P_1 = \frac{1}{\mathrm{Kn}}\frac{N-1}{N}\int_{\mathbb{R}^3}\int_{S_+^2} |(v-v_*)\cdot n|$$
$$[P_2(x,v',x-\delta n,v_*',t) - P_2(x,v,x+\delta n,v_*,t)]\, dn\, dv_*\,, \quad (1.21)$$

with the dimensionless *Knudsen number*

$$\mathrm{Kn} = \frac{\lambda}{l_0}\,, \qquad \lambda = \frac{1}{\varrho_0\sigma^2}\,,$$

which can be interpreted as a scaled version of the *mean free path* λ, and with the second dimensionless parameter $\delta = \sigma/l_0$. The *kinetic scaling* $l_0 = \lambda$ produces $\mathrm{Kn} = 1$ and $\delta = \varrho_0\sigma^3$. This quantity is a measure for the fraction of space occupied by the spheres. The limits $N \to \infty$ and $\delta \to 0$ will be carried out. Because of the latter, we shall obtain a model for a *rarefied gas.* The derivation of the Boltzmann equation is then completed by the so called *molecular chaos assumption*: In (1.21) the two-particle probability distribution is evaluated for situations just before a collision. Since the collision between two particular particles (out of N) typically is a very rare event, the probability distributions of the two particles before their collision can be expected to be independent (as if they would have never seen each other before), with the consequence that, in the limit $N \to \infty$, the two-particle probability density can be written as the product of the one-particle densities:

$$P_1 \to f\,, \quad P_2(x, v', x - \delta n, v'_*, t) \to f' f'_*\,, \quad P_2(x, v, x + \delta n, v_*, t) \to f f_*\,,$$

as $N \to \infty$, $\delta \to 0$, where the abbreviations

$$f' = f(x, v', t)\,, \quad f'_* = f(x, v'_*, t)\,, \quad f = f(x, v, t)\,, \quad f_* = f(x, v_*, t)\,,$$

have been used.

Finally, since, in the limit $\delta \to 0$, the integrands in (1.18) and (1.19) are even functions of n, both integrations with respect to n can be extended to S^2 (and the resulting factor $1/2$ absorbed by a rescaling of x and t in (1.21)). With all these considerations applied in (1.21), the derivation of the *Boltzmann equation for hard spheres* is complete:

$$\partial_t f + v \cdot \nabla_x f = \int_{\mathbb{R}^3} \int_{S^2} |(v - v_*) \cdot n| (f' f'_* - f f_*) dn\, dv_*\,, \tag{1.22}$$

The distribution function f can also be interpreted as the *expected particle density* in phase space in terms of the *empirical distribution*:

$$f(x, v, t) = \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N \delta(x - x_i(t))\delta(v - v_i(t))\right] .$$

With a test function $\varphi \in C_0^\infty(\Omega \times \mathbb{R}^3)$, we verify

$$\int_{\Omega\times\mathbb{R}^3} f\varphi\, dx\, dv = \frac{1}{N}\sum_{i=1}^N \mathbb{E}[\varphi(x_i(t), v_i(t))] = \frac{1}{N}\sum_{i=1}^N \int_{\Omega\times\mathbb{R}^3} f\varphi\, dx\, dv\,.$$

The independence from i of the term under the sum is a consequence of the indistinguishability of the particles.

Finally, the *specular reflection* boundary condition (1.9)

$$f(x, v, t) = f(x, v - 2(\nu \cdot v)\nu, t)\,, \qquad \text{for } x \in \partial\Omega\,, \tag{1.23}$$

remains valid for the one-particle distribution, and the formulation of a (hopefully) well posed problem for the determination of f is completed by the prescription of initial conditions

$$f(x,v,0) = f_I(x,v)\,, \qquad \text{for } x \in \Omega\,,\ v \in \mathbb{R}^3\,. \tag{1.24}$$

Variants of the problem (1.22), (1.23) (see [CIP], [StR]) should be mentioned, although they will not be dealt with here. One point, where alternative models are adequate, is the so called *collision kernel* $|(v - v_*) \cdot n|$ in (1.22). The Boltzmann equation is also valid for other two-particle interactions than hard spheres. Other interaction potentials produce different collision kernels. A second possibility of variations is in the boundary conditions. Specular reflection models a perfectly smooth impenetrable container wall. Many other choices are possible. A mathematical well posedness condition comes from studying the left hand side of (1.22). Boundary conditions should allow to obtain the value of f at boundary points with incoming characteristics, i.e. points $(x,v) \in \partial\Omega \times \mathbb{R}^3$ with $\nu(x) \cdot v < 0$. The specular reflection boundary condition is obviously satisfied, even if it is only required for $\nu \cdot v < 0$. Then it prescribes the values of f on incoming characteristics in terms of the values on outgoing characteristics. Situations without boundary conditions, where Ω is either the whole space $\mathbb{R}^3$ or a torus are also of (at least mathematical) interest.

As a final remark we emphasize the subtleties in the above procedure. If for example, (v', v'_*) were changed back to (v, v_*) in (1.21), then the right hand side would formally converge to zero. If additionally, (v, v_*) in the loss term were replaced by (v', v'_*) (meaning that only post-collisional situations are used), then the limit would be (1.22) with a minus sign in front of the right hand side. Thus, the representation in terms of pre-collisional states is essential and the question, why this is the correct choice, is highly nontrivial (see [CIP]).

Another question is concerned with the molecular chaos assumption. Molecular chaos cannot be expected to be produced by the dynamics, it rather has to be assumed as a property of the initial state. A difficult step in making the above procedure rigorous is to prove *propagation of chaos,* i.e. that in the limit $N \to \infty$ this property is conserved in time, although it is destroyed by the dynamics for finite N.

Chapter 2

Formal properties and macroscopic limits

Rotational invariance and conservation laws

The right hand side of the Boltzmann equation (1.22) is called the *collision operator* and denoted by $Q(f, f)$, referring to the bilinear form

$$Q(f,g)(v) = \int_{\mathbb{R}^3} \int_{S^2} |(v - v_*) \cdot n|(f' g'_* - f g_*) dn\, dv_* \,,$$

where

$$\begin{aligned} f' = f(v')\,, \quad g'_* = g(v'_*)\,, \quad f = f(v)\,, \quad g_* = g(v_*)\,, \\ v' = v - (n \cdot (v - v_*))n\,, \quad v'_* = v_* + (n \cdot (v - v_*))n\,. \end{aligned}$$

A rotation is described by a matrix R, satisfying $R^{tr} = R^{-1}$. With the notation $f_R(x, v, t) = f(Rx, Rv, t)$ it is straightforward to check the rotational invariance of the terms in the Boltzmann equation:

$$(\partial_t f)_R = \partial_t f_R\,, \qquad (v \cdot \nabla_x f)_R = v \cdot \nabla_x f_R\,, \qquad Q(f, f)_R = Q(f_R, f_R)\,.$$

We deduce *rotational invariance* of the equation in the sense that, if f satisfies the Boltzmann equation then the same is true for f_R. In other words, the Boltzmann equation is independent from the choice of Euclidean coordinates.

We recall from the previous section that the above transformation from pre-collisional velocities (v, v_*) to post-collisional velocities (v', v'_*) preserves the measure in $\mathbb{R}^6$ and leaves the collision kernel invariant (because of $(v' - v'_*) \cdot n = -(v - v_*) \cdot n$). Therefore, the weak formulation of the collision operator can be conveniently rewritten as

$$\begin{aligned} \int_{\mathbb{R}^3} Q(f,f)\varphi\, dv &= \frac{1}{2} \int_{\mathbb{R}^6} \int_{S^2} |(v - v_*) \cdot n|(f' f'_* - f f_*)(\varphi + \varphi_*) dn\, dv_*\, dv \\ &= \frac{1}{4} \int_{\mathbb{R}^6} \int_{S^2} |(v - v_*) \cdot n|(f' f'_* - f f_*)(\varphi + \varphi_* - \varphi' - \varphi'_*) dn\, dv_*\, dv\,, \quad (2.1) \end{aligned}$$

whereas $\varphi(v)$ is a continuous test function and we assume that f and φ are such that all integrals exist. Because of conservation of momentum and of energy in two-particle collisions, the five functions

$$\varphi(v) = 1,\, v_1,\, v_2,\, v_3,\, |v|^2\,, \tag{2.2}$$

called *collision invariants*, satisfy

$$\varphi(v) + \varphi(v_*) - \varphi(v') - \varphi(v'_*) = 0\,, \qquad \text{for all } (v, v_*, n) \in \mathbb{R}^6 \times S^2\,, \tag{2.3}$$

and therefore $\int_{\mathbb{R}^3} Q(f,f)\varphi\, dv = 0$. It can be shown that linear combinations of the collision invariants (2.2) are the only functions satisfying (2.3), a question we shall return to in the following section.

