Lecture Notes on

# Quantum Field Theory

Kevin Zhou

kzhou7@gmail.com

These notes constitute a year-long course in quantum field theory. The primary sources were:

- David Tong's Quantum Field Theory lecture notes. A clear, readable, and entertaining set of notes, good for a first pass through first-semester quantum field theory.
- Timo Weigand's Quantum Field Theory lecture notes. Covers similar material, but from a more careful, formal point of view; it opens with LSZ reduction and handles the canonical quantization of gauge theories in far more detail. May be too dry for a first pass, but very useful for clarifying points on a second pass.
- David Skinner's Advanced Quantum Field Theory lecture notes. An excellent second-semester quantum field theory course with differential geometry and the Wilsonian point of view baked in throughout. Also contains a thorough list of QFT books and resources.
- Sidney Coleman's Quantum Field Theory lecture notes. A classic course from the 1980s delivered by a legendary physicist. When I was starting out, I had a very hard time understanding the middle third, which introduces renormalization in a rather formal way. However, the first third is a great introduction to the basics.
- Peskin and Schroeder, *Quantum Field Theory*. The standard book with all the standard conventions, from which many sets of lecture notes above draw inspiration. Part I is a standard, clear introduction to the basics, though slightly sketchy when introducing interactions. Part II covers renormalization, using the Wilsonian point of view more than other books. Part III is exceptional, with great physical explanations and a deep exploration of practical computations in QCD and the Standard Model at large.
- Srednicki, *Quantum Field Theory*. A newer book with a focus on the path integral. Distinguished by its clean, modular style and nontraditional ordering of topics by spin, which allows one to encounter the conceptual novelties in interacting field theory, spinor representations, and gauge symmetry separately. Has little on canonical quantization, but briefly covers topics beyond the Standard Model. One downside is that it starts rather formal, and takes a long time to make contact with familiar physics; the first third of the book covers just $\phi^3$ theory in six dimensions.

- Schwartz, *Quantum Field Theory and the Standard Model*. A new book with a breezy, conversational tone. Performs many concrete calculations with modern methods, and applications to collider physics. Has clear and fresh explanations, but also sometimes sweeps issues under the rug using vague arguments or ambiguous notation (according to a reviewer: "the logic can be off-shell"). In my opinion, the main problem with this book is that it is especially sloppy in the first 250 pages, which is exactly where students are most likely to get confused.

- Zee, *Quantum Field Theory in a Nutshell*. A fun book which focuses on the path integral and includes applications to condensed matter. Covers a lot of ground extremely briefly. It might not be literally impossible to learn quantum field theory by doing every calculation in the book (as Zee continuously implores the reader), but it wouldn't be an efficient route. For those with some background already, the book is a great way to broaden knowledge.

- Weinberg, *The Quantum Theory of Fields*. A massive three-volume tome that addresses many subtle points. The notation is dense and clunky, but this is done intentionally in the service of making the logic as transparent and explicit as possible. There are many natural questions one wants to ask when learning quantum field theory, that all the usual books seem to completely ignore. Weinberg often has the answers; it is the book the books above refer to for the ground truth. However, it is completely unsuitable as an introduction.

- Ryder, *Quantum Field Theory*. A friendly book which serves as a general introduction to particle theory, deemphasizing cross sections but incorporating topics like monopoles, supersymmetry, and differential geometry. It's very readable, but has little detail on the more advanced parts and no problems; best used as a supplement.

- Collins, *Renormalization*. A monograph focused on renormalization methods, which covers many technical points skimmed over in standard books.

- Banks, *Modern Quantum Field Theory*. A brief summary from a string theorist's point of view; similar spirit to Skinner's lecture notes. Completely useless if you don't already know quantum field theory (e.g. a full pass through Peskin, Srednicki, or Schwartz), but a fun and enlightening read if you do. The chapter on renormalization is especially good.

- Weinberg, *Classical Solutions in Quantum Field Theory*. Contains clear and insightful discussions of solitons, anomalies, and instantons.

- David Tong's Gauge Theory lecture notes. An absolutely exceptional set of notes on special topics in quantum field theory, with many references to original literature.

- Many insightful questions and answers on Physics.SE. I particularly recommend the answers by Qmechanic, ACuriousMind, and Chiral Anomaly.

Starting to learn quantum field theory can be very rough, so some people say the standard books are intentionally confusing, to make the authors look smart. Nothing could be further from the truth. Writing a good textbook requires thousands of hours of work and years teaching with drafts of it, which implies real devotion to students. The real issue is that these books were written for capstone

courses, taken by theoretical particle physicists only, after four years of undergraduate courses and two years of rigorous graduate courses. They assume you've already seen topics like relativistic classical fields, representation theory, Green's functions, and scattering theory. Nowadays, many people want to learn quantum field theory earlier, and this mismatch makes it harder to start. Below are some "student friendly" ways to help bridge the gap.

- David Tong's Quantum Field Theory lecture notes. Yes, again. These notes follow the first 5 chapters of Peskin and Schroeder, but with lower density, keeping about half the detail and adding double the explanation. This is a good balance for most new students. If you have a strong understanding of undergraduate physics, I think this is the best place to start.

- Donoghue and Sorbo, *A Prelude to Quantum Field Theory*. This slim new book is adapted from the lecture notes of an undergraduate course. It clearly explains some of the trickiest topics for beginners, including renormalization, symmetry breaking, and the interpretation of virtual particles, by relentlessly focusing on the simplest possible examples. The book is not meant to be comprehensive – for instance, spinors are relegated to a few pages near the back. But it is a fantastic source for supplemental reading.

- Greiner, *Field Quantization*. This crisp, clear book is used for introductory courses in Europe. It roughly corresponds to the first 5 chapters of Peskin, but removes all handwaving, lays out definitions and rules clearly, and includes many concrete worked examples, solved at the level of detail a beginning student would want. It can be rather dry and tiring to read, but if you prefer precision and have the time and energy, this might be the best choice.

- Lancaster and Blundell, *Quantum Field Theory for the Gifted Amateur*. A fun book which briefly introduces a wide range of topics in both relativistic and nonrelativistic field theory. Compared to the books above, this one goes in the opposite direction: it covers *more* topics than your typical textbook, but suppresses a huge amount of detail. My main complaint with this book is that it contains just enough to make you think you can flesh out the handwavy arguments, but it's missing just enough so that this would be nearly impossible for a new student. As a result, I don't recommend studying it closely if you're a beginner, but it's a great book for bedtime reading.

- Schwichtenberg, *No-Nonsense Quantum Field Theory*. This book again follows the first 5 chapters of Peskin and Schroeder, but it keeps about 1/4 of the detail and adds quadruple the explanation, including reviews of basic quantum mechanics, special relativity, and Lagrangian mechanics. In my opinion, if you plan on eventually getting a full understanding of quantum field theory, e.g. at the level of Peskin and Schroeder, this book is not the right place to start. It has no exercises, and it would be more efficient in the long run to back up and review undergraduate physics directly. If you're self-studying for fun and want to read just one book to see the basic ideas of quantum field theory, it would be better to go with:

- Zee, *Quantum Field Theory, as Simply as Possible*. This is an upgraded popular science book, aimed at people who know calculus but essentially no physics. It's packed full of charming

historical anecdotes and vivid analogies. Like every popular book, these analogies are fragile and not a replacement for real mathematics; the difference is that Zee points this out, and constantly implores the reader to eventually graduate to real textbooks. This is a great option for the casual reader who wants to get an idea of what particle physicists do.

- Klauber, *Student Friendly Quantum Field Theory*. A dense book where half of the first 5 chapters of Peskin and Schroeder are spelled out in extreme detail. Notation is kept *completely* explicit throughout, leading to single equations that take up three whole lines in small font. This book is praised by online reviewers, but I think keeping all details explicit actually inhibits learning, because you'll use a ton of mental energy to juggle boilerplate which should be left off the page. With its obsessive focus on the mechanics of calculations, the book says very little about the bigger picture. Worse, when it does address concepts, it makes some dubious claims, and rants against strawman arguments from "established physicists." Not recommended.

Further aspects of quantum field theory, such as discrete symmetries and spontaneous symmetry breaking, are covered in the notes on the Standard Model. The most recent version is here; please report any errors found to kzhou7@gmail.com.

# Contents

# 1 Scalar Fields

## 1.1 Classical Field Theory

We begin by reviewing the basics of classical field theory.

- We assume the field is described by an action of the form

$$S[\phi(x)] = \int d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)$$

where $\mathcal{L}$ is called the Lagrangian density. The absence of explicit $x$-dependence of $\mathcal{L}$ is required by translational invariance. The fact that $S$ can be written in terms of a Lagrangian density with a finite number of derivatives means that it is local; fields are only coupled to each other at the same spacetime point.

- Under a field variation $\phi \to \phi + \delta\phi$, let the action change by $\delta S$. Then we define the functional derivative by

$$\delta S = \int d^4x \, \frac{\delta S}{\delta \phi(x)} \delta\phi(x).$$

If we assume the variation $\delta\phi$ goes to zero at infinity, typically done by fixing the field at temporal endpoints and demanding it vanish at spatial infinity, then we can integrate by parts neglecting boundary terms, giving

$$\frac{\delta S}{\delta \phi} = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}.$$

Classically, the field minimizes its action, and the equation of motion $\delta S/\delta\phi = 0$ is the Euler–Lagrange equation.

- As an example, for the free real scalar field we have

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2$$

from which we identify the kinetic energy density as $\dot{\phi}^2/2$ and the potential energy density as $(m^2\phi^2 + (\nabla\phi)^2)/2$. The equation of motion is the Klein–Gordan equation,

$$(\partial_\mu \partial^\mu + m^2)\phi = 0.$$

- We will often use the Hamiltonian formulation, defining the canonical momentum and Hamiltonian by

$$\pi(\mathbf{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(\mathbf{x})}, \quad \mathcal{H} = \pi(\mathbf{x})\dot{\phi}(\mathbf{x}) - \mathcal{L}, \quad H = \int d^3x\, \mathcal{H}.$$

Since we have chosen a preferred time direction, we will have to carefully check for Lorentz invariance as we go. Note that the canonical momentum isn't spatial momentum; it instead measures momentum "in the field direction". For example, for transverse waves on a horizontal string, the canonical momentum is the vertical momentum.

- Suppose an infinitesimal transformation $\phi \to \phi + \delta\phi$ modifies the Lagrangian by a total derivative, $\mathcal{L} \to \mathcal{L} + \partial_\mu F^\mu$, so that the action remains unchanged. Then

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi) = \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\delta\phi + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) = \partial_\mu F^\mu.$$

- Therefore, we have
$$\partial_\mu j^\mu = -\frac{\delta S}{\delta \phi(x)}\delta\phi(x), \quad j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi - F^\mu.$$
When the equation of motion is satisfied, $\partial_\mu j^\mu = 0$, so $j^\mu$ is a conserved current; the spatial integral of $j^0$ is the conserved charge. This is Noether's theorem.

We now give some example applications of Noether's theorem.

- Some symmetry transformations, such as Lorentz symmetry, can be viewed either actively or passively. The active framework is more general and more straightforward, since it involves only one "coordinate system", so we focus on it.
- Sometimes, passive thinking can be useful to help write down the active transformation. For example, if a passive transformation is $x \to x'$, then the equivalent active transformation maps $\phi \to \phi'$ where $\phi'(x') = \phi(x)$. More generally, all our active transformations will take the form $\phi'(x') = F(\phi(x))$ by locality.
- For an infinitesimal passive translation $x \to x - a$, we equivalently have $\phi \to \phi'$ where
$$\phi'(x) = \phi(x + a) = \phi(x) + a^\mu \partial_\mu \phi(x).$$
The Lagrangian changes by the total derivative $a^\mu \partial_\mu \mathcal{L}$ and the conserved current is
$$j^\mu = a_\nu(\pi_\mu \partial^\nu \phi - \delta^\mu_\nu \mathcal{L}).$$
The four conserved currents above may be combined into the stress-energy tensor,
$$T^\mu{}_\nu = \pi^\mu \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}, \quad \partial_\mu T^\mu{}_\nu = 0.$$
The conserved charges are the physical energy and momenta of the field; $T^{00} = \mathcal{H}$ is the energy density and $T^{0i}$ is the momentum density. Note that these quantities are bilinear in the fields, as physical observables generally are.
- It can be shown that Noether charges are tensorial. That is, if $T^{\mu\nu\cdots}$ is a tensor, then
$$Q^{\nu\cdots} = \int d^3x\, T^{0\nu\cdots}$$
is a tensor as well, which is independent of the spatial surface used for the integral. For example, the total energy and momentum form a four-vector. This fact is proven in the general setting of curved spacetime, in the lecture notes on General Relativity.
- The energy-momentum tensor is ambiguous, as we could also take
$$\Theta^{\mu\nu} = T^{\mu\nu} + \partial_\rho \Gamma^{\rho\mu\nu}$$
where $\Gamma$ is antisymmetric in its first two indices, so $\Theta$ is also conserved. It is convenient to choose $\Gamma$ so that $\Theta$ is symmetric, yielding the so-called Belinfante tensor, but this still leaves further freedom; it is removed in general relativity, where the physical stress-energy tensor is the one which appears in the Einstein field equation.

- Consider a passive Lorentz transformation $x \to x' = \Lambda x$. Infinitesimally we have

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu.$$

  Then the Lorentz condition $\Lambda^\mu{}_\sigma \eta^{\sigma\tau} \Lambda^\nu{}_\tau = \eta^{\mu\nu}$ reduces to $\omega^{\mu\nu} = -\omega^{\mu\nu}$, so Lorentz transformations are parametrized by antisymmetric matrices.

- As a result, we have the corresponding infinitesimal active transformation

$$\phi'(x) = \phi(x) - \omega^\mu{}_\nu x^\nu \partial_\mu \phi(x)$$

  with a similar transformation for $\mathcal{L}$ by analogy with our results for translations, with $\omega^\mu{}_\nu x^\nu$ replacing $a^\mu$. Then similarly the current is

$$j^\mu = -\omega^\rho{}_\nu T^\mu{}_\rho x^\nu = \frac{1}{2}\omega_{\rho\nu}(T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu)$$

  where we used the antisymmetry of $\omega_{\rho\nu}$. Note that this current is only conserved if $T^{\mu\nu}$ is properly symmetrized.

- The six symmetries can hence be packaged into a rank three tensor, representing the relativistic analogue of angular momentum density,

$$(\mathcal{J}^\mu)^{\rho\sigma} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho}.$$

  For each choice of $\rho$ and $\omega$, there is a conserved current. For spacelike indices we get angular momentum, while choosing $\rho = 0$ and $\sigma = i$ gives "conservation of the velocity of the center of energy". When we work with spinor and vector fields, we'll get more terms due to the transformation of the fields themselves, which will correspond to spin angular momentum.

- Stepping back, a field is a map $\phi : M \to T$ where $M$ is the base manifold and $T$ is the target space. Translational symmetry can be thought of as either due to a "horizontal" change in $M$ (by $x \to x + \delta x$) or due to a "vertical" change in $T$ (by $\phi \to \phi + \delta x\, \phi'$). Above we've always chosen the vertical option, and all infinitesimal changes can be written in this form. Allowing horizontal transformations can be occasionally useful for clarity, and adds a factor $\delta x_\nu T^{\mu\nu}$ to the Noether current.

**Note.** Passive transformations need to be interpreted differently to count as symmetries. It is trivially true that physics is invariant under a change of coordinates. The content of symmetry by a passive transformation is that after such a transformation, the *form* of the equations remains the same. For example, Maxwell's equations look the same after a Lorentz transformation, but not after a Galilean transformation. We won't worry about this because we'll only use active transformations.

**Example.** Consider the dilation symmetry

$$x \to x' = \lambda x, \quad \phi'(x') = \frac{1}{\lambda}\phi(x).$$

More generally, the power of $\lambda$ in the denominator is called the scaling dimension of the field. (It cannot be intuitively interpreted passively since $F$ is nontrivial, unless we imagine the "rulers" we use to measure $\phi$ change as well.) The dilation leaves the action

$$S = \int d^4x \, \frac{1}{2}(\partial\phi)^2 - g\phi^4$$

invariant, though a mass term would not be invariant. For an infinitesimal transformation $\lambda = 1 - \epsilon$,

$$\delta\phi = \phi + x^\mu \partial_\mu \phi.$$

Since the Noether current is linear in $\delta\phi$, we can consider the two terms separately. The first term straightforwardly gives $\pi^\mu \delta\phi$. The second term is just a translation, with $x^\mu$ replacing $a^\mu$, so

$$j^\mu = \pi^\mu \delta\phi + x_\nu T^{\mu\nu} = (\partial^\mu \phi)(\phi + x_\nu \partial^\nu \phi) - x^\mu \mathcal{L}.$$

**Example.** An internal symmetry. We consider the complex scalar field

$$\mathcal{L} = |\partial_\mu \phi|^2 - m^2 |\phi|^2$$

which has a $U(1)$ internal symmetry $\phi \to e^{i\alpha}\phi$. Treating $\phi$ and $\phi^*$ as independent fields, the equation of motion is

$$(\partial_\mu \partial^\mu + m^2)\phi = 0$$

along with its conjugate, and under the $U(1)$ symmetry,

$$\alpha \Delta\phi = i\alpha\phi, \quad \alpha\Delta\phi^* = -i\alpha\phi^*, \quad \Delta\mathcal{L} = 0$$

which gives the conserved current

$$j^\mu = i[(\partial^\mu \phi^*)\phi - \phi^*(\partial^\mu \phi)].$$

This quantity corresponds to the particle number current; it is reminiscent of the probability current in quantum mechanics, but $j^0$ can be positive or negative. This causes problems when interpreting the Klein–Gordan equation as a relativistic particle equation; as a field equation the negative sign corresponds to antiparticles.

**Note.** Why can we treat $\phi$ and $\phi^*$ as independent? Consider the variation

$$\Delta S \sim A\,\delta\phi + A^*\,\delta\phi^*.$$

Treating the variations as independent gives $A = A^* = 0$. On the other hand, taking real and imaginary variations gives $A \pm A^* = 0$ which gives the exact same conclusion.

Another way to think about this is to write $\phi = \phi_1 + i\phi_2$ where the $\phi_i$ are real. Then the 'extra' degrees of freedom we get are the same as those when we use complex variations $\delta\phi_i$. This gives no extra constraints because $\mathcal{L}$ is analytic in the $\phi_i$, so if $\delta\mathcal{L}$ vanishes along the real direction, it also vanishes along the imaginary direction.

**Note.** Why is there no $1/2$ in the complex scalar field Lagrangian? Classically, it doesn't matter because scaling the action has no effect. At the quantum level, fields must be canonically normalized (for free fields, have a kinetic term with coefficient $1/2$) for the fields to create conventionally normalized particle states. We may define

$$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}$$

where the $\sqrt{2}$ factor ensures that $\phi$ is canonically normalized if the $\phi_i$ are. Then the Lagrangian in terms of the real scalar fields $\phi_i$ has the desired factors of $1/2$.

**Example.** Consider longitudinal vibrations of a spring, described by

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\partial_x \phi)^2.$$

There are two symmetries, corresponding to shifts in $M$ and shifts in $T$,

$$\delta x = a, \quad \delta\phi = a.$$

This is subtle, because both of these symmetries would appear to be translational symmetry, as $\phi$ is simply the displacement of a point on the spring; in this case it is essential to distinguish $M$ and $T$. The first symmetry corresponds to $\delta\phi = a\,\partial_x\phi$ and corresponds to translating everything; it yields the momentum. The second symmetry heuristically to "translating the wave within the spring" and yields the pseudomomentum $\int \pi\, dx$.

**Example.** Electromagnetism is described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

The factor of $-1/4$ ensures that we have canonically normalized kinetic terms of the form $\dot{A}_i^2/2$. However, the $A_0$ field has no kinetic term at all, which will present some subtleties later. As currently written, $\mathcal{L}$ doesn't depend on $A_\mu$, so

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{2}\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)}F^{\alpha\beta} = -\frac{1}{2}(F^{\mu\nu} - F^{\nu\mu}) = -F^{\mu\nu}$$

giving the equation of motion

$$\partial_\mu F^{\mu\nu} = 0.$$

We can also integrate by parts, so that the canonical momenta change, and forces $\partial\mathcal{L}/\partial A_\mu$ appear. Of course, the equations of motion and all observable quantities remain the same. The naive stress-energy tensor is neither symmetric or gauge invariant, and must be improved, as discussed in the lecture notes on General Relativity.

**Example.** A massive vector field instead has the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu.$$

There is still no kinetic term for $A_0$, but we no longer have a gauge symmetry. The Euler–Lagrange equations are

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0$$

and taking the divergence gives $m^2\partial_\mu A^\mu = 0$, so $\partial_\mu A^\mu = 0$ automatically; in the case $m^2 = 0$ we must impose this constraint by hand as a gauge fixing condition. Then the equation of motion reduces to the Klein–Gordan equation $(\partial^2 + m^2)A_\mu = 0$ as expected.

**Note.** Basic dimensional analysis. We have

$$[\text{length}^{-1}] = [\text{time}^{-1}] = [\text{mass}] = 1.$$

Starting from the action $[S] = 0$, we conclude

$$[\mathcal{L}] = 4, \quad [d^4x] = -4, \quad [\partial_\mu] = 1, \quad [\phi] = [A_\mu] = 1.$$

However, spinor fields will have dimension $3/2$.

## 1.2 Canonical Quantization

We begin by quantizing the real scalar field introduced previously.

- Motivated by the canonical commutators in quantum mechanics, we promote $\phi$ and $\pi$ to operators satisfying
$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y})$$
with all other commutators equal to zero. Pointwise, these operators behave like $x$ and $p$ in quantum mechanics, but the position/momentum is in field space, not physical space.

- Classically, the real scalar field is analogous to an infinite set of coupled harmonic oscillators, which have independent normal modes. Thus we are motivated to Fourier transform. Now,
$$\int d\mathbf{x}\, e^{i\mathbf{p}\cdot\mathbf{x}} = (2\pi)^3\delta(\mathbf{p}), \quad \int d\mathbf{p}\, e^{i\mathbf{p}\cdot\mathbf{x}} = (2\pi)^3\delta(\mathbf{x})$$
which tells us that a net factor of $2\pi$ must be in the Fourier transform. We choose
$$\tilde{f}(\mathbf{p}) = \int d\mathbf{x}\, f(\mathbf{x})e^{-i\mathbf{p}\cdot\mathbf{x}}, \quad f(\mathbf{x}) = \int \text{đ}\mathbf{p}\, \tilde{f}(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}, \quad \text{đ}\mathbf{p} = \frac{d\mathbf{p}}{(2\pi)^3}$$
To save space we drop the tilde, identifying the function by its argument. We also write $\text{đ}(\mathbf{p}) = (2\pi)^3\delta(\mathbf{p})$ and $\text{đ}(p) = (2\pi)^4\delta(p)$.

- Expanding the Hamiltonian, we have
$$H = \frac{1}{2}\int d\mathbf{x}\, \left(\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2\right).$$
We may simplify these terms by plugging in the Fourier transform and using $\phi(\mathbf{p}) = \phi^\dagger(-\mathbf{p})$ and $\pi(\mathbf{p}) = \pi^\dagger(-\mathbf{p})$, which holds because $\phi(\mathbf{x})$ and $\pi(\mathbf{x})$ are self-adjoint, giving
$$H = \frac{1}{2}\int \text{đ}\mathbf{p}\, \left(|\pi(\mathbf{p})|^2 + \omega_p^2|\phi(\mathbf{p})|^2\right), \quad \omega_p = \sqrt{p^2 + m^2}$$
where for an operator $A$, $|A^2| = A^\dagger A$. This describes decoupled harmonic oscillators with frequency $\omega_p$.

- Now, for the ordinary harmonic oscillator, $H = (1/2)(p^2 + \omega^2 q^2)$, we define
$$q = \frac{a + a^\dagger}{\sqrt{2\omega}}, \quad p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger).$$
By analogy, here we define
$$a_\mathbf{p} = \frac{1}{2}\left(\sqrt{2\omega_p}\,\phi(\mathbf{p}) + i\sqrt{\frac{2}{\omega_p}}\,\pi(\mathbf{p})\right)$$
and find, for instance, that
$$\phi(\mathbf{p}) = \frac{a_\mathbf{p} + a^\dagger_{-\mathbf{p}}}{\sqrt{2\omega_p}}.$$
The calculations are a little more complicated because both $\mathbf{p}$ and $-\mathbf{p}$ terms are present.

- After working through the algebra, we find

$$\phi(\mathbf{x}) = \int \frac{đ\mathbf{p}}{\sqrt{2\omega_p}}(a_\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{x}} + a_\mathbf{p}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}), \quad \pi(\mathbf{x}) = \int đ\mathbf{p}\,(-i)\sqrt{\frac{\omega_p}{2}}(a_\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{x}} - a_\mathbf{p}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}).$$

  We'll use this form of the field operators almost exclusively.

- Some calculation gives the commutation relations

$$[a_\mathbf{p}, a_\mathbf{q}^\dagger] = đ(\mathbf{p} - \mathbf{q})$$

  with all other commutators equal to zero. Applying these commutation relations in a slightly long but straightforward calculation gives

$$H = \int đ\mathbf{p}\,\omega_p a_\mathbf{p}^\dagger a_\mathbf{p} + \frac{1}{2}\int d\mathbf{p}\,\omega_p \delta(\mathbf{0})$$

  with a large divergent counterterm.

**Note.** We find a divergent constant for two reasons. The $\delta(\mathbf{0})$ term is an IR divergence due to the infinite size of our space. Generally, IR divergences signal that we are asking an unphysical question. The integral of $\omega_\mathbf{p}$ also diverges in the UV because the theory must break down at some upper scale. For example, if we take the continuum limit of a lattice with spacing $a$, it should break down at energy $\Lambda \sim 1/a$. We will be able to perform calculations at $E \ll \Lambda$ using the machinery of regularization and renormalization.

As an explicit example, we can instead quantize the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - V_0$$

where $V_0$ is a counterterm, and regularize by introducing a cutoff $\Lambda$. We choose $V_0(\Lambda)$ so that the vacuum energy density is a finite constant for every choice of $\Lambda$, then take the limit $\Lambda \to \infty$ without issue. The price is that our theory now no longer predicts the value of the vacuum energy density. Generically, we would expect it to be some arbitrary number between zero and $M_P^4$ (the cutoff for quantum gravity), and to lie somewhere in the middle of the range. But we instead observe a vacuum energy density about $10^{122}$ times smaller than the maximum value. This is the "old fashioned" way to explain the cosmological constant problem.

As we'll see later, the reasoning here is not quite correct, but in the modern Wilsonian framework the problem remains. Furthermore, we cannot simply ignore the vacuum energy entirely, defining it to always be zero, because we can measure differences in it via the Casimir effect.

We now investigate the Hilbert space of our theory.

- In accordance with the remark above, we set the vacuum energy to zero. Another way to view the vacuum energy is to think of it as an artifact of our quantization procedure; the map from $H(q,p)$ to $\hat{H}(\hat{q},\hat{p})$ is not unique because $q$ and $p$ commute but $\hat{q}$ and $\hat{p}$ don't. We may choose to map $H(a,a^\dagger)$ to $\hat{H}(\hat{a},\hat{a}^\dagger)$ so that all the $\hat{a}$'s are to the right, giving zero vacuum energy. The resulting Hamiltonian is said to be normal ordered.

- The spatial momentum is

$$P^i = \int d\mathbf{x}\, \dot{\phi}(\mathbf{x})\partial^i\phi(\mathbf{x}) = \int đ\mathbf{p}\, p^i a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + \frac{1}{2}\int d\mathbf{p}\, p^i \delta(\mathbf{0})$$

  where the constant is zero by "symmetry". (This can also be justified more carefully by regulating the delta function in any way.) Thus in general we define the four-momentum

$$P^\mu = \int đ\mathbf{p}\, p^\mu a^\dagger_{\mathbf{p}} a_{\mathbf{p}}, \quad p^\mu = (\omega_p, \mathbf{p})$$

- The four-momentum operator satisfies the commutation relations

$$[P^\mu, a^\dagger_{\mathbf{p}}] = p^\mu a^\dagger_{\mathbf{p}}, \quad [P^\mu, a_{\mathbf{p}}] = -p^\mu a_{\mathbf{p}}.$$

  Since $P^\mu$ is self-adjoint, it has real eigenvalues. The commutation relations imply that $a^\dagger_{\mathbf{p}}$ raises the $P^\mu$ eigenvalue by $p^\mu$.

- Finally, we note that $H = P^0$ is nonnegative, so we cannot apply $a_{\mathbf{q}}$ indefinitely. In particular, there must exist a state satisfying

$$a_{\mathbf{q}}|0\rangle = 0$$

  for all $\mathbf{q}$. We postulate this state is unique and call it the vacuum state. By direct computation, it has zero four-momentum, $P^\mu|0\rangle = 0$.

- The state $a^\dagger_{\mathbf{p}}|0\rangle$ has the correct relativistic dispersion relation for a particle of mass $m$, so we interpret the state as containing one particle. Similarly, acting with $n$ creation operators gives an $n$-particle state. We can show that acting with only creation operators can't take us back to the vacuum. Thus, we have pinned down the structure of the Hilbert space using only the commutation relations and the postulated vacuum state.

- As we'll see later, the fact that the creation operators commute means that the particles obey Bose statistics. The spin-statistics theorem states that all scalar particles (in a four-dimensional relativistic field theory) must obey Bose statistics.

- We can directly promote all the quantities we found in the previous section (e.g. the stress-energy tensor and the six Lorentz currents) to operators. For example, we can show that the angular momentum of the one-particle state with zero momentum is zero, which shows that our particles have zero spin.

We now explicitly define the one-particle states with relativistic normalization.

- We choose to normalize the one-particle states as

$$|\mathbf{p}\rangle = \sqrt{2E_p}\, a^\dagger_{\mathbf{p}}|0\rangle$$

  and more generally multiply for $\sqrt{2E_p}$ for each creation operator $a^\dagger_{\mathbf{p}}$ for a multiparticle state. Applying the commutation relations gives

$$\langle\mathbf{q}|\mathbf{p}\rangle = (2E_p)đ(\mathbf{p}-\mathbf{q})$$

  which implies that the identity on the one-particle Hilbert space is

$$1_{\text{one-particle}} = \int \frac{đ\mathbf{p}}{2E_p}|\mathbf{p}\rangle\langle\mathbf{p}|.$$

- The reason we prefer this form is the appearance of the Lorentz invariant measure. Note that

$$\int đp\, \delta(p^2 - m^2)\Theta(p^0) = \int \frac{đ\mathbf{p}}{2E_p}.$$

  Since the left-hand side is Lorentz invariant, $đ\mathbf{p}/2E_p$ is as well.

- Similarly, $(2E_p)\delta(\mathbf{p} - \mathbf{q})$ is Lorentz-invariant, since it gives one when integrated against the Lorentz invariant measure. Thus the normalization of the one-particle states is Lorentz invariant; when a Lorentz transformation $\Lambda$ is represented as a unitary operator $U(\Lambda)$, it will satisfy $U(\Lambda)|\mathbf{p}\rangle = |\Lambda\mathbf{p}\rangle$, where $\Lambda\mathbf{p}$ is defined as the spatial part of $\Lambda(E_p, \mathbf{p})$.

- Finally, we define the states $|\mathbf{x}\rangle = \phi(\mathbf{x})|0\rangle$. Using the commutation relations gives

$$\langle\mathbf{p}|\mathbf{x}\rangle = e^{-i\mathbf{p}\cdot\mathbf{x}}$$

  which is similar to how momentum and position eigenstates are related in quantum mechanics. Then $|\mathbf{x}\rangle$ represents a particle localized near $\mathbf{x}$.

- There are no perfectly localized states in relativistic quantum field theory. Using the definition above, $\langle\mathbf{y}|\mathbf{x}\rangle$ is not a delta function, but rather has range on the order of the Compton wavelength $1/m$, which stems from the Lorentz invariant measure factor of $1/2E_p$. Alternatively, one could omit this factor, yielding perfectly localized "Newton–Wigner" states. However, these states don't transform nicely under Lorentz transformations, and in particular, they do not remain localized states in other frames.

- Note that the quantum field has a factor of $1/\sqrt{2E_p}$ rather than the Lorentz invariant $1/2E_p$. This is because the creation operators have to be multiplied by $\sqrt{2E_p}$ to give relativistically normalized states, i.e. the expression is really $(đ\mathbf{p}/2E_p)\sqrt{2E_p}\, a^\dagger$.

**Example.** The complex scalar field. We take

$$\mathcal{L} = (\partial_\mu\phi)^\dagger(\partial^\mu\phi) - m^2\phi^\dagger\phi$$

and treat $\phi$ and $\phi^\dagger$ as independent fields. The conjugate momenta of $\phi$ and $\phi^\dagger$ are $\pi = \dot{\phi}^\dagger$ and $\pi^\dagger = \dot{\phi}$ giving Hamiltonian

$$\mathcal{H} = \pi^\dagger\pi + \nabla\phi^\dagger\cdot\nabla\phi + m^2\phi^\dagger\phi.$$

We perform canonical quantization by demanding

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = [\phi^\dagger(\mathbf{x}), \pi^\dagger(\mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y})$$

with other commutators zero. As a result, the mode expansion is

$$\phi(\mathbf{x}) = \int \frac{đ\mathbf{p}}{\sqrt{2E_p}}(a_\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{x}} + b^\dagger_\mathbf{p} e^{-i\mathbf{p}\cdot\mathbf{x}})$$

with a conjugate expression for $\phi^\dagger$, and $\pi$ determined by $\pi = \dot{\phi}^\dagger$. The quickest way to derive this expression is to write $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$ and simply add the real field results together, identifying $a_\mathbf{p}$ and $b^\dagger_\mathbf{p}$ as the coefficients of the $e^{i\mathbf{p}\cdot\mathbf{x}}$ and $e^{-i\mathbf{p}\cdot\mathbf{x}}$ terms. We can also do this for $\pi$, but note that $\pi = (\pi_1 - i\pi_2)/\sqrt{2}$. Finally, this method gives the commutators

$$[a_\mathbf{p}, a^\dagger_\mathbf{q}] = [b_\mathbf{p}, b^\dagger_\mathbf{q}] = \delta(\mathbf{p} - \mathbf{q})$$

with all other commutators vanishing. Note that $\phi$ is now no longer Hermitian and therefore not observable; this is acceptable as we generally observe only field bilinears. Constructing the Hilbert space as before gives two independent types of particles. The momentum has mode expansion

$$P^\mu = \int đ\mathbf{p}\, p^\mu (a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + b_{\mathbf{p}}^\dagger b_{\mathbf{p}})$$

so that both particles have mass $m$. Finally, the Noether charge under the $U(1)$ symmetry is

$$Q = i\int d\mathbf{x}\,(\pi\phi - \phi^*\pi^*) = \int đ\mathbf{p}\,(b_{\mathbf{p}}^\dagger b_{\mathbf{p}} - a_{\mathbf{p}}^\dagger a_{\mathbf{p}})$$

where we implicitly normal ordered $Q$. Then the two types of particles carry opposite Noether charge. When we study QED, we will identify this charge with the electric charge.

**Example.** Now suppose we have two complex scalar fields $\phi_a$ of equal mass. Then we have a $U(2)$ symmetry, with the generators of the $SU(2)$ part being

$$Q^i = i\int d\mathbf{x}\,(\pi_a \sigma^i_{ab}\phi_b - \phi_a^*\sigma^i_{ab}\pi_b^*).$$

These generators satisfy the $\mathfrak{su}(2)$ commutation relations; in the case of the proton and neutron, or up and down quark, they generate isospin transformations.

## 1.3 Heisenberg Picture

To restore some of the Lorentz invariance, we switch to Heisenberg picture.

- Recall that in Heisenberg picture,

  $$A^H(t) = e^{iH^S t} A^S e^{-iH^S t}$$

  where we have assumed all Schrodinger operators are time-independent. In particular, the Hamiltonian satisfies $H^S = H^H$, and the time evolution is

  $$\frac{d}{dt}A^H(t) = i[H, A^H(t)].$$

- Switching to Heisenberg picture preserves relations between operators as long as all operators are evaluated at the same time. Then the commutators become equal time commutators,

  $$[q_i^H(t), p_j^H(t)] = i\delta_{ij}.$$

- For fields, we will drop the indices. A Schrodinger field will have argument $\mathbf{x}$, while a Heisenberg field will have argument $x = (t, \mathbf{x})$. The equal time commutators are

  $$[\phi(t,\mathbf{x}), \pi(t,\mathbf{y})] = i\delta(\mathbf{x}-\mathbf{y}), \quad [\phi(t,\mathbf{x}), \phi(t,\mathbf{y})] = [\pi(t,\mathbf{x}), \pi(t,\mathbf{y})] = 0.$$

- The Heisenberg equation of motion for the fields is

  $$\dot\phi(t,\mathbf{x}) = i[H, \phi(t,\mathbf{x})].$$

Now, $H$ doesn't evolve in time, so we choose to evaluate it at time $t$, where

$$\mathcal{H}(t,\mathbf{x}) = \frac{1}{2}\pi^2(t,\mathbf{x}) + \frac{1}{2}(\nabla\phi(t,\mathbf{x}))^2 + \frac{1}{2}m^2\phi(t,\mathbf{x})^2$$

so we can apply the equal time commutation relations. We thus find

$$\dot{\phi}(t,\mathbf{x}) = \pi(t,\mathbf{x}), \quad \dot{\pi}(t,\mathbf{x}) = (\nabla^2 - m^2)\phi(t,\mathbf{x})$$

which correspond to the classical Hamilton's equations. The first equation gives an easy way of recovering the mode expansion for $\pi$. The two together show the field $\phi$ obeys the Klein-Gordon equation.

- The Heisenberg equations also 'covariantize', i.e. we can show that

$$\partial^\mu\phi(x) = i[P^\mu, \phi(x)], \quad \phi(x+a) = e^{iaP}\phi(x)e^{-iaP}.$$

- Finally, we compute the mode expansion in Heisenberg picture; we choose to keep the creation and annihilation operators the same. We get factors like $e^{iHt}a_{\mathbf{p}}e^{-iHt}$ in the field expansion; to handle them note that

$$Ha_{\mathbf{p}} = a_{\mathbf{p}}(H - \omega_p), \quad f(H)a_{\mathbf{p}} = a_{\mathbf{p}}f(H - \omega_p).$$

Commuting the creation and annihilation operators in this way gives

$$\boxed{\phi(x) = \int \frac{đ\mathbf{p}}{\sqrt{2\omega_p}}(a_{\mathbf{p}}e^{-ipx} + a_{\mathbf{p}}^\dagger e^{ipx}).}$$

Note that $px$ stands for $p^\mu x_\mu$, with an apparent sign flip because of the $(+---)$ metric.

- The inverse expression is

$$a_{\mathbf{p}} = \frac{i}{\sqrt{2\omega_p}}\int d\mathbf{x}\, e^{iqx}\overleftrightarrow{\partial_0}\phi(x)$$

where $f\overleftrightarrow{\partial}g = f\partial g - g\partial f$ and the time in $\phi(x)$ is arbitrary.

Finally, we return to the question of causality.

- Using the Heisenberg picture, we can think of measurements as taking place at different times; conceptually $O^H(t)$ represents a measurement of $O$ at time $t$.
- To understand this, suppose that $O_1(t_1)$ and $O_2(t_2)$ commute. Then they have a common eigenbasis $|\psi_i\rangle$ where the $|\psi_i\rangle$ are nonevolving Heisenberg states. In Schrodinger picture,

$$|\psi_i(t_j)\rangle \text{ is an eigenvector of } O_j.$$

Thus, if $O_1$ is measured at $t_1$, there is no statistical effect on a measurement of $O_2$ at time $t_2$.

- Therefore, we say operators $O_1(x)$ and $O_2(y)$ on spacetime are local if

$$[O_1(x), O_2(y)] = 0 \text{ when } (x-y)^2 < 0.$$

This enforces causality: a measurement of $O_1$ cannot affect $O_2$ superluminally.

- Since the elementary observables are the fields $\phi$, we compute

$$\Delta(x-y) = [\phi(x), \phi(y)] = \int \frac{đ\mathbf{p}}{2\omega_p}(e^{-ip(x-y)} - e^{-ip(y-x)}).$$

  Now, the integration measure and integrand are both Lorentz invariant, so if the separation is spacelike we can switch to a frame where $x^0 = y^0$, giving factor $(e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})})$, which integrates to zero as desired.

- More generally, the same calculation goes through for commutators of $\partial_\mu\phi$, and all local operators $O(x)$ can be built out of $\phi(x)$ and $\partial_\mu\phi(x)$. Thus quantum fields maintain causality.

- For interacting fields, we cannot use the free mode expansion; the commutator turns out to be an operator rather than a number. Working casually, we could show it also vanishes at spacelike separation, but formally, we would take this as one of our axioms.

Next, we consider propagators and Green's functions.

- Define $|x\rangle = \phi^H(x)|0\rangle$. To interpret $|x\rangle$, switch back to Schrodinger picture for

$$|x\rangle = U(-t)\phi(\mathbf{x})U(t)|0\rangle = U(-t)\phi(\mathbf{x})|0\rangle = U(-t)|\mathbf{x}\rangle.$$

  Since $|\mathbf{x}\rangle$ is roughly a state with a particle at $\mathbf{x}$, $|x\rangle$ is a state that has a particle at $\mathbf{x}$ after evolution for a time $t$, so $|x\rangle$ has a particle at $x$. This particle exists for all times, both before and after $t$, but is only well-localized at time $t$.

- Therefore, we define the propagator

$$D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{đ\mathbf{p}}{2\omega_p} e^{-ip(x-y)}$$

  which describes the amplitude for a particle to go from $y$ to $x$.

- The commutator is $D(x-y) - D(y-x)$. While the propagator is nonzero for spacelike separations, the commutator is, because the amplitudes for propagation from $x$ to $y$ and $y$ to $x$ cancel out! Note the interplay between the field intuition for $\phi$ (in the commutator) and the particle intuition for $\phi$ (in the propagator).

- When we generalize this to the complex scalar field, we see that the amplitude for a particle to go from $x$ to $y$ is canceled by the amplitude for an antiparticle to go from $y$ to $x$. (For the real scalar field, the particles are their own antiparticles.) This is generally the reason that a multi-particle picture is necessary to preserve causality in relativistic quantum mechanics.

- We define the Feynman propagator

$$D_F(x-y) = \langle 0|T\phi(x)\phi(y)|0\rangle$$

  where time ordering means to order fields "later on left", so that earlier fields are applied to the state first. Then

$$\begin{aligned} D_F(x-y) &= \theta(x^0-y^0)D(x-y) + \theta(y^0-x^0)D(y-x) \\ &= \int \frac{đ\mathbf{p}}{2\omega_p} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \left(\theta(x^0-y^0)e^{-i\omega_p(x^0-y^0)} + \theta(y^0-x^0)e^{i\omega_p(x^0-y^0)}\right). \end{aligned}$$

- One can show that this is equal to the contour integral

$$\boxed{D_F(x-y) = \int \text{đ}p \, \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}}$$

  where the $p^0$ integral is along the real axis. The two step function cases come from how we close the contour, which affects which pole we pick up. To remember the sign, note that the $i\epsilon$ shifts the poles so we can Wick rotate to a contour that goes up the imaginary axis.

- Working in Fourier space makes the Feynman propagator easy to differentiate. We find

$$(\partial_x^2 + m^2) D_F(x-y) = -i\delta(x-y)$$

  so it is a Green's function for the Klein–Gordan equation. More generally, any of the four possible pole prescriptions gives an independent Green's function, including the retarded and advanced Green's functions.

- There are some inconsistencies in terminology here. In quantum mechanics, a propagator obeys the equations of motion, while Green's functions obey it up to a delta function driving. Thus a propagator is used to propagate a solution for a homogeneous equation forward in time, while a Green's function is used to solve an inhomogeneous equation.

- However, in quantum field theory, we use the term "propagator" to denote an amplitude for a particle to propagate from one point to another. Thus both $D$ and $D_F$ are called propagators, though only $D$ is a propagator in the quantum mechanical sense; $D_F$ is a Green's function.

- The retarded propagator propagates modes to the future, while the Feynman propagator propagates positive frequency modes forward in time and negative frequency modes backward in the time. The Feynman propagator is not often used in classical field theory, since it is not causal, and not even real, but plays an important role in perturbative quantum field theory, which centers around time-ordered correlation functions.

- The very rough intuition for the Feynman propagator is that after quantization, an excitation of a negative frequency mode going backwards in time corresponds to an antiparticle going forwards in time. So the Feynman propagator propagates both particles and antiparticles forwards in time.

**Note.** The classical complex scalar field has both positive and negative frequency modes. At the classical level, this isn't a problem, as both modes have positive energy. However, in the quantum theory, we run into trouble because of the relation $E = \hbar\omega$, which ties frequency to energy. If the field is regarded as a single-particle wavefunction, we find arbitrarily negative energy levels, which plagued early attempts at relativistic quantum mechanics.

This problem is avoided in quantum field theory because each mode is associated not with an eigenstate of the Hamiltonian, as in relativistic quantum particle mechanics, but with a set of creation and annihilation operators. For instance, for the complex scalar field, we associate a positive frequency mode with an annihilation operator but a negative frequency mode with a creation operator, in the mode expansion

$$\boxed{\phi(x) = \int \frac{\text{đ}\mathbf{p}}{\sqrt{2E_p}} \, (a_{\mathbf{p}} e^{-ipx} + b_{\mathbf{p}}^\dagger e^{ipx}).}$$

This is forced by canonical quantization. As a result, both $a_{\mathbf{p}}^\dagger$ and $b_{\mathbf{p}}^\dagger$ increase the energy, avoiding the negative energy levels.

In all cases, the quantization of a relativistic field yields pairs of positive and negative frequency modes. These both must appear in the mode expansion; if one is associated with a creation operator, the other must be associated with an annihilation operator. If there is an internal symmetry under which the field has a definite charge, this implies that the positive and negative frequency modes are associated with particles with opposite charges. This is how relativistic quantum field theory predicts the existence of antimatter.

**Note.** Keeping track of a tricky sign. In Heisenberg picture, we have $a^\dagger \sim e^{i\omega t}$ because $a^\dagger$ increases the energy, i.e. it makes a state behave like $e^{-i\omega t}$, but in Heisenberg picture $a^\dagger(t)$ makes a particle with no phase at time $t$, which means we need to run time backwards to $t = 0$ to see how it affects the Heisenberg state, flipping the sign. Since we take $a^\dagger$ to be time-independent in Heisenberg picture, the $e^{i\omega t}$ is absorbed into the exponent. Thus the coefficient of a positive frequency solution (i.e. one proportional to $e^{-i\omega t}$) is an annihilation operator. For the same reason, if the field carries a quantum number $Q$, the action of the field on states generally lowers $Q$. This minus sign will be mostly invisible in the discussion below, but will occasionally pop up to cause confusion.

## 1.4 Quantum Mechanics

Finally, we connect our results above to 'ordinary' quantum mechanics. We begin with describing attempts at nonrelativistic quantum mechanics.

- The Schrodinger equation has a conserved current

$$\rho = \psi^*\psi, \quad \mathbf{j} = -\frac{i}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*).$$

  If $\psi$ is interpreted as a particle wavefunction, then it implies probability is conserved. In particular, a Schrodinger equation for particles can't account for particle creation or annihilation.

- On the other hand, if $\psi$ is interpreted as a field, this conserved current comes from the $U(1)$ symmetry of the first-order Schrodinger Lagrangian

$$\mathcal{L} = i\psi^*\dot{\psi} - \frac{1}{2m}\nabla\psi^* \cdot \nabla\psi - V(x)\psi^*\psi.$$

  In the early days, this conceptual distinction was not clearly made. Note that here the momentum conjugate to $\psi$ is simply $i\psi^*$, so only $\psi$ is needed as an initial condition.

- Historically, Schrodinger first invented the Klein–Gordan equation, with conserved current

$$j^\mu = i(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*).$$

  The issue is that if $\phi$ is interpreted as a wavefunction, then the probability density $j^0$ is not positive definite. Moreover, there are negative energy states.

- In the context of the Klein–Gordan field theory, these problems are both fixed. There, $j^\mu$ is interpreted as a charge current, not a probability current. Moreover, negative energy states are avoided by the conceptual replacement of states by modes, as mentioned above.

- Dirac attempted to solve these problems by constructing an equation that was first order in time, giving the Dirac equation. The conserved current

$$j^\mu = \overline{\psi}\gamma^\mu\psi$$

  has positive definite density $j^0 = \psi^\dagger\psi$. There remain negative energy solutions, which Dirac resolved by postulating that all negative energy states were already occupied, by a "Dirac sea".

- Despite these odd features, the Dirac equation was extremely successful. For instance, it automatically accounted for all fine structure corrections, as well as the $g$-factor of the electron, which is determined by minimal coupling rather than put in by hand. We further describe these successes in the notes on Undergraduate Physics.

**Note.** Some further discussion of the Dirac sea. In the years following the Dirac equation, interpreting the negative energy solutions was a major issue. One obvious fix is to just dismiss them as unphysical, but this was unsuitable for several reasons. First, the negative energy subspaces for free electrons and, e.g. the electron in hydrogen were different, so the prescription was not well-defined to begin with. Second, the positive energy states did not form a complete set, and if one does not include the negative energy states as intermediate states in perturbation theory, one gets the wrong results. And finally, the Zitterbewegung caused by interference between positive and negative energy states was necessary to produce the Darwin term, which contributed to the experimentally observed fine structure.

Dirac solved all of these problems at once by postulating the Dirac sea, interpreting positrons as holes in the Dirac sea. It's worth seeing how this is resolved within quantum field theory; it is different from the resolution we saw earlier, which works for bosonic fields. For fermionic fields, the modes are described by fermionic quantum harmonic oscillators, obeying $\{a, a^\dagger\} = 1$. However, the occupied and unoccupied states are completely symmetric: one can just as easily define $b = a^\dagger$ and $b^\dagger = a$. Hence in quantum field theory, we simply *define* particles in terms of the excitations about the lowest-energy state, which is defined to be the vacuum. So by fiat, the vacuum has no particles in it, and is energetically stable. The problematic negative energy modes of the Dirac equation are reinterpreted as negative frequency modes of the Dirac field, for which we need to perform the swap above. (This idea does not work for bosonic fields, as exchanging creation and annihilation operators would yield the wrong commutation relation, $[a, a^\dagger] = -1$. If one insisted on interpreting this as a set of creation and annihilation operators, then $a^\dagger$ would create states with negative norm, leading right back to the negative probabilities fixed by passing to quantum field theory.)

In other words, in the Dirac equation we postulate the unwanted states are already occupied, while in the Dirac field, we define the notion of "occupation" so that in the lowest energy state, nothing is occupied. This might look like mathematical slight of hand. Indeed, in spirit the two are just the same core idea described with different words. For example, one problem with the Dirac sea is the resulting infinite charge density, which was thought to cancel out an infinite "bare" charge of empty space. In quantum field theory, the exact same problem appears, because the naive charge operator diverges when evaluated on the vacuum state. We remove this infinity by normal ordering, which is essentially just subtracting out the contribution we don't want. As another example, the Dirac sea also provides an infinite mass density, which produces unwanted gravity. The exact same problem appears in quantum field theory, where we must tune the cosmological constant to cancel such contributions.

However, the Dirac field has legitimate conceptual advantages over the Dirac sea, especially in situations where particles are created or destroyed. For example, one might be led by the Dirac sea

to think that electron number must be conserved, though it isn't in weak processes. (One might think that beta decay, observed by the time of the Dirac sea, should have made the idea obviously wrong. But at this time the neutron was not known: nuclei were thought to be made of protons and "nuclear electrons", so beta decay *did* conserve electron number.) It is also puzzling how the Dirac sea is formed in the first place, since one needs just enough electrons to fill the sea and no more; it seems the Dirac sea breaks the symmetry between matter and antimatter. But this problem lives on in field theory, as the problem of baryogenesis.

Next, we recover the Schrodinger equation from quantum field theory.

- In order to revert to 'ordinary' quantum mechanics described by a Schrodinger equation, we must take the nonrelativistic limit to avoid particle creation. We also need a conserved current which will eventually serve as the probability current, which means the simplest option of a real scalar field doesn't work; we instead choose a complex scalar field.

- Taking the Klein–Gordan equation $(\partial^2 + m^2)\phi = 0$ and setting $\phi = e^{-imt}\chi/\sqrt{2m}$ yields

$$i\dot{\chi} = \frac{\ddot{\chi} - \nabla^2\chi}{2m}.$$

  Noting that $\ddot{\chi}/m \ll \dot{\chi}$, since the field is nonrelativistic, we recover the Schrodinger equation. Similarly, we can go from the Klein–Gordan Lagrangian to the Schrodinger Lagrangian,

$$\mathcal{L} = i\chi^*\dot{\chi} - \frac{1}{2m}\nabla\chi^* \cdot \nabla\chi.$$

  Note that the current $j^\mu$ reduces to the nonrelativistic expressions for $\rho$ and $\mathbf{j}$, where the factor of $\sqrt{2m}$ converts between relativistic and nonrelativistic normalization. The reason $\rho$ and $\mathbf{j}$ look superficially different is that $\phi$ and $\chi$ are related by only a time-varying phase; as usual the deeper, relativistic theory "makes more sense", uniting the two expressions.

- However, we are not done, because $\chi$ is a classical field, not a quantum wavefunction. To put it another way, the position remains a parameter, rather than an operator. However, we do have a chance of recovering quantum mechanics if we quantize this field.

- Turning the crank of canonical quantization, we have

$$\pi = i\chi^*, \quad \mathcal{H} = \frac{1}{2m}\nabla\chi^* \cdot \nabla\chi, \quad [\chi(\mathbf{x}), \chi^\dagger(\mathbf{y})] = \delta(\mathbf{x} - \mathbf{y}).$$

  Unlike the usual canonical commutation relations, these can be solved by

$$\chi(\mathbf{x}) = \int \text{đ}\mathbf{p}\, a_\mathbf{p} e^{ipx}, \quad [a_\mathbf{p}, a_\mathbf{q}^\dagger] = \text{đ}(\mathbf{p} - \mathbf{q}).$$

  Hence antiparticles do not appear in the nonrelativistic limit, because the field equation is not relativistic. Of course one could include them, but the formalism does not require them.

- The key structural difference is the following: the Lagrangian must be second order in space derivatives to be a scalar (or, upon integrating by parts, quadratic in first spatial derivatives). Lorentz invariance requires the Lagrangian to be second order in time, while without it the Lagrangian may be first order in time, giving the simpler commutation relations above.

