Maths for Physics

University of Birmingham

Mathematics Support Centre

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Foreword

Mathematics is an integral component of all of the scientific disciplines, but for physics, it is a vital and essential skill that anyone who chooses to study this subject must master. Mathematics allows a physicist to understand a range of important concepts, model physical scenarios, and solve problems. In your pre-university studies you will have encountered mathematics: perhaps when considering Newton's laws of motion and gravitation, exploring the laws of electricity and magnetism, or when comparing the absolute magnitudes of stars. As you move through your university studies you will see the mathematical concepts underpinning physical ideas develop in increasingly sophisticated ways; you will need to ensure you not only have a highly developed knowledge of algebra and calculus but you can apply these effectively to a range of different and complex scenarios. Although there are many different branches of physics, the ability to understand and apply mathematics will be important regardless of which you choose to study. Mathematics forms the entire basis for physics, and is a reason why physics graduates are so highly sought by a range of businesses and industries.

For some time it has become apparent that many students struggle with their mathematical skills and knowledge as they make the transition to university in a wide range of subjects. It may perhaps be surprising that this includes physics, however this was one of the original disciplines where a problem was first identified in a seminal report back in 2000. Despite a range of activities, interventions and resources, a mathematics problem within physics still remains. A 2011 report from the Institute of Physics indicated many physics and engineering academic members of staff feel new undergraduates within their disciplines are underprepared as they commence their university studies due to a lack of fluency in mathematics. In addition, the report also highlights the concerns that students themselves are now beginning to articulate in relation to their mathematical skills prior to university entry. This is despite the evidence that they are typically arriving at university with increased mathematical grades.

In the summer of 2015 we set out to try to address this problem by working with two dedicated and talented undergraduate interns to develop a supporting resource for those studying physics. There are already a range of textbooks available that aim to help physics students develop their mathematical knowledge and skills. This guide isn't intended to replace those, or indeed the notes provided by your lecturers and tutors, but instead it provides an additional source of material presented in a quick reference style allowing you to explore key mathematical ideas quickly and succinctly. Its structure is mapped to include the key mathematical content most undergraduate physics students encounter during their first year of study. Its key feature is that it contains numerous examples demonstrating how the mathematics you will learn is applied directly within a physics context. Perhaps most significantly, it has been developed by students for students.

While this guide can act as a very useful reference resource, it is essential you work to not only understand the mathematical ideas and concepts it contains, but that you also continually practice your mathematical skills throughout your undergraduate studies. Understanding key mathematical ideas and being able to apply these to problems in physics is an essential part of being a competent and successful physicist. We hope this guide provides a useful and accessible resource as you begin your study of physics within higher education. Enjoy, and good luck!

Michael Grove & Joe Kyle
October 2015

Acknowledgements

First we would like to thank Michael Grove for providing support whenever it was needed, as well as for entrusting us with this project and giving us the opportunity to make this document our own work.

Thank you also to Dr Joe Kyle whose feedback on our work has been essential in making our booklet professional and accurate, and his commitment to providing diagrams and advice has been invaluable.

We would like to express our gratitude to the University of Birmingham physics department, in particular Professor Ray Jones, for their willingness to help and for allowing us access to their resources.

Deepest gratitude are also due to the University of Birmingham Mathematics Support Centre as this project would not have been possible without their funding.

Many thanks to Samantha Pugh from the University of Leeds for helping set up the project and Rachel Wood from the University of Birmingham for providing operational support, whenever needed.

Further, we would like to thank Rory Whelan and Allan Cunningham for allowing us to access and use their templates and LaTeX code from the *Maths for Chemistry* booklet which they produced in 2014. This allowed us to quickly get to grips with the coding and therefore focus on the content of our booklet.

Finally, a huge thank you to our fellow interns Amy Turner, Patrick Morton, Calum Ridyard, Heather Collis, and Abdikarim Timer who not only gave us support and advice throughout the entire project, but also provided us with great company.

0 Introduction

This booklet has been produced to assist second year physics students with the mathematics content of their course. It has been designed as an interactive resource to compliment lecture material with particular focus on the application of maths in physics. The content in this booklet has been developed using resources, such as lecture notes, lecture slides and past papers, provided to us by the University of Birmingham.

0.1 About the Authors

DANIEL BRETT is currently in his third year, studying a joint honours course in Theoretical Physics and Applied Mathematics at the University of Birmingham. He has enjoyed producing this learning resource and hopes that it will help students with their studies.
JOSEPH VOVROSH has just finished his first year of University, also studying Theoretical Physics and Applied Mathematics. He has a keen interest in maths and physics and hopes to pursue it as a career.

0.2 How to use this Booklet

- The contents contains hyper-links to the sections and subsections listed and they can be easily viewed by clicking on them.
- The book on the bottom of each page will return you to the contents when clicked. Try it out for yourself now:
- In the booklet, important equations and relations appear in boxes, as shown below:

$$\boxed{\text{Physics} > \text{Maths}}$$

- Worked examples of the mathematics contained in this booklet are in the red boxes as shown below:

Example: What is the sum $1 + 1$?

Solution: $1 + 1 = 2$

- Worked physics examples that explain the application of mathematics in a physics-related problem are in the blue boxes as shown below:

Physics Example: A particle travels 30 metres in 3 seconds. What is the velocity of the particle, v?

Solution: We know that $velocity = \dfrac{distance}{time}$. Therefore $v = \frac{30}{3} = 10\text{ms}^{-1}$.

- Some examples can also be viewed as video examples. They have hyper-links that will take you to the webpage the video is hosted on. Try this out for yourself now by clicking on the link below:

Physics Example: A particle is falling under gravity. After a time t the particle's velocity has increased from u to v. The acceleration is a and is described by the equation $v = u + at$. Rearrange the equation to make a the subject.

Click here for a video example

Solution: ...

1 Functions and Geometry

1.1 Properties of Functions

This section will explain what is meant by a function, as well as some of its properties, giving formal definitions and notation. You should already be familiar with much of this content, however it is included here for completeness and to provide a more formal approach.

Sets

A set is a collection of elements denoted by: $\{x_1, x_2, x_3, \ ...\}$ or $\{x \, : \,$ requirement on $x\}$. You should already be familiar with the standard sets of numbers:

- Natural numbers: $\mathbb{N}$ =$\{1,\ 2,\ 3,...\}$.
- Integer numbers: $\mathbb{Z}$ =$\{0,\ \pm1,\ \pm2,\ \pm3,...\}$.
- Rational numbers: $\mathbb{Q}$ =$\{\frac{m}{n} \, : \, m, n \in \mathbb{Z},\ n \neq 0\}$.
- Real numbers: $\mathbb{R}$. Real numbers may be rational or irrational.
- Complex numbers: $\mathbb{C} = \{a + bi \, : \, a, b \in \mathbb{R}\}$.
  These will be covered in their own section later on in the booklet.

However a set can contain any collection of numbers or objects. In this document we will only consider sets of numbers.

Example:
Some examples of sets:

1. $\{apple, pear, 3, x\}$
2. $\{1, 4, 9, 2\}$
3. $\{x : \frac{x}{2} \in \mathbb{N}\}$
4. $\{2, 7, blue, 8, 2\}$

Note that the third set is the set of positive multiples of 2. Also note that the fourth set is the same as writing $\{2, 7, blue, 8\}$ since we ignore repeated elements.

Functions

A function is a rule that transforms each element, x, of a set X to a unique element, y, of a second set Y, or formally:
A function f from a set X to a set Y is a rule that associates to each number $x \in X$ exactly one number $y \in Y$. The function is denoted as $y = f(x)$, or $f : X \to Y$, where X is the domain of the function, Y is the codomain of the function and all possible values of y make up the image.

- The **domain** of a function is the set X of all the values which we put into the function. The domain cannot include values for which the function is undefined.
- The **codomain** is the set Y of all the values which may possibly come out of the function.
- The **image** is the set of all the values that actually come out of the function.

This means that the codomain contains the image, but they are not necessarily equal sets. If we are not specifically given a domain then the **domain convention** tells us that we choose the domain as the set of all real numbers for which the function defines a real number.

Example: Let $f : \mathbb{N} \to \mathbb{R}$ such that $f(x) = x^2$. What are the domain, the codomain and the image of f?

Click here for a video example

Solution: The domain is the set of natural numbers, the codomain is the set of real numbers and the image is the set of positive square numbers.

Note: The image is a subset of the codomain.

It is very important to note that functions can only be 'one-to-one' or 'many-to-one' and NOT 'one-to-many'. In other words, for any value of x in our domain, a function must only return one value of y. This becomes important when considering inverse functions, as we are sometimes forced to restrict our domain so that the inverse function does not become 'one-to-many'.

Example:
The function $f(x) = x^2$ is 'many-to-one', since both $x = 1$ and $x = -1$ give $f(x) = 1$. However, the inverse function of this, $g(x) = \sqrt{x}$ is 'one-to-many', since $x = 1$ gives both $g(x) = 1$ and $g(x) = -1$. To make the inverse function 'one-to-one', we define $\sqrt{x}$ carefully as having the domain $\{x : x \geq 0\}$ and being the unique, non-negative number $\sqrt{x}$ such that $(\sqrt{x})^2 = x$. Thus $\sqrt{x}$ always represents a positive (or zero) number.

Composite Functions

Consider two functions $f(y)$ and $g(x)$ (or $f : Y' \to Z$ and $g : X \to Y$) where the codomain of g is a subset of the domain of f. A composite function of these two functions is written as $f(g(x))$, which applies the function f to $g(x)$. We can also write this composite function as $f(g(x)) = f \circ g(x)$. Note that this is not the same as a product of two functions, $gf(x)$. To use a composite function, we first calculate $g(x)$ to obtain a value x', say, and then calculate $f(x')$.

Example: Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$ and $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = x + 1$. Find the value of $g \circ f(2)$.

Solution: $g \circ f(x) = g(f(x))$
$= g(x^2)$
$= x^2 + 1$
$\Rightarrow g \circ f(2) = 2^2 + 1 = 5.$

Inverse Functions

An inverse function is formally defined as follows:
For a function $f : X \to Y$ the function $f^{-1} : Y \to X$ is called the inverse function of f if $f^{-1}(f(x)) = x$ for any $x \in X$ and $f(f^{-1}(y)) = y$ for any $y \in Y$.
In less formal terms, this means that given a function f which turns a value of x into a particular value of y, the inverse function f^{-1} converts that value of y back into the original value of x.
We must be careful when working out an inverse of a function as only 'one-to-one' functions have an inverse.
A very simple example is as follows:

Example: Given the function $f(x) = 2x + 3$ from $\mathbb{R}$ to $\mathbb{R}$, what is the inverse function of $f(x)$?

Solution: To apply the original function, we multiply x by 2 and then add 3. To find the inverse of f, we need to do the **opposite** of this. Therefore we need to subtract 3 from x and then divide by 2 so that our inverse function is $f^{-1}(x) = \frac{x-3}{2}$.

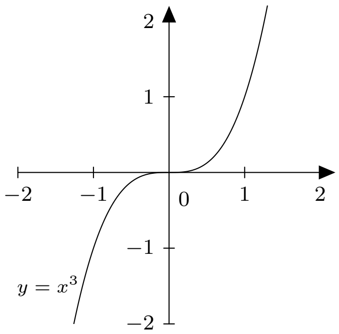
To check our answer, we can calculate $f^{-1}(f(x))$ and make sure that it is equal to x:
$f^{-1}(f(x)) = \frac{(2x+3)-3}{2}$
$= \frac{2x}{2}$
$= x$ as required.

Odd and Even Functions

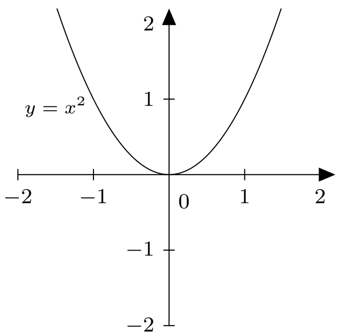
Given a domain $X \subseteq \mathbb{R}$ which is symmetric about zero and a codomain $Y \subseteq \mathbb{R}$,

- A function $f : X \to Y$ is said to be **odd** if $f(-x) = -f(x)$ for all x in $\mathbb{R}$.
- A function $f : X \to Y$ is said to be **even** if $f(-x) = f(x)$ for all x in $\mathbb{R}$.

You may find this easier to understand graphically.



Above is the graph of $y = x^3$. We can see this is an odd function as the negative x portion of the graph is an upside down reflection of the positive x portion.



Above is the graph of $y = x^2$. We can see that this is an even function as the graph is symmetric about the y-axis.

It is also useful to note that:

- The product of two odd functions is an even function.
- The product of two even functions is also an even functions
- The product of an odd function and an even function is an odd function.

This property is important as it allows us to predict the behaviour of functions, and in some cases take shortcuts during integration, which will be covered in a later chapter.

Example: Determine whether the following functions are odd or even.

1. $f(x) = 2x^6 + 13x^2$
2. $f(x) = \frac{1}{4x^3 - 2x}$

Solution:

1. $f(-x) = 2(-x)^6 + 13(-x)^2$
   $= 2x^6 + 13x^2$
   $= f(x)$
   $\Rightarrow f(-x) = f(x)$
   Thus the function is even.
2. $f(-x) = \frac{1}{4(-x)^3 - 2(-x)}$
   $= \frac{1}{-4x^3 + 2x}$
   $= -f(x)$
   $\Rightarrow f(-x) = -f(x)$
   Thus the function is odd.

Increasing and Decreasing Functions

Another important property of functions is whether they are increasing or decreasing.

- An increasing function $f(x)$ is such that $f(x_1) < f(x_2)$ for all $x_1, x_2 \in X$ such that $x_1 < x_2$.
- A decreasing function $f(x)$ is such that $f(x_1) > f(x_2)$ for all $x_1, x_2 \in X$ such that $x_1 < x_2$.

Example: Find the restrictions of the domain, $X = \mathbb{R}$, for the function

$$f(x) = (x + 1)^2$$

which make $f(x)$

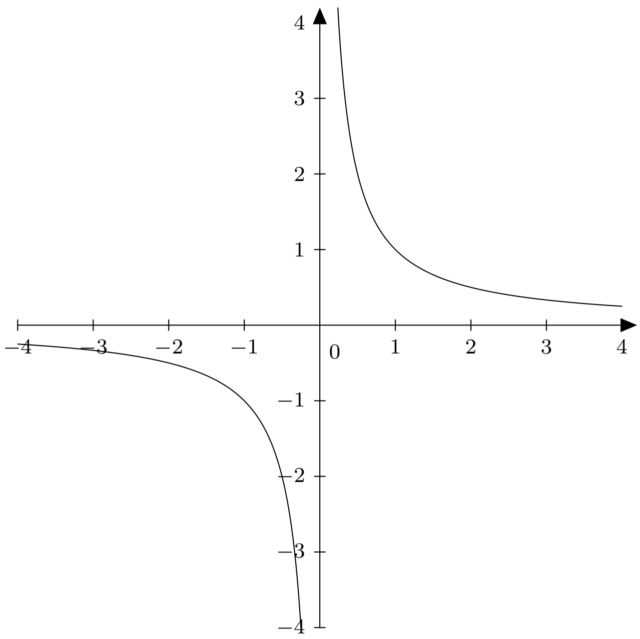
1. an increasing function.
2. a decreasing function.

Click here for a video example

Solution: See the video example above.

Undefined Functions

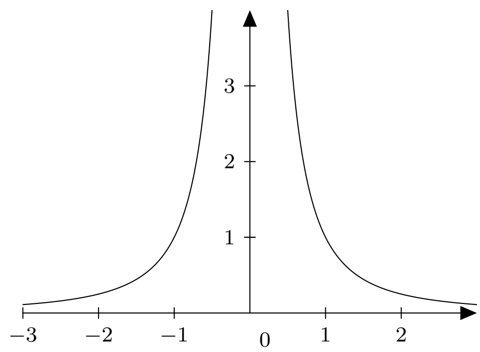
When the rule of a function cannot be calculated at a point x' we call it 'undefined at the point x''. An example of this is $f(x) = \frac{1}{x}$ at the point where $x = 0$, i.e. $f(0) = \frac{1}{0}$. As x gets closer to zero from above the function will grow without bound to ∞. However, if the initial value of x was negative and was increased to 0 the function would tend to $-\infty$.



From above we can see that at $x = 0$, this function tends to both $+\infty$ as well as $-\infty$. The function is thus undefined and so division by zero is forbidden.

Continuity

Informally, we can say a function $f(x)$ is continuous if there are no 'jumps' in the function. Graphically, this means that a function is continuous if there are no breaks in the line, and discontinuous if there is at least one break in the line. In physics, the most common discontinuity that you will encounter is an asymptotic discontinuity, which occurs when the curve approaches $\pm\infty$ at a point. To find the discontinuous points of a function, we find the values of x for which the function is undefined. Functions may have more than one discontinuous point.
An example of an asymptotically discontinuous function is $y = \frac{1}{x^2}$, shown below. It has a discontinuous point at $x = 0$ since at this point, y tends to ∞ so the function is undefined.



Functions with discontinuities must be handled with extra care, especially when differentiating and integrating, which will be covered later. To see the reason for this, think about what happens to the gradient of the function at the discontinuous point shown above.

Example: Find the discontinuous point(s) of the following functions:

1. $f(x) = \frac{1}{x-4}$
2. $f(x) = \frac{1}{(x^2-1)(x+3)}$

Solution:

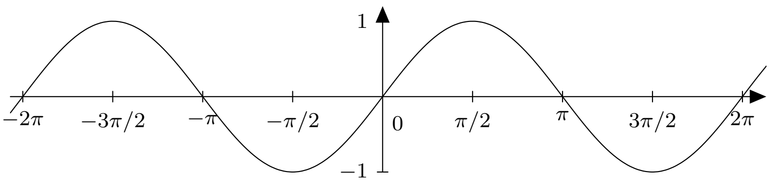
1. We look for the values of x for which the denominator is zero. At the point $x = 4$, the function becomes $\frac{1}{0}$ which is undefined. Therefore $f(x) = \frac{1}{x-4}$ has a discontinuous point at $x = 4$
2. At the point $x = -3$ there is a discontinuous point, since $(x + 3)$ becomes zero, and hence the denominator becomes zero. Now we look at the values of x for which $(x^2 - 1)$ is zero. Since there is an x^2, both $x = 1$ and $x = -1$ satisfy this condition. Therefore there exist three discontinuous points: $x = -3$, $x = -1$ and $x = 1$.

Periodicity

Some functions are 'periodic', meaning that they repeat their values in regular intervals or periods. The general expression for these functions is

$$f(x + P) = f(x)$$

A function of this type has fundamental period P. In other words, after every interval of length P, the function 'resets' and traces out the same pattern for each interval. An important example of a periodic function is the 'sine' function, which will be covered in more detail in the next section. By looking at the graph of $y = \sin x$, you can see that the pattern is repeated after each interval of 2π.

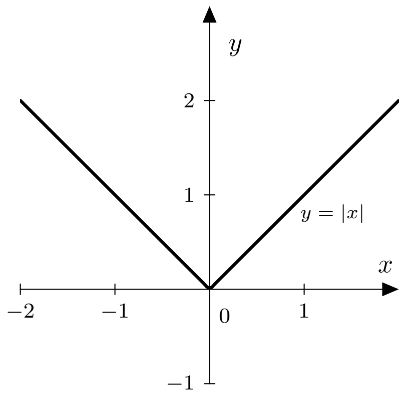


The Modulus Function

The modulus function is denoted $|x|$ and is defined by:

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

This just means that given a number x, if it's negative, multiply it by minus one. The modulus function will only output non-negative numbers. The modulus function is extremely useful in cases where we would like to know the distance between two real numbers. For example, the distance between a and b can be written as $|a - b|$. Below is the graph of $y = |x|$.



Example:

1. What is the modulus of -324?
2. Solve $|x+1| = 3$.

Solution:

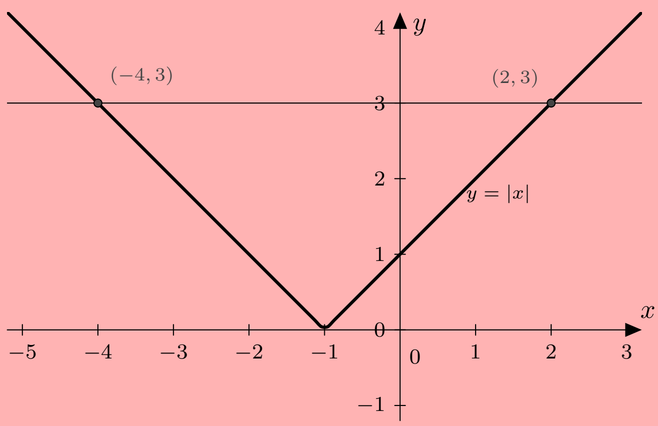
1. The modulus of -324 is written as $|-324|$
$= -1 \times -324$
$= 324$
2. To solve this requires an understanding of possible cases of $|x+1|$.

   If $x < -1$ this means $x+1 < 0$ thus $|x+1| = -(x+1)$.
   However if $x \geq -1$ then $x+1 \geq 0$ and $|x+1| = x+1$
   So to achieve a full solution to this problem one needs to consider all cases separately.

   (a) Case 1: $x < -1$.
   As seen above, in this case $|x+1| = -(x+1)$
   Therefore $-(x+1) = 3$
   $\Rightarrow -x - 1 = 3$
   $\Rightarrow x = -4$

   (b) Case 2: $x \geq -1$.
   As seen above, in this case $|x+1| = x+1$.
   Therefore $x+1 = 3$
   $\Rightarrow x = 2$

   This means that the full solution is $x = 2$ or $x = -4$. The solutions can also be shown on a graph.



Note: This idea of case analysis is very powerful and can be used to solve inequalities involving the modulus function as well as simple equations like above.

1.2 Curve Sketching

If we are given a function $f(x)$, then it is often beneficial to sketch the function in order to understand how it changes with x. In order to do this, there are a number of properties of the function that we can look for:

1. **Asymptotes:** We look for how $f(x)$ behaves as x tends to $\pm\infty$.
2. **Zeros:** Find the values of x for which $f(x) = 0$, then we know that this is where the function crosses the x-axis.
3. **Singularities:** Find the values of x for which $f(x) = \pm\infty$.
4. **Sign Change:** Find where $f(x)$ changes from positive to negative, then we can deduce for what ranges of x the graph is above or below the x-axis.
5. **Stationary Points:** We find where the function reaches a local maximum, a local minimum or a point of inflexion as this gives an idea of the shape of the graph. The method for this is described in detail in the section on 'Stationary Points'.

Example: Sketch the graph of

$$y = x + \frac{1}{x-2}$$

Solution:

- First we look at how the equation acts at x tends to $\pm\infty$:
  - We can see that as x tends to $+\infty$ the term $\frac{1}{x-2}$ will tend to $\frac{1}{\infty} = 0$ therefore y will tend to the line $y = x$.
  - As x tends to $-\infty$ we can see that $\frac{1}{x-2}$ will tend to zero again so y will tend to the line $y = x$.
- To find out if the equation has any zeros we need to solve

$$0 = x + \frac{1}{x-2}$$

$$\Rightarrow x(x-2) + 1 = 0$$

$$\Rightarrow (x-1)^2 = 0$$

  Therefore $x = 1$ is a zero of the equation.
- We can see the equation has a singularity at $x = 2$.
  - As x tends to 2 from negative values of y, $\frac{1}{x-2}$ will tend to $-\infty$ so y tends to $-\infty$.
  - As x tends to 2 from large positive values of y, $\frac{1}{x-2}$ will tend to ∞ so y tends to ∞.
- The sign of the function only changes either side of $x = 2$
- $\frac{dy}{dx} = 1 - \frac{1}{(x-2)^2}$ so $\frac{dy}{dx} = 0$ at:

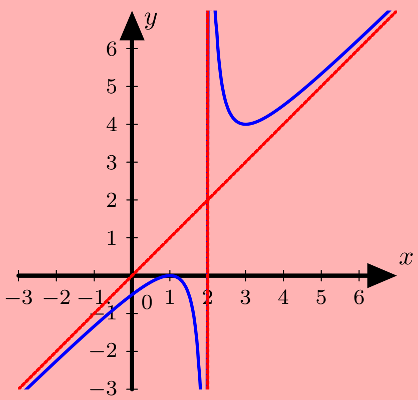
$$1 = \frac{1}{(x-2)^2}$$

  This expands to give

$$x^2 - 4x + 3 = (x-1)(x-3) = 0$$

  So there are turning points at $x = 1$ and $x = 3$.
  $\frac{d^2y}{dx^2} = \frac{2}{(x-2)^2}$
  At $x = 1$, $\frac{d^2y}{dx^2} > 0$ therefore this point is a minimum.
  At $x = 3$, $\frac{d^2y}{dx^2} < 0$ therefore this point is a maximum.

Putting all this together gives the following graph (red lines are the asymptotes):

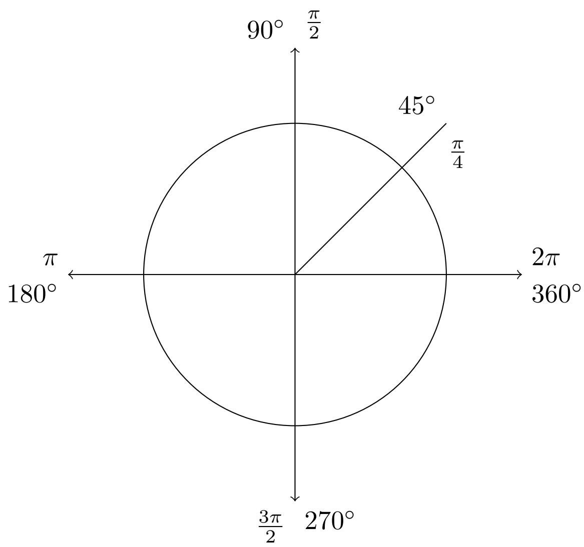


1.3 Trigonometry

In this section we will recap trigonometry and discuss some important ideas involving properties of the trigonometric functions.

Radians

We often use degrees to measure angles, however we also have the option of measuring angles using **radians** which can be much more convenient. Instead of splitting a full circle into $360°$, it is split into 2π radians. We must use radians in calculus (differentiation and integration), complex numbers and polar coordinates.
Below is shown a full circle showing some angles in degrees and what they are in radians.



We may wish to convert between degree and radians. In order to turn x degrees into radians we use the formula:

$$\boxed{\frac{x°}{360°} \times 2\pi}$$

In order to convert x radians into degrees we use the formula:

$$\boxed{\frac{x}{2\pi} \times 360°}$$

Example: Convert the following:

1. $60°$ into radians.
2. $\dfrac{3\pi}{8}$ radians into degrees.

Solution:

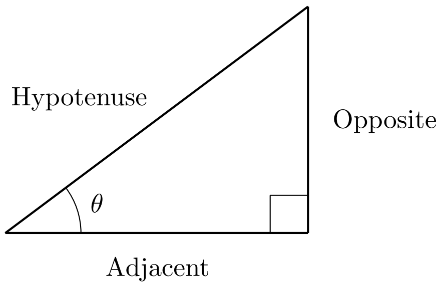
1. We need to use the formula $\dfrac{x°}{360°} \times 2\pi$

$$\frac{x°}{360°} \times 2\pi = \frac{60°}{360°} \times 2\pi = \frac{\pi}{3} \text{ radians.}$$

2. We need to use the formula $\dfrac{x}{2\pi} \times 360°$.

$$\frac{x}{2\pi} \times 360° = \frac{\left(\frac{3\pi}{8}\right)}{2\pi} \times 360° = \frac{3}{16} \times 360° = 67.5°$$

SOHCAHTOA



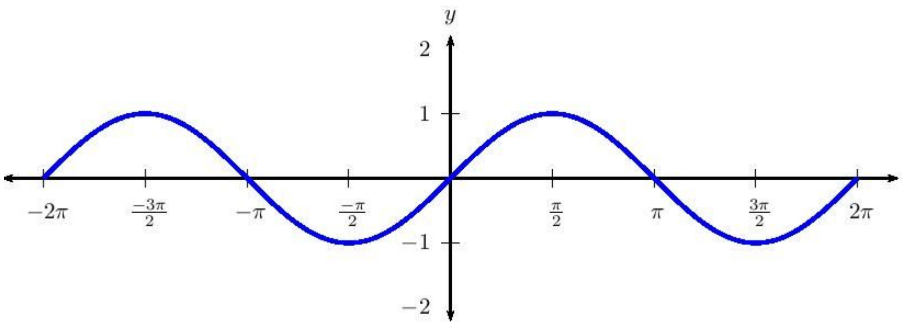
Given our right-angled triangle we start by defining sine, cosine and tangent before reviewing what their graphs look like.
Note: The angle θ is measured in radians in the following graphs on the x-axis.

Sine Function

We define sine as the ratio of the opposite to the hypotenuse.

$$\boxed{\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}}}$$

The graph of $y = \sin(\theta)$ is shown below:



The domain of $\sin(\theta)$ is the set of real numbers ($\mathbb{R}$)
The range of $\sin(\theta)$ is $\{y : -1 \leq y \leq 1\}$

The sine function has two very useful properties:

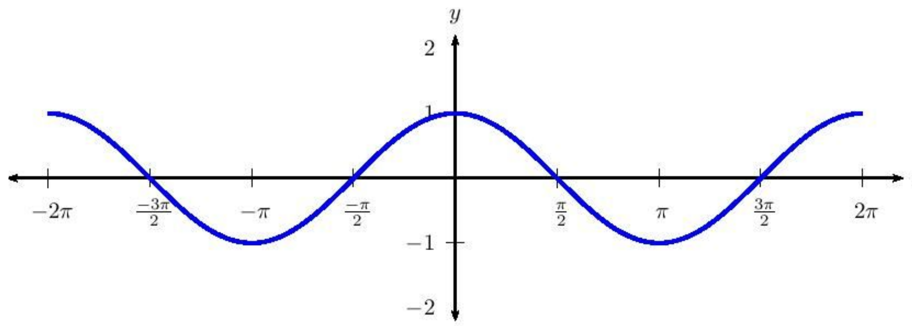
- $\sin\theta$ is an odd function, i.e. $\sin(-\theta) = -\sin(\theta)$.
- $\sin\theta$ is periodic with period 2π radians, as mentioned in the section on periodicity, ie. $\sin(\theta+2\pi n) = \sin(\theta)$.

Cosine Function

We define cosine as the ratio of the adjacent to the hypotenuse.

$$\boxed{\cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}}}$$

The graph of $y = \cos(\theta)$ is shown below:



The domain of $\cos(\theta)$ is the set of real numbers ($\mathbb{R}$)
The range of $\cos(\theta)$ is $\{y : -1 \leq y \leq 1\}$

As with the sine function, the cosine function also has two very useful properties:

- $\cos\theta$ is an even function, i.e. $\cos(-\theta) = \cos(\theta)$.
- $\cos\theta$ is periodic with period 2π radians, as mentioned in the section on periodicity, ie. $\cos(\theta+2\pi n) = \cos(\theta)$.

Tangent Function

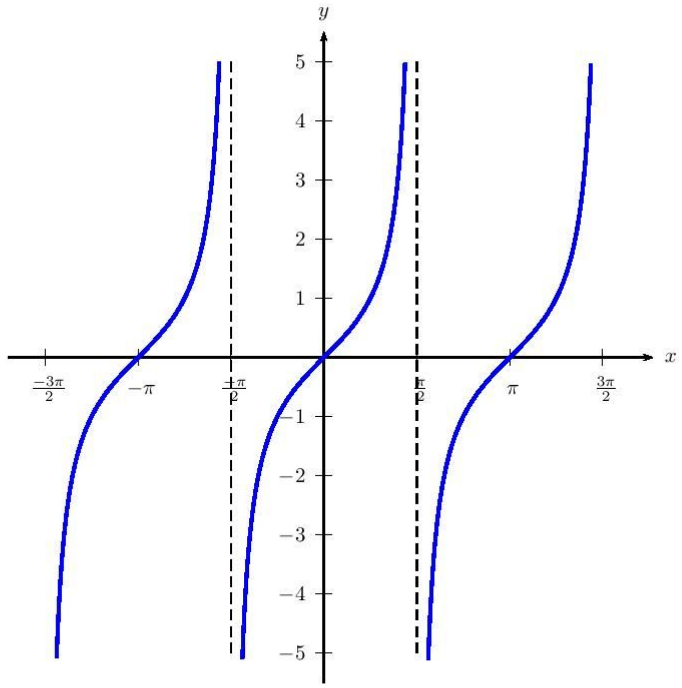
We define tangent as the ratio of the opposite to the adjacent.

$$\boxed{\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}}}$$

This leads to the relationship $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$ shown below:

$$\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{\text{Opposite} \times \text{Hypotenuse}}{\text{Adjacent} \times \text{Hypotenuse}} = \frac{\left(\dfrac{\text{Opposite}}{\text{Hypotenuse}}\right)}{\left(\dfrac{\text{Adjacent}}{\text{Hypotenuse}}\right)} = \frac{\sin(\theta)}{\cos(\theta)}$$

The graph of $y = \tan(\theta)$ is shown below:



The domain of $\tan(\theta)$ is $\{\theta : \theta \neq \frac{(2n+1)\pi}{2}, n \in \mathbb{Z}\}$
The range of $\tan(\theta)$ is the set of real numbers ($\mathbb{R}$)

The function $\tan(\theta)$ has the useful property that it is periodic with period π.

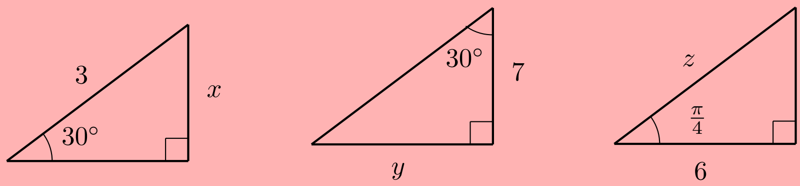
The definitions of sine, cosine and tangent are often more easily remembered by using SOHCAHTOA. We can use the formulas in SOHCAHTOA to calculate the size of angles and the length of sides in **right-angled triangles**.

$$\underbrace{\mathrm{S\ O\ H}}_{\sin(\theta)=\frac{\text{Opposite}}{\text{Hypotenuse}}}\underbrace{\mathrm{C\ A\ H}}_{\cos(\theta)=\frac{\text{Adjacent}}{\text{Hypotenuse}}}\underbrace{\mathrm{T\ O\ A}}_{\tan(\theta)=\frac{\text{Opposite}}{\text{Adjacent}}}$$

There are some common values of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ which are useful to remember, presented in the table below.

Angle (θ)	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
0	**0**	**1**	**0**
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	**1**
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	**1**	**0**	**undefined**
π	**0**	-1	**0**

Example: Using SOHCAHTOA find the lengths x, y and z in the right-angled triangles below.



Solution: For shorthand H is used to denote the hypotenuse, O the opposite and A the adjacent.

1. We know the length of the hypotenuse and wish to find the opposite hence we need to use

$$\sin(\theta) = \frac{\text{O}}{\text{H}} \Longrightarrow \sin(30°) = \frac{\text{O}}{3} \Longrightarrow \text{O} = \sin(30°) \times 3 = \frac{3}{2}$$

   Note: This question is in degrees.

2. We know the length of the adjacent and wish to find the opposite hence we need to use

$$\tan(\theta) = \frac{\text{O}}{\text{A}} \Longrightarrow \tan(30°) = \frac{\text{O}}{7} \Longrightarrow \text{O} = \tan(30°) \times 7 = \frac{7\sqrt{3}}{3}$$

   Note: This question is in degrees.

3. We know the length of the adjacent and wish to find the hypotenuse hence we need to use

$$\cos(\theta) = \frac{\text{A}}{\text{H}} \Longrightarrow \cos\left(\frac{\pi}{4}\right) = \frac{6}{\text{H}} \Longrightarrow \text{H} = \frac{6}{\cos\left(\frac{\pi}{4}\right)} = 6\sqrt{2}$$

   Note: This question is in radians.

Trigonometric Formulae and Identities

A very useful tool in many areas of physics is to be able to simplify and manipulate trigonometric expressions. There are a number of formulae that can be used to achieve this, however the most important of these by far is the identity:

$$\boxed{\cos^2(\theta) + \sin^2(\theta) = 1}$$

This means that for ANY value of θ, the sum of the squares of the sine and cosine functions is ALWAYS equal to 1. From this identity and the two following formulae, we can derive many useful equations which can help us to simplify some of the complicated trigonometric expressions that appear in physics. The following formulae can be proved geometrically. For the sake of interest, here is a link to the proofs. Two useful identities are:

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$$
$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$

From these we can show that:

$$\tan(A \pm B) = \frac{\sin(A \pm B)}{\cos(A \pm B)} = \frac{\sin(A)\cos(B) \pm \cos(A)\sin(B)}{\cos(A)\cos(B) \mp \sin(A)\sin(B)}$$

And dividing through by $\cos(A)\cos(B)$ gives:

$$\tan(A \pm B) = \frac{\tan(A) \pm \tan(B)}{1 \mp \tan(A)\tan(B)}$$

We can also show that if $B = A$ then,

$$\boxed{\sin(2A) = 2\sin(A)\cos(A)}$$

And that

$$\cos(A+B) = \cos(2A) = \cos^2(A) - \sin^2(A)$$

Using $\cos^2(\theta) + \sin^2(\theta) = 1$ we can further show that:

$$\boxed{\cos(2A) = 1 - 2\sin^2(A) = 2\cos^2(A) - 1}$$

You only need to remember the first three identities, as long as you can derive the rest of the equations as above.

Reciprocal Trigonometric Functions

As well as the standard trigonometric functions there are functions that are defined by the reciprocals of these functions:

- The Secant function:

$$\boxed{\sec(\theta) = \frac{1}{\cos(\theta)}}$$

  The domain is $\{\theta : \theta \neq \frac{(2n+1)\pi}{2}, n \in \mathbb{Z}\}$
  The range is $\{y : -1 \leq y \text{ or } y \geq 1\}$
  The period is 2π

- The Cosecant function:

$$\boxed{\text{cosec}(\theta) = \frac{1}{\sin(\theta)}}$$

  The domain is $\{\theta : \theta \neq n\pi, n \in \mathbb{Z}\}$
  The range is $\{y : -1 \leq y \text{ or } y \geq 1\}$
  The period is 2π

- The Cotangent function:

$$\boxed{\cot(\theta) = \frac{1}{\tan(\theta)}}$$

  The domain is $\{\theta : \theta \neq n\pi, n \in \mathbb{Z}\}$
  The range is the set of real numbers ($\mathbb{R}$)
  The period is π

An easy way to remember which reciprocal function corresponds to each trig function is to look at the third letter of each function:

- se<u>c</u> $\rightarrow$ <u>c</u>osine.
- co<u>s</u>ec $\rightarrow$ <u>s</u>ine.
- co<u>t</u> $\rightarrow$ <u>t</u>angent.

From these we can expand the list of identities above, using $\cos^2(\theta) + \sin^2(\theta) = 1$, and dividing through by $\cos^2(\theta)$ and $\sin^2(\theta)$ separately:

- Dividing through by $\cos^2(\theta)$ gives
$1 + \tan^2(\theta) = \sec^2(\theta)$.

- Dividing through by $\sin^2(\theta)$ gives
$1 + \cot^2(\theta) = \text{cosec}^2(\theta)$

Example:

1. Simplify $\dfrac{\sin(\theta)\sec(\theta)}{\cos^2(\theta)}$

2. Solve $\tan(\theta) + \sec^2(\theta) = 1$.

Solution:

1. $\dfrac{\sin(\theta)\sec(\theta)}{\cos^2(\theta)} = \tan(\theta)\dfrac{\sec(\theta)}{\cos(\theta)}$

   $= \tan(\theta)\sec^2(\theta)$

   $= \tan(\theta)(\tan^2(\theta) + 1)$

   $= \tan^3(\theta) + \tan(\theta)$

2. Using $1 + \tan^2(\theta) = \sec^2(\theta)$ we get:

   $\tan(\theta) + \tan^2(\theta) + 1 = 1$

   $\Rightarrow \tan(\theta) + \tan^2(\theta) = 0$

   $\Rightarrow \tan(\theta)\big(\tan(\theta) + 1\big) = 0$

   $\Rightarrow \tan(\theta) = 0$ or $\tan(\theta) = -1$

   If $\tan(\theta) = 0$, $\theta = n\pi$ where $n \in \mathbb{Z}$

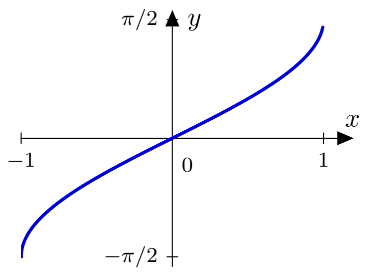
   If $\tan(\theta) = -1$, $\theta = -\dfrac{\pi}{4} + n\pi$ where $n \in \mathbb{Z}$

Inverse Trigonometric Functions

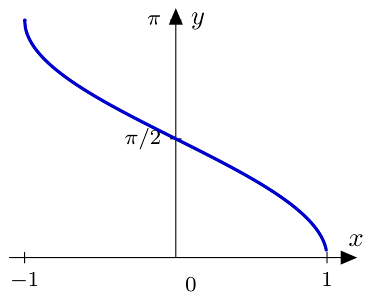
We may be given the lengths of two sides of a right-angled triangle and be asked to find the angle θ between them or be asked to rearrange an equation with a trigonometric function. In order to do these we need to introduce the inverse trigonometric functions. As the trigonometric functions are 'one-to-many' we also need to restrict the ranges of these inverses. The y-axis is measured in radians.

Note: We should also remark that whilst $\left(\sin(\theta)\right)^{-1} = \dfrac{1}{\sin(\theta)}$, the notation $\sin^{-1}(\theta)$ is **not** the same since it represents the inverse function of $\sin(\theta)$ such that $\sin\left(\sin^{-1}(\theta)\right) = \theta$.

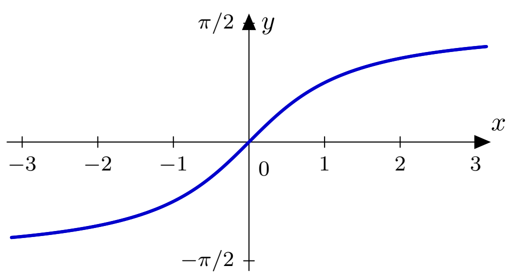
- The inverse function of $\sin(x)$ is $\sin^{-1}(x)$ and can also be denoted as $\arcsin(x)$.
  Domain: $-1 \leq x \leq 1$
  Range: $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$



- The inverse function of $\cos(x)$ is $\cos^{-1}(x)$ and can also be denoted as $\arccos(x)$.
  Domain: $-1 \leq x \leq 1$
  Range: $0 \leq y \leq \pi$



- The inverse function of $\tan(x)$ is $\tan^{-1}(x)$ and can also be denoted as $\arctan(x)$.
  Domain: $x \in \mathbb{R}$
  Range: $\frac{-\pi}{2} < y < \frac{\pi}{2}$
  Asymptotes at $y = \pm\frac{\pi}{2}$



We use the inverse trigonometric functions with SOHCAHTOA to find the values of angles in right-angled triangles in which two of the lengths of the sides have been given. Suppose we know that:

$$\sin(\theta) = \frac{1}{2}$$

To find θ we can apply the inverse sine function to both sides of the equation.

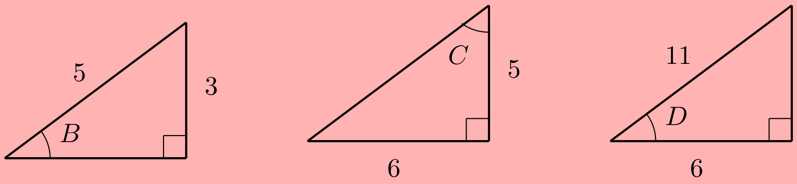
$$\sin^{-1}(\sin(\theta)) = \sin^{-1}\left(\frac{1}{2}\right).$$

$$\implies \theta = \sin^{-1}\left(\frac{1}{2}\right).$$

$$\implies \theta = 30°.$$

We can use a calculator to find $\sin^{-1}\left(\frac{1}{2}\right) = 30°$ or $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ radians.

Example: Using SOHCAHTOA find the size of the angles B, C and D in the right-angled triangles below.



Solution: For shorthand we have used H to denote the hypotenuse, O the opposite and A the adjacent.

1. We know the length of the hypotenuse and the opposite.

$$\sin(\theta) = \frac{\text{O}}{\text{H}} \implies \sin(B) = \frac{3}{5} \implies B = \sin^{-1}\left(\frac{3}{5}\right) = 36.9° \text{ or } 0.64 \text{ radians.}$$

2. We know the length of the adjacent and the opposite.

$$\tan(\theta) = \frac{\text{O}}{\text{A}} \implies \tan(C) = \frac{6}{5} \implies C = \tan^{-1}\left(\frac{6}{5}\right) = 50.2° \text{ or } 0.88 \text{ radians.}$$

3. We know the length of the hypotenuse and the adjacent.

$$\cos(\theta) = \frac{\text{A}}{\text{H}} \implies \cos(D) = \frac{6}{11} \implies D = \cos^{-1}\left(\frac{6}{11}\right) = 56.9° \text{ or } 0.99 \text{ radians.}$$

1.4 Hyperbolics

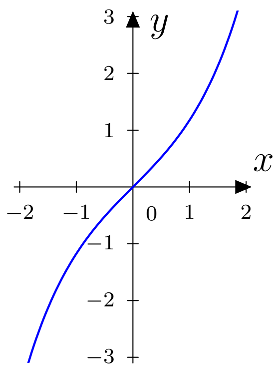
Basics

Another set of functions which appear often in physics are the hyperbolic functions. They are convenient combinations of e^x and e^{-x} which have some useful properties and even share some properties with the trigonometric functions. If you are unfamiliar with the functions e^x and $\ln(x)$, refer to the following resource for a detailed explanation: Exponential and Logarithmic Functions

The three main hyperbolic functions are:

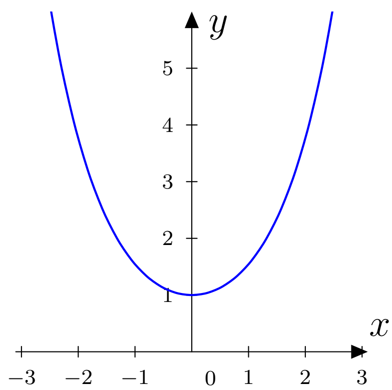
- $\sinh(x)$ (pronounced 'sinch' or 'shine'), which can also be written as

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$



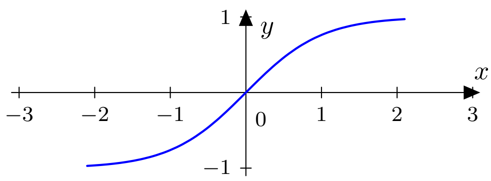
- $\cosh(x)$ (pronounced 'cosh'), which can also be written as

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$



- $\tanh(x)$ (pronounced 'tanch' or 'than'), which can also be written as

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$$



Some useful properties:

- $\sinh(x)$ is an odd function
- $\cosh(x)$ is an even function
- $\tanh(x)$ is an odd function
- $\tanh(x)$ has asymptotes at $y = \pm 1$

Example:

1. Evaluate $\tanh(\ln\sqrt{3})$
2. Solve $\sinh(x) + \cosh(x) = 2$
3. Using the exponential form of $\cosh(x)$ show that $2\cosh^2(x) - \cosh(2x) = 1$

Solution:

1. Using the exponential form of $\tanh(x)$:

$$\begin{aligned}\tanh(\ln\sqrt{3}) &= \frac{e^{2\ln\sqrt{3}} - 1}{e^{2\ln\sqrt{3}} + 1} \\ &= \frac{3-1}{3+1} \\ &= \frac{1}{2}\end{aligned}$$

2. Using the exponential forms we get:

$$\begin{aligned}\frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2} &= 2 \\ \Rightarrow e^x - e^{-x} + e^x + e^{-x} &= 4 \\ \Rightarrow 2e^x &= 4 \\ \Rightarrow e^x &= 2 \\ \Rightarrow x &= \ln(2)\end{aligned}$$

3. Using the exponential form of $\cosh(x)$ we get:

$$\begin{aligned}&2\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^{2x} + e^{-2x}}{2}\right) \\ &\equiv \frac{e^{2x} + 2 + e^{-2x} - e^{2x} - e^{-2x}}{2} \\ &\equiv \frac{2}{2} = 1\end{aligned}$$

Reciprocals and Inverses

Again these functions all have their inverse functions:

- $\sinh^{-1}(x) = \text{arsinh}(x) = \ln(x + \sqrt{x^2+1})$
  $\text{arsinh}(x)$ has domain $\{x : x \geq 1\}$
- $\cosh^{-1}(x) = \text{arcosh}(x) = \ln(x + \sqrt{x^2-1})$
  $\text{arcosh}(x)$ has domain $\{x : x \geq 1\}$

- $\tanh^{-1}(x) = \text{artanh}(x) = \frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(1-x)$
  artanh(x) has domain $\{x : -1 < x < 1\}$

As well as their reciprocal functions:

- $\text{cosech}(x) = \frac{1}{\sinh(x)}$
- $\text{sech}(x) = \frac{1}{\cosh(x)}$
- $\coth(x) = \frac{1}{\tanh(x)}$

Identities

Hyperbolic function identities have very similar forms to the trigonometric identities however there is one key difference outlined in Osborn's rule. This rule states that all the identities for the hyperbolic functions are exactly the same as the trigonometric identities, except whenever a product of two sinh functions is present we put a minus sign in front. For example if a trigonometric formula involved a $\sin^2(x)$, then the corresponding hyperbolic formula would contain a $-\sinh^2(x)$ instead.
Remember : $\tanh(\theta) = \frac{\sinh(\theta)}{\cosh(\theta)}$ so Osborn's rule applies for a $\tanh^2(\theta)$ as well!

Trigonometric	Hyperbolic
$\cos^2(\theta) + \sin^2(\theta) = 1$	$\cosh^2(x) - \sinh^2(x) = 1$
$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$	$\sinh(A \pm B) = \sinh(A)\cosh(B) \pm \cosh(A)\sinh(B)$
$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$	$\cosh(A \pm B) = \cosh(A)\cos(B) \mp \sinh(A)\sinh(B)$
$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$	$\cosh(2\theta) = \cosh^2(\theta) + \sinh^2(\theta)$
$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$	$\sinh(2\theta) = 2\sinh(\theta)\cosh(\theta)$
$1 + \tan^2(\theta) = \sec^2(\theta)$	$1 - \tanh^2(\theta) = \text{sech}^2(\theta)$
$1 + \cot^2(\theta) = \text{cosec}^2(\theta)$	$1 - \coth^2(\theta) = -\text{cosech}^2(\theta)$

Example:

1. Show that $\tanh^2(x) + \text{sech}^2(x) = 1$
2. Simplify $1 - \dfrac{\sinh(x)\sqrt{\cosh^2(x) - 1}}{\cosh^2(x)}$

Solution:

1. Using the exponential forms we get:

$$\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 + \left(\frac{2}{e^x + e^{-x}}\right)^2$$

$$= \frac{e^{2x} - 2 + e^{-2x}}{e^{2x} + 2 + e^{-2x}} + \frac{4}{e^{2x} + 2 + e^{-2x}}$$

$$= \frac{e^{2x} - 2 + e^{-2x} + 4}{e^{2x} + 2 + e^{-2x}}$$

$$= \frac{e^{2x} + 2 + e^{-2x}}{e^{2x} + 2 + e^{-2x}} = 1$$

2. Using $\cosh^2(x) - \sinh^2(x) = 1$ we can see that $\sinh(x) = \sqrt{\cosh^2(x) - 1}$

$$\Rightarrow 1 - \frac{\sinh(x)\sqrt{\cosh^2(x) - 1}}{\cosh^2(x)} = 1 - \frac{\sinh^2(x)}{\cosh^2(x)}$$

$$\Rightarrow 1 - \tanh^2(x)$$

$$= \text{sech}^2(x)$$

1.5 Parametric Equations

We can also express x and y in terms of a parameter t, which gives us two equations; $x = f(t)$ and $y = g(t)$. We call these parametric equations. For example given the equation:

$$y = 9x^2 + 5$$

We can convert this to a Parametric form such as:

$$x = \frac{1}{3}t$$

$$y = t^2 + 5$$

Another trivial parametric form of the same equation is:

$$x = t$$

$$y = 9t^2 + 5$$

There are actually infinitely many ways we can paramaterise each Cartesian equation but some are generally more useful than others.

In a physical system, it is often useful to represent trajectories of particles in terms of the time, using time as the parameter. For example we can say that at time t, a particle has velocity $v(t)$ and acceleration $a(t)$.

Example: Parametrise the equation of a unit circle:

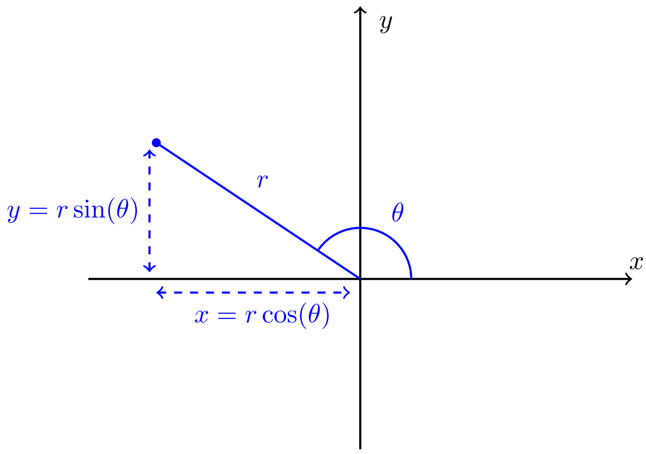
$$x^2 + y^2 = 1$$

Solution: By considering the identity $\cos^2(\theta) + \sin^2(\theta) = 1$, we can parameterise as:

$$x = \cos(t)$$
$$y = \sin(t)$$

1.6 Polar Coordinates

You will already be familiar with coordinates in the form (x, y) meaning that we move x units in the x-direction (along the x-axis) and y in the y-direction (along the y-axis). These are Cartesian coordinates on the x, y-plane. Although Cartesian coordinates are very useful, there are sometimes situations where it is much easier to use another coordinate system called Polar coordinates. These are coordinates in the form (r, θ) where r is the distance to the point from the origin $(0, 0)$ and θ is the angle, in **radians**, between the positive x-axis and the line formed by r. This is all shown in the diagram below:



So we could represent the above coordinate as (x, y) in Cartesian form or as (r, θ) in Polar form. From trigonometry and Pythagoras' theorem there are the following relationships:

- $x = r\cos(\theta)$
- $y = r\sin(\theta)$
- $r = \sqrt{x^2 + y^2}$

We can use the formulae above to allow us to convert between Polar and Cartesian coordinates.

Example: $\left(3, \frac{-\pi}{4}\right)$ is a Polar coordinate. What is the corresponding Cartesian coordinate?

Solution: So $r = 3$ and $\theta = \frac{-\pi}{4}$. So using the formulae above gives:

$x = r\cos(\theta) = 3 \times \cos\left(\frac{-\pi}{4}\right) = \frac{3\sqrt{2}}{2}$

$y = r\sin(\theta) = 3 \times \sin\left(\frac{-\pi}{4}\right) = \frac{-3\sqrt{2}}{2}$

So our Cartesian coordinate is $\left(\frac{3\sqrt{2}}{2}, \frac{-3\sqrt{2}}{2}\right)$.

Example: $(-2, 3)$ is a Cartesian coordinate. What is the corresponding Polar coordinate?

Solution: So we can write $x = -2$ and $y = 3$. Using the formulae above gives:
$r = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$
Now we can find θ.
$x = r\cos(\theta) \Rightarrow \theta = \cos^{-1}\left(\frac{x}{r}\right) = \cos^{-1}\left(\frac{-2}{\sqrt{13}}\right) = 2.158\cdots = 2.16$ radians
So our Polar coordinate is $(\sqrt{13}, 2.16)$.

It is also very useful to be able to convert from Cartesian equations to Polar equations and vice versa. We do this by substituting the above relationships into the equation, so that the new equation only contains terms in x and y, or r and θ. Sometimes it is necessary to first rearrange the relationships to make the substitution easier. For example, some integrals become much easier, and often trivial, once we change the coordinate systems. It is also much easier to represent a radial field using polar coordinates.

Example:

1. Convert the Cartesian equation $3x^2 - x = 12y + 1$ into a Polar equation.
2. Convert the Polar equation $r = -3\sin(\theta)$ into a Cartesian equation.

Solution:

1. We substitute $x = r\cos(\theta)$ and $y = r\sin(\theta)$ into the equation.

$$3(r\cos(\theta))^2 - (r\cos(\theta)) = 12(r\sin(\theta)) + 1$$

$$= 3r^2\cos^2(\theta) - r\cos(\theta) = 12r\sin(\theta) + 1$$

This is now in Polar form, as all the terms are in r or θ and do not contain x or y.

2. We rearrange the relationship $y = r\sin(\theta)$ so that $\sin(\theta) = \frac{y}{r}$. We can then rewrite the equation as

$$r = -3\frac{y}{r}$$

Multiplying through by r gives

$$r^2 = -3y$$

Using the relationship $r = \sqrt{x^2 + y^2}$, we can write

$$x^2 + y^2 = -3y$$

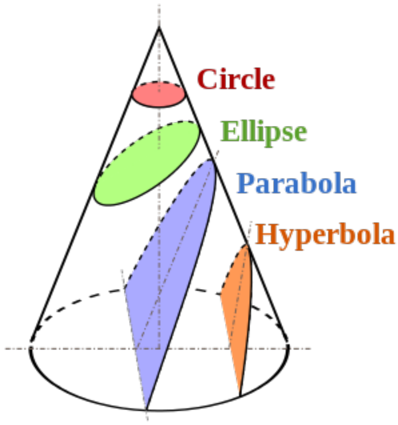
$$x^2 + y^2 + 3y = 0$$

This is now in Cartesian form, as all the terms are in x or y and do not contain r or θ.

