

Nuclear Physics Lecture Notes
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Lecture 1

Nuclear Sizes and Isotope Shifts

Assume a uniform distribution of charge Ze in a spherical nucleus of radius R . Then calculate the potential inside nucleus V_{inside} .

E_{inside} by Gauss' law:

$$E_{inside} = -\frac{Ze}{4\pi\epsilon_0 r^2} \left(\frac{r}{R}\right)^3 \quad (1.1)$$

Find V_{inside} by integrating E_{inside} and applying boundary conditions at $r=R$ to match V_{inside} to usual $1/r^2$ potential:

$$V_{inside}(r) = -\frac{Zer^2}{8\pi\epsilon_0 R^3} + \frac{3Ze}{8\pi\epsilon_0 R} \quad (1.2)$$

The difference between actual potential and Coulomb:

$$\Delta V(r) = -\frac{Zer^2}{8\pi\epsilon_0 R^3} + \frac{3Ze}{8\pi\epsilon_0 R} - \frac{Ze}{4\pi\epsilon_0 r} \quad (r < R) \quad (1.3)$$

Using 1st order perturbation theory to calculate energy shift ΔE :

$$\Delta E = \int_0^R 4\pi r^2 \psi^*(r) [-e\Delta V(r)] \psi(r) dr \quad (1.4)$$

Inserting approximate Hydrogenic ground state wave function:

$$\begin{aligned} \psi(r) &= 2 \left(\frac{Z}{a_0}\right)^{3/2} \exp(-Zr/a_0) \approx 2 \left(\frac{Z}{a_0}\right)^{3/2} \\ \Rightarrow \Delta E &= \int_0^R 4\pi r^2 4 \left(\frac{Z}{a_0}\right)^3 (-e) \left[-\frac{Zer^2}{8\pi\epsilon_0 R^3} + \frac{3Ze}{8\pi\epsilon_0 R} - \frac{Ze}{4\pi\epsilon_0 r} \right] dr \\ \Delta E &= -4e \left(\frac{Z}{a_0}\right)^3 \frac{Ze}{4\pi\epsilon_0} \pi R^2 \left[-\frac{4}{10} + \left(\frac{4}{3}\right) \left(\frac{3}{2}\right) - 2 \right] \\ \Delta E &= \frac{2Ze^2 R^2}{5\epsilon_0} \left(\frac{Z}{a_0}\right)^3 \quad (1.6) \end{aligned}$$

Note how ΔE is proportional to Z^4 and R^2 therefore the most noticeable effect is deep inside large Z nuclei. To see the isotope shift, look at transitions from $l=1$ (no isotope shift) to $l=0$ (large isotope shift), preferably look for transitions at low n .

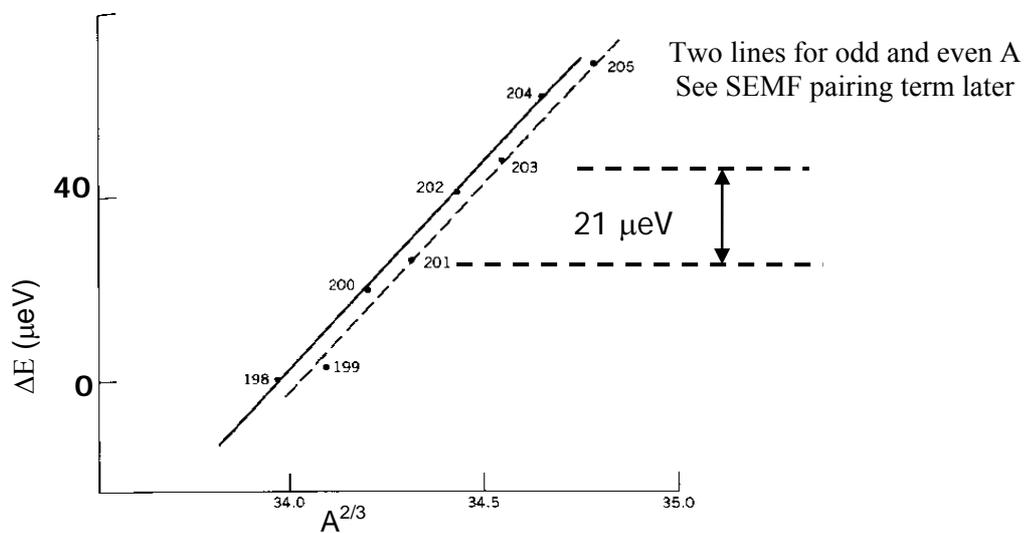
- Types of isotope shifts in increasing shift order:
 - Isotope shift for optical spectra: $\Delta E = O(meV)$
 - Isotope shift for X-ray spectra (bigger effect than optical because electrons closer to nucleus): $\Delta E = O(0.1 eV)$

- Isotope shift for X-ray spectra for muonic atoms. Effect greatly enhanced because $m_m \sim 207 m_e$ and $a_0 \sim 1/m$. $\Delta E = O(keV)$
- All data consistent with $R = R_0 A^{1/3}$ using $R_0 = 1.25 \text{ fm}$.

Isotope Shift in Optical Spectra:

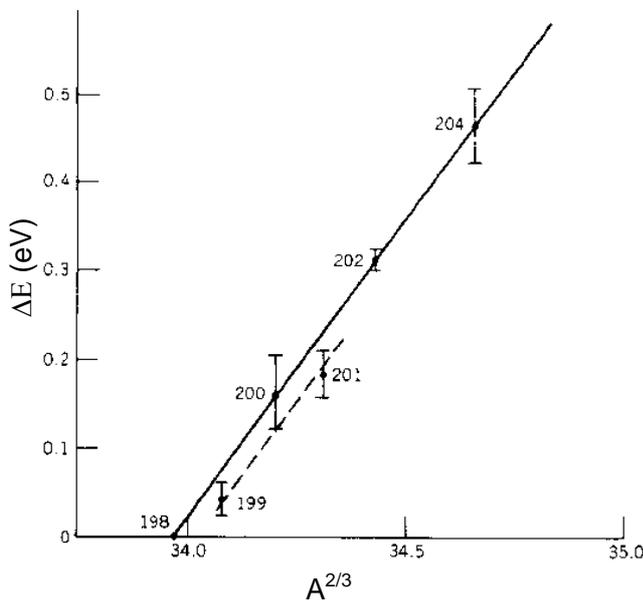
- Need to use higher n wave functions to calculate this
- Use $Z_{\text{eff}} \approx Z - n$
- expect $(Z_{\text{eff}}/Z)^4$ dependence in ΔE
- Why is $\Delta E \sim A^{2/3}$?
- $\Delta E \sim R^2$ (see before) and $R = R_0 A^{1/3}$

Energy shift of an optical transition in Hg at $\lambda = 253.7 \text{ nm}$ for different A relative to $A = 198$. Data was obtained by Doppler free laser spectroscopy. The effect is about 1 in 10^7 (Note the even/odd structure).

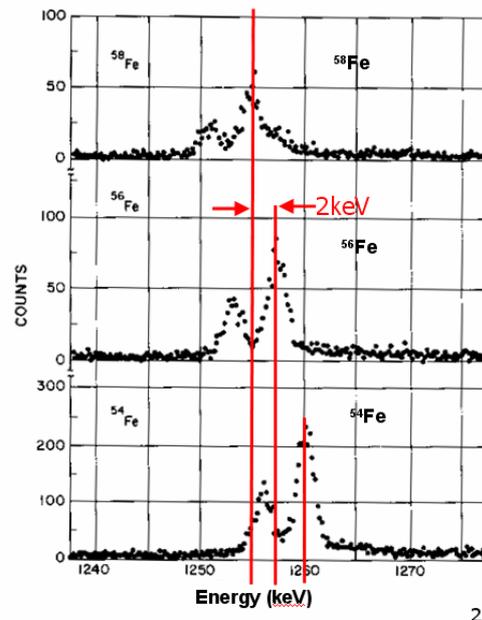


Isotope Shift in X-Ray Spectra:

Data on the isotope shift of K X-ray lines in Hg. The effect is about 1 in 10^6 . Again the data shows the $R^2 = A^{2/3}$ dependence and the even/odd effect.



Isotope Shift in Muonic Atoms:



Because $a_0 \sim 1/m$ the effect is $\sim 0.4\%$, i.e. much larger than for an electron. Changing R_{nucl} by increasing A gives changes in isotope shifts of 2 keV. Data on isotope shift of K X-rays from muonic atoms [in which a muon with $m = 207m_e$ takes the place of the atomic electron]. The large peak is $2p_{3/2}$ to $1s_{1/2}$. The small peak is $2p_{1/2}$ to $1s_{1/2}$. The size comes from the $2j+1$ statistical weight.

Isotope Shift Summary:

All types of isotopes shifts show $\sim A^{2/3}$ as expected for a R_{nucl}^2 dependence. This holds for all types of nuclei. When fitting the slopes we find the same R_0 in $R_{nucl}^2 = A^{2/3}$, this tells us that the nuclear density is a universal constant.

Lecture 2

The Semi Empirical Mass Formula - SEMF

- Assumptions:
 - Nuclear density is constant (see lecture 1).
 - We can model effect of short range attraction due to strong interaction by a liquid drop model.
 - Coulomb corrections can be computed using electromagnetism (even at these small scales)
 - Nucleons are fermions at $T = 0$ in separate wells (Fermi gas model leads to the asymmetry term)
 - QM holds at these small scales leading to pairing term.

SEMF = Liquid Drop Model + Fermi Gas Model + phenomenology + QM + EM

Liquid Drop Nucleus:

Phenomenological model is used to understand binding energies. Consider a liquid drop with no rotation and no influence of gravity. Can assume that intermolecular forces are repulsive at short distances, attractive at intermediate distances and negligible at large distances also that the nucleus is of constant density.

n =number of molecules, T =surface tension, B =binding energy, E =total energy of the drop, a, b =free constants

$$E = -an + 4pR^2T \quad \rightarrow \quad B = an - bn^{2/3}$$

The liquid drop is analogous to the nucleus, which has constant density. From nucleon-nucleon scattering experiments we know that nuclear force has short range repulsion and is attractive at intermediate distances. Assume charge independence of nuclear force, neutrons and protons - have same strong interactions this can be checked with experiment (Mirror Nuclei).

Coulomb Term:

The nucleus is electrically charged with total charge Ze . Assuming that the charge distribution is spherical and computing the reduction in binding energy due to the Coulomb interaction.

$$\begin{aligned} E_{Coulomb} &= \int_0^{Ze} \frac{Q(r)}{4\pi\epsilon_0 r} dQ \\ Q(r) &= Ze \left(\frac{r}{R} \right)^3 \\ dQ &= \frac{3Ze r^2}{R^3} dr \\ E_{Coulomb} &= \int_0^R \frac{3(Ze)^2}{4\pi\epsilon_0 r} \frac{r^5}{R^6} dr = \frac{3}{5} \frac{(Ze)^2}{4\pi\epsilon_0 R} \end{aligned} \quad (2.1)$$

However this includes self interaction of last proton with itself. To correct this replace Z^2 with $Z(Z-1)$. Using $R = R_0 A^{1/3}$. This leads to the equation below. In principle could take d from the above calculation but it is more accurate to take it from the overall fit of the SEMF to data (since nuclei not totally spherical or homogeneous)

$$B_{Coulomb}(Z, A) = -d \frac{Z(Z-1)}{A^{1/3}} \quad (2.2)$$

Does the assumption of the drop model of constant binding energy for every constituent of the drop actually hold for nuclei? Compare binding energies of mirror nuclei (nuclei with $n \leftrightarrow p$). Eg ${}^7_3\text{Li}$ and ${}^7_4\text{Be}$. Then if the assumption holds the mass difference should be due to n/p mass difference and Coulomb energy alone.

$$\begin{aligned} \Delta E_{coulomb}(Z, Z-1) &= \frac{3}{5} \frac{e^2}{4\pi\epsilon_0 R} [Z(Z-1) - (Z-1)(Z-2)] \\ &= \frac{3}{5} \frac{e^2}{4\pi\epsilon_0 R} 2(Z-1) \\ Z &\sim A/2; R = R_0 A^{1/3} \\ \Rightarrow \Delta E_c(Z, Z-1) &\propto A^{2/3} \end{aligned} \quad (2.3)$$

Now lets measure mirror nuclei mass, assuming that the model holds and derive $\Delta E_{coulomb}$ from the measurement. This should show a $A^{2/3}$ dependence and the scaling factor should yield the correct R_0 of 1.2 fm if the assumptions were right.

nn and **pp** interaction same (apart from Coulomb).

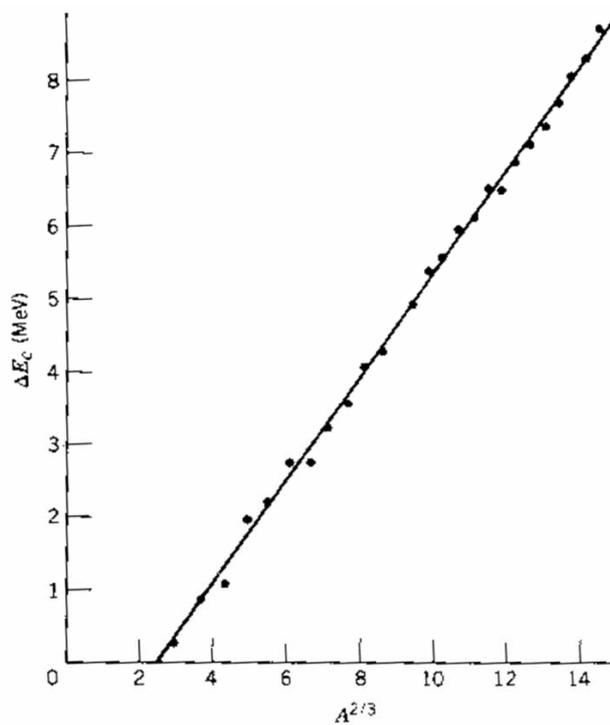
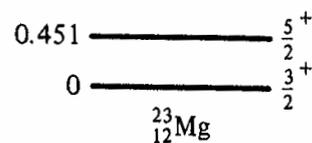
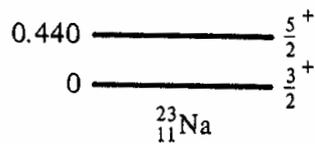
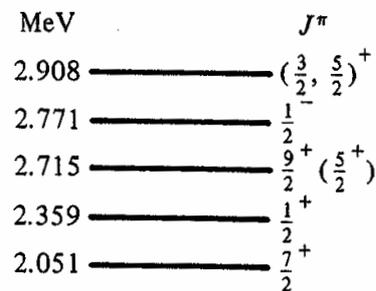
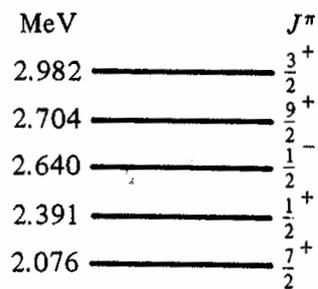
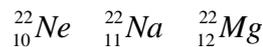


Figure 3.10 Coulomb energy differences of mirror nuclei. The data show the expected $A^{2/3}$ dependence, and the slope of the line gives $R_0 = 1.22$ fm.

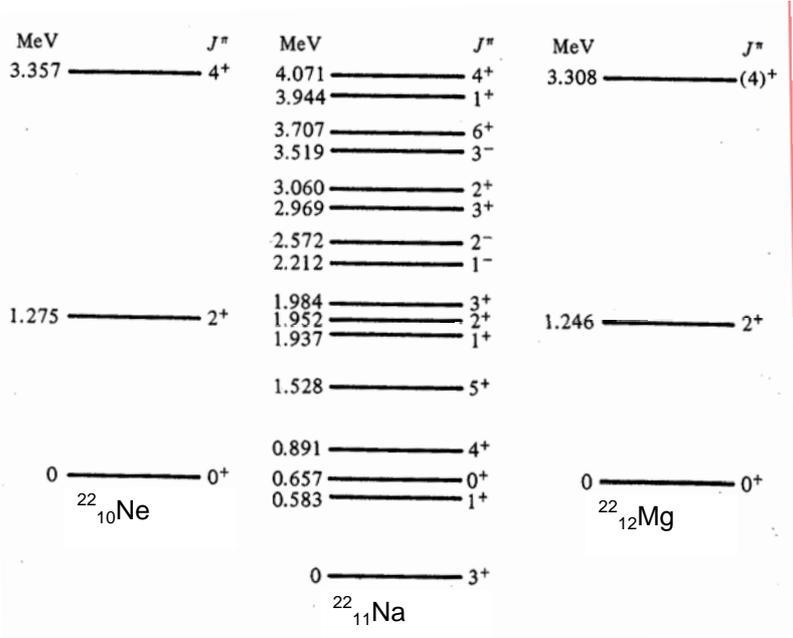
Energy levels of two mirror nuclei for a number of excited states corrected for n/p mass difference and Coulomb energy.



Mirror nuclei showed that strong interaction is the same for nn and pp. What about np? Compare energy levels in “triplets” with same A, different number of n and p. e.g.



Find the same energy levels for the same spin states and therefore the strong interaction is the same for np as nn and pp. Same spin/parity states should have the same energy. Indeed when $np=nn=pp$ this is true. Note that there are far more states in ${}^{22}_{11}\text{Na}$. Because it has more np pairs than the others, np pairs can be in any spin-space configuration, but pp or nn pairs are excluded from the totally symmetric ones by Pauli’s exclusion principle. Also ${}^{22}_{11}\text{Na}$ has the lowest (most bound) state. (The diagram for the elements is shown on the next page)



We have shown by measurement that if we correct for n/p mass difference and Coulomb interaction, then energy levels in nuclei are unchanged under $n \leftrightarrow p$ exchange and we must change nothing else i.e. spin and space wavefunctions must remain the same. Strong two-body interaction must be the same for pp, pn and nn if nucleons are in the same quantum state. However by Pauli exclusion principle there are bound state of pn but not pp or nn, this is because the strong force is spin dependent and the most strongly bound spin-space configurations (deuteron) are not available to nn or pp. Just like $^{22}_{11}\text{Na}$ on the previous triplet level schema.

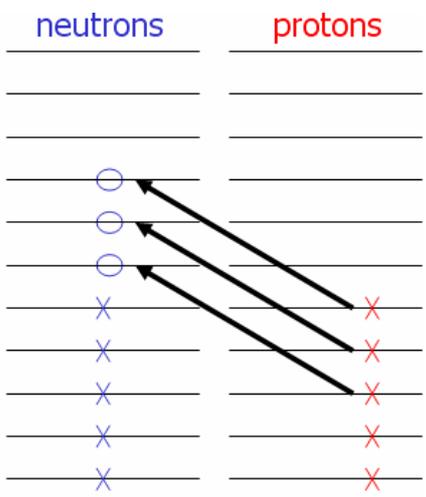
Volume and Surface Term:

Now we have all we need to trust that we can apply the liquid drop model to a nucleus - constant density and same binding energy for all constituents. Since we are building a phenomenological model in which the coefficients a and b will be determined by a fit to measured nuclear binding energies we must include any further terms we may find with the same A dependence together.

$$B_{Volume}(A) = aA \tag{2.4}$$

$$B_{Surface}(A) = -bA^{2/3}$$

Asymmetry Term:



Neutrons and protons are spin $\frac{1}{2}$ fermions and hence must obey Pauli exclusion principle. If all other factors were equal nuclear ground state would have equal numbers of n & p.

In the diagram n and p states are with same spacing Δ . Crosses represent initially occupied states in ground state. If 3 protons were turned into neutrons, the extra energy required would be $3 \times 3\Delta$. In general if there are $Z - N$ excess protons over neutrons the extra energy is $((Z - N)/2)2\Delta$ relative to $Z = N$.