Multiplication of the Boltzmann equation (1.22) with a collision invariant and integration with respect to v gives a *local conservation law*:

$$\partial_t \int_{\mathbb{R}^3} \varphi f\, dv + \nabla_x \cdot \int_{\mathbb{R}^3} v\varphi f\, dv = 0\,.$$

The densities corresponding to the collision invariants (2.2) have physical meanings:

$$\begin{aligned}
\text{Mass density:} \qquad & \varrho_f(x,t) = \int_{\mathbb{R}^3} f(x,v,t)dv\,, \\
\text{Momentum density:} \qquad & \varrho_f(x,t)u_f(x,t) = \int_{\mathbb{R}^3} v\, f(x,v,t)dv\,, \\
\text{Energy density:} \qquad & E_f(x,t) = \int_{\mathbb{R}^3} \frac{|v|^2}{2} f(x,v,t)dv\,,
\end{aligned}$$

with the mean velocity $u_f(x,t)$. The corresponding local conservation laws are

$$\partial_t \varrho_f + \nabla_x \cdot (\varrho_f u_f) = 0\,, \tag{2.4}$$

$$\partial_t(\varrho_f u_f) + \nabla_x \cdot \int_{\mathbb{R}^3} v \otimes v\, f\, dv = 0\,, \tag{2.5}$$

$$\partial_t E_f + \nabla_x \cdot \int_{\mathbb{R}^3} v \frac{|v|^2}{2}\, f\, dv = 0\,. \tag{2.6}$$

These are the basic equations of continuum mechanics. In general, the momentum and energy fluxes cannot be computed in terms of ϱ_f, u_f, and E_f. If this were the case (i.e. for a complete set of *constitutive relations*), the conservation laws would be a closed system.

Three more local conservation laws follow from the conservation of momentum and from the symmetry of the momentum flux tensor. Taking the vector product of (2.5) with x, conservation of *angular momentum*,

$$\partial_t(\varrho(x \times u)) + \nabla_x \cdot \int_{\mathbb{R}^3} v \otimes (x \times v) f\, dv = 0\,,$$

follows from $\nabla_x \cdot (v \otimes (x \times v)) = 0$.

If a local conservation law is integrated with respect to x and the divergence theorem is applied, the normal component of the flux has to be evaluated along the boundary $\partial\Omega$. Using the specular reflection boundary condition

$$f(x,v,t) = f(x,v',t)\,, \quad v' = v - 2(\nu(x)\cdot v)\nu(x)\,, \qquad \text{for } x \in \partial\Omega\,, \tag{2.7}$$

we obtain for the mass flux

$$\begin{aligned} \nu\cdot(\varrho u) &= \int_{\nu\cdot v>0} (\nu\cdot v) f\,dv + \int_{\nu\cdot v<0} (\nu\cdot v) f'\,dv \\ &= \int_{\nu\cdot v>0} (\nu\cdot v) f\,dv - \int_{\nu\cdot v'>0} (\nu\cdot v') f'\,dv' = 0\,. \end{aligned} \tag{2.8}$$

This implies the *global conservation* of mass:

$$\int_\Omega \varrho_f(x,t)dx = \int_\Omega \varrho_{f_I}(x)dx\,, \qquad \text{for all } t>0\,.$$

As a consequence of $|v'|^2 = |v|^2$, an analogous computation shows that the normal component of the energy flux also vanishes along $\partial\Omega$, giving global conservation of energy:

$$\int_\Omega E_f(x,t)dx = \int_\Omega E_{f_I}(x)dx\,, \qquad \text{for all } t>0\,.$$

The collisions with the container wall do not conserve momentum. Thus, for the specular reflection boundary conditions, momentum is not a globally conserved quantity. However, angular momentum can be globally conserved. Suppose for example, that the container is a ball, i.e. $\Omega = \{x \in \mathbb{R}^3 : |x| < R\}$. This is equivalent to the requirement $x = R\nu(x)$ for all $x \in \partial\Omega$. Then $x \times \nu = 0$ and the angular momentum of a particle is conserved by specular reflection:

$$x \times v' = x \times v - 2(\nu\cdot v)x\times\nu = x\times v\,.$$

Then again the argument used for the normal component of the mass flux can be repeated with the consequence of global conservation of angular momentum:

$$\int_\Omega \varrho_f(x,t)x\times u_f(x,t)dx = \int_\Omega \varrho_{f_I}(x)x\times u_{f_I}(x)dx\,, \qquad \text{for all } t>0\,.$$

Similarly it can be shown that for cylinder symmetric containers the component of angular momentum in the direction of the rotation axis is globally conserved.

Entropy – the H-theorem

With the choice $\varphi = \log f$ in the weak formulation (2.1) of the collision operator, we arrive at

$$\begin{aligned} D(f) &= -\int_{\mathbb{R}^3} Q(f,f)\log f\,dv \\ &= \frac{1}{4}\int_{\mathbb{R}^6}\int_{S^2} |(v-v_*)\cdot n|(f'f'_* - ff_*)\log\frac{f'f'_*}{ff_*}\,dn\,dv_*\,dv\,, \end{aligned} \tag{2.9}$$

The important observation (already made by Boltzmann) is that this quantity has a sign. Because of $(a-b)\log(a/b) \ge 0$, it is nonnegative. Multiplication of the Boltzmann equation by $\log f$ and integration with respect to v gives

$$\partial_t \int_{\mathbb{R}^3} f\log f\,dv + \nabla_x \cdot \int_{\mathbb{R}^3} vf\log f\,dv = -D(f) \le 0\,. \tag{2.10}$$

With the specular reflection boundary conditions, it can be shown analogously to the previous section that the normal component of the flux vanishes along the boundary. As a consequence, the total (physical) *entropy* given by *Boltzmann's H-functional*

$$H(t) = -\int_{\Omega\times\mathbb{R}^3} f(x,v,t)\log f(x,v,t)\,dv\,dx$$

is nondecreasing in time, which is the statement of Boltzmann's H-theorem. It shows that the Boltzmann equation violates mechanical reversibility, since the value of the H-functional is invariant under the transformation $v \to -v$. Since $f\log f$ is convex, the decay of the *mathematical entropy* $-H$ (simply called 'entropy' in the following) can be used for the control of solutions of the Boltzmann equation.

Because of the definiteness of the expression $(a-b)\log(a/b)$, $Q(f,f) = 0$ implies that $\log f$ has to satisfy (2.3). We claimed above (without giving a proof) that this implies that $\log f$ is a linear combination of the collision invariants. The proof presented here is due to Perthame [15].

Lemma 1 *Let the function $f(v) \ge 0$ satisfy $Q(f,f) = 0$ and*

$$\int_{\mathbb{R}^3} (1+|v|^2)f(v)dv < \infty\,.$$

Then there exist $\varrho \ge 0$, $T > 0$ and $u \in \mathbb{R}^3$, such that

$$f(v) = \mathcal{M}_{\varrho,u,T}(v) := \frac{\varrho}{(2\pi T)^{3/2}}\exp\left(-\frac{|v-u|^2}{2T}\right)\,. \tag{2.11}$$

Beweis: By the boundedness assumption, there exist

$$\varrho = \int_{\mathbb{R}^3} f(v)dv \geq 0 \quad \text{and} \quad u = \frac{1}{\varrho}\int_{\mathbb{R}^3} vf(v)dv \in \mathbb{R}^3 ,$$

with u arbitrary if $\varrho = 0$, in which case the proof is finished. So $\varrho > 0$ is assumed for the following. We define the normalized distribution $f_0(v) := f(v+u)/\varrho$, satisfying

$$\int_{\mathbb{R}^3} f_0(v)dv = 1 , \qquad \int_{\mathbb{R}^3} vf_0(v)dv = 0 .$$

Then the Fourier transform

$$g(k) = \int_{\mathbb{R}^3} f_0(v)e^{ik\cdot v}dv$$

satisfies $g(0) = 1$ and $\nabla g(0) = 0$.