1.7 Conics

Standard Equations of Conics

Conic sections are the family of curves obtained by intersecting a cone with a plane. This intersection can take different forms according to the angle the intersecting plane makes with the side of the cone. The standard conic sections are the circle, the parabola, the ellipse and the hyperbola. There are also special cases, such as a point or a line, however these are trivial (sometimes called degenerate) so we shall not cover them.



Conic	Cartesian Equation	Parametric Equation
Circle	$x^2 + y^2 = a^2$	$x = \cos(t),\ y = \sin(t)$
Parabola	$x = 4ay^2$	$x = 4at^2,\ y = t$
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$x = a\cos(t),\ y = b\sin(t)$
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$x = b\tan(t),\ a\sec(t)$

In the case of an ellipse with centre at the origin (0,0), we can use that fact that it crosses the x-axis at $\pm a$ and crosses the y-axis at $\pm b$. This fact can be extended to ellipses with a centre other than (0,0) by translating these axes intercepts according to it's position.
Note: The circle is a special case of the ellipse, when $a = b$

Translating Conics

In order to translate a conic (or any equation) in the x or y direction, we just have to add or subtract a number from x or y respectively.

For example, if we wished to translate the conic $y = 2x^2$ from the origin $(0, 0)$ to the point $(3, -1)$ then we would **subtract** 3 from x and **add** 1 to y, ie $x \to x - 3$ and $y \to y + 1$.
This would give the equation $(y + 1) = 2(x - 3)^2 \Rightarrow y = 2x^2 - 12x + 17$.

Remember: To move a distance a in the x-direction, we substitute $x = (x - a)$ into the equation. To move a distance b in the y-direction, we substitute $y = (y + b)$ into the equation.
Of course if we want to move a distance $-a$ in the x-direction, for example, then we would substitute $x = (x + a)$ into the equation instead.

Recognising Conics

The general equation of any conic is $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. When we see an equation like this we know that it describes a conic, but to find out which conic it describes one has to manipulate the equation into one of the equations given above. This is helpful in areas such as chaos, as we are often faced with equations that we cannot solve and so we can only extract some of the information. One method is to sketch a phase plot which is a graph drawn up from the given equation to allow us to try understand what the equation describes in a physical system. In many cases we can manipulate the given equation to take the form of a conic.

Example: Determine the conic described by the equation: $4x^2 - 8x + 8y^2 + 32y + 20 = 0$.
Solution: Notice that we can divide the entire equation by 4:

$$4x^2 - 8x - 8y^2 - 32y + 20 = x^2 - 2x + 2y^2 + 4y + 5 = 0$$

The next step is to complete the square for both x and y:

$$((x-1)^2 - 1) - 2((y+2)^2 - 2 + 5 = 0$$

$$\Rightarrow (x-1)^2 - 2(y+2)^2 = 4$$

$$\Rightarrow \frac{(x-1)^2}{4} - \frac{(y+2)^2}{2} = 1$$

This is a hyperbola with its centre at $(1, -2)$.

Physics Example: A particle's motion is described by $9x^2 + 4v^2 + 10x^6 = 36$, where v is its velocity and x is its displacement. Describe the motion of the particle for small displacements.

Solution: As the question only asks us to describe the particle's motion for small displacements, we can say that x is much less than one, so that $x^6 \simeq 0$. We can simplify this equation to give $9x^2 + 4v^2 - 18x + 8v \simeq 36$ which is in the general form of a conic.

$$9x^2 + 4v^2 = 36$$

$$\Rightarrow 9x^2 + 4v^2 = 36$$

$$\Rightarrow \frac{x^2}{4} + \frac{v^2}{9} = 1$$

This is an ellipse. Since $a = \sqrt{4} = 2$ and $b = \sqrt{9} = 3$, we can deduce that the ellipse crosses the x-axis at $x = -1$ and $x = 3$, and crosses the v-axis at $v = -4$ and $v = 2$.
Physically, this situation describes Simple Harmonic Motion.

1.8 Review Questions

Easy Questions

Question 1: If

$$f(x) = 3x^2 + 2$$

and

$$g(x) = \cos(4x)$$

find $g \circ f(x)$ and $f \circ g(x)$.

Answer:

$$g \circ f(x) = \cos(12x^2 + 8)$$

$$f \circ g(x) = 3\cos^2(4x) + 2$$

Question 2: Find the inverse of the function $f(x)$ if

$$f(x) = \frac{\sec^{\frac{2}{3}}(x)}{4}.$$

Answer:

$$f^{-1}(x) = \sec^{-1}(8x^{\frac{3}{2}})$$

Question 3: Let

$$f(x) = e^{-x^2}\cos(x)$$

is $f(x)$ even or odd?

Answer:

$f(x)$ is an even function.

Question 4: What is the period of the functions

1. $\cos(4x + 2)$
2. $\tan(3x)$
3. $\sin\left(\frac{4x}{5}\right)$

Answer:

1. $\frac{\pi}{2}$
2. $\frac{\pi}{3}$
3. $\frac{5\pi}{2}$

Question 5: Combine the parametric equations

$$y = 3t^2 + 7$$

$$x = 4t + 1$$

to give y as a function of x.

Answer:

$$y = \frac{3x^2}{16} - \frac{3x}{2} + 10$$

Medium Questions

Question 6: Solve the inequality

$$|x+2|+3 \geq 4x$$

Answer:

$$x \leq \frac{5}{3}$$

Question 7: Use the definitions of the hyperbolic functions to prove that

$$\sinh(2x) = 2\sinh(x)\cosh(x)$$

Question 8: Prove that

$$(\sec\theta - \operatorname{cosec}\theta)(1+\tan\theta+\cot\theta) = \tan\theta\sec\theta - \cot\theta\operatorname{cosec}\theta$$

Question 9: The eccentricity of a hyperbola with center $(0,0)$ and focus $(5,0)$ is $\frac{5}{3}$. What is the standard equation for the hyperbola?

Answer: The planes intersect on the line

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

Question 10: What is the eccentricity of the ellipse $81x2 + y2 - 324x - 6y + 252 = 0$?

Answer:

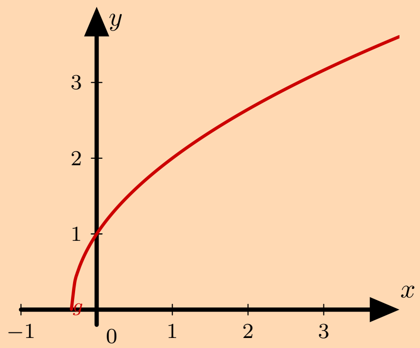
$$\frac{\sqrt{80}}{9}$$

Hard Questions

Question 11: Sketch the curve

$$y = \sqrt{3x+1}.$$

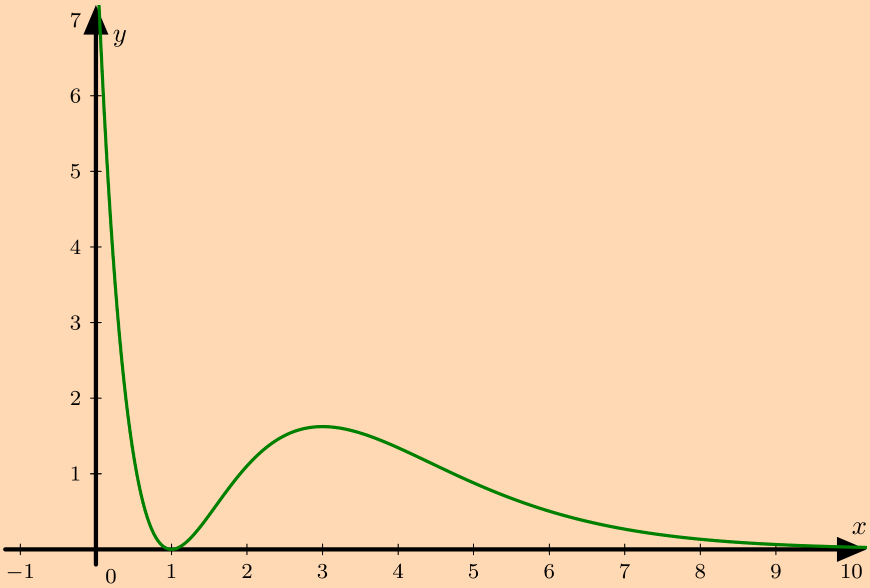
Answer:



Question 13: Shetch the graph of the function

$$f(x) = e^{-x+1}(x+1)^2.$$

Answer:



Question 12: Convert the polar equation

$$r = -8\cos\theta$$

into cartesian coordinates.

Answer:

$$y = \sqrt{-8x - x^2}$$

2 Complex Numbers

2.1 Imaginary Numbers

Imaginary numbers allow us to find an answer to the question 'what is the square root of a negative number?' We define i to be the square root of minus one.

$$\boxed{i = \sqrt{-1}}$$

We find the other square roots of a negative number say $-x$ as follows:

$$\boxed{\sqrt{-x} = \sqrt{x \times -1} = \sqrt{x} \times \sqrt{-1} = \sqrt{x} \times i}$$

Example: Simplify the following roots:

1. $\sqrt{-9}$
2. $\sqrt{-13}$

Solution:

1. $\sqrt{-9} = \sqrt{9 \times -1} = \sqrt{9} \times \sqrt{-1} = 3i$
2. $\sqrt{-13} = \sqrt{13 \times -1} = \sqrt{13} \times \sqrt{-1} = \sqrt{13}\, i$

2.2 Complex Numbers

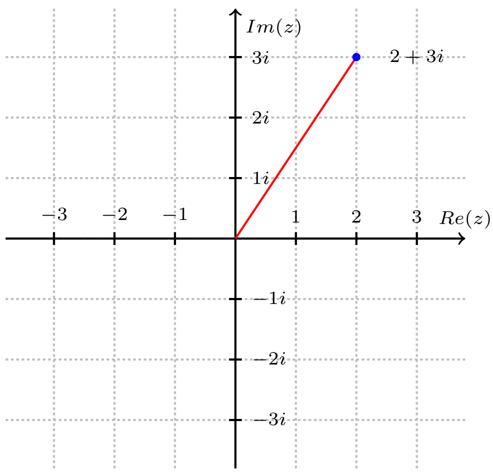
A complex number is a number z that has both a real and an imaginary component. They can be written in the form:

$$\boxed{z = a + bi}$$

where a and b are real numbers and $i = \sqrt{-1}$.

- a is the real part of z we denote this $Re(z) = a$.
- b is the imaginary part of z we denote this $Im(z) = b$.
- $z^* = a - bi$ is called the complex conjugate of $z = a + bi$. To find the complex conjugate, we just replace 'i' with '$-i$'. (Note that some books will use $\bar{z}$ to represent a complex conjugate, rather than z^*)

Just as there is a number line for real numbers, we can draw complex numbers on an x, y axis called an Argand diagram with imaginary numbers on the y-axis and real numbers on the x-axis. Below is a sketch of $2 + 3i$ on the Argand diagram below.



Example: Find the imaginary and real parts of the following complex numbers along with their complex conjugates.

1. $z = -3 + 5i$
2. $z = 1 - 2\sqrt{3}\,i$
3. $z = i$

Solution:

1. For $z = -3 + 5i$ we obtain $Re(z) = -3$, $Im(z) = 5$ and $z^* = -3 - 5i$
2. For $z = 1 + 2\sqrt{3}\,i$ we obtain $Re(z) = 1$, $Im(z) = -2\sqrt{3}$ and $z^* = 1 + 2\sqrt{3}\,i$
3. For $z = i$ we obtain $Re(z) = 0$, $Im(z) = 1$ and $z^* = -i$

Different Forms for Complex Numbers

When a complex number z is in the form $z = a + bi$ we say it is written in Cartesian form. We can also write complex numbers in:

- Modulus-Argument (or Polar) form: $z = r(\cos(\phi) + i\sin(\phi))$
- Exponential form: $z = re^{i\phi}$

where the magnitude $|z|$ (or modulus) of z is the length r. The reason for this is due to Euler's Formula which is presented below.

$$\boxed{|z| = r = \sqrt{a^2 + b^2}}$$

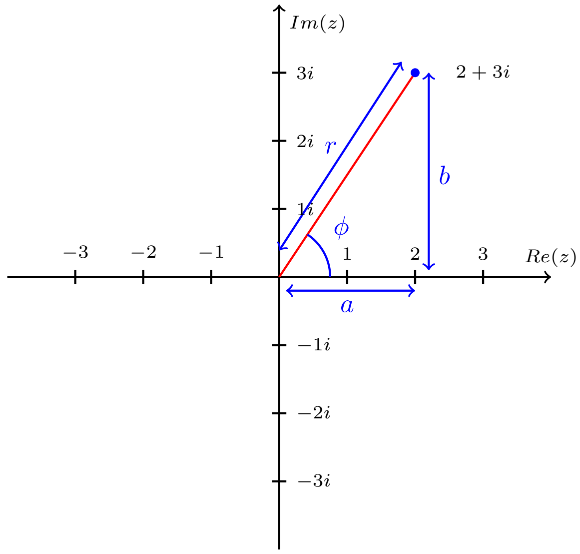
and the argument of z is the angle ϕ (this must be in radians).

$$\boxed{\arg(z) = \phi = \tan^{-1}\left(\frac{b}{a}\right)}$$

This argument ϕ is usually between $-\pi$ and π. Expressed mathematically this is $-\pi < \phi \leq \pi$. It is very important that we check which quadrant of the Argand diagram our complex number lies in however, since the above formula calculates the angle from the appropriate side of the x-axis whereas we usually want the angle from the **positive** x-axis.

1. If z is in the first quadrant $(+x, +y)$ then $arg(z)$ is simply ϕ.
2. If z is in the second quadrant $(-x, +y)$ then $arg(z) = \pi - |\phi|$.
3. If z is in the third quadrant $(+x, -y)$ then $arg(z)$ is simply ϕ.
4. If z is in the fourth quadrant $(-x, -y)$ then $arg(z) = |\phi| - \pi$.

We can see on an Argand diagram the relationship between the different forms for representing a complex number.


Im(z)
3i
2i
1i
−1i
−2i
−3i
−3
−2
−1
1
2
3
Re(z)
2 + 3i
r
b
a
ϕ

Example: Find the modulus and argument of the following complex numbers:

1. $3 + 3i$
2. $-4 + 3i$
3. $-3 + 3i$
4. $-3i$
5. 4

Solution:

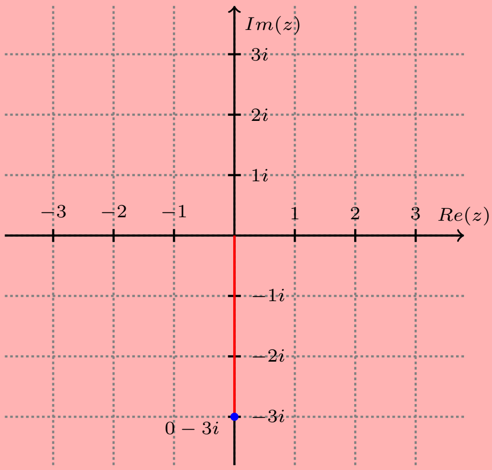
1. For $3 + 3i$ we obtain $|z| = \sqrt{3^2 + 3^2} = \sqrt{18}$ and $\phi = \tan^{-1}\left(\frac{3}{3}\right) = \frac{\pi}{4}$ radians. Since z is in the first quadrant, $arg(z) = \phi = \frac{\pi}{4}$ radians.

2. For $-4 + 3i$ we obtain $|z| = \sqrt{(-4)^2 + (3)^2} = 5$ and $\phi = \tan^{-1}\left(\frac{3}{-4}\right) = -0.64$ radians. But z is in the second quadrant so $arg(z) = \pi - |\phi| = 2.5$ radians

3. For $-3 - 3i$ we obtain $|z| = \sqrt{(-3)^2 + (-3)^2} = \sqrt{18}$ and $\phi = \tan^{-1}\left(\frac{-3}{-3}\right) = \frac{\pi}{4}$ radians. Since z is in the fourth quadrant, $arg(z) = |\phi| - \pi = \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$ radians.

4. For $-3i$ we obtain $|z| = \sqrt{0^2 + (-3)^2} = 3$ and $arg(z) = \tan^{-1}\left(\frac{-3}{0}\right)$

   This calculation does not make sense since we can't divide by zero. However from plotting $-3i$ on an Argand diagram the argument is clearly $arg(z) = -\frac{\pi}{2}$ radians. We could use the fact that we are essentially solving $0 = r\cos(\phi)$ and $-3 = \sin(\phi)$ simultaneously.



5. For 4 we obtain $|z| = \sqrt{4^2 + 0^2} = 4$ and $arg(z) = \tan^{-1}\left(\frac{0}{4}\right) = 0$ radians.

Example: Express $4 - 5i$ in modulus-argument form and exponential form.

Solution: First we need to find the modulus and argument.
$|z| = r = \sqrt{4^2 + (-5)^2} = \sqrt{41}$ and $arg(z) = \phi = \tan^{-1}\left(\frac{-5}{4}\right) = -0.90$ radians
So in modulus-argument form $r(\cos(\phi) + i\sin(\phi))$ we obtain $\sqrt{41}(\cos(-0.90) + i\sin(-0.90))$.
In exponential form $re^{i\phi}$ we obtain $\sqrt{41}e^{-0.90\,i}$.

De Moivre's Theorem

An important theorem when using complex numbers is De Moivre's Theorem, which provides a formula for calculating powers of complex numbers. The theorem states that when a complex number, z, in the modulus-argument form, i.e. $z = r(\cos\phi + i\sin\phi)$ is raised to the power n, where $n \in \mathbb{N}$, we can rearrange the expression by bringing the modulus to the power n and multiplying the argument by the power n:

$$z = r(\cos\phi + i\sin\phi)$$

$$\Rightarrow z^n = r^n(\cos n\phi + i\sin n\phi)$$

This also holds when the complex number is in exponential form:

$$z = re^{i\phi}$$

$$\Rightarrow z^n = r^n e^{in\phi}$$

Example: Using De Moivre's Theorem, show that $\cos(2\phi) = 2\cos^2(\phi) - 1$.

Solution: Notice $Re(\cos(2\phi) + i\sin(2\phi)) = \cos(2\phi)$.
By De Moivre's Theorem,

$$\cos(2\phi) + i\sin(2\phi) = (\cos\phi + i\sin\phi)^2$$

$$= \cos^2\phi + i\sin\phi\cos\phi + i^2\sin^2\phi$$

The real part of this is $\cos^2\phi - \sin^2\phi$ (as $i^2 = -1$).

$$\therefore \cos(2\phi) = \cos^2\phi - 1 + \cos^2\phi$$

$$= 2\cos^2\phi - 1$$

Euler's Formula

We saw in the Hyperbolics section that we can write hyperbolic functions in terms of exponentials. There is a parallel to this involving the trigonometric functions, however we needed to understand complex numbers first.

To determine these formulae, we begin with Euler's formula:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

If we substitute $-\theta$ instead of θ we get:

$$e^{-i\theta} = \cos(\theta) - i\sin(\theta)$$

since $\sin(\theta)$ is an odd function and $\cos(\theta)$ is an even function.

Now we can compute

$$e^{i\theta} + e^{-i\theta} = 2\cos(\theta)$$

and hence,

$$\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

Similarly for sine, remembering the factor of i, we can write:

$$\sin(\theta) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

Arithmetic of Complex Numbers

If we are given two complex numbers $a + bi$ and $c + di$ then the following rules for the arithmetic of complex numbers apply:

1. **Addition:** $(a + bi) + (c + di) = (a + c) + (b + d)i$

   Add the real parts and add the complex parts separately.

2. **Subtraction:** $(a + bi) - (c + di) = (a - c) + (b - d)i$

   Same as addition but with subtraction.

3. **Multiplication:** $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$

   Treat it as two sets of brackets and multiply out as normal. Then collect the real terms and imaginary terms. Remember that $i^2 = -1$.

4. **Division:** $\dfrac{a + bi}{c + di} = \dfrac{ac + bd}{c^2 + d^2} + \dfrac{bc - ad}{c^2 + d^2}i$

   Multiply the numerator and denominator by the conjugate of the denominator. This makes the denominator a real number and the numerator the multiplication of two complex numbers.

Example: Find the following:

1. $(2 + i) + (1 - i)$
2. $(-4 - 4i) - (-5i)$
3. $(1 + i)(2 - 3i)$
4. $\dfrac{1 - i}{2 + 3i}$

Solution:

1. $(2 + i) + (1 - i) = (2 + 1) + (1 - 1)i = 3$
2. $(-4 - 4i) - (-5i) = (-4 - 0) + (-4 - (-5))i = -4 + i$
3. $(1 + i)(2 - 3i) = ((1 \times 2) - (1 \times -3)) + ((1 \times -3) + (1 \times 2))i = 5 - i$
4. $\dfrac{1 - i}{2 + 3i} = \dfrac{(1 \times 2) + (-1 \times 3)}{2^2 + 3^2} + \dfrac{(-1 \times 2) - (1 \times 3)}{2^2 + 3^2}i = \dfrac{-1}{13} - \dfrac{5}{13}i$

Physics Example: A particle is described by a wavefunction Ψ. To calculate the probability of finding this particle in a particular region, $\Psi\Psi^*$ needs to be calculated. When Ψ is in the form:

$$\Psi = a + bi$$

calculate $\Psi\Psi^*$.

Solution: Since $\Psi = a + bi$, this means that $\Psi^* = a - bi$. Multiplying them together gives:

$$\begin{aligned}
\Psi\Psi^* &= (a + bi)(a - bi) && \text{Expand the brackets} \\
&= a^2 + abi - abi - (bi)^2 && \text{Simplify this} \\
&= a^2 - b^2 i^2 && \text{Simplify the } i^2 \\
&= a^2 + b^2
\end{aligned}$$

Note: $a^2 + b^2$ is also r^2 if the complex number was in polar or exponential form. Since r is called the modulus, $\Psi\Psi^*$ is often called the mod-squared distribution since it produces the modulus squared.

2.3 Applications of Complex Numbers

nth Roots of Unity

An nth root of unity, where n is a positive integer, is a complex number $z = a + bi$ such that

$$z^n = 1$$

Before meeting complex numbers, the only solution to this equation that we could obtain is $z = 1$ (and $z = -1$ if n is even). Now that we can apply complex numbers to this situation, we see that there are actually n roots to this equation, however 1 is the only **real** root (in the case of an even n we have two real roots $z = \pm 1$). In order to work out the other roots, we must apply de Moivre's Theorem.

Example: Find all the solutions of the equation $z^3 = 1$.

Click here for a video example

Solution: Begin by writing z in exponential form

$$z^3 = (re^{i\phi})^3$$
$$= r^3 e^{3i\phi}$$

Now, since $z = 1$ is clearly one of the solutions, we know that the modulus of z is 1. The reason for this is that $r \geq 1$ and $r^3 = 1$ so $r = 1$. A consequence of this is that the roots all lie on a circle of radius 1. We also know that 1 can be written as $1e^{i(2\pi k)}$, where $k = 0, \pm 1, \pm 2 ....$ Thus the equation reduces to

$$e^{3i\phi} = e^{i(2\pi k)}$$

By comparing the indices we can say that

$$3i\phi = 2\pi k i$$
$$\Rightarrow \phi = \frac{2\pi}{3} k$$

Therefore $r = 1$ and $\phi = 0, \pm\frac{2\pi}{3}, \pm\frac{4\pi}{3} ...$
Then

$$z = 1, e^{i\frac{2\pi}{3}}, e^{-i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}, e^{-i\frac{4\pi}{3}}, ...$$

Finally, by thinking of these positions on a circle, note that

$$e^{i\frac{4\pi}{3}} = e^{-i\frac{2\pi}{3}}$$

and similarly for other combinations of solutions. This leaves us with the three distinct solutions:

$$z = 1, e^{i\frac{2\pi}{3}}, e^{-i\frac{2\pi}{3}}$$

Note: For $z^n = 1$, we will always get n complex roots. It is also possible to apply this method to complex numbers other than 1.

We can extend this technique to find roots of any complex number.

Example: Find all the solutions of the equation $z^2 = 2i$.

Solution: If we write this in modulus argument form we get

$$z^2 = 2i = 2\Big(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\Big)$$

As $\sin(x)$ and $\cos(x)$ are periodic functions with a period of 2π it is true to say

$$z^2 = 2i = 2\Big(\cos\left(\frac{\pi}{2} + 2\pi k\right) + i\sin\left(\frac{\pi}{2} + 2\pi k\right)\Big)$$

By de Moivre's theorem we know

$$z = \sqrt{2}\Big(\cos\left(\frac{\pi}{4}\pi k\right) + i\sin\left(\frac{\pi}{4} + \pi k\right)\Big)$$

At $k = 0$ we have the solution

$$z = \sqrt{2}\Big(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\Big)$$

At $k = -1$ we obtain

$$z = \sqrt{2}\Big(\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\Big)$$

These are our solutions.

Note: As we had z^2, we have two solutions. The reason for the k values used above is that they give arguments in the range $-\pi < \theta \leq \pi$, which is the range for the **principle argument** for complex numbers. With other values of k we would obtain either of the solutions we had already achieved but with an argument out of the principle argument range.

Polynomials with Real Coefficients

We can use what we know about complex numbers and their complex conjugates to help us solve polynomial equations with real coefficients that we could not previously solve.

When using the quadratic formula:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

we would not have known what to do if $b^2 - 4ac$ was negative, however now we are able to solve this square root using imaginary numbers.

Example: Solve the equation $x^2 + 3x + 12$ using the quadratic formula

Solution: So $a = 1$, $b = 3$ and $c = 12$ so the quadratic formula gives:

$$\begin{aligned}\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &= \frac{-3 \pm \sqrt{3^2 - 4 \times 1 \times 12}}{2 \times 1} \\ &= \frac{-3 \pm \sqrt{-39}}{2} \\ &= \frac{-3 \pm i\sqrt{39}}{2}\end{aligned}$$

This gives us $x = \dfrac{-3 + i\sqrt{39}}{2}$ and $x = \dfrac{-3 - i\sqrt{39}}{2}$.
Note: The two roots are complex conjugates of each other.

The above example shows that we obtain two complex roots which are complex conjugates of each other. In fact, if we obtain a complex root for any polynomial equation with real coefficients, the complex conjugate of this root is **always** another root of the polynomial. This means that the roots of a polynomial with real coefficients are either real numbers, pairs of complex conjugates, or a combination of these. No polynomial with real coefficients has just one single complex root; they always come in pairs.

Quadratic Equations with Complex Coefficients

When dealing with a quadratic equation with complex coefficients, the best way to approach solving the equation is to complete the square.

Example: Given that the square roots of $3+4i$ are given by $2-i$ and $-2+i$, solve the equation $z^2+(4+6i)z-8+16i=0$ by completing the square.

Solution: By completing the square we obtain

$$\big(z+(2+3i)\big)^2-(2+3i)^2-8+16i=0$$

$$\big(z+(2+3i)\big)^2-3-4i=0$$

$$\big(z+(2+3i)\big)^2=3+4i$$

As we are given the square roots of $3+4i$ it is clear to see that

$$z+(2+3i)=2-i$$

$$z=-4i$$

OR

$$z+(2+3i)=-2+i$$

$$z=-4-2i$$

Phasors

Using the idea that a complex number can be written in the form Ae^{ix} allows us to understand *phasors*, which can simplify mathematical tasks we face in physics, particularly in optics and AC circuits.
Consider Ae^{ix} and replace the x with ωt, $Ae^{i\omega t}$, where ω is an angular frequency, this allows us to model the complex number as a vector spinning around an origin. To understand this, think about plotting Ae^{ix} on an Argand diagram. Now if x is time dependent, the angle to the real axis will also be time dependent and this angle will increase as time moves forward but the magnitude of the complex number will stay constant. i.e. the complex number will spin about the origin. Since $\omega=\dfrac{2\pi}{T}$, where T is the time period, we can say that $t=T$ corresponds to one spin of the phasor as:

$$Ae^{i\omega t}=Ae^{i\frac{2\pi}{T}t}=Ae^{i2\pi}$$

The idea of phasors makes addition of sinusoidal functions a simple task as the real part of the complex number on the phasor at any t value of $Ae^{i\omega t-\theta}$ is completely equivalent to the value of $A\cos(\omega t-\theta)$ at the same point. Here is an animated picture that shows this. Since we model the complex number as a spinning vector, when adding more than one function together on a phasor we use vector addition. Here is an animated picture that shows how why adding the complex numbers like vectors on a phasor works.

Example: Simplify:

$$\cos(\omega t + 30) + \cos(\omega t + 150) + \cos(\omega t - 90)$$

1. By trigonometric identities
2. By use of phasors

Solution:

1. Using that $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ we obtain

$$\cos(\omega t)\cos(30) - \sin(\omega t)\sin(30) + \cos(\omega t)\cos(150) -$$

$$\sin(\omega t)\sin(150) + \cos(\omega t)\cos(-90) - \sin(\omega t)\sin(-90)$$

   This is equivalent to:

$$\left(\frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{3} + 0\right)\cos(\omega t) + \left(-\frac{1}{2} - \frac{1}{2} + 1\right)\sin(\omega t) = 0$$

2. Using phasors we can simply write:

$$e^{30i} + e^{150i} + e^{-90i} = 0$$

   Think about the Argand diagram; we end on a purely imaginary number and as we only consider the real part of the phasor, the answer must be zero.

2.4 Review Questions

Easy Questions

Question 1: Simplify the following roots:

1. $\sqrt{-101}$
2. $\sqrt{-256}$

Answer:

1. $\sqrt{101}i$
2. $16i$

Question 2: Find the complex conjugate of the following complex numbers:

1. $z = 2 - 3i$
2. $z = 3$

Answer:

1. $z^* = 2 + 3i$
2. $z^* = 3$

Question 3: Let $z = 4 - 2i$ and $w = 7 + i$. Evaluate the following:

1. $z + w$
2. $w - z^*$
3. $w^* z$

Answer:

1. $11 - i$
2. $3 + i$
3. $26 - 28i$

Question 4: Simplify

$$z = \frac{5+i}{2+3i}$$

Answer:

$$z = 1 - i$$

Question 5: Sketch the following complex numbers on an Argand diagram:

1. $z = 1 + 5i$
2. $w = -2 + 4i$

Medium Questions

Question 6: Write the following complex numbers in modulus argument form:

1. $z = 1 - \sqrt{3}$
2. $w = -\frac{3}{2} + \frac{\sqrt{3}}{2}i$

Answer:

1. $z = 2(\cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3}))$
2. $w = \sqrt{3}(\cos(\frac{5\pi}{6}) + i\sin(\frac{5\pi}{6}))$

Question 7: Write the following complex numbers in exponential form:

1. $z = -2 - 2\sqrt{3}i$
2. $w = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

Answer:

1. $z = 4e^{-\frac{2\pi}{3}i}$
2. $w = e^{\frac{\pi}{4}i}$

Question 8: If

$$z = 3e^{\frac{\pi}{3}i}$$

find z^2 and z^3 in the form $a + bi$.

Answer:

$$z^2 = \frac{9}{2} + \frac{\sqrt{3}}{2}i$$
$$z^3 = -27$$

Hard Questions

Question 9: Find, in exponential form, all the solutions of the equation

$$z^5 = 1$$

Answer:

$$z = 1$$
$$z = e^{\frac{2\pi}{5}i}$$
$$z = e^{\frac{4\pi}{5}i}$$
$$z = e^{-\frac{2\pi}{5}i}$$
$$z = e^{-\frac{4\pi}{5}i}$$

Question 10: Find, in exponential form, all the fourth roots of the complex number

$$z = -2 + 2\sqrt{3}i$$

Answer:

$$z = \sqrt{2}e^{\frac{\pi}{6}i}$$
$$z = \sqrt{2}e^{\frac{2\pi}{3}i}$$
$$z = \sqrt{2}e^{-\frac{\pi}{3}i}$$
$$z = \sqrt{2}e^{-\frac{5\pi}{6}i}$$

Question 11: Given that $z = -3 + 2i$ is a root of the equation

$$z^4 + 5z^3 + z^2 - 49z - 78$$

factorise the polynomial in terms of real factors.

Answer:

$$(z^2 + 6z + 13)(z + 2)(z - 3) = 0$$

3 Matrices

3.1 Introduction to Matrices

A matrix is defined as an array of numbers such as:

$$\begin{pmatrix} 3 & 1 & 8 \\ 0 & -4 & 5 \end{pmatrix}$$

- Matrices come in different sizes. We write the size of the matrix as 'rows $\times$ columns'. So the matrix above is a 2×3 matrix.
- We use capital letters such as A or B to denote matrices. (Often you will see matrices denoted as a capital letter with double underlining, e.g. $\underline{\underline{A}}$)
- The numbers contained in a matrix are known as matrix elements. They are denoted a_{ij} where i represents the row and j the column in which the element is in. For example in the above matrix $a_{11} = 3$ and $a_{23} = 5$.
- Matrices with an equal number of rows and columns are called square matrices. For example the 2×2 matrix below:

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

Note: Ordinary scalar numbers such as 14 can be considered as 1×1 matrices, e.g. (14).

Example: For the matrices below find the size of the matrix and the value of a_{21} and a_{33} if possible.

1. $A = \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$
2. $B = \begin{pmatrix} 1 \\ -1 \\ 40 \end{pmatrix}$
3. $C = \begin{pmatrix} a & b & c \\ e & d & f \\ g & h & i \end{pmatrix}$

Solution:

1. The matrix A has 4 rows and 3 columns, hence is a 4×3 matrix. Here $a_{21} = 1$ and $a_{33} = 6$.
2. The matrix B has 3 rows and 1 column, hence is a 3×1 matrix. Here $a_{21} = -1$ and a_{33} does not exist as this is a 3×1 matrix.
3. The matrix C has 3 rows and 3 columns, hence it is a 3×3 matrix (also a square matrix). Here $a_{21} = d$ and $a_{33} = i$

Matrices are an excellent way of expressing large amounts of information or data in a small space.

3.2 Matrix Algebra

Addition and Subtraction

Matrices can only be added or subtracted if they are of the same size. Suppose we are given the 2×2 matrices below:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

Then their sum is found by adding together corresponding matrix elements:

$$\boxed{A + B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}}$$

Subtraction works in exactly the same way:

$$\boxed{A - B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11} - b_{11} & a_{12} - b_{12} \\ a_{21} - b_{21} & a_{22} - b_{22} \end{pmatrix}}$$

In general if we are adding or subtracting two $m \times n$ matrices:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \pm \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} = \begin{pmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & \cdots & a_{1n} \pm b_{1n} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & \cdots & a_{2n} \pm b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} \pm b_{m1} & a_{m2} \pm b_{m2} & \cdots & a_{mn} \pm b_{mn} \end{pmatrix}$$

Note: Matrix addition and subtraction are commutative e.g. $A + B = B + A$.

Example: Calculate the following:

1. $\begin{pmatrix} 1 & 1 \\ 4 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 4 \\ 2 & -3 \end{pmatrix}$

2. $\begin{pmatrix} 10 \\ -3 \\ 0 \end{pmatrix} - \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}$

3. $\begin{pmatrix} 9 \\ -2 \end{pmatrix} + \begin{pmatrix} -9 & 0 & 1 \\ 13 & 13 & 0 \\ 9 & 0 & 9 \end{pmatrix}$

Solution:

1. $\begin{pmatrix} 1 & 1 \\ 4 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 4 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 1 + (-1) & 1 + 4 \\ 4 + 2 & 0 + (-3) \end{pmatrix} = \begin{pmatrix} 0 & 5 \\ 6 & -3 \end{pmatrix}$

2. $\begin{pmatrix} 10 \\ -3 \\ 0 \end{pmatrix} - \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 10 - (-3) \\ -3 - 2 \\ 0 - 0 \end{pmatrix} = \begin{pmatrix} 13 \\ -5 \\ 0 \end{pmatrix}$

3. We can not add these matrices as they are of different sizes!

Multiplication by a Constant

The simplest form of multiplication is where we multiply a matrix by a constant or scalar. We show the rule on a 2×2 matrix but it follows in the same way for any sized matrix we simply multiply every element by the scalar.

$$\boxed{\lambda A = \lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}}$$

Example: Find the following:

1. $2\begin{pmatrix} -1 & 2 \\ 0 & 4 \end{pmatrix}$

2. $\dfrac{\sqrt{3}}{3}\begin{pmatrix} -9 & 0 & 3 \\ 12 & 12 & 0 \\ 3 & 0 & 3 \end{pmatrix}$

Solution:

1. $2\begin{pmatrix} -1 & 2 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} -1 \times 2 & 2 \times 2 \\ 0 \times 2 & 4 \times 2 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ 0 & 8 \end{pmatrix}$

2. $\dfrac{\sqrt{3}}{3}\begin{pmatrix} -9 & 0 & 3 \\ 12 & 12 & 0 \\ 3 & 0 & 3 \end{pmatrix} = \begin{pmatrix} -9 \times \frac{\sqrt{3}}{3} & 0 \times \frac{\sqrt{3}}{3} & 3 \times \frac{\sqrt{3}}{3} \\ 12 \times \frac{\sqrt{3}}{3} & 12 \times \frac{\sqrt{3}}{3} & 0 \times \frac{\sqrt{3}}{3} \\ 3 \times \frac{\sqrt{3}}{3} & 0 \times \frac{\sqrt{3}}{3} & 3 \times \frac{\sqrt{3}}{3} \end{pmatrix} = \begin{pmatrix} -3\sqrt{3} & 0 & \sqrt{3} \\ 4\sqrt{3} & 4\sqrt{3} & 0 \\ \sqrt{3} & 0 & \sqrt{3} \end{pmatrix}$

Matrix Multiplication

Suppose we want to find the product AB of the matrices A and B. Then:

- The number of columns of A must be equal to the number of rows of B.
- The product matrix has the same number of rows as A and the same number of columns as B.

Below is the rule for find the product of two 2×2 matrices:

$$\boxed{AB = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}}$$

In general if we want to find the product AB of a $m \times l$ matrix A and $k \times n$ matrix B:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1l} \\ a_{21} & a_{22} & \cdots & a_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{ml} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1} & b_{k2} & \cdots & b_{kn} \end{pmatrix}$$

$$\text{so that } AB = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{pmatrix}$$

We calculate c_{ij} to be the scalar product of row i from the first matrix and column j from the second matrix. (The scalar dot product will be covered more extensively in the section on 'Vectors'.)

Example: Find the following products:

1. $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$

2. $\begin{pmatrix} 10 & 2 & 2 \\ 8 & -1 & 4 \end{pmatrix}\begin{pmatrix} -5 & -3 \\ 2 & 2 \end{pmatrix}$

Solution:

1. As we are multiplying a 2×2 matrix by another 2×2 matrix the product matrix will also be a 2×2 matrix in the form below:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

We now calculate the elements of the product matrix above:

To find a_{11} we take the dot product of row 1 and column 1:

$$a_{11} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 1 \times 3 + 0 \times 2 = 3$$

To find a_{12} we take the dot product of row 1 and column 2:

$$a_{12} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \times 1 + 0 \times 1 = 1$$

To find a_{21} we take the dot product of row 2 and column 1:

$$a_{21} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 2 \times 3 + 1 \times 2 = 8$$

To find a_{22} we take the dot product of row 2 and column 2:

$$a_{22} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 \times 1 + 1 \times 1 = 3$$

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 8 & 3 \end{pmatrix}$$

2. The number of columns of the matrix on the left does not equal to the number of rows of the matrix on the right hence this product can not be found!

Note: Matrix multiplication is **NOT** commutative i.e. in general $AB \neq BA$

Example:

Let $A = \begin{pmatrix} 1 & 1 \\ 4 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 4 \\ 2 & -3 \end{pmatrix}$ Then $AB = \begin{pmatrix} 1 & 1 \\ -4 & 16 \end{pmatrix}$ and $BA = \begin{pmatrix} 15 & -1 \\ -10 & 2 \end{pmatrix}$ This shows that matrix multiplication is not commutative as $AB \neq BA$.

3.3 The Identity Matrix, Determinant and Inverse of a Matrix

The Identity Matrix

The Identity Matrix (or Unit Matrix) is a square matrix which is denoted I and has '1' along the leading diagonal (from top left to bottom right) with '0' in all other positions.
The 2×2 identity matrix is:

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The 3×3 identity matrix is:

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In general the $n \times n$ identity matrix is:

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

The Identity matrix has the property that when multiplied with another matrix it leaves the other matrix unchanged:

$$\boxed{AI = A = IA}$$

Example:

$$\begin{pmatrix} 3 & 4 \\ 8 & 9 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 8 & 9 \end{pmatrix}$$

Transpose of a Matrix

The transpose of a matrix is denoted A^T and is obtained by interchanging the rows and columns of the matrix. Visually, we may visualise this as 'reflecting' elements about the leading diagonal. The rule for a 3×3 matrix A is:

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$A^T = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

Note: This formula can be extended to any square matrix.

Example: For the matrix A below find A^T.

$$A = \begin{pmatrix} 3 & 4 & 7 \\ 8 & 9 & 11 \\ 1 & 2 & 6 \end{pmatrix}$$

Solution:

$$A^{-T} = \begin{pmatrix} 3 & 8 & 1 \\ 4 & 9 & 2 \\ 7 & 11 & 6 \end{pmatrix}$$

Determinant of a Square Matrix

The determinant of a square matrix is denoted $|A|$ or 'det(A)'. To find the determinant of a 2×2 matrix, A, the following formula applies:

$$\boxed{\text{For } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ the determinant is } |A| = ad - bc}$$

There exist more complicated methods for finding the determinants of 3×3 matrices and square matrices of larger size. The first method shown below uses definitions and a provides more formal, general approach, whereas the second method gives a very practical procedure which can be followed in almost any case.

Method 1 (Formal):

First we will require some definitions:

- The (i,j)-**minor** of a matrix A is the determinant of the submatrix obtained by deleting the ith row and jth column of A. We denote this submatrix as $M_{ij}(A)$.
- The (i,j)-**cofactor** of a matrix A is the matrix $C_{ij}(A) = (-1)^{i+j}M_{ij}(A)$.

Now, in order to calculate the determinant of an $n \times n$ matrix, we calculate:

$$\boxed{\det(A) = a_{11}C_{11}(A) + a_{12}C_{12}(A) + a_{13}C_{13}(A) + ... + a_{1n}C_{1n}(A)}$$

This is the formal explanation of how to calculate a determinant, however we can explain this in a less technical way.

Method 2 (Informal):

If we have a 3×3 matrix:

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

we can write the above formula in a more practical way:

$$\boxed{\det(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}}$$

This formula is obtained by working along the top row of the matrix and, for each element, following this procedure:

1. Check the matrix of signs:
$$\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$
and write down the sign that corresponds to the position of our element.
2. Write down the element we are considering.
3. Cover up the row and column containing the element.
4. Calculate the determinant for the submatrix of elements that are not covered up and multiply our element by this determinant.
5. Move along the row to the next element and repeat steps 1-4.

We can also produce other formulae by completing the above procedure on different rows and columns, however only one formula is adequate. In addition, this approach can be very easily extended to $n \times n$ matrices.

Two useful definitions are as follows:

- A matrix whose determinant is zero ($|A| = 0$) is said to be singular.
- A matrix whose determinant is non-zero ($|A| \neq 0$) is said to be non-singular.

Example: Find the determinant of the matrix A:

$$A = \begin{pmatrix} 1 & -4 \\ 2 & 5 \end{pmatrix}$$

Solution:

$$|A| = \begin{vmatrix} 1 & -4 \\ 2 & 5 \end{vmatrix} = (1 \times 5) - (-4 \times 2) = 13$$

Note: The matrix A is non-singular.

Example: Find the determinant of the matrix B:

$$B = \begin{pmatrix} 6 & 3 & 2 \\ 9 & 11 & -3 \\ 7 & 8 & 4 \end{pmatrix}$$

Solution:

$$\begin{aligned} \det(B) &= 6\begin{vmatrix} 11 & 3 \\ 8 & 4 \end{vmatrix} - 3\begin{vmatrix} 9 & 3 \\ 7 & 4 \end{vmatrix} + 2\begin{vmatrix} 9 & 11 \\ 7 & 8 \end{vmatrix} \\ &= 6\big[1 \times 4 - (-3) \times 8\big] - 3\big[9 \times 4 - (-3) \times 7\big] + 2\big[9 \times 8 - 11 \times 7\big] \\ &= 408 - 171 - 10 \\ &= 227 \end{aligned}$$

Note: The matrix B is non-singular.

Inverse of a Matrix

Only non-singular matrices have an inverse matrix. The inverse of a matrix A is denoted A^{-1} and has the following property:

$$\boxed{AA^{-1} = A^{-1}A = I}$$

To find the inverse of a 2×2 matrix A we have the following formula:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\text{where } |A| = ad - bc \neq 0$$

Note: There exists more complicated methods for finding the inverses of 3×3 matrices and square matrices of larger size, which we will not discuss here.

Example: Find the inverse A^{-1} of the matrix A.

$$A = \begin{pmatrix} 1 & -4 \\ 2 & 5 \end{pmatrix}$$

Solution: From the previous example on determinants we know that $|A| = 13$. So the inverse of A is:

$$A^{-1} = \frac{1}{13}\begin{pmatrix} 5 & 4 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{5}{13} & \frac{4}{13} \\ -\frac{2}{13} & \frac{1}{13} \end{pmatrix}$$

3.4 Review Questions

Easy Questions

In this section we will let

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 3 & -9 & 12 \\ 4 & 4 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 5 & 5 & 5 \\ 16 & -4 & -8 \\ 7 & 0 & 0 \end{pmatrix}$$

Question 1: Find $A + B$ and $B - A$.

Answer:

$$A + B = \begin{pmatrix} 6 & 8 & 10 \\ 19 & -13 & 4 \\ 11 & 4 & 1 \end{pmatrix}$$

$$B - A = \begin{pmatrix} -4 & -2 & 0 \\ -13 & -5 & 20 \\ -3 & 4 & 1 \end{pmatrix}$$

Question 2: Show that $AB \neq BA$.

Question 3: What is $4 \times A$.

Answer:

$$4A = \begin{pmatrix} 4 & 12 & 20 \\ 12 & -36 & 48 \\ 16 & 16 & 4 \end{pmatrix}$$

Question 4: What matrix X, when multiplied by A, gives A, i.e.

$$XA = A$$

Question 5: Find AB^T.

Answer:

$$AB^T = \begin{pmatrix} 45 & -36 & 7 \\ 30 & -12 & 21 \\ 45 & 40 & 28 \end{pmatrix}$$

Medium Questions

In this section we will let

$$A = \begin{pmatrix} 5 & 3 \\ 1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 3 & 9 \\ 1 & -5 \end{pmatrix}$$

$$C = \begin{pmatrix} 7 & -6 & 7 \\ 0 & -2 & 1 \\ 4 & 0 & 8 \end{pmatrix}$$

Question 6: Find the inverse of B.

Answer:

$$B^{-1} = \begin{pmatrix} \frac{5}{24} & \frac{3}{8} \\ \frac{1}{24} & -\frac{1}{8} \end{pmatrix}$$

Question 7: Find the determinant of C.

Answer:

$$\det(C) = -80$$

Question 8: Find X such that $AX = B$.

Answer:

$$X = \begin{pmatrix} 1 & -5 \\ \frac{1}{3} & \frac{34}{3} \end{pmatrix}$$

4 Vectors

4.1 Introduction to Vectors

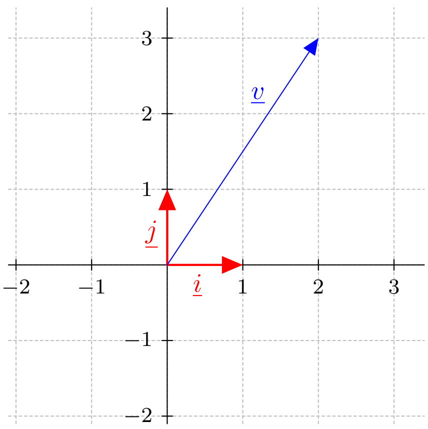
When describing some quantities, just a number and units aren't enough. Say we are navigating a ship across the ocean and we are told that the port is 5km away but we don't know in which direction we have to sail. 5km due east would give us all the information that we need. This is an example of a **vector**, a quantity that has *magnitude* and *direction*.

- Some physics related vector quantities are velocity, force, acceleration, linear and angular momentum and electric and magnetic fields.
- Quantities that have just a *magnitude* are **scalars**. Examples of these are temperature, mass and speed.

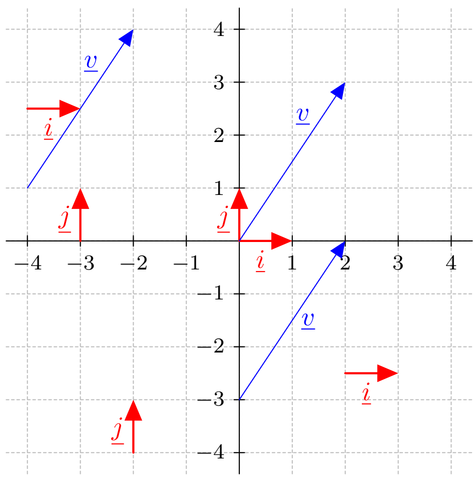
Notation: To distinguish vectors from scalars we write them with a line underneath, e.g. $\underline{\mathrm{F}}$ is used as the symbol for a force vector. We can also write them in bold type, e.g. $\mathbf{F}$, or with an arrow above, e.g. $\vec{F}$. In addition to this, if we want to specify a vector $\underline{a}$ from the point A to the point B, we can write $\underline{a} = \vec{AB}$. Using this notation, a vector $\underline{p}$ from the origin to the point P would be $\underline{p} = \vec{OP}$.

Vectors in 2-D Space

A vector in 2-D space can be written as a combination of 2 base vectors, $\underline{i}$ being a horizontal vector (in the x direction) and $\underline{j}$ being a vertical vector (in the y direction). They are both unit vectors meaning that their length (magnitude) is 1. For an example the vector $\underline{v} = 2\underline{i} + 3\underline{j}$ is shown below:



Note: The vector $\underline{v}$ above also describes all vectors that move 2 $\underline{i}$ units and 3 $\underline{j}$ units, they don't have to start at the origin. All the vectors in the graph below are exactly the same:



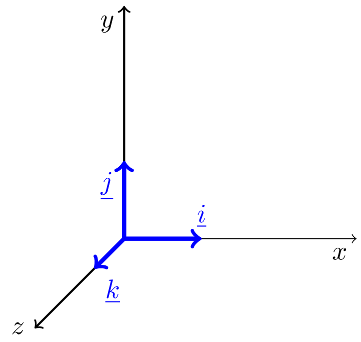
Writing a general 2-D vector $\underline{u}$ as a combination of the unit vectors would give:

$$\boxed{\begin{array}{c}\underline{u} = a\underline{i} + b\underline{j} \\ \text{Where } a \text{ and } b \text{ are real numbers}\end{array}}$$

However, if we wish to specify a single point in space then we can use position vectors which always start at the origin, for example $\underline{p} = \vec{OP}$.

Vectors in 3-D Space

Any vector in 3-D space can be written as a combination of 3 base vectors; $\underline{i}$, $\underline{j}$ and $\underline{k}$, each being the unit vector in the x, y and z directions. Below is a diagram of the 3 axes with their respective unit vectors:



Writing a general vector $\underline{v}$ as a combination of the unit vectors would give:

$$\boxed{\begin{array}{c}\underline{v} = a\underline{i} + b\underline{j} + c\underline{k} \\ \text{Where } a,\ b \text{ and } c \text{ are real numbers}\end{array}}$$

Representation of Vectors

A vector $\underline{v}$ can be written in the following forms:

1. **Unit vector notation:**

$$\boxed{\underline{v} = a\underline{i} + b\underline{j} + c\underline{k}}$$

   The vector is displayed as the clear sum of it's base vectors. In 3-D space those are the $\underline{i}$, $\underline{j}$ and $\underline{k}$ vectors.

2. **Ordered set notation:**

$$\boxed{\underline{v} = (a, b, c)}$$

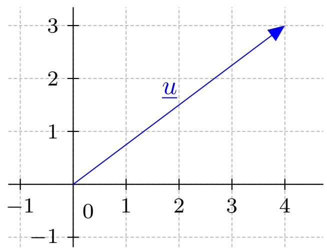
   The vector is in the same form as unit vector notation (a, b and c are the same numbers) but it is more compact.

3. **Column notation:**

$$\boxed{\underline{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}}$$

   This is identical to ordered set notation apart from it being vertical. This form makes some calculations easier to visualise, such as the dot product.

For example, the 2-D vector $\underline{u}$ shown below can be written as $\underline{u} = 4\underline{i} + 3\underline{j}$ or $\underline{u} = (4, 3)$ or $\underline{u} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$



Magnitude of Vectors

If we know the lengths of each of the component vectors we can find the length (or magnitude) of the vector using Pythagoras:

$$\text{For a vector } \underline{v} = a\underline{i} + b\underline{j} + c\underline{k}$$
$$\text{Magnitude of } \underline{v} = |\,\underline{v}\,| = \sqrt{a^2 + b^2 + c^2}$$

For vector $\underline{u} = 4\underline{i} + 3\underline{j}$ above we can calculate the magnitude $|\,\underline{u}\,|$ to be $\sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5$

Example: Find the magnitude of the following vectors:

1. $(2, 6, 3)$
2. $\underline{i} + 7\underline{j} + 4\underline{k}$
3. $\begin{pmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{pmatrix}$

Solution:

1. $|\,(2, 6, 3)\,| = \sqrt{2^2 + 6^2 + 3^2} = \sqrt{4 + 36 + 9} = \sqrt{49} = 7$
2. $|\underline{i} + 7\underline{j} + 4\underline{k}\,| = \sqrt{1^2 + 7^2 + 4^2} = \sqrt{1 + 49 + 16} = \sqrt{66}$
3. $\left|\begin{pmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{pmatrix}\right| = \sqrt{\sqrt{2}^2 + \sqrt{2}^2} = \sqrt{2 + 2} = \sqrt{4} = 2$

Physics Example: A particle is moving with a velocity $\underline{v}$ of $20\underline{i} - 15\underline{j}$ ms^{-1}. What is it's speed?

Solution: Since speed is the magnitude of velocity, we take the magnitude of the velocity vector. This gives:

$$|\,20\underline{i} - 15\underline{j}\,| = \sqrt{20^2 + 15^2} = \sqrt{625} = 25$$

The particle's speed is 25 ms^{-1}

Unit Vectors

In an equation such as Newton's Law of Gravitation:

$$F = -\frac{Gm_1m_2}{r^2}$$

we may wish to specify in which direction the force F is acting.
In order to do this, we can introduce the idea of 'unit vectors'. We have already defined a unit vector as a quantity with magnitude (or length) of 1 and a direction. We denote a unit vector using the normal notation for a vector, but with a 'hat', e.g. $\hat{\underline{a}}$ is a vector with magnitude 1, in the direction of $\underline{a}$. This is very useful in specifying the direction of a physical quantity, since multiplying a quantity by a unit vector does not change the magnitude of the quantity, but simply gives it a direction.
In the example of Newton's Law of Gravitation above, we can multiply the right hand side of the equation by the unit vector $\hat{\underline{r}}$ to signify that the force due to gravity, $\underline{F}$, acts in the opposite direction (because of the minus sign) to the radius $\underline{r}$. The equation now reads:

$$\underline{F} = -\frac{Gm_1m_2}{r^2}\hat{\underline{r}}$$

To calculate a unit vector we use the formula

$$\boxed{\hat{\underline{u}} = \frac{\underline{u}}{|\underline{u}|}}$$

where $\hat{\underline{u}}$ is a unit vector of $\underline{u}$ and $|\underline{u}|$ is the magnitude of $\underline{u}$.

4.2 Operations with Vectors

Multiplication by Scalar

Scalar multiplication is when we multiply a vector by a scalar. We do this by multipling each component of the vector by the scalar. This can be expressed as:

$$\lambda \times (x, y, z) = (\lambda x, \lambda y, \lambda z)$$

Where λ is a real number

Note: Scalar multiplication only changes the magnitude of the vector while the direction stays the same.

Example: Simplify $7(-12, 4)$

Solution: We multiply each component of the vector by 7 giving:

$$(7 \times -12,\ 7 \times 4) = (-84, 28)$$

Vector Addition and Subtraction

When we add or subtract two vectors we add or subtract each of the individual components of the vectors. This can be expressed as:

$$(a, b, c) + (x, y, z) = (a + x,\ b + y,\ c + z)$$

Where (a, b, c) and (x, y, z) are both vectors using $\underline{i}$, $\underline{j}$ and $\underline{k}$.

If we are given two points, A and B, and wish to find the vector from A to B, i.e. $\vec{AB}$, then we calculate

$$\boxed{\vec{AB} = \vec{OB} - \underline{OA} = \underline{b} - \underline{a}}$$

where $\underline{a}$ and $\underline{b}$ are the position vectors or A and B, respectively.

IMPORTANT: If the vectors have a different number of base components, for example $(3, 6) + (1, 0, 7)$ then we cannot complete the addition. For all vector-on-vector operations both vectors need to have the same number of base components.

Example: Calculate the following:

1. $(2, 2, 0) + (5, -1, 3)$
2. $3(a, 4a, 0) - a(7, 0, -1)$
3. $(1, 3, 8) + (2, 1, -5) + (-1, 2, -1)$

Solution:

1. $(2, 2, 0) + (5, -1, 3) = (2 + 5,\ 2 - 1,\ 0 + 3) = (7, 1, 3)$
2. $3(a, 4a, 0) - a(7, 0, -1) = (3a, 12a, 0) - (7a, 0, -a) = (3a - 7a,\ 12a - 0,\ 0 + a) = (-4a, 12a, a)$
3. $(1, 3, 8) + (2, 1, -5) + (-1, 2, -1) = (1 + 2 - 1,\ 3 + 1 + 2,\ 8 - 5 - 1) = (2, 6, 2)$

Example: Given $A = (1, 3, 4)$ and $B = (9, -3, 6)$ find the direction vector $\vec{AB}$

Solution: We use the formula $\vec{AB} = \underline{b} - \underline{a}$, so

$$\begin{aligned}\vec{AB} &= (9, -3, 6) - (1, 3, 4)\\ &= (8, -6, 2)\end{aligned}$$

Vector Multiplication: Dot Product

There are two ways of multiplying two vectors together. The first is the dot (or scalar) product which is defined to be:

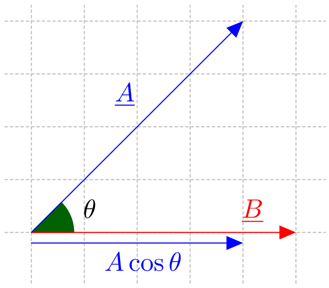
$$\underline{A} \cdot \underline{B} = |\,\underline{A}\,|\,|\,\underline{B}\,|\cos\theta$$

Where θ is the angle between vectors $\underline{A}$ and $\underline{B}$

IMPORTANT: Taking the dot product of two vectors produces a scalar quantity.
Given two vectors, the dot product can also be calculated in the following way:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} x \\ y \\ z \end{pmatrix} = ax + by + cz$$

Note: The dot product can be thought of as how much one vector is pointing in the direction of the other.



