### LECTURES ON ORIENTIFOLDS AND DUALITY a

## ATISH DABHOLKAR

Department of Theoretical Physics
School of Natural Sciences
Tata Institute of Fundamental Research
Homi Bhabha Road, Mumbai (Bombay), India 400005

This is an introduction to orientifolds with emphasis on applications to duality.

### 1 Introduction

These lecture notes are intended as a pedagogical introduction to orientifolds. Aspects of orbifolds and F-theory are also discussed in brief to provide the necessary background. The emphasis is on the applications of these constructions to duality. The approach is based on simple examples that can be easily worked out in detail but which, at the same time, illustrate the main ingredients of the general procedure.

## 1.1 Motivation

Orientifolds are intrinsically perturbative. By contrast, much of the recent work in string theory has focused on explorations of nonperturbative aspects of the theory using the idea of 'duality'. In view of these developments, it is natural to ask, before embarking on the details of the construction, why orientifolds are interesting. Let me begin by addressing this question.

From the perspective of duality, the motivation for studying orientifolds is twofold.

1) New dualities: It is usually much easier to establish the duality between two theories that possess a lot of supersymmetry because, with more supersymmetry, the structure of the theory is more tightly constrained. On the other hand, theories with less supersymmetry contain more interesting dynamical phenomena that are not merely consequences of supersymmetry. Moreover, to be closer to the real world, one would like as little supersymmetry as possible. Orientifolds and orbifolds are very useful tools for establishing new dualities with less supersymmetry starting with known dualities with more

 $<sup>^</sup>a$ Based on lectures given at the 1997 Trieste Summer School on Particle Physics and Cosmology, Italy.

supersymmetry. To illustrate this point, I discuss two dual pairs, each with 16 supercharges:

- (a) Heterotic string on a 4-torus and Type IIA string on a  $K_3$  surface,
- (b) Heterotic string on a 2-torus and F-theory on a  $K_3$  surface. We shall see how these two dualities can be 'derived' starting with the  $SL(2, \mathbf{Z})$  duality of Type-IIB theory which has 32 supercharges, and the duality between Type-I string and heterotic string in ten dimensions.
- 2) New Compactifications: The space of string compactifications can have many disconnected pieces. Orientifolds have proved to be very useful for exploring different parts of this moduli space that were not accessible before as perturbative string vacua. These new compactifications are often nonperturbatively connected with known compactifications and have interesting duals in M and F theory.

As an illustration I discuss orientifolds in six dimensions with 8 supercharges. Many phenomena such as multiple tensor multiplets or small instantons which appear as exotic strong coupling effects in the conventional Calabi-Yau compactifications can be described perturbatively and more explicitly in the corresponding orientifold duals. Orientifolds and orbifolds are exact conformal field theories. Therefore, in principle, one can calculate not only the spectrum but many other more detailed quantities such as quantum corrections to scattering amplitudes by evaluating various correlation functions in the conformal field theory. These constructions are thus complementary to the geometric Calabi-Yau compactifications which are simpler to deal with if one is interested only in the massless spectrum.

More generally, orientifolds and orbifolds provide us with 'discrete' constructions which may not have any geometric interpretation as strings moving on some smooth manifold. Such non-geometric constructions are potentially very interesting for discovering new regions of the moduli space which may be disconnected from known compactifications even at a nonperturbative level.

# 1.2 Outline

My objective will be to give a reasonably self-contained account of the basic construction of orientifolds starting with elementary considerations of free string theory. After reviewing aspects of various related topics, I discuss some applications to duality and compactification. As we shall see, one can get surprisingly far with this formalism by keeping track of a few discrete symmetries.

Our starting point will be the Type-IIB string in ten space-time dimensions which will be reviewed in section §2. In the sections §3 and §4 the orbifold and orientifold construction will be described by working through the examples of

an orbifold of the Type-IIB string to get Type-IIA string and an orientifold of Type-IIB string to get the Type-I string. Aspects of the K3 surface and Type-II and F-theory compactifications on K3 will be discussed in section §5. Section §6 deals with applications of orientifolds to duality of theories with 16 supercharges. Some orientifolds that give six-dimensional compactifications with 8 supercharges will be surveyed in section §7 along with their duals.

### 1.3 Orientation

The remarkable progress in recent years in our understanding of string theories has made the subject more exciting and challenging but also somewhat less easily accessible for beginners. With the rapidly growing literature on the subject, it is not possible to be completely self-contained in this short review. Many excellent reviews already exist which cover some of the background material used here and which complement these lecture notes. I give below a representative but not a very complete list of reviews as well as some original articles that can orient the reader. The two volumes of 'Superstring Theory' by Green, Schwarz, and Witten [28] discuss quantization of free superstring in both GS and NSR formalism, low energy supergravity equations of Type-IIB, Calabi-Yau compactifications along with the relevant algebraic geometry. A recent review by Sen on duality contains an introduction to various dualities used here and aspects of nonperturbative string theory. D-branes are described in detail in the TASI lectures of Polchinski [46] and in [44]. More details on orbifolds can be found in the papers by Dixon, Harvey, Vafa, and Witten [19] and in the Les Houches lectures of Ginsparg [30]. Various aspects of the K3surface are discussed in the review by Aspinwall [1]. F-theory is introduced and elaborated upon in the papers by Vafa [62] and Morrison and Vafa [39, 40].

## 2 Type-IIB String

## 2.1 Worldsheet Action and Spectrum

The gauge-fixed, physical spectrum of the ten-dimensional Type-IIB string is easiest to calculate in the light-cone gauge. In the light-cone gauge, the transverse group of rotations is SO(8) whose covering group is Spin(8). The three representations of Spin(8) that will be relevant to us are the vector representation  $\mathbf{8v}$ , the spinor representation  $\mathbf{8s}$ , and the conjugate spinor representation  $\mathbf{8c}$  which are all eight-dimensional. The spinor  $\mathbf{8s}$  with right-handed chirality is related to the conjugate spinor  $\mathbf{8c}$  with left-handed chirality by parity transformation that flips the sign of one of the components of the vector  $\mathbf{8v}$ . We

shall use the letters i, j, k as the **8v** indices, the letters a, b, c as the **8s** indices and the letters,  $\dot{a}, \dot{b}, \dot{c}$  as the **8c** indices.

In the Green-Schwarz formalism in the light-cone gauge, the worldsheet action of the Type-IIB string is given by [28]

$$S_{l.c.} = \frac{-1}{2\pi} \int d\sigma d\tau \, (\partial_+ X^i \partial_- X^i - i S^a \partial_- S^a - i \tilde{S}^a \, \partial_+ \tilde{S}^a), \tag{1}$$

where  $\sigma$  is the coordinate along the string,  $0 \le \sigma < 2\pi$ , and  $\tau$  is the worldsheet time. We have set  $\alpha' = \frac{1}{2}$ . In addition to bosonic fields on the string worldsheet  $X^i$ , which are the transverse spatial coordinates of the string, there are additional fermionic fields on the worldsheet: left-moving  $S^a$  and right-moving  $\tilde{S}^a$  both of which transform as 8s. Since both right and left movers have the same spacetime transformation properties, this theory is non-chiral on the worldsheet. But since only the right-handed chirality of spacetime fermions appears and not the parity transform, the theory is chiral in spacetime. Another inequivalent choice is to take left-moving  $S^a$  which transforms as a left-handed conjugate spinor and right-moving  $\tilde{S}^a$  which transforms as the right-handed spinor. This choice gives Type-IIA theory which has opposite chirality properties. To summarize, we have,

$$S^a$$
  $\tilde{S}^a$  II B chiral in spacetime nonchiral on world sheet  $S^{\dot{a}}$   $\tilde{S}^a$  II A nonchiral in spacetime chiral on worldsheet. (2)

Quantization of this 1+1 dimensional free field theory is straightforward. The bosons  $X^i$  satisfy periodic boundary condition along  $\sigma$ , and by spacetime supersymmetry so do the fermions:  $S^a(\sigma + 2\pi) = S^a(\sigma)$  etc. With this boundary condition, the mode expansion is

$$X^{i} = x^{i} + \frac{1}{2}p^{i}\tau + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_{n}^{i}e^{-in(\tau+\sigma)} + \frac{1}{n}\tilde{\alpha}_{n}^{i}e^{-in(\tau-\sigma)},$$

$$S^{a} = \frac{1}{\sqrt{2}}\sum_{-\infty}^{\infty}S_{n}^{a}e^{-in(\tau+\sigma)}, \qquad \tilde{S}^{a} = \frac{1}{\sqrt{2}}\sum_{-\infty}^{\infty}\tilde{S}_{n}^{a}e^{-in(\tau-\sigma)}.$$
 (3)

Canonical quantization of the fields implies standard commutation and anticommutation relations [28] for the oscillator modes:

$$[\alpha_m^i, \alpha_n^j] = m\delta^{ij}\delta_{m+n}, \qquad [\tilde{\alpha}_m^i, \tilde{\alpha}_n^j] = m\delta^{ij}\delta_{m+n}$$
  
$$\{S_n^a, S_m^b\} = \delta^{ab}\delta_{m+n}, \qquad \{\tilde{S}_n^a, \tilde{S}_m^b\} = \delta^{ab}\delta_{m+n}. \tag{4}$$

The zero modes of the  $X^i$  fields satisfy the Heisenberg commutation relations  $[x^i, p^j] = i\delta^{ij}$ , and the ground state is therefore labeled by the momentum

eigenvalue  $|p\rangle$ . Note that there are fermionic zero modes as well,  $S_0^a$  and  $\tilde{S}_0^a$ . The ground state should furnish a representation of the zero mode algebra

$$\{S_0^a, S_0^b\} = \delta^{ab}, \qquad \{\tilde{S}_0^a, \tilde{S}_0^b\} = \delta^{ab}.$$
 (5)

Let us look at the left-movers since right-movers can be treated similarly. Let us rewrite the anticommutations by defining four fermionic oscillators  $\sqrt{2}b_m = (S^{2m-1} + iS^{2m}), m = 1, \dots, 4$ , which satisfy the usual anticommutation relations

$$\{b_m, b_n^{\dagger}\} = \delta_{mn}, \quad \{b_m, b_n^{\dagger}\} = 0, \quad \{b_m^{\dagger}, b_n^{\dagger}\} = 0.$$
 (6)

This rewriting amounts to choosing a particular embedding  $SO(8) \supset SU(4) \times U(1)$ , so that  $\{b_m\}$  transform in the fundamental representation 4 of SU(4) with  $\frac{1}{2}$  unit of U(1) charge, which we denote as  $4(\frac{1}{2})$ , and the  $\{b_m^{\dagger}\}$  transform in the complex conjugate representation. With this embedding various representations decompose as

$$8\mathbf{v} = \mathbf{6}(0) + \mathbf{1}(1) + \mathbf{1}(-1)$$

$$8\mathbf{s} = \mathbf{4}(\frac{1}{2}) + \mathbf{\bar{4}}(-\frac{1}{2})$$

$$8\mathbf{c} = \mathbf{4}(-\frac{1}{2}) + \mathbf{\bar{4}}(\frac{1}{2}).$$
(7)

This embedding is more obvious if we use the fact that  $SO(6) \sim SU(4)$  and  $SO(2) \sim U(1)$ . Then the above is a decomposition of the SO(8) spinor in terms of the SO(6) spinor and its conjugate under the embedding  $SO(8) \supset SO(6) \times SO(2)$ . The representation of Eq. 5 can now be worked out easily by starting with the completely 'empty' Fock space vacuum  $|0\rangle$  which is annihilated by all  $b_m$ 's and then obtaining various filled states by acting with the creation operators. One obtains a 16-dimensional representation:

$$\begin{array}{ccc} |0\rangle & & \mathbf{1}(1) \\ b_{m}^{\dagger}|0\rangle & & \mathbf{\bar{4}}(\frac{1}{2}) \\ b_{m}^{\dagger}b_{n}^{\dagger}|0\rangle & & \mathbf{6}(0) \\ b_{m}^{\dagger}b_{n}^{\dagger}b_{p}^{\dagger}|0\rangle & & \mathbf{4}(-\frac{1}{2}) \\ b_{m}^{\dagger}b_{n}^{\dagger}b_{p}^{\dagger}b_{p}^{\dagger}|0\rangle & & \mathbf{1}(-1) \end{array} \tag{8}$$

where the labels in the second column indicate the dimensions of the SU(4) representation and the U(1) charges. We see from the  $SU(4) \times U(1)$  quantum numbers that 16-dimensional representation of the left-moving ground states reduces as a sum of two representations  $\mathbf{8v+8c}$ . Similarly, for the right-movers, the ground states are given by the sum of  $\mathbf{8v+8c}$ .

A string state  $|\psi\rangle$  is constructed by acting with various creation operators on this 16 × 16-dimensional ground state carrying some spacetime momentum p. A physical state is subject to the on-shell conditions:

$$\alpha' M^{2} = -\alpha' p^{\mu} p_{\mu} = 4(N \equiv \sum_{n=0}^{\infty} n \alpha_{-n}^{i} \alpha_{n}^{i} + S_{-n}^{a} S_{n}^{a})$$

$$= 4(\tilde{N} \equiv \sum_{n=0}^{\infty} n \tilde{\alpha}_{-n}^{i} \tilde{\alpha}_{n}^{i} + \tilde{S}_{-n}^{a} \tilde{S}_{n}^{a}). \tag{9}$$

We see that the massless states have no oscillator excitations and can therefore be read off by tensoring the left-moving and right-moving ground states:

$$(|i\rangle \oplus |\dot{a}\rangle) \otimes (|j\rangle \oplus |\dot{b}\rangle).$$
 (10)

In the Neveu-Schwarz-Ramond formalism of the superstring, which will be reviewed in §2.4, and which is equivalent to the Green-Schwarz formalism that we have used here, the  $8\mathbf{v}$  comes from the Neveu-Schwarz (NS) sector, whereas the (NS) and the  $8\mathbf{c}$  comes from the Ramond (R) sector. The Neveu-Schwarz states are spacetime bosons whereas Ramond states are spacetime fermions. In the tensor product of left-moving and right-moving states, the NS-R and R-NS sector give rise to spacetime fermions  $\psi_{i\dot{a}}$  and  $\psi_{j\dot{b}}$  which are the two gravitini of Type-IIB string. The NS-NS states  $|i\rangle \otimes |j\rangle$  can be reduced in terms of the symmetric traceless, antisymmetric, and scalar combinations which give rise to the metric  $g_{ij}$ , the 2-form  $B_{ij}$ , and the dilaton  $\phi$ , respectively. The R-R states  $|\dot{a}\rangle \otimes |\dot{b}\rangle$  can be reduced as

$$\lambda_1^{\dot{a}}\lambda_2^{\dot{b}} \sim \lambda_1^T \lambda_2 \oplus \lambda_1^T \Gamma^{ij} \lambda_2 \oplus \lambda_1^T \Gamma^{ijkl} \lambda_2, \tag{11}$$

in terms of the Gamma matrices  $\Gamma^i$ , and their totally antisymmetrized products  $\Gamma^{ij}$  and  $\Gamma^{ijkl}$ . Because  $\lambda_1$  and  $\lambda_2$  have the same chirality, products such as  $\Gamma^i$  and  $\Gamma^{ijk}$  do not appear, and moreover, the combination  $\lambda_1^T \Gamma^{ijkl} \lambda_2$  is required to be self-dual. Altogether we obtain a scalar  $\chi$ , a 2-form  $B'_{ij}$ , and a self-dual 4-form  $D_{ijkl}$  from the R-R sector. In summary, the massless spectrum of Type-IIB is as follows.

### Bosons:

NS-NS: metric  $g_{ij}$ , 2-form  $B_{ij}$ , dilaton  $\phi$ ,

R-R: scalar  $\chi$ , 2-form  $B'_{ij}$ , self-dual 4-form  $D_{ijkl}$ .

### Fermions:

NS-R: gravitino  $\psi_{i\dot{a}}$ . R-NS: gravitino  $\psi_{j\dot{b}}$ . There are two perturbative  $\mathbb{Z}_2$  symmetries of the Type-IIB string which will be of particular interest to us  $^b$ .

1)  $\Omega$ : As we have seen, Type-IIB theory is non-chiral on the worldsheet. Hence, worldsheet parity  $\Omega$  which reverses the orientation of the string ( $\sigma \to 2\pi - \sigma$ ) is a symmetry of the theory. Orientation reversal takes right-movers to left-movers. Therefore, of the states  $|i\rangle \otimes |j\rangle$  coming from the NS-NS sector, the symmetric combinations are even and antisymmetric combinations are odd under  $\Omega$ . For the states  $|a\rangle \otimes |b\rangle$  coming from the R-R sector, we have to remember Fermi statistics under exchange. To summarize,

 $g_{ij}, \phi, B'_{ij}$  are even under  $\Omega$ ,

 $\chi, B_{ij}, D_{ijkl}$  are odd under  $\Omega$ .

 $\Omega$  takes the NS-R states to R-NS states, thus one combination of the two gravitini is even under  $\Omega$  and the other is odd.

2)  $(-1)^{F_L}$ : The action in Eq. 1 is invariant under  $S^a \to -S^a$ . This symmetry can be written as  $(-1)^{F_L}$ , where  $F_L$  is spacetime fermion number coming from left-movers. Only left-moving fermions are odd under this symmetry, so R-NS and R-R states are odd whereas the NS-R and NS-NS states are even. To summarize,

 $g_{ij}, \phi, B_{ij}$  are even under  $(-1)^{F_L}$ ,

 $\chi, B'_{ij}, D_{ijkl}$  are odd under  $(-1)^{F_L}$ .

The two elements do not commute with each other. In particular,

$$\Omega(-1)^{F_L}\Omega = (-1)^{F_R}, (12)$$

where  $F_R$  is the right-moving spacetime fermion number and  $(-1)^{F_R}$  takes  $\tilde{S}^a$  to  $-\tilde{S}^a$ . If we consider all distinct products of these elements, we get an eight-element nonabelian group as the group of discrete perturbative symmetries of Type-IIB. This group is isomorphic to  $D_4$ — the group of symmetries of a square—which is a subgroup of O(2), the group of rotations and reflections in the x-y plane.  $D_4$  is generated by two elements: A reflection,  $(x,y) \to (-x,y)$ , which we identify with  $\Omega$ , and a rotation through  $\pi/2$ :  $(x,y) \to (-y,x)$ , which we identify with  $\Omega(-1)^{F_L}$ . The full group of perturbative symmetries is

$$G = \{1, \Omega(-1)^{F_L}, (-1)^F, \Omega(-1)^{F_R}, (-1)^{F_L}, \Omega(-1)^F, (-1)^{F_R}\}.$$
(13)

 $<sup>^</sup>b$ By perturbative symmetry we mean here a symmetry that is evident at the perturbative level but which is believed to be unbroken even nonperturbatively. A nonperturbative symmetry by contrast is not evident at the perturbative level.

## 2.3 Nonperturbative Symmetries

Apart from the perturbative symmetries, the Type-IIB string has an  $SL(2, \mathbf{Z})$  duality symmetry. Part of this duality symmetry is nonperturbative because as we shall see it relates a theory at strong coupling to the dual theory at weak coupling. The great utility of duality, is that we can learn about the strong coupling behavior of a theory from the weak coupling behavior of the dual theory [53].