- At this point, we have the so-called Schrodinger field theory. This formalism is useful in many-body problems in atomic, nuclear, and condensed matter physics, even in situations where the particle number is fixed, because it is second quantized and hence automatically accounts for the symmetrization postulate.

- To describe fermions, we could instead quantize using the canonical anticommutation relations, as we will for the Dirac field below. We can quantize either way, as statistics is only determined by spin within relativistic field theory.

- The $U(1)$ symmetry conserves the number of particles. To recover the Schrodinger equation, we must define position and momentum operators. As before, we must have

$$H = \int đ\mathbf{p}\, \frac{p^2}{2m} a_{\mathbf{p}}^\dagger a_{\mathbf{p}}, \quad \mathbf{P} = \int đ\mathbf{p}\, \mathbf{p}\, a_{\mathbf{p}}^\dagger a_{\mathbf{p}}$$

  which are simply the usual definitions in nonrelativistic quantum field theory.

- We focus on single particle states, defining $|\mathbf{x}\rangle = \chi^\dagger(\mathbf{x})|0\rangle$ and

$$\mathbf{X} = \int d\mathbf{x}\, \mathbf{x}\, \chi^\dagger(\mathbf{x})\chi(\mathbf{x}), \quad \mathbf{X}|\mathbf{x}\rangle = \mathbf{x}|\mathbf{x}\rangle$$

  where the latter property is just as expected for a position operator. Note that this step doesn't work in the relativistic case, because the analogues of the $|\mathbf{x}\rangle$ states are not perfectly localized.

- Define a wavefunction by

$$|\psi\rangle = \int d\mathbf{x}\, \psi(\mathbf{x})|x\rangle.$$

  Then it is straightforward to verify that

$$\mathbf{X}|\psi\rangle = \int d\mathbf{x}\, \mathbf{x}\psi|\mathbf{x}\rangle, \quad \mathbf{P}|\psi\rangle = \int d\mathbf{x}\, (-i\nabla\psi)|\mathbf{x}\rangle, \quad H|\psi\rangle = \int d\mathbf{x}\, \left(-\frac{1}{2m}\nabla^2\psi\right)|\mathbf{x}\rangle.$$

  Hence, by working with the wavefunctions directly, the canonical commutation relations are satisfied and the wavefunction obeys the Schrodinger equation.

- A similar procedure works for the Dirac equation. We now can use either commutation relations or anticommutation relations to quantize the field; choosing the latter, we preserve the commutation relation between $\mathbf{X}$ and $\mathbf{P}$ because these operators are bilinear in the fields. However, returning from the Dirac field to the Dirac equation for a single particle is a bit more subtle, because of the issues of interpreting negative energy solutions. **(add more detail)**

- Note that the equation satisfied by the wavefunction of a single particle is the same as the equation satisfied by the field as a whole! This was responsible for much confusion in the early days of quantum mechanics, where it was thought that field theory resulted from quantizing the wavefunction itself. That was a reasonable idea, since in those days it was also unclear how the wavefunction was to be interpreted. Now we know that *neither* single particle wavefunctions or classical fields come first; they both emerge from quantum fields, and this common origin is the reason for their similarity.

# 2 Interacting Scalar Fields

## 2.1 Spectral Representation

We now carefully treat interacting scalar fields.

- We consider an interacting Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}, \quad \mathcal{L}_{\text{int}} = -V_{\text{int}}$$

  where $V_{\text{int}}$ is a power series in $\phi$ starting at cubic order.

- We still have Poincare symmetry, i.e. we still have unitary operators realizing these symmetries. Thus we can still talk, for example, about the momentum of a state. We also still have the equal-time commutation relations.

- We assume the full Hamiltonian has a unique vacuum $|\Omega\rangle$ with zero four-momentum,

$$P^\mu|\Omega\rangle = 0.$$

  Hence the vacuum is translation invariant, $e^{ixP}|\Omega\rangle = |\Omega\rangle$.

- Since we still have translation invariance, the Hamiltonian and 3-momentum commute, so we can choose a basis of states with definite four-momentum,

$$H|\lambda_{\mathbf{p}}\rangle = E_p(\lambda)|\lambda_{\mathbf{p}}\rangle, \quad \mathbf{P}|\lambda_{\mathbf{p}}\rangle = \mathbf{p}|\lambda_{\mathbf{p}}\rangle.$$

  One can show that, by applying a Lorentz boost, we get a family of states $|\lambda_{\mathbf{p}}\rangle$ for every $\lambda$, whose energies are $E_p = (\mathbf{p}^2 + m_\lambda^2)^{1/2}$.

- The index $\lambda$ labels irreps of the Poincare group in the interacting Hilbert space. In a weakly interacting theory, the sum includes "dressed" versions of the particles in the free theory, bound states of those particles, and unbound multiparticle states. In the latter case, $\lambda$ has a continuum of values, as necessary to specify the relative motion of the particles.

- In a strongly interacting theory, there is no sharp distinction between dressed particles and bound states. The only meaningful distinction we can make is between multiparticle states and others, because the continuous values of $\lambda$ for multiparticle states will have analytic consequences.

- Since the equation of motion is now nonlinear, the mode decomposition of $\phi$ is no longer useful. Even at the classical level, such a decomposition doesn't work because the interaction mixes the modes. At the quantum level, we could formally expand $\phi(x)$ in terms of ladder operators as in the free field case, but as we'll shortly see, these operators wouldn't have a simple interpretation in terms of creating and annihilating particles.

- The completeness relation, with relativistic normalization, is

$$1 = |\Omega\rangle\langle\Omega| + \sum_\lambda \int \frac{\bar{d}\mathbf{p}}{2E_p(\lambda)} |\lambda_{\mathbf{p}}\rangle\langle\lambda_{\mathbf{p}}|.$$

- Translation operators and Lorentz boost act on the field as

$$\phi(x) = e^{ixP}\phi(0)e^{-ixP}, \quad \phi(x) = U^{-1}(\Lambda)\phi(x')U(\Lambda)$$

  where $U(\Lambda)$ carries out the Lorentz transformation $x' = \Lambda x$. To check the signs, we can take matrix elements on both sides and act on the states instead.

With this setup, we now compute the Feynman propagator.

- Consider $\langle\Omega|\phi(x)|\Omega\rangle$. Using the translational invariance of the vacuum, we find that $\langle\Omega|\phi(x)|\Omega\rangle = \langle\Omega|\phi(0)|\Omega\rangle$ for all $x$. We can set the constant to zero by shifting the field, as this preserves the canonical commutators.

- Therefore, inserting the identity in a correlation function,

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \sum_\lambda \int \frac{đ\mathbf{p}}{2E_p(\lambda)} \langle\Omega|\phi(x)|\lambda_\mathbf{p}\rangle\langle\lambda_\mathbf{p}|\phi(y)|\Omega\rangle$$

  where the vacuum term vanishes. Using the fact that the $|\lambda_\mathbf{p}\rangle$ have definite momentum,

$$\langle\Omega|\phi(x)|\lambda_\mathbf{p}\rangle = \langle\Omega|\phi(0)|\lambda_\mathbf{p}\rangle e^{-ipx}.$$

  Next, applying a Lorentz boost such that $|\lambda_\mathbf{p}\rangle = U^{-1}|\lambda_0\rangle$,

$$\langle\Omega|\phi(0)|\lambda_\mathbf{p}\rangle = \langle\Omega|U^{-1}U\phi(0)U^{-1}U|\lambda_\mathbf{p}\rangle e^{-ipx} = \langle\Omega|\phi(0)|\lambda_0\rangle$$

  where we used the Lorentz invariance of the zero vector and the vacuum.

- Putting these results together, we have

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \sum_\lambda \int \frac{đ\mathbf{p}}{2E_p(\lambda)} e^{-ip(x-y)} |\langle\Omega|\phi(0)|\lambda_0\rangle|^2.$$

  Doing the exact same manipulations for the time ordered correlator gives

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \sum_\lambda \int đp \, \frac{i}{p^2 - m_\lambda^2 + i\epsilon} e^{-ip(x-y)} |\langle\Omega|\phi(0)|\lambda_0\rangle|^2.$$

- It is useful to parametrize this result by mass,

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \int_0^\infty đ(M^2)\, \rho(M^2) D_F^0(x-y, M^2)$$

  where

$$D_F^0(x-y, M^2) = \int đp \, \frac{i}{p^2 - M^2 + i\epsilon} e^{-ip(x-y)}, \quad \rho(M^2) = \sum_\lambda đ(M^2 - m_\lambda^2) |\langle\Omega|\phi(0)|\lambda_0\rangle|^2.$$

  The quantity $\rho(M^2)$ is called the spectral function, and by its definition, it is real and positive.

- Typically the spectral function will have poles for one-particle states and bound states. Multiparticle states have continuously "smeared out" poles, i.e. branch cuts. Separating out the one-particle contribution,

$$\rho(M^2) = \not{\delta}(M^2 - m^2)Z + \text{higher}, \quad Z = |\langle\Omega|\phi(0)|1_0\rangle|^2$$

so that

$$\boxed{\int dx\, e^{ip(x-y)}\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \frac{iZ}{p^2 - m^2 + i\epsilon} + \int_{m^2_{\text{bound}}}^{\infty} \not{d}(M^2)\,\rho(M^2)\frac{i}{p^2 - M^2 + i\epsilon}.}$$

The quantity $Z$ is called the wavefunction renormalization, where $Z = 1$ when the theory is free. We see that in an interacting theory, $\phi(x)$ creates not only a one-particle state, but a superposition of many kinds of states. However, as long as $Z \neq 0$, it has at least some amplitude to create the single-particle states that we need to calculate scattering amplitudes.

- Note that this derivation was very general, and never required an expansion into creation and annihilation operators; it works for any field operator, e.g. also for composite operators. However, one can show that for bare fields (i.e. fields that satisfy $[\phi_0(t,\mathbf{x}), \partial_t\phi_0(t,\mathbf{y})] = i\delta(\mathbf{x}-\mathbf{y})$), the spectral function has unit integral, and hence $Z \leq 1$.

- There is also a similar spectral representation for the Dirac field,

$$\int dx\, e^{ip(x-y)}\langle\Omega|T\psi(x)\overline{\psi}(y)|\Omega\rangle = \frac{iZ_2\sum_s u^s(p)\overline{u}^s(p)}{p^2 - m^2 + i\epsilon} + \ldots, \quad \sqrt{Z_2}u^s(p) = \langle\Omega|\psi(0)|p,s\rangle.$$

To derive it, we also need to keep track of how the Lorentz boosts above affect the field.

- Interactions could also render one-particle states unstable, e.g. by allowing them to decay to lighter particles. Such states are not energy eigenstates at all. They instead appear in the Fourier transform of the two-point function as poles off the real axis, as we'll discuss later. Another possible subtlety is that in a theory with massless particles, the branch cut extends down to $M^2 = 0$.

**Note.** A cheap RG analysis. Consider interactions of the form $\lambda_n\phi^n$. Since $[\mathcal{L}] = 4$ and $[\phi] = 1$, we have $[\lambda_n] = 4 - n$. We consider a few examples.

- For the cubic interaction, $[\lambda_3] = 1$, so the effect of the interaction is described by the dimensionless parameter $\lambda_3/E$, where $E$ is the energy scale of the process. (This is clearest in the path integral picture, where $\lambda_3/E$ estimates the action contribution.) Then the interaction is strong at low energies, so it is called relevant.

- For the quartic interaction, $[\lambda_4] = 0$, and we say it is marginal.

- For the quintic interaction, $[\lambda_5] = -1$, so its effect is described by $\lambda_5 E$, which is weak at low energies, so it is called irrelevant.

- Now suppose we have a fundamental theory defined at some high scale $\Lambda$. Then the dimensionless coupling constant for $\phi^5$ would be $g_5 = \Lambda\lambda_5$, where by naturalness we expect $g_5 = O(1)$. Then the effect of the interaction at scale $E$ is $g_5(E/\Lambda)$.

- If $\Lambda$ is taken to be the GUT scale or the Planck scale, then $E/\Lambda$ is extremely small, so it is an excellent approximation to ignore irrelevant terms.
- But there is a subtlety: consider the dimension 2 mass term. If the above argument worked, then we would expect it to scale as $\Lambda^2\phi^2$, which implies a mass of order $\Lambda$. But clearly, there exist plenty of particles with masses well below the Planck scale. This violation of naive dimensional analysis can result from approximate symmetries, tuning, or nontrivial dynamics.

Note that this analysis depends on the spatial dimension. In six dimensions, $[\mathcal{L}] = 6$, $[\phi] = 2$, and it is $\lambda_6$ that is marginal.

## 2.2 The LSZ Reduction Formula

We now carefully define scattering amplitudes in terms of "in" and "out" states.

- We are still working in Heisenberg picture, so all states are "fixed in time". Note that a better way of thinking about Heisenberg picture is that all the states "extend through time", so they may be described by how they look at any given time.
- An in state $|i, \text{in}\rangle$ is a state which looks like several widely separated, incoming particles as $t \to -\infty$. Similarly, an out state $|f, \text{out}\rangle$ is a state which has widely separated, outgoing particles as $t \to \infty$. Here, $i$ and $f$ are shorthand for a specification like "two particles, of momenta $p^\mu$ and $q^\mu$". The asymptotic vacua are $|\text{vac}, \text{in}\rangle$ and $|\text{vac}, \text{out}\rangle$.
- The $S$ matrix maps out states to in states,

$$|i, \text{in}\rangle = S|i, \text{out}\rangle.$$

  In most reasonable physical theories, we have

$$|\text{vac, in}\rangle = |\text{vac, out}\rangle = |\Omega\rangle, \quad S|\Omega\rangle = |\Omega\rangle, \quad S^\dagger = S^{-1}.$$

  We are interested in calculating the transition amplitudes

$$\langle f, \text{out}|i, \text{in}\rangle = \langle f, \text{in}|S|i, \text{in}\rangle.$$

  Hence these transition amplitudes are also called $S$-matrix elements. Note that all of these states live in the full interacting Hilbert space.

- To formally construct the in states, we define an "in field" $\phi_{\text{in}}$ with the following properties.
  - The in field is a free Klein–Gordan field $\phi(x)$ which creates in states from $|\text{vac, in}\rangle$ as $t \to -\infty$. Hence we have the mode expansion

$$\phi_{\text{in}}(x) = \int \frac{đ\mathbf{p}}{\sqrt{2\omega_p}} (a_{\text{in},\mathbf{p}} e^{-ipx} + a^\dagger_{\text{in},\mathbf{p}} e^{ipx}).$$

  - Since the in states have mass $m$, where $m$ is the single-particle mass in the full theory, the in field must have mass $m$.

– The in field approaches the interacting field as $t \to -\infty$ in the sense that

$$\langle\alpha|\phi(x)|\beta\rangle \to \sqrt{Z}\langle\alpha|\phi_{\text{in}}(x)|\beta\rangle$$

for all states as $t \to -\infty$. The constant $\sqrt{Z}$ is determined by setting $|\beta\rangle = |\Omega\rangle$ and $|\alpha\rangle = |1_{\mathbf{p}}\rangle$ and using the spectral form.

- The out field $\phi_{\text{out}}$ has analogous properties as $t \to \infty$. The two are related by

$$\phi_{\text{in}}(x) = S\phi_{\text{out}}(x)S^{-1}.$$

We now apply our setup to derive the LSZ reduction formula.

- For concreteness, consider the $S$-matrix element $\langle p_1, p_2, \text{out}|q_1, q_2, \text{in}\rangle$. We have

$$\begin{aligned}\langle p_1, p_2, \text{out}|q_1, q_2, \text{in}\rangle &= \sqrt{2\omega_{q_1}}\langle p_1, p_2, \text{out}|a^\dagger_{\text{in},\mathbf{q}_1}|q_2, \text{in}\rangle \\ &= \frac{1}{i}\int d\mathbf{x}\, e^{-iq_1x}\overleftrightarrow{\partial_0}\langle p_1, p_2, \text{out}|\phi_{\text{in}}(x)|q_2, \text{in}\rangle\big|_{x^0=t} \\ &= \lim_{t\to-\infty} Z^{-1/2}\frac{1}{i}\int d\mathbf{x}\, e^{-iq_1x}\overleftrightarrow{\partial_0}\langle p_1, p_2, \text{out}|\phi(x)|q_2, \text{in}\rangle\end{aligned}$$

where we used the expression for $a^\dagger_{\text{in}}$ in terms of $\phi_{\text{in}}$, which holds for arbitrary $t$, then took $t \to -\infty$ to match with the interacting field.

- The next step is to act with $\phi(x)$ on the left to get rid of one of the out particles. This is only possible if we have an out field, which requires a limit $t \to +\infty$, so we use

$$\lim_{t\to-\infty} f(t) = \lim_{t\to+\infty} f(t) - \int dt\, \partial_0 f(t).$$

The first term will give us terms like $\langle p_2, \text{out}|q_2, \text{in}\rangle\delta(\mathbf{p}_1 - \mathbf{q}_1)$ with another term with $p_1$ and $p_2$ swapped. These represent "disconnected" contributions to the $S$-matrix, i.e. one where the first particle doesn't scatter at all. We don't care about them here because it can be recursively computed from $S$-matrix elements with fewer particles.

- The connected term yields an integral over spacetime, and directly carrying out the derivatives and integrating by parts yields

$$iZ^{-1/2}\int dx_1\, e^{-iq_1x_1}(\Box_1 + m^2)\langle p_1, p_2, \text{out}|\phi(x_1)|q_2, \text{int}\rangle.$$

There is a sticky point here, which is that the integration by parts is not legal, because $e^{-iq_1x}$ doesn't go to zero at infinity. This occurs because momentum eigenstates unphysically cover all of space. To define our in and our states properly, we should have constructed finite wavepackets and worked with them from the start, but we sweep this under the rug to avoid the complication.

- Next, we would like to repeat this process for all of the other incoming and outgoing particles. The only snag is that when we flip $t \to -\infty$ to $t \to +\infty$ with the identity above, the new fields may be in the "wrong place" and can't be commuted past the others. The fix is to replace the correlation functions with time-ordered correlation functions, which automatically put the in and out fields on the right side.

- Carrying out this procedure and generalizing to $n+r$ particles,

$$\langle p_1,\dots,p_n,\text{out}|q_1,\dots,q_4,\text{in}\rangle = \sum \text{disconnected}$$
$$+(iZ^{-1/2})^{n+r}\int dy_1\dots dx_1\dots \prod_k e^{ip_k y_k}(\Box_{y_k}+m^2)\prod_\ell e^{-iq_\ell x_\ell}(\Box_{x_\ell}+m^2)\langle\Omega|T\phi(y_1)\dots\phi(x_1)\dots|\Omega\rangle.$$

  This is the LSZ reduction formula, relating $S$-matrix elements to time ordered correlation functions. Note that all momenta here are on-shell, since they are the momenta of asymptotic particles, so we can't treat the scattering of unstable particles.

- In momentum space, the connected part becomes

$$(iZ^{-1/2})^{n+r}\prod_k(-p_k^2+m^2)\prod_\ell(-q_\ell^2+m^2)\langle\Omega|T\phi(p_1)\dots\phi(q_1)\dots|\Omega\rangle.$$

  For the $S$-matrix element to be nonsingular, the time-ordered correlators must have a corresponding pole structure, with a Feynman propagator factor giving poles for on-shell particles.

- Another useful rewriting is

$$\prod_k\int dy_k\, e^{ip_k y_k}\prod_\ell\int dx_\ell\, e^{-iq_l x_l}\,\langle\Omega|T\phi(y_1)\dots\phi(x_1)\dots|\Omega\rangle$$
$$=\prod_k\frac{i\sqrt{Z}}{p_k^2-m^2}\prod_\ell\frac{i\sqrt{Z}}{q_\ell^2-m^2}\langle p_1,\dots|S|q_1,\dots\rangle\Big|_{\text{connected}}.$$

  Our goal now is to understand how to compute time-ordered correlators.

**Note.** In summary, in free field theory, the fields always create or destroy a particle near a point. Then position-space correlation functions describe amplitudes for particle propagation between those points, and LSZ states that momentum-space correlation functions are essentially equal to $S$-matrix elements after removing the poles.

In an interacting field theory, $\phi(x)$ still creates some kind of excitation near the point $x$ (as can be seen by its transformation properties) but it is generally a complicated combination of single-particle and multiparticle states, as shown in the spectral representation. The point of the LSZ reduction formula is that we can project out the part we want (i.e. above, single-particle states of mass $m$) by looking at the residues of the appropriate poles, up to a $\sqrt{Z}$ correction factor.

More generally, the LSZ reduction theorem means that any field can be used to compute $S$-matrix elements of any kind of particle, as long as that field has nonzero overlap with the particle. For example, we can compute $S$-matrix elements using the canonical momentum $\pi(x)$, because it has some amplitude to make single-particle states. For a complex scalar field, we can compute $S$-matrix elements for bound particle-antiparticle pairs using the field $\psi^\dagger(x)\psi(x)$.

## 2.3 Time-Ordered Correlators

We compute time-ordered correlators in the interaction picture.

- In the interaction picture, we split

$$H = H_0 + H_{\text{int}}$$

and evolve the states with $H_{\text{int}}$ and the operators with $H_0$. In particular, this means the interaction picture field

$$\phi_I(x) = e^{iH_0 t}\phi(0,\mathbf{x})e^{-iH_0 t}$$

act exactly like the free Heisenberg fields which we introduced earlier. They satisfy the Klein–Gordan equation with the bare mass $m_0$, not the physical mass $m$. Moreover, using the free mode expansion to define $a_{I,\mathbf{p}}$, the interacting annihilation operator annihilates the free vacuum $|0\rangle$, not the interacting vacuum $|\Omega\rangle$.

- At general times, the Heisenberg picture operators are related by

$$\phi(t,\mathbf{x}) = U^\dagger(t)\phi_I(t,\mathbf{x})U(t), \quad U(t) = e^{iH_0 t}e^{-iHt}.$$

Taking matrix elements of both sides shows that states evolve in interaction picture via $U(t)$.

- Differentiating, $U(t)$ obeys the equation

$$i\frac{\partial}{\partial t}U(t) = H_I(t)U(t), \quad H_I(t) = e^{iH_0 t}H_{\text{int}}e^{-iH_0 t}$$

where $H_I(t)$ is the interacting Hamiltonian in the interaction picture. The solution is

$$\begin{aligned} U(t) &= T\exp\left(-i\int_0^t dt'\, H_I(t')\right) \\ &= 1 - i\int_0^t dt_1\, H_I(t_1) + (-i)^2\int_0^t\int_0^t dt_1\, dt_2\, T H_I(t_1)H_I(t_2) + \dots \end{aligned}$$

which is called Dyson's equation. In general, we have

$$\boxed{U(t,t_0) = T\exp\left(-i\int_{t_0}^t dt'\, H_I(t')\right)}$$

and the time ordering ensures $U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3)$.

- Next, we relate the interacting and free vacuum. Let $H|n\rangle = E_n|n\rangle$. Then

$$e^{-iHt}|0\rangle = \sum_n e^{iE_n t}|n\rangle\langle n|0\rangle$$

by inserting the energy basis. The vacuum state will have some nonzero energy $E_\Omega$ (as we already used up our vacuum energy counterterm to set $E_0 = 0$), but all other states will have higher energy. Thus, sending time to infinity with a small damping,

$$|\Omega\rangle = \lim_{t\to\infty(1-i\epsilon)} \frac{e^{-iHt}|0\rangle}{e^{-iE_\Omega t}\langle\Omega|0\rangle}.$$

Note that we must assume $\langle\Omega|0\rangle$ is nonzero, which should be true if $H_{\text{int}}$ is 'weak'. This trick also fails if the theory has massless particles, which yield states of arbitrarily low energy.

- Next, note that

$$e^{-iHt}|0\rangle = e^{-iHt}e^{iH_0 t}|0\rangle = U(-t)^\dagger|0\rangle = U(t)|0\rangle = U(0,-t)|0\rangle$$

where we used $H_0|0\rangle = 0$. Plugging this into a correlation function,

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \lim_{t\to\infty(1-i\epsilon)}(|\langle 0|\Omega\rangle|^2 e^{-2iE_\Omega t})^{-1} \times \langle 0|U(t,0)\,U(x^0,0)^\dagger\phi_I(x)U(x^0,0)\,U(y^0,0)^\dagger\phi_I(y)U(y^0,0)\,U(0,-t)|0\rangle$$

where we simply switched to interaction picture.

- Using the multiplication rule for $U(t,t')$, we have

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \lim_{t\to\infty(1-i\epsilon)}(|\langle 0|\Omega\rangle|^2 e^{-2iE_\Omega t})^{-1} \times \langle 0|U(t,x^0)\phi_I(x)U(x^0,y^0)\phi_I(y)U(y^0,-t)|0\rangle.$$

We see that as long as the fields were time ordered, everything inside the expectation value is automatically timed ordered!

- To deal with the prefactor, note that

$$1 = \langle\Omega|\Omega\rangle = \lim_{t\to\infty(1-i\epsilon)} \frac{\langle 0|U(t,-t)|0\rangle}{|\langle 0|\Omega\rangle|^2 e^{-2iE_\Omega t}}.$$

Combining these results together and generalizing to $n$ fields, we find

$$\boxed{\langle\Omega|T\phi(x_1)\dots\phi(x_n)|\Omega\rangle = \lim_{t\to\infty(1-i\epsilon)} \frac{\langle 0|T\phi_I(x_1)\dots\phi_I(x_n)U(t,-t)|0\rangle}{\langle 0|U(t,-t)|0\rangle}.}$$

We use Wick's theorem to compute these interaction picture correlators. Since we'll be working exclusively with interaction picture fields from now on, we drop the subscript.

- We decompose the interaction picture field into positive and negative free modes,

$$\phi(x) = \phi^+(x) + \phi^-(x) = \int \frac{đ\mathbf{p}}{\sqrt{2\omega_p}} a_\mathbf{p} e^{-ipx} + \int \frac{đ\mathbf{p}}{\sqrt{2\omega_p}} a_\mathbf{p}^\dagger e^{ipx}.$$

The plus and minus fields are easier to deal with because $\phi^+(x)|0\rangle = \langle 0|\phi^-(x) = 0$.

- Given an operator $O$ define as a string of creation and annihilation operators, its normal ordering $:O:$ is the same string but with all the annihilation operators moved to the right. As long as $O$ does not have a c-number piece $cI$,

$$\langle 0|\ :O:\ |0\rangle = 0.$$

- Note that normal ordering is *not* a map on operators, but a map on strings of creation and annihilation operators. For example, $aa^\dagger - a^\dagger a = 1$ is an operator equation, but normal ordering both sides gives $0 = 1$.

- Now, for the product of two fields, we have

$$\phi(x)\phi(y) = \phi^-(x)\phi^+(y) + \phi^-(x)\phi^+(y) + \phi^+(y)\phi^+(y) + \phi^+(x)\phi^-(y).$$

The first three terms are in normal order, but the last is not. Normal ordering both sides (which is legal because we haven't done anything nontrivial like applying commutation relations) gives

$$\phi(x)\phi(y) = :\phi(x)\phi(y):\ + [\phi^+(x), \phi^-(y)].$$

- Applying this to a time-ordered pair of fields, we have

$$T\phi(x)\phi(y) = :\phi(x)\phi(y): + \theta(x^0 - y^0)[\phi^+(x), \phi^-(y)] + \theta(y^0 - x^0)[\phi^+(y), \phi^-(x)].$$

  The last two terms are c-numbers by the commutation relations. By taking the vev of both sides, we see they must sum to the vev of $T\phi(x)\phi(y)$, i.e. the (bare) Feynman propagator, so

$$T\phi(x)\phi(y) = :\phi(x)\phi(y): + D_F^0(x - y).$$

- More generally, for a string of $n$ fields, we get $2^n$ terms in total, $(n!)^2$ of which are already normal ordered. For the other terms, we have to do some number of commutations to reach normal ordering, each of which produces a two-point function.

- The full result can be written in terms of contractions, where contracting two fields removes them from the normal ordering and produces a factor of their two-point function. Wick's theorem states that

$$T(\phi_1 \dots \phi_n) = :\phi_1 \dots \phi_n: + \text{all possible contractions}$$

  which includes terms with any number of contractions; here we write $\phi_i = \phi(x_i)$.

- The sum includes terms which are not fully contracted, such as $D_F^0(x_1 - x_2)\ :\phi_3\phi_4:$, which vanish when we take the vev. The fully contracted terms all come from the term that begins anti-normal ordered. We thus conclude

$$\langle 0|T\phi_1 \dots \phi_{2n}|0\rangle = D_F^0(x_1 - x_2) \cdots D_F^0(x_{2n-1}, x_{2n}) + \text{all other full contractions}$$

  while the time-ordered correlator of an odd number of fields vanishes.

## 2.4 Feynman Diagrams

We can represent the terms in Wick's theorem diagrammatically.

- For $\langle 0|T\phi_1 \dots \phi_n|0\rangle$, draw $n$ points, then draw all possible diagrams made by connecting disjoint pairs of points. For an edge from $i$ to $j$, write down the factor $D_F^0(x_i - x_j)$.

- Interactions can be expanded order by order, e.g. for a $\phi^4$ interaction,

$$\langle 0|T\phi(x)\phi(y)e^{-i(\lambda/4!)\int dz\, \phi^4(z)}|0\rangle = \langle 0|T\phi(x)\phi(y)|0\rangle + \langle 0|T\phi(x)\phi(y)\int dz \left(\frac{-i\lambda}{4!}\right)\phi^4(z)|0\rangle + \dots.$$

  The $1/n!$ factor in the Taylor series is always canceled by the permutation symmetry of the vertices. Naively, the $1/4!$ factors are canceled because we have 4! contractions possible for each factor of $\phi^4(z)$, but this is not true for symmetric diagrams; the overall result is that we must divide by the symmetry factor of the diagram.

- This gives the position-space Feynman rules for $\langle 0|T\phi(x_1) \dots \phi(x_n)U(\infty, -\infty)|0\rangle$.
  - Draw a point for every 'external' point $x_i$. These are the original fields in the correlator.
  - Draw some internal vertices $z_i$ to account for the interaction. For every vertex, multiply by $-i\lambda$ and integrate over $z_i$.

  - Draw directed lines between points until every external point is connected to one line, and every internal vertex is connected to four.
  - For a line from $x$ to $y$, write down the Feynman propagator

$$D_F^0(x-y) = \int đp \, \frac{i}{p^2 - m_0^2 + i\epsilon} e^{-ip(x-y)}.$$

    The orientation of the line doesn't matter, since $D_F^0(x-y) = D_F^0(y-x)$.
  - Divide by the symmetry factor of the diagram and sum over all diagrams.

- Note that we can contract fields at a point to other fields at the same point, giving factors of $D_F^0(0)$ which diverges. This is a loop diagram, which must be addressed by renormalization.

- To simplify this, we can perform the position integrals, leaving behind only momentum integrals. If lines with momentum $p_1$ and $p_2$ point into a vertex $z$ and lines with momentum $p_3$ and $p_4$ point out, the position integral is

$$\int dz \, e^{ip_1 z} e^{ip_2 z} e^{-ip_3 z} e^{-ip_4 z} = \not\delta(p_1 + p_2 - p_3 - p_4).$$

  Therefore, we have the momentum-space Feynman rules shown below.

  - Give each line a directed momentum $p_i$ and write down the factor

$$\tilde{D}_F^0(p) = \frac{i}{p^2 - m_0^2 + i\epsilon}.$$

  - For each vertex, multiply by

$$(-i\lambda)\not\delta\left(\Sigma_{\text{in}}\, p_i - \Sigma_{\text{out}}\, p_j\right).$$

  - For each external point, multiply by $e^{-ipx}$ if momentum $p$ points out of the point, and $e^{ipx}$ if it points in.
  - Integrate $đp$ over each internal momentum.
  - For each connected component that is a vacuum bubble, multiply by the volume of spacetime $VT$.

  Note that we are not computing the Fourier transforms of the correlators; we're computing the exact same thing as before, but with some of the work already done for us.

- To explain the last rule, note that for vacuum bubbles, not all of the position integrals can be performed. We are left with integrals parametrizing the location of each bubble,

$$\int dz = \not\delta(0) = VT$$

  where $VT$ is the volume of the spacetime. This is generally what $\delta(0)$ means when it appears.

Next, we discuss how to handle disconnected diagrams.

- The nomenclature is a bit confusing. We say a diagram is connected if every vertex is connected, in the graph theory sense, to an external point, i.e. there are no vacuum bubbles. We say a diagram is fully connected if it has one connected component. Neither condition implies the other. For example, a diagram with only a vacuum bubble is fully connected but not connected.

- Every diagram can be factorized into the product of a connected diagram and a diagram with no external points, which we call a vacuum bubble. Then

$$\sum \text{all diagrams} = \left(\sum \text{connected diagrams}\right)\left(\sum \text{vacuum bubbles}\right).$$

- By applying our rules above to the case of zero external points, we have

$$\lim_{t\to\infty(1-i\epsilon)} \langle 0|U(t,-t)|0\rangle = \sum \text{vacuum bubbles} \equiv Z$$

  which tells us that the correlator of Heisenberg fields is

$$\langle\Omega|T\phi_1\dots\phi_n|\Omega\rangle = \sum \text{connected diagrams with } n \text{ external points.}$$

  We call the sum of the vacuum bubbles $Z$ the partition function.

- The partition function has further structure. Let $\{V_i\}$ be the set of fully connected vacuum bubbles. Then a vacuum bubble with $n_i$ copies of $V_i$ comes with the additional symmetry factor $\prod_i 1/n_i!$, so

$$Z = \exp\left(\sum \text{fully-connected vacuum bubbles}\right)$$

- On the other hand, by our previous work we have

$$\lim_{t\to\infty(1-i\epsilon)} \langle 0|U(t,-t)|0\rangle = \lim_{t\to\infty(1-i\epsilon)} |\langle\Omega|0\rangle|^2 e^{-iE_\Omega T}.$$

  Taking the logarithm, the $|\langle\Omega|0\rangle|^2$ term is finite and vanishes in the limit, giving

$$E_\Omega = \lim_{t\to\infty(1-i\epsilon)} \frac{i}{T}\log Z.$$

  Thus the sum of fully-connected vacuum bubbles gives the vacuum energy.

- We can divide by $V$ to get the vacuum energy density. On the right-hand side, this combines with $T$ to give the volume of the spacetime, which cancels with the factor of $\not{\delta}(0)$ which comes with every fully-connected vacuum bubble. This removes the IR divergence, though we still have a UV divergence, which must be handled by renormalization.

Next, we introduce 1PI diagrams to compute the propagator.

- In perturbation theory, the exact propagator can be expanded as,

$$D_F(x-y) = \langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \sum \text{connected diagrams with 2 external points.}$$

  To organize the perturbation series, define a one-particle irreducible (1PI) diagram to be one which cannot be separated into two separate nontrivial diagrams by cutting a single line.

- We define
$$-iM^2(p^2) = \sum \text{all non-trivial 1PI diagrams}$$
where the 1PI diagrams have incoming and outgoing momentum $p^2$, and do not include factors for the incoming/outgoing propagators or external points.

- We define the Fourier transform
$$D_F(x-y) = \int đp\, e^{-ip(x-y)} D_F(p^2).$$
Then $D_F(p^2)$ is computed with the same Feynman rules as $D_F(x-y)$, but there are no factors for external points.

- Therefore, we have the expansion
$$D_F(p^2) = D_F^0(p^2) + D_F^0(p^2)(-iM^2(p^2))D_F^0(p^2) + \ldots = \frac{i}{p^2 - m_0^2 + i\epsilon} \frac{1}{1 - \frac{M^2(p^2)}{p^2 - m_0^2 + i\epsilon}}$$
which gives the compact expression
$$D_F(p^2) = \frac{i}{p^2 - (m_0^2 + M^2(p^2)) + i\epsilon}.$$
This procedure is called Dyson resummation, and $M^2(p^2)$ is the self-energy of the particle.

- Let $m^2$ be the first analytic pole of $D_F(p^2)$. Then we may write
$$D_F(p^2) = \frac{iZ}{p^2 - m^2 + i\epsilon} + \text{terms regular at } m^2.$$
Comparing this with the spectral representation, $Z$ is the wavefunction renormalization and $m$ is the physical mass. Thus we have found a way to compute $Z$ and $m$ perturbatively.

## 2.5 Scattering Amplitudes

We now use LSZ to relate our correlators back to $S$-matrix elements.

- We return to our statement of LSZ, with $n+r$ total incoming and outgoing particles,
$$\prod_k \int dy_k\, e^{ip_k y_k} \prod_\ell \int dx_\ell\, e^{-iq_l x_l} \langle \Omega | T\phi(y_1) \ldots \phi(x_1) \ldots | \Omega \rangle$$
$$= \prod_k \frac{i\sqrt{Z}}{p_k^2 - m^2} \prod_\ell \frac{i\sqrt{Z}}{q_\ell^2 - m^2} \langle p_1, \ldots | S | q_1, \ldots \rangle \Big|_{\text{connected}}.$$
Above, all momenta are on-shell. The $S$-matrix element cannot cancel any of the poles, because then it would be proportional to $p_k^2 - m^2$ or $q_\ell^2 - m^2$, and hence vanish entirely. Thus contributions to $S|_{\text{connected}}$ come from correlation functions with $n+r$ poles at $m^2$.

- The only diagrams which produce such poles are fully connected diagrams. For example, if one particle was in its own connected component, it would only contribute one factor of $iZ/(p^2 - m^2)$ from its propagator rather than the two it should. This lines up with our intuition that $S|_{\text{connected}}$ gives scattering amplitudes where all particles participate together.

- Every fully connected diagram can be decomposed as $n + r$ exact propagators attaching to an "amputated" diagram. Specifically, to amputate a leg, we start at some external point and cut off as much as possible so that the diagram splits into two components, one of which contains only that external point. Note that all 1PI diagrams are amputated but not vice versa.

- As we've seen above, each resummed propagator contributes $iZ/(p^2 - m^2 + i\epsilon)$, plus a regular piece which is not relevant to our calculation; these cancel with the factors on the right-hand side of the LSZ reduction formula. Taking the Fourier transform has the sole effect of removing the exponential factors for the external points.

- Therefore, the Feynman rules for $\langle p_1, \ldots, p_n | S | q_1, \ldots, q_r \rangle|_{\text{connected}}$ are

  – Add internal points and lines as specified earlier, giving every line a directed momentum.

  – For each internal line of momentum $p$, multiply by

  $$\frac{i}{p^2 - m_0^2 + i\epsilon}.$$

  – For each vertex, multiply by

  $$(-i\lambda)\not{\delta}\left(\Sigma_{\text{in}}\, p_i - \Sigma_{\text{out}}\, p_j\right).$$

  – Integrate $đp$ over each internal momentum and divide by the symmetry factor.

  – Sum over fully-connected, amputated diagrams and multiply by $\sqrt{Z}^{n+r}$.

  Formally, $\sqrt{Z} = 1 + O(\lambda)$, so we can set $Z = 1$ for a lowest-order computation. We will always get an overall factor of $\not{\delta}(q_1 + \ldots + q_r - p_1 + \ldots - p_n)$.

- For the physical computations which follow, we will almost never be interested in vacuum bubbles, i.e. all diagrams will be connected. Hence we will rename "fully connected" to "connected" for the rest of these notes; note that we've already been using this nomenclature to refer to the connected $S$-matrix.

We will now introduce a naive version of perturbation theory which is easier to handle at tree level.

- Working in Schrodinger picture, we suppose that $H = H_0 + f(t)H_{\text{int}}$, where $f(t)$ is chosen so the interaction adiabatically turns on and off. This makes calculations much easier, though we lose generality. For example, if $H_0$ is just taken to be the free part of $H$, then it is impossible to describe the scattering of bound states, which fall apart when the interaction is turned off; however, it is possible if $H_0$ is chosen appropriately.

- We define an "in" state $|i^{\text{in}}(t)\rangle$ to be a state which approaches a free state for $t \to -\infty$,

  $$|i^{\text{in}}(-\infty)\rangle = |i(-\infty)\rangle$$

  where $|i^{\text{in}}\rangle$ evolves by $H$ and $|i\rangle$ evolves by $H_0$, and $i$ stands for a state specification like 'two one-particle wavepackets with momenta $\mathbf{p}$ and $\mathbf{q}$ approaching each other'.

- We define the $S$-matrix by
$$\langle f(0)|S|i(0)\rangle = \langle f^{\text{out}}(0)|i^{\text{in}}(0)\rangle.$$
Thus, when the particles in the free state $|i(0)\rangle$ are about to meet and pass through each other, $S$ takes them to a superposition of other free particle states in order to reproduce the dynamics of an interaction. This definition is equivalent to our earlier one, though everything is evaluated at time $t = 0$ on the left rather than at $t \to -\infty$.

- Letting $U$ and $U_0$ be the full and free time evolution operators,
$$\begin{aligned}\langle f^{\text{out}}(0)|i^{\text{in}}(0)\rangle &= \langle f^{\text{out}}(\infty)|U(\infty,-\infty)|i^{\text{in}}(-\infty)\rangle \\ &= \langle f(\infty)|U(\infty,-\infty)|i(-\infty)\rangle \\ &= \langle f(0)|U_0(\infty,0)U(\infty,-\infty)U_0(-\infty,0)|i(0)\rangle \\ &= \langle f(0)|U_I(\infty,0)U_I^\dagger(-\infty,0)|i(0)\rangle \\ &= \langle f(0)|U_I(\infty,-\infty)|i(0)\rangle\end{aligned}$$
which shows that $S = U_I(\infty,-\infty)$. In other words, the point of the adiabatic turn-off is that it allows us to get a simple formal expression for the $S$-matrix itself. We can also show that $[S, H_0] = 0$, so scattering does not change the free particle energy.

Calculations can also be done with the adiabatic approach introduced above.

- For concreteness we consider scalar Yukawa theory,
$$\mathcal{L} = |\partial_\mu\psi|^2 + \frac{1}{2}(\partial^\mu\phi)^2 - M^2|\psi|^2 - \frac{1}{2}m^2\phi^2 - g\psi^\dagger\psi\phi$$
in which case
$$S = U_I(\infty,-\infty) = T\exp\left(-ig\int dx\,\psi^\dagger\psi\phi\right)$$
where the fields above are interaction picture fields.

- We would like to calculate the amplitudes $\langle f(0)|S|i(0)\rangle$. For convenience, we construct the states in interaction picture, i.e. we view $|i(0)\rangle_S = |i(0)\rangle_I$, so that particles in the initial state are also created by interaction picture operators. Since the interaction picture states don't change at all (as the states are free), we drop the $t = 0$ argument.

- Concretely, we call the real particles mesons and the complex particles nucleons, with
$$\phi \sim a + a^\dagger, \quad \psi \sim b + c^\dagger, \quad \psi^\dagger \sim b^\dagger + c$$
where $b^\dagger$ creates a nucleon and $c^\dagger$ creates an antinucleon. Then for meson decay,
$$|i\rangle = \sqrt{2E_{\mathbf{p}}}a^\dagger_{\mathbf{p}}|0\rangle, \quad |f\rangle = \sqrt{4E_{\mathbf{q}_1}E_{\mathbf{q}_2}}b^\dagger_{\mathbf{q}_1}c^\dagger_{\mathbf{q}_2}|0\rangle.$$

- To lowest order in $g$, we have
$$\langle f|S-1|i\rangle = -ig\langle f|\int dx\,\psi^\dagger(x)\psi(x)\phi(x)|i\rangle.$$

This can be simplified using the commutation relations for free field creation and annihilation operators; only one combination of operators gives a nonzero contribution. We find

$$\langle f|S-1|i\rangle = -ig\not{\phi}(q_1+q_2-p)$$

in accordance with the Feynman rules above. To save space, we define

$$\langle f|S-1|i\rangle = \langle f|iT|i\rangle = i\mathcal{M}\not{\phi}(\cdots)$$

since the delta function always appears, so that $i\mathcal{M} = -ig$.

- For more complicated processes, we use Wick's theorem, but slightly differently. Since the initial and final states aren't vacuum, we don't care about full contractions; instead we want enough fields left over to turn the initial and final states to vacuum. In this formalism, contractions correspond to only internal lines in diagrams.

**Example.** Consider nucleon-nucleon scattering, $\psi\psi \to \psi\psi$, with

$$|i\rangle = |p_1,p_2\rangle = \sqrt{4E_{\mathbf{p}_1}E_{\mathbf{p}_2}}b^\dagger_{\mathbf{p}_1}b^\dagger_{\mathbf{p}_2}|0\rangle, \quad |f\rangle = |q_1,q_2\rangle = \sqrt{4E_{\mathbf{q}_1}E_{\mathbf{q}_2}}b^\dagger_{\mathbf{q}_1}b^\dagger_{\mathbf{q}_2}|0\rangle$$

and the $O(g^2)$ contribution to the $S$-matrix element is

$$\frac{(-ig)^2}{2}\langle f|\int dx_1dx_2\, T(\psi^\dagger(x_1)\psi(x_1)\phi(x_1)\psi^\dagger(x_2)\psi(x_2)\phi(x_2))|i\rangle.$$

Then the only contraction which contributes is

$$T(\psi^\dagger(x_1)\psi(x_1)\phi(x_1)\psi^\dagger(x_2)\psi(x_2)\phi(x_2)) \supset :\psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2): D_F^0(x_1-x_2).$$

Since all the annihilation operators are moved to the right, we have

$$\langle f|\ :\psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2):\ |i\rangle = \langle f|\psi^\dagger(x_1)\psi^\dagger(x_2)|0\rangle\langle 0|\psi(x_1)\psi(x_2)|i\rangle.$$

Expanding the fields into creation and annihilation operators, each piece has two terms, giving

$$\frac{(-ig)^2}{2}\int dx_1dx_2\,(e^{i(q_1x_1+q_2x_2)}+e^{i(q_1x_2+q_2x_1)})(e^{-i(p_1x_1+p_2x_2)}+e^{-i(p_1x_2+p_2x_1)})\int \frac{đk\, ie^{ik(x_1-x_2)}}{k^2-m^2+i\epsilon}.$$

Doing the position integrals gives momentum-conserving delta functions, giving

$$i\mathcal{M} = (-ig)^2\left(\frac{i}{(p_1-q_1)^2-m^2}+\frac{i}{(p_1-q_2)^2-m^2}\right)$$

We dropped the $i\epsilon$ because the denominator is never zero; to see this, work in the center of mass frame where $|\mathbf{p}_1| = |\mathbf{q}_1|$. Then the timelike component of $p_1-q_1$ is zero, so $(p_1-q_1)^2 < 0$. There is also an $O(g^0)$ contribution when $|i\rangle = |f\rangle$, which is not included in $S-1$.

We can generalize this calculation into Feynman rules.

- The Feynman rules for computing $i\mathcal{M}$, valid at tree level, are the following.
  - Add internal vertices and lines as specified earlier, giving every line a directed momentum, and conserving momentum at every vertex.

  - For each internal line of momentum $p$ and mass $m$, multiply by $i/(p^2 - m^2 + i\epsilon)$.
  - For every vertex, multiply by $-ig$.
  - Divide by the symmetry factor, and sum over tree-level diagrams, excluding the totally disconnected diagram.

  For complex scalar fields, we draw arrows on the lines to indicate the flow of particle number, but define the direction of the momentum so that $p^0$ is positive for external legs. There is no fixed convention for internal lines; there we might as well align the two.

- As another example, for $\psi\overline{\psi} \to \psi\overline{\psi}$, we have

$$i\mathcal{M} = (-ig)^2 \left( \frac{i}{(p_1 - q_1)^2 - m^2} + \frac{i}{(p_1 + p_2)^2 - m^2 + i\epsilon} \right)$$

  where the $i\epsilon$ cannot be dropped in the second term, as it diverges when the incoming nucleons form an on-shell meson. As we'll describe in more detail later, this appears as a resonance peak in the scattering cross section.

**Note.** The calculational formalism we have built up in this chapter is very different from the material one learns in ordinary quantum mechanics. It's important to keep in mind that quantum field theory is in principle a special case of quantum mechanics, so the same techniques still work. For example, one can calculate cross sections using the standard techniques of time-independent perturbation theory, in a procedure particle physicists call "old-fashioned perturbation theory", but which atomic and optical physicists still commonly use. For example, one gets expressions like

$$\frac{dP}{dt} \sim |\mathcal{M}|^2, \quad \mathcal{M} \sim \sum_k \frac{\langle f|H_i|k\rangle\langle k|H_i|i\rangle}{E_i - E_k}$$

for a process that occurs at second order. Old-fashioned perturbation theory doesn't require learning complex, abstract new tools such as the LSZ reduction theory. However, it relies on the Hamiltonian and hence breaks Lorentz invariance. In particular, single Feynman diagrams break into multiple diagrams in this formalism, one for each possible time ordering of the interaction vertices, making calculations more complicated. (By combining pairs of such diagrams, one combines the energy denominators $1/(E_i - E_k)$ into Lorentz invariant Feynman propagators.) Old-fashioned perturbation theory is covered in detail in these lecture notes, as well as Sakurai's book *Advanced Quantum Mechanics*, whose introduction contains forceful arguments in favor of studying such methods.

**Note.** Above, we have seen that the connected part of the $S$-matrix contains a single overall momentum conserving delta function, so the connected amplitude $\mathcal{M}$ contains no delta functions. In our treatment, this followed automatically from the structure of the perturbative expansion, in terms of Feynman diagrams.

More generally, this result can be understood as a requirement of the cluster decomposition principle, the notion that distant experiments should give uncorrelated results. To see this, first note that the connected $S$-matrix can be defined independently of the dynamics; it is simply the part of the $S$-matrix, for $n$-particle scattering, that cannot be constructed from combining $S$-matrix elements for $(n-1)$-particle scattering and lower. Thus, the connected $S$-matrix tells us about the part of the scattering that involves all $n$ particles together. Now, the $S$-matrix is unchanged under translating all of the particles, which is equivalent in momentum space to requiring an overall momentum conserving delta function. If there were further delta functions in the connected

amplitude, it would imply that the connected $S$-matrix was invariant under translating a *subset* of the particles, i.e. that these particles would continue to have an effect on the others if they were translated arbitrarily far away. Hence this is forbidden by the cluster decomposition principle.

Following Weinberg, the fact that this result follows automatically from a local field description is one of the key motivations for introducing quantum fields at all, rather than focusing on just the particle states. For a masterful account with interesting historical asides, see his article *What is Quantum Field Theory, and What Did We Think It Is?*

## 2.6 Physical Observables

We now link our scattering amplitudes to physical quantities. We begin with the potential energy $U(r)$ of two nucleons.

- At the classical level, a delta-function source for the meson field yields

$$\phi(\mathbf{x}) = \frac{e^{-mr}}{4\pi r}$$

  where $m$ is the meson mass. We would like to think of nucleons as such delta-function sources, so that this gives the potential energy of two nucleons.

- To see this picture at the quantum level, we consider the process $\psi\psi \to \psi\psi$ at tree level. We work in the center of mass frame, so $\mathbf{p} = \mathbf{p}_1 = -\mathbf{p}_2$ and $\mathbf{q} = \mathbf{q}_1 = -\mathbf{q}_2$. Then

$$i\mathcal{M} = ig^2 \left( \frac{1}{(\mathbf{p}-\mathbf{q})^2 + m^2} + \frac{1}{(\mathbf{p}+\mathbf{q})^2 - m^2} \right).$$

  On the other hand, we know from nonrelativistic quantum mechanics that the amplitude for this scattering is $\widetilde{U}(\mathbf{p}-\mathbf{q}) + \widetilde{U}(\mathbf{p}+\mathbf{q})$ in the first Born approximation, where we took into account the fact that the $\psi$ particles are identical. Then

$$\widetilde{U}(\mathbf{p}-\mathbf{q}) = \frac{-\lambda^2}{(\mathbf{p}-\mathbf{q})^2 + m^2}, \quad U(\mathbf{r}) = -\frac{\lambda^2}{4\pi r} e^{-mr}$$

  where $\lambda = g/2M$ and the extra factors of $2M$ are from the relativistic normalization. To get the result for $U(\mathbf{r})$, we have to work in spherical coordinates and perform a contour integral.

- The exact same logic holds for $\psi\overline{\psi} \to \psi\overline{\psi}$. In this case, the $s$-channel doesn't contribute because it vanishes in the nonrelativistic limit, as long as $M \gg m$, but the $t$-channel contributes in the same way, giving the exact same potential. Therefore, the force between a nucleon and antinucleon is also attractive. In general, forces mediated by particles of even spin are universally attractive.

- Similarly, in $\phi^4$ theory with $H_{\text{int}} = \lambda\phi^4/4!$ the tree-level amplitude is

$$i\mathcal{M} = -i\lambda, \quad \widetilde{U}(\mathbf{p}) \propto \lambda, \quad U(\mathbf{r}) \propto \delta(\mathbf{r}).$$

  That is, the potential in $\phi^4$ theory has zero range.

- In general, the Born approximation is a useful way of relating the scattering amplitudes we know how to compute to "position-space" quantities such as potentials. For example, we can use it to relate the magnetic moment of the electron to its scattering amplitude off a fixed background vector potential, which can be computed diagrammatically.

- For $2 \to 2$ scattering, the Mandelstam variables

$$s = (p_1 + p_2)^2, \quad t = (p_1 - q_1)^2, \quad u = (p_1 - q_2)^2$$

appear frequently. We say that $\psi\psi \to \psi\psi$ has $t$-channel and $u$-channel diagrams at $O(g^2)$, while $\psi\overline{\psi} \to \psi\overline{\psi}$ has $s$-channel and $t$-channel diagrams. If the particles have masses $m_i$,

$$s + t + u = \sum_i m_i^2.$$

Suppose two identical particles scatter by an angle $\theta$ in the center-of-mass frame. Then

$$s = 4E_{\text{cm}}^2, \quad t = -2p^2(1 - \cos\theta), \quad u = -2p^2(1 + \cos\theta).$$

In this frame, $s$ determines the energy of each particle, while $t$ and $u$ give directional information.

Next, we turn to the calculation of cross sections and decay rates.

- Let $V$ be the spatial volume and $T$ be the total time separating the asymptotic past and future. Then $\not{\delta}(0) = VT$ and $\not{\delta}(\mathbf{0}) = V$, as we've seen earlier. We'll need these results because we will encounter squared delta functions, which arise from working in an infinite space.