The part of $\underline{A}$ that goes in the same direction as $\underline{B}$ has a length of $|\underline{A}|\cos\theta$, making the dot product the length of $\underline{B}$ times the length of $\underline{A}$ that is in the same direction as $\underline{B}$

Example: Calculate the following:

1. $\begin{pmatrix} 3 \\ 9 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -2 \\ 10 \end{pmatrix}$

2. $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix}$

Solution:

1. $\begin{pmatrix} 3 \\ 9 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -2 \\ 10 \end{pmatrix} = 3 \times 1 + 9 \times (-2) + 1 \times 10 = 3 - 18 + 10 = -5$

2. $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix} = 1 \times \sqrt{2} + 0 \times \sqrt{2} = \sqrt{2}$

Physics Example: A charged particle has 2 forces acting on it: a resistive force from the medium with a vector of $(3, \sqrt{7}, 3)$ and an electromagnetic force from an electric field with a vector of $(-5, 3, 4)$. What is the angle between these two vectors?

Solution: To calculate this we find the dot product of the two vectors and then employ the definition of the dot product to find the angle. The dot product of the two vectors is:

$$\begin{pmatrix} 3 \\ \sqrt{7} \\ 3 \end{pmatrix} \bullet \begin{pmatrix} -5 \\ 3 \\ 4 \end{pmatrix} = 3 \times (-5) + \sqrt{7} \times 3 + 3 \times 4 \qquad \text{Simplify this}$$

$$= -15 + 3\sqrt{7} + 12 = -3 + 3\sqrt{7} = 3(\sqrt{7} - 1) \approx 4.937$$

Using the definition $\underline{A} \cdot \underline{B} = |\,\underline{A}\,|\,|\,\underline{B}\,| \cos\theta$ we can now find out what θ is. First we must find the magnitudes of the two vectors:

$$|(3, \sqrt{7}, 3)| = \sqrt{3^2 + 7 + 3^2} = \sqrt{25} = 5 \qquad \text{For the first vector}$$

$$|(-5, 3, 4)| = \sqrt{(-5)^2 + 3^2 + 4^2} = \sqrt{50} = 5\sqrt{2} \qquad \text{For the second vector}$$

Putting all this information into the equation $\underline{A} \cdot \underline{B} = |\,\underline{A}\,|\,|\,\underline{B}\,| \cos\theta$ gives us:

$$3(\sqrt{7} - 1) = 5 \times 5\sqrt{2} \cos\theta \qquad \text{Rearrange for } \cos\theta$$

$$\frac{3(\sqrt{7} - 1)}{25\sqrt{2}} = \cos\theta \qquad \text{Take the arccos of both sides}$$

$$\theta = \arccos(0.1396\dots) = 81.97^\circ \dots = 82.0^\circ \text{ to 3 s.f.}$$

Note: The next example involves the use of integration. Since we have not yet covered integration in this booklet, you may want to wait until you have some basic knowledge of integration before attempting this question.

Physics Example: What is the work done w by the vector force $\underline{F} = (3t\underline{i} + 3\underline{j})$ N on a particle of velocity $\underline{v} = (5\underline{i} - t\underline{j})\,\text{ms}^{-1}$ in the time interval $0 < t < 3$ s given that:

$$w = \int_0^3 F \cdot v \, dt$$

Solution: First we must find the dot product $F \cdot v$.

$$F \cdot v = (3t\underline{i} + 3\underline{j}) \cdot (5\underline{i} - t\underline{j}) = 3t \times 5 + 3 \times (-t) = 12t$$

Now we have found the dot product we can substitute it in the integral and solve.

$$\begin{aligned} w &= \int_0^3 F \cdot v \, dt \\ &= \int_0^3 12t \, dt \\ &= \Big[6t^2\Big]_0^3 \\ &= \Big[6 \times 3^2\Big] - \Big[6 \times 0^2\Big] = 54\,\text{J} \end{aligned}$$

Vector Multiplication: Cross Product

The cross product (or vector product) is defined to be:

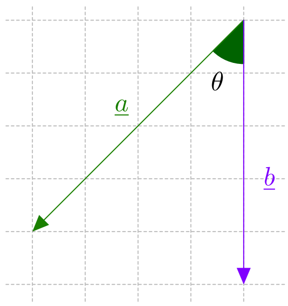
$$\boxed{\underline{A} \times \underline{B} = |\,\underline{A}\,|\,|\,\underline{B}\,|\sin(\theta)\ \underline{\hat{n}}}$$

Where θ is the angle between the two vectors and $\underline{\hat{n}}$ is the unit vector in the direction perpendicular to both A and B. Two perpendicular vectors are said to be **orthogonal** and two perpendicular unit vectors are called **orthonormal**.
IMPORTANT: The cross product takes two vectors and produces a new vector. This new vector is in the direction **perpendicular** to the plane that $\underline{A}$ and $\underline{B}$ are in. It is only possible to take the cross product of two 3-D vectors (and technically 7-D vectors as well).

The cross product is also dependent on the order of the vectors that appear in the product. If we change the order of the vectors in a cross product then our answer becomes negative. The vector points in the opposite direction.

$$\boxed{\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}}$$

The right hand rule determines which way the vector produced will point. We use the rule as follows: curl your fingers from the first vector in the cross product to the second vector and the direction your thumb points is the direction of the vector produced by the cross product.



In this case, $\underline{a} \times \underline{b}$ will point out of the page and $\underline{b} \times \underline{a}$ will point into the page.

Calculating the Cross Product (Method 1)

When using the standard base vectors of $\underline{\hat{i}}$, $\underline{\hat{j}}$ and $\underline{\hat{k}}$, we can calculate the cross product by considering the cross product of the base vectors with each other:

$$\boxed{\underline{i} \times \underline{i} = \underline{j} \times \underline{j} = \underline{k} \times \underline{k} = \underline{0}} \quad \boxed{\underline{i} \times \underline{j} = -\underline{j} \times \underline{i} = \underline{k}} \quad \boxed{\underline{j} \times \underline{k} = -\underline{k} \times \underline{j} = \underline{i}} \quad \boxed{\underline{k} \times \underline{i} = -\underline{i} \times \underline{k} = \underline{j}}$$

Taking the cross products of two vectors is just like expanding brackets. The cross product of two general vectors, $\big(a\underline{i} + b\underline{j} + c\underline{k}\big) \times \big(x\underline{i} + y\underline{j} + z\underline{k}\big)$, would give $ax(\underline{i} \times \underline{i}) + ay(\underline{i} \times \underline{j}) + az(\underline{i} \times \underline{k}) + \ldots$ Fully expanded and simplified gives:

$$\boxed{\begin{array}{c}\big(a\underline{i} + b\underline{j} + c\underline{k}\big) \times \big(x\underline{i} + y\underline{j} + z\underline{k}\big) = \big(bz - cy\big)\underline{i} + \big(cx - az\big)\underline{j} + \big(ay - bx\big)\underline{k} \\ \text{Where } a,\ b,\ c,\ x,\ y \text{ and } z \text{ are real numbers}\end{array}}$$

Example: Calculate the following cross product using the method above:

$$\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Solution: So to find this we use the formula below:

$$(a\underline{i} + b\underline{j} + c\underline{k}) \times (x\underline{i} + y\underline{j} + z\underline{k}) = (bz - cy)\underline{i} + (cx - az)\underline{j} + (ay - bx)\underline{k}$$

Now we can write the vectors in the question in the form of the rule so $(a\underline{i}+b\underline{j}+c\underline{k}) = (3\underline{i}+1\underline{j}-2\underline{k})$ and $(x\underline{i} + y\underline{j} + z\underline{k}) = (2\underline{i} - 1\underline{j} + 1\underline{k})$
Hence $a = 3$, $b = 1$, $c = -2$, $x = 2$, $y = -1$ and $z = 1$ and therefore:

$$\begin{aligned} &(3\underline{i} + 1\underline{j} - 2\underline{k}) \times (2\underline{i} - 1\underline{j} + 1\underline{k}) \\ &= (1 \times 1 - (-2) \times (-1))\underline{i} + ((-2) \times 2 - 3 \times 1)\underline{j} + (3 \times (-1) - 1 \times 2)\underline{k} \\ &= -1\underline{i} - 7\underline{j} - 5\underline{k} \end{aligned}$$

This could also be written:

$$\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -7 \\ -5 \end{pmatrix}$$

Calculating the Cross Product (Method 2)

We can also calculate the cross product of two vectors, $(a\underline{i} + b\underline{j} + c\underline{k}) \times (x\underline{i} + y\underline{j} + z\underline{k})$, by considering the determinant:

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a & b & c \\ x & y & z \end{vmatrix}$$

We calculate this determinant as discussed in the section on 'Matrices'.

This results in the same formula as in the previous method, but using this method means that we do not need to remember the formula since it is included in the expression for the determinant!

Example: Calculate the following cross product using the determinant method:

$$\begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 6 \\ -2 \end{pmatrix}$$

Solution: We can form a determinant and evaluate:

$$\begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 6 \\ -2 \end{pmatrix} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 4 & 3 & 1 \\ 1 & 6 & -2 \end{vmatrix}$$

$$\begin{aligned} &= \underline{i}\begin{vmatrix} 3 & 1 \\ 6 & -2 \end{vmatrix} - \underline{j}\begin{vmatrix} 4 & 1 \\ 1 & -2 \end{vmatrix} + \underline{k}\begin{vmatrix} 4 & 3 \\ 1 & 6 \end{vmatrix} \\ &= \underline{i}\big[3 \times (-2) - 6 \times 1\big] - \underline{j}\big[4 \times (-2) - 1 \times 1\big] + \underline{k}\big[4 \times 6 - 1 \times 3\big] \\ &= -12\underline{i} - (-9)\underline{j} + 21\underline{k} \\ &= -12\underline{i} + 9\underline{j} + 21\underline{k} \end{aligned}$$

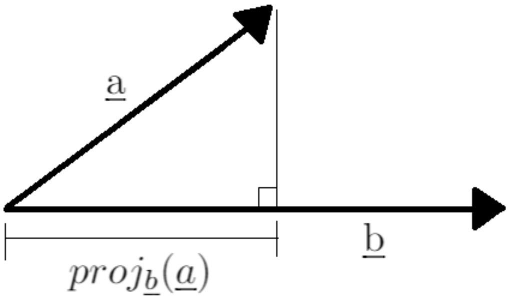
We can also write:

$$\begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 6 \\ -2 \end{pmatrix} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 4 & 3 & 1 \\ 1 & 6 & -2 \end{vmatrix} = \begin{pmatrix} -12 \\ 9 \\ 21 \end{pmatrix}$$

Projection of one Vector onto another

If we have two non-parallel vectors $\underline{u}$ and $\underline{v}$, we can find the component of $\underline{u}$ which is in the direction of $\underline{v}$. This is called the **projection** of $\underline{v}$ onto $\underline{u}$. This is calculated by the following formula:

$$\boxed{proj_{\underline{u}}(\underline{v}) = \left(\underline{v} \cdot \frac{\underline{u}}{|\underline{u}|}\right)\frac{\underline{u}}{|\underline{u}|}}$$



Triple Products

Now that we have defined the dot product and the cross product, we can look at what happens when we want to multiply three vectors using different combinations of these operations.

Scalar Triple Product

The scalar triple product between three vectors $\underline{a} = (a_1, a_2, a_3)$, $\underline{b} = (b_1, b_2, b_3)$ and $\underline{c} = (c_1, c_2, c_3)$ is defined as

$$\boxed{\underline{a} \cdot (\underline{b} \times \underline{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}}$$

which produces a scalar. We can interpret the scalar triple product as the volume of a parallelepiped, which is a cuboid where every side is a parallelogram.

We can use the scalar triple product to calculate the following volumes, where $\underline{a}$, $\underline{b}$ and $\underline{c}$ are any three mutually adjacent edges of the given shape.

- The volume of a parallelepiped is
$$V = |\underline{a} \cdot \underline{b} \times \underline{c}|$$
- The volume of a pyramid is
$$V = \frac{1}{3}|\underline{a} \cdot \underline{b} \times \underline{c}|$$
- The volume of a tetrahedron is
$$V = \frac{1}{6}|\underline{a} \cdot \underline{b} \times \underline{c}|$$

We can also use the scalar triple product to show that three vectors, $\underline{a}$, $\underline{b}$ and $\underline{c}$, are **coplanar**, i.e. they are all parallel to a common plane. To do this, we only need to show that

$$\underline{a} \cdot \underline{b} \times \underline{c} = 0$$

Vector Triple Product

The vector triple product between three vectors $\underline{a} = (a_1, a_2, a_3)$, $\underline{b} = (b_1, b_2, b_3)$ and $\underline{c} = (c_1, c_2, c_3)$ is defined as

$$\boxed{\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}}$$

which produces a scalar.

Note: the third possibility, $\underline{a} \cdot \underline{b} \cdot \underline{c}$, is completely meaningless; the first product produces a scalar quantity, which makes the second product impossible since we cannot perform a dot product between a scalar and a vector.

4.3 Vector Equations

Lines

Vector Equation of a Line

We can describe a straight line in space using vectors. Suppose we have a line L and wish to find an expression for a general point on the line, R, then we start from a point on the line, P, and move along the line until we get to R. In terms of an equation, this can be written as

$$\boxed{\underline{r} = \underline{p} + \lambda \underline{q}}$$

where $\underline{r}$ is a general position vector on the line, $\underline{p}$ is any position vector that gets us onto the line, $\underline{q}$ is a vector in the direction of the line and λ is a constant. This is the **vector** equation of a line.
This approach is very intuitive. Essentially we are starting somewhere on the line, and moving in the direction of the line, associating each point on the line to a different value of λ. Note that λ can be positive or negative, which corresponds to moving forward along the line, or backwards, respectively.

We can write the equation for a line in a number of other forms, which each have advantages in different situations.

Parametric Equation of a Line

By writing the vectors above in terms of their components, i.e. $\underline{r} = (x, y, z)$, $\underline{p} = (p_1, p_2, p_3)$ and $\underline{q} = (q_1, q_2, q_3)$, we can obtain a **parametric** form of the equation in terms of the parameter λ:

$$\boxed{\begin{aligned} x &= p_1 + \lambda q_1 \\ y &= p_2 + \lambda q_2 \\ z &= p_3 + \lambda q_3 \end{aligned}}$$

Cartesian Equation of a Line

From this, we can obtain the **standard** equation of a line by rearranging each of these equations to make λ the subject and then equating all the expressions for λ:

$$\boxed{\lambda = \frac{x - p_1}{q_1} = \frac{y - p_2}{q_2} = \frac{z - p_3}{q_3}}$$

Example: Find the equations of a line which passes through the points $A = (2, 3, -1)$ and $B = (5, 2, 2)$

Solution: We will first find the vector equation of the line. We begin by finding a direction vector $\vec{AB} = (5, 2, 2) - (2, 3, -1) = (3, -1, 3)$.
Now we formulate the equation choosing a point on the line. Since we are told that the line passes through A and B, we can choose either of these. We will choose A. This gives us the vector equation of the line:

$$\underline{r} = (2, 3, -1) + \lambda(3, -1, 3)$$

Now to find the parametric equation we separate this into its three equations:

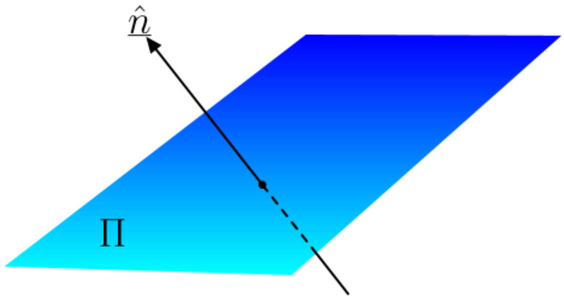
$$\begin{aligned} x &= 2 + 3\lambda \\ y &= 3 - \lambda \\ z &= -1 + 3\lambda \end{aligned}$$

To find the standard equation we rearrange these three equations to make λ the subject. This gives

$$\lambda = \frac{x - 2}{3} = \frac{y - 3}{-1} = \frac{z + 1}{3}$$

Planes

A plane in 3-D space is a flat surface with zero thickness, often denoted as Π. This can be visualised as a sheet of paper. A normal to a plane is a vector which is perpendicular to the plane. The orientation of a plane can be specified by a normal vector.



A plane has two normal vectors in opposite directions. We denote a unit normal vector as $\hat{\underline{n}}$.
We can find a normal vector to a plane by calculating the cross product of two direction vectors in the plane.

Vector Equation of a Plane

If we have three points A, B and C which do not all lie in the same straight line (i.e. not collinear), then they lie in a unique plane. We can write down a general equation for a plane by considering a point on the plane, A, with position vector $\underline{a}$, and deducing how we can get to any point on the plane, R. To reach any point in 2-D space, we require two direction vectors. Think about an x, y-plane: we can get anywhere on this surface by specifying a distance in the x-direction and a distance in the y-direction. Similarly, by using the fact that we have three points in the plane, we can calculate two direction vectors within the plane, for example $\vec{AB}$ and $\vec{AC}$, and state that any linear combination of these will get us to any point on the plane from an initial point A.
Using this analysis, we can see that the general **vector** equation of a plane is

$$\boxed{\begin{aligned}\underline{r} &= \underline{a} + \lambda\vec{AB} + \mu\vec{AC} \\ &= \underline{a} + \lambda(\underline{b} - \underline{a}) + \mu(\underline{c} - \underline{a})\end{aligned}}$$

Scalar Product Equation of a Plane

Using the concept of the normal vector to a plane, we can formulate another equation of the plane. Given a point A, with position vector $\underline{a}$, on the plane and the general point on the plane R, we know that the direction vector $\vec{AR}$ also lies on the plane. Thus we know that the normal vector to the plane, $\underline{n}$ is orthogonal to $\vec{AR}$, so we can write that $\vec{AR} \cdot \underline{n} = 0$. Therefore $(\underline{r} - \underline{a}) \cdot n = 0$. Now expanding the brackets gives us the **scalar product** equation of a plane:

$$\boxed{\underline{r} \cdot \underline{n} = \underline{a} \cdot \underline{n} = d}$$

where d is a scalar calculated from $\underline{a} \cdot \underline{n}$.

Cartesian Equation of a Plane

We can now easily deduce another equation of the plane by calculating the scalar products. If $\underline{n} = (n_1, n_2, n_3)$, and $\underline{r} = (x, y, z)$, then the **cartesian** equation of the plane is:

$$\boxed{n_1x + n_2y + n_3z = \underline{a} \cdot \underline{n} = d}$$

where d is a scalar calculated from $\underline{a} \cdot \underline{n}$.

4.4 Intersections and Distances

Shortest Distance from a Point to a Plane

The shortest distance between a point and a plane is always the length of the line that comes out of the plane at a right angle and meets the point in question. If you are struggling to visualise this try using a piece of paper as the plane and thinking about the shortest distance to points not on that piece of paper. Therefore the distance we are trying to work out is parallel to the normal vector. If we work out a unit vector in the direction of the normal and calculate the dot product between this unit vector and a line from the point q to a point on the plane, this will give us the distance we want (The understanding of this is tough and not the important part at this level, but make sure you remember the formulae!).

Consider the plane $\underline{n}.p = d$ and the point q not on the plane, the unit vector for the vector $\underline{n}$ is equal to $\frac{n}{|\underline{n}|}$. So the magnitude of the distance we want, D, is equal to:

$$D = \left|\frac{n}{|\underline{n}|}.\vec{qp}\right| = \frac{|\underline{n}.\vec{qp}|}{|\underline{n}|}$$

$\vec{qp}$ is the line from the point q to a point on the plane, p. If we let $p = (x_0,\, y_0,\, z_0)$ and $q = (x_1,\, y_1,\, z_1)$ this simplifies further to:

$$D = \frac{|(x_1 - x_0, y_1 - y_0, z_1 - z_0).\underline{n}|}{|\underline{n}|}$$

$$D = \frac{|(x_1 - x_0, y_1 - y_0, z_1 - z_0).\underline{n}|}{|\underline{n}|}$$

If $\underline{n} = (A, B, C)$ our equation above becomes:

$$D = \frac{\left|A(x_1 - x_0) + B(y_1 - y_0) + C(z_1 - z_0)\right|}{\sqrt{A^2 + B^2 + C^2}}$$

As $\underline{n}.p = d$ we can rearrange this using that $p = (x_0,\, y_0,\, z_0)$ and $\underline{n} = (A, B, C)$ to give $d = -Ax_0 - By_0 - Cz_0$. Therefore:

$$D = \frac{\left|Ax_1 + By_1 + Cz_1 + d\right|}{\sqrt{A^2 + B^2 + C^2}}$$

Example: Find the shortest distance between the plane $\Pi = (1, 2, 3) + \lambda(2, 5, 7) + \mu(2, 3, 6)$ and the point $q = (4, 5, 6)$

Solution: We know the equation for the distance we want is:

$$D = \frac{\left|Ax_1 + By_1 + Cz_1 + d\right|}{\sqrt{A^2 + B^2 + C^2}}$$

To find A, B, C and D we need to find the normal vector. To do this we need to do the cross product of the two direction vectors:

$$\underline{n} = \begin{vmatrix} i & j & k \\ 2 & 5 & 7 \\ 2 & 3 & 6 \end{vmatrix} = (9, 2, -4)$$

To find D we us that $\underline{n}.p = d$ and therefore:

$$(9, 2, -4).(1, 2, 3) = d = 1$$

Therefore the distance we want is equal to:

$$D = \frac{\left|9(4) + 2(5) + -4(6) + 1\right|}{\sqrt{9^2 + 2^2 + (-4)^2}}$$

$$= \frac{23}{\sqrt{101}}$$

Shortest Distance between Two Skew Lines

When two lines are not parallel and do not intersect we call them skew lines. To find the distance between two skew lines there are various techniques but we will do it by constructing two parallel planes from these skew lines and using this to find the perpendicular distance between them.

If we have two lines $L_1 = \underline{A} + \lambda\vec{B}$ and $L_2 = \underline{A'} + \lambda\vec{B'}$, to find a normal vector to both of these we just need find the cross product between the direction vector of L_1 and L_2, $\vec{B} \times \vec{B'}$. After this we can use the plane equation $\underline{n}.p = d$ with the normal vector we worked out and a point on each line to construct two parallel planes, Π_1 and Π_2.

To work out the distance between these planes, which is equal to the distance between the original skew lines, we first note that Π_1 can be described by $\underline{n} \cdot p = d$ and Π_2 can be described by the equation $\underline{n} \cdot p = d'$ and therefore the distance between them is:

$$\frac{|d - d'|}{|n|}$$

Example: Find the shortest distance between the lines $L_1 : x - 2 = 1 - y = \frac{z+2}{3}$ and $L_2 : \frac{x+17}{3} = -y + 3 = \frac{z+1}{2}$.

Click here for a video example

Solution: We can write the equations of the lines in the form:

$$L_1 = (2, 1, -2) + t(1, -1, 3)$$

$$L_2 = (-17, 3, -1) + s(3, -1, 2)$$

The cross product of the direction vectors is:

$$\underline{n} = \begin{vmatrix} i & j & k \\ 1 & -1 & 3 \\ 3 & -1 & 2 \end{vmatrix} = (1, 7, 2)$$

Constructing these into planes:
Finding Π_1:

$$(1, 7, 2) \cdot (2, 1, -2) = 5$$

Therefore

$$\Pi_1 : (1, 7, 2) \cdot p = 5$$

Finding Π_2:

$$(1, 7, 2) \cdot (-17, 3, -1) = 2$$

Therefore

$$\Pi_1 : (1, 7, 2) \cdot q = 2$$

Therefore the distance we want is:

$$\frac{|5 - 2|}{\sqrt{1^2 + 7^2 + 2^2}} = \frac{3}{\sqrt{54}} = \frac{\sqrt{6}}{6}$$

Intersection between a Line and a Plane

If we wish to find the point of intersection, $\underline{p}$, between a line and a plane then we must obtain the equations for the line and the plane, and then solve as them as we would for simultaneous equations to find a point which lies on both structures.

Example: Find the point of intersection P between the line

$$\underline{r} = (2,3,1) + t(3,3,2)$$

and the plane

$$\underline{r} \cdot (1,-2,4) = 5$$

Solution: Since we already have the equations for the line and plane, we can proceed to solve as simultaneous equations, by substituting the equation of a point on the line into the equation for the plane.

$$[(2,3,1) + t(3,3,2)] \cdot (1,-2,4) = 5$$
$$(2+3t, 3+3t, 1+2t) \cdot (1,-2,4)$$
$$(2+3t) - 2(3+3t) + 4(1+2t) = 5$$
$$2 - 6 + 4 + 3t - 6t + 8t = 5$$
$$\Rightarrow 5t = 5$$
$$\Rightarrow t = 1$$

This value of the parameter t corresponds to the point on the line which intersect with the plane. Now we simply substitute this t value into the equation for the line.

$$\underline{p} = (2,3,1) + (3,3,2) = (5,6,2)$$

Example: Given the plane

$$2x + 3y + 3z = 6$$

and a line which is perpendicular to the plane and passes through the point $Q : (5,3,7)$, find the point where the line intersects the plane, P.

Solution: From the general equation of a plane $\underline{r} \cdot \underline{n} = d$, we can deduce that a normal vector to the plane is $(2,3,3)$. Since the line is perpendicular to the plane and passes through the point Q, we can write the vector equation of the line as

$$\underline{r} = (5,3,7) + t(2,3,3)$$

Now, as before, we can solve these equations simultaneously to find the parameter t and hence find P.

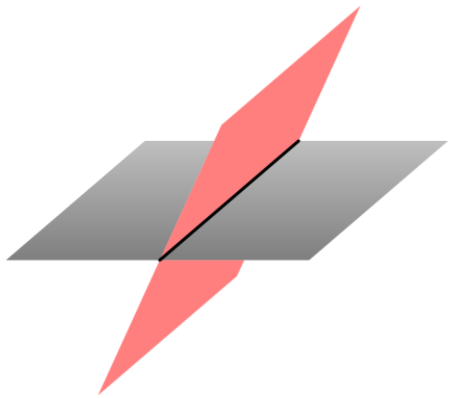
$$2(5+2t) + 3(3+3t) + 3(7+3t) = 6$$
$$10 + 4t + 9 + 9t + 21 + 9t = 6$$
$$10 + 9 + 21 + 4t + 9t + 9t = 6$$
$$\Rightarrow 22t = -34$$
$$\Rightarrow t = \frac{34}{22} = \frac{17}{11}$$

Hence the point of intersection is

$$\underline{p} = \left(\frac{89}{11}, \frac{84}{11}, \frac{128}{11}\right)$$

Intersection between Two Planes

Two planes, Π_1 and Π_2, which are not parallel to each other will intersect along a straight line l. In order to find the direction vector of l, we can calculate the cross product of the normals of the two planes, giving us a vector parallel to both planes, and hence the direction of the line of intersection. We can then find a point of intersection, i.e. a point on the line, by setting $y = 0$ and solving the equations of the two planes simultaneously to find x and z values. This will give us our equation for the line of intersection of the two planes.



Example: Find an equation for the line of intersection of the two planes

$$\Pi_1 : x - y - z = 1$$

and

$$\Pi_2 : 2x + 4y - 5z = 5$$

Click here for a video example

Solution: By considering the standard equation for a plane, $\underline{r} \cdot n = d$, we can deduce that the normal vector to Π_1 is

$$\underline{n_1} = (1, -1, -1)$$

and the normal vector to Π_2 is

$$\underline{n_2} = (2, 4, -5)$$

Now we take the cross product of these normal vectors to find the direction vector for the line of intersection

$$\underline{n_1} \times \underline{n_2} = \begin{vmatrix} i & j & k \\ 1 & -1 & -1 \\ 2 & 4 & -5 \end{vmatrix}$$

$$= (5 + 4, -2 + 5, 4 + 2)$$

$$= (9, 3, 6)$$

So a direction vector of the line of intersection is (3,1,2). Now we find a point which lies on both planes. We set $y = 0$, and by looking at the equations of the planes, we get the two equations

$$x - z = 1$$

and

$$2x - 5z = 5$$

And we can solve these to get $x = 2$ and $z = -1$, so a point on the line is (2,0,-1). Finally we have our equation for the line of intersection of the two planes.

$$\underline{r} = (2, 0, -1) + t(3, 1, 2)$$

Types of Intersection between Three Planes

If we have three planes, there are four possibilities:

- The three planes intersect each other at a single point.
- The three planes intersect along a single straight line.
- The planes do not all share any common points but they all intersect.
- The planes do not intersect and are parallel to each other.

4.5 Review Questions

Easy Questions

Question 1: If
$$U = (1, 3, -1), V = (4, 5, -3), W = (-7, 3, 4)$$
find $U\vec{W}$ and $U\vec{V}$.

Answer:
$$U\vec{W} = (-8, 0, 5)$$
$$U\vec{V} = (3, 2, -2)$$

Question 2: Find the magnitude of the vector $\underline{u} = (2, -4, \sqrt{5})$.

Answer:
$$|\underline{u}| = 5$$

Question 3: Find unit vector $\underline{u}$ which has magnitude 6 and has the same direction as the vector $\underline{v} = (1, -2, 1)$.

Answer:
$$\underline{u} = (\sqrt{6}, 2\sqrt{6}, \sqrt{6})$$

Question 4: Find the unit vector, $\underline{u_1}$, with the same direction as $\underline{u}$ and the unit vector$\underline{u_2}$ with the opposite direction vector to $\underline{u}$ if
$$\underline{v} = (0, \sqrt{11}, 5).$$

Answer:
$$\underline{u_1} = \left(0, \frac{\sqrt{11}}{6}, \frac{5}{6}\right)$$
$$\underline{u_2} = \left(0, -\frac{\sqrt{11}}{6}, -\frac{5}{6}\right)$$

Question 5: If $\underline{u} = (2, 3, -6), \underline{v} = (3, 11, 0)$ find
$$2\underline{v} - 3\underline{u}.$$

Answer:
$$(0, 13, 18)$$

Medium Questions

Question 6: For each of the following pairs of vectors, determine whether they are perpendicular:

1. $\underline{u} = (2, 4, -1)$, $\underline{v} = (6, 0, 12)$
2. $\underline{u} = (3, 9, 1)$, $\underline{v} = (2, -\frac{2}{3}, 1)$
3. $\underline{u} = (4, 0, 0)$, $\underline{v} = (0, 5, -5)$

Answer:

1. Yes
2. No
3. Yes

Question 7: Find the value of lambda which makes the two vectors, $\underline{u}$ and $\underline{v}$, parrallel if

$$\underline{u} = (1, \lambda, 4), \underline{v} = (16, -2\lambda, 4).$$

Answer:

$$\lambda = \pm 4$$

Question 8: Find the $\underline{v} \times \underline{u}$ if

$$\underline{v} = (2, 3, -1)$$

$$\underline{u} = (-2, -2, 2).$$

Answer:

$$(5, -2, 4)$$

Question 9: Find the intersection between the planes Π_1 and Π_2 such that

$$\Pi_1 : 3x + y - 4z = 2$$

$$\Pi_2 : x + y = 18.$$

Answer: The planes intersect on the line

$$\frac{x+8}{4} = \frac{26-y}{4} = \frac{z}{2}$$

Question 10: Find the projection of $\underline{v}$ onto $\underline{u}$ if

$$\underline{v} = (1, 4, 2)$$

$$\underline{u} = (2, 5, 7).$$

Answer:

$$\left(\frac{12}{13}, \frac{30}{13}, \frac{42}{13}\right)$$

Hard Questions

Question 11: If a plane, Π, is parallel to the yz-plane and passes through the point $(1, 2, 3)$, what is the equation of the plane?

Answer:

$$x = 1$$

Question 12: If a plane, Π, passes through the points $U = (1, 1, 3), V = (-1, 3, 2), W = (1, -2, 5)$, what is the equation of the plane?

Answer:

$$x + 4y + 6z = 23$$

Question 13: The mid-points of the sides AB, BC, CD and DA of a quadrilateral $ABCD$ are P, Q, R, S respectively. Show, using vectors, that $PQRS$ is a parallelogram.

Hint: Take position vectors relative to an origin O and use the result that a quadrilateral with a pair of opposite sides which are parallel and equal in length is a parallelogram.

Question 14: Find the angle, θ, between the vectors $\underline{u} = (-1, 1, 0)$ and $\underline{v} = (-1, 0, 1)$.

Answer:

$$\theta = \frac{\pi}{3}$$

Question 15: Find the shortest distance between the skew lines

$$L_1 = (1, 2, 2) + s(4, 3, 2)$$

$$L_2 = (1, 0, -3) + t(4, -6, -1).$$

Answer:

$$4 \ \text{ units}$$

5 Limits

5.1 Notation and Definitions

We often wish to find the value that a function $f(x)$ approaches as x tends to a particular value. We call this the limit of $f(x)$ as x tends to a and it is denoted by

$$\lim_{x\to a} f(x)$$

Limits have many uses including determining the behaviour of a function, calculating the value of an infinite series, and providing a basis for all of calculus. They can also help us when plotting graphs.

As an example, we consider the function

$$f(x) = \frac{x^2-1}{x-1}$$

at the point $x = 1$. Normally, we would just say that the function is undefined, since we would be dividing by zero. However, as x approaches 1, $f(x)$ actually approaches the value 2. In other words, if we are infinitely close to the point $x = 1$, then $f(x)$ is infinitely close to 2, however it never actually reaches it. To represent this, we can write

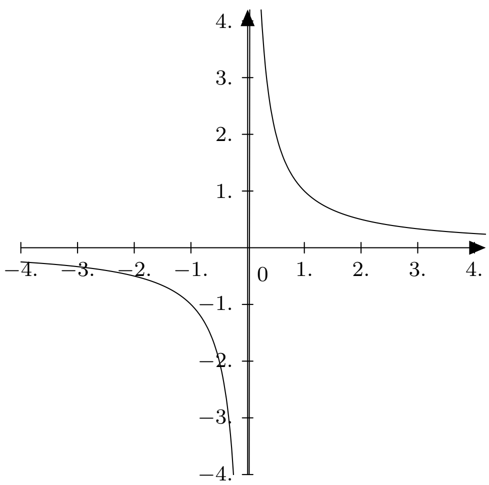
$$\lim_{x\to 1}\left(\frac{x^2-1}{x-1}\right) = 2$$

In the next section we will describe methods for proving this.

We can also have 'one-sided limits', which is where a function approaches two different values depending on from which direction we approach the limit point. For example, the function

$$f(x) = \frac{1}{x}$$

tends to minus infinity at $x = 0$ if we approach $x = 0$ from the left hand side, but it also tends to plus infinity if we approach $x = 0$ from the right hand side. We can see this clearly by considering the graph of $f(x) = \frac{1}{x}$:



We can write these limits as

$$\lim_{x\to 0-}\frac{1}{x} = -\infty$$

$$\lim_{x\to 0+}\frac{1}{x} = \infty$$

5.2 Algebra of Limits

We can use the algebra of limits to help us find the limits of functions. The rules included in the algebra of limits are as follows:

- The limit of the sum of two functions is the same as the sum of the limits of the two functions:

$$\lim_{x\to a}\big(f(x)g(x)\big) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

- The limit of the product of two functions is the same as the product of the limits of the two functions:

$$\lim_{x\to a}\big(f(x)+g(x)\big) = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x)$$

- The limit of the quotient of two functions is the same as the quotient of the limits of the two functions (provided $\lim_{x\to a} g(x) \neq 0$) :

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

Essentially this means that operations with limits are linear.

5.3 Methods for Finding Limits

This section will consist of examples describing different methods of calculating limits.

Example: Calculate the limit:

$$\lim_{x\to\infty}\frac{x^2+x+1}{2x^2+1}$$

Solution: We can use the algebra of limits and the fact that

$$\lim_{x\to\infty}\frac{1}{x} = 0$$

to find this limit.
First, dividing top and bottom of the fraction by the highest power of x, we get

$$\lim_{x\to\infty}\frac{x^2+x+1}{2x^2+1} = \lim_{x\to\infty}\frac{1+\frac{1}{x}+\frac{1}{x^2}}{2+\frac{1}{x^2}}$$

Now use the algebra of limits to separate this into individual limits:

$$\frac{1+\lim_{x\to\infty}\left(\frac{1}{x}\right)+\lim_{x\to\infty}\left(\frac{1}{x^2}\right)}{2+\lim_{x\to\infty}\left(\frac{1}{x^2}\right)}$$

$$= \frac{1+0+0}{2+0} = \frac{1}{2}$$

This technique only works when we can write the expression in terms of limits which tend to zero.

Example: Calculate the limit:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

Solution: We can rewrite this as

$$\begin{aligned} \lim_{x \to 1} \frac{x^2 - 1}{x - 1} &= \lim_{x \to 1} \frac{(x+1)(x-1)}{x-1} \\ &= \lim_{x \to 1} (x+1) \\ &= 2 \end{aligned}$$

This technique works where we can eliminate the part of the function that causes the function to become undefined.

L'Hôpital's Rule

In some cases we can use L'Hôpital's rule to help us find the limit of a function, which states that for two functions $f(x)$ and $g(x)$ such that

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$$
or
$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \pm\infty$$

where the limit

$$\lim_{x \to a} \frac{f'(x)}{g'(x)}$$

exists (and $f'(x)$ is the derivative of $f(x)$ as explained in the section on Differentiation), we can use the formula

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}}$$

Example: Calculate the limit:

$$\lim_{x \to 0} \frac{\sin(x)}{x}$$

Solution: We must first test the conditions for the rule to work. Let $f(x) = \sin(x)$ and $g(x) = x$. We can easily check that

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} g(x) = 0$$

Now we need to check that the limit

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)}$$

exists, and if it does then by the rule, this limit is equal to the original limit.

$$\frac{f'(x)}{g'(x)} = \frac{\cos(x)}{1}$$

$$\lim_{x \to 0} \cos(x) = 1$$

Hence the limit exists and so

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \cos(x) = 1$$

Example: Calculate the limit:

$$\lim_{x\to 0} \frac{1-\cos(2x)}{x^2}$$

Solution: We know that $\cos(2x) = 1 - 2sin^2(x)$ so our limit simplifies to:

$$\lim_{x\to 0} \frac{1-\left(1-2\sin^2(x)\right)}{x^2} = 2\left[\lim_{x\to 0} \frac{\sin^2(2x)}{x^2}\right]$$

$$= 2\left[\lim_{x\to 0} \frac{\sin(x)}{x}\right]^2$$

Using the above we know that

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

Therefore

$$2\left[\lim_{x\to 0} \frac{\sin(x)}{x}\right]^2 = 2[1]^2 = 2$$

5.4 Review Questions

Easy Questions

Question 1: What is the limit of the function

$$f(x) = \frac{\tan(x)}{x}$$

as x tends 0?

Answer:

$$\lim_{x\to 0} \frac{\tan(x)}{x} = 1$$

Question 2: What is the limit of the function

$$f(x) = \frac{3x^3 - 2x^2 + x + 4}{5x^3 + 34x^2 + 4x + 9}$$

as x tends ∞?

Answer:

$$\lim_{x\to\infty} \frac{3x^3 - 2x^2 + x + 4}{5x^3 + 34x^2 + 4x + 9} = \frac{3}{5}$$

Question 3: What is the limit of the function

$$f(x) = \frac{x}{\sin(x)}$$

as x tends 0?

Answer:

$$\lim_{x\to 4} \frac{x}{\sin(x)} = 1$$

Question 4: What is the limit of the function

$$f(x) = \frac{x^2 + x - 2}{x - 1}$$

as x tends 1?

Answer:

$$\lim_{x\to 1} \frac{x^2 + x - 2}{x - 1} = 3$$

Medium Questions

Question 5: What is the limit of the function

$$f(x) = \frac{\sin(2x)}{\sin(x)}$$

as x tends 0?

Answer:

$$\lim_{x\to 0} \frac{\sin(2x)}{\sin(x)} = 2$$

Question 6: What is the limit of the function

$$f(x) = \sqrt{x+10} - \sqrt{x}$$

as x tends ∞?

Hint: Use the difference of two squares.

Answer:

$$\lim_{x\to\infty} \left(\sqrt{x+10} - \sqrt{x}\right) = 0$$

Question 7: What is the limit of the function

$$f(x) = \frac{\sqrt{5x-16} - \sqrt{x}}{x-4}$$

as x tends 4?

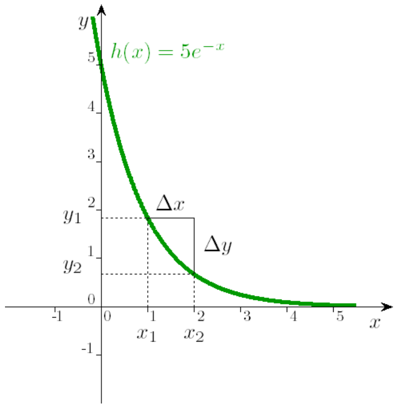
Answer:

$$\lim_{x\to 4} \frac{\sqrt{5x-16} - \sqrt{x}}{x-4} = 1$$

6 Differentiation

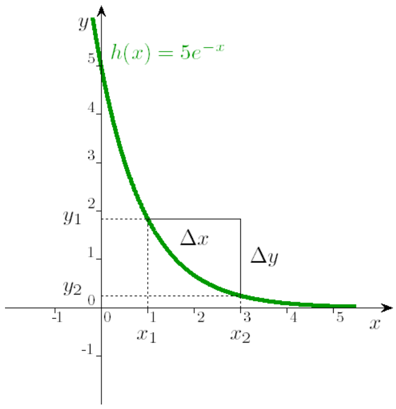
6.1 Introduction to Differentiation

Recall that the equation of a straight line is of the form of $y = mx + c$ where m is the gradient and c is the y-intercept. We can find the gradient m of a straight line using the triangle method with the formula $m = \frac{\Delta x}{\Delta y}$, where Δx means 'change in x'
Note: Remember that the gradient is a measure of how steep the graph is at a certain point.
As the gradient of a straight line is the same at every point on the line it is easy to find. With a curve this is not the case. Consider the graph of $y = 5e^{-x}$ shown below. We will use the triangle method with two different triangles and show that we get two different values for the gradient.
Using the triangle method with the triangle in the graph below gives:



$$\text{gradient} = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0.68 - 1.85}{3 - 1} = -1.17$$

Now if we use the triangle method again but this time with a larger triangle as in the graph below:



$$\text{gradient} = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0.25 - 1.85}{3 - 1} = -0.8$$

We have obtained two different values for the gradient of the curve. This is because the value of the gradient is not the same at all points on the curve, unlike for a straight line. This should be obvious as we can see that the steepness of the graph changes along the curve. This means it is quite hard to find the gradient of a curve using the triangle method and so we need a new method.
Differentiation allows us to find the gradient of almost any graph. This works because when we differentiate we are calculating how one variable changes due to a change in another (in other words the rate of change). This is useful in physics; for example we might wish to find the acceleration of a particle by considering the rate of change of velocity.

Differentiation from First Principles

The properties of differentiation are born from the idea of calculating the gradient between two points at the limit that the distance between them tends to zero. As shown in the previous section a gradient is calculated by $\frac{f(x_2)-f(x_1)}{x_2-x_1}$, but for differentiation we consider the points $f(x+dx)$ and $f(x)$, where dx is the infinitesimal distance between the two points i.e. The derivative, or gradient, of a point on a curve is:

$$\lim_{dx\to 0}\left(\frac{f(x+dx)-f(x)}{dx}\right)$$

Using this idea, it is possible to derive all the standard derivatives.

Example: Differentiate from first principles $4x^2$:

Solution: The derivative of $f(x)$ is defined above as

$$\lim_{dx\to 0}\left(\frac{f(x+dx)-f(x)}{dx}\right)$$

so if we let $f(x)=4x^2$ we get that the derivative is

$$\lim_{dx\to 0}\left(\frac{(x+dx)^2-x^2}{dx}\right)$$

Consider

$$\frac{(x+dx)^2-x^2}{dx}$$

Expanding the brackets we obtain

$$\frac{x^2+2xdx+dx^2-x^2}{dx}$$

This simplifies to

$$2x+dx^2$$

Therefore the derivative of $4x^2$ is

$$\lim_{dx\to 0}(2x+dx^2)$$

$$=2x$$

Example: Differentiate from first principles x^n

Solution: The derivative of x^n =

$$\lim_{dx \to 0} \left(\frac{(x+dx)^n - x^n}{dx} \right)$$

Consider

$$\frac{(x+dx)^n - x^n}{dx}$$

Using a binomial expansion (see section 11.4) of the bracketed terms gives

$$\frac{(x^n + nx^{n-1}dx + \frac{1}{2}n(n-1)x^{n-2}dx^2 + ...) - x^n}{dx}$$

This is equivalent to

$$nx^{n-1} + \frac{1}{2}n(n-1)x^{n-2}dx + ...$$

Therefore the derivative of x^n is

$$\lim_{dx \to 0} (nx^{n-1} + \frac{1}{2}n(n-1)x^{n-2}dx + ...)$$

$$= nx^{n-1}$$

Notation

When we use the triangle method we are finding $\frac{\Delta y}{\Delta x}$ where Δ means 'the difference in'. Similarly, when we differentiate we are effectively drawing an infinitely small triangle to calculate the rate of change (or gradient) of a tiny slice of the function. When differentiating y with respect to x we use the notation:

$$\frac{dy}{dx}$$

where the d means an infinitely small difference. (We don't use Δ any more since this represents a finite difference).

- We say that dx is the **differential** of x.
- We say that $\frac{dy}{dx}$ is the **derivative** of y with respect to x.

Example:

The derivative of x^2, as shown above, is $2x$. Using the notation above we would write this as:

$$\text{If } y = x^2 \text{ then } \frac{dy}{dx} = 2x$$

Recall that we can write a function as either y or $f(x)$, leading to other notation for differentiation displayed in the table below:

Function	Derivative	Pronunciation
y	$\frac{dy}{dx}$	'd-y by d-x' **or** 'd-y d-x'
$f(x)$	$f'(x)$	'f prime of x' **or** 'f dash of x'
$f(x)$	$\frac{d}{dx}(f(x))$	'd by d-x of f of x'

Note: We may sometimes refer to $\dfrac{d}{dx}$ as the differential operator.

Example:

So if $f(x) = x^2$ then we can write the derivative in the following ways:

$$f'(x) = 2x$$

OR

$$\frac{d}{dx}(x^2) = 2x$$

IMPORTANT: When we find $\frac{dy}{dx}$ we say we are 'differentiating y with respect to x'. Now suppose instead that we have the equation $a = b^2$ then:

$$\frac{da}{db} = 2b$$

This works in the same way as we have seen for x and y but in this case we have it in terms of a and b and would say we are 'differentiating a with respect to b'.

6.2 Standard Derivatives

Differentiating Polynomials

A polynomial is an expression in the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where

- $a_n, \ldots, a_2, a_1, a_0$ are constant coefficients.
- $x^n, x^{n-1}, \ldots, x^2$ are powers of a variable x.

A rather simple example of a polynomial is the quadratic $x^2 + 3x + 1$.
Differentiation finds the gradient of a curve at a given point. Suppose we want to find the gradient of a polynomial function such as $y = x^3 + 1$. How do we find this derivative?
As differentiation may be new to many readers we will begin slowly and first explain how to differentiate the individual terms of a polynomial, for example x^2, before moving on in the section on 'how to differentiate a sum' to help us differentiate polynomial sums such as $4x^3 + 1$.
The general rule for differentiating polynomials is as follows:

$$\boxed{\text{If } y = x^n \text{ then } \frac{dy}{dx} = nx^{n-1}}$$

You can remember this using the phrase: 'power down the front, one off the power'.
For example:

- If we have $y = x$ then this is the same as $y = x^1$ so we have $\dfrac{dy}{dx} = 1 \times x^{1-1} = 1$.
- If we have $y = x^2$ then we have $\dfrac{dy}{dx} = 2 \times x^{2-1} = 2x$.

Another important rule to remember is as follows:

$$\boxed{\text{If } y = c \text{ for any constant } c \text{ then } \frac{dy}{dx} = 0.}$$

We have only considered examples of polynomials up to the power of 2. The next example shows that we can apply this rule to polynomials of higher powers.

Example: Differentiate $y = x^5$.

Solution: From above we know that if $y = x^n$ then $\dfrac{dy}{dx} = nx^{n-1}$. So in this example we have $n = 5$. This gives the following answer:

$$\frac{dy}{dx} = 5x^{5-1} = 5x^4$$

We can also differentiate using this rule when x is raised to a negative power as shown in the next example.

Example: Differentiate $y = x^{-1}$.

Solution: From above we know that if $y = x^n$ then $\dfrac{dy}{dx} = nx^{n-1}$. So in this example we have $n = -1$. This gives the following answer:

$$\frac{dy}{dx} = -1 \times x^{-1-1} = -x^{-2}$$

We can differentiate using this rule when x is raised to a fractional power as shown in the next example.

Example: Differentiate $y = x^{\frac{1}{2}}$.

Solution: So using the rule that if $y = x^n$ then $\frac{dy}{dx} = nx^{n-1}$. We have $n = \frac{1}{2}$. This gives the answer:

$$\frac{dy}{dx} = \frac{1}{2} \times x^{\frac{1}{2}-1} = \frac{x^{-\frac{1}{2}}}{2} = \frac{1}{2\sqrt{x}}$$

So far we have only considered polynomials which have a coefficient of 1. But how do we differentiate $y = 4x^2$ where the coefficient is equal to 4? When we have a coefficient a we have the rule:

$$\boxed{\text{If } y = ax^n \text{ then } \frac{dy}{dx} = a \times n \times x^{n-1}}$$

Note: The first rule that we discussed is just a special case of the above rule when $a = 1$.
So going back to $y = 4x^2$ we can see that the coefficient $a = 4$ and x is raised to the power of $n = 2$. So using the rule, if $y = ax^n$ then $\frac{dy}{dx} = a \times n \times x^{n-1}$. We have:

$$\frac{dy}{dx} = 4 \times 2 \times x^{2-1} = 8x^1 = 8x$$

IMPORTANT: When differentiating any function $f(x)$ multiplied by some constant a, finding $\frac{d}{dx}[af(x)]$, we can take the constant a out of the derivative so:

$$\boxed{\frac{d}{dx}\left[af(x)\right] = a\frac{d}{dx}\left[f(x)\right]}$$

For example $\frac{d}{dx}(2x^3) = 2 \times \frac{d}{dx}(x^3) = 2 \times 3x^2 = 6x^2$.

Example: Differentiate $y = 4x^{-\frac{1}{2}}$

Solution: This example combines everything we have looked at so far. So using the rule:

$$\text{If } y = ax^n \text{ then } \frac{dy}{dx} = a \times n \times x^{n-1}$$

we have $a = 4$ and $n = -\frac{1}{2}$. This gives the answer:

$$\frac{dy}{dx} = 4 \times -\frac{1}{2} \times x^{-\frac{1}{2}-1} = -2x^{-\frac{3}{2}}$$

Physics Example: Coulomb's Law states that the magnitude of the attractive force F between two charges decays according to the inverse square of the separating distance r:

$$F = \frac{1}{r^2}$$

If we plot a graph of F on the y-axis against r on the x-axis, what is the gradient?

Solution: To find the gradient we find the derivative with respect to r denoted $\frac{dF}{dr}$ of:

$$F = \frac{1}{r^2} = r^{-2}$$

All the examples so far have only contained x and y however in this question we have the variables F and r. We know $y = x^n$ then $\frac{dy}{dx} = nx^{n-1}$, so with y as F and x as r and $n = -2$. We have:

$$\frac{dF}{dr} = -2 \times r^{-2-1} = -2 \times r^{-3} = -\frac{2}{r^3}$$

Below we have the rules for differentiating trigonometric functions:

$$\boxed{\text{If } y = \sin(x) \text{ then } \frac{dy}{dx} = \cos(x)}$$

$$\boxed{\text{If } y = \cos(x) \text{ then } \frac{dy}{dx} = -\sin(x)}$$

$$\boxed{\text{If } y = \tan(x) \text{ then } \frac{dy}{dx} = \sec^2(x)}$$

When we have constants multiplying the trigonometric functions and coefficients multiplying our x we can use the following rules:

$$\boxed{\text{If } y = a\sin(bx) \text{ then } \frac{dy}{dx} = a \times b \times \cos(bx)}$$

$$\boxed{\text{If } y = a\cos(bx) \text{ then } \frac{dy}{dx} = -a \times b \times \sin(bx)}$$

$$\boxed{\text{If } y = a\tan(bx) \text{ then } \frac{dy}{dx} = a \times b \times \sec^2(bx)}$$

Note: These rules only work when x is being measured in radians.

Example: Differentiate:

1. $y = \sin(9x)$
2. $y = 4\cos\left(\frac{x}{2}\right)$
3. $y = -2\tan(-7x)$

Solution: Using the rules above:

1. If $y = a\sin(bx)$ then $\frac{dy}{dx} = a \times b \times \cos(bx)$

   Note that we have $y = \sin(9x)$ so $a = 1$ and $b = 9$. So $\frac{dy}{dx} = 9\cos(9x)$

2. If $y = a\cos(bx)$ then $\frac{dy}{dx} = -a \times b \times \sin(bx)$

   Note that we have $y = 4\cos\left(\frac{x}{2}\right)$ so $a = 4$ and $b = \frac{1}{2}$ $\frac{dy}{dx} = -4 \times \times \frac{1}{2} \times \sin\left(\frac{x}{2}\right) = -2\sin\left(\frac{x}{2}\right)$

3. If $y = a\tan(bx)$ then $\frac{dy}{dx} = a \times b \times \sec^2(bx)$

   Note that we have $y = -2\tan(-7x)$ so $a = -2$ and $b = -7$ $\frac{dy}{dx} = (-2) \times \left(-7\sec^2(-7x)\right) = -14\sec^2(-7x) = 14\sec^2(7x)$

Physics Example: In X-ray crystallography, the Bragg equation relates the distance d between successive layers in a crystal, the wavelength of the X-rays λ, an integer n and the angle through which the X-rays are scattered θ in the equation:

$$\lambda = \frac{2d}{n}\sin\theta$$

What is the rate of change of λ with θ?

Solution: The question is asking us to find $\frac{d\lambda}{d\theta}$. So as n and d are constants we can use the rule:

$$\text{If } y = a\sin(bx) \text{ then } \frac{dy}{dx} = a \times b \times \cos(bx)$$

So we have $a = \left(\frac{2d}{n}\right)$ and $b = 1$ with λ as y and θ as x therefore:

$$\frac{d\lambda}{d\theta} = \left(\frac{2d}{n}\right) \times 1 \times \cos\theta = \frac{2d}{n}\cos\theta.$$

Differentiating Exponential Functions

The exponential function e^x is special, since when we differentiate e^x, we just obtain e^x again. This is shown in the rule below:

$$\boxed{\text{If } y = e^x \text{ then } \frac{dy}{dx} = e^x}$$

We can also differentiate when the x has a coefficient, for example $y = e^{4x}$, and also when the exponential is multiplied by a constant, for example $y = 3e^x$. We combine these into the rule below:

$$\boxed{\text{If } y = ae^{bx} \text{ then } \frac{dy}{dx} = a \times b \times e^{bx}}$$

Example: Differentiate the following with respect to x:

1. $y = e^{2x}$
2. $y = 8\exp\left(-\frac{x}{3}\right)$

Solution:

1. Using the rule

$$\text{If } y = ae^{bx} \text{ then } \frac{dy}{dx} = a \times b \times e^{bx}$$

note $y = e^{2x}$ so $a = 1$ and $b = 2$ hence $\frac{dy}{dx} = 2e^{2x}$

2. Using the rule

$$\text{If } y = ae^{bx} \text{ then } \frac{dy}{dx} = a \times b \times e^{bx}$$

note $y = 8\exp\left(-\frac{x}{3}\right)$ so $a = 8$ and $b = -\frac{1}{3}$ hence $\frac{dy}{dx} = 8 \times -\frac{1}{3} \times \exp\left(-\frac{x}{3}\right) = -\frac{8}{3}e^{-\frac{x}{3}}$

Differentiating Logarithmic Functions

We differentiate logarithmic functions using the rule shown below:

$$\boxed{\text{If } y = \ln(x) \text{ then } \frac{dy}{dx} = \frac{1}{x}}$$

We can differentiate when the x has a coefficient, for example $y = \ln(4x)$, and also when the logarithmic function has a coefficient, for example $y = 3\ln(x)$. We combine these into the rule below:

$$\boxed{\text{If } y = a\ln(bx) \text{ then } \frac{dy}{dx} = \frac{a}{x}}$$

Note: In the above rule, the b disappears and it forms a nice exercise in the practice of using the laws of logarithms hence we have included the algebra in the next example. We do have to differentiate a sum however so if you are unsure on that step then check out the next section for more information.

Example: For $y = a\ln(bx)$ find $\dfrac{dy}{dx}$

Solution: Using laws of logarithms we can write $y = a\ln(bx)$ as:

$$y = a\left(\ln(x) + \ln(b)\right)$$

Now differentiating this as a sum gives:

$$\frac{dy}{dx} = \frac{d}{dx}\left[a(\ln(x) + \ln(b))\right]$$

$$\implies \frac{dy}{dx} = a\frac{d}{dx}\left[\ln(x)\right] + a\frac{d}{dx}\left[\ln(b)\right]$$

Now as $\dfrac{d}{dx}\left[\ln(x)\right] = \dfrac{1}{x}$ and $\dfrac{d}{dx}\left[\ln(b)\right] = 0$ as $\ln(b)$ is a constant, we have:

$$\frac{dy}{dx} = \frac{a}{x}$$

Note: This derivation is included as it provides an advanced example of the use of the laws of logarithms with differentiation.