Unlike the usual perturbative symmetries, which are valid order by order in perturbation theory, a nonperturbative quantum symmetry is not apparent at the perturbative level. Establishing the existence of a nonperturbative symmetry, in general, would require the knowledge of of the full quantum theory including its strong coupling behavior. What makes duality useful is that, in a supersymmetric theory, it is often possible to discover the duality symmetries only from the weak coupling, semiclassical data without having to know all the details of the full quantum theory. Supersymmetry places powerful constraints on the structure of the quantum theory. With enough supersymmetry, many semiclassical quantities receive no quantum corrections-either perturbative or nonperturbative—and are exact. One can analyze such quantities in the weak coupling regime, and then analytically continuing them in the strong coupling regime. For two theories to be dual to each other, all such nonrenormalized quantities must match. This requirement provides many nontrivial consistency checks on the possible duality symmetry. The verification of these consistency checks is often sufficiently compelling to establish duality even though one cannot actually 'prove' it.

The Type-IIB theory has N=2 chiral supersymmetry in ten dimensions with 32 real supercharges. This supersymmetry is highly restrictive. In fact, Type-IIB string is the only string theory, and the Type-IIB supergravity is the only possible supergravity with N=2 supersymmetry in ten dimensions. As long as the strong coupling effects do not break supersymmetry, the only theory that a Type-IIB theory can possibly be dual to is Type-IIB theory itself. The Type-IIB theory is indeed self-dual with duality group  $SL(2, \mathbf{Z})$ . We now indicate some evidence for this claim.

There are two semiclassical quantities that are expected not to get renormalized which must exhibit the  $SL(2, \mathbf{Z})$  symmetry. These are,

- 1) the massless spectrum and their equations of motion,
- 2) the spectrum of all BPS-saturated supersymmetric states.

We shall discuss the first point in this subsection, and return to the second point in subsection §2.6.

The low energy equations of motion of Type-IIB string are given by the

Type-IIB supergravity [28]. These equations are indeed invariant under the action of a noncompact group of symmetry  $SL(2,\mathbf{R})$ , which is the group of  $2\times 2$  real matrices with determinant one. A general element of  $SL(2,\mathbf{R})$  is  $\Lambda=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  with a,b,c,d real and ad-bc=1. To exhibit the action of the symmetry on massless fields, let us define a complex scalar  $\lambda=\chi+ie^{-\phi}$  and Einstein metric  $G_{MN}=e^{\phi/2}g_{MN}$ . The field  $\lambda$  parametrizes the upper half plane. The action of  $SL(2,\mathbf{R})$  on the bosonic fields is given by

$$\lambda \to \frac{a\lambda + b}{c\lambda + d}, \quad \begin{pmatrix} B \\ B' \end{pmatrix} \to \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} B \\ B' \end{pmatrix}, \quad D \to D, \quad G \to G.$$
 (14)

At the quantum level, the full  $SL(2, \mathbf{R})$  symmetry does not survive. The reason is that the theory contains states that are charged with respect to the fields B and B'. The charges are integers satisfying the Dirac quantization condition and they transform linearly in the same way as the gauge fields B and B' that they couple to. After a general  $SL(2, \mathbf{R})$  transformation, the charges would no longer be integers and would not respect the quantization condition. However, an  $SL(2, \mathbf{Z})$  subgroup that consists of  $SL(2, \mathbf{R})$  matrices with a, b, c, d all integers does not change the integrality of charges. This subgroup can be, and in fact is, an exact duality symmetry of Type-IIB theory. The  $SL(2, \mathbf{Z})$  is generated by the elements:

$$T : \lambda \to \lambda + 1, \qquad \Lambda = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$S : \lambda \to -1/\lambda, \qquad \Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$R : \lambda \to \lambda, \qquad \Lambda = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{15}$$

### Observations

- 1) A given vacuum of the Type-IIB theory is characterized by an expectation value of the scalar field  $\lambda$ .  $\lambda$  parametrizes the upper half-plane, which can also be written as a coset  $SL(2,\mathbf{R})/SO(2)$ . The duality symmetry  $SL(2,\mathbf{Z})$  is a discrete gauge symmetry; it says that a theory at a given  $\lambda$  is nonperturbatively equivalent to all theories at the images of  $\lambda$  under  $SL(2,\mathbf{Z})$ . All gauge equivalent theories must be identified. The moduli space of Type-IIB is therefore  $SL(2,\mathbf{Z})\backslash SL(2,\mathbf{R})/SO(2)$  which is as shown in Figure 1.
- 2) The expectation value of  $e^{\phi}$  is the string coupling constant. Weak coupling corresponds to  $e^{-\phi} = Im(\lambda) \to \infty$ . If we set  $\chi = 0$  then the element

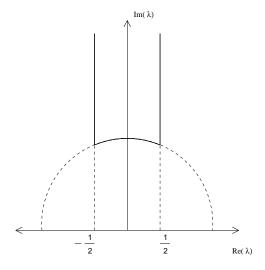


Figure 1: The moduli Space of Type-IIB

S takes  $e^{\phi}$  to  $e^{-\phi},$  and thus relates a theory at strong coupling to a theory at weak coupling.

- 3)  $R = (-1)^{F_L}\Omega$ . This can be checked easily from their action on the massless spectrum. Under R, the two 2-forms B and B' are odd, and all other fields are even, which we see, from §2.2, is the same action as  $(-1)^{F_L}\Omega$ .
- 4)  $S(-1)^{F_L}S^{-1}=\Omega$ . This can be immediately verified from §2.2 and Eq. 14 and will be important later when we discuss F-theory.

# ${\it 2.4} \quad Neveu\text{-}Schwarz\text{-}Ramond\ Formalism$

In the NSR formalism in the light-cone gauge, the worldsheet fermions  $\psi^i$  and  $\tilde{\psi}^i$  transform as a spinor on the worldsheet but as a vector,  $\mathbf{8v}$ , of Spin(8). The worldsheet action of the Type-II string is given by [28]

$$S_{l.c.} = \frac{-1}{2\pi} \int d\sigma d\tau \, (\partial_{+} X^{i} \partial_{-} X^{i} - i \psi^{i} \partial_{-} \psi^{i} - i \tilde{\psi}^{i} \, \partial_{+} \tilde{\psi}^{i}). \tag{16}$$

The bosons satisfy periodic boundary condition and are treated as before. Fermions can be either periodic or antiperiodic on the left and on the right. In each sector one has to perform the GSO projection to obtain the superstring [28].

Let us look at the left-movers. Antiperiodic boundary condition for the fermions gives the Neveu-Schwarz sector of the theory. The bosons are integer-moded but the fermions are half-integer moded. The ground state  $|{\rm NS}\rangle$  has energy  $-\frac{1}{2}$  and is tachyonic. A state with oscillator number N has mass M which satisfies the mass-shell condition

$$\alpha' M^2 = 4(N - \frac{1}{2}). \tag{17}$$

The ground state is therefore tachyonic.

The GSO projection projects out states with odd worldsheet fermion number f. The ground state is assigned odd worldsheet fermion number,  $(-1)^f = -1$ . This assignment of the fermion number follows from the fact that the 'vacuum'  $|NS\rangle$  actually has a fermionic ghost excitation [46]. It is also the choice that projects out the tachyon after the GSO projection. The GSO-projected spectrum contains a massless state

$$\psi_{-\frac{1}{2}}^{i}|\text{NS}\rangle,$$
 (18)

which transforms as a vector  $\mathbf{8v}$  of Spin(8).

Periodic boundary condition for the fermions gives the Ramond sector. Now, the bosons and fermions are both integer moded, and the ground state energy is zero. There are 8 zero modes  $\psi_0^i$  with anticommutations

$$\{\psi_0^i, \psi_0^j\} = \delta^{ij}. (19)$$

With  $\psi^i = \sqrt{2}\Gamma^i$ , Eq. 19 defines the usual Clifford algebra of Dirac matrices. The representation of this algebra can be found by following steps similar to those that led to Eq. 8 in §2.1. Now the fermionic oscillators are defined by  $\sqrt{2}d_m = \psi^{2m-1} + i\psi^{2m}, m = 1, \ldots, 4$ , which satisfy the usual anticommutation relations

$$\{d_m,d_n^\dagger\}=\delta_{mn},\quad \{d_m,d_n^\dagger\}=0,\quad \{d_m^\dagger,d_n^\dagger\}=0.$$

This definition amounts to a different embedding  $SO(8) \supset SU(4) \times U(1)$  than Eq. 7. Various representations now decompose as

$$8\mathbf{v} = 4(\frac{1}{2}) + \overline{4}(-\frac{1}{2})$$

$$8\mathbf{s} = 4(-\frac{1}{2}) + \overline{4}(\frac{1}{2}).$$

$$8\mathbf{c} = 6(0) + \mathbf{1}(1) + \mathbf{1}(-1)$$
(20)

The Fock space of the fermionic oscillators furnishes a 16-dimensional representation. One can define the chirality matrix

$$\Gamma = \Gamma^1 \Gamma^2 \dots \Gamma^8, \qquad (\Gamma)^2 = 1, \qquad \{\Gamma, \Gamma^i\} = 0. \tag{21}$$

On left handed fermions,  $\Gamma = 1$ , and on right-handed fermions  $\Gamma = -1$ . The 16-dimensional representation reduces as  $\mathbf{8s} + \mathbf{8c}$ . The similarity between the the algebra of  $\psi_0^i$ 's and of  $S_0^a$ 's is a reflection of the triality symmetry of the Spin(8) algebra which interchanges the three eight-dimensional representations  $\mathbf{8v}$ ,  $\mathbf{8s}$ , and  $\mathbf{8c}$  into each other. The Clifford algebra and the representations are related by triality:

$$\Gamma^{i} \in \mathbf{8v}, \quad \{\Gamma^{i}, \Gamma^{j}\} = 2\delta^{ij} \quad \text{gives} \quad \mathbf{8s} \oplus \mathbf{8c}$$

$$\Gamma^{a} \in \mathbf{8s}, \quad \{\Gamma^{a}, \Gamma^{b}\} = 2\delta^{ab} \quad \text{gives} \quad \mathbf{8v} \oplus \mathbf{8c}$$

$$\Gamma^{\dot{a}} \in \mathbf{8c}, \quad \{\Gamma^{\dot{a}}, \Gamma^{\dot{b}}\} = 2\delta^{\dot{a}\dot{b}} \quad \text{gives} \quad \mathbf{8s} \oplus \mathbf{8v}$$
(22)

Indeed, the triality of the Spin(8) algebra is what makes the equivalence between the NSR and the GS formalism possible [28].

Similarly, there is a NS and R sector for the right-movers. GSO projection in the R sector keeps only one the two spinors. The relative choice of the GSO projection for the right-movers and for the left -movers is significant: we can keep either fermions of the same chirality or of opposite chirality in the two sectors. Depending on the choice, we get either Type-IIA theory or Type-IIB theory:

Type IIA: 
$$(8\mathbf{v} \oplus 8\mathbf{s}) \otimes (8\mathbf{v} \oplus 8\mathbf{c})$$
  
Type IIB:  $(8\mathbf{v} \oplus 8\mathbf{c}) \otimes (8\mathbf{v} \oplus 8\mathbf{c})$  (23)

## 2.5 T-duality

Consider a single periodic boson X with period  $2\pi R$ , which can be thought of as a coordinate of a string on a circle of radius R. The momentum along the circle is now quantized, p=n/R. Moreover, the string can wind around the circle before closing, so there are different topological sectors labeled by the winding number w. In sector with winding number w, X satisfies the boundary condition,  $X(\sigma + 2\pi, \tau) = X(\sigma) + 2\pi wR$ . The mode expansion of X in each sector is similar to Eq.3:

$$X(\sigma,\tau) \sim x + \frac{n\alpha'}{R}\tau + wR\sigma + \text{oscillators.}$$
 (24)

We can write  $X(\sigma, \tau) = X_L(\sigma^+) + X_R(\sigma^-)$ , where  $X_L$  and  $X_R$  are left-moving and right-moving fields respectively with  $\sigma^+ = \tau + \sigma$  and  $\sigma^- = \tau - \sigma$ . Their mode expansion is given by

$$X_{L}(\sigma^{+}) = x_{L} + \sqrt{\frac{\alpha'}{2}} (q\sigma^{+} + i \sum_{n \neq 0} \frac{1}{n} \alpha_{n} e^{-in\sigma^{+}}),$$

$$X_{R}(\sigma^{-}) = x_{R} + \sqrt{\frac{\alpha'}{2}} (\tilde{q}\sigma^{-} + i \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_{n} e^{-in\sigma^{-}}),$$
(25)

where

$$q = \left(\frac{n}{R} + \frac{wR}{\alpha'}\right)\sqrt{\frac{\alpha'}{2}}$$

$$\tilde{q} = \left(\frac{n}{R} - \frac{wR}{\alpha'}\right)\sqrt{\frac{\alpha'}{2}}.$$
(26)

The Hamiltonian for the boson is

$$H = (\frac{q^2}{2} + N) + (\frac{\tilde{q}^2}{2} + \tilde{N}), \tag{27}$$

where N and  $\tilde{N}$  are the oscillator numbers. From Eqs. 26 and 27, it is easy to check that the spectrum is invariant under the 'T-duality' transformation,  $R \to \alpha'/R$  and  $n \leftrightarrow w$ , which takes  $q \to q$  and  $\tilde{q} \to -\tilde{q}$ . At the level of the field X, T-duality can be thought of as a one sided parity transform,

$$X_L \to -X_L, \qquad X_R \to X_R,$$
 (28)

taking R to  $\alpha'/R$  at the same time.

This symmetry of a free-boson in two dimensions has a remarkable interpretation in Type II string theory. It relates Type-IIB string compactified on a circle of radius R to Type-IIA string compactified on a circle of radius  $\alpha'/R$ . In string theory, there are additional fields on the worldsheet. If we T-dualize along a direction 9, then other bosonic coordinates are not affected, but the fermions must transform in accordance with spacetime supersymmetry. By spacetime supersymmetry, T-duality must act like a left-sided parity transform even for spacetime fermions:

$$X_L^9 \to -X_L^9, \qquad S^a \to \Gamma \Gamma^9 S^a$$
  
 $X_R^9 \to X_R^9, \qquad \tilde{S}^a \to \tilde{S}^a,$  (29)

where the Gamma matrices are as defined in Eq. 22 and Eq. 21.  $\Gamma\Gamma^9$  represents the action of the parity transformation  $X^9 \to X^9$  on the  $\Gamma$  matrices, because it anticommutes with  $\Gamma^9$  ( $\Gamma^9 \to -\Gamma^9$ ) and commutes with  $\Gamma^j$  ( $\Gamma^j \to \Gamma^j$ ) for  $j \neq 9$ . This operation changes the chirality of right-moving fermions, because  $\Gamma\Gamma^9S^a$  transforms as a conjugate spinor with a dotted index  $\dot{a}$ . We thus get, on the worldsheet a left-moving conjugate spinor (8c) and right-moving spinor (8s), which is Type-IIA string theory. To summarize, the T-duality  $T_9$  takes Type IIB on a circle of radius R in the  $X_9$  direction to Type IIA on the dual circle of radius  $\alpha'/R$ .

T-duality is a symmetry not only of the free string theory but also of the interacting theory. Indeed, the worldsheet path integral of the boson on a higher genus Riemann surface can be shown to be invariant under this transformation [26]. Therefore, by factorization, not only the free spectrum, but also string interactions respect this symmetry.

Let us see what happens to various symmetries of Type II theory under T-duality.

$$\Omega \text{ in II B} \xrightarrow{T_9} I_9 \Omega \text{ in II A}$$
 (30)

where  $I_9$  is the inversion of the 9-th coordinates:

$$I_9: (X_L^9, X_R^9) \to (-X_L^9, -X_R^9).$$
 (31)

In other words,

$$T_9\Omega T_9^{-1} = I_9\Omega, (32)$$

as can be seen from

$$(X_L^9,X_R^9) \xrightarrow{T_9^{-1}} (-X_L^9,X_R^9) \xrightarrow{\Omega} (X_R^9,-X_L^9) \xrightarrow{T_9} (-X_L^9,-X_R^9). \tag{33}$$

On fermions,  $I_9$  is a parity transformation for both left-movers and right-movers

$$I_9: (S^a, \tilde{S}^b) = (\Gamma \Gamma^9 S^a, \tilde{\Gamma} \tilde{\Gamma}^9 \tilde{S}^b), \tag{34}$$

which flips the chirality of both fermions. Note that  $\Omega$  by itself is *not* a symmetry of Type-IIA because starting with a left-moving spinor  $S^a$  and right-moving conjugate spinor  $\tilde{S}^{\dot{a}}$  we get a left-moving conjugate spinor  $S^{\dot{a}}$  and right-moving spinor  $\tilde{S}^a$ . To flip the chiralities this operation has to be followed by the parity transformation  $I_9$  to get a genuine symmetry:

$$(8s, 8c) \xrightarrow{\Omega} (8c, 8s) \xrightarrow{I_9} (8s, 8c).$$
 (35)

In §2.3 we discussed the  $SL(2, \mathbf{Z})$  invariance of the effective action of massless states. Let us now turn to the spectrum of massive BPS-saturated states. BPS states are special states in the spectrum that preserve some of the spacetime supersymmetries. The mass of a BPS-state is proportional to its charge, and because of supersymmetry their spectrum is not quantum corrected [65]. If  $SL(2, \mathbf{Z})$  is to be the duality symmetry, then the spectrum of BPS states must be invariant under the  $SL(2, \mathbf{Z})$ .

The spectrum of perturbative BPS-states by itself is certainly not invariant under  $SL(2,\mathbf{Z})$ . For example, take a string that winds around the  $X^9$  direction as in §2.4 but carries no momentum along that direction. Such a state is a BPS-state and its mass is proportional to the winding number [13]. The winding number is in fact the quantized charge of NS-NS 2-form field  $B_{MN}$ . This can be checked easily from a vertex operator calculation. Now, the element S of the duality group (Eq. 15) takes the NS-NS field B to the R-R field B'. Therefore, for duality to hold, we must find BPS-states that that are charged with respect to B'. But there are no such states in the perturbative spectrum. This follows from a general fact that the vertex operator for the R-R fields involves their field strengths and not the potential [46]. As a result, all perturbative states couple to the R-R field strength and not to the potential. This coupling of R-R forms to perturbative string states is analogous to the coupling of a photon to a neutron. A neutron has a magnetic moment that couples to the field strength but has no charge that can couple minimally to the vector potential.

The 2-form B' can couple to a string or a one-dimensional brane. In general, a (p+1)-form from the R-R sector would couple to an extended soliton which is a p-dimensional membrane or a p-brane. Even though there are no such states in the perturbative spectrum that couple minimally to the R-R fields, they do exist in the spectrum as nonperturbative solitons. What is more, these solitons have an amazingly simple description in terms of free open strings with mixed Dirichlet and Neumann boundary conditions [42, 44, 46].

Let us recall some facts about open strings. The mode expansion for an open string is very similar to Eq. 3, but at the end points of the string the left-moving wave gets reflected and turns into a right-moving wave. It can reflect back either in phase or out of phase with in the incoming wave, which corresponds to either Neumann or Dirichlet boundary condition respectively. Let us take the worldsheet coordinate  $\sigma$  along open string to run from 0 to  $\pi$ . Then the boundary conditions at both ends of the open string is

Neumann : 
$$\partial_{+}X = \partial_{-}X$$
,  
Dirichlet :  $\partial_{+}X = -\partial_{-}X$ , (36)

at the ends  $\sigma = 0, \pi$ . The mode expansion is

$$X(\sigma,0) = x + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\alpha_n}{n} (e^{in\sigma} \pm e^{-in\sigma}), \tag{37}$$

where the + sign is for Neumann boundary condition at both ends (NN sector), and the - sign is for Dirichlet boundary condition at both ends (DD sector). An important difference between these boundary conditions is that with Dirichlet boundary condition, the zero mode term  $p\tau$  in Eq. 3 is absent in the mode expansion. This reflects the fact that the string cannot move in this direction because the end points are stuck at the position x.