Assume:

- p and n form two independent, non-interacting gases occupying their own square Fermi wells
- $kT \ll \Delta$ so we can neglect kT and assume $T=0$, this should be obvious as nuclei don't suddenly change state at room temperature
- Nucleons move non-relativistically (check later if this makes sense)

Fermi Gas Model

Define the momentum associated with the Fermi level through

$$E_F = \frac{p_F^2}{2m} \quad (2.5)$$

Where m is the mass of a nucleon. Ignoring the presence of fermions beyond the Fermi level, we can write the volume for states in momentum space as

$$V_{p_F} = \frac{4\pi}{3} p_F^3 \quad (2.6)$$

If V denotes the physical nuclear volume, then the total volume for the states in what can be called 'phase space' will be given by the product

$$\begin{aligned} V_{TOT} &= VV_{p_F} = \frac{4\pi}{3} r_0^3 A \frac{4\pi}{3} p_F^3 \\ &= \left(\frac{4\pi}{3}\right)^2 A (r_0 p_F)^3 \end{aligned} \quad (2.7)$$

which is proportional to the total number of quantum states of the system. The Heisenberg uncertainty principle provides the restriction on the minimum volume that can be associated with any physical state of the system, which can be shown to be

$$V_{state} = (2\pi\hbar)^3 = h^3 \quad (2.8)$$

it follows that the number of fermions that can fill states up to and including the Fermi level is

$$\begin{aligned} n_F &= 2 \frac{V_{TOT}}{(2\pi\hbar)^3} = \frac{2}{(2\pi\hbar)^3} \left(\frac{4\pi}{3}\right)^2 A (r_0 p_F)^3 \\ &= \frac{4}{9\pi} A \left(\frac{r_0 p_F}{\hbar}\right)^3 \end{aligned} \quad (2.9)$$

where the factor of 2 arises because each state can be occupied by 2 fermions with opposite spins. Considering for simplicity a nucleus with $N = Z = A/2$ and assume that all the states up to and including the Fermi level are filled, then

$$\begin{aligned} N = Z &= \frac{A}{2} = \frac{4}{9\pi} A \left(\frac{r_0 p_F}{\hbar}\right)^3 \\ \Rightarrow p_F &= \frac{\hbar}{r_0} \left(\frac{9\pi}{8}\right)^{\frac{1}{3}} \end{aligned} \quad (2.10)$$

In this case, the Fermi momentum is constant and independent of the nucleon number. This leads to the equation for the Fermi energy

$$E_F = \frac{p_F^2}{2m} = \frac{1}{2m} \left(\frac{\hbar}{r_0} \right)^2 \left(\frac{9\pi}{8} \right)^{\frac{2}{3}} \approx 33 \text{ MeV} \quad (2.11)$$

From statistical mechanics, know that

$$\langle E \rangle = \frac{3}{5} E_F \quad (2.12)$$

Return to equation (2.10) and rearrange for Fermi momentum to find Fermi energy

$$\begin{aligned} Z &= \frac{4}{9\pi} A \left(\frac{r_0 p_F}{\hbar} \right)^3 \\ p_F &= \frac{\hbar}{r_0} \left(\frac{9\pi}{4} \frac{Z}{A} \right)^{\frac{1}{3}} \\ E_F^Z &= \frac{\hbar^2}{2mr_0^2} \left(\frac{9\pi}{4} \frac{Z}{A} \right)^{\frac{2}{3}} \end{aligned} \quad (2.13)$$

Therefore can now calculate the total energy of protons and neutrons

$$\begin{aligned} E_{tot} &= Z \langle E^Z \rangle + N \langle E^N \rangle \\ &= \frac{3}{5} E_F (Z + N) \end{aligned} \quad (2.14)$$

$$E_{tot} = \frac{3}{5} \frac{\hbar^2}{2mr_0^2} \left(\frac{9\pi}{4} \right)^{\frac{2}{3}} \left[\frac{Z^{\frac{5}{3}} + N^{\frac{5}{3}}}{A^{\frac{2}{3}}} \right] \quad (2.15)$$

Introduce new variable $y = N - Z$ and substitute into the above equation

$$\begin{aligned} E_{tot} &= \frac{3}{5} \frac{\hbar^2}{2mr_0^2} \left(\frac{9\pi}{4} \right)^{\frac{2}{3}} \frac{1}{2^{\frac{5}{3}}} \left[\frac{(A-y)^{\frac{5}{3}} + (A+y)^{\frac{5}{3}}}{A^{\frac{2}{3}}} \right] \\ &= \frac{KA}{2^{\frac{5}{3}}} \left[\left(1 - \frac{y}{A} \right)^{\frac{5}{3}} + \left(1 + \frac{y}{A} \right)^{\frac{5}{3}} \right] \end{aligned}$$

use binomial expansion, defined as

$$\begin{aligned} (1+x)^n &\approx 1 + nx + \frac{n(n-1)}{2!} x^2 + \dots \\ \left(1 - \frac{y}{A} \right)^{\frac{5}{3}} &\approx 1 - \frac{5}{3} \frac{y}{A} + \frac{5}{9} \left(\frac{y}{A} \right)^2 - \dots \\ \left(1 + \frac{y}{A} \right)^{\frac{5}{3}} &\approx 1 + \frac{5}{3} \frac{y}{A} + \frac{5}{9} \left(\frac{y}{A} \right)^2 + \dots \\ E_{tot} &\approx \frac{KA}{2^{\frac{2}{3}}} \left[1 + \frac{5}{9} \left(\frac{y}{A} \right)^2 \right] \\ &\approx \frac{KA}{2^{\frac{2}{3}}} + \frac{5}{9} \frac{K}{2^{\frac{2}{3}}} \frac{(N-Z)^2}{A} \end{aligned} \quad (2.16)$$

$$E_{tot} = \frac{3}{5}E_F A + \frac{1}{3}E_F \frac{(N-Z)^2}{A} \quad (2.17)$$

The first term of equation (2.16) is only proportional to A , it has already been captured by the volume term of the liquid drop model. Therefore the Fermi gas model leads to the Asymmetry term:

$$B_{Asymmetry}(N, Z) = -c \frac{(N-Z)^2}{A} \quad (2.18)$$

Pairing Term:

From observation, nuclei with even Z and N are more tightly bound than when Z and/or N are odd. This is due to 2 protons or 2 neutrons in the same energy level having opposite spin, this results in having to have a symmetric wavefunction which is more likely for the particle to have a maximum overlap and therefore be more bound. Empirical fit gives

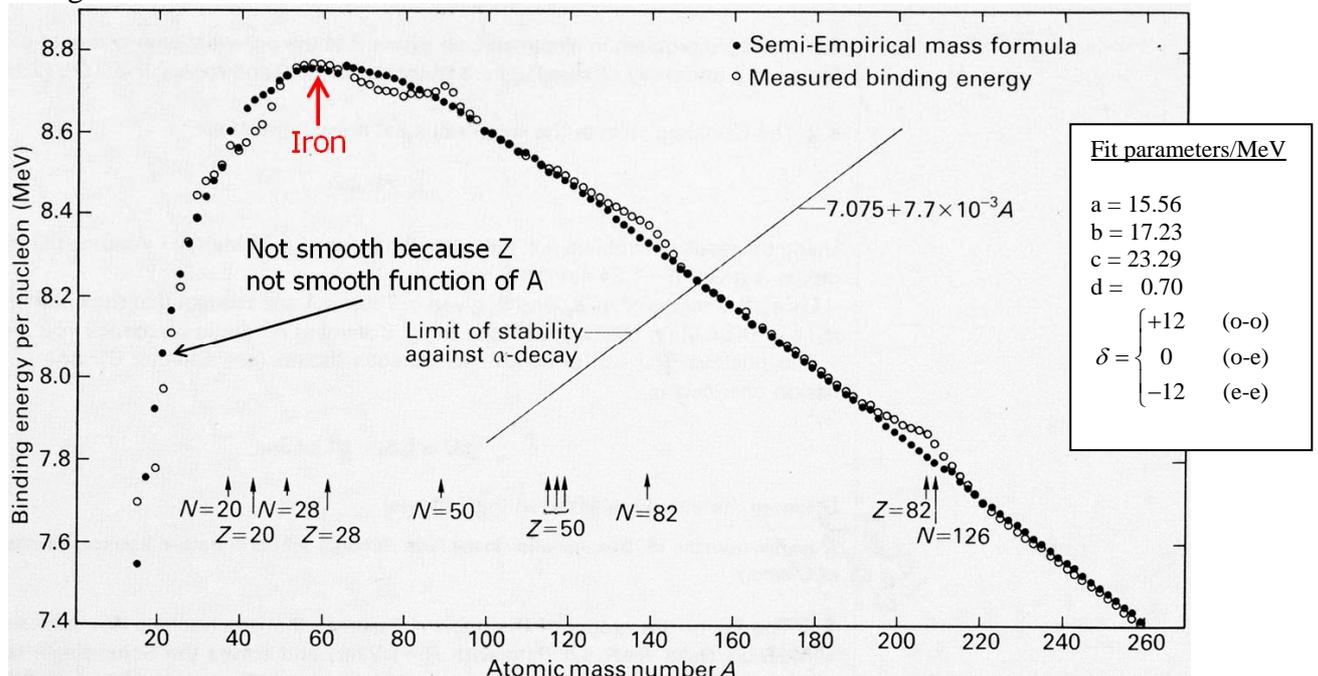
$$B_{Pairing}(A) = -\frac{\delta}{A^{1/2}} \quad (2.19)$$

	δ
e-e	+ive
e-o	0
o-o	-ive

Putting the results for the Semi Empirical Mass Formula together, find

$$B(N, Z) = aA - bA^{2/3} - c \frac{(N-Z)^2}{A} - d \frac{Z^2}{A^{1/3}} - \frac{\delta}{A^{1/2}} \quad (2.20)$$

Please note that the Coulomb term has been changed from $Z(Z-1)$ to just Z^2 as the formula is only applicable for $A > 20$ and for heavy atoms a difference of 1 nucleon is not going to be significant.



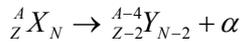
Lecture 3 (8 & 9)

Stability of Atoms

- From observation, stable nuclei do not lie on a straight line in N-Z plane. The SEMF predicts this:
 - Coulomb term pulls them down (prefers $Z < N$) and
 - wins with increasing Z over Asymmetry term (prefers $Z = N$)
- Rich structure in location of stable elements
 - more stable isotopes of e-e than o-o nuclei (see β -decay)
 - No "life" beyond $Z=92$ (U) and a big gap from $Z=82$ to 92 (the region of natural radio activity)
 - Funny magic numbers for Z and N (see SEMF limitations)

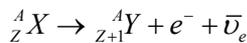
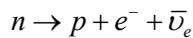
α -decay:

- emission of Helium nucleus
- $Z \rightarrow Z - 2$
- $N \rightarrow N - 2$
- $A \rightarrow A - 4$



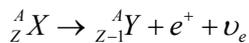
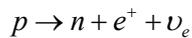
β^- decay

- emission of e^- and $\bar{\nu}_e$
- $Z \rightarrow Z + 1$
- $N \rightarrow N - 1$
- $A = \text{const}$



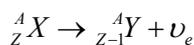
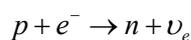
β^+ decay

- emission of e^+ and ν_e
- $Z \rightarrow Z - 1$
- $N \rightarrow N + 1$
- $A = \text{const}$



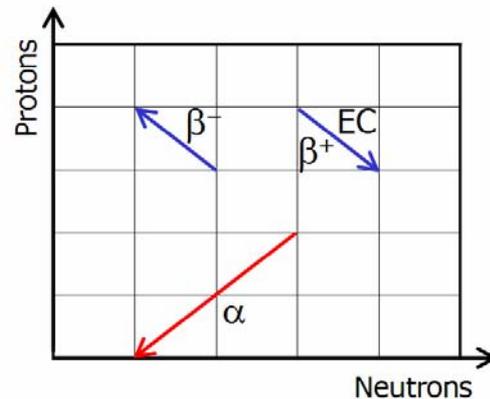
electron capture (EC)

- absorption of e^- and emission of $\bar{\nu}_e$
- $Z \rightarrow Z - 1$
- $N \rightarrow N + 1$
- $A = \text{const}$



γ decay

- emission of gamma ray
- Z, N, A all constant

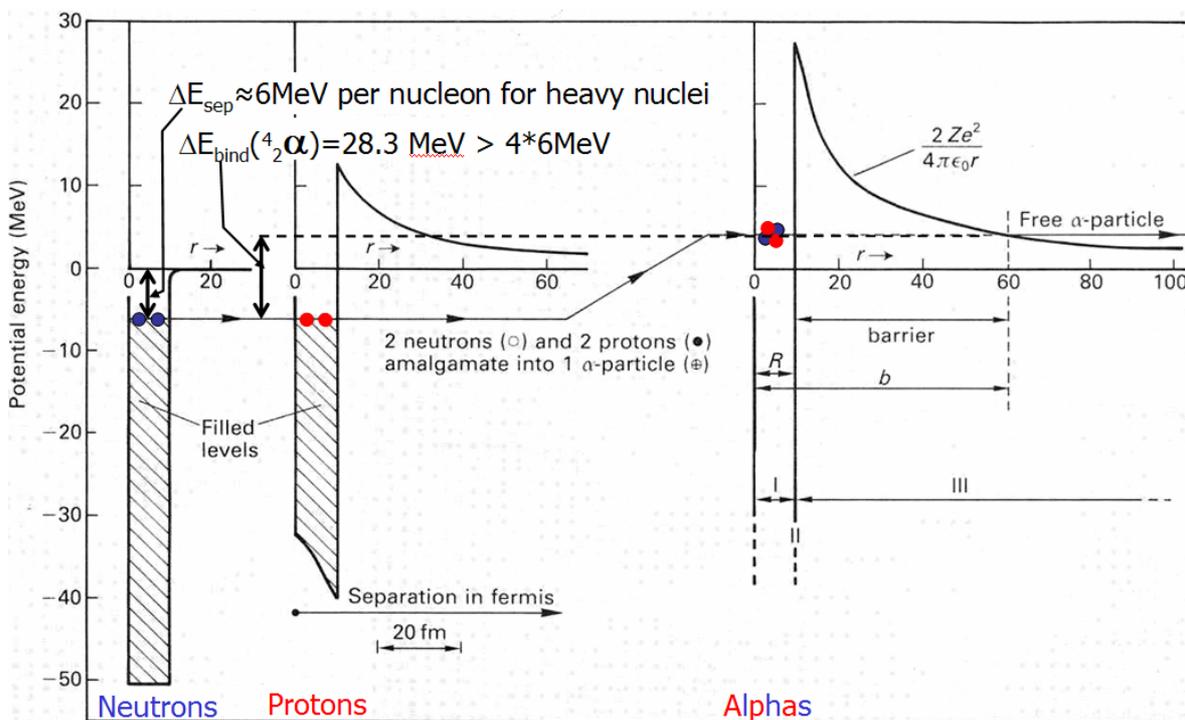


α decay (Cottingham 2nd Ed. pg. 74; Williams pg. 83; Krane pg. 251)

- α decay is due to the emission of a ${}^4_2\text{He}$ nucleus
- ${}^4_2\text{He}$ is doubly magic and very tightly bound
- α decay is energetically favourable for almost all nuclei having $A \geq 190$ and for many $A \geq 150$
- but the world is full of isotopes with $A > 151$
- and only 7 natural α -emitters observed with $A < 206$ because
 - barrier penetration has $\tau \sim \exp(-Q)$
 - energies are too low to get $\tau \ll \text{age of earth } (4 \times 10^9 \text{ years})$
- Note: Shell effects $O(1 \text{ MeV})$ make the life times of α -emitters deviate by several orders of magnitude from SEMF predictions
- The kinetic energy release, $Q(A, Z)$ in an α decay of a nucleus (A, Z) is given in terms of binding energies of the parent and daughter nuclei by:

$$Q(A, Z) = B(A-4, Z-2) - B(A, Z) + 28.3 \text{ MeV} \quad (3.1)$$

where 28.3 MeV is the experimental binding energy of a ${}^4_2\text{He}$ nucleus. Negative values of Q imply absolute stability against α decay.



Consider decay of Thorium:

- ${}^{232}_{90}\text{Th}$ emits α with $Q \approx 4 \text{ MeV}$
- $R_{\text{Th}} \approx 1.2 \times 232^{1/3} \text{ fm} = 7.36 \text{ fm}$
- α has $V(R_{\text{Th}}) = 24 \text{ MeV}$
- Classically the α particle cannot penetrate the barrier

Conclusion:

- α must tunnel out of the nucleus
- half lives should have $\exp(-Q)$ dependence (true over 24 orders)

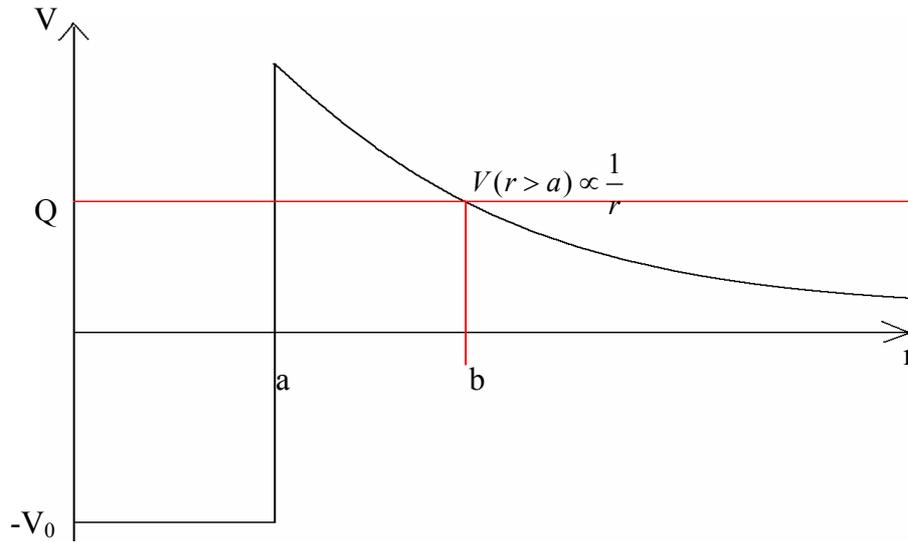


Figure above shows the relative potential energy of α particle, daughter nucleus system as a function of their separation. Inside the nuclear surface at $r = a$, the potential is represented as a square well; beyond the surface, only the Coulomb repulsion operates. The α particle must tunnel through the Coulomb barrier from a to b . The horizontal line, Q is the disintegration energy.

In the theory developed in 1928 simultaneously by Gamow and Gurney an α particle is assumed to move in a spherical region determined by the daughter nucleus. The central feature of this one-body model is that the α particle is preformed inside the parent nucleus. Actually there is not much reason to believe that α particles exist separately within heavy nuclei; nonetheless the theory works quite well, especially for even-even nuclei. This success of the theory does not prove that α particles are preformed, merely that they behave as if they were.

Consider a rectangular potential barrier that a particle of energy $Q (< V_0)$ incident from $x = -\infty$ experiences.

$$V(x) = \begin{cases} 0 & x < 0 \\ V_0 & 0 \leq x \leq t \\ 0 & x > t \end{cases}$$

The solutions in the three regions respectively are then:

$$\begin{aligned} \psi_1 &= Ae^{ik_1x} + Be^{-ik_1x} \\ \psi_2 &= Ce^{k_2x} + De^{-k_2x} \\ \psi_3 &= Fe^{ik_3x} \end{aligned} \quad \text{where:} \quad (3.2)$$

$$k_1 = k_3 = \sqrt{\frac{2mQ}{\hbar^2}}$$

$$k_2 = \sqrt{\frac{2m(V_0 - Q)}{\hbar^2}}$$

Setting the boundary condition on the equations that ψ and $d\psi/dx$ are continuous at $x = 0$ and $x = t$, find the transmission coefficient, $T = |F|^2/|A|^2$ as:

$$T = \left[1 + \frac{1}{4} \frac{V_0^2}{Q(V_0 - Q)} \sinh^2 k_2 t \right]^{-1} \quad (3.3)$$

From the figure on page 12, the Coulomb barrier has height B at $r = a$, such that:

$$B = \frac{1}{4\pi\epsilon_0} \frac{zZ'e^2}{a} \quad (3.4)$$

where the α particle has charge ze and the daughter nucleus, which provides the Coulomb repulsion, has charge $Z'e = (Z - z)e$. The height of the barrier thus varies from $(B - Q)$ above the particle energy at $r = a$ to zero at $r = b$. A representative average height is then $\frac{1}{2}(B - Q)$. Similarly the average step size is $\frac{1}{2}(b - a)$. The factor of k_2 in Eq. (3.2) then becomes $\sqrt{(2m/\hbar^2)(B - Q)/2}$. For a typical heavy nucleus ($Z = 90$, $a = 7.5 \text{ fm}$), the barrier height, $B \sim 34 \text{ MeV}$, so k_2 is approximately 1.6 fm^{-1} .

$$b = \frac{1}{4\pi\epsilon_0} \frac{zZ'e^2}{Q} \quad (3.5)$$

For typical values of $Q \sim 6 \text{ MeV}$, $b \sim 42 \text{ fm}$, $\therefore k_2 t \sim k_2 \frac{1}{2}(b - a) \gg 1$. Eq. (3.3) can then be approximated as

$$P \cong e^{-2k_2(1/2)(b-a)} \quad (3.6)$$

The actual Coulomb potential varies with r and therefore k_2 changes. Dividing the potential into small strips and multiplying together the exponents leads to:

$$\begin{aligned} dP &= \exp \left\{ -2dr \left[\frac{2m(V(r) - Q)}{\hbar^2} \right]^{1/2} \right\} \\ P &= \prod_i e^{-2k_i \Delta r} \\ &= \exp \left\{ -2 \int_a^b dr \left[\frac{2m(V(r) - Q)}{\hbar^2} \right]^{1/2} \right\} \\ &= \exp \{-2G\} \end{aligned} \quad (3.7)$$

where G is the Gamow factor which can be evaluated as:

$$V(r > a) = \frac{zZ'e^2}{4\pi\epsilon_0 r} \equiv \frac{Qb}{r} \quad (3.8)$$

$$\begin{aligned} G &= \left(\frac{2mQ}{\hbar^2} \right)^{1/2} \int_a^b dr \left[\frac{b}{r} - 1 \right]^{1/2} \\ &\left| \begin{array}{l} \text{let:} \\ r = b \cos^2 \theta; \quad dr = -2b \cos \theta \sin \theta d\theta \end{array} \right. \\ &= \left(\frac{2mQ}{\hbar^2} \right)^{1/2} \int_{\cos^{-1}(a/b)^{1/2}}^0 d\theta \left[\frac{1}{\cos^2 \theta} - 1 \right]^{1/2} (-2b \cos \theta \sin \theta d\theta) \end{aligned} \quad (3.9)$$

$$\begin{aligned}
G &= \left(\frac{2mQ}{\hbar^2} \right)^{1/2} \int_0^{\cos^{-1}\left(\frac{a}{b}\right)^{1/2}} 2b \sin^2 \theta d\theta \\
&= \left(\frac{2mQ}{\hbar^2} \right)^{1/2} b \left[\theta - (1 - \cos^2 \theta)^{1/2} \cos \theta \right]_0^{\cos^{-1}\left(\frac{a}{b}\right)^{1/2}} \\
&= \left(\frac{2m}{\hbar^2 Q} \right)^{1/2} b \left[\cos^{-1}\left(\frac{a}{b}\right)^{1/2} - \sqrt{\left(1 - \frac{a}{b}\right) \frac{a}{b}} \right]
\end{aligned} \tag{3.10}$$

for the case where $a \ll b$:

$$G \approx \left(\frac{2m}{\hbar^2 Q} \right)^{1/2} \frac{zZ'e^2 \pi}{4\pi\epsilon_0 2} \tag{3.11}$$