A straightforward computation shows $0 = Q(f,f)(v+u) = \varrho^2 Q(f_0,f_0)(v)$. As mentioned above, the expression $(a-b)\log(a/b) \geq 0$ is definite, whence, by (2.9),

$$f_0(v)f_0(v_*) = f_0(v')f_0(v'_*) \quad \text{for all } v, v_* \in \mathbb{R}^3 ,\ n \in S^2 ,$$

follows. Fourier transformation with respect to v and v_* of this equation gives

$$\begin{aligned} g(k)g(k_*) &= \int_{\mathbb{R}^3} f_0(v')f_0(v'_*)e^{i(k\cdot v + k_*\cdot v_*)}dv\,dv_* \\ &= \int_{\mathbb{R}^3} f_0(v)f_0(v_*)e^{i(k\cdot v + k_*\cdot v_* - (k-k_*)\cdot n\,(v-v_*)\cdot n)}dv\,dv_* , \end{aligned}$$

by the measure preserving change of variables $(v,v_*) \to (v',v'_*)$. Now we choose $n = n_0 + \eta$ with $|n_0| = 1$, $(k-k_*)\cdot n_0 = 0$, and with η small, implying $(k-k_*)\cdot\eta \neq 0$. Using the $O(\eta)$-term in the Taylor expansion

$$e^{-i(k-k_*)\cdot n\,(v-v_*)\cdot n} = 1 - i(k-k_*)\cdot\eta\,(v-v_*)\cdot n_0 + O(|\eta|^2) ,$$

the above equation implies

$$n_0 \cdot (g(k_*)\nabla g(k) - g(k)\nabla g(k_*)) = 0 . \tag{2.12}$$

The Taylor expansion is justified since the boundedness assumption on f implies $g \in C^2(\mathbb{R}^3)$. Setting $k_* = 0$ gives $n_0 \cdot \nabla g(k) = 0$ for all n_0 orthogonal to k. Therefore ∇g is parallel to k and, thus, $g(k) = \hat{g}(|k|^2)$. Then (2.12) implies

$$(n_0 \cdot k)[\hat{g}(|k_*|^2)\hat{g}'(|k|^2) - \hat{g}(|k|^2)\hat{g}'(|k_*|^2)] = 0 .$$

For every choice of $r = |k|^2$ and $r_* = |k_*|^2$, k and k_* can be chosen not collinear and, thus, there exists an appropriate n_0 such that $n_0 \cdot k \neq 0$. Therefore $\hat{g}(r)$ has to be an exponential function implying $g(k) = e^{-T|k|^2/2}$ with a positive T (for Fourier-transformability). As a consequence, f_0 is the Gaussian $f_0(v) = \mathcal{M}_{1,0,T}(v)$, and the proof is complete. ∎

In the context of kinetic transport theory, the Gaussian (2.11) is called the *Maxwellian equilibrium distribution*, which is the only distribution which is not changed by the collision effects. The use of the symbols ϱ and u is compatible with our earlier notation, since a straightforward computation (using $\int_{\mathbb{R}^3} \exp(-|w|^2/2)dw = (2\pi)^{3/2}$) shows that they are the mass density and, respectively, mean velocity of the Maxwellian. The energy density of the Maxwellian is given by

$$\int_{\mathbb{R}^3} \frac{|v|^2}{2} \mathcal{M}_{\varrho,u,T}(v)dv = \frac{\varrho|u|^2}{2} + \frac{3}{2}\varrho T\,,$$

which reflects the (in continuum mechanics) usual splitting into macroscopic kinetic energy and internal energy and motivates to call T the *temperature.* Actually, from the microscopic point of view also internal energy is kinetic energy (due to velocity fluctuations).

The Euler limit

A typical value for the mean free path, used as the reference length in the derivation of the Boltzmann equation, is 10^{-4}cm $= 1\mu$m (for air at atmospheric pressures). This is very small compared to the typical lengths scales in many applications. We therefore assume in this section that the relevant length scales are much larger and introduce a rescaling $x \to x/\varepsilon$, where ε is a small positive dimensionless parameter. Actually ε is the Knudsen number in the new scaling. As a consequence, there are two time scales relevant for the dynamics: the original *kinetic* scale and a *macroscopic scale*, introduced by $t \to t/\varepsilon$. The rescaled Boltzmann equation then has the form

$$\varepsilon\partial_t f + \varepsilon v \cdot \nabla_x f = Q(f,f)\,. \tag{2.13}$$

In a formal asymptotic analysis, we assume convergence as $\varepsilon \to 0$ of the solution: $\lim_{\varepsilon\to 0} f = f_0$ (without specification of a topology). Obviously, the limit has to satisfy $Q(f_0, f_0) = 0$ and, thus, there exist $\varrho(x,t)$, $u(x,t)$, and $T(x,t)$ such that

$$f_0(x,v,t) = \mathcal{M}_{\varrho(x,t),u(x,t),T(x,t)}(v)\,.$$

Equations for ϱ, u, and T are found by multiplication of (2.13) by the collision invariants and by integration with respect to v. The resulting equations can be divided by ε, and therefore provide additional information as $\varepsilon \to 0$.

Actually we obtain the conservation laws (2.4)–(2.6) with f replaced by f_0. This allows for computing the momentum and energy fluxes in terms of ϱ, u, and T. The result are the *compressible Euler equations for an ideal gas:*

$$
\begin{aligned}
&\partial_t \varrho + \nabla_x \cdot (\varrho u) = 0\,, && (2.14)\\
&\partial_t(\varrho u) + \nabla_x \cdot (\varrho u \otimes u) + \nabla_x p = 0\,, && (2.15)\\
&\partial_t \left(\varrho \left(|u|^2/2 + e\right)\right) + \nabla_x \cdot \left(\varrho u \left(|u|^2/2 + e\right) + up\right) = 0\,, && (2.16)
\end{aligned}
$$

where the *specific internal energy* and the *pressure* satisfy $e = 3T/2$ and, respectively, the *ideal gas law* $p = \varrho T$.

The Euler equations can be understood as a wave propagation model. The wave speeds are determined by rewriting the system in the form

$$
\partial_t U + \sum_{i=1}^{3} A_i(U)\partial_{x_i} U = 0\,,
$$

where the vector $U(x,t) \in \mathbb{R}^5$ is composed of the conserved quantities or other convenient sets of variables (e.g., $U = (\varrho, u, T)$). The eigenvalues of the matrix A_i,

$$
u_i\,, \quad u_i + c\,, \qquad u_i - c\,, \qquad \text{with } c = \sqrt{5T/3}\,,
$$

(which are independent from the choice of U) can then be interpreted as speeds of waves in the x_i-direction. The first eigenvalue has multiplicity 3. Waves corresponding to the second and third eigenvalues are called *sound waves*, and c is the *speed of sound.* The relative size of these speeds is measured by the *Mach number* $\mathrm{Ma} = |u|/c$.

There is no completely satisfactory solution theory for initial value problems for the three-dimensional compressible Euler equations. This is partially due to the fact that in general smooth solutions only exist for finite time and can only be extended as weak solutions, possibly containing jump discontinuities *(shocks).* These weak solutions are typically not unique, but uniqueness can be recovered by an additional requirement: In terms of the macroscopic scaling of this section, the entropy balance equation (2.10) takes the form

$$
\partial_t \int_{\mathbb{R}^3} f \log f \, dv + \nabla_x \cdot \int_{\mathbb{R}^3} v f \log f \, dv = -\frac{1}{\varepsilon} D(f) \le 0\,.
$$

It is hard to tell, what the limit of the right hand side as $\varepsilon \to 0$ might be. However, its sign will be preserved in the limit, leading to the *macroscopic entropy inequality*

$$
\partial_t \left(\varrho \log \frac{\varrho}{T^{3/2}}\right) + \nabla_x \cdot \left(\varrho u \log \frac{\varrho}{T^{3/2}}\right) \le 0\,.
$$

A straightforward computation shows that for smooth solutions of (2.14)–(2.16) this inequality becomes an equality. However, for weak solutions it poses an additional criterion (called the *entropy condition*) making them unique.