To describe a p-dimensional soliton, consider a p-dimensional hyperplane along the directions  $X^1, \ldots, X^p$ . Take the longitudinal coordinates  $X^{\mu}, \mu =$  $0, \ldots, p$  to satisfy NN boundary conditions, and the transverse coordinates  $X^m, m = p + 1, \dots, 9$  to satisfy DD boundary conditions. These boundary conditions break translational invariance. Open strings are allowed to end on the p-dimensional hyperplane which can be viewed as a p-brane at a location determined by the zero mode of the coordinates  $X^m$ . This configuration, called a Dirichlet p-brane, behaves in every respect like a BPS soliton. It couples to gravity and the R-R (p+1)-form field. Its mass is proportional to the charge with respect to the RR field. The D-brane worldvolume carries a U(1)supersymmetric gauge theory that is obtained by dimensional reduction of N=1 super Yang-Mills theory in ten dimensions to p+1 dimensions. This can be seen from quantization of the superstring subject to the Dirichlet and Neumann boundary conditions [46]. The states  $\psi^{\mu}_{-\frac{1}{2}}|NS\rangle$  in the NS sector give the a vector of U(1) on the worldvolume and the states  $\psi^m_{-\frac{1}{2}}|\text{NS}\rangle$  are the scalar superpartners in the worldvolume. The Ramond sector gives the fermionic superpartners.

If there are n identical parallel D-branes, then the open string can begin on a D-brane labeled by i and end on one labeled by j (Figure 2). The label of



Figure 2: An open string beginning on the i-th D-brane and ending on the j-th D-brane.

the D-brane is what in early string theory was called the Chan-Paton index at each end. Let us denote a general state in the open string sector by  $|\psi, ij\rangle \lambda_{ij}$ .

Here i, j are Chan-Paton indices,  $\lambda_{ij}$  is the Chan-Paton wave-function,  $\psi$  is the state of the worldsheet fields, and by reality of the string wave function,  $\lambda^{\dagger} = \lambda$ . The massless excitations of the open string now give rise to a supersymmetric U(n) gauge theory on the worldvolume.

The spectrum of these nonperturbative states provides many non-trivial checks of duality. For example,  $SL(2, \mathbf{Z})$  predicts a whole tower of 'dyonic' (p,q) strings that that have charge p with respect to B and charge q with respect to B' [50, 69]. Many of these predictions have now been confirmed providing substantial evidence for the correctness of duality.

T-duality has a simple action on D-branes [46]. T-duality along a longitudinal direction of a p-brane turns it into a (p-1)-brane, and T-duality along a transverse direction turns it into a (p+1) brane. This follows from the observation that T-duality is a one-sided parity transform so it turns Dirichlet and Neumann boundary conditions into each other.

### 3 Orbifolds

### 3.1 General Remarks

Given a manifold  $\mathcal{M}$  with a discrete symmetry G, one can construct an orbifold  $\mathcal{M}' = \mathcal{M}/G$ . If the symmetry acts freely on  $\mathcal{M}$ , *i.e.*, without any fixed points, then  $\mathcal{M}'$  is also a smooth manifold. If there are fixed points then  $\mathcal{M}'$  is singular near the fixed points. If we now consider strings moving on a target space  $\mathcal{M}$ , then we are naturally led to the concept of orbifolds in conformal field theory.

Consider a theory A with a discrete symmetry group G. One can construct a new theory A' = orbifold of A by G,

$$A' = A/G$$
.

For simplicity we shall take G to be a  $\mathbf{Z}_2 \equiv \{1, \alpha\}$  generated by an involution  $\alpha$  because in fact all examples used in these lectures are  $\mathbf{Z}_2$  orbifolds.

In point particle theory, we simply take the Hilbert space of A and keep only those states that are invariant under G to obtain the Hilbert space of A'. However, the particle propagation would be singular near the fixed points of G. In closed string theory, we must also add the "twisted sectors" that are localized near the fixed points. In twisted sectors, the string is closed only up to an action by an element of the group. What is surprising is that after the inclusion of twisted sectors, string propagation on the orbifold is nonsingular even near the fixed points.

In string theory, there is a well-defined procedure for adding twisted sectors. Twisted sectors are necessary for modular invariance which is the requirement that the string path integral be invariant under the modular group.

For a torus, the modular group is  $SL(2, \mathbf{Z})$ . The modular group is the group of global diffeomorphisms of the surface. Invariance with respect to this group is essential to avoid global gravitational anomalies which would render the theory inconsistent. This requirement necessitates the inclusion of twisted sectors. We refer the reader to [19, 30] for details of modular invariance. Physically, unitarity is what requires twisted sectors. Even if you excluded twisted states at tree level, once you include interactions, they will appear in loops because an untwisted string can split into a string twisted by an element  $\hat{g}$  and another string twisted by  $\hat{g}^{-1}$ .

For  $\mathbf{Z}_2$  orbifolds there are only two sectors: one untwisted and the other twisted by  $\alpha$ . In each sector we must perform the projection onto  $\mathbf{Z}_2$  invariant states with the projector  $\frac{1}{2}(1+\hat{\alpha})$ . Here  $\hat{\alpha}$  is the operator that represents the action of  $\alpha$  on the Hilbert space. In the sector twisted by  $\alpha$ , all worldsheet fields, which we generically refer to as  $\Phi$ , satisfy the boundary condition

$$\Phi(\sigma + 2\pi, \tau) = \hat{\alpha}\Phi(\sigma, \tau). \tag{38}$$

For  $\mathbb{Z}_2$  orbifolds, level matching is necessary and sufficient to ensure modular invariance at one loop. Level matching requires that

$$E_L - E_R = 0 \bmod \frac{1}{2} \tag{39}$$

where  $E_L$  and  $E_R$  are the energies of any two states of the left-moving and right-moving Hilbert spaces respectively.

In the next subsection I illustrate this procedure by constructing Type IIA theory as a  $\mathbb{Z}_2$  orbifold of Type-IIB.

# 3.2 Type-IIA theory as an orbifold

We orbifold Type-IIB theory by the symmetry group

$$\mathbf{Z}_2 \equiv \{1, (-1)^{F_L}\}. \tag{40}$$

## Untwisted Sector:

After the projection  $\frac{1}{2}(1+(-1)^{F_L})$ , all R-R and R-NS states are removed but the NS-NS and NS-R states  $||i\rangle \otimes (|j\rangle \oplus |\dot{b}\rangle)$  survive. We are left with  $g_{ij}, B_{ij}, \phi$  and a single gravitino  $\psi i\dot{b}$ .

# Twisted Sector:

The twisting of boundary conditions affects only the left-moving fermion  $S^a$  because other fields are invariant under  $(-1)^{F_L}$ .

$$S^{a}(\sigma + 2\pi) = (-1)^{F_L} S^{a}(\sigma) = -S^{a}(\sigma)$$
  

$$\tilde{S}^{a}(\sigma + 2\pi) = \tilde{S}^{a}(\sigma), \quad X^{i}(\sigma + 2\pi) = X^{i}(\sigma).$$
(41)

Therefore, the mode expansion of the coordinates  $X^i$  and  $\tilde{S}^a$  in the twisted sector is the same as in the untwisted sector. The oscillators are integer moded as before and, in particular, the right-moving ground states are given by the representation of the zero mode algebra  $\{\tilde{S}^a_0, \tilde{S}^b_0\} = \delta^{ab}$ . We thus obtain, as in the untwisted sector,

$$|j\rangle \oplus |\dot{b}\rangle$$
 (42)

as the right-moving ground states.

The oscillators of the left-moving fields are moded with half-integer modings so as to satisfy the boundary condition 41

$$S^{a} = \frac{1}{\sqrt{2}} \sum_{r=\mathbf{Z}+\frac{1}{2}} S_{r}^{a} e^{-in(\tau+\sigma)}, \tag{43}$$

The ground state energy is a sum of zero point energies of the oscillators. For a single complex boson twisted by a phase  $e^{2\pi i\eta}$ , the ground state energy is given by the formal sum  $\sum_{n=0}^{\infty} \frac{1}{2}(n+\eta)$ . It can be evaluated as  $\zeta(0,\eta)$ , where  $\zeta(k,\eta) = \sum_{n=0}^{\infty} \frac{1}{2}(n+\eta)^{-k}$  is the Riemann zeta-function which regularizes the sum. The ground state energy of a single complex boson is [19]

$$-\frac{1}{12} + \frac{1}{2}\eta(1-\eta). \tag{44}$$

The ground state energy of a fermion with the same twisting is negative of the above. Now, in the left-moving twisted sector, there are 4 (complex) bosons which are untwisted  $(\eta=0)$  and and integer moded, and 4 complex fermions that are twisted with  $(\eta=0)$  and are integer moded. Adding the zero point energies of these fields we get that the ground state energy in the twisted sector is  $-\frac{1}{2}$ . The ground state is therefore tachyonic because The mass-shell condition is  $M^2=4/\alpha'(N-\frac{1}{2})$  and the level matching condition for physical states is

$$N - \frac{1}{2} = \tilde{N} \tag{45}$$

The ground state does not satisfy the physical state condition. Moreover, it is odd under the action of  $(-1)^{F_L}$  and is any way projected out by the  $\mathbb{Z}_2$  projection. The first excited state

$$S^a_{-\frac{1}{2}}|0\rangle$$

satisfies the constraints and the  $\mathbb{Z}_2$  invariance. It gives rise to massless states

$$|a\rangle\otimes(|j\rangle\oplus|\dot{b}\rangle).$$
 (46)

We see that an additional gravitino  $\psi_{j\alpha}$  has appeared in the twisted sector with chirality opposite to the one that was projected out. The product  $|a\rangle \otimes |\dot{b}\rangle$  can be reduced as in Eq.11

$$\lambda_1^a \lambda_2^b \sim \lambda_1^T \Gamma^i \lambda_2 \oplus \lambda_1^T \Gamma^{ijk} \lambda_2. \tag{47}$$

Now, because  $\lambda_1$  and  $\lambda_2$  have opposite chirality, only products such as  $\Gamma^i$  and  $\Gamma^{ijk}$  appear. We thus obtain a vector  $A_i$  and a 3-form  $C_{ijk}$ . Altogether what we have obtained is precisely the spectrum of Type-IIA theory, which has two spinors  $S^{\dot{a}}$  and  $\tilde{S}^b$  to begin with:

### Bosons

NS-NS: metric  $g_{ij}$ , 2-form  $B_{ij}$ , dilaton  $\phi$ ,

R-R: vector  $A_i$ , 3-form  $C_{ijk}$ ,

## Fermions:

NS-R: gravitino  $\psi_{i\dot{b}}$ , R-NS: gravitino  $\psi_{ja}$ .

# 4 Type-I String as an Orientifold

An important and simple example which illustrates most of the features of the orientifold construction is Type-I theory in ten dimensions. In this section we shall work through this example in detail, after some general remarks about orientifolds.

# 4.1 General Remarks About Orientifolds

In general, a symmetry operation of a string theory A can be a combination of target spacetime symmetry and orientation-reversal on the world sheet. The group of symmetry can then be written as a union

$$G = G_1 \cup \Omega G_2$$
.

Given such a symmetry of A, one can construct a new theory A' = A/G. In section §3 we had implicitly assumed that  $G_2$  is empty and that the orbifold symmetry consists of only target space symmetries. If  $G_2$  is non-empty, the resulting theory A' is called an "orientifold" of A [17, 47, 4, 32, 33, 31, 48]. In most examples discussed recently, one starts typically with a  $\mathbf{Z}_N$  orbifold of toroidally compactified Type IIB theory and then orientifolds it further by a symmetry  $\mathbf{Z}_2 = \{1, \Omega\beta\}$ , where  $\beta$  is a  $\mathbf{Z}_2$  involution of the orbifold. If

the orbifold group  $\mathbf{Z}_N$  is generated by the element  $\alpha$ , then the total orientifold symmetry is  $G = \{1, \alpha, \dots, \alpha^{N-1}, \Omega\beta, \Omega\beta\alpha, \dots, \Omega\beta\alpha^{N-1}\}$  or symbolically,  $G = \mathbf{Z}_N \cup \Omega(\beta\mathbf{Z}_N)$ . We describe below some general features of the orientifold construction.

# (1) Unoriented Surfaces:

An orientifold is obtained, like an orbifold, by gauging the symmetry G. A non-empty  $\Omega G_2$  means that orientation reversal, accompanied by an element of  $G_2$ , is a local gauge symmetry; a string and its orientation reversed image are gauge equivalent and must be identified. Therefore, the string perturbation theory of the orientifold includes unoriented surfaces like the Klein bottle.

# (2) Closed String Sector:

The closed string sector of the theory A' consists of states in the Hilbert space of A that are invariant under G and which survive the orientifold projection. It is completely analogous to the untwisted sector of an orbifold after the projection. Typically, starting with oriented closed strings, one gets unoriented closed strings after the projection.

### (3) Tadpole Cancellation and Orientifold Planes:

Orientifolds often but not always have open strings in addition to the closed strings. The open string sector in orientifolds is analogous to, but not exactly the same as, the twisted sectors in orbifolds. In the case of orbifolds, twisted sectors are necessitated by the requirement of modular invariance. In the case of orientifolds, the one-loop diagrams in string perturbation theory include unoriented and open surfaces for which there is no analog of the modular group. There is, however, a consistency requirement for these surfaces that is analogous to the requirement of modular invariance for the torus. This is the requirement of 'tadpole cancellation'. These loop diagrams can have a divergence in the tree channel corresponding to a tadpole of a massless particle. Cancellation of all tadpoles is necessary for obtaining a stable string vacuum. This requirement is very restrictive and it more or less completely determines when and how the open string should be added.

Physically, nonzero tadpoles imply that the equations of motion of some massless fields are not satisfied. They occur for the following reason. The planes that are left invariant by the elements of  $G_2$  are called the 'orientifold planes'. Like a D-brane, an orientifold plane is a p-dimensional hyperplane which couples to an R-R (p+1)-form which we generically refer to as  $A_{p+1}$ . The charge of the orientifold plane can be calculated by looking the R-R tadpole, *i.e.*, emission of an R-R closed string state in the zero momentum limit. If the orientifold plane has a nonzero charge then it acts as a source term in the equations of motion for the (p+1)-form field  $A_{p+1}$ :

$$dH_{p+2} = *J_{7-p} d*H_{p+2} = *J_{p+1}, (48)$$

where  $H_{p+2}$  is the (p+2)-form field strength of  $A_{p+1}$ ,  $J_{p+1}$  and  $J_{7-p}$  are the 'electric' and 'magnetic' sources.

Consistency of the field equations requires that  $\int_{\Sigma_k} *J_{10-k} = 0$ , for all surfaces  $\Sigma_k$  without a boundary. In particular, there can be no net charge on a compact space. This is the analog of Gauss law in electrodynamics. The field lines emanating from a charge must either escape to infinity or end on an opposite charge. In a compact space, the field lines have nowhere to go to and hence must end on an equal and opposite charge. The only way the negative charge of a p-dimensional orientifold plane in a compact transverse space can be neutralized is by adding the right-number of Dirichlet p-branes so that Gauss law is satisfied and all tadpoles cancel.

### (4) Open String Sector and Surfaces with Boundaries:

D-branes are hyperplanes where open strings can end. Inclusion of D-branes introduces the open string sector in the theory. The action of the group G is represented in the D-brane sector by some matrices, which we denote by  $\gamma$ . The  $\gamma$  matrices act on the Chan-Paton indices:

$$g: |\psi, ij\rangle\lambda_{ij} \to |\hat{g}(\psi), ij\rangle\lambda'_{ij}; \quad \lambda \to \lambda' = \gamma_q^{-1}\lambda\gamma_g$$
 (49)

$$\Omega h: \qquad |\psi, ij\rangle \lambda_{ij} \to |\hat{\Omega h}(\psi), i'j'\rangle \lambda'_{ij}; \quad \lambda \to \lambda' = \gamma_{\Omega h}^{-1} \lambda^T \gamma_{\Omega h}$$
 (50)

Tadpole cancellation together with the requirement that the  $\gamma$  matrices furnish a representation of the symmetry G in the D-brane sector determine not only the number of D-branes but also the form of the  $\gamma$  matrices. When n D-branes coincide, the worldvolume gauge group is U(n). After the projection onto G-invariant states, we are left with a subgroup of U(n). The group as well as the representations are usually uniquely determined by the consistency requirements discussed above.

## 4.2 Orientifold Group and Spectrum of Type-I

Let me illustrate the statements in the previous subsection in the context of Type-I theory. Let me first give the orientifold group and the closed and open string spectrum before discussing tadpole cancellation and consistency conditions.

Type-I theory is an orientifold of Type-IIB theory with orientifold symmetry group

$$\mathbf{Z}_2 = \{1, \Omega\}. \tag{51}$$

## Closed String Sector:

The closed string sector of Type-I theory contains unoriented strings that are invariant under orientation-reversal. The massless states are simply the states of Type-IIB that are invariant under  $\Omega$ . From §2.2 we see that only  $g_{ij}$ ,  $\phi$ ,  $B'_{ij}$ , and a symmetric combination of the two gravitini survive the projection. Open String Sector:

Open string sector arises from the addition of D-branes that are required to cancel the charge of the orientifold plane. Orientation reversal is a purely worldsheet symmetry, so it leaves the entire nine-dimensional space invariant. Thus, the orientifold plane is a 9-plane. It turns out to have -32 units of charge with respect to the 10-form non-propagating field from the R-R sector. This charge can be canceled by adding 32 Dirichlet 9-branes which each have unit charge. The world-volume theory of the D9-branes gives rise to gauge group U(32) but only an SO(32) subgroup is invariant under the action of  $\Omega$ .

Type-I supergravity super Yang-Mills theory is anomaly free only if the gauge group is SO(32) or  $E_8 \times E_8$ . It is satisfying that the spectrum determined by the requiring worldsheet consistency is automatically anomaly free [11, 9, 10].

### 4.3 Loop Channel and Tree Channel

A massless tadpole leads to a divergence in tree channel. For calculating tadpoles it is useful to keep a field theory example in mind. Let us consider a very massive charged particle in field theory with charge Q. At low momentum, the charge acts as a stationary source for a massless photon. One can calculate the charge Q of the particle by calculating the amplitude for vacuum going into a single photon in the background of this charge. (Figure 3) Alternatively, one can calculate the interaction between two particles each of



Figure 3: A massless tadpole leads to a divergence in tree channel

charge Q at zero momentum exchange. The Feynman diagram has  $1/q^2$  where q is momentum exchange and the residue is proportional to  $Q^2$ . If we write  $1/q^2$  as  $\int_0^\infty dl \exp{(-q^2 l)}$ , then the zero momentum divergence corresponds to the divergence of this integral coming from very long propagation times l.

D-branes and orientifold planes can be treated similarly. A D-brane is like a very massive charged particle. The interaction between the i-th D-brane and the j-th D-brane due to closed string exchanges between the two branes

can be computed by evaluating a cylinder diagram with one boundary on the i-th brane and the other boundary on the j-th brane. In string theory, unlike in particle theory, because of conformal invariance the tree channel and loop channel diagrams are related. For example, as shown in Figure 6, the tree channel cylinder diagram can also be viewed as a loop-channel diagram that evaluates the loop of an open string with one end stuck at the i-th brane and the other end at the j-th brane. Similarly, the interaction between an orientifold plane and the i-th D-brane is given by the Möbius strip diagram which has one boundary stuck at the i-th brane and one crosscap stuck at the orientifold plane. Recall that a crosscap is a circular boundary with opposite points on the boundary identified. Because some of the elements of the orientifold group leave the orientifold plane invariant, the closed string that emanates from the plane has further identifications under the symmetry and it looks like a crosscap.