- By definition, the probability to transition from state $|i\rangle$ (with particles of momentum $p_i$) to $|f\rangle$ (with particles of momentum $q_j$) is

$$P = \frac{|\langle f|S|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle}.$$

Under the relativistic normalization convention,

$$\langle f|f\rangle = \prod_j 2E_{\mathbf{q}_i}\not{\delta}(\mathbf{0}) = \prod_j 2E_{\mathbf{q}_j}V, \quad \langle i|i\rangle = \prod_i 2E_{\mathbf{p}_i}V$$

and the $S$ matrix element is

$$|\langle f|S|i\rangle|^2 = |\mathcal{M}_{fi}|^2\not{\delta}(p_I - p_F)^2 = |\mathcal{M}_{fi}|^2\not{\delta}(p_I - p_F)VT$$

where $p_I$ and $p_F$ are the total initial and final momenta.

- Next, we must integrate over the momenta of the particles in the final states, using

$$\frac{1}{V}\sum_{\mathbf{p}} \to \int đ\mathbf{p}, \quad V\delta_{\mathbf{p},\mathbf{q}} \to \not{\delta}(\mathbf{p} - \mathbf{q})$$

which gives

$$P = \left(\prod_i \frac{1}{2E_{\mathbf{p}_i}V}\right) VT \int |\mathcal{M}_{fi}|^2\, d\Pi, \quad d\Pi = \not{\delta}(p_F - p_I)\prod_j \frac{đ\mathbf{q}_j}{2E_{\mathbf{q}_j}}.$$

If some of the particles in the final state are identical, the Lorentz invariant phase space $d\Pi$ should contain $1/N!$ factors to avoid overcounting.

- First, consider the decay rate of a particle with mass $m$. Taking a rate cancels the $T$. The result is not Lorentz invariant, because of the factor of $1/E_{\mathbf{p}}$, but this is correct because it accounts for time dilation. Usually we will work in the rest frame, where $E_{\mathbf{p}} = m$, giving

$$\boxed{\Gamma = \frac{1}{2m}\int |\mathcal{M}_{fi}|^2 \, d\Pi}$$

  If there are multiple decay channels, we must also sum over types of final states.

- Next, consider scattering of two particles with energies $E_1$ and $E_2$. Then we wish to calculate the differential cross section

$$d\sigma = \frac{\text{differential probability}}{\text{unit time} \times \text{unit flux}} = \frac{1}{4E_1E_2V}\frac{1}{F}|\mathcal{M}_{fi}|^2 \, d\Pi.$$

  The flux $F$ is the number of incoming particles per area per unit time.

- Working in the center of mass frame, the probability per volume is $1/V$ for each initial particle, and the relative velocity is $|\mathbf{v}_1 - \mathbf{v}_2|$, giving

$$\boxed{d\sigma = \frac{1}{4E_1E_2|\mathbf{v}_1 - \mathbf{v}_2|}|\mathcal{M}_{fi}|^2 \, d\Pi.}$$

- The denominator is Lorentz-invariant under boosts along $\mathbf{v}_1 - \mathbf{v}_2$, but not others, since the cross section can length contract. To see this, note that for velocities parallel to $\hat{\mathbf{z}}$,

$$E_1E_2|\mathbf{v}_1 - \mathbf{v}_2| = |E_2\mathbf{p}_1 - E_1\mathbf{p}_2| = |\epsilon_{\mu xy\nu}p_1^\mu p_2^\nu|.$$

  Thus, the inverse of this quantity has precisely the Lorentz transformation quantities of the $xy$ component of an antisymmetric rank 2 tensor, which is precisely the geometrical representation of an area element in the $xy$ plane.

- Some other sources define the cross section in a slightly different, Lorentz invariant way,

$$d\sigma = \frac{1}{4v_{\text{rel}}(p_1 \cdot p_2)}|\mathcal{M}_{fi}|^2 \, d\Pi$$

  where the relative velocity (i.e. the speed of one particle in the frame of the other) is

$$v_{\text{rel}} = \sqrt{1 - \frac{m_1^2m_2^2}{(p_1 \cdot p_2)^2}}.$$

  This matches the boxed definition above in the center of mass frame, so it will make no difference for these notes. On the other hand, in situations where the particles may not have opposite momenta (such as in a thermal average) one has to be careful about which definition is used.

- For scattering with two particles in the final state, we are interested in the quantity $d\sigma/d\Omega$, which can be found by writing $d\Pi = (d\Pi/d\Omega)\, d\Omega$. One can show, in the CM frame,

$$d\Pi = \frac{|\mathbf{q}_1|}{(2\pi)^2(4E_{\text{cm}})}\, d\Omega$$

  where $\mathbf{q}_1$ is the final momentum, which gives

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{4E_1E_2|\mathbf{v}_1 - \mathbf{v}_2|}\frac{|\mathbf{q}_1|}{(2\pi)^2(4E_{\text{cm}})}|\mathcal{M}_{fi}|^2.}$$

- For all equal masses, this reduces to

$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}_{fi}|^2}{64\pi^2 E_{\text{cm}}^2}.$$

  For example, we see that $s$-wave scattering is isotropic, and $\sigma \propto 1/s$ for $\phi^4$ theory. This is a general phenomenon for scattering off pointlike sources.

- The traditional unit for a cross section is the typical cross sectional area of a uranium nucleus,

$$1\,\text{barn} = 10^{-28}\,\text{m}^2 = 100\,\text{fm}^2$$

  which corresponds to a scatterer whose radius is a few femtometers. For context, the total cross section for proton-proton scattering is of order 100 mb, and the instantaneous luminosity of the LHC is about $10\,\text{nb}^{-1}\text{s}^{-1}$. The total integrated luminosity of the LHC so far is about $200\,\text{fb}^{-1} = 0.2\,\text{ab}^{-1}$, while the future HL-LHC will accumulate $3\,\text{ab}^{-1}$.

- Finally, another useful case is a final state with three particles. In the CM frame, the momenta $\mathbf{q}_i$ lie in a common plane, called the event plane. It is useful to define

$$x_i = \frac{2E_i}{E_{\text{cm}}}, \quad x_1 + x_2 + x_3 = 2.$$

  In these variables, a laborious calculation shows that the phase space takes a very simple form,

$$d\Pi = \frac{E_{\text{cm}}^2}{128\pi^3}\, dx_1 dx_2$$

  where the right-hand side is implicitly supported only over the kinematically allowed region.

- The result is also simple in terms of the invariant masses of pairs,

$$m_{12}^2 = (p_1 + p_2)^2, \quad m_{23}^2 = (p_2 + p_3)^2$$

  in which case

$$d\Pi = \frac{1}{128\pi^3 E_{\text{cm}}^2}\, dm_{12}^2 dm_{23}^2.$$

- Again, there are implicit and messy theta functions on the right-hand side to mark the kinematically allowed region. Thus, the utility of this parametrization is really on the experimental side. Plotting events in the $(m_{12}^2, m_{23}^2)$ plane yields a Dalitz plot, and features of the plot can be attributed to the scattering amplitude itself, and hence indicate physically interesting effects.

# 3 Spinor Fields

## 3.1 Dirac Spinors

A set of relativistic classical fields transforms under a representation of the Lorentz group,

$$\phi_i(x) \to R(\Lambda)_{ij}\phi_j(\Lambda^{-1}x).$$

We would thus like to classify representations of $SO(1,3)$.

- As we've seen, an infinitesimal Lorentz transformation takes the form

  $$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$$

  where $\omega^{\mu\nu} = -\omega^{\nu\mu}$. We may index the six independent Lorentz transformations with two antisymmetric indices,

  $$(J^{\rho\sigma})^{\mu\nu} = \eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu}.$$

- Then any infinitesimal Lorentz transformation can be written as

  $$\omega^\mu{}_\nu = \frac{1}{2}\Omega_{\rho\sigma}(J^{\rho\sigma})^\mu{}_\nu$$

  and finite Lorentz transformations are given by exponentiation,

  $$\Lambda^\mu{}_\nu = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}(J^{\rho\sigma})^\mu{}_\nu\right).$$

  The factors of $1/2$ are canceled by the antisymmetry of $\Omega$ and $J$. Explicitly, $\Omega_{12} = -\Omega_{21} = \theta$ gives an active $\theta$ rotation about $\hat{\mathbf{z}}$, while $\Omega_{0i} = \chi_i$ gives an active boost with rapidity $\boldsymbol{\chi}$.

- Note that while the generators are antisymmetric with indices raised, they aren't with mixed indices, so the $\Lambda^\mu{}_\nu$ are not unitary. Alternatively, Lorentz transformations preserve $\eta_{\mu\nu}V^\mu V^\nu$ and not $\delta_{\mu\nu}V^\mu V^\nu$. In general, there are no finite-dimensional unitary representations of a noncompact Lie group like the Lorentz group.

- Some explicit computation yields the Lorentz algebra $\mathfrak{so}(1,3)$,

  $$[J^{\rho\sigma}, J^{\tau\nu}] = \eta^{\sigma\tau}J^{\rho\nu} - \eta^{\rho\tau}J^{\sigma\nu} + \eta^{\rho\nu}J^{\sigma\tau} - \eta^{\sigma\nu}J^{\rho\tau}.$$

  We seek representations of this algebra. When we exponentiate them, we will generally find a projective representation of the Lorentz group, i.e. a representation of its double cover $\mathrm{Spin}(1,3) \cong SL(2,\mathbb{C})$. This is a bit strange, since classical fields don't have phase ambiguities like quantum states, but acceptable since the spinor field will not be directly measurable.

We construct the Dirac spinor representation starting from the Clifford algebra.

- The Clifford algebra is a set of four matrices $\gamma^\mu$ satisfying

  $$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

  That is, all of the $\gamma^\mu$ anticommute with each other, and they square as

  $$(\gamma^0)^2 = 1, \quad (\gamma^i)^2 = -1.$$

- The lowest-dimensional representation of the Clifford algebra is four-dimensional. We will use the Weyl/chiral representation,

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

  We note the useful Pauli matrix identities

$$\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k, \quad \{\sigma^i, \sigma^j\} = 2\delta^{ij}, \quad [\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k.$$

  Other representations can be constructed by replacing $\gamma^\mu$ with $V\gamma^\mu V^{-1}$.

- The commutators of the $\gamma^\mu$ are

$$S^{\rho\sigma} = \frac{1}{4}[\gamma^\rho, \gamma^\sigma] = \begin{cases} 0 & \rho = \sigma \\ (1/2)\gamma^\rho\gamma^\sigma & \rho \neq \sigma \end{cases} = \frac{1}{2}\gamma^\rho\gamma^\sigma - \frac{1}{2}\eta^{\rho\sigma}.$$

  We claim the $S^{\mu\nu}$ form a representation of the Lorentz algebra. First, note that

$$[S^{\mu\nu}, \gamma^\rho] = \frac{1}{2}[\gamma^\mu\gamma^\nu, \gamma^\rho] = \frac{1}{2}\gamma^\mu\gamma^\nu\gamma^\rho - \frac{1}{2}\gamma^\rho\gamma^\mu\gamma^\nu = \gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\rho\mu}$$

  where we used anticommutation to cancel the two terms, picking up $\gamma\eta$ terms. Next,

$$[S^{\mu\nu}, S^{\rho\sigma}] = \frac{1}{2}[S^{\mu\nu}, \gamma^\rho\gamma^\sigma] = \frac{1}{2}(\gamma^\mu\gamma^\sigma\eta^{\nu\rho} - \gamma^\nu\gamma^\sigma\eta^{\rho\mu} + \gamma^\rho\gamma^\mu\eta^{\nu\sigma} - \gamma^\rho\gamma^\nu\eta^{\sigma\mu})$$

  where we used the commutator product rule and the previous result. Finally, applying $\gamma^\mu\gamma^\nu = 2S^{\mu\nu} + \eta^{\mu\nu}$ gives the result. Note that the $\gamma$ and $S$ matrices contain complex numbers, so we have a complex representation.

Next, we exponentiate the $S^{\mu\nu}$ to find a projective representation of $SO(1,3)$.

- The $S^{\mu\nu}$ act on a Dirac spinor field, i.e. a field with four complex components $\psi^\alpha(x)$ so that

$$\psi^\alpha(x) \to S[\Lambda]^\alpha{}_\beta\psi^\beta(\Lambda^{-1}x), \quad S[\Lambda] = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right)$$

- To see this is a projective representation, we explicitly work out $S[\Lambda]$ for rotations. For $i \neq j$,

$$S^{ij} = \frac{1}{2}\gamma^i\gamma^j = -\frac{i}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}.$$

  Defining the angle by $\Omega_{ij} = \epsilon_{ijk}\varphi^k$, we have

$$S[\Lambda] = \begin{pmatrix} e^{-i\,\boldsymbol{\varphi}\cdot\boldsymbol{\sigma}/2} & 0 \\ 0 & e^{-i\,\boldsymbol{\varphi}\cdot\boldsymbol{\sigma}/2} \end{pmatrix}$$

  which is $-1$ for a $2\pi$ rotation.

- Similarly, for boosts, we define $\Omega_{0i} = \chi_i$ and

$$S^{0i} = \frac{1}{2}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}, \quad S[\Lambda] = \begin{pmatrix} e^{-\boldsymbol{\chi}\cdot\boldsymbol{\sigma}/2} & 0 \\ 0 & e^{\boldsymbol{\chi}\cdot\boldsymbol{\sigma}/2} \end{pmatrix}.$$

  We see the Dirac spinor splits into two components which transform identically under rotations but oppositely under boosts.

- Sources may differ on the signs of $\varphi$ and $\chi$. Here we have taken the active point of view, so $\chi$ refers to the amount we actively change the spinor's velocity. But $\chi$ can also label the relative velocity of a frame we transform into, which would give an extra minus sign.
- The Dirac spinor representation is not unitary. To see this at the level of the Clifford algebra, note that unitarity requires $S^{\mu\nu}$ to be anti-Hermitian, which works if the $\gamma^\mu$ are all Hermitian or all anti-Hermitian. But since $(\gamma^0)^2 = 1$ and $(\gamma^i)^2 = -1$, $\gamma^0$ can only be picked to be Hermitian and $\gamma^i$ can only be picked to be anti-Hermitian. Again, the issue comes from the indefinite signature of $\eta$.

## 3.2 The Dirac Action

We now build a Lorentz-invariant action using our Dirac spinor.

- Suppressing position arguments, we have

$$\psi \to S[\Lambda]\psi, \quad \psi^\dagger \to \psi^\dagger S^\dagger[\Lambda].$$

  Therefore, the naive guess $\psi^\dagger\psi$ does not work because the $S[\Lambda]$ are not unitary.
- For concreteness, we use a representation of the Clifford algebra where $\gamma^0$ is Hermitian and the $\gamma^i$ are anti-Hermitian, such as the chiral representation.
- In any specific representation, it is straightforward to show that

$$\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger$$

  which means $\gamma^0$ takes the adjoint of gamma matrices when pulled through them. (Since all representations are related by unitary transformations, it suffices to show this result in any representation – but it's not straightforward to do it without ever picking one.) Then

$$(S^{\mu\nu})^\dagger = -\gamma^0 S^{\mu\nu}\gamma^0, \quad S[\Lambda]^\dagger = \gamma^0 S[\Lambda]^{-1}\gamma^0.$$

- Defining the Dirac adjoint $\overline{\psi} = \psi^\dagger\gamma^0$, we have

$$\overline{\psi} \to \psi^\dagger S[\Lambda]^\dagger\gamma^0 = \psi^\dagger\gamma^0 S[\Lambda]^{-1} = \overline{\psi}S[\Lambda]^{-1}$$

  which implies that $\overline{\psi}\psi$ is a Lorentz scalar.
- Moreover, $\overline{\psi}\gamma^\mu\psi$ is a Lorentz vector. Since we have

$$\overline{\psi}\gamma^\mu\psi \to \overline{\psi}S[\Lambda]^{-1}\gamma^\mu S[\Lambda]\psi.$$

  we need to show $S[\Lambda]^{-1}\gamma^\mu S[\Lambda] = \Lambda^\mu{}_\nu\gamma^\nu$. For infinitesimal Lorentz transformations, this becomes

$$[-S^{\rho\sigma}, \gamma^\mu] = (J^{\rho\sigma})^\mu{}_\nu\gamma^\nu.$$

  We can directly verify this is true using computations we've already done.
- Similarly, one can show that $\overline{\psi}\gamma^\mu\gamma^\nu\psi$ transforms as a Lorentz tensor, as does any combination of more gamma matrices. Taking the symmetric part yields $\eta^{\mu\nu}\overline{\psi}\psi$, which is simply a multiple of our Lorentz scalar. The new feature is the antisymmetric part, $\overline{\psi}S^{\mu\nu}\psi$.

- We use our scalar and vector to build the Dirac Lagrangian

$$\mathcal{L} = \overline{\psi}(i\not{\partial} - m)\psi, \quad \not{\partial} = \gamma^\mu \partial_\mu.$$

  Note that the factor of $i$ is necessary to make $\mathcal{L}$ real, i.e. invariant under conjugation, to cancel out a minus sign from integration by parts. We haven't seen this issue before because our other Lagrangians were second-order, requiring two integrations by parts.

- Using the trick of treating $\psi$ and $\overline{\psi}$ as independent fields, varying with respect to $\overline{\psi}$ gives the Dirac equation

$$(i\not{\partial} - m)\psi = 0.$$

  Varying with respect to $\psi$ gives the conjugate equation

$$i\partial_\mu \overline{\psi}\gamma^\mu + m\overline{\psi} = \overline{\psi}(i\overleftarrow{\not{\partial}} + m) = 0$$

  where the arrow indicates the derivative acts to the left.

- Each component of the Dirac spinor satisfies the Klein–Gordan equation, as

$$(i\gamma^\nu \partial_\nu + m)(i\gamma^\mu \partial_\mu - m)\psi = -(\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + m^2)\psi = -(\partial^2 + m^2)\psi = 0.$$

  Thus the Dirac spinor describes four types of particles of mass $m$. The left-hand side contains a 'factorization' of the Klein–Gordan operator $\partial^2 + m^2$, so the Dirac equation is sometimes called the 'square root' of the Klein–Gordan equation. Here it's clear that the definition of a Clifford algebra was originally chosen just to make this work out.

Next, we decompose the Dirac spinor into Weyl spinors.

- As we've seen, in the chiral representation of the Clifford algebra, the Lorentz transformations are block-diagonal. Hence the Dirac spinor splits into two irreps, $\psi = (u_+, u_-)$, called Weyl spinors, which transform as

$$u_\pm \to e^{-i\boldsymbol{\varphi}\cdot\boldsymbol{\sigma}/2} u_\pm, \quad u_\pm \to e^{\mp\boldsymbol{\chi}\cdot\boldsymbol{\sigma}/2} u_\pm.$$

  The Weyl spinors are $(1/2, 0)$ and $(0, 1/2)$, or left-handed and right-handed respectively, and the Dirac spinor is $(1/2, 0) \oplus (0, 1/2)$.

- Introducing the notation

$$\sigma^\mu = (1, \sigma^i), \quad \overline{\sigma}^\mu = (1, -\sigma^i)$$

  the Dirac Lagrangian can be written in terms of Weyl spinors as

$$\mathcal{L} = iu_-^\dagger \sigma^\mu \partial_\mu u_- + iu_+^\dagger \overline{\sigma}^\mu \partial_\mu u_+ - m(u_+^\dagger u_- + u_-^\dagger u_+).$$

  Therefore the mass term couples the Weyl spinors together. To get a theory with only one Weyl spinor, we must set it to zero. The individual Weyl spinors satisfy the Weyl equations

$$i\overline{\sigma}^\mu \partial_\mu u_+ = 0, \quad i\sigma^\mu \partial_\mu u_- = 0.$$

- Suppose a classical field theory has $n$ real degrees of freedom in configuration space at each spatial point. Then it has $2n$ degrees of freedom per point in phase space, which become $2n/2 = n$ particles upon quantization. This yields $n = 1$ for a real scalar field and $n = 2$ for a complex scalar field, as expected.

- Now, the Dirac spinor naively has 16 real degrees of freedom in phase space. However, since the Dirac equation is first-order, $\pi_\psi = i\psi^\dagger$, so the phase space is parametrized by $\psi$ and $\psi^\dagger$. Then the phase space has real dimension 8, giving 4 particles upon quantization, i.e. a spin up/down particle/antiparticle. Similarly the Weyl spinor yields 2 particles.

- Another way of saying this is that physical fermions must have constraints in their phase space; it can be shown that we otherwise inevitably get negative-norm states.

- To define the Weyl spinors in an invariant way, define

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3.$$

  Then one can show

$$\{\gamma^5, \gamma^\mu\} = 0, \quad (\gamma^5)^2 = 1.$$

  Given this definition, the matrices $(\gamma^\mu, -i\gamma^5)$ form a five-dimensional Clifford algebra. Moreover, choosing $\gamma^0$ to be Hermitian and $\gamma^i$ to be anti-Hermitian as before, ${\gamma^5}^\dagger = \gamma^5$.

- We can also show that $[S_{\mu\nu}, \gamma^5] = 0$, so $\gamma^5$ is a 'Lorentz scalar'. Then we have

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_L = \frac{1}{2}(1 - \gamma^5), \quad P_R = \frac{1}{2}(1 + \gamma^5)$$

  where $P_L$ and $P_R$ project on the left-handed and right-handed Weyl spinors.

## 3.3 Symmetries and Conserved Quantities

Next, we define parity symmetry.

- Conjugating a rotation by parity leaves it invariant, but conjugating a boost with parity flips its sign. Thus we want parity to exchange the Weyl spinors,

$$P : \psi_\pm(\mathbf{x}, t) \mapsto \psi_\mp(-\mathbf{x}, t).$$

  Thus, in the chiral representation,

$$P : \psi(\mathbf{x}, t) \mapsto \gamma^0\psi(-\mathbf{x}, t).$$

  The Dirac equation is parity invariant, since if $\psi(\mathbf{x}, t)$ satisfies it, so does $\gamma^0\psi(-\mathbf{x}, t)$.

- Note that $\overline{\psi}\psi$ transforms as a scalar under parity because $(\gamma^0)^2 = 1$. However, for the vector $\overline{\psi}\gamma^\mu\psi$, the spacelike parts flip sign,

$$\overline{\psi}\gamma^i\psi \to \overline{\psi}\gamma^0\gamma^i\gamma^0\psi = -\overline{\psi}\gamma^i\psi$$

  which implies that $\overline{\psi}\gamma^\mu\psi$ transforms as a vector under parity.

- Similarly, $\overline{\psi}S^{\mu\nu}\psi$ transforms as a tensor under parity. We also have $\overline{\psi}\gamma^5\psi$, which transforms as a pseudoscalar, and $\overline{\psi}\gamma^5\gamma^\mu\psi$, which transforms as a pseudovector; both of these pick up an extra sign flip due to the $\gamma^5$.

- As we've seen above, the Dirac spinor is a certain representation of the Lorentz group and the set of spinor bilinears is that representation's tensor product with itself,

  $$((1/2,0)+(0,1/2))\times((1/2,0)+(0,1/2)) = (0,0)+(1/2,1/2)+(1,0)+(0,1)+(1/2,1/2)+(0,0)$$

  which decompose as $1+4+3+3+4+1$. The two 3's are packaged together into the rank 2 antisymmetric tensor and correspond to self-dual and anti-self-dual tensors.

- The pseudoscalar and axial vector terms treat the $u_\pm$ asymmetrically since they are not parity invariant. In general, any theory that treats the $u_\pm$ asymmetrically is called a chiral theory (vs. a vector-like theory). However, chiral theories can be parity invariant; for example, the term $\phi\overline{\psi}\gamma^5\psi$ respects parity if $\phi$ is a pseudoscalar.

**Note.** Sometimes people ask: why don't the gamma matrices transform under Lorentz transformations, like everything else with a four-vector index does? The representation theory above shows the right way to think about this. A gamma matrix is a set of Clebsch–Gordan coefficients which extracts the Lorentz vector representation from the tensor product of two Dirac spinor representations. These coefficients don't change under Lorentz transformations, for exactly the same reason that ordinary Clebsch–Gordan coefficients for spin don't. For example, for spin we have $1 \subset 1/2 \times 1/2$, and the state $|\ell, m_z\rangle = |1,0\rangle$ is *always* of the form $(|1/2,1/2\rangle|1/2,-1/2\rangle + |1/2,-1/2\rangle|1/2,1/2\rangle)/\sqrt{2}$. This doesn't depend on how you orient the $z$-axis, as long as you use the same $z$-axis to define the $m_z$ states in the 1 and 1/2 representations, which of course we always do.

We now define charge conjugation symmetry and Majorana fermions.

- Naively, we would like to define charge conjugation of a classical field as just the complex conjugate, as is suitable for a complex scalar field. But charge conjugation should commute with Lorentz transformations, so the naive option doesn't work since $S[\Lambda]$ is not real.

- Instead, we define

  $$\psi^{(c)} = C\psi^*$$

  where $C$ is a $4\times 4$ unitary matrix satisfying

  $$C^\dagger \gamma^\mu C = (-\gamma^\mu)^*$$

  in a general representation of the Clifford algebra where $(\gamma^0)^\dagger = \gamma^0$ and $(\gamma^i)^\dagger = -\gamma^i$.

- To check the transformation properties, we have

  $$\psi^{(c)} \to CS[\Lambda]^*\psi^* = S[\Lambda]C\psi^* = S[\Lambda]\psi^{(c)}$$

  as desired. The Dirac equation is also invariant. Complex conjugating it gives

  $$(-i\not{\partial}^* - m)\psi^* = 0.$$

  Multiplying on the left by $C$ and pulling it through shows that $\psi^{(c)}$ obeys the Dirac equation.

- A Majorana spinor obeys the reality condition $\psi^{(c)} = \psi$. This constraint relates the two Weyl spinors. Majorana spinors can have a mass, but they cannot have charges.

- Finally, we can write $C$ explicitly given a fixed basis for the Clifford algebra. In the chiral basis,

$$C = i\gamma^2$$

because only $\gamma^2$ is imaginary. In terms of the Weyl spinors, this implies $u_- = -i\sigma^2 u_+^*$.

- Alternatively, in the Majorana basis,

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}.$$

Then all of the gamma matrices are pure imaginary, so the $S[\Lambda]$ are real and $C = 1$.

**Note.** This definition of the $C$ matrix differs from the $C$ matrix appearing in the quantum charge conjugation operation $\widehat{C}$. This is because this $C$ arises from the classical charge conjugation on fields, which in this case acts like $\widehat{C}$ with an additional parity factor $\gamma_0$. This is explained further in the notes on the Standard Model.

**Note.** We check the Lagrangian is invariant under charge conjugation. Consider the mass term,

$$\overline{\psi}\psi = \psi^\dagger\gamma^0\psi \to \psi^T C^\dagger \gamma^0 C\psi^* = -\psi^T(\gamma^0)^*\psi^*.$$

This quantity is, naively, equal to its own transpose. However, we must account for the fact that spinors are inherently anticommuting, and hence the classical fields here are really Grassmann numbers; taking the transpose flips the sign because of the anticommutation. Then we get

$$\overline{\psi}\psi \to \psi^\dagger {\gamma^0}^\dagger \psi = \overline{\psi}\psi$$

as desired. Demanding the invariance of the $\overline{\psi}\not{D}\psi$ term forces

$$A_\mu \to -A_\mu$$

under charge conjugation, which intuitively is because $\partial^2 A_\mu = j_\mu$ and $j_\mu \to -j_\mu$. More generally, for a non-abelian gauge theory, the gauge interaction term $\overline{\psi}_i \not{A}_{ij}\psi_j$ is invariant if

$$A^\mu_{ij} \to -A^\mu_{ji}.$$

This is sensible, as it's just the usual rule for conjugating a Lie algebra representation.

Finally, we consider the continuous symmetries and conserved currents of the Dirac action.

- We begin with translational symmetry, treating $\psi$ and $\overline{\psi}$ as independent as usual. Since the Lagrangian does not depend on $\partial\overline{\psi}$, we have

$$T^{\mu\nu} = i\overline{\psi}\gamma^\mu\partial^\nu\psi - \eta^{\mu\nu}\mathcal{L}.$$

We can further simplify this using the equations of motion, which state that $\mathcal{L}$ vanishes on shell. The total energy is then

$$E = \int d\mathbf{x}\, T^{00} = \int d\mathbf{x}\, i\overline{\psi}\gamma^0\dot{\psi} = \int d\mathbf{x}\, \psi^\dagger\gamma^0(-i\gamma^i\partial_i + m)\psi.$$

- An infinitesimal Lorentz transformation gives

$$\delta\psi^\alpha = -\omega^\mu{}_\nu x^\nu \partial_\mu \psi^\alpha + \frac{1}{2}\Omega_{\rho\sigma}(S^{\rho\sigma})^\alpha{}_\beta \psi^\beta.$$

  Expanding out the definitions yields $\omega^{\mu\nu} = \Omega^{\mu\nu}$. Now, the first term contributes the same thing we saw for the scalar field, while the second term contributes a "spin" term,

$$(\mathcal{J}^\mu)^{\rho\sigma} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho} - i\overline{\psi}\gamma^\mu S^{\rho\sigma}\psi.$$

  Upon quantization the latter will yield a spin of $\pm 1/2$ for each particle. A similar spin/orbit decomposition appears for general fields with spin.

- The Dirac action has an internal phase symmetry, $\psi \to e^{-i\alpha}\psi$ which yields

$$j_V^\mu = \overline{\psi}\gamma^\mu\psi$$

  where $j_V^\mu$ is called the vector current. The conserved quantity is

$$Q = \int d\mathbf{x}\, \overline{\psi}\gamma^0\psi = \int d\mathbf{x}\, \psi^\dagger\psi$$

  which we will see can be interpreted as the total particle number.

- When $m = 0$, the Weyl spinors decouple, so there is an additional independent symmetry given by rotating the Weyl spinors in opposite directions,

$$\psi \to e^{i\alpha\gamma^5}\psi, \quad \overline{\psi} \to \overline{\psi}e^{i\alpha\gamma^5}.$$

  This gives the 'axial current',

$$j_A^\mu = \overline{\psi}\gamma^\mu\gamma^5\psi$$

  which is an axial vector. Later, we will see that the axial symmetry is anomalous. In QED with massless fermions, the Lagrangian has axial symmetry, but the current is not conserved in the quantum theory.

## 3.4 Plane Wave Solutions

In this section, we find the classical plane wave solutions of the Dirac equation in the chiral basis.

- We consider a positive frequency plane wave $\psi(x) = u(\mathbf{p})e^{-ipx}$ where $p^0 > 0$. Then the Dirac equation becomes

$$\begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\overline{\sigma} & -m \end{pmatrix} u(\mathbf{p}) = 0.$$

  Since the individual components satisfy the Klein–Gordan equation, $p^2 = m^2$. Now we have

$$u(\mathbf{p}) = \begin{pmatrix} mu_1 \\ (p\cdot\overline{\sigma})\, u_1 \end{pmatrix}.$$

  for some spinor $u_1$. Defining $mu_1 = \sqrt{p\cdot\sigma}\xi$ and using the identity

$$(p\cdot\sigma)(p\cdot\overline{\sigma}) = p\cdot p = m^2$$

  on the bottom spinor, we arrive at the general form

$$u(\mathbf{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\overline{\sigma}}\xi \end{pmatrix}.$$

  We will conventionally normalize $\xi$ by $\xi^\dagger\xi = 1$.

- Similarly, there are negative frequency solutions. Letting $\psi(x) = v(\mathbf{p})e^{ipx}$ with $p^0 > 0$,

$$v(\mathbf{p}) = \begin{pmatrix} \sqrt{p \cdot \sigma}\eta \\ -\sqrt{p \cdot \overline{\sigma}}\eta \end{pmatrix}$$

  where we normalize $\eta^\dagger \eta = 1$. Note these classical field solutions vary in space as if they had four-momentum $-p$. However, we will fix this sign discrepancy after quantization, so that the corresponding particles indeed have four-momentum $p$.

- As a simple example, consider the positive frequency solution with mass $m$ and $\mathbf{p} = 0$,

$$u(\mathbf{p}) = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}.$$

  To interpret $\xi$, recall that the angular momentum operator is $J^i = \text{diag}(\sigma^i/2, \sigma^i/2)$. Then $\xi$ describes the spin, with $\xi = (1, 0)^T$ yielding spin up.

- Next, consider a boost along the $x^3$ direction, $p^\mu = (E, 0, 0, p)$. Then for spin up,

$$u(\mathbf{p}) = \begin{pmatrix} \sqrt{E - p^3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \sqrt{E + p^3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}.$$

  In particular, in the massless case we have $u(\mathbf{p}) = (0, 0, 1, 0)^T$. Similarly, for a massless spin down field we have $u(\mathbf{p}) = (0, 1, 0, 0)^T$.

- Define the helicity as the projection of spin along the direction of momentum,

$$h = \frac{i}{2}\epsilon_{ijk} p^i S^{jk} = \frac{1}{2} \begin{pmatrix} p_i \sigma^i & 0 \\ 0 & p_i \sigma^i \end{pmatrix}.$$

  Then the spin up solution above has positive/right-handed helicity and the spin down solution has negative/left-handed helicity. In the massless case helicity coincides with chirality. When a mass is added, helicity is no longer Lorentz invariant, and chirality is no longer conserved. However, we're playing a dangerous game here because we're assigning helicity to classical solutions, while it is really a property of quantum states; this will lead to some extra sign flips.

- Now pick orthonormal bases $\xi_s$ and $\chi_s$ for the positive and negative frequency solutions. Then

$$u^{r\dagger}(\mathbf{p})u^s(\mathbf{p}) = 2p_0\delta^{rs}, \quad \overline{u}^r(\mathbf{p})u^s(\mathbf{p}) = 2m\delta^{rs}, \quad v^{r\dagger}(\mathbf{p})v^s(\mathbf{p}) = 2p_0\delta^{rs}, \quad \overline{v}^r(\mathbf{p})v^s(\mathbf{p}) = -2m\delta^{rs}$$

  by direct expansion, using the identity $(p \cdot \sigma)(p \cdot \overline{\sigma}) = m^2$. Another useful identity is

$$u^{r\dagger}(\mathbf{p})v^s(-\mathbf{p}) = v^{r\dagger}(\mathbf{p})u^s(-\mathbf{p}) = 0.$$

- Finally, for outer products we have

$$\sum_s u^s(\mathbf{p})\overline{u}^s(\mathbf{p}) = \not{p} + m, \quad \sum_s v^s(\mathbf{p})\overline{v}^s(\mathbf{p}) = \not{p} - m.$$

  These combinations will appear in the numerator of the propagator.

## 3.5 Quantizing the Dirac Field

Now we turn to the quantization of the Dirac field. We begin by naively imposing canonical commutation relations.

- We recall the conjugate momentum is

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\overline{\psi}\gamma^0 = i\psi^\dagger.$$

  Perform the Legendre transform by $\mathcal{H} = \pi\dot{\psi} - \mathcal{L}$, the Hamiltonian is

$$H = \int d\mathbf{x}\, \overline{\psi}(-i\gamma^i\partial_i + m)\psi$$

  as we found earlier when computing $T^{\mu\nu}$.

- Now, we impose the usual canonical commutators

$$[\psi_\alpha(\mathbf{x}), \psi^\dagger_\beta(\mathbf{y})] = \delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{y})$$

  with all other commutators zero, and perform the mode expansion

$$\psi(\mathbf{x}) = \int \frac{đ\mathbf{p}}{\sqrt{2E_p}} (b^s_\mathbf{p} u^s(\mathbf{p}) e^{i\mathbf{px}} + c^{s\,\dagger}_\mathbf{p} v^s(\mathbf{p}) e^{-i\mathbf{px}}).$$

  Taking the conjugate gives

$$\psi^\dagger(\mathbf{x}) = \int \frac{đ\mathbf{p}}{\sqrt{2E_p}} (b^{s\,\dagger}_\mathbf{p} u^s(\mathbf{p})^\dagger e^{-i\mathbf{px}} + c^s_\mathbf{p} v^s(\mathbf{p})^\dagger e^{i\mathbf{px}}).$$

  Here, the spin indices $s$ run from 1 to 2, the spinor indices $\alpha$ and $\beta$ run from 1 to 4, and the summation convention is used on the spin indices.

- By a similar computation to the scalar field case, we have the commutators

$$[b^r_\mathbf{p}, b^{s\,\dagger}_\mathbf{q}] = \delta^{rs}đ(\mathbf{p}-\mathbf{q}), \quad [c^r_\mathbf{p}, c^{s\,\dagger}_\mathbf{q}] = -\delta^{rs}đ(\mathbf{p}-\mathbf{q})$$

  with all others zero; note the appearance of the minus sign.

- To simplify the Hamiltonian, we apply the on-shell spinor condition

$$(-\gamma^i p_i + m)u^s(\mathbf{p}) = \gamma^0 p_0 u^s(\mathbf{p}), \quad (\gamma^i p_i + m)v^s(\mathbf{p}) = -\gamma^0 p_0 v^s(\mathbf{p})$$

  so we simply have

$$(-i\gamma^i\partial_i + m)\psi = \int đ\mathbf{p} \sqrt{\frac{E_p}{2}} \gamma^0 (b^s_\mathbf{p} u^s(\mathbf{p}) e^{i\mathbf{px}} - c^{s\,\dagger}_\mathbf{p} v^s(\mathbf{p}) e^{-i\mathbf{px}}).$$

  Note there is an extra minus sign here because $e^{i\mathbf{px}}$ looks like $e^{-ipx}$. Applying our spinor inner product identities gives

$$H = \int đ\mathbf{p}\, E_p (b^{s\,\dagger}_\mathbf{p} b^s_\mathbf{p} - c^s_\mathbf{p} c^{s\,\dagger}_\mathbf{p}) = \int đ\mathbf{p}\, E_p (b^{s\,\dagger}_\mathbf{p} b^s_\mathbf{p} - c^{s\,\dagger}_\mathbf{p} c^s_\mathbf{p} + đ(\mathbf{0})).$$

- Now, note that $[H, c^\dagger(\mathbf{p})] = E_p c^\dagger(\mathbf{p})$ as usual for a creation operator; the extra minus sign in the commutation relations cancels out the minus sign in the Hamiltonian. The problem is that the particles created by $c^\dagger$ have negative norm, as

$$\|c_{\mathbf{p}}^{r\,\dagger}|0\rangle\|^2 = \langle 0|[c_{\mathbf{p}}^r, c_{\mathbf{p}}^{r\,\dagger}]|0\rangle < 0$$

  which violates unitarity, as we get negative probabilities. We can fix this by interpreting $c^\dagger$ as an annihilation operator and $c$ as a creation operator, but then $c$ creates particles with negative energy, so the spectrum is unbounded below. Either way, the theory is sick.

**Note.** The problem above can be traced backwards to the fact that the Hamiltonian for a spinor field is linear in time derivatives, while that of a scalar field is quadratic, which can in turn be traced back to the Lagrangian. This generalizes to higher spins. An integer spin field can be represented using a number of vector indices, while a half-integer spin field must additionally have one spinor index. The kinetic term of the Lagrangian must contract the spinor indices by a factor of $\gamma_\mu$ (or more properly, for a Weyl spinor index, a factor of $\sigma_\mu$), whose Lorentz index must then be contracted with a single derivative $\partial_\mu$. The remaining Lorentz indices simply come in pairs, so half-integer spin fields have equations of motion with an odd number of derivatives.

The problem is fixed by switching to anticommutation relations.

- The spin-statistics theorem states that half-integer spin particles in a relativistic quantum field theory must be fermions, i.e. must be quantized with anticommutation relations. Thus we instead impose the canonical anticommutation relations

$$\{\psi_\alpha(\mathbf{x}), \psi_\beta^\dagger(\mathbf{y})\} = \delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{y})$$

  which are equivalent to the anticommutation relations

$$\{b_{\mathbf{p}}^r, b_{\mathbf{q}}^{s\,\dagger}\} = \delta^{rs}\not{\delta}(\mathbf{p}-\mathbf{q}), \quad \{c_{\mathbf{p}}^r, c_{\mathbf{q}}^{s\,\dagger}\} = \delta^{rs}\not{\delta}(\mathbf{p}-\mathbf{q})$$

  with all other anticommutators zero.

- All of our manipulations above go through unchanged, except that at the last step,

$$H = \int đ\mathbf{p}\, E_p(b_{\mathbf{p}}^{s\,\dagger} b_{\mathbf{p}}^s - c_{\mathbf{p}}^s c_{\mathbf{p}}^{s\,\dagger}) = \int đ\mathbf{p}\, E_p(b_{\mathbf{p}}^{s\,\dagger} b_{\mathbf{p}}^s + c_{\mathbf{p}}^{s\,\dagger} c_{\mathbf{p}}^s - \not{\delta}(\mathbf{0})).$$

  Then both types of particles have positive energy and positive norm. The vacuum energy contribution also comes out negative, canceling part of the boson contribution.

- We can construct the Hilbert space exactly as before, acting with the creation operators on the vacuum. They both raise the energy, as

$$[H, b_{\mathbf{p}}^{s\,\dagger}] = E_{\mathbf{p}} b_{\mathbf{p}}^{s\,\dagger}, \quad [H, c_{\mathbf{p}}^{s\,\dagger}] = E_{\mathbf{p}} c_{\mathbf{p}}^{s\,\dagger}.$$

  The anticommutation relations imply the Pauli exclusion principle: every mode can have either zero or one particle. The anticommutation makes the multiparticle wavefunction antisymmetric, as postulated in nonrelativistic quantum mechanics.

- As an example, the conserved charge from the internal vector symmetry is

$$Q = \int đ\mathbf{p}\,(b_{\mathbf{p}}^{s\,\dagger} b_{\mathbf{p}}^{s} - c_{\mathbf{p}}^{s\,\dagger} c_{\mathbf{p}}^{s})$$

  indicating that the $c$ particles have negative charge. We thus interpret them as antimatter.

- One might wonder how the $c$ particles can have positive energy when they are the quantization of a classical mode with negative frequency. Each of these modes can have occupancy either 0 or 1, and it is arbitrary which of these two states is labeled 'unoccupied'. We thus define the vacuum to be the lowest energy state, and for the vacuum to have all modes unoccupied. In this new picture, the excitations created by $c^\dagger$ are 'holes' in the Dirac sea.

- The Dirac sea also explains why the vacuum energy comes out negative for fermionic fields. All the bosonic degrees of freedom get zero-point energy $\hbar\omega/2$, while all the negative frequency fermionic degrees of freedom get Dirac sea energy $-\hbar\omega$.

- Since $c^\dagger$ makes a hole, every quantum number is flipped, so that $c_{\mathbf{p}}^{s\,\dagger}$ creates a particle with momentum $\mathbf{p}$ and negative charge. It's convenient to define the basis spinors so that $c_{\mathbf{p}}^{s\,\dagger}$ and $b_{\mathbf{p}}^{s\,\dagger}$ have the same spin.

Next, we switch to Heisenberg picture, establish causality, and find the fermionic propagator.

- In Heisenberg picture, the field simply becomes

$$\boxed{\psi(\mathbf{x}) = \int \frac{đ\mathbf{p}}{\sqrt{2E_p}}\,(b_{\mathbf{p}}^{s} u^{s}(\mathbf{p}) e^{-ipx} + c_{\mathbf{p}}^{s\,\dagger} v^{s}(\mathbf{p}) e^{ipx}).}$$

  Next, we compute the anticommutator

$$iS_{\alpha\beta}(x-y) = \{\psi_\alpha(x), \overline{\psi}_\beta(y)\}.$$

  Suppressing spinor indices and using the anticommutation relations, we have

$$\begin{aligned} iS(x-y) &= \int \frac{đ\mathbf{p}}{2E_p}\,(u^s(\mathbf{p})\overline{u}^s(\mathbf{p}) e^{-ip(x-y)} + v^s(\mathbf{p})\overline{v}^s(\mathbf{p}) e^{ip(x-y)}) \\ &= \int \frac{đ\mathbf{p}}{2E_p}\,((\not{p} + m) e^{-ip(x-y)} + (\not{p} - m) e^{ip(x-y)}) \\ &= (i\not{\partial}_x + m)(D(x-y) - D(y-x)) \end{aligned}$$

  where $D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle$ is the propagator for a free scalar field.

- Since $D(x-y)$ vanishes for spacelike separations, $S(x-y)$ and hence the anticommutator vanishes for spacelike separations. Since all observables are fermion bilinears or combinations thereof, observables commute at spacelike separations.

- Next, we define the time-ordering symbol as

$$T(\psi(x)\overline{\psi}(y)) = \begin{cases} \psi(x)\overline{\psi}(y) & x^0 \geq y^0, \\ -\overline{\psi}(y)\psi(x) & y^0 > x^0. \end{cases}.$$

  To see one reason the minus sign is necessary, note that at spacelike separations we have $\psi(x)\overline{\psi}(y) = -\overline{\psi}(y)\psi(x)$ by the above calculation. In such cases we can perform a boost to change the time ordering of $x$ and $y$, so the two cases in the time ordering must be equal.

- We define the Feynman propagator as

$$S_F(x-y) = \langle 0|T\psi(x)\overline{\psi}(y)|0\rangle.$$

  By a similar calculation to above, we find

$$S_F(x-y) = \theta(x^0-y^0)\int \frac{đ\mathbf{p}}{2E_p}(\not{p}+m)e^{-ip(x-y)} - \theta(y^0-x^0)\int \frac{đ\mathbf{p}}{2E_p}(\not{p}-m)e^{ip(x-y)}$$

  from which we conclude

$$S_F(x-y) = (i\not{\partial}_x + m)D_F(x-y) = i\int đp\,\frac{\not{p}+m}{p^2-m^2+i\epsilon}e^{-ip(x-y)}$$

  where $D_F$ is the Feynman propagator for a free scalar theory. If we had used commutation relations to quantize the field, we would have found an ugly, non-Lorentz invariant result.

- As expected, $S_F$ is a Green's function for the Dirac equation, since applying $i\not{\partial}_x - m$ gives $(\not{p}-m)(\not{p}+m) = p^2-m^2$ in the numerator. Because of this identity, the Feynman propagator is sometimes formally written as $S_F(p) = i(\not{p}-m)^{-1}$. Alternatively, we can think of the Feynman propagator as a sum of Feynman propagators for each degree of freedom; in this picture the numerator is a sum over polarizations. We will prove this more generally later.

- Normal ordering for fermionic fields is also defined with an extra minus sign for every interchange. Thus the general method for normal and time ordering is to simply commute or anticommute, neglecting all extra resulting terms. The proof of Wick's theorem goes through as before, with contractions yielding a factor of $S_F$.

- Contractions now come with signs, so that swapping any two $\psi$ or $\overline{\psi}$ fields in the contraction structure yields a factor of $-1$. If we draw lines connecting contracted fields, the number of sign flips is the number of intersections of the lines.

- Note that the contraction of $\psi$ with $\psi$, or of $\overline{\psi}$ with $\overline{\psi}$, is automatically zero. Then $n$-point functions of spinor fields can only be nonzero if there are an equal number of $\psi$ and $\overline{\psi}$ fields, reflecting charge conservation.

- The proof of the LSZ reduction formula also goes through mostly as before, expect that the poles are projected out by factors of $i\not{\partial} - m$ rather than $\partial^2 + m^2$.

## 3.6 Feynman Rules

In this section, we will illustrate the Feynman rules by a direct calculation in the interaction picture. The same results can also be found by the same rigorous route followed for the scalar field, starting with the LSZ reduction theorem.

- For concreteness we consider Yukawa theory,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}\mu^2\phi^2 + \overline{\psi}(i\not{\partial} - m)\psi - \lambda\phi\overline{\psi}\psi$$

  where $\phi$ is a scalar field. We think of the $\psi$ particles as nucleons and the $\phi$ particles as mesons (specifically, pions). The Yukawa coupling here also appears in the same form between fermions and the Higgs boson.

- We now carefully work out $\psi\psi \to \psi\psi$ scattering, with initial and final states

$$|i\rangle = \sqrt{4E_pE_q}\, b_{\mathbf{p}}^{s\,\dagger} b_{\mathbf{q}}^{r\,\dagger}|0\rangle, \quad |f\rangle = \sqrt{4E_{p'}E_{q'}}\, b_{\mathbf{p}'}^{s'\,\dagger} b_{\mathbf{q}'}^{r'\,\dagger}|0\rangle.$$

  Note that taking the adjoint to get $\langle f|$ reverses the order of the operators; we were careless about this before because the operators commuted.

- The $O(\lambda^2)$ contribution to $\langle f|S-1|i\rangle$ is

$$\frac{(-i\lambda)^2}{2}\int dx_1 dx_2\, T(\overline{\psi}(x_1)\psi(x_1)\phi(x_1)\overline{\psi}(x_2)\psi(x_2)\phi(x_2))$$

  where the two $\phi$ fields must be contracted, just as in scalar Yukawa theory.

- Next, we need to be careful about how the spinor indices are contracted. While we've kept this implicit, there is a contraction between $\overline{\psi}$ and $\psi$ in the interaction term, so explicitly

$$\begin{aligned}&:\overline{\psi}(x_1)\psi(x_1)\overline{\psi}(x_2)\psi(x_2):\ b_{\mathbf{p}}^{s\,\dagger} b_{\mathbf{q}}^{r\,\dagger}|0\rangle \\ &\qquad = -\int \frac{đ\mathbf{k}_1 đ\mathbf{k}_2}{\sqrt{4E_{k_1}E_{k_2}}}(\overline{\psi}(x_1)\cdot u^m(\mathbf{k}_1))(\overline{\psi}(x_2)\cdot u^n(\mathbf{k}_2))e^{-i(k_1x_2+k_2x_2)} b_{\mathbf{k}_1}^m b_{\mathbf{k}_2}^n b_{\mathbf{p}}^{s\,\dagger} b_{\mathbf{q}}^{r\,\dagger}|0\rangle\end{aligned}$$

  where the dot indicates spinor contraction, and we picked up a sign from moving $\psi(x_1)$ past $\overline{\psi}(x_2)$. Applying the anticommutation relations gives two terms which differ by a sign.

- Finally, we need to take care of the final state. Each of the two terms above yields two possible contraction structures, canceling the $1/2$ in front of the amplitude. We are left with the position integrations, which yield delta functions, finally giving

$$\mathcal{M} = (-i\lambda)^2\left(\frac{(\overline{u}^{s'}(\mathbf{p}')\cdot u^s(\mathbf{p}))(\overline{u}^{r'}(\mathbf{q}')\cdot u^r(\mathbf{q}))}{(p'-p)^2-\mu^2+i\epsilon} - \frac{(\overline{u}^{s'}(\mathbf{p}')\cdot u^r(\mathbf{q}))(\overline{u}^{r'}(\mathbf{q}')\cdot u^s(\mathbf{p}))}{(q'-p)^2-\mu^2+i\epsilon}\right)$$

- We can summarize our results with the following Feynman rules.
  - For every incoming fermion with momentum $p$ and spin $r$, write down $u^r(\mathbf{p})$. For outgoing fermions, write $\overline{u}^r(\mathbf{p})$.
  - For incoming and outgoing antifermions write $\overline{v}^r(\mathbf{p})$ and $v^r(\mathbf{p})$, respectively,
  - Every vertex gives a factor of $-i\lambda$.
  - Every internal line gets a Feynman propagator,

$$\text{scalar: } \frac{i}{p^2-\mu^2+i\epsilon}, \quad \text{fermion: } \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}.$$

  - Conserve momentum at every vertex, and integrate $đp$ over loop momenta.
  - For every fermion line, draw an arrow indicating the flow of particle number, and contract spinor indices along these lines. This is independent of the momentum flow arrows, though the two can always be aligned for internal lines.

  An easy way to remember the external fermion factors is to note that the matrix multiplications, read left to right, always go opposite the particle flow lines.

- There are also extra minus signs in various places.
  - Interchange of identical particles gives a factor of $-1$. For example, the $t$-channel and $u$-channel diagrams have opposite signs for $\psi\psi \to \psi\psi$, as we saw above.
  - Every closed fermion loop has a factor of $-1$. This is because we will always need an odd number of swaps to 'untangle' all of the contractions. We write 'tr' to indicate the spinor indices are contracted in a loop.
  - Unfortunately, there are further signs that are more subtle, for which these two rules don't suffice. The most reliable way is to simply return to the level of contractions and count the anticommutations necessary to form one. An equivalent diagrammatic method is to redraw the diagram so that all fermion lines start on the left (in a standard order) and end on the right; the sign is the sign of the permutation of the lines on the right.
- Finally, we can find the Yukawa potential in the nonrelativistic limit. For $\psi\psi \to \psi\psi$ scattering, note that the spinors become

$$u(\mathbf{p}) \to \sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}, \quad v(\mathbf{p}) \to \sqrt{m}\begin{pmatrix}\xi\\ -\xi\end{pmatrix}$$

  so the spin-dependence in the terms is just $\delta^{ss'}\delta^{rr'}$ and $\delta^{rs'}\delta^{sr'}$, so the interaction conserves spin and is spin-independent. Besides that, the amplitude is the same as in Yukawa theory, so the potential is attractive.
- Next, consider the process $\psi\overline{\psi} \to \psi\overline{\psi}$. There is a sign flip from fermionic statistics as shown above, but another sign flip from $\overline{v}^{s'} \cdot v^s = -2m\delta^{ss'}$, which means nucleons and antinucleons still attract. This is because forces mediated by spin 0 particles are universally attractive.
- More realistically, the pion is a pseudoscalar, so we should considering the coupling

$$\mathcal{L}_{\text{Yuk}} = -\lambda\phi\overline{\psi}\gamma^5\psi$$

  which obeys parity. Then the interaction vertex becomes $-i\lambda\gamma^5$. In this case, the calculation becomes significantly more complicated, since taking the naive nonrelativistic limit above just yields zero. The result is a spin-dependent potential which goes as $1/r^3$ for massless pions. For details, see the paper *New Macroscopic Forces?*

# 4 Vector Fields

## 4.1 Gauge Symmetry

In this section, we quantize the electromagnetic field, finding several new obstacles due to the gauge symmetry. For background on quantization with constraints used here, see the lecture notes on String Theory. **(finish, see Weinberg)**

- Electromagnetism is described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

  and satisfies the equations of motion and Bianchi identity

$$\partial_\mu F^{\mu\nu} = 0, \quad \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0.$$

  These each produce four equations, i.e. two of Maxwell's equations each.

- Many of the degrees of freedom in $A_\mu$ are redundant, as we know the photon has only two polarization states. The first issue is that $A_0$ has no kinetic term, as the Lagrangian does not depend on $\dot{A}_0$. It is a 'non-dynamical' field, whose value is totally determined by initial conditions for $A_i$ and $\dot{A}_i$.