Concerning boundary conditions, passing to the limit in (2.8) gives the zero-flux condition $u(x,t) \cdot \nu(x) = 0$ for $x \in \partial\Omega$.

The macroscopic limit of this section has been rigorously verified so far (2010) only for situations where the limiting solutions of the Euler equations are smooth and where no boundary is present (see e.g. [4]).

The incompressible Navier-Stokes limit

Dynamics on a different (slower) time scale than in the preceding section can be expected close to steady states of the Euler equations, e.g. for vanishing velocity and pressure gradient. Therefore, in this section small perturbations of a steady state equilibrium solution will be considered, where the equilibrium is a Maxwellian with constant density and temperature and vanishing velocity. With the main perturbation only in the mean velocity, the evolution of a distribution satisfying

$$f(x,v,0) = \mathcal{M}_{1,\delta u_0(x),1}(v)\,, \tag{2.17}$$

will be considered, where, without loss of generality, the constant values of the scaled density and temperature have been set equal to one. If the rescaled initial mean velocity u_0 takes moderate values, the small parameter δ measures the size of the Mach number $\mathrm{Ma} = \delta|u_0|\sqrt{3/5}$ (since the speed of sound is $c = \sqrt{5/3}$).

If the macroscopic scaling (2.13) of the Boltzmann equation is used with the initial condition (2.17), one could try to compute a two-parameter (ε and δ) asymptotic expansion of the solution. Since, for $\delta = 0$,

$$M(v) := \mathcal{M}_{1,0,1}(v)$$

is the exact solution of the problem, no terms of the pure orders ε^k will appear. On the other hand, the problem with $\varepsilon = 0$ is trivial as well, since the exact solution

$$\mathcal{M}_{1,\delta u_0,1} = M\left(1 + \delta u_0 \cdot v + \delta^2/2\left((u_0 \cdot v)^2 - |u|^2\right) + O(\delta^3)\right)$$

is known again.

If the initial condition is substituted in the left hand side of the Boltzmann equation, a term of the order $\varepsilon\delta$ arises, which can be balanced by the right hand side, if a term of this order occurs in the Taylor expansion of the solution. Since, by these arguments, terms of the orders δ^2 and $\varepsilon\delta$ are necessary, the question of the relative sizes of ε and δ arises and, obviously, the choice

$$\varepsilon = \delta \ll 1$$

corresponds to a *significant limit.* We shall therefore consider the initial condition

$$f(x,v,0) = \mathcal{M}_{1,\varepsilon u_0(x),1}(v)\,, \tag{2.18}$$

and look for an asymptotic expansion of the form

$$f(x,v,t) = M(v)\left(1 + \varepsilon u(x,t)\cdot v + \varepsilon^2 g_1(x,v,t) + O(\varepsilon^3)\right)\,.$$

Since the momentum flux of the $O(\varepsilon)$-term vanishes, the appropriate time scale for the momentum conservation is achieved by the rescaling $t \to t/\varepsilon$ in (2.13):

$$\varepsilon^2\partial_t f + \varepsilon v\cdot\nabla_x f = Q(f,f)\,.$$

Motivated by the asymptotic ansatz above, the new unknown g is introduced by $f = M(1+\varepsilon g)$, leading to the equation

$$\varepsilon^2\partial_t g + \varepsilon v\cdot\nabla_x g = \mathcal{L}_M(g) + \varepsilon Q_M(g,g)\,, \tag{2.19}$$

with

$$\mathcal{L}_M(g) = \frac{1}{M}\Big(Q(M,Mg) + Q(Mg,M)\Big)\,, \qquad Q_M(g,g) = \frac{1}{M}Q(Mg,Mg)\,.$$

The initial condition becomes

$$g(x,v,0) = \frac{\mathcal{M}_{1,\varepsilon u_0(x),1}(v) - M(v)}{\varepsilon M(v)} = u_0(x)\cdot v + O(\varepsilon)\,.$$

The linearized collision operator and the quadratic remainder can be written as

$$\begin{aligned}
\mathcal{L}_M(g) &= \int_{\mathbb{R}^3}\int_{S^2} |(v-v_*)\cdot n| M_*(g' + g'_* - g - g_*)dn\,dv_*\,,\\
Q_M(g,g) &= \int_{\mathbb{R}^3}\int_{S^2} |(v-v_*)\cdot n| M_*(g'g'_* - gg_*)dn\,dv_*\,,
\end{aligned}$$

where the equilibrium identity $MM_* = M'M'_*$ has been used. The linearized collision operator satisfies an H-theorem similarly to Q:

$$-\int_{\mathbb{R}^3}\mathcal{L}_M(g)gM\,dv = \frac{1}{4}\int_{\mathbb{R}^6}\int_{S^2}|(v-v_*)\cdot n|MM_*(g'+g'_*-g-g_*)^2dn\,dv_*\,dv$$

The proof is analogous to (2.1), and the symmetry property

$$\int_{\mathbf{R}^3} \mathcal{L}_M(g)hM\,dv = \int_{\mathbf{R}^3} \mathcal{L}_M(h)gM\,dv$$

is shown in the same way. The H-theorem shows that the null space of $\mathcal{L}_M$ consists of the collision invariants:

$$\mathcal{N}(\mathcal{L}_M) = \mathrm{span}\{1, v_1, v_2, v_3, |v|^2\}\,.$$

Not quite as simple (and omitted here, see [1]) is the proof of the coercivity estimate

$$-\int_{\mathbf{R}^3} \mathcal{L}_M(g)gM\,dv \geq C\int_{\mathbf{R}^3} (g - \Pi g)^2 M\,dv\,,$$

where Π is the orthogonal (with respect to $L^2(M\,dv)$) projection to $\mathcal{N}(\mathcal{L}_M)$. These results make it plausible (see [StR] for the complete proof) that for $h \in \mathcal{N}(\mathcal{L}_M)^\perp$, the equation $\mathcal{L}_M(g) = h$ has a unique solution $g \in \mathcal{N}(\mathcal{L}_M)^\perp$.

The initial condition suggests to look for an asymptotic expansion of the form

$$g = g_0 + \varepsilon g_1 + O(\varepsilon^2)\,, \qquad \text{with } g_0(x,v,t) = u(x,t)\cdot v\,.$$

The leading order term satisfies the necessary condition $g_0 \in \mathcal{N}(\mathcal{L}_M)$, and the initial condition requires

$$u(x,0) = u_0(x)\,. \tag{2.20}$$

Comparing coefficients of ε in (2.19) gives

$$v\cdot\nabla_x g_0 = \mathcal{L}_M(g_1) + Q_M(g_0,g_0)\,, \tag{2.21}$$

which we consider as an equation for g_1. Its solution will be facilitated by the following result (see, e.g. [8]; here we provide a very simple proof):

Lemma 2 *Let $g_0 \in \mathcal{N}(\mathcal{L}_M)$. Then*

$$Q_M(g_0,g_0) = -\mathcal{L}_M(g_0^2/2)\,.$$

Beweis: The result is a straightforward consequence of the identity

$$\begin{aligned} Q_M(g_0,g_0) \;&=\; -\frac{1}{2}\int_{\mathbf{R}^3}\int_{S^2} |(v-v_*)\cdot n| M_*\Big(-(g_0' + g_{0*}')^2 + (g_0+g_{0*})^2 \\ &\qquad + (g_0')^2 + (g_{0*}')^2 - (g_0)^2 - (g_{0*})^2\Big) dn\,dv_*\,. \end{aligned}$$

∎

For solvability of (2.21), the left hand side has to be in $\mathcal{N}(\mathcal{L}_M)^\perp$:

$$\int_{\mathbf{R}^3} (v\cdot\nabla_x g_0) M dv = \int_{\mathbf{R}^3} v(v\cdot\nabla_x g_0) M dv = \int_{\mathbf{R}^3} |v|^2 (v\cdot\nabla_x g_0) M dv = 0\,.$$