Thus the result of the quantum mechanical calculation for the lifetime of α decay:

$$\tau = \frac{1}{\lambda} = \frac{1}{fP} \tag{3.12}$$

where λ is the disintegration constant and f is the frequency at which the α particle strikes the barrier with probability of transmission of P . The time between collisions is $2a/v$ (v is the velocity $\sim (2Q/m)^{1/2}$).

$$\begin{aligned}
\tau &\approx \frac{2a}{v} e^{2G} \\
&\approx \frac{2a}{v} \exp \left\{ \left(\frac{2m}{\hbar^2 Q} \right)^{1/2} \frac{zZ'e^2 \pi}{4\pi\epsilon_0} \right\}
\end{aligned} \tag{3.13}$$

Experimental tests:

- Predict exponential decay rate proportional to $(Q)^{1/2}$
- Agrees approximately with data for even-even nuclei.
- The approximation that the nucleus is spherical with mean radius of $1.25A^{1/3}$ has a substantial influence on the half-lives. The nuclei with $A \geq 230$ have strongly deformed shapes and the calculated half-lives are extremely sensitive to small changes in assumed radius.
- Assumed the existence of one α particle in nucleus and have not taken into account the probability of formation.
- Assumed “semi-classical” estimate for escape frequency, f .
- Also angular momentum effects complicate the picture:
 - Additional angular momentum barrier (as in atomic physics)

$$E_l = \frac{l(l+1)(\hbar c)^2}{2mc^2 r^2} \tag{3.14}$$

- E_l is small compared to $E_{Coulomb}$
- E.g. $l = 1, R = 15 \text{ fm} \rightarrow E_l \sim 0.05 \text{ MeV}$ compared to
- $Z = 90 \rightarrow E_{Coulomb} \sim 17 \text{ MeV}$.
- The emission of α particles is suppressed for high l states.
- but still generates noticeable extra exponential suppression.
- Spin (ΔJ) and parity (ΔP) change from parent to daughter
 $\Delta J = L_\alpha \quad \Delta P = (-1)^L$

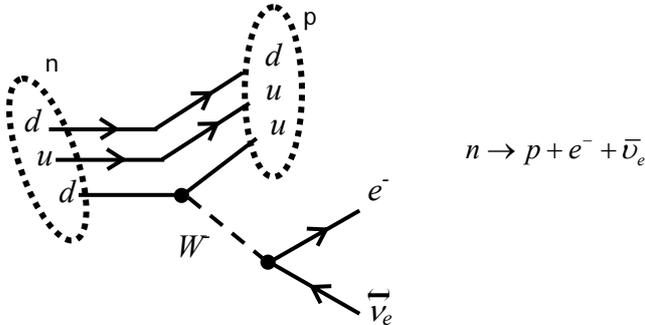
β decay

The continuous energy distribution of β decay electrons was confusing experimental result in the 1920's. Alpha particles are emitted with sharp, well defined energies, equal to the difference in mass energy between the initial and final states (less recoil corrections). Beta particles have a continuous distribution of energies, from zero up to an upper limit which is equal to the energy difference between the initial and final states. If β were like α decay, a two body process, we would expect all β particles to have a unique energy. The neutrino proposed by Fermi carries away the 'missing' energy and because it is highly penetrating radiation, it is not stopped within the calorimeter.

In all there are 3 processes that are referred to as β decay:

$$\begin{array}{ll} \beta^- : & n \rightarrow p + e^- + \bar{\nu}_e & {}^A_Z X \rightarrow {}^A_{Z+1} Y + e^- + \bar{\nu}_e \\ \beta^+ : & p \rightarrow n + e^+ + \nu_e & {}^A_Z X \rightarrow {}^A_{Z-1} Y + e^+ + \nu_e \\ EC : & p + e^- \rightarrow n + \nu_e & {}^A_Z X \rightarrow {}^A_{Z-1} Y + \nu_e \end{array}$$

A nucleus with overabundance of neutrons or protons can become more stable isotope by emitting and electron/positron or absorbing one.



Beta decay energetics:

Consider:

$$n \rightarrow p + e^- + \bar{\nu}_e$$

Define the Q value as follows:

$$\begin{aligned} Q &= (m_n - m_p - m_e - m_{\bar{\nu}})c^2 \\ &= 939.573 \text{ MeV} - 938.280 \text{ MeV} - 0.511 \text{ MeV} - m_{\bar{\nu}}c^2 \\ &= 0.782 \text{ MeV} - m_{\bar{\nu}}c^2 \end{aligned} \quad (3.15)$$

For decay of neutron at rest:

$$Q = T_p + T_e + T_{\bar{\nu}} \quad (3.16)$$

The proton recoil energy can be considered as negligible, which gives the maximum energy electrons, $Q \sim (T_e)_{\max} = 0.782 \pm 0.013 \text{ MeV}$ (according to measurements) in agreement with Eq. (3.15).

Valley of stability

Can use SEMF to find a stable value of Z for a given A . The mass of a neutral atom is given by Eq. (3.17) with nucleus of Z protons and electrons and $(A - Z)$ neutrons.

$$M(Z, A) = Z(m_p + m_e) + (A - Z)m_n - B(Z, A)/c^2 \quad (3.17)$$

From SEMF have:

$$B(Z, A) = aA - bA^{2/3} - s \frac{(A - 2Z)^2}{A} - d \frac{Z^2}{A^{1/3}} - \frac{\delta}{A^{1/2}} \quad (3.18)$$

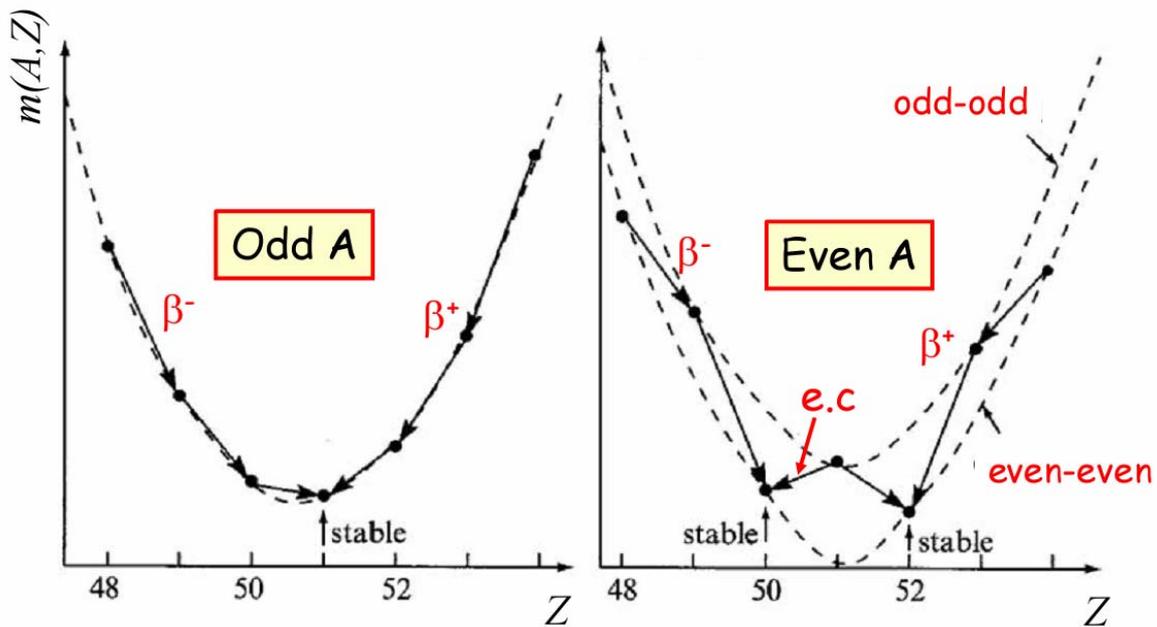
$$\begin{aligned} \Rightarrow M(Z, A)c^2 &= (Am_n c^2 - aA + bA^{2/3} + sA + \delta A^{-1/2}) \\ &\quad - (4s + (m_n - m_p - m_e)c^2)Z \\ &\quad + (4sA^{-1} + dA^{-1/3})Z^2 \end{aligned} \quad (3.19)$$

$$M(Z, A)c^2 \equiv \alpha - \beta Z + \gamma Z^2$$

Minimum found at:

$$\begin{aligned} \left(\frac{\partial M}{\partial Z} \right)_A &= 0 \\ \Rightarrow Z_{\min} &= \frac{\beta}{2\gamma} \\ &= \frac{(4s + (m_n - m_p - m_e)c^2)A}{2(4s + dA^{2/3})} \end{aligned} \quad (3.20)$$

The plots of the parabolae of Eq. (3.19) are shown below for odd and even A :



- Due to the pairing term, for even A there are 2 parabolae
- β decay is a weak interaction mediated by the W boson
- Angular momentum consideration influence whether β decay occurs readily or not

Fermi theory of β decay (Cottingham 2nd Ed. pg 166; Krane pg 277; Williams pg 294)

Assumptions:

- Not considering parity non-conservation
- Nucleons in closed shells do not take part in the decay by Pauli's exclusion principle
- Ignoring spin states of particles
- Neglecting recoil of nucleus
- Lifetime of W boson is of short range and therefore at the moment of interaction all particles can be considered at the same point in space

- Described by Fermi's Golden Rule which is from non-relativistic perturbation theory

The transition rate, λ from initial state of the system Ψ_i to final state, Ψ_f is given by Fermi's golden rule:

$$\begin{aligned}\lambda &= \frac{1}{\tau} = \frac{\Gamma}{\hbar} \\ &= \frac{2\pi}{\hbar} |H_{fi}|^2 \rho(E_f)\end{aligned}\quad (3.21)$$

where H_{fi} is the matrix element linking the initial and final quasi-stationary states and $\rho(E_f)$ is the density of specified Ψ_f states. For β^+ decay have:

$$\begin{aligned}\Psi_i &= \psi_p(\mathbf{r}_p), \\ \Psi_f &= \psi_n(\mathbf{r}_n)\psi_e(\mathbf{r}_e)\psi_\nu(\mathbf{r}_\nu)\end{aligned}\quad (3.22)$$

$$\begin{aligned}H_{fi} &= \int d\nu \Psi_f^* H \Psi_i \\ (d\nu &= d^3\mathbf{r}_n d^3\mathbf{r}_p d^3\mathbf{r}_e d^3\mathbf{r}_\nu)\end{aligned}\quad (3.23)$$

Eq. (3.23) could be of the form:

$$H_{fi} = G_W \int d^3\mathbf{r} (\psi_n(\mathbf{r})\psi_e(\mathbf{r})\psi_\nu(\mathbf{r}))^* \psi_p(\mathbf{r})\quad (3.24)$$

where G_W is the strength of the weak interaction. The constant G_W is given in terms of more fundamental constants of particle physics by:

$$G_W = G_F V_{ud}\quad (3.25)$$

where G_F is the Fermi constant and V_{ud} is an element of the 'Kobayashi-Maskawa matrix' their values are found to be experimentally as:

$$\begin{aligned}G_F &= 1.16639(2) \times 10^{-11} (\hbar c)^3 \text{ MeV}^{-2} \\ V_{ud} &= 0.9744(10)\end{aligned}$$

Taking neutrino and positron to have plane wave states, normalised in a volume V .

$$\psi_\nu(\mathbf{r}) = \frac{1}{V^{1/2}} e^{i\mathbf{k}_\nu \cdot \mathbf{r}_\nu}, \quad \psi_e(\mathbf{r}) = \frac{1}{V^{1/2}} e^{i\mathbf{k}_e \cdot \mathbf{r}_e}\quad (3.26)$$

Therefore matrix element H_{fi} of Eq. (3.24) becomes:

$$H_{fi} = \frac{G_W}{V} |M_F| = \frac{G_W}{V} \int d^3\mathbf{r} \psi_n^*(\mathbf{r}) \psi_p(\mathbf{r}) e^{-i(\mathbf{k}_\nu + \mathbf{k}_e) \cdot \mathbf{r}}\quad (3.27)$$

Energies in β decay are generally of order $\sim \text{MeV}$ and the corresponding wavevectors are $\sim \text{MeV}/\hbar c \sim 10^2 \text{ fm}^{-1}$. To a good approximation can expand the integral (3.27) such that:

$$H_{fi} = \frac{G_W}{V} \int d^3\mathbf{r} \psi_n^*(\mathbf{r}) \psi_p(\mathbf{r}) - \frac{iG_W}{V} (\mathbf{k}_\nu + \mathbf{k}_e) \cdot \mathbf{r} \int d^3\mathbf{r} \psi_n^*(\mathbf{r}) \psi_p(\mathbf{r}) + \dots\quad (3.28)$$

The density of states factor determines (to lowest order) the shape of the beta energy spectrum. To find the density of states, we need to know the number of final states accessible to the decay products. Consider the locus of points representing momenta in the range dp at p is a spherical shell of radius p and thickness dp thus having volume $4\pi p^2 dp$. If the electron is confined to a box of volume V , then the number of final electron states dn_e , corresponding to momenta in the range $[p, p + dp]$ is:

$$dn_e = \frac{4\pi p^2 dp V}{h^3} \quad (3.29)$$

And similarly for the neutrino states:

$$dn_\nu = \frac{4\pi q^2 dq V}{h^3} \quad (3.30)$$

where q is the neutrino momentum. The available spatial and momentum states are counted in 6-dimensional (x, y, z, p_x, p_y, p_z) phase space; the unit volume in phase space is h^3 . The final number of states which have simultaneously an electron and a neutrino with proper momenta is:

$$d^2n = dn_e dn_\nu = \frac{(4\pi)^2 V^2 p^2 q^2 dp dq}{h^6} \quad (3.31)$$

To find the differential decay rate, $d\lambda/dp$ need to find $d^2n/dpdE_f$. The total energy of the final state, E_f , assuming the neutron and proton do not recoil in the decay is given by the sum of the neutrino and electron energies: $E_f = E_e + E_\nu = E_e + qc$. Now p is fixed as want to find the $d\lambda/dp$ for a given p , therefore:

$$dE = dE_f = cdq \quad (3.32)$$

Eq. (3.31) then becomes:

$$\frac{d^2n}{dpdE_f} = \frac{(4\pi)^2 V^2 p^2 q^2}{h^6 c} \quad (3.33)$$

Differentiating Eq. (3.21) wrt p

$$\frac{d\lambda}{dp} = \frac{2\pi}{\hbar} |H_{fi}|^2 \frac{\rho(E_f)}{dp} = \frac{2\pi}{\hbar} |H_{fi}|^2 \frac{d^2n}{dpdE_f} \quad (3.34)$$

Now substitute Eq. (3.33) and (3.27):

$$\frac{d\lambda}{dp} = \frac{2\pi}{\hbar} \left(\frac{G_W}{V} |M_F| \right)^2 \frac{(4\pi)^2 V^2 p^2 q^2}{h^6 c} \quad (3.35)$$

$$= \frac{64\pi^4}{h^7 c^3} G_W^2 |M_F|^2 p^2 (E_f - E_e)^2 \quad (3.36)$$

Using the relativistic energy-mass relation can rewrite Eq. (3.36) in the usual form.

$$p^2 = \frac{1}{c^2}(E_e^2 - m_e^2 c^4)$$

$$\frac{dp}{dE_e} = \frac{E_e}{pc^2} \quad (3.37)$$

$$\frac{d\lambda}{dE_e} = \frac{dp}{dE_e} \frac{64\pi^4}{h^7 c^3} G_W^2 |M_F|^2 p^2 (E_f - E_e)^2$$

$$= \frac{64\pi^4}{h^7 c^6} G_W^2 |M_F|^2 S_0(E_e) \quad (3.38)$$

$$S_0(E_e) = (E_e^2 - m_e^2 c^4)^{1/2} E_e (E_f - E_e)^2 \quad (3.39)$$

As expected the arbitrary normalisation volume, V cancels out from the final result. The formula can be improved by allowing for the interaction between the electron and the Coulomb field of the daughter nucleus of charge Z_d . Since only the electron (positron) wavefunction at the nucleus is important, $S_0(E_e)$ is modified to

$$S_e(E_e) = F(Z_d, E_e) S_0(E_e)$$

$$F(Z_d, E_e) = \left| \frac{\psi_e(Z_d, 0)}{\psi_e(0, 0)} \right|^2 \quad (3.40)$$

where $\psi_e(Z_d, \mathbf{r})$ is the electron wavefunction at energy E_e in the Coulomb potential $\pm Z_d e^2 / 4\pi\epsilon_0 r$.

Selection Rules in β decay

From Fermi theory, cf Eq.(3.27).

Superaligned transitions:

$$M_F \sim \int \psi_p^* \psi_n d^3\mathbf{r} \approx 1 \quad (3.41)$$

Allowed transitions:

- Angular momentum of e, ν pair relative to nucleus, $l = 0$.

$$e^{-i(\mathbf{k}_e + \mathbf{k}_\nu) \cdot \mathbf{r}} \sim 1 \quad (3.42)$$

There are two types of allowed/superaligned transitions depending on the relative spin states of the emitted e and ν . If both have spin $1/2$ then the total spin of $e\nu$ system is $S_{e\nu} = 0$ or 1 .

Fermi transitions $S_{e\nu} = 0$:

$$n \uparrow \rightarrow p \uparrow + \underbrace{e^- \uparrow + \bar{\nu}_e \downarrow}_{S_{e\nu}=0}$$

$$J_x = J_y; \quad \Delta J = 0$$

Gamow-Teller transitions $S_{ev} = I$

$$\begin{aligned}
 n \uparrow &\rightarrow p \uparrow + \underbrace{e^- \uparrow + \bar{\nu}_e \uparrow}_{S_{ev}=\pm 1} \\
 J_x &= J_y; \Delta J = \pm 1 \\
 n \uparrow &\rightarrow p \uparrow + \underbrace{e^- \uparrow + \bar{\nu}_e \uparrow}_{S_{ev}=\pm 1} \\
 J_x &= J_y \pm 1; \Delta J = 0 \quad (0 \rightarrow 0 \text{ forbidden})
 \end{aligned}$$

- Total number of spin states of $e\nu$ is 4 (3 G-T and 1 Fermi)
- No change in angular momentum of the $e\nu$ pair relative to the nucleus, $l = 0 \rightarrow$ parity is unchanged.

Forbidden transitions:

- Angular momentum of $e\nu$ pair relative to the nucleus $l > 0$

$$\begin{aligned}
 e^{-i(\mathbf{k}_e + \mathbf{k}_\nu) \cdot \mathbf{r}} &= 1 - i(\mathbf{k}_e + \mathbf{k}_\nu) \cdot \mathbf{r} + [(\mathbf{k}_e + \mathbf{k}_\nu) \cdot \mathbf{r}]^2 - \dots \\
 l & \qquad \qquad \qquad 0 \qquad 1 \qquad 2 \qquad \dots \\
 P = (-1)^l & \qquad \text{even} \quad \text{odd} \quad \text{even} \\
 & \qquad \text{Allowed; 1}^{\text{st}} \text{ forbidden; 2}^{\text{nd}} \text{ forbidden}
 \end{aligned}$$

Transition probabilities for larger l are small \rightarrow forbidden transitions. Forbidden transitions are only competitive if an allowed transition cannot occur (selection rules). The lowest permitted order of ‘forbiddenness’ will dominate.

Electron capture

The initial state is:

$$\Psi_i = \psi_p(\mathbf{r}_p)\psi_e(\mathbf{r}_e) \tag{3.43}$$

To a good approximation the electron is most likely to be in the K-shell, since the K-shell wavefunction has the greatest overlap with the nucleus. This wavefunction can be considered as hydrogen-like:

$$\psi_e(\mathbf{r}_e) = \frac{1}{\sqrt{\pi}} \left(\frac{Z}{a_0} \right)^{3/2} e^{-\frac{Zr}{a_0}} \tag{3.44}$$

where a_0 is the Bohr radius ($= m_e e^2 / (4\pi\epsilon_0 \hbar^2)$). Final state is:

$$\Psi_f = \psi_n(\mathbf{r}_n)\psi_\nu(\mathbf{r}_\nu) \tag{3.45}$$

The neutrino wavefunction is the same as in Eq. (3.26) that takes form of a plane wave state. The appropriate equation now for the matrix element is:

$$H_{fi} = G_W \int d^3\mathbf{r} \psi_n^*(\mathbf{r}_n)\psi_\nu^*(\mathbf{r}_\nu)\psi_p(\mathbf{r}_p)\psi_e(\mathbf{r}_e) \tag{3.46}$$

which for an allowed transition reduces to:

$$\begin{aligned}
 H_{fi} &= \frac{G_W}{\sqrt{V}} \psi_e(0) \int d^3\mathbf{r} \psi_n^*(\mathbf{r}_n) \psi_p(\mathbf{r}_p) \\
 &= \frac{G_W}{\sqrt{V}\pi} \left(\frac{Z}{a_0} \right)^{3/2} M_F
 \end{aligned} \tag{3.47}$$

where we have treated the electron and neutrino wavefunctions as constant over the nuclear volume. Once again neglecting nuclear recoil, the emitted neutrino has energy E_ν :

$$E_\nu \approx E_f + m_e c^2 \tag{3.48}$$

To find the density of neutrino states, ρ_ν , use result from Eq. (3.30) and the following property from statistical mechanics:

$$\begin{aligned}
 \frac{dn_\nu}{dq} dq &= \frac{dn_\nu}{dE_\nu} dE_\nu \\
 \therefore \rho_\nu(E_\nu) &= \frac{dn_\nu}{dq} \frac{dq}{dE_\nu}
 \end{aligned} \tag{3.49}$$

$$\begin{aligned}
 \rho_\nu(E_\nu) &= \frac{4\pi V}{h^3} \left(\frac{E_\nu}{c} \right)^2 \left(\frac{1}{c} \right) \\
 &= \frac{4\pi V}{h^3 c^3} E_\nu^2
 \end{aligned} \tag{3.50}$$

The decay rate, λ_{EC} for electron capture using Fermi's golden rule in Eq. (3.21) and inserting Eq. (3.47) and (3.50) becomes:

$$\begin{aligned}
 \lambda_{EC} &= \frac{2\pi}{\hbar} |H_{fi}|^2 \rho_\nu(E_\nu) \\
 &= \frac{2\pi}{\hbar} \left[\frac{G_W}{\sqrt{V}\pi} \left(\frac{Z}{a_0} \right)^{3/2} \right]^2 |M_F|^2 \frac{4\pi V}{h^3 c^3} E_\nu^2 \\
 &= \frac{G_W^2 |M_F|^2 E_\nu^2}{\pi^2 \hbar^4 c^3} \left(\frac{Z m_e e^2}{4\pi \epsilon_0 \hbar^2} \right)^3
 \end{aligned} \tag{3.51}$$

Since we have neglected electron spin, only one K electron is included in the calculation. At high Z , the Z^3 factor and increasing Coulomb barrier for positron emission will make electron capture the dominant process.