- Intuitively, a kinetic term $\dot{A}_0^2$ gives the action a 'penalty' for changing $A_0$, and hence gives the field $A_0$ some 'inertia'. Without this inertia, action minimization always takes $A_0(t)$ to whatever minimizes the Lagrangian given $A_i(t)$ and $\dot{A}_i(t)$ at every time $t$.

- More explicitly, we can solve for $A_0$ (up to gauge symmetry) using its equation of motion,

$$\nabla^2 A_0 + \nabla \cdot \frac{\partial \mathbf{A}}{\partial t} = 0, \quad A_0(\mathbf{x}) = \int d\mathbf{x}' \frac{(\nabla \cdot (\partial \mathbf{A}/\partial t))(\mathbf{x}')}{4\pi|\mathbf{x} - \mathbf{x}'|}.$$

  Note that this equation of motion is equivalent to the constraint $\nabla \cdot \mathbf{E} = 0$. We could plug this expression for $A_0$ back into the Lagrangian, eliminating $A_0$ entirely, but this would be much messier than just keeping $A_0$ explicit.

- The Lagrangian also has a gauge symmetry

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x)$$

  where we assume $\alpha(x) \to 0$ as $x \to \infty$, under which the field strength is invariant. Classically, we think of states related by a gauge transformation as the same physical state. The gauge symmetry also takes away one degree of freedom because the gauge parameter $\alpha$ has one value at every point, leaving two.

- Gauge symmetry places powerful constraints on the action. For example, as we will see in detail for the non-abelian case, the Maxwell action is essentially the only possible kinetic term for a photon which is gauge invariant, not irrelevant, and not a total derivative.

- Since $A_0$ is non-dynamical its conjugate momentum vanishes,

$$\pi^0 = 0, \quad \pi^i = -F^{0i} = E^i.$$

The Hamiltonian can be straightforwardly computed to be

$$H = \int d\mathbf{x}\, \frac{1}{2}E^2 + \frac{1}{2}B^2 - A_0(\nabla \cdot \mathbf{E}).$$

We hence recover the familiar electromagnetic field energy, with $A_0$ serving as a Lagrange multiplier which enforces Gauss's law. Thus we have first-class constraints $\pi^0 = \nabla \cdot \mathbf{E} = 0$.

We will look at a few different gauges.

- Lorenz gauge is the condition $\partial_\mu A^\mu = 0$. This is always achievable since we can set $\partial^2 \alpha = -\partial_\mu A^\mu$ for any starting $A^\mu$. This is not a complete gauge fixing, because $\partial^2 \alpha = 0$ has nontrivial solutions, but Lorenz gauge is Lorentz invariant.
- Coulomb gauge is the additional restriction $\nabla \cdot \mathbf{A} = 0$, which means $A_0$ is time-independent in vacuum; it is achievable by the same logic as Lorenz gauge. This is still not a complete gauge fixing, because we may still apply gauge transformations with parameters $\alpha$ so that

$$\ddot{\alpha} = \nabla^2 \alpha = 0.$$

Usually, we further require $A_0 = 0$, which can be achieved by performing a final gauge transformation with $\alpha = A_0 t$. This brings us to Coulomb gauge, which breaks Lorentz invariance, but makes it easy to see the two independent polarizations.

- In Lorenz gauge and hence in Coulomb gauge, the equation of motion is $\partial^2 A_\mu = 0$.
- The counting of degrees of freedom can be a little puzzling. Intuitively, there is only "one degree of freedom" in $\alpha(x)$, which is a function on spacetime. Lorenz gauge is also intuitively "one constraint", so it seems nothing should be left over. The point is that the remaining freedom is of measure zero, as a solution to $\partial^2 \alpha = 0$ can be specified by initial conditions on a hypersurface. Similarly, after fixing Coulomb gauge, we have the freedom to perform gauge transformations with $\dot{\alpha} = \nabla^2 \alpha = 0$, which again is of measure zero relative to what we started with.
- For a fixed four-vector $n^\mu$, the axial gauge is $n^\mu A_\mu = 0$. We will not deal with axial gauge here, but it is useful in Yang–Mills theory because ghost fields are not required.

## 4.2 Quantization in Coulomb Gauge

We begin with quantization in Coulomb gauge. This gauge is popular in atomic physics, where the breaking of Lorentz invariance is not problematic, since the matter is typically nonrelativistic anyway. However, it is especially difficult to perform renormalization with it.

- Since we have set $A_0 = 0$, the classical plane wave solutions are

$$\mathbf{A}(\mathbf{x}) = \int \text{đ}\mathbf{p}\, \boldsymbol{\xi}(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}}, \quad p^2 = 0$$

where $\boldsymbol{\xi}(\mathbf{p})$ is the polarization. The constraint $\nabla \cdot \mathbf{A} = 0$ yields $\boldsymbol{\xi} \cdot \mathbf{p} = 0$.

- For each momentum, there are two independent polarization vectors $\boldsymbol{\epsilon}_r(\mathbf{p})$ satisfying

$$\boldsymbol{\epsilon}_r(\mathbf{p}) \cdot \mathbf{p} = 0, \quad \boldsymbol{\epsilon}_r(\mathbf{p}) \cdot \boldsymbol{\epsilon}_s(\mathbf{p}) = \delta_{rs}.$$

  Note that it is impossible to choose the polarization vectors continuously by the hairy ball theorem. However, this topological issue has no impact on scattering calculations.

- Naively, we would impose the canonical commutators

$$[A_i(\mathbf{x}), E_j(\mathbf{y})] = i\delta_{ij}\delta(\mathbf{x}-\mathbf{y})$$

  with all other commutators zero, but we must account for the constraints. Before gauge fixing, we had the first-class constraints

$$\pi^0 = 0, \quad \nabla \cdot \mathbf{E} = 0.$$

  In the process of gauge fixing, we imposed the additional constraints

$$A^0 = 0, \quad \nabla \cdot \mathbf{A} = 0$$

  and now $\pi^0 = A^0 = 0$ are automatically satisfied since we have eliminated those variables entirely. The remaining constraints are now second-class. They are not obeyed by the naive commutators, as, for example,

$$[\nabla \cdot \mathbf{A}(\mathbf{x}), \nabla \cdot \mathbf{E}(\mathbf{x})] = i\nabla^2\delta(\mathbf{x}-\mathbf{y}) \neq 0.$$

- The fix is to use the Dirac bracket, i.e. the modified bracket

$$[A_i(\mathbf{x}), E_j(\mathbf{y})] = i\left(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2}\right)\delta(\mathbf{x}-\mathbf{y}).$$

  Then in momentum space, we have

$$[A_i(\mathbf{x}), E_j(\mathbf{y})] = i\int đ\mathbf{p}\,\left(\delta_{ij} - \frac{p_i p_j}{|\mathbf{p}|^2}\right)e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}$$

  so that $\partial_i A_i$ and $\partial_i E_i$ each have zero bracket with everything else.

- Formally, because our theory has constraints, we would like to define it on a subset of the full phase space. However, explicitly reducing the three degrees of freedom to two would be quite ugly. Thinking geometrically of Poisson brackets $\{f, g\}$ as the rate of change of $g$ under the flow generated by $f$, the Dirac bracket amounts to adjusting the flows so that they stay on the constraint manifold, giving the same result.

- Now, the usual mode expansion gives

$$\mathbf{A}(\mathbf{x}) = \int \frac{đ\mathbf{p}}{\sqrt{2|\mathbf{p}|}}\sum_{r=1}^{2}\boldsymbol{\epsilon}_r(\mathbf{p})(a^r_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + a^{r\,\dagger}_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}})$$

  and

$$\mathbf{E}(\mathbf{x}) = \int đ\mathbf{p}\,(-i)\sqrt{\frac{|\mathbf{p}|}{2}}\sum_{r=1}^{2}\boldsymbol{\epsilon}_r(\mathbf{p})(a^r_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} - a^{r\,\dagger}_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}).$$

  The polarizations $\boldsymbol{\epsilon}_r(\mathbf{p})$ remain vectors, not operators, obeying the same constraints as before.

- One can show that the commutation relations above lead to the usual result

$$[a^r_{\mathbf{p}}, a^{s\,\dagger}_{\mathbf{q}}] = \delta^{rs}\not{\delta}(\mathbf{p}-\mathbf{q})$$

with all others zero. This can be derived using the completeness relation for the polarizations,

$$\sum_{r=1}^{2} \epsilon^i_r(\mathbf{p})\epsilon^j_r(\mathbf{p}) = \delta^{ij} - \frac{p^i p^j}{|\mathbf{p}|^2}.$$

- Finally, we substitute our mode expansion back into the Hamiltonian, which gives

$$H = \int đ\mathbf{p}\,|\mathbf{p}| \sum_{r=1}^{2} a^{r\,\dagger}_{\mathbf{p}} a^r_{\mathbf{p}}$$

which confirms the excitations all have energy $p$.

- Switching to Heisenberg picture and applying the completeness relation gives the propagator

$$D_{ij}(x-y) = \langle 0|TA_i(x)A_j(y)|0\rangle = \int đp \frac{i}{p^2+i\epsilon}\left(\delta_{ij} - \frac{p_i p_j}{|\mathbf{p}|^2}\right) e^{-ip(x-y)}.$$

Unfortunately, the result is far from Lorentz invariant. Instead we'll redo the quantization procedure keeping Lorentz invariance explicit throughout.

**Note.** When treating the electromagnetic field, we typically either consider the quantized version, or the classical limit, which arises out of states with many photons. However, it is also valid to consider states with one photon, and see what the quantum theory says about them. For massive fields, the one-particle states would be useful in the nonrelativistic limit, where there isn't enough energy to make more particles. For photons, this doesn't apply, but considering one photon at a time is still interesting since many processes involve only one, or a few photons.

We can write a general state containing one photon as

$$|\psi\rangle = \int đ\mathbf{p} \sum_{r=1}^{2} \boldsymbol{\epsilon}_r(\mathbf{p})\psi_r(\mathbf{p}) a^{r\,\dagger}_{\mathbf{p}}|0\rangle.$$

Here, we are working with a basis of one-photon states that diagonalizes momentum $\mathbf{p}$ and helicity, so $\psi_r(\mathbf{p})$ is the "wavefunction in momentum/helicity space". It is normalized in the usual way,

$$\int đ\mathbf{p} \sum_{r=1}^{2} |\psi_r(\mathbf{p})|^2 = 1.$$

Unlike nonrelativistic quantum mechanics, in quantum field theory it is simpler to start in momentum space. We can define a position operator by noting that, by the canonical commutation relations, we should have $\mathbf{x} = -i\nabla$ where the gradient is in momentum space. (This operator is formally called the Newton–Wigner position operator. One can show that this is the unique candidate for $\mathbf{x}$ which transforms appropriately under the Poincare group.) There is no problem in defining a position space wavefunction, by taking the Fourier transform. Next, we can define the orbital angular momentum $\mathbf{L} = \mathbf{x}\times\mathbf{p}$, which has the same value on one-photon states as the orbital angular

momentum derived by Noether's theorem for the fields. The spin $\mathbf{S}$ is already defined, taking values $\pm\hat{\mathbf{p}}$ depending on the polarization, and the helicity is $h = \hat{\mathbf{p}} \cdot \mathbf{S}$.

However, this has a serious problem, which can be seen by defining a vector-valued wavefunction

$$\boldsymbol{\psi}(\mathbf{p}) = \sum_{r=1}^{2} \boldsymbol{\epsilon}_r(\mathbf{p})\psi_r(\mathbf{p}).$$

This wavefunction obeys the transverse constraint, $\mathbf{p} \cdot \boldsymbol{\psi}(\mathbf{p}) = 0$, which means the position space wavefunction obeys $\nabla \cdot \boldsymbol{\psi}(\mathbf{x}) = 0$. But in position space, the position operator just multiplies by $\mathbf{x}$, which does not preserve this property. More formally, Wigner's classification tells us that photons only occupy helicity states $\lambda = \pm 1$, with $\lambda = 0$ forbidden. But $\mathbf{x}$ does not commute with helicity, since it does not commute with $\mathbf{p}$, so it introduces a $\lambda = 0$ component. One can restrict $\mathbf{x}$ to the $\lambda = \pm 1$ subspace, but then it no longer has the right transformation properties. This problem also applies to $\mathbf{L}$ and $\mathbf{S}$. Intuitively, these rotate the field rigidly in space, and the field value itself, and rotating the two independently will turn transverse polarizations into longitudinal ones. More generally, this problem does not occur for massive particles, where one can boost to the rest frame and straightforwardly construct the operators there. For massless particles, the problem only appears for helicities of $|\lambda| \geq 1$, since otherwise there are no missing helicities.

Note that $(\mathbf{p}, h)$ is not the only choice for a complete set of commuting observables for one-photon states. For example, if one wants to talk about angular momentum, one can diagonalize $(p, J^2, J_z, h)$ or $(p, J^2, J_z, P)$ where $P$ is parity. The position space wavefunctions involve spherical Bessel functions and spherical harmonics, and are known as vector multipole fields.

The obstruction above is usually described by mathematical physicists as saying that "the position of a photon is meaningless", which is probably overenthusiastic. There clearly is no problem with confining a photon in a cavity or a beam.

**Note.** There are yet more subtleties even when the Newton–Wigner position operator exists. In this case, there exist states that are perfectly localized in a given reference frame, but localization is not Lorentz invariant: the very same state is not perfectly localized upon applying a boost. (However, this might not be relevant in an experimental situation where the detector defines a preferred frame.) Furthermore, the wavefunction spreads out faster than the speed of light, a result known as the Hegerfeldt paradox. The easiest way to see that causality is not actually violated is to return to the field picture. This is one of the reasons modern particle physics sources often avoid the topic of one-photon wavefunctions: they have historically led to much confusion.

## 4.3 Gupta–Bleuler Quantization

Alternatively, we may impose the constraint at the level of the Hilbert space.

- We instead consider the new, "gauge-fixed" Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

  where $\xi$ is a number. Naively, we have changed the theory, and this will be justified below by showing the $S$-matrix is independent of $\xi$. But it's more correct to say that the non-gauge-fixed theory isn't defined as a quantum theory at all, and we're really checking that the specific way we gauge fix doesn't matter. We'll derive this again in a more satisfying way with the path integral, where the extra term falls out naturally by restricting the integral to gauge-inequivalent configurations.

- To motivate this choice, note that if $\xi$ were a field, the equations of motion would be

$$\partial^2 A^\mu - (1 - 1/\xi)\partial^\mu \partial^\nu A_\nu = 0, \quad \partial_\mu A^\mu = 0$$

so $\xi$ acts as a Lagrange multiplier that enforces Lorenz gauge.

- However, we don't want the extra auxiliary field $\xi$, so we instead let $\xi$ be a number. A natural choice is $\xi = 1$, where the equation of motion becomes $\partial^2 A^\mu = 0$ as it is in Lorenz gauge. However, note that the Lorenz gauge condition is not enforced, i.e. we don't have $\partial_\mu A^\mu = 0$.

- Next, we perform canonical quantization. The new term ensures all fields are dynamical,

$$\pi^0 = -\partial_\mu A^\mu, \quad \pi^i = \partial^i A^0 - \dot{A}^i.$$

Below, it will be more convenient to integrate by parts, yielding the new canonical momenta,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu, \quad \pi^\mu = -\dot{A}^\mu.$$

Next, we impose the Lorentz-invariant canonical commutation relations

$$[A_\mu(\mathbf{x}), \pi_\nu(\mathbf{y})] = i\eta_{\mu\nu}\delta(\mathbf{x} - \mathbf{y})$$

with all other commutators zero; note the extra minus sign, because of the minus sign in $\pi^\mu$. We haven't imposed any constraints, so there are no issues with Dirac brackets.

- Next, we perform the usual mode expansion, giving

$$\boxed{A_\mu(x) = \int \frac{đ\mathbf{p}}{\sqrt{2|\mathbf{p}|}} \sum_{\lambda=0}^{3} \epsilon_\mu^\lambda(\mathbf{p})(a_{\mathbf{p}}^\lambda e^{-ipx} + a_{\mathbf{p}}^{\lambda\dagger} e^{ipx})}$$

in Heisenberg picture, and

$$\pi_\mu(x) = \int đ\mathbf{p}\sqrt{\frac{|\mathbf{p}|}{2}}(+i) \sum_{\lambda=0}^{3} \epsilon_\mu^\lambda(\mathbf{p})(a_{\mathbf{p}}^\lambda e^{-ipx} - a_{\mathbf{p}}^{\lambda\dagger} e^{ipx})$$

where the canonical momentum has a $+i$ rather than a $-i$ because $\pi^\mu = -\dot{A}^\mu$.

- Consider a photon of momentum $p^\mu$ and let $n^\mu = (1, 0, 0, 0)$. Then we define
  - $\epsilon_\mu^0 = n_\mu$ is the timelike polarization
  - $\epsilon_\mu^1$ and $\epsilon_\mu^2$ are the transverse polarizations, obeying

$$\epsilon^i \cdot n = \epsilon^i \cdot p = 0, \quad \epsilon^i \cdot \epsilon^j = -\delta^{ij}$$

  - $\epsilon_\mu^3$ is the longitudinal polarization, defined to be orthogonal to all the others,

$$\epsilon_\mu^3 \propto p_\mu - n_\mu(p \cdot n)$$

and normalized to have norm $-1$.

For example, when $p^\mu = (E, 0, 0, E)$, these are simply the standard basis for $\mathbb{R}^4$. Using these definitions, we have the orthogonality and completeness relations

$$\epsilon^\lambda \cdot \epsilon^{\lambda'} = \eta^{\lambda\lambda'}, \quad \epsilon^\lambda_\mu \epsilon^{\lambda'}_\nu \eta_{\lambda\lambda'} = \eta_{\mu\nu}.$$

- The resulting commutation relations for the creation/annihilation operators are

$$[a^\lambda_{\mathbf{p}}, a^{\lambda'\dagger}_{\mathbf{q}}] = -\eta^{\lambda\lambda'} \not{\delta}(\mathbf{p} - \mathbf{q}).$$

Then there is a minus sign for the timelike polarizations, which have negative norm, as we saw for the incorrectly quantized spinors. That is, our naive canonical quantization procedure hasn't given us a Hilbert space at all, because the inner product is not positive definite.

Next, we remove the unphysical degrees of freedom by imposing the Lorenz gauge constraint on the naive Fock space.

- The simplest idea is to set $\partial_\mu A^\mu = 0$ as an operator equation in Heisenberg picture, but this is unacceptable as it sets $\pi^0 = 0$. We can also attempt to identify a set of physical states by

$$\partial_\mu A^\mu |\Psi\rangle = 0.$$

However, this is also too restrictive since not even the vacuum is a physical state.

- Finally, consider writing $A_\mu = A^+_\mu + A^-_\mu$ where $A^+_\mu$ has positive frequency, and hence contains only annihilation operators. Then we define physical states by the Gupta-Bleuler condition

$$\partial^\mu A^+_\mu |\Psi\rangle = 0$$

so that the vacuum is indeed physical. This is equivalent to

$$\langle \Psi' | \partial_\mu A^\mu |\Psi\rangle = 0.$$

That is, we demand the operator $\partial_\mu A^\mu$ is zero when restricted to the physical states.

- Explicitly, we define the one-photon states by

$$|\mathbf{p}, \xi\rangle = \sqrt{2|\mathbf{p}|}\, \alpha_\lambda a^\dagger_\lambda(\mathbf{p})|0\rangle, \quad \xi^\mu = \sum_{\lambda,\lambda'} \alpha_\lambda \eta_{\lambda\lambda'} \epsilon^{\mu\lambda'}$$

where $\xi$ is a polarization vector. Then we have

$$\langle \mathbf{q}, \xi' | \mathbf{p}, \xi\rangle = -2|\mathbf{p}| \not{\delta}(\mathbf{p} - \mathbf{q})\, (\xi' \cdot \xi).$$

- Using the strange convention above, the Gupta-Bleuler condition is simply

$$p^\mu \xi_\mu = 0.$$

This eliminates the negative-norm timelike photons, but we still have the zero-norm states

$$|\mathbf{p}, \xi_S\rangle \propto (a^\dagger_0(\mathbf{p}) - a^\dagger_3(\mathbf{p}))|0\rangle$$

which are combinations of timelike and longitudinal photons, with $p^\mu \propto \xi^\mu$. We need a relative minus sign between the terms above, even though $p^\mu \propto (1, 0, 0, 1)$, because of the minus sign in the commutation relations for timelike photons.

- More generally, a basis for our Fock space contains elements of the form $|\Psi\rangle = |\psi\rangle|\phi\rangle$ where $|\psi\rangle$ contains transverse photons and $|\phi\rangle$ contains timelike and longitudinal photons. Then the allowed $|\phi\rangle$ states satisfy the Gupta-Bleuler condition

$$(a^3_{\mathbf{p}} - a^0_{\mathbf{p}})|\phi\rangle = 0$$

so that the allowed $|\phi\rangle$ are of the form $|\phi_n\rangle = (a^{0\,\dagger}_{\mathbf{p}} - a^{3\,\dagger}_{\mathbf{p}})^n|0\rangle$, which have zero norm for $n > 0$. The presence of these states reflects the fact that Lorenz gauge is not a complete gauge fixing.

- We claim the zero-norm states decouple, in the sense that they have zero expectation value for all gauge-invariant observables. For example, the Hamiltonian is

$$H = \int đ\mathbf{p}\,|\mathbf{p}| \left( \sum_{i=1}^{3} a^{i\,\dagger}_{\mathbf{p}} a^i_{\mathbf{p}} - a^{0\,\dagger}_{\mathbf{p}} a^0_{\mathbf{p}} \right).$$

However, the Gupta-Bleuler condition implies $\langle\phi|a^{3\,\dagger}_{\mathbf{p}} a^3_{\mathbf{p}} - a^{0\,\dagger}_{\mathbf{p}} a^0_{\mathbf{p}}|\phi\rangle = 0$, so the $|\phi_n\rangle$ have zero energy. Thus, for a free theory, one can think of the states $|\psi\rangle|\phi_n\rangle$ as all being 'gauge equivalent' to $|\psi\rangle|\phi_0\rangle$, yielding a Hilbert space where all states have positive norm.

- On the other hand, zero-norm states do affect the expectation value of $A_\mu$, as we have

$$\langle\phi_n|A_\mu(x)|\phi_n\rangle = \partial_\mu\alpha, \quad \partial^\mu\partial_\mu\alpha = 0.$$

This is acceptable since $A_\mu$ is not gauge-invariant. Indeed, this is the exact analogue of the remaining classical gauge freedom $A_\mu \to A_\mu + \partial_\mu\alpha$ with $\partial^\mu\partial_\mu\alpha = 0$.

- We cannot ignore the zero-norm states entirely. In an interacting theory, they appear as intermediate states in scattering processes, though we will see that the Ward identity guarantees they decouple from initial and final states. They are also required to maintain Lorentz invariance, because a boost can turn a transverse polarization into a partly longitudinal one.

- Finally, we have the Lorentz-invariant propagator

$$\boxed{\langle 0|TA_\mu(x)A_\nu(y)|0\rangle = \int đp \frac{i}{p^2 + i\epsilon} \left( -\eta_{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2} \right) e^{-ip(x-y)}}$$

where above we worked with $\xi = 1$, the numerator comes from the sum of all four polarizations, and the minus sign ensures the right sign for the spacelike polarizations. The choice $\xi = 1$ is called Feynman gauge, while $\xi \to 0$ is called Landau gauge. Landau gauge is the quantum version of Lorenz gauge, as the Lorenz gauge condition is imposed exactly in this limit.

## 4.4 Coupling to Matter

Finally, we couple our theory of light to matter.

- While naively we can write down any Lorentz-invariant interaction term involving $A_\mu$, we must be careful to maintain decoupling of the negative-norm and zero-norm states. By the Feynman rules, a scattering amplitude involving an external photon of momentum $k$ and polarization $\zeta_\mu(k)$ has the form

$$\mathcal{M} = \zeta_\mu(k)\mathcal{M}^\mu.$$

Note that by the Feynman rules, $\mathcal{M}^\mu$ depends on $k$ but not on $\zeta$.

- The Ward identity states that
$$k_\mu \mathcal{M}^\mu = 0.$$
This ensures the amplitude to produce a photon with $\zeta_\mu(k) \propto k_\mu$ is zero, which means the zero-norm states decouple as desired. We'll show later that the Ward identity implies that all unphysical polarizations decouple.

- Suppose the coupling to matter takes the form $A_\mu j^\mu$. The classical equation of motion is
$$\partial_\mu F^{\mu\nu} = j^\nu$$
which implies that $\partial_\mu j^\mu = 0$, so $j^\mu$ must be a conserved current. Heuristically, at the quantum level, since $j^\mu$ creates photons we have
$$M^\mu(k) \sim \int dx\, e^{ikx} \langle f | j^\mu(x) | i \rangle$$
so the Ward identity follows from conservation of the current $j^\mu$.

- More generally, it can be shown that $j^\mu$ must be the conserved current arising from a global $U(1)$ symmetry of the action. To maintain gauge invariance, this symmetry must be gauged.

We will justify these statements more carefully later; now we turn to some examples. Note that different sources will differ on the sign of $e$ below.

- First, consider coupling to Dirac fermions. We have a global $U(1)$ symmetry from phase rotation $\psi(x) \to e^{-ie\alpha}\psi(x)$, where we have introduced a dimensionless coupling constant $e$, which yields the conserved current $j^\mu = e\overline{\psi}\gamma^\mu\psi$.

- The interaction term can be written neatly with the covariant derivative,
$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\not{D} - m)\psi, \quad D_\mu\psi = \partial_\mu\psi + ieA_\mu\psi.$$
We gauge the global symmetry to
$$A_\mu \to A_\mu + \partial_\mu\alpha, \quad \psi \to e^{-ie\alpha}\psi$$
where $\alpha$ is a general function of $x$. Then the action is gauge-invariant because
$$D_\mu\psi \to \partial_\mu(e^{-ie\alpha}\psi) + ie(A_\mu + \partial_\mu\alpha)(e^{-ie\alpha}\psi) = e^{-ie\alpha}D_\mu\psi.$$

- Given the coupling $A_\mu j^\mu$, the component $j^0$ can be interpreted as electric charge. Tracing back to our earlier results, this shows that particles and antiparticles have charge $\pm e$. One can also show that charge conjugation turns $e$ to $-e$, as expected.

- Note that a gauge transformation with constant parameter is just the same as a global transformation, which does *not* take us to the same physical state; this must be true because we can infer a conservation law from the global symmetry. Only gauge transformations where the parameter goes to zero at infinity are 'true' gauge transformations.

- Next, we turn to scalar fields. A real scalar field has no suitable conserved current, so we focus on a complex scalar field $\varphi$, which has a symmetry $\varphi \to e^{-ie\alpha}\varphi$ and conserved current

$$j^\mu = ie(\varphi^\dagger \partial^\mu \varphi - (\partial^\mu \varphi^\dagger)\varphi).$$

  However, the naive coupling $j^\mu A_\mu$ doesn't work, because $j^\mu$ depends on $\partial^\mu \varphi$. Then if we add the term $j^\mu A_\mu$ to the Lagrangian, the conserved quantity associated with the $U(1)$ global symmetry changes, so $j^\mu$ is no longer conserved!

- It's possible to fix this by manually adding terms, but a simpler, reliable method is 'minimal coupling', i.e. replacing partial derivatives with covariant derivatives as above. We define

$$D_\mu \phi = (\partial_\mu + ieA_\mu)\phi, \quad \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^\dagger D^\mu \phi - m^2\phi^\dagger\phi$$

  which is gauge-invariant by the same reasoning. The interaction terms are

$$\mathcal{L}_{\text{int}} = -\left(ie(\phi^\dagger\partial^\mu\phi - (\partial^\mu\phi^\dagger)\phi)A_\mu - e^2 A_\mu A^\mu \phi^\dagger\phi\right)$$

  which includes a two-photon vertex with a factor of 2. The equation of motion can be read off by minimal coupling,

$$(D_\mu D^\mu + m^2)\phi = 0.$$

- The new conserved current associated with the $U(1)$ global symmetry is

$$j'^\mu = ie(\phi^\dagger D^\mu \phi - (D^\mu\phi)^\dagger\phi)$$

  and one can check that $\mathcal{L}_{\text{int}} = -j'^\mu A_\mu$ as required. But this final result would have been rather difficult to guess without the benefit of the covariant derivative. Also note that to lowest order in $e$, the coupling is $-j^\mu A_\mu$. This is a universal result for gauge theories.

**Note.** What if we wished to introduce a mass term for the photon field? The Proca action is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu.$$

The second term is not gauge invariant, so the theory does not have a gauge symmetry. The equation of motion is

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0$$

so that, as long as the mass term is nonzero, we automatically have $\partial_\mu A^\mu = 0$. Plugging this back into the equation of motion yields the Klein–Gordan equation,

$$(\partial^2 + m^2)A^\mu = 0.$$

Hence we have three massive degrees of freedom. The quantization of this theory is much more straightforward than the quantization of QED. We have only the second-class constraints $\partial_\mu A^\mu = 0$ and $\pi^0 = 0$, and Dirac brackets get the job done.

In fact, it is possible to obtain the Proca action from a gauge invariant action by gauge fixing; this is called the Stuckelberg trick. The Stuckelberg action is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(mA_\mu - \partial_\mu\phi)^2.$$

The action has the gauge symmetry

$$\delta A_\mu = \partial_\mu \epsilon, \quad \delta\phi = m\epsilon.$$

We recover the Proca action by gauge fixing $\phi = 0$. The benefit of starting from the Stuckelberg action is that the gauge symmetry constrains the divergences that may appear in perturbation theory, allowing one to prove the theory is renormalizable. Since the $\phi$ field is eaten, the Stuckelberg theory can be thought of as the abelian Higgs model in the limit where the Higgs is very heavy, and decouples. The Stuckelberg trick doesn't work for a non-abelian gauge theory, at least in this form, as in that case $\delta A_\mu$ is more complicated.

## 4.5 Feynman Rules for QED

Now we write down the Feynman rules for scattering amplitudes QED, in Feynman gauge.

- Each interaction vertex has two fermion legs and one photon leg, and gives the factor $-ie\gamma^\mu$. Here, the spinor indices are contracted along the fermion lines as usual, while the Lorentz indices are contracted along photon lines. Momentum is conserved at each vertex and loop momenta are integrated as usual.

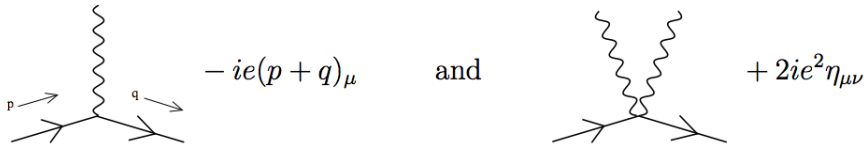
- Internal lines get the Feynman propagators,

$$\text{fermion: } \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}, \quad \text{photon: } \frac{-i\eta^{\mu\nu}}{p^2 + i\epsilon}.$$

- External fermion receive the usual factors, while incoming and outgoing photons of momentum $\mathbf{p}$ receive factors $\epsilon_\mu(\mathbf{p})$ and $\epsilon^*_\mu(\mathbf{p})$, respectively. As with the fermion factors, this can be derived from the LSZ analysis for photon fields or less rigorously in the interaction picture.

- Strictly speaking, both external fermion and photon lines should receive factors of $\sqrt{Z_e}$ and $\sqrt{Z_A}$, respectively. We ignore these issues for now because we are focusing on tree-level QED.

- For charged scalars, we have the Feynman rules




  The factor of 2 arises because the $A^\mu A_\mu \phi^\dagger \phi$ term has two identical particles but no compensating factor of 1/2!. Note that for the sign to be right in the first term, the momentum arrows need to be oriented along the particle flow arrows.

**Note.** If we had quantized in Coulomb gauge, we would have more work to do. In the presence of matter, we no longer have $A_0 = 0$. Instead we find $\nabla^2 A_0 = -ej^0$ and we may eliminate $A_0$ by substituting

$$A_0(\mathbf{x}, t) = e \int d\mathbf{x}' \, \frac{j^0(\mathbf{x}', t)}{4\pi|\mathbf{x} - \mathbf{x}'|}.$$

However, this makes the action nonlocal in $j$. Thus we have both a non-Lorentz invariant propagator and a nonlocal current-current interaction. With some effort, they can be combined into the Lorentz invariant propagator above.

Symmetries give us some useful identities.

- If we had quantized with a general $\xi$ above, we would have found the propagator

$$-\frac{i}{p^2+i\epsilon}\left(\eta^{\mu\nu}+(\xi-1)\frac{p^\mu p^\nu}{p^2}\right).$$

  We will show later that we always get the same quantum theory regardless of the value of $\xi$.

- For now, we can show that $\xi$ doesn't matter at tree level. For concreteness, consider $s$-channel electron-positron scattering, with incoming momenta $p$ and $q$. Then the photon has momentum $p+q$, and the extra term above contributes

$$\overline{v}(p)\gamma^\mu(p_\mu+q_\mu)u(q)=(\overline{v}(p)\not{p})u(q)+\overline{v}(p)(\not{q}u(q))=(m-m)\overline{v}(p)u(q)=0$$

  where we used the on-shell spinor conditions,

$$(\not{p}-m)u(\mathbf{p})=\overline{u}(\mathbf{p})(\not{p}-m)=(\not{p}+m)v(\mathbf{p})=\overline{v}(\mathbf{p})(\not{p}+m)=0.$$

  Similar reasoning holds for all tree-level diagrams.

- The Ward identity gives a useful result for polarization sums. Define $\mathcal{M}^\mu$ as above and let $k^\mu\propto(1,0,0,1)$. Then the Ward identity implies

$$k_\mu\mathcal{M}^\mu=0,\quad \mathcal{M}^0=\mathcal{M}^3.$$

  Now, the sum over physical polarizations is

$$\sum_{\lambda=1}^{2}|\epsilon_\mu^\lambda\mathcal{M}^\mu|^2=|\mathcal{M}^1|^2+|\mathcal{M}^2|^2=|\mathcal{M}^1|^2+|\mathcal{M}^2|^2+|\mathcal{M}^3|^2-|\mathcal{M}^0|^2=-\eta_{\mu\nu}\mathcal{M}^\mu\mathcal{M}^\nu$$

  by the Ward identity, so we can replace a sum over physical polarizations with a sum over all polarizations; this will be useful in some QED calculations.

- This result shows why it is possible to ignore the negative-norm states: the amplitudes to produce the unphysical states cancel out. Let $P$ be the projector onto the space of transverse polarizations. Then our result above heuristically says

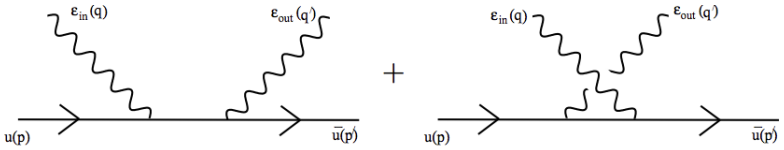
$$S^\dagger PS=S^\dagger S.$$

  Using the unitarity of the $S$-matrix on the entire Fock space, we have $(PSP)^\dagger(PSP)=P$. Then the restriction of the $S$-matrix to the subspace of transverse polarizations is unitary, as desired.

- Note that unphysical polarizations are still produced in intermediate states, as shown by the photon propagator being proportional to $\eta_{\mu\nu}$. We cannot simply ignore these unphysical polarizations entirely, because the transverse polarizations alone are not Lorentz invariant; one can perform a Lorentz transformation on the field to convert a transverse polarization into an unphysical one. That is, gauges that restrict to transverse polarizations only, such as Coulomb gauge, are necessarily not Lorentz invariant.

- In the case of a massive vector field, there is no gauge symmetry; instead we have three physical polarizations and a single negative-norm polarization with $\epsilon^\mu(k) \propto k^\mu$. In this case, a direct application of the Ward identity $k_\mu \mathcal{M}^\mu = 0$, which still holds, is enough to show the unphysical state decouples.

**Example.** Compton scattering. Two diagrams contribute to the amplitude $i\mathcal{M}$, as shown below.




$$= \; i(-ie)^2 \bar{u}^{r'}(\vec{p}') \left( \frac{\gamma_\mu (\not{p} + \not{q} + m)\gamma_\nu}{(p+q)^2 - m^2} + \frac{\gamma_\nu (\not{p} - \not{q}' + m)\gamma_\mu}{(p-q')^2 - m^2} \right) u^s(\vec{p}) \, \epsilon_{\text{in}}^\nu \, \epsilon_{\text{out}}^\mu$$

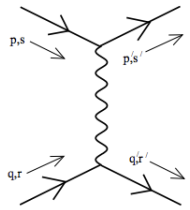
If we wish to sum over final polarizations and average over initial polarizations, we could use the identity derived above from the Ward identity. Alternatively, we can use this result to check the Ward identity. Setting $\epsilon_{\text{in}} = q$, we get

$$i\mathcal{M} \propto \frac{2p \cdot q}{(p+q)^2 - m^2} + \frac{2p' \cdot q}{(p'-q)^2 - m^2} = 1 - 1 = 0$$

where we used the on-shell spinor equations and $q^2 = \not{q}\not{q} = 0$. Note that the Ward identity doesn't hold for each diagram individually; it is a result for amplitudes, not diagrams.

Finally, we recover the Coulomb potential.

- First, consider electron-electron scattering, with the following amplitude.




$$= -i(-ie)^2 \, \frac{[\bar{u}(\vec{p}')\gamma^\mu u(\vec{p})]\,[\bar{u}(\vec{q}')\gamma_\mu u(\vec{q})]}{(p'-p)^2}$$

- In the nonrelativistic limit, the spinors become $u(p) \to \sqrt{m}(\xi, \xi)^T$. Then the only term in the spinor contractions that contributes is $\mu = 0$ (which means the interaction is mediated by a timelike photon) giving a numerator of $(2m)^2$, canceling the relativistic normalization.

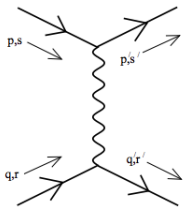
- Working in the center-of-mass frame, the process is elastic, so the denominator gives a factor of $-(\Delta \mathbf{p})^2$ where $\Delta \mathbf{p}$ is the momentum transfer. The nonrelativistic amplitude for scattering is $\mathcal{M} \sim -i\widetilde{U}(\Delta \mathbf{p})$ by the Born approximation, giving

$$U(\mathbf{p}) = \frac{e^2}{p^2}$$

  which is a repulsive Coulomb potential.

- Next, we consider electron-positron scattering. In this case, we pick up a sign flip from fermionic statistics, as in Yukawa theory. In Yukawa theory, there was an second sign flip since $\overline{v}v \to -2m$, but here we instead have $\overline{v}\gamma^0 v \to 2m$. Then all the logic above goes through as before, and the result is an attractive Coulomb potential.

- Thus, we have shown that while forces mediated by spin-0 particles are universally attractive, forces mediated by spin-1 particles are not. One might say the sign flip really comes from the negative norm of the timelike photon.
- The same sign flip can be found in scalar QED. For example, we have:




$$= -i\eta_{\mu\nu}(-ie)^2 \frac{(p+p')^\mu (q+q')_\nu}{(p'-p)^2} \;\to\; -i(-ie)^2 \frac{(2m)^2}{-(\vec{p}-\vec{p}')^2}$$

  Then particles with the same charge repel, and again only $A_0$ contributes. If we flipped the charge of one of the incoming scalars, then we would get a sign flip because the momentum factors are correlated with the charge flow, giving attraction.

**Note.** More intuition for the sign. We consider coupling to a current $j^\mu$, so that at lowest order the energy of interaction of two currents is

$$E \sim \int dk\, J^\mu(k)^* D_{\mu\nu}(k) J^\nu(k).$$

We work with a massive photon for simplicity. Naively, since each component of $A_\mu$ satisfies the Klein–Gordan equation, we should sum over all four polarizations, giving a numerator of $\eta_{\mu\nu}$. However, since we automatically have $\partial_\mu A^\mu = 0$, the polarizations satisfy $k^\mu \epsilon_\mu = 0$, or equivalently we are missing the "timelike" polarization $\epsilon^\mu \sim k^\mu$, called so because it is $(1,0,0,0)$ in the particle's rest frame. Then in a general frame, we have

$$\sum_\lambda \epsilon_\lambda^{\mu *}(k)\epsilon_\lambda^\nu(k) = -\eta^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}.$$

This gives the propagator for a massive vector,

$$D_{\mu\nu}(k) = \frac{-i\eta_{\mu\nu} + ik_\mu k_\nu/m^2}{k^2 - m^2 + i\epsilon}.$$

Now we take the mass to zero. For a massless photon, we can only couple to a conserved current, so that $k_\mu J^\mu(k) = 0$ and the singular second term vanishes; then the spacelike components have the usual sign and the timelike component has the opposite sign. This is why parallel currents attract in electromagnetism but like charges repel!

**Note.** Guessing the propagator for gravity. In general, the numerator of a propagator is simply the sum of projections onto polarizations. Define the spin 1 numerator as $-G_{\mu\nu}(k)$, we have

$$\sum_a \epsilon_\mu^{(a)}(k)\epsilon_\nu^{(a)}(k) = -G_{\mu\nu}(k)$$

as can be checked by explicit computation in the rest frame. Now for spin 2, there are five matrix polarizations satisfying

$$k^\mu \epsilon_{\mu\nu}^{(a)} = 0, \quad \eta^{\mu\nu}\epsilon_{\mu\nu}^{(a)} = 0$$

which project away the spin 1 and spin 0 components. Applying these constraints to the most general candidate built out of $G_{\mu\nu}$ and $k_\mu$ gives

$$D_{\mu\nu,\lambda\sigma}(k) = i\frac{G_{\mu\lambda}G_{\nu\sigma} + G_{\mu\sigma}G_{\nu\lambda} - (2/3)G_{\mu\nu}G_{\lambda\sigma}}{k^2 - m^2 + i\epsilon}$$

where the overall constant can be found by evaluating in the rest frame. Gravity couples to the stress-energy tensor, and its conservation means we can replace $G_{\mu\nu}$ with $\eta_{\mu\nu}$. It turns out that in proper GR, the massless limit converts the 2/3 to a 1.

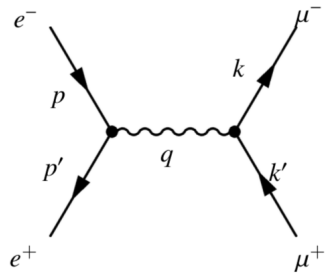
The sign of the gravitational force is determined by the sign of $D_{00,00}$, which is indeed positive. (Another way of understanding this is that the sign flip for spin 1 occurs twice; universal attraction holds for even spins.) Pressure is also attractive by the same amount as energy density, since $D_{ii,ii} = D_{00,00}$, as we know from the notes on General Relativity.

# 5 Quantum Electrodynamics

## 5.1 Cross Sections and Spin Sums

In this section, we do an extended example to demonstrate 'technology' for QED calculations.

- We consider $e^+e^- \to \mu^+\mu^-$, which is simple since there is only one diagram, the $s$-channel.




  We set the initial spin polarizations to $s$ and $s'$ and the final polarizations to $r$ and $r'$.

- Applying the Feynman rules, we have

$$i\mathcal{M}(s,s',r,r') = \frac{ie^2}{q^2}(\overline{v}_{s'}(p')\gamma^\mu u_s(p))(\overline{u}_r(k)\gamma_\mu v_{r'}(k')), \quad q = p + p' = k + k'.$$

  Often, the initial state is unpolarized, and the detector cannot measure the polarization of the final state. Then the squared amplitude relevant for the cross section is

$$|\mathcal{M}|^2_{\text{tot}} = \frac{1}{2}\sum_s \frac{1}{2}\sum_{s'}\sum_r\sum_{r'} |\mathcal{M}(s,s',r,r')|^2.$$

- Using the identities $(\gamma^0)^\dagger = \gamma^0$ and $\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger$, we have

$$(\overline{v}_{s'}\gamma^\mu u_s)^* = \overline{u}_s\gamma^\mu v_{s'}.$$

  A similar rule holds in general: the gamma matrices just reverse in order, and there's an additional sign flip for every factor of $\gamma^5$ since $\gamma^0\gamma^5\gamma^0 = -\gamma^5 = (-\gamma^5)^\dagger$. For example,

$$(\overline{v}_{s'}\gamma^\mu\gamma^\nu\gamma^\rho u_s)^* = \overline{u}_s\gamma^\rho\gamma^\nu\gamma^\mu v_{s'}.$$

  In our current case, the total squared amplitude has two factors,

$$|\mathcal{M}|^2_{\text{tot}} = \frac{1}{4}\frac{e^4}{q^4}\sum_{ss'rr'}(\overline{u}_s\gamma^\mu v_{s'}\overline{v}_{s'}\gamma^\nu u_s)(\overline{v}_{r'}\gamma_\mu u_r\overline{u}_r\gamma_\nu v_{r'})$$

  where we suppress momentum arguments.

- We can simplify this using the completeness relations

$$\sum_s u_s(p)\overline{u}_s(p) = \not{p} + m_e, \quad \sum_s v_s(p)\overline{v}_s(p) = \not{p} - m_\mu.$$

  Carefully keeping track of spinor indices, we find spinor traces, giving

$$|\mathcal{M}|^2_{\text{tot}} = \frac{1}{4}\frac{e^4}{q^4}\operatorname{tr}((\not{p} + m_e)\gamma^\mu(\not{p}' - m_e)\gamma^\nu)\operatorname{tr}((\not{k}' - m_\mu)\gamma_\mu(\not{k} + m_\mu)\gamma_\nu).$$

  More generally, we will have multiple diagrams that contribute, so the squared amplitude will have cross-terms.

- Thus, we need to evaluate the traces of gamma matrices. We use the identities

$$\text{tr}(\text{odd number of } \gamma^\mu) = 0, \quad \text{tr}\,\gamma^\mu\gamma^\nu = 4\eta^{\mu\nu}, \quad \text{tr}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}).$$

  For pseudoscalar and pseudovector interactions, we will also need

$$\text{tr}(\gamma^5(\text{zero to three } \gamma^\mu)) = 0, \quad \text{tr}\,\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = -4i\epsilon^{\mu\nu\rho\sigma}.$$

- These identities can generally be proven by inserting the identity, in terms of $(\gamma^0)^2$ or $(\gamma^5)^2$, and using the anticommutation relations; the $\gamma^5$ identities can be shown by noting that the result must be a totally antisymmetric tensor.

- A more general technique is to recall that $S[\Lambda]^{-1}\gamma^\mu S[\Lambda] = \Lambda^\mu{}_\nu\gamma^\nu$. Therefore, if we "transform" each gamma matrix on the left-hand side by conjugating by $S[\Lambda]$, the trace remains invariant, but the right-hand side transforms like a Lorentz tensor, or a pseudotensor if $\gamma^5$ appears. Thus, the right-hand side must be a Lorentz invariant (pseudo)tensor, which can only be built from $\eta^{\mu\nu}$ and $\epsilon^{\mu\nu\rho\sigma}$. This fixes the answer in all cases above, and is also useful for larger traces.

- We can express antisymmetrized products of gamma matrices as

$$\gamma^5 = -\frac{i}{24}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma, \quad \gamma^{[\nu}\gamma^\rho\gamma^{\sigma]} = i\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma^5.$$

  In fact, the identity, $\gamma^\mu$, $\gamma^{[\mu}\gamma^{\nu]}$, $\gamma^\mu\gamma^5$, and $\gamma^5$ form a basis for the space of $4 \times 4$ matrices; we used this when analyzing all Dirac bilinears above.

- Finally, we have

$$\gamma^\mu\gamma_\mu = 4, \quad \gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu, \quad \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4\eta^{\nu\rho}, \quad \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu.$$

  However, these particular identities change in dimensional regularization, as we'll see below.

- Sometimes it's useful to rearrange things using the Clifford algebra,

$$\not{p}\not{q} = 2p \cdot q - \not{q}\not{p}.$$

- Note that the $\gamma^5$ identities use the convention

$$\epsilon^{0123} = 1, \quad \epsilon_{0123} = -1.$$

  This is the opposite of the convention in general relativity, where the volume form naturally has lowered indices, so $\epsilon_{0123} = \sqrt{|g|} > 0$.

- Applying these identities to our result and setting $m_e$ to zero gives

$$|\mathcal{M}|^2_{\text{tot}} = \frac{8e^4}{q^4}\left((p \cdot k)(p' \cdot k') + (p' \cdot k)(p \cdot k') + (p \cdot p')m_\mu^2\right).$$

  Working in the center of mass frame, let the initial energies of the particles be $E$, and let the outgoing muon come out at an angle $\theta$ from the incoming electron. Then

$$|\mathcal{M}|^2_{\text{tot}} = e^4\left(1 + m_\mu^2/E^2 + (1 - m_\mu^2/E^2)\cos^2\theta\right).$$

- Finally, we may substitute this into the $2 \to 2$ scattering cross section and integrate for

$$\sigma = \frac{4\pi\alpha^2}{3E_{\text{cm}}}\sqrt{1 + m_\mu/E}\,(1 + m_\mu^2/2E^2), \quad \alpha = \frac{e^2}{4\pi}.$$

  Here, the factor in parentheses is determined by the dynamics of QED, while everything else is determined by the "kinematics", i.e. by dimensional analysis and the volume of available phase space. For example, for $E_{\text{cm}} \gg m_\mu$ we must have $\sigma \propto 1/E_{\text{cm}}^2$ by dimensional analysis.

**Note.** A taste of real experimental particle physics. Currently, there are many experiments searching for the dark photon, a hypothetical particle whose motivation is described in the notes on Cosmology. The dark photon acts like the regular photon, but it has a mass $m_{A'}$ and couplings $\epsilon e$ to fermions, where $\epsilon$ is the kinetic mixing parameter. Accelerator experiments attempt to produce the dark photon and detect its decay. The dominant production channel depends on the experiment, such as whether it uses lepton or hadron beams, but some important production channels are:

- Electron-positron annihilation, $e^+e^- \to \gamma A'$. At high energies, the rate for this is $\epsilon^2$ smaller than the rate for $e^+e^- \to \gamma\gamma$.
- $A'$ Bremsstrahlung. Most processes can also produce an $A'$ by attaching it to a fermion line.
- Meson decay. For example, pions can decay as $\pi^0 \to \gamma\gamma$, though we won't be able to calculate this rate for some time. However, it is simply related to the rate for $\pi^0 \to \gamma A'$.

The $A'$ can then decay, e.g. through $A' \to e^+e^-$, which is trivial to calculate given what we already know, or through $A' \to \pi^+\pi^-$, which is somewhat more tricky. The way we detect this decay depends on the lifetime of the $A'$, which generally scales as $\tau \sim 1/\Gamma \sim 1/\alpha\epsilon^2 m_{A'}$. The dark photon will travel a distance $\gamma c\tau$, but the opening angle of the decay products is $\sim 1/\gamma$, so the quantity that determines experimental resolution turns out to be

$$\ell \sim c\tau \sim (10^{-6}\,\text{m})\left(\frac{10^{-3}}{\epsilon}\right)^2 \frac{10\,\text{MeV}}{m_{A'}}.$$

For $\ell \lesssim 10\,\mu$m, the decay is "prompt", i.e. it happens at the same point as production, up to the detector resolution. In this case, individual dark photon events look like background events, but they can be found by computing the invariant mass of the decay products and looking for a statistically significant bump at $m_{A'}$. For larger $\ell$, we enter the "displaced" regime, where the decay products can be measured to be coming from a point away from the interaction point. Here the signal rate tends to be lower, but measuring a displaced decay dramatically cuts down on backgrounds. Finally, if $\ell$ is larger than the detector, we enter the "long-lived" regime, where the dark photon flies out undetected. These can be detected by looking for "missing energy/momentum", but this procedure introduces its own backgrounds and uncertainties. Alternatively, one can search for the dark photon's decay inside a shielded secondary detector, far downstream of the main detector. These events would be very rare, but backgrounds are very low. Furthermore, in this case the dark photon doesn't need to be traced back to the event that produced it, so we can enhance the luminosity by using a "beam dump", crashing a beam into a wall rather than colliding it with another beam. Every one of these options is being considered by several completed, ongoing and proposed experiments around the world, including BaBar, Belle II, APEX, MAMI, MESA, DarkLight, HPS, DarkQuest, NA48, and NA42 (for prompt and displaced decays) and GAZELLE, CODEXb, FASER, MATHUSLA, E137, E141, E774, NA64, CHARM, NuCal, and NA62 (for long-lived particles).

## 5.2 Ward–Takahashi Identity

In this section, we justify the statements about the Ward identity made earlier. We begin by covering the Schwinger–Dyson equations.

- For simplicity, consider a free real scalar field. The classical equation of motion is

$$\frac{\delta S}{\delta \phi(x)} = -(\partial^2 + m^2)\phi(x) = 0.$$

  The quantum Heisenberg fields also satisfy the same equation.

- For the two-point function, we have

$$(\partial_x^2 + m^2) i\langle 0|T\phi(x)\phi(x_1)|0\rangle = \delta(x - x_1)$$

  where the delta function comes from differentiating a step function, and we use the equal-time commutation relations. By extending this reasoning, we have the Schwinger–Dyson equation

$$\boxed{\langle 0|T\frac{\delta S}{\delta \phi(x)}\phi(x_1)\dots\phi(x_n)|0\rangle = i\sum_{j=1}^{n}\langle 0|T\phi(x_1)\dots\hat{\phi}(x_j)\delta(x - x_j)\dots\phi(x_n)|0\rangle}$$

  where the hat indicates a missing argument; it is the analog of the classical 'equations of motion' for correlation functions.

- Heuristically, the Schwinger–Dyson equation says that correlation functions in classical field theory obey the classical equations of motion, while correlation functions in quantum field theory are corrected by 'contact terms' which correspond to the emission and absorption of virtual particles; accordingly, classical field theory Feynman diagrams have no loops.

- Off-shell, the Noether current associated with the symmetry $\delta\phi(x)$ satisfies

$$\partial_\mu j^\mu = -\frac{\delta S}{\delta \phi(x)}\delta\phi(x)$$

  so the Schwinger–Dyson equations become

$$\boxed{\partial_\mu\langle 0|Tj^\mu(x)\phi(x_1)\dots\phi(x_n)|0\rangle + i\sum_{j=1}^{n}\langle 0|T\phi(x_1)\dots\hat{\phi}(x_j)\delta\phi(x)\delta(x - x_j)\dots\phi(x_n)|0\rangle = 0.}$$

  This result is called the Ward–Takahashi identity; it is the analog of classical current conservation for correlation functions. They are generalized in Yang–Mills to the Slavnov–Taylor identities.