Example: Differentiate:

1. $y = \ln(3x)$
2. $y = -\dfrac{4}{5}\ln\left(\dfrac{x}{2}\right)$

Solution:

1. We use the rule:

$$\text{If } y = a\ln(bx) \text{ then } \frac{dy}{dx} = \frac{a}{x}$$

   Note that we have $y = \ln(3x)$ so $a = 1$ and $b = 3$. Hence $\dfrac{dy}{dx} = \dfrac{1}{x}$

2. We use the rule:

$$\text{If } y = a\ln(bx) \text{ then } \frac{dy}{dx} = \frac{a}{x}$$

   Note that we have $y = -\dfrac{4}{5}\ln\left(\dfrac{x}{2}\right)$ so $a = -\dfrac{4}{5}$ and $b = \dfrac{1}{2}$. Hence $\dfrac{dy}{dx} = -\dfrac{4}{5} \times \dfrac{1}{x} = -\dfrac{4}{5x}$

Differentiating Hyperbolic Functions

Below we have the rules for differentiating hyperbolic functions:

$$\boxed{\text{If } y = \sinh(x) \text{ then } \frac{dy}{dx} = \cosh(x)}$$

$$\boxed{\text{If } y = \cosh(x) \text{ then } \frac{dy}{dx} = \sinh(x)}$$

$$\boxed{\text{If } y = \tanh(x) \text{ then } \frac{dy}{dx} = \text{sech}^2(x)}$$

When we have constants multiplying the hyperbolic functions and coefficients multiplying our x we can use the following rules:

$$\boxed{\text{If } y = a\sinh(bx) \text{ then } \frac{dy}{dx} = a \times b \times \cosh(bx)}$$

$$\boxed{\text{If } y = a\cosh(bx) \text{ then } \frac{dy}{dx} = a \times b \times \sinh(bx)}$$

$$\boxed{\text{If } y = a\tanh(bx) \text{ then } \frac{dy}{dx} = a \times b \times \text{sech}^2(bx)}$$

Note: that the derivative of $\cosh(x)$ is $\sinh(x)$ and NOT $-\sinh(x)$. We can prove these derivatives by differentiating the exponential forms of the hyperbolic functions:

Example: Differentiate $\cosh(x)$ by converting to the exponential form.

Solution: Begin by writing $\cosh(x)$ as

$$\cosh(x) = \frac{e^x}{2} + \frac{e^{-x}}{2}$$

Differentiating each term:

$$\begin{aligned}\frac{d}{dx}\cosh(x) &= \frac{e^x}{2} - \frac{e^{-x}}{2}\\ &= \sinh(x)\end{aligned}$$

6.3 Differentiation Techniques

Differentiating a Sum

If we have multiple terms in the form of a sum, for example $y = x^3 + 8x + 1$, then in order to find $\frac{dy}{dx}$, we apply the differential operator on each of the terms in turn. So in other words we differentiate each part of the sum one by one.
Consider $y = x^3 + 8x + 1$. If we want to find $\frac{dy}{dx}$ then we would apply the operator $\frac{d}{dx}$ to both sides of the equation giving us:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(x^3 + 8x + 1) && \text{Now we apply the } \frac{d}{dx} \text{ operator to each term.} \\ \implies \frac{dy}{dx} &= \frac{d}{dx}(x^3) + \frac{d}{dx}(8x) + \frac{d}{dx}(1) && \text{Now we differentiate each of the terms individually.} \\ \implies \frac{dy}{dx} &= 3x^2 + 8 + 0 && \text{Now simplify.} \\ \implies \frac{dy}{dx} &= 3x^2 + 8 \end{aligned}$$

Example: Differentiate the following:

1. $y = e^x + 4x^6$
2. $y = \ln(2x) + \frac{1}{3}\sin(9x)$

Solution:

1. To differentiate y we differentiate all of the terms individually. We use the rules for differentiating exponentials and powers to get:

$$y = e^x + 4x^6 \implies \frac{dy}{dx} = e^x + 4 \times 6x^5 = e^x + 24x^5$$

2. Again, we differentiate each term.

$$y = \ln(2x) + \frac{1}{3}\sin(9x) \implies \frac{dy}{dx} = \frac{1}{x} + \frac{1}{3} \times 9\cos(9x) = \frac{1}{x} + 3\cos(9x)$$

Example: Differentiate $z = \cos(4t) + 6e^t + t^5 + 2t^2 + 7$ with respect to t.

Solution: This question is different as our equation is not in terms of y and x, but instead in terms of z and t. We treat this in the same way as if the terms were y and x, so we differentiate each term individually.

$$\frac{dz}{dt} = -4\sin(4t) + 6e^t + 5t^4 + 4t$$

Physics Example: An equation of motion is

$$s = ut + \frac{1}{2}at^2$$

where s is the displacement of a body, t is the time that has passed and u and a are constants. Find the velocity $v = \dfrac{ds}{dt}$ of the body.

Solution: As with the previous example, we aren't using x and y. We need to differentiate s with respect to t to find the velocity.

$$\begin{aligned} v = \frac{ds}{dt} &= \frac{d}{dt}(ut) + \frac{d}{dt}\left(\frac{1}{2}at^2\right) && \text{We can now differentiate each of the terms} \\ \frac{ds}{dt} &= u + \frac{1}{2}(2at) && \text{Then simplify} \\ \frac{ds}{dt} &= u + at \end{aligned}$$

This confirms the equation of motion:

$$v = u + at$$

The Product Rule

Now suppose we want to differentiate $y = x^2 \sin(x)$. We encounter the problem: How do we differentiate a function that is the product of two other functions, in this case x^2 and $\sin(x)$?
To do this we need the product rule:

If the function $y = f(x)$ is written as the product of two functions say $u(x)$ and $v(x)$ so

$$y = f(x) = u(x)v(x) \quad \text{then} \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

So returning to our example we have $f(x) = x^2 \sin(x)$ with $u = x^2$ and $v = \sin(x)$.
First we find that $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = \cos(x)$ then we substitute into $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\implies \quad \frac{dy}{dx} = x^2 \cos(x) + \sin(x)2x$$
$$\implies \quad \frac{dy}{dx} = x(x\cos(x) + 2\sin(x))$$

Example: Differentiate using the product rule:

1. $y = x\ln(x)$
2. $y = 6e^x \cos(x)$

Solution:

1. Take $u = x$ and $v = \ln(x)$.
   We can now differentiate these to find that $\frac{du}{dx} = 1$ and that $\frac{dv}{dx} = \frac{1}{x}$ Using the product rule:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \qquad \text{Substitute in the quantities}$$
$$\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln(x)\Big(1\Big) \qquad \text{Expand and simply}$$
$$\frac{dy}{dx} = 1 + \ln(x)$$

2. For this take $u = 6e^x$ and $v = \cos(x)$.
   Now we differentiate those to find that $\frac{du}{dx} = 6e^x$ and $\frac{dv}{dx} = -\sin(x)$ By the product rule we obtain:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \qquad \text{Substitute in the expressions}$$
$$\frac{dy}{dx} = 6e^x\Big(-\sin(x)\Big) + \cos(x)\Big(6e^x\Big) \qquad \text{Expand the brackets}$$
$$\frac{dy}{dx} = -6e^x\sin(x) + 6e^x\cos(x) \qquad \text{Take a factor of } 6e^x \text{ out}$$
$$\frac{dy}{dx} = 6e^x\Big(\cos(x) - \sin(x)\Big)$$

Example: Differentiate $y = \frac{1}{3}(x^2 + 5x)\sin(x)$.

Solution: This is a product of two functions and we must use the product rule.
Set $u = \frac{1}{3}(x^2 + 5x)$ and $v = \sin(x)$

We can differentiate to find that $\frac{du}{dx} = \frac{1}{3}(2x + 5)$ and $\frac{dv}{dx} = \cos(x)$

We now use the product rule:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \qquad \text{Substitute in our expressions}$$

$$\frac{dy}{dx} = \frac{1}{3}(x^2 + 5x)\Big(\cos(x)\Big) + \sin(x)\Big(\frac{1}{3}(2x + 5)\Big) \qquad \text{Simplify by taking a factor of } \frac{1}{3} \text{ out}$$

$$\frac{dy}{dx} = \frac{1}{3}\Big((x^2 + 5x)\cos(x) + (2x + 5)\sin(x)\Big)$$

The end result may appear complicated and ugly but it is nicer than having all the brackets expanded.

The Quotient Rule

A quotient is a fraction of two functions, for example:

$$y = \frac{x^3}{e^{3x}}$$

When we have a function in the form of a quotient we differentiate it using the quotient rule:

> If the function $f(x)$ is written as a quotient such as
>
> $$y = \frac{u(x)}{v(x)} \quad \text{then} \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

So returning to $y = \dfrac{x^3}{e^{3x}}$ we have $u = x^3$ and $v = e^{3x}$.

Hence $\dfrac{du}{dx} = 3x^2$ and $\dfrac{dv}{dx} = 3e^{3x}$ and $v^2 = e^{6x}$. Then substitute into quotient rule $\dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\implies \frac{dy}{dx} = \frac{e^{3x}3x^2 - x^3 3e^{3x}}{e^{6x}}$$

$$\implies \frac{dy}{dx} = 3x^2e^{-3x}(1-x)$$

Example: Differentiate $y = \dfrac{\sin(x)}{3\ln(x)}$.

Solution: Just as in the previous example, we have u in the numerator and v in the denominator of the quotient so $u = \sin(x)$ and $v = 3\ln(x)$. We can now differentiate these to get $\dfrac{du}{dx} = \cos(x)$ and $\dfrac{dv}{dx} = \dfrac{3}{x}$. Using the quotient rule we get that:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \qquad \text{Substitute in what we have}$$

$$\frac{dy}{dx} = \frac{3\ln(x) \times \cos(x) \ - \ \sin(x) \times \frac{3}{x}}{\left(3\ln(x)\right)^2} \qquad \text{We can simplify}$$

$$\frac{dy}{dx} = \frac{3\ln(x)\cos(x) \ - \ \frac{3\sin(x)}{x}}{9\ln(x)^2} \qquad \text{A factor of 3 can be taken out}$$

$$\frac{dy}{dx} = \frac{\ln(x)\cos(x) \ - \ \frac{\sin(x)}{x}}{3\ln(x)^2}$$

Example: Differentiate $y = \dfrac{x}{e^{2x}}$

Solution: We have a function x divided by a function e^{2x} so to find the derivative the quotient rule needs to be used. So $u = x$ and $v = e^{2x}$ and therefore $\dfrac{du}{dx} = 1$ and $\dfrac{dv}{dx} = 2e^{2x}$. We can now employ the quotient rule to get:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \qquad \text{Substitute what we know in}$$

$$\frac{dy}{dx} = \frac{e^{2x} \times 1 - x \times 2e^{2x}}{(e^{2x})^2} \qquad \text{Simplifying this by taking a factor of } e^{2x} \text{ out}$$

$$\frac{dy}{dx} = \frac{e^{2x}(1-2x)}{(e^{2x})^2} \qquad \text{The } e^{2x} \text{ on top cancels with one on the bottom}$$

$$\frac{dy}{dx} = e^{-2x}(1-2x)$$

Now that we know the quotient rule, we can prove the derivative of $y = \tan(x)$

Example: Differentiate $y = \tan(x)$.

Solution: We begin by writing

$$y = \tan(x) = \frac{\sin(x)}{\cos(x)}$$

Now apply the quotient rule:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\cos(x)\frac{d}{dx}(\sin(x)) - \sin(x)\frac{d}{dx}(\cos(x))}{\cos^2(x)} \\
&= \frac{\cos^2(x) - (-\sin^2(x))}{\cos^2(x)} \\
&= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\
&= \frac{1}{\cos^2(x)} \\
&= \sec^2(x)
\end{aligned}$$

as we expected.

The Chain Rule

When we have a function inside of a function (called a composite function) we differentiate using the chain rule. An example of a composite function is $y = \sin(x^2 + 1)$. There are two functions $\sin(..)$ and $x^2 + 1$. We are applying the sin function to $x^2 + 1$, thus making it a function inside a function, or a composite function.
As was discussed in an earlier section, the general form of a composite function is:

$$\boxed{y = f(g(x))}$$

where f and g are both functions. In the above example $y = \sin(x^2+1)$ we would have $f(g(x)) = \sin(g(x))$ and $g(x) = x^2 + 1$.

To use the chain rule we use the following steps:

1. Introduce a new variable u to be equal to $u = g(x)$.
2. Substitute $u = g(x)$ into the expression $y = f(g(x))$ so that $y = f(u)$.
3. Find $\dfrac{dy}{du}$ and $\dfrac{du}{dx}$.
4. The derivative $\dfrac{dy}{dx}$ can be found by this equation $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ and then substitute $u = g(x)$ back in.

The chain rule is summarised in the box below:

$$\boxed{\begin{gathered}\text{If } y = f(g(x)) \quad \text{let} \quad u = g(x) \quad \text{hence} \quad y = f(g(x)) = f(u) \\ \text{then} \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}\end{gathered}}$$

Below we follow the steps of the chain rule to differentiate $y = \sin(x^2 + 1)$:

1. Introduce a new variable u to be equal to $g(x)$ (the inside function). For this example $u = x^2 + 1$.
2. Substitute $u = g(x)$ into the expression $y = f(g(x))$ so that $y = f(u)$. As $u = x^2 + 1$ this would make $y = \sin(u)$.
3. Find $\dfrac{dy}{du}$ and $\dfrac{du}{dx}$. In this example we have:

$$y = \sin(u) \Longrightarrow \frac{dy}{du} = \cos(u)$$

and

$$u = x^2 + 1 \Longrightarrow \frac{du}{dx} = 2x$$

4. The derivative $\dfrac{dy}{dx}$ can be found by this equation $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ and then substitute back in $u = g(x)$.

$$\Longrightarrow \frac{dy}{dx} = \cos(u) \times 2x = 2x\cos(x^2 + 1)$$

Example: Differentiate $y = (x^2+2)^3$ using the chain rule:
Solution: We take $u = x^2+2$ and therefore $y = u^3$.
Then we differentiate each of those to find:

$$\frac{du}{dx} = 2x \qquad \frac{dy}{du} = 3u^2$$

From the chain rule:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} && \text{Substitute in our expressions for } \frac{dy}{du} \text{ and } \frac{du}{dx} \\ \frac{dy}{dx} &= 3u^2 \times 2x && \text{Substitute } u = x^2+2 \\ \frac{dy}{dx} &= 6x(x^2+2)^2 \end{aligned}$$

Example: Given $y = \ln(x^3)$ find $\dfrac{dy}{dx}$
Solution: We take $u = x^3$ which means that $y = \ln(u)$.
Then differentiate to find that:

$$\frac{du}{dx} = 3x^2 \qquad \frac{dy}{du} = \frac{1}{u}$$

From the chain rule $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ we can see that:

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{u} \times 3x^2 && \text{Substitute } u = x^3 \\ \frac{dy}{dx} &= \frac{1}{x^3} \times 3x^2 && \text{Simplify further} \\ \frac{dy}{dx} &= \frac{3}{x} \end{aligned}$$

Note: A faster solution would have been to use the laws of logarithms to notice that $\ln(x^3) = 3\ln(x)$ and then differentiate, avoiding the use of the chain rule.

Example: Differentiate $y = \cos(e^{2x})$

Solution: Let $u = e^{2x}$ therefore $y = \cos(u)$.
Differentiate to find:

$$\frac{du}{dx} = 2e^{2x} \text{ and } \frac{dy}{du} = -\sin(u)$$

Using the chain rule $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ we can show that:

$$\begin{aligned} \frac{dy}{dx} &= -\sin(u) \times 2e^{2x} && \text{Substitute that } u = e^{2x} \\ \frac{dy}{dx} &= -2e^{2x}\sin(e^{2x}) \end{aligned}$$

Physics Example: The Maxwell Boltzmann Distribution is a probability distribution of finding particles at certain speed v in 3 dimensions. It has the form of:

$$f(v) = Av^2 e^{-Bv^2}$$

where A and B are positive constants. Using the chain rule and the product rule find $\frac{d}{dv}f(v)$.

Solution: This example involves two steps, first of all the product rule is required to differentiate the whole thing as $f(v)$ is a product of two functions. The chain rule is also needed to differentiate the exponential. Let's call $P = Av^2$ and $Q = e^{-Bv^2}$, thus making $f(v) = P \times Q$. From the product rule:

$$\frac{d}{dv}f(v) = P\frac{dQ}{dv} + Q\frac{dP}{dv}$$

We have $\frac{dP}{dv} = 2Av$ but we will need to use the chain rule to find $\frac{dQ}{dv}$.

We let $u = -Bv^2$, making $Q = e^u$. We can then calcuate that $\frac{du}{dv} = -2Bv$ and that $\frac{dQ}{du} = e^u$. From here the chain rule tells us that:

$$\frac{dQ}{dv} = \frac{dQ}{du} \times \frac{du}{dv} \qquad \text{We substitute our expressions for for } \frac{dQ}{du} \text{ and } \frac{du}{dv}$$

$$\frac{dQ}{dv} = e^u \times (-2Bv) \qquad \text{Substitute } u = -Bv^2$$

$$\frac{dQ}{dv} = -2Bv \cdot e^{-Bv^2}$$

Substituting into the previous equation (shown below) gives us:

$$\frac{d}{dv}f(v) = P\frac{dQ}{dv} + Q\frac{dP}{dv} \qquad \text{Substitute in what we have found}$$

$$\frac{d}{dv}f(v) = Av^2(-2Bv \times e^{-Bv^2}) + e^{-Bv^2} \times 2Av \qquad \text{This can be simplified}$$

$$\frac{d}{dv}f(v) = -2ABv^3 e^{-Bv^2} + 2Av\, e^{-Bv^2} \qquad \text{Take a factor of } 2Av\, e^{-Bv^2} \text{ out}$$

$$\frac{d}{dv}f(v) = 2Av\, e^{-Bv^2}(1 - Bv^2)$$

Note: This example is more complicated since we are using both the product rule and the chain rule. Even more advanced functions might require the use of many differentiation rules in a calculation of their derivatives.

Implicit Differentiation

Consider two variables x and y which are linked through an implicit formula, $f(x, y) = 0$. If we are asked to find $\frac{dy}{dx}$ there are many ways to achieve this:

- Firstly, we may be able rearrange to make y the subject of the equation then differentiate as normal.
- Secondly, if it is easier we could make x the subject and find $\frac{dx}{dy}$ and, as seen in the next section, the reciprocal of this is $\frac{dy}{dx}$.
- However neither of the above may be feasible, so we use **implicit differentiation**.

In the case above, where we have an implicit formula relating x and y, it may not be easy to rearrange the equation to get $y = f(x)$ or $x = g(y)$. In this case, if we wish to find $\frac{dy}{dx}$, we use implicit differentiation. To do this, we treat y as a function of x, i.e. $y = y(x)$ and then simply differentiate each term by x. Note that if we are differentiating a term involving $y(x)$, we must use the chain rule where appropriate.

For example, consider our method for differentiating $(x + 1)^2$ with respect to x. We use the chain rule to obtain:

$$2(x + 1) \times 1$$

Now, if $y = x + 1$, we can write $(x + 1)^2 = y^2$ and differentiate this with respect to x to obtain

$$2y \times \frac{dy}{dx}$$

Note that since $y = x + 1$,

$$2(x + 1) \times 1 = 2y \times \frac{dy}{dx}$$

and this equivalence allows us to see that implicit differentiation works.

Example: Find $\frac{dy}{dx}$ if

$$y + x = 3x^3 + \frac{1}{3}y^3$$

Click here for a video example

Solution: Differentiating the left hand side gives

$$1 \times \frac{dy}{dx} + 1$$

Differentiating the right hand side gives

$$9x^2 + y^2 \times \frac{dy}{dx}$$

Therefore

$$\frac{dy}{dx} + 1 = 9x^2 + y^2\frac{dy}{dx}$$

Thus

$$\frac{dy}{dx} = \frac{9x^2 - 1}{1 - y^2}$$

Example: Find the value of $\frac{dx}{dt}$ if

$$xt = \cos(3t) + 4$$

at $t = \pi$.

Solution: Consider x as a function of t, i.e. $x(t)$. On the left hand side of the equation we must use the product rule to differentiate as we have $x(t) \times t$. This gives

$$x + t \times \frac{dx}{dt}$$

Differentiating the right hand side of the equation gives

$$-3\sin(3t)$$

Therefore

$$x + t \times \frac{dx}{dt} = -3\sin(3t)$$

At $t = \pi$ we have that

$$x\pi = \cos(3\pi) + 4$$

$$x\pi = 0 + 4$$

$$x = \frac{4}{\pi}$$

Thus at $t = \pi \quad x = \frac{4}{\pi}$.

Substituting these values in we obtain

$$\frac{4}{\pi} + \pi\frac{dy}{dx} = -3sin(3\pi)$$

Thus

$$\frac{dx}{dt} = -\frac{4}{\pi^2}$$

Differentiating Inverse Functions

Here we look at how the derivative of the inverse of a function is related to the derivative of the original function. The relationship is given in the following rule.

If $y = y(x)$, find $x(y)$ and calculate $\dfrac{dx}{dy}$

$$\text{then } \frac{dy}{dx} = \frac{1}{\left(\dfrac{dx}{dy}\right)}$$

then substitute $y = y(x)$ into the equation we obtain.

This will become more clear with the help of an example:

Example: If we have $y = \frac{1}{3}x^{\frac{1}{5}}$, we may find it easier to find the inverse function and differentiate this instead. Using the rule for differentiating inverse functions, find $\dfrac{dy}{dx}$.

Solution: First of all, we need to find $x(y)$. To do this, we rearrange the function $y = \frac{1}{3}x^{\frac{1}{5}}$ to make x the subject.

$$\begin{aligned} y &= \tfrac{1}{3}x^{\frac{1}{5}} && \text{Multiply both sides by 3} \\ 3y &= x^{\frac{1}{5}} && \text{Raise both sides to the power 5} \\ 243y^5 &= x \end{aligned}$$

Now we need to calculate $\dfrac{dx}{dy}$:

$$\begin{aligned} \frac{dx}{dy} &= 5 \times 243y^4 \\ &= 1215y^4 \end{aligned}$$

Next, we use the rule above to find $\dfrac{dy}{dx}$:

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\left(\dfrac{dx}{dy}\right)} \\ &= \frac{1}{1215y^4} \\ &= \frac{1}{1215}y^{-4} \end{aligned}$$

Finally, we must substitute $y = \frac{1}{3}x^{\frac{1}{5}}$ back into the equation so that we have $\dfrac{dx}{dy}$ in terms of x:

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{1215}\left(\frac{1}{3}x^{\frac{1}{5}}\right)^{-4} \\ &= \frac{1}{15}x^{-\frac{4}{5}} \end{aligned}$$

Although this method is not incredibly useful in this particular example, there are cases when the derivative of an inverse function is far easier than that of the original function, and so this method can save us a lot of time. The next section will give an important application of this method.

Derivatives of Inverse Trigonometric and Hyperbolic Functions

Now that we have learnt the rule for differentiating inverse functions, we can find the derivatives of inverse trigonometric functions:

$$\boxed{\text{If } y = \sin^{-1}(x) \text{ then } \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}}$$

$$\boxed{\text{If } y = \cos^{-1}(x) \text{ then } \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}}$$

$$\boxed{\text{If } y = \tan^{-1}(x) \text{ then } \frac{dy}{dx} = \frac{1}{1+x^2}}$$

We will only provide the proof for one of these derivatives, as the proofs are all rather similar.

Example: Using the method for differentiating inverse functions, differentiate $y = \sin^{-1}(x)$.

Solution: First of all, we need to write x in terms of y:

$$y = \sin^{-1}(x) \Rightarrow x = \sin(y)$$

Now we need to calculate $\dfrac{dx}{dy}$:

$$\frac{dx}{dy} = \cos(y)$$

Next, we use the rule above to find $\dfrac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$$

$$= \frac{1}{\cos(y)}$$

Notice that we can use the identity $\cos^2(\theta) + \sin^2(\theta) = 1$ to rewrite $\cos(y)$ as

$$\cos(y) = \sqrt{1 - \sin^2(y)}$$

and using our equation $x = \sin(y)$ we can write:

$$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-x^2}}$$

Similarly, we can find the derivatives of inverse hyperbolic functions:

$$\boxed{\text{If } y = \sinh^{-1}(x) \text{ then } \frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}}$$

$$\boxed{\text{If } y = \cosh^{-1}(x) \text{ then } \frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}}}$$

$$\boxed{\text{If } y = \tanh^{-1}(x) \text{ then } \frac{dy}{dx} = \frac{1}{1-x^2}}$$

Note that we could also have found these derivatives by writing the inverse hyperbolic functions in logarithm form.

Differentiating Parametric Equations

We can also differentiate parametric equations to find $\frac{dy}{dx}$. The technique used to do this is to differenctiate both y and x with respect to the parameter, t say, then use the rule:

$$\boxed{\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}}$$

Example: Find $\frac{dy}{dx}$ for the parametric equation:

$$y = 12t^4 + 3$$

$$x = \cos(t)$$

Solution: Fist we need to find $\frac{dy}{dt}$ and $\frac{dt}{dx}$, it is easy to see that:

$$\frac{dy}{dt} = 48t^3$$

$$\frac{dx}{dt} = -\sin(t)$$

Using the rule for differentiating inverse functions we can see that:

$$\frac{dt}{dx} = -\operatorname{cosec}(t)$$

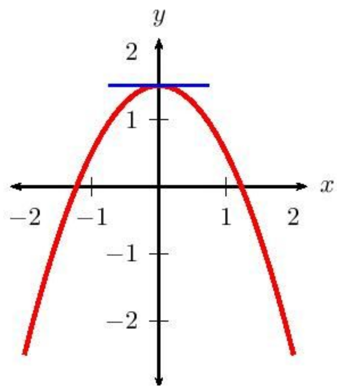
Then using $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$, we can see that:

$$\frac{dy}{dx} = -48t^3 \operatorname{cosec}(t)$$

6.4 Stationary points

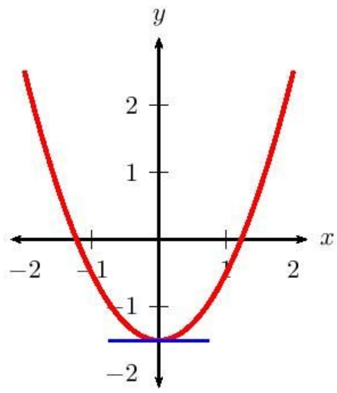
Stationary points occur when we have $\dfrac{dy}{dx} = 0$. This represents when the gradient of a curve is horizontal. There are three types of stationary points:

1. Local maximum



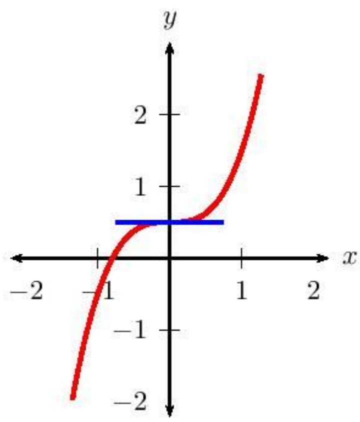
   We can see that when we pass through a maximum, the gradient goes from being positive i.e. $f'(x) > 0$ (or $\frac{dy}{dx} > 0$) to being negative i.e. $f'(x) < 0$ (or $\frac{dy}{dx} < 0$).

2. Local minimum



   We can see that when we pass through a minimum the gradient goes from being negative i.e $f'(x) < 0$ (or $\frac{dy}{dx} < 0$) to being positive i.e $f'(x) > 0$ (or $\frac{dy}{dx} > 0$).

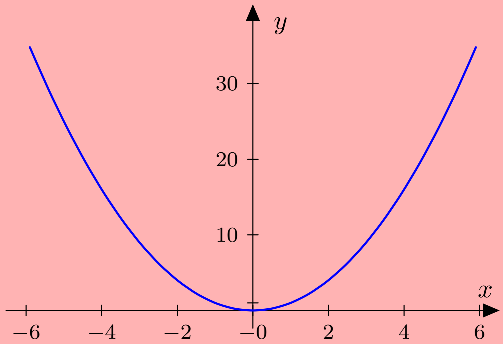
3. Point of inflection (or Saddle Point)



   When we pass through a point of inflection we should note that the gradient does not change sign.

Example: Find the stationary point of $y = x^2$

Solution: From the graph of $y = x^2$ we can see that there is a minimum at $x = 0$ however we need to find this algebraically.



First we need to find when $\frac{dy}{dx} = 0$. Now $\frac{dy}{dx} = 2x$ and therefore $\frac{dy}{dx} = 0$ when $x = 0$.
So we know we have a stationary point at $x = 0$. In the next section we will discuss how to decide whether this is a maximum, minimum or point of inflection.

Classifying Stationary Points

We now need to introduce the second derivative, which is denoted by:

$$\frac{d^2y}{dx^2} \quad \text{or} \quad f''(x)$$

This means that we differentiate a function twice, so for example if we have $y = x^3$ we would first find:

$$\frac{dy}{dx} = 3x^2$$

which we would then differentiate again to find:

$$\frac{d^2y}{dx^2} = 6x$$

First Test

The second derivative is used to decide whether a stationary point is a maximum or a minimum. If we have the point x where there is a stationary point then we substitute this value into the second derivative.

- If $\frac{d^2y}{dx^2} > 0$ we have a minimum.
- If $\frac{d^2y}{dx^2} < 0$ we have a maximum.

Unfortunately this test fails if we find that $\frac{d^2y}{dx^2} = 0$. In this case we could have a maximum, minimum or point of inflection and we must carry out the following test in order to decide:

Second Test

First find two x values just to the left and right of the stationary point and calculate $f'(x)$ for each. Let use denote the x value just to the left as x_L and the one to the right as x_R then:

- If $f'(x_L) > 0$ and $f'(x_R) < 0$ we have a maximum.
- If $f'(x_L) < 0$ and $f'(x_R) > 0$ we have a minimum.
- If $f'(x_L)$ and $f'(x_R)$ have the same sign (so both are either positive or negative) then we have a point of inflection.

Example: Find and classify the stationary points of $y = \frac{x^4}{4} - \frac{x^2}{2}$

Solution: First we find where $\frac{dy}{dx} = 0$. To find where the stationary points are.

$$\begin{aligned} \frac{dy}{dx} &= x^3 - x = 0 \\ \implies x(x^2-1) &= 0 \\ \implies x(x-1)(x+1) &= 0 \end{aligned}$$

Therefore we have stationary points at $x = 0$, $x = 1$ and $x = -1$ to classify them we must find the second derivative:

$$\frac{d^2y}{dx^2} = 3x^2 - 1$$

Now we substitute our x-values for the location of the stationary points into $\frac{d^2y}{dx^2}$.

- Now when $x = 0$ we have $\frac{d^2y}{dx^2} = -1$ hence $\frac{d^2y}{dx^2} < 0$ (negative) so this is a maximum.
- Now when $x = 1$ we have $\frac{d^2y}{dx^2} = 2$ hence $\frac{d^2y}{dx^2} > 0$ (positive) so this is a minimum.
- Now when $x = -1$ we again have $\frac{d^2y}{dx^2} = 2$ hence $\frac{d^2y}{dx^2} > 0$ (positive) so this is a minimum.

Example: Find and classify the stationary points of $y = x^3$

Solution: First we find when $\frac{dy}{dx} = 0$ to find where the stationary points are.

$$\begin{aligned} \frac{dy}{dx} &= 3x^2 = 0 \\ \implies x &= 0 \end{aligned}$$

Therefore we have one stationary point at $x = 0$. To classify this point we need to find the second derivative:

$$\frac{d^2y}{dx^2} = 6x$$

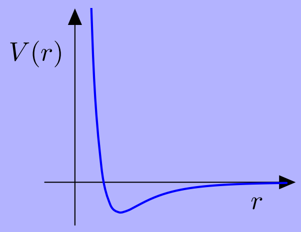
Now when $x = 0$ we have $\frac{d^2y}{dx^2} = 0$ hence our test has failed and we must carry out the further test.
Now we need to pick x values close to either side of the stationary point at $x = 0$. So an x value just to the left would be $x_L = -0.5$ and one just to the right would be $x_R = 0.5$.
Now as $\frac{dy}{dx} = f'(x) = 3x^2$ we obtain:

- $f'(x_L) = 3(x_L)^2 = 3 \times -0.5^2 = 0.75 > 0$
- $f'(x_R) = 3(x_R)^2 = 3 \times 0.5^2 = 0.75 > 0$

So both $f'(x_L) > 0$ and $f'(x_R) > 0$ are positive so are of the same sign. Hence the stationary point at $x = 0$ is a point of inflection.

Physics Example: The Lennard-Jones potential decribes the potential energy V between two atoms separated by a distance r. The equation and graph of this function are shown below:

$$V(R) = \frac{A}{r^{12}} - \frac{B}{r^6}$$



where A and B are constants. The two particles are at their equilibrium separation when the potential is at a minimum (V is at a minimum). By differentiating this equation with respect to r, find the equilibrium separation.

Solution: To find the minimum V we need to differentiate it and then set it equal to zero (find $\frac{dV}{dr} = 0$). The value of r that this equation gives will be where the potential is a minimum. Differentiating V with respect to r gives:

$$\frac{dV}{dr} = \frac{-12A}{r^{13}} + \frac{6B}{r^7}$$

We need to find when $\dfrac{dV}{dr} = 0$ so we need to solve the equation:

$$\begin{aligned}
\frac{-12A}{r^{13}} + \frac{6B}{r^7} &= 0 && \text{Add } \frac{-12A}{r^{-13}} \text{ to both sides} \\
\frac{6B}{r^7} &= \frac{12A}{r^{13}} && \text{Multiply each side by } r^{13} \text{ and divide by } 6B \\
\frac{r^{13}}{r^7} &= \frac{12A}{6B} && \text{Simplify both of the fractions} \\
r^6 &= \frac{2A}{B} && \text{Take the 6th root of both sides} \\
r &= \left(\frac{2A}{B}\right)^{\frac{1}{6}}
\end{aligned}$$

Hence at the equilibrium separation $r = \left(\frac{2A}{B}\right)^{\frac{1}{6}}$.

6.5 Review Questions

Easy Questions

Question 1: Find $\frac{dy}{dx}$ if

$$y = \frac{x^2+5}{x^2-5}$$

Answer:

$$\frac{dy}{dx} = \frac{-20x}{(x^2-5)^2}$$

Question 2: Find $\frac{dy}{dx}$ if

$$y = \sqrt{1+2\tan(x)}$$

Answer:

$$\frac{dy}{dx} = \frac{\sec^2(x)}{\sqrt{1+2\tan(x)}}$$

Question 3: Find $\frac{dy}{dx}$ if

$$y = \ln\left(\tanh(\frac{x}{3})\right)$$

Answer:

$$\frac{dy}{dx} = \operatorname{cosech}(x)\operatorname{sech}(x)$$

Question 4: Find $\frac{dy}{dx}$ if

$$y = \ln\left(\frac{1+\cosh(x)}{\cosh(x)-1}\right)$$

Answer:

$$\frac{dy}{dx} = -2\operatorname{cosech}(x)$$

Question 5: Find $\frac{dy}{dx}$ if

$$y = \frac{x^2+5x-3}{3x^{\frac{1}{2}}}$$

Answer:

$$\frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}} + \frac{5}{6}x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}$$

Medium Questions

Question 6: Find the gradient of the circle

$$y^2 + x^2 = 25$$

at the point $(3, -4)$

Answer:

$$\frac{3}{4}$$

Question 7: Find $\frac{dy}{dx}$ if

$$y = \ln(x^3 e^x)$$

Answer:

$$\frac{dy}{dx} = \frac{3}{x} + 1$$

Question 8: Find $\frac{dy}{dx}$ if

$$y = \ln(x \ln(x))$$

Answer:

$$\frac{dy}{dx} = \frac{\ln(x) + 1}{x \ln(x)}$$

Question 9: Find $\frac{dy}{dx}$ if

$$y = \ln(e^x + e^{-x})$$

Answer:

$$\frac{dy}{dx} = \tanh(x)$$

Question 10: Find $\frac{dz}{dp}$ if

$$z = a^p$$

where a is a constant.

Answer:

$$\frac{dz}{dp} = a^p \ln(a)$$

Hard Questions

Question 11: For $x + y = 100$ prove that the product $P = xy$ is a maximum when $x = y$ and find the maximum value of P.

Hint: Try finding $\frac{dP}{dx}$ and $\frac{dP}{dy}$.

Answer:

$$P_{max} = 2500$$

Question 12: Two positive quantities p and q vary in such a way that $p^3 q = 9$. Another quantity z is defined by $z = 16p + 3q$. Find values of p and q that make z a minimum and hence find the minimum value of z.

Answer:

$$z_{min} = 32$$

Question 13: Find $\frac{dy}{dx}$ if

$$y = x^x$$

Answer:

$$\frac{dy}{dx} = x^x(\ln(x) + 1)$$

Question 14: Find $\frac{dy}{dx}$ if

$$y = \arccos(x) + 2\arcsin(x)$$

Hint: Consider each term separately

Answer:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Question 15: Show that if $y = 3x^2 + \sin(2x)$, then

$$\frac{d^2y}{dx^2} + 4y = 12x^2 + 6$$

7 Partial Differentiation and Multivariable Differential Calculus

7.1 Partial Differentiation

We often encounter quantities that depend on more than one variable. For example, the temperature T in a room depends on where you are in the room, so T depends on the three directional vectors x, y and z. We write $T = T(x, y, z)$, which means that T is a function of three variables x, y and z. Another case is the ideal gas equation:

$$p = \frac{nRT}{V}$$

The pressure p of a gas depends on its volume V and its temperature T. This can be written as $p = p(V, T)$. We often find it useful to calculate how one variable changes with another while the remaining variables are taken to be constant.

As we have just seen, we can use the notation $y = f(x, z)$ to mean that y is a function of x and z where x, y, z are variables. For example:

$$y = x^2 + zx$$

We use partial derivatives to differentiate functions with multiple variables. We do this by differentiating with respect to one variable and treating the other variable as constant. So for $y = f(x, z)$ the partial derivative of a y with respect to x is denoted:

$$\left(\frac{\partial y}{\partial x}\right)_z$$

This is pronounced 'the partial derivative of y with respect to x', or 'partial y, partial x'.

The ∂x term informs us that we perform partial differentiation with respect to x while the subscript variable, in this case z, is being considered as a constant (not every textbook uses the subscript notation).

We can use another notation for partial derivatives as follows:

If we have a function $f = f(x, y, z)$, i.e. f depends on three variables x, y and z, then the partial derivatives of f with respect to its three variables are:

$$\begin{aligned}
&\left(\frac{\partial f}{\partial x}\right)_{y,z} \quad &\text{or} \quad &f_x \\
&\left(\frac{\partial f}{\partial y}\right)_{x,z} \quad &\text{or} \quad &f_y \\
&\left(\frac{\partial f}{\partial z}\right)_{x,y} \quad &\text{or} \quad &f_z
\end{aligned}$$

The second notation is often more convenient and saves time when we are solving questions involving lots of partial derivatives.

In many cases, we can save time by dropping the subscript and the brackets and just writing

$$\left(\frac{\partial y}{\partial x}\right)_z = \frac{\partial y}{\partial x}$$

as it is often implicit which variables are being kept constant, however you should always make sure that you understand that we are only **treating** these variables as constant for the sake of partial differentiation.

All of the rules for ordinary differentiation described in the previous chapter also apply to partial differentiation.

Example: Using our example $y = x^2 + zx$, find $\dfrac{\partial y}{\partial x}$

Solution: We note:

- The curvy symbol ∂ informs us that we perform partial differentiation.
- We are differentiating y with respect to x.
- We treat z as a constant.

So we differentiate the x^2 term as usual to get $2x$. Then we differentiate the zx term treating z as a constant so this gives z. Hence:

$$\frac{\partial y}{\partial x} = 2x + z$$

Example: For $y = 2\ln(z) + \sin(zx)$ find $\dfrac{\partial y}{\partial x}$ and $\dfrac{\partial y}{\partial z}$

Solution: For $\dfrac{\partial y}{\partial x}$ we treat z as a constant and differentiate with respect to x.

$$\frac{\partial y}{\partial x} = \frac{\partial}{\partial x} 2\ln(z) + \frac{\partial}{\partial x}\sin(zx) \qquad \text{Treating } z \text{ as a constant}$$

$$\frac{\partial y}{\partial x} = 0 + z\cos(zx) = z\cos(zx)$$

For $\dfrac{\partial y}{\partial z}$ we treat x as a constant and then differentiate with respect to z.

$$\frac{\partial y}{\partial x} = \frac{\partial}{\partial z} 2\ln(z) + \frac{\partial}{\partial z}\sin(zx) \qquad \text{Treating } x \text{ as a constant}$$

$$\frac{\partial y}{\partial x} = \frac{2}{z} + z\cos(zx)$$

Physics Example: The ideal gas equation is $pV = nRT$. Find:

1. $\left(\frac{\partial V}{\partial T}\right)_p$

2. $\left(\frac{\partial T}{\partial p}\right)_V$

Solution:

1. To be able to find $\left(\frac{\partial V}{\partial T}\right)_p$ we must first make V the subject in the equation. This is done by dividing both sides by p to produce $V = \frac{nRT}{p}$.
   Now we differentiate with respect to T, taking p to be a constant.

$$\left(\frac{\partial V}{\partial T}\right)_p = \frac{\partial}{\partial T}\left(\frac{nRT}{p}\right) \qquad \text{Note } \frac{nR}{p} \text{ is a constant.}$$

$$\left(\frac{\partial V}{\partial T}\right)_p = \frac{nR}{p}$$

   This is Charles's law; at constant pressure, the volume of a gas is directly proportional to the temperature.

2. We do the same process to find $\left(\frac{\partial T}{\partial p}\right)_V$. First make T the subject by dividing both sides by nR to make $T = \frac{PV}{nR}$.
   As before we then differentiate, treating V as a constant this time to get:

$$\left(\frac{\partial T}{\partial p}\right)_V = \frac{\partial}{\partial p}\left(\frac{pV}{nR}\right) \qquad \text{Note } \frac{V}{nR} \text{ is a constant}$$

$$\left(\frac{\partial T}{\partial p}\right)_V = \frac{V}{nR}$$

   This is Gay-Lussac's (or, the pressure) law; under fixed volume the pressure is proportional to the temperature.

7.2 Higher Order Partial Derivatives

As with ordinary differentiation, we can calculate second order partial derivatives or even higher orders. This is done in exactly the same way as before, however, since we are handling functions which depend on multiple variables, we can now differentiate functions with respect to one variable, and then with respect to a different variable. For example, if we have a function $f = f(x, y, z)$ then we can differentiate with respect to x, and then differentiate with respect to y. This is written as:

$$\frac{\partial^2 f}{\partial y \partial x} \quad \text{or} \quad f_{xy}$$

where we have dropped the subscript 'z' to simplify the notation.
We pronounce this as 'the second partial derivative of f with respect to x and y'.
It is important to notice that in the first notation we write the order in which we are differentiating backwards. We can understand the reason for this by writing out exactly what this notation means:
First we are differentiating f with respect to x:

$$\frac{\partial f}{\partial x}$$

We then differentiate this with respect to y:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

And so after expanding the brackets we have:

$$\frac{\partial^2 f}{\partial y \partial x}$$

This is analogous to our notation for composite functions such as $fg(x)$ where we apply g to x first, and then apply f.
On the other hand, in the second notation given above, we write the subscripts in the same order in which we differentiate.

Example: Given the function $f = x\cos(z) + xz^2 + zy^2$, find:

1. $\dfrac{\partial^2 f}{\partial z^2}$
2. $\dfrac{\partial^2 f}{\partial x \partial z}$
3. $\dfrac{\partial^2 f}{\partial z \partial x}$

Click here for a video example

Solution:

1. First we need to differentiate with respect to z:

$$\frac{\partial f}{\partial z} = -x\sin(z) + 2xz + y^2$$

Now to get the second partial derivative with respect to z, we differentiate with respect to z again:

$$\begin{aligned}\frac{\partial^2 f}{\partial z^2} &= -x\cos(x) + 2x + 0\\ &= -x\cos(x) + 2x\end{aligned}$$

2. First we need to differentiate with respect to z:

$$\frac{\partial f}{\partial z} = -x\sin(z) + 2xz + y^2$$

Now to get the second partial derivative with respect to z and x, we now differentiate with respect to x:

$$\begin{aligned}\frac{\partial^2 f}{\partial x \partial z} &= -\sin(x) + 2z + 0\\ &= -\sin(x) + 2z\end{aligned}$$

3. First we need to differentiate with respect to x:

$$\begin{aligned}\frac{\partial f}{\partial z} &= \cos(z) + z^2 + 0\\ &= \cos(z) + z^2\end{aligned}$$

Now to get the second partial derivative with respect to x and z, we now differentiate with respect to z:

$$\begin{aligned}\frac{\partial^2 f}{\partial z \partial x} &= -\sin(x) + 2z\\ &= -\sin(x) + 2z\end{aligned}$$

Notice that in this example:

$$\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x}$$

and so the order of partial differentiation does not matter. However this is not always the case! When a function does not have continuous second order partial derivatives, then the order of partial differentiation **does** matter!

7.3 The Chain Rule with Partial Differentiation

We have already described the chain rule in a previous section, however it can be used with partial differentiation to greatly simplify problems.
Say for example that we have a function $f = f(x, y, z)$ which depends on the three variables x, y and z. Now suppose that each of these x, y and z all depend on a single variable, t. If we want to find $\frac{df}{dt}$, then we can solve the problem by substituting our expressions for x, y and z into f, so that we get an equation in just one variable, t, and then differentiating to get $\frac{df}{dt}$. However, in many cases this method would be extremely laborious and difficult, so we can use the chain rule to make the calculation simpler. The following formula is incredibly useful for solving these types of problems:

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{df}{dt}}$$

Although the formula looks rather complicated, in practice it is rather simple to compute.

Example: Given the function $f = 2xy - y^2$, where $x = t^2 + 1$ and $y = t^2 - t$, find $\frac{df}{dt}$.

Solution: First, notice that we do not have z in our expression for f, so we can ignore the 'z' part of the chain rule formula as the derivative of zero is just zero.
Begin by calculating all the derivatives and partial derivatives that we will need. These are:

$$\frac{dx}{dt} = 2t$$

$$\frac{dy}{dt} = 2t - 1$$

$$\frac{\partial f}{\partial x} = 2y$$

$$\frac{\partial f}{\partial y} = 2x - 2y$$

For the partial derivatives containing x or y, we should now substitute in our expressions for x and y in terms of t:

$$\frac{\partial f}{\partial x} = 2y = 2t^2 - 2t$$

$$\frac{\partial f}{\partial y} = 2x - 2y$$

$$= 2t^2 + 2 - 2t^2 - 2t$$

$$= 2 - 2t$$

Now that everything is in terms of t, we can apply the chain rule formula:

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$2t \times (2t^2 - 2t) + (2 - 2t) \times (2t - 1)$$

$$= 4t^3 - 8t^2 + 6t - 2$$

7.4 Gradients and ∇f

Note: In this section we will be using **vectors**. These are covered in their own section earlier on in the booklet, so feel free to return to that section and recap the key concepts. In this section we will be using the notation $\vec{u}$ with an arrow to represent a vector.
You should now be confident with calculating partial derivatives, however we have not yet mentioned what a partial derivative actually **means**.
Similar to ordinary differentiation, a partial derivative represents the gradient of a function **in a particular direction**. For example, $\dfrac{\partial f}{\partial x}$ represents the magnitude of the gradient of a function f parallel to the x-axis.

Vector Fields

A vector field is an assignment of a vector to each point in space. This means that, for each point in space, we can place an arrow representing the magnitude and direction of the field at that point. We can denote a vector field as the vector quantity $\vec{F}$ where

$$\begin{aligned}\vec{F} &= F_x\vec{i} + F_y\vec{j} + F_z\vec{k}\\ &= (F_x, F_y, F_z)\end{aligned}$$

In other words, $\vec{F}$ consists of three functions F_x, F_y and F_z which each correspond to a different perpendicular (or orthogonal) direction.
As you can imagine, vector fields are very useful for modelling fluids or other media where there is some sort of 'flow'.

The Gradient Operator, ∇

We can now define the **gradient vector field** as:

$$\boxed{\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}}$$

where ∇f is pronounced 'grad f' or 'nabla f'. The ∇ is known as the gradient operator.
This quantity gives us the rate of change of f at any point.

Since the operator ∇ is defined as

$$\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

we can consider it as a vector in its own right. This allows us to perform operations between ∇ and vector fields.
Note: Although ∇f is a vector quantity, we do not worry about underlining it or putting an arrow above it (i.e. $\underline{\nabla}$ or $\vec{\nabla}$). This is because ∇ always represents a vector and so we do not need to specify that it is a vector.

Example: Given the function $f = \sin(xy) + x^2z^2$, find ∇f.

Solution: We start be calculating all the partial derivatives that we need:

$$\frac{\partial f}{\partial x} = y\cos(xy) + 2xz^2$$

$$\frac{\partial f}{\partial y} = x\cos(xy)$$

$$\frac{\partial f}{\partial z} = 2x^2z$$

Now we substitute these derivatives into the formula for ∇f:

$$\nabla f = (y\cos(xy) + 2xz^2)\vec{i} + (x\cos(xy))\vec{j} + (2x^2z)\vec{k}$$

Physics Example: The electric potential at a point (x, y, z) of a point charge placed at the origin is given by:

$$V = \frac{Q}{4\pi\epsilon_o}\frac{1}{r} = \frac{Q}{4\pi\epsilon_o}\frac{1}{\sqrt{x^2+y^2+z^2}}$$

where Q, π and ϵ_0 are constants. Calculate ∇V.

Solution: As before, we first calculate all the partial derivatives that we will need:

$$\frac{\partial V}{\partial x} = \frac{Q}{4\pi\epsilon_o} \times 2x \times -\frac{1}{2}\frac{1}{(x^2+y^2+z^2)^{-\frac{3}{2}}} = \frac{Q}{4\pi\epsilon_o}\frac{x}{r^3}$$

$$\frac{\partial V}{\partial y} = \frac{Q}{4\pi\epsilon_o} \times 2y \times -\frac{1}{2}\frac{1}{(x^2+y^2+z^2)^{-\frac{3}{2}}} = \frac{Q}{4\pi\epsilon_o}\frac{y}{r^3}$$

$$\frac{\partial V}{\partial z} = \frac{Q}{4\pi\epsilon_o} \times 2z \times -\frac{1}{2}\frac{1}{(x^2+y^2+z^2)^{-\frac{3}{2}}} = \frac{Q}{4\pi\epsilon_o}\frac{z}{r^3}$$

Now we substitute these derivatives into the formula for ∇V:

$$\begin{aligned}\nabla V &= \frac{Q}{4\pi\epsilon_o}\frac{x}{r^3}\vec{i} + \frac{Q}{4\pi\epsilon_o}\frac{y}{r^3}\vec{j} + \frac{Q}{4\pi\epsilon_o}\frac{z}{r^3}\vec{k} \\ &= \frac{Q}{4\pi\epsilon_o r^3}(x\vec{i} + y\vec{j} + z\vec{k}) \\ &= \frac{Q}{4\pi\epsilon_o r^3}\vec{r} \\ &= \frac{Q}{4\pi\epsilon_o r^2}\hat{\vec{r}}\end{aligned}$$

Notice that this is the same as the equation for the electric field strength, so we have essentially proved the relation $\vec{E} = \nabla V$.

Directional Derivatives

Now that we can use ∇f, we can find the gradient of a function f in any direction. We call this a **directional derivative**, and it is calculated as follows:
The directional derivative of f at a point $\vec{a}$, in the direction of the unit vector $\vec{u}$ is:

$$\boxed{D_u f(\vec{a}) = \vec{u} \cdot \nabla f(\vec{a})}$$

Note: $\vec{u} \cdot \nabla f(\vec{a})$ is a scalar product (or dot product) and so the directional derivative is a scalar quantity, not a vector.

Example: Let

$$f(x, y, z) = xy^2z^3$$

Find the directional derivative of f at the point $(1, 1, 1)$ in the direction to the point $(3, 2, 3)$.

Solution: We first find the gradient of f.

$$\nabla f = y^2z^3\vec{i} + 2xyz^3\vec{j} + 3xy^2z^2\vec{k}$$

At the point $(1, 1, 1)$,

$$\nabla f(1, 1, 1) = \vec{i} + 2\vec{j} + 3\vec{k}$$

Now we must find the unit vector in the direction from $(1, 1, 1)$ to $(3, 2, 3)$.

$$(3, 2, 3) - (1, 1, 1) = (2, 1, 2)$$

$$\Rightarrow \vec{u} = \frac{2\vec{i} + \vec{j} + 2\vec{k}}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{1}{3}(2\vec{i} + \vec{j} + 2\vec{k})$$

Finally, we use the formula for the directional derivative to obtain

$$\begin{aligned} D_u f(1, 1, 1) &= \vec{u} \cdot \nabla f(1, 1, 1) \\ &= \frac{1}{3}(2, 1, 2) \cdot (1, 2, 3) \\ &= \frac{2}{3} + \frac{2}{3} + 2 \\ &= \frac{10}{3} \end{aligned}$$

7.5 Other Operations with ∇

The Laplace operator, ∇^2

Since ∇f is a vector, we can take the scalar product $\nabla \cdot (\nabla f)$. We often write this as $\nabla^2 f$. Using our method for calculating scalar products, we can interpret this in the following way:

$$\begin{aligned}\nabla^2 f = \nabla \cdot (\nabla f) &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) \\ &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\end{aligned}$$

In other words, we calculate the second derivative of f with respect to each of its variables, and then add these derivatives together.
This is known as the 'Laplacian of f' or 'del squared of f'. We call ∇^2 the 'Laplacian operator'.
Note: The Laplacian is a scalar.

The Divergence of $\vec{F}$

Given a vector field $\vec{F}$, we can calculate the object $\nabla \cdot \vec{F}$. This is often written as $div(\vec{F})$ and is called the '**divergence** of $\vec{F}$'. It represents how much the vector field expands (or contracts) about a particular point. It can also be thought of as the rate of change of volume of a flow.
We define the divergence of $\vec{F}$ as:

$$\boxed{div(\vec{F}) = \nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}}$$

We differentiate each part of $\vec{F}$ with respect to its corresponding direction variable, i.e. F_x is the part of $\vec{F}$ that acts in the x direction, so we differentiate F_x with respect to x.
We often call $\nabla\cdot$ the 'divergence operator'.
Note: The divergence is a scalar quantity.

Example: Calculate the divergence of the vector field $\vec{F} = (x^3y^2, z\cos(y), z^2 - x - y)$

Solution: Using the formula:

$$\begin{aligned}div(\vec{F}) &= \frac{\partial}{\partial x}(x^3y^2) + \frac{\partial}{\partial y}(z\cos(y)) + \frac{\partial}{\partial z}(z^2 - x - y) \\ &= 3x^2y - z\sin(y) + 2z\end{aligned}$$

The Curl of $\vec{F}$

Given a vector field $\vec{F}$, we can calculate the object $\nabla \times \vec{F}$. This is often written as $curl(\vec{F})$ and is called the '**curl** of $\vec{F}$'. It represents how much the vector field rotates (or swirls) about a particular point.
We define the curl of $\vec{F}$ as:

$$\boxed{curl(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\vec{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\vec{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\vec{k}}$$

We often call $\nabla\times$ the 'curl operator'.
Note: The curl is a vector quantity.

Example: Calculate the curl of the vector field $\vec{F} = (y+z-2, xyz^2, y^3\tan(z))$

Solution: Using the formula:

$$curl(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y+z-2 & xyz^2 & y^3\tan(z) \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(y^3\tan(z)) - \frac{\partial}{\partial z}(xyz^2)\right)\vec{i} + \left(\frac{\partial}{\partial z}(y+z-2) - \frac{\partial}{\partial x}(y^3\tan(z))\right)\vec{j} + \left(\frac{\partial}{\partial x}(xyz^2) - \frac{\partial}{\partial y}(y+z-2)\right)\vec{k}$$

$$= (3y^2\tan(z) - 2xyz)\vec{i} + (1-0)\vec{j} + (yz^2-1)\vec{k}$$

$$= (3y^2\tan(z) - 2xyz)\vec{i} + \vec{j} + (yz^2-1)\vec{k}$$

For first year physics, an understanding of the proofs for the divergence and curl of a vector field is not required, however it is useful to remember and be able to apply the formulae.

Identities with div, grad and curl

Given vector fields $\vec{F}$ and $\vec{G}$, a scalar function f and the scalars α and β, some useful identities involving the new operations we have just discussed are as follows:

- The divergence operator is linear, i.e.

$$\nabla \cdot (\alpha\vec{F} + \beta\vec{G}) = \alpha\nabla \cdot (\vec{F}) + \beta\nabla \cdot (\vec{G})$$

- The curl operator is linear, i.e.

$$\nabla \times (\alpha\vec{F} + \beta\vec{G}) = \alpha\nabla \times (\vec{F}) + \beta\nabla \times (\vec{G})$$

- The curl of $grad(f)$ is zero, i.e.

$$curl(\nabla f) = \nabla \times (\nabla f) = 0$$

- The divergence of $grad(f)$ is the Laplacian, i.e.

$$div(\nabla f) = \nabla \cdot (\nabla f) = \nabla^2 f$$

- The divergence of $curl(\vec{F})$ is zero, i.e.

$$div(curl\vec{F}) = \nabla \cdot (\nabla \times \vec{F}) = 0$$

7.6 Applications of Partial Differentiation

Lagrange Multipliers

If we have a problem in which we wish to minimise or maximise a function $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$, then we can use the method of Lagrange Multipliers. Examples of when this is useful is when we wish to find the maximum distance from a point to a sphere, or the largest rectangular box which fits inside an ellipsoid. We call these types of problems **constrained extreme value problems**.