Inverse β decay (Krane pg. 296)

$$\begin{aligned}
 \beta^- \text{ decay:} & \quad n \rightarrow p e^- \bar{\nu}_e \\
 \text{inv. } \beta^- \text{ decay:} & \quad \bar{\nu}_e p \rightarrow n e^+
 \end{aligned}$$

Define the reaction cross section for the inverse β decay as:

$$\sigma = \frac{\text{probability/target atom for reaction to occur}}{\text{incident flux of } \bar{\nu}_e} \tag{3.52}$$

The reaction probability can be again calculated from Fermi's golden rule. The matrix element, H_{fi} is again as in Eq. (3.27), $G_w|M_F|/V$. The neutron recoil is neglected and the density of final states comes only from the positron. Eq. (3.52) becomes:

$$\sigma = \frac{\frac{2\pi}{\hbar} \frac{G_w^2}{V^2} |M_F|^2 \left(\frac{4\pi V p_{e^+}^2}{h^3} \right) dp_{e^+}}{c/V} \quad (3.53)$$

From Eq. (3.37): $dp/dE = E/c^2 p$ using which gives the result:

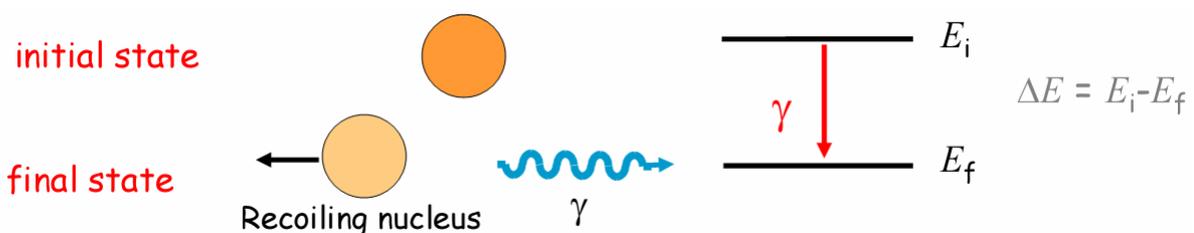
$$\sigma = 16\pi^3 G_w^2 |M_F|^2 \frac{p_{e^+} E_{e^+}}{h^4 c^3} \quad (3.54)$$

γ decay

- Very similar to atomic physics transitions
- When do they occur?
 - Nuclei have excited states similar to atoms (need a proper shell model to understand).
 - when there is not enough energy to emit a strongly interacting particle (nucleon), often after other nuclear decays
 - $E_{\gamma}^{\text{atomic}} < 100 \text{ keV}$; $E_{\gamma}^{\text{nuclear}} < O(1 \text{ MeV})$
 - Heavy nuclear rotational states can have $E_{\gamma}^{\text{nuclear, rot}} < O(10 \text{ keV})$
 - EM interaction less strong than the strong (nuclear) interaction
 - Low energy excited states $E < 6 \text{ MeV}$ above ground state can't usually decay by nuclear interaction $\rightarrow \gamma$ -decays
- γ decays important in cascade decays following α and β decays.
- Practical consequences
 - Fission. Significant energy released in γ decays
 - Radiotherapy: γ from Co60 decays
 - Medical imaging eg Tc
 - Studying γ emission and its competing process, internal conversion, allows to deduce the spins and parities of the excited states of the nucleus

Mössbauer effect (Eisberg 2nd ed. pg 584; Krane pg 328/361)

A source nucleus in an excited state makes a transition to its ground state, emitting a γ ray. The γ ray could be subsequently absorbed by another nucleus (previously in ground state) which is excited. Small changes to the source energy, absorber energy or γ ray energy will destroy the resonance. The problem has to do with the recoil of the nuclei upon emission and absorption of the γ ray.



The total linear momentum of the decay is zero before emission. The nuclear recoil momentum is p_n after the emission and equal to momentum carried away by the γ ray, p_γ .

If the nuclear mass, M is high and the recoil velocity is low, may use the classical expression for kinetic energy of the nuclear recoil, K as:

$$K = \frac{p_n^2}{2M} \quad (3.55)$$

For the γ ray, $p_\gamma = E_\gamma/c$, therefore from conservation of angular momentum have

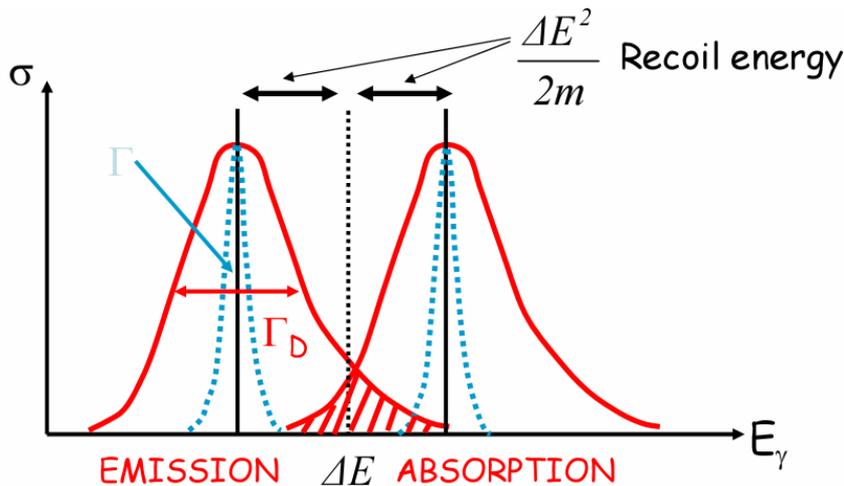
$$K = \frac{1}{2M} \left(\frac{E_\gamma}{c} \right)^2 \quad (3.56)$$

For γ emission have:

$$\begin{aligned} \Delta E &= E_\gamma + K \\ E_\gamma &= \Delta E - \frac{E_\gamma^2}{2Mc^2} \approx \Delta E - \frac{\Delta E^2}{2Mc^2} \end{aligned} \quad (3.57)$$

For absorption:

$$E_\gamma \approx \Delta E + \frac{\Delta E^2}{2Mc^2} \quad (3.58)$$



- E_γ varies due to natural width of energy levels.
- If initial state is in thermal motion, need to consider Doppler shift.
- Absorption of γ rays can only occur for energies in the overlap region shown above
- If the atoms are put into a lattice, the strong bonding allows the recoil of a nucleus from emission/absorption to be distributed between many atoms. Effectively this corresponds to greater value of mass in Eq. (3.56). If mass is very large, the energies of emission/absorption will be approximately the same.
- The above will be true in a lattice provided the recoil energy is smaller than the phonon energy – such that there is insufficient energy to excite the lattice to a higher vibrational energy state.

Lecture 4 & 5

Shortcomings of the SEMF:

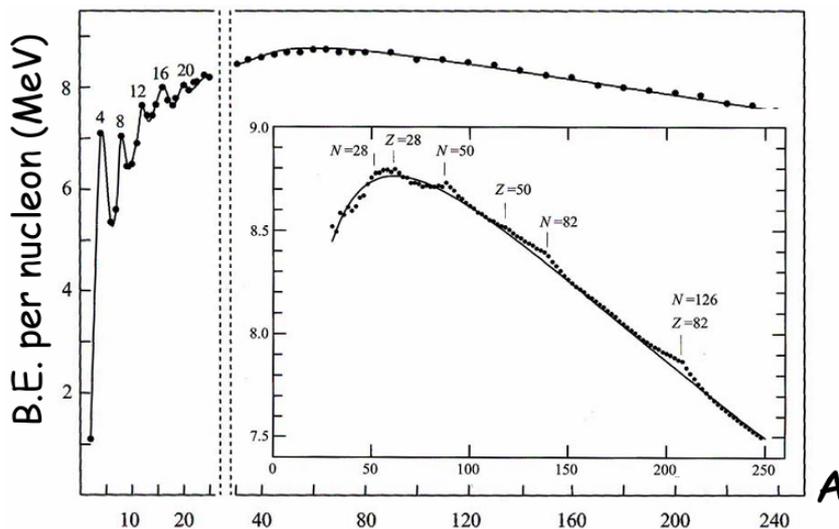
- spin & parity of nuclei do not fit into a drop model
- magnetic moments of nuclei are incompatible with drops
- actual value of nuclear density is unpredicted
- values of the SEMF coefficients except Coulomb and Asymmetry are completely empirical
- Magic numbers unaccounted for

Magic numbers

Nuclei with values of

Z and/or N = 2, 8, 20, 28, 50, 82, 126... **MAGIC NUMBERS**

are very stable and show significant departures from the average nucleus behaviour. They represent the effects of the filled major shells analogous to the atomic shell model. The binding energy per nucleon is large for magic numbers.



- Doubly magic nuclei extremely stable (where Z and N are magic)
- Energies in alpha and beta decay high when daughter nucleus is magic
- Nuclear radius is not changed much with Z , N at magic numbers
- 1st excited states for magic numbers higher than neighbours
- Spontaneous neutron emitters have magic number $+1$
- Terrestrial nuclear abundances for Z or N magic are greater than those for non-magic elements.
- Elements with Z/N magic have many more isotopes than with Z/N non-magic
- Odd A nuclei have small quadrupole moment when magic, etc, etc

Shell model (Williams pg. 131; Cottingham 2nd ed. pg. 56; Krane pg. 116)

The atomic theory based on the shell model has provided remarkable clarification of the complicated details of the atomic structure. Nuclear physicists therefore attempted to use a similar theory to solve the problem of nuclear structure. A major difference is that in the atomic case the potential is supplied by the Coulomb field of the nucleus; the orbits are

established by external agent. In nucleus there is no such external agent; the nucleons move in a potential that they themselves create. This is overcome by the fundamental assumption of the shell model: the motion of a single nucleon is governed by a potential caused by all of the other nucleons. Treating each nucleon individually allows for the nucleons to be occupying the energy levels of a series of sub-shells.

Another difficulty is that electrons move in orbits free of collisions with other electrons. Nucleons on the other hand have relatively large diameter compared to the size of the nucleus. However, the existence of spatial orbits depends on the Pauli principle. For example in a heavy nucleus a collision between nucleons in a state near the bottom of the potential well will result in a transfer of energy to one another. But if all the energy levels are filled up to the level of the valence nucleon, there is no way for one of the nucleons to gain energy except to move to the valence level as other low lying levels are filled. The transition thus requires more energy than the nucleons are likely to transfer in a collision. Thus collisions cannot occur and nucleons orbit as if they were transparent to one another.

Assumptions:

- Each nucleon moves in an averaged potential
 - neutrons see average of all nucleon-nucleon nuclear interactions
 - protons see same as neutrons plus proton-proton electric repulsion
 - the two potentials for n and p are wells of some form (nucleons are bound)
- Each nucleon moves in single particle orbit corresponding to its state in the potential
 - → We are making a **single particle shell model**
 - Q: why does this make sense if **nucleus full of nucleons** and typical **mean free paths** of nuclear scattering projectiles = $O(2\text{fm})$
 - A: Because nucleons are fermions and stack up. They can not lose energy in collisions since there is **no state to drop into** after collision
- Use Schrödinger Equation to compute energies (i.e. non-relativistic), justified by simple infinite square well energy estimates
- Aim to get the correct magic numbers (shell closures)

The Saxon-Woods potential can be used to approximate the potential as experienced by an individual nucleon.

$$V(r) = \frac{-V_0}{1 + e^{(r-R)/a}} \quad (4.1)$$

The parameters R and a give, respectively, the mean radius and skin thickness. Their values are chosen with accordance to measurements such that: $R = 1.25A^{1/3} \text{ fm}$ and $a = 0.524 \text{ fm}$. V_0 is adjusted to give the proper separation energies and is of order 50 MeV . This potential is then substituted into the Schroedinger Equation and the energy levels found. However the central potential alone cannot reproduce the magic numbers, need to account for the spin-orbit interactions.

Spin-orbit potential

In atomic physics, the spin-orbit interaction causes the observed fine structure of spectral lines, comes about because of the electromagnetic interaction of the electron's magnetic moment with the magnetic field generated by its motion about the nucleus. This concept is adopted in nuclear physics. From scattering experiments there is strong evidence of nucleon-nucleon spin-orbit force. The potential is altered such that:

$$V(r) \rightarrow V(r) + W(r)\mathbf{L}\cdot\mathbf{S} \quad (4.2)$$

where L and S are orbital and spin angular momentum operators and $W(r)$ is a function of radial position.

where:
$$W(r) = -|V_{LS}| \left(\frac{\hbar}{m_{\pi} c} \right)^2 \frac{1}{r} \frac{dV}{dr} \tag{4.3}$$

and
$$V_{LS} = V_{LS}(E_{nucleon})$$

$V(r)$ is the Saxon-Woods potential of Eq. (4.1). As with atomic physics, the total angular momentum operator is defined below:

$$\mathbf{J} = \mathbf{L} + \mathbf{S} \tag{4.4}$$

The eigenvalue of $\mathbf{L} \cdot \mathbf{S}$ for a stationary state with good quantum numbers l, j and $s (=1/2)$ is

$$\frac{\hbar^2}{2} [j(j+1) - l(l+1) - s(s+1)] \tag{4.5}$$

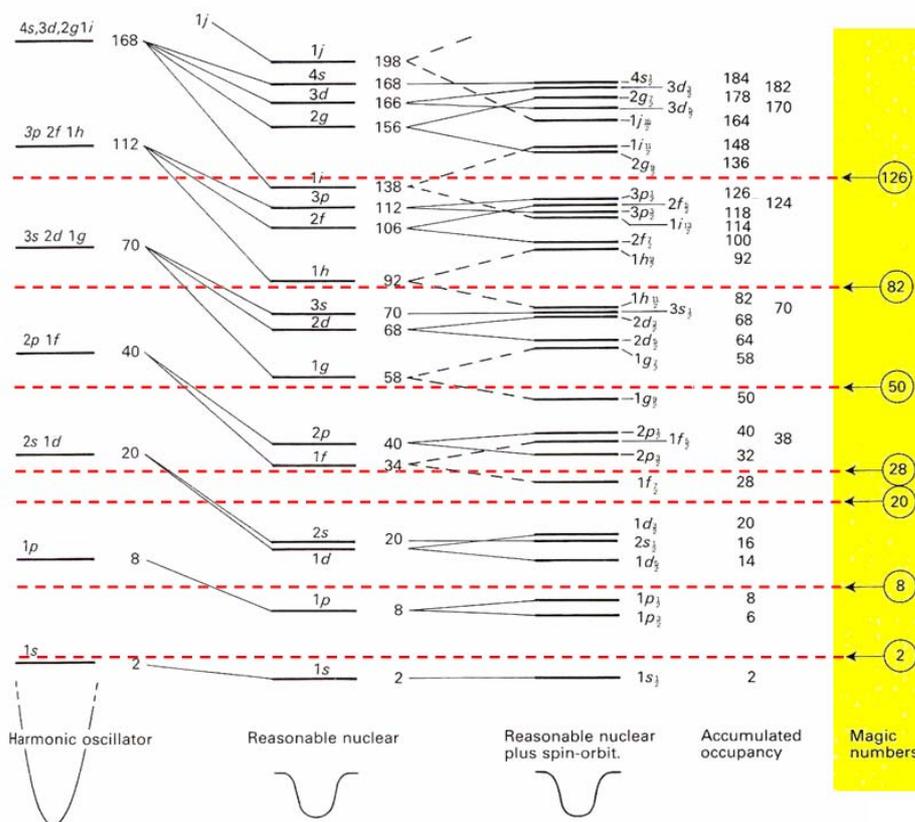
Therefore, the potential for $j = l + 1/2$ is:

$$V(r) + \frac{1}{2} \hbar^2 W(r) \tag{4.6}$$

and for $j = l - 1/2$:

$$V(r) - \frac{1}{2} \hbar^2 W(r) \tag{4.7}$$

Since $W(r)$ is negative (to obtain agreement with observation), the $j = l + 1/2$ level will be below that with $j = l - 1/2$. The resultant energy structure is shown below.



Summary of successful predictions of the Shell model:

- Origin of magic numbers
- Spins and parities of ground states
- Trend in magnetic moments
- Some excited states near closed shells, small excitations in odd A nuclei
- In general not good far from closed shells and non-spherically symmetric potentials
- Collective properties of nuclei can be incorporated into the nuclear shell model by replacing the spherically symmetric potential by a deformed one. This improves description for
 - Even A excited states
 - Electric quadrupole and magnetic dipole moments

Shortcomings of the Shell model:

- Cannot predict spin or parity for odd-odd nuclei – do not have a very good model for the LS interactions
- A consequence of the above is that the shell model predictions for nuclear magnetic moments are very imprecise
- Cannot predict accurate energy levels because:
 - we only use one “well” to suit all nuclei
 - we ignore the fact that n and p should have separate wells of different shape
- As a consequence of the above we cannot reliably predict much (configuration, excitation energy) about excited states other than an educated guess of the configuration of the lowest excitation

Lecture 6 & 7

Partial decay widths

Particles can often decay with more than one decay mode, each with its own transition rate, λ as given by Fermi’s golden rule (cf lecture 3):

$$\lambda = \frac{2\pi}{\hbar} |H_{fi}|^2 \rho(E_f) \quad (5.1)$$

The **total decay rate** is given by:

$$\lambda = \sum_i \lambda_i$$

This determines the **average lifetime**:

$$\tau = \frac{1}{\lambda}$$

The **total width** of particle state is:

$$\Gamma = \hbar\lambda = \hbar \sum_i \lambda_i$$

Define the **partial widths**:

$$\Gamma_i = \hbar\lambda_i \Rightarrow \Gamma = \sum_i \Gamma_i$$

The proportion of decays to a particular

decay mode is called the **branching fraction**:

$$B_i = \frac{\Gamma_i}{\Gamma} \quad \sum_i B_i = 1$$

Cross section

The strength of a particular reaction between two particles is specified by the interaction cross-section. A cross-section is an effective target area presented to the incoming particle

for it to cause the reaction. Often use units of ‘barns’ defined as $1 \text{ barn } (b) = 10^{-28} \text{ m}^2$. The cross-section, σ is defined as the reaction rate per target particle, λ per incident flux, Φ that is the number of beam particles passing through unit area per second.

$$\lambda = \Phi \sigma \quad (5.2)$$

Consider a beam of N particles/sec of area A incident upon a target width dx and n nuclei/unit volume.