- We have only justified these results for a free theory, but the same results hold for an interacting theory, where $\delta S/\delta\phi(x)$ simply gets more complicated; we will justify this with the path integral later. One caveat is that correlation functions generally involve divergent loop integrals that must be regulated. If we cannot find a regulator that respects the classical symmetry, then the symmetry is anomalous and does not hold at the quantum level.

Now we see how these results imply the Ward identity.

- By the LSZ reduction formula, a scattering amplitude involving an external photon $|k^\mu, \xi^\mu(k)\rangle$ and $n$ other external particles has the form

$$\langle f|i\rangle \sim \xi^\mu(k) \int dx\, e^{-ikx} \partial_x^2 \dots \langle 0|TA_\mu(x) \dots |0\rangle.$$

- In Feynman gauge, the classical equation of motion for $A^\mu$ is $\partial^2 A^\mu = j^\mu$. Then we have

$$\partial_x^2 \langle 0|TA_\mu(x) \dots |0\rangle = \langle 0|Tj_\mu(x) \dots |0\rangle + \text{contact terms}$$

  where the contact terms contract our photon with another external photon. This represents a disconnected contribution and hence doesn't contribute to the connected $S$-matrix.

- Now set $\xi^\mu = k^\mu$. Integrating by parts turns the momentum into a derivative, giving

$$\partial^\mu \langle 0|Tj_\mu(x) \dots |0\rangle = \text{contact terms}$$

  by the Ward–Takahashi identity. Since $j^\mu$ is the conserved current associated with the $U(1)$ symmetry of the electron field, we get a contact term for each external electron. None of these terms contribute to the $S$-matrix because in each term, the momentum of one external leg will be displaced by $\pm k$ from where it should be; this proves the Ward identity.

- Alternatively, backing up a step and removing the LSZ pole cancellation factors, we have derived an identity between off-shell momentum-space correlation functions, where a correlation function involving an external photon is equal to a sum of correlation functions without it, where the external electron momenta are shifted by $\pm k$. This is the form of the Ward–Takahashi identity presented in Peskin, where it is proven diagrammatically.

**Note.** More generally, any amplitude $\mathcal{M}^{\mu\nu\cdots}\epsilon_\mu^{(1)}\epsilon_\nu^{(2)}\dots$ will vanish if any nonzero number of the $\epsilon^{(i)}$ are made longitudinal. However, in non-abelian gauge theory the equations of motion are more complicated, and the above argument is not applicable. It turns out that the amplitude still vanishes if any one of the $\epsilon^{(i)}$ are made longitudinal, but the rest must be transverse.

This leads to some subtleties. For processes involving off-shell gluons, we cannot naively replace sums over physical polarizations with $\eta_{\mu\nu}$ as we did for QED. However, the extra contribution due to including the extra unphysical polarizations can be cancelled by ghosts. For example, in the amplitude for $q\overline{q} \to gg$, the extra unphysical contribution can be cancelled by adding the amplitude for $q\overline{q} \to c\overline{c}$. This latter amplitude contributes *negatively* to the cross section, which is possible because the ghosts have negative norm.

We now explicitly link current conservation and gauge invariance.

- Write the action as

$$S = S_A[A] + S_{\text{int}}[A, \phi] + S_\phi[\phi].$$

  In retrospect, we see that our derivation of the Ward identity only requires $S$ to have a global continuous symmetry with conserved current $j^\mu$ that couples to $A^\mu$ as

$$\frac{\delta S_{\text{int}}[A, \phi]}{\delta A_\mu} = -j^\mu.$$

  This is the first step in the derivation, and conservation of $j^\mu$ is the second.

- We claim that if $A^\mu$ couples to a conserved current in a massless vector theory, the action $S$ must be gauge invariant. To see this, consider the global continuous symmetry $\phi(x) \to \phi(x) + \epsilon\delta\phi(x)$, which leaves $S$ invariant. If we promote $\epsilon$ to $\epsilon(x)$, then

$$\delta S = \int dx\, \tilde{j}^\mu \partial_\mu \epsilon(x) = -\int dx\, (\partial_\mu \tilde{j}^\mu)\epsilon(x)$$

  where the first step follows by Lorentz invariance, and because $\delta S$ is zero when $\epsilon(x)$ is constant. Comparing this to our earlier expression shows that $\tilde{j}^\mu = j^\mu$.

- Next, using our original hypothesis we have

$$\int dx \left(\frac{\delta S_{\text{int}}}{\delta A_\mu} + j^\mu\right) \partial_\mu \epsilon(x) = 0.$$

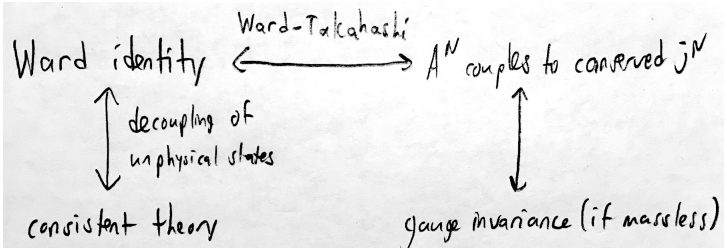
  As we've just seen, the second term is the change in the action due to the local transformation $\phi(x) \to \phi(x) + \epsilon(x)\delta\phi(x)$. The first term is the change in the action due to the local transformation $A_\mu \to A_\mu + \partial_\mu \epsilon(x)$, where we used the fact that $S_A[A]$ is gauge invariant. Thus we have shown that the full action has a gauge symmetry.

- The converse of this statement follows from running the steps in reverse. If there is a gauge symmetry, taking $\epsilon$ constant gives a global symmetry with conserved current $j^\mu$, and the variation of the action under the gauge transformation is

$$\int dx \left(\frac{\delta S_{\text{int}}}{\delta A_\mu} + j^\mu\right) \partial_\mu \epsilon(x) = 0$$

  as before. Since $\partial_\mu \epsilon(x)$ is arbitrary, the term in brackets is zero.

- In the case of a massive vector theory, there is no gauge invariance, but the Ward identity still holds if $A^\mu$ couples to a conserved current $j^\mu$. A map of equivalences is given below.




  All of these statements are so closely related that the Ward identity is sometimes called a statement of 'current conservation' or 'gauge invariance'.

## 5.3 Electron Self-Energy

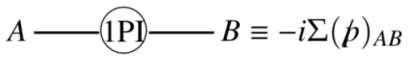
Next, we give a quick overview of radiative corrections.

- We will call the parameters in the Lagrangian bare parameters, and write the bare fermion mass as $m_0$. The bare propagator is the tree-level propagator,

$$S_F(x-y) = \int dp\, e^{-ip(x-y)} \frac{i(\not{p} + m_0)}{p^2 - m_0^2 + i\epsilon} = \int dp\, e^{-ip(x-y)} \frac{i}{\not{p} - m_0 + i\epsilon}$$

  where the latter notation is just suggestive, motivated by 'factoring the difference of squares'.

- We define the electron self-energy $\Sigma(\not{p})$ by the sum of 1PI diagrams,

  $$\equiv -i\Sigma(\not{p})_{AB}$$

  By Dyson resummation, the full propagator is

  $$\int dp\, e^{-ip(x-y)} \frac{i}{\not{p} - m_0 - \Sigma(\not{p}) + i\epsilon}$$

  and hence we call $\Sigma(\not{p})$ the self-energy of the electron. Note the self-energy has two spinor indices which contract with the spinor indices of the incoming and outgoing electron.

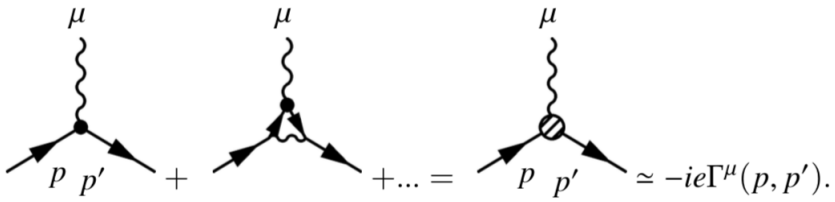
- Similarly, we consider the full photon propagator. It is corrected by Dyson resummation of the 1PI contribution,

  $$\equiv i\Pi^{\mu\nu}(q^2)$$

  and $\Pi^{\mu\nu}(q^2)$ is called the self-energy of the photon or the vacuum polarization.

- Finally, the interaction vertex receives radiative corrections, and summing all these corrections gives the effective vertex,

  $$\simeq -ie\Gamma^{\mu}(p, p').$$

  We only count amputated diagrams, which in this case coincide with 1PI diagrams.

- Generally, we will find UV divergences from virtual particles with arbitrarily high momenta; they must be tamed by regularization. There are also IR divergences for loops with photons; these cancel against IR divergences from the radiation of soft photons.

**Note.** One strange feature of Dyson resummation is that it involves summing over diagrams of *all* orders, even though we only calculate the 1PI diagrams (and most other things) to a fixed order. For instance, if we calculate the sum of 1PI diagrams up to $O(e^2)$, then the Dyson sum includes additional contributions of $O(e^4)$ and higher. But this extra accuracy seems pointless, because there are also 1PI diagrams of $O(e^4)$ we're neglecting!

The real point is that in processes where the intermediate propagator is nearly on-shell, the higher-order terms in the Dyson sum become important. To see this, note that $\Sigma \sim m_0$ on dimensional grounds, so each term in the Dyson series is suppressed relative to the last by a factor

$$\Sigma(p^2)D_F^0(p^2) \sim \frac{e^2 m_0}{\not{p} - m_0} \sim \frac{e^2 m_0^2}{p^2 - m_0^2}.$$

This ratio can be large when $p^2 - m_0^2$ is small enough, in which case the *entire* Dyson series has to be summed to get an accurate answer, even at $O(e^2)$. That is, in some situations we must sum over diagrams of all orders merely to ensure accuracy at a fixed order.

We need to perform Dyson resummation for the photon propagator for a similar reason. For processes with low-energy photons, the full Dyson series needs to be summed to get an accurate result, and the result is the "running coupling" where the effective strength of the electromagnetic interaction varies with energy, as described below. A more obscure example of this phenomenon occurs when two nonrelativistic charged particles scatter. The propagator for photon exchange in the $t$-channel can be very large, since $1/t \sim 1/v^2$. In this case, all the "ladder" diagrams involving $n$ photon exchanges are comparable in size, and must be summed to get an accurate result. The result is a complex structure with resonant peaks corresponding to the bound states of the particles, which matches what one finds using the nonrelativistic Schrodinger equation. These kinds of calculations are not covered in standard introductory textbooks, but can be found in *Quantum Electrodynamics* by Berestetskii, Lifshitz, and Pitaevskii.

In the rest of this section, we calculate the one-loop electron self-energy in detail.

- The one-loop contribution is due to emission and absorption of a virtual photon, with

$$-i\Sigma_2(\not p) = (-ie)^2 \int đk \, \frac{\gamma^\mu (i(\not k + m_0))\gamma_\mu (-i)}{(k^2 - m_0^2 + i\epsilon)((p-k)^2 - \mu^2 + i\epsilon)}$$

  where we have applied the Feynman rules in Feynman gauge and introduced a small photon mass $\mu$ to regulate the IR divergence.

- To handle the denominator, we turn it into a square with the identity

$$\frac{1}{AB} = \int_0^1 \frac{dx}{(xA + (1-x)B)^2}$$

  which generalizes to

$$\boxed{\frac{1}{A_1 \dots A_n} = \int_0^1 dx_1 \dots dx_n \, \delta\left(\sum_i x_i - 1\right) \frac{(n-1)!}{(x_1 A_1 + \dots + x_n A_n)^n}.}$$

  The measure is normalized so that

$$\int_0^1 dx_1 \dots dx_n \, (n-1)! \, \delta\left(\sum_i x_i - 1\right) = 1.$$

- After some simplification and a change of variable, we find

$$i\Sigma_2(\not p) = e^2 \int_0^1 dx \int đ\ell \, \frac{\gamma^\mu (\not k + m_0)\gamma_\mu}{(\ell^2 - \Delta + i\epsilon)^2}, \quad \ell = k - xp, \quad \Delta = -x(1-x)p^2 + x\mu^2 + (1-x)m_0^2.$$

  This form is useful because the denominator has a simple dependence on the integration variable. Note that the $\not \ell$ term in the numerator vanishes because it is odd in $\ell$.

- Next, we perform a Wick rotation to Euclidean space. Note that the $d\ell^0$ integral above can be viewed as a contour integral, with poles in the second and fourth quadrants. Then we can rotate the contour to go along the positive imaginary axis.

- We parametrize the rotated contour with the real variable $\ell_E^0$, so that $\ell^0 = i\ell_E^0$. Hence $d\ell^0 = id\ell_E^0$, picking up a factor of $i$, and

$$\int \frac{\bar{d}\ell}{(\ell^2 - \Delta + i\epsilon)^m} = i(-1)^m \int \frac{\bar{d}\ell_E}{(\ell_E^2 + \Delta - i\epsilon)^m}, \quad \ell_E^2 = (\ell_E^0)^2 + \boldsymbol{\ell} \cdot \boldsymbol{\ell}$$

  where the sign flip occurs because the metric for $(\ell_E^0, \boldsymbol{\ell})$ is negative definite. Finally, we drop the $i\epsilon$ and regard the integral as an integral in real Euclidean space $\mathbb{R}^4$.

- In our case, we would like to evaluate the UV-divergent integral

$$I_4 = \int \frac{\bar{d}\ell_E}{(\ell_E^2 + \Delta)^2}.$$

  The simplest method is a hard UV cutoff at $r = \Lambda$, which gives

$$I_4 \sim \log(\Lambda^2/\Delta)$$

  which is a log-divergence. However, this method breaks gauge invariance, since a gauge transformation can introduce arbitrarily high Fourier modes.

We instead regulate the integral with dimensional regularization (DR), using $d = 4 - \epsilon$ dimensions.

- For the angular part, note that for integer $d$, we have

$$\boxed{\int d\Omega_d \equiv \mathrm{Vol}(S^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)}}$$

  where $\Gamma(n) = (n-1)!$ and $\Gamma(z)$ has poles at $z = 0, -1, -2, \ldots$. This formula may be derived by taking the identity

$$\pi^{d/2} = \left( \int dx\, e^{-x^2} \right)^d$$

  and evaluating the right-hand side in $d$-dimensional spherical coordinates. We then define this to hold for all $d \in \mathbb{C}$. Some useful special cases are

$$\mathrm{Vol}(S^1) = 2\pi, \quad \mathrm{Vol}(S^2) = 4\pi, \quad \mathrm{Vol}(S^3) = 2\pi^2.$$

- Therefore, using this result, the measure in $d$ dimensions with spherical symmetry is

$$\bar{d}^d p = \mathrm{Vol}(S^{d-1}) \frac{p^{d-1}\, dp}{(2\pi)^d} = \frac{(p^2)^{d/2-1}\, d(p^2)}{(4\pi)^{d/2}\Gamma(d/2)}.$$

  Thus our integral generalized to $d$ dimensions is

$$I_d = \frac{1}{(4\pi)^{d/2}\Gamma(d/2)} \int_0^\infty \frac{u^{d/2-1}}{(u+\Delta)^2}\, du = \frac{\Delta^{d/2-2}}{(4\pi)^{d/2}\Gamma(d/2)} \int_0^1 dx\, x^{1-d/2}(1-x)^{d/2-1}$$

  where we substituted $x = \Delta/(u+\Delta)$. This integral can be evaluated using the beta function,

$$\boxed{B(\alpha,\beta) \equiv \int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.}$$

- More generally, for an arbitrary power of $p$ in the numerator, we have

$$\boxed{\int \bar{d}^d p \frac{p^{2a}}{(p^2-\Delta)^b} = i(-1)^{a-b} \frac{1}{(4\pi)^{d/2}} \frac{1}{\Delta^{b-a-d/2}} \frac{\Gamma(a+\frac{d}{2})\Gamma(b-a-\frac{d}{2})}{\Gamma(b)\Gamma(\frac{d}{2})}}$$

by similar logic, where the sign factors come from flipping the negative-definite Euclidean signature. Some useful special cases of this result are

$$\begin{aligned}
\int \bar{d}^d p \frac{1}{p^2-\Delta} &= \frac{-i}{(4\pi)^{d/2}} \frac{1}{\Delta^{1-d/2}} \Gamma\left(\frac{2-d}{2}\right), \\
\int \bar{d}^d p \frac{1}{(p^2-\Delta)^2} &= \frac{i}{(4\pi)^{d/2}} \frac{1}{\Delta^{2-d/2}} \Gamma\left(\frac{4-d}{2}\right), \\
\int \bar{d}^d p \frac{p^2}{(p^2-\Delta)^2} &= \frac{d}{2} \frac{-i}{(4\pi)^{d/2}} \frac{1}{\Delta^{1-d/2}} \Gamma\left(\frac{2-d}{2}\right), \\
\int \bar{d}^d p \frac{p^2}{(p^2-\Delta)^3} &= \frac{d}{4} \frac{i}{(4\pi)^{d/2}} \frac{1}{\Delta^{2-d/2}} \Gamma\left(\frac{4-d}{2}\right).
\end{aligned}$$

Note that we generally end up with a factor of $1/(4\pi)^2$, which suppresses the contribution of the loop; this is called a loop factor.

- In Euclidean signature, we have the same result with no $i$ or sign factors,

$$\int \bar{d}^d k \frac{k^{2a}}{(k^2+\Delta)^b} = \frac{1}{(4\pi)^{d/2}} \frac{1}{\Delta^{b-a-d/2}} \frac{\Gamma(a+\frac{d}{2})\Gamma(b-a-\frac{d}{2})}{\Gamma(b)\Gamma(\frac{d}{2})}$$

- An alternate way to derive this result is to use Schwinger parametrization. Before doing the angular integral, we note that

$$\frac{1}{a^n} = \frac{1}{\Gamma(n)} \int_0^\infty du\, u^{n-1} e^{-ua}.$$

If we apply this to the denominator, then we would have

$$\int \bar{d}^d k \frac{k^{2a}}{(k^2+\Delta)^b} = \frac{1}{\Gamma(b)} \int du\, u^{b-1} e^{-u\Delta} \int \bar{d}^d k\, e^{-uk^2} k^{2a}$$

in Euclidean signature. After doing the angular integral, the inner integral is merely a gamma function after substitution, and so is the outer integral, giving the same result.

- Finally, we need to handle numerators, which involve spinors and the metric. We define

$$g^{\mu\nu} g_{\mu\nu} = d$$

and the Lorentz-invariant phase space as

$$d\Pi_{\text{LIPS}} = \bar{\not{\delta}}^d \left(\sum p_j\right) \prod_j \frac{\bar{d}^{d-1} p_j}{2E_{p_j}}.$$

Formally, we suppose there are $d$ four-dimensional gamma matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, so that the trace of the identity in spinor space is still 4.

- As a result, spinor identities that involve metric contractions change, such as

$$\gamma^\mu\gamma_\mu = d, \quad \gamma^\mu\gamma^\nu\gamma_\mu = (2-d)\gamma^\nu$$

and

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4\eta^{\nu\rho} + (d-4)\gamma^\nu\gamma^\rho, \quad \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu + (4-d)\gamma^\nu\gamma^\rho\gamma^\sigma.$$

We also assume there exists a matrix $\gamma_5$ so that $\{\gamma_5, \gamma_\mu\} = 0$, though this runs into some subtleties with anomalies, as we'll see below.

- For the final step, we substitute $d = 4 - \epsilon$ and take the limit $\epsilon \to 0^+$, using

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + O(\epsilon), \quad x^\epsilon = 1 + \epsilon \log x + O(\epsilon^2).$$

where $\gamma$ is the Euler–Mascheroni constant. For the other poles of $\Gamma$, we use $\Gamma(z+1) = z\Gamma(z)$,

$$\Gamma(\epsilon - 1) = -\frac{1}{\epsilon} - 1 + \gamma + O(\epsilon), \quad \Gamma(\epsilon - 2) = \frac{1}{2\epsilon} + \frac{3}{4} - \frac{\gamma}{2} + O(\epsilon)$$

- Note that dimensional regularization is just a formal technique. It is merely a perturbative regularization scheme, and it does not provide a definition of the path integral in $d = 4$.

**Note.** A proof of our identity for the beta function. Let us define

$$C(\alpha, \beta) = \int_0^1 dx\, x^{-\alpha-1}(1-x)^{\beta-1}.$$

We wish to prove that this coincides with the definition of $B(\alpha, \beta)$ in terms of the gamma function. By integration by parts, it is straightforward to see that

$$C(\alpha - 1, \beta + 1) = \frac{\beta}{\alpha - 1} C(\alpha, \beta).$$

Furthermore, by writing $x = 1 - (1 - x)$, we have

$$C(\alpha + 1, \beta) = C(\alpha, \beta) - C(\alpha, \beta + 1).$$

It is easy to see using the gamma function recursion relation $\Gamma(z+1) = z\Gamma(z)$ that the gamma function obeys the same relations. Since $C(1,1) = B(1,1)$, the above relations imply that $B$ and $C$ agree on integers; using their similar asymptotic behaviors suffices to show they are equal.

We now apply this technique to our divergent integral.

- Applying our results, we find

$$I_d = \frac{\Gamma(2 - d/2)}{(4\pi)^{d/2}\Delta^{2-d/2}}.$$

We now set $d = 4 - \epsilon$ and taking the limit $\epsilon \to 0$. The dimensions of $I_d$ depend on $d$, so we introduce a compensating mass scale $\widetilde{M}$, giving

$$I_4 = \lim_{\epsilon\to 0} \frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon} + \log\frac{M^2}{\Delta} + O(\epsilon)\right), \quad M^2 = 4\pi e^{-\gamma}\widetilde{M}^2.$$

Simplifying the numerator with spinor identities gives

$$\Sigma_2(\not{p}) = \lim_{\epsilon\to 0} \frac{\alpha}{2\pi}\int_0^1 dx\,((2-\epsilon/2)m_0 - (1-\epsilon/2)x\not{p})\left(\frac{2}{\epsilon} + \log\frac{M^2}{\Delta}\right).$$

- The same result can be carried out in Pauli–Villars regularization, where we subtract off a loop diagram containing a fictitious massive particle of mass $\Lambda$ in place of the photon. This is a better approach than the hard-cutoff, and introducing an arbitrary mass scale $M$ gives

$$\Sigma_2(\not{p}) = \lim_{\Lambda\to\infty} \frac{\alpha}{2\pi} \int_0^1 dx\, (2m_0 - x\not{p}) \left( \log \frac{x\Lambda^2}{M^2} + \log \frac{M^2}{\Delta} \right).$$

  The momentum-dependent terms are the same, while the log-divergence matches with that seen with the hard cutoff, and translates to a $1/\epsilon$ pole in DR.

To understand the consequences of the self-energy, we recall the spectral representation.

- Applying the spectral representation, we have

$$\frac{i}{\not{p} - m_0 - \Sigma(\not{p})} = \frac{iZ_2}{\not{p} - m} + \text{terms analytic at } m.$$

  The physical mass $m$ is the location of the lowest-lying pole, and hence satisfies

$$\not{p} - m_0 - \Sigma(\not{p})\Big|_{\not{p}=m} = 0.$$

  We can extract the residue $Z_2$ by differentiating with respect to $\not{p}$ here, giving

$$\boxed{Z_2^{-1} = 1 - \frac{d\Sigma(\not{p})}{d\not{p}}\Big|_{\not{p}=m}}.$$

  In both equations above, the equality $\not{p} = m$ looks rather strange, though it's straightforward to use. It is a valid shortcut for a more correct procedure, as justified here.

- Since we are working to lowest order in $\alpha$, the mass shift is

$$\delta m = m - m_0 = \Sigma_2(\not{p} = m_0) + O(\alpha^2)$$

  where we evaluated $\Sigma_2$ at $m_0$ rather than $m$ since the resulting error is second-order.

- We need to interpret the fact that the correction $\delta m$ is divergent. The parameter $m$ is physical, while $m_0$ can never be measured directly. Hence we interpret our mass renormalization as a definition of $m_0$ in terms of $m$ in terms of the regularization parameters (e.g. $\epsilon$ and $M$). As the regularization is removed, $m_0$ diverges, but this is acceptable since it is not physical. The cost of this interpretation is that $m$ is now an input rather than an output of the theory.

**Note.** In classical electrodynamics, the UV divergence of the mass is even worse, as

$$\delta m \sim \int d\mathbf{x}\, E^2 \sim \alpha \int \frac{r^2 dr}{r^4} \sim \alpha\Lambda.$$

The reason the divergence is less strong in quantum electrodynamics is that when $m_0 = 0$, the electron separates into two uncoupled Weyl spinors, which must remain massless. Then assuming $\delta m$ is analytic in $m_0$, the leading term is linear in $m_0$, and for that term the dependence on $\Lambda$ must be logarithmic; indeed, we have $\delta m \sim m_0 \log(\Lambda^2/m_0^2)$.

## 5.4 Photon Self-Energy

Next, we consider the photon self-energy on general grounds.

- By Lorentz invariance, its tensorial structure can only depend on $\eta^{\mu\nu}$ and $q^\mu q^\nu$. By the Ward identity, the photon self-energy is also transverse, $q^\mu \Pi_{\mu\nu}(q) = 0$. (Note that this does *not* imply the photon propagator is transverse; instead, this depends on the gauge.) We give a more detailed justification of this fact in section 10.3.

- Therefore, the relative magnitude of the two terms is fixed, as

$$i\Pi^{\mu\nu}(q) = \left(\eta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) f(q^2).$$

  Heuristically, $i\Pi^{\mu\nu}(q)$ is regular at $q^2 = 0$ because poles can only come from single-particle massless intermediate states. (This is a bit too quick; there are subtleties here, but they don't apply to QED in four dimensions.)

- Therefore, we can pull out a $q^2$ factor from $f(q^2)$ to give

$$i\Pi^{\mu\nu}(q) = (q^2\eta^{\mu\nu} - q^\mu q^\nu)\Pi(q^2)$$

  with $\Pi(q^2)$ also regular at $q^2 = 0$. Note that when we talk about regularity, we always imagine a UV cutoff, since $\Pi^{\mu\nu}(q)$ is UV divergent.

- As a result, when we apply Dyson resummation, only the transverse part of the propagator is affected. In general $R_\xi$ gauge, the exact propagator is

$$\begin{aligned}\Delta_{\mu\nu}(q) &= \Delta^0_{\mu\nu}(q) + \Delta^0_{\mu\rho}(q)\Pi^{\rho\sigma}(q)\Delta^0_{\sigma\nu}(q) + \dots \\ &= \frac{-i}{q^2(1-\Pi(q^2))}\left(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \xi\frac{-i}{q^2}\frac{q_\mu q_\nu}{q^2}.\end{aligned}$$

  Then in Landau gauge $\xi = 0$, we have propagator

$$\Delta_{\mu\nu}(q) = \frac{-i}{q^2(1-\Pi(q^2))}\left(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right).$$

  We see the photon remains massless, since $\Pi(q^2)$ is regular at $q = 0$, and

$$\boxed{Z_3 = \frac{1}{1-\Pi(0)}}.$$

- Explicitly, one can show that at one-loop, under DR,

$$\Pi_2(q^2) = -\frac{2\alpha}{\pi}\lim_{\epsilon\to 0}\int_0^1 dx\, x(1-x)\left(\frac{2}{\epsilon} + \log\frac{M^2}{\Delta}\right), \quad \Delta = m_0^2 - x(1-x)q^2.$$

  If we had instead used a hard cutoff, we would have found a correction to $\Pi_2^{\mu\nu}(q)$ proportional to $\eta^{\mu\nu}$, violating the Ward identity. We will see later than in the path integral formulation, such anomalous symmetries occur because the path integral measure is not invariant under the classical symmetry.

The photon field renormalization leads to a running coupling.

- Consider electron-electron scattering by an intermediate photon of momentum $q$. Then the amplitude is proportional to
$$\frac{(-ie_0)^2}{1-\Pi(q^2)}$$
and we are thus motivated to define the effective coupling
$$e(q^2) = \frac{e_0}{\sqrt{1-\Pi(q^2)}}.$$
Note that there are other processes that contribute, e.g. by the exchange of two photons. However, to 'lowest order' this is a good estimate of the total scattering amplitude.

- As with the electron mass, $\Pi(q^2)$ is divergent, so we define the physical/renormalized coupling
$$e = \frac{e_0}{\sqrt{1-\Pi(0)}} = e_0\sqrt{Z_3}.$$
As before, we then flip this around and regard it as an expression for the divergent bare quantity $e_0$ in terms of the physical quantity $e$.

- Therefore, the effective coupling and physical coupling are related at one-loop order by
$$e^2(q^2) = \frac{1-\Pi_2(0)}{1-\Pi_2(q^2)}e^2 + O(\alpha^2) = \frac{e^2}{1-\widehat{\Pi}_2(q^2)} + O(\alpha^2)$$
where
$$\widehat{\Pi}_2(q^2) = \Pi_2(q^2) - \Pi_2(0) = -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\log\frac{m^2}{m^2 - x(1-x)q^2}$$
where we replaced $m_0$ with $m$, which is accurate up to $O(\alpha^2)$ terms. Physically, this result tells us that the interaction between electrons gets stronger at high energies; note that the unphysical renormalization scale $M$ has dropped out.

- The $e$ that appears in Coulomb's law, which corresponds to the typical quoted value $\alpha \approx 1/137$, applies at very low energies. At energies $q^2 \sim m_W^2$, we instead have $\alpha \approx 1/128$.

**Note.** Heuristically, the vertex renormalization is really what renormalizes the charge $e$, if we think of $e$ as the coefficient of the vertex factor. But as we'll see below, this effect is exactly canceled by the renormalization of the electron field $Z_2$. The effect we've computed above is really a renormalization of the electromagnetic field, but since its effects always appear in tandem with an electron-field interaction, it can be thought of as renormalizing the charge.

**Note.** Interpreting the running coupling. In the nonrelativistic limit, the potential is
$$V(\mathbf{q}) = -\frac{e^2}{|\mathbf{q}|^2(1-\hat{\Pi}_2(-|\mathbf{q}|^2))}$$
and the real-space potential is found by Fourier transform. Taking the inverse Fourier transform is tricky, since for high $|\mathbf{q}|$ the $\hat{\Pi}_2$ factor provides a branch cut; doing this yields the Uehling potential.

Alternatively, for low $|\mathbf{q}|$ we can simply expand $\hat{\Pi}_2$ in a Taylor series. The zeroth order term slightly modifies $e$, while the $|\mathbf{q}|^2$ term contributes a constant to $V(\mathbf{q})$ and hence a delta function to the potential. This slightly lowers the energy of the 2s state in hydrogen relative to the 2p state, producing the Lamb shift.

In the opposite limit, we consider a high-energy scattering where the photon is in the $t$-channel or $u$-channel, so $-q^2 \gg m^2$. Then

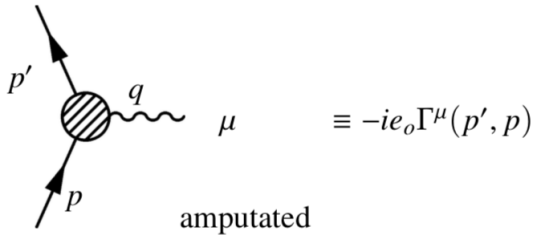
$$\alpha_{\text{eff}}(q^2) = \frac{e^2(q^2)}{4\pi} \approx \alpha \left(1 - \frac{\alpha}{3\pi} \log \frac{-q^2}{\exp(5/3)m^2}\right)^{-1}$$

by straightforward calculation. This diverges for very high energies at the Landau pole of QED.

## 5.5 Vertex Renormalization

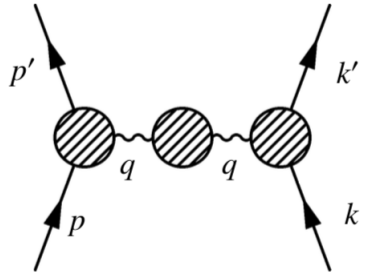
Finally, we turn to the renormalization of the QED vertex.

- We denote the amputated resummed QED vertex as




$$\equiv -ie_o\Gamma^\mu(p', p)$$

- Keeping track of the fermion field renormalization, the following amplitude




  is of the form

$$i\mathcal{M} \sim \overline{u}(p')(-ie_0 Z_2 \Gamma^\mu(p', p))u(p) \frac{-i(\eta_{\mu\nu} - q^\mu q^\nu/q^2)}{q^2(1 - \Pi(q^2))} \overline{u}(k')(-ie_0 Z_2 \Gamma^\nu(k', k))u(k).$$

- The most general form possible is

$$\Gamma^\mu = A\gamma^\mu + B(p'^\mu + p^\mu) + C(p'^\mu - p^\mu).$$

  Since the $\Gamma^\mu$ is sandwiched between on-shell spinors, we can replace all factors of $\not{p}$ and $\not{p}'$ with $m$. Then the coefficients can only depend on $q^2$. The Ward identity requires the amplitude to vanish when $\Gamma^\mu$ is contracted with $q_\mu$, setting $C = 0$.

- Next, we use the Gordan identity

$$\overline{u}(p')\gamma^\mu u(p) = \overline{u}(p')\left(\frac{p'^\mu + p^\mu}{2m} + \frac{iS^{\mu\nu}q_\nu}{2m}\right)u(p), \quad S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

  to arrive at the conventional form

$$\Gamma^\mu(p', p) = \gamma^\mu F_1(q^2) + i\frac{S^{\mu\nu}q_\nu}{2m_0}F_2(q^2),$$

  where $F_1(q^2)$ and $F_2(q^2)$ are called form factors.

- The first form factor $\delta F_1$ is both UV and IR divergent. At one loop with PV regularization,

$$\delta F_1(q^2) = \frac{\alpha}{2\pi}\int_0^1 dxdydz\,\delta(x+y+z-1)\left(\log\frac{z\Lambda^2}{\Delta} + \frac{(1-x)(1-y)}{\Delta}q^2 + (1-4z+z^2)m_0^2\right)$$

  where $\Delta = -xyq^2 + (1-z)^2m_0^2 + \mu^2 z$, and $\mu$ is a photon mass that regulates the IR divergence.

- Similarly, one finds that

$$\delta F_2(q^2) = \frac{\alpha}{2\pi}\int_0^1 dxdydz\,\delta(x+y+z-1)\frac{2m_0^2 z(1-z)}{\Delta}$$

  which is luckily finite. This remains true at all orders in perturbation theory.

- In the limit of small momentum transfer $q \to 0$, only $F_1$ contributes, so we define the vertex renormalization

$$\boxed{\lim_{q\to 0}\Gamma^\mu(p+q,p) = Z_1^{-1}\gamma^\mu}.$$

  The Ward–Takahashi identity can be used to show that $Z_1 = Z_2$, so

$$\lim_{q\to 0} Z_2\Gamma^\mu(p+q,p) = \gamma^\mu$$

  and at each order in perturbation theory,

$$\delta F_1(0) = -\delta Z_2.$$

- Comparing this with our scattering amplitude, the physical charge at $q = 0$ is $e_0(Z_2/Z_1)\sqrt{Z_3} = e_0\sqrt{Z_3}$ as found earlier. As a physical application, this means that electrons and muons have the same bare charge if and only if they have the same physical charge.

- One can show that the $F_2$ form factor affects the magnetic moment of the electron, by

$$\boldsymbol{\mu} = \frac{ge}{2m}\mathbf{S}, \quad g = 2 + 2F_2(0), \quad F_2(0) = \frac{\alpha}{2\pi}.$$

  This is one of the most famous predictions of QED.

**Note.** When one speaks of the physical charge of a particle, one almost always means the charge $e(\mu)$ evaluated at zero momentum, $e(0)^2 \approx 1/137$. This is the quantity relevant to all experiments besides those in high energy physics.

At this point one could make a metaphysical objection: what does it mean for a "fundamental constant of nature" to be scale-dependent? Isn't the amount of charge on the electron fixed? The point is that there is no such thing as a direct measurement of the charge itself; instead we only measure how strongly the electron interacts in various situations. We've found that the effective interaction strength changes depending on the scale $\mu$ of the process, and there's nothing strange, physically or metaphysically, about that. The only reason one might think it was strange is that the dependence happens to be very weak at low $\mu$, where most experiments historically took place. But once we frame it this way, $e(0)$ stops appearing particularly fundamental at all; the truly fundamental constants are defined at high scales.

## 5.6 Renormalized Perturbation Theory

Next, we step back and review the structure of the loop corrections.

- For a systematic treatment of UV divergent diagrams, it suffices to consider amputated, 1PI UV divergent diagrams; all other UV divergent diagrams can be built from them.
- As we'll show later, it is possible to classify the divergent 1PI diagrams by power counting. In the case of QED, the only ones are the photon propagator, the electron propagator, and the vertex, all of which we've studied above. At one-loop order we found, in PV,

$$\Pi_2(q^2) = c_0^{(1)} \log \frac{\Lambda}{M} + \text{finite}, \quad -i\Sigma_2(\not{p}) = a_0^{(1)} m_0 \log \frac{\Lambda}{M} + a_1^{(1)} \not{p} \log \frac{\Lambda}{M} + \text{finite}$$

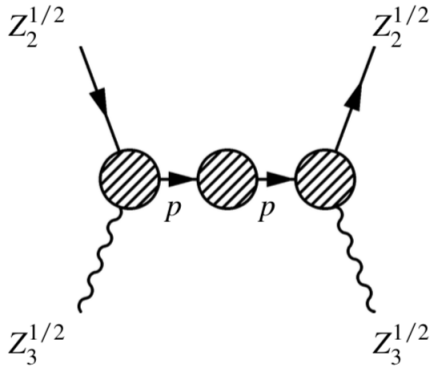
  and

$$-i\Gamma_2^\mu(p', p) = b_0^{(1)} \gamma^\mu \log \frac{\Lambda}{M} + \text{finite}$$

  where $M$ is the renormalization scale and $\Lambda$ is the cutoff.
- Later, we will show that this structure persists to all orders in perturbation theory. Then the UV divergences are completely described by the four divergent constants $a_0$, $a_1$, $b_0$, and $c_0$, where e.g. $a_0^{(1)}$ is the $O(\alpha)$ term of $a_0$.
- In bare perturbation theory, we write the Lagrangian in terms of bare parameters $m_0$ and $e_0$ and impose a finite cutoff $\Lambda$, or in DR, a nonzero $\epsilon$. We then compute the physical electron mass $m$, the physical coupling $e$, and the field renormalizations $Z_2$ and $Z_3$ in terms of $m_0$, $e_0$, and $\Lambda$. The renormalization step is to regard $m_0$ and $e_0$ as functions of $m$, $e$, and $\Lambda$, and vary them as $\Lambda \to \infty$ so that $m$ and $e$ remain fixed.
- Finally, we perform the usual perturbation theory, e.g. expanding in powers of $e_0$ with a cutoff $\Lambda$. If the preceding step has been performed correctly, then when the answers are written in terms of $m$ and $e$, all divergences will cancel out of scattering amplitudes.
- In practice, we only compute $m(m_0, e_0, \Lambda)$ and $e(m_0, e_0, \Lambda)$ to some loop order in $e_0$. Then we must compute the scattering amplitudes to the same order in $e_0$.
- It might be disturbing that we are expanding in the formally divergent parameter $e_0$. In fact, this is acceptable because $e_0$ diverges only logarithmically, and the theory of QED has a reasonable cutoff so that $e_0(\Lambda)$ is still small. However, we will reformulate perturbation theory without $e_0$ below.

- In general, a theory is renormalizable if there are only a finite number of UV divergent resummed amputated 1PI diagrams. If there are $m$ fields and $n$ such diagrams, then $m$ divergent constants can be absorbed into field renormalizations, while $n-m$ are absorbed into bare couplings. The $n-m$ bare couplings must then be specified by measurements of physical couplings.
- Philosophically, the UV divergences signal that our theory must break down at some high scale, where some unknown physics takes over. The process of renormalization parametrizes our ignorance about this new physics into $n-m$ observables that must be measured. A renormalizable theory is one where this process only requires a finite number of experimental inputs. A truly fundamental theory must be UV finite, as string theory is believed to be.
- Finally, recall that our regularization schemes introduce an arbitrary mass scale $M$. The results don't depend on $M$, but higher-order corrections for a process with energy scale $E$ contain factors of $\log E/M$. Then the perturbation theory converges fastest when $M \sim E$.

**Example.** Consider the following contribution to the amplitude for Compton scattering.




The amplitude takes the form

$$(Z_3^{1/2} i e_0 \Gamma^\mu Z_2) \frac{1}{Z_2} \frac{i}{\not p - m_0 - \Sigma(\not p)} (Z_3^{1/2} i e_0 \Gamma_\mu Z_2).$$

As we've already seen, $e = e_0 Z_3^{1/2}$ is the finite physical charge. The factor $\Gamma^\mu Z_2$ is also finite, since

$$\Gamma^\mu(p+q,p) = \gamma^\mu F_1(q^2) + \text{finite}, \quad F_1(q^2) = 1 + \delta F_1(0) + \text{finite}$$

so that the divergent part of $\Gamma^\mu Z_2$ is $\delta Z_2 + \delta F_1(0) = 0$ at each order in perturbation theory, as argued earlier. Finally, the propagator is

$$\frac{1}{Z_2} \frac{i}{\not p - m_0 - \Sigma(\not p)} = \frac{i}{\not p - m} + \frac{1}{Z_2}(\text{finite terms analytic in } \not p = m).$$

The extra terms are finite because they come from the second and higher derivatives of $\Sigma(\not p)$, which are finite, as the divergences in $\Sigma(\not p)$ are proportional to $m_0$ and $\not p$.

We can also use renormalized perturbation theory, which works explicitly with physical parameters. In this formulation, the action depends on the cutoff/renormalization parameters just as before, but we split it into a finite, cutoff-independent "renormalized" piece and an infinite, cutoff-dependent "counterterm" piece.

- The Lagrangian, called the bare Lagrangian, is

$$\mathcal{L} = -\frac{1}{4}F^2 + \overline{\psi}(i\partial\!\!\!/ - m_0)\psi - e_0\overline{\psi}\gamma^\mu\psi A_\mu.$$

We recall the exact two-point functions for photons and electrons have coefficients of $Z_3$ and $Z_2$ respectively. We thus define the renormalized fields

$$A^\mu = Z_3^{1/2}A_r^\mu, \quad \psi = Z_2^{1/2}\psi_r$$

so that two-point functions of the renormalized fields have no such factors.

- We can thus rewrite the Lagrangian in terms of the renormalized fields as

$$\mathcal{L} = -\frac{1}{4}Z_3F_r^2 + Z_2\overline{\psi}_r(i\partial\!\!\!/ - m_0)\psi_r - Z_2Z_3^{1/2}e_0\overline{\psi}_r\gamma^\mu\psi_r A_\mu^r.$$

This is simply a change of variables; the Lagrangian itself is unchanged. The Feynman rules for the renormalized fields have no $Z$ factors, the fermion has mass $m = Z_2m_0$, and the coupling is $e' = Z_2Z_3^{1/2}e_0$. Applying these rules yields the same results as the original Feynman rules.

- Next, we split the (bare) Lagrangian into the renormalized Lagrangian and the counterterms. We further define the parameters $e$ and $m$ by

$$e' = Z_1e, \quad Z_2m_0 = m + \delta_m$$

to be given a physical interpretation below. Letting $Z_i = 1+\delta_i$, and suppressing the $r$ subscript, the Lagrangian splits as

$$\begin{aligned}\mathcal{L} &= \mathcal{L}^{(1)} + \mathcal{L}^{(2)} \\ &= \left(-\frac{1}{4}F^2 + \overline{\psi}(i\partial\!\!\!/ - m)\psi - e\overline{\psi}\gamma^\mu\psi A_\mu\right) + \left(-\frac{1}{4}\delta_3F^2 + \overline{\psi}(i\delta_2\partial\!\!\!/ - \delta_m)\psi - e\delta_1\overline{\psi}\gamma^\mu\psi A^\mu\right).\end{aligned}$$

The perturbation parameter is now $e$. The four counterterms $\delta_1$, $\delta_2$, $\delta_3$, and $\delta_m$ are are defined order by order in $\alpha = e^2/4\pi$. The new arbitrary parameters $e$ and $m$ will be chosen to be more physically meaningful than $e_0$ and $m_0$.

- The Feynman rules for $\mathcal{L}^{(1)}$ have no $Z$ factors, mass $m$, and coupling $e$. They are UV divergent in the exact same way as before, but with $m_0$ and $e_0$ replaced with $m$ and $e$. The terms in $\mathcal{L}^{(2)}$ lead to additional diagrams, such as




The dependence on the bare parameters is entirely captured by the counterterms.

- The one-loop electron self-energy takes the form

$$-i\Sigma_r^{(1)}(p\!\!\!/) = -i\Sigma_2^{(1)}(p\!\!\!/) + i(p\!\!\!/\delta_2^{(1)} - \delta_m^{(1)})$$

where $\Sigma_2^{(1)}$ is the piece due to $\mathcal{L}_r^{(1)}$. Similarly, the photon self-energy in Landau gauge is

$$i\Pi_r^{(1)\mu\nu}(q) = i(q^2\eta^{\mu\nu} - q^\mu q^\nu)(\Pi_2^{(1)}(q^2) - \delta_3^{(1)}).$$

Finally, the vertex factor is

$$-ie\Gamma_r^{(1)\mu}(p+q,p) = -ie\Gamma^{(1)\mu}(p+q,p) - ie\gamma^\mu\delta_1^{(1)}.$$

- Next, we determine the counterterms, which depend on the cutoff, by renormalization conditions. In the on-shell scheme, we let $m$ be the physical mass of the electron and $e$ be the physical charge measured at $q = 0$. By construction, we also require the propagators to have no $Z$-factors at the physical mass poles. Then we have four conditions,

$$\Sigma_r(\not{p})\Big|_{\not{p}=m} = 0, \quad \frac{d}{d\not{p}}\Sigma_r(\not{p})\Big|_{\not{p}=m} = 0, \quad \Pi_r(q^2)\Big|_{q^2=0} = 0, \quad -ie\Gamma_r^\mu(p,p) = -ie\gamma^\mu$$

  which fix the values of the counterterms order by order in $\alpha$ in terms of $e$, $m$, and the cutoff. We have removed all dependence on $e_0$ and $m_0$, and when the regularization is removed, all amplitudes are finite functions of $e$ and $m$.

- At two-loop order, we have two-loop diagrams involving renormalized fields and second-order contributions to the counterterms such as $\delta_1^{(2)}$. In addition, one-loop diagrams containing first-order counterterms like $\delta_1^{(1)}$ also contribute.

- We assumed that the counterterms above were sufficient to absorb all UV divergences. As before, we can do this with finitely many counterterms only for a renormalizable theory.

**Note.** There's a fair deal of historical confusion about renormalization. Many sources phrase the procedure above in terms of "adding counterterms" to "cancel divergences". This is not a good way of thinking about it, because adding terms to a Lagrangian changes the theory. We are simply working with a single Lagrangian the entire time, whose terms may be formally divergent but whose physical predictions are always finite.

However, this naive picture is redeemed in Wilsonian renormalization. As we will see later, the bare Lagrangian is the fundamental Lagrangian, while the renormalized Lagrangian can be thought of as the effective Lagrangian after integrating out high-energy modes. The parameters in the effective Lagrangian are closely related to physically measurable ones, because loop integrals are cut off at a low scale and hence have little effect. The counterterm consists of the terms added to the fundamental Lagrangian as we perform Wilsonian RG flow down to the low scale.

**Note.** There are a wide variety of ways to set the counterterms, called subtraction schemes. For example, in minimal subtraction, we use dimensional regularization and let the counterterms have zero finite part, so they're only series in $1/\epsilon$. This is more computationally efficient, and the standard in modern calculations. Alternatively, in the off-shell scheme, we do the same thing as the on-shell scheme, but let $m$ above be an arbitrary parameter rather than the physical mass. Note that the bare Lagrangian doesn't depend on the subtraction scheme; it only affects the split into $\mathcal{L}_r^{(1)} + \mathcal{L}_r^{(2)}$.

Whenever a subtraction scheme contains a mass parameter, such as the $\mu$ parameter in DR, we have a continuum RG flow by changing the parameter. In the on-shell scheme, there is no continuum RG, though we have the Wilsonian RG as always.

**Note.** The point of these other subtraction schemes is to improve perturbation theory. Generically, a loop integral will give logarithmic factors like $\log(s/\Lambda^2)$, so

$$\mathcal{M}(s) \sim \lambda + \lambda^2 \log\frac{s}{\Lambda^2} + \dots.$$

We may define the renormalized coupling to be the physical coupling at some arbitrary scale $s_0$,

$$\lambda_R = \mathcal{M}(s_0)$$

so that the perturbation series becomes

$$\mathcal{M}(s) \sim \lambda_R + \lambda_R^2 \log \frac{s}{s_0} + \dots.$$

Then if $s$ is very far from $s_0$, the perturbation series doesn't converge quickly. Thus it's very useful to be able to adjust $s_0$ as needed. The continuum RG gives a differential equation for $\mathcal{M}(s_0)$ as $s_0$ is varied, essentially resumming the perturbation series.

## 5.7 Physical Interpretation

We now give some insight into the meaning of renormalization using simple toy models.

- We consider the general problem of fitting a quantum mechanical model to data. We try a candidate class of Hamiltonians $H(\theta)$ with parameters $\theta$, which output predictions $v(\theta)$, such as energy eigenvalues. We then compute the predictions to data to infer $\theta$.
- An important point is that the parameters $\theta$ need not have a direct physical meaning. For example, different models of quarks will have significantly different quark masses in $\theta$, and this is acceptable because quark masses are not physically measurable.
- Parameters that are model-independent and directly measurable by experiment, such as the physical electron mass, are called renormalized or dressed parameters, in contrast to bare parameters. The purpose of renormalization is to reparametrize $H(\theta)$ as $H_{\text{ren}}(\theta_{\text{ren}})$ so that ideally the $\theta_{\text{ren}}$ are directly measurable, or at least can be easily inferred from the data.
- As a trivial example, consider a two-state system with

  $$H(g,\Lambda) = H_0 + gV_\Lambda, \quad H_0 = \begin{pmatrix} 0 & 0 \\ 0 & \omega \end{pmatrix}, \quad V_\Lambda = \begin{pmatrix} -1 & \Lambda \\ \Lambda & 0 \end{pmatrix}$$

  where $\Lambda$ plays a role like a UV cutoff. The energy eigenvalues are

  $$E_\pm = \frac{1}{2}\left(\omega - g \pm \sqrt{(\omega+g)^2 + 4g^2\Lambda^2}\right).$$

  If $\Lambda$ is a very large number, the outputs $E_\pm$ are extremely sensitive to changes in $g$, indicating a poor choice of parametrization.
- To avoid this problem, we could reparametrize in terms of $E_\pm$, but these quantities are somewhat complicated. A simpler option is to use $g_{\text{ren}} = g\Lambda$, giving the renormalized Hamiltonian

  $$H_{\text{ren}}(g_{\text{ren}},\Lambda) = \begin{pmatrix} -g_{\text{ren}}\Lambda^{-1} & g_{\text{ren}} \\ g_{\text{ren}} & \omega \end{pmatrix}.$$

  In this case, the energy levels are less sensitive to $g_{\text{ren}}$ for large $\Lambda$. The reparametrization is so nice that we can even take the limit $\Lambda \to \infty$ and find a finite result.
- Situations of this form typically arise when there is a naturally large parameter, such as a volume or energy cutoff. Note that taking $\Lambda \to \infty$ is not logically necessary; in the Wilsonian picture we never do this because realistic theories will have some finite cutoff. The point is that after a successful renormalization we have removed the extreme sensitivity to $\Lambda$, so we could take $\Lambda \to \infty$ if we wanted to; either way it shouldn't affect the experimental outputs.

Next, we turn to perturbation theory.

- Usually, for practical reasons, we expand perturbatively about a reference Hamiltonian $H_0(\mu)$, where $\mu = \mu(\theta)$. For example, the decomposition $H = H_0 + gV_\Lambda$ expands about $H_0(\omega)$. The energy levels are

$$E_- = -g - \frac{g^2\Lambda^2}{\omega} + O(g^3), \quad E_+ = \omega + \frac{g^2\Lambda^2}{\omega} + O(g^3).$$

- We see that $g$ is a bad expansion parameter because for large $\Lambda$, we require $g \ll \Lambda^{-1}$ instead of the typical $g \ll 1$. Alternatively, it is impossible to expand in $g$ at all in the limit $\Lambda \to \infty$.

- On the other hand, after renormalization and taking the limit $\Lambda \to \infty$, we have

$$E_- = -\frac{g_{\text{ren}}^2}{\omega} + O(g_{\text{ren}}^4), \quad E_1 = \omega + \frac{g_{\text{ren}}^2}{\omega} + O(g_{\text{ren}}^4)$$

  which is a perfectly well-behaved series. Thus the quality of the perturbation series depends on the parametrization $\theta$.

- As another example of this phenomenon, consider the Taylor series of the logarithm,

$$\log(1+g) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} g^k.$$

  This only converges for $|g| < 1$. But if we expand in terms of $z = g/(g+2)$, we have

$$\log(1+g) = \log\left(\frac{1+z}{1-z}\right) = \sum_{k=0}^{\infty} \frac{2}{2k+1} z^{2k+1}$$

  which converges for $|z| < 1$ and hence for all $\operatorname{Re} g > -1$, much faster than the original series. The singularity at $g = -1$ remains at $z = -1$, but all $\operatorname{Re} g > -1$ are 'sucked into' the region $|z| < 1$.

- The quality of the perturbation series also depends on the choice $H_0(\mu)$ as well as the specific value of $\mu$. Since the total Hamiltonian is always the same, these choices do not affect the exact results, but they can affect the accuracy of a truncated series expansion. One might worry that this reduces predictivity; thus a good choice of $\mu$ is one which minimizes the sensitivity of the results to changes in $\mu$.

- As an example, consider finding $x(t)$ for a particle in the potential $V(x) = ax^2 + \epsilon x^4$ at a large time $t = T$, so that $T$ is the large parameter. Then $x(T)$ depends very sensitively upon the bare parameters $a$ and $b$. Now suppose we attempt a perturbative expansion. Taking $H_0 = p^2/2m$ is clearly a very poor choice; a more reasonable choice is $H_0 = p^2/2m + ax^2$. However, expanding in $\epsilon$ gives a bad perturbation series; the terms in the series depend on $T$ and diverges for $T \to \infty$.