In summary, we can imagine that a crosscap is stuck at the orientifold plane and the boundary is stuck at a D-brane. With an orientifold with charge Q and with N D-branes of unit charge, the total charge is  $(Q+N)^2$ , which can be written as  $Q^2+N^2+2QN$ . The term  $N^2$  is proportional to the interaction between the D-branes and is computed by the cylinder diagram, the interaction 2QN between the D-branes and orientifold planes is computed by the Möbius strip diagram and the interaction between orientifold planes  $Q^2$  is computed by the Klein bottle diagram. An efficient way to evaluate these diagrams is to compute them in loop channel and then factorize them in tree channel.

The loop-counting parameter in string theory is the Euler character. A k-th order term in string perturbation theory which goes as the k-th power of the string coupling constant  $\lambda$  corresponds to Riemann surfaces with Euler



Figure 4: A Surface with two boundaries, one crosscap and one handle

character k-1. The Euler character of a Riemann surface with b boundaries, c crosscaps, and h handles is given by

$$\chi = 2 - 2h - b - c. \tag{52}$$

A surface with no crosscaps is orientable, otherwise it is nonorientable. We are interested in the first quantum correction, *i. e.*, Riemann surfaces with  $\chi = 0$ . There are four surfaces that contribute: a torus (one handle), a Klein Bottle (two crosscaps), a Möbius strip (one boundary, one crosscap), and a cylinder (two boundaries) ( Figure 5). Let  $\sigma_1$  and  $\sigma_2$  be the coordinate of the surface.

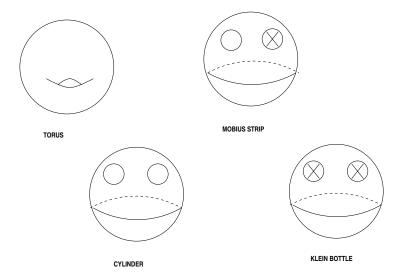


Figure 5: Surfaces with  $\chi = 0$  in the orientifold perturbation theory

Then we may time slice along constant  $\sigma_1$  or along constant  $\sigma_2$ . For a tours, both two time-slicings give a loop diagram, but for the other three surfaces one time slicing give a loop diagram and the other time slicing gives a tree diagram. For these three surfaces, we would like to determine, for later use, what a closed string of length  $2\pi$  propagating for time  $2\pi l$  in the tree channel corresponds to in the loop channel.

The simplest surface is the cylinder. Consider, as shown on the left in Figure 6, a closed string of length  $2\pi$  in tree channel, propagating between two D-branes for a Euclidean time  $2\pi l$ . Time runs sideways in the diagram. Now, use the conformal invariance of string theory to conformally rescale the coordinates by  $\frac{1}{2l}$  and take time to run upwards to get the diagram on the right in Figure 6. This diagram represents an open string of length  $\pi$  beginning on one D-brane and ending on the other D-brane, propagating in a loop for a time  $2\pi t \equiv \pi/l$ . We conclude that t = 1/2l for the cylinder.

For the Klein bottle, consider the double cover of the bottle, viz., a torus

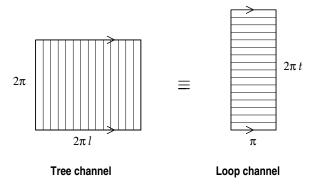


Figure 6: Two ways to view a cylinder.

with coordinates  $0 \le \sigma_1 \le 4\pi l$  and  $0 \le \sigma_2 \le 2\pi$ , with the identifications  $\sigma_1 \sim \sigma_1 + 4\pi l$  and  $\sigma_2 \sim \sigma_2 + 2\pi$ . The Klein bottle is obtained by a  $\mathbb{Z}_2$  identification of the torus:

$$(\sigma_1, \sigma_2) \sim (4\pi l - \sigma_1, \sigma_2 + \pi). \tag{53}$$

We can choose two different fundamental regions. If we choose the fundamental region as on the top of Figure 7, then we get the tree channel diagram. It represents a closed string propagating between two orientifold planes for time  $2\pi l$ . If we choose the fundamental domain taking time to run upwards now, as on the bottom of Figure 7, then we have a closed string propagating in a loop and undergoing a twist  $\Omega$ . We have to rescale by 1/2l to obtain a closed string of length  $2\pi$  in the loop, which gives t = 1/4l.

Similarly, for the Möbius strip we consider the double cover, viz., a cylinder with coordinates  $0 \le \sigma_1 \le 4\pi l$  and  $0 \le \sigma_2 \le 2\pi$ , with the identification  $\sigma_1 \sim \sigma_1 + 4\pi l$ . The Möbius strip is obtained by the same  $\mathbb{Z}_2$  identification as in Eq. 53. Again, we can choose two different fundamental regions. The fundamental region as on the top of Figure 8 gives the tree channel diagram which represents a closed string propagating between an orientifold plane and

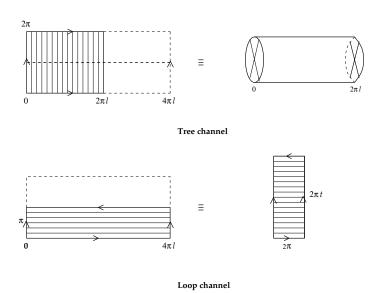


Figure 7: Two ways to view the Klein Bottle.

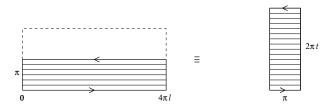
a D-brane. The fundamental domain as on the bottom of Figure 8, taking time to run upwards now, represents an open string, with both ends on the D-brane, propagating in a loop and undergoing a twist  $\Omega$ . We have to rescale by 1/4l to obtain an open string of length  $\pi$  in the loop, which gives t = 1/8l.

To summarize, a closed string of length  $2\pi$  propagating for time  $2\pi l$  in the tree channel corresponds to an open string of length  $\pi$ , or a closed string of length  $2\pi$  propagating for time  $2\pi t$  in the loop channel. For fixed l in the tree channel, the loop channel time t for different surfaces is given by

Cylinder:  $t = \frac{1}{2l}$ KleinBottle:  $t = \frac{1}{4l}$ 



Tree channel



Loop channel

Figure 8: Two ways to view a Möbius strip.

MobiusStrip: 
$$t = \frac{1}{8l}$$
. (54)

We also need to know how the boundary conditions in tree channel map onto boundary conditions in loop channel. In the tree channel, ( $0 \le \sigma^1 \le 2\pi l$ ,  $0 \le \sigma^2 \le 2\pi$ ) the periodicity and boundary conditions on a generic world-sheet field  $\phi$  in the g-twisted sector (see Figure 9) are as follows:

KB: 
$$\phi(0, \pi + \sigma^2) = \Omega \tilde{h}_1 \phi(0, \sigma^2), \quad \phi(2\pi l, \pi + \sigma^2) = \Omega \tilde{h}_2 \phi(2\pi l, \sigma^2)$$
$$\phi(\sigma^1, 2\pi + \sigma^2) = \tilde{g}\phi(\sigma^1, \sigma^2)$$
MS: 
$$\phi(0, \sigma^2) \in \tilde{M}_i, \quad \phi(2\pi l, \pi + \sigma^2) = \Omega \tilde{h}\phi(2\pi l, \sigma^2)$$
$$\phi(\sigma^1, 2\pi + \sigma^2) = \tilde{g}\phi(\sigma^1, \sigma^2)$$

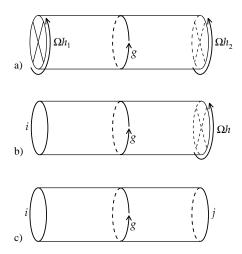


Figure 9: a) Klein bottle. b) Möbius strip. c) Cylinder.

C: 
$$\phi(0, \sigma^2) \in \tilde{M}_i, \quad \phi(2\pi, \sigma^2) \in \tilde{M}_j,$$
  

$$\phi(\sigma^1, 2\pi + \sigma^2) = \tilde{g}\phi(\sigma^1, \sigma^2). \tag{55}$$

Here  $M_i$  is the submanifold where the *i*-th D-brane is located. The tilde on the group elements allows for additional  $\pm$  signs that depend on the GSO projection to accompany the action of the group element for world-sheet fermions. The definitions in Eq. 55 are consistent only if

KB: 
$$(\Omega \tilde{h}_1)^2 = (\Omega \tilde{h}_2)^2 = \tilde{g}$$
  
MS:  $(\Omega \tilde{h})^2 = \tilde{g}, \quad \tilde{g}\tilde{M}_i = \tilde{M}_i$   
C:  $\tilde{g}\tilde{M}_i = \tilde{M}_i, \quad \tilde{g}\tilde{M}_j = \tilde{M}_j$ ; (56)

otherwise the corresponding path integral vanishes.

The loop channel for the Klein bottle and the Möbius strip  $(0 \le \sigma^1 \le 4\pi l, 0 \le \sigma^2 \le \pi)$  is obtained geometrically by taking the upper strip  $\pi \le \sigma^2 \le 2\pi$ , inverting it from right to left, multiplying the fields by  $(\Omega \tilde{h}_2)^{-1}$ , and gluing it to the right side of the lower strip. This construction ensures that the fields are smooth at  $\sigma^1 = 2\pi l$ . The periodicity conditions are

KB: 
$$\phi(\sigma^{1}, \pi + \sigma^{2}) = \Omega \tilde{h}_{2} \phi(4\pi l - \sigma^{1}, \sigma^{2}), \quad \phi(4\pi l, \sigma^{2}) = \tilde{g}' \phi(0, \sigma^{2})$$
MS: 
$$\phi(\sigma^{1}, \pi + \sigma^{2}) = \Omega \tilde{h} \phi(4\pi l - \sigma^{1}, \sigma^{2}), \quad \phi(0, \sigma^{2}) \in \tilde{M}_{i},$$

$$\phi(4\pi l, \sigma^{2}) \in \tilde{M}_{i}$$
(57)

where  $\tilde{g}' = \Omega \tilde{h}_2(\Omega \tilde{h}_1)^{-1}$ . Rescaling the coordinates to standard length for string in loops the respective amplitudes are

where the closed string trace is labeled by the spacelike twist g' and the open string traces are labeled by the Chan-Paton labels.

## 4.4 Tadpole Calculation

We have followed in these lectures the formalism and notations of Gimon and Polchinski [20]. A similar formalism was used for bosonic orientifolds by Pradisi and Sagnotti in earlier work [47]. The tadpole constraints that we are about to describe were applied to orientifolds also in refs. [32, 37, 5, 4, 49]. An equivalent but technically different method for calculating tadpoles is to construct the boundary state and the crosscap state. We do not use the boundary state method in these lectures but the details can be found in [9, 10, 11, 38].

One-loop amplitude calculates the one loop cosmological constant in spacetime as the sum of zero point energies of all the fields in the spectrum of the string. Let us look, for example, the sum of zero point energies of the fields in the open string sector:

$$\sum_{\text{bosons}} \frac{\hbar \omega_{\vec{p}}}{2} - \sum_{\text{fermions}} \frac{\hbar \omega_{\vec{p}}}{2} = -\sum_{i} \frac{V_{10}}{(2\pi)^{10}} \int d^{10}p \, \frac{1}{2} \log(p^2 + m_i^2) (-1)^{F_i}, \quad (59)$$

where  $m_i$  is the mass and  $F_i$  the spacetime fermion number of a state i. Now we use the identity

$$\log A = -\lim_{\epsilon \to 0} \frac{d}{d\epsilon} A^{-\epsilon} = -\lim_{\epsilon \to 0} \frac{d}{d\epsilon} \left(\epsilon \int \frac{dt}{t^{1-\epsilon}} e^{-2\pi At}\right) = -\int \frac{dt}{t} e^{-2\pi At}, \quad (60)$$

and

$$\alpha'(p^2 + m_i^2) = L_0 = \alpha' p^2 + \sum_{i=1}^8 \alpha_{-n}^i \alpha_{-n}^i + \sum_{i=1}^8 \psi_{-r}^i \psi_r^i + a, \tag{61}$$

where we have included, for convenience, the normal ordering constant a in the definition of  $L_0$ . The sum in Eq. 59 then equals

$$\int_0^\infty \frac{dt}{2t} \operatorname{Tr}_{NS-R} \exp^{-2\pi t L_o} \left( 1 + \frac{(-1)^f}{2} \right). \tag{62}$$

Here f is the worldsheet fermion number,  $\frac{1+(-1)^f}{2}$  performs the GSO projection, and the combination NS-R for the trace takes into  $(-1)^{F_i}$  in Eq. 59. The Trace includes the momentum integration  $\frac{V_{10}}{(2\pi)^{10}}\int d^{10}p$ , where  $V_{10}$  is, as usual, the regularized volume of a 10-torus that is taken to be very large to get the theory in 10 flat spacetime dimensions.

The trace Eq. 62 in the canonical formalism equals the path integral on a cylinder by the usual relation between the canonical formalism and the path integral formalism. An open string propagates in a loop for Euclidean time  $2\pi t$  with time evolution operator  $\exp^{-2\pi t L_o}$  giving a cylinder diagram.

To obtain the orientifold we have to project onto states that are invariant under  $\Omega$ , which is achieved by inserting the projector  $\frac{1+\Omega}{2}$  in Eq. 62. The term proportional to 1/2 corresponds to the cylinder and the term proportional to  $\Omega/2$  corresponds to the Möbius strip.

The resulting partition sums for the Klein bottle, the Möbius strip, and the cylinder are respectively  $\int_0^\infty dt/2t$  times

KB: 
$$\frac{\text{Tr}}{\text{NSNS+RR}} \left\{ \frac{\Omega}{2} \frac{1 + (-1)^f}{2} e^{-2\pi t (L_0 + \tilde{L}_0)} \right\}$$
MS: 
$$\frac{\text{Tr}}{\text{NS-R}} \left\{ \frac{\Omega}{2} \frac{1 + (-1)^f}{2} e^{-2\pi t L_0} \right\}$$
C: 
$$\frac{\text{Tr}}{\text{NS-R}} \left\{ \frac{1}{2} \frac{1 + (-1)^f}{2} e^{-2\pi t L_0} \right\}.$$
(63)

For the Klein bottle, the sectors NS-R and R-NS are mapped into each other by  $\Omega$  and therefore do not contribute to the trace. In the closed string sector, the Virasoro generators  $L_0$  and  $\tilde{L}_0$  are

$$L_{0} = \frac{\alpha' p^{2}}{4} + \sum_{i=1}^{8} \alpha_{-n}^{i} \alpha_{-n}^{i} + \sum_{i=1}^{8} \psi_{-r}^{i} \psi_{r}^{i} + a,$$

$$\tilde{L}_{0} = \frac{\alpha' p^{2}}{4} + \sum_{i=1}^{8} \tilde{\alpha}_{-n}^{i} \tilde{\alpha}_{-n}^{i} + \sum_{i=1}^{8} \tilde{\psi}_{-r}^{i} \tilde{\psi}_{r}^{i} + \tilde{a},$$
(64)

where n is summed over integers; r is summed over integers in the Ramond sector and over half-integers in the Neveu-Schwarz sector. The normal ordering constant a is 0 in the R sector, -1/2 in the NS sector, and similarly for  $\tilde{a}$ .

The action of  $\Omega$  on the modes of the closed string is

$$\Omega \alpha_r \Omega^{-1} = \tilde{\alpha}_r, \qquad \Omega \psi_r \Omega^{-1} = \tilde{\psi}_r, \qquad \Omega \tilde{\psi}_r \Omega^{-1} = -\psi_r$$
 (65)

for integer and half-integer r. The minus sign included in the last equation gives the convenient result  $\Omega \psi_M \tilde{\psi}_M \Omega^{-1} = \psi_M \tilde{\psi}_M$  for any product  $\psi_M$  of mode operators. If this sign can be omitted it just corresponds to  $\Omega \to (-1)^f \Omega$ , which has the same action on physical states. In open string, the mode expansions for a boson X are as in Eq. 37. Orientation reversal,  $X(\sigma,0) \to X(\pi-\sigma,0)$ , takes

$$\alpha_m \to \pm e^{i\pi m} \alpha_m.$$
 (66)

with the upper sign for NN boundaries conditions and lower for DD. For fermions, the mode expansions are

$$\psi(\sigma,0) = \sum_{r} e^{ir\sigma} \psi_r, \qquad \tilde{\psi}(\sigma,0) = \sum_{r} e^{-ir\sigma} \psi_r.$$
 (67)

Orientation reversal,  $\psi(\sigma, 0) \to \pm \tilde{\psi}(\pi - \sigma, 0)$ , takes

$$\psi_r \to \pm e^{i\pi r} \psi_r$$
 (68)

for integer and half-integer r. As for the closed string there is some physically irrelevant sign freedom. Following Gimon and Polchinski [20] we define

$$f_1(q) = q^{1/12} \prod_{n=1}^{\infty} (1 - q^{2n}), \qquad f_2(q) = q^{1/12} \sqrt{2} \prod_{n=1}^{\infty} (1 + q^{2n})$$

$$f_3(q) = q^{-1/24} \prod_{n=1}^{\infty} (1 + q^{2n-1}), \qquad f_4(q) = q^{-1/24} \prod_{n=1}^{\infty} (1 - q^{2n-1}), \quad (69)$$

which satisfy the Jacobi identity

$$f_3^8(q) = f_2^8(q) + f_4^8(q) (70)$$

and have the modular transformations

$$f_1(e^{-\pi/s}) = \sqrt{s} f_1(e^{-\pi s}), \ f_3(e^{-\pi/s}) = f_3(e^{-\pi s}), \ f_2(e^{-\pi/s}) = f_4(e^{-\pi s}).$$
 (71)

These combinations are so defined that the normal ordering constant are automatically taken into account. The relevant amplitudes are then given by  $(1-1)\frac{v_{10}}{256}\int_0^\infty \frac{dt}{t^6}$  times

KB: 
$$32 \frac{f_4^8(e^{-2\pi t})}{f_1^8(e^{-2\pi t})}$$
MS: 
$$-\frac{f_2^8(e^{-2\pi t})f_4^8(e^{-2\pi t})}{f_1^8(e^{-2\pi t})f_3^8(e^{-2\pi t})} \left\{ \text{Tr}(\gamma_{\Omega}^{-1}\gamma_{\Omega}^T) \right\}$$
C: 
$$\frac{f_4^8(e^{-\pi t})}{f_1^8(e^{-\pi t})} \left\{ (\text{Tr}(\gamma_1))^2 \right\}. \tag{72}$$

We have defined  $v_{10} = V_{10}/(4\pi^2\alpha')^3$  where  $V_{10}$  is the regulated spacetime. The factor (1-1) corresponds to NSNS-RR exchange.

The total amplitude in Eq. 72 is zero by supersymmetry. In the loop channel, the vanishing is because of the cancellation between the bosonic and fermionic zero point energies. In the tree channel, there is a different interpretation. The amplitude vanishes because the graviton-dilaton exchange is attractive but the RR exchange is repulsive so the net force is zero. We are of course interested in making sure that the tadpole of the RR field by iteself is zero.

Let me indicate where all the factors come from. Consider the cylinder amplitude. Exchange of R-R field in the tree channel means a periodic boundary condition for the worldsheet fermions in the NSR formalism. In the loop channel this corresponds to periodic boundary condition in the Euclidean time direction which calculates  $\underset{NS-R}{\text{Tr}} (-1)^f$ . In the Ramond sector  $Tr(-1)^f = 0$  because the ground state has equal number of states that have odd and even fermion number. In the NS sector the fermions are half-integer moded and the bosons are integer moded. Therefore,

$$\operatorname{Tr}_{NS} e^{-2\pi t N} (-1)^f \sim \frac{\Pi(1 - e^{-2\pi t (n+1/2)})^8}{\Pi(1 - e^{-2\pi t n})^8} \sim \frac{f_4^8(e^{-\pi t})}{f_1^8(e^{-\pi t})}$$
(73)

The momentum integration gives a factor of  $(1/2\alpha')^5$  and there is a factor of 1/16 because of all the 1/2's in the projectors in Eq. 63.