- Number of target particles, N_T in area A , $= nAdx$
- Effective area for absorption $= \sigma nAdx$
- Rate at which particles are removed from the beam $= -dN = (N/A)\sigma nAdx$

$$\therefore -\frac{dN}{N} = \sigma ndx \quad (5.3)$$

$$\sigma = \frac{N^\circ \text{ scattered particles/sec}}{Nndx} \quad (5.4)$$

For a thick target ($\sigma nL \gg 1$) the Eq. (5.3) can be integrated such that:

$$-\int_{N_i}^{N_f} \frac{dN}{N} = \int_0^L \sigma ndx \quad (5.5)$$

$$\Rightarrow N_f = N_i e^{-\sigma nL}$$

The mean free path between interaction is then $1/n\sigma$. For a thin target ($\sigma nL \ll 1$):

$$N_f = N_i (1 - \sigma nL) \quad (5.6)$$

The total reaction rate per unit surface area in the thin target will be $= \sigma nL\Phi$. To rewrite the cross-section in terms of incident flux use the relation, $\Phi = N/A$. Eq. (5.4) can then be rewritten as:

$$\sigma = \frac{N^\circ \text{ scattered particles/sec}}{(\Phi A)(N_T/(Ax)) dx} \quad (5.7)$$

$$\sigma = \frac{N^\circ \text{ scattered particles/sec}}{\text{Flux} \times \text{Number target particles}} \quad (5.8)$$

$$\Rightarrow \lambda = \Phi \sigma$$

There are usually several reaction channels that incident particle can sustain:

- 1) elastic scattering by the target
- 2) inelastic scattering or
- 3) absorption by the target

Given that a reaction occurs, each reaction channel, ‘ i ’ has definite probability p_i , the partial cross-section σ_i is defined as $\sigma_i = p_i \sigma_{tot}$. The incident particle in general can scatter at any angle with respect to the incident direction. The elastic differential cross-section, σ_e for a particle scattering into solid angle $d\Omega$ is defined as:

$$\int \frac{d\sigma_e}{d\Omega} d\Omega = \sigma_e \quad (5.9)$$

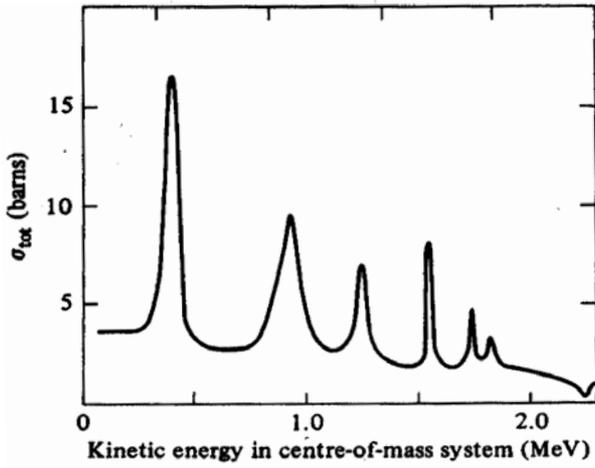


Figure to the left shows the total cross-section for neutron to interact with $^{16}_8\text{O}$ as a function of kinetic energy (in the centre of mass frame). The principal features are narrow resonance peaks, superposed on a slowly varying background. These peaks are due to the formation of excited states of ^{17}O from the neutron and ^{16}O at resonance energies. When the energy of the incident neutron is such that the total energy of the system matches to within width Γ one of the excited states of ^{17}O , the neutron is readily accepted. The shape of peaks at resonance is described by the Breit-Wigner formula whose derivation is outlined below.

The wavefunction of the unstable state is denoted by $|0\rangle$ and it decays to states $|1\rangle, |2\rangle, \dots, |m\rangle$ where $m > 0$. The states can be chosen to be orthonormal. The state of the system, which is $|0\rangle$ at $t = 0$ can be expressed as a superposition of states $|m\rangle$.

$$\Psi(t) = \sum_{m=0}^{\infty} a_m(t) e^{-iE_m t/\hbar} |m\rangle \quad (5.10)$$

where

$$E_m = H_{mm} = \langle m|H|m\rangle \quad (5.11)$$

Assume:

- The Hamiltonian, H is known that perturbs the resonance and lets it decay
- Dealing with a spinless resonance

Now insert Eq. (5.10) into the Schrödinger equation:

$$i\hbar \frac{d\Psi}{dt} = H\Psi \quad (5.12)$$

which gives,

$$\sum_m (i\hbar \dot{a}_m + E_m a_m) e^{-iE_m t/\hbar} |m\rangle = \sum_m a_m H e^{-iE_m t/\hbar} |m\rangle \quad (5.13)$$

Multiplying by $\langle n|$, the orthogonality picks out the time dependence of a_n .

$$\begin{aligned} \langle n| \sum_m (i\hbar \dot{a}_m + E_m a_m) e^{-iE_m t/\hbar} |m\rangle &= \langle n| \sum_m a_m H e^{-iE_m t/\hbar} |m\rangle \\ (i\hbar \dot{a}_n + E_n a_n) e^{-iE_n t/\hbar} &= \sum_{m \neq n} a_m \langle n|H|m\rangle e^{-iE_m t/\hbar} + E_n a_n e^{-iE_n t/\hbar} \\ i\hbar \dot{a}_n &= \sum_{m \neq n} a_m H_{nm} e^{-i(E_m - E_n)t/\hbar} \end{aligned} \quad (5.14)$$

The initial conditions at $t = 0$ are $a_0(0) = 1$, $a_m(0) = 0$ for $m \geq a$. To first order the quantities H_{nm} can be regarded as small when $n \neq m$, then for $n \geq 1$ have

$$i\hbar\dot{a}_n \approx a_0 H_{n0} e^{-i(E_0 - E_n)t/\hbar} \quad (5.15)$$

The state $|0\rangle$ is unstable, so make the following ansatz

$$\begin{aligned} a_0(t) &= e^{-\Gamma t/2\hbar} \\ |a_0(t)|^2 &= e^{-t/\tau} \end{aligned} \quad (5.16)$$

where Γ is the energetic width (uncertainty) of our initial resonance. The probability of finding the system in $|0\rangle$ state decays exponentially with time. Eq. (5.15) can now be integrated to give

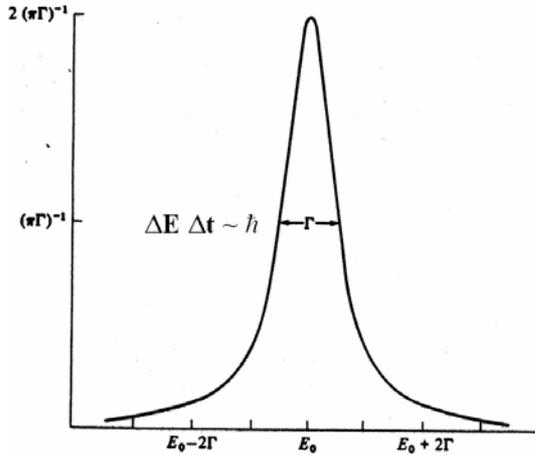
$$\begin{aligned} i\hbar a_n(t) &= H_{n0} \int_0^t dt' \left\{ e^{-i[(E_0 - E_n) - i\Gamma/2]t'/\hbar} \right\} \\ &= \frac{\hbar}{i} H_{n0} \left[\frac{e^{-i[(E_0 - E_n) - i\Gamma/2]t/\hbar} - 1}{(E_n - E_0) + i\Gamma/2} \right] \end{aligned} \quad (5.17)$$

For times $t \gg \tau$, the probability of decay to the state $|n\rangle$ is:

$$|a_n(t \gg \tau)|^2 = \frac{2\pi}{\Gamma} |H_{n0}|^2 P(E_n - E_0) \quad (5.18)$$

where

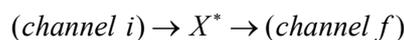
$$P(E_n - E_0) = \frac{\Gamma}{2\pi} \frac{1}{(E_n - E_0)^2 + \Gamma^2/4} \quad (5.19)$$



The function in Eq. (5.19) is normalised and is regarded as the probability distribution in energy of the state $|0\rangle$ it is shown in figure below. The energy of the final state E_n is not identically equal to E_0 and is not absolutely determined. This is a consequence that the state $|0\rangle$ does not have definite energy. The instability implies that it has a small spread of energy Γ about the mean energy E_0 .

Breit-Wigner cross-sections

Consider a channel i which consists of two particles, for example a neutron interacting with nucleus, I at an energy close to an energy at which the two can combine to form the unstable excited state X^* . The X^* then decays into one of its decay channels f .



Use the result obtained in Eq. (5.18):

$$|a_n(t \gg \hbar/\Gamma)|^2 = \frac{|H_{n0}|^2}{(E_n - E_0)^2 + \Gamma^2/4} \quad (5.20)$$

The probability that X^* is formed from i is:

$$\begin{aligned} P(i \rightarrow X^*) &= |a_x(t \gg \hbar/\Gamma_{x,tot})|^2 \\ &= \frac{|H_{xi}|^2}{(E_x - E_i)^2 + \Gamma_{x,tot}^2/4} \end{aligned} \quad (5.21)$$

Use Fermi's golden rule to substitute for $|H_{xi}|^2$. Since H is hermitian; $|H_{xi}|^2 = |H_{ix}|^2$.

$$\frac{\Gamma_{x \rightarrow i}}{\hbar} = \frac{2\pi}{\hbar} |H_{ix}|^2 \rho_i(E_x) \Leftrightarrow |H_{ix}|^2 = \frac{\Gamma_{x \rightarrow i}}{2\pi\rho_i(E_x)} = |H_{xi}|^2 \quad (5.22)$$

$$P(i \rightarrow X^*) = \frac{1}{2\pi\rho_i(E_x)} \frac{\Gamma_{x \rightarrow i}}{(E_x - E_i)^2 + \Gamma_{x,tot}^2/4} \quad (5.23)$$

The rate of the reaction from channel i to channel f , $\lambda(i \rightarrow x \rightarrow f)$ is:

$$\lambda_{i \rightarrow f} = P(i \rightarrow X^*) \lambda_{x \rightarrow f} \quad (5.24)$$

$$\lambda_{i \rightarrow f} = \frac{1}{2\pi\hbar\rho_i(E_x)} \frac{\Gamma_{x \rightarrow f} \Gamma_{x \rightarrow i}}{(E_x - E_i)^2 + \Gamma_{x,tot}^2/4} \quad (5.25)$$

Using the reaction rate can find the cross-section using the relation in Eq. (5.2). For a free particle in initial state i . Where wavefunction is normalised to 1 particle per volume V and v is the velocity of the particle.

$$\begin{aligned} \Psi &= \frac{1}{\sqrt{V}} e^{ik \cdot r} \\ \Phi &= \frac{1}{V} v \end{aligned} \quad (5.26)$$

The density of states in the i th channel is given by:

$$\rho_i(E)dE = \frac{V}{(2\pi)^3} 4\pi k^2 \frac{dk}{dE} dE \quad (5.27)$$

$$\frac{dE}{dk} = \frac{\hbar^2 k}{m} = \hbar v \quad (5.28)$$

where m is the reduced mass of the particles. And have made use of the relation $p = mv = \hbar v$. Eq. (5.27) is then:

$$\rho_i(E)dE = \frac{V}{(2\pi)^3} \frac{4\pi k^2}{\hbar v} dE \quad (5.29)$$

$$\begin{aligned}
\sigma_{i \rightarrow f} &= \frac{\lambda_{i \rightarrow f}}{\Phi} \\
&= \frac{1}{\Phi} \frac{1}{\rho_i(E_x)} \frac{1}{2\pi\hbar} \frac{\Gamma_{x \rightarrow f} \Gamma_{x \rightarrow i}}{(E_x - E_i)^2 + \Gamma_{x, tot}^2 / 4} \\
&= \frac{V}{v} \frac{(2\pi)^3}{V} \frac{\hbar v}{4\pi k^2} \frac{1}{2\pi\hbar} \frac{\Gamma_{x \rightarrow f} \Gamma_{x \rightarrow i}}{(E_x - E_i)^2 + \Gamma_{x, tot}^2 / 4}
\end{aligned}$$

$$\begin{aligned}
\sigma_{i \rightarrow f} &= \frac{\pi}{k_i^2} \bigg|_{E_i=E_x} \frac{\Gamma_{x \rightarrow f} \Gamma_{x \rightarrow i}}{(E_x - E_i)^2 + \Gamma_{x, tot}^2 / 4} \\
&= \frac{\pi \hbar^2}{2mE_x} \frac{\Gamma_{x \rightarrow f} \Gamma_{x \rightarrow i}}{(E_x - E_i)^2 + \Gamma_{x, tot}^2 / 4}
\end{aligned} \tag{5.30}$$

This result is the *Breit-Wigner formula* for the special case when the incoming and the compound nucleus have zero spin. For small Γ , the cross-section peaks sharply at $E_x = E_i$. The phenomenon is known as resonance scattering and is common in nuclear physics; experimental resonance peaks can often be well fitted by an expression of this form.

Rutherford scattering (Sakurai pg. 386)

What do we want to describe:

- Scattering between two spin less nuclei due to Coulomb interactions
- Non relativistic scattering energies ($E_{cm} \ll$ smallest of the two nuclear masses)
- The correct approach to Rutherford scattering is by use of quantum mechanics, although classically would get the same result it is only accidental
- Use the Born approximation
 - plane waves going into and coming out of scattering
 - no disturbance of wave functions during the scattering
 - acceleration happens at one instance in time
 - nuclei stay what they were (no break-up or emission of other particles etc.)
- First nucleus, denoted by $i1$ and $f1$
 - is light compared to the first one to guarantee no recoil
 - has charge Z_1
- Second nucleus denoted by $i2$ and $f2$
 - is heavy \rightarrow no recoil
 - is stationary in the lab frame before collision
 - has charge Z_2

The scattering potential, $V(r)$ is the Coulomb potential which has been modified to allow for solutions to the integration

$$\begin{aligned}
V(r) &= \frac{Z_1 Z_2 \alpha \hbar c}{r} e^{-r/a} \\
\alpha &= \frac{e^2}{4\pi\epsilon_0 \hbar c}
\end{aligned} \tag{5.31}$$

The wavefunctions of the incoming and outgoing first nucleus are:

$$\psi_i = \frac{1}{\sqrt{V}} e^{ik_{i1} \cdot r}; \quad \psi_f = \frac{1}{\sqrt{V}} e^{ik_{f1} \cdot r} \tag{5.32}$$

The matrix element of the Coulomb interaction Hamiltonian as in Eq. (3.23) is:

$$H_{fi} = \frac{1}{V} \int_{all\ space} e^{i\mathbf{k}_{i1}\cdot\mathbf{r}} V(r) e^{-i\mathbf{k}_{f1}\cdot\mathbf{r}} d\mathbf{v} \quad (5.33)$$

Use $\mathbf{K} = \mathbf{k}_{i1} - \mathbf{k}_{f1}$ such that the above is rewritten as

$$H_{fi} = \frac{1}{V} \int_{all\ space} V(r) e^{i\mathbf{K}\cdot\mathbf{r}} d\mathbf{v} \quad (5.34)$$

Choose z-axis to be parallel to \mathbf{K}

$$\begin{aligned} H_{fi} &= \frac{Z_1 Z_2 \alpha \hbar c}{V} \int_0^\infty \int_0^\pi \int_{-1}^1 e^{iKr \cos \theta} \frac{e^{-(r/a)}}{r} r^2 d(\cos \theta) d\phi dr \\ &= 2\pi \frac{Z_1 Z_2 \alpha \hbar c}{V} \int_0^\infty \frac{e^{iKr} - e^{-iKr}}{iKr} \frac{e^{-(r/a)}}{r} r^2 dr \\ &= 2\pi \frac{Z_1 Z_2 \alpha \hbar c}{iKV} \int_0^\infty e^{\left(\frac{1}{a} - iK\right)r} - e^{\left(\frac{1}{a} + iK\right)r} dr \\ &= 2\pi \frac{Z_1 Z_2 \alpha \hbar c}{iKV} \left[\frac{1}{-\frac{1}{a} + iK} - \frac{1}{-\frac{1}{a} - iK} \right] \\ &= 2\pi \frac{Z_1 Z_2 \alpha \hbar c}{iKV} \frac{2iK}{\left(\frac{1}{a^2} + K^2\right)} \end{aligned} \quad (5.36)$$

Now taking the limit as $a \rightarrow \infty$ find the matrix element as

$$H_{fi} = 4\pi \frac{Z_1 Z_2 \alpha \hbar c}{K^2 V} \quad (5.37)$$

Using Fermi's golden rule have the expression for cross-section:

$$d\sigma = \frac{2\pi}{\hbar} |H_{fi}|^2 \frac{V}{v} d\rho_f(E_f) \quad (5.38)$$

where v is the velocity of the incident particles. Now need to find the density of states in solid angle $d\Omega$. The number of states between p and $p + dp$ in solid angle $d\Omega$ is:

$$\begin{aligned} d^2 N &= \frac{V}{h^3} p^2 dp d\Omega \\ d\rho &= \frac{V}{h^3} p^2 \frac{dp}{dE} d\Omega \end{aligned} \quad (5.39)$$

where p and E are centre of mass momentum of one of the final state particles and total energy in the final state respectively. In the non-relativistic limit, Eq. (3.37) becomes $dp/dE = 1/v$. Now substitute the results Eq. (5.37) and (5.39) into (5.38).

$$d\sigma = \frac{2\pi}{\hbar} \left| 4\pi \frac{Z_1 Z_2 \alpha \hbar c}{K^2 V} \right|^2 \frac{V}{v} \frac{V}{h^3} p^2 \frac{d\Omega}{v} \quad (5.40)$$

$$\left(\frac{d\sigma}{d\Omega} \right) = \frac{4p^2}{v^2 \hbar^4} \left| \frac{Z_1 Z_2 \alpha \hbar c}{K^2} \right|^2 \quad (5.41)$$

By energy conservation, $|\mathbf{k}_{i1}| = |\mathbf{k}_{f1}|$:

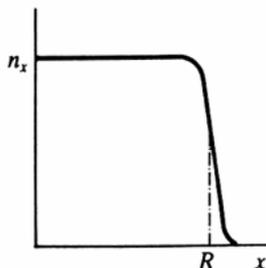
$$K = |\mathbf{k}_{i1} - \mathbf{k}_{f1}| = 2k_1 \sin\left(\frac{\theta}{2}\right) \quad (5.42)$$

$$\left(\frac{d\sigma}{d\Omega} \right) = \frac{(2m)^2}{\hbar^4} \frac{(Z_1 Z_2 \alpha \hbar c)^2}{16k_1^4 \sin^4(\theta/2)} \quad (5.43)$$

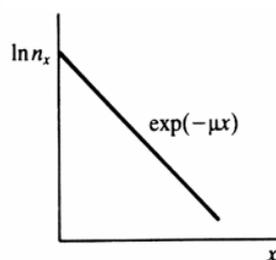
Lecture 9 & 10

Interaction with matter

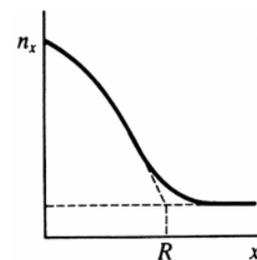
- Measure properties of nuclei through **decay** products
- Measure energy, momentum, mass & charge of particles with
 - $M \in [0 (\gamma) ; \text{few } 100 \text{ GeV (fission fragment)}]$
 - $E_{\text{kin}} \in [\text{keV (Radioactivity)} ; \text{few GeV (accelerator experiments)}]$
 - $Q/e \in [0 (\gamma, n) ; O(100) \text{ (fission fragments)}]$
- Need to translate microscopic particle properties into quantitatively measurable macroscopic signals
- Do this by interactions between particles and matter
- Which interactions would be useful?
 - Weak? \rightarrow Too weak at low (nuclear) interaction energies
 - Strong? \rightarrow Some times useful but often noisy (strong fluctuations, few interactions per distance)
 - **EM?** \rightarrow Underlies most nuclear and particle physics detectors (L9&10)
- Energies released $\leq E_{\text{kin}}$ (particle) often too small for direct detection \rightarrow need **amplification** of signals (see detector section L11)
- Particle Ranges
 - a) If smooth energy loss via many steps (i.e. ionisation from light ions) \rightarrow sharply defined range, useful for rough energy measurement
 - b) If a few or a single event can stop the particle (i.e. photo-effect) \rightarrow exponential decay of particle beam intensity, \rightarrow decay constant can have useful energy dependence \rightarrow No range but mean free path defined
 - c) Sometimes several types of processes happen (i.e. high energy electrons) \rightarrow mixed curves, extrapolated maximum range



a)



b)



c)

Particles we are interested in

- photons
 - exponential attenuation at low E , often get absorbed in single events
 - detect secondary electrons and ions liberated in absorption process.
- charged particles
 - sharper range (continuously lose energy via ionisation)
 - leave tracks of ionisation in matter → measure momentum in B field
 - sometimes radiate photons → can be used to identify particle type
- neutrons
 - electrically neutral → no first-order em-interaction → difficult to detect
 - react only via strong force (at nuclear energies)
 - long exponential range (lots of nuclear scattering events followed by absorption or decay)
 - need specific nuclear reactions to convert them into photons and/or charged particles when captured by a target nucleus
 - if stopped, measure decay products, $e^- + p + \nu$

Charged particles in matter

- If particle or medium emit photons, coherent with incoming particle → radiation process
 - Bremsstrahlung, Synchrotron-radiation: emitted by **particle**
 - Cherenkov-radiation: emitted by **medium**
- If no coherent radiation → non-radiating process
 - Ionisation, scattering of nuclei or atoms
- Note: Scintillation is a secondary process in which the light is emitted after ionisation or atomic excitation. It is **NOT** a radiation process

Charged particle can collide with:

Atomic electrons (“free”)

- large energy loss $\Delta E \approx q^2/2m_e$ (small m_e , q = momentum transfer)
- small scattering angle

Nuclei

- small energy loss ($\Delta E \approx q^2/2m_{\text{nucleus}}$)
- large scattering angle

Unresolved atoms (predominant at low energies)

- medium energy loss $\Delta E < q^2/2m_e^{\text{eff}}$ because: $m_e^{\text{eff}}(\text{bound}) > m_e(\text{free})$
- medium scattering angle
- atoms get excited and will later emit photons (scintillation)

	■ Rutherford Scattering	■ Bethe-Bloch situation
commonalities	▪ any charged particle X (original used α 's) scatters of nucleus	▪ any charged particle X scatters of electron (in matter)
	▪ Charge(X)= Ze	▪ Charge(X)= Ze
	▪ Charge(nucleus)= $Z'e$	▪ Charge(electron)= $1e$
	▪ $M_{\text{nuc}} \gg M_X \rightarrow$ no nuclear-recoil	▪ $M_X \gg m_e \rightarrow$ no X -recoil (not true for $X=e^-$)
	▪ first order perturbation theory ($Z^*Z^*a_{em} \ll I$)	▪ first order perturbation theory ($Z^*I^*a_{em} \ll I$)
	▪ point ↔ point scattering → no form-factors	▪ point ↔ point scattering → no form-factors
differences	▪ spin-0 scatters of spin-0	▪ spin-0 scatters of spin-1/2
	▪ non-relativistic	▪ could be relativistic
	▪ nucleus assumed unbound	▪ electron is often bound

Bethe-Bloch formula (Williams pg. 234; Das pg. 114)

Charged particle ionises or excites the atoms as it transverses through a material. The mean rate of energy loss is given by the Bethe-Bloch formula the derivation of which is outlined below. The Mott scattering formula for differential cross-section for an electron of momentum p and velocity V by a heavy nucleus of charge $z|e|$ is given by:

$$\frac{d\sigma}{d\Omega} = \underbrace{\left(\frac{z\alpha\hbar c}{2pV}\right)^2 \frac{1}{\sin^4(\theta/2)}}_{\text{Rutherford term}} \left[\underbrace{1 - \left(\frac{V}{c} \sin^2(\theta/2)\right)^2}_{\text{Mott term}} \right] \quad (6.1)$$

Now want to change variables from $d\Omega$ to q^2 , where q is the momentum transferred to the scattered electron.

$$\begin{aligned} \mathbf{q} &= \mathbf{p} - \mathbf{p}' \\ q^2 &= p^2 + p'^2 - 2pp' \cos \theta \end{aligned} \quad (6.2)$$

For elastic scattering:

$$q^2 = 4p^2 \sin^2\left(\frac{\theta}{2}\right) \quad (6.3)$$

$$\Rightarrow \frac{dq^2}{d\theta} = 2p^2 \sin \theta \quad (6.4)$$

If have no ϕ dependence:

$$\begin{aligned} d\Omega &= 2\pi \sin \theta d\theta \\ \frac{d\theta}{d\Omega} &= \frac{1}{2\pi \sin \theta} \end{aligned} \quad (6.5)$$

Putting it together gives:

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{d\theta}{d\Omega} \frac{dq^2}{d\theta} \frac{d\sigma}{dq^2} \\ &= \frac{1}{2\pi \sin \theta} 2p^2 \sin \theta \frac{d\sigma}{dq^2} \\ \frac{d\sigma}{dq^2} &= \frac{\pi}{p^2} \frac{d\sigma}{d\Omega} \end{aligned} \quad (6.6)$$

Substitute Eq. (6.1) and (6.3) into (6.6)

$$\frac{d\sigma}{dq^2} = \frac{\pi}{p^2} \left(\frac{z\alpha\hbar c}{2pV}\right)^2 \left(\frac{2p}{q}\right)^4 \left[1 - \left(\frac{Vq}{2cp}\right)^2 \right] \quad (6.7)$$

$$\frac{d\sigma}{dq^2} = 4\pi \left(\frac{z\alpha\hbar c}{q^2V}\right)^2 \left[1 - \left(\frac{Vq}{2cp}\right)^2 \right] \quad (6.8)$$

Now change frame from one where nucleus is at rest and electron is moving to the frame in which the heavy nucleus of mass M is moving with velocity V towards collision with stationary electron. The momentum p in Eq. (6.8) is still the momentum of the electron, $p = \gamma m_e V$. Here q^2 is the same in both frames. This is true non-relativistically and if it is correctly defined also relativistically. For an electron initially at rest, the energy transfer, ν to the electron is $2m_e \nu = q^2$.