- The solution is to work in terms of a renormalized frequency. We set

$$H_0 = \frac{p^2}{2m} + cx^2, \quad c = a + \sum_n c_n \epsilon^n$$

so that $H_0$ describes a harmonic oscillator with the same period as the anharmonic oscillator. Then if we perform a first-order expansion in $\epsilon$, we can set the 'counterterm' $c_1$ so that the first-order correction is small for all $T$. Similarly to perform a second-order expansion we would have to also determine $c_2$.

- The sensitivity to $T$ has been reduced; changing variables from $(a, \epsilon)$ to $(c, \epsilon')$, where $c$ is the renormalized frequency and $\epsilon$ describes the amount of anharmonicity. The result $x(T)$ depends sensitively on $\epsilon$, but not on $\epsilon'$.

# 6 Amplitudes

## 6.1 Introduction

In this section, we consider some of the properties of amplitudes, i.e. the quantities $\mathcal{M}(i \to f)$ simply related to $S$-matrix elements.

- Amplitudes are constrained by Poincare invariance. Note that

$$\langle f, \text{out}|i, \text{in}\rangle = \langle f, \text{out}|U^\dagger(\Lambda, a)U(\Lambda, a)|i, \text{in}\rangle$$

  where $U(\Lambda, a)$ is our representation of the Poincare group. We know how it must act on the in and out states, since these are just combinations of independent single particles, and using this gives us a relation between two amplitudes.

- Specifically, let us label the incoming particles with momenta $p_i$, little group indices $\sigma_i$, and discrete labels $n_i$ for the particle type, and likewise with primes for the outgoing particles. Then

$$S_{\{p_i,\sigma_i,n_i\},\{p_i',\sigma_i',n_i'\}} = \exp\left(ia_\mu \Lambda^\mu{}_\nu (\Sigma p_i' - \Sigma p_i)^\nu\right) \times \left(\sum_{\{\overline{\sigma}_i\}} \prod_i D^{(j_i)}_{\overline{\sigma}_i \sigma_i}(W(\Lambda, p_i))\right) \left(\sum_{\{\overline{\sigma}_i'\}} \prod_i D^{(j_i')}_{\overline{\sigma}_i' \sigma_i'}{}^*(W(\Lambda, p_i'))\right) S_{\{\Lambda p_i,\overline{\sigma}_i,n_i\},\{\Lambda p_i',\overline{\sigma}_i',n_i'\}}$$

  In other words, we just pick up the expected transformations for the momenta and little group labels. Here, we've omitted a convention-dependent normalization factor.

- We pick up additional constraints from internal symmetries, which are defined to be those that change the particle type label $n_i$, without changing the momenta.

- Scattering amplitudes are analytic in the momenta, so we can analytically continue them "off-shell". At the perturbative level, we can define this analytic continuation by simply demanding that off-shell amplitudes can be computed using the same Feynman rules as on-shell ones.

- A simple reason this is useful is that diagrammatically, off-shell amplitudes could appear as internal parts of larger, on-shell amplitudes.

- Another reason is that they appear when one has an external source,

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}} - J(x)\phi(x).$$

  In this case, one has a Feynman rule where a particle can appear out of the source, with a factor $iJ(x)$. The diagrams that contribute to the vacuum-to-vacuum transition amplitude $Z[J]$, at $n^{\text{th}}$ order in $J$, have $n$ copies of this vertex.

- Taking the Fourier transform, since $\tilde{J}(k)$ can have support off-shell, the resulting amplitude is just an $n$-particle off-shell scattering amplitude times factors of $i\tilde{J}(k_i)$. This is important because, as we'll see later, $Z[J]$ is what the path integral naturally computes.

- There is one more direct interpretation of off-shell amplitudes. We recall that the $S$-matrix in the adiabatic formalism is $S = U_I(\infty, -\infty)$. With the external source,

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}} + J(x)\phi(x).$$

Now, suppose we work in interaction picture by treating $\mathcal{H}_0 + \mathcal{H}_{\text{int}}$ as the "free" Hamiltonian. In this case, the interaction picture fields are what we usually call the Heisenberg fields, and the vacuum-to-vacuum amplitude is

$$Z[J] = \langle \Omega | S | \Omega \rangle = \langle \Omega | T \exp\left( -i \int dx\, J(x) \phi_H(x) \right) | \Omega \rangle.$$

By Taylor expanding the right-hand side, we see that $n$-particle off-shell amplitudes are just the Fourier transforms of the vev of the time-ordered product of $n$ Heisenberg fields.

- This point is important because we usually take for granted that symmetries work for off-shell amplitudes or amplitudes with external currents just as they do for on-shell amplitudes, which is not strictly justified. If we define off-shell amplitudes in terms of Heisenberg fields, then we can establish this rigorously. For example, this ensures that momentum is conserved in off-shell amplitudes, beyond perturbation theory, where it is clear from the Feynman diagram expansion.

Another useful consequence of analytic continuation is crossing symmetry.

- Crossing symmetry is the statement that

$$\mathcal{M}(\phi(p) + \ldots \to \ldots) = \mathcal{M}(\ldots \to \ldots + \tilde{\phi}(-p))$$

where $\phi(p)$ represents a particle with four-momentum $p$, and $\tilde{\phi}(-p)$ represents an antiparticle with momentum $-p$. Note that it is impossible for both sides above to be on-shell, so this statement must be understood in terms of analytic continuations.

- For scalars, crossing symmetry can easily be shown perturbatively with the Feynman rules, as for every diagram with $\phi$ incoming there is a corresponding diagram with $\tilde{\phi}$ outgoing. For spinors and vectors, one picks up factors for the external polarizations, which can lead to extra sign flips; however, these can be removed if one uses appropriate conventions for the off-shell polarization vectors.

- Crossing symmetry can also be shown nonperturbatively using the LSZ reduction formula. This gives an analytic formula for $S$-matrix elements which only distinguishes between in and out particles by the sign of the energy in the Fourier transform. Analytically continuing a positive energy to a negative one thus yields a crossed $S$-matrix element.

- For $2 \to 2$ scattering, crossing symmetry is especially simple in terms of the Mandelstam variables. For example, for the process $a + b \to c + d$ with Mandelstam variables $s$, $t$, and $u$ and scattering amplitude $f(s,t,u)$, crossing symmetry implies that the amplitude for $\overline{c} + b \to \overline{a} + d$, in terms of its Mandelstam variables $s'$, $t'$, $u'$, is $f(u',t',s')$. An equivalent way of saying this is that upon expressing $(s',t',u')$ back in terms of $(s,t,u)$, the amplitude is the exact same analytic function $f(s,t,u)$.

- Also note that applying crossing symmetry to switch all incoming particles to outgoing ones, and vice versa, just yields the $S$-matrix element related by CPT.

## 6.2 The Optical Theorem

The unitarity of the $S$-matrix leads to useful constraints.

- We recall that we have defined

$$S = 1 + iT, \quad T = \mathcal{M}\not{\phi}(\Sigma p_{\text{in}} - \Sigma p_{\text{out}}).$$

  The unitarity condition $S^\dagger S = 1$ gives

$$-i(T - T^\dagger) = T^\dagger T.$$

- To make this more explicit, we take matrix elements of both sides, suppressing "in" and "out" labels, and introduce a resolution of the identity on the right-hand side. Canceling a delta function from both sides, and considering a $2 \to 2$ scattering for concreteness, we have

$$\begin{aligned} &-i(\mathcal{M}(k_1 k_2 \to p_1 p_2) - \mathcal{M}^*(p_1 p_2 \to k_1 k_2)) \\ &\qquad = \sum_n \left( \prod_{i=1}^n \int \frac{\not{d}\mathbf{q}_i}{2E_i} \right) \mathcal{M}^*(p_1 p_2 \to \{q_i\}) \mathcal{M}(k_1 k_2 \to \{q_i\}) \not{\phi}(k_1 + k_2 - \sum_i q_i). \end{aligned}$$

  In a more abbreviated notation, this is

$$2 \operatorname{Im} \mathcal{M}(a \to b) = \sum_f \int d\Pi_f \, \mathcal{M}^*(b \to f) \mathcal{M}(a \to f).$$

  This is the optical theorem.

- The optical theorem is most useful when $a$ and $b$ are the same state. For a one-particle state, the right-hand side is just the decay rate, up to some constants, so

$$\operatorname{Im} \mathcal{M}(a \to a) = m_A \sum_f \Gamma(a \to f) = m_a \Gamma_a.$$

  It's not clear if the left-hand side makes sense for an unstable particle, because such particles can't appear in asymptotic states. One needs to do more work to resolve this, but we'll just ignore it. **(resolve)**

- For a two-particle state, the right-hand side gives the total scattering cross section, up to some constants, and we have

$$\operatorname{Im} \mathcal{M}(a \to a) = 2 E_{\text{cm}} p_{\text{cm}} \sigma_{\text{tot}}(a \to \text{anything})$$

  where $p_{\text{cm}}$ is the momentum of either particle in the center of mass frame.

- These results have precisely the same interpretation as in the notes on Undergraduate Physics. The imaginary part of the forward scattering amplitude is precisely the amount by which the amplitude for remaining in the same state is depleted, so it is sensible that this is related to the probability to end up in a different state.

- The optical theorem is especially useful when we work perturbatively, because it relates contributions from different orders in perturbation theory. For example, we will see that the leading contribution to $\operatorname{Im} \mathcal{M}(a \to a)$ is at loop level, while the right-hand sides have leading contributions at tree level. Thus, even if we didn't know the full theory, unitarity can still be used to constrain loop level amplitudes given tree level ones. This will be formalized below by "cutting rules".

Next, we give an explicit example of the optical theorem. We focus on a $2 \to 2$ scattering and consider $\mathcal{M}$ as an analytic function of $s$, defined by the Feynman rules.

- First, it's worth noting that typically, we expect the imaginary part of $\mathcal{M}$, for real $s$, to vanish. This is a bit confusing because the Feynman rules have many factors of $i$, but we can trace them as follows.
  - Vertices always come with a factor of $i$, e.g. because they come from expanding $e^{iH_{\text{int}}t}$, along with a momentum conserving delta function.
  - Edge (i.e. Feynman propagators) always come with a factor of $i$ in the numerator, along with an integral over four-momentum $đp$.
  - Each seemingly real integral over four-momentum is secretly a purely imaginary number, because the integral over $p^0$ can be thought of as a contour integral, which yields $2\pi i$ times the residues of relevant poles. This might sound unfamiliar but it isn't strange, because this is precisely the reverse of the logic that let us introduce four-momentum integrals in the first place.
  - Thus, each edge secretly comes with an extra factor of $i$, as does every vertex (since delta functions remove these four-momentum integrals).

  Since edges and vertices each come with *two* factors of $i$, the amplitude is real. Alternatively, if we use the delta functions to do as many four-momentum integrals as possible, then the number of remaining integrals is the number of loops. Then the total number of factors of $i$ is the number of vertices, plus edges, plus faces (loops) mod 2, which is zero by Euler's formula.

- The idea that a loop gives a factor of $i$ is also consistent with our previous treatment of renormalization. All of our renormalization schemes (dimensional regularization, Pauli–Villars, or a Wilsonian cutoff) *begin* by Wick rotating the integral over $p^0$, giving a factor of $i$.

- Another way of thinking about this is that, as a distribution,

$$\operatorname{Im} \frac{1}{p^2 - m^2 \pm i\epsilon} = \mp \pi \delta(p^2 - m^2).$$

  In a typical loop integral, where only one propagator can be on-shell at once, this yields an imaginary number, making the amplitude overall real. But there is a chance for the amplitude to become imaginary if *multiple* propagators go on-shell at once. And if the intermediate states in a diagram are on-shell, then we can "cut" the diagram in half, giving diagrams contributing to the right-hand side of the optical theorem.

- Now we treat this explicitly. First, we note that at low $s$, we are "below threshold" to produce on-shell intermediate states, so we have

$$\mathcal{M}(s) = (\mathcal{M}(s^*))^*.$$

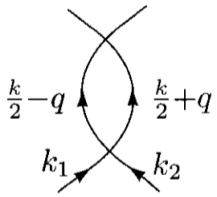
Both sides are analytic, so by analytic continuation we have, for all real $s$,

$$\operatorname{Re}\mathcal{M}(s+i\epsilon) = \operatorname{Re}\mathcal{M}(s-i\epsilon), \quad \operatorname{Im}\mathcal{M}(s+i\epsilon) = -\operatorname{Im}\mathcal{M}(s-i\epsilon).$$

At threshold, $\mathcal{M}(s)$ picks up a branch cut along the real axis, across which the imaginary part is discontinuous. The physically measured quantity corresponds to $\mathcal{M}(s+i\epsilon)$, which is related to the discontinuity across the cut by

$$\operatorname{Disc}\mathcal{M}(s) = 2i\operatorname{Im}\mathcal{M}(s+i\epsilon).$$

- As an example, we consider $\phi^4$ theory. For $2 \to 2$ scattering, at loop level we have $s$, $t$, and $u$-channel diagrams. The latter two don't give rise to branch cuts, so we focus on the $s$-channel.




Let $k = k_1 + k_2$ be the total incoming momentum.

- Applying the Feynman rules, this diagram yields

$$i(\delta\mathcal{M}) = \frac{\lambda^2}{2}\int \frac{đq}{((k/2-q)^2 - m^2 + i\epsilon)((k/2+q)^2 - m^2 + i\epsilon)}.$$

Now we consider the (contour) integral over $q^0$. In the center of mass frame, $k = (k^0, \mathbf{0})$, and defining $E_{\mathbf{q}} = \sqrt{|\mathbf{q}|^2 + m^2}$ as usual, the integral has poles at

$$q^0 = \frac{1}{2}k^0 \pm (E_{\mathbf{q}} - i\epsilon), \quad q^0 = -\frac{1}{2}k^0 \pm (E_{\mathbf{q}} - i\epsilon).$$

To perform the integral, we close the contour downward, picking up the residues of the poles in the lower half-plane. Only the pole at $q^0 = -(1/2)k^0 + E_{\mathbf{q}}$ will contribute to the discontinuity, and accounting for it is equivalent to replacing

$$\frac{1}{(k/2+q)^2 - m^2 + i\epsilon} \to -i\not{\phi}((k/2+q)^2 - m^2)$$

in the original integral.

- Straightforwardly performing the integral gives

$$i(\delta\mathcal{M}) = -i\frac{\lambda^2}{2}\int \frac{đ\mathbf{q}}{2E_{\mathbf{q}}}\frac{1}{(k^0 - E_{\mathbf{q}})^2 - E_{\mathbf{q}}^2} = -i\frac{\lambda^2}{2}\frac{1}{2\pi^2}\int_m^\infty dE_{\mathbf{q}}\, E_{\mathbf{q}}|\mathbf{q}|\frac{1}{2E_{\mathbf{q}}}\frac{1}{k^0(k^0 - 2E_{\mathbf{q}})}.$$

The integrand has a pole at $E_{\mathbf{q}} = k^0/2$, which thus produces a branch cut if the integral goes over it, $k^0 > 2m$. To find the discontinuity across the branch cut, note that

$$\frac{1}{k^0 - 2E_{\mathbf{q}} \pm i\epsilon} = P\frac{1}{k^0 - 2E_{\mathbf{q}}} \mp i\pi\delta(k^0 - 2E_{\mathbf{q}})$$

so we may replace this factor with a delta function. This is equivalent to replacing

$$\frac{1}{(k/2-q)^2-m^2+i\epsilon} \to -i\not{\phi}((k/2-q)^2-m^2)$$

in the original integral. (Of course, the original integral is also divergent and must be regularized. We don't need to talk about a specific regulator here, however, because any reasonable regulator must preserve unitarity.)

- Now we return to the original integral, rewriting the loop integral as

$$\int đq = \int đp_1 \int đp_2 \not{\phi}(p_1+p_2-k).$$

If we make the two replacements above, we simply put the momenta $p_i$ on shell, and integrating the $p_i^0$ against the delta functions produces the standard factors for Lorentz invariant phase space, giving

$$\operatorname{Disc}\mathcal{M}(k) = 2i\operatorname{Im}\mathcal{M}(k) = \frac{i}{2}\int \frac{đ\mathbf{p}_1}{2E_1}\frac{đ\mathbf{p}_2}{2E_2}\lambda^2\not{\phi}(p_1+p_2-k).$$

This is precisely the prediction of the optical theorem, where the $1/2$ in front is the phase space factor for identical final particles, and the $\lambda^2$ is the squared matrix element.

- This logic above can be generalized to yield "Cutkosky cutting rules" to compute the discontinuity of any diagram:
  - Cut through the diagram, so that it falls apart into two pieces, separating the initial and final particles.
  - For each cut, replace $1/(p^2-m^2+i\epsilon) \to -i\not{\phi}(p^2-m^2)$.
  - Sum over all possible cuts.

  This can be used to prove the optical theorem to all orders in perturbation theory.

- Another way to arrive at this is to note that the Feynman propagator and retarded propagator differ only in that the retarded propagator has both poles in the upper half plane, so

$$D_F(p) = D_R(p) + \frac{\pi}{E_\mathbf{p}}\delta(p^0-E_\mathbf{p}).$$

If we apply this formula to all propagators in a loop, then the resulting term consisting of a product of only retarded propagators has zero contour integral, by closing the contour in the lower half plane. By applying the formula in reverse, we recover terms with Feynman propagators, but each term has at least one delta function, thereby reducing a loop amplitude to a tree amplitude.

- We could also treat the example by just evaluating the integral explicitly in, e.g. dimensional regularization, in which case the discontinuity appears through a logarithm.

- Now consider applying the same logic to particles with spin. The right-hand side of the optical theorem will involve spin sums, e.g. for decay into two fermions $\psi(p_1)\overline{\psi}(p_2)$ we get $\operatorname{tr}((\not{p}_1+m)(\not{p}_2-m))$, as shown in the previous section. The optical theorem hence requires that the numerator of a propagator must involve a sum over physical spin states.

- For Dirac fermions this is clearly true, as $\not{p} + m$ indeed appears in the numerator. The result is also clearly true for massive vectors. But for massless vectors, the spin sum is over the two physical polarizations, while the propagator numerator is $\xi$-dependent! However, as we've seen, we can replace the spin sum with $\eta^{\mu\nu}$ by the Ward identity, and we can freely choose $\xi = 1$ by gauge invariance to make the propagator match.

Next, we apply the optical theorem to unstable particles.

- Recall that we wrote the exact propagator for a scalar particle as

$$D_F(p^2) = \frac{i}{p^2 - m_0^2 - M^2(p^2)}$$

  where $-iM^2(p^2)$ was the sum of all 1PI diagrams. On the other hand, these same diagrams can be used to compute $1 \to 1$ forward scattering, and the LSZ reduction formula gives

$$\mathcal{M}(p \to p) = -ZM^2(p^2).$$

- Suppose that the scalar particle is stable. Then by the optical theorem, $\mathcal{M}(p \to p)$ is real, so the self-energy is real, which implies that the pole of the propagator is simply shifted along the real axis, as we've seen before.

- If the particle can decay, then $M^2(p^2)$ acquires an imaginary part, shifting the pole off the real axis. In this case, the particle is not an asymptotic state, so we can't assign it an unambiguous mass; equivalently, we have some freedom over what to call the mass. A reasonable choice for a slowly decaying particle is to let $m$ be the "real pole mass",

$$m^2 - m_0^2 - \operatorname{Re} M^2(m^2) = 0.$$

  Then for $p^2 \approx m^2$, the exact propagator is

$$D_F(p^2) \sim \frac{iZ}{p^2 - m^2 - iZ \operatorname{Im} M^2(p^2)}.$$

  For particles such as the bottom quark, which immediately hadronize, the standard choice is to instead use the $\overline{\text{MS}}$ mass.

- If the propagator appears in the $s$-channel of a Feynman diagram, then the cross section obeys

$$\sigma(s) \propto \left| \frac{1}{s - m^2 - iZ \operatorname{Im} M^2(s)} \right|^2 = \frac{1}{(s - m^2)^2 + (Z \operatorname{Im} M^2(s))^2}$$

  which gives a Breit–Wigner resonance peak. The optical theorem thus tells us that the width of this peak is related to the particle's decay rate,

$$\Gamma = -\frac{Z}{m} \operatorname{Im} M^2(m^2).$$

  Above we have treated $\operatorname{Im} M^2(p^2)$ as a constant near the peak, so this holds in the case of narrow resonances, $\Gamma \ll m$. For broad resonances, there can be deviations from the Breit–Wigner shape.

- Suppose we are searching for a weakly coupled particle produced in the $s$-channel of a scattering process. There is naively a paradox here. If we consider just the first half of the Feynman diagram, involving the on-shell production of the new particle, then there is only one vertex, so the cross section scales as $g^2$. But if we consider the full Feynman diagram, which includes the particle's subsequent decay, then there are two vertices and the cross section scales as $g^4$.
- The resolution is that in the full diagram, we should perform the integral $\int \sigma(s)\,ds$ to get the full rate, and the sharp peak of the Breit–Wigner distribution gives a factor of $1/\Gamma \propto 1/g^2$. Therefore, the two calculations give equivalent results. This enhancement was used by LEP when it ran at the $Z$-pole.
- The above story does not hold if the kinematics are such that the weakly coupled particle is always far off-shell; in that case the diagram cannot be cut, and the rate scales as $g^4$. Also, there is a limit to how long in $g$ one can probe, because for sufficiently low $g$ the particle becomes so long-lived that it does not decay before exiting the detector.
- If we trace through the definitions above, the shifted pole in $s$ is below the real axis, while for stable particles, the physical values of $s$ were $i\epsilon$ above the real axis. So if we place the multiparticle branch cut along the real axis, we reach this pole by analytically continuing downward through it, so the pole is really on the "second sheet".

Finally, we discuss unitarity bounds.

- Consider a $2 \to 2$ elastic scattering in the CM frame. The total cross section is

$$\sigma_{\text{el}} = \frac{1}{32\pi E_{\text{cm}}^2} \int d\cos\theta\, |\mathcal{M}(\theta)|^2$$

  where $\theta$ is the final angle between the particles. In quantum field theory, the partial wave expansion is simply the decomposition

$$\mathcal{M}(\theta) = 16\pi \sum_{j=0}^{\infty} a_j (2j+1) P_j(\cos\theta).$$

- By logic similar to the nonrelativistic version of the partial wave expansion,

$$\sigma_{\text{el}} = \frac{16\pi}{E_{\text{cm}}^2} \sum_{j=0}^{\infty} (2j+1)|a_j|^2.$$

- The optical theorem gives us the relation

$$\operatorname{Im} \mathcal{M}(\theta = 0) = 2E_{\text{cm}} p_{\text{cm}} \sigma_{\text{tot}} \geq 2E_{\text{cm}} p_{\text{cm}} \sigma_{\text{el}}$$

  where $\sigma_{\text{tot}}$ is the cross section for scattering into any final state. This yields

$$\sum_{j=0}^{\infty} (2j+1) \operatorname{Im} a_j \geq \frac{2p_{\text{cm}}}{E_{\text{cm}}} \sum_{j=0}^{\infty} (2j+1)|a_j|^2.$$

  In fact, the sum on $j$ can be dropped, by considering scattering of angular momentum eigenstates rather than plane waves.

- For simplicity, consider the ultrarelativistic limit, where $p_{\text{cm}} \approx E_{\text{cm}}/2$. Then

$$\operatorname{Im} a_j \geq |a_j|^2$$

which is the interior of a circle of unit diameter in the complex plane, centered at $i/2$, so

$$|a_j| \leq 1, \quad 0 \leq \operatorname{Im} a_j \leq 1, \quad |\operatorname{Re} a_j| \leq \frac{1}{2}.$$

- The coefficients $a_j$ can be computed order by order in perturbation theory. If the partial wave unitarity bound is violated at some order, we say the theory violates "perturbative unitarity". This doesn't mean that the theory violates unitarity; rather, it means that the results of perturbation theory cannot be trusted, and we must switch to a new description that can be.

- This was one of the main motivations for searching for the Higgs boson. If one calculates the amplitude for $W^+W^- \to W^+W^-$ in the Standard Model without including the diagrams with the Higgs boson, the amplitude scales as $\mathcal{M} \sim g^2E^2/m_W^2$, leading to a breakdown of perturbative unitarity above a few hundred GeV. However, adding the diagrams involving the Higgs precisely cancels this growth.

- Another useful fact is the Froissart bound: the total cross section cannot grow faster than $\log^2(E_{\text{cm}})$ at high energies.

- Finally, recall that the spectral representation writes the exact time-ordered two-point function as a combination of bare Feynman propagators with positive spectral weight. This implies that propagators cannot fall off faster than $1/p^2$ at large $p^2$.

- For example, the theory

$$\mathcal{L} \supset -\frac{1}{2}\phi\left(\partial^2 + \frac{\partial^4}{\Lambda^2} + m^2\right)\phi$$

has a propagator that falls off as $1/p^4$ at large $p^2$. Thus, it is tempting to use this to construct a UV finite quantum field theory, but the point above forbids this. As we'll see later, terms like this do appear in effective theories, but this is acceptable because this behavior only kicks in as we approach the cutoff, at which point infinitely many other terms become important and the naive calculation of the propagator falls apart.

## 6.3 Pololoy (TODO)

## 6.4 Soft Theorems

In this section, we use the soft limit and Lorentz invariance to establish strong constraints on how massless particles can interact.

- Consider an arbitrarily complicated amplitude with an incoming particle of momentum $p_i$, $\mathcal{M}_0(p_i)$. Now suppose a photon of momentum $q$ is emitted off that particle's incoming line, giving the amplitude $\mathcal{M}_i(p_i, q)$.

- For concreteness, we suppose the particle is a scalar with charge $e$. Using the Feynman rules of scalar QED, and assuming the photon polarization is real to avoid having to write $\epsilon_\mu^*$ everywhere,

$$\mathcal{M}_i(p_i, q) = (ie)\frac{i(p_i^\mu + (p_i^\mu - q^\mu)}{(p_i - q)^2 - m^2}\epsilon_\mu \mathcal{M}_0(p_i - q).$$

Since external particles are on-shell, we have $p_i^2 = m^2$ and $q^2 = 0$. Since the photon polarizations are transverse, $q^\mu \epsilon_\mu = 0$. Thus, we have

$$\mathcal{M}_i(p_i, q) = e \frac{p_i \cdot \epsilon}{p_i \cdot q} \mathcal{M}_0(p_i - q).$$

- Next, we take the soft limit of small $q$, which strictly speaking means $|q \cdot p_i| \ll |p_j \cdot p_k|$ for all external momenta $p_j$ and $p_k$. Then we have

$$\mathcal{M}_i(p_i, q) \approx \left( e \frac{p_i \cdot \epsilon}{p_i \cdot q} \right) \mathcal{M}_0(p_i).$$

- Now, consider the total scattering amplitude $\mathcal{M}(q)$ involving all of the original particles in $\mathcal{M}_0$, and an extra outgoing soft photon with momentum $q$. To compute this, we need to sum over all possible attachment points for the photon. However, the "soft factor" that we pick up is large only when the photon attaches to an on-shell particle. For generic amplitudes, this can only happen on external lines or in loops, but in loops the on-shell part is softened by an integration over momenta, so the contributions of external lines dominate.

- Repeating the reasoning above, we find that when we attach the photon on incoming particles, we get a factor of the charge $eQ_i$, while attaching the photon on outgoing particles gives the same result with the opposite sign. Thus,

$$\mathcal{M}(q) \approx e \left( \sum_{i \text{ incoming}} Q_i \frac{p_i \cdot \epsilon}{p_i \cdot q} - \sum_{i \text{ outgoing}} Q_i \frac{p_i \cdot \epsilon}{p_i \cdot q} \right) \mathcal{M}_0.$$

- Now, the amplitude $\mathcal{M}(q)$ must be Lorentz invariant. Under a Lorentz transformation, the amplitude can only pick up a trivial phase factor due to the little group action on the photon. But we also know that in general, a Lorentz transformation shifts the polarization vector, $\epsilon_\mu \to \epsilon_\mu + q_\mu$, which changes the amplitude by

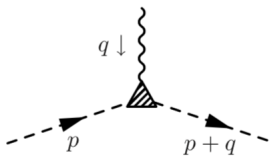
$$e \left( \sum_{i \text{ incoming}} Q_i - \sum_{i \text{ outgoing}} Q_i \right) \mathcal{M}_0.$$

  Demanding this be zero implies conservation of the charge the photon couples to.

- Note that we have already shown that a photon must couple to a conserved current using the canonical formalism, but this derivation is much more minimal, requiring only Lorentz invariance of the $S$-matrix and the soft limit.

Next, we generalize this derivation.

- First, it's not clear that the logic above holds for any type of external particle. Let's suppose that the particle is a scalar, but the interaction vertex is arbitrary,




$$= -ie\Gamma_\mu(p, q).$$

- In general, we can write the vertex in terms of form factors,

$$\Gamma^\mu = 2p^\mu F(p^2, q^2, p\cdot q) + q^\mu G(p^2, q^2, p\cdot q).$$

  The form factor $G$ is irrelevant because $q^\mu \epsilon_\mu = 0$.

- Since $p^2 = m^2$ and $q^2 = 0$, the dependence of $F$ can be written as

$$\Gamma^\mu = 2p^\mu F\left(\frac{p\cdot q}{m^2}\right)$$

  by dimensional analysis. In the soft limit, the vertex simplifies further to $2p^\mu F(0)$. Thus everything goes through as before, with $F(0)$ being the definition of the conserved charge, as we've already seen in QED.

- For a minimally coupled Dirac spinor, we would instead pick up the factor

$$\overline{u}^s(\mathbf{p})\gamma^\mu(\not{p} + m) = \sum_{s'} \overline{u}^s(\mathbf{p})\gamma^\mu u^{s'}(\mathbf{p})\overline{u}^s(\mathbf{p})$$

  in the numerator. However, we end up getting precisely the same result, by the identity

$$\overline{u}^s(\mathbf{p})\gamma^\mu u^{s'}(\mathbf{p}) = 2p^\mu \delta^{ss'}.$$

  The analysis proceeds similarly for a non-minimally coupled Dirac spinor.

- Next, we consider a massless spin 2 field. Such a field has two polarizations $\epsilon^i_{\mu\nu}$, satisfying

$$q^\mu \epsilon_{\mu\nu} = 0, \quad \eta^{\mu\nu}\epsilon_{\mu\nu} = 0.$$

  Under a little group transformation, we have

$$\epsilon_{\mu\nu} \to \epsilon_{\mu\nu} + \Lambda_\mu q_\nu + q_\mu \Lambda_\nu + \Lambda q_\mu q_\nu$$

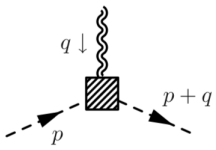
  where $\Lambda_\mu$ and $\Lambda$ depend on the little group transformation in a way we do not write out here.

- Therefore, the Ward identity must be that for any amplitude involving such an external particle, $\mathcal{M} = \epsilon_{\mu\nu}\mathcal{M}^{\mu\nu}$, we must have

$$q_\mu \mathcal{M}^{\mu\nu} = q_\nu \mathcal{M}^{\mu\nu} = 0$$

  so that the amplitude is properly Lorentz invariant.

- Now we can repeat our above argument. By similar logic, an arbitrary interaction vertex is




$$= -i\Gamma^{\mu\nu}(p,q) = -2ip^\mu p^\nu \tilde{F}\left(\frac{p\cdot q}{m^2}\right),$$

  where the index structure must be $p^\mu p^\nu$, because we cannot use factors of $q^\mu$ as the polarizations are transverse, and we cannot use factors of $\eta^{\mu\nu}$ as the polarizations are traceless.

- Therefore, in the soft limit,

$$\mathcal{M}(q) \approx \left( \sum_{i \text{ incoming}} \tilde{F}_i(0) \frac{p_i^\mu \epsilon_{\mu\nu} p_i^\nu}{p_i \cdot q} - \sum_{i \text{ outgoing}} \tilde{F}_i(0) \frac{p_i^\mu \epsilon_{\mu\nu} p_i^\nu}{p_i \cdot q} \right) \mathcal{M}_0.$$

  By Lorentz invariance, this must vanish if we substitute $\epsilon_{\mu\nu} \to q_\mu \Lambda_\nu$, so

$$\mathcal{M}_0 \Lambda_\nu \left( \sum_{i \text{ incoming}} \tilde{F}_i(0) p_i^\nu - \sum_{i \text{ outgoing}} \tilde{F}_i(0) p_i^\nu \right) = 0.$$

  However, we already know that the sum of incoming and outgoing momenta must be equal, by momentum conservation. So the quantity above can only vanish in general if the $\tilde{F}_i(0)$ are all equal, which means that massless spin 2 particles must couple with universal strength. This is indeed the case for the only massless spin 2 particle we know of, the graviton.

- For higher spin, the soft limit becomes even more restrictive. For massless spin 3,

$$\sum_{i \text{ incoming}} \tilde{F}_i(0) p_i^\mu p_i^\nu = \sum_{i \text{ outgoing}} \tilde{F}_i(0) p_i^\mu p_i^\nu$$

  but when combined with momentum conservation, this only has solutions for general momenta if the $\tilde{F}_i(0)$ are all zero. As a result, higher spin massless particles cannot mediate long-range forces, though they can mediate interactions with a faster falloff.

- One loophole is the case of continuous spin particles, which contain all possible helicities. These evade the reasoning above because their little group transformation is more general than just a phase, but rather can mix different helicity states, thereby allowing them to interact.

- By applying this reasoning to two soft particles, one can get further constraints. For example, it is possible to show that Yang–Mills theory is essentially the unique theory of interacting massless spin 1 particles (e.g. the quartic gauge boson interaction that appears in Yang–Mills is required by the soft limit). Similar reasoning can be used to constrain the graviton self-coupling, in a way that matches what appears in the Einstein–Hilbert action.

**(what about half-integer helicity?) (what about scalar QED, where you can couple to 2 soft photons?)**

# 7 Path Integrals in Zero Dimensions

## 7.1 Introduction

We begin by specifying what a quantum field theory is, from the path integral perspective.

- First, we specify the space where the fields live, usually a smooth manifold with metric $(M, d)$.
  - For most applications to particle physics, we choose $(M, g) = (\mathbb{R}^4, \eta)$.
  - In cosmology, we choose a different background metric, e.g. an FRW metric.
  - For most applications in condensed matter, we choose $(M, g) = (\mathbb{R}^3, \delta)$ where $\delta$ is the Euclidean metric.
  - The worldsheet description of string theory uses $(\Sigma, [g])$ where $\Sigma$ is a Riemann surface and $g$ is a metric defined up to conformal equivalence.
  - Applications of quantum field theory to knot theory typically use an orientable three-manifold $M$ with no metric at all.

  In all cases, we regard the metric as fixed; making it dynamical requires quantum gravity.

- The fields are maps $\phi : M \to N$ where $N$ is called the target space.
  - Ordinary nonrelativistic quantum mechanics can be thought of as a $d = 1$ quantum field theory living on the interval $I = [0, 1]$ called the worldline, with target space $\mathbb{R}^3$.
  - In particle physics, pions live in a coset space, so $N \cong G/H$ for Lie groups $G$ and $H$.
  - In string theory, some of the worldsheet fields map to a Calabi-Yau manifold $N$.
  - In gauge theories, the gauge field is a connection on a principal $G$-bundle $P \to M$, and matter charged under this field is a section of a an associated vector bundle $E \to M$ specified by a representation of $G$.

- Let $\mathcal{C}$ denote the space of field configurations on $M$. Typically, $\mathcal{C}$ is an infinite-dimensional function space. An action is a function $S : \mathcal{C} \to \mathbb{R}$. The critical set where $\delta S = 0$ correspond to classical solutions, where the variation $\delta$ should be thought of as an exterior derivative.

- We usually assume the action is local, so that, e.g. for a single scalar field $\phi$,

$$S[\phi] = \int_M d^d x \, \sqrt{g} \mathcal{L}(\phi(x), \partial\phi(x), \ldots).$$

  This is a very strong constraint; local actions are a very small subset of the set of all actions. Classically, the Euler–Lagrange equations become integro-differential equations, leading to action at a distance, which we think of as unphysical. However, nonlocal actions do appear in quantum field theory and must be treated carefully. We'll often suppress the $\sqrt{g}$.

Next, we consider what we want to compute.

- The most important quantity to compute is the partition function

$$\mathcal{Z} = \int_{\mathcal{C}} \mathcal{D}\phi \, \exp\left(-\frac{S[\phi]}{\hbar}\right)$$

  where we are working in Euclidean signature, $\mathcal{D}\phi$ is some sort of measure over the set of field configurations, and the integral makes sense if $M$ is closed and compact. This depends only on $S$, the space $M$, the metric $g$, and $\hbar$, not on the fields, which are the integration variables.

- Conceptually, the exponential factor damps the impact of 'wild' field configurations with rapid fluctuations. We've seen the 'fight' between the two in statistical mechanics, where the $e^{-\beta H}$ factor favors smooth field configurations, but rapidly varying configurations are favored by sheer numbers; it is "the eternal struggle between energy and entropy". This leads to rich and unexpected phenomena such as phase transitions.

- In the case of quantum field theory, the issues are even worse, since the configuration spaces are infinite-dimensional. We won't deal with these issues, but we will pay respect to them by using only Euclidean path integrals, which are somewhat nicer.

- We also may want to compute correlation functions, defined as

$$\left\langle \prod_i \mathcal{O}_i[\phi] \right\rangle = \frac{1}{\mathcal{Z}} \int_C \mathcal{D}\phi \, \exp\left(-\frac{S[\phi]}{\hbar}\right) \prod_i \mathcal{O}_i[\phi].$$

  Mathematically, we are computing moments of the probability distribution $\mathcal{D}\phi e^{-S/\hbar}/\mathcal{Z}$. We call the $\mathcal{O}_i$ operator insertions, which will make sense once we relate path integration to the canonical picture.

- Often we are interested in local operators, such as $\phi^3(x)\partial^\mu\phi(x)\partial_\mu\phi(x)$ or $e^{\phi(x)}$, but we can also consider nonlocal operators. For example, the Wilson loop

$$W_\Gamma[A] = \operatorname{tr} P \exp\left(-\oint A\right)$$

  depends on the value of the gauge field $A$ along the loop $\Gamma$. Again, correlation functions of such operators don't depend on the fields themselves, only on the theory.

- Formal differentiation of the partition function allows us to recover correlation functions of operators in the action. For example, if the action has a term

$$\mathcal{O} = \lambda \int_M d^d x \, \phi^4(x)$$

  then differentiating with respect to $\lambda$ gives

$$\frac{\partial \mathcal{Z}}{\partial \lambda} = -\frac{1}{\hbar} \int_{\mathcal{C}} \mathcal{D}\phi \, e^{-S[\phi]/\hbar} \int_M d^d x \, \phi^4 = -\frac{\mathcal{Z}\langle \mathcal{O} \rangle}{\hbar}.$$

  Thus, correlation functions tell us about the response of the partition function to changes in the parameters of the theory.

- Similarly, suppose we add a source term to the action,

$$S[\phi] \to S[\phi] + \int_M d^d x \, J_i(x) \mathcal{O}_i(x).$$

  Then the sources $J_i(x)$ are fields, and $\mathcal{Z}$ is a functional of it; varying with respect to it gives correlation functions of local operators,

$$-\hbar \frac{\delta \mathcal{Z}[J_i]}{\delta J_i(x_i)} = \int_{\mathcal{C}} \mathcal{D}\phi \, e^{-(S[\phi] + \int J_j \mathcal{O}_j)/\hbar} \mathcal{O}_i(x_i) = \mathcal{Z}\langle \mathcal{O}_i(x_i) \rangle.$$

  Repeating this procedure, we have

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle = \frac{(-\hbar)^n}{\mathcal{Z}} \frac{\delta^n \mathcal{Z}[J]}{\delta J_1(x_1) \dots \delta J_n(x_n)} \bigg|_{J=0}.$$

- The most common case is when the operators $\mathcal{O}_i$ are just the fields; for example, with one scalar field we have
$$\langle \phi(x_1) \dots \phi(x_n) \rangle = \frac{(-\hbar)^n}{\mathcal{Z}} \frac{\delta^n \mathcal{Z}[J]}{\delta J(x_1) \dots \delta J(x_n)}\bigg|_{J=0}.$$
However, there's no reason we can't do this for composite operators, i.e. operators that are nonlinear in the fields. For example, under a metric variation
$$\delta S[\phi] = \frac{1}{2} \int_M d^d x \, \sqrt{g} \, T_{\mu\nu}(x) \delta g^{\mu\nu}(x)$$
which implies that
$$-\frac{2\hbar}{\sqrt{g(x)}} \frac{\delta \log \mathcal{Z}}{\delta g^{\mu\nu}(x)} = \langle T_{\mu\nu}(x) \rangle$$
where everything is evaluated at the background metric, $\delta g = 0$.

Next, we link path integrals to the Hilbert space.

- Suppose our manifold $M$ has boundaries, $\partial M = \cup_i B_i$. Then to specify the path integral, we must choose boundary conditions on the $B_i$. For each $B_i$, the set of boundary conditions forms a Hilbert space $\mathcal{H}_i$, so the path integral defines a map $\mathcal{Z} : \otimes_i \mathcal{H}_i \to \mathbb{C}$ by
$$\mathcal{Z}[B_i] = \int_{\phi|_{B_i} = \varphi_i} \mathcal{D}\phi \, e^{-S[\phi]/\hbar}.$$
We won't specify an inner product on the $\mathcal{H}_i$, because it's unknown how to define it in general.

- An important special case is $M = N \times I$ where $I$ is an interval of length $T$ with respect to the metric $g$ on $M$. Then the path integral gives a map
$$U(T) : \mathcal{H} \to \mathcal{H}, \quad \langle \varphi_1 | U(T) | \varphi_0 \rangle = \int_{\phi|_{N\times\{0\}} = \varphi_0}^{\phi|_{N\times\{T\}} = \varphi_1} \mathcal{D}\phi \, e^{-S[\phi]/\hbar}.$$
This is simply the time evolution operator. However, it is difficult to show that this map is unitary; that's one of the advantages of the canonical picture.

- Now we take a detour in classical field theory, i.e. considering only the action and not integrating over the fields. The variation of the action takes the form
$$\delta S[\phi] = (\text{bulk eom}) \, \delta\phi + \sum_i \int_{B_i} n_i^\mu \frac{\delta \mathcal{L}}{\delta(\partial^\mu \phi)} \delta\phi \, \sqrt{g} \, d^{d-1} x$$
where we allow the variation to be nonzero on the boundary. Define the field momentum
$$\pi_i = \sqrt{g} \, n_i^\mu \frac{\delta \mathcal{L}}{\delta(\partial^\mu \phi)}.$$
For example, when $\partial M$ is a constant time slice of flat spacetime, then $\pi(t) = \delta\mathcal{L}/\delta\dot{\phi}(t)$.

- Now, the variation $\delta$ maps functions on $\mathcal{H}$ to one-forms, since $\delta S[\phi]$ is a number, so we may extend its definition so it is an exterior derivative on differential forms. Then

$$0 = \delta^2 S[\phi]|_{\text{eom}} = \int_{N\times\{T\}} \delta\pi \wedge \delta\phi \, d^{d-1}x - \int_{N\times\{0\}} \delta\pi \wedge \delta\phi \, d^{d-1}x.$$

  This implies that there is a conserved quantity

$$\Omega = \int_N \delta\pi \wedge \delta\phi \, d^{d-1}$$

  which is a symplectic form on the space of fields. The symplectic form defines a Poisson bracket, and its conservation implies Liouville's theorem. This is the structure that we aim to quantize in canonical quantization, replacing brackets with commutators.

- For example, for the canonical quantization on $(\mathbb{R}^4, \eta)$ of the free real scalar field,

$$\phi(x) = \int \frac{đ\mathbf{p}}{\sqrt{2E}} \left(e^{ipx} a(\mathbf{p}) + e^{-ipx} a^\dagger(\mathbf{p})\right), \quad [\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta(\mathbf{x}-\mathbf{y}).$$

  The commutation relations are only canonical at equal times, since the symplectic form is defined on timeslices. We defined $\phi(x)$ on all of spacetime, rather than timeslices, but this isn't a big difference since the values of $\phi$ and $\pi$ on one timeslice determine it everywhere, so boundary values and full solutions are equivalent.

- Next, we look at the state space. A general state on the boundary Hilbert space is

$$|\Psi\rangle = \int_{\mathcal{C}[B]} [d\varphi] \, |\varphi\rangle\langle\varphi|\Psi\rangle, \quad \Psi[\varphi] = \langle\varphi|\Psi\rangle$$

  where we are vague about the inner product and measure. That is, a general state is a superposition of $|\varphi\rangle$ states, each of which correspond to a function on $N$, representing a definite field configuration.

- In canonical quantization, we used wavefunctions that were polynomials, with monomials

$$\int_{N^{\otimes n}} d\mathbf{x}_1 \ldots d\mathbf{x}_n \, \psi(\mathbf{x}_1, \ldots, \mathbf{x}_n)\phi(\mathbf{x}_1)\ldots\phi(\mathbf{x}_n)$$

  interpreted as $n$-particle states. For example, the monomial $\phi(\mathbf{x})$ corresponds to the state $\widehat{\phi}(\mathbf{x})|0\rangle$, i.e. a single particle at $\mathbf{x}$. This state is a superposition of the states $|\varphi\rangle$, weighting states with larger $\phi(\mathbf{x})$ more.

- Assuming our particles are bosons, $n$-particle states live in the Fock space

$$\mathcal{H} = \mathcal{C} \oplus V \oplus \text{Sym}^2 V \oplus \text{Sym}^3 V \oplus \ldots$$

  where $V$ is the one-particle Hilbert space. Working in the Fock basis, which is not always available, is analogous to expanding a solution to the quantum harmonic oscillator in terms of Hermite polynomials (times Gaussian weights). The general problem is analogous to working in the position basis in quantum mechanics, which is mathematically much trickier.

- If $M$ is non-compact, there may be a region that is asymptotically far away. Then to define the path integral, we must specify asymptotic values of the fields. The simplest choice is $\phi \to 0$, but another important choice is $\phi \to \phi_i$ in the asymptotic past and $\phi \to \phi_f$ in the asymptotic future, in which case the path integral gives the scattering amplitude $\langle\phi_f|\phi_i\rangle$. These are related to correlation functions by the LSZ reduction formula.

## 7.2 Zero Dimensional Field Theory

We begin with the case of zero dimensions, where some important ideas will appear simply.

- If our spacetime $M$ is zero-dimensional and connected, it must be a single point. There is no metric, and the Lorentz group is trivial, so all of its representations are; all fields are scalars. In the path integral, fields are just maps from the single point to $\mathbb{R}$, so $\mathcal{C} \cong \mathbb{R}$.
- The path integral measure $\mathcal{D}\phi$ becomes the usual measure $d\phi$ on $\mathbb{R}$ and the partition function

$$\mathcal{Z} = \int_{\mathbb{R}} d\phi \, e^{-S(\phi)/\hbar}$$

  is an ordinary real integral.
- The action can't have derivatives, so there are no kinetic terms. We'll usually choose it to be a polynomial, such as $S(\phi) = m^2\phi^2/2 + g\phi^4/4!$, where the highest term must have even degree and positive coefficient for the partition function to converge. We can think of the coupling constants as coordinates on an infinite-dimensional 'space of theories'.
- The partition function is a function of these coupling constants, $\mathcal{Z} = \mathcal{Z}(m^2, g, \ldots)$. We denote the free partition function $\mathcal{Z}_0$ to be the partition function where the action is quadratic.
- Correlation functions are defined as

$$\langle f \rangle = \frac{1}{\mathcal{Z}} \int_{\mathbb{R}} d\phi \, f(\phi) e^{-S(\phi)/\hbar}.$$

  The only restriction is that $f$ cannot grow so fast the integral diverges; in practice we restrict $f$ to be polynomial. Thinking of $e^{-S(\phi)/\hbar}$ as a probability distribution, $\langle f \rangle$ is simply the expectation value of $f$.
- Again, we can recover correlation functions from the partition function. For example,

$$\frac{1}{p!}\langle \phi^p \rangle = -\frac{\hbar}{\mathcal{Z}} \frac{\partial}{\partial g_p} \mathcal{Z}(m^2, \lambda_i)$$

  where the action has terms $g_p\phi^p/p!$.

Now we turn to some explicit computations in a free field theory.

- We consider $n$ fields $\phi^a$ with action

$$S(\phi) = \frac{1}{2} M(\phi, \phi) = \frac{1}{2} M_{ab}\phi^a\phi^b$$

  where $M_{ab}$ is real, positive-definite and symmetric. Then the partition function

$$\mathcal{Z}_0 = \int_{\mathbb{R}^n} d^n\phi \, e^{-M(\phi,\phi)/2\hbar}$$

  is a Gaussian integral.

- We may diagonalize $M$ using an orthogonal transformation, which leaves the measure invariant. Then applying the standard Gaussian integral,

$$\int_{\mathbb{R}} d\chi\, e^{-m\chi^2/2\hbar} = \sqrt{\frac{2\pi\hbar}{m}}, \quad \mathcal{Z}_0 = \frac{(2\pi\hbar)^{n/2}}{\sqrt{\det M}}.$$

- More generally, we can consider a linear source term,

$$S(\phi) = \frac{1}{2}M(\phi,\phi) + J\cdot\phi = \frac{1}{2}M(\tilde{\phi},\tilde{\phi}) - \frac{1}{2}M^{-1}(J,J), \quad \tilde{\phi} \equiv \phi + M^{-1}(J,\cdot)$$

where the shift from $\phi$ to $\tilde{\phi}$ removes the linear term and leaves the measure invariant. Then

$$\mathcal{Z}(J) = \exp\left(\frac{M^{-1}(J,J)}{2\hbar}\right)\mathcal{Z}_0.$$

- Now we consider the correlation function $\langle P(\phi)\rangle$ where $P$ is a polynomial in the $n$ fields. By linearity, it suffices to let $P$ be a monomial. We will treat the case where $P$ is a product of linear factors $\ell(\phi) = \ell\cdot\phi$, so we compute

$$\langle \ell_1(\phi)\dots\ell_p(\phi)\rangle = \frac{1}{\mathcal{Z}_0}\int_{\mathbb{R}^n} d^n\phi\, e^{-M(\phi,\phi)/2\hbar}\prod_{i=1}^{p}\ell_i(\phi).$$

If $p$ is odd, the integrand is odd in $\phi$ and hence the integral vanishes. For $p = 2k$,

$$\begin{aligned}
\langle \ell_1(\phi)\dots\ell_{2k}(\phi)\rangle &= \frac{1}{\mathcal{Z}_0}\int_{\mathbb{R}^n} d^n\phi \prod_{i=1}^{2k}\ell_i(\phi)\, e^{-M(\phi,\phi)/2\hbar - J(\phi)/\hbar}\Big|_{J=0} \\
&= \frac{(-\hbar)^{2k}}{\mathcal{Z}_0}\int_{\mathbb{R}^n} d^n\phi \prod_{i=1}^{2k}\ell_i\cdot\partial_J\, e^{-M(\phi,\phi)/2\hbar - J(\phi)/\hbar}\Big|_{J=0} \\
&= \prod_{i=1}^{2k}\ell_i\cdot\partial_J \frac{(-\hbar)^{2k}}{\mathcal{Z}_0}\int_{\mathbb{R}^n} d^n\phi\, e^{-M(\phi,\phi)/2\hbar - J(\phi)/\hbar}\Big|_{J=0} \\
&= \hbar^{2k}\prod_{i=1}^{2k}\ell_i\cdot\partial_J\left(e^{M^{-1}(J,J)/2\hbar}\right)\Big|_{J=0}.
\end{aligned}$$

Every factor of $\partial_J$ pops out a factor of $M^{-1}(\ell_i, J)$. Each of these factors must be differentiated with respect to $J$, to leave a factor of $M^{-1}(\ell_i,\ell_j)$, since we set $J = 0$ at the end.

- For example, for a two-point function

$$\langle \ell_1(\phi)\ell_2(\phi)\rangle = M^{-1}(\ell_1,\ell_2)$$

and more generally

$$\langle \ell_1(\phi)\dots\ell_{2k}(\phi)\rangle = \hbar^k \sum_{\text{pairs }(a_i,b_i)}\prod_{i=1}^{k} M^{-1}(\ell_{a_i},\ell_{b_i})$$

where, for example, the four-point function has three terms. We call $M^{-1}$ the (free) propagator; it is simply the inverse of the kinetic term. When all the $\ell_i$ are the same,

$$\langle(\ell\cdot\phi)^{2k}\rangle = \frac{(2k)!}{2^k k!}(\hbar M^{-1}(\ell,\ell))^k.$$

The terms can be represented with free field Feynman diagrams with $2k$ external points.

- The result we've just proven is called Wick's theorem. In the context of probability theory, it's a result about moments of the Gaussian distribution called Isserlis' theorem.
- For fermionic variables that anticommute, $\theta^a\theta^b = -\theta^b\theta^a$, the starting point is

$$\int d^n\theta d^n\overline{\theta}\, \exp(M(\overline{\theta},\theta) + \overline{\eta}\cdot\theta + \overline{\theta}\cdot\eta) = e^{M^{-1}(\overline{\eta},\eta)} \det M$$

  and we can use this to derive an analogue of Wick's theorem.

## 7.3 Perturbation Theory

Next, we turn to perturbation theory.