A straightforward computation shows that for $g_0 = u\cdot v$ these requirements are equivalent to the *incompressibility condition*

$$\nabla_x \cdot u = 0\,. \tag{2.22}$$

Assuming this, the left hand side of (2.21) can be rewritten as

$$v\cdot\nabla_x(u\cdot v) = A(v) : \nabla_x u\,, \qquad \text{with } A(v) = v\otimes v - \frac{|v|^2}{3} I\,.$$

Each entry of the matrix A is in $\mathcal{N}(\mathcal{L}_M)^\perp$. With these observations, (2.21) implies

$$g_1 = \overline{g}_1 + \frac{(u\cdot v)^2}{2} + \hat{A} : \nabla_x u\,,$$

where the matrix $\hat{A}(v)$ is the unique solution of $\mathcal{L}_M(\hat{A}) = A$ with entries in $\mathcal{N}(\mathcal{L}_M)^\perp$, and $\overline{g}_1 \in \mathcal{N}(\mathcal{L}_M)$. As a consequence of the rotational symmetry of $\mathcal{L}_M$ (following from the rotational symmetry of Q and of M), there exists a scalar function $a(|v|)$ such that $\hat{A}(v) = A(v)a(|v|)$ (for a proof see [6]).

The final step in the asymptotics is to pass to the limit in the momentum balance equation derived by multiplication of (2.19) by vM, integration with respect to v, and division by ε^2:

$$\partial_t \int_{\mathbf{R}^3} vgM dv + \frac{1}{\varepsilon}\nabla_x \cdot \int_{\mathbf{R}^3} v\otimes vgM dv = 0\,. \tag{2.23}$$

In the limit, this gives

$$\partial_t \int_{\mathbf{R}^3} vg_0 M dv + \nabla_x \cdot \int_{\mathbf{R}^3} v\otimes vg_1 M dv = 0\,.$$

It is easily seen that

$$\int_{\mathbf{R}^3} vg_0 M dv = u\,, \qquad \int_{\mathbf{R}^3} v\otimes v\left(\overline{g}_1 + \frac{(u\cdot v)^2}{2}\right) M dv = u\otimes u + p_1 I\,,$$

where $p_1(x,t)$ is a scalar function depending on $\overline{g}_1$. For the computation of the remaining term

$$\int_{\mathbf{R}^3} v\otimes v(\hat{A} : \nabla_x u) M dv = \int_{\mathbf{R}^3} v\otimes v((v\otimes v) : \nabla_x u) a M dv$$

we define

$$\mu = -\int_{\mathbf{R}^3} v_i^2 v_j^2 a M dv\,,$$

where $i \neq j$ can be chosen arbitrarily. With this notation

$$\int_{\mathbf{R}^3} v \otimes v(\hat{A} : \nabla_x u) M dv = -\mu(\nabla_x u + (\nabla_x u)^{tr}) + p_2 I$$

holds with another scalar function p_2. Combining our results leads to the *incompressible Navier-Stokes equations*

$$\partial_t u + (u \cdot \nabla_x) u + \nabla_x p = \mu \Delta_x u \,, \qquad (2.24)$$

where the differential operators $u \cdot \nabla_x$ and Δ_x are applied componentwise, and the *pressure* $p = p_1 + p_2$ can be seen as Lagrange multiplier for the incompressibility condition (2.22). We observe that with $h(v) = v_i v_j a(|v|)$, the *viscosity* satisfies

$$\mu = -\int_{\mathbf{R}^3} \mathcal{L}_M(h) h M dv > 0$$

by the coercivity of $\mathcal{L}_M$, since $h \notin \mathcal{N}(\mathcal{L}_M)$.

Finally, boundary conditions will be derived. It is easily seen that g, such as f, satisfies the specular reflection boundary conditions (2.7):

$$g(x,v,t) = g(x,v',t) \,, \quad v' = v - 2(\nu(x)\cdot v)\nu(x) \,, \qquad \text{for } x \in \partial\Omega \,.$$

Obviously, this holds for $g_0 = u.v$, if the zero-flux boundary condition

$$u \cdot \nu = 0 \,, \qquad \text{on } \partial\Omega \,, \qquad (2.25)$$

is satisfied. Since the Navier-Stokes equations are a parabolic system for the velocity, this condition, only prescribing one component of the velocity vector on the boundary, is not sufficient. Additional information will be derived from a weak formulation of the momentum conservation equation (2.23). Multiplication of (2.23) by a smooth vector field $w(x)$, satisfying $w \cdot \nu = 0$ on $\partial\Omega$ and $\nabla_x \cdot w = 0$, and integration (by parts) with respect to x gives

$$\frac{d}{dt}\int_{\Omega\times\mathbf{R}^3} w.v\, gM\, dv\, dx + \frac{1}{\varepsilon}\int_{\partial\Omega\times\mathbf{R}^3} (w\cdot v)(\nu\cdot v) gM\, dv\, d\sigma$$
$$-\frac{1}{\varepsilon}\int_{\Omega\times\mathbf{R}^3} \nabla_x w : (v\otimes v) gM\, dv\, dx = 0$$

The coordinate change $v \to v'$ in the boundary integral, together with the facts $w \cdot v = w \cdot v'$ (because of $w \cdot \nu = 0$), $\nu \cdot v' = -\nu \cdot v$ and the boundary condition for g, implies that this integral vanishes, and the limit $\varepsilon \to 0$ gives

$$\int_\Omega w \cdot \partial_t u\, dx = \int_\Omega \nabla_x w : \left[u \otimes u - \mu \left(\nabla_x u + (\nabla_x u)^{tr} \right) \right] dx \,.$$

An integration by parts on the right hand side and (2.25) imply that the viscous stress $\mu\,(\nabla_x u + (\nabla_x u)^{tr}) \cdot \nu$ on the boundary has to be in the direction of the normal vector (no shear force), which can be written as the *Navier boundary condition*

$$\nu \times \Big[\Big(\nabla_x u + (\nabla_x u)^{tr}\Big) \cdot \nu\Big] = 0\,, \qquad \text{on } \partial\Omega\,. \tag{2.26}$$

Initial-boundary value problems for the Navier-Stokes equations with the boundary condtions (2.25), (2.26) can be expected to be well posed problems.

Chapter 3

Velocity averaging

One of the main difficulties in the analysis of the Boltzmann equation is to obtain enough compactness to be able to deal with the nonlinear collision operator. It is important to observe that this nonlinearity is nonlocal in the velocity and local in position and time. Compactness is therefore only necessary with respect to the latter variables. It is provided by velocity averaging lemmas, which are a consequence of dispersion properties of the free streaming operator. This chapter mostly follows [9], neglecting some of the proofs.

Averaging lemmas in L^2

For L^2-based Sobolev spaces of functions defined on the whole space, we use the norm

$$\|u\|_{H^s(\mathbb{R}^3_x)} = \||\xi|^s \mathcal{F}u\|_{L^2(\mathbb{R}^3_\xi)}\,, \qquad s > 0\,,$$

where $(\mathcal{F}u)(\xi)$ is the Fourier transform of $u(x)$. The following result shows compactness with respect to the position variable of velocity averages.

Satz 1 *Let $f \in L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v \times \mathbb{R}_t)$ be a solution of*

$$\varepsilon\partial_t f + v \cdot \nabla_x f = S \in L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v \times \mathbb{R}_t)\,,$$

with $\varepsilon \geq 0$, and let $\varphi \in L^\infty(\mathbb{R}^3_v)$ have compact support. Then

$$m(x,t) := \int_{\mathbb{R}^3} f(x,v,t)\varphi(v)dv$$

satisfies $m \in L^2(\mathbb{R}_t;\, H^{1/2}(\mathbb{R}^3_x))$ and

$$\|m\|_{L^2(\mathbb{R}_t;\, H^{1/2}(\mathbb{R}^3_x))} \leq C\|f\|^{1/2}_{L^2(\mathbb{R}^3_x\times\mathbb{R}^3_v\times\mathbb{R}_t)}\|S\|^{1/2}_{L^2(\mathbb{R}^3_x\times\mathbb{R}^3_v\times\mathbb{R}_t)}\,,$$

with C only depending on φ (and not on ε).