In order to solve these problems, we introduce the **Lagrangian function**:

$$\boxed{L(x, y, z; \lambda) = f(x, y, z) + \lambda g(x, y, z)}$$

We then find the stationary equations of this function and find each variable x, y and z if terms of the constant λ. We then substitute our equations for the variables into our constraint and calculate the possible values for λ. Finally, we substitute the values for λ into our equations for the three variables and hence calculate the (x, y, z) which correspond to the maxima and minima.

The stationary equations of the Lagrangian function are:

$$\boxed{\begin{aligned} L_x &= f_x + \lambda g_x = 0 \\ L_y &= f_y + \lambda g_y = 0 \\ L_z &= f_z + \lambda g_z = 0 \\ L_\lambda &= g(x, y, z) = 0 \end{aligned}}$$

where L_x is the partial derivative of L with respect to x and so on.

Example: Using the method of Lagrange multipliers, find the maximum and minimum distance from the point (-2,1,2) to the sphere $x^2 + y^2 + z^2 = 1$

Solution: We know that the distance from (-2,1,2) to the point (x, y, z) is given by

$$d = \sqrt{(x+2)^2 + (y-1)^2 + (z-2)^2}$$

When d approaches a maximum or minimum, so does d^2, so let us choose

$$f(x, y, z) = d^2 = (x+2)^2 + (y-1)^2 + (z-2)^2$$

with the constraint

$$g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$$

Then our Lagrangian function is

$$L(x, y, z) = (x+2)^2 + (y-1)^2 + (z-2)^2 + \lambda(x^2 + y^2 + z^2 - 1)$$

Now we write out our stationary equations:

$$\begin{aligned} L_x = 2(x+2) + 2\lambda x = 0 &\Rightarrow x = -\frac{2}{1+\lambda} \\ L_y = 2(y-1) + 2\lambda y = 0 &\Rightarrow y = \frac{1}{1+\lambda} \\ L_z = 2(z-2) + 2\lambda z = 0 &\Rightarrow z = -\frac{2}{1+\lambda} \\ L_\lambda = x^2 + y^2 + z^2 - 1 = 0 & \end{aligned}$$

Substituting our three equations for the variables into L_λ allows us to find the values of λ which correspond to the minima and maxima of the problem.

$$\left(-\frac{2}{1+\lambda}\right)^2 + \left(\frac{1}{1+\lambda}\right)^2 + \left(-\frac{2}{1+\lambda}\right)^2 = 1$$

$$\Rightarrow 4 + 1 + 4 = (1+\lambda)^2$$

$$\Rightarrow \lambda = 2 \quad \text{or} \quad \lambda = -4$$

So by substituting these values of λ into our equations for x, y and z, we find that the point on the sphere which gives the minimum distance is $(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3})$ and the point which gives the maximum distance is $(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3})$

Finally, we can substitute these points into our expression for the distance $f(x, y, z)$ to get $d_{min} = 2$ and $d_{max} = 4$.

7.7 Review Questions

Easy Questions

Question 1: Find f_x, f_y and f_z if

$$f = z\cos(y) + x$$

Answer:

$$f_x = 1$$
$$f_y = -z\sin(y)$$
$$f_z = \cos(y)$$

Question 2: Find f_x if

$$f = x\sin(xy)$$

Answer:

$$f_x = \sin(xy) + xy\cos(xy)$$

Question 3: Find f_{xx}, f_{yy} and f_{xy} if

$$f = x\sin(y) + yx^2$$

Answer:

$$f_{xx} = 2y$$
$$f_{yy} = -x\sin(y)$$
$$f_{xy} = \cos(y) + 2x$$

Question 4: Find $\dfrac{\partial z}{\partial x}$ if the equation

$$yz - \ln(z) = x + y$$

defines z as a function of x and y.

Hint: You will need to use implicit partial differentiation in this question. Refer to the section on implicit differentiation in the previous chapter if you need to.

Answer:

$$\frac{\partial z}{\partial x} = \frac{z}{yz - 1}$$

Question 5: Using the chain rule, find $\dfrac{\partial z}{\partial t}$ if

$$z = 2xy - y^2$$

$$x = t^2 + 1,\ y = t^2 - 1$$

and express the result in terms of t.

Answer:

$$\frac{\partial z}{\partial t} = 4t(t^2 + 1)$$

Medium Questions

Question 6: Find ∇f if

$$f = \big(y^2 + \sin(z)\big)e^{-x}$$

Answer:

$$\nabla f = -\big(y^2 + \sin(z)\big)e^{-x}\vec{i} + 2ye^{-x}\vec{j} + e^{-x}\cos(z)\vec{k}$$

Question 7: Let

$$f(x, y, z) = x^2 z + yz^3$$

Find the directional derivative of f at the point $(1, 3, 2)$ in the direction to the point $(2, 1, 3)$.

Answer:

$$D_u f(1, 3, 2) = \frac{25}{\sqrt{6}}$$

Question 8: Calculate $div\vec{F}$ if

$$\vec{F} = (x^2 y, -xze^{2y}, x^2 y^2)$$

Answer:

$$div\vec{F} = 2xy - 2xze^{2y}$$

Question 9: Calculate $curl\vec{F}$ if

$$\vec{F} = (xyz, x^2\cos(z), e^{xy})$$

Answer:

$$curl\vec{F} = (xe^{xy} + x^2\sin(z))\vec{i} - (ye^{xy} - xy)\vec{j} + (2x\cos(z) - xz)\vec{k}$$

Question 10: Find $\nabla^2 f$ if

$$f = 3x^3 y^2 z^3$$

Answer:

$$\nabla^2 f = 18xy^2z^3 + 6x^3z^3 + 18x^3y^2z$$

Hard Questions

Question 11: Using the chain rule, find $\dfrac{\partial w}{\partial t}$ if

$$w = e^x yz$$

$$x = ts,\ \ y = s^2,\ \ z = t - s$$

Hint: You may wish to calculate all the partial derivatives you will need and substitute them into the formula for the chain rule.

Answer:

$$\frac{\partial w}{\partial t} = e^{ts}s^2\big((t - s)s + 1\big)$$

Question 12: By considering the function

$$f = x^2yz + e^{xyz}$$

show that $div(\nabla f) = \nabla^2 f$.

Question 13: Recall the relations between Cartesian coordinates and polar coordinates:

$$x = r\cos(\theta) \text{ and } y = r\sin(\theta)$$

Find r_x and θ_x.

Hint: Use implicit partial differentiation on two formulae $r^2 = x^2 + y^2$ and $\tan(\theta) = \dfrac{x}{y}$.

Answer:

$$r_x = \cos(\theta)$$

$$\theta_x = -\frac{\sin(\theta)}{r}$$

Question 14: Consider $f\big(g(x,y), h(x,y)\big)$, where $x = x(t)$ and $y = y(t,s)$. Find f_t and f_s.

Hint: You will need to use the chain rule multiple times.

Answer:

$$f_t = \frac{\partial f}{\partial g}\left(\frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial t}\right) + \frac{\partial f}{\partial h}\left(\frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial y}\frac{\partial y}{\partial t}\right)$$

$$f_s = \frac{\partial f}{\partial g}\frac{\partial g}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial h}\frac{\partial h}{\partial y}\frac{\partial y}{\partial s}$$

Question 15: Use Lagrange multipliers to find the maximum and minimum values of the function

$$f(x,y) = x^2y$$

subject to the constraint

$$g(x,y) = x^2 + y^2 - 3 = 0$$

Answer:

$$f_{min} = -2 \quad \text{and} \quad f_{max} = 2$$

8 Integration

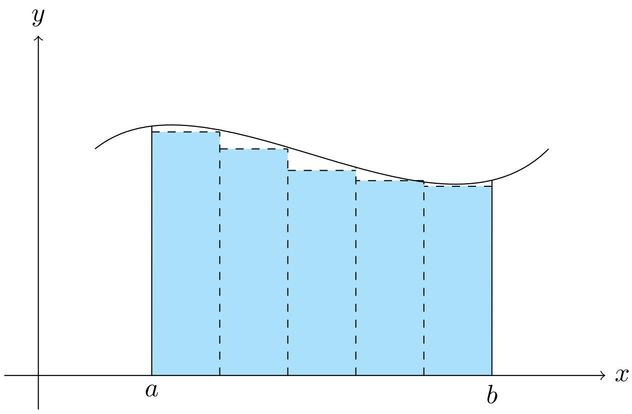
8.1 Introduction to Integration

Integration is the opposite of differentiation so it helps us find the answer to the question:

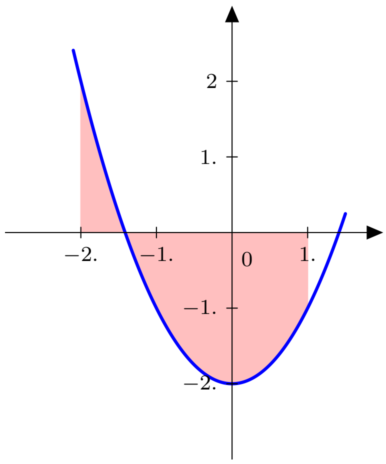
$$\text{‘Suppose we have } \frac{dy}{dx} = 2x \text{ then what is } y\text{?’}$$

From the previous section we know the answer is $y = x^2 + C$, where C is a constant. So we say that the **integral** of $2x$ is $x^2 + C$.
Note: We have an infinite number of solutions to the above question as the constant C could be any number!

Graphically, we know that differentiation gives us the gradient of a curve. Integration, on the other hand, finds the area beneath a curve.



Suppose we wish to find the area under the curve between the x values a and b in the graph above. Then we could find an approximate answer by creating the dashed rectangles above and adding their areas. Now, we can see this does not give a perfect answer, however the smaller the width of the rectangles the smaller the error in our answer would be.
So if we could use rectangles of an infinitely small width we would find an answer with no error. This is what integration does. It creates an infinite number of rectangles under the curve all with an infinitely small width and sums their area to find the total area under the curve.



So integration could be used to find the area shaded (between $x = -2$ and $x = 1$) on the graph of $y = x^2 - 2$ above. Note that integration always finds the area between the curve and the x-axis, so when the curve is below the x-axis, integration finds the area above the curve and below the x-axis as depicted above.
Note: If the area we are finding is below the x-axis then we get a **negative** answer when we integrate.

This means that if we are integrating to find the area shaded in blue above, we will need to separate our calculation into two parts: one to calculate the area under the x-axis and one to calculate the area above the x-axis. We would then have to add up the modulus of our two areas to get the total positive area.

Notation

We use the fact that the integral of $2x$ is $x^2 + C$ to help introduce the following notation for integration. If we want to integrate $2x$ we write this as:

$$\int 2x\,dx = x^2 + C \quad \text{or} \quad \int dx\,2x = x^2 + C$$

- The $\int$ tells us we need to integrate.
- The dx tells use to integrate with respect to x. So we could write $\int 2z\,dz = z^2 + C$ but in this case we have integrated with respect to z.
- We call $\int 2x\,dx = x^2 + C$ the integral.
- We call the expression that we are integrating the integrand. So in this case the integrand is $2x$.

Another notation which we can use is the following:
If we have the function $f(x)$ then we write the integral of this as $F(x)$. So if $f(x) = 2x$ then $F(x) = x^2+C$. Integration finds the area under the curve. So if we want to find the area between two points on the x-axis, say a and b, we use the following notation:

$$\int_a^b f(x)\,dx$$

This tell us we want to find the integral of $f(x)$ and then use that information to find the area under the curve between a and b. We call a and b the 'limits' of this integral. We find the area by using the formula below:

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

Note: This is explained in detail in the section on definite integrals.

- When we are given limits then we have a definite integral.
- When we are not given limits then we have an indefinite integral.

Rules for Integrals

Integration is a **linear** operation. This means that:

1. We can split up a sum into separate integrals. For example $\int (x^3 + 2x + 7)\,dx = \int x^3\,dx + \int 2x\,dx + \int 7\,dx$. Note that each of these integrals will result in a constant, however since these are all arbitrary numbers, we can combine them into a single constant C.
2. We can take constants out of the integral. For example $\int 6x^4\,dx = 6\int x^4\,dx$. Note that the constant resulting from the integral will be multiplied my 6 too, however as it is arbitrary we can incorporate the 6 into the constant and just have one constant C.

8.2 Standard Integrals

Integrating Polynomials

When we are differentiating x^n we multiply by n and then reduce the power on x by 1. When integrating we do the opposite. First we 'add 1 to the power, then divide by this new power'. In general we have the rule:

$$\boxed{\int x^n \, dx = \frac{x^{n+1}}{n+1} + C}$$

Note: This rule works for all values of n apart from $n = -1$. If we were to use the rule with $n = -1$ we would have $\frac{x^0}{0} + C$. This would have us dividing by 0 which would be undefined. The integral of x^{-1} is discussed on the next page.

Example: Find the integrals of the following:

1. x^2
2. $2x + 6x^2$
3. $x(x+3)$

Solution:

1. We are being asked to find $\int x^2 \, dx$, so using the rule $\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + C$

$$\begin{aligned} \int x^2 \, dx &= \frac{x^{2+1}}{2+1} + C \quad \text{This can be simplified} \\ &= \frac{x^3}{3} + C \end{aligned}$$

2. We are being asked to find $\int (2x + 6x^2) \, dx$.
When integrating multiple terms we do each of them individually in turn. This means that:

$$\begin{aligned} \int (2x + 6x^2) \, dx &= \int 2x \, dx + \int 6x^2 \, dx \quad \text{Constants are taken out of the integral} \\ &= 2\int x \, dx + 6\int x^2 \, dx \quad \text{We can now integrate} \\ &= 2\,\frac{x^{1+1}}{1+1} + 6\,\frac{x^{2+1}}{2+1} + C \quad \text{This can be simplified} \\ &= x^2 + 2x^3 + C \end{aligned}$$

3. First we expand the brackets to produce $x^2 + 3x$. Then we integrate this:

$$\begin{aligned} \int (x^2 + 3x) \, dx &= \frac{x^{2+1}}{2+1} + 3\frac{x^{1+1}}{1+1} + C \quad \text{This can be simplified} \\ &= \frac{1}{3}x^3 + \frac{3}{2}x^2 + C \end{aligned}$$

Integrating x^{-1}

We recall that if we differentiate $\ln(x)$ we get x^{-1}. Since integration is the opposite of differentiation, when we integrate x^{-1} we get $\ln(x)$.

$$\boxed{\int x^{-1}\, dx = \ln|x| + C}$$

Note: The $|$ signs on either side of the x mean the modulus (or the absolute value) of x as we discussed on the section on the modulus function. For example $|5| = 5$ and $|-5| = 5$. It is important in this integral as we can input a negative value of x into x^{-1} however we cannot take the log of a negative x.

Example: Find the following integral:

$$\int \frac{x^2 + 2x^3 - 4x^4}{x^3}\, dx$$

Solution: The fraction can be simplified, by dividing each term on the numerator by x^3. This reduces the original integral to:

$$\begin{aligned}
\int \frac{x^2 + 2x^3 - 4x^4}{x^3}\, dx &= \int \left(\frac{1}{x} + 2 - 4x\right) dx && \text{Integrate each term individually} \\
&= \ln|x| + 2x + \frac{-4}{2}x^2 + C && \text{Simplify the terms} \\
&= \ln|x| + 2x - 2x^2 + C
\end{aligned}$$

Integrating Exponential Functions

We can recall that:

$$\boxed{\text{If } y = e^{ax} \text{ then } \frac{dy}{dx} = a \times e^{ax}}$$

So when we are differentiating e^{ax} we multiply the exponential by the coefficient of the power. When integrating we do the opposite so we divide by the coefficient of the power. In general we have:

$$\boxed{\int e^{ax}\,dx = \frac{e^{ax}}{a} + C}$$

Example: Evaluate the following integrals:

1. $\displaystyle\int e^{6x}\,dx$
2. $\displaystyle\int (e^{x} + 4e^{-3x}\ )\,dx$
3. $\displaystyle\int 9e^{3z}\,dz$

Solution:

1. Using the rule $\displaystyle\int e^{ax}\,dx = \frac{e^{ax}}{a} + C$ and dividing by the coefficent of the power we find:

$$\int e^{6x}\,dx = \frac{e^{6x}}{6} + C$$

2. First we should break up the integral across the sum:

$$\int (e^{x} + 4e^{-3x})\,dx = \int e^{x}\,dx + 4\int e^{-3x}\,dx$$

Now we can apply the rule above to get:

$$\int e^{x}\,dx + 4\int e^{-3x}\,dx = \frac{e^{x}}{1} + 4 \times \frac{e^{-3x}}{-3} + C = e^{x} - \frac{4e^{-3x}}{3} + C$$

3. Using the rule above but this time with respect to z, we find:

$$\int 9e^{3z}\,dz = 9\int e^{3z}\,dz = 9 \times \left(\frac{e^{3x}}{3} + C\right) = 3e^{3x} + C'$$

where $C' = 9C$ is just another constant.

Integrating Trigonometric Functions

By recalling the derivatives of $\sin(x)$, $\cos(x)$ and $\tan(x)$, we can formulate the following rules by considering the reverse operations:

$$\boxed{\int \cos(ax)\,dx = \frac{\sin(ax)}{a} + C}$$

$$\boxed{\int \sin(ax)\,dx = \frac{-\cos(ax)}{a} + C}$$

$$\boxed{\int \sec^2(ax)\,dx = \frac{\tan(ax)}{a} + C}$$

Notice that when we integrate $\sin(x)$ we get $-\cos(x)$.
Note: The integral of $\tan(x)$ is more complicated and will be discussed in a later section.

Example: Evaluate the following integrals:

1. $\displaystyle\int \cos(4x)\,dx$
2. $\displaystyle\int 6\sin(3x)\,dx$
3. $\displaystyle\int \left(5\cos(-x) + \sec^2(3x)\right)dx$

Solution:

1. Using the rule $\displaystyle\int \cos(ax)\,dx = \frac{\sin(ax)}{a} + C$ a we find:

$$\int \cos(4x)\,dx = \frac{\sin(4x)}{4} + C$$

2. First we should note $\displaystyle\int 6\sin(3x)\,dx = 6\int \sin(3x)\,dx$

Now we can apply the rules above to get:

$$\int 6\sin(3x)\,dx = 6\int \sin(3x)\,dx = 6 \times \frac{-\cos(3x)}{3} + C = -2\cos(3x)$$

3. Splitting up the integral and extracting constants gives:

$$\int \left(5\cos(-x) + \sec^2(3x)\right)dx = 5\int \cos(-x)\,dx + \int \sec^2(3x)\,dx$$

Now by applying the above rules we get:

$$\begin{aligned}\int 5\cos(-x) + \sec^2(3x)\,dx = \quad & 5\int \cos(-x)\,dx + \int \sec^2(3x)\,dx \\ & = 5 \times \frac{\sin(-x)}{-1} + \frac{\tan(3x)}{3} + C \\ & = -5\sin(-x) - \frac{\tan(3x)}{3} + C\end{aligned}$$

Integrating Hyperbolic Functions

By recalling the derivatives of $\sinh(x)$, $\cosh(x)$ and $\tanh(x)$, we can formulate the following rules by considering the reverse operations:

$$\boxed{\int \cosh(ax)\,dx = \frac{\sinh(ax)}{a} + C}$$

$$\boxed{\int \sinh(ax)\,dx = \frac{\cosh(ax)}{a} + C}$$

$$\boxed{\int \operatorname{sech}^2(ax)\,dx = \frac{\tanh(ax)}{a} + C}$$

Note: The integral of $\tanh(x)$ is more complicated and will be discussed in a later section.

Example: Evaluate the following:

1. $\displaystyle\int 2\sinh(2x)\,dx$
2. $\displaystyle\int \big(\operatorname{sech}^2(-3x) - 9\cosh(2x)\big)\,dx$

Solution:

1. First we should note $\displaystyle\int 2\sinh(2x)\,dx = 2\int \sinh(2x)\,dx$

   Using the rule $\displaystyle\int \sinh(ax)\,dx = \frac{\cosh(ax)}{a} + C$, we find:

$$\begin{aligned} 2\int \sinh(2x)\,dx &= \frac{2\cosh(2x)}{2} + C \\ &= \cosh(2x) + C \end{aligned}$$

2. Splitting up the integral and extracting constants gives:

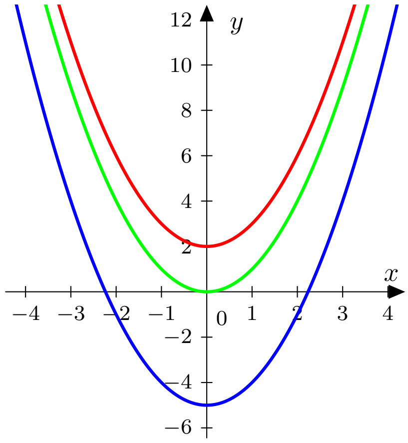
$$\int \big(\operatorname{sech}^2(-3x) - 9\cosh(2x)\big)\,dx = \int \operatorname{sech}^2(-3x)\,dx - 9\int \cosh(2x)\,dx$$

   Now by applying the above rules we get:

$$\begin{aligned} \int \big(\operatorname{sech}^2(-3x) - 9\cosh(2x)\big)\,dx &= \int \operatorname{sech}^2(-3x)\,dx - 9\int \cosh(2x)\,dx \\ &= \frac{\tanh(-3x)}{-3} - 9\frac{\sinh(2x)}{2} + C \\ &= -\frac{1}{3}\tanh(-3x) - \frac{9}{2}\sinh(2x) + C \end{aligned}$$

8.3 Finding the Constant of Integration

In the integrals so far we have always had a constant of integration denoted $+C$ at the end. Consider the functions $y = x^2$, $y = x^2 + 2$ and $y = x^2 - 5$, as shown on the graph below. These all differentiate to $2x$. When integrating $2x$ we don't know whether the actual answer is $y = x^2$, $y = x^2 + 2$ or $y = x^2 - 5$ so we give the general answer of $y = x^2 + C$.



If we are given the gradient of a curve and a point that the curve goes through, we can find the full equation of the curve and therefore a value of C. This is called using boundary conditions.

Example: A curve of gradient $4x^5$ passes through the point $(1, 2)$. What is the full equation of the line?

Solution: We know the gradient is $4x^5$ meaning $\frac{dy}{dx} = 4x^5$. Hence we have:

$$\begin{aligned} y &= \int 4x^5\,dx && \text{Now we integrate} \\ y &= \frac{4}{6}x^6 + C && \text{Simplify the fraction} \\ y &= \frac{2}{3}x^6 + C \end{aligned}$$

We now have an expression for y in terms of x however it has a constant C in it. Since we know a point $(1, 2)$ that the curve passes through we can find this constant. So when $x = 1$ we have $y = 2$. Substituting this into the equation gives:

$$\begin{aligned} 2 &= \frac{2}{3} \times 1^6 + C && \text{Rearrange to find } C \\ \frac{4}{3} &= C \end{aligned}$$

We now substitute the value of C into the equation from before to find our answer:

$$y = \frac{2}{3}x^6 + \frac{4}{3}$$

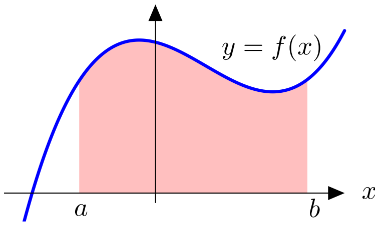
8.4 Integrals with Limits

Integrals represent the area under a graph. The probability of finding an electron between two points or the energy needed to separate two atoms can be represented as the area under a curve and we can use integration to find their value. **Definite integrals** are ones evaluated between two values (or limits):

$$\boxed{\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a)}$$

- The braces $\Big[..\Big]_a^b$ with a and b signify that we will evaluate the integral between these two limits.
- The numerical answer produced by $F(b) - F(a)$ represents the area under the curve.

Below is a function $f(x)$ integrated between the limits a and b with the shaded area being $\displaystyle\int_a^b f(x)\,dx$



Note: There is no need to find C when doing this since the constants will cancel out. When evaluating between the two points we will have $\big(F(b) + C\big) - \big(F(a) + C\big) = F(b) - F(a) + C - C = F(b) - F(a)$

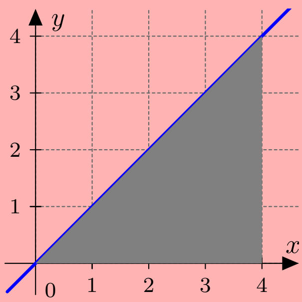
Example: Calculate $\displaystyle\int_0^4 x\,dx$

Solution: We first integrate with respect to x.

$$\int_0^4 x\,dx = \left[\frac{1}{2}x^2\right]_0^4 \qquad \text{Substitute the limits in}$$

$$\left[\frac{1}{2}x^2\right]_0^4 = \left[\frac{1}{2}\times 4^2\right] - \left[\frac{1}{2}\times 0\right] \qquad = 8$$

We can verify this answer by seeing that the area in question is a triangle:



We could have easily found the area of the triangle without integration but when we have more complicated functions then we must use integration.

Example: Find $\displaystyle\int_0^\pi \sin(x)\,dx$

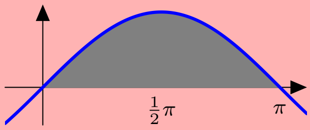
Solution: To begin we will integrate the function:

$$\int_0^\pi \sin(x)\,dx = \Big[-\cos(x)\Big]_0^\pi \qquad \text{Evaluate this at the limits}$$

$$\Big[-\cos(x)\Big]_0^\pi = \big(-\cos(\pi)\big) - \big(-\cos(0)\big) \qquad \text{Evaluate the cosines}$$

$$\big(-\cos(\pi)\big) - \big(-\cos(0)\big) = -(-1) + 1 = 2$$

The integral is 2, which means that the area under the curve from 0 to π is 2 as shown below.



Physics Example: The force on a charged particle, A, due to a second charged particle, B, is given by

$$F = \frac{qq_0}{4\pi\varepsilon_0 r^2}$$

Where Q is the charge of A, q is the charge of B and r is the distance of A from B. The subsequent motion of A means it travels from $r = 1$ to $r = 4$, what is the work done on A.

Solution: The work done on the particle is given by

$$W = \int_a^b F(r)\,dr$$

Therefore

$$\begin{aligned} W &= \int_1^4 \frac{Qq}{4\pi\varepsilon_0 r^2}\,dr \\ &= \left[\frac{-Qq}{4\pi\varepsilon_0 r}\right]_1^4 \\ &= \frac{-Qq}{16\pi\varepsilon_0} + \frac{Qq}{4\pi\varepsilon_0 r} \\ &= \frac{-3Qq}{16\pi\varepsilon_0 r} \end{aligned}$$

8.5 Integration Techniques

Integrating Odd and Even Functions over Symmetric Limits

In some cases where we have symmetric limits (i.e. both limits are the same distance from the origin but in opposite directions) we can use a trick to simplify integrals of odd and even functions.

Generally, we can use the rules:

- If the function $f(x)$ is **even** then

$$\int_{-a}^{+a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

- If the function $f(x)$ is **odd** then

$$\int_{-a}^{+a} f(x)\,dx = 0$$

If the integrand is neither odd nor even then we must solve using other methods and cannot simplify using these rules.

Example: Evaluate the following integrals using the simplification rules above:

1. $\displaystyle\int_{-\pi}^{\pi} \sin(x)\,dx$
2. $\displaystyle\int_{-3}^{3} x^2\,dx$

Solution:

1. Note that the integrand $\sin(x)$ is an odd function. Therefore we can easily solve this using the rule for integrating an odd function over symmetric limits.

$$\int_{-\pi}^{\pi} \sin(x)\,dx = 0$$

2. Note that the integrand x^2 is an even function, so we can use the rule for integrating an even function over symmetric limits which simplifies this to

$$\begin{aligned}\int_{-3}^{3} x^2\,dx &= 2\int_{0}^{3} x^2\,dx\\ &= 2\left[\frac{1}{3}x^3\right]_0^3\\ &= 18\end{aligned}$$

Using Identities to Solve Integrals

We can use our trigonometric and hyperbolic identities to convert some expressions into a form that we can easily integrate. For example, we may find it difficult to integrate $\cos^2(x)$, however we can use the double angle formula to rearrange this into an expression in terms of $\cos(2x)$, which is something we can integrate. This kind of technique is something that you will become comfortable with the more you practice!

Example: Evaluate the following integrals:

1. $\displaystyle\int \sin^2(x)\,dx$
2. $\displaystyle\int 2\cos(x)\sin(x)\,dx$
3. $\displaystyle\int 2\tanh^2(x)\,dx$

Solution:

1. We can rearrange the double angle formula $\cos(2x) = \cos^2(x) - \sin^2(x)$ as follows:
$$\cos(2x) = \cos^2(x) - \sin^2(x) \Rightarrow \cos(2x) = 1 - 2\sin^2(x)$$
$$\Rightarrow \sin^2(x) = \frac{1}{2}\big(1 - \cos(2x)\big)$$
and use this to simplify the integral.
$$\int \sin^2(x)\,dx = \int \frac{1}{2}\big(1 - \cos(2x)\big)\,dx$$
Splitting up the integral and extracting constants gives:
$$\int \frac{1}{2}\big(1 - \cos(2x)\big)\,dx = \frac{1}{2}\int 1\,dx - \frac{1}{2}\int \cos(2x)\,dx$$
$$= \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$$
2. We can use the double angle formula $\sin(2x) = 2\cos(x)\sin(x)$ to simplify the integral.
$$\int 2\cos(x)\sin(x)\,dx = \int \sin(2x)\,dx$$
$$= -\frac{1}{2}\cos(2x) + C$$
3. We can use the identity $1 - \tanh^2(x) = \text{sech}^2(x)$ to simplify the integral.
$$\int 2\tanh^2(x)\,dx = \int 2\big(1 - \text{sech}^2(x)\big)\,dx$$
$$= 2\int 1\,dx - 2\int \text{sech}^2(x)\,dx$$
$$= 2x - 2\tanh(x) + C$$

Integration by Substitution

If we have an integral that we cannot directly integrate, then we can transform it into something that we can more easily integrate using a substitution, or change of variable. This approach arises from the fact that we can differentiate a function of a function using the formula:

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}}$$

where $y = (u(x))$, or in other words, y is a function of u and u is a function of x. This formula was discussed in the section on the Chain Rule.

This method relies heavily on intuition and knowledge of differentiation and standard integrals, as we choose our substitution so that our integral becomes closer to something that we know how to integrate. We do this by choosing a substitution $x = x(u)$ and then writing our integral in terms of only u. This gives us an expression of the form:

$$\int y(x)\,dx = \int y(u)\,dx$$

However, notice that we are now integrating a function of u with respect to x! This would of course give us that the integral is equal to zero. But, since we have a relationship between x and u, we can find $\frac{dx}{du}$. Although we technically cannot do ordinary algebra with differentials such as dx, in this case it is okay to think of this in terms of normal multiplication and division. Of course we know that we can write $x = \frac{x}{u}u$, so we can also write $dx = \frac{dx}{du}du$. This enables us to write the formula:

$$\boxed{\int y(x)\,dx = \int y(u)\frac{dx}{du}\,du}$$

Now we are integrating with respect to u. In this way, we have changed the variable of our integral from x to u.

We can also make a substitution $u = u(x)$, however then we must use the rule for differentiating an inverse function to find $\frac{dx}{du}$.

Remember: Once we have our solution in terms of u, we must always remember to return back to our original variable x.

This method will become clearer with the help of some examples.

Example: Evaluate the integral:

$$\int \frac{1}{2x+3}\,dx$$

Solution: We make the substitution $u = 2x+3$. To calculate $\frac{dx}{du}$ we need to first calculate $\frac{du}{dx}$ and then use the rule for differentiating an inverse product. (alternatively we could find x in terms of u, i.e. $x(u)$, however in general it is simpler to use the method that we are using in this example)

$$\frac{du}{dx} = 2$$

$$\Rightarrow \frac{dx}{du} = \frac{1}{\left(\frac{du}{dx}\right)} = \frac{1}{2}$$

Now we can write:

$$\begin{aligned}\int \frac{1}{2x+3}\,dx &= \int \frac{1}{u}\frac{dx}{du}du \\ &= \int \frac{1}{u} \times \frac{1}{2}\,du \\ &= \frac{1}{2}\ln|u| + C\end{aligned}$$

However, since our original integral was in terms of x, we must now substitute our expression $u = 2x+3$ back into our solution, so that our solution is also in terms of x.

$$\int \frac{1}{2x+3}\,dx = \frac{1}{2}\ln|2x+3| + C$$

Example: Evaluate the integral:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx$$

Solution: For integrals such as this, intuition tells us that we should use the substitution $x = \sin(u)$. Now we need to calculate $\frac{dx}{du}$.

$$\frac{dx}{du} = \cos(u)$$

Substituting these into the formula and using the identity $\cos^2(\theta) + \sin^2(\theta) = 1$ gives

$$\begin{aligned}\int \frac{1}{\sqrt{1-x^2}}\,dx &= \int \frac{1}{\sqrt{1-\sin^2(u)}} \times \cos(u)\,du \\ &= \int \frac{1}{\sqrt{\cos^2(u)}} \times \cos(u)\,du \\ &= \int \frac{\cos(u)}{\cos(u)}\,du \\ &= \int 1\,du = u + C\end{aligned}$$

Now rewrite $x = \sin(u)$ as $u = \sin^{-1}(x)$ and substitute into our solution so that everything is in terms of x:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + C$$

Example: Evaluate the integral:

$$\int 2xe^{x^2+1}\,dx$$

Solution: By picking an intelligent substitution, we can make this integral very simple. Choose our substitution to be $u = x^2+1$. Now calculate $\frac{dx}{du}$ using the rule for differentiating inverse functions:

$$\frac{dx}{du} = \frac{1}{\left(\frac{du}{dx}\right)} = \frac{1}{2x}$$

Substitute these into the formula:

$$\int 2xe^{x^2+1}\,dx = \int 2xe^{u}\frac{1}{2x}\,du$$

$$= \int e^{u}\,du$$

$$= e^{u} + C$$

Notice that our choice of substitution allowed us to eliminate the $2x$.
Now substitute $u = x^2 + 1$ into our solution

$$\int 2xe^{x^2+1}\,dx = e^{x^2+1} + C$$

Example: Evaluate the integral:

$$\int \tan(x)\,dx$$

Solution: First, we write

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx$$

Now make the substitution $u = \cos(x)$, so that

$$\frac{dx}{du} = \frac{1}{\left(\frac{du}{dx}\right)} = -\frac{1}{\sin(x)}$$

Plug these expressions into the formula:

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{u} \times -\frac{1}{\sin(x)}\,dx$$

$$= \int -\frac{1)}{u}\,dx$$

$$= -\ln|u| + C$$

Now substitute $u = \cos(x)$ back in:

$$\int \tan(x)\,dx = -\ln|\cos(x)| + C$$

The integral of $\tan(x)$ is an example of a 'Logarithmic Integral'. This is when we have an integral of the form

$$\int \frac{f'(x)}{f(x)}\,dx$$

In other words, the numerator is the derivative of the denominator. In this case, we can just use the rule:

$$\boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C}$$

Example: Evaluate the integral:

$$\int \frac{1}{\sqrt{25+4x^2}}\,dx$$

Solution: First, we must rearrange the integrand into the form $\dfrac{1}{\sqrt{1+\alpha^2x^2}}$ by dividing top and bottom of the fraction by 5:

$$\frac{1}{5\sqrt{1+\frac{4}{25}x^2}}$$

For integrals such as this, intuition tells us that we should use the substitution $x = \frac{5}{2}\sinh(u)$ (where $\frac{5}{2} = \frac{1}{\alpha}$). Now we need to calculate $\dfrac{dx}{du}$.

$$\frac{dx}{du} = \tfrac{5}{2}\cosh(u)$$

Substituting these into the formula and using the identity $\cosh^2(\theta) - \sinh^2(\theta) = 1$ gives

$$\begin{aligned}
\int \frac{1}{5\sqrt{1+\frac{4}{25}x^2}}\,dx &= \int \frac{1}{5\sqrt{1+\sinh^2(u)}} \times \frac{5}{2}\cosh(u)\,du \\
&= \int \frac{1}{5\sqrt{\cosh^2(u)}} \times \tfrac{5}{2}\cosh(u)\,du \\
&= \int \frac{\cosh(u)}{2\cosh(u)}\,du \\
&= \int \frac{1}{2}\,du = \frac{1}{2}u + C
\end{aligned}$$

Now rewrite $x = \frac{5}{2}\sinh(u)$ as $u = \operatorname{arsinh}(\frac{2}{5}x)$ and substitute into our solution so that everything is in terms of x:

$$\int \frac{1}{\sqrt{25+4x^2}}\,dx = \tfrac{1}{2}\operatorname{arsinh}(\tfrac{2}{5}x) + C$$

Example: Evaluate the integral:

$$\int \frac{2}{\sqrt{4x^2 - 8x + 3}}\, dx$$

Solution: First, we must rearrange the integrand into a form for which we can make a substitution by completing the square on the polynomial.

$$\int \frac{2}{\sqrt{4x^2 - 8x + 3}}\, dx = \int \frac{2}{\sqrt{4(x-1)^2 - 1}}\, dx$$

Now, we can see that the substitution $x = \frac{1}{2}\cosh(u) - 1$ will be suitable so we will also need $\frac{dx}{du}$.

$$\frac{dx}{du} = \frac{1}{2}\sinh(u)$$

Now substitute these expressions into the formula and use the identity $\cosh^2(u) - \sinh^2(u) = 1$.

$$\begin{aligned}\int \frac{2}{\sqrt{4x^2 - 8x + 3}}\, dx &= \int \frac{2}{\sqrt{\cosh^2(u) - 1}} \times \frac{1}{2}\sinh(u)\, dx \\ &= \int \frac{\sinh(u)}{\sinh(u)}\, dx \\ &= \int 1\, dx \\ &= u + C\end{aligned}$$

Now rearrange $x = \frac{1}{2}\cosh(u) - 1$ to make u the subject and substitute this in for u:

$$\int \frac{2}{\sqrt{4x^2 - 8x + 3}}\, dx = \operatorname{arsinh}(2(x-1)) + C$$

If we are integrating between limits, i.e. finding a definite integral, then we have two approaches to dealing with these:

- **Method 1:** We can leave the limits in terms of the original variable x and then evaluate the limits **after** we have converted our solution back in terms of x.
- **Method 2:** Alternatively, we can use our substitution to change our limits so that they correspond to the new variable u. If we choose to do this, then we **do not** need to rewrite our solution in terms of the original variable; we can just evaluate the limits directly.

Whichever method we use, it is very important that we indicate which variable the limits are in terms of.

In the next example, we will evaluate an indefinite integral using both methods.

Example: Evaluate the definite integral:

$$\int_{x=1}^{x=\sqrt{3}} \frac{1}{1+x^2}\,dx$$

Solution:

- **Method 1:** We can use the substitution $x = \tan(u)$. Then $\dfrac{dx}{du} = \sec^2(u)$.
  Substituting these into the formula and using the identity $1 + \tan^2(\theta) = \sec^2(\theta)$ gives:

$$\begin{aligned}\int_{x=1}^{x=\sqrt{3}} \frac{1}{1+x^2}\,dx &= \int_{x=1}^{x=\sqrt{3}} \frac{1}{1+\tan^2(u)} \times \sec^2(u)\,du \\ &= \int_{x=1}^{x=\sqrt{3}} \frac{\sec^2(u)}{\sec^2(u)}\,du \\ &= \int_{x=1}^{x=\sqrt{3}} 1\,du = [u]_{x=1}^{x=\sqrt{3}}\end{aligned}$$

  Then rearranging $x = \tan(u)$ to $u = \tan^{-1}(x)$ and substituting gives:

$$= \Big[\tan^{-1}(x)\Big]_{x=1}^{x=\sqrt{3}}$$

  Then evaluate the limits as usual

$$\begin{aligned}\int_{x=1}^{x=\sqrt{3}} \frac{1}{1+x^2}\,dx &= \Big[\tan^{-1}(x)\Big]_{x=1}^{x=\sqrt{3}} \\ &= \tan^{-1}(\sqrt{3}) - \tan^{-1}(1) \\ &= \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}\end{aligned}$$

- **Method 2:** Again, use the substitution $x = \tan(u)$. Then $\dfrac{dx}{du} = \sec^2(u)$. Now substitute the x-limits into $x = \tan(u)$ and find the u-limits:
  When $x = \sqrt{3}$, $u = \frac{\pi}{3}$. When $x = 1$, $u = \frac{\pi}{4}$.
  Substituting all of these into the formula and using the identity $1 + \tan^2(\theta) = \sec^2(\theta)$ gives:

$$\begin{aligned}\int_{u=\frac{\pi}{4}}^{u=\frac{\pi}{3}} \frac{1}{1+x^2}\,dx &= \int_{u=\frac{\pi}{4}}^{u=\frac{\pi}{3}} \frac{1}{1+\tan^2(u)} \times \sec^2(u)\,du \\ &= \int_{u=\frac{\pi}{4}}^{u=\frac{\pi}{3}} \frac{\sec^2(u)}{\sec^2(u)}\,du \\ &= \int_{u=\frac{\pi}{4}}^{u=\frac{\pi}{3}} 1\,du = [u]_{u=\frac{\pi}{4}}^{u=\frac{\pi}{3}}\end{aligned}$$

  Then evaluate the limits as usual

$$\begin{aligned}\int_{x=1}^{x=\sqrt{3}} \frac{1}{1+x^2}\,dx &= [u]_{u=\frac{\pi}{4}}^{u=\frac{\pi}{3}} \\ &= \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}\end{aligned}$$

  As you can see, both methods give the same answer, however there is generally less room for error in the second method.

Integration by Parts

In the section on differentiation we discussed the product rule. Since integration is the inverse of differentiation, we can use the product rule in a different form in order to integrate certain types of functions. You will remember the product rule as:

$$\boxed{\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}}$$

where $u = u(x)$ and $v = v(x)$ are functions of x.
We can rearrange this rule and integrate the whole equation with respect to x to obtain:

$$\boxed{\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx}$$

The trick to using this formula is to write the function we want to integrate as $u\dfrac{dv}{dx}$, choosing u and $\dfrac{dv}{dx}$ so that we can use our standard rules for integration and differentiation to find $v(x)$ and $\dfrac{du}{dx}$. Again, this is a technique that you will get better at with practice.

Example: Evaluate the following integrals:

1. $\int x\cos(x)\,dx$
2. $\int xe^x\,dx$

Solution:

1. Begin by choosing $u(x) = x$ and $\frac{dv}{dx} = \cos(x)$.

   Now we need to find $\frac{du}{dx}$ by differentiating $u(x) = x$ and $v(x)$ by integrating $\frac{dv}{dx} = \cos(x)$.

$$\frac{du}{dx} = 1$$

$$v(x) = \sin(x)$$

   Then we substitute these into the formula

$$\int x\cos(x)\,dx = x\sin(x) - \int 1 \times \sin(x)\,dx$$

   And integrate the last term

$$= x\sin(x) - (-\cos(x)) + C$$

$$= x\sin(x) + \cos(x) + C$$

2. Begin by choosing $u(x) = x$ and $\frac{dv}{dx} = e^x$.

   Now we need to find $\frac{du}{dx}$ by differentiating $u(x) = x$ and $v(x)$ by integrating $\frac{dv}{dx} = e^x$.

$$\frac{du}{dx} = 1$$

$$v(x) = e^x$$

   Then we substitute these into the formula

$$\int xe^x\,dx = xe^x - \int 1 \times e^x\,dx$$

   And integrate the last term

$$= xe^x - e^x + C$$

We can also use integration by parts to integrate $\ln(x)$.

Example: Evaluate the integral:

$$\int \ln(x)\, dx$$

Solution: We cannot directly integrate $\ln(x)$, however we can differentiate it, so choose $u(x) = \ln(x)$ and $\frac{dv}{dx} = 1$.

Now we need to find $\frac{du}{dx}$ by differentiating $u(x) = \ln(x)$ and $v(x)$ by integrating $\frac{dv}{dx} = 1$.

$$\frac{du}{dx} = \frac{1}{x}$$

$$v(x) = x$$

Then we substitute these into the formula

$$\int \ln(x)\, dx = x\ln(x) - \int \frac{x}{x}\, dx$$

$$= x\ln(x) - \int 1\, dx$$

And integrate the last term

$$= x\ln(x) - x + C$$

In some cases, we may be required to apply integration by parts multiple times, as shown in the next example.

Example: Evaluate the integral:

$$\int x^2 e^x\, dx$$

Solution: Begin by choosing $u(x) = x^2$ and $\frac{dv}{dx} = e^x$.

Now we need to find $\frac{du}{dx}$ by differentiating $u(x) = x^2$ and $v(x)$ by integrating $\frac{dv}{dx} = e^x$.

$$\frac{du}{dx} = 2x$$

$$v(x) = e^x$$

Then we substitute these into the formula

$$\int x^2 e^x\, dx = x^2 e^x - \int 2xe^x\, dx$$

$$= x^2 e^x - 2\int xe^x\, dx$$

Now we must apply integration by parts again to the resulting integral. Note that we have already calculated $\int xe^x\, dx$ in a previous example, so we will just give the final solution here.

$$= \int x^2 e^x\, dx = x^2 e^x - 2\big(xe^x - e^x\big) + C$$

$$= x^2 e^x - 2xe^x + 2e^x + C$$

Integration using Partial Fractions

Sometimes we will be presented with an integral of the form

$$\int \frac{S+Tx}{P+Qx+Rx^2}\,dx$$

where P, Q, R, S and T are constants.
We cannot integrate this directly, so we need to find a way of 'breaking it down' into smaller parts which we can integrate.
Suppose we want to add the two expressions:

$$\frac{6}{x+3}+\frac{2}{x-4}$$

$$=\frac{6(x-4)}{(x+3)(x-4)}+\frac{2(x+3)}{(x-4)(x+3)}$$

$$=\frac{6(x-4)+2(x+3)}{(x+3)(x-4)}$$

$$=\frac{8x-18}{x^2-x-12}$$

As you can see, this has resulted in an expression in the form above. We also note that we can integrate

$$\int\left(\frac{6}{x+3}+\frac{2}{x-4}\right)dx=\ln|x+3|+\ln|x-4|+C$$

so we would like to have a method for obtaining A and B in the following equation:

$$\frac{S+Tx}{P+Qx+Rx^2}=\frac{A}{x-a}+\frac{B}{x-b}$$

which would allow us to manipulate the first function into a form which we can integrate.

Example: Resolve the expression

$$\frac{1}{x^2+3x+2}$$

into partial fractions and hence find its integral.

Solution: First, factorise the denominator:

$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)}$$

Now write in the form:

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

We now need to solve this for A and B. Multiply both sides by $(x+1)(x+2)$ to give

$$1 = A(x+2) + B(x+1)$$

There are two methods for solving this:

Method 1: Collect the terms in each power of x and compare coefficients:

$$1 = x(A+B) + (2A+B)$$

$$\Rightarrow A+B=0 \text{ and } 2A+B=1$$

$$\Rightarrow A=1 \text{ and } B=-1$$

Method 2: Notice that when $x=-2$, we have an expression only in terms of A. So we can easily find B:

$$1 = 0 + B(-2+1) \Rightarrow B=-1$$

Similarly for $x=-1$:

$$1 = A(-1+2) + 0 \Rightarrow A = 1$$

Regardless of which method we use, the result is that we can now write:

$$\frac{1}{x^2+3x+2} = \frac{1}{x+1} - \frac{1}{x+2}$$

and therefore

$$\int \frac{1}{x^2+3x+2}\,dx = \ln|x+1| - \ln|x+2| + C$$

$$= \ln\left|\frac{x+1}{x+2}\right| + C$$

Note: It is important to remember which numerator (A or B) corresponds to which denominator, as it can be easy to get confused between them.

We can actually apply this approach to ANY fraction of two polynomials, provided the denominator is of higher order than the numerator (i.e. includes a higher power of x).
Three important points to note are the following:

- We can perform the same procedure if the numerator is of the form $Px+Q$.
- If we have an x^2 inside one of the denominator brackets, for example $\dfrac{1}{(x^2+4)(x-5)}$ then we must write our partial fractions equation as:

$$\frac{1}{(x^2+4)(x-5)} = \frac{A}{(x-5)} + \frac{Bx+C}{(x^2+4)}$$

  and solve for A, B and C. (Note: If using Method 2, we can find A as before, and then set $x=0$ to find C, and consequently find B)

- If we have a repeated root of the denominator, such as $\frac{1}{(x-1)^2(x+5)}$, then we must write our partial fractions equation as:

$$\frac{1}{(x-1)^2(x+5)} = \frac{A}{(x+5)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$$

and solve for A, B and C as before.

Example: Resolve the expression

$$\frac{x+2}{(x-1)^2(x+5)}$$

into partial fractions and hence find its integral.

Solution: First, write in the form:

$$\frac{x+2}{(x-1)^2(x+5)} = \frac{A}{(x+5)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$$

We now need to solve this for A, B and C. Multiply both sides by $(x-1)^2(x+5)$ to give

$$x+2 = A(x-1)^2 + B(x-1)(x+5) + C(x+5)$$

We will use Method 2 in this example. Notice that when $x = 1$, we have

$$1+2 = 0+0+6C \Rightarrow C = \frac{1}{2}$$

When $x = -5$:

$$-5+2 = A(-1-5)^2 + 0 + 0 \Rightarrow A = -\frac{1}{12}$$

Now since we cannot find B by eliminating the other variables, we will just have to choose a value of x and solve for B (since we now know A and C). For convenience let's pick $x = 0$.
When $x = 0$:

$$0+2 = A - 5B + 5C \Rightarrow 2 = -\frac{1}{12} - 5B + \frac{5}{2}$$

$$\Rightarrow B = \frac{7}{60}$$

So finally we can write:

$$\frac{x+2}{(x-1)^2(x+5)} = \frac{-1}{12(x+5)} + \frac{7}{60(x-1)} + \frac{1}{2(x-1)^2}$$

Then integrating gives:

$$\int \frac{x+2}{(x-1)^2(x+5)}\,dx = -\frac{1}{12}\ln|x+5| + \frac{7}{60}\ln|x-1| + \frac{1}{2(1-x)} + C$$

where we have found the third term using standard integration of polynomials.

Reduction Formulae

Reduction formulae provide a means to hugely simplify some integrals which would otherwise be very demanding to solve. In particular, this approach applies to the integration of functions which are raised to a power $n \geq 0$.
If we have an integral

$$\int f(x)^n \, dx$$

then we use the notation:

$$I_n = \int f(x)^n \, dx$$

e.g.

$$I_2 = \int f(x)^2 \, dx$$

$$I_5 = \int f(x)^5 \, dx$$

Reduction formulae allow us to find a formula which can be used to calculate I_n for any $n \geq 0$, however we are required to obtain an initial integral. For example if we know I_0, a reduction formula could enable us to find I_1, I_2 and so on. This type of relation, where we find a value in terms of previous values in the sequence, is called a **recurrence relation**.
Consider the integral

$$I_n = \int x^n e^x \, dx$$

If we can find a formula which allows us to find I_n given I_{n-1}, then by finding an initial value, say I_0, we will be able to find I_n for all $n \geq 0$.
In order to find a reduction formula we are required to use integration by parts one or more times to reduce the integral. Recall the formula for this technique:

$$\boxed{\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}}$$

We will now find a reduction formula for the above integral.
We use integration by parts, choosing $u = x^n$ and $\dfrac{dv}{dx} = e^x$.
Now we can find $\dfrac{du}{dx} = nx^{n-1}$ and $v = e^x$.
Use the integration by parts formula

$$I_n = \int x^n e^x \, dx = x^n e^x - \int e^x (nx^{n-1}) \, dx$$

$$= x^n e^x - n \int e^x x^{n-1} \, dx$$

Now notice that the integral in the second term is the same as the initial integral, but replacing n with $n-1$. This means that we can write the following:

$$= x^n e^x - nI_{n-1} \, dx$$

So we have deduced the reduction formula:

$$I_n = x^n e^x - nI_{n-1}$$

Now, if we wished to find $I_2 = \displaystyle\int x^2 e^x \, dx$ then we can use this formula, rather than integrating by parts twice.
We begin by calculating

$$I_0 = \int e^x \, dx = e^x$$

Then it follows from the reduction formula that

$$I_1 = xe^x - I_0 = (x-1)e^x$$

And so finally

$$I_2 = x^2e^x - 2I_1 = x^2e^x - (x-1)e^x = (x^2 - x + 1)e^x$$

Example: Find a reduction formula for

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n(x)\,dx$$

Solution: We begin by rewriting the integral as

$$I_n = \int_0^{\frac{\pi}{2}} \sin(x)\sin^{n-1}(x)\,dx$$

Now using integration by parts with $u = \sin^{n-1}(x)$ and $\dfrac{dv}{dx} = \sin(x)$ gives

$$I_n = -\Big[\cos(x)\sin^{n-1}(x)\Big]_0^{\frac{\pi}{2}} - (n-1)\int_0^{\frac{\pi}{2}} -\sin^{n-2}\cos^2(x)\,dx$$

$$= 0 + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\cos^2(x)\,dx$$

Using the identity $\cos^2(x) + \sin^2(x) = 1$, we get

$$I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}(x)(1 - \sin^2(x))\,dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}(x)\,dx + (n-1)\int_0^{\frac{\pi}{2}} \sin^2(x)\,dx$$

Notice that the integral in the third term is I_n, and so moving this onto the left hand side of the equation, we have

$$nI_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}(x)\,dx$$

$$\Rightarrow I_n = \frac{n-1}{n} I_{n-2}$$

Note that this reduction formula only applies when we have the limits 0 and $\frac{\pi}{2}$. The general reduction formula for $\displaystyle\int \sin^n(x)\,dx$ is actually:

$$I_n = -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n} I_{n-2}$$

Example: Find the reduction formula for the integral:

$$\int_0^{\frac{\pi}{2}} \operatorname{sech}^n(x)\,dx$$

Solution: Notice

$$\int_0^{\frac{\pi}{2}} \operatorname{sech}^n(x)\,dx = \int_0^{\frac{\pi}{2}} \operatorname{sech}^2(x)\operatorname{sech}^{n-2}(x)\,dx$$

In this form we can use integration by parts ($\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$) to simplify the integrand. Choosing $\operatorname{sech}^2(x)$ to be $\frac{dv}{dx}$, and $\operatorname{sech}^{n-2}$ to be u.
We know that the integral of $\operatorname{sech}^2(x)$ is $\tanh(x)$, and the derivative of $\operatorname{sech}^{n-2}$ is $(n-2)(\operatorname{sech}^{n-3}(x))(-\operatorname{sech}(x)\tanh(x))$.
Therefore

$$\int_0^{\frac{\pi}{2}} \operatorname{sech}^2(x)\operatorname{sech}^{n-2}(x)\,dx$$

$$= \Big[\operatorname{sech}^{n-2}(x)\tanh(x)\Big]_0^{\frac{\pi}{2}} - (n-2)\int_0^{\frac{\pi}{2}} \tanh(x)(\operatorname{sech}^{n-3}(x))(-\operatorname{sech}(x)\tanh(x))\,dx$$

$$= \frac{3}{25}\times 4^{n-2} - \int_0^{\frac{\pi}{2}} \tanh^2(x)\operatorname{sech}^{n-2}(x)\,dx$$

Using that $\tanh^2(x) = \operatorname{sech}^2(x) - 1$ we get

$$\int_0^{\frac{\pi}{2}} \tanh^2(x)\operatorname{sech}^{n-2}(x)\,dx = \int_0^{\frac{\pi}{2}} \operatorname{sech}^n(x) - \operatorname{sech}^{n-2}(x)\,dx$$

$$= \int_0^{\frac{\pi}{2}} \operatorname{sech}^n(x)\,dx - \int_0^{\frac{\pi}{2}} \operatorname{sech}^{n-2}(x)\,dx$$

Therefore

$$\int_0^{\frac{\pi}{2}} \operatorname{sech}^n(x)\,dx = \frac{3}{25}\times 4^{n-2} - (n-2)\left(\int_0^{\frac{\pi}{2}} \operatorname{sech}^n(x)\,dx - \int_0^{\frac{\pi}{2}} \operatorname{sech}^{n-2}(x)\,dx\right)$$

By taking the term $-(n-2)\int_0^{\frac{\pi}{2}} \operatorname{sech}^n(x)\,dx$ over to the left hand side, we get that

$$(n-1)\int_0^{\frac{\pi}{2}} \operatorname{sech}^n(x)\,dx = \frac{3}{25}\times 4^{n-2} + (n-2)\int_0^{\frac{\pi}{2}} \operatorname{sech}^{n-2}(x)\,dx$$

Finally if

$$I_n = \int_0^{\frac{\pi}{2}} \operatorname{sech}^n(x)\,dx$$

we have shown that

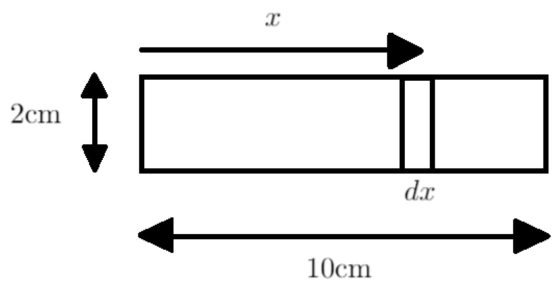
$$(n-1)I_n = \frac{3}{25}\times 4^{n-2} + (n-2)I_{n-2}$$

8.6 Constructing Equations Using Integrals

Using the idea of infinitesimals we can set up integrals to solve a variety of problems. Considering a system as infinitely many small segments and using integration to add up all these segments is a powerful tool in physics.

Consider a strip of paper of width 2cm and length 10cm, it is trivial to say that the area of the paper is $2 \times 10 = 20\text{cm}^2$ but we will use this example to understand how an integral can ba used to solve problems.