If this nucleus loses energy $-dT$ in a distance dx in a material containing n atoms of atomic number Z per unit volume, then

$$-dT = \underbrace{nZdx}_{\substack{\text{n. of collisions with} \\ e^- \text{ in length } dx \text{ per} \\ \text{unit cross-sectional} \\ \text{area}}} \underbrace{\int_{v_{\min}}^{v_{\max}} v \frac{d\sigma}{dv} dv}_{\substack{\text{cross-section weighted} \\ \text{average energy lost per} \\ \text{collision}}} \quad (6.9)$$

$$-dT = nZdx \int_{v_{\min}}^{v_{\max}} v 2m_e 4\pi \left[\frac{z\alpha\hbar c}{(2m_e v)V} \right]^2 \left[1 - (2m_e v) \left(\frac{V}{2cm_e V} \right)^2 \left(1 - \frac{V^2}{c^2} \right) \right] dv \quad (6.10)$$

$$-dT = nZdx \int_{v_{\min}}^{v_{\max}} v 2\pi \frac{(z\alpha\hbar c)^2}{m_e V^2} \frac{1}{v^2} \left[1 - \frac{v}{2m_e c^2} \left(1 - \frac{V^2}{c^2} \right) \right] dv$$

$$-\frac{dT}{dx} = 2\pi nZ \frac{(z\alpha\hbar c)^2}{m_e V^2} \left[\ln \left(\frac{v_{\max}}{v_{\min}} \right) - \frac{v_{\max} - v_{\min}}{2m_e c^2} \left(1 - \frac{V^2}{c^2} \right) \right] \quad (6.11)$$

where v_{\max} and v_{\min} are the maximum and minimum values of energy transfer. For a heavy incident particle:

$$v_{\max} = \frac{2m_e V^2}{\left(1 - \frac{V^2}{c^2} \right)} \quad (6.12)$$

Although v_{\max} is much greater than v_{\min} , do not know the latter. At large v the bound electrons can be assumed to be free, but at low v that assumption is no longer valid. It becomes possible to have an energy, ν and momentum, q transfer that do not satisfy the constraint ($2m_e \nu = q^2$) imposed if those quantities impact on a free electron. The momentum and energy transfer can now lead not only to ionisation but also excitation of an atom. Thus the integral is not correct and has to be done over variables ν and q for differential cross-sections that depend on the detailed atomic structure. This gives an important contribution to the whole energy loss and expect it to differ by some factor from the result that has been obtained thus far. The Bethe-Bloch formula parameterises these problems by defining a quantity I , the mean excitation potential, for all Z atomic electrons – this is an element dependant parameter which has to be determined from experimental data. The correct result is thus (note a correction factor of 2 has been added, but will not be justified here):

$$-\frac{dT}{dx} = 4\pi nZ \frac{(z\alpha\hbar c)^2}{m_e V^2} \left[\ln \left(\frac{2m_e V^2}{I(1-V^2/c^2)} \right) - \frac{V^2}{c^2} \right] \quad (6.13)$$

δ = density correction: dielectric properties of medium shield growing range of Lorentz-compacted E field that would reach more atoms laterally. Without this the stopping power would logarithmically diverge at large projectile velocities. Only relevant at very large $\beta\gamma$.

$$\frac{1}{\rho} \frac{dT}{dx} = \frac{4\pi N_A Z (z\alpha\hbar c)^2}{A m_e \beta^2 c^2} \left[\ln \left(\frac{2m_e \beta^2 c^2 \gamma^2}{I} \right) - \beta^2 - \frac{\delta}{2} \right] \quad (6.14)$$

$$\left\langle -\frac{1}{\rho} \frac{dT}{dx} \right\rangle, \text{ called the stopping power} \quad (6.15)$$

where ρ is mass density of material, A is the atomic weight, N_A , Avogadro's number.

Limitations:

- totally wrong for very low V (\ln goes negative \rightarrow particle gains energy)
- correct but not useful for very large V (particle starts radiating)

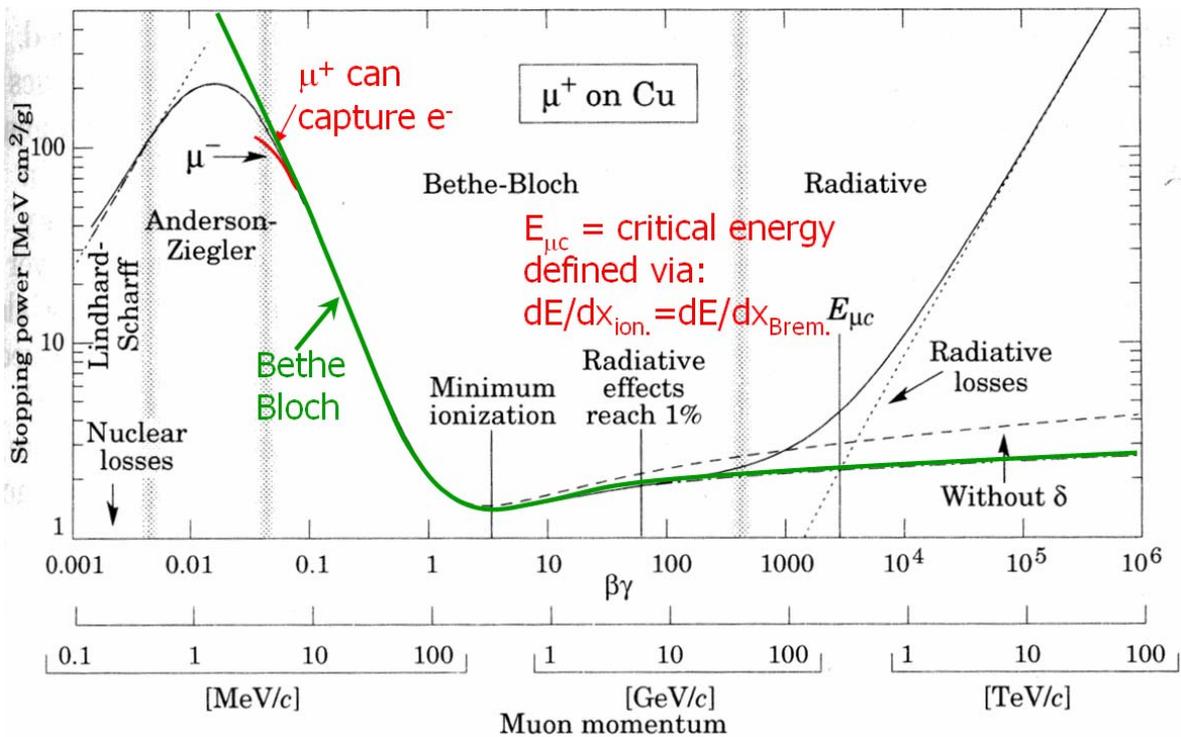
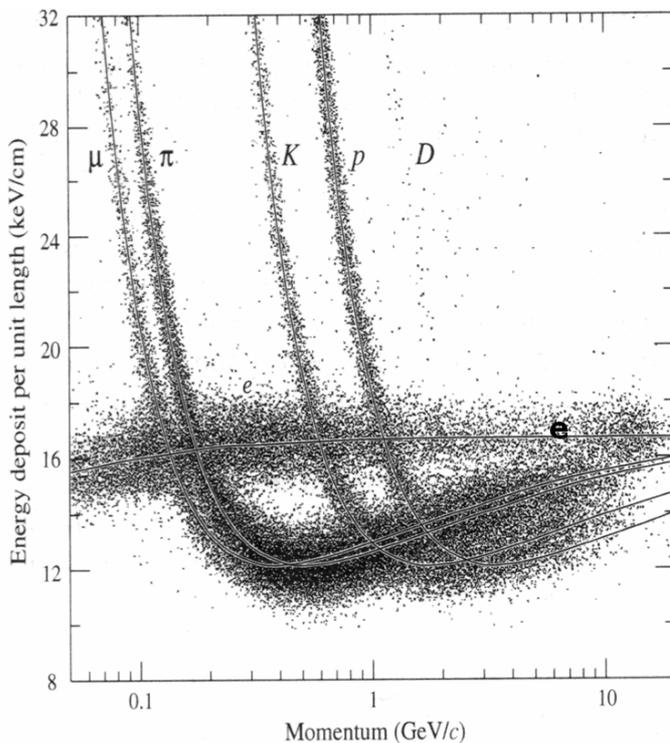


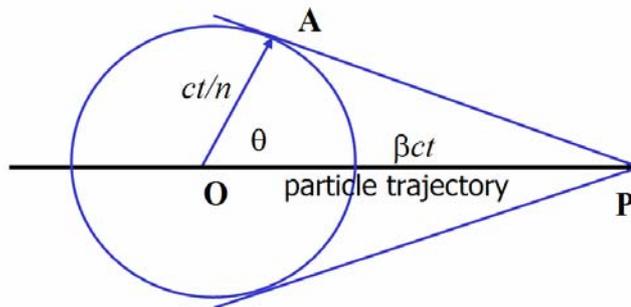
Figure to show the ionization and Bethe-Bloch formula variation with $\beta\gamma$

- Broad minimum at $\beta\gamma \approx 3.0$ (3.5) for $Z = 100$ (7)
- At minimum, stopping power is nearly independent of particle type and material
- Stopping Power at minimum varies from 1.1 to $1.8 \text{ MeV g}^{-1} \text{ cm}^2$
- Particle is called minimum ionising (MIP) when at minimum



- variation in dT/dx is useful for particle identification
- variation is most pronounced in low energy falling part of curve
- if determine momentum, p and dT/dx you can determine the particle mass and thus its type

Cherenkov radiation (Das pg. 148; Shaw 2nd ed. pg. 64)



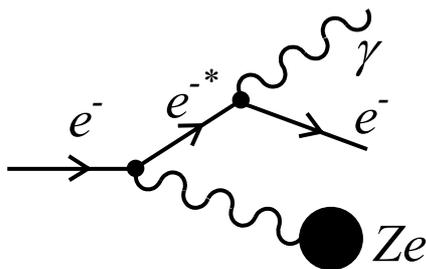
- Source of E-field (charge) passing through medium at a $v > v_{phase}$ (light in medium) creates conical shock wave. Similar to a sonic boom.
- Not possible in vacuum since $v < c$. Possible in a medium when $\beta n > 1$.
 - The Cherenkov threshold at $\beta = 1/n$ can be used to measure β and thus to identify particles if measure the momentum as well.
- Huygens secondary wavelet construction gives angle of shockwave as

$$\cos \theta_c = \frac{ct}{\beta ct} = \frac{1}{\beta n} \quad (6.16)$$

this can be used to measure particle direction and β .

- In time that the particle goes from O to P, light goes from O to A.
- Cherenkov radiation first used in discovery of antiproton (1954).
- Now often used in large water-filled neutrino detectors and for other particle physics detectors
- Total energy emitted as Cherenkov Radiation is $\sim 0.1\%$ of other dT/dx .

Bremsstrahlung (Williams pg. 247; Shaw 2nd ed. pg. 55)



- Due to acceleration of incident charged particle in nuclear Coulomb field
- Radiative correction to Rutherford Scattering.
- Continuum part of x-ray emission spectra.
- The intensity is proportional to inverse mass squared
- Lorentz transformation of dipole radiation from incident particle frame to laboratory frame gives “narrow” (not sharp) cone of blue-shifted radiation centred around cone angle of $\theta = 1/\gamma$.
- Radiation spectrum falls as $1/E$ (E = photon energy) because particles loose many low-E photons and few high-E photons. i.e. it is rare to hit nuclei with small impact parameter because most of matter is vacuum
- Photon energy limits:
 - low energy (large impact parameter) limited through shielding of nuclear charge by atomic electrons.
 - high energy limited by maximum incident particle energy.

Energy loss due to bremsstrahlung is:

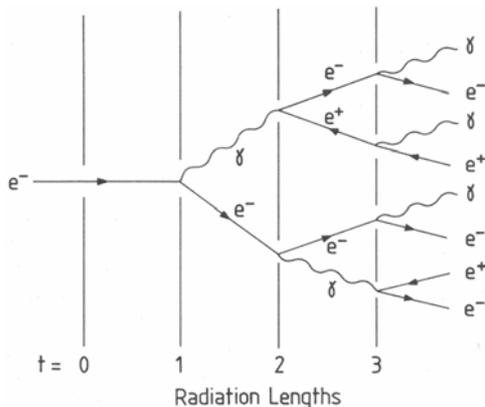
$$-\left(\frac{dT}{dx}\right)_{brem} \propto T \quad (6.17)$$

$$T(x) = T_0 e^{-x/L_B}$$

$$\frac{1}{L_B} = \left[\frac{4nZ^2\alpha^3(\hbar c)^2}{(m_e c^2)^2} \ln\left(\frac{183}{Z^{1/3}}\right) \right] \quad (6.18)$$

where L_0 is the radiation length, n is the number per unit volume of the nuclei, atomic number Z .

- $dT/dx|_{brem} \sim T$ dominates over $dT/dx|_{ionise} \sim \ln(T)$ at high T .
- E_{crit} is the energy at which bremsstrahlung losses exceed ionization losses
- For electrons Bremsstrahlung dominates in nearly all materials above few 10 MeV. $E_{crit}(e^-) \approx 600 \text{ MeV}/Z$
- Radiation Length L_0 of a medium is defined as:
 - distance over which electron energy reduced to $1/e$ via many small bremsstrahlung-losses
 - $L_B \sim Z^2$ approximately as it is the charge that particles interact with
- Bremsstrahlung photon can produce e^+e^- -pair and start an em-shower
- The development of em-showers, whether started by primary e or γ is measured in L_B .



- Simple shower model assumes:
 - $e \approx 2$
 - $E_0 \gg E_{crit}$
 - only single Brem- γ or pair production per L_B
- The model predicts:
 - after 1 L_B , $1/2$ of E_0 lost by primary via Bremsstrahlung
 - after next L_B both primary and photon lose $1/2 E$ again
 - until E of generation drops below E_{crit}
 - At this stage remaining Energy lost via ionisation (for e^{\pm}) or Compton scattering, photo-effect (for γ) etc.
 - Abrupt end of shower happens at $E_0 = E_{crit}$, when $L_B^{max} = \ln(E_0/E_{crit})/\ln 2$
 - Indeed observe logarithmic dependence of shower depth on E_0
 - The main features of this model are observed experimentally.
 - The physical sizes of calorimeters need increase only slowly with the maximum energies of particles to be detected.
 - The energy resolution of calorimeter depends on statistical fluctuations which are neglected in this model, for em calorimeter they are typically: $\Delta E/E = 0.05E^{-1/2} \text{ GeV}$

Synchrotron Radiation (Das pg. 163; Shaw 2nd ed. pg. 46)

- Appears mainly in circular accelerators (mainly to electrons) and limits maximum energy achievable.
- Similar to Bremsstrahlung
- Replace microscopic force from E-field in Bremsstrahlung with macroscopic force from $\mathbf{v} \times \mathbf{B}$ to keep electron on circular orbit
- Electrons radiate only to the outside of circle because they are accelerated inward
- Angle of maximum intensity of synchrotron radiation with tangent of ring $\theta = 1/\gamma$
- Synchrotron radiation = very bright source of broad range of photon energies up to few 10 keV used in many areas of science
- Many astrophysical objects emit synchrotron radiation from relativistic electrons in strong magnetic fields

For relativistic energies, the equation of motion for a charged particle mass, m and charge e in a magnetic field B becomes:

$$\frac{d\mathbf{p}}{dt} = e \frac{\mathbf{v} \times \mathbf{B}}{c} \quad (6.19)$$

Equating to the centripetal force with $|\mathbf{v}| \sim c$:

$$m\gamma \frac{d\mathbf{v}}{dt} = m\gamma \mathbf{v} \times \boldsymbol{\omega} = e \frac{\mathbf{v} \times \mathbf{B}}{c} \quad (6.20)$$

For magnetic field and axis of circular motion perpendicular to the direction of motion:

$$\omega = \frac{eB}{m\gamma c} \quad (6.21)$$

or

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi} \left(\frac{eB}{mc} \right) \left(1 - \frac{v^2}{c^2} \right)^{1/2} \quad (6.22)$$

For this relation to hold during an acceleration cycle either the alternating frequency has to decrease or the magnetic field to increase, or both must happen as $v \rightarrow c$.

- Synchrocyclotrons – magnetic field kept constant, frequency varied.
- Synchrotrons – magnetic field changed, irrespective of whether the frequency is changed. In electron synchrotrons, the frequency is held constant and magnetic field varied, whereas in proton synchrotrons, both frequency and magnetic field are altered.

With relativistic effects taken into account, Eq. (6.22) can be used to obtain the parameters for accelerating particles to any desired energy. Rewrite Eq. (6.22) in terms of momentum and radius of final orbit. For $v \rightarrow c$, frequency of the motion can be expressed as:

$$f = \frac{1}{2\pi} \frac{v}{R} \approx \frac{1}{2\pi} \frac{c}{R} \quad (6.23)$$

Eq. (6.22) is then:

$$\frac{1}{2\pi} \frac{c}{R} = \frac{1}{2\pi} \left(\frac{eB}{mc} \right) \frac{1}{\gamma} \quad (6.24)$$

let $p = m\gamma v \approx m\gamma c$, the equation for radius, R is:

$$R = \frac{pc}{eB} \quad (6.25)$$

The above equation is often written in mixed units of accelerator science as:

$$R \approx \frac{P}{0.3B} \quad (6.26)$$

where R is in meters, p in GeV/c and e has magnitude corresponding to charge on an electron.