Beweis: Introducing the partial Fourier transforms $\hat{f}(\xi, v, \tau)$, $\hat{S}(\xi, v, \tau)$ of f and, respectively, S with respect to position and time,

$$i(\varepsilon\tau + v \cdot \xi)\hat{f} = \hat{S}$$

holds. The Fourier transform of the average can then be estimated by

$$\begin{aligned}
|\hat{m}| &\leq \int_{|\varepsilon\tau+v\cdot\xi|<\alpha} |\hat{f}\varphi|\,dv + \int_{|\varepsilon\tau+v\cdot\xi|\geq\alpha} \frac{|\hat{S}\varphi|}{|\varepsilon\tau + v\cdot\xi|}\,dv \\
&\leq \|\hat{f}\|_{L^2_v} \left(\int_{|\varepsilon\tau+v\cdot\xi|<\alpha} \varphi^2\,dv\right)^{1/2} + \|\hat{S}\|_{L^2_v} \left(\int_{|\varepsilon\tau+v\cdot\xi|\geq\alpha} \frac{\varphi^2}{(\varepsilon\tau + v\cdot\xi)^2}\,dv\right)^{1/2},
\end{aligned}$$

where the Cauchy-Schwarz inequality has been used, and $\alpha > 0$ will be chosen later. The inequality $|\varepsilon\tau + v \cdot \xi| < \alpha$ defines a region between to parallel planes at the distance $2\alpha/|\xi|$ in $\mathbb{R}^3_v$. The intersection of this region with the support of φ therefore has a volume, which can be estimated by $C\alpha/|\xi|$, where from now on C denotes constants only depending on φ. These observations and the boundedness of φ imply

$$\int_{|\varepsilon\tau+v\cdot\xi|<\alpha} \varphi^2\,dv \leq \frac{C\alpha}{|\xi|}\,.$$

On the other hand, the coordinate change $v \to (y, v^\perp)$, defined by

$$v = \left(y - \frac{\varepsilon\tau}{|\xi|}\right)\frac{\xi}{|\xi|} + v^\perp\,,$$

implies

$$\int_{|\varepsilon\tau+v\cdot\xi|\geq\alpha} \frac{\varphi^2}{(\varepsilon\tau + v\cdot\xi)^2}\,dv \leq C\int_{|y|\geq\alpha/|\xi|} \frac{dy}{y^2|\xi|^2} = \frac{2C}{\alpha|\xi|}\,.$$

Collecting our results, we obtain

$$|\xi|^{1/2}|\hat{m}| \leq C\left(\alpha\|\hat{f}\|_{L^2_v} + \frac{1}{\alpha}\|\hat{S}\|_{L^2_v}\right),$$

and, as a consequence,

$$\|m\|_{L^2_t(H^{1/2}_x)} \leq C\left(\alpha\|f\|_{L^2_{x,v,t}} + \frac{1}{\alpha}\|S\|_{L^2_{x,v,t}}\right),$$

where the Plancherel identity has been used. The proof is completed by the choice

$$\alpha = \sqrt{\frac{\|S\|_{L^2_{x,v,t}}}{\|f\|_{L^2_{x,v,t}}}}\,.$$

■

Compactness with respect to time will be a consequence of technical estimates:

Lemma 3 *With the notation of Theorem 1,*

$$a) \qquad \int_{|\varepsilon\tau+v\cdot\xi|<\alpha} \varphi^2\,dv \le \frac{C\alpha}{\sqrt{\varepsilon^2\tau^2+|\xi|^2}}\,,$$

$$b) \qquad \int_{|\varepsilon\tau+v\cdot\xi|\ge\alpha} \frac{\varphi^2}{(\varepsilon\tau+v\cdot\xi)^2}\,dv \le \frac{C}{\alpha\sqrt{\varepsilon^2\tau^2+|\xi|^2}}\,,$$

holds.

Beweis: We choose R large enough, such that $\mathrm{supp}(\varphi) \subset B_R(0)$ and introduce $\varrho = \sqrt{\varepsilon^2\tau^2+|\xi|^2}$, $\tau_0 = \varepsilon\tau/\varrho$, $\xi_0 = \xi/\varrho$, implying $\tau_0^2 + |\xi_0|^2 = 1$. In both integrals we introduce the change of coordinates $v \mapsto (y, v^\perp)$, defined by

$$v = \left(y - \frac{\tau_0}{|\xi_0|}\right)\frac{\xi_0}{|\xi_0|} + v^\perp\,.$$

The restrictions $|\varepsilon\tau + v\cdot\xi| < \alpha$ and $v \in B_R$ then imply $|y| < \alpha/(\varrho|\xi_0|)$ and, respectively, $|y - \tau_0/|\xi_0|| < R$.
a) For estimating the first integral I_1 we distinguish between small and large values of α/ϱ. We obviously have

$$I_1 \le C \le 4C\frac{\alpha}{\varrho} \qquad \text{for } \frac{\alpha}{\varrho} \ge \frac{1}{4}\,.$$

For $\alpha/\varrho < 1/4$, we observe that

$$|\xi_0| < C_1 := \min\left\{\frac{1}{2R}, \frac{\sqrt{7}}{4}\right\}$$

implies $|\tau_0| > 3/4$ and, thus,

$$\frac{|\tau_0|}{|\xi_0|} - R - \frac{\alpha}{\varrho|\xi_0|} > \frac{1}{2|\xi_0|} - R > 0\,.$$

As a consequence $I_1 = 0$ in this case, and we can assume $|\xi_0| \ge C_1$ in the following:

$$I_1 \le C\int_{|y|<\alpha/(\varrho|\xi_0|)} dy = \frac{2C\alpha}{\varrho|\xi_0|} \le \frac{2C}{C_1}\,\frac{\alpha}{\varrho}\,,$$

completing the proof of a).
b) The second integral can be estimated by

$$I_2 \le \frac{C}{\varrho^2|\xi_0|^2}\int_{|y|>\alpha/(\varrho|\xi_0|),\,|y-\tau_0/|\xi_0||<R} \frac{dy}{y^2} = \frac{C}{\varrho^2|\xi_0|}\int_{|z|>\alpha/\varrho,\,|z-\tau_0|<|\xi_0|R} \frac{dz}{z^2}\,.$$

In the case

$$|\tau_0| - R|\xi_0| > \frac{\alpha}{\varrho}\,,$$

we obtain

$$
\begin{aligned}
I_2 &\leq \frac{C}{\varrho^2|\xi_0|}\int_{|z-\tau_0|<|\xi_0|R}\frac{dz}{z^2} = \frac{2RC}{\tau_0^2-R^2|\xi_0|^2}\,\frac{1}{\varrho^2} \leq \frac{2RC}{|\tau_0|+R|\xi_0|}\,\frac{1}{\alpha\varrho} \\
&\leq \frac{2RC}{\min\{1,R\}}\,\frac{1}{\alpha\varrho}\,.
\end{aligned}
$$

On the other hand, for

$$
|\tau_0| - R|\xi_0| \leq \frac{\alpha}{\varrho}\,,
$$

we get

$$
\begin{aligned}
I_2 &\leq \frac{2C}{\varrho^2|\xi_0|}\int_{\alpha/\varrho}^{|\tau_0|+|\xi_0|R}\frac{dz}{z^2} = \frac{2C}{|\xi_0|}\left(1-\frac{\alpha/\varrho}{|\tau_0|+R|\xi_0|}\right)\frac{1}{\alpha\varrho} \\
&\leq \frac{2C}{|\xi_0|}\left(1-\frac{|\tau_0|-R|\xi_0|}{|\tau_0|+R|\xi_0|}\right)\frac{1}{\alpha\varrho} \leq \frac{2RC}{\min\{1,R\}}\,\frac{1}{\alpha\varrho}\,,
\end{aligned}
$$

completing the proof. ∎

Repeating the proof of Theorem 1 with these new estimates, a stronger result is derived:

Satz 2 *With the assumptions of Theorem 1, $m \in H^{1/2}(\mathbb{R}_t \times \mathbb{R}_x^3)$ holds and*

$$
\|m\|_{H^{1/2}(\mathbb{R}_t\times\mathbb{R}_x^3)} \leq \frac{C}{\min\{1,\sqrt{\varepsilon}\}}\|f\|^{1/2}_{L^2(\mathbb{R}_x^3\times\mathbb{R}_v^3\times\mathbb{R}_t)}\|S\|^{1/2}_{L^2(\mathbb{R}_x^3\times\mathbb{R}_v^3\times\mathbb{R}_t)}\,.
$$

Averaging lemmas in L^1

Theorems 1 and 2 can be extended to L^p-spaces with $1 < p < \infty$. Unfortunately, no control of solutions of the Boltzmann equation in any of these spaces is available. For L^1-based averaging lemmas, stronger assumptions are required. A first result is concerned with stationary transport equations.