- In general, we can't evaluate the path integral exactly for an interacting theory, even in zero dimensions. We might hope to Taylor expand in the small parameter $\hbar$. However, the radius of convergence is zero, because the integral diverges in the region $\mathrm{Re}(\hbar) < 0$.
- However, it is possible to obtain an asymptotic expansion. Suppose $S(\phi)$ has a global minimum at $\phi_0$. Then the method of stationary phase gives

$$\int_{\mathbb{R}^n} d^n\phi\, f(\phi) e^{-S(\phi)/\hbar} \sim (2\pi\hbar)^{n/2} \frac{f(\phi_0)e^{-S(\phi_0)/\hbar}}{\sqrt{\det(\partial_a\partial_b S(\phi_0))}}(1 + \hbar A_1 + \hbar^2 A_2 + \ldots)$$

  as $\hbar \to 0^+$, where the first term is the 'semiclassical' term. This is an asymptotic series.
- In general, an asymptotic series $\sum_n a_n\hbar^n$ for $I(\hbar)$ means that

$$\lim_{\hbar\to 0^+} \frac{1}{\hbar^N}\left| I(\hbar) - \sum_{n=0}^{N} a_n\hbar^n \right| = 0$$

  so the truncated series is arbitrarily good for fixed $N$ and sufficiently small $\hbar$. But we cannot make the series arbitrarily good for fixed $\hbar$. Instead, for sufficiently high $N$, the error goes back up; the entire series diverges.
- Consider a single scalar field with action $S(\phi) = m^2\phi^2/2 + \lambda\phi^4/4!$ where we need $\lambda > 0$ for convergence and $m^2 > 0$ to get a unique minimum at $\phi_0 = 0$. The leading term is

$$(2\pi\hbar)^{1/2}\frac{e^{-S(\phi_0)}}{\sqrt{S''(\phi_0)}} = \frac{\sqrt{2\pi\hbar}}{m}.$$

  Going further, we can construct an asymptotic series by expanding the action,

$$\mathcal{Z}(m^2,\lambda) = \int_{\mathbb{R}} d\phi\, e^{-(m^2\phi^2/2 + \lambda\phi^4/4!)/\hbar} = \int_{\mathbb{R}} d\phi\, e^{-m^2\phi^2/2\hbar} \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-\lambda}{4!\hbar}\right)^n \phi^{4n}.$$

  It is illegitimate to exchange the sum and integral because the integral does not converge when $\lambda$ is negative. However, we can cut off the sum and then exchange the finite sum, for

$$\mathcal{Z}(m^2,\lambda) \sim \sum_{n=0}^{N} \frac{1}{n!}\left(\frac{-\lambda}{4!\hbar}\right)^n \int_{\mathbb{R}} d\phi\, e^{-m^2\phi^2/2}\phi^{4n} = \frac{\sqrt{2\hbar}}{m}\sum_{n=0}^{N}\frac{1}{n!}\left(\frac{-\hbar\lambda}{3!m^4}\right)^n \Gamma(2n+1/2)$$

  where we recognized the integral representation of the gamma function, and the $\sim$ denotes asymptotic equality, i.e. equality in the limit $\hbar \to 0^+$.

- Expanding the gamma function, we have the asymptotic series

$$\mathcal{Z}(m^2,\lambda) \sim \frac{\sqrt{2\pi\hbar}}{m}\sum_{n=0}^{N}\left(\frac{-\hbar\lambda}{m^4}\right)^n \frac{(4n)!}{(4!)^n n! 4^n (2n)!} = \frac{\sqrt{2\pi\hbar}}{m}\left(1 - \frac{\hbar\lambda}{8m^4} + \frac{35}{384}\frac{\hbar^2\lambda^2}{m^8} - \dots\right).$$

  Note that up to the overall factor of $\sqrt{2\pi\hbar}$, the series only depends on the combination $\hbar\lambda$. Thus our series can either be thought of as a semiclassical expansion in $\hbar$, or a weak coupling expansion in $\lambda$, and this is generally true for graphs with a fixed number of external lines by some graph theory.

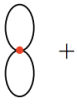
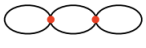
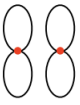
- The series is asymptotic, because $n! \sim e^{n\log n}$, so the numerical factor goes as $e^{n\log n}$. Then eventually it overwhelms the exponential suppression, and the series has zero radius of convergence. However, it is possible to recover $\mathcal{Z}$ from this asymptotic series using Borel summation.

- The theory is well-defined if $m^2 < 0$, but then our asymptotic series is invalid, because we end up integrating a divergent Gaussian. The issue is that we expanded about a maximum instead of a minimum. In physical terminology, we have expanded about an 'unstable vacuum' and found tachyons.

We can also find this series via Feynman diagrams.

- There are two Feynman rules: a propagator is $\hbar/m^2$ and a four-point vertex is $-\lambda/\hbar$. Note that various factors of $i$ are different since we are in Euclidean signature. Here, since the path integral's integrand is proportional to $e^{-S}$, every vertex comes with a factor of $-1$.

- The partition function is the sum of graphs with zero external points. For instance,

$$\begin{aligned}\mathcal{Z}/\mathcal{Z}_0 &= \emptyset + \text{  } + \text{  } + \text{  } + \text{  } + \cdots \\ &= 1 - \frac{\hbar\lambda}{8m^4} + \frac{\hbar^2\lambda^2}{48m^8} + \frac{\hbar^2\lambda^2}{16m^8} + \frac{\hbar^2\lambda^2}{128m^8} + \cdots\end{aligned}$$

  The hard part is explaining the numerical factors, which we do below.

- We let a graph be a Feynman diagram where every vertex and edge has a distinguishable label, and let $D_n$ be the set of graphs with $n$ vertices. Adding the labels introduces overcounting; formally the group $G_n = (S_4)^n \rtimes S_n$ performs arbitrary relabelings, and we have overcounted by a factor of $|G_n|$. Then

$$\frac{\mathcal{Z}}{\mathcal{Z}_0} \sim \sum_{n=0}^{\infty}\left(\frac{-\hbar\lambda}{m^4}\right)^n \frac{|D_n|}{|G_n|}, \quad |G_n| = (4!)^n n!, \quad \mathcal{Z}_0 = \mathcal{Z}(m^2,0)$$

  where the dimensional factors are just from the Feynman rules. Meanwhile, we have

$$|D_n| = (4n-1)(4n-3)\dots(3)(1) = \frac{(4n)!}{4^n(2n)!}$$

  which reproduces the coefficients we found before.

- There is another way to find the factors $|D_n|/|G_n|$. Note that $G_n$ acts on $D_n$, partitioning it into orbits $O_n$. Each orbit $\Gamma \in O_n$ is a 'topological graph', i.e. a graph without labellings. Then the orbit-stabilizer theorem states that

$$\frac{|D_n|}{|G_n|} = \sum_{\Gamma \in O_n} \frac{1}{|\operatorname{Aut}\Gamma|}$$

  where $\operatorname{Aut}\Gamma$ is the stabilizer of any element of $\Gamma$. The quantity $|\operatorname{Aut}\Gamma|$ is also called the symmetry factor. For example, for the lowest order term, we can flip within each loop and interchange the loops, giving a symmetry factor of 8.

- Finally, we can then express our sum as a sum over orbits,

$$\frac{\mathcal{Z}}{\mathcal{Z}_0} = \sum_{\Gamma} \frac{\hbar^{e(\Gamma)-v(\Gamma)}}{|\operatorname{Aut}\Gamma|} \frac{(-\lambda)^{v(\Gamma)}}{(m^2)^{e(\Gamma)}}.$$

  This recovers the usual Feynman diagram expansion.

- It is useful to define the free energy or Wilsonian effective action by $\mathcal{W} = -\hbar \log \mathcal{Z}$, so

$$\frac{\mathcal{Z}}{\mathcal{Z}_0} = e^{(-\mathcal{W}+\mathcal{W}_0)/\hbar}.$$

  Then by the standard combinatoric argument, $\mathcal{W}$ contains only connected diagrams; note that the empty diagram is not connected. Note that because of the sign, $\mathcal{W}$ is really the *opposite* of the sum of connected diagrams.

**Note.** The explicit combinatoric argument. Let $F(\Gamma)$ be the contribution of a Feynman graph without the symmetry factor. We may expand any graph as the product of disconnected pieces,

$$|\operatorname{Aut}(\Gamma_1^{n_1} \dots \Gamma_k^{n_k})| = \prod_{j=1}^{k} (n_j!) |\operatorname{Aut}(\Gamma_j)|^{n_j}, \quad F\left(\prod_j \Gamma_j^{n_j}\right) = \prod_j F(\Gamma_j)^{n_j}.$$

Therefore we have

$$\frac{\mathcal{Z}}{\mathcal{Z}_0} = \sum_{\Gamma} \frac{F(\Gamma)}{|\operatorname{Aut}(\Gamma)|} = \sum_{\{n_j\}} \frac{F\left(\prod_j \Gamma_j^{n_j}\right)}{\left|\operatorname{Aut}\left(\prod_j \Gamma_j^{n_j}\right)\right|} = \sum_{\{n_j\}} \prod_j \frac{1}{n_j!} \left(\frac{F(\Gamma_j)}{|\operatorname{Aut}(\Gamma_j)|}\right)^{n_j} = \exp\left(\sum_j \frac{F(\Gamma_j)}{|\operatorname{Aut}(\Gamma_j)|}\right).$$

Therefore, $\mathcal{W}$ is the opposite of the sum of connected vacuum graphs,

$$-\mathcal{W} + \mathcal{W}_0 = \hbar \sum_{\text{connected } \Gamma} \frac{F(\Gamma)}{|\operatorname{Aut}(\Gamma)|}.$$

In a general quantum field theory, since $\mathcal{Z}$ sums over vacuum graphs, it measures the 'vacuum to vacuum' transition amplitude $e^{-iEt/\hbar}$ where $E$ is the vacuum energy. Then $\mathcal{W}$ directly measures the vacuum energy.

**Note.** Counting powers of $\hbar$. Propagators are proportional to $\hbar$ and vertices are inversely proportional to $\hbar$, so a graph contributes $\hbar^{\ell(\Gamma)}$ where $\ell(\Gamma) = e(\Gamma) - v(\Gamma) + 1$ in the expansion for $\mathcal{W}$. For a

connected graph, this is the number of loops by Euler's theorem, as loops are just faces of a planar graph, and the 'outside face' doesn't count.

To count powers of $\hbar$ in detail, let there be $v_a$ vertices with $a$ fields. Then for vacuum graphs,

$$2e = \sum_a a v_a$$

where we suppress the $\Gamma$ argument, so

$$\ell = 1 + e - \sum_a v_a = 1 + \sum_a \left(\frac{a}{2} - 1\right) v_a.$$

The $\hbar^0$ term is from classical field theory, and corresponds to tree-level diagrams; there are none here because we're focusing on vacuum diagrams. The $\hbar^1$ term comes from one-loop diagrams, and we see above that such diagrams only have quadratic vertices. This makes sense, because the semiclassical contribution comes from integrating over fluctuations in the Gaussian (quadratic) ensemble, and hence only depends on the quadratic part of the action. Finally, all vacuum graphs with an interaction vertex are two-loop or higher. Note that while loops correspond to powers of $\hbar$ in general, the result may be shifted by a constant by field insertions, external states, and external sources.

## 7.4 Effective Actions

Zero-dimensional field theory also provides simple examples of effective theories.

- Consider two real scalar fields with

  $$S(\phi, \chi) = \frac{m^2}{2}\phi^2 + \frac{M^2}{2}\chi^2 + \frac{\lambda}{4}\phi^2\chi^2.$$

  Then we have propagators $\hbar/m^2$ and $\hbar/M^2$, and a vertex $-\lambda/\hbar$.

- We can then expand the partition function as we did before,

  $$\ln\left[\frac{\mathcal{Z}}{\mathcal{Z}_0}\right] = \quad + \quad + \quad + \quad$$
  $$= -\frac{\hbar\lambda}{4m^2M^2} + \frac{\hbar^2\lambda^2}{16m^4M^4} + \frac{\hbar^2\lambda^2}{16m^4M^4} + \frac{\hbar^2\lambda^2}{8m^4M^4}$$

  We can also compute expectation values of powers of $\phi$, represented by external points,

  $$\langle\phi^2\rangle = \quad + \quad + \quad + \quad + \quad$$
  $$= \frac{\hbar}{m^2} - \frac{\lambda\hbar^2}{2m^4M^2} + \frac{\lambda^2\hbar^3}{4m^6M^4} + \frac{\lambda^2\hbar^3}{2m^6M^4} + \frac{\lambda^2\hbar^3}{4m^6M^4}$$

- Now suppose that we only want to evaluate correlation functions of $\phi$. For example, the mass $M$ of the $\chi$ may be very high, so we cannot produce it in an experiment. Then it is useful to "integrate out" the $\chi$, defining the effective action

  $$S_{\text{eff}}(\phi) \equiv -\hbar \log\left(\int_{\mathbb{R}} d\chi\, e^{-S(\phi,\chi)/\hbar}\right).$$

  Then correlation functions of $\phi$ can be evaluated with the weight $e^{-S_{\text{eff}}(\phi)/\hbar}$.

- In this case, the action is quadratic in $\chi$, so we can explicitly compute

$$\int_{\mathbb{R}} d\chi\, e^{-S(\phi,\chi)/\hbar} = e^{-m^2\phi^2/2\hbar}\sqrt{\frac{2\pi\hbar}{M^2+\lambda\phi^2/2}}.$$

  However, in general we must use perturbation theory. Here we have, exactly,

$$S_{\text{eff}}(\phi) = \frac{m^2}{2}\phi^2 + \frac{\hbar}{2}\log\left(1+\frac{\lambda}{2M^2}\phi^2\right) + \frac{\hbar}{2}\log\frac{M^2}{2\pi\hbar}.$$

  We ignore the final term, which is a constant, but its appearance is a manifestation of the cosmological constant problem.

- By expanding the logarithm, the effective action takes the form

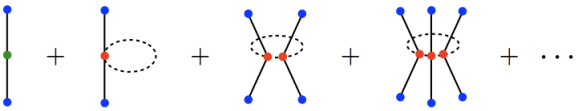
$$S_{\text{eff}}(\phi) \equiv \frac{m_{\text{eff}}^2}{2}\phi^2 + \frac{\lambda_4}{4!}\phi^4 + \frac{\lambda_6}{6!}\phi^6 + \ldots, \quad m_{\text{eff}}^2 = m^2 + \frac{\hbar\lambda}{2M^2}$$

  so we have a mass renormalization, and an infinite set of couplings

$$\lambda_4 = -\frac{3\hbar}{2}\frac{\lambda^2}{M^4}, \quad \lambda_6 = 15\hbar\frac{\lambda^3}{M^6}, \quad \lambda_{2k} = (-1)^{k+1}\hbar\frac{(2k)!}{2^{k+1}k}\frac{\lambda^k}{M^{2k}}.$$

  We note that all these effectives are quantum effects, since they are proportional to $\hbar$, and they are suppressed by powers of $M$.

- We can compute the effective action diagrammatically. It is a polynomial in $\phi$, where the contribution to the $\phi^n$ term comes from diagrams with $n$ external $\phi$ fields, only connected diagrams are used since we take the logarithm, and only $\chi$ propagators appear since the $\phi$ is not dynamical:

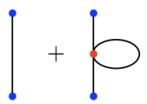
$-S_{\text{eff}}$ =  + $\cdots$

$$= -\frac{m^2}{2}\phi^2 \; -\frac{\lambda}{4M^2}\phi^2 \; +\frac{\lambda^2}{16M^4}\phi^4 \; -\frac{\lambda^3}{48M^6}\phi^6 + \cdots$$

  Terms that depend only on $\phi$ can be simply pulled out of the integral; here we have represented the original mass term using a diagram with a special green vertex. The solid lines aren't $\phi$ propagators; they're for visual convenience, to separate the $\phi$ fields from their vertices.

- In this theory, all contributions to $S_{\text{eff}}(\phi)$ have one $\chi$ loop, so the corrections to the couplings are $O(\hbar)$. In general, the corrections will each themselves be asymptotic series in $\hbar$. We can imagine the process of integrating out $\chi$ as 'zooming out' until we can't see the loop.

- Using the effective action, it is much easier to compute $\langle\phi^2\rangle$ to $O(\lambda^2)$,

$\langle\phi^2\rangle$ =  + $\cdots$

$$= \frac{\hbar}{m_{\text{eff}}^2} - \frac{\lambda_4\hbar^2}{2m_{\text{eff}}^6} + \cdots$$

In general, we think of the world as a series of nested effective field theories. As a result, *every* field theory has an infinite amount of terms, i.e. all terms allowed by symmetry, and every computation is actually using an effective action. However, if the new physics is at a much higher energy scale, the effective action will look like a low-order polynomial because only renormalizable terms are significant.

**Note.** We've talked about two kinds of effective action, $\mathcal{W}$ and $S_{\text{eff}}$. These are essentially the same; consider adding an external current, so the action becomes $S[\phi, J]$. Then $\mathcal{W}[J]$ is the effective action for $J$ after integrating out $\phi$. Since we've done everything with $J = 0$, only vacuum diagrams appear in the expansion for $\mathcal{W}$, but generally there would be external $J$ points just like in $S_{\text{eff}}$. Both $\mathcal{W}$ and $S_{\text{eff}}$ are examples of Wilsonian effective actions.

There's yet another kind of effective action, the 1PI effective action.

- As we've seen above, the effective action $\mathcal{W}[J]$ is the effective action for $J$ once the field $\phi$ has been integrated out; it is analogous to the Helmholtz free energy $F(H)$ for a magnet. However, for an isolated system there is no clear external source.
- In thermodynamics, we deal with this by switching to the Gibbs free energy $G(M)$. By minimizing $G(M)$, we may find the equilibrium magnetization $M = \langle s(\mathbf{x}) \rangle$. We can perform a simple minimization because $G(M)$ already accounts for thermal effects, unlike $H(s(\mathbf{x}))$.
- Similarly, in quantum field theory we Legendre transform from the effective action $\mathcal{W}(J)$ to the 1PI effective action $\Gamma(\Phi)$, where the 'classical field' $\Phi$ is the vev in the presence of a source,

  $$\Phi = \langle \phi \rangle_J.$$

  We will see that minimizing $\Gamma(\Phi)$ can tell us about phase transitions, just like $G(M)$.
- Explicitly, note that

  $$\frac{\partial \mathcal{W}}{\partial J} = -\frac{\hbar}{\mathcal{Z}(J)} \frac{\partial}{\partial J} \int d\phi \, e^{-(S+J\phi)/\hbar} = \frac{1}{\mathcal{Z}(J)} \int d\phi \, e^{-(S+J\phi)/\hbar} \phi = \langle \phi \rangle_J$$

  so that $\Phi = \partial \mathcal{W}/\partial J$, analogous to the relation $M = -\partial F/\partial H$.
- We define the quantum effective action $\Gamma(\Phi)$ by the Legendre transformation

  $$\boxed{\Gamma(\Phi) = \mathcal{W}(J) - \Phi J.}$$

  Then by the usual logic for Legendre transformations,

  $$J = -\frac{\partial \Gamma}{\partial \Phi}$$

  and assuming the Legendre transformation is invertible, we can transform back and forth between $\mathcal{W}(J)$ and $\Gamma(\Phi)$.
- To understand the meaning of $\Gamma(\Phi)$, note that when $J = 0$, possible values for $\Phi$ correspond to extrema of $\Gamma$. Then in higher dimensions, the action $S$ yields the classical equations of motion $\delta S/\delta \phi = 0$ while $\Gamma$ yields the 'quantum' equations of motion for $\delta \Gamma/\delta \Phi = 0$ which account for quantum effects. In zero dimensions these are simply algebraic equations, but in higher dimensions we can have nontrivial spatial dependence, yielding solitons.

- Next, we define a quantum theory by a path integral of $\Phi$, where we let $\Gamma(\Phi)$ play the role of the classical action. We define

$$e^{-W_\Gamma(J)/g} = \int d\Phi\, e^{-(\Gamma(\Phi)+J\Phi)/g}$$

where $g$ takes the formal role of $\hbar$, which is in $\Gamma(\Phi)$. As before, this means

$$W_\Gamma(J) = \sum_{\ell=0}^{\infty} g^\ell \mathcal{W}_\Gamma^{(\ell)}(J)$$

where $W_\Gamma^{(\ell)}(J)$ is the sum of all $\ell$-loop connected Feynman graphs built from $\Gamma(\Phi)$.

- Next, suppose we take the limit $g \to 0$. Then only the tree-level connected Feynman graphs contribute; alternatively, the path integral is dominated by the minimum of the argument of the exponential, so

$$\lim_{g\to 0} \mathcal{W}_\Gamma(J) = W_\Gamma^{(0)}(J) = \Gamma(\Phi) + J\Phi\Big|_{J=-\frac{\partial\Gamma}{\partial\Phi}} = \mathcal{W}(J).$$

Therefore, the connected graphs built from the classical action $S(\phi) + J\phi$ are equal to the tree graphs built from the effective action $\Gamma(\Phi) + J\Phi$, another sense in which $\Gamma$ includes quantum corrections to $S$.

- Diagrammatically, every connected diagram is a tree whose vertices are one-particle irreducible (1PI) graphs. Therefore, we can interpret $\Gamma(\Phi)$ as the sum of 1PI graphs built from $S(\phi)$, where a 1PI diagram with $n$ external $\phi$ fields contributes to the $\Phi^n$ term/vertex in $\Gamma(\Phi)$.

- Explicitly, suppose we have fields $\phi^a$ and sources $J_a$. Then

$$-\hbar\frac{\partial^2\mathcal{W}}{\partial J_a \partial J_b} = -\hbar\frac{\partial}{\partial J_a}\left(\frac{1}{\mathcal{Z}(J)}\int d^n\phi\, e^{-(S(\phi^a)+J_a\phi^a)/\hbar}\phi^b\right) = \langle\phi^a\phi^b\rangle_J - \langle\phi^a\rangle_J\langle\phi^b\rangle_J \equiv \langle\phi^a\phi^b\rangle_J^{\text{conn}}.$$

The first term contains both connected and disconnected diagrams, but the disconnected ones are canceled out by the second term, leaving the connected correlation function, also called the exact propagator. It is the quadratic term in $\mathcal{W}[J]$, as expected.

- On the other hand, note that

$$\langle\phi^a\phi^b\rangle_J^{\text{conn}} = -\hbar\frac{\partial^2\mathcal{W}}{\partial J_a\partial J_b} = -\hbar\frac{\partial\Phi^b}{\partial J_a} = -\hbar\left(\frac{\partial J_a}{\partial\Phi_b}\right)^{-1} = \hbar\left(\frac{\partial^2\Gamma}{\partial\Phi^b\partial\Phi^a}\right)^{-1}.$$

That is, the exact propagator is the inverse of the quadratic term of the 1PI effective action, which is exactly what we expect.

- To connect this to diagrams, note that we have the diagrammatic recurrence relation

$$\text{connected} \sim \text{connected} \times \text{1PI} \times \text{connected}$$

which is indeed compatible with what we found.

- For more than two external fields, we can continue differentiating $\mathcal{W}$ with respect to $J$. The details are messy, but the key identity is
$$\frac{\partial}{\partial \Phi} = \frac{\partial J}{\partial \Phi}\frac{\partial}{\partial J} = (\langle \phi^2 \rangle^{\text{conn}})^{-1}\frac{\partial}{\partial J}.$$
That is, adding another external $\Phi$ leg to $\Gamma$ is the same as adding another external $J$ to $\mathcal{W}$ and amputating the resulting leg; this is exactly how 1PI diagrams are produced.
- In general dimension, $\Gamma$ is extensive and dividing it by the spacetime volume gives the effective potential $V_{\text{eff}}(\Phi)$. As in thermodynamics, the effective potential must be convex; taking the convex hull is the same as the Maxwell construction for the var der Waals gas.

**Note.** There are various sign flips above, so $\mathcal{W}[J]$ and $\Gamma[\Phi]$ are not really the sum of connected and 1PI diagrams, but rather the sum up to a sign. Explicitly, we have
$$W[J] = \sum_n \frac{(-1)^{n-1}}{n!} J^n \langle \phi^n \rangle^{\text{conn}}, \quad \Gamma[\Phi] = -\sum_n \frac{1}{n!}\Phi^n \langle \phi^n \rangle^{\text{1PI}}$$
as can be shown using the identities above. The only exception is the case $n = 2$ for the 1PI effective action. Note that for $n > 2$, the $n^{\text{th}}$ order terms in the action contribute directly to $\langle \phi^n \rangle^{\text{1PI}}$, but the quadratic terms don't contribute directly to $\Pi$. This is because we treated the quadratic part of the action as 'free' and hence differently from the 'interacting' part when setting per perturbation theory. By summing a geometric series, we hace
$$\langle \phi^2 \rangle^{\text{conn}}(p^2) = \frac{1}{p^2 - m_0^2 - \Pi(p^2)}, \quad \Pi(p^2) = \sum \text{1PI diagrams}$$
which implies that the quadratic term of the 1PI effective action is
$$\langle \phi^2 \rangle^{\text{conn}}(p^2)^{-1} = p^2 - m^2 - \Pi(p^2)$$
with an extra inverse bare propagator.

**Example.** In the case of no external sources, $J = 0$, we have
$$W[0] = \Gamma[\Phi]\Big|_{\partial\Gamma/\partial\Phi = 0}.$$
Thus, by plugging in the equilibrium value of $\Phi$, we get a relationship between the set of connected vacuum bubbles and the set of 1PI diagrams. For example, in the simple case $\Phi = 0$ we find
$$\sum \text{connected vacuum bubbles} = \sum \text{1PI vacuum bubbles}$$
which holds because all non-1PI connected vacuum bubbles have tadpoles, which vanish when $\Phi = 0$.

**Note.** Here we make the analogy with statistical mechanics more explicit.

- We consider a magnet with spin field $s(x)$ and energy $E(s)$. Then the partition function is
$$\mathcal{Z} = \int ds\, e^{-\beta(E(s) - Hs)}$$
where $H$ is an external field. We define the Helmholtz free energy by
$$F(H) = -\frac{1}{\beta}\log \mathcal{Z}$$
where we integrated out the spin field, leaving a function of the external field. We leave the dependence on $T$ implicit, because in quantum field theory its analogue $\hbar$ can't be varied.

- We define the magnetization $M = \langle s \rangle = -\partial F/\partial H$. This is the spin seen on a macroscopic scale, accounting for thermal corrections. We define the Gibbs free energy

$$G(M) = F(H) + MH, \quad \frac{\partial G}{\partial M} = H$$

by Legendre transform so it is directly a function of $M$.

- To compute the equilibrium value of $M$, we simply find the minima of $G$. That is, we don't have to do a path integral over configurations of $M$, so $G$ 'accounts for thermal corrections'. In particular, it's easy to tell where phase transitions occur.

- For example, in the ferromagnetic phase transition, the energy is minimized by having the spins aligned, but thermal effects favor disorder; we can tell which wins by looking at $G(T, M)$. In quantum field theory, an analogue is the Coleman-Weinberg potential: a $\phi^4$ theory can exhibit spontaneous symmetry breaking because the effective potential has a nonzero minimum.

## 7.5 Fermionic Fields

We now turn to the description of fermionic fields.

- A fermionic field is an anticommuting field. In zero dimensions, there is no notion of spin, but in $d = 3 + 1$ fermionic fields have half-integer spin by the spin-statistics theorem.

- The Grassmann algebra is an algebra over the complex numbers, generated by $n$ elements $\theta^a$ that anticommute, $\theta^a\theta^b = -\theta^b\theta^a$. Note that scalar multiplication still commutes, $\theta^a\alpha = \alpha\theta^a$ for $\alpha \in \mathbb{C}$. Also, products of an even number of Grassmann numbers commute with everything, so they are 'bosonic'.

- Every Grassmann number squares to zero, so a general element of the algebra is

$$F(\theta) = \alpha + \rho_a\theta^a + \frac{1}{2!}g_{a_1a_2}\theta^{a_1}\theta^{a_2} + \ldots + \frac{1}{n!}h_{a_1\ldots a_n}\theta^{a_1}\ldots\theta^{a_n}$$

where we can take the coefficients to be totally antisymmetric.

- Next, we define differentiation and integration. The derivative is defined by

$$\frac{\partial\theta^a}{\partial\theta^b} = \delta^a_b, \quad \frac{\partial\alpha}{\partial\theta^a} = 0, \quad \alpha \in \mathbb{C}$$

so it anticommutes with multiplication by a Grassmann number,

$$\frac{\partial}{\partial\theta^a}\theta^b + \theta^b\frac{\partial}{\partial\theta^a} = \delta^b_a.$$

where the derivatives act on everything to the right, i.e. on a test function. Then the Grassmann derivative obeys a product rule with an extra minus sign. (Note the spatial derivative of a product of Grassmann fields obeys the product rule, with no extra signs.)

- To define the integral, we consider the case of one Grassmann number, so the most general function is $f(\theta) = \alpha + \rho\theta$. We demand the integral be linear, so

$$\int d\theta\,(\alpha + \rho\theta) = \alpha\int d\theta + \rho\int d\theta\,\theta.$$

We also demand invariance under a shift of the integration variable by $\eta$. For complex $\eta$,

$$\int d\theta\, f(\theta) = \int d\theta\, f(\theta+\eta) = (\alpha+\rho\eta)\int d\theta + \rho\int d\theta\,\theta.$$

- Using our choice of normalization for the measure, we conclude

  $$\int d\theta\, 1 = 0, \quad \int d\theta\,\theta = 1.$$

  More generally, $\eta$ can be Grassmann valued, which shows that the $d\theta$ integral of anything not linear in $\theta$ vanishes; in particular, Grassmann differentiation and integration are equal!

- Using the definition of integration, we find

  $$\int d\theta\,\frac{\partial}{\partial\theta}F(\theta) = 0$$

  since the derivative removes the term linear in $\theta$. This allows us to integrate with parts, though there's an extra sign flip.

- If we have $n$ Grassmann variables $\theta^a$, we define $d^n\theta = d\theta^n \dots d\theta^1$, so

  $$\int d^n\theta\,\theta^1\dots\theta^n = 1$$

  with all lower-order terms integrating to zero, and in general,

  $$\int d^n\theta\,\theta^{a_1}\dots\theta^{a_n} = \epsilon^{a_1\dots a_n}.$$

- Next, consider a change of variables $\theta'^a = N^a_b\theta^b$. Then by linearity,

  $$\int d^n\theta\,\theta'^{a_1}\dots\theta'^{a_n} = N^{a_1}_{b_1}\dots N^{a_n}_{b_n}\int d^n\theta\,\theta^{b_1}\dots\theta^{b_n} = N^{a_1}_{b_1}\dots N^{a_n}_{b_n}\epsilon^{b_1\dots b_n} = (\det N)\epsilon^{a_1\dots a_n}$$

  which shows that

  $$d^n\theta = (\det N)d^n\theta'.$$

  This is the exact opposite of the transformation for bosons, e.g. $d\theta = 2d(2\theta)$.

- It is also occasionally useful to consider Grassmann Dirac delta functions, which obey

  $$\int d\eta\,\delta(\eta-\theta)f(\eta) = f(\theta).$$

  By considering the general form $f(\eta)$ could take, we see that $\delta(\eta-\theta) = \eta-\theta$.

**Note.** Concretely, the $\theta^a$ can be thought of as one-forms, and the product can be thought of as the wedge product. The Grassmann integral $\int d^n\theta$ can be thought of as integration of a differential form over $\mathbb{R}^n$, where only the integral of a top-dimensional form is nonzero.

Next, we consider a simple calculation in a fermionic free field theory.

- To get a nontrivial theory, we need two fermionic fields, $\theta^1$ and $\theta^2$. The action must be bosonic, so it must be even in the fields, so we must have

$$S(\theta) = \frac{1}{2} A \theta^1 \theta^2.$$

Then the partition function is simply

$$\mathcal{Z}_0 = \int d^2\theta \, e^{-S(\theta)/\hbar} = \int d^2\theta \left(1 - \frac{A}{2\hbar}\theta^1\theta^2\right) = -\frac{A}{2\hbar}$$

where we took a Taylor expansion. Note that $S$ is Grassmann-valued, but $\mathcal{Z}_0$ is not.

- More generally, consider $2m$ fermionic fields $\theta^a$ with

$$S(\theta) = \frac{1}{2} A_{ab} \theta^a \theta^b$$

where $A$ is antisymmetric. Then the partition function is

$$\mathcal{Z}_0 = \int d^{2m}\theta \, e^{-A(\theta,\theta)/2\hbar} = \left(-\frac{1}{2\hbar}\right)^m \frac{1}{m!} \int d^{2m}\theta \, A_{a_1 a_2} \dots A_{a_{2m-1} a_{2m}} \theta^{a_1} \dots \theta^{a_{2m}}$$

where we used the fact that only the $m^{\text{th}}$ order term can contribute.

- We define the Pfaffian of a $2m \times 2m$ antisymmetric matrix $A$ by

$$\text{Pfaff } A = \frac{1}{2^m m!} \epsilon^{a_1 \dots a_{2m}} A_{a_1 a_2} \dots A_{a_{2m-1} a_{2m}}$$

and one can show that

$$(\text{Pfaff } A)^2 = \det A.$$

Thus we conclude that

$$\mathcal{Z}_0 = \pm \sqrt{\frac{\det A}{\hbar^n}}$$

while for free bosons we had the $\hbar$'s in the numerator and the determinant in the denominator.

- We can also consider sources,

$$S(\theta, \eta) = \frac{1}{2} A_{ab} \theta^a \theta^b + \eta_a \theta^b$$

where the source $\eta$ must be fermionic. As before, we may complete the square for

$$S(\theta, \eta) = \frac{1}{2}(\theta^a + \eta_c (A^{-1})^{ca}) A_{ab} (\theta^b + \eta_d (A^{-1})^{db}) + \frac{1}{2} \eta_a (A^{-1})^{ab} \eta_b$$

where the two cross-terms are equal because $A$ is antisymmetric and $\theta$ and $\eta$ anticommute.

- Therefore, the partition function is

$$\mathcal{Z}_0(\eta) = \exp\left(-\frac{1}{2\hbar} A^{-1}(\eta, \eta)\right) \mathcal{Z}_0(0)$$

which allows us to compute the two-point function as

$$\langle \theta^a \theta^b \rangle = \frac{\hbar^2}{\mathcal{Z}_0(0)} \frac{\partial^2 \mathcal{Z}_0(\eta)}{\partial \eta_a \partial \eta_b}\bigg|_{\eta=0} = \hbar (A^{-1})^{ab}$$

where we use Taylor expansion and the fact that derivatives anticommute with Grassmann numbers. This is just the same as the bosonic result: the free propagator is always the inverse of the kinetic term.

- The fact that functions of a finite number of Grassmann variables can always be represented as polynomials means that in $d = 0$, all fermionic path integrals can be evaluated exactly. In higher dimensions, we use Grassmann-valued fields which contain infinitely many independent Grassmann variables, and turn to perturbation theory with Feynman diagrams.

**Note.** Strictly speaking, *all* fermionic fields in relativistic quantum field theory must be Grassmann numbers, even when we are performing canonical quantization. For example, as we will see in the notes on the Standard Model, a Majorana mass term cannot be written down in a classical Lagrangian unless the spinors are anticommuting variables; otherwise the term vanishes identically.

One might worry our classical analysis of the Dirac Lagrangian missed some sign flips. The results are unchanged as long as we define $\pi = \partial\mathcal{L}/\partial\dot{\psi}$ as a derivative to the right of $\mathcal{L}$ acting to the left, and $\mathcal{H} = \pi\dot{\psi} - \mathcal{L}$. Practically speaking, just as there are ambiguities when going from a classical to a quantum theory (which are, e.g. fixed by normal ordering and other prescriptions), there are ambiguities when going from a classical to a semiclassical theory, where the spinors become Grassmann valued. As with normal ordering, we pick the conventions so that the results work.

We now give an example where supersymmetry makes a path integral much easier to evaluate.

- Consider a theory with one bosonic field $\phi$ and two fermionic fields $\psi_1$ and $\psi_2$, normalizing the measure so that
$$\mathcal{Z} = \int \frac{d\phi d\psi_1 d\psi_2}{\sqrt{2\pi}} e^{-S(\phi,\psi_i)}.$$
We suppose the action takes the special form
$$S(\phi, \psi_1, \psi_2) = \frac{1}{2}(\partial h)^2 - \psi_1\psi_2\partial^2 h$$
where $h(\phi)$ is a real-valued polynomial and $\partial h$ is its derivative with respect to $\phi$. This action has the most general possible fermionic term, so this just restricts the purely bosonic piece.

- Now consider the zero-dimensional supersymmetry transformations
$$\delta\phi = \epsilon_1\psi_1 + \epsilon_2\psi_2, \quad \delta\psi_1 = \epsilon_2\partial h, \quad \delta\psi_2 = -\epsilon_1\partial h$$
where the $\epsilon_i$ are fermionic. Then the action transforms as
$$\delta S = \partial h\, \partial^2 h\, (\epsilon_1\psi_1 + \epsilon_2\psi_2) - (\epsilon_2\partial h)\psi_2\partial^2 h - \psi_1(-\epsilon_1\partial h)\partial^2 h = 0$$
and is thus invariant; one can show the measure is invariant as well. We also have $\delta_1^2 = \delta_2^2 = [\delta_1, \delta_2] = 0$.

- Now consider the variation $\delta\mathcal{O}$ of some operator $\mathcal{O}(\phi, \psi_i)$. Then
$$\langle\delta\mathcal{O}\rangle = \frac{1}{\mathcal{Z}_0}\int d\phi d^2\psi\, e^{-S}\delta\mathcal{O} = \frac{1}{\mathcal{Z}_0}\int d\phi d^2\psi\, \delta(e^{-S}\mathcal{O}).$$
Now, the quantity in parentheses is at most linear in the $\psi_i$, so the terms that come from varying $\psi_i$ don't depend on $\psi_i$ and hence integrate to zero. The terms that come from varying $\phi$ are total derivatives in $\phi$. Then $\langle\delta\mathcal{O}\rangle = 0$.

- Now suppose we choose $\mathcal{O} = (\partial g)\psi_1$ for some $g(\phi)$. Setting $\epsilon_1 = -\epsilon_2 = \epsilon$, we have

$$0 = \langle \delta \mathcal{O} \rangle = \epsilon \langle \partial g \partial g - \partial^2 g \psi_1 \psi_2 \rangle.$$

  But this is also the first-order change of the action under the deformation $h \to h + g$. That is, we can replace $h$ in the action with any $h + g$ as long as we don't change the behavior at infinity. In particular, if $g = h$, then we can scale $h$ up. Then the path integral only depends on regions right around extrema of $\phi$, a phenomenon known as localization.

- We can explicitly find the partition function. Near an extremum $\phi_*$, we have

$$h(\phi) = h(\phi_*) + \frac{c_*}{2}(\phi - \phi_*)^2, \quad S(\phi, \psi_i) \approx \frac{c_*^2}{2}(\phi - \phi_*)^2 + c_* \psi_1 \psi_2.$$

  The contribution to the partition function is

$$\frac{1}{\sqrt{2\pi}} \int d\phi d^2\psi \, e^{-c_*^2(\phi-\phi_*)^2/2}(1 - c_* \psi_1 \psi_2) = \frac{c_*}{\sqrt{2\pi}} \int d\phi \, e^{-c_*^2(\phi-\phi_*)^2/2} = \frac{c_*}{\sqrt{c_*^2}} = \text{sign}(c_*)$$

- Then the full partition function is

$$\mathcal{Z}[h] = \sum_{\text{extrema } \phi_*} \text{sign}(\partial^2 h|_{\phi_*}).$$

  This is a very simple result: if $h$ has odd degree, then the partition function is zero, and if $h$ has even degree, the partition function is $\pm 1$.

- Localization is also useful for computing correlation functions of operators $\mathcal{O}_i$. We are interested in supersymmetry-invariant operators, $\delta \mathcal{O} = 0$. On the other hand, if $\mathcal{O} = \delta \mathcal{O}'$ then the correlator is automatically zero. Thus the nontrivial observables in the theory correspond to the cohomology of $\delta$. That is, one can use supersymmetric quantum field theories to compute the cohomologies of interesting spaces.

# 8 Path Integrals in Higher Dimensions

## 8.1 Quantum Mechanics

Finally, we turn to quantum mechanics, a one-dimensional quantum field theory.

- We recall that we usually consider fields of the form $x : M \to N$ where $(N, g)$ is a Riemannian manifold. In the one-dimensional case, $M$ is either the circle $S^1$, parametrized by $t \in [0, T)$ with endpoints identified, or the interval $I$, parametrized by $t \in [0, T]$.

- For every time $t$, $x(t)$ is a point in $N$. We let $x^a(t)$ denote its coordinates; specifically this is the pullback to $M$ by $x$ of coordinates on a patch of $N$. A standard choice of action is

$$S[\phi] = \int_M \frac{1}{2} g_{ab}(x) \dot{x}^a \dot{x}^b + V(x)\, dt$$

  where $g_{ab}(x(t))$ is the pullback to $M$ of the Riemannian metric on $N$, and $V : N \to \mathbb{R}$ is a 'potential' and $V(x(t))$ is its pullback. We've implicitly chosen the metric on $M$ to be the Euclidean metric $\delta_{tt} = 1$.

- Directly varying this action gives the Euler–Lagrange equation

$$\frac{d^2 x^a}{dt^2} + \Gamma^a_{bc} \dot{x}^b \dot{x}^c = g^{ab}(x) \frac{dV}{dx^b}, \quad \Gamma^a_{bc} = \frac{1}{2} g^{ad} (\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc})$$

  which is simply the geodesic equation. However, there is no minus sign on the right-hand side because of the Wick rotation $t \to it$. In Euclidean signature, $\mathbf{F} = -m\mathbf{a}$ because $\mathbf{a}$ picks up a factor of $i^2$ in the Wick rotation.

- We interpret the target space $N$ as being the space where we live, and $x(t)$ as the trajectory of a quantum particle; we call either $M$ or $x(M) \subset N$ the worldline of the particle. However, note that we usually interpret $M$ as being the space where we live, and $N$ as an abstract space of fields, e.g. a Grassmann algebra for fermion fields. Even within quantum mechanics, with two particles $N$ is no longer space, but a configuration space with twice the dimension.

Next, we recover the path integral picture from standard quantum mechanics.

- We work in the Hilbert space $\mathcal{H} = L^2(N)$ with Hamiltonian

$$H = \frac{1}{2}\Delta + V, \quad \Delta = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^a} \left( \sqrt{g} g^{ab} \frac{\partial}{\partial x^b} \right)$$

  where $\Delta$ is the Laplacian on $(N, g)$.

- We define the heat kernel

$$K_T(y_0, y_1) = \langle y_1 | e^{-HT} | y_0 \rangle = \langle y_1, T | y_0, 0 \rangle$$

  representing the amplitude for a particle to travel from $y_0$ to $y_1$ in time $T$, in Schrodinger and Heisenberg picture respectively. In the Heisenberg picture expression, $|y_i, t\rangle$ is defined as an eigenvector of $\hat{y}(t)$ with eigenvalue $y_i$.

- The heat kernel obeys the differential equation

$$\frac{\partial}{\partial t}K_t(x,y) + HK_t(x,y) = 0$$

which is the Schrodinger equation in Euclidean signature, with $\hbar = 1$.

- In the case where $N \cong \mathbb{R}^n$, the metric is $g_{ab} = \delta_{ab}$, and $V = 0$, we have the usual heat kernel

$$K_t(x,y) = \frac{1}{(2\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{2t}\right).$$

It can be shown that for a general metric, the heat kernel has this form for small $t$, specifically

$$\lim_{\Delta t \to 0} K_{\Delta t}(x,y) \sim \frac{1}{(2\pi\Delta t)^{n/2}} a(x) \exp\left(-\frac{d(x,y)^2}{2\Delta t}\right)$$

where $d(x,y)$ is the geodesic distance from $x$ to $y$ and $a(x)$ is a polynomial in the Riemann curvature tensor.

- The heat kernel can be rewritten by inserting copies of the identity, giving

$$\langle y_1|e^{-HT}|y_0\rangle = \lim_{N\to\infty}\left(\frac{1}{2\pi\Delta t}\right)^{nN/2} \int \prod_{i=1}^{N-1} d^n x_i \, a(x_i) \exp\left[-\frac{\Delta t}{2}\left(\frac{d(x_{i+1},x_i)}{\Delta t}\right)^2\right].$$

This recovers the path integral, if we may define

$$\mathcal{D}x = \lim_{N\to\infty}\left(\frac{1}{2\pi\Delta t}\right)^{nN/2} \int \prod_{i=1}^{N-1} d^n x_i \, a(x_i).$$

If the paths are differentiable, then the sum over $(d(x_{i+1},x_i)/\Delta t)^2$ converges to the integral over $g_{ab}\dot{x}^a\dot{x}^b$ and we recover the path integral expression

$$\boxed{K_T(y_0,y_1) = \int_{C_T[y_0,y_1]} \mathcal{D}x \, e^{-S}}$$

where $S$ is the action we wrote down earlier with $V = 0$, and $C_T[y_0,y_1]$ specifies boundary conditions $x(0) = y_0$ and $x(T) = y_1$.

Next, we turn to the computation of correlation functions.

- A local operator is one which depends on the field (in this case, $\mathbf{x}$) at only one point of the worldline. For example, any function $\mathcal{O} : N \to \mathbb{R}$ corresponds to a local operator $\hat{\mathcal{O}}$. Let $|y,t\rangle$ be the Heisenberg state that will be peaked at $y$ at time $t$. Then in Heisenberg picture,

$$\langle y_1, T|\hat{\mathcal{O}}(t)|y_0, 0\rangle = \langle y_1, 0|e^{-H(T-t)}\hat{\mathcal{O}}(0)e^{-Ht}|y_0,0\rangle = \langle y_1|e^{-H(T-t)}\hat{\mathcal{O}}e^{-Ht}|y_0\rangle$$

where the final expression is in Schrodinger picture.

- Inserting a complete set of states, we find this is equal to

$$\int d^n x\, \mathcal{O}(x) K_{T-t}(y_1, x) K_t(x, y_0) = \int_{C_T[y_1,y_0]} \mathcal{D}x\, e^{-S} \mathcal{O}(x(t)).$$

  Therefore, in general correlation functions are computed from the path integral by

$$\langle y_1, T | \hat{\mathcal{O}}_n(t_n) \dots \hat{\mathcal{O}}_1(t_1) | y_0, 0 \rangle = \int_{C_T[y_0,y_1]} \mathcal{D}x\, e^{-S} \prod_{i=1}^{n} \mathcal{O}_i(x(t_i)).$$

  Note that we don't need to divide by a partition function here, because the normalization is implicitly in the definition of the path integral measure.

- Note that when we insert complete sets of states, we find a path of the form $(y_0, 0) \to (x_1, t_1) \to \dots \to (x_n, t_n) \to (y_1, T)$ and integrate out the internal points to get an arbitrary path $(y_0, 0) \to (y_1, T)$. This is only possible if the points are in time order, as otherwise we'd end up with an integral over paths that go forward and backward in time. Then

$$\boxed{\langle y_1, T | \mathcal{T} \prod_{i=1}^{n} \hat{\mathcal{O}}_i(t_i) | y_0, 0 \rangle = \int_{C_T[y_0,y_1]} \mathcal{D}x\, e^{-S} \prod_{i=1}^{n} \mathcal{O}_i(x(t_i)).}$$

  This is fitting, as we know the $\mathcal{O}_i$ commute, but the $\hat{\mathcal{O}}_i$ do not, and time ordering makes everything inside it effectively commute. Time-ordered products appear ubiquitously when dealing with the path integral, and we will often drop the time-ordering symbol.

- Note that if there are no derivative terms, positions at arbitrarily close times 'decouple', and the path integral splits into individual independent integrals for each time. Then all correlation functions split as

$$\langle \hat{\mathcal{O}}_1(t_1) \hat{\mathcal{O}}_2(t_2) \rangle = \langle \hat{\mathcal{O}}_1(t_1) \rangle \langle \hat{\mathcal{O}}_2(t_2) \rangle$$

  so there are no nontrivial correlations. This corresponds to the statement that fields without kinetic terms are nondynamical and do not propagate.

- More generally, we can consider operators that depend on derivatives of $\mathbf{x}$. In particular, the canonical momentum for our action is

$$p_a = \frac{\partial L}{\partial \dot{x}^a} = g_{ab} \dot{x}^b$$

  so we could replace the function $\mathcal{O}(x^a, \dot{x}^a)$ with the operator $\hat{\mathcal{O}} = \mathcal{O}(\hat{x}^a, g^{ab}(\hat{x}) \hat{p}_b)$. However, this is also puzzling because the latter depends on an ordering prescription while the former does not. One can simply define $\hat{\mathcal{O}}$ to be normal ordered, but we would like to understand where this comes from in the path integral.

**Note.** Such ordering problems also occur in canonical quantization. Here, given the Poisson bracket structure $\{f, g\} = h$ we would like to define operators satisfying $[\hat{f}, \hat{g}] = i\hbar \hat{h}$ which act irreducibly on the Hilbert space. However, the Groenewald-van Hove theorem states that this is impossible in general; in fact, it's even impossible for polynomials in the positions and momenta, once we go beyond quadratics. The idea of quantizing a classical system in general remains ambiguous.

To understand ordering problems, we carefully examine the continuum limit.

- For a free particle in one dimension, let $0 < t_- < t < t_+ < T$. Then

$$\int_{C_T[y_0,y_1]} \mathcal{D}x\, e^{-S} x(t)\dot{x}(t_-) = \langle y_1|e^{-H(T-t_+)}\hat{x}e^{-H(t_+-t)}\hat{p}e^{-Ht}|y_0\rangle$$

and

$$\int_{C_T[y_0,y_1]} \mathcal{D}x\, e^{-S} x(t)\dot{x}(t_+) = \langle y_1|e^{-H(T-t)}\hat{x}e^{-H(t-t_-)}\hat{p}e^{-Ht}|y_0\rangle.$$

Taking the limits $t_+ \to t$ from above and $t_- \to t$ from below, the difference of the right-hand sides is nonzero, as

$$\langle y_1|e^{-H(T-t)}[\hat{x},\hat{p}]e^{-Ht}|y_0\rangle = \langle y_1|e^{-HT}|y_0\rangle.$$

- To take the same limits on the left-hand side, we explicitly restore the discretization $\Delta t$. Then the furthest we can take the limit is

$$x(t)\dot{x}(t_-) - x(t)\dot{x}(t_+) \to x_t\frac{x_t - x_{t-\Delta t}}{\Delta t} - x_t\frac{x_{t+\Delta t} - x_t}{\Delta t}.$$

This is a discretized second derivative times $x_t\Delta t$, which would vanish in the limit $\Delta t \to 0$ if our paths were smooth. However, we integrate over paths that are not even differentiable.

- More explicitly, the part of the path integral that depends on $x_t$ is

$$\int dx_t\, K_{\Delta t}(x_{t+\Delta t}, x_t)\left(x_t\frac{x_t - x_{t-\Delta t}}{\Delta t} - x_t\frac{x_{t+\Delta t} - x_t}{\Delta t}\right) K_{\Delta t}(x_t, x_{t-\Delta t})$$

but using the known form of the heat kernel, this is equal to

$$-\int dx_t\, x_t\frac{\partial}{\partial x_t}\left(K_{\Delta t}(x_{t+\Delta t}, x_t)K_{\Delta t}(x_t, x_{t-\Delta t})\right) = K_{2\Delta t}(x_{t+\Delta t}, x_{t-\Delta t})$$

by integration by parts. Performing the other integrals gives $\langle y_1|e^{-HT}|y_0\rangle$ as desired. Similarly, the operator ordering in the Hamiltonian is also determined by the discretization procedure.

- To understand the result more quantitatively, note that

$$K_t(x,y) = \frac{1}{2\pi t}e^{-(x-y)^2/2t}$$

has $\langle(x-y)^2\rangle = t$, typical for a diffusion process. Then a typical derivative is actually $O(\sqrt{\Delta t})$ instead of $O(1)$, allowing it to contribute in ways it naively should not. For example, changing a forward derivative to a backward derivative within a single timeslice yields a change of

$$x_{t+\Delta t}\frac{x_{t+\Delta t} - x_t}{\Delta t} - x_t\frac{x_{t+\Delta t} - x_t}{\Delta t} = \Delta t\left(\frac{x_{t+\Delta t} - x_t}{\Delta t}\right)^2 = O(1).$$

Hence the operator ordering is hidden in the discretization procedure.

Next, we consider the issue of the path integral measure.

- Naively, we may take the limit of infinitely many position-space integration measures to yield the path integral measure $\mathcal{D}x$. Alternatively, one might want to do this in Fourier space, taking the limit of arbitrarily high frequencies; in this picture it is clear we only need a countable infinity of integrations. However, both these limits do not exist.

- We say $d\mu$ is a Lebesgue measure on $\mathbb{R}^D$ if it assigns a strictly positive volume

$$\text{vol}(U) \equiv \int_U d\mu$$

to every non-empty open set $U$, is translationally invariant, and if for every $x \in \mathbb{R}^D$ there is at least one open neighborhood $U_x$ of $x$ with finite volume. A standard choice is $d\mu = d^D x$.

- There is no non-trivial Lebesgue measure on a vector space with countably infinite dimension. Let $C(L)$ denote an open hypercube of side length $L$. Dividing it gives

$$\text{vol}(C(L)) \geq 2^D \text{vol}(C(L/2)).$$

Then if $D \to \infty$, the only way for the measure to remain finite is for $\text{vol}(C(L/2))$ to go to zero, so the measure of any hypercube must be zero. Since the dimension is countably infinite any open set can be covered with countably many cubes, so the measure is identically zero.

- For the path integral in one dimension, there is a nontrivial measure,

$$d\mu = \lim_{N\to\infty} \left(\frac{1}{2\pi\Delta t}\right)^{nN/2} \int \prod_{i=1}^{N-1} d^n x_i \, \exp\left(-\frac{\Delta t}{2}\left(\frac{x_{t_{i+1}} - x_{t_i}}{\Delta t}\right)^2\right).$$

This is called the Wiener measure, and it necessarily involves the $\mathcal{D}x$ and $e^{-S[x]}$ factor together. It evades the above theorem because it is not translationally invariant, i.e. shifting one of the $x_i$ changes the measure.

- In higher dimensions, the naive generalization of the Wiener measure does not generally exist, but it is believed that quantum field theories that are asymptotically free, such as Yang–Mills, do have a continuum limit. However, a continuum limit almost certainly doesn't exist for general relativity or quantum electrodynamics, and probably doesn't exist for the Standard Model. This is acceptable in practice because we can just treat them as effective theories.

## 8.2 Effective Quantum Mechanics

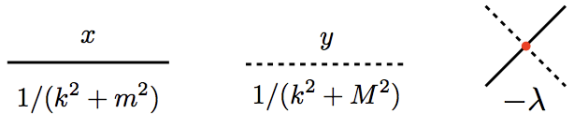
In this section, we give some examples of calculations in quantum mechanics.

- First, we consider a circular worldline with two fields $x$ and $y$, with action

$$S[x,y] = \int_{S^1} \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + V(x,y)\,dt, \quad V(x,y) = \frac{1}{2}(m^2x^2 + M^2y^2) + \frac{\lambda}{4}x^2y^2.$$

As a quantum mechanical system, it's two coupled harmonic oscillators in periodic time $T$. In quantum field theory language, it's a theory of two interacting fields with masses $m$ and $M$.