We consider a small segment of the paper, a distance x from the left hand side of the strip, with width $2cm$ and length dx. We can say that the area of this segment is $2 \times dx = 2dx$. Now using the



idea that the area of the whole strip of paper could be made up of infinitely many of these infinitesimal strips we can add up all the infinitesimal areas using integration to get an answer for the total area of the strip of paper, i.e.

$$\text{Area} = \int_0^{10} 2\, dx$$

The limits are obtained from the fact that the distance of the first segment is $x = 0\text{m}$ and the distance of the last segment is at $x = 10\text{m}$. Solving this integral gives

$$\text{Area} = \Big[2x\Big]_0^{10} = \Big[2 \times 10\Big] - \Big[2 \times 0\Big] = 20\text{cm}^2$$

Example: Use the fact that the circumference of a circle is $2\pi r$, where r is the radius of the circle, and the technique of integration to obtain an equation for the area of the circle.

Click here for a video example

Solution: Consider the circle as made up of an infinite amount of rings centred at the centre of the circle with a infinitesimal width dx.

A general ring a distance x from the centre of the circle has circumference is $2\pi x$.
We approximate the area of this ring as

$$2\pi x \times dx$$

To find the area of the circle we must add up all the areas of the rings that make up the circle i.e.

$$\int_0^r 2\pi x\, dx$$

The limits are chosen at $x = 0$ as this is the smallest possible ring and at $x = r$ as this is the largest possible ring.

$$\begin{aligned}\int_0^r 2\pi x\, dx &= \Big[\pi x^2\Big]_0^r \\ &= \Big[\pi r^2\Big] - \Big[0\Big] \\ &= \pi r^2\end{aligned}$$

This idea can be stretched to volumes of shapes, for example we can consider a sphere. We already know that for a sphere:

$$\text{Volume} = \frac{4}{3}\pi r^3$$

where r is the radius of the sphere. However, using integration and the fact that we know the surface area of the sphere is $4\pi r^2$ we can derive this equation for the volume.

Consider an elementary shell centred at a distance x from the centre of the sphere with width dx. We can approximate the volume of this shell as:

$$4\pi r^2 \times dx$$

Using integration, we add up all the elementary shells from the centre of the sphere to the edge, i.e.

$$\text{Volume} = \int_0^r 4\pi x^2 \, dx$$

$$= \left[\frac{4}{3}\pi x^3\right]_0^r = \left[\frac{4}{3}\pi \times r^3\right] - \left[\frac{4}{3}\pi \times 0^2\right] = \frac{4}{3}\pi r^3$$

Example: Use the fact that the area of a circle is πr^2, where r is the radius of the circle, and the technique of integration to obtain an equation for the volume of a cone of base radius R and height h.

Click here for a video example

Solution: Consider an elementary disk with width dx and a radius r at a distance x from the point of a cone.
As the width of the disk is infinitesimal we can approximate the volume of the disk to be $\pi r^2 \times dx$. Therefore to add up all the volumes of the infinitesimal disks we need to compute the integral:

$$\text{Volume} = \int_0^h \pi r^2 \, dx$$

However r is not a constant so we cannot solve this integral yet. To allow us to continue we need to find r as a function of x. Note that

$$\tan(\theta) = \frac{r}{x} = \frac{R}{h}$$

Therefore

$$r = \frac{Rx}{h}$$

Substituting this back into our integral gives

$$\text{Volume} = \int_0^h \pi \frac{R^2 x^2}{h^2} \, dx$$

$$= \left[\pi \frac{R^2}{3h^2} \times x^3\right]_0^h = \left[\pi \frac{R^2}{3h^2} \times h^3\right] - \left[\pi \frac{R^2}{3h^2} \times 0^3\right] = \frac{1}{3}\pi R^2 h$$

8.7 Geometric Applications of Integration

Arc Length

We can use integration to help us find the exact length of a curve between the points a and b. In order to do this we aim to sum up all the infinitesimal lengths ds, i.e. we want to calculate

$$s = \int_a^b ds$$

Since we will usually be working in 2-D space, this is not simple to calculate, so we can use Pythagoras to write

$$ds = \sqrt{dx^2 + dy^2}$$

and hence

$$s = \int_a^b ds = \int_a^b \sqrt{dx^2 + dy^2}$$

Then pulling a dx out of the square root, we obtain the formula:

$$\boxed{s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$

Example: Find the arc length between the points $x = 0$ and $x = \ln(3)$ on the curve $y = \cosh(x)$

Solution: We know that the formulae for the arc length on a curve is

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

So first we need to find out what $\frac{dy}{dx}$ is. In this case, where $y = \cosh(x)$, $\frac{dy}{dx} = \sinh(x)$. Therefore the arc length that we are asked to work out is equal to

$$\int_0^{\ln(3)} \sqrt{1 + \sinh^2(x)}\, dx$$

Using the identity $\cosh^2(x) = 1 + \sinh^2(x)$ this is equivalent to

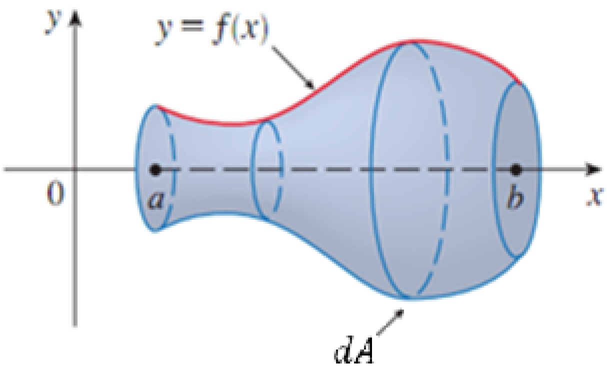
$$\int_0^{\ln(3)} \cosh(x)\, dx$$

$$= \Big[\sinh(x)\Big]_0^{\ln(3)} = \frac{1}{2}(e^{\ln(3)} - e^{-\ln(3)} - e^0 + e^0)$$

$$= \frac{1}{2}\left(3 - \frac{1}{3}\right) = \frac{8}{6} = \frac{4}{3} \text{ units}$$

Surface Area of a Revolution

When we rotate a curve once around the x-axis, we form a solid 3-D shape. We can find the surface area of this shape between $x = a$ and $x = b$ using integration.



If we consider an infinitesimal piece of the curve ds at a point x at height y, then the contribution to the total surface area of the revolution from this piece is

$$dA = ds \times \text{circumference} = ds \times 2\pi \times \text{radius}$$

Therefore we can write

$$A = \int_a^b ds \times 2\pi \times \text{radius}$$

Using the relation $ds = \sqrt{dx^2 + dy^2}$ from the previous section,

$$A = 2\pi \int_a^b y\sqrt{dx^2 + dy^2}$$

and then pulling a dx out of the square root, as before, we get the formula:

$$\boxed{A = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$

Example: Find the surface area of revolution between the points $x = 0$ and $x = 1$ on the curve $y = \frac{1}{3}x^3$

Solution: We know that the formula for the surface area of revolution of a curve is

$$A = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

So first we need to find out what $\frac{dy}{dx}$ is. In this case, where $y = \frac{1}{3}x^3$, $\frac{dy}{dx} = x^2$.
Therefore the surface area of revolution is equal to

$$2\pi \int_0^1 \frac{1}{3}x^3\sqrt{1 + x^4}\, dx$$

Using the substitution $u = 1 + x^4$ we get that $\frac{du}{dx} = 4x^3$, therefore $\frac{dx}{du} = \frac{1}{4x^3}$. This allows us to write

$$2\pi \int_0^1 \frac{1}{3}x^3\sqrt{1 + x^4}\, dx = 2\pi \int_1^2 \frac{1}{12}\sqrt{u}\, du$$

$$= 2\pi\left[\frac{1}{18}u^{\frac{3}{2}}\right]_1^2 = \frac{1}{9}(2\sqrt{2} - 2) \text{ square units}$$

Volume of a Revolution

We can also find the volume of a revolution between $x = a$ and $x = b$ using integration. By looking at the diagram in the previous section and considering an infinitesimal slice of the volume, we can deduce that

$$\begin{aligned} dV &= \text{area} \times \text{width of slice} \\ &= \pi y^2 \times dx \end{aligned}$$

Thus we have the following formula for the volume of a revolution:

$$\boxed{V = \pi \int_a^b y^2 dx}$$

Example: Find the volume of revolution between the points $x = 0$ and $x = \frac{\pi}{4}$ on the curve $y = \sec(x)$

Solution: We know that the formulae for the volume of revolution of a curve is

$$V = \pi \int_a^b y^2 \, dx$$

. Therefore the volume of revolution that we are asked to work out is equal to

$$\pi \int_0^{\frac{\pi}{4}} \sec^2(x) \, dx$$

This is a standard integral so we can quote the answer as

$$\pi \Big[\tan(x) \Big]_0^{\frac{\pi}{4}} = \pi(1 - 0) = \pi \text{ cubic units}$$

8.8 Review Questions

Easy Questions

Question 1: Evaluate the definite integral

$$\int_1^3 \frac{5x-4}{x^3}\,dx$$

Answer:

$$\frac{14}{9}$$

Question 2: Evaluate the integral

$$\int \frac{2}{4x^2-9}\,dx$$

Answer:

$$\frac{1}{6}\ln\left|\frac{3-2x}{3+2x}\right| + C$$

Question 3: Evaluate the definite integral

$$\int_0^{\frac{3\pi}{4}} 4\cos(-4x+\pi)\,dx$$

Answer:

$$0$$

Question 4: Evaluate the integral

$$\int 4e^{3x}(3x-7)^2\,dx$$

Answer:

$$\frac{4}{3}e^{3x}(9x^2-48x+65)+C$$

Question 5: Evaluate the integral

$$\int 4e^{3x}(3x-7)^2\,dx$$

Answer:

$$\frac{4}{3}e^{3x}(9x^2-48x+65)+C$$

Medium Questions

Question 6: Use integration to obtain an expression for the volume of a cylinder in terms of it's radius and height.

Answer:

$$V=\pi r^2 h$$

Question 7: Evaluate the integral

$$\int \sin^3(3x)\cos^5(3x)\,dx$$

Hint: Use the substitution $u = \sin(3x)$

Answer:

$$\frac{1}{12}\sin^4(3x) - \frac{2}{18}\sin^6(3x) + \frac{1}{24}\sin^8(3x) + C$$

Question 8: Evaluate the integral

$$\int \sqrt{a^2 - x^2}\,dx$$

Hint: Use the substitution $u = a\sin(\theta)$

Answer:

$$\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \left(\frac{x}{2}\right)\sqrt{a^2 - x^2} + C$$

Question 9: Evaluate the integral

$$\int e^x \cos(x)\,dx$$

Answer:

$$\frac{1}{2}e^x\big(\sin(x) + \cos(x)\big) + C$$

Question 10: Evaluate the integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx$$

Hint: Consider the graph of the integrand

Answer:

$$0$$

Hard Questions

Question 11: Find the reduction formulae for

$$I_n = \int \tan^n(x)\,dx$$

Answer:

$$I_n = -I_{n-2} + \frac{1}{n-1}\tan^{n-1}(x)$$

Question 12: Evaluate the integral

$$\int \frac{x+1}{\sqrt{1+x^2}}\,dx$$

Hint: Split the integrand up into two separate terms.

Answer:

$$\sqrt{1+x^2} + \sinh^{-1}(x) + C$$

Question 13: Evaluate the integral

$$\int a^x\,dx \qquad (a>0)$$

Answer:

$$\frac{a^x}{\ln(a)} + C$$

Question 14: Evaluate the integral

$$\int \frac{\arctan(x)}{x^2+1}\,dx$$

Hint: Try integration by parts.

Answer:

$$\frac{1}{2}\arctan^2(x) + C$$

Question 15: Evaluate the integral

$$\int \frac{1}{\sqrt{x-1}+\sqrt{(x-1)^3}}\,dx$$

Hint: Rearrange the integrand before trying to Evaluate the integral.

Answer:

$$2\arctan(\sqrt{x-1}) + C$$

9 Multiple Integration

We have already discussed the definite integral of a function of one variable, $f(x)$, between $x = a$ and $x = b$, and how it can be used to calculate the area enclosed by the function and the x-axis within the interval. We explained how integration essentially separates the area into many small slices and calculates the sum of these slices as their size becomes infinitely small. We can now extended this idea to functions of more than one variable, such as $f(x, y)$ and $f(x, y, z)$.

9.1 Double Integrals (Plane Surface Integrals)

Whereas with single integrals we are calculating an area, double (or plane surface) integrals allow us to calculate the volume enclosed by a surface by integrating a function of two variables. In order to do this we consider a region S of the x, y-plane and the function $z = f(x, y)$ which describes a surface, where z gives the height of the surface at the point (x, y). We can then define the plane surface integral of $f(x, y)$ as the volume enclosed by the surface above the region S. We can write this as

$$\boxed{V = \iint_S f(x, y)\, dS}$$

where dS is an infinitesimal section of the area S. We can imagine this by thinking of separating the volume into infinitely many pillars, each with volume $z\, dS = f(x, y)\, dS$, and then adding up all of these volumes to get the total volume. This works in exactly the same way as with single integration, however now we are calculating a volume and will have two sets of limits: limits on x and limits on y.

Note that if $f(x, y) = 1$ then the height of the surface is constant and hence

$$\iint_S 1\, dS$$

simply calculates the area of the surface S, and so is effectively a single integral.

Double Integrals in Cartesian Coordinates

When working with Cartesian coordinates, we can write the infinitesimal area dS as

$$dS = dx\, dy$$

meaning it is a square area with length dx in the x-direction and length dy in the y-direction. Now if we wish to find the plane surface integral of $f(x, y)$ over the area

$$S = \big\{(x, y) : a \leq x \leq b,\ c(x) \leq y \leq d(x)\big\}$$

then we calculate the integral

$$V = \iint_S f(x, y)\, dS = \int_a^b \int_{c(x)}^{d(x)} f(x, y)\, dy dx$$

Physicists are often lazy and so we may also write this as

$$\int_a^b dx \int_{c(x)}^{d(x)} dy\, f(x, y)$$

Notice that in all cases where the region of integration is not rectangular, y will also depend on x, and so our limits on y must be written in terms of x.

In order to calculate the above integral, we simply integrate with respect to one variable, treating the other as constant, and then integrate with respect to the other variable. So if we are integrating with respect to y and then with respect to x, we can think of this as calculating the area under the curve $z = f(c, y)$ as in the previous chapter, where c is a constant, and then adding up all the areas for every single value of the constant c by integrating with respect to x, giving us the total volume of the shape. Similarly, we could perform the exact same procedure in reverse, integrating with respect to x and then with respect to y, however it is often important to choose the order of integration intelligently. For example, in the above equation, since our y limits depend on x whilst our x limits are simple numbers, and must integrate with respect to y first. We can then simply calculate

$$\int_a^b \left(\int_{c(x)}^{d(x)} f(x, y)\, dy \right) dx$$

using our techniques of integration from before. Notice that this is why we have written $dydx$ in this order above: because we are integrating with respect to y and then with respect to x.

Example: Calculate

$$\iint_S (3 - x - y)\, dS$$

where S is the region bounded by the x and y axes and the lines $x = 1$ and $y = 2$

Solution: By inserting the limits, we can write this integral as

$$\iint_S (3 - x - y)\, dS = \int_0^1 \int_0^2 (3 - x - y)\, dydx$$

now we integrate with respect to y:

$$= \int_0^1 \left[3y - xy - \frac{y^2}{2} \right]_0^2 dx$$

$$= \int_0^1 (4 - 2x)\, dx$$

Now we integrate with respect to x, since y is no longer present in our equation.

$$= \left[4x - x^2 \right]_0^1$$

$$= 4 - 1$$

$$= 3$$

Notice that since none of our limits depended on the other variable we could have integrated with respect to x first and then y to give the same answer, i.e.

$$\int_0^2 \int_0^1 (3 - x - y)\, dxdy$$

Example: Calculate

$$\iint_S (x^2y + 2x)\, dS$$

where $S = \{(x, y) : 1 \leq x \leq 2y,\ 0 \leq y \leq 3\}$

Solution: By inserting the limits, we can write this integral as

$$\iint_S (x^2y + 2x)\, dS = \int_0^3 \int_1^{2y} (x^2y + 2x)\, dxdy$$

Since our x limits depend on y, we must integrate with respect to x first, treating y as a constant.

$$= \int_0^3 \left[\frac{1}{3}x^3y + x^2\right]_1^{2y} dy$$

$$= \int_0^3 \left(\frac{1}{3}y^4 + y^2 - \frac{1}{3}y - 1\right) dy$$

Now we integrate with respect to y, since x is no longer present in our equation.

$$= \left[\frac{1}{15}y^5 + \frac{1}{3}y^3 - \frac{1}{6}y^2 - y\right]_0^3$$

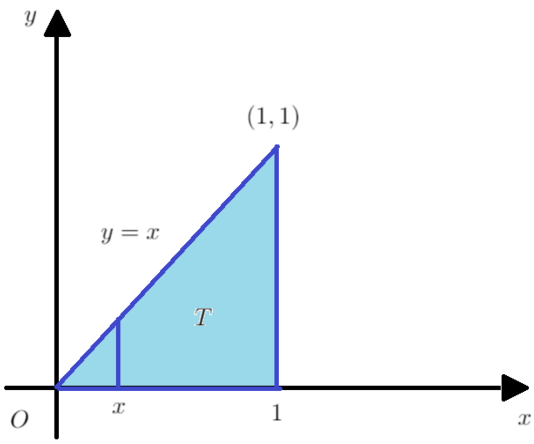
$$= \frac{81}{5} + 9 - \frac{3}{2} - 3$$

$$= \frac{207}{10}$$

We can also deduce what the limits of integration are if we are given a region over which we want to integrate. For example, we wish to find the integral

$$\iint_S f(x, y)\, dS$$

where S is the region bounded by the triangle with vertices (0,0), (1,0), (1,1). We can sketch this triangle on the x, y-plane very easily. From this sketch we can see that x goes from $x = 0$ to $x = 1$. Now, we



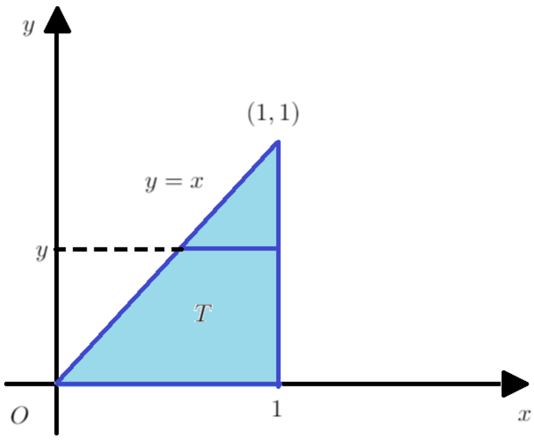
cannot write that y goes from $y = 0$ to $y = 1$ because y depends on x. Think about how the limits on y change as x changes, for example when $x = 0.5$, y can only go from $y = 0$ to $y = 0.5$. By looking at the diagram we can deduce that our region is defined by:

$$S = \{(x, y) : 0 \leq x \leq 1,\ 0 \leq y \leq x\}$$

Thus our integral can be written as

$$\iint_S f(x,y)\,dS = \int_0^1 \int_0^x f(x,y)\,dydx$$

Equally, we could have calculated different limits by looking at how the x limits change as y changes. which would give



$$S = \{(x,y) : y \le x \le 1,\, 0 \le y \le 1\}$$

$$\Rightarrow \iint_S f(x,y)\,dS = \int_0^1 \int_y^1 f(x,y)\,dxdy$$

So we can write

$$\int_0^1 \int_0^x f(x,y)\,dydx = \int_0^1 \int_y^1 f(x,y)\,dxdy$$

We can also use the idea that

$$\iint_S 1\,dS$$

is the area of the surface S to find the area enclosed by two functions. For example, say we want to find the area enclosed between the two functions $y = x^2$ and $y = 2x - x^2$, then we can find the limits of the surface S and simply compute the above integral. Sketching a graph can help to find limits, however we can also do this algebraically. First we want to find the points where these functions intersect, so we solve them simultaneously to find that they intersect at $(0,0)$ and $(1,1)$. Now we know that x has the limits $x = 0$ and $x = 1$ however, as before, we must remember that the limits for y will depend on x since the region is not rectangular. we notice that $x^2 > 2x - x^2$ for all x within $[0,1]$, and so we can write that the limits on y are $y = x^2$ to $y = 2x - y^2$. We now have our limits so can write our integral in full and solve:

$$\int_0^1 \int_{x^2}^{2x-x^2} 1\,dydx$$

$$= \int_0^1 [y]_{x^2}^{2x-x^2}\,dx$$

$$= \int_0^1 (2x - x^2) - x^2\,dx$$

$$= \int_0^1 2x - 2x^2\,dx$$

$$= \left[x^2 - \frac{2}{3}x^3\right]_0^1$$

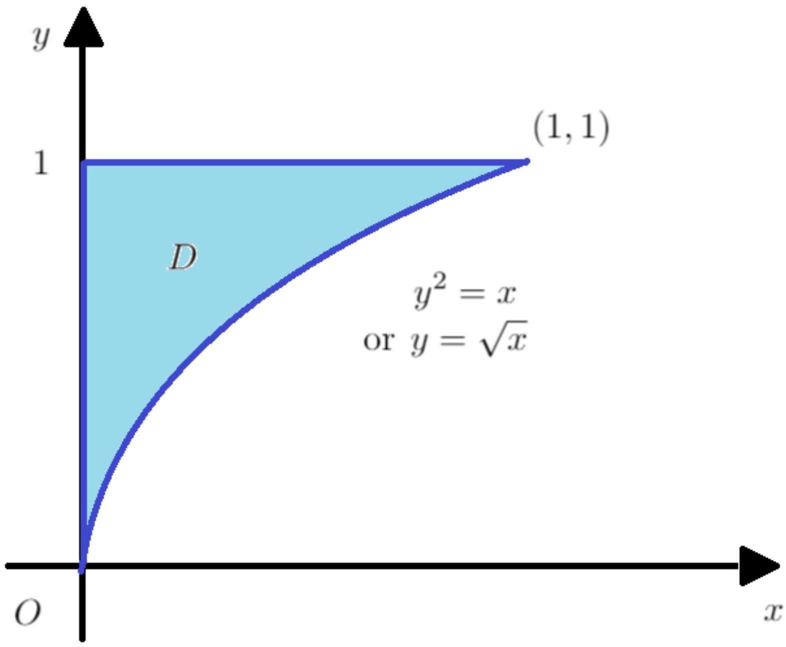
$$= \frac{1}{3}$$

Changing the Order of Integration

Whilst the order of integration does not actually change the result, sometimes it is essential to choose the most appropriate order when integrating a function which cannot be integrated with respect to some variables. Being able to change the order of integration can also help simplify calculations. The above approach of using a diagram can allow us to do this. For example, consider the integral

$$\int_0^1 dx \int_{\sqrt{x}}^1 e^{y^3}\, dy$$

We cannot integrate e^{y^3}, however by changing the order of integration, we can get it into a form which we can integrate. Consider the sketch of the region on the x, y-plane represented by the limits Now we



can work out how the x limits change as y changes. We can see that y goes from $y = 0$ to $y = 1$. Note that the curve $y = \sqrt{x}$ is the same as $x = y^2$. Now look at a general point $y = y$, then x goes from $x = 0$ to $x = y^2$. So we can rewrite our integral as

$$\int_0^1 dy \int_0^{y^2} e^{y^3}\, dx$$

Since our function does not contain x, we can write the integral as follows, however note that even though this notation looks as if there are two separate integrals, we are still integrating the **entire** function with respect to both x and y.

$$= \int_0^1 e^{y^3}\, dy \int_0^{y^2} 1\, dx$$

$$= \int_0^1 e^{y^3} \left[x\right]_0^{y^2} dy$$

$$= \int_0^1 y^2 e^{y^3}\, dy$$

We now have an integral that we can compute, by using the substitution $u = y^3$, giving

$$\int_0^1 dx \int_{\sqrt{x}}^1 e^{y^3}\, dy = \int_0^1 e^{y^3}\, dy \int_0^{y^2} 1\, dx = \left[\frac{e^{y^3}}{3}\right]_0^1$$

$$= \frac{e-1}{3}$$

Double Integrals in Polar Coordinates

As you might expect, we can also compute double integrals using polar coordinates by specifying the region S in terms of an angle θ and a radius r. It is beneficial to understand that if we hold θ constant and change r, we simple get a straight half-line stretching out from the origin at an angle θ to the x-axis. Alternatively, if we hold r constant and change θ, we get a circle of radius r with its centre at the origin. The lines formed by holding one variable constant and changing the other are called **coordinate lines**. In addition, since these two coordinate lines intersect at right angles, we call polar coordinates an **orthogonal coordinate system**. Note that Cartesian coordinates are orthogonal also.

Now we must deduce how to find an area dS of a polar coordinate system. Clearly, if we start at a point on the surface and increase r by dr, (keeping θ constant) then we have generated an element of length dr, just as we do with Cartesian coordinates. If we keep r constant and increase θ by $d\theta$, then we actually create a small arc of length $rd\theta$ (note that this is **not** just $d\theta$) and so the element of area which we form is $dS = rdrd\theta$. This means that when we are integrating with respect to polar coordinates, we have the double integral

$$\boxed{\iint_S f(r,\theta)\, rdrd\theta}$$

In other words, we must always multiply our function by r before we integrate when using polar coordinates.

Remember that we can convert between Cartesian and Polar coordinates using the relations:

Cartesian → Polar

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

Polar → Cartesian

$$r = \sqrt{x^2+y^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Example: Evaluate the integral

$$\iint_S (x^2 + y^2)\, dS$$

where $S = \{(r, \theta) : 1 < r < 2, \frac{\pi}{3} < \theta < \frac{\pi}{2}\}$.

Solution: Since our region of integration is given in terms of polar coordinates, we must change our integral from Cartesian to polar coordinates, by using the formulae above. Using the relation $r^2 = x^2 + y^2$, remembering that in polar coordinates $dS = r\, dr\, d\theta$ and inserting the limits, we can write the integral as

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_1^2 r^2 \times r\, dr\, d\theta$$

Remember that we can integrate in any ordersince all th limits are constant, however in this example we will integrate with respect to r first.

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(\int_1^2 r^3\, dr \right) d\theta = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left[\frac{r^4}{4} \right]_1^2 d\theta$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{15}{4}\, d\theta$$

Now we integrate with respect to θ

$$= \left[\frac{15}{4}\theta \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= \frac{15}{4}\left(\frac{\pi}{2} - \frac{\pi}{3} \right)$$

$$= \frac{5\pi}{8}$$

9.2 Triple Integrals (Volume Integrals)

We can extend this further to functions of three variables. An integral of a function of three variables is called a triple (or volume) integral and can represent the mass of a body. In order to compute these we consider a solid enclosed by the region V of the x, y, z-space and the function $f(x, y, z)$ which describes the density of the volume. We can then define the volume integral of $f(x, y, z)$ as the total mass of the body in the region V. We can write this as

$$V = \iiint_V f(x, y, z)\, dV$$

where dV is an infinitesimal section of the volume V. A volume integral is far harder to visualise than a single or double integral, however the calculation is almost identical, albeit with an extra variable of integration. Essentially, we are adding up the mass of infinitesimal sections of the volume to get the total mass. Of course, in the case of triple integrals we will have three sets of limits: limits on x, limits on y and limits on z.

Note that if $f(x, y, z) = 1$ then the density throughout the volume is constant and hence

$$\iiint_V 1\, dV$$

simply calculates the volume of the solid, and so is essentially a double integral.

Triple Integrals in Cartesian Coordinates

We treat a triple integral with respect to Cartesian coordinates in exactly the same way as we do with double integrals in Cartesian coordinates, except that we have an extra variable, z. This means that our infinitesimal volume is $dV = dxdydz$, and if $V = \big\{(x, y, z) : a \leq x \leq b,\, c(x) \leq y \leq d(x), g(x, y) \leq x \leq h(x, y)\big\}$, then our integral takes the form

$$\int_{x=a}^{x=b} \int_{y=c(x)}^{y=d(x)} \int_{z=g(x,y)}^{z=h(x,y)} f(x, y, z)\, dzdydx$$

Note that our limits on z depend on both x and y, so in this case we must integrate with respect to z, then with respect to y and finally with respect to x. With triple integrals in particular, it is often helpful to label our limits so that we know which limits correspond to each variable.

If our region of integration is a cuboid, i.e. is bounded by six rectangular planes with edges parallel to the axes, then our limits for all three variables will just be numbers and so we can integrate in any order we choose.

Example: Calculate

$$\iiint_V xyz\, dV$$

where $V = \left\{(x,y,z) : 1 \le x \le 2,\ 2 \le y \le 3,\ 0 \le z \le 2\right\}$

Solution: By inserting the limits, we can write this integral as

$$\iiint_V xyz\, dV = \int_{z=0}^{z=2} \int_{y=2}^{y=3} \int_{x=1}^{x=2} xyz\, dxdydz$$

We can now integrate this in a similar way to a double integral with rectangular limits. First we will integrate with respect to x.

$$= \int_{z=0}^{z=2} \int_{y=2}^{y=3} \left[\frac{1}{2}x^2yz\right]_1^2 dydz$$

$$= \int_{z=0}^{z=2} \int_{y=2}^{y=3} \frac{3}{2}yz\, dydz$$

Now we can integrate with respect to y, since x no longer appears in our integral.

$$= \int_{z=0}^{z=2} \left[\frac{3}{4}y^2z\right]_2^3 dz$$

$$= \int_{z=0}^{z=2} \frac{15}{4}z\, dz$$

And finally since our function only includes z, we can integrate with respect to z

$$= \left[\frac{15}{8}z^2\right]_0^2$$

$$= \frac{60}{8} = \frac{15}{2}$$

Note that we could have integrated in any order to obtain the same answer.

As with double integrals, we can usually find the limits of a region of integration from a given shape. For example, we want to evaluate the integral

$$\iiint_V 3x\, dV$$

where V is the region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + 2y + x = 3$. We first need to find the limits on x, y and z. Clearly the lower limit of all the variables is zero, so now we just need to find the upper limits. By sketching or imagining the diagram, we can see that x takes its maximum value when $z = y = 0$, thus our upper x limit is $x = 3$. Now, since the upper limits of y and z depend on x, they vary as x varies. Again by consideration of the diagram, the upper y limit occurs when $z = 0$, so by substituting this into $x + 2y + z = 3$, and rearranging to make y the subject gives the upper y limit as $y = \frac{1}{2}(3-x)$. Finally, to find the upper z limit, we simply rearrange the equation of the slanted plane to make z the subject, i.e. $z = 3 - x - 2y$. We now have all our limits and can write the integral as

$$\int_{x=0}^{x=3} \int_{y=0}^{y=\frac{1}{2}(3-x)} \int_{z=0}^{z=3-x-2y} 3x\, dzdydx$$

Now we integrate with respect to z (since our z limits contain x and y)

$$= \int_{x=0}^{x=3} \int_{y=0}^{y=\frac{1}{2}(3-x)} 3x(3 - x - 2y)\, dydx$$

$$= \int_{x=0}^{x=3} \int_{y=0}^{y=\frac{1}{2}(3-x)} 9x - 3x^2 - 6xy\, dydx$$

Then we integrate with respect to y now that z no longer appears in the integral

$$= \int_{x=0}^{x=3} \frac{9}{2}x(3-x) - \frac{3}{2}x^2(3-x) - \frac{3}{4}xy(3-x)^2 \, dx$$

$$= \int_{x=0}^{x=3} \frac{3}{4}x^3 - \frac{9}{2}x^2 + \frac{27}{4}x \, dx$$

And finally we integrate with respect to x

$$= \left[\frac{3}{16}x^4 - \frac{3}{2}x^3 + \frac{27}{8}x^2\right]_0^3$$

$$= \frac{81}{16}$$

Example: Using a triple integral with cartesian coordinates, find the volume of a sphere with radius 2, centred at the origin.

Solution: Remember that we can use the integral

$$\iiint_V 1 \, dV$$

to find the volume of the solid represented by V. To do this, we need to find the limits of the three variables x, y and z. Similarly to the example above, we note that since our solid is a sphere at the origin with radius 2, the limits on x are $x = -2$ and $x = 2$. The equation of this sphere is

$$x^2 + y^2 + z^2 = 4$$

and by considering the diagram of this, we can deduce that the limits on y occur when $z = 0$, giving $y = \pm\sqrt{4-x^2}$. Finally our limits on z are found by rearranging the equation of the sphere: $z = \pm\sqrt{4-x^2-y^2}$. Thus our integral can be rewritten as:

$$\int_{x=-2}^{x=2} \int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} \int_{z=-\sqrt{4-x^2-y^2}}^{z=\sqrt{4-x^2-y^2}} dzdydx$$

$$= \int_{x=-2}^{x=2} \int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} 2\sqrt{4-x^2-y^2} \, dydx$$

We can then integrate with respect to y using the substitution $y = \sqrt{4-x^2}\sin(u)$ to give

$$= \int_{x=-2}^{x=2} \pi(4-x^2) \, dx$$

where we have left out the calculation as a similar example has already been covered in a previous section. We can then finally integrate with respect to x since x is the only remaining variable.

$$\left[4\pi x - \frac{\pi x^3}{3}\right]_{-2}^{2}$$

$$= \pi(16 - \frac{16}{3})$$

$$= \frac{32\pi}{3}$$

as we would expect from the standard formula $V = \dfrac{4\pi a^3}{3}$

Triple Integrals in Cylindrical Coordinates

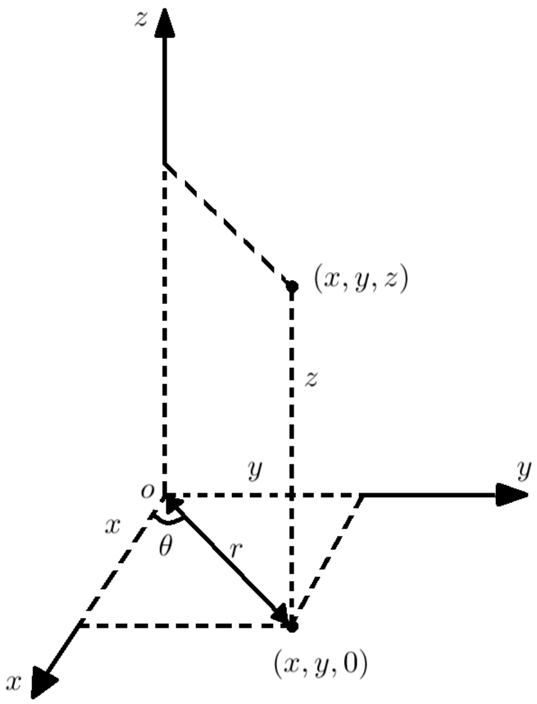
When we wish to change variables when using triple integrals, we have two options: **cylindrical coordinates** or **spherical coordinates**. Cylindrical coordinates are a straightforward extension of two dimensional polar coordinates into three dimensions, where we simply add a z axis extending from the origin of the polar circle and perpendicular to r and θ. The relations between Cartesian and cylindrical coordinates are thus:

Cartesian → Cylindrical

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$
$$z = z$$

Cylindrical → Cartesian

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$z = z$$



Note: When using $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ remember to check which quadrant θ belongs to.

By comparison with polar coordinates, we must remember that the element of volume $dV = r\,drd\theta dz$, so that a triple integral using cylindrical coordinates is

$$\boxed{\iiint_V f(r,\theta,z)\, r\, drd\theta dz}$$

Example: The region bounded by the surface of the paraboloid $z = 2 - x^2 - y^2$ and the surface of a cone $z = x^2 + y^2$ is given by

$$D + \{(x, y, z) : x^2 + y^2 \leq 1, \sqrt{x^2 + y^2} \leq z \leq 2 - x^2 - y^2\}$$

Use a triple integral to find the volume of this region.

Solution: We can see that a change of coordinate system to cylindrical coordinates would be wise in this example. To help with the limits, we will rewrite the region D in terms of cylindrical coordinates. Since $r^2 = x^2 + y^2$, we can deduce that $r \leq 1$. Also remember that r is always non-negative as it is a radius. Now θ rotates around a full circle and z is restricted by the surfaces of the paraboloid and cone.

$$D = \{(r, \theta, z) : 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi, r \leq z \leq 2 - r^2\}$$

We can now form an integral to find the volume of D.

$$\int_0^{2\pi} \int_0^1 \int_r^{2-r^2} 1\, r\, dz\, dr\, d\theta$$

Integrating with respect to z first gives

$$\int_0^{2\pi} \int_0^1 r(2 - r^2 - r)\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 (r - r^3)\, dr\, d\theta$$

Now we integrate with respect to θ.

$$= \int_0^1 2\pi(r - r^3)\, dr$$

Finally we integrate with respect to r.

$$= 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1$$

$$= \frac{\pi}{2}$$

Triple Integrals in Spherical Coordinates

An alternative coordinate system is spherical coordinates which is less easy to visualise than previous coordinate systems, however it is not all that much more difficult. We imagine a radius of a sphere ρ and two angles: θ which is the same angle as θ in cylindrical coordinates and rotates around the z axis, and then an azimuthal angle ϕ which represents the angle of our radius from the positive z-axis to the negative z-axis. Note that ϕ can only take values between 0 and π since it only rotates through a half-circle. Our relations between the various coordinate systems are:

Cartesian → Spherical

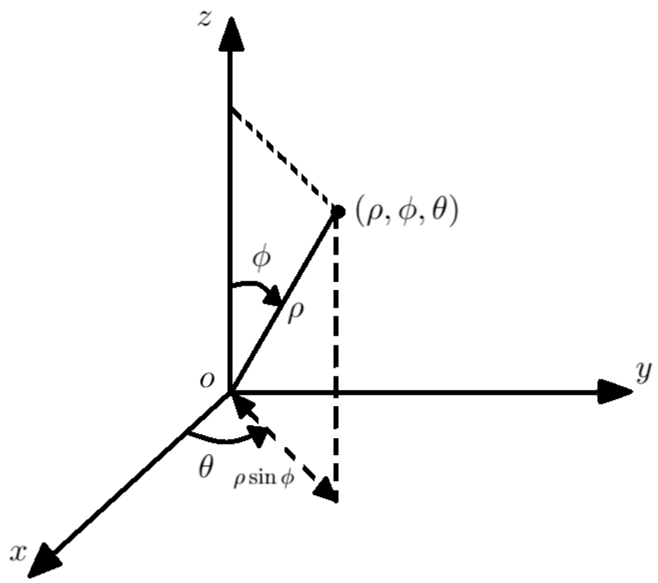
$$x = \rho\sin(\phi)\cos(\theta)$$
$$y = \rho\sin(\phi)\sin(\theta)$$
$$z = \rho\cos(\phi)$$

Spherical → Cartesian

$$\rho = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$\phi = \cos^{-1}\left(\frac{z}{\rho}\right)$$

Cylindrical → Spherical

$$r = \rho\sin(\phi)$$
$$\theta = \theta$$
$$z = \rho\cos(\phi)$$



Note: When using $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ remember to check which quadrant θ belongs to.

Example:

1. Find the cylindrical coordinate and spherical coordinate of a point P with Cartesian coordinate (1,1,1).
2. We can express the surface of an infinite cylinder C of radius R in cylindrical coordinates using the limits $C = \{(r, \theta, z) : r = R,\ 0 \leq \theta \leq 2\pi\ -\infty \leq z \leq \infty\}$. Find this surface in terms of Cartesian and spherical coordinates.

Solution:

1. **Cylindrical:** We can use the relations between Cartesian and cylindrical coordinates to find

$$r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$z = z = 1$$

$$\theta = \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

Since the point $(1, 1)$ lies in the first quadrant. So the point P in cylindrical coordinates is $(\sqrt{2}, \frac{\pi}{4}, 1)$

Spherical: We can use the relations between Cartesian and spherical coordinates to find

$$\rho = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$\phi = \arccos\left(\frac{1}{3}\right) = \frac{\pi}{4}$$

$$\theta = \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

Since the point $(1, 1)$ lies in the first quadrant. So the point P in spherical coordinates is $(\sqrt{3}, \frac{\pi}{4}, \frac{\pi}{4})$

We could also have solved this problem by considering a diagram of the point, and writing on all our variables.

2. **Cartesian:** We can use the relations between Cartesian and cylindrical coordinates to find

$$r = R = \sqrt{x^2 + y^2}$$

$$z = z$$

So the surface of C in Cartesian coordinates can be described by $C = \{(x, y, z) : x^2 + y^2 = R^2, -\infty \leq z \leq \infty\}$

Spherical: We can use the relations between Cartesian and spherical coordinates to find

$$R = \rho \sin(\phi) \Rightarrow \rho = \frac{\sin(\phi)}{R}$$

Now, as before θ rotates through a full circle, and because we have an infinite cylinder, ϕ rotates through a full half-circle, giving the surface of C in spherical coordinates as $C = \{(\rho, \phi, \theta) : \rho = \frac{\sin(\phi)}{R},\ 0 \leq \phi \leq \pi,\ 0 \leq \theta \leq 2\pi\}$

Once again, we must now reconsider what our element of volume is by increasing each variable by an infinitesimal amount and working out the total volume produced. First, by increasing ρ by $d\rho$ whilst keeping θ and ϕ constant, we generate a length $d\rho$. Now increase θ by $d\theta$ and keeping ρ and ϕ constant we obtain a length $\rho \sin(\phi) d\theta$. Finally we increase ϕ by $d\phi$ whilst keeping the other variables constant, which gives an arc of length $\rho d\phi$. Putting all of this together we obtain the element of volume

$dV = \rho^2 \sin(\phi)\, d\rho d\theta d\phi$, so that a triple integral using spherical coordinates is

$$\boxed{\iiint_V f(\rho, \theta, \phi)\, \rho^2 \sin(\phi)\, d\rho d\phi d\theta}$$

We will now repeat a previous example question on calculating the volume of a sphere to emphasise the usefulness of spherical coordinates in certain situations.

Example: Using a triple integral with spherical coordinates, find the volume of a sphere with radius 2, centred at the origin.

Solution: Remember that we can use the integral

$$\iiint_V 1\, dV$$

to find the volume of the solid represented by V. To do this, we need to find the limits of the three variables ρ, θ and ϕ. Starting with θ, we can see that the limits are $\theta = 0$ to $\theta = 2\pi$ as it describes a full circle. Now the radius ρ is orthogonal to θ and so its limits are just numbers, in this case $r = 0$ to $r = 2$. Finally, we consider ϕ and notice that once again, ϕ is orthogonal to the other variables and so the limits are $\phi = 0$ to $\phi = \pi$. We can now write our integral as:

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{2} 1\rho^2 \sin(\phi) d\rho d\phi d\theta$$

Since our limits are all independent of each other, we integrate with respect to each variable in any order. We will start with θ.

$$= \int_0^{\pi} \int_0^{2} 2\pi\rho^2 \sin(\phi) d\rho d\phi$$

Then integrate with respect to ϕ

$$= \int_0^{2} 2\pi\rho^2 \Big[-\cos(\phi) \Big]_0^{\pi} d\rho$$

$$= \int_0^{2} 4\pi\rho^2 d\rho$$

Finally integrate with respect to ρ

$$= 4\pi \left[\frac{1}{3}\rho^3 \right]_0^2$$

$$= \frac{8 \times 4\pi}{3}$$

$$= \frac{32\pi}{3}$$

as we would expect from the standard formula $V = \dfrac{4\pi a^3}{3}$ and have obtained from a previous example. This example demonstrates that a change of coordinate system can massively simplify calculations.

Example: Using spherical coordinates, find the mass of a solid sphere of radius a which is centred on the origin, given that it has a mass density given by

$$f(x, y, z) = b\left(\frac{z}{a}\right)^4$$

Solution: We can write the mass as an integral

$$M = b\iiint_V \left(\frac{z}{a}\right)^4 dV$$

where in spherical coordinates,

$$V = \{(\rho, \phi, \theta) : 0 \le r \le a,\ 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi\}$$

and

$$z = \rho\cos(\phi)$$

Thus our integral can be written as

$$M = \frac{b}{a^4}\int_0^a \int_0^\pi \int_0^{2\pi} \rho^4\cos^4(\phi) \times \rho^2\sin(\phi)\, d\theta\, d\phi\, d\rho$$

Since each of our sets of limits do not depend on one of the integration variables, the order of integration does not matter so we can separate this integral into three separate integrals as follows, and then evaluate each part individually.

$$= \frac{b}{a^4}\left(\int_0^a \rho^6\, d\rho\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^\pi \sin(\phi)\cos^4(\phi)\, d\phi\right)$$

$$= \frac{b}{a^4}\left[\frac{\rho^7}{7}\right]_0^a \left[\theta\right]_0^{2\pi} \int_0^\pi \sin(\phi)\cos^4(\phi)\, d\phi$$

We can solve the third integral using the substitution $u = \cos(\phi)$ to obtain

$$= 2\pi\frac{ba^3}{7}\left[\frac{\cos^5(\phi)}{5}\right]_0^\pi$$

$$= \frac{4\pi\rho a^3}{35}$$

9.3 Review Questions

Easy Questions

Question 1: Evaluate

$$I = \iint_S xy\, dS$$

where

$$S = \{(x, y) : 3 \le x \le 4,\ 0 \le y \le 2\}$$

Answer:

$$I = 7$$

Question 2: Evaluate

$$I = \iint_A (x^2 + y)\, dA$$

where A is the triangle with vertices at $(4, -1)$, $(7, -1)$ and $(7, 5)$.

Hint: First find the equations of the lines of the triangle.

Answer:

$$I = \frac{675}{2}$$

Question 3: Using a double integral, find the area enclosed between the two curves

$$y = x^2 \text{ and } y = 2x - x^2$$

Hint: Remember to find where the curves intersect first.

Answer:

$$Area = \frac{1}{3}$$

Note: This question can also be done using single integration.

Question 4: By considering the region of integration, change the limits on the integral

$$I = \int_0^{\ln(3)} \int_{e^x}^{3} f(x, y)\, dydx$$

so that we could integrate with respect to x first.

Answer:

$$I = \int_1^{3} \int_0^{\ln(y)} f(x, y)\, dxdy$$

Question 5: By converting to polar coordinates, evaluate

$$I = \iint_R \frac{y^2}{x^2}\, dA$$

where R is part of the ring $0 < a^2 \leq x^2 + y^2 \leq b^2$ lying in the first quadrant and below the line $y = x$.

Hint: Remember $dA = r\, dr d\theta$ in polar coordinates.

Answer:

$$I = \frac{4-\pi}{8}(b^2 - a^2)$$

Medium Questions

Question 6: Evaluate the integral

$$I = \int_0^1 \int_2^3 \int_0^2 x^2 z + y\, dx dy dz$$

Answer:

$$I = \frac{19}{3}$$

Question 7: Let T be the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + 2y + z = 2$. Evaluate the integral

$$I = \iiint_T x\, dV$$

Answer:

$$I = \frac{1}{3}$$

Question 8: The point P is given in spherical coordinates as $\left(3, \frac{\pi}{4}, \frac{\pi}{3}\right)$. Find the Cartesian and cylindrical coordinates of the point P.

Answer:

$$\text{Cartesian: } \left(\frac{3\sqrt{2}}{4}, \frac{3\sqrt{6}}{4}, \frac{3\sqrt{2}}{2}\right)$$

$$\text{Cylindrical: } \left(\frac{3\sqrt{2}}{2}, \frac{\pi}{3}, \frac{3\sqrt{2}}{2}\right)$$

Question 9: Using a triple integral with the appropriate coordinate system, find the volume of a body bounded by the surfaces $x^2 + y^2 = 4$, $z = 0$ and $z = 2$.

Answer:

$$Volume = 8\pi$$

Question 10: By changing the order of integration, evaluate

$$I = \int_0^1 \int_{x^{1/4}}^1 e^{y^5}\, dxdy$$

Answer:

$$I = \frac{e-1}{5}$$

Hard Questions

Question 11: Evaluate

$$I = \iiint_D (y^2 + x^2)\, dV$$

where D is the region bounded by the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ and the planes $x = y$, $x = 0$, $z = 0$ and $z = 1$.

Hint: Use cylindrical coordinates.

Answer:

$$I = \frac{15\pi}{16}$$

Question 12: A solid cone has boundary surfaces $x^2 + y^2 = z^2$ and $z = 1$. The density of the cone at point (x, y, z) is z. Find the mass of the cone using a triple integral in both spherical coordinates and cylindrical coordinates.

Answer:

$$Volume = \frac{2\pi}{3}$$

Question 13: By changing to cylindrical coordinates, evaluate the integral

$$I = \iiint_R z e^{-(x^2+y^2+z^2)} dV$$

where

$$R = \{(x, y, z) : -\infty < x, y < \infty,\ 0 \le z \le 1\}$$

Answer:

$$I = \frac{\pi}{2}(1 - e^{-1})$$

Question 14: Evaluate the integral

$$I = \int_0^1 \int_0^y \frac{xy}{x^2 + y^2}\, dxdy$$

Answer:

$$I = \frac{1}{4}\ln 2$$

Question 15: Using a triple integral in spherical coordinates, find the volume of a cone of height h and radius a.

Answer:

$$Volume = \frac{1}{3}\pi a^2 h$$

10 Differential Equations

10.1 Introduction to Differential Equations

Definitions and Classifications

A differential equation is an equation involving an unknown function and one or more of its derivatives. Solving a differential equations involves looking for a function (or functions) that satisfies the equation. Some simple examples of differential equations are:

- $\dfrac{dy}{dx} + x = y$
- $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + x = 0$
- $(x+y)\dfrac{dy}{dx} = 2$

The general form of an Ordinary Differential Equation is:

$$f(x, y, y', y'', ..., y^{(n)}) = 0$$

where x is the independent variable, y is the dependent variable and $y' = \dfrac{dy}{dx}, y'' = \dfrac{d^2y}{dx^2}$, etc... are derivatives of y. Our aim when solving an ordinary differential equation is to find the function $y(x)$ which satisfies this equation.

We will often refer to Ordinary Differential Equations as 'ODEs'. (There are other types of differential equation, such as Partial Differential Equations, or 'PDEs', which will not be discussed in this booklet.)

In order to characterize differential equations, we will use the following definitions:

- The **order** of a differential equation is the highest derivative, $y^{(n)}$, that appears in the equation.
- A differential equation is **linear** if the power on the dependent variable and all its derivatives is 1 and there are no terms which contain a product of these. In other words, we can write the differential equation as a linear combination of the dependent variable and its derivatives. The equation is called **non-linear** if the power on either the dependent variable or any of its derivative is higher than 1 or the equation contains terms which include a product of these.
- The **degree** of a differential equation is the power on the highest order derivative.
- A differential equation is **homogeneous** if the dependent variable, y, or any of its derivatives are present in all the terms. Consequently, $y = 0$ is always a solution of a homogeneous equation.
- A differential equation is **autonomous** if the independent variable, x, is not explicitly present in the equation.

Examples of classifications of ODEs:

1. $(x+y)^2\dfrac{dy}{dx} = 3$ is a non-linear inhomogeneous ODE of first order and first degree.
2. $\dfrac{d^2y}{dx^2} + y = x^3$ is a linear inhomogeneous ODE of second order and first degree.
3. $2\dfrac{d^2s}{dt^2} + \dfrac{ds}{dt} = 2s + s\cos(t)$ is a linear homogeneous ODE of second order and first degree.
4. $\left(\dfrac{d^3y}{dx^3}\right)^2 = y^2$ is a non-linear homogeneous autonomous ODE of third order and second degree.

Conditions

We have two types of extra conditions:

- If we specify the dependent variable and/or some of its derivatives at a particular point then we call these **Initial Conditions**

- If we specify these conditions at different points we call these **Boundary Conditions**

In this booklet, however, we will just refer to both types of condition as boundary conditions.
Boundary conditions are often in the form $y(x_0) = y_0$ which means that when $x = x_0$, then $y = y_0$. They can also be in the form $y^{(n)}(x_0) = y_0$, which means that when $x = x_0$, then the nth derivative of y is equal to y_0. When we are solving a differential equation subject to a boundary condition, we call the problem an **initial-value problem**.

The Superposition Principle

The Superposition Principle states that if we have any two solutions, $y = u_1(x)$ and $y = u_2(x)$, to a linear homogeneous ordinary differential equation, then $y = \alpha u_1(x) + \beta u_2(x)$ is also a solution for any constants α and β.
We must adapt this theorem slightly when considering linear inhomogeneous ordinary differential equations as follows: For any two solutions, $y = u_1(x)$ and $y = u_2(x)$, to a linear inhomogeneous ordinary differential equation, then $y = y_P + \alpha u_1(x) + \beta u_2(x)$ is also a solution for any constants α and β and y_P is the particular solution.
This theorem enables us to use a variety of techniques to find solutions to linear ODEs, which we will encounter later on in this chapter.
Important: The Superposition Principle only applies to **linear** ODEs.

10.2 First Order ODEs

Solution by Direct Integration

The most simple form of an ODE is:

$$\boxed{\frac{dy}{dx} = f(x)}$$

In this case we simply need to integrate the function $f(x)$, as we saw in the previous chapter on Integration.
For example, given the ODE:

$$\frac{dy}{dx} = 3x^2$$

we would integrate to get

$$y = x^3 + C$$

where C is a constant of integration, as usual.
This is called the **general solution** of the ODE. It is general because we have an arbitrary constant and so this solution describes an infinite number of curves corresponding to an infinite number of constants of integration. To find a **particular solution**, in which we would have no unknown constants, we require an additional piece of information in the form of a boundary condition.
We do this by simply substituting in our boundary conditions and thereby calculating C from the equation. For example, in the above problem, if we are given the initial condition $y(0) = 2$, then we substitute $x = 0$ and $y = 2$ into the general solution, giving:

$$2 = 0 + C \Rightarrow C = 2$$

Hence our particular solution to this initial-value problem is

$$y = x^3 + 2$$

Note: We can always check if our answer is a solution by substituting it into the ODE.

Solution by direct integration is actually a special case of the following method, separation of variables, which occurs when the function $g(y) = 1$.

Separation of Variables

If we have a differential equation in the form:

$$\boxed{\frac{dy}{dx} = f(x)g(y)}$$

where $f(x)$ is a function of only x and $g(y)$ is a function of only y, then we can use the method of separating the variables which works by rewriting our differential equation into the form:

$$\boxed{\frac{dy}{g(y)} = f(x)\,dx}$$

so that we have all the y's on the left hand side of the equation and all the x's on the right hand side so that we can integrate each side with respect to one variable.

$$\boxed{\int \frac{dy}{g(y)} = \int f(x)\,dx}$$

Example: If $\dfrac{dy}{dx} = \dfrac{4x^3}{y^2}$, use the method of separation of variables to find the function y.

Solution: First we need to get the differential equation into the correct form with all the x's on the right hand side and all the y's on the left. Multiplying both sides by y^2 gives:

$$y^2\,dy = 4x^3\,dx$$

We can now integrate both sides; the left with respect to y and the right with respect to x:

$$\int y^2\,dy = \int 4x^3\,dx$$

$$\frac{1}{3}y^3 = \frac{4x^4}{4} + C$$

$$y^3 = 3x^4 + C'$$

$$y = \sqrt[3]{3x^4 + C'}$$

where $C' = 3C$ and is a constant.

Example: If $\dfrac{dy}{dx} = \dfrac{\sin(-x) + e^{4x}}{y^2}$ then use the method of separation of variables to find an expression for y in terms of x.

Solution: So first we need to get the differential equation into the correct form will all the x's on the right hand side and all the y's on the left. So we have:

$$y^2\,dy = \left(\sin(-x) + e^{4x}\right)dx$$

We can now integrate both sides, the left with respect to y and the right with respect to x:

$$\int y^2\,dy = \int \left(\sin(-x) + e^{4x}\right)dx \qquad \text{Split up the integral on the right}$$

$$\int y^2\,dy = \int \sin(-x)\,dx + \int e^{4x}\,dx \qquad \text{Integrate both sides}$$

$$\frac{y^3}{3} = \cos(-x) + \frac{e^{4x}}{4} + C \qquad \text{Multiply both sides by 3}$$

$$y^3 = \frac{3}{4}e^{4x} + 3\cos(-x) + 3C \qquad \text{Take the cube root of both sides}$$

$$y = \left(\frac{3}{4}e^{4x} + 3\cos(-x) + C'\right)^{\frac{1}{3}} \qquad \text{Where } C' = 3C$$

Physics Example: The force between two particles is modelled to be:

$$F = \frac{12\varepsilon}{a_0}\left[\left(\frac{a_0}{r}\right)^{13} - \left(\frac{a_0}{r}\right)^{7}\right]$$

Given that force is the negative derivative of potential, i.e. $F = -\frac{d}{dr}U$, and at $r = a_0$, $U = -\varepsilon$ calculate the potential between the two particles.

Solution: This is a problem of seperating variables so first we set up the problem:

$$-\frac{d}{dr}U = \frac{12\varepsilon}{a_0}\left[\left(\frac{a_0}{r}\right)^{13} - \left(\frac{a_0}{r}\right)^{7}\right] \quad \text{Turn this into a nicer form with negative powers}$$

$$-\frac{d}{dr}U = \frac{12\varepsilon}{a_0}\left[a_0^{13}\,r^{-13} - a_0^{7}\,r^{-7}\right] \quad \text{Multiply by } dr \text{ and integrate both sides}$$

$$-\int dU = \int \frac{12\varepsilon}{a_0}\left[a_0^{13}\,r^{-13} - a_0^{7}\,r^{-7}\right] dr \quad \text{Take the constant } \frac{12\varepsilon}{a_0} \text{ out of the integral}$$

$$-U = \frac{12\varepsilon}{a_0}\int \left[a_0^{13}\,r^{-13} - a_0^{7}\,r^{-7}\right] dr \quad \text{Now integrate the right hand side}$$

$$-U = \frac{12\varepsilon}{a_0}\left(-\frac{a_0^{13}}{12}r^{-12} + \frac{a_0^{7}}{6}r^{-6}\right) + C \quad \text{This simplifies down}$$

$$U = \varepsilon\left(a_0^{12}\,r^{-12} - 2a_0^{6}\,r^{-6}\right) + C$$

Now find C by substitution ($r = a_0, U = -\varepsilon$)

$$-\varepsilon = \varepsilon\left(a_0^{12}\,a_0^{-12} - 2a_0^{6}\,a_0^{-6}\right) + C \quad \text{Simplify this}$$

$$-\varepsilon = \varepsilon\left(1 - 2\right) + C$$

$$-\varepsilon = -\varepsilon + C$$

$$0 = C$$

Back into our equation we get, $U = \varepsilon\left(a_0^{12}\,r^{-12} - 2a_0^{6}\,r^{-6}\right)$.