Satz 3 *Let the set $\mathcal{F} \subset L^1(\mathbb{R}_x^3 \times \mathbb{R}_v^3)$ be weakly relatively compact and let $\{v\cdot\nabla_x f : f \in \mathcal{F}\}$ be bounded in $L^1(\mathbb{R}_x^3 \times \mathbb{R}_v^3)$ and uniformly integrable. Then the set*

$$
\left\{\int_{\mathbb{R}^3} f\,dv : f \in \mathcal{F}\right\}
$$

is relatively compact in $L^1(\mathbb{R}_x^3)$.

In the proof, a general criterion for relative compactness in Banach spaces and the Dunford-Pettis criterion for weak relative compactness in L^1 will be used:

Lemma 4 *The subset H of a Banach space is relatively compact, if for each $\varepsilon > 0$ there exists a compact set K_ε, such that $H \subset K_\varepsilon + B_\varepsilon(0)$.*

Lemma 5 *(Dunford-Pettis) A bounded subset $\mathcal{F}$ of $L^1(\mathbb{R}^d)$ is weakly relatively compact, if and only if it is* uniformly integrable, *i.e.*

$$\int_A |f(z)|dz \to 0 \quad \textit{as } |A| \to 0\,, \quad \textit{uniformly in } \mathcal{F}\,,$$

and it is tight, *i.e.*

$$\int_{|z|>R} |f(z)|dz \to 0 \quad \textit{as } R \to \infty\,, \quad \textit{uniformly in } \mathcal{F}\,.$$

Finally, we state a useful criterion for uniform integrability:

Lemma 6 *A subset $\mathcal{F}$ of $L^1(\mathbb{R}^d)$ is uniformly integrable, if and only if*

$$\int_{|f(z)|\ge c} |f(z)|dz \to 0 \quad \textit{as } c \to \infty\,, \quad \textit{uniformly in } \mathcal{F}\,.$$

Beweis: (of Theorem 3): By the Dunford-Pettis criterion, for each $\varepsilon > 0$, each $f \in \mathcal{F}$ can be split in $f = f_1 + f_2$ with $f_1 = 0$ for $|x| + |v| > R$ and with

$$\int_{\mathbb{R}^6} |f_2| dx\, dv < \varepsilon\,. \tag{3.1}$$

Then $\{g := f_1 + v \cdot \nabla_x f_1 \,:\, f \in \mathcal{F}\}$ is uniformly integrable, and for $c > 0$ we introduce the further splitting $g = g\mathbf{1}_{|g|\le c} + g\mathbf{1}_{|g|>c}$. With the resolvent operator

$$R_\lambda g(x,v) = \int_0^\infty e^{-\lambda s} g(x - sv, v) ds\,,$$

(where $f = R_\lambda g$ solves $\lambda f + v \cdot \nabla_x f = g$,) we obtain

$$f_1 = R_1(g\mathbf{1}_{|g|\le c}) + R_1(g\mathbf{1}_{|g|>c})\,.$$

The resolvent operator satisfies the estimate

$$\|R_\lambda g\|_{L^p} \le \int_0^\infty e^{-\lambda s} \|g(x - sv, v)\|_{L^p_{x,v}} ds = \|g\|_{L^p} \int_0^\infty e^{-\lambda s} ds = \frac{1}{\lambda}\|g\|_{L^p}\,, \tag{3.2}$$

for $p \ge 1$. Therefore, by the uniform integrability, we can choose c such that

$$\|R_1(g\mathbf{1}_{|g|>c})\|_{L^1} < \varepsilon\,. \tag{3.3}$$

On the other hand, $g\mathbf{1}_{|g|\le c}$ has compact support, is therefore bounded in $L^2_{x,v}$, and the same is true by (3.2) for $R_1(g\mathbf{1}_{|g|\le c})$. As a consequence of the L^2-averaging lemma, in the splitting

$$\int_{\mathbb{R}^3} f\, dv = \int_{\mathbb{R}^3} R_1(g\mathbf{1}_{|g|\le c}) dv + \int_{\mathbb{R}^3} \Big(R_1(g\mathbf{1}_{|g|>c}) + f_2\Big)\, dv\,,$$

the first term is relatively compact and the second small in L^1 by (3.1) and (3.3). An application of the compactness criterion Lemma 4 concludes the proof. ∎

Theorem 3 can be improved in two ways: The uniform integrability assumption can be removed, and the time dependent case can be covered:

Satz 4 *Let the set $\mathcal{F} \subset L^1((0,T) \times \mathbb{R}^3_x \times \mathbb{R}^3_v)$ be weakly relatively compact and let $\{\partial_t f + v \cdot \nabla_x f : f \in \mathcal{F}\}$ be bounded in $L^1((0,T) \times \mathbb{R}^3_x \times \mathbb{R}^3_v)$. Then the set*

$$\left\{ \int_{\mathbb{R}^3} f\, dv : f \in \mathcal{F} \right\}$$

is relatively compact in $L^1((0,T) \times \mathbb{R}^3_x)$.

Chapter 4

Global existence for the Boltzmann equation

4.1 Renormalized solutions

In this chapter, the initial value problem for the Boltzmann equation in whole space, i.e. for $x, v \in \mathbb{R}^3$, will be considered. Like in the previous chapter, we shall essentially follow [9].

For solutions decaying fast enough as $|x|, |v| \to \infty$, the conservation laws and the H-theorem imply that

$$\begin{aligned} &\int_{\mathbb{R}^6} (1 + |v|^2 + \log f) f \, dv \, dx \,\Big|_{t=T} \\ &+ \frac{1}{4} \int_0^T \int_{\mathbb{R}^9} \int_{S^2} |(v - v_*) \cdot n| (f' f'_* - f f_*) \log \frac{f' f'_*}{f f_*} \, dn \, dv_* \, dv \, dx \, dt \\ &= \int_{\mathbb{R}^6} (1 + |v|^2 + \log f) f \, dv \, dx \,\Big|_{t=0}, \end{aligned} \tag{4.1}$$

providing a bound for solutions, if the right hand side is finite. Since no better estimates are available for solutions without smallness restrictions, the fundamental difficulty occurs of giving sense to the collision operator with its quadratic nonlinearity. The essential idea for dealing with this problem is due to P.-L. Lions and R. DiPerna [7]. For the loss term $Q_-(f, f)$ in the collision integral, the expression

$$\frac{Q_-(f, f)}{1 + f} = \frac{f}{1 + f} \int_{\mathbb{R}^3 \times S^2} |(v - v_*) \cdot n| f_* \, dn \, dv_* = \frac{2\pi f}{1 + f} \int_{\mathbb{R}^3} |v - v_*| f_* \, dv_*$$

is at least locally integrable with respect to (x, v, t) for an f, such that the first term in (4.1) is bounded uniformly in time. This simple observation can be improved, when also the bound on the entropy dissipation (second line in (4.1)) resulting from (4.1) is used.