Photons in matter (Das pg. 124; Cottingham 2nd ed. pg. 207; Shaw 2nd ed. pg. 56; Krane pg. 198)

- Rayleigh scattering
 - Coherent, elastic scattering on the entire atom (the blue sky)
 - $\gamma + \text{atom} \rightarrow \gamma + \text{atom}$
 - dominant at $\lambda_\gamma > \text{size of atoms}$
- Compton scattering
 - Incoherent scattering on electron from atom
 - $\gamma + e^-_{\text{bound}} \rightarrow \gamma + e^-_{\text{free}}$
 - possible at all $E_\gamma > \min(E_{\text{bind}})$
 - to properly call it Compton requires $E_\gamma \gg E_{\text{bind}}(e^-)$ to approximate free e^-
 - Consider an incident photon striking an electron, initially at rest, mass m . The photon's energy after collision is E'_γ travelling in the exact opposite direction to incidence and electron gains momentum, p . From energy (6.27) and momentum (6.28) conservation have (in natural units):

$$E_\gamma + m_0 = E'_\gamma + \underbrace{\gamma m_0}_{E_e} \quad (6.27)$$

$$E_\gamma = p - E'_\gamma \quad (6.28)$$

Use invariant:

$$E_e^2 = p^2 + m^2 \quad (6.29)$$

Substitute into (6.29) Eq. (6.27) for E_e and Eq. (6.28) for p :

$$\begin{aligned} (E_\gamma - E'_\gamma + m_0)^2 &= (E_\gamma + E'_\gamma)^2 + m^2 \\ \Rightarrow E_\gamma - E'_\gamma &= \frac{2E_\gamma E'_\gamma}{m} \\ E'_\gamma &= \frac{mE_\gamma}{m + 2E_\gamma} \\ \Delta E_\gamma^{\text{max}} &= \frac{2E_\gamma^2}{m + 2E_\gamma} \end{aligned} \quad (6.30)$$

- Photoelectric effect
 - absorption of photon and ejection of single atomic electron
 - $\gamma + \text{atom} \rightarrow \gamma + e^-_{\text{free}} + \text{ion}$
 - possible for $E_\gamma < \max(E_{\text{bind}}) + dE(E_{\text{atomic-recoil}} \text{ line width})$ (just above K-edge)

- Free electrons cannot absorb a photon and recoil. Energy and momentum cannot both be conserved in such a process; a heavy atom is necessary to absorb the momentum at little cost to the energy.
- This can be shown as follows. Consider an electron at rest colliding with a photon, the rest mass of the electron cannot change (otherwise would not be an electron). By energy and momentum conservation have the following:

$$m_e c^2 + hf = \gamma m_e c^2 \quad (6.31)$$

$$\frac{hf}{c} = \gamma m_e v \quad (6.32)$$

$$\begin{aligned} \therefore m_e c^2 + \gamma m_e v c &= \gamma m_e c^2 \\ \Rightarrow 1 - \frac{v}{c} = \frac{1}{\gamma} &= \left[\left(1 - \frac{v}{c} \right) \left(1 + \frac{v}{c} \right) \right]^{1/2} \\ \Rightarrow v &= 0 \end{aligned}$$

therefore must have scattering rather than absorption by a free electron.

- The kinetic energy of the electron is equal to the photon energy less the binding energy of the electron:

$$T_e = E_\gamma - B_e \quad (6.33)$$

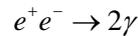
- The absorption of a photon for photoelectric effect is most significant for low energy photons (~ 100 keV), it increases rapidly with atomic number Z of absorber atoms (as Z^4) and decreases rapidly with increasing photon energy (of order E_γ^{-3}). There are discontinuous jumps in the probability for photoelectric absorption at energies corresponding to binding energies of particular electronic shells.
- Pair production
 - absorption of γ in atom and emission of $e^+ e^-$ pair
 - The energy balance is:

$$E_\gamma = T_+ + m_e c^2 + T_- + m_e c^2 \quad (6.34)$$

- Two varieties:
 - a) dominant: $\gamma + \text{nucleus} \rightarrow e^+ + e^- + \text{nucleus}_{recoil}$
 - b) weak: $\gamma + Z \cdot \text{atomic } e^- \rightarrow e^+ + e^- + Z \cdot \text{atomic } e^-_{recoil}$
- Both variants need: $E_\gamma > 2m_e c^2 + E_{recoil}$
 - bigger M_{recoil} gives lower threshold because $E_{recoil} = P_{recoil}^2 / 2M_{recoil}$
 - type a) has lower threshold than type b) because $M_{nucl} \gg M_e^{eff}$
- Nucleus/atom has to recoil to conserve momentum \rightarrow coupling to nucleus/atom needed \rightarrow strongly charge-dependent cross-section (i.e. growing with Z)
 - type a) has approximately Z times larger coupling \rightarrow dominant
- Like photoelectric absorption require the presence of a nearby atom for momentum conservation.
- The threshold for this process is $2m_e c^2$, or 1.022 MeV and in general pair production is important for high energy photons.
- Pair production becomes dominant only for energies above 5 MeV.
- At very high energies (>100 MeV) the $e^+ e^-$ pair cross-section saturates and can be characterised by a constant mean free path for conversion (or by constant absorption coefficient. The mean free path is given by, where L_B is the radiation length:

$$L_{pair} \approx \frac{9}{7} L_B \quad (6.35)$$

- After pair production, the positrons transverse the medium much like electrons and deposit their energies through ionisation or bremsstrahlung. Once a positron loses most of its kinetic energy, it captures an electron to form a hydrogen-like system, referred to as positronium, where proton is replaced by positron. However this system is unstable and decays (annihilate) with lifetimes of $\sim 10^{-10}$ sec to form two photons:

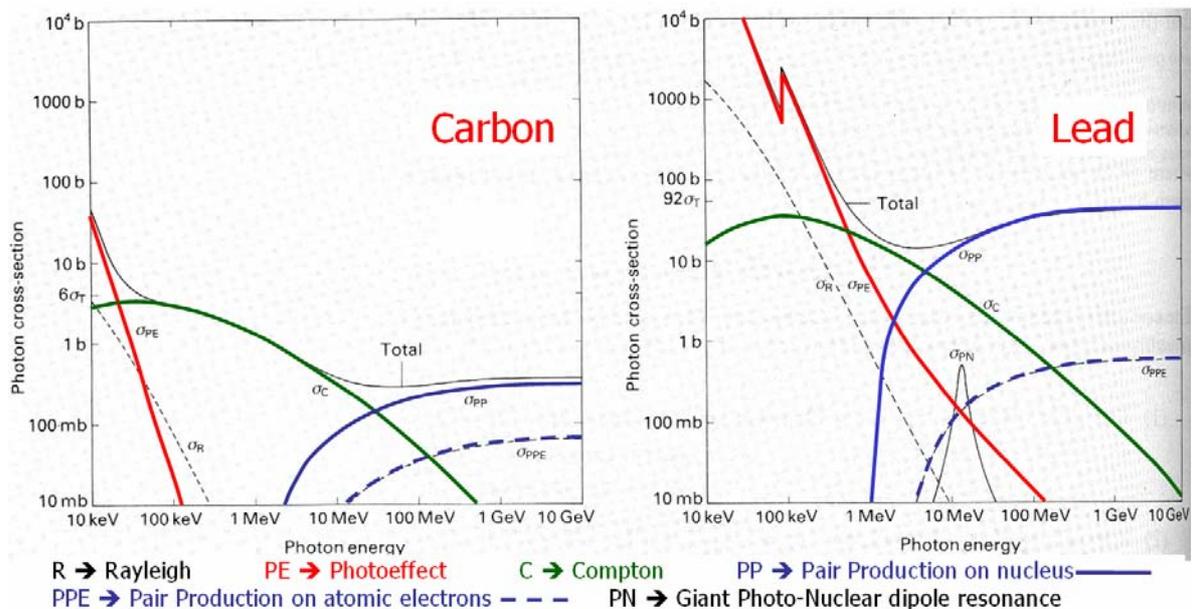


to conserve momentum-energy, each photon carries away 0.511 MeV. Thus pair production provides a very clean signal for detecting positrons as well as calibrating the low energy response of the detector.

The three main processes – photoelectric effect, Compton scattering and pair production provide independent contributions to the absorption of photons in any medium. The total absorption coefficient is given as the sum of the three separate coefficients:

$$\mu = \mu_{pe} + \mu_{comp} + \mu_{pair} \quad (6.36)$$

The sum of this is shown in the figure below as a function of photon energy



- As Z increases
 - PE extends to higher E due to stronger atomic e^- binding
 - PP & PPE extend to lower E due to stronger coupling of projectile to target
 - Threshold for PPE decreases as nucleus contributes more to recoil via stronger atomic electron-nucleus bond
- As A increases E_{recoil} (nucleus) decreases and threshold for PP gets closer to minimum of $2m_e c^2$

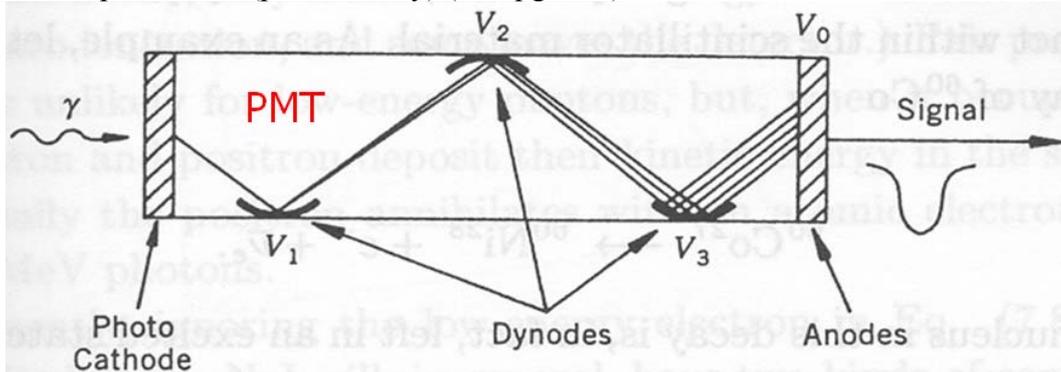
The fractional loss in intensity in crossing any thickness dx of material is

$$\frac{dI}{I} = -\mu dx \quad (6.37)$$

$$I = I_0 e^{-\mu x} \quad (6.38)$$

Lecture 11 – Detectors

Photomultiplier, PMT (photons only) (Das pg. 143)



- primary electrons liberated by photon from photo-cathode (low work function, high photo-effect cross-section, metal)
- visible photons have sufficiently large photo-effect cross-section
- acceleration of electron in electric field 100 – 200 eV per stage
- create secondary electrons upon impact onto dynode surface (low work function metal) → multiplication factor 3 to 5
- 6 to 14 such stages give total gain of 10^4 to 10^7 (electron amplification factor)
- fast amplification times (few ns, due to differing electron transit times – different paths/velocities) → good for triggers or veto's
- signal on last dynode proportional to number of photons impacting
- can have large area photo-cathode with smaller acceleration tube → large area applications

Avalanche Photo Diode, APD (photons only)

- solid state alternative to PMT for photons up to $\lambda < 1600$ nm
- strongly reverse biased (30-70 V) photo diode gives “limited” avalanche when hit by photon
- Avalanche in APD:
 - electrons and holes accelerated by high E-field inside photo diode
 - in one mean free path electrons gain enough energy to generate another electron hole pair in their next collision
 - Multiplication of electrons (and holes) every time an electron collides
 - Dynode separation in PMT corresponds to mean free path in APD
- advantages over PMT's - very much smaller, relatively low voltage, cheap
- often gets used for amplification of light delivered via fibres because this suits their small area
- multiple diodes in one chip for imaging applications

Scintillators (em interacting particles)

- Particle (charged or γ) excites atom through ionisation or photo-effect or Compton scattering
- Observe photon from de-excitation of atomic electron using eye or PMT or APD
- Takes approximately 10 times more energy to produce a scintillation photon than one electron-ion pair in the same material because there are many other ways of losing energy. Typical 1 photon per 100eV of dT/dx
- Very old style: Zinc sulphite screens viewed by eye (Rutherford)
- Scintillators today on the front of every CRT TV-tube.
- Problem: normally materials re-absorb their own scintillation light
- Two solutions to this problem exists

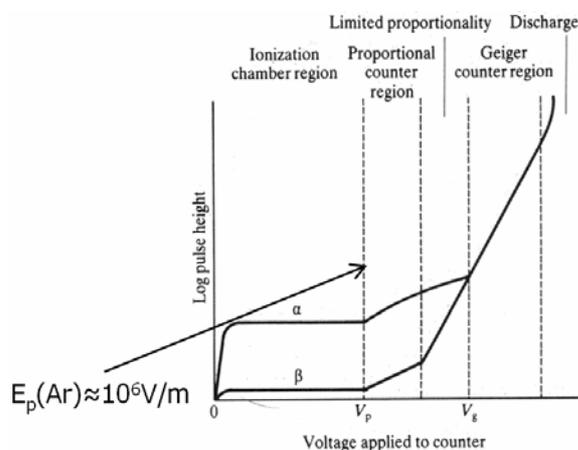
Solution 1: Organic scintillators

- Naphthalene, anthracene are organic molecules, low density ($\rho \approx 1.3$)
- excitation \rightarrow non-radiating de-excitation to first excited state \rightarrow scintillating transition to one of many vibrational sub-states of the ground state (direct transition to ground state is forbidden)
- low cross-section to re-absorbing this photon unless molecule already in this particular vibrational state
- often used together with wavelength shifters to further reduce re-absorption and attenuation in light guides
- Wavelength shifter: low concentration of absorber which absorbs one high E_{in} γ and emit 2 or more low E_{out} γ in cascade decay which can not be re-absorbed by bulk of scintillator
- Organic scintillators give fast scintillation light, de-excitation time $O(10^{-8}$ s)
- Organic scintillators are cheap \rightarrow large area panels

Solution 2: Inorganic scintillators

- NaI activated (doped) with Thallium, semi-conductor, high density: $\rho(\text{NaI}) = 3.6$, $\rho(\text{PbWO}_4) = 8.3 \rightarrow$ high stopping power
- Dopant atom creates energy level (luminescence centre) in band-gap of the semi-conductor
- Electron excited by passing particle into conduction band can fall into luminescence level (non radiative, phonon emission)
- Note: electron must live long enough (not recombine with holes) to reach luminescence centre
- From luminescence level falls back into valence band under photon emission
- this photon can only be re-absorbed by another dopant atom \rightarrow crystal remains transparent to the scintillation light
- High density of inorganic crystals \rightarrow good for totally absorbing calorimetry even at very high particle energies (many 100 GeV)
- de-excitation time $O(10^{-6}$ s) slower than organic scintillators

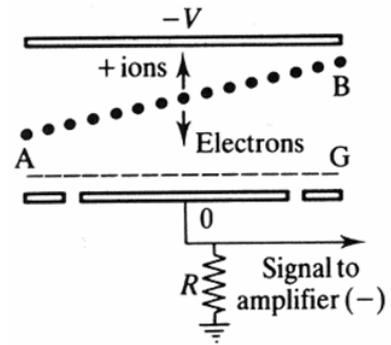
Gas-filled counters (em interacting particles) (Das pg. 136; Krane pg. 204)



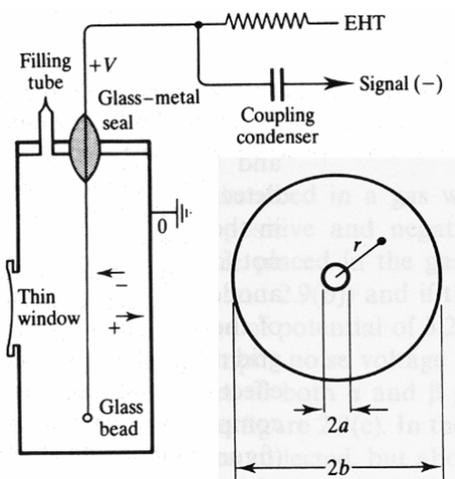
- 6 MeV α particle stopped in gas gives typically 2×10^5 ion pairs (30 eV/ion pair – for air) = 3.2×10^{-14} C negative charge
- Release into $C = 10$ pF \rightarrow 3.2 mV $\gg V_{noise}$ (typ. ampl.) \rightarrow small, but detectable
- Amount of collected charge depends on collection voltage
- Low voltage \rightarrow Ionisation chamber, collect only primary ionisation
- Medium voltage \rightarrow proportional counter \rightarrow avalanche (secondary collision ionisation) \rightarrow signal is proportional to primary ionisation
- High voltage \rightarrow Geiger counter \rightarrow each particle produces the same amount of charge in an unlimited avalanche
- Too high voltage \rightarrow continuous spark (breakdown)

- Ionisation Chambers – now obsolete

- Essentially a parallel plate capacitor in which region between plates is filled by gas, often air.
- Used for single particle and flux measurements
- Can be used to measure particle energy up to few MeV. At higher energies the particle will not be stopped in the gas.
- Measure energy with accuracy of 0.5% (mediocre), limited due to fluctuations of energy loss
- In the gas electrons are more mobile than ions → detect electrons earlier than ions. Collection time = $O(\mu s)$
- Slow recovery from ion drift
- The amplitude of the signal is proportional to the number of ions formed (and thus to the energy deposited by the radiation), and is independent of the voltage between the plates. The applied voltage determines the speed at which the electron and ion clouds drift to their respective electrodes.
- Replaced by solid state detectors



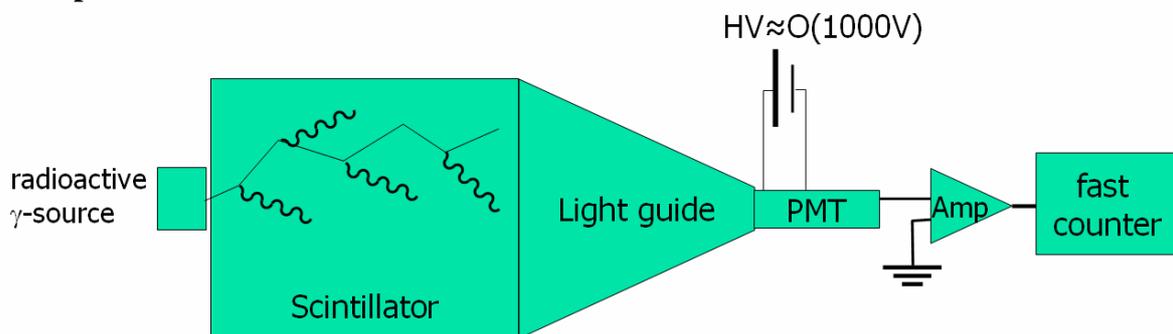
- Proportional chambers (Krane pg. 205; Das pg. 138)



- To use gas-filled detectors to observe individual pulses, must provide considerable amplification.
- A large electric field is able to accelerate the electrons that result from ionisation processes; rather than drifting slowly towards the anode, making occasional elastic collisions with gas atoms, the accelerated electrons can acquire enough energy to make inelastic collisions and even create new ionised atoms – this is known as Townsend avalanche.
- Even though there is a large number ($\sim 10^3 - 10^5$) of secondary events for each original ion, the chamber is always operated such that the number of secondary events is proportional to the number of primary events and hence the name of the device.
- Geometry of a proportional chamber/counter is usually cylindrical as shown in figure above.
- Use small wire as positive electrode (anode)
- $E = V/[r \ln(b/a)]$ high field close to wire
 - local avalanche near wire
 - most electrons released close to wire
 - short average drift distance
 - fast signal rise time $O(\mu s)$
- Use avalanche amplification to measure small ionisation
- Problem: uv-photons from recombination spread through volume → catch them on large organic molecules (quencher) → quenchers vibrationally de-excite
- Many such detectors (MWPC – multiwire proportional chamber) used as large-area position sensitive device
- Can add drift time measurement to increase position resolution → Drift chamber

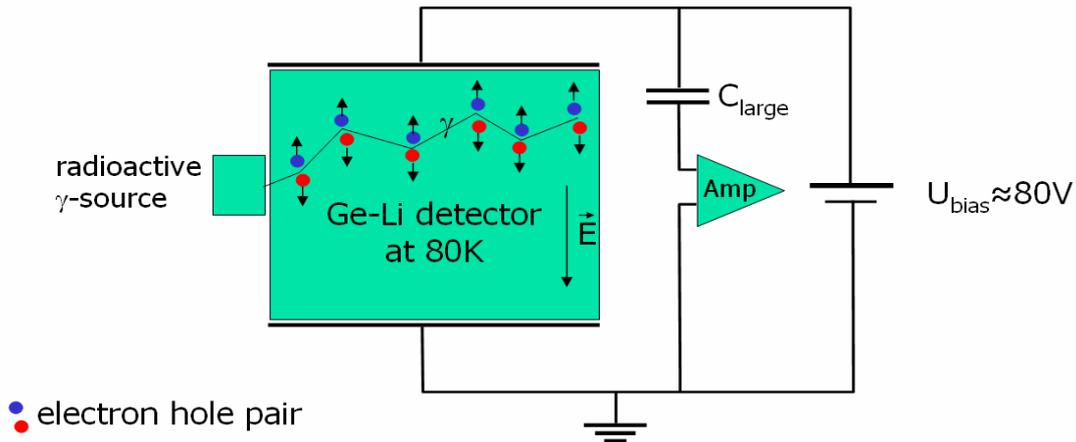
- Geiger counters (Das pg. 141; Krane pg. 206)
 - If the electric field is increased to even larger values, secondary avalanches can occur. These can be triggered by photons emitted by atoms excited in the original (or in subsequent) avalanche. These photons can travel relatively far from the region of original avalanche, and soon the entire tube is participating in the process.
 - Amplification factor as large as 10^{10} .
 - Because entire tube participates for every incident event, there is no information on the energy of the original radiation – all incident radiations produce identical output pulses.
 - ‘Voltage pulse is large and easily detectable ~ 1 V.
 - The cycle would be completed once the positive ions have drifted to the cathode and become neutralised, but during their travel they can be accelerated and strike the cathode with enough energy to release the electrons from the cathode to begin the process again (it takes only one electron to create an output pulse). To avoid this, a second type of gas – quenching gas is added to the tube. As charge consisting mainly of argon ions begins to drift towards the cathode, collisions occur with quenching gas in which there is high probability of transfer of an electron so that the argon is neutralised and the ionised quenching gas (such as ethanol) drifts back to the cathode. It is neutralised at the cathode. The quenching gas is gradually used up and has to be replaced.
 - Geiger counter is advantageous in the simplicity of construction and its insensitivity to small voltage fluctuations.
 - Construction nearly same as proportional counter
 - Operate with $V_g < V < V_{discharge}$
 - Detection here means counting of particles
 - Long recovery time limits counting range $O(100\text{Hz})$
 - Not used much for nuclear physics
 - Some use in radiation protection where you only want to know whether or not there is radiation of any sort