Note: This is another form of the Lennard-Jones 6-12 potential, with a_0 being the equilibrium seperation and ε being the energy needed to move the particles apart to infinity.

The Integrating Factor Method

Equations of the form

$$\boxed{\frac{dy}{dx} + g(x)y = f(x)}$$

are linear and can be solved using the method of integrating factor. In order to use this method, we must always ensure that our differential equation is in the form above and that there is nothing multiplied by the $\frac{dy}{dx}$.

Our objective is to write our equation in the form

$$\frac{d}{dx}\big(I(x)y\big) = I(x)f(x)$$

by finding an **integrating factor** $I(x)$. This form is beneficial because we can now directly integrate this to find the solution.

The formula for the integrating factor is:

$$\boxed{I(x) = e^{\int g(x)}}$$

We then simply need to find

$$\frac{1}{I(x)}\int I(x)f(x)\,dx$$

. For example, given the equation

$$x\frac{dy}{dx} + 2y = x^2 - x + 1$$

we begin by dividing everything by x to get the equation in the correct form:

$$\frac{dy}{dx} + 2\frac{y}{x} = x - 1 + \frac{1}{x}$$

Now we can compare this equation to our general form for these equations, and note that $f(x) = x-1+\frac{1}{x}$ and $g(x) = \frac{2}{x}$. Now, we can find the integrating factor as follows:

$$I(x) = e^{\int g(x)\,dx} = e^{\int \frac{2}{x}\,dx}$$
$$= e^{2\ln|x|} = x^2$$

Now we can write the ODE as

$$\frac{d}{dx}\left(x^2y\right) = x^2(x - 1 + \frac{1}{x}) = x^3 - x^2 + x$$

and so we only need to solve

$$y = \frac{1}{x^2}\int (x^3 - x^2 + x)\,dx$$
$$= \frac{1}{x^2}(\frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{2}x^2 + C)$$
$$= \frac{1}{4}x^2 - \frac{1}{3}x^1 + \frac{1}{2}x + Cx^{-2}$$

Example: Find the general solution of the following differential equation:

$$\frac{dy}{dx} + 2xy = 6x$$

Solution: The integrating factor is

$$I(x) = e^{\int 2x\,dx} = e^{x^2}$$

And as the theory tells us that

$$\frac{d(yI(x))}{dx} = 6xI(x)$$

(Where $f(x) = 6x$)

This implies that

$$ye^{x^2} = \int 6xe^{x^2}\,dx$$

Knowing that e^{x^2} differentiates to $2xe^{x^2}$ allows us to assume that $2xe^{x^2}$ integates to give e^{x^2}. Therefore we if write the equation above as

$$ye^{x^2} = 3\int 2xe^{x^2}\,dx$$

Then is it easy to see that

$$ye^{x^2} = 3e^{x^2} + C$$

Therefore

$$y = 3 + Ce^{-x^2}$$

Example: Find the solution to the initial value problem:

$$\frac{dy}{dx} + \frac{-2y}{x} = 8x^2$$

If $x = 1$ at $y = 0$

Solution: The integrating factor is

$$I(x) = e^{\int \frac{-2}{x}\,dx} = e^{-2\ln x} = x^{-2}$$

And as

$$\frac{d(yI(x))}{dx} = f(x)I(x)$$

An $f(x) = 8x^2$ This implies that

$$yx^{-2} = \int 8\,dx$$

Therefore

$$yx^{-2} = 8x + C$$

Using the Boundary Conditions, we ge that $0 = 8 + C$ and therefore $C = -8$

Our final answer is then

$$y = 8(x^3 - x^2)$$

Equations of the Form $\dfrac{dy}{dx} = f(\frac{y}{x})$

If we have a homogeneous differential equation of the form

$$\boxed{\frac{dy}{dx} = f(\tfrac{y}{x})}$$

then we cannot always use separation of variables, so we use the change of variable $V = y/x$ in order to get our equation into a more manageable form. We can then usually separate the variables and integrate with respect to this new variable.

Note: In this case, the word homogeneous has a different meaning to the one described on page 194. We can write these types of equations in an alternate form:

$$\boxed{\frac{dy}{dx} = \frac{F(x,y)}{G(x,y)}}$$

In order to use this method, the functions $F(x,y)$ and $G(x,y)$ must be homogeneous functions of degree one, meaning that they satisfy the condition:

$$F(kx, ky) = kF(x,y)$$

For example, if we have the differential equation

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2 - y^2}}{x}$$

then $F(x,y) = y + \sqrt{x^2 - y^2}$ satisfies the above relation since $ky + \sqrt{k^2x^2 - k^2y^2} = k(y + \sqrt{x^2 - y^2})$, and clearly so does $G(x,y) = x$. Now that we have checked the condition, we can then use the substitution $V = y/x \Rightarrow y = Vx$. We then have:

$$\frac{dy}{dx} = \frac{dVx}{dx} = V + x\frac{dV}{dx}$$

So the equation becomes:

$$x\frac{dV}{dx} = \frac{Vx + \sqrt{x^2 - V^2x^2}}{x} - V = V + \sqrt{1 - V^2} - V$$

$$\Rightarrow x\frac{dV}{dx} = \sqrt{1 - V^2}$$

This equation is separable and so we can write:

$$\int \frac{1}{\sqrt{1 - V^2}}\, dV \int \frac{1}{x}\, dx$$

and then integrate this using the methods in the Integration chapter to give:

$$\arcsin(V) = \ln|x| + C \Rightarrow V = \sin(\ln|x| + C)$$

Now we must remember to return to the original variable y by substituting $V = y/x$:

$$y = x\sin(\ln|x| + C)$$

Example: Find the general solution of the following differential equation:

$$5xy^2\frac{dy}{dx} = x^3 + 2y^3, x > 0$$

Solution: Rearranging the equation we get

$$\frac{dy}{dx} = \frac{x^3 + 2y^3}{5xy^2}$$

which is an equation in the form $\frac{dy}{dx} = \frac{F(x,y)}{G(x,y)}$ where F and G are homogeneous equations. Using the substitution $v = \frac{y}{x}$ we obtain get

$$\frac{dy}{dx} = v + \frac{dv}{dx} \text{ and } y = xv$$

Using this to re-write the equation gives

$$v + x\frac{dv}{dx} = \frac{x^3 + (xv)^3}{5x(xv)^2}$$

Dividing by x^3 on the top and bottom of the fraction gives

$$v + x\frac{dv}{dx} = \frac{1 + v^3}{5v^2}$$

Using the technique of separating the variables gives

$$\int \frac{1}{x}\, dx = \int \frac{5v^2}{1 - 3v^3}\, dx$$

The integral on the left hand side is a standard integral but the right hand side is more difficult. However it can be done simply enough with the right trick. Notice that the derivative of $1 - 3v^3$ with respect to v is $-9x^3$. Therefore we can get this integrand to a standard inspection integral by rewriting it as

$$-\frac{5}{9}\int \frac{-9v^2}{1 - 3v^3}\, dx$$

(An integral in the form of $\int \frac{f'(x)}{f(x)}\, dx$)
Integrating both sides gives

$$\ln|x| = -\frac{5}{9}\ln|1 - 3v^3| + c$$

Rewriting c as $\ln|k|$ and using log rules this equation simplifies further to

$$\ln\left|\frac{x(1 + 3v^3)^{5/9}}{k}\right| = 0$$

As $\ln(1) = 0$ we know

$$\frac{x(1 + 3v^3)^{5/9}}{k} = 1$$

and with further rearranging

$$v = \left(\frac{(\frac{k}{x})^{\frac{9}{5}} - 1}{3}\right)^{\frac{1}{3}}$$

Equations of the Form $\frac{dy}{dx} + f(x)y = g(x)y^k$

Equations of the form

$$\boxed{\frac{dy}{dx} + f(x)y = g(x)y^k}$$

are often called Bernoulli equations. Clearly we cannot simply separate the variables, but if we use the substitution $z = y^{1-k}$ we can get round this. Differentiating with respect to x gives:

$$\frac{dz}{dx} = (1-k)y^{-k}\frac{dy}{dx}$$

On rewriting the original equation as

$$y^{-k}\frac{dy}{dx} + f(x)y^{1-k} = g(x)$$

and now substituting our substitution in, we obtain:

$$\frac{dz}{dx} + (1-k)f(x)z = (1-k)g(x)$$

We can now solve this equation using an integrating factor as before.
For example, given the equation

$$\frac{dy}{dx} + xy = \frac{y}{x}$$

, we rewrite as

$$y\frac{dy}{dx} + xy^2 = x$$

Now, use the substitution $z = y^2$ and differentiate this to get:

$$\frac{dz}{dx} = 2y\frac{dy}{dx}$$

Substituting these into our equations gives:

$$\frac{dz}{dx} + 2xz = 2x$$

We can now solve this using an integrating factor $e^{\int 2x\,dx}$ to get

$$z = 1 + Ce^{-x^2}$$

Then, returning to the original variable

$$y^2 = 1 + Ce^{-x^2}$$

Example: Find the general solution of the following differential equation:

$$2\frac{dy}{dx} + 4xy = 2xy^4 = 0$$

Solution: Rearranging the equation we get

$$2y^{-4}\frac{dy}{dx} + 2\frac{2x}{y^3} = -2x$$

Dividing by 2 gives

$$y^{-4}\frac{dy}{dx} + \frac{2x}{y^3} = -x$$

Using the substitution $z = y^{1-k}$ with $k = 4$ gives $\frac{dz}{dx} = -3y^{-4}\frac{dy}{dx}$, which allows us to get

$$\frac{dz}{dx} - 6xz = 3$$

Using that the integrating factor is

$$e^{\int -6x\,dx} = e^{-3x^2}$$

we obtain

$$ze^{-3x^2} = \int 3xe^{-3x^2}\,dx$$

Rewriting this as

$$ze^{-3x^2} = -\frac{1}{2}\int -6xe^{-3x^2}\,dx$$

allows us to integrate by inspection

$$ze^{-3x^2} = -\frac{1}{2}e^{-3x^2} + C$$

Which implies

$$z = \frac{-1}{2} + Ce^{3x^2}$$

And in turn

$$y = \frac{1}{(Ce^{3x^2} - \frac{1}{2})^{\frac{1}{3}}}$$

10.3 Second Order Differential Equations

A second order differential equation is an equation of the form $f(y, y', y'') = 0$. In this section we will only be considering linear second order ODE and our aim when given this type of equation is to find $y(x)$.

When we are tasked with solving a second order ODE the first question we must ask is whether the equation we're given is **homogeneous** or not. A general second order ODE is:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

where a, b and c are constants.

This is a **homogeneous** function if $f(x) = 0$ since the dependent variable y then appears in all the terms. i.e.

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

Homogeneous Second Order Constant Coefficient Differential Equations

To solve a homogeneous second order ODE we predict a general solution of $y = e^{mx}$. From this we can determine that

$$\frac{dy}{dx} = me^{mx}$$

and that

$$\frac{d^2y}{dx^2} = m^2e^{mx}$$

Substituting this in to our general equation of a homogeneous second order ODE we obtain

$$am^2e^{mx} + bme^{mx} + ce^{mx} = 0$$

OR

$$e^{mx}(am^2 + bm + c) = 0$$

As we know that an exponential term (in this case the e^{mx}) cannot equal zero we know that am^2+bm+c must equal zero. Solving this quadratic equation (called the **auxiliary equation** or **characteristic equation** of the ODE) will give us the values of m for which $y = e^{mx}$ is a solution of the second order ODE given.

However there are three cases when solving a quadratic equation: it can have two real roots, one repeated root or no real roots. Each of these cases requires a different trial function in order to solve the equation.

Case 1: Two Real Roots

For the case when the equation has two real roots, we have two solutions to the second order ODE $y = e^{m_1x}$ and $y = e^{m_2x}$. We simplify this to one solution by adding them together using the superposition principle. However to allow this to be a general solution which encompasses all different possible solutions dependent on initial conditions, we put an arbitrary constant in front of both terms.

i.e. For a homogeneous second order ODE with two solutions to the auxiliary equation, m_1 and m_2, we give the general solution as

$$y = Ae^{m_1x} + Be^{m_2x}$$

where A and B are arbitrary.

Example: Solve the initial value problem

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 24 = 0$$

if $x = 0$ at $y = 8$ and $\dfrac{dy}{dx} = 122$

Solution: The auxiliary equation is

$$m^2 + 2m - 24y = 0$$

This factorises to

$$(m-4)(m+6) = 0$$

So has two real roots;

$$m = 4$$

$$m = -6$$

Therefore the general solution is

$$y = Ae^{4x} + Be^{-6x}$$

If at $x = 0$, $y = 8$

$$8 = A + B$$

$$\frac{dy}{dx} = 4Ae^{4x} - 6Be^{-6x}$$

If at $x = 0$, $\dfrac{dy}{dx} = 122$

$$122 = 4A - 6B$$

Solving these equations simultaneously yields the answer

$$A = 17$$

$$B = -9$$

Then the particular solution to this problem is

$$y = 68e^{4x} + 54e^{-6x}$$

Case 2: One Repeated Root

For this case, we use a general solution of the form

$$y = e^{mx}(A + Bx)$$

An understanding of why this is the case is not necessary but it is important to remember the general form.

Example: Find the general solution to the second order ODE:

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9 = 0$$

Solution: The auxiliary equation is

$$m^2 + 6m + 9 = 0$$

This factorizes to

$$(m+3)^2 = 0$$

So has one repeated root

$$m = -3$$

Therefore the general solution is

$$y = e^{-3x}(A + Bx)$$

Case 3: No Real Roots (Complex Roots)

In the case where the roots are $m = \alpha \pm i\beta$, using the solution from case one we obtain

$$y = Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x}$$

Taking out the common factor of e^{α} we obtain

$$e^{\alpha x}(Ae^{i\beta x} + Be^{-i\beta x})$$

Converting the complex parts to modulus-argument form

$$e^{\alpha x}(A(\cos(\beta x) + i\sin(\beta x)) + B(\cos(-\beta x) + i\sin(-\beta x))$$

As cosine is an even function and sine is an odd function we can simplify this further

$$e^{\alpha x}(A(\cos(\beta x) + i\sin(\beta x)) + B(\cos(\beta x) - i\sin(\beta x)))$$

$$e^{\alpha x}((A+B)\cos(\beta x) + (A-B)i\sin(\beta x))$$

Combining the constants we obtain

$$e^{\alpha x}(A'\cos(\beta x) + B'\sin(\beta x))$$

This is the general form we use when solving a second order ODE of this type.

Example: Solve the initial value problem

$$\frac{d^2y}{dx^2} + 64 = 0$$

if at $x = \pi$, $y = 8$ and at $x = \frac{\pi}{8}$, $y = 3$

Solution: The auxiliary equation is

$$m^2 + 64 = 0$$

This has complex roots of

$$m = \pm 8i$$

Therefore the general solution is

$$y = A\cos(8x) + B\sin(8x)$$

If at $x = \pi$, $y = 8 \Rightarrow A = 1$
If at $x = \frac{\pi}{8}$, $y = 3 \Rightarrow B = -3$
Therefore the solution to the problem is:

$$y = \cos(8x) + -3\sin(8x)$$

Note: There is no need to derive the general formulae each time you do a question, all three are quotable.

Inhomogeneous Second Order Differential Equations

An inhomogeneous second order ODE has the general from

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

where $f(x) \neq 0$.
To solve these is slightly more laborious however is not particularly difficult. The first step is to solve the corresponding homogeneous equation (which has the same left hand side, but setting $f(x)$ to 0). The solution we get with this technique is called the **complementary function**. After this, to complete the solution, we need to find the **particular integral**.
To find the particular integral we need to consider the $f(x)$ term on the right hand side of the equation. We choose a 'trial function' dependent on the form of $f(x)$. Here is a table showing which trial function to make for each case:

$f(x)$	**Trial Function**
Polynomial of degree n	$Ax^n + Bx^{n-1} + Cx^{n-2} + ...$
e^{kx}	Ce^{kx}
$\cos(kx)$ **and/or** $\sin(kx)$	$C\cos(kx) + D\sin(kx)$

Note: If $f(x)$ is a product of two or three of the terms given in the left hand column of the table, we use a trial function that is the product of the corresponding trial functions given in the right hand column of the table.
After choosing the right trial function we proceed to differentiate it one and then twice, and substitute these into the equation as $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$ and y correspondingly. By comparing the coefficients of the terms on either side of the equation we can get values for all arbitrary constants. Once these are found we know the particular integral.
The final general solution is

$$y(x) = y_{CF}(x) + y_{PI}(x)$$

Where $y_{CF}(x)$ is the complementary function and $y_{PI}(x)$ is the particular integral.

Example: Solve the second order ODE

$$\frac{d^2y}{dx^2} - 9y = 2x^2 + x$$

Solution: Consider the corresponding homogeneous equation:

$$\frac{d^2y}{dx^2} - 9y = 0$$

and solve as before.

$$m^2 - 9 = 0$$

Therefore $m = \pm 3$
Therefore the complementary function is

$$y_{CF} = Ae^{3x} + Be^{-3x}$$

For the particular integral we will make the trial function $Cx^2 + Dx + E$.

$$y_{PI} = Cx^2 + Dx + E$$
$$\frac{d(y_{PI})}{dx} = 2Cx + D$$
$$\frac{d^2(y_{PI})}{dx^2} = 2C$$

Substituting these into the original equation we get:

$$2C - 9Cx^2 - 9Dx - 9E = 2x^2 + x$$

Then equating the coefficients of sine and cosine:

$$-9C = 2 \Rightarrow C = -\frac{2}{9}$$

$$-9D = 1 \Rightarrow D = -\frac{1}{9}$$

$$2C - 9E = 0 \Rightarrow E = \frac{4}{81}$$

The final solution is:

$$y = Ae^{3x} + Be^{-3x} - \frac{2}{9}x^2 - \frac{1}{9}x + \frac{4}{81}$$

Example: Solve the second order ODE

$$\frac{d^2y}{dx^2} + 15\frac{dy}{dx} + 50y = 4\cos(x)$$

Solution: Consider the corresponding homogeneous equation:

$$\frac{d^2y}{dx^2} + 15\frac{dy}{dx} + 50y = 0$$

and solve as before.

$$m^2 + 15m + 50 = 0$$

This factorizes to give $(m+5)(m+10) = 0$
Therefore $m = -5$ or $m = -10$
Therefore the complementary function is

$$y_{CF} = Ae^{-5x} + Be^{-10x}$$

For the particular integral we will make the trial function $C\cos(x) + D\sin(x)$.

$$y_{PI} = C\cos(x) + D\sin(x)$$
$$\frac{d(y_{PI})}{dx} = -C\sin(x) + D\cos(x)$$
$$\frac{d^2(y_{PI})}{dx^2} = -C\cos(x) - D\sin(x)$$

Substituting these into the original equation we get:

$$-C\cos(x) - D\sin(x) + 15(-C\sin(x) + D\cos(x)) + 50(C\cos(x) + D\sin(x)) = 4\cos(x)$$

Then equating the coefficients of sine and cosine:

$$15D - 49C = 4$$
$$49D - 15C = 0$$
$$D = -\frac{15}{544}$$
$$C = -\frac{49}{544}$$

The final solution is:

$$y = Ae^{-5x} + Be^{-10x} - \frac{49}{544}\cos(x) - \frac{15}{544}\sin(x)$$

Example: Solve the second order ODE

$$12\frac{d^2z}{dy^2} + 32\frac{dz}{dy} + 5 = 25e^{4x}\cos(x)$$

Solution: Solving the corresponding homogeneous equation: The auxiliary equation is

$$12m^2 + 32m + 5 = 0$$

This factorizes to give $(2m+5)(6m+1) = 0$
Therefore the complementary function is

$$z_{CF} = Ce^{-\frac{5}{2}y} + De^{-\frac{1}{6}y}$$

For the particular integral we will use the trial function $e^{4y}(A\cos(y) + B\sin(y))$.

$$z_{PI} = e^{4y}(A\cos(y) + B\sin(y))$$

$$\frac{d(z_{PI})}{dx} = 4e^{4y}(A\cos(y) + B\sin(y)) + e^{4y}(-A\sin(y) + B\cos(y))$$

$$\frac{d^2(z_{PI})}{dy^2} = 16e^{4y}(A\cos(y) + B\sin(y)) + 4e^{4y}(-A\sin(y) + B\cos(y)) +$$
$$4e^{4y}(-A\sin(y) + B\cos(y)) + e^{4y}(-A\cos(y) - B\sin(y))$$

Substituting these into the original equation we get:

$$12(16e^{4y}(A\cos(y) + B\sin(y)) + 4e^{4y}(-A\sin(y) + B\cos(y)) + 4e^{4y}(-A\sin(y) + B\cos(y)) +$$
$$e^{4y}(-A\cos(y) - B\sin(y))) + 32(4e^{4y}(A\cos(y) + B\sin(y)) + e^{4y}(-A\sin(y) + B\cos(y))) +$$
$$5(e^{4y}(A\cos(y) + B\sin(y))) = 25e^{4y}\cos(y)$$

$$313A + 128B = 25$$
$$128A + 313B = 0$$

$$A = 0.0684 \ \ (3sf)$$

$$B = 0.0279 \ \ (3sf)$$

$$z_{PI} = e^{4y}(0.0684\cos(y) + 0.0279\sin(y))$$

The final solution is:

$$z = Ce^{-\frac{5}{2}y} + De^{-\frac{1}{6}y} + e^{4x}(0.0684\cos(y) + 0.0279\sin(y))$$

Physics Example: The motion of a damped driven oscillator is described by

$$\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} = \lambda\cos(\omega t)$$

where θ is the angle of the pendulum from equilibrium, t is time, ω is the angular frequency and λ and b are constants. Find the general solution for this ODE.

Solution: Solving the corresponding homogeneous equation:

$$m^2 + bm = 0$$

This factorizes to give $m(m+b) = 0$
Therefore $m = 0$ or $m = -b$
Therefore the complementary function is

$$\theta_{CF} = Ae^{0t} + Be^{-bt} = A + Be^{-bt}$$

For the particular integral we will make the trial function $C\cos(\omega t) + D\sin(\omega t)$.

$$\theta_{PI} = C\cos(\omega t) + D\sin(\omega t)$$
$$\frac{d(\theta_{PI})}{dt} = -C\omega\sin(\omega t) + D\omega\cos(\omega t)$$
$$\frac{d^2(\theta_{PI})}{dt^2} = -C\omega^2\cos(\omega t) - D\omega^2\sin(\omega t)$$

Substituting these into the original equation we get:

$$-C\omega^2\cos(\omega t) - D\omega^2\sin(\omega t) + b(-C\omega\sin(\omega t) + D\omega\cos(\omega t)) = \lambda\cos(\omega t)$$

Then equating the coefficients of sine and cosine:

$$-C\omega^2 + bD\omega = \lambda$$
$$-D\omega^2 - Cb\omega = 0$$
$$C = \frac{\lambda}{b^2 - \omega^2}$$
$$D = \frac{\lambda b}{\omega(b^2 - \omega^2)}$$

The final solution is:

$$\theta = A + Be^{-bt}\frac{\lambda}{b^2 - \omega^2}\cos(\omega t) + \frac{\lambda b}{\omega(b^2 - \omega^2)}\sin(x)$$

Equations with Dangerous Terms

The technique described above only works when $f(x)$ in not proportional to a term in the complementary function. For example if we have a complementary function of

$$y = Ae^{12x} + Be^{7x}$$

but

$$f(x) = 6e^{12x}$$

then as one of the terms in the complementary function is Ae^{12x} which ***is*** proportional to $f(x)$ we cannot choose our trial function as Ce^{12x} as the table would suggest.

When this happens, to solve the problem, we use the trial function multiplied by x. So in this example we would use:

$$Cxe^{12x}$$

However, in our example, if the complementary function had been $y = e^{12x}(A + Bx)$ (in the case of having a repeated root), even when the trial function was multiplied by x it would still be proportional to Bxe^{12x} and therefore our trial function would fail again, our technique to solve this is to multiply by x again and therefore we would use:

$$Cx^2e^{12x}$$

Example: Solve the second order ODE

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4 = 7e^{-2x}$$

Solution: Solving the corresponding homogeneous equation: The auxiliary equation is

$$m^2 + 4m + 4 = 0$$

This factorizes to give $(m+2)^2 = 0$
Therefore the complementary function is

$$y_{CF} = e^{-2x}(A + Bx)$$

For the particular integral we would normally make the trial function Ce^{-2x}. But as $f(x)$ is proportional to the first term in the complementary function, so we have to multiple it by an x.

$$Cxe^{-2x}$$

However, this is proportional to the second term in our complementary function so again we must multiplied by an x.

$$Cx^2e^{-2x}$$

$$y_{PI} = Cx^2e^{-2x}$$

$$\frac{d(y_{PI})}{dx} = 2Cxe^{-2x} - 2Cx^2e^{-2x}$$

$$\frac{d^2(y_{PI})}{dx^2} = 2Ce^{-2x} - 4Cxe^{-2x} - 4Cxe^{-2x} + 4Cx^2e^{-2x}$$

Substituting these into the original equation we get:

$$2Ce^{-2x} - 4Cxe^{-2x} - 4Cxe^{-2x} + 4Cx^2e^{-2x} + 4(2Cxe^{-2x} - 2Cx^2e^{-2x}) + 4(Cx^2e^{-2x}) = 7e^{-2x}$$

$$2Ce^{-2x} = 7Ce^{-2x}$$

$$\Rightarrow 2C = 7$$

$$C = \frac{7}{2}$$

$$y_{PI} = \frac{7}{2}x^2e^{-2x}$$

The final solution is:

$$y = e^{-2x}(A + Bx) + \frac{7}{2}x^2e^{-2x}$$

10.4 Setting Up Differential Equations

Knowing how to solve a differential equation is an essential skill, however in physics one must be able to create the differential equation from the physical situation in the first place. As explained previously, a derivative is the rate of change of a function with respect to the given variable. A strong understanding of this concept is key to success in this section. It is hard to picture how the gradient is the rate of change of the dependent variable with respect the independent variable so we can understand it with examples.

Physically we understand velociy as how fast something moves, but mathematically we define it as the rate of change of displacement with respect to time. The two ideas are actually one and the same when one understands rate of changes; how fast something moves is a measure of how far they have travelled (the change in displacement) in a specified time (with respect to time).

Physics Example: A particle is released from rest at $t = 0$ and $x = 0$, its subsequent velocity is described by the equation

$$v = 3t - 45t^2 + 3$$

How what the particles displacement function of time, x(t)?

Solution: We know that the velocity is equal to the rate of change of displacement with respect to time, i.e.

$$v = \frac{dx}{dt}$$

So

$$\frac{dx}{dt} = 3t - 45t^2 + 3$$

By using the method of separation of variables we obtain:

$$\int x\,dx = \int 3t - 45t^2 + 3\,dt$$

and therefore:

$$\frac{x^2}{2} = \frac{3t^2}{2} - 15t^3 + 3t + C$$

Using the boundary conditions we deduce that C must equal 0.
Our final answer is:

$$x = \sqrt{3t^2 - 30t^3 + 6t}$$

With the understanding of what a derivative is we are able to convert physical situations into mathematical equations which increases our understanding of the situation at hand and allows us to predict what will happen if the variables alter. For example if we are told 'in radioactive decay the rate of change of one nuclear species into another depends on the amount of un-decayed material present.' We can say that:

$$\frac{dN}{dt} = -\lambda N$$

Where N is the number or un-decayed nuclei and λ is the proportionality constant (there is a minus sign as N is decreasing). Now we have this equation, we could solve for $N(t)$ and make predictions for the amount of un-decayed material at any time.

Physics Example: Determine the time dependence of the temperature of a cup of tea, T(t).

Solution: If we assume that the loss of heat of the cup of tea is proportional to its temperature we can say that:

$$\frac{dT}{dt} = -\alpha T$$

Where α is a constant of proportionality. Separating the variable we get:

$$\int \frac{1}{T}\,dT = \int -\alpha\,dt$$

$$\ln|T| = -\alpha t + C$$

If we let $T = T_0$ at $t = 0$ we obtain that

$$C = \ln|T_0|$$

Finally we re-arrange:

$$\ln|T| - \ln|T_0| = -\alpha t$$

$$\ln\left|\frac{T}{T_0}\right| = -\alpha t$$

$$T = T_0 e^{-\alpha t}$$

Physics Example: What happens to an un-powered boat which has an initial velocity v_0 and is moving in a viscous medium such that the viscous force on the boat is proportional to the velocity of the boat? Produce an equation for the time dependence of the boat's velocity.

Solution: Note that 'the viscous force on the boat is proportional to the velocity of the boat'. This suggests that the resistive force on the boat, R, can be described by:

$$R = \lambda v$$

Where λ is a constant of proportionality and v is the velocity of the boat.
From Newton's second law of motion we know that 'resultant force = mass × acceleration' or $F = ma$. Using this we say:

$$-R = -\lambda v = ma$$

Acceleration is the rate of change of velocity (how much the velocity changes in a unit time) so we can say that:

$$-\lambda v = m\frac{dv}{dt}$$

Separating the variables gives:

$$\int -\lambda\,dt = \int \frac{m}{v}\,dv$$

$$-\lambda t = m\ln|v| + C$$

At $t = 0$ we know that $v = v_0$, therefore:

$$-\lambda \times 0 = m\ln|v_0| + C$$

$$C = -m\ln|v_0|$$

Substituting this in and rearranging gives

$$v = v_0 e^{\frac{-\lambda t}{m}}$$

The question asks what happens to the boat as time goes on so we need to use this equation to predict what happens as t increases. It is clear to see that an increase of t will mean that the exponential term will be less than one and decreasing, i.e. the boat will slow down and stop at time $t = \infty$.

The following example is harder than the previous two but it is built on the same idea of converting a situation to equations using rates of change.

Physics Example: A bath tub of volume of $100L$ is filled with a mixture of 90% water and 10% alcohol. If the plug hole is opened, 1 litre of fluid is lost every minute. A hose is turned on and put in the tub at $t = 0$. It sprays 1 litre of a mixture of 50% water and 50% alcohol into the tub every 30 seconds. At $t = 0$ the plug hole is also opened. How long does it take for the mixture in the tub to be 50% water and 50% alcohol?

Solution: The best way to tackle this problem is to consider each liquid a separately. If we consider water only and define W as the total water in the bath, W_h as the net water from the hose and W_p as the net water through the plug hole:
The hose is putting a constant volume of half a litre into the tub every 30 seconds, so we can state that:

$$\frac{dW_h}{dt} = \frac{1}{2} \times \frac{1}{30} = \frac{1}{60}$$

The plug hole is releasing a constant volume of liquid but the amount of water released is dependent on the ratio of water to alcohol in the tub:

$$\frac{dW_p}{dt} = -\frac{1}{30} \times \frac{W}{100} = -\frac{W}{3000}$$

So the net loss/increase of water is described by:

$$\frac{dW}{dt} = \frac{dW_h}{dt} + \frac{dW_p}{dt} = \frac{1}{60} - \frac{W}{3000}$$

Solving this with separation of variables we obtain:

$$\int \frac{-dW}{\frac{W}{3000} - \frac{1}{60}} = \int dt$$

$$= -3000 \ln |W - 50| = t + C$$

This rearranges to give:

$$W = e^{\frac{-t+C}{3000}} + 50$$

Or

$$W = C' e^{\frac{-t}{3000}} + 50$$

At t=0 there are 90L of water so

$$90 = C' e^{\frac{0}{3000}} + 50$$

$\Rightarrow C' = 40$ So we obtain

$$W = 40e^{\frac{-t}{3000}} + 50$$

If $W = 50$

$$50 = 40e^{\frac{-t}{3000}} + 50$$

$$40e^{\frac{-t}{3000}} = 0$$

Therefore the split of water and alcohol will be even only as t tends to ∞.

You can check this by considering the alcohol rather than the water.

This idea is very useful in all areas of physics but for a 1st year physics course it is mainly seen in the skills course.

10.5 Review Questions

Easy Questions

Question 1: Find the solution to this initial value problem

$$\frac{dy}{dx} = e^x, \quad y(0) = 1$$

Answer:

$$y(x) = e^x$$

Question 2: Solve

$$\frac{dy}{dx} = \frac{1}{2}(1 - y^2)$$

Subject to the initial condition $y(0) = -2$

Answer:

$$\coth\left(\frac{x}{2} - \frac{\ln(3)}{2}\right)$$

Question 3: Solve the second order differential equation

$$\frac{dx}{dt} - 3\frac{dx}{dt} + 2y = 0$$

Answer:

$$x(t) = Ae^t + Be^2t$$

Question 4: Solve the differential equation

$$(y + 1)^2\frac{dy}{dx} = -x^3$$

Answer:

$$\frac{(y + 1)^3}{3} = -\frac{x^4}{4} + C$$

Question 5: Solve the first order differential equation

$$\frac{dx}{dt} = 5x - 3$$

Answer:

$$x(t) = \frac{1}{5}e^{5t+5C} + \frac{3}{5}$$

Medium Questions

Question 6: Solve the differential equation

$$x\frac{dy}{dx} = y + x^3 + 3x^2 - 2x$$

Answer:

$$y(x) = \frac{x^3}{2} + 3x^2 - 2x\ln(x) + Cx$$

Question 7: Solve the differential equation

$$x\frac{dy}{dx} + y = xy^3(1 + \ln(x))$$

Answer:

$$y(x) = \left(-\frac{4}{9}x - \frac{2}{3}x\ln(x) + \frac{C}{x^2}\right)^{-\frac{1}{2}}$$

Question 8: Taking x and the dependent variable to solve the equation

$$y\ln(y)dx + (x - \ln(y))dy = 0$$

Answer:

$$x(y) = \frac{\ln(y)}{2} + \frac{C}{\ln(y)}$$

Question 9: Find $y$in terms of $x$if

$$\frac{dy}{dx} + xy = \frac{x}{y}$$

Answer:

$$y(x) = \sqrt{1 + Ce^{-}x^2}$$

Question 10: Find the solution of the second order differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = e^x$$

Answer:

$$y(x) = e^x(A\cos(3x) + B\sin(3x)) + \frac{1}{9}e^x$$

Hard Questions

Question 11: Solve the differential equation

$$x\frac{dy}{dx} - (y + \sqrt{x^2 - y^2}) = 0$$

Answer:

$$y(x) = x\sinh(\ln(x) + C)$$

Question 12: Solve second order ODE

$$\frac{d^2y}{dx^2} + 9y = x\cos(x)$$

Hint: Try the particular integral $(ax+b)\cos(x) + (cx+d)\sin(x)$.

Answer:

$$y(x) = A\cos(3x) + B\sin(3x) + \frac{x}{8}\cos(x) + \frac{1}{32}\sin(x)$$

Question 13: Solve the second order differential equation

$$\frac{d^2y}{dx^2} - 4y = e^{2x}$$

Hint: Be careful with the particular integral.

Answer:

$$y(x) = Ae^{2x} + Be^{-2x} + \frac{x}{4}e^{2x}$$

Question 14: Solve the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = (x^3 + x)e^{2x}$$

Hint: Be careful with the particular integral.

Answer:

$$y(x) = \left(\frac{x^5}{20} + \frac{x^3}{6} + Bx + A\right)e^{2x}$$

11 Series and Expansions

11.1 Sequences

For completeness, we will briefly discuss what we mean by 'sequences', as this will lead directly into the next section on 'series'. A sequence is an ordered list of numbers such as

$$1, 4, 9, 16, ...$$

or

$$3, 5, 7, 9, ...$$

where the '...' means that the list does not end. This should appear familiar from the section on 'sets', however note that in the case of sequences, the numbers are **ordered**. We can generalise a sequence by writing

$$a_1, a_2, a_3, ..., a_n, ...$$

where a_n is called the nth term of the sequence.
Formally, a sequence is a function between the natural numbers $\mathbb{N}$ and the real numbers $\mathbb{R}$. For example, the functions displayed above would represent the functions $a_n = n^2$ and $b_n = 2n + 1$ respectively. So for example if we wish to find the nth term of the first sequence, we just need to calculate n^2. We can also find the function corresponding to a simple sequence by inspection, by looking at the differences between the terms in the sequence and using our knowledge of different functions.

Example:

1. Find the first four terms of the sequence $a_n = \frac{1}{n+1}$
2. Find the 6th term, the 18th term and the 2015th term of the sequence $b_n = 3n - 4$.
3. Find the formula which yields the sequence $9, 11, 13, 15, \ldots$

Solution:

1. Since we are only asked to find the first four terms, we just need to substitute $n = 1$, $n = 2, n = 3$ and $n = 4$ into the formula.

$$a_1 = \frac{1}{1+1} = \frac{1}{2}$$

$$a_2 = \frac{1}{2+1} = \frac{1}{3}$$

$$a_3 = \frac{1}{3+1} = \frac{1}{4}$$

$$a_4 = \frac{1}{4+1} = \frac{1}{5}$$

2. We must substitute the values $n = 6, n = 18$ and $n = 2015$ into the formula.

$$b_6 = 3 \times 6 - 4 = 14$$

$$b_{18} = 3 \times 18 - 4 = 50$$

$$b_{2015} = 3 \times 2015 - 4 = 6041$$

3. First we look at the difference between each consecutive term and see that to get from one term to the next we add 2. This means that to get from each value of n to the nth term in our sequence, we have to multiply n by 2. So far our formula looks like $c_n = 2n + \alpha$. Now we need to find α by substituting the first term into this equation:

$$9 = 2 \times 1 + \alpha$$

$$\Rightarrow 9 - 2 = \alpha$$

$$\Rightarrow \alpha = 7$$

So our formula is $c_n = 2n + 7$.
It is a good idea to check that the formula works for the first few terms in the sequence:

$$2 \times 2 + 7 = 4 + 7 = 11$$

$$2 \times 3 + 7 = 6 + 7 = 13$$

We are often interested in the behaviour of sequences as n gets increasingly large. In particular, it is useful to know if a function tends to $\pm\infty$ or converges to a real number. We will give a brief, informal description of what these behaviours mean, however we will not give methods for proving them.

- A sequence **tends to infinity** if after some point A the sequence can be made as large as we like. Note that a sequence **tends to minus infinity** if after some point A the sequence can be made as small as we like.
- A sequence **converges to a real number** ℓ if for any $\epsilon > 0$, $|a_n - \ell| < \epsilon$ for all $n \geq N$. In other words, after some point N, each consecutive term in the sequence can be made as close as we like to ℓ.

11.2 Summations

A finite sum of real numbers,

$$a_1 + a_2 + a_3 + a_4 + ... + a_N$$

is denoted by

$$\sum_{n=1}^{N} a_n$$

where the '$n = 1$' below the summation symbol declares which value of n we begin summing from and the 'N' above the symbol declares which value of n we stop summing at. Notice also that the 'a_n' represents the formula which we use to calculate each term in the sum, just as in the previous section on sequences. We call n the 'index of summation'.

Example:

1. Calculate the sum:

$$\sum_{n=2}^{8} (-1)^n n$$

2. Calculate the sum:

$$\sum_{n=1}^{4} \frac{1}{n}$$

Solution:

1. We are required to add up all terms in the sequence $a_n = (-1)^n n$ from $n = 2$ to $n = 8$. Note that the $(-1)^n$ is positive for even n and negative for odd n.

$$\sum_{n=2}^{8} n = 2 - 3 + 4 - 5 + 6 - 7 + 8 = 5$$

2. We are required to add up all terms in the sequence $a_n = \frac{1}{n}$ from $n = 1$ to $n = 4$.

$$\sum_{n=1}^{4} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$$

11.3 Series

A 'series' is an **infinite sum** of real numbers and is denoted by

$$a_1 + a_2 + a_3 + a_4 + ...$$

where, as before, the '...' means that the series does not end.
We can also represent a series using the notation for a summation, ie.

$$\sum_{n=1}^{\infty} a_n$$

where the '$n = 1$' below the summation symbol declares which value of n we begin summing from and the '∞' above the symbol tells us that this is an infinite sum. Notice that the notation here is the same as in the section on 'summations', however we have replaced N with ∞.

As with sequences, it can be useful to know the behaviour of a series as n gets increasingly large. To this end, we use the idea of a partial sum s_N, where

$$s_N = \sum_{n=1}^{N} a_n$$

A series can do one of two things:

- A series may **converge** to a real number s. This happens when a series can be made as close as we like to s as we add more and more terms to the summation. If the sequence of partial sums

$$s_N = \sum_{n=1}^{N} a_n$$

  converges, then the infinite series also converges.

- If a series does not converge, then it is said to **diverge**. This is when a series either becomes increasingly large (or small) as we add more terms to it, or else does something where it does not converge to a single real number. If the sequence of partial sums

$$s_N = \sum_{n=1}^{N} a_n$$

  diverges, then the infinite series also diverges.

Arithmetic and Geometric Series

Arithmetic Series One special type of sequence is called an arithmetic sequence. The general form of this type of sequence is:

$$a_n = a_{n-1} + d\ , \qquad a_1 = \alpha$$

where α is a constant. In words, this means that to get to the next term in the sequence, we add d to the previous term.

The terms of this sequence can be added up to give an arithmetic series:

$$S_N = \sum_{n=1}^{N} [a_1 + (n-1)d]$$

This sum can be can be simplified:

$$S_N = \sum_{n=1}^{N} a_1 + \sum_{n=1}^{N} (n-1)d$$

$$= Na_1 + d\sum_{n=2}^{N} (n-1)$$

$$= Na_1 + d\sum_{n=1}^{N-1} n$$

We now need to consider a different sum $S' = \sum_{n=1}^{M} n$. By writing out this sum we obtain

$$S' = 1 + 2 + ... + (M-1) + M$$

Writing this again backwards gives

$$S' = M + (M-1) + ... + 2 + 1$$

Adding these together produces

$$2S' = M \times (M+1)$$

and hence

$$S' = \sum_{n=1}^{M} n = \frac{1}{2}M(M+1)$$

By replacing M in the above formula with $(N-1)$ as in S_N we deduce:

$$S_N = \sum_{n=1}^{N}[a_1 + (n-1)d] = Na_1 + \frac{1}{2}d(N-1)(N)$$

Therefore

$$\boxed{S_N = \tfrac{1}{2}N\big(2a_1 + d(N-1)\big)}$$

Note

$$a_1 + a_N = 2a_1 + d(N-1)$$

So

$$\boxed{S_N = \tfrac{1}{2}N(a_1 + a_N)}$$

Example: Find the sum of the arithmetic series

$$S_N = \sum_{n=0}^{N}\big[4 + (n-1)3\big]$$

when

1. $N = 7$
2. $N = 28$

Solution: Using $S_N = \frac{1}{2}N\big(2a_1 + d(N-1)\big)$:

1. $S_7 = \sum_{n=0}^{7}\big[4 + (n-1)3\big] = \frac{1}{2} \times 7\big(2 \times 4 + 3(7-1)\big) = 91$
2. $S_{28} = \sum_{n=0}^{28}\big[4 + (n-1)3\big] = \frac{1}{2} \times 28\big(2 \times 4 + 3(28-1)\big) = 1246$

Example: Find the sum of the arithmetic series

$$S_5 = \sum_{n=0}^{5} [3 + (n-1)9]$$

Solution: As $a_5 = 3 + (5-1)9 = 39$ and using $S_N = \frac{1}{2}N(a_1 + a_N)$ we can deduce that

$$S_5 = \frac{1}{2}N(3 + 39) = 21$$

Geometric Series

Another special sequence is a geometric sequence. This type of sequence has the general form:

$$a_n = a^n$$

Again the terms of this sequence can be summed, this gives a geometric series:

$$S_N = \sum_{n=0}^{N} a^n$$

This sum can be re-written to give the formula for the sum of the geometric series:

$$\boxed{S_N = \sum_{n=0}^{N} a^n = \frac{1 - a^N}{1 - a}}$$

It is also useful to know that the 'sum to infinity' of this series is:

$$\boxed{S_\infty = \sum_{n=0}^{\infty} a^n = \frac{1}{1 - a} \quad , \quad |a| < 1}$$

Example: Find the sum of the geometric series

$$S_N = \sum_{n=0}^{N} 7(0.5^n)$$

when

1. $N = 5$
2. $N = \infty$

Solution:

1. $S_5 = \sum_{n=0}^{5} 7(0.5^n) = 7\left(\sum_{n=0}^{5} (0.5^n)\right) = 7 \times \frac{1 - 0.5^5}{1 - 0.5} = 7 \times 1.9375 = 13.5625$
2. $S_\infty = \sum_{n=0}^{\infty} 7(0.5^n) = 7\left(\sum_{n=0}^{\infty} (0.5^n)\right) = \frac{7}{1 - 0.5} = 14$

Power Series

An important type of series, which will come in handy in the next section, is power series. A power series about the point $x = x_0$ is an expression of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

$$= a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + ...$$

where x is a variable and the a_n are constants.
Power series always have a radius of convergence. This is used to describe for exactly which values the series will converge. A power series with radius of convergence R will converge for any x such that $|x - x_0| < R$ and will diverge for any x such that $|x - x_0| < R$. For $|x - x_0| = R$, we cannot say whether the power series converges or not. Note that if $R = \infty$, the power series converges for all x, and if $R = 0$, the power series converges only when $x = x_0$.

Note: A strong understanding of the radius of convergence is not needed.

Adding and Subtracting Power Series

It is possible to add and subtract power series. If we consider the series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ with the radius of convergence $R_1 > 0$ and $\sum_{n=0}^{\infty} b_n(x - x_0)^n$ with the radius of convergence $R_2 > 0$, then:

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n \pm \sum_{n=0}^{\infty} b_n(x - x_0)^n = \sum_{n=0}^{\infty} (a_n \pm b_n)(x - x_0)^n$$

This new sum has the radius of convergence $R \geq min\{R_1, R_2\}$

Example: Add the two series

$$\sum_{n=0}^{\infty} (n + 2)x^n$$

$$\sum_{n=0}^{\infty} (3n - 2)x^n$$

Solution: These add to give:

$$\sum_{n=0}^{\infty} (n + 2 + 3n - 2)(x^n) = \sum_{n=0}^{\infty} 4nx^n$$

Multiplying Power Series

It is also possible to multiply power series. If we consider the series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ with the radius of convergence $R_1 > 0$ and $\sum_{n=0}^{\infty} b_n(x - x_0)^n$ with the radius of convergence $R_2 > 0$, then:

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n \times \sum_{n=0}^{\infty} b_n(x - x_0)^n = \sum_{n=0}^{\infty} (c_n)(x - x_0)^n$$

Where c_n is given by:

$$a_0b_n + a_1b_{n-1} + a_2b_{n-2} + ... + a_nb_0$$

This new sum has the radius of convergence $R \geq min\{R_1, R_2\}$.

Differentiating Power Series

Finally it is possible to differentiate power series.

If we consider the series $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ which converges for $|x - x_0| < R$, where $R > 0$ then f is differentiable on $(x_0 - R,\, x_0 + R)$ and

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}$$

This means that for all values of x for which $f(x)$ converges, $f(x)$ is differentiable and can be differentiated by differentiating each term separately.
This idea is used to calculate means and help calculate variances of generation physics in probability and random processes.

Example: Differentiate the following series:

$$\sum_{n=0}^{\infty} (\frac{n^4 + 34}{11n}) x^n$$

Solution: This differentiates to give:

$$\sum_{n=0}^{\infty} n(\frac{n^4 + 34}{11n}) x^{n-1}$$

$$= \sum_{n=0}^{\infty} (\frac{n^4 + 34}{11}) x^{n-1}$$

11.4 Expansions

Order and Rate of Growth

A useful technique when approximating functions is to collect terms by their 'order' and eliminate terms that can be considered negligible. For example, if we have a function $f(x)$ which can be expanded to $f(x) = ax + bx^3 + cx^5$ and we a considering how the function behaves for small values of x (i.e. $x \approx 0$) then we can approximate the function by removing terms that do not contribute much. Since x is small, x^3 will be very small and x^5 will be incredibly small. Therefore we can remove the term cx^5 since it will not contribute to the total function. We may even wish to remove the term bx^3, depending on how small the values of x that we are considering are and how accurate we want our function to be. This technique is especially useful where we have an infinite series of terms and we wish to truncate it.

We say that higher powers of x are of higher order and lower powers of x are of lower order. We can also produce a hierarchy on rate of growth of different functions. In general we can express this as follows:

$$\boxed{c < x^s < \lambda^x < x!}$$

where c, s and λ are constants and we are considering the rate of growth as we increase the variable x. In other words, this means that, for example, $x!$ will always eventually beat an exponential term λ^x as x increases, and so on.

Binomial Expansions

In physics, when trying to describe the world around us with mathematics, the equations that are most accurate are often very complicated. To allow us to be able to simplify these equations we can use a binomial expansion. It is a way to eliminate terms of an equation that affect the result in such a small way that we can consider them negligible. The general binomial expansion for the integer n is:

$$\boxed{(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k}$$

The binomial co-efficient $\binom{n}{k}$ is defined as $\dfrac{n!}{k!(n-k)!}$. Therefore this expands to:

$$\boxed{(a+b)^n = \frac{n!}{n!0!}a^n + \frac{n!}{1!(n-1)!}a^{n-1}b + \frac{n!}{2!(n-2)!}a^{n-2}b^2 + ... + \frac{n!}{(n-1)!1!}a^2b^{n-1} + \frac{n!}{n!0!}b^n}$$

Note: $0! = 1$. If you are interested to find out why click here.
However in physics it is often helpful to use the fact that we can take out a factor of a from the bracketed term:

$$a\left(1+\frac{b}{a}\right)^n = a\left(\frac{n!}{n!0!}1^n + \frac{n!}{1!(n-1)!}1^{n-1}\left(\frac{b}{a}\right) + \frac{n!}{2!(n-2)!}1^{n-2}\left(\frac{b}{a}\right)^2 + ... + \frac{n!}{n!0!}\left(\frac{b}{a}\right)^n\right)$$

Simplifying to:

$$a\left(1+\frac{b}{a}\right)^n = a\left(1 + \frac{n!}{1!(n-1)!}\left(\frac{b}{a}\right) + \frac{n!}{2!(n-2)!}\left(\frac{b}{a}\right)^2 + ... + \left(\frac{b}{a}\right)^n\right)$$

$$= a\left(1 + n\left(\frac{b}{a}\right) + \frac{1}{2}n(n-1)\left(\frac{b}{a}\right)^2 + \frac{1}{6}n(n-1)(n-2)\left(\frac{b}{a}\right)^3 + ...\right)$$

The reason behind using this form of the expansion rather than the general form is that it allows us to use the physical relationship between the variables a and b to simplify the expression. The most common use of this is for when $b << a$ (b is much smaller than a) which allows us to confidently say that $\frac{b}{a}$ is very small, and each time the power is increased on this term it will become increasing small until after a certain point the influence of this term is negligible and we can approximate the equation by taking out all the negligible terms as discussed in the previous section on 'order'.

Generally we can use the formula:

$$(1+x)^n = 1 + nx + \frac{1}{2}n(n-1)x^2 + \frac{1}{6}n(n-1)(n-2)x^3 + ...$$

Physics Example: The electric field due to an electric dipole is described by:

$$E = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{(r-\frac{a}{2})^2} - \frac{q}{(r+\frac{a}{2})^2}\right)$$

Where q is the charge of each particle, a is the separation of the particles and r is the distance from the center of the dipoles. Simplify the expression for distances where $a << r$.

Solution: By taking out a factor of q, then putting both fractions over a common denominator we obtain :

$$E = \frac{q}{4\pi\varepsilon_0}\left(\frac{(r+\frac{a}{2})^2 - (r-\frac{a}{2})^2}{(r-\frac{a}{2})^2 \times (r+\frac{a}{2})^2}\right)$$

Expanding the top of the fraction gives:

$$E = \frac{q}{4\pi\varepsilon_0}\left(\frac{r^2 + ar + \frac{a^2}{4} - r^2 + ar - \frac{a^2}{4}}{(r-\frac{a}{2})^2 \times (r+\frac{a}{2})^2}\right)$$

$$= \frac{2arq}{4\pi\varepsilon_0}\left(\frac{1}{(r-\frac{a}{2})^2 \times (r+\frac{a}{2})^2}\right)$$

Notice this is equivalent to:

$$E = \frac{2arq}{4\pi\varepsilon_0}\left(\frac{1}{\left((r-\frac{a}{2}) \times (r+\frac{a}{2})\right)^2}\right)$$

Which is equivalent to:

$$E = \frac{2arq}{4\pi\varepsilon_0}\left(\frac{1}{(r^2-\frac{a^2}{4})^2}\right)$$

Consider the term:

$$\frac{1}{(r^2-\frac{a^2}{4})^2}$$

This is equivalent to:

$$r^{-4}\left(1-\frac{a^2}{4r^2}\right)^{-2}$$

Using a binomial expansion gives:

$$r^{-4}\left(1-\frac{a^2}{4r^2}\right)^{-2} = r^{-4}\left(1 - 2\left(\frac{a^2}{4r^2}\right) + 3\left(\frac{a^2}{4r^2}\right)^2 - 4\left(\frac{a^2}{4r^2}\right)^3 + ...\right)$$

As $a << r$ we can assume that $\frac{a^2}{4r^2}$ and the higher order terms, $\left(\frac{a^2}{4r^2}\right)^2$, $\left(\frac{a^2}{4r^2}\right)^3$, $\left(\frac{a^2}{4r^2}\right)^4$ etc, are negligible.

Therefore $r^{-4}\left(1-\frac{a^2}{4r^2}\right)^{-2}$ simplifies to just r^{-4}.

In turn this means that

$$E = \frac{2arq}{4\pi\varepsilon_0}\left(\frac{1}{(r^2-\frac{a^2}{4})^2}\right)$$

$$\approx \frac{2arq}{4\pi\varepsilon_0}\left(\frac{1}{r^4}\right)$$

$$= \frac{aq}{2\pi\varepsilon_0 r^3}$$

Therefore:

$$E = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{(r-\frac{a}{2})^2} - \frac{q}{(r+\frac{a}{2})^2}\right) \approx \frac{aq}{2\pi\varepsilon_0 r^3}$$

Using this simplified form of the equation saves a lot of time when calculating predictions for experiments done with electric dipoles.

Taylor Series

One of the most powerful techniques we can use in physics is Taylor series. This enables us to write many functions as an infinite power series. As in the previous section on the binomial expansion, we can then find an approximation of the function by considering small values of x and ignoring terms which are negligible. Just as mentioned in the section on power series, it is worth noting that many Taylor series of functions only work for a certain range of x values and so has a radius of convergence R. In the case where $x_0 = 0$, we call the expansion a Maclaurin series.

We can usually find a Taylor expansion for a function $f(x)$ using the following formula:

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n}$$

where $f^{(n)}(x_0)$ is the nth derivative of $f(x)$, evaluated at x_0.

By expanding out the sum, this can be written as:

$$\boxed{f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + ...}$$

This means that if we can differentiate a function multiple times then we can find its Taylor expansion. However, in some cases, calculating the Taylor series of a function using this formula can be very laborious and so it is often useful to know by heart the Taylor series for some simple functions, and use products of these to find Taylor series of more complicated function. Here are the expansions of some common functions:

- For all $x \in \mathbb{R}$,
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + ...$$
- For all $x \in \mathbb{R}$,
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + ... + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + ...$$
- For all $x \in \mathbb{R}$,
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ... + (-1)^n \frac{x^{2n}}{(2n)!} + ...$$
- We can use the binomial expansion to find the Taylor series for $(1 + x)^n$, however it only works for $|x| < 1$.
$$(1 + x)^n = 1 + nx + \frac{1}{2!}n(n - 1)x^2 + \frac{1}{3!}n(n - 1)(n - 2)x^3 + ...$$

Example:

1. Using the appropriate Taylor series, find e^2 up to the first six terms.
2. Using the general formula for the Taylor series of a function, prove the above Taylor series for $\cos(x)$ about $x = 0$ for the first three non-zero terms.
3. Find the first six terms of the Taylor series for e^{ix} and hence deduce Euler's formula: $e^{ix} = \cos(x) + i\sin(x)$.

Solution:

1. We use the above formula for e^x and substitute $x = 2$.

$$\begin{aligned} e^2 &= 1 + 2 + \frac{2^2}{2} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + ... \\ &= 1 + 2 + \frac{4}{2} + \frac{8}{6} + \frac{16}{24} + \frac{32}{120} + ... \\ &= 1 + 2 + 2 + \frac{4}{3} + \frac{2}{3} + \frac{4}{15} + ... \\ &\approx \frac{109}{15} \approx 7.27 \end{aligned}$$

Comparing this to the actual value of $e^2 \approx 7.39$, we see that our approximation is not far off and would converge to the actual value as we include more terms in our series.

2. Use the formula for the general Taylor series of a function. By differentiating $f(x) = \cos(x)$ we have:

$$\begin{aligned} f'(x) &= -\sin(x) \\ f''(x) &= -\cos(x) \\ f'''(x) &= \sin(x) \\ f''''(x) &= \cos(x) \end{aligned}$$

Then

$$\begin{aligned} \cos(x) &= \cos(0) - \sin(0)x - \frac{\cos(0)}{2}x^2 + \frac{\sin(0)}{3!} + \frac{\cos(0)}{4!} + ... \\ &= 1 - 0 + \frac{x^2}{2} + 0 + \frac{x^4}{4!} + ... = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + ... \end{aligned}$$

as required.

3. Using the Taylor series for e^x with $x = ix$, and by considering powers of i, we get:

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + ... \\ &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + ... \end{aligned}$$

Collecting the real and imaginary terms:

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ...\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...\right)$$

Now by comparing this with the Taylor series for $\sin(x)$ and $\cos(x)$, we have

$$e^{ix} = \cos(x) + i\sin(x)$$

Example: Find the power series of $e^{4x}\sin(x)$ up to and including x^4.