Lemma 7 *Assume that $f(x,v,t)$ is a measurable function such that (4.1) holds with the right hand side being finite. Then for every $T>0$, $R>0$, there exists $C>0$, such that*

$$\int_0^T \int_{\mathbb{R}^3} \int_{|v|<R} \frac{|Q(f,f)|}{\sqrt{1+f}} dv\, dx\, dt \le C\,.$$

Beweis: We shall make use of the inequalities

$$|a-b| \le \left(\sqrt{a}-\sqrt{b}\right)^2 + 2\sqrt{b}\left|\sqrt{a}-\sqrt{b}\right|\,,$$
$$\left(\sqrt{a}-\sqrt{b}\right)^2 \le \frac{1}{4}(a-b)\log\frac{a}{b}\,,$$

implying

$$\frac{|f'f'_*-ff_*|}{\sqrt{1+f}} \le \frac{1}{4}(f'f'_*-ff_*)\log\frac{f'f'_*}{ff_*} + \sqrt{f_*}\sqrt{(f'f'_*-ff_*)\log\frac{f'f'_*}{ff_*}}\,.$$

Multiplication by $|(v-v_*)\cdot n|$ and integration gives

$$\int_0^T \int_{\mathbb{R}^3} \int_{|v|<R} \frac{|Q(f,f)|}{\sqrt{1+f}} dv\, dx\, dt$$
$$\le E + 2\sqrt{E}\left(\int_0^T \int_{\mathbb{R}^6} \left(\int_{|v|<R}\int_{S^2} |(v-v_*)\cdot n| dn\, dv\right) f_* dv_*\, dx\, dt\right)^{1/2}\,,$$

where the Cauchy-Schwarz inequality has been used and E is the value of the right hand side of (4.1). With the estimate

$$\int_{|v|<R}\int_{S^2} |(v-v_*)\cdot n| dn\, dv = 2\pi \int_{|v|<R} |v-v_*| dv \le C_R\left(1+|v_*|^2\right)\,,$$

and again using (4.1), we finally obtain

$$\int_0^T \int_{\mathbb{R}^3} \int_{|v|<R} \frac{|Q(f,f)|}{\sqrt{1+f}} dv\, dx\, dt \le \left(1+2\sqrt{C_R T}\right) E\,.$$

■

This result motivates the following definition.

Definition 1 *A nonnegtive function $f \in C([0,\infty); L^1(\mathbb{R}^3\times\mathbb{R}^3)$ is a* renormalized solution *of the Boltzmann equation, iff*

$$\frac{Q(f,f)}{\sqrt{1+f}} \in L^1_{loc}([0,\infty)\times\mathbb{R}^3\times\mathbb{R}^3)\,,$$

and for every $\beta \in C^1([0,\infty))$ such that $\beta'(z) \le C(1+z)^{-1/2}$,

$$\partial_t \beta(f) + v\cdot\nabla_x \beta(f) = \beta'(f) Q(f,f)\,,$$

holds in the sense of distributions.

The following sections will be devoted to the proof of the DiPerna-Lions existence theorem [7]:

Satz 5 *Let $f_0(x,v) \geq 0$ satisfy*

$$\int_{\mathbf{R}^6} (1 + |x|^2 + |v|^2 + |\log f_0|) f_0 \, dv \, dx < \infty \, .$$

Then there exists a renormalized solution f of the Boltzmann equation satisfying $f(t=0) = f_0$. The renormalized solution also satisfies
1) <u>Local conservation laws</u>: conservation of mass:

$$\partial_t \int_{\mathbf{R}^3} f \, dv + \nabla_x \cdot \int_{\mathbf{R}^3} v f \, dv = 0 \, ,$$

conservation of momentum (with defect measure):

$$\partial_t \int_{\mathbf{R}^3} v f \, dv + \nabla_x \cdot \int_{\mathbf{R}^3} v \otimes v f \, dv + \nabla_x \cdot m = 0 \, ,$$

where m is nonnegative, symmetric, with entries in $L^\infty((0,\infty); \mathcal{M}(\mathbb{R}^3))$ ($\mathcal{M}(\mathbb{R}^3)$ denotes the space of Radon measures on $\mathbb{R}^3$).
2) <u>Global conservation laws</u>: conservation of mass and momentum:

$$\int_{\mathbf{R}^6} f \, dv \, dx = \int_{\mathbf{R}^6} f_0 \, dv \, dx \, , \qquad \int_{\mathbf{R}^6} v f \, dv \, dx = \int_{\mathbf{R}^6} v f_0 \, dv \, dx \, ,$$

conservation of energy (with defect measure):

$$\int_{\mathbf{R}^6} |v|^2 f \, dv \, dx + \int_{\mathbf{R}^3} \mathrm{trace}(m) dx = \int_{\mathbf{R}^6} |v|^2 f_0 \, dv \, dx \, .$$

3) <u>Entropy inequality</u>:

$$\begin{aligned} &\int_{\mathbf{R}^6} f \log f \, dv \, dx - \int_{\mathbf{R}^6} f_0 \log f_0 \, dv \, dx \\ &\leq \frac{1}{4} \int_0^t \int_{\mathbf{R}^9} \int_{S^2} |(v - v_*) \cdot n| (f' f'_* - f f_*) \log \frac{f' f'_*}{f f_*} \, dn \, dv_* \, dv \, dx \, ds \, . \end{aligned}$$

The theorem has some weaknesses concerning the conservation laws. Only the results on local conservation of mass and global conservation of mass and momentum are satisfactory. Note that, in particular, nothing is known concerning local conservation of energy. Also the entropy inequality should be an equality from a formal point of view.

4.2 Approximative solutions

The proof of a simplified version of the DiPerna-Lions theorem will be presented. In particular, the collision cross section $|(v-v_*)\cdot n|$ will be replaced by a bounded function $b((v-v_*)\cdot n)$.

A regularized collision operator can be defined by

$$Q_N(f,f) := \left(1+\frac{\varrho_f}{N}\right)^{-1} \int_{\mathbb{R}^3\times S^2} b((v-v_*)\cdot n)(f'f'_* - ff_*)dn\, dv_* \,.$$

Approximative solutions of the Boltzmann equation are then obtained by solving

$$\partial_t f_N + v\cdot\nabla_x f_N = Q_N(f_N,f_N)\,, \qquad f_N(x,v,0) = f_0(x,v)\,, \tag{4.2}$$

for $N\in\mathbb{N}$.

Satz 6 *Under the assumptions of Theorem 5 on the initial data f_0, there exists, for every $N\in\mathbb{N}$, a unique global solution $f_N\in C([0,\infty);L^1(\mathbb{R}^6))$ of (4.2). It satisfies the entropy relation*

$$\begin{aligned}&\int_{\mathbb{R}^6} f_N\log f_N\, dv\, dx - \int_{\mathbb{R}^6} f_0\log f_0\, dv\, dx \\ &= \frac14\int_0^t\int_{\mathbb{R}^9}\int_{S^2}\frac{b}{1+\varrho_{f_N}/N}(f'_Nf'_{N*} - f_Nf_{N*})\log\frac{f'_Nf'_{N*}}{f_Nf_{N*}}\, dn\, dv_*\, dv\, dx\, ds\,,\end{aligned}$$

and the bound

$$\int_{\mathbb{R}^6}(1+|x|^2+|v|^2+|\log f_N|)f_N\, dv\, dx \le C_0(1+t^2)\,,$$

with a constant $C_0\ge 0$ depending only on f_0.

Beweis: (some ideas only, see [7] for details):
1) $Q_N: L^1\to L^1$ is Lipschitz.
2) Mild formulation:

$$f_N(x,v,t) = f_0(x-vt,v) + \int_0^t Q_N(f_N,f_N)(x-v(t-\tau),v,\tau)d\tau\,.$$

Local existence by Picard iteration, global existence by bound in L^1.
3) Global conservation laws and

$$\frac{d}{dt}\int_{\mathbb{R}^6}|x-vt|^2 f_N dv\, dx = 0$$

imply

$$\int_{\mathbb{R}^6}|x|^2 f_N dv\, dx \le C_0(1+t^2)\,.$$

4)

$$\int_{f_N\le 1} f_N|\log f_N|dv\,dx \quad\le\quad \int_{\exp(-|x-vt|^2-|v|^2)\le f_N\le 1} f_N|\log f_N|dv\,dx + \int_{f_N\le \exp(-|x-vt|^2-|v|^2)} f_N|\log f_N|dv\,dx\,.$$

The integrand in the first integral on the right hand side is bounded by $(|x-vt|^2+|v|^2)f_N$. The second integral is also bounded. ∎

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