The Möbius strip and the Klein bottle amplitude can be computed similarly. It is important to keep track of the boundary conditions for fermions as we go from the loop channel to tree channel (Eq. 57).

To factorize in tree channel, we use the modular transformations Eq. 71 and the Poisson resummation formula

$$\sum_{n=-\infty}^{\infty} e^{-\pi(n-b)^2/a} = \sqrt{a} \sum_{s=-\infty}^{\infty} e^{-\pi a s^2 + 2\pi i s b}.$$
 (74)

Using the relations Eq. 54 between t and l, the total amplitude for large l becomes

$$(1-1)\frac{v_{10}}{16} \int_0^\infty dl \left\{ 32^2 - 64 \text{Tr}(\gamma_{\Omega}^{-1} \gamma_{\Omega}^T) + (\text{Tr}(\gamma_1))^2 \right\}. \tag{75}$$

# 4.5 Determination of the Gauge Group

When we gauge a symmetry group G, we identify field configurations that are gauge-equivalent. To be able to to gauge a symmetry, the group must furnish a

proper representation, and not merely a projective (i. e. representation up to a phase) representation of a group. In the open string sector, this requirement places restrictions on the  $\gamma$  matrices. For example, in our case,  $\Omega^2$  should equal 1.

$$\Omega^{2}: \qquad |\psi, ij\rangle\lambda_{ij} \to |\psi, i'j'\rangle\lambda'_{ij}; \quad \lambda \to \lambda' = (\gamma_{\Omega}^{-1}\gamma_{\Omega}^{T})\lambda(\gamma_{\Omega}^{-1} \ \gamma_{\Omega}^{T})^{-1}$$
 (76)

implying that

$$\gamma_{\Omega}^{-1} \gamma_{\Omega}^{T} = 1, \tag{77}$$

or

$$\gamma_{\Omega}^{T} = \pm \gamma_{\Omega}. \tag{78}$$

Furthermore,  $\gamma_1 = 1$ , so that  $Tr(\gamma_1) = n_9$  is the number of D9-branes.

If  $\gamma_{\Omega}$  is symmetric, then by a unitary change of basis  $\gamma_{\Omega} \to U \gamma_{\Omega} U^T$ , we can make  $\gamma_{\Omega} = 1$ . If  $\gamma_{\Omega}$  is antisymmetric,  $n_9$  must be even and we can choose a basis such that  $\gamma_{\Omega}$  is the symplectic matrix

$$J = \begin{bmatrix} 0 & iI \\ -iI & 0 \end{bmatrix}. \tag{79}$$

Lets look at the open-string sector. Tachyon is projected out by the GSO projection. For the massless open string vector

$$\psi_{-1/2}^{i}|0 \ ij\rangle\lambda_{ij},\tag{80}$$

the  $\Omega$  eigenvalue of the oscillator state is -1.  $^c$  For  $\gamma_{\Omega}$  symmetric, the Chan-Paton wavefunction of the vector is then antisymmetric, giving the gauge group  $SO(n_9)$ . For  $\gamma_{\Omega}$  antisymmetric, the massless vectors form the adjoint of  $USp(n_9)$ .

Tadpole cancellation requires that the integrand in Eq. 75 must vanish, giving us  $n_9 = 32$  and  $\gamma_{\Omega}^T = \gamma_{\Omega}$ . Therefore, the Chan-Paton wave function satisfies  $\lambda^T = -\lambda$  and the gauge group of Type-I theory is SO(32).

Let us recapitulate why we obtained SO(32), i.e., why we needed 32 D-branes. To get the number 32, the Klein bottle diagram should be  $32^2$  times larger than the cylinder diagram. This factor comes as follows. There is a factor of  $2^5$  that comes from momentum integration in the loop channel because of the difference between the Hamiltonians for closed strings and for open strings. There is an additional factor of  $2^5$  in going from the variable t to variable t in Eq. 72 using Eq. 54. We see therefore that the gauge group is closely linked with the number of spacetime dimensions.

<sup>&</sup>lt;sup>c</sup>The overall sign of  $\Omega$  is fixed by the requirement that string interactions, or equivalently, the correlation functions in the conformal field theory preserve  $\Omega$ . The minus sign is easiest to see for the bosonic string [46] and in the ghost number zero picture for the superstring [23].

## 5 Some Compactifications on K3

## 5.1 K3 as an Orbifold

"Kummer's third surface" or K3 has played an important role in many developments concerning duality. Let us recall some of its properties. K3 is a four dimensional manifold which has SU(2) holonomy. To understand what this means, consider a generic 4d real manifold. If you take a vector in the tangent space at point P, parallel transport it, and come back to point P, then, in general, it will be rotated by an SO(4) matrix:

$$V_i(P) \to O_{ij} \ V_i(P) \quad O_{ij} \in SO(4).$$
 (81)

Such a manifold is then said to to have SO(4) holonomy. In the case of K3, the holonomy is a subgroup of SO(4), namely SU(2). The smaller the holonomy group, the more "symmetric" the space. For example, for a torus, the holonomy group consists of just the identity because the space is flat and Riemann curvature is zero; so, upon parallel transport along a closed loop, a vector comes back to itself. For a K3, there is nonzero curvature but it is not completely arbitrary: the Riemann tensor is non-vanishing but the Ricci tensor  $R_{ij}$  vanishes. Therefore, K3 can alternatively be defined as the manifold of compactification that solves the vacuum Einstein equations.

Only other thing about K3 that we need to know is the topological information. A surface can have nontrivial cycles which cannot be shrunk to a point. For example, a torus has two nontrivial 1-cycles. The number of nontrivial k-cycles which cannot be smoothly deformed into each other is given by the k-th Betti number  $b_k$  of the surface. The number of non-trivial k-cycles is in one to one correspondence with the number of harmonic k-forms on the surface given by the k-th de-Rham cohomology [28]. A harmonic k-form  $F_k$  satisfies the Laplace equation, or equivalently satisfies the equations

$$d^*F_k = 0, \qquad dF_k = 0 \tag{82}$$

A manifold always has a harmonic 0-form, viz., a constant, and a harmonic 4-form, viz., the volume from, assuming we can integrate on it. K3 has no harmonic 1-forms or 3-forms, but has 22 harmonic 2-forms. So, the Betti numbers for K3 are:

$$b_0 = 1,$$
  $b_1 = 0,$   $b_2 = 22,$   $b_3 = 0,$   $b_4 = 1.$  (83)

Out of the 22 2-forms, 19 are anti-self-dual, and 3 are self-dual. In other words,

$$b_2^s = 3, b_2^a = 19. (84)$$

This is all the information one needs to compute the massless spectrum of compactifications on K3.

K3 has a simple description as a  $\mathbb{Z}_2$  orbifold of a 4-torus. Let  $(x_1, x_2, x_3, x_4)$  be the real coordinates of the torus  $\mathbb{T}^4$ . Let us further take the torus to be a product  $\mathbb{T}^4 = \mathbb{T}^2 \times \mathbb{T}^2$ . Let us introduce complex coordinates  $(z_1, z_2)$ ,  $z_1 = x_1 + ix_2$  and  $z_2 = x_3 + ix_4$ . The 2-torus with coordinate  $z_1$  is defined by the identifications  $z_1 \sim z_1 + 1 \sim z_1 + i$ , and similarly for the other torus. The tangent space group is  $Spin(4) \equiv SU(2)_1 \times SU(2)_2$ , and the vector representation is  $4\mathbf{v} \equiv (\mathbf{2}, \mathbf{2})$ . If we take a subgroup  $SU(2)_1 \times U(1)$  of Spin(4), then the vector decomposes as

$$4\mathbf{v} = \mathbf{2}_+ \oplus \mathbf{\bar{2}}_-. \tag{85}$$

The coordinates  $(z_1, z_2)$  transform as the doublet  $\mathbf{2}_+$  and  $(\bar{z}_1, \bar{z}_2)$  as the  $\bar{\mathbf{2}}_-$ . The  $\mathbf{Z}_2 = \{1, I\}$  is generated by

$$I: (z_1, z_2) \to (-z_1, -z_2).$$
 (86)

This  $\mathbf{Z}_2$  is a subgroup and in fact the center of  $SU(2)_1$ . Consequently, as we shall see, the resulting manifold has SU(2), indeed a  $\mathbf{Z}_2$  holonomy. For a torus coordinatized by  $z_1$ , there are 4 fixed points of  $z_1 \to -z_1$  (Figure 10). Altogether, on  $\mathbf{T}^4/\mathbf{Z}_2$ , there are 16 fixed points. Let us calculate the number

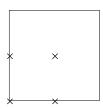


Figure 10: The four fixed point of reflection of a torus

of harmonic forms on this orbifold. To begin with, we have on the torus  $\mathbf{T}^4$ , the following harmonic forms:

$$1 1 4 dx^{i} 6 dx^{i} \wedge dx^{j} 4 dx^{i} \wedge dx^{j} \wedge dx^{l} 1 dx^{i} \wedge dx^{j} \wedge dx^{k} \wedge dx^{l}. (87)$$

The first column gives the number of forms indicated in the second column where the indices i, j, k, l take values  $1, \dots 4$ . Under the reflection I, only the even forms  $1, dx^i \wedge dx^j$ , and  $dx^i \wedge dx^j \wedge dx^k \wedge dx^l$  survive.

where the second column give the number of forms on the torus and the third column the number of forms that survive the projection. Let us look at the 2-forms from the torus that survive the  $\mathbf{Z}_2$  projection. By taking the combinations

$$dx^i \wedge dx^j \pm \frac{1}{2} \epsilon^{ijkl} dx^k \wedge dx^l$$

we see that three of these 2-forms are self-dual and the remaining three are anti-self-dual.

At the fixed point of the orbifold symmetry there is a curvature singularity. The singularity can be repaired as follows. We cut out a ball of radius R around each point, which has a boundary  $S^3/\mathbb{Z}_2$ , replace it with a noncompact smooth manifold that is also Ricci flat and has a boundary  $S^3/\mathbb{Z}_2$ , and then take the limit  $R \to 0$ . The required noncompact Ricci-flat manifold with boundary  $S^3/\mathbb{Z}_2$  is known to exist and is called the Eguchi-Hanson space [22]. The Betti number of the Eguchi Hanson space are  $b_0 = b_4 = 1$  ad  $b_2^a = 1$ . Therefore, each fixed point contributes an anti-self-dual 2-form which corresponds to a nontrivial 2-cycle in the Eguchi-Hanson space that would be stuck at the fixed point in the limit  $R \to 0$ .

Altogether, we get  $b_0 = 1$ ,  $b_2^s = 3$ ,  $b_2^a = 3 + 16 = 19$ ,  $b_4 = 1$ , and  $b_1 = b_3 = 0$  giving us the cohomology of K3. It obviously has SU(2) holonomy. Away from the fixed point, a parallel transported vector goes back to itself, because all the curvature is concentrated at the fixed points. As we go around the fixed point a vector is returned to its reflected image (for instance,  $(dz_1, dz_2) \rightarrow -(dz_1, dz_2)$ ), i.e., transformed by an element of SU(2).

In string theory there is no need to repair the singularity by hand. We shall see in §5.3 and §5.4 that the twisted states in the spectrum of Type-II string moving on an orbifold automatically take care of the repairing. The twisted states somehow know about the Eguchi-Hanson manifold that would be necessary to geometrically repair the singularity. A general method of computing the cohomology of orbifolds in conformal field theory is described in [70].

### 5.2 K3 as an Elliptic Fibration over a 2-sphere

Let us first recall the description of a 2-torus as an elliptic curve. An elliptic curve is a complex 1-dimensional curve defined by a polynomial equation involving two complex variables x and y,

$$y^2 = x^3 + fx + g, (89)$$

where f and g are complex numbers that determine the parameters of the torus.

There are a number of ways to see that this equation defines a torus. Before proceeding it will be useful to recall some relevant facts from algebraic geometry [28]. A Calabi-Yau n-fold ( $\mathbf{CY}_n$ ) is an n complex dimensional manifold with SU(n) holonomy. For example, K3 is a Calabi-Yau 2-fold, since it has SU(2) holonomy. A simple way to obtain a Calabi-Yau n-fold is to define a special hypersurface in a weighted projective variety of dimension n+1. Recall that a weighted projective variety of dimension n+1, is defined by n+2 complex coordinates  $(z_1, z_2, \ldots, z_{n+2})$ , not all zero, subject to the an equivalence relation

$$(z_1, z_2, \dots, z_{n+2}) \sim (\lambda^{r_1} z_1, \lambda^{r_2} a_2, \dots, \lambda^{r_{n+2}} z_{n+2})$$
 (90)

where  $\lambda$  is any nonzero complex number, and the integers  $r_1, \ldots, r_{n+2}$  are called the weights. The projective variety defined in this manner is denoted by  $\mathbf{WP}_{r_1,\ldots,r_{n+2}}^{n+1}(z_1,\ldots,z_{n+2})$ . For example, consider  $\mathbf{WP}_{1,1}^1(z_1,z_2)$ , also known as  $\mathbf{CP}^1$ . It is defined by two complex coordinates satisfying the equivalence relation  $(z_1,z_2) \sim \lambda(z_1,z_2)$ . When  $z_2 \neq 0$ , we choose  $\lambda = 1/z_2$  so that  $(z_1,z_2)$  is equivalent to  $(w_1,1)$  with  $w_1 = z_1/z_2$ . The points  $(w_1,1)$  define the complex plane. When  $z_2 = 0$ , we choose  $\lambda = 1/z_1$  so that  $(z_1,0)$  is equivalent to (1,0). This additional point (1,0) can be regarded as the 'point at infinity' that needs to be added to 'compactify' the  $w_1$  complex plane to get the Riemann sphere. In general  $\mathbf{WP}^{n+1}$  gives a suitable compact n+1-dimensional complex manifold. A hypersurface defined by the vanishing of a complex homogeneous polynomial equation of degree k would be a complex submanifold of dimension n. To obtain a Calabi Yau manifold there is an additional requirement: the degree of the polynomial k must equal the sum of the weights of the coordinates.

In summary, a Calabi-Yau manifold is defined by a homogeneous polynomial of degree k, in a weighted projective variety,  $\mathbf{WP}_{r_1,\dots,r_{n+2}}^{n+1}(z_1,\dots,z_{n+2})$  such that

$$k = \sum_{i=1}^{n+2} r_i. (91)$$

If Eq. 91 is satisfied, then the first Chern class of the hypersurface vanishes ensuring that it has SU(n) holonomy.

Let us return to the torus after this digression. A torus is the simplest Calabi-Yau manifold, viz.,  $\mathbf{CY}_1$ . It has SU(1) holonomy, that is to say, no holonomy at all. In other words it is flat. In particular we see that the equation

$$wy^2 = x^3 + fw^2x + gw^3 (92)$$

defines it as a cubic in  $\mathbf{WP}_{1,1,1}^3(w,x,y)$  which obviously satisfies Eq. 91. For  $w \neq 0$  we can scale it out to get Eq. 89. The point w = 0 is the point at infinity which is needed to ensure compactness.

A more geometric way to recognize Eq. 89 as a torus is to note that

$$y = \pm \sqrt{(x - c_1)(x - c_2)(x - c_3)}$$
(93)

is a function of x defined on the double cover the Riemann sphere with four branch points:  $c_1, c_2, c_3$  and the point at infinity. From Figure 11 we recognize that the resulting Riemann surface has the topology of a torus. In equation 89

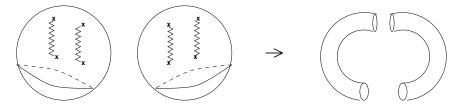


Figure 11: A double cover of branched sphere.

the parameters f and g are constants. The modular parameter  $\tau$  of the torus which determines the shape of the torus up to conformal rescaling (Fig 12) is given by the elliptic j function. The modular parameter  $\tau$  of the torus is given by

$$j(\tau) = \frac{4(24f)^3}{27g^2 + 4f^3} \tag{94}$$

where  $j(\tau)$  is the well known j-function,

$$j(\tau) = \frac{\theta_1^8(\tau) + \theta_2^8(\tau) + \theta_3^8(\tau)^3}{\eta(\tau)^{24}},\tag{95}$$

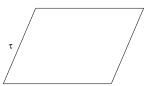


Figure 12: A torus with modular parameter  $\tau$ 

where  $\theta_i, i=1,2,3,$  are the well-known Jacobi  $\theta$  functions [21] and  $\eta$  is the Dedekind  $\eta$  function

$$\eta(\tau) = q^{1/24} \prod_{n} (1 - q^n), \qquad q = \exp 2\pi i \tau.$$
(96)

The j function gives a one-one map from the fundamental domain (the moduli space of the torus) to the complex plane. Given f and g, we can obtain  $\tau$  by inverting j.

To obtain a 2 complex dimensional fiber bundle with an elliptic curve as a fiber and a sphere as a base, locally we can take the parameters f and g of the elliptic curve in Eq. 89 to be functions of a complex coordinate z that takes values in  $\mathbf{CP}^1$ . Then for every value of z i.e. at each point, we get a torus. As we move around on the base by varying z, the parameters f and g and consequently the modular parameter  $\tau$  of the fiber will vary. The functions f and g should be so chosen that globally we obtain a smooth K3 that is elliptically fibered. This is achieved by taking f(z) to be an arbitrary polynomial in z of degree 8 and g(z) to be an arbitrary polynomial of degree 12. Given a polynomial  $f(z) = \sum_{k=0}^8 \alpha_k z^k$  we can define  $f(w,z) = \sum_{k=0}^8 \alpha_k w^{8-k} z^k$  which is a polynomial of degree 8 that is homogeneous if we assign weight 1 to both w and z. The coordinates w and z are nothing but the projective coordinates of  $\mathbf{CP}^1$  Now, the equation

$$y^{2} = x^{3} + f(z, w)x + g(z, w)$$
(97)

is a polynomial of degree 12 in

$$\mathbf{WP}_{1,4,6,1}^{3}(w,x,y,z) \tag{98}$$

The sum of weights 1+4+6+1=12 equals the degree of the polynomial showing that the first Chern class vanishes. Hence Eq. 97 defines a CY<sub>2</sub>, i.e., a K3. Using the equivalence relation  $(w, x, y, z) \sim (\lambda w, \lambda^4 x, \lambda^6 y, \lambda z)$  we can set w=1 by choosing  $\lambda=(1/w)$  when  $w\neq 0$  to get

$$y^{2} = x^{3} + f(z)x + g(z). (99)$$

We shall use Eq. 99 as the defining equation of a K3 elliptically fibered over  $\mathbb{CP}^1$ .

# 5.3 Type IIB string on K3

Consider II-B compactified on K3. The resulting theory in the remaining 6dimensional Minkowski space has (0,2) chiral supersymmetry. To discuss the spectrum let us recall that massless states are labeled by the representations of the little group in six dimensions which is  $Spin(4) = SU(2) \times SU(2)$ . With (0,2) supersymmetry, only two massless supermultiplets are possible. In terms of representations of the little group the supermultiplets are given by

1. The gravity multiplet: a graviton (3,3),

five self-dual 2-forms 5(1,3), gravitini 4(2,3),

2. The tensor multiplet:

an anti-self-dual 2-form (3, 1), fermions 4(2,1), five scalars (1,1).

The gravitini are right-handed whereas the fermions in the tensor multiplets are left-handed.