Solution: We know that:

$$e^x = 1 + 4x + \frac{16x^2}{2!} + \frac{64x^3}{3!} + ... + \frac{(4x)^n}{n!} + ...$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + ... + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + ...$$

Therefore:

$$e^{4x}\sin(x) = (1 + 4x + \frac{16x^2}{2!} + \frac{64x^3}{3!} + ... + \frac{(4x)^n}{n!} + ...)(x - \frac{x^3}{3!} + \frac{x^5}{5!} + ... + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + ...)$$

We can see that when we expand the brackets there will be no term including x^0 (or constant term).

We can also see that the only term that includes x^1 will be when the first term in each series is multiplied together, 1 and x. So the first term in our answer will be

$$x$$

To get a term including x^2 we can multiply together $4x$ with x. Therefore our second term in our answer is

$$4x^2$$

The term including x^3 can be found by multiplying 1 with $\frac{-x^3}{3!}$ as well as multiplying $\frac{16x^2}{2!}$ with x. Therefore the third term in our answer is

$$\frac{-x^3}{3!} + \frac{16x^3}{2!} = \frac{47x^3}{6}$$

To find the last term in our series we need to look how we will get terms including x^4. We will obtain this from multiplying the terms $4x$ with $\frac{-x^3}{3!}$ and $\frac{64x^3}{3!}$ with x. Therefore the fourth and final term we are looking for is

$$\frac{-2x^4}{3} + \frac{32x^4}{3} = 10x^4$$

Therefore:

$$e^{4x}\sin(x) = x + 4x^2 + \frac{47x^3}{6} + 10x^4$$

Example: What is the Taylor series up to and including the term of order x^3 for

$$\frac{e^x}{1-x}$$

if $|x| < 1$.

Solution: We know that the Taylor series for e^x is:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + ...$$

And that the Taylor series for $\frac{1}{1-x}$ is:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... + x^n + ...$$

Therefore

$$\frac{e^x}{1-x} = (1 + x + x^2 + x^3 + ... + x^n + ...)(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + ...)$$

Expanding this and collecting like terms gives:

$$\frac{e^x}{1-x} = 1 + 2x + \frac{5x^2}{2} + \frac{8x^3}{3} + ...$$

11.5 Review Questions

Easy Questions

Question 1: What are the first 4 terms of the sequence given by

$$a_n = (a_{n-1})^2 + 3$$

if:

- $a_1 = 4$,
- $a_1 = -2$,
- $a_1 = 0$?

Question 2: Find the sum of the finite series:

$$S = \sum_{n=0}^{4} n^3 + 2n + 1$$

Answer:

$$125$$

Question 3: Find the sum of the finite series:

$$S = \sum_{n=0}^{6} 2^n$$

Answer:

$$127$$

Question 4: Find the sum of the infinite series:

$$S = \sum_{n=0}^{\infty} 0.7^n$$

Answer:

$$\frac{10}{3}$$

Question 5: What is

$$S_N = \sum_{n=1}^{17} [3 + (n-1)2]$$

Answer:

$$323$$

Medium Questions

Question 6: Find first 4 terms of the binomial expansion of

$$(2+3x)^6$$

.

Answer:

$$64+576x+2160x^2+4320x^3$$

Question 7: Find the Taylor expansion of $\sin(x)$ and e^x using the formulae to find a Taylor expansion.

Answer:

$$\sin(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}+...+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+...$$

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+...+\frac{x^n}{n!}+...$$

Question 8: Find the first 4 terms of

$$e^x\sin(x)$$

using the Taylor expansions derived in the previous question.

Answer:

$$x+x^2+\frac{x^3}{3}-\frac{x^5}{30}$$

Hard Questions

Question 11: Use a Taylor expansion to prove that

$$\lim_{x\to 0}\frac{\sin(x)}{x}=1$$

12 Operators

12.1 Introduction to Operators

Operators are covered more extensively in the second year, however it is useful to have a basic understanding of what an operator is and why they are used. An operator takes its name from the fact that it is something that acts on a function as opposed to acting on numbers. In first year we are mainly concerned with Hermitian operators, as all physical observables (quantities) can be represented by a Hermitian operator in quantum mechanics. This is written as:

$$\hat{A}\Psi = a\Psi$$

The above equation is an example of an **eigenvalue equation**, where a is an eigenvalue of the operator $\hat{A}$ and Ψ is an eigenfunction. We denote operators with a 'hat', and eigenvalues are always real constants. An operator may have multiple eigenvalues, each representing a possible measurement of the physical observable in question. However, these eigenvalues are the **only** values that the observable can take for that particular wavefunction. If an operator acts on a wavefunction and does not return a constant multiplied by the wavefunction, then we cannot measure the observable represented by the operator for this wavefunction and the wavefunction is said to not be an eigenfunction of the observable. This is due to the Heisenberg Uncertainty Principle, which will not be covered here. Some examples of important operators in quantum mechanics are:

- The momentum operator (in the x-direction):

$$\hat{p_x} = i\hbar\frac{\partial}{\partial x}$$

- The kinetic energy operator:

$$\hat{T} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial^2 x}$$

- The total energy operator:

$$\hat{E} = i\hbar\frac{\partial}{\partial t}$$

So if we have a wave represented by a wavefunction, $\Psi(x)$, and we wish to find the kinetic energy of this wave, we would calculate

$$\hat{T}\Psi(x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial^2 x}\Psi(x)$$

If this returned an expression of the form $a\Psi$, then we know that the kinetic energy of this wave is a.

Physics Example:

1. Using the operators defined above, show that $\Psi(x,t) = A\sin(kx-\omega t)$ is an eigenfunction of kinetic energy and find the eigenvalue.

2. Using the operators defined above, show that $\Psi(x,t) = Ae^{-i(kx-\omega t)}$ is an eigenfunction of momentum (in the x-direction) and find the eigenvalue.

Solution:

1. We are required to calculate

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial^2 x}\Psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial^2 x}A\sin(kx-\omega t)$$

$$= -\frac{\hbar^2}{2m}Ak\cos(kx-\omega t)$$

$$= \frac{\hbar^2}{2m}Ak^2\sin(kx-\omega t)$$

$$= \frac{\hbar^2 k^2}{2m}\Psi(x,t)$$

Therefore $\Psi(x,t)$ is an eigenfunction of kinetic energy with eigenvalue $\frac{\hbar^2k^2}{2m}$. The physical significance of this is that if we measure the kinetic energy of a wave with the given wavefunction, it will have a kinetic energy of $= \frac{\hbar^2k^2}{2m}$.

2. We are required to calculate

$$i\hbar\frac{\partial}{\partial x}\Psi(x,t) = i\hbar\frac{\partial}{\partial x}Ae^{-i(kx-\omega t)}$$
$$= i\hbar \times -ikAe^{-i(kx-\omega t)}$$
$$= -(i)^2\hbar kAe^{-i(kx-\omega t)}$$
$$= \hbar kAe^{-i(kx-\omega t)}$$
$$= \hbar k\Psi(x,t)$$

Therefore $\Psi(x,t)$ is an eigenfunction of momentum with eigenvalue $\hbar k$. The physical significance of this is that if we measure the momentum of a wave with the given wavefunction, it will have a momentum of $\hbar k$.

12.2 Review Questions

? **Question 1:** What will be obtained if the operator

$$\hat{O} = 2x\frac{d}{dx}$$

acts on the function

$$f(x) = 3\cos(x^2).$$

Answer:

$$-12x^2\sin(x^2)$$

? **Question 2:** If

$$\hat{P} = 13\cos(x)\frac{d}{dx}$$

and

$$f(x) = 4x^2$$

what is $\hat{P}^2 f(x)$?

Answer:

$$\hat{P}^2 f(x) = 1352\cos^2(x)$$

? **Question 3:** If

$$\hat{Q} = \frac{d}{dx}$$

what is the general form of the eigenfunctions for this operator?

Hint: What differentiates to give itself?

13 Mechanics

In many mechanics questions it can be instructive to draw diagrams of the scenario in order to visualise what is happening and avoid making mistakes. The examples in this booklet do not include diagrams as being able to draw a diagram from a piece of text is a useful skill that comes with practice.

13.1 Dimensional Analysis

Physicists measure quantities in different units which depends on what we are measuring, for example we measure distance in metres and time in seconds. In physics, we often use a large variety of equations which provide relationships between physical quantities and we can use dimensional analysis to check that these equations make sense physically. Physical quantities have units (such as metres) and dimensions which tell us what sort of quantity we are measuring (for example length or time). Below is a table of some typical quantities in physics with their units and dimensions.

Quantity	Units	Dimensions
Distance (s)	**metres (m)**	L **(length)**
Time (t)	**seconds (s)**	T **(time)**
Mass (m)	**kilograms (kg)**	M **(mass)**
Velocity (v)	$\mathbf{ms^{-1}}$	LT^{-1}
Acceleration (a)	$\mathbf{ms^{-2}}$	LT^{-2}
Force (F)	**Newtons (N) or** $\mathbf{kg\,ms^{-2}}$**)**	MLT^{-2}
Density (ρ)	$\mathbf{kg\,m^{-3}}$	ML^{-3}
Viscosity (μ)	$\mathbf{kg\,m^{-1}}s^{-1}$	$ML^{-1}T^{-1}$

Occasionally we may wish to derive or come up with our own equation for a particular scenario however it can be difficult to check if an equation really does represent the correct scenario, and so it is useful to have a means of checking that the equation makes physical sense. One method which we can use to do this is dimensional analysis, which uses the idea that any physical equation must be dimensionally correct, in other words it must have the same units on both sides. For example, $v = u + at$ is dimensionally correct, since the dimensions on both sides are ms^{-1} and so are the same, however the equation $v = a$ has dimensions of velocity, ms^{-1}, on the left but units of acceleration, ms^{-2} on the right hand side, therefore this second equation makes no sense and cannot be physically correct.

We can check that an equation is dimensionally correct by writing the equation in terms of its dimensions and checking that everything matches up. For example, we could write $v = u + at$ as

$$[LT^{-1}] = [LT^{-1}] + [LT^{-2}][T]$$

where we write the dimensions of each variable in square brackets. By usual algebra we can obtain

$$[LT^{-1}] = 2[LT^{-1}]$$

however since we are only considering dimensions, we can ignore dimensionless constants and so the equation is dimensionally correct. A dimensionless quantity has no units and so we can write its dimensions as [1]. Examples of dimensionless quantities are π, amount of substance (mol) and angles measured in radians (rad).

This technique can tell us if an equation does not make physical sense, however it **cannot** necessarily confirm that the equation is correct for the scenario. This is because there are often multiple ways for an equation to be dimensionally correct and because dimensional analysis does not take into account constants.

We can use dimensional analysis to determine an estimate for a relation between some quantities by considering the dimensions of the quantities.

Physics Example: A student predicts that the drag force F experienced by a sphere of radius R when immersed in a fluid of viscosity μ and travelling at a velocity v is:

$$F = k\mu R^2 v$$

where k is a dimensionless constant. Check whether this equation is correct or not. If it is incorrect, find a relation between these variables which is dimensionally correct

Solution: Write the equation in terms of its dimensions

$$[MLT^{-2}] = [1][ML^{-1}T^{-1}][L^2][LT^{-1}]$$

Now use ordinary multiplication to simplify the right hand side and check if it is equal to the left hand side.

$$\begin{aligned} RHS &= [ML^{-1}T^{-1}][L^3T^{-1}] \\ &= [ML^2T^{-2}] \end{aligned}$$

But

$$[MLT^{-2}] \neq [ML^2T^{-2}]$$

so the equation is not dimensionally correct. By simply consideration of the dimensions of the quantities, we can easily see that we can correct the equation by reducing the power of R by 1. Thus our correct equation is

$$F = k\mu Rv$$

which we can check has the correct dimensions in a similar way to above.

We can also use dimensional analysis to find the dimensions of a quantity, given an equation for which we know the dimensions of the other quantities.

Physics Example: The following equation for the gravitational attractive force is dimensionally correct:

$$F = -\frac{Gm_1m_2}{r^2}$$

Find the dimensions of the constant G.

Solution: We can rearrange the equation to make G the subject and then write the equation in terms of its dimensions whilst ignoring scalar multiples (in this case the '-1'):

$$G = \frac{Fr^2}{m_1m_2}$$

$$G = \frac{[MLT^{-2}][L^2]}{[M][M]}$$

We then use normal multiplication to simplify the dimensions

$$= [ML^3T^{-2}][M^{-2}]$$

$$= [M^{-1}L^3T^{-2}]$$

Thus the dimensions of G are $\mathrm{kg}^{-1}\mathrm{m}^3\mathrm{s}^{-2}$

Physics Example: A researcher expects that the frequency of oscillation ω of a star depends on the gravitational constant G, the density ρ of the star and the radius R of the star R.

Solution: We can write this above statement as

$$\omega = kG^a\rho^b R^c$$

where k is a dimensionless constant and a, b and c are constant powers of the quantities which we need to find. We now write our equation in terms of dimensions

$$\begin{aligned}[T^{-1}] &= [1][M^{-1}L^3T^{-2}]^a[ML^{-3}]^b[L]^c \\ &= M^{-a+b}L^{3a-3b+c}T^{-2a}\end{aligned}$$

Then we compare the powers of each of our quantities to get a set of simultaneous equations:

$$\begin{aligned}-a+b &= 0 \\ 3a-3b+c &= 0 \\ -2a &= -1\end{aligned}$$

Which gives

$$\begin{aligned}a &= \tfrac{1}{2} \\ b &= \tfrac{1}{2} \\ c &= 0\end{aligned}$$

Now substituting these values in, we find that a dimensionally correct equation is

$$\omega = k\sqrt{G\rho}$$

Notice that the frequency of oscillation does not depend on the radius of the star!

13.2 Kinematics

The basic ideas of kinematics evolve from the concepts of velocity and acceleration. In mathematics, average velocity is defined as

$$v = \frac{s(t_2) - s(t_1)}{t_2 - t_2}$$

where t_1 is the initial time, t_2 is the final time, s is the displacement from the origin and v is the velocity. Both velocity v and displacement s are vectors, whereas their magnitudes are called speed and distance respectively and these are scalar quantities. By taking the limit of this as $t_2 \to t_1$ the instantaneous velocity is obtained and is defined as

$$v = \frac{ds}{dt}$$

or the rate of change of displacement with respect to time. Similarly acceleration is defined as

$$a = \frac{dv}{dt}$$

or the rate of change of velocity with respect to time. Combining both of these definitions together gives:

$$a = \frac{dv}{dt} = \frac{d}{dt}\frac{ds}{dt} = \frac{d^2s}{dt^2}$$

For a body travelling at velocity v and not accelerating you will have been taught that $v = \frac{s}{t}$. When considering the same body but with a constant acceleration a, this equation still holds but using the average velocity during the movement, which yields the equation

$$\boxed{s = \frac{(u+v)}{2}t}$$

Keeping the assumption that acceleration is constant and using the definitions above it is true to say

$$\int \frac{d^2s}{dt^2}\,dt = \int a\,dt$$

$$\Rightarrow v = at + C$$

If at $t = 0$ the motion we are describing has initial velocity u this implies

$$\boxed{v = u + at}$$

Furthermore:

$$\int v\,dt = \int \frac{ds}{dt}\,dt = \int u + a\,dt$$

$$\Rightarrow s = ut + \frac{1}{2}at^2 + C$$

If it is assumed that $s = 0$ and $t = 0$ then it follows $C = 0$, so:

$$\boxed{s = ut + \tfrac{1}{2}at^2}$$

Finally by combining both of these results one more useful equation falls out. By solving for t in both equations, substituting one into the other and re-arranging the result we arrive at:

$$\boxed{v^2 = u^2 + 2as}$$

The equations in boxes are often called SUVAT equations, and they are invaluable in mechanics problems where there is a constant acceleration.

Example: A particle is dropped from rest at a height 5 metres from the floor, with what speed will the particle hit the floor?

Solution: The information we know about when the ball hits the floor is (taking downwards to be positive):

$$s = 5\text{m}$$
$$u = 0\text{ms}^{-1}$$
$$a = 9.81\text{ms}^{-2}$$

Using that

$$v^2 = u^2 + 2as$$

And substituting what we know gives

$$v^2 = 0^2 + 2 \times 9.81 \times 5$$

$$\Rightarrow v = \sqrt{98.1} = 9.90\text{ms}^{-1}\ (3sf)$$

Example: A stunt car drives up a ramp angled at 30° to the horizontal. The car leaves the top of the ramp, at a height 10m from the ground, at 30ms^{-1}. What will be the greatest height the car reaches?

Solution: At the top of the car's motion the vertical velocity will be zero since it will stop moving upwards and begin to fall. Using this we can deduce the height we want. The information we know about the vertical motion of the car at its highest point in its motion is (taking upwards to be positive):

$$v = 0\text{ms}^{-1}$$
$$u = 30\sin(30) = 15\text{ms}^{-1}$$
$$a = -9.81\text{ms}^{-2}$$

Using that

$$v^2 = u^2 + 2as$$

and substituting what we know gives

$$0^2 = 15^2 + 2 \times -9.81 \times s$$

$$\Rightarrow s = 11.5\text{m}\ (3sf)$$

However this is the height above the ramp and not the total height we want. So our final answer is:

$$10 + 11.5 = 21.5\text{m}$$

Example: A parkour runner jumps horizontally off a building at $v = 3\text{ms}^{-1}$. If the height of the building is 2.5 meters, how far will the parkour runner land from the base of the building?

Solution: If we consider the vertical motion first we can work out the how long the parkour runner is in the air for and use this with horizontal motion to find the distance in question. The information we know the runner's vertical motion at the point he hits the floor is (taking downwards to be positive):

$$s = 2.5\text{m}$$

$$u = 0\text{ms}^{-1}$$

$$a = 9.81\text{ms}^{-2}$$

As we are trying to work out the time of the jump we will use

$$s = ut + \frac{1}{2}at^2$$

Substituting what we know gives

$$2.5 = 0t + \frac{1}{2} \times 9.81 \times t^2$$

$$\Rightarrow t = \sqrt{\frac{5}{9.81}} = 0.713\text{s } (3sf)$$

Now that we know the time of the flight we know enough about the horizontal at the point he hits the floor to work out the distance we want.

$$u = 3\text{ms}^{-1}$$

$$a = 0\text{ms}^{-2}$$

$$t = 0.713\text{s}$$

Using

$$s = ut + \frac{1}{2}at^2$$

We can deduce that

$$s = 3 \times 0.713 + \frac{1}{2} \times 0 \times 0.713^2$$

$$\Rightarrow s = 2.14\text{m } (3sf)$$

13.3 Newton's Laws

Newton's Laws of Motion

In order to understand what makes an object move, we consider Newton's laws of motion, which introduce the concept of force and provide some very important relations between different forces and acceleration. Force (usually denoted as F) is a vector quantity and is measured in Newtons, N.

The First Law

Newton's first law of motion states that

A body will remain at rest, or continue to move at constant velocity, unless an external unbalanced force acts on it.

This means that a force acting on a body is what causes the body to move. The law also explains that a force causes a change in velocity and does not sustain a velocity, as Aristotle wrongly deduced. By unbalanced force, we mean there is a resultant force acting on the object that is not zero.

The Second Law

Newton's second law of motion describes the relationship between the mass m of a body, the acceleration a of the body and the resultant force F acting on the body as

$$\boxed{F = ma}$$

This is a vector equation (i.e. $\underline{F} = m\underline{a}$) and since mass is a scalar and force and acceleration are vectors, this means that the acceleration of a body is in the same direction as the force acting on it. The resultant force acting on a body is the vector sum of all the individual forces acting on it. For example if we had an object being pushed in two opposite directions with the same magnitude of force, the resultant force on the object would be zero, even though there are two forces acting on it.

The Third Law

Finally, Newton's third law of motion states that

For every action there is an equal and opposite reaction

meaning that if a body pushes against another object and exerts a force on it, the object will exert a force of equal magnitude against the body, in the opposite direction to the original force.

Gravity

In mechanics, the force due to gravity is called the **weight** which we denote as W. Using Newton's second law we can obtain the relation

$$\boxed{W = mg}$$

where we call g the acceleration due to gravity and is the acceleration of an object due to it's weight. On Earth, g generally takes the value $g = 9.81\text{ms}^{-1}$ however on other planets and at extreme heights on Earth, g can take other values. It is important to understand the difference between mass and weight. Mass, measured in kg, is a quantity representing 'how much stuff' a body contains and is constant regardless of where the object is. On the other hand, weight is a force which depends on the mass of a body and the strength of the gravitational field which the body is in. For example, an object on the Moon would have the same mass as an identical object on Earth, but its weight would be less since the Moon's gravitational field is weaker. Newton also formulated another law which gives the force due to gravity which acts between two masses m_1 and m_2. Newton's law of gravitation states that

$$\boxed{F = -\frac{Gm_1m_2}{r^2}}$$

where F is the force acting on each mass (it acts on both masses but in opposite directions due to Newton's third law), r is the distance between the masses and G is called the gravitational constant with the value $G = 6.67 \times 10^{-23}$. Note that the minus sign indicates that the force is **attractive** rather than repulsive.

Friction

Aristotle theorised that a force acting on a body causes a body to maintain its velocity and accelerate when the force is increased. Newton proved this to be incorrect and stated in his first law of motion that an external force causes a body to change its velocity, and a body will maintain its velocity without the presence of a force. Aristotle's approach initially appears to agree with physical intuition. Consider a ball rolling along a carpet; unless we continuously push it along, it will slow down, whereas Newton's law suggests that it should keep moving at a steady pace without anything helping it along. The reason for this is that there is a **decelerating force** due to the ball getting slowed by the tiny ridges and bumps in the carpet. We call this force **friction**, and the strength of this force depends on the smoothness of the surface the object is moving across, and the reaction force acting on the object. Thus the inequality for the force due to friction is

$$\boxed{F_N \leq \mu R}$$

where μ is called the coefficient of friction of the surface and is a constant between zero and one. R is the reaction force described in Newton's third law which acts perpendicular away from the surface and F_N is the force due to friction and is a vector which acts to oppose the movement of the object.

Example: A particle of mass 3kg is projected up a rough slope, inclined at 30° to the horizontal, with velocity 3ms^{-1}. If the coefficient of friction between the particle and the slope is 0.7, what distance s up the slope does the particle travel?

Solution: As the particle is in motion the friction takes its maximum value.

$$F_N = \mu R$$

Finding the resultant force up the slope and using Newton's second law of motion will allow us to find the acceleration, a, of the particle. The resultant force up the slope is:

$$-3g\sin(30) - 0.7 \times 3g\cos(30) = -32.6N \ (3sf)$$

Thus

$$-32.6 = ma = 3a$$

$$\Rightarrow a = -10.9\text{ms}^{-1} \ (3sf)$$

Using $v^2 = u^2 + 2as$ and the information we have we obtain

$$0^2 = 3^2 + 2 \times -10.9 \times s$$

Therefore

$$s = 4.12\text{m} \ (3sf)$$

Example: A box of mass 5kg lies on a rough slope inclined at an angle θ to the horizontal. If the coefficient of friction between the particle and the slope is 0.5, What is the greatest value of θ for which the box will not slip down the slope?

Solution: At the point of slipping the friction acting on the box takes its maximum value.

$$F_N = \mu R$$

Resolving forces acting parallel to the slope gives:

$$-5g\sin(\theta) + 0.5 \times 5g\cos(\theta) = 0 \ (3sf)$$

Thus

$$\tan(\theta) = 0.5$$

$$\Rightarrow \theta = 26.6° \ (3sf)$$

Air Resistance

In all the examples when using the SUVAT equations air resistance was ignored, this was due to the assumption that acceleration was constant. When including air resistance this is no longer the case as this resistance is proportional to velocity, the two simple cases we will consider are either directly proportional or proportional to the square of the velocity. In reality it can be very hard to obtain an equation for the air resistance. To solve these types of problems it is best to use Newton's second law of motion.

Case 1: An air resistance of $R = mkv$
Consider a particle of mass m falling in the Earth's gravitational field. There is an air resistance of magnitude $R = mkv$ where v is the particle's velocity and k is a constant. Find an equation for the velocity of the particle in terms of time, $v(t)$.

Resolving the forces vertically, taking downwards to be positive, we obtain that the resultant force is $mg - mkv$ and therefore by Newton's second law of motion we obtain:

$$mg - mkv = ma$$

$$g - kv = a$$

$$\frac{dv}{dt} = g - kv$$

$$\int \frac{1}{g - kv}\, dv = \int dt$$

$$-\frac{1}{k}\ln(g - kv) = t + C$$

If we assume that $v = 0$ at $t = 0$ then $C = \frac{-1}{k}\ln(g)$
Substituting this in we get:

$$-\frac{1}{k}\ln(g - kv) = t - \frac{1}{k}\ln(g)$$

And upon rearranging we obtain:

$$v = \frac{g}{k}(1 - e^{-kt})$$

It is worth mentioning that in this equation as t tends to infinity the velocity of the particle tends $\frac{g}{k}$, which is the terminal velocity of the particle, the maximum speed the particle will achieve when falling. This occurs due to the air resistance increasing until it equals the weight of the particle and by Newton's first law of motion it will no longer accelerate and will continue with constant velocity.

Case 2: An air resistance of $R = mkv^2$
Consider the same particle of mass m falling in the Earth's gravitational field. This time there is an air resistance of magnitude $R = mkv^2$ acting on it, where v is the particle's velocity and k is a constant. Find an equation for the velocity of the particle in terms of time, $v(t)$.

Resolving the forces vertically taking downward to be positive we obtain that the resultant force is $mg - mkv^2$ and therefore by Newton's second law of motion we can say:

$$mg - mkv^2 = ma$$

$$g - kv^2 = a$$

$$\frac{dv}{dt} = g - kv^2$$

$$\int \frac{1}{g - kv^2}\, dv = \int dt$$

Rearranging this gives:

$$\frac{1}{k}\int \frac{1}{\frac{g}{k} - v^2}\, dv = \int dt$$

Therefore
$$\frac{1}{k}\sqrt{\frac{k}{g}}\,\mathrm{artanh}\left(v\sqrt{\frac{k}{g}}\right) = t + C$$
If we assume that $v = 0$ at $t = 0$ then $C = 0$
$$\frac{1}{k}\sqrt{\frac{k}{g}}\,\mathrm{artanh}\left(v\sqrt{\frac{k}{g}}\right) = t$$
Upon rearranging we obtain
$$v = \sqrt{\frac{g}{k}}\left(\frac{1 - e^{-2\sqrt{kg}t}}{1 + e^{-2\sqrt{kg}t}}\right)$$
If we let t tend to infinity to find the terminal velocity, the equation yields $v = \sqrt{\frac{g}{k}}$.

13.4 Conservation Laws

In physics, there are some quantities which are **conserved**, meaning that there is a constant amount of them in any closed system. A closed system is a system in which no external forces or changes affect the system, and so it can be thought of as 'cut off' from the rest of the universe. Two examples of 'conservation laws' will be considered here, which will enable us to solve physical problems.

Conservation of Momentum

Momentum can be thought of as 'mass in motion' or the 'unstoppability' of a body and is defined by the formula

$$\boxed{p = mv}$$

where p is the momentum, m is the mass of the body and v is the velocity of the body. A body which is at rest has zero momentum.

We can also define the impulse of a force as the force multiplied by the time over which the force acts. This is equivalent to the change in momentum of the object the force is acting on. We can write this as an equation:

$$\boxed{I = Ft = \Delta p}$$

Conservation of momentum states that the total momentum before an event is equal to the total momentum after an event, this idea is specifically useful when dealing with questions on collisions or explosions. In this booklet, we will only consider collisions of two objects.

Note: Momentum is a vector quantity.

Example: Two particles, A and B, collide. Before the collision A was travelling with 3ms^{-1} and B was travelling with 5ms^{-1} in the opposite direction. After the collision, A travels in the same direction as before the collision but with speed 1ms^{-1} and B's direction of motion has reversed and it still travels with 5ms^{-1}. If B has a mass of 3kg, what is the mass, m, of A?

Solution: Taking the direction of A's motion to be positive we get the initial momentum of the system as:

$$m \times 3 - 3 \times 5 = 3m - 15$$

The final momentum of the system is:

$$m \times 1 + 3 \times 5 = m + 15$$

Using conservation of momentum we obtain:

$$3m - 15 = m + 15$$

$$\Rightarrow m = 15\text{kg}$$

Example: Two particles, L_1 and L_2 have a head on collision. Particle L_1 was travelling at 30ms^{-1} just before the collision and had a mass 4kg, L_2 was travelling at 15ms^{-1} in the opposite direction to L_1 just before the collision and had a mass 7kg. After the collision the two particles coalesce, what is the speed of the combined two particles, v? Which direction are they moving in after the collision?

Solution: Taking the original direction of L_1's motion to be positive we have that the initial momentum of the system is:

$$30 \times 4 - 15 \times 7 = 15\text{kg ms}^{-1}$$

The final momentum of the system is:

$$(4 + 7) \times v = 15\text{kg ms}^{-1}$$

Therefore

$$v = \frac{15}{11} = 1.36\text{ms}^{-1}\ (3sf)$$

As the velocity is positive we know that the direction of the coalesced particles is in the same direction as L_1's original motion.

An **elastic** collision is one in which there is no loss of kinetic energy. In physics, the vast majority of collisions are inelastic, meaning that some kinetic energy is lost during the collision (however we can often model a collision as being elastic for simplification). In order to do calculations for these inelastic collisions, we must know the **coefficient of restitution**, e, of the collision. This is a property which depends on the materials of both objects in the collision and is defined as:

$$\boxed{e = \frac{\text{speed of separation of objects}}{\text{speed of approach of objects}} = \frac{v_b - v_a}{u_a - u_b}}$$

where u_a and u_b are the velocities of the two particles A and B before the collision, and v_a and v_b are the velocities of the two particles A and B after the collision. For elastic collisions, $e = 1$.

Example: A particle of mass 2kg is moving at a constant velocity of 5ms^{-2}. It collides with a wall and bounces back in the opposite direction to its original motion. If the coefficient of restitution of the collision is $e = 0.7$, with what speed, v, does the ball bounce back with? What is the impulse that acts on the particle?

Solution: As we know

$$e = \frac{\text{speed of separation of the particles}}{\text{speed of approach of particles}}$$

In this example the wall can be considered to be a particle that doesn't move before or after the collision. So the equation becomes (taking the direction of the particle's original motion as positive):

$$0.7 = \frac{0 - (v)}{5 - 0}$$

Therefore

$$v = -(0.7 \times 5) = -3.5\text{ms}^{-1}$$

So the velocity of the particle after the collision is 3.5ms^{-1} in the negative direction. To find the impulse that acts on the particle we need to find the change in momentum, i.e.

$$I = \text{final momentum} - \text{initial momentum}$$

$$I = (2 \times -3.5) - (2 \times 5) = -17\text{kg ms}^{-1}$$

Therefore the impulse that acts on the particle is 17kg ms^{-1} in the negative direction.

Example: A particle A of mass 6kg is travelling in the positive direction with velocity 5ms^{-1} when it collides with a particle B of mass m, travelling in the opposite direction with velocity 2ms^{-1}. After the collision, both particles are travelling in the positive direction, A with velocity v and B with velocity 4ms^{-1}. If the coefficient between the two particles is $e = 0.4$, what is the mass of B?

Solution: In order to use the conservation of momentum to find the mass of B, we need to know all the initial and final velocities of the two particles. To find the final velocity of A, v, we can use restitution:

$$e = \frac{v_b - v_a}{u_a - u_b}$$
$$\Rightarrow 0.4 = \frac{4 - v}{5 - (-2)}$$
$$\Rightarrow v = 1.2\text{ms}^{-1}$$

Now we can use the conservation of momentum to find the mass of B

$$6 \times 5 - m \times 2 = 6 \times 1.2 + m \times 4$$

$$\Rightarrow m(2 + 4) = 6(5 + 1.2)$$
$$\Rightarrow m = 6.2\text{kg}$$

Conservation of Energy

If a body applies a constant force to an object, the work done by the body on the object is defined as the force the body applies to the object, multiplied by the distance travelled by the object in the direction of the force whilst the force is being applied. As an equation, this is written as

$$\boxed{W = Fd}$$

Energy is a measure of an object's ability to do work, measured in joules (J), and it can be transferred to other objects or converted into different forms. Energy cannot be created or destroyed. There are many types of energy however the two main types we will consider are kinetic and potential energy. Kinetic energy is the energy a body possesses by virtue of its motion. The equation we use to quantify it is:

$$\boxed{\text{KE} = \tfrac{1}{2}mv^2}$$

where KE is kinetic energy, m is mass and v is velocity.
In mechanics we often encounter two main types of potential energy: elastic potential energy and gravitational potential energy. Elastic potential energy is the stored energy in a stretched elastic material, this energy is released when the elastic material is allowed to return to it's original shape. The equation we use to quantify this type of energy is:

$$\boxed{\text{EPE} = \tfrac{1}{2}kx^2}$$

where EPE is elastic potential energy, k is the spring constant of the extension material and x is the extension of the material.
Gravitational potential energy is the stored energy in a body due to its position in a gravitational field. When quantifying gravitational potential energy we consider the energy between two points. To do this we set an appropriate reference height at which we say that the gravitational potential energy is zero and calculate the gravitational potential energy relative to this height. For example, we often choose the Earth's surface as the reference height and calculate the gravitational potential energy at a distance h above the surface.

$$\boxed{GPE = mgh}$$

where GPE is gravitational potential energy, m is the mass of the body, g is the acceleration due to gravity at that point and h is the height above the reference point.

Note: Normally all types of potential energy can just be referred to as PE, however to distinguish between them clearly in this booklet we will use EPE for elastic potential energy and GPE for gravitational potential energy.

Conservation of energy allows us to state that in a closed system the total energy before an event (the initial energy) equals the total energy after the event (the final energy). This idea provides an alternative approach to solving many problems in mechanics.

Note: Energy is a scalar quantity.

Example: A ball of mass 1kg is thrown vertically upwards from the ground with speed 10ms^{-1}. What is the maximum height the ball reaches?

Solution: We set the reference height as the ground level at which we state the ball has no gravitational potential energy. This allows us to say that at this point the ball has only kinetic energy at the moment it's thrown, the magnitude of which is:

$$KE = \frac{1}{2}mv^2 = \frac{1}{2} \times 1 \times 10^2 = 50\text{J}$$

When the ball is at its maximum height during its motion, h say, the velocity of the ball is 0ms^{-1}. Therefore it's energy is purely gravitational potential energy with magnitude:

$$\text{GPE} = mgh = 1 \times 9.81 \times h = 9.81h$$

From conservation of energy we can say that the kinetic energy the ball was thrown with is equal to the gravitational potential energy it has at the maximum height of it's motion, i.e.

$$50 = 9.81h$$

Therefore the height we are trying to work out, h, is:

$$h = \frac{50}{9.81} = 5.10\text{m } (3sf)$$

Example: A man pushes a wagon, of mass 10kg from rest up a slope inclined 60° to the horizontal. After the man has pushed the wagon 10m, they are travelling at a velocity of 3ms^{-1}. What is the work done by the man in pushing the wagon 10m up the slope?

Solution: If we set our reference height (at which we consider the gravitational potential energy to be zero) as the man and wagon's starting point, we can see that the initial energy of the system is zero; this is because they 'start from rest'. However the final energy (the energy when they are 10m up the slope) is not zero as the man and wagon now have kinetic energy (they are now moving) as well as gravitational potential energy (their vertical height has increased). The increase in energy of the wagon is equal to the work done by the man.

The increase in kinetic energy of the wagon is:

$$\frac{1}{2}mv^2 = \frac{1}{2} \times 10 \times 3^2 - 0 = 45\text{J}$$

The increase in gravitational potential energy of the wagon is:

$$mgh = 10 \times 9.81 \times 10\sin(60^\circ) = 85.0\text{J}$$

Note: The height used to work out the gravitational potential energy is the vertical height the man travels only.

Therefore the work done by the man is:

$$45 + 85.0 = 130\text{J}$$

Note: We used $10\sin(60^\circ)$ for h when calculating the final gravitational potential energy as this is the *vertical height* that the wagon rose.

13.5 Oscillatory Motion

Definitons and Concepts

Oscillatory (or periodic) motion occurs often in physics, in particular in areas such as astrophysics and mechanics. In order to effectively study periodic motion, we will need to identity and define some new quantities:

- The **period** (T) is the time required to complete a full cycle, measured in seconds (s).
- The **frequency** (f) is the number of cycles completed in one second. Frequency is measured in Hertz (Hz) and is related to the period by

$$\boxed{f = \frac{1}{T}}$$

- The **displacement** (x) is the vector distance moved from equilibrium and is measured in metres (m). It is either positive or negative, depending on its direction from equilibrium.
- The **amplitude** (A) is the maximum distance from equilibrium and is measured in metres (m).
- The **angular frequency** (ω) is a measure of the rate of rotation of the cycle. It is measured in radians per second (rad s^{-1}) and is related to frequency and period by

$$\boxed{\omega = 2\pi f = \frac{2\pi}{T}}$$

Simple Harmonic Motion

One of the most simple types of oscillatory motion occurs when there is a 'restoring force' acting which is directly proportional to the displacement. This type of oscillation is called **simple harmonic motion** (SHM) and an oscillator which undergoes SHM is called a **harmonic oscillator**. In the case of a particle attached to a spring, this occurs when the spring obeys Hooke's law. If the spring has spring constant k, then the restoring force acting on the particle is

$$\boxed{F = -kx}$$

where there is a minus sign because the restoring force acts in the opposite direction to the displacement from the origin, towards the equilibrium position.

From Newton's second law of motion, we can then state that the acceleration also acts to oppose the displacement, and is given by

$$a = -\frac{k}{m}x$$

By consideration of dimensional analysis, we can define the angular frequency as

$$\boxed{\omega = \sqrt{\frac{k}{m}}}$$

We know that acceleration is the second derivative of displacement with respect to time, so we can write the above as

$$\boxed{\ddot{x} = -\omega^2 x}$$

where $\ddot{x}$ is a short notation for $\frac{d^2x}{dt^2}$ (equally $\dot{x} = \frac{dx}{dt}$). An object whose motion is described by this equation is undergoing simple harmonic motion. Now we can find an equation for the displacement x of the oscillator by solving this differential equation. We can use the methods for solving second order ODEs as described in a previous section, or we can simply appeal to our knowledge of derivatives to deduce that a solution to this ODE is

$$\boxed{x = A\cos(\omega t + \phi)}$$

where the oscillator begins its motion at the furthest point from the origin. This then gives:

$$\dot{x} = -A\omega \sin(\omega t + \phi)$$

$$\ddot{x} = -A\omega^2 \cos(\omega t + \phi) = -\omega^2 x$$

which satisfies the differential equation. The constant ϕ represents the **phase angle** of the oscillator which tells us at what point in the cycle the motion was at $t = 0$.

Example: A particle moves along the x-axis and its displacement from equilibrium is described by

$$x = \sin\left(\frac{\pi}{3}t\right)$$

1. Show that the particle is a simple harmonic oscillator.
2. Find the period and amplitude of the particle's motion.
3. Find the maximum velocity of the particle during its motion.

Solution:

1. We need to show that the particle obeys the equation

$$\ddot{x} = -\omega^2 x$$

We differentiate our equation for x twice to find $\ddot{x}$.

$$\dot{x} = 5 \times \frac{\pi}{3}\cos(\frac{\pi}{3}t)$$

$$\ddot{x} = -5 \times \left(\frac{\pi}{3}\right)^2 \sin(\frac{\pi}{3}t)$$

$$= -\left(\frac{\pi}{3}\right)^2 x$$

Hence the equation for simple harmonic motion is satisfied and $\omega = \frac{\pi}{3}$.

2. The amplitude is the maximum displacement which we can find from maximising $\sin\left(\frac{\pi}{3}t\right)$. The maximum value of sin(..) is 1 and so the amplitude is 5m. The period is found using the formula $\omega = \frac{2\pi}{T}$. Thus

$$T = \frac{2\pi}{\frac{\pi}{3}} = 6\text{s}$$

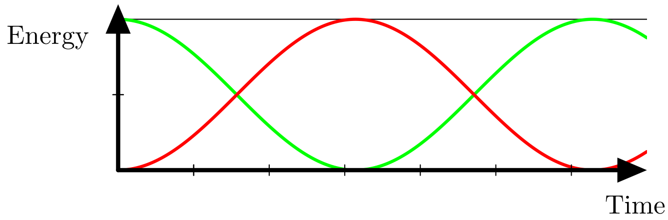
3. The velocity of the particle is calculated in the first part of this question and is

$$v = \dot{x} = 5 \times \frac{\pi}{3}\cos(\frac{\pi}{3}t)$$

Using a similar technique to how we found the amplitude, the maximum velocity is when $\cos(\frac{\pi}{3}t) = 1$, and hence the maximum velocity is

$$v_{max} = \frac{5\pi}{3}$$

Another note of interest is that the energy of a simple harmonic oscillator continuously changes between kinetic energy and potential energy. At $x = 0$, the oscillator has only kinetic energy and no potential energy since it is moving at maximum velocity and there is no acceleration or deceleration, whereas at $x = A$, the oscillator has only potential energy and no kinetic energy because it is not moving and is experiencing its greatest restoring force. Due to conservation of energy, this means that the maximum kinetic energy is equal to the maximum potential energy, and so we can use this to solve problems. The graph below shows the kinetic energy (green line) and potential energy (red line) of a simple harmonic oscillator which begins at the equilibrium position. The straight black line represents the total energy of the oscillator.



Example: A particle of mass 3kg is attached to a spring of spring constant $k = 10\,\text{Nm}^{-1}$ and is displaced horizontally by 0.3m from equilibrium. The particle then undergoes horizontal simple harmonic motion. Find the maximum velocity of the particle.

Solution:
Since the motion all happens in a horizontal direction, the maximum potential energy is entirely elastic potential energy and occurs at the maximum displacement. Thus

$$EPE = \frac{1}{2}kx^2$$

$$= \frac{1}{2} \times 10 \times (0.3)^2$$

$$= 0.45\text{J}$$

Using conservation of energy for SHM, the maximum potential energy is equal to the maximum kinetic energy. This gives

$$0.45 = \frac{1}{2}mv^2$$

$$\Rightarrow v = \sqrt{\frac{2 \times 0.45}{3}}$$

$$= \frac{\sqrt{30}}{10} \approx 0.55\text{ms}^{-1}\ (3sf)$$

Damping

We have only considered idealised oscillators so far but in real life, all oscillators experience a **damping force**. A damping force is one that opposes the motion of the oscillator and reduces, restricts or prevents the oscillations. A linear damping force is proportional to the oscillator's velocity and acts in the opposite direction to the velocity. An example of a linear damping force is friction, however other, non-linear, damping forces are possible but are not considered in first year physics. If we add a linear damping term into the SHM equation we obtain:

$$\frac{d^2x}{dt} = -\omega^2 x - \alpha\frac{dx}{dt}$$

where α is a constant of proportionality.

There are different types of linear damping forces:

- **Overdamped** - The amplitude of the oscillator exponentially decreases to zero without oscillating.
- **Critically damped** - The oscillator returns to equilibrium as quickly as possible without oscillating.
- **Underdamped** - The system oscillates (at reduced frequency compared to an undamped case) with the amplitude gradually decreasing to zero.

Forced Oscillations

In first year you will also encounter a forcing term in an equation describing an oscillator. This term describes a driving force which supplies energy to the oscillator and can take different forms. Two common forms of the driving force are as follows:

- A sinusoidal driving force with the general form $F(t) = \frac{1}{m}F_0 \sin(\omega t)$.
- A step input with the general form

$$F(t) = \begin{cases} \omega_0^2, & \text{if } t \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

If we were to have an oscillator with a linear damping force and a sinusoidal driving force, the equation describing its motion would have the form:

$$\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} + \omega^2 x = \frac{1}{m}F_0 \sin(\omega t)$$

Resonance

If a system is displaced from equilibrium and the left alone, it will oscillate with a **natural frequency**. If we apply a driving force to the system then we will obtain a forced oscillation which will oscillate with a **driving frequency**. The driving frequency does not have to be equal to the natural frequency, however if the driving frequency is close to the natural frequency of the system, then the system will **resonate**. This causes the amplitude of the oscillation to increase. The maximum amplitude occurs when the driving frequency is exactly equal to the natural frequency and thus in this case the system has the maximum energy it could obtain due to the driving force.

Resonance can occur in any physical system where there is periodic motion and a driving force. Physicists and engineers often try to avoid resonance in machinery and electronics as it can be destructive; for example if a company of soldiers marches in step across a bridge at the natural frequency of the bridge, the bridge will resonate and the resulting increase in amplitude may tear the bridge apart. Similarly, the reason that a tumble dryer vibrates and makes a loud noise is because the driving force due to the rotation has a similar frequency to the tumble dryer itself, and hence there is resonance.

13.6 Circular Motion

Moment and Torque of a Force

The torque, or moment of a force, is defined as the magnitude of the force multiplied by the perpendicular distance from the pivot to the line of action of the force. It is a measure of the turning effect of the force and can be calculated using the equation:

$$\boxed{\text{Moment} = Fd}$$

where F is the magnitude of the force and d is the perpendicular distance from the pivot to the line of action of the force.

Note: The units of a moment are newton metres (Nm).

Centripetal Force and Motion in a Circle

Many situations in physics involve objects moving in a circular motion around a point, for example charged particles travelling perpendicular to a magnetic field, satellites orbiting the Earth at a constant height and even cars driving around corners. The motion in these cases is described using many of the same techniques and definitions as in the section on oscillatory motion, however we are now considering motion in two dimensional space, rather than one dimensional.

An object travelling in circular motion does so because it experiences a constant force which acts perpendicular to its motion. This is called the **centripetal force** and causes the object to accelerate towards the centre of a circular path. Since this force does not have a component acting in the same direction as the motion, the object travels around the circle at a constant speed, however since velocity is a vector and has a direction, the velocity is constantly changing, and therefore the object is accelerating despite moving at a constant speed.

The centripetal force is given by

$$\boxed{F = \frac{mv^2}{r}}$$

where m is the mass of the object, v is the velocity of the object and r is the radius of the circle which the object describes. This force **always** acts towards the centre of the circle, regardless of where the object is on the circle.

From Newton's second law, we can determine the formula for the acceleration due to change of direction:

$$\boxed{a = \frac{v^2}{r} = \omega^2 r}$$

Another useful formula for centripetal motion can be obtained by considering how quickly the object moves around the circle. The distance the object travels is the circumference of a circle, given by

$$C = 2\pi r$$

and the time taken to complete one cycle is the time period T. Therefore, the speed with which the object moves around the circle is the circumference divided by the time period, or:

$$\boxed{v = \frac{2\pi r}{T}}$$

Physics Example: A woman attaches a ball of mass 1kg to a string of length 0.75m and swings the ball horizontally above her head 90 times in one minute. Calculate the centripetal force that acts on the ball due to the string.

Solution:
First, we notice that the length of the string will be equivalent to the radius of the circle that the ball describes and so $r = 0.75$. We now deduce the frequency of rotation by remembering that frequency is the number of rotations **per second** and so $f = \frac{90}{60} = 1.5$Hz. We can use the equation relating frequency and period to obtain the velocity of the ball:

$$v = \frac{2\pi r}{T} = 2\pi r f = 2.25\pi \text{ms}^{-1}$$

Finally we use the equation for the centripetal force to find the force acting on the ball due to the string

$$F = \frac{mv^2}{r} = \frac{1 \times (2.25\pi)^2}{0.75} = 66.6\text{N } (3sf)$$

Physics Example: If the moon makes a complete orbit around the Earth every 28 days calculate the distance between the Moon and the Earth's centre. You may assume the mass of the Earth is 6×10^{24}kg and $G = 6.67 \times 10^{-11}$.

Solution: The centripetal force of the moon's orbit is the force due to gravity, therefore we can say

$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$

Where M is the mass of the Earth, m is the mass of the moon, r is the separation of the Earth and the Moon and v is the velocity of the moon.

$$r = \frac{GM}{v^2}$$

We know that $v = \frac{2\pi r}{T}$ where T is the time of one orbit. Therefore

$$r = \frac{GMT^2}{4\pi^2 r^2}$$

$$r^3 = \frac{(6.67 \times 10^{-11})(6 \times 10^{24})(28 \times 24 \times 60 \times 60)^2}{4\pi^2}$$

$$r = \left(\frac{(6.67 \times 10^{-11})(6 \times 10^{24})(28 \times 24 \times 60 \times 60)^2}{4\pi^2}\right)^{\frac{1}{3}}$$

Therefore

$$r \approx 3.9 \times 10^8\text{m}$$

13.7 Variable Mass

In all above sections, we have only considered bodies with a constant mass. However, in some situations this is not the case, for example a rocket loses mass as it travels from the surface of the earth due to burning fuel. In the event that mass is not a constant we turn to the **impulse-momentum principle** which states that the change of linear momentum of the whole system in a time interval, δt, will be equal to impulse of external forces acting on the whole system in time interval δt.

To understand this principle it is useful to consider a simple example. A body, B, of mass m is falling due to gravity with velocity v picking up mass as it travels (the extra mass is not moving in this example).

We consider the body B to pick up a mass of δm in a time δt while its traveling. The initial momentum (before the mass, δm has been collected) is

$$mv + \delta m \times 0 = mv$$

After the mass has been collected the new mass is $m + \delta m$ and the velocity of the particle will have changed, we will call this new velocity $v + \delta v$. Therefore the final momentum (after the mass dm has been collected) is

$$(m + \delta m) \times (v + \delta v)$$

The force that is acting during this event is the weight of both M and δm, i.e. $mg + (\delta m)g$. So the impulse of external forces acting on the system is

$$(mg + \delta m)g \times \delta t$$

Using the impulse-momentum principle we can state:

$$(m + \delta m) \times (v + \delta v) - mv = (mg + \delta m)g \times \delta t$$

Expanding this gives

$$m\delta v + v\delta m + \delta m \times \delta v = mg\delta t + g\delta m \times \delta t$$

As δm and δv are infinitesimal (very small) when they are multiplied together we they are even smaller and thus we ignore these terms. This yields

$$m\delta v + v\delta m = mg\delta t + g\delta m \times \delta t$$

Dividing each term by dt gives

$$m\frac{\delta v}{\delta t} + v\frac{\delta m}{\delta t} = mg + g\delta m$$

If we take limits as $\delta t \to 0$ and using the fact that $\delta m \to 0$ we have

$$m\frac{dv}{dt} + v\frac{dm}{dt} = mg$$

Now we have an equation in a useful form, for example this could be modelled to predict the size or speed of hail stones when they hit the ground.

Example: A rocket has initial mass M_0. It propels itself by ejecting mass at a constant rate k per unit time with speed u relative to the rocket. the rocket is launched from rest vertically upwards. Determine an expression for the velocity of the rocket at time t and show your workings.

Solution: We say that at a time t the rocket is travelling at a velocity v and has mass m, after a short time, δt, the rocket loses a small amount of mass, $-\delta m$ (negative because it's a loss of mass). The initial momentum (before the ejection of mass) is mv.
The velocity of the rocket after the ejection of mass has changed so we will call it $v + \delta v$. The ejected mass, $-\delta m$ has a speed u relative to the rocket new velocity so the ejected mass has velocity $(v + \delta v - u)$. Therefore the final momentum of the system (when the mass is ejected) is

$$(m - (-\delta m))(v + \delta v) + (-\delta m)(v + \delta v - u)$$

$$(m + \delta m)(v + \delta v) - \delta m(v + \delta v - u)$$

The only external force acting on the system is the weight of the rocket and the ejected mass, so the impulse of external forces acting on the system is

$$\big[-(m - (-\delta m))g - (-\delta m)g\big]\delta t = -mg\delta t$$

Both terms are negative as the weight acts downwards.
Using the impulse-momentum principle it is true to say:

$$(m + \delta m)(v + \delta v) - \delta m(v + \delta v - u) - mv = -mg\delta t$$

Expanding and simplifying this gives

$$m\delta v + \delta m u = -mg\delta t$$

Dividing by δt gives

$$m\frac{\delta v}{\delta t} + \frac{\delta m}{\delta t}u = -mg$$

If we take limits as $\delta t \to 0$ and using the fact that $\delta m \to 0$ and $\delta v \to 0$ we have

$$m\frac{dv}{dt} + \frac{dm}{dt}u = -mg$$

Now we have an equation that we can use to answer the question asked. First note that

$$\frac{dm}{dt} = -k$$

Therefore

$$m = -kt + C$$

Using this and the fact that at $t = 0$, $m = M_0$ we obtain that

$$m = M_0 - kt$$

Substituting this into our equation gives

$$(M_0 - kt)\frac{dv}{dt} - ku = -(M_0 - kt)g$$

Rearranging this gives

$$\frac{dv}{dt} = \frac{ku}{M_0 - kt} - g$$

Solving this and using that $v = 0$ at $t = 0$ gives

$$v = u\ln\left(\frac{M_0}{M_0 - kt}\right) - gt$$

13.8 Review Questions

Easy Questions

Question 1: Using dimensional analysis, show that the following equation is dimensionally correct.

$$v_s = k\sqrt{\frac{p}{\rho}}$$

where v_s is the speed of sound through a gaseous medium, p is the pressure of the medium, ρ is the density of the medium and k is a dimensionless constant.

Hint: You may need to look up the dimensions of some of these quantities.

Question 2: The radius, R, of a shockwave from a nuclear explosion depends on the energy of the explosion, E, the density of the air, ρ, and the time that has elapsed, t. Use dimensional analysis to express R as a function of the other quantities.

Answer:

$$R = kE^{\frac{1}{5}}\rho^{-\frac{1}{5}}t^{\frac{2}{5}}$$

Question 3: A ball is thrown upwards from a height of 1.5m with a velocity of 6ms^{-1}. What is the maximum height s that the ball will reach?

Answer:

$$s = 3.34\text{m } (3sf)$$

Question 4: On the planet Zabrox, an alien throws a space-rock off the edge of a space-cliff so that it leaves his hand 2m above the clifftop. The space-cliff is 13m high. The space-rock initially has a vertical velocity of 8ms^{-1}. Given that the acceleration due to gravity on the planet Zabrox is two-thirds that of Earth, and considering only the vertical motion of the space-rock, find the time taken t for the space-rock to reach the bottom of the space-cliff and find its instantaneous velocity v when it reaches the bottom.

Answer:

$$t = 3.69\text{s } (3sf)$$
$$v = 12.7\text{ms}^{-1} \quad (3sf)$$

Question 5: A girl throws a paper aeroplane horizontally with an initial velocity of 2ms^{-1}. Calculate the horizontal distance s the paper aeroplane travels before it reaches the ground, 5m below.

Hint: You will have to consider the vertical motion and the horizontal motion separately.
Answer:

$$s = 2.02\text{m} \quad (3sf)$$

Medium Questions

Question 6: A rod of length 5m, and uniform mass 5kg, is balancing on a pivot such that the pivot is a distance 2m from one end of the rod. In order for the system to be in equilibrium a force, F, is applied to the end of the rod that is closest to the pivot. Find the magnitude and direction of F.
Answer:

$$F = \frac{5}{4}g \quad \text{N}$$

Question 7: Two 64kg stick figures are performing an extreme blob jump. One stick figure stands atop a 7.0m high platform with a 256kg boulder. A second stick figure stands on a partially inflated air bag known as a blob (or water trampoline). The first stick figure rolls the boulder off the edge of the platform. It falls onto the blob, catapulting the second stick figure into the air. What is the maximum height, h, to which the second stick figure can rise? Assume that stick figures, boulders, and blobs obey the law of conservation of energy.

Answer:

$$h = 8\text{m}$$

Question 8: A particle, P, with mass 2kg is travelling at 5ms^{-1} when it collides with a second particle, Q, with mass $m$kg, which is at rest. After the collision P reverses its direction of travel and moves with a velocity 2ms^{-1} and Q travels in the same direction to P's original velocity with velocity 4ms^{-1}. What is the mass of Q?

Answer:

$$m = \frac{7}{2}\text{kg}$$

Question 9: A particle, A, with mass 4kg is travelling at 3ms^{-1} when it collides with a second particle, B, with mass $m$kg, which is traveling in the opposite direction to A but with an equal speed. After the collision A reverses its direction of travel and moves with a velocity 2ms^{-1}. What is the mass of B and the velocity, v, it travels at after the collision if the coefficient of restitution betweeen the particles is $\frac{1}{3}$?

Answer:

$$m = \frac{20}{3}\text{kg}$$

$$v = 0\text{ms}^{-1}$$

Question 10: Predict the mass of the Earth given that the distance from the center of the Earth's core to the center of the moon is $384,000$km.

Answer:

$$5.73 \times 10^{24}\text{kg}$$

Question 11: A ferris wheel has a diameter of 80m and takes 12 minutes to complete one revolution. Find the centripetal acceleration a of each capsule.

Answer:

$$a = 3.05 \times 10^{-3}\text{ms}^{-2} \quad (3sf)$$

Hard Questions

Question 12: A rain drop falls under gravity and gains mass in a way such that

$$\frac{dm}{dx} = km.$$

Find velocity as a function of time.

Answer:

$$\sqrt{\frac{g}{k}}\frac{1 - e^{-2t\sqrt{gk}}}{1 + e^{-2t\sqrt{gk}}}$$

Question 13: A particle, P, of mass 15kg, is dropped from rest. If the resistance to motion felt by P is $3v$, where v is the velocity P falls at, and it takes 2 seconds for P to reach the ground, what is the velocity of P when it hits the ground?

Answer:

$$v = 14.3\text{ms}^{-1}$$

Question 14: A particle, Q, of mass 2kg, is dropped from rest. If the resistance to motion felt by Q is $10v^2v$, where v is the velocity Q falls at, and it takes 15 seconds for Q to reach the ground, what is the velocity of Q when 1 second before it hits the ground?

Answer:

$$v = 1.4\text{ms}^{-1}$$

Question 15: A crate of mass 3kg rests in limiting equilibrium on a rough ramp inclined at 30° to the horizontal. The coefficient of friction between the crate and the ramp is 1/3. A horizontal force of magnitude F Newtons is applied to the crate so that the equilibrium is broken and the crate just begins to move up the ramp. Calculate the force F.

Answer:

$$F = 33.2\text{N}$$