Set-up 1 – Scintillator

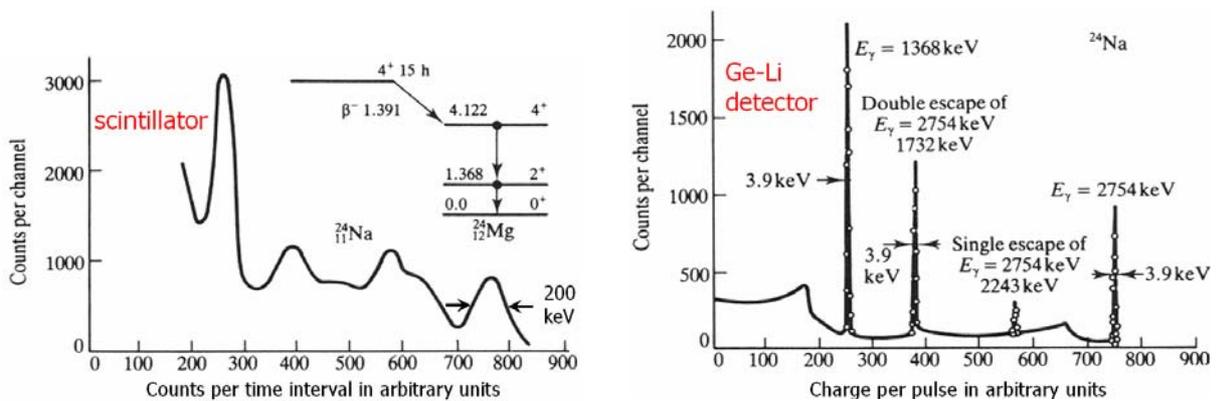


- Scintillator makes number of visible photons proportional to energy lost by γ -ray
- Light guide collects them to PMT photo cathode
- PMT makes electron pulse for each photon
- Counter counts pulses
- Number of pulses in short time window is proportional to γ -ray Energy

Set-up 2 – Germanium detector



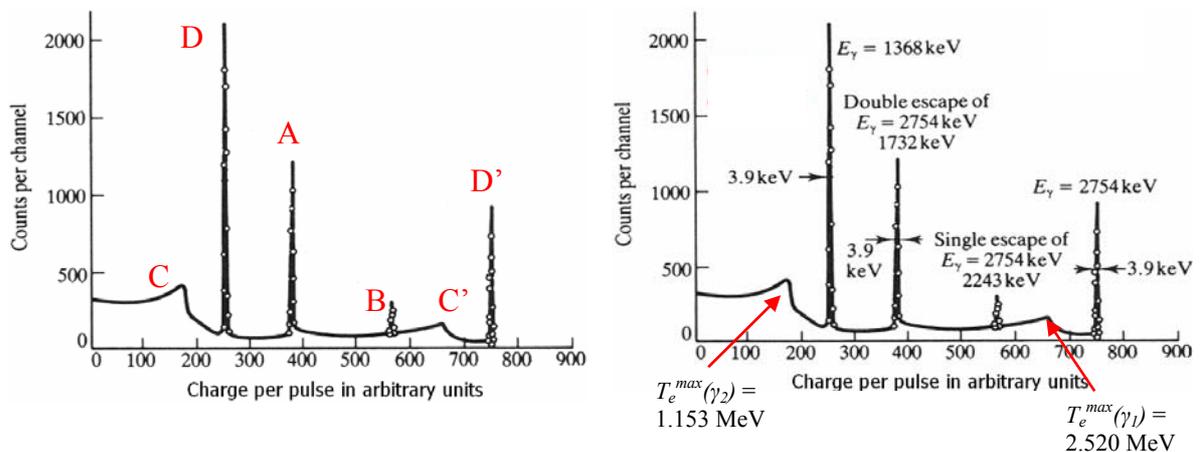
- Move electrons from valence to conduction band via collision with particle \rightarrow electron-hole pair
 - Band gaps O(eV) \rightarrow Energy per electron-hole pair = typical 3-4 eV \rightarrow 1 MeV lost by particle \rightarrow 3×10^5 pairs \rightarrow only 0.2% statistical fluctuation according to \sqrt{n} \rightarrow excellent energy resolution
 - Lowest band gap for Ge = 0.64 eV per pair
 - Ge detectors have highest energy resolution (few keV)
 - Ge-Li detector generates electron hole pairs proportional to energy lost by γ -ray and acts as a source of current pulses
 - One pulse per γ -ray
 - Amplifier measures integrated charge of the pulse which is proportional to energy of γ -ray
- Source contains $^{24}_{11}\text{Na}$, $\rho(\text{Na}) \sim 1 \text{ g/cm}^3$
 - β -decay of $^{24}_{11}\text{Na}$ goes to excited state of $^{24}_{12}\text{Mg}$
 - $E_{\text{kin}}(\beta) = 1.391 \text{ MeV}$ and the β is stuck in the source because according to Bethe-Bloch formula electron will lose O(10 MeV/cm) and thus only has a range of O(1mm)
 - Daughter nucleus $^{24}_{12}\text{Mg}$ decays in two steps via γ -decay
 - Gamma rays escape from source and are observed by the two different detectors
 - $\Delta E_{\gamma_1} = 2.754 \text{ MeV}$; $\Delta E_{\gamma_2} = 1.368 \text{ MeV}$



- Example γ -ray spectra from the two detectors

- Scintillator:
 - approx. 100eV/scintillation photon
 - O(10%) of photons reach photo detector
 - O(10%) quantum efficiency of photo detector
 - 27000 photons for $E_{\gamma 1} = 2.754 \text{ MeV}$
 - 270 reach detector
 - $\sqrt{270} \approx 16.4 \leftrightarrow 6\%$ of $E_{\gamma 1}$
 - consistent with poorly resolved peak width of 7%
- Ge-Li detector
 - 0.64 eV per e-hole pair
 - 4.3×10^6 pairs for $E_{\gamma 1} = 2.754 \text{ MeV}$
 - O(10%) of pairs make it across large detector to the electrodes
 - $\sqrt{4.3 \times 10^5} \approx 656 \leftrightarrow 0.1\%$ of $E_{\gamma 1}$
 - consistent with observed peak width of 0.14%

Response of a Ge-Li detector (Krane pg. 220)



When a γ -ray photon enters a solid detector, the photon can Compton scatter several times; after each scattering, the photon loses some energy and a free electron is produced. Gradually the photon either continues the repeated Compton scattering, eventually becoming so low in energy that photoelectric absorption occurs and the photon vanishes, or it wanders too close to the edge of the crystal and scatters out of the detector. The energy of the photon is converted into electrons (photoelectrons or Compton scattered electrons), which have a very short range in the crystal (by Bethe-Bloch formulation), lose energy rapidly by creating light photons in a scintillator or electron-hole pairs in semiconductor detector. If the original photon eventually suffers photoelectric absorption, the energy deposited is equal to the original γ -ray energy. If it scatters out of the crystal, the energy deposited is less than the original photon energy.

In a single Compton scattering event, the electron gains the following amount of kinetic energy, T_e (by generalised derivation as Eq. (6.30)):

$$T_e = \frac{E_{\gamma}^2 (1 - \cos \theta)}{mc^2 + E_{\gamma} (1 - \cos \theta)} \quad (7.1)$$

Since all scattering angles can occur in the detector, the scattered electron ranges in energy from 0 for $\theta = 0^\circ$ and to $2E_{\gamma}^2 / (mc^2 + 2E_{\gamma})$ for $\theta = 180^\circ$ (Eq. (6.30)). These electrons will normally be totally absorbed in the detector, and (if the scattered photons escape) the contribute to the energy response of the detector a continuum called the Compton

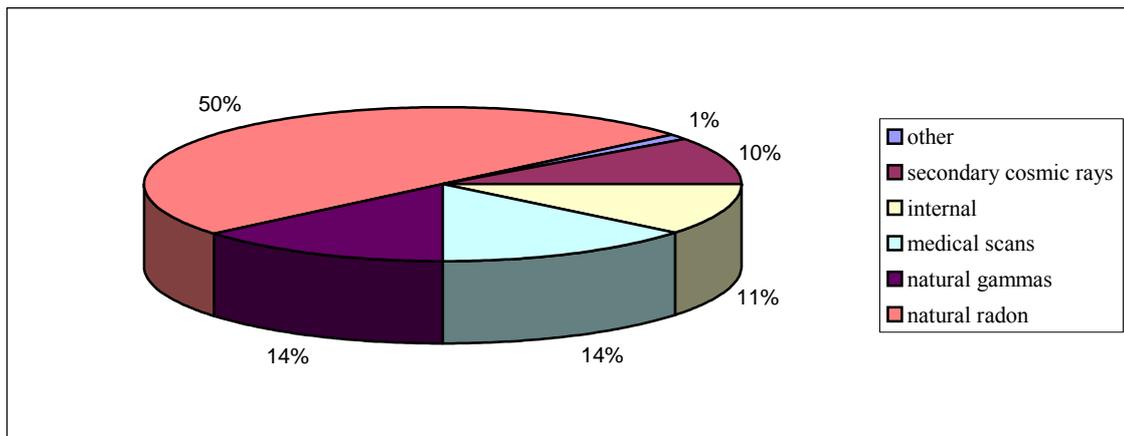
continuum ranging from zero to a maximum known as the Compton edge (due to Compton scattering probability varying with angle, the continuum is not flat). This can be seen on the figure above as features C and C' corresponding to the Compton edges for 2 different E_γ 's. This gives $T_e^{max}(\gamma_1 = 2.754 \text{ MeV}) = 2.520 \text{ MeV}$ and $T_e^{max}(\gamma_2 = 1.368 \text{ MeV}) = 1.153 \text{ MeV}$. These peaks are slightly rounded as electrons are not exactly free but slightly bound.

The peaks at $E = E_{\gamma_1}$ and $E = E_{\gamma_2}$ (D' and D) correspond to complete photoelectric absorption.

The final process is that of pair production by the γ -ray photon. The electron-positron pair are created with total kinetic energy of $E_\gamma - 2m_e c^2$, by Eq. (6.34). The loss of this energy in the detector would result in a peak at full energy. However, once the positron slows down to energy near to that of atomic electron, annihilation takes place and $e^+e^- \rightarrow 2\gamma$ with the new photons each of energy $m_e c^2$, 0.511 MeV. Should both photons escape would expect to have peaks at $E_\gamma - 2m_e c^2$ (peak A for γ_1) if one escapes and the other is absorbed then peaks at $E_\gamma - m_e c^2$ (peak B for γ_1) and finally peaks at E_γ if both are absorbed (photopeaks – D and D').

The relative amplitudes of the photopeak, Compton continuum and escape peaks depend on the size and shape of the detector. In general, the larger the detector, the smaller the Compton continuum relative to the photopeak, for there is a smaller chance of Compton scattered photon surviving from the center to the surface without interacting again. Similarly the annihilation photons are more likely to be captured in a larger detector.

Radiation units (Krane pg. 184)



Radiation	w_R
X- and γ -rays, all energies	1
Electrons and muons, all energies	1
Neutrons < 10 keV	5
10–100 keV	10
> 100 keV to 2 MeV	20
2–20 MeV	10
> 20 MeV	5
Protons (other than recoils) > 2 MeV	5
Alphas, fission fragments, & heavy nuclei	20

- **Activity of a source**
 - Becquerel (Bq) is the number of disintegrations per second.
 - $1 \text{ Bq} = 2.7 \times 10^{11} \text{ Curie (Ci)}$
 - radiation levels sometimes quoted in Bq m^{-3} .
- **Absorbed Dose**
 - 1 Gray (Gy) = 1 joule of deposited energy per kg of irradiated mass
- $1 \text{ Gy} = 100 \text{ rad} = 6.24 \times 10^{12} \text{ MeV/kg}$.
- **Equivalent Dose for biological damage**
 - 1 Sievert (Sv) = absorbed dose equivalent in damage to 1 Gy of x-rays, β or γ .
 - per unit energy deposited:
 - some particles have larger dT/dx than β or γ & strong interactions \rightarrow localised damage \rightarrow more long term biological risk \rightarrow higher weight w_R than β or γ
 - $1 \text{ Sv} = 100 \text{ rem}$ (**R**oentgen equivalent for **m**an)

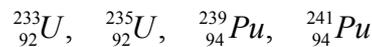
- **Examples of Sv**
 - Lethal whole-body dose 2.5 - 3.0 Sv → death in 30 days without treatment
 - Limit for radiation workers: 15 mSv yr⁻¹ (UK) or 50 mSv yr⁻¹ (US)
 - Chest x-ray 0.04 mSv
 - CT scan 8 mSv
 - Average UK whole body dose rate 2.6 mSv yr⁻¹ (world from 0.4 – 4 mSv yr⁻¹)
- Weight expresses risk from low levels of chronic exposure
- Main consequences in risk evaluation are cancer and leukaemia
- Average breakdown of 2.6 mSv yr⁻¹ taken from NRPB report (1995).
- Internally released = (⁴⁰K, ¹⁴C)
- Cosmic flux at sea level, $\Phi_{\text{cosmic}} \approx 1 \text{ min}^{-1} \text{ cm}^{-2} \text{ sr}^{-1}$

Lecture 12

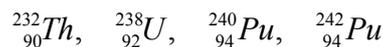
Fission (Cottingham 2nd ed. pg. 115; Krane pg. 501; Williams pg. 123)

The Coulomb barriers inhibiting spontaneous fission are in the range 5 – 6 MeV for nuclei with $A \sim 240$. If a neutron of zero kinetic energy enters a nucleus to form a compound nucleus, the compound nucleus will have excitation energy above its ground state equal to the neutron's binding energy in that ground state. For example, a zero energy neutron entering ²³⁵U forms a state of ²³⁶U with an excitation energy of 6.46 MeV. This energy is above the fission barrier and the compound nucleus quickly undergoes fission. To induce fission in ²³⁸U on the other hand requires a neutron with a kinetic energy of approximately 1.4 MeV.

The differences in binding energy of the last neutron in even- A and odd- A nuclei are incorporated in the SEMF in the pairing term. The odd- A nuclei:



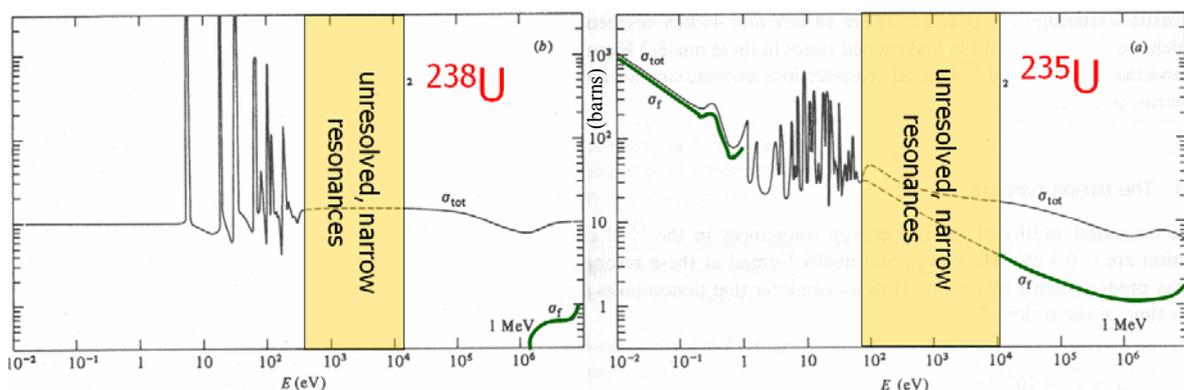
are fissile nuclei – the fission can be induced by zero energy neutron. The even- A nuclei:



require an energetic neutron to induce fission. Note that all Pu isotopes are manmade.

Neutron cross-sections for ²³⁵U and ²³⁸U

The principal isotopes of naturally occurring uranium are ²³⁸U (99.27%) and ²³⁵U (0.72%).



Total cross-section σ_{tot} and fission cross-section σ_f as function of neutron energy.

The figure on page 53 shows the total cross-sections of ^{235}U and ^{238}U for incident neutrons of energy E from 0.01 eV to 10 MeV. At very low energies, below 0.1 eV in ^{235}U the law $1/v$ can be observed and the total and fission cross-sections are large because of an excited state of ^{236}U lying just below $E = 0$. The fission fraction $\sigma_f/\sigma_{tot} \sim 84\%$, the remaining 16% of σ_{tot} corresponds mostly to radiative capture. In contrast, the cross-section for ^{238}U is very much smaller and nearly constant in this region and is due almost entirely to elastic scattering.

Region between 1 eV and 1 keV, resonances are prominent in both isotopes. These resonances are very narrow and radiative capture gives a significant fraction of the total widths. This is particularly true of resonances in ^{238}U , which are below the fission threshold. For example, γ -decays account for 95% of the width of the resonance at 6.68 eV.

In the final region, between 1 keV and 3 MeV, the resonances are not resolved by measured cross-sections. Compound nuclear states at these energies are more dense and wider. The fission cross-section for ^{238}U appears above 1.4 MeV and the ^{235}U fission fraction σ_f/σ_{tot} remains significant. However in both isotopes at these higher energies the result of a neutron interaction is predominantly scattering, either elastic or at higher energies inelastic with neutron energy lost in exciting the nucleus.

Fission process

A single fission event of ^{235}U will on average produce 2.5 neutrons. This number will depend somewhat on the energy of the incident neutron. In addition there are on average 0.02 delayed neutrons produced per fission, emitted following chains of β^- decays of neutron-rich fission products. Each of these 'second-generation' neutrons is capable of producing another fission event and so on. This is known as a chain reaction. The total energy release on the induced fission of a ^{235}U nucleus, is on average 205 MeV. The break down of which is:

	MeV
Prompt:	
Kinetic energy of fission fragments	167
Kinetic energy of fission neutrons	5
Energy of prompt γ -rays	6
Delayed:	
Electrons from subsequent β decays	8
γ -rays following β decays	7
Neutrino Energy	12
	205

The delayed energy release is regarded as a nuisance, some of it is delayed by decades or more and presents potential biological hazard in discarded nuclear waste. The radiative capture yields a further 3 – 12 MeV of useful energy in emitted γ -rays, which are not included in the table.

Consider a sample of pure ^{235}U on which a neutron of energy 2 MeV is incident. The nuclear number density, n of uranium metal is 4.8×10^{28} nuclei/m³. The mean free path of a neutron in the mixture is

$$\lambda = \frac{1}{n \langle \sigma_{tot} \rangle} \quad (7.2)$$

For 2 MeV neutron, the total cross-section from figure a) on page 53 is ~ 7 barns and so $\lambda \sim 3$ cm. Not all neutrons will induce fission. The 2 MeV neutron has an 18% chance of inducing fission (again from figure a)). The probable number of collisions before fission is induced is therefore ~ 6 . For a random walk, the neutron will travel $\sim \lambda\sqrt{6} = 7$ cm from its

starting point. The time for it to travel this distance, t_p is approximately 10^{-8} s. Some will escape from the surface and some will undergo radiative capture. Let probability that a neutron will induce fission be q and average number of prompt neutrons be ν , then on average addition $(\nu q - 1)$ neutrons will be created in time t_p . If number of neutrons at time t is $n(t)$, then at $t + \delta t$ there will be:

$$n(t + \delta t) = n(t) + (\nu q - 1)n(t) \frac{\delta t}{t_p}$$

$$\Rightarrow \frac{dn}{dt} = \frac{(\nu q - 1)}{t_p} n(t) \quad (7.3)$$

$$n(t) = n(0)e^{(\nu q - 1)t/t_p} \quad (7.4)$$

For ^{235}U , $\nu = 2.5$, therefore if $q > 0.4$, there will be an exponential increase in number of neutrons and hence the reaction becomes supercritical. For a small amount of ^{235}U , much less than 7 cm, there will be a good chance of escape and therefore q will be small and chain reaction will be damped out. If sufficiently large mass of uranium is brought together at $t = 0$ will have $q > 0.4$. There will be neutrons present at $t = 0$ due to spontaneous fission, a large amount of energy will be released even in a microsecond. The bare sphere of ^{235}U will have critical radius of 8.7 cm and critical mass of 52 kg for this to occur.

For nuclear reactors need to control the chain reaction, to maintain a steady state release of energy would like for the pile (lattice of blocks of uranium alternating with graphite) to be exactly critical ($\nu q = 1$). In a fission process after fragmentation, the neutrons will have energy \sim MeV, such neutrons have a relatively low probability of inducing new fissions, they will scatter rather than be absorbed therefore they must be slowed using a moderator – where neutrons can scatter and reduce their energy to thermal energies \sim 0.1 eV, where the ^{235}U cross-section is much larger than of ^{238}U . These thermal neutrons, if captured in the fuel rods are predominantly captured by ^{235}U , the large cross-section compensates for the low number density. The neutrons are slowed to thermal energies in the moderator rather than in the fuel rods, capture into ^{238}U resonances is avoided. The captures into ^{235}U lead to fission with a probability of $\sigma_f^{235}/\sigma_{tot}^{235} \sim 84\%$ at thermal energies and the chain reaction can be sustained in this way.

The neutrons are most effectively slowed down by using light elements/materials as moderator – as the atoms will recoil more and energy of neutrons will be reduced more efficiently. Most effective moderator would be hydrogen; however neutrons are likely to be captured: $p + n \rightarrow {}^2_1\text{H} + \gamma$ in hydrogen and water. Deuterium is better as a moderator as it has a very low neutron absorption cross-section. Therefore deuterium can use ordinary uranium as fuel as more neutrons are available. Carbon is a light material which is solid, stable and abundant; it too has a low neutron absorption cross-section.