We can explicitly work out the spectrum of Type-IIB on a K3 that is a  $\mathbb{Z}_2$ orbifold. Let us take  $X^m$ , m=6,7,8,9 to be the coordinates of the internal torus and  $X^{i}$ , i = 2, 3, 4, 5, to be the noncompact light-cone coordinates. It is convenient to decompose the little group in ten dimensions SO(8) as

$$Spin(8) \supset Spin(4)_I \times Spin(4)_E$$

$$\equiv SU(2)_{1I} \times SU(2)_{2I} \times SU(2)_{1E} \times SU(2)_{2E}, \qquad (100)$$

where the subscript I is for internal, E is for external. With this embedding, the representations decompose as

$$8\mathbf{v} = (4\mathbf{v}, \mathbf{1}) \oplus (\mathbf{1}, 4\mathbf{v}) \equiv (\mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2}),$$

$$8\mathbf{s} = (2\mathbf{s}, 2\mathbf{s}) \oplus (2\mathbf{c}, 2\mathbf{c}) \equiv (\mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2}),$$

$$8\mathbf{c} = (2\mathbf{s}, 2\mathbf{c}) \oplus (2\mathbf{c}, 2\mathbf{s}) \equiv (\mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}).$$
(101)

The orbifold group is a  $\mathbb{Z}_2$  subgroup of  $SU(2)_{LI}$  which acts as -1 on the doublet representation 2.

Untwisted sector:

The states in the untwisted sector are obtained by keeping the  $\mathbf{Z}_2$  invariant states of the original 10-dimensional states.

$$(8\mathbf{v} \oplus 8\mathbf{c}) \otimes (8\mathbf{v} \oplus 8\mathbf{c}). \tag{102}$$

For example, the bosons (labeled by  $SU(2)_{1E} \times SU(2)_{2E}$  quantum numbers are

$$[4(1,1) \otimes 4(1,1)] \oplus [(2,2) \otimes (2,2)]$$
$$[2(1,2) \otimes 2(1,2)] \oplus [2(2,1) \otimes 2(2,1)]$$
(103)

This gives rise to a graviton, 25 scalars, 5 self-dual and 5 anti-self-dual 2-forms. The fermions can be obtained similarly which give the superpartners required by supersymmetry. Together, we get the gravity multiplet and five tensor multiplets.

Twisted Sector:

There are 16 twisted sectors coming from the 16 fixed points. The bosonic fields and fermionic fields are twisted according to their transformation property under the  $\mathbb{Z}_2$ . We see from that four fermions that transform as 2(2,1) and four bosons that transform as (2,2) are  $\mathbb{Z}_2$  invariant and are not twisted where as the four other are twisted. The ground state energy is zero because there are equal number of bosons and fermions that are twisted. The untwisted fermions have zero modes. By steps analogous to those that led to Eq. 8 in §2.1, the zero mode algebra gives rise to a four dimensional representation  $(2,1) \oplus 2(1,1)$ . Therefore the massless representation is

$$[(\mathbf{2},\mathbf{1}) \oplus 2(\mathbf{1},\mathbf{1})] \otimes [(\mathbf{2},\mathbf{1}) \oplus 2(\mathbf{1},\mathbf{1})]$$
 (104)

which gives precisely the particle content of a tensor multiplet. Therefore, the twisted sector contributes 16 tensor multiplets.

The massless spectrum of Type-IIB on a K3 orbifold thus consists of a gravity multiplet and 21 tensor multiplet together from the untwisted and the untwisted sector. There are 105 scalars that parametrizes the moduli space  $O(21,5; \mathbf{Z}) \setminus O(21,5; \mathbf{R}) / O(21; \mathbf{R}) \times O(5; \mathbf{R})$ .

The spectrum of Type-IIB is chiral. A chiral theory can have gravitational anomalies In 4k + 2 dimensions Up to overall normalization the gravitational anomalies are

$$I_{3/2} = -\frac{43}{288}(trR^2)^2 + \frac{245}{360}trR^4,$$

$$I_{1/2} = +\frac{1}{288}(trR^2)^2 + \frac{1}{360}trR^4,$$

$$I_A = -\frac{8}{288}(trR^2)^2 + \frac{28}{360}trR^4.$$
(105)

Here  $I_{3/2}$ ,  $I_{1/2}$ , and  $I_A$  refer to the anomalies for the gravitino, a right-handed fermion, and a self-dual two-form (1,3) respectively  $^d$ .

To get a consistent theory, the total gravitational anomalies must cancel. This requirement is very restrictive and in fact completely determines the spectrum for the theory with (0, 2) supersymmetry. Using the formulae from Eq. 105 it is easy to check that gravitational anomalies cancel only when there are precisely 21 tensor multiplets along with the gravity multiplet [61]. This, as we have seen, is the spectrum of II-B compactified on K3.

### 5.4 Type IIA string on K3

Type-IIA compactified on a K3 gives a non-chiral theory in six-dimensional Minkowski spacetime with (1,1) supersymmetry. There are only two supermultiplets that are possible.

1. The gravity multiplet:

```
a graviton (3,3), a scalar (1,1), four vectors 4(2,2), a 2-form (3,1) \oplus (1,3), gravitini 2(2,3) \oplus 2(3,2) two fermions 2(2,1),2(1,2)
```

2. The vector multiplet:

```
a vector (\mathbf{2}, \mathbf{2}), four scalars 4(\mathbf{1}, \mathbf{1}), gauginoes 2(\mathbf{2}, \mathbf{1}), 2(\mathbf{1}, \mathbf{2}).
```

The spectrum can be found as in the previous section.

Untwisted sector:

Now the ten-dimensional states are

$$(8\mathbf{v} \oplus 8\mathbf{s}) \otimes (8\mathbf{v} \oplus 8\mathbf{c}). \tag{106}$$

Keeping  $\mathbb{Z}_2$  invariant states we obtain the gravity multiplet and 4 vector multiplet.

Twisted Sector:

Now, the fermions that not twisted have different quantum number on the

 $<sup>^{</sup>d}$ In 4k+2 dimensions, the CPT conjugate of a left-handed fermion is also left-handed. Therefore, gravitational couplings can be chiral and consequently gravitational anomalies are possible. Contrast this with the 4k dimensions as in the familiar case of four dimensions where the CPT conjugate of a left-handed fermion is right-handed and a CPT-invariant theory is automatically nonchiral unless there are gauge charges in addition to gravity that distinguish between left and right.

left and on the right. Therefore, the representation of the fermion zero mode algebra is different on the left and the right. The massless representation are given by the product

$$[(\mathbf{1}, \mathbf{2}) \oplus 2(\mathbf{1}, \mathbf{1})] \otimes [(\mathbf{2}, \mathbf{1}) \oplus 2(\mathbf{1}, \mathbf{1})]$$
 (107)

which gives precisely the particle content of a vector multiplet. Therefore, the twisted sector contributes 16 tensor multiplets one from each of the fixed points.

The massless spectrum of Type-IIA on a K3 orbifold thus consists of the gravity multiplet and 20 vector multiplets. There are 80 scalars that parametrize the moduli space  $O(20, 4; \mathbf{Z}) \setminus O(20, 4; \mathbf{R}) / O(20; \mathbf{R}) \times O(4; \mathbf{R})$ .

### 5.5 F-theory on K3

Until recently string compactifications basically solved vacuum Einstein equations in the low-energy limit for some compact manifold K,

$$R_{ij} = 0, (108)$$

In fact, unbroken supersymmetry in the remaining noncompact dimensions requires that the compact manifold be a Calabi-Yau manifold with SU(n) holonomy, in particular with vanishing first Chern class.

Instead of solving the vacuum Einstein equations, one can imagine solving the equations with some nonzero background fields. In particular, one can ask if there are consistent solutions of the Type-IIB string where the complex field  $\lambda = \chi + ie^{-\phi}$  varies. When all other massless fields of Type-IIB theory are set to zero, the equations of motion for the graviton  $g_{MN}$  and the scalar  $\lambda$  can be derived from the action

$$\int d^{10}x \sqrt{g} \left( R - \frac{1}{2} \frac{g^{MN} \partial_M \bar{\lambda} \partial_N \lambda}{Im(\tau)^2} \right). \tag{109}$$

A particularly interesting nontrivial solution of this action is Type-IIB compactified on a 2-sphere  $(S^2 \equiv \mathbf{CP^1})$ . The spacetime of this compactification is of the form

$$\mathbf{M}^8 \times S^2, \tag{110}$$

where  $\mathbf{M}^8$  is flat Minkowski spacetime with coordinates  $X^0, X^1, \dots, X^7$  and  $S^2$  is the compactification sphere with coordinates  $X^8, X^9$ . Now, the sphere which has nonzero curvature  $R_{ij} \neq 0$ . In fact, the first Chern class of  $S^2$  is

the Euler character of the sphere which is nonzero, so  $S^2$  is obviously not a Calabi-Yau manifold. The way equations of motion are still satisfied is that the spacetime contains 24 7-branes. A 7-brane, as we shall see, can be thought of a special topological defect which couples to  $\lambda$ . The worldvolume of the 7-brane fills the noncompact  $\mathbf{M}^8$ , so in the transvers  $S^2$  it looks like a point. The energy momentum tensor  $T^\lambda_{ij}$  of  $\lambda$  and the metric are precisely such that they solve the Einstein equation

$$R_{ij} - \frac{1}{2}g_{ij}R = T_{ij}^{\lambda}.$$
 (111)

To describe the solution, let us first discuss a single 7-brane. Let  $z=X^8+iX^9$  be the complex coordinate on the plane transverse to the 7-brane. The coordinates  $X^0,X^1,\ldots,X^7$  are along the worldvolume of the 7-brane. The equation of motion for  $\lambda$  that follows from Eq. 109 is

$$(\lambda - \bar{\lambda})\partial\bar{\partial}\lambda - 2\partial\lambda\bar{\partial}\lambda = 0. \tag{112}$$

This equation is solved by any  $\lambda$  that is a holomorphic function of z

$$\bar{\partial}\lambda(z,\bar{z}) = 0. \tag{113}$$

Not any holomorphic function will do. Recall that  $\lambda$  parametrizes, after  $SL(2,\mathbf{Z})$  identification, the fundamental domain of the moduli space of a torus (Figure 1). To get a well-defined solution we want a one-to-one map from the fundamental domain to the complex plane. We have already seen that the j-function defined in Eq. 95 gives precisely such a map. Therefore, instead of looking for  $\lambda$  as a function of z it is convenient to look for j(l) as function of z. Furthermore, the resulting configuration should have finite energy to be an acceptable solution.

The simplest solution that satisfies all the requirement is

$$j(\lambda(z)) = \frac{1}{z} \tag{114}$$

which has the right properties.

If we suppress 6 of the coordinates along the 7-brane (say  $X^2, \ldots, X^7$ ) then the 7-brane looks like a cosmic string in four dimensions  $X^0, X^1, X^8, X^9$ . This in fact is nothing but the "stringy" cosmic string solution discussed by Greene et. al.[29]. Near z=0 j has a pole. The only pole of j is at  $q=\exp 2\pi i\lambda=0$  at  $\lambda_2\to\infty$ . For large  $\lambda_2$  we have,

$$j(\lambda) \sim \exp{-2\pi i \lambda}.$$
 (115)

The solution looks like

$$\lambda = \frac{1}{2\pi i} \log z \tag{116}$$

near z = 0.

If we go around the origin on a circle at infinity in the z plane with  $z \to ze^{2\pi i}$  then  $\lambda \to \lambda + 1$ . This is very much like a global cosmic string or a vortex line in superfluid helium. A cosmic string is a topological defect in which the phase angle  $\theta$  of the order-parameter field has a winding number.

$$\theta(ze^{2\pi i}) \to \theta(z) + 2\pi$$
 (117)

The RR-field a in Type-II string is very similar to the phase of the order parameter  $\theta/2\pi$ . An important difference is that the total energy of global string or a vortex line has an infrared divergence because very far from the core the superfluid in a vortex undergoes huge rotation. By contrast, the energy density of the stringy cosmic string is finite. This is possible because the  $\lambda$  field can undergo  $SL(2, \mathbf{Z})$  jumps away from the core. Near the core z=0, the  $\lambda$  field has a nontrivial monodromy or jump under the element T of  $SL(2, \mathbf{Z})$ ,  $T: \lambda \to \lambda + 1$ , but far away it can undergo jumps under other elements of  $SL(2, \mathbf{Z})$ .

The nontrivial monodromy of  $\lambda$  around the point z=0 means in string theory that there is a 7-brane at this point that is magnetically charged with respect to the scalar  $\lambda$ . Indeed, near the origin, this is exactly like a D7-brane in Type-IIB which is magnetically charged with respect to the RR scalar. In other words, it couples to the 8-form RR potential  $A_8$  that is dual to a,

$$dA_8 = *da. (118)$$

Let us now look at the effect of the 7-brane on the metric. Because of the energy density contained in the field  $\lambda$ , the metric in z plane has a conical deficit near z=0 with conical deficit angle  $\delta$ . The metric near such a point  $z_i$  can be found explicitly [29], and has the form

$$ds^{2} = \frac{dzd\bar{z}}{|z - z_{i}|^{1/6}} \sim r^{-1/6}(dr^{2} + r^{2}d\psi^{2}), \qquad 0 \le \psi \le 2\pi.$$
 (119)

A metric of the form

$$r^{-2\lambda}(dr^2 + r^2d\psi^2) \tag{120}$$

can be written in the form

$$dt^2 + t^2(1 - \lambda)^2 d\psi^2, (121)$$

with  $t = r^{1-\lambda}/(1-\lambda)$ . We can nowredefine the angle

$$\phi = (1 - \lambda)\psi \tag{122}$$

to bring it to the standard flat metric of the plane in polar coordinates  $dt^2 + t^2 d\phi^2$ . But then  $\phi$  goes from 0 to  $2\pi - 2\pi\lambda$ . Therefore, the deficit angle is  $\delta = 2\pi\lambda$  which for the metric in Eq. 119 is  $\pi/6$ . The deficit angle measures the total curvature or  $\delta = \int R$  where R is the Ricci scalar.

So far, z is a coordinate on the noncompact complex plane. If we put precisely 24 7-branes on the plane then the plane curls up into a sphere. This is because the total contribution to the deficit angle from all the cosmic strings now adds up to  $4\pi$  to make up the solid angle of a sphere. For the sphere, the Euler character 2 and  $\int R = 2\pi\chi = 4\pi$ .

A sphere with a collection of 24 F-theory 7-branes is a compact manifold. In fact, one can associate with it an elliptically fibered K3, with the sphere as the base, if the  $\tau$  parameter of the fiber torus is identified with  $\lambda$ . The elliptic K3 is described by Eq. 99 in §5.3

$$y^{2} = x^{3} + f(z)x + g(z). {123}$$

where f and g are polynomials of degree 8 and 12 respectively. The locations of 7-branes are determined by the zeroes of the discriminant  $\Delta \equiv 4f^3 + 27g^2$  which is the denominator of the j function. Since f is a polynomial of order 8 and g is a polynomial of order 12, there will, in general, be 24 zeroes of the discriminant which correspond to the locations of the 24 7-branes. Near every zero  $z_i$ , we have

$$j(\lambda) \sim \frac{c}{(z - z_i)} \sim \frac{1}{q}$$
 (124)

$$\lambda(z) \simeq \frac{1}{2\pi i} \log(z - z_i),$$
 (125)

corresponding to a single 7-brane.

The compactification of Type-IIB of  $S^2$  that we have described has been called an 'F-theory' compactification on K3. 'F-theory' refers to a possible 12-dimensional theory which when compactified on  $\mathbf{T}^2$  would give Type-IIB. In general, consider an elliptically fibered Calabi-Yau manifold K which is a fiber bundle over a base manifold K with a torus as a fiber whose complex structure parameter is T. Even-though K is a smooth manifold, there will be points in the base manifolds where the fiber becomes singular, and the parameter T can have a nontrivial SL(2,Z) monodromy around these points. An F-theory compactification on K refers to a compactification of Type-IIB theory on K0, where the coupling K1 is identified with T2.

### 6 Applications of Orientifolds to Duality

#### 6.1 General Remarks on Duality

Apart from T-duality, two other dualities will be relevant to us in the following discussion. Both are 'strong-weak' dualities which relate the strong coupling limit of a ten-dimensional string theory to the weak coupling limit of the dual theory.

- (1) Duality between the Type-I string and the SO(32) Heterotic string. The massless spectrum as well as the form of the low energy effective action agrees [66] under the duality transformation which takes the coupling constant of Type-I string to the inverse coupling constant of the Heterotic string. The spectrum of some of the solitons in the two theories has also been checked to be in agreement with duality [12, 35, 43].
  - (2) Self-duality of the Type-IIB string.

We have already discussed this. The element S of the  $SL(2, \mathbf{Z})$  duality group takes the coupling constant to the inverse coupling constant in the dual theory.

For a detailed discussion of the evidence for these dualities, see [53]. In this section we would like to use these dualities, T-duality, and our knowledge of orbifolds and orientifolds, to deduce two more dualities in lower dimensions.

### 6.2 Duality of Type-IIA on K3 and Heterotic on T<sup>4</sup>

The main principle that we use in this subsection is 'fiberwise application of duality', which we explain below.

Consider a theory A compactified on  $K_A$  that is dual to another theory B compactified on  $K_B$ . This duality can be used to deduce some further dualities. Consider A and B compactified respectively on  $E_A$  and  $E_B$ , which are obtained by fibering  $K_A$  and  $K_B$  over  $\Sigma$ . By this, we mean that locally  $E_A$  looks like  $K_A \times \Sigma$ . The moduli  $m_\alpha$  of the fiber  $K_A$  can vary as a function of the coordinates of the base manifold  $\Sigma$ . As long as the moduli of  $K_A$  vary slowly we expect to be able to use the original duality to derive a new duality between A on  $E_A$  and B on  $E_B$ . There are two possibilities that aries.

(i) The first possibility is that the fiber is smooth at all points on the base manifold  $\Sigma$ . In this case, the duality between A on  $E_A$  and B on  $E_B$  follows from the 'adiabatic argument' of Vafa and Witten [64]. The idea is that we can choose the size of  $\Sigma$  to be very large. Then locally A compactified on  $E_A$  has  $K_A \times \mathbf{M}^n$  as the target spacetime and and B on  $E_B$  has  $K_B \times \mathbf{M}^n$  as the target spacetime. Knowing the duality between the two, we can assert the new duality. The fibered structure will become apparent to a local observer only after circumnavigating the (very large) manifold  $\Sigma$  and so will not be relevant

to local physics. We can thus establish the duality between A on  $E_A$  and B on  $E_B$  in the limit of large  $\Sigma$ . Now, if we adiabatically reduce the volume of  $\Sigma_2$  we expect that the duality will continue to hold.

(ii) The second possibility is that the fiber is smooth everywhere on  $\Sigma$  except at a few discrete points where it degenerates ( Figure 13). The total

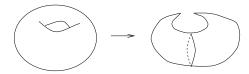


Figure 13: A torus degenerates into a sphere by pinching a handle.

manifold  $E_A$  can still be smooth, it is only the fibration that becomes singular. In this case, the adiabatic argument is strictly not applicable. Near the points where the fiber degenerates, the argument breaks because the moduli vary rapidly. However, in a large number of examples constructed so far, the resulting theories do appear to be dual as long as the number of singular points is a set of "measure zero". Heuristically, even with the singular points, duality is forced by the duality in the bulk.

It is instructive to apply these arguments to the special case of fibered manifolds that are orbifolds. Take a smooth manifold  $\mathcal{M}$  with a  $\mathbb{Z}_2$  symmetry  $\{1, s\}$ . Let  $K_A$  and  $K_B$  have some  $\mathbb{Z}_2$  symmetry  $\{1, h_A\}$  and  $\{1, h_B\}$ . Take

$$E_A = \frac{K_A \times \mathcal{M}}{\{1, sh_A\}}, \qquad E_B = \frac{K_B \times \mathcal{M}}{\{1, sh_B\}}$$
 (126)

Here there are two possibilities.

(i) If s has no fixed points on  $\mathcal{M}$ , then we have the possibility (i) above. The orbifold  $E_A$  can be viewed at all points as a fiber bundle with  $\Sigma =$ 

 $\mathcal{M}/\{1,s\}$  as the base space and  $K_A$  as the fiber. The fiber is smooth everywhere. There is a twist  $h_A$  along the fiber as we move along a closed curve from a point p on  $\Sigma$  and its image s(p).

(ii) If s leaves some points on  $\mathcal{M}$  fixed, then we have the possibility (ii). The orbifold  $E_A$  still has the structure of a fiber bundle with base manifold  $\Sigma = \mathcal{M}/\{1, s\}$  and fiber  $K_A$  everywhere except the fixed points. At the fixed points the fiber degenerates to  $K_A/\{1, h_A\}$  giving us a singular fibration. e

As an illustration of the second possibility we now 'derive' the duality between Type-IIA on K3 and Heterotic on  $T^4$ . Let us take A to be the Type-IIB theory in ten dimensions and B to be the Type-IIB theory that is S-dual to A ( $K_A$  and  $K_B$  are null sets). Compactify both sides on a torus  $\mathbf{T}^4$  with coordinates  $X_6, X_7, X_8, X_9, i.e.$ ,  $\mathcal{M} = \mathbf{T}^4$ . Now, take  $h_A = (-1)^{F_L}$ ,  $h_B = \Omega$ , and s to be the reflection

$$I_{6789}: (X_6, X_7, X_8, X_9) \rightarrow (-X_6, -X_7, -X_8, -X_9)$$

We have used the duality relations

$$IIB \xrightarrow{S} IIB, \qquad (-1)^{F_L} \xrightarrow{S} \Omega,$$
 (127)

from §2.3. Therefore, from the earlier arguments, we get the duality

IIB on 
$$\frac{\mathbf{T}^4}{\{1, (-1)^{F_L} I_{6789}\}} \equiv \text{IIB on } \frac{\mathbf{T}^4}{\{1, \Omega I_{6789}\}}$$
 (128)

Now we can use T-dualities to turn these orientifolds into more familiar ones. We use two observations.

(1) If we T-dualize one of the coordinates, say  $X^6$ , we get Type-IIA theory on the dual circle. Moreover, the symmetry  $I_{6789}(-1)^{F_L}$  in IIB maps under this T-duality onto  $I_{6789}$  in the dual IIA. To see this, recall that if we T-dualize in a direction  $X^i$ 

$$(X_L^i, X_R^i) \to (-X_L^i, X_R^i) \tag{129}$$

then the left-moving fermions transform as

$$T_i: S^a \to P_i S^a,$$
 (130)

<sup>&</sup>lt;sup>e</sup>There is a third possibility that s leaves the entire manifold  $\mathcal{M}$  invariant. In this case the resulting orbifolds do not have the fibered structure even locally. In such a situation orbifolding does not commute with duality [56, 55] and cannot be used to deduce new dualities. For example, Type-I is an orientifold of Type-IIB by  $\{1, \Omega\}$  and Type-IIA is an orbifold of Type-IIB by  $\{1, (-1)^{F_L}\}$ . Now,  $\Omega$  is conjugate to  $(-1)^{F_L}$  by the element S of the  $SL(2, \mathbb{Z})$  group but Type-I is not dual to Type-IIA.

where  $P_i = \Gamma \Gamma^i$ . Now using the properties of the  $\Gamma$  matrices, we see that  $P_i$  does not square to identity but instead  $P_i^2 = (-1)^{F_L}$ . Furthermore  $P_i P_j = -P_j P_i$ . Note that the reflection  $I_{6789}$  acts as  $P_6 P_7 P_8 P_9 S^a$  on the left-moving fermions, and similarly on the right-moving fermions. Using the properties of  $P_i$  matrices we see that

$$I_{6789}(-1)^{F_L} = T_6^{-1} I_{6789} T_6 (131)$$

Therefore, the orbifold Type-IIB on  $\mathbf{T}^4/\{1,(-1)^{F_L}I_{6789}\}$  is T-dual to Type-IIA on the K3 orbifold  $\mathbf{T}^4/\{1,I_{6789}\}$ .

(2) T-duality of all four coordinates maps  $I_{6789}\Omega$  in IIB into  $\Omega$  in IIB by a reasoning similar to the above.

$$I_{6789}\Omega = T_{6789}^{-1} \Omega T_{6789} \tag{132}$$

Hence, Type-IIB on the orientifold  $\mathbf{T}^4/\{1,\Omega I_{6789}\}$  is T-dual to Type-IIB on the orientifold  $\mathbf{T}^4/\{1,\Omega\}$ , which is nothing but Type-I on  $\mathbf{T}^4$ . From the duality between Type-I and Heterotic in ten dimensions, it is dual to Heterotic on  $\mathbf{T}^4$ .

We conclude from Eq.128 and points (A) and (B) above that

IIA on K3 
$$\equiv$$
 Heterotic on  $\mathbf{T}^4$ . (133)

This equivalence has been established at a special point in the moduli space where the K3 becomes the orbifold  $\mathbf{T}^4/\mathbf{Z}_2$ . The duality gives a one-one map between the massless fields on both sides. By giving expectation values to the massless scalar field we can move around in the moduli space and establish the duality at all points in the moduli space.

Thus, starting with the  $SL(2, \mathbf{Z})$  duality of II-B with 32-supercharges (and the duality between Type-I and Heterotic), we can get all the structure of the more interesting 'string-string' duality [36] between II-A on K3 Heterotic on  $\mathbf{T}4$  with 16-supercharges. This is quite an explicit construction, and all we needed to do was to keep track of a few discrete symmetries and follow the orientifold and orbifold construction.

Let us quickly check if the spectrum of Heterotic on  $\mathbf{T}^4$  matches with the dual Type-IIA spectrum. The moduli space of Heterotic on  $\mathbf{T}^4$  is the Narain moduli space [41]  $O(20,4,\mathbf{Z})\backslash O(20,4,\mathbf{R})/O(20,\mathbf{R})\times O(4,\mathbf{R})$  which is identical to the moduli space of Type-IIA on K3. At a generic point in the moduli space, the gauge group SO(32) is broken to  $U(1)^{16}$ . In six dimensions we get 4 vector bosons  $g_{\mu m}$ , 4 vectors from  $B_{\mu m}$ , and 16 vectors from the original gauge fields in ten dimensions,  $A^I_{\mu}$  ( $I=1,\ldots,16$  is the gauge index,  $m=6,\ldots,9$  is the internal index, and  $\mu$  is the Minkowski index  $\mu=0,\ldots,5$ ).

Altogether there are 24 vector bosons, exactly as in the case of Type-IIA ( $\S5.4$ ), which transform in the vector representation of the duality group  $SO(4,20,\mathbf{R})$ .

I shall now describe one more duality with 16 supersymmetries before moving onto theories with 8 supercharges.

# 6.3 Duality of F-theory on K3 and Heterotic on $T^2$

Another interesting application of orientifolds is in connection with F-theory. In this subsection we concern ourselves with F-theory compactification on K3 to eight-dimensional Minkowski spacetime, but these considerations are applicable to more general compactifications.

We have seen in  $\S5.5$  that to obtain an F-theory compactification, we start with an elliptically fibered K3 that is described by

$$y^{2} = x^{3} + f(z)x + g(z), (134)$$

where f is a polynomial of degree 8 and g is a polynomial of degree 12. Such a K3 represents 24 stringy cosmic strings on a 2-sphere located at the 24 points where the torus degenerates. Typically, the coupling constant field  $\lambda$  will vary as we move from point to point in the base manifold. Consequently, there will be a non-vanishing RR background. Moreover, the field is allowed to undergo  $SL(2, \mathbf{Z})$  jumps. Some of the elements of the  $SL(2, \mathbf{Z})$  like S are nonperturbative. Therefore, for a generic K3, such backgrounds cannot be described perturbatively as conformal field theories.

There is a special limit of the K3 for which the modular parameter  $\lambda$  of the fiber torus does not vary as we move on the sphere. This is achieved by choosing  $f^3/g^2 = \text{constant}$ .

$$g = \phi^3$$
  $f = \alpha \phi^2$   $\phi(z) = \text{polynomial of degree 4}$ 

by rescaling y and x we can set  $\phi = \prod_{a=1}^{4} (z - z_a)$ . Then the j function is given by

$$j(\lambda) = \frac{4(24\alpha)^3}{27 + 4\alpha 3} = constant, \tag{135}$$

therefore,  $\lambda$ , which is the image under  $j^{-1}$ , is also a constant at all points over the sphere. However there is a nonzero  $SL(2, \mathbf{Z})$  monodromy

$$R = \begin{pmatrix} -1 & 0\\ 0 & -1 \end{pmatrix}. \tag{136}$$

as we go around a point  $z=z_a$ . This is the hyperelliptic involution of the torus which reflects both periods of the torus without changing its modular parameter. The discriminant is

$$\Delta = (4\alpha^3 + 27) \prod_{a=1}^{4} (z - z_a)^6$$
 (137)

which shows that the 24 7-branes are bunched in groups of six at four points  $\{z_a\}$ . The metric on the base can be read of from 119 by putting 6 7-branes at one point.

$$ds^{2} = \frac{dzd\bar{z}}{\prod_{a}|z - z_{a}|^{1/2}|\bar{z} - \bar{z}_{a}|^{1/2}}$$
(138)

There is a conical deficit angle of  $\pi$  at four points, otherwise the metric is flat. In other words, the base is  $\mathbf{T}^2/\mathbf{Z}_2$  and the fiber is  $\mathbf{T}^2$  at all points except at the fixed points.

It is easy to see that this K3 that is nothing but the orbifold  $T^4/\mathbb{Z}_2$ . Conversely, the K3 orbifold  $T_4/\mathbb{Z}_2$  that we constructed in §5.1 can be viewed as an elliptic fibration over  $S^2$ . The K3 has coordinates  $(z_1, z_2)$  and the orbifold symmetry is  $(z_1, z_2) \sim (-z_1, -z_2)$ . It can be viewed as

$$\frac{\mathbf{T}^4}{\mathbf{Z}_2} = \frac{\mathbf{T}_{(1)}^2 \times \mathbf{T}_{(2)}^2}{\{1, R_1 R_2\}} \tag{139}$$

Let us take  $z_1$  to be the coordinate of the first torus  $\mathbf{T}_1^2$ , and  $z_2$  to be the coordinate of the base. Let  $R_i$  be the operation  $z_i \to -z_i$ . Then the orbifold Eq. 139 can be viewed as a fiber bundle with  $\mathbf{T}_1^2$  as the fiber and  $\mathbf{T}_2^2/\{1,R_2\}$  as the base. The base manifold  $\mathbf{T}^2/\mathbf{Z}_2$  is nothing but a sphere. To see this note that the  $\mathbf{Z}_2$  symmetry acts as  $z_2 \to -z_2$  has four fixed points each with deficit angle  $180^\circ = \pi$ . So the total deficit angle is  $4\pi$  giving us  $\int R = 4\pi$  i.e. the correct Euler character 2 for a sphere  $S^2$ . So locally, away from the fixed points of  $R_2$  the orbifold looks like  $\mathbf{T}_{(1)}^2 \times S^2$  If we go around a fixed point of  $R_2$  then the coordinate  $z_1$  of the  $\mathbf{T}_{(1)}^2$  is twisted by  $R_1$ , i.e., it is inverted  $(z_1 \to -z_1)$ , but its modular parameter  $\lambda$  is unchanged. This is precisely the  $\mathbf{Z}_2$  monodromy R in Eq. 136. Therefore, in this limit, for this special configuration of 7-branes, the field  $\lambda$  is constant everywhere in space and this F-theory compactification can be described as an perturbative orientifold [57]. Such an identification of F-theory with an orientifold is very useful.

F-theory on the K3 orbifold above is nothing but the orientifold

IIB on 
$$\frac{\mathbf{T}^2}{\{1, RI_{89}\}} \equiv \frac{\mathbf{T}^2}{\{1, \Omega(-1)^{F_L}I_{89}\}},$$
 (140)

if we identify  $R_1$  in Eq. 139 with  $R = \Omega(-1)^{F_L}$  and  $R_2$  in Eq. 139 with  $I_{89}$ .

There are 4 orientifold fixed planes and 16 7-branes are required to cancel the charge of the orientifold planes. This fact is obvious if we note further that after T-dualizing in the 89 directions,  $\Omega(-1)^{F_L}I_{89}$  goes to  $\Omega$ :

$$\Omega(-1)^{F_L} I_{89} = T_{89}^{-1} \Omega T_{89}. \tag{141}$$

Therefore, this orientifold Eq. 140 is T-dual to the orientifold

$$IIB \quad on \quad \frac{\mathbf{T}^2}{\{1,\Omega\}},\tag{142}$$

which is nothing but Type-I theory compactified on  $\mathbf{T}^2$ . After two T-dualities the 32 9-branes turns into 7-branes. Because of the identification  $I_{89}$  they have to move in pairs so effectively there are only 16 of them on the orientifold.

Thus, at this special point in the moduli space, F-theory is nothing but a T-dual of Type-I on  $\mathbf{T}^2$  which in turn is dual to Heterotic on  $\mathbf{T}^2$ . We have thus established the duality

F theory on 
$$K3 \equiv \text{Heterotic on } \mathbf{T}^2$$
. (143)

Apart from its use in understanding this duality, the orientifold limit of Ftheory has other very interesting applications. To obtain the orientifold limit, we had to place sixteen 7-branes at the four orientifold planes in four bunches of four. On the other hand, on the F-theory side, we have 24 F-theory 7-branes in four bunches of six. What happens is that as we move the D-7branes away from the orientifold 7-branes, then the orientifold 7-plane splits nonperturbatively into two 7-planes [57]. Thus, in F-theory, the orientifold planes and the D-branes are on an equal footing and are related nonperturbatively. An orientifold plane and four D7-branes turn into six F-theory 7-branes. This splitting of the orientifold plane is very similar to the splitting of the SU(2)point into the monopole point and the dyon point in Seiberg-Witten theory in 3+1 dimensions with gauge group SU(2) and four quark flavors [52]. This similarity is not an accident but a precise consequence of using D3-brane probes to probe the geometry near the orientifold plane. The worldvolume theory of D3-brane probe near an orientifold plane and four D-7branes has exactly the same structure as a Seiberg-Witten theory [3].

## 7 Orientifolds in Six Dimensions with (0,1) Supersymmetry.

One important application of orientifolds is in the construction of models in six dimensions with (0,1) supersymmetry which has only 8 supercharges. With

only 8 supercharges, instead of 16 or 32, supersymmetry is much less restrictive and therefore much more interesting dynamics is possible. At the same time, supersymmetry is still sufficiently restrictive to be a useful guide for checking the properties of these theories as well their possible duals.

The massless supermultiplets of (0,1) supersymmetry in terms of representations of the little group  $SU(2) \times SU(2)$  are as follows:

```
1. The gravity multiplet:
    a graviton (3, 3), a self-dual two-form (1, 3),
    a gravitino 2(2, 3).
2. The vector multiplet:
    a gauge boson (2, 2),
    a gaugino 2(1, 2).
3. The tensor multiplet:
    an anti-self-dual two-form (3, 1),
    a fermion 2(2,1), a scalar (1,1).
4. The hypermultiplet:
    four scalars 4(1,1).
    a fermion 2(2,1).
```

Cancellation of gravitational anomalies places restrictions on the matter content. Consider V vector multiplets, H hypermultiplets and T tensor multiplets. Then the  $(trR^4)$  term in the anomaly polynomial Eq. 105 cancels only if

$$\mathbf{H} - \mathbf{V} = 273 - 29\mathbf{T}.\tag{144}$$

The  $(trR^2)^2$  term is in general nonzero, and needs to be canceled by the Green-Schwarz mechanism. For example, if we compactify the heterotic string or Type-I string on a smooth K3 we obtain T = 1 and H = V + 244.

The dynamics of (0,1) theories in six dimensions offers many surprises like the possibility of exotic phase transitions in which the number of (chiral) tensor multiplets changes, or the appearance of enhanced gauge symmetry when an instanton shrinks to zero scale size. Given the limitations of time it is not possible to discuss the detailed construction in the same depth as we did in earlier sections. In this section I briefly survey two interesting phenomena:

- (i) enhanced gauge symmetry and appearance of USp(2k) symmetry, and
- (ii) appearance of multiple tensor multiplets.

and describe the utility of orientifolds in this context.

To organize the discussion, a useful starting point is the heterotic string compactified on K3. To begin with, in ten dimensions there are two consistent heterotic strings that have N=1 supersymmetry.

- (1) The heterotic string with gauge group SO(32): The strong coupling limit of this theory is Type-I string. We have already used this duality in §6.
- (2) The heterotic string with gauge group  $E_8 \times E_8$ : The strong coupling limit of this theory is M-theory on  $\mathbf{M}^{10} \times (\mathbf{S}^1/\mathbf{Z}_2)$  [34]. The generator of  $\mathbf{Z}_2$  reflects the coordinate of the circle so the resulting orbifold  $(\mathbf{S}^1/\mathbf{Z}_2)$  is nothing but a line segment  $\mathbf{I}:[0,\pi R]$ .  $\mathbf{M}^{10}$  is ten-dimensional Minkowski spacetime. The two  $E_8$  factors live on the two 'end of the world' boundaries of this manifold. Gravity lives in the bulk and on the boundary. In the bulk we have eleven dimensional supergravity which corresponds to N=2 Type-IIA supergravity in ten dimensions upon dimensional reduction on a circle. Boundary conditions break it to N=1 supergravity of heterotic string.

The supergravity sector of the heterotic string is identical to the Type-I theory. It contains, along with a graviton, a dilaton, and the fermions, a 2-form field B with field strength H which satisfies the Bianchi identity

$$dH = tr(R \wedge R) - tr(F \wedge F), \tag{145}$$

where R is the curvature 2-form and F is the gauge field strength 2-form. Integrating the Bianchi identity gives the constraint that the integral of the right hand side of Eq. 145 should vanish on a manifold without a boundary. In particular, when we compactify the string on K3, the integral of the right hand side over K3 should vanish. Now  $\int_{K3} tr(R \wedge R)$  is the Euler character of K3 which equals 24. Therefore, the integral  $\int_{K3} tr(F \wedge F)$ , which is the Pontryagin index or the instanton number of the gauge field on K3, should also equal 24. A consistent heterotic compactification on K3 is possible only if the gauge field is also nontrivial such that there are 24 instantons on K3. An instanton on K3 looks like a solitonic 5-brane [60] in ten dimensions that fills flat six-dimensional Minkowski spacetime.

The heterotic compactification can become singular in certain limits. One singularity that will concern us here is when the scale size of an instanton shrinks to zero. When a 5-brane instanton shrinks, the geometry in the transverse space becomes singular and it develops a long throat. Deep down the throat the coupling constant grows and the perturbative description of the 5-brane breaks down. We would like understand what happens near such a singularity.

From our experience with the moduli space of supersymmetric gauge theories and string theories, typically a singularity in the moduli space indicates that additional states are becoming massless at that point in the moduli space. The singularity occurs because we have incorrectly integrated out these states

that are massless. We expect that the singularities in the instanton moduli space also has a similar physical explanation. The small instanton singularity has a completely different physical resolutions in the two heterotic strings, each remarkable in its own way.

### (i) Small instantons in SO(32) heterotic string:

If there are k small instantons that coincide, then there are additional massless gauge bosons which give rise to an additional USp(2k) gauge symmetry. In heterotic theory, this remarkable conclusion was arrived at from considerations of the ADHM construction of instantons[68].

This phenomenon, which cannot be described in a weakly coupled conformal field theory in heterotic compactifications, has a simple perturbative description in terms of a Dirichlet 5-brane in the dual Type-I orientifold theory. In particular, the enhanced USp(2k) symmetry when k small instantons coincide can be understood in terms of coincident 5-branes with a specific symplectic projection in the open string sector that is determined by the consistency of the world-sheet theory.

### (ii) Small instantons in $E_8 \times E_8$ heterotic string:

These are even more unusual to understand. The picture is clearer in the dual M-theory. Consider the dual compactification of M-theory on K3  $\times$  ( $\mathbf{S}^1/\mathbf{Z}_2$ ). The gauge fields and the corresponding instantons in the two  $E_8$  factors are confined to the two boundaries. M-theory contains a solitonic 5-brane which carries unit charge exactly like the heterotic instanton 5-brane. When one of the instanton 5-brane in the boundary  $E_8$  gauge theory shrinks to zero size, the resulting singularity corresponds to an M-theory solitonic 5-brane stuck to the boundary. One of the possibilities consistent with the Bianchi identity of H and anomaly cancellation is that the M-theory 5-brane can be emitted from the 'end of the world' into into the bulk [51]. The worldvolume theory of an M-theory 5-brane contains a tensor multiplet in its worldvolume theory [8]. The 5-brane fills the noncompact six-dimensional space-time  $\mathbf{M}^6$ . Therefore, in the process of emission of an M-theory 5-brane into the bulk, the number of tensor multiplets in the six-dimensional theory can increase by one.

Multiple tensor multiplets are not possible with usual Calabi-Yau compactifications. However, as we shall see in §7.2, one can easily construct orientifolds that have this property.

#### 7.1 Symplectic Gauge Groups

Let us consider Type-IIB theory on a K3 orbifold  $\mathbf{T}^4/\mathbf{Z}_2$  and orientifold this theory further by  $\{1,\Omega\}$  to obtain Type-I theory on the K3 orbifold. Details of this orientifold can be found in [20]. Here we shall point out some of the

salient features.

(a) The orientifold group is

$$G = \{1, I_{6789}, \Omega, \Omega I_{6789}\}. \tag{146}$$

- (b) Fixed planes of  $\Omega$  are orientifold 9-planes filling all space, which are identical to those in Type-I theory in ten dimensions discussed in §4. The charge of the orientifold plane is -32 as before requiring 32 D9-branes to cancel it. Fixed planes of  $I_{6789}\Omega$  are orientifold 5-planes located at the 16 fixed planes of  $I_{6789}$  each of charge -2. The net charge is again -32 requiring 32 D5-branes.
- (c) There are now four open string sector: 55, 59, 95, 99, depending on what type of brane the two ends of an open string end on.
- (d) A D-5 brane has the same charge as a small instanton and is dual of the solitonic 5-brane in the heterotic string.
- (e) Tadpole cancellation determines the matrix  $\gamma_{\Omega,9}$ , which implements the  $\Omega$  projection in the 99 sector, to be symmetric as in §4. Therefore, the gauge group of the 32 9-branes is an orthogonal subgroup of U(32) after the  $\Omega$  projection. This group is further broken by the  $I_{6789}$  projection.
- (f) Consistency requirements determine that the matrix  $\gamma_{\Omega,5}$ , which implements the  $\Omega$  projection in the 55 sector, must be antisymmetric if  $\gamma_{\Omega,9}$  is symmetric. This follows from a somewhat subtle argument by [20] that involves considerations of factorization and the action of  $\Omega^2$  in the 59 sector. We do not repeat the argument here and refer the reader to [20] for details. When 2k 5-branes coincide, we get a symplectic subgroup USp(2k) of U(2k) after the  $\Omega$  projection, by the arguments discussed at the end of §4.5. Thus, the small instantons and the enhanced USp(2k) gauge symmetry have a very simple perturbative description in terms of D 5-branes in Type I.

There are many interesting aspects of this model which have been analyzed in great detail in [6]. A nonperturbative description using F-theory can be found in [58].

### 7.2 Multiple tensor Multiplets

We now describe models with (0,1) supersymmetry with multiple tensor multiplets. With conventional Calabi-Yau compactifications, the only way to obtain (0,1) supersymmetry is to compactify either the Heterotic or the Type-I theory on a K3. These compactifications give only a single tensor multiplet.

For Type-II strings, compactification on a K3 leads to N=2 supersymmetry as we saw in §5.3. One cannot obtain lower supersymmetry with Calabi-Yau compactification. One way to reduce supersymmetry further is to

take an orientifold so that only one combination of the left-moving and the right-moving supercharges that is preserved by the orientation-reversal survives. If we wish to obtain a large number of tensor multiplets, a natural starting point for orientifolding is the Type-IIB theory compactified on K3 which has 21 tensor multiplets of (0,2) supersymmetry. A tensor multiplet of (0,2) supersymmetry is a sum of a tensor-multiplet  $(\mathbf{T})$  and a hyper-multiplet  $(\mathbf{H})$  of (0,1) supersymmetry.

If we use the projection  $(1+\Omega)/2$  then we get Type I theory on K3. Under  $\Omega$ , the 4-form  $D_{ijkl}$  and the 2-form  $B_{ij}$  are odd. Therefore all zero modes of these fields are also projected out. Only the zero modes of the field  $B'_{ij}$  survive which gives one self-dual and one anti-self-dual tensor in six dimensions. The self-dual tensor is required in the gravity multiplet. So, we end up with a single tensor multiplet.

This counting suggests a generalization. If the K3 orbifold has a  $Z_2$  symmetry with generator S, then we can consider taking  $\{1,\Omega S\}$  as the orientifold group. If we want N=1 supersymmetry, then the requirement is that S should not break supersymmetry further. This is ensured if it leaves the holomorphic two-form of the K3 invariant. But, S can have nontrivial action on other harmonic forms of the K3. Some of the zero modes of  $D_{ijkl}$ , can be even with respect to the combined action of  $\Omega S$  even if they are odd with respect to  $\Omega$  alone. These can give rise to additional tensor multiplets that we are interested in.

Concretely, let us consider an example of such a symmetry discussed in [14]. Consider a K3 orbifold  $\mathbf{T}^4/\mathbf{Z}_2$  that we have been discussing in §5. Such a K3 admits a  $\mathbf{Z}_2$  involution

$$S: (z_1, z_2) \to (-z_1 + \frac{1}{2}, -z_2 + \frac{1}{2}).$$
 (147)

S leaves the holomorphic 2-form  $dz^1 \wedge dz^2$  of the K3 orbifold, and in fact all forms coming from the untwisted sector of the orbifold invariant. Let us look at its action on the twisted sector. It is easy to see from Figure 10 that S takes the 16 fixed points of  $I_{6789}$  into each other so out of the  $16_a$  anti-self-dual 2-forms coming from the twisted sector, eight are odd and eight are even.

To obtain an anti-self-dual 2-form in six dimensions as a zero mode of the 4-form in ten dimensions, we use separation of variables to write the 4-form as  $D^{(4)} = B_{\alpha}^{(2)} \wedge f_{\alpha}^2$ ,  $\alpha = 1, \ldots, 19$  where  $f_{\alpha}^2$  is one of the anti-self-dual harmonic 2-forms on K3 which depends only on the coordinates of K3 and  $B_{\alpha}^{(2)}$  depends only on the non-compact coordinates. Because  $f_{\alpha}^2$  is harmonic,  $B_{\alpha}^{(2)}$  is a massless field in six dimensions. By the self-duality of  $D^{(4)}$  in ten dimensions,  $B_{\alpha}^{(2)}$  is anti-self-dual in the six Minkowski dimensions. Now, if

we use the combined projection,  $\frac{(1+\Omega S)}{2}$  instead of  $\frac{(1+\Omega)}{2}$  then eight tensors coming from the eight  $f_{\alpha}^2$ 's that are odd under S survive and the remaining are projected out. In addition there is one more tensor multiplet that comes from the zero mode of  $B'_{ij}$  as in Type-I. Altogether, we get  $\mathbf{T}=9$ .

The orientifold group is

$$G = \{1, I_{6789}, \Omega S, \Omega I_{6789} S\}. \tag{148}$$

We shall not discuss the open string sector here but it can be found in [14]. One interesting aspect of models with multiple tensors is worth mentioning. The cancellation of gauge and gravitational anomalies in these models requires an extension of of the Green-Schwarz mechanism found by Sagnotti [49] in which more than one tensors participate in the anomaly cancellation. Details of anomaly cancellation for the model described above can be found in [14]).

The model has an M-theory dual [54] that makes use of the observation that Type-IIB on K3 is dual to M-theory on  $\mathbf{T}^5/\mathbf{Z}_2$  [18, 67].

There are a number of other ways to obtain multiple tensors in string models. Orientifolds of K3 orbifolds where the orbifold group is other than  $\mathbb{Z}_2$  typically give multiple tensors [24, 15, 25]. Yet another interesting variation is to accompany the action of  $\Omega$  with additional phases in the twisted sectors of the  $\mathbb{Z}_2$  orbifold symmetry [45, 7, 16, 27]. This is the analog of discrete torsion for  $\mathbb{Z}_N \times \mathbb{Z}_N$  orbifolds [63].

### Acknowledgments

I would like to thank the organizers of the ICTP Summer school for inviting me deliver these lectures, and Rajan Pawar and M. R. Shinde for help with figures and with TeXing the manuscript.

### References

- 1. P. Aspinwall, Nucl. Phys. B 460, 1996 (57).
- 2. L. Alvarez-Gaumé and E. Witten, Nucl. Phys. B234 (1983) 269.
- 3. T. Banks, M. Douglas, and N. Seiberg, 'Probing F-theory With Branes," *Phys. Lett.* B **387**, 1996 (278), hep-th/9605199.
- M. Bianchi and A. Sagnotti, Phys. Lett. B 247, 1990 (517); Nucl. Phys. B 361, 1991 (519)
- Z. Bern, and D. C. Dunbar, Phys. Lett. **B242** (1990) 175; Phys. Rev. Lett. **64** (1990) 827; Nucl. Phys. **B319** (1989) 104; Phys. Lett. **203B** (1988) 109.

- M. Berkooz, R. Leigh, J. Polchinski, J. Schwarz, N. Seiberg, and E. Witten, "Anomalies, Dualities, and Topology of D=6 N=1 Superstring Vacua," Nucl. Phys. B 475, 1996 (115), hep-th/9605184
- 7. J. Blum and A. Zaffaroni, "An Orientifold from F Theory," *Phys. Lett.* **B387** (1996) 71 hep-th/9607019.
- 8. C. Callan, J. Harvey, and A. Strominger, "Worldbrane Actions for String Solitons" *Nucl. Phys.* B **359**, 1991 (611).
- C. G. Callan, C. Lovelace, C. R. Nappi and S.A. Yost, Nucl. Phys. B308 (1988) 221.
- C. G. Callan, C. Lovelace, C. R. Nappi and S.A. Yost, Nucl. Phys. B293 (1987) 83.
- 11. J. Polchinski and Y. Cai, Nucl. Phys. B296, 91 (1988)
- 12. A. Dabholkar, "Ten-dimensional Heterotic String as a Soliton," *Phys. Lett.* B **357**, 1995 (307), hep-th/9506160.
- A. Dabholkar and J. A. Harvey, "Nonrenormalization of Superstring Tension," *Phys. Rev. Lett.* **63**, 1989 (478); A. Dabholkar, G. Gibbons, J. A. Harvey, and F. R. Ruiz-Ruiz, "Superstrings and Solitons," *Nucl. Phys.* B **340**, 1990 (33).
- A. Dabholkar and J. Park, "An Orientifold of Type IIB theory on K3," Nucl. Phys. B 472, 1996 (207), hep-th/9602030.
- A. Dabholkar and J. Park, "Strings on Orientifolds," Nucl. Phys. B 477, 1996 (701)hep-th/9604178.
- A. Dabholkar and J. Park, "Note on F-theory and Orientifolds," Phys. Lett. B394 (1997) 302, hep-th/9607041.
- J. Dai, R. G. Leigh, and J. Polchinski, Mod. Phys. Lett. A4 (1989) 2073.
- 18. K. Dasgupta and S. Mukhi, "Orbifolds of M-Theory," hep-th/9512196.
- L. Dixon, J. Harvey, C. Vafa, and E. Witten, "Strings on Orbifolds I and II," Nucl. Phys. B 261, 1985 (678); Nucl. Phys. B 274, 1986 (285).
- E. G. Gimon and J. Polchinski, "Consistency Conditions for Orientifolds and D-manifolds," hep-th/9601038.
- 21. A. Erdelyi et. al., Higher Transcendental Functions, vol. 2, (McGraw-Hill, 1958).
- 22. T. Eguchi, P. B. Gilkey, and A. J. Hanson, Phys. Rept. 66 (1980) 213.
- 23. D. Friedan, E. Martinec, and S. Shenker, Nucl. Phys. B 271, 1986 (93).
- E. Gimon and C. Johnson, "K3 Orientifolds," Nucl. Phys. B477 (1996) 715, hep-th/9604129.
- 25. E. Gimon and C. Johnson, "Multiple Realizations of N=1 Vacua in Six-Dimensions,"
  - Nucl. Phys. **B479** (1996) 285, hep-th/9606176.

- A. Giveon, M. Porrati and E. Rabinovici, Phys. Rep. 244 (1994) 77, hep-th/9401139.
- 27. R. Gopakumar and S. Mukhi, *Nucl. Phys.* **B479** (1996) 260, hep-th/9607057.
- M. B. Green, J. H. Schwarz, and E. Witten, Superstring Theory, Vol. I and II, Cambridge University Press (1987).
- B. R. Greene, A. Shapere, C. Vafa, and S. T. Yau, "Stringy Cosmic Strings and Noncompact Calabi-Yau Manifolds," Nucl. Phys. B 337, 1990 (1).
- 30. P. Ginsparg, "Applied Conformal Field Theory," Les Houches, Session XLIX, 1988, Fields, Strings, and Critical Phenomena, ed. E. Brezin and J. Zin-Justin, Elsevier Science Publishers B. V. (1989).
- 31. J. Govaerts, *Phys. Lett.* B **220**, 1989 (461).
- 32. P. Horava, "Strings on Worldsheet Orbifolds," Nucl. Phys. B **327**, 1989 (461).
- 33. P. Horava, "Background Duality of Open String Models, *Phys. Lett.* **B231** (1989) 251.
- 34. P. Horava and E. Witten, "Heterotic and Type I String Dynamics from Eleven Dimensions," NPB **460**, 1989 (506).
- 35. C. M. Hull, *Phys. Lett.* B **357**, 1995 (345), hep-th/9506194.
- C. M. Hull and P. K. Townsend, "Unity of Superstring Dualities," Nucl. Phys. B 438, 1995 (109), hep-th/9410167.
- 37. N. Ishibashi and T. Onogi, Nucl. Phys. **B318** (1989) 239.
- 38. M. Li, "Boundary States of D-Branes and Dy-Strings," Nucl. Phys. B460 (1996) 351, hep-th/9510161
- 39. D. Morrison and C. Vafa, "Compactifications of F-Theory on Calabi-Yau Threefolds-I," hep-th/9602114.
- 40. D. Morrison and C. Vafa, "Compactifications of F-Theory on Calabi-Yau Threefolds-II," hep-th/9603161.
- K. S. Narain, *Phys. Lett.* B **169**, 1986 (41);
   K. S. Narain, M. H. Sarmadi, and E. Witten, *Nucl. Phys.* B **279**, 1987 (369)
- 42. J. Polchinski, "Dirichlet Branes and Ramond-Ramond Charges," *Phys. Rev. Lett.* **75**, 1995 (4724), hep-th/9510017.
- 43. J. Polchinski and E. Witten, "Evidence for Heterotic-Type I String Duality," *Nucl. Phys.* B **460**, 1996 (525), hep-th/9510169.
- 44. J. Polchinski, S. Chaudhuri and C. Johnson, "Notes on D-Branes," hep-th/9602052.
- 45. J. Polchinski, "Tensors From K3 Orientifolds," hep-th/9606165.
- 46. J. Polchinski, "TASI Lectures On D-branes," hep-th/9611050.

- 47. G. Pradisi and A. Sagnotti, em Phys. Lett. **B216** (1989) 59.
- 48. A. Sagnotti, in Cargese '87, "Non-perturbative Quantum Field Theory," ed. G. Mack et. al. (Pergamon Press, 1988) p. 521
- A. Sagnotti, Phys. Lett. **B294** (1992) 196;
   A. Sagnotti, Some Properties of Open-String Theories, preprint ROM2F-95/18, hep-th/9509080.
- 50. J. Scwharz, "An  $SL(2, \mathbf{Z})$  Multiplet of Type IIB Superstrings," hep-th/9508143.
- N. Seiberg and E. Witten, "Comments on String Dynamics in Six Dimensions," Nucl. Phys. B 471, 1996 (121), hep-th/9603003.
- N. Seiberg and E. Witten, "Electric-Magnetic Duality, Monopole Condensation, And Confinement In N = 2 Supersymmetric Yang-Mills Theory," Nucl. Phys. B426 (1994) 19, hep-th/9407087;
   "Monopoles, Duality, And Chiral Symmetry Breaking In N = 2 Supersymmetric QCD," Nucl. Phys. B431 (1994) 484, hep-th/9408099.
- 53. A. Sen, "Non-perturbative String Theory," hep-th/9802051.
- 54. A. Sen, "M-Theory on  $(K3 \times S^1)/Z_2$ ," Phys. Rev. D **53**, 1996 (6725), hep-th/9602010.
- A. Sen, "Orbifolds of M-Theory and String Theory," Mod. Phys. Lett. 11, 1996 (1339), hep-th/9603113.
- A. Sen, "Duality and Orbifolds," Nucl. Phys. B 474, 1996 (361), hepth/9604070.
- 57. A. Sen, "F Theory and Orientifolds," Nucl. Phys. B **475**, 1996 (562), hep-th/9605150.
- 58. A. Sen, "A non-perturbative description of the Gimon-Polchinski Orientifold, *Nucl. Phys.* B **489**, 1997 (139), hep-th/9611186.
- 59. A. Sen, "Unification of String Dualities," hep-th/9609176.
- 60. A. Strominger, "Heterotic Solitons," Nucl. Phys. B 343, 1990 (167).
- P. Townsend, *Phys. Lett.* **B139** (84) 283;
   N. Seiberg, *Nucl. Phys.* **B303** (88) 286.
- 62. C. Vafa, "Evidence for F Theory," HUTP-96-A004, hep-th/9602022.
- 63. C. Vafa, "Modular Invariance and Discrete Torsion on Orbifolds," *Nucl. Phys.* **B273** (1986) 592.
- 64. C. Vafa and E. Witten, "Dual String Pairs with N=1 and N=2 Supersymmetry in Four Dimensions, hep-th/9507050.
- 65. E. Witten and D. Olive, *Phys. Lett.* **B78** (1978) 97.
- 66. E. Witten, "String Theory Dynamics in Various Dimensions," Nucl. Phys. B 443, 1995 (85), hep-th/.
- 67. E. Witten, "Five-branes And M-Theory On An Orbifold," Nucl. Phys. B  ${\bf 463}$ , 1996 (383), hep-th/9512219

- 68. E. Witten, "Small Instantons in String Theory,"  $\it Nucl. Phys.$ B 460, 1996 (541), hep-th/9511030
- 69. E. Witten, "Bound States Of Strings And p-Branes," Nucl. Phys. B  ${\bf 469},\,1996$  (335)
- 70. E. Zaslow, "Topological Orbifold Couplings and Quantum Cohomology Rings," Comm. Math. Phys. 156, 1993 (301).