- Using the usual procedure, we arrive at the Feynman rules




where $k$ is the worldline momentum, which is quantized in units $2\pi/T$.

- Alternatively, we can integrate out the $y$ field, giving

$$\int \mathcal{D}y \exp\left(-\frac{1}{2}\int_0^T dt\, y\left(-\frac{d^2}{dt^2}+M^2+\frac{\lambda}{2}x^2\right)y\right) \sim \det\left(-\frac{d^2}{dt^2}+M^2+\frac{\lambda}{2}x^2\right)^{-1/2}$$

where we integrated by parts with the boundary terms canceling by periodicity, and then performed a formal Gaussian integral. Therefore the effective action is

$$S_{\text{eff}}[x] = \int_0^T dt\, \left(\frac{1}{2}\dot{x}^2+\frac{m^2}{2}x^2\right)+\frac{1}{2}\log\det\left(-\frac{d^2}{dt^2}+M^2+\frac{\lambda}{2}x^2\right)$$

- We also know that $\log\det(AB) = \log\det A + \log\det B$, so

$$\log\det(AB) = \operatorname{tr}\log A + \operatorname{tr}\log B.$$

We also define the worldline Green's function $G(t,t')$ by

$$\left(\frac{d^2}{dt^2}-M^2\right)G(t,t') = \delta(t-t').$$

Explicitly, it can be shown that

$$G(t,t') = \frac{1}{2M}\sum_{n\in\mathbb{Z}} e^{-M|t-t'+nT|}$$

- Then we can expand the new term perturbatively as

$$\operatorname{tr}\log\left(-\frac{d^2}{dt^2}+M^2+\frac{\lambda}{2}x^2\right) = \operatorname{tr}\log\left(-\frac{d^2}{dt^2}+M^2\right)+\operatorname{tr}\log\left(1-\lambda\left(\frac{d^2}{dt^2}-M^2\right)^{-1}\frac{x^2}{2}\right).$$

The first term is a divergent constant, while the next term can be expanded in a series. The inverse is simply the Green's function, and the trace is over time, giving

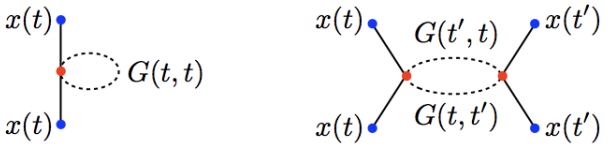
$$-\frac{\lambda}{2}\int_{S^1} dt\, G(t,t)x^2(t) - \frac{\lambda^2}{8}\int_{S^1\times S^1} dtdt'\, G(t',t)x^2(t)G(t,t')x^2(t') + \dots$$

with the general term

$$-\sum_{n=1}^{\infty}\frac{\lambda^n}{2^n n}\int_{(S^1)^n} dt_1\dots dt_n\, G(t_n,t_1)x^2(t_1)G(t_1,t_2)x^2(t_2)\dots G(t_{n-1},t_n)x^2(t_n)$$

so that we generate an infinite series of new interactions as usual.

- The striking new feature is that the new interactions are non-local in time. To understand this, we consider the Feynman diagrams for the first two terms,




Since the $y$ field is dynamical, it has its own propagator which allows it to move around on the worldline, so integrating it out gives a nonlocal interaction.

- To estimate the size of the nonlocal interactions, we Taylor expand in $t - t'$ for

$$\int dt dt' \, G(t,t')^2 x^2(t) x^2(t') = \int dt dt' \, G(t,t')^2 x^t(t) \left( x^2(t) + 2x(t)\dot{x}(t)(t-t') + \ldots \right).$$

  Note that $G(t,t')$ only depends on $t - t'$ through the combination $M(t-t')$. Then we can integrate over $t - t'$, giving a power of $1/M$ for every time derivative. The terms look like

$$\int dt \, \frac{x^4(t)}{M} + \frac{\text{two derivatives}}{M^3} + \frac{\text{four derivatives}}{M^5} + \ldots$$

  where we've suppressed numerical factors, and all terms are quartic in $x$. Therefore, as long as the derivatives are small in units of $M^{-1}$, we can truncate the series and get a local action.

- Note that we have a double expansion in $\lambda$ and $M$. The former is relatively innocuous, but the latter breaks down for energies on the scale of $M$, where all terms are important; if we continue to try to use only a few terms, we'll find violations of unitarity, signaling that perturbation theory is breaking down. To fix it in a tractable way, we must 'un-integrate out' the $y$ field.

- This same scenario was important in the discovery of the $W$ boson. Fermi's theory of weak decay was an effective theory that had integrated out the $W$ boson, but which led to a violation of unitarity at high energies.

**Note.** Perturbative unitarity violation is a typical feature of nonrenormalizable theories. For example, consider the innocuous case of massless $\phi^6$ theory in $d = 4$, with interaction term $\lambda\phi^6/M^2$. The coupling constant gives a mass scale $M$ where perturbation theory must break down. Specifically, the lowest-order contribution to $2\phi \to 4\phi$ scattering is

$$\mathcal{M} \sim \frac{\lambda}{M^2}, \quad |\mathcal{M}|^2 \sim \frac{\lambda^2}{M^4}, \quad \sigma \sim \frac{\lambda^2}{M^4} p^2$$

where we included the $p^2$ factor by dimensional analysis. Then for $p \gtrsim M$ the cross section violates unitarity bounds. The full nonperturbative result could still be unitary, but in reality the most common case is that some new physics takes over. This situation looks a bit different from the infinite tower of terms found above, but it's really the same, since these terms will be generated by renormalization.

Next, we show the worldline approach to perturbative quantum field theory. In this approach, we think about the particles of a quantum field theory in terms of ordinary quantum mechanics.

- The simplest way to describe a relativistic particle is to consider maps $x : [0,T] \to \mathbb{R}^n$ with

$$S[x] = \int_0^T dt \, \sqrt{G_{ab}(x)\dot{x}^a\dot{x}^b}$$

  where $G$ is the metric on $\mathbb{R}^n$ and $t$ is interpreted as a parameter; this is just the standard path length action, and it is invariant under diffeomorphisms of the worldline.

- We claim that this is equivalent to the action

$$S[g,x] = \int_M dt \, \sqrt{g} \left( \frac{1}{2} G_{ab}(x) g^{tt}(t) \dot{x}^a \dot{x}^b + \frac{1}{2} V(x) \right)$$

  where we vary with respect to both the worldline metric $g$ on $M$ and the fields $x$. Without the square root, this naively looks like the nonrelativistic action. The difference is that now we're dealing with a one-dimensional quantum gravity theory, since $g$ is varied too.

- The metric $g$ is specified by a single function $\sqrt{g} = |e|$ with $g^{tt} = e^{-2}$. The Riemann tensor vanishes, so the Einstein-Hilbert term vanishes. The Einstein equation is simply

$$T_{tt} = \frac{2}{\sqrt{g}} \frac{\delta(\sqrt{g}\mathcal{L})}{\delta g^{tt}} \propto \frac{1}{|e|} \left( G_{ab}(x)\dot{x}^a\dot{x}^b - e^2(t)V(x) \right) = 0$$

  which yields

$$g_{tt}(t) = e^2(t) = \frac{1}{V(x)} G_{ab}(x)\dot{x}^a\dot{x}^b.$$

  We see the metric is non-dynamical; it is simply determined by a constraint.

- Plugging this into our action $S[g, x]$ in the case $V(x) = V_0$ gives

$$S[x] = \sqrt{V_0} \int_M dt \, \sqrt{G_{ab}(x)\dot{x}^a\dot{x}^b}$$

  which is the geometrically natural action we encountered earlier.

- To see this another way, the momentum conjugate to the field $x^a$ is

$$p_a = \frac{\partial \mathcal{L}}{\partial \dot{x}^a} = \frac{1}{|e|} G_{ab}(x)\dot{x}^b$$

  so the Einstein equation gives the constraint

$$G^{ab}(x)p_a p_b + V(x) = 0$$

  which becomes the Klein–Gordan equation, where $m^2 = V_0$. The Hamiltonian is

$$H = p_a\dot{x}^a - \mathcal{L} = p^2 + m^2$$

  which vanishes on shell; this occurs generically for reparametrization-invariant theories. **(typos here?)**

- To see this a third way, we can perform the path integral explicitly in the case where $G$ is the Minkowski metric. In one dimension, diffeomorphisms of the worldline, which are gauge transformations, let us set $g_{tt} = 1$. Then integrating over all possible metrics is equivalent to integrating over $T$, which is called a Schwinger parameter. In formal language, we are integrating over the moduli space of Riemannian metrics.

- With fixed $T$, the propagator is

$$\langle y|e^{-HT}|x\rangle = \int dp\, dq\, \langle y|p\rangle\langle p|e^{-HT}|q\rangle\langle q|x\rangle = \int đp\, e^{ip(x-y)} e^{-T(p^2+m^2)}$$

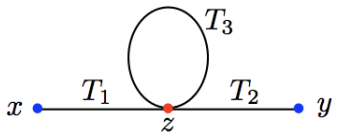
  Integrating over $T$, we indeed find the propagator for a free scalar field in the target space,

$$\int_0^\infty dT \int_{C_I[x,y]} \mathcal{D}x\, e^{-S} = \int_0^\infty dT\, \langle y|e^{-HT}|x\rangle = \int đp\, \frac{e^{ip(x-y)}}{p^2+m^2}.$$

- We would like to use one-dimensional quantum gravity to describe interacting scalar fields. We can do this by summing over possible topologies of $M$, replacing the worldline $I$ with a worldgraph $\Gamma$. More elaborate setups allow us to include spin.

- Specifically, to compute an $r$-point correlation function, we sum over all graphs with $r$ endpoints, where, e.g. for $\phi^4$ theory all vertices have four edges. Each edge $e$ has a Schwinger parameter $T_e$ integrated over $(0,\infty)$. Each vertex has a position, which is also integrated over, producing momentum-conserving delta functions, and an appropriate factor involving the coupling constant. As usual, we also divide by the symmetry factor.

- Explicitly, in $\phi^4$ theory one contribution to the propagator is




  which yields the factor

$$-\frac{\lambda}{4}\int_0^\infty dT_1 \int_{C_{T_1}[x,z]} \mathcal{D}x\, e^{-S} \int_0^\infty dT_2 \int_{C_{T_2}[y,z]} \mathcal{D}x\, e^{-S} \int dz \int_0^\infty dT_3 \int_{C_{T_3}[z,z]} \mathcal{D}x\, e^{-S}$$

  which simplifies to what we know from the Feynman rules,

$$-\frac{\lambda}{4}\int dz\, đp\, đq\, đ\ell\, \frac{e^{ip(x-z)}}{p^2+m^2}\frac{e^{iq(y-z)}}{q^2+m^2}\frac{e^{i\ell(z-z)}}{\ell^2+m^2} = -\frac{\lambda}{4}\int đp\, đ\ell\, \frac{e^{ip(x-y)}}{(p^2+m^2)^2(\ell^2+m^2)}.$$

- The formalism we've used here is called the worldline approach to quantum field theory, a predecessor to the modern path integral formulation using fields. (The quantum gravity interpretation came much later.) It is explicitly perturbative, since it directly deals with particles and Feynman diagrams.

- In $d=2$ and $d=3$, we have a Riemann tensor but no Weyl tensor, so gravitational degrees of freedom do not propagate and the path integral for quantum gravity can be performed. The standard approach to string theory is just the case $d=2$, where the graph becomes a Riemann surface, and we sum over the topologies of this surface. This approach is thus a perturbative picture of "string field theory", a mysterious subject about which little is known.

## 8.3 Quantum Statistical Mechanics

Our results above are also useful for thermal/statistical field theory (SFT). First we'll lay out the analogies between SFT and QFT, distinguishing Euclidean and Lorentzian signature explicitly.

- Consider a QFT with fields $\phi : M \to N$ where $\dim M = D$, so

$$Z_{\text{QFT}} = \int \mathcal{D}\phi\, e^{iS/\hbar}.$$

  Wick rotation flips the sign the relative signs of the potential and kinetic terms, giving

$$Z_{\text{QFT}} = \int \mathcal{D}\phi\, e^{-S_E/\hbar}, \quad \beta = \frac{1}{\hbar}.$$

  That is, we have

$$\text{QFT in } D \text{ spacetime dimensions} \sim \text{classical SFT in } D \text{ spatial dimensions}$$

in the sense that their partition functions are simply related. For example, for $D = 1$ we have

$$\text{quantum mechanics} \sim \text{classical SFT in 1 spatial dimension.}$$

We've been making this correspondence implicitly above.

- A different correspondence starts with a quantum SFT defined on $D$ spatial dimensions. Partition functions in SFT have the form of a path integral, but with a periodic imaginary "time",

$$Z_{\text{SFT}} = \operatorname{tr} e^{-\beta H} = \sum_n \langle n|e^{-\beta H}|n\rangle = \sum_n \int_{C_t[n,n]} \mathcal{D}\phi\, e^{-S_E/\hbar} = \int_{\mathcal{C}[M\times S^1]} \mathcal{D}\phi\, e^{-S_E/\hbar}.$$

  Here $S_E$ is the Euclidean action in $D+1$ spatial dimensions. Hence we have

$$\text{quantum SFT in } D \text{ spatial dimensions} \sim \text{classical SFT in } D+1 \text{ spatial dimensions}$$

  where one dimension in the classical SFT has period $t = \hbar\beta$.

- To compute an amplitude in such a theory, integrals $\int dk$ are replaced with sums over discrete frequencies in the periodic time direction. In the high temperature limit $\beta \to 0$, only the zero frequency matters, so we recover classical SFT in $D$ dimensions. Hence the high temperature limit of a quantum SFT in $D$ dimensions is a classical SFT in $D$ dimensions, as expected.

- As we saw in the notes on Undergraduate Physics, amplitudes in quantum mechanics are related to matrix elements of the thermal density matrix $\rho = e^{-\beta H}$ in quantum statistical mechanics, if the time evolution is taken to be imaginary. The diagonal elements of $\rho$ are of course the occupancies, but the off-diagonal elements are harder to interpret.

- Conceptually, the reason that $Z_{\text{QFT}}$ doesn't have boundary conditions but $Z_{\text{SFT}}$ does is because the time direction in a QFT is infinite, and the $i\epsilon$ damping automatically projects out the vacuum at temporal infinity. By contrast, $Z_{\text{SFT}}$ doesn't have a dynamical time at all.

Next we show how to perform computations in quantum statistical mechanics.

- In quantum statistical mechanics, our goal is to compute thermal expectation values. Quantum statistical mechanics takes place in zero spatial dimensions and periodic time, so

$$\left\langle \prod_i \hat{O}_i \right\rangle = \frac{\int_{\mathcal{C}[S^1]} \mathcal{D}\phi \prod_i \hat{O}_i\, e^{-S_E/\hbar}}{\int_{\mathcal{C}[S^1]} \mathcal{D}\phi\, e^{-S_E/\hbar}}.$$

- More explicitly, in the case of a one-dimensional harmonic oscillator,

$$\langle x^n \rangle = \frac{\int dx\, x^n \int_{C_t[x,x]} \mathcal{D}\phi\, e^{-S_E/\hbar}}{\int dx \int_{C_t[x,x]} \mathcal{D}\phi\, e^{-S_E/\hbar}}$$

  where the time period $t = \hbar\beta$, and the Euclidean action is

$$-\frac{1}{\hbar} S_E[x(\tau)] = -\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau\, \frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + \frac{k}{2}x^2 = -\frac{1}{2}\int_0^{\hbar\beta} d\tau\, x(\tau)\hat{D}x(\tau), \quad \hat{D} = -\frac{m}{\hbar}\frac{d^2}{d\tau^2} + \frac{\kappa}{\hbar}.$$

  Here we pulled the $x^n$ out of the functional integral, because it doesn't depend on $\tau$. However, we can formally consider $\tau$-dependent operators, which behave just as in quantum mechanics. In particular, we get $\tau$-ordered correlators on the left-hand side.

- As before, we may define a generating functional

$$Z[J] = \int \mathcal{D}x(\tau)\, \exp\left(\int_0^{\hbar\beta} d\tau\, -\frac{1}{2}x(\tau)\hat{D}x(\tau) + J(\tau)x(\tau)\right).$$

  Then $\langle x^n \rangle$ may be obtained by differentiating with respect to $J(0)$ and then setting $J(\tau) = 0$.

- By the same logic as in the zero-dimensional theory, for a free theory $Z[J]$ may be written in terms of the imaginary time Green's function,

$$Z[J] = Z[0] \exp\left(\frac{1}{2}\int d\tau d\tau'\, J(\tau)G(\tau-\tau')J(\tau')\right), \quad \hat{D}_\tau G(\tau-\tau') = \delta(\tau-\tau'), \quad G(0) = G(\beta\hbar).$$

- Since the Green's function takes place in periodic time, we take a Fourier series,

$$G(\tau) = \frac{1}{\sqrt{\beta\hbar}}\sum_n g_n e^{i\omega_n\tau}, \quad \omega_n = \frac{2\pi n}{\beta\hbar}, \quad n \in \mathbb{Z}$$

  where the $\omega_n$ are the Matsubara frequencies. Plugging this into the defining equation,

$$G(\tau) = \frac{1}{\beta\kappa}\sum_n \frac{\omega^2}{\omega^2+\omega_n^2}e^{i\omega_n\tau}, \quad \omega = \sqrt{\frac{\kappa}{m}}.$$

  This is equivalent to

$$G(\tau) = \frac{\hbar\omega}{2\kappa}\left(\frac{e^{\omega|\tau|}}{e^{\hbar\beta\omega}-1} + \frac{e^{-\omega|\tau|}}{1-e^{-\hbar\beta\omega}}\right)$$

  as can be shown by contour integrating the latter.

Now we give some explicit examples of computations.

- Correlators may be expanded by Wick's theorem. For example, we have

$$\langle \hat{x}^{2n+1}\rangle = 0, \quad \langle \hat{x}^2\rangle = G(0).$$

  To extend to an arbitrary imaginary time, we define the "Heisenberg" operators

$$\overline{x}(\tau) = e^{H\tau/\hbar}\hat{x}e^{-H\tau/\hbar}.$$

  One must take some care, since $\overline{x}(\tau)$ is not Hermitian. Now we have

$$\langle T\overline{x}(\tau_1)\overline{x}(\tau_2)\rangle = G(\tau_1-\tau_2)$$

  while higher correlators are also decomposed by Wick's theorem.

- Using the form of the Green's function, we have

$$\langle \hat{x}^2\rangle = \frac{\hbar\omega}{\kappa}\left(\frac{1}{e^{-\beta\hbar\omega}-1}+\frac{1}{2}\right).$$

  Using $\langle H\rangle = 2\langle V\rangle = \kappa G(0)$ and $\hat{H} = \hbar\omega(\hat{n}+1/2)$, we find the Bose-Einstein distribution.

- We can compute the position probability distribution by using our knowledge of the moments. Alternatively, we can perform the computation in one step, as the probability distribution is

$$\langle \delta(\hat{x} - x_0) \rangle \equiv \int đk \, \langle e^{ik(\hat{x}-x_0)} \rangle = \int đk e^{-ikx_0} \frac{Z[ik\delta(\tau)]}{Z[0]} = \int đk \, e^{-ikx_0} e^{-k^2 G(0)/2}$$

  where we used the explicit form of $Z[J]$. Thus, we conclude

$$\langle \delta(\hat{x} - x_0) \rangle = \frac{e^{-x_0^2/2G(0)}}{\sqrt{2\pi G(0)}}.$$

  In the low temperature limit this gives the ground state probability distribution, while in the high temperature limit this gives the Boltzmann distribution.

- For the harmonic oscillator, the computations done above can also be done in the "Heisenberg" picture. Starting with the usual creation and annihilation operators, define

$$\overline{a}(\tau) = e^{H\tau/\hbar} a e^{-H\tau/\hbar}, \quad \overline{a}^\dagger = e^{H\tau/\hbar} a^\dagger e^{-H\tau/\hbar}.$$

  Again we must be careful, as $\overline{a}^\dagger(\tau) \neq (\overline{a}(\tau))^\dagger$. Integrating the Heisenberg equation of motion,

$$\overline{a}(\tau) = e^{-\omega\tau} a, \quad \overline{a}^\dagger(\tau) = e^{\omega\tau} a^\dagger.$$

- Now we may straightforwardly compute imaginary time-dependent thermal averages,

$$\langle \overline{a}(\tau) \overline{a}^\dagger(0) \rangle = \frac{1}{Z(\beta)} \operatorname{tr} e^{-\beta H} \overline{a}(\tau) \overline{a}^\dagger(0) = \frac{e^{-\omega\tau}}{1 - e^{-\beta\hbar\omega}}.$$

  Using the relationship between $\hat{x}$ and $\hat{a}^\dagger$ and $\hat{a}$, which continues to hold with $\tau$-dependence, we may compute $\langle \overline{x}(\tau) \overline{x}(0) \rangle$, which agrees with the Green's function computed above.

- We can set up perturbation theory with Feynman diagrams just as we did earlier. For example, for the anharmonic oscillator with a $\lambda \hat{x}^4/4!$ term, we have

$$Z_\lambda[J] = \int \mathcal{D}x(\tau) \, \exp\left( -\frac{S_E}{\hbar} + \int_0^{\hbar\beta} d\tau \, Jx - \frac{\lambda}{4!\hbar} x^4 \right) = \exp\left( -\frac{\lambda}{4!\hbar} \int_0^{\hbar\beta} d\tau \frac{\delta^4}{\delta J(\tau)^4} \right) Z_0[J]$$

  where $S_E$ is the Euclidean action for the harmonic oscillator. We can then expand the exponential in a series, using our simple expression for $Z_0[J]$, and simplify each term by Wick's theorem. Each set of contractions yields a Feynman diagram.

- Hence we have the following Feynman rules for computing the partition function $Z_\lambda[0]$.
  - Draw $n$ internal points at times $\tau_i$.
  - For every point, multiply by $-\lambda/\hbar$ and integrate over $\tau_i$.
  - Contract the $4n$ edges pairwise. For each edge, write $G(\tau_i - \tau_j)$.
  - Multiply by $Z_0[0]$ and sum over all contractions.

  Summing over contractions, we get equivalence classes of diagrams, where we must divide by the symmetry factor as usual. We can evaluate correlators similarly, though now there are external points.

## 8.4 Quantum Fields

Finally, we generalize to quantum fields, returning to Lorentzian signature, and tie up some loose ends from the previous sections.

- For a scalar field, we may define field eigenstates by

$$\hat{\phi}(x)|\phi(x)\rangle = \phi(x)|\phi(x)\rangle$$

  and similarly field momentum eigenstates $|\pi(x)\rangle$, and use them to construct a path integral as before using the formal completeness relation

$$\int \mathcal{D}\phi(\mathbf{x})\,|\phi(\mathbf{x})\rangle\langle\phi(\mathbf{x})| = \int \mathcal{D}\pi(\mathbf{x})\,|\pi(\mathbf{x})\rangle\langle\pi(\mathbf{x})| = 1.$$

  The path integral then yields the transition amplitude between two states $|\phi_a\rangle$ and $|\phi_b\rangle$.

- In relativistic quantum field theory, we are usually interested in vacuum expectation values, since they appear in the LSZ reduction formula. If we let the vacuum be $|\Omega\rangle$, then we can isolate it from arbitrary boundary conditions, as long as they have some overlap with the vacuum state, by taking the time to infinity in a slightly imaginary direction. The result is

$$\langle\Omega|\mathcal{T}\prod_{i=1}^{n}\hat{\mathcal{O}}_i(x_i)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)}\frac{\displaystyle\int \mathcal{D}\phi(x)\,e^{iS}\prod_{i=1}^{n}\mathcal{O}_i(x_i)}{\displaystyle\int \mathcal{D}\phi(x)\,e^{iS}}$$

  where the paths on the right-hand side run from $-T$ to $T$, and the denominator cancels out the phase and overlap factors, so that $\langle\Omega|\Omega\rangle = 1$. We will keep this implicit below.

- Taking the time to infinity this way effectively provides an infinitesimal damping which makes the path integral converge. It is always necessary in the definition of the path integral, even in Lorentzian signature, and it is what makes the transition to Euclidean signature natural. It is equivalent to add an $i\epsilon$ term in the action like

$$\mathcal{L} \supset (m^2 - i\epsilon)\phi^2.$$

  This manifestly yields the expected $i\epsilon$ in the Feynman propagator. In all cases, whenever we write $m^2$, we really mean $m^2 - i\epsilon$.

- A general state of the quantum field is a superposition of the field eigenstates, so it is a wavefunctional of the classical field configurations. For example, the vacuum wavefunctional can be computed from the path integral as

$$\langle\phi(x)|\Omega\rangle = \lim_{t_i\to-\infty(1-i\epsilon)}\int \mathcal{D}\phi(\mathbf{x})\,e^{iS}$$

  where the final boundary condition is $\phi(x)$ and the initial boundary condition is arbitrary.

- The partition function can be interpreted as the vacuum to vacuum amplitude,

$$\langle\Omega|e^{-iHT}|\Omega\rangle = \int_{\text{time}\,T}\mathcal{D}\phi(x)\,e^{iS} = \mathcal{Z}[0] = \text{sum of vacuum diagrams}$$

  which implies that it is equal to $e^{-iET}$ where $E$ is the vacuum energy, which is generally divergent. Thus dividing by $\mathcal{Z}[0]$ to compute expectation values is equivalent to renormalizing the vacuum energy to zero. Then $\mathcal{W}[0]$ is simply the vacuum energy times $T$.

Since Dirac fields are complex, we'll also have to deal with complex Grassmann numbers.

- We can construct complex Grassmann numbers from real Grassmann numbers, e.g.

$$\theta = \frac{1}{\sqrt{2}}(\eta_1 + i\eta_2), \quad \theta^* = \frac{1}{\sqrt{2}}(\eta_1 - i\eta_2).$$

  Using the change of variables formula derived earlier,

$$\int d\theta^* d\theta \, \theta\theta^* = (-i)(i) \int d\eta_2 d\eta_1 \, \eta_1\eta_2 = 1.$$

  For multiple Grassmann variables we define $d^n\theta d^n\theta^* = d\theta_n d\theta_n^* \dots d\theta_1 d\theta_1^*$. Note that $d\theta_i d\theta_i^*$ is Grassmann-even.

- By convention, we take complex conjugation to reverse the order of products, $(\theta\eta)^* = \eta^*\theta^*$.

- A complex Gaussian integral takes the form

$$\int d^n\theta d^n\theta^* \, e^{-\theta_i^* M_{ij} \theta_j} = \int d^n\theta d^n\theta^* \, e^{-\sum_i \theta_i^* m_i \theta_i} = \prod_i m_i = \det M$$

  where the $m_i$ are the eigenvalues of $M$. Note that the derivation of this formula does not require $\theta^*$ to be the conjugate of $\theta$, and that the final result has no square root.

- The partition function with anticommuting sources $\chi$ and $\chi^*$ is

$$Z(\chi) = \int d^n\theta d^n\theta^* \, e^{-\theta_i^* M_{ij} \theta_j + \chi_i^* \theta_i + \theta_i^* \chi_i}$$

  and performing the shift $\theta \to \theta + M^{-1}\chi$ yields

$$Z(\chi) = (\det M) \, e^{\chi_i^* (M^{-1})_{ij} \chi_j}.$$

  Note that $M$ must be antisymmetric. When $n$ is odd, $M$ has a zero eigenvalue and hence $M^{-1}$ does not exist, so the above formula does not apply.

- In the case of a Dirac field, there is an independent Grassmann algebra at every point in spacetime, and we write

$$\psi(x) = \sum_i \psi_i \phi_i(x)$$

  where the $\phi_i(x)$ are a basis of four-component spinors, and the $\psi_i$ form a basis of the Grassmann algebra at $x$. In particular, this is how one should think of relativistic spinors even on the classical level; otherwise, the anticommutativity must be put in 'by hand' upon quantization.

- As an example, in $d = 4$, every interaction of the form $(\overline{\psi}\psi)^5$ is trivial, because there are only eight independent Grassmann variables at every point. This is physically sensible, because interactions are contact interactions, and we cannot put five Dirac fermions at the same point.

Next, we explicitly construct the fermionic path integral using coherent states.

- We consider a single fermionic degree of freedom, e.g. a single mode that could be occupied by one kind of fermion. We construct the path integral using coherent states,

$$\hat{\psi}|\psi\rangle = \psi|\psi\rangle$$

where $\psi$ is a complex Grassmann number and $\hat{\psi}$ is an annihilation operator. The operators $\hat{\psi}$ and $\hat{\psi}^\dagger$ obey

$$\{\hat{\psi}, \hat{\psi}^\dagger\} = 1$$

with all other anticommutators zero. Note that we are allowing 'scalar multiplication' with Grassmann numbers in our Hilbert space, though our final answers will be ordinary numbers.

- We postulate a unique ground state $|0\rangle$ so that $\hat{\psi}|0\rangle = 0$ and define $\hat{\psi}^\dagger|0\rangle = |1\rangle$. Then there are no other states. The desired coherent state is

$$|\psi\rangle = |0\rangle - \psi|1\rangle.$$

To show this very explicitly, note that

$$\hat{\psi}|\psi\rangle = -\hat{\psi}\psi|1\rangle = \psi\hat{\psi}|1\rangle = \psi\hat{\psi}\hat{\psi}^\dagger|0\rangle = \psi|0\rangle = \psi|\psi\rangle.$$

It is easy to make a mistake, as everything can have Grassmann parity. The matrix elements of $\hat{\psi}$ are Grassmann numbers, and the state $|\psi\rangle$ itself has Grassmann coefficients.

- To make the analogy clearer, note that

$$|\psi\rangle = e^{-\psi\hat{\psi}^\dagger}|0\rangle.$$

In the bosonic case, coherent states are defined as

$$a|\mu\rangle = \mu|\mu\rangle, \quad |\mu\rangle = e^{\mu a^\dagger}|0\rangle.$$

The latter expression is identical up to a sign.

- The complex conjugate state $\langle\psi|$ is defined by

$$\langle\psi|\hat{\psi}^\dagger = \langle\psi|\psi^*, \quad \langle\psi| = \langle 0| - \langle 1|\psi^* = \langle 0|e^{-\hat{\psi}\psi^*}.$$

The inner product of two coherent states is

$$\langle\psi|\psi'\rangle = \langle 0|0\rangle + \langle 1|\psi^*\psi'|1\rangle = 1 + \psi^*\psi' = e^{\psi^*\psi'}.$$

- Finally, the completeness relation is

$$\int d\psi^* d\psi \, |\psi\rangle e^{-\psi^*\psi}\langle\psi| = 1.$$

This result holds because the left-hand side is equal to $|0\rangle\langle 0| + |1\rangle\langle 1|$.

We now use this setup to compute transition amplitudes.

- We compute the transition amplitude

$$\langle\psi_f|e^{-i\hat{H}(t_f-t_i)}|\psi_i\rangle, \quad \hat{H}=\hat{\psi}^\dagger M\hat{\psi}$$

for a quadratic Hamiltonian. As in the bosonic path integral, we split the time interval by inserting copies of the identity.

- Each copy comes with a factor of

$$\int d\psi_j^* d\psi_j \, \langle\psi_{j+1}|e^{-i\hat{H}\delta t}|\psi_j\rangle e^{-\psi_j^*\psi_j}\langle\psi_j|.$$

Expanding the exponential, we find

$$\langle\psi_{j+1}|e^{-i\hat{H}\delta t}|\psi_j\rangle = (1-i\psi_{j+1}^* M\psi_j\delta t)\langle\psi_{j+1}|\psi_j\rangle = e^{\psi_{j+1}^*\psi_j}e^{-i\psi_{j+1}^* M\psi_j\delta t}.$$

Then some of the phase factors telescope, and we find

$$\begin{aligned}\langle\psi_f,t_f|\psi_i,t_i\rangle &= \lim_{N\to\infty}\int\prod_{j=1}^N d\psi_j^* d\psi_j \, \exp\left(\sum_{j=0}^N -\psi_{j+1}^*(\psi_{j+1}-\psi_j) - i\delta t H(\psi_{j+1}^*,\psi_j)\right)\\ &= \int \mathcal{D}\psi^*\mathcal{D}\psi \, e^{i\int_{t_i}^{t_f} dt\,(\psi^* i\frac{\partial\psi}{\partial t}-H)}\end{aligned}$$

and in the usual case the quantity in parentheses is the Lagrangian.

- Now, the Dirac field is a relativistic field, so the field operator necessarily contains both creation and annihilation operators. Hence the interpretation is somewhat different from above, but the same manipulations work. In this case we integrate $\mathcal{D}\overline{\psi}\mathcal{D}\psi$ and define the generating functional

$$Z[\eta,\overline{\eta}] = \int \mathcal{D}\overline{\psi}\mathcal{D}\psi \, e^{iS[\psi,\overline{\psi}]+i\overline{\eta}\psi+i\overline{\psi}\eta}$$

and by the same manipulations in the bosonic case, we have

$$\langle\Omega|T\psi(x_1)\overline{\psi}(x_2)|\Omega\rangle = \frac{1}{Z[0,0]}\frac{\delta}{i\delta\overline{\eta}(x_1)}\frac{-\delta}{i\delta\eta(x_2)}Z[\eta,\overline{\eta}]\bigg|_{\eta=\overline{\eta}=0}$$

where we pick up an extra minus sign by anticommutation.

- By our earlier result for a Gaussian integral with linear sources, the Feynman propagator is again the inverse of the kinetic term $-i(i\not{\partial}-m)$, recovering the usual result in Fourier space.

**Note.** As mentioned earlier, spinors should be Grassmann-valued in classical field theory, leading to tricky signs. We defined the complex conjugate so that $(\theta_1\theta_2)^* = \theta_2^*\theta_1^*$, and define the transpose of a single Grassmann number to do nothing. Then for spinor fields $\psi_i$,

$$(\overline{\psi}_1\psi_2)^T = -\psi_2^T\psi_1$$

because the terms in the sum have the order of the Grassmann variables flipped. But more nicely,

$$(\overline{\psi}_1\psi_2)^\dagger = -(\psi_2^T\,\overline{\psi}_1^T)^* = \psi_2^\dagger\,\overline{\psi}_1^\dagger$$

so the adjoint simply applies to everything in flipped order.

## 8.5 Symmetries of the Path Integral

In this section, we derive the Ward–Takahashi identities. They are exact, nonperturbative relations between correlation functions that result from symmetries of the path integral. We begin by reviewing Noether's theorem.

- We work in curved spacetime using the language of differential forms, restating every result in coordinates. Consider the infinitesimal transformation

$$\delta_\epsilon \phi(x) = \epsilon f(\phi, \partial_\mu \phi).$$

  The transformation is local if $f$ depends only on the fields and their derivatives at $x$, in which case it is generated by the vector

$$V_f = \int_M d^d x \, \sqrt{g} \, f(\phi, \partial\phi) \frac{\delta}{\delta \phi(x)}$$

  acting on the space of fields.

- This transformation is a symmetry if it leaves the action invariant. Now, we introduce the so-called "Noether trick", promoting $\epsilon$ to be spacetime-dependent. Since we have a symmetry for constant $\epsilon$, the variation of the action should depend only on derivatives of $\epsilon$, so we can always find some $j$ so that

$$\delta_\epsilon S[\phi] = -\int_M *j \wedge d\epsilon = -\int_M d^d x \, \sqrt{g} \, g^{\mu\nu} j_\mu(x) \partial_\nu \epsilon(x).$$

  One can check $j$ is just the conserved current found in the usual proof of Noether's theorem.

- When the equations of motion hold, the action is invariant under any infinitesimal change in the fields, $\delta_\epsilon S[\phi] = 0$. Integrating by parts and choosing $\epsilon(x)$ to have compact support,

$$d * j = \partial_\mu (\sqrt{g} g^{\mu\nu} j_\nu) = 0.$$

- We define the charge $Q$ on a hypersurface $N$ of codimension one by

$$Q[N] = \int_N *j = \int_N d^{d-1} x \, \sqrt{g_N} \, j_\mu n^\mu$$

  where $g_N$ is the metric pulled back to $N$ and $n^\mu$ is a unit normal.

- Now consider two such hypersurfaces $N_0$ and $N_1$ bounding a region $M'$. Then

$$Q[N_1] - Q[N_0] = \int_{\partial M'} *j = \int_{M'} d * j = 0.$$

  That is, the charge depends on $N$ only through its homology class. In the simple case where the $N_i$ are constant time-slices of Minkowski space, this means $Q$ is conserved in time.

**Example.** For the complex scalar field we have

$$S[\phi] = \int_M d\overline{\phi} \wedge *d\phi + *V(|\phi|^2)$$

where we write conjugation with a bar to avoid confusion with the Hodge star. Then by direct computation, the current is $j = i(\phi d\overline{\phi} - \overline{\phi} d\phi)$ and for a time-slice,

$$Q[N] = i \int_N *(\phi d\overline{\phi} - \overline{\phi} d\phi) = i \int d\mathbf{x}\, \phi \partial_0 \overline{\phi} - \overline{\phi} \partial_0 \phi.$$

The canonical momenta are

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \overline{\phi}, \quad \overline{\pi} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \overline{\phi})} = \partial_0 \phi$$

so the charge is

$$Q = i \int d\mathbf{x}\, (\phi \pi - \overline{\phi}\overline{\pi})$$

which indeed generates the transformations by Poisson brackets.

**Note.** Dynamical symmetries are symmetries of the equations of motion that are not symmetries of the Lagrangian; they are associated with integrable systems. For example, for a free particle, the Lagrangian has an $O(n)$ symmetry, while the equation of motion has a $\mathrm{GL}_n(\mathbb{R})$ symmetry because it only requires the particle to move uniformly on a line.

Next, we turn to symmetries in the quantum theory.

- In the quantum theory, the local field transformation $\phi \to \phi'(\phi)$ must leave the product of the path integral weight and measure invariant to be a symmetry,

  $$\mathcal{D}\phi\, e^{-S[\phi]} = \mathcal{D}\phi'\, e^{-S[\phi']}.$$

  This is necessary and sufficient, but since we have more experience with symmetries of the action from classical field theory, we often use those and hope the measure is invariant as well.

- For example, suppose $S[\phi]$ only depends on the derivatives of $\phi$. Then classically we have the shift symmetry $\phi(x) \to \phi(x) + \phi_0$. It would appear that the measure would obviously be invariant, but the measure is not defined without regularization, and the regularized measure might not be invariant.

- If $M = T^d$, we might expand $\phi(x)$ in a Fourier series, then cut off the sum. Since the constant translation only affects the lowest Fourier mode, and the regularized measure integrates over all values of this coefficient, the measure is indeed invariant.

- As another example, consider rotational symmetry in Euclidean space. The action is $SO(d)$ invariant if it is a scalar under rotation. We can regularize the path integral by integrating over all Fourier modes where the $SO(d)$ invariant quantity $p^\mu p_\mu$ is less than some cutoff; this is slightly more subtle for $SO(d-1,1)$ in Minkowski space. Alternatively, we can regularize by replacing space with a lattice, but this breaks $SO(d)$ down to the lattice's point group.

- Sometimes we may run into tradeoffs, where only some symmetries can be preserved, and we may intentionally choose which to lose. A worse situation is when there doesn't exist *any* regulator that preserves a symmetry, in which case the symmetry is said to be anomalous; it is said to be incompatible with quantum mechanics itself.

- For example, QED in four dimensions with a massless fermion is conformally invariant, but there is a conformal anomaly; indeed, the beta functions are nonzero in DR, with a hard cutoff, and on a lattice. It's possible to prove that a symmetry is anomalous by considering the geometry and topology of the space of fields, using sophisticated mathematics such as the Atiyah-Singer index theorem.

- The Standard Model contains a number of anomalies, such as the global symmetry of baryon number. In general, it's acceptable for global symmetries to be anomalous, but not gauge symmetries, since we lose the Ward identity and hence unitarity. The hypercharges in the Standard Model are fixed by requiring gauge anomalies to vanish.

- Consider operators whose only variation under a symmetry transformation is through the transformation of the field, e.g. scalar operators for rotations. Then $\mathcal{O}(\phi) \to \mathcal{O}(\phi')$, and on a compact manifold $M$ we have

$$\int \mathcal{D}\phi\, e^{-S[\phi]} \prod_i \mathcal{O}_i(\phi(x_i)) = \int \mathcal{D}\phi'\, e^{-S[\phi']} \prod_i \mathcal{O}_i(\phi'(x_i)) = \int \mathcal{D}\phi\, e^{-S[\phi]} \prod_i \mathcal{O}_i(\phi'(x_i))$$

  where we simply renamed the dummy variable in the first step, then defined $\phi'$ in terms of $\phi$ and used the definition of a symmetry in the second.

- Therefore, we conclude (leaving the time ordering implicit)

$$\langle \mathcal{O}_1(\phi(x_1)) \dots \mathcal{O}_n(\phi(x_n)) \rangle = \langle \mathcal{O}_1(\phi'(x_1)) \dots \mathcal{O}_n(\phi'(x_n)) \rangle.$$

  In general, we will call any identity between correlation functions derived from a symmetry a Ward or Ward–Takahashi identity. This is an example of a 'global' Ward–Takahashi identity.

**Example.** Consider the $U(1)$ symmetry of a complex scalar field,

$$\phi \to \phi' = e^{i\alpha}\phi, \quad \overline{\phi} \to \overline{\phi}' = e^{-i\alpha}\overline{\phi}.$$

The path integral measure is invariant under this symmetry as long as we integrate over as many modes of $\overline{\phi}$ as we do of $\phi$. Therefore for $\mathcal{O}_i = \phi^{r_i}\overline{\phi}^{s_i}$, we have

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle = e^{i\alpha \sum_i (r_i - s_i)} \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle.$$

Then correlators vanish unless they contain the same number of $\phi$ and $\overline{\phi}$ fields.

**Example.** If the action and path integral measure are translationally invariant, and the operators $\mathcal{O}_i$ depend on $x$ only through their dependence on $\phi(x)$, then

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle = \langle \mathcal{O}_1(x_1 - a) \dots \mathcal{O}_n(x_n - a) \rangle$$

so correlators only depend on position differences. Similarly, Lorentz invariance ensures that correlators can only depend on the invariant interval between the insertion points.

**Note.** Formally, the path integral measure picks up a Jacobian factor,

$$\mathcal{D}\phi' = \det\left(\frac{\delta\phi'(x)}{\delta\phi(y)}\right)\mathcal{D}\phi.$$

In the case of the transformation described above, we have

$$\frac{\delta}{\delta\phi(y)}(\phi(x)+\epsilon f(x)) = \delta(x-y) + \epsilon\frac{\partial f(\phi(x))}{\partial\phi(y)}$$

and hence the Jacobian is

$$\det\left(\frac{\delta\phi'(x)}{\delta\phi(y)}\right) = 1 + \operatorname{tr}\left(\epsilon\frac{\delta f(\phi(x))}{\delta\phi(y)}\right).$$

Then classical symmetries linear in the fields should remain symmetries in the quantum theory, because the argument of the determinant is field-independent, so the Jacobian is just a constant that cancels out. However, this formal argument is deceptive, because we are working with an undefined, "ideal" path integral measure. Instead we must check invariance directly.

These results roughly correspond to charge conservation, but we can derive more powerful identities that correspond to current conservation, usually called the Ward–Takahashi identities. To warm up, we derive the Schwinger–Dyson and Ward–Takahashi identities in standard notation.

- We consider the partition function of a generic theory $Z[J]$ and a transformation

  $$\phi(x) \to \phi'(x) = \phi(x) + \epsilon\Delta\phi(x),$$

  not necessarily a symmetry, which leaves the measure invariant. Then infinitesimally

  $$Z[J] = \int \mathcal{D}\phi\, e^{i(S[\phi]+\phi\cdot J)} = \int \mathcal{D}\phi'\, e^{i(S[\phi']+\phi'\cdot J)} = \int \mathcal{D}\phi\, e^{i(S[\phi]+\phi\cdot J)}\left(1 + i\left(\frac{\delta S}{\delta\phi} + J\right)\epsilon\Delta\phi\right)$$

  which shows that

  $$\int \mathcal{D}\phi\, e^{i(S[\phi]+\phi\cdot J)} \int dy\, \left(\frac{\delta S}{\delta\phi(y)} + J(y)\right)\Delta\phi(y) = 0.$$

- Next, we consider the specific transformation that shifts the field at one point,

  $$\Delta\phi(y) = \delta(y-x)$$

  which should leave a reasonable measure invariant; this shift-invariance was the motivation behind our definition of Grassmann integration. Then we find

  $$\int \mathcal{D}\phi\, \left(\frac{\delta S}{\delta\phi(x)} + J(x)\right) e^{i(S[\phi]+J\cdot\phi)} = 0$$

  which states that the classical equations of motion hold in vacuum expectation.

- To get relationships between correlation functions, we act by $n$ field derivatives $\delta/i\delta J(x_i)$. This yields one term with $n$ factors of $\phi$, and $n$ terms with $n-1$ factors of $\phi$, where one of the derivatives hits the $J(x)$ factor. Setting $J=0$ gives

  $$\left\langle \frac{\delta S}{\delta\phi(x)}\prod_{i=1}^{n}\phi(x_i)\right\rangle = i\sum_{i=1}^{n}\left\langle \phi(x_1)\ldots\widehat{\phi}(x_i)\delta(x-x_i)\ldots\phi(x_n)\right\rangle$$

  where the hat denotes a missing argument. This is the Schwinger–Dyson equation, as we derived earlier. Letting $X$ stand for all the fields, it can be concisely written as

  $$\left\langle \frac{\delta S}{\delta\phi(x)}X\right\rangle = i\left\langle \frac{\delta X}{\delta\phi(x)}\right\rangle.$$

  Indeed, here $X$ can be any function of the fields.

- The classical equivalent of a time-ordered correlator is just a product of numbers,

$$\langle \phi(x_1) \dots \phi(x_n) \rangle \to \phi(x_1) \dots \phi(x_n)$$

so in the classical limit, the left-hand side is zero as $\delta S / \delta \phi(x) = 0$. Thus the contact terms on the right encode the difference between classical products and quantum correlation functions.

- As an example, consider a free scalar field. Then the Schwinger–Dyson equation for $n = 1$ reduces to the statement that the Feynman propagator is a Green's function for the Klein–Gordan operator; the contact term supplies the required delta function. For an interacting theory, we would instead find a relationship between the two-point correlator and higher correlators, which we can use to organize a perturbative expansion.

- To derive the Ward–Takahashi identities, we do the same procedure, but for a classical global symmetry associated with a field change $\delta\phi$. Here we have

$$\Delta\phi(y) = \delta(y - x)\delta\phi(x), \quad \frac{\delta S}{\delta \phi(x)} \delta\phi(x) = -\partial_\mu j^\mu(x).$$

Assuming this transformation leaves the measure invariant as well, we have

$$\int \mathcal{D}\phi \, e^{i(S[\phi] + \phi \cdot J)} (-\partial_\mu j^\mu(x) + J(x)\delta\phi(x)) = 0.$$

Taking derivatives as before gives the Ward–Takahashi identity

$$\partial_\mu \left\langle j^\mu(x) \prod_{i=1}^n \phi(x_i) \right\rangle = -i \sum_{i=1}^n \left\langle \phi(x_1) \dots \widehat{\phi}(x_i) \delta\phi(x) \delta(x - x_i) \dots \phi(x_n) \right\rangle$$

where the derivative acts on $x$. In the case $n = 0$, we see the current $j^\mu$ is conserved in expectation, $\partial_\mu \langle j^\mu \rangle = 0$.

**Note.** It is conventional in theoretical physics to write the Schwinger–Dyson equation as, e.g.

$$\frac{\delta S}{\delta \phi(x)} \phi(y) = i\delta(x - y)$$

and call this an "operator equation", even though the two sides are *not* equal as operators on the Hilbert space. The sociological reason behind this is that statements about operators are difficult to make, since they are fraught with short-distance singularities. In the mathematical physics community, these difficulties are addressed in a formalism which replaces operators with operator-valued distributions. In the theoretical physics community, these issues are avoided by simply redefining the phrase "operator equation" to mean something weaker. The operator equation $\mathcal{O}_1 = \mathcal{O}_2$ means that

$$\langle \mathcal{O}_1 X \rangle = \langle \mathcal{O}_2 X \rangle$$

for any product of operators in $X$ involving fields evaluated at times different from those of $\mathcal{O}_1$ and $\mathcal{O}_2$. (If the times matched, we would pick up extra contact terms.)

Next, we rederive the Ward–Takahashi identity a bit more generally. We will work in curved spacetime, and use symmetries of the path integral, which need not be classical symmetries.

- We consider an infinitesimal symmetry of the path integral, $\phi \to \phi' = \phi + \epsilon\delta\phi$, where $\epsilon$ is constant. Then if we allow $\epsilon$ to vary in space, falling off at infinity to avoid boundary terms,

$$\mathcal{Z} = \int \mathcal{D}\phi'\, e^{-S[\phi']} = \int \mathcal{D}\phi\, e^{-S[\phi]} \left(1 - \int_M *j \wedge d\epsilon\right)$$

as we saw classically, but the variation of the measure may also contribute to $j$. Then

$$0 = -\int_M *\langle j(x)\rangle \wedge d\epsilon = \int_M \epsilon(x)\, d * \langle j(x)\rangle$$

which implies that $j^\mu(x)$ is a conserved current in expectation, $\partial_\mu \langle j^\mu(x)\rangle = 0$.

- Now, as before, we introduce local operators which transform as

$$\mathcal{O} \to \mathcal{O}' = \mathcal{O} + \epsilon\delta\mathcal{O}.$$

Accounting for both the change in the action and measure, and the operators, we find

$$\int_M \epsilon(x) \wedge d * \left\langle j(x) \prod_{i=1}^n \mathcal{O}_i(x_i) \right\rangle = -\sum_{i=1}^n \left\langle \epsilon(x_i)\delta\mathcal{O}_i(x_i) \prod_{j\neq i} \mathcal{O}_j(x_j) \right\rangle.$$

Note that the exterior derivative $d$ acts on $x$, and the correlators are time-ordered as usual.

- To finish up, we would like to strip off the parameter $\epsilon(x)$. Note that

$$\epsilon(x_i)\delta\mathcal{O}_i(x) = \int_M *\delta(x - x_i)\epsilon(x)\delta\mathcal{O}_i(x_i) = \int_M \delta(x - x_i)\epsilon(x)\delta\mathcal{O}_i(x_i)\sqrt{g}\, dx.$$

Having expressed both sides as integrals, we simply have

$$d * \left\langle j(x) \prod_{i=1}^n \mathcal{O}_i(x_i) \right\rangle = - * \sum_{i=1}^n \delta(x - x_i) \left\langle \delta\mathcal{O}_i(x_i) \prod_{j\neq i} \mathcal{O}_j(x_j) \right\rangle.$$

This is the Ward–Takahashi identity in a curved spacetime. In more pedestrian notation, in flat spacetime, this reduces to

$$\partial_\mu \left\langle j^\mu(x) \prod_{i=1}^n \mathcal{O}_i(x_i) \right\rangle = -\sum_{i=1}^n \delta(x - x_i) \left\langle \delta\mathcal{O}_i(x_i) \prod_{j\neq i} \mathcal{O}_j(x_j) \right\rangle.$$

- Next, we integrate over $x$ to investigate charge conservation. Consider integrating over some region $M' \subset M$ with $\delta M' = N_1 - N_0$, as we studied classically, where $M'$ contains all the points $x_i$. Then

$$\langle Q[N_1] \prod_i \mathcal{O}_i(x_i)\rangle - \langle Q[N_0] \prod_i \mathcal{O}_i(x_i)\rangle = -\sum_{i=1}^n \langle \delta\mathcal{O}_i(x_i) \prod_{j\neq i} \mathcal{O}_j(x_j)\rangle.$$

If $M$ is compact without boundary and $M' = M$, the left-hand side vanishes, and

$$\boxed{\sum_{i=1}^n \langle \delta\mathcal{O}_i(x_i) \prod_{j\neq i} \mathcal{O}_j(x_j)\rangle = 0}$$

which is just the infinitesimal form of the global Ward–Takahashi identity under the symmetry generated by $Q$.

- If $M$ has a boundary, we pick up extra terms. For example, we saw earlier that correlation functions of $\phi$ vanish if they have nonzero $U(1)$ charge. In this more general context, they vanish unless their charge is equal to the difference of the initial and final $U(1)$ charges.
- A more subtle example is when $M$ is non-compact. In this case we must impose boundary conditions at infinity, e.g. that the field approaches a constant that is a minimum of the effective potential. This can result in spontaneous symmetry breaking.

**Note.** We may use the Ward–Takahashi identity to show that the conserved charge generates the symmetry transformation. We integrate over a spacetime region bounded by $t = t_-$ and $t = t_+$, where the time interval contains $x_1^0$ but none of the other $x_i^0$. Then

$$\langle Q(t_+)\mathcal{O}_1(x_1)Y\rangle - \langle Q(t_-)\mathcal{O}_1(x_1)Y\rangle = -\langle \delta\mathcal{O}_1(x_1)Y\rangle, \quad Y = \prod_{i=2}^{n}\mathcal{O}_i(x_i).$$

Taking the limits $t_- \to x_1^0$ and $t_+ \to x_1^0$, the left-hand side becomes a commutator because the correlators are time-ordered. Since $Y$ is arbitrary, we conclude

$$[Q, \mathcal{O}] = -\delta\mathcal{O}$$

as an operator equation in the Hilbert space.

Finally, we relate quantum symmetries to symmetries of the 1PI effective action.

- Under the infinitesimal symmetry $\phi \to \phi' = \phi + \epsilon f(\phi, \partial\phi)$ we have

$$\begin{aligned}\mathcal{Z}[J] &= \int \mathcal{D}\phi' \, \exp\left(-\frac{1}{\hbar}\left(S[\phi'] + \int_M dx\, J(x)\phi'(x)\right)\right) \\ &= \int \mathcal{D}\phi \, \exp\left(-\frac{1}{\hbar}\left(S[\phi] + \int_M dx\, J(x)\phi'(x)\right)\right) \\ &= \mathcal{Z}[J]\left(1 - \frac{\epsilon}{\hbar}\int_M dx\, J(x)\langle f(\phi, \partial\phi)\rangle_J\right)\end{aligned}$$

  where the expectation value is taken in the presence of the source $J$. Then we have

$$\int_M dx\, J(x)\langle f(\phi, \partial\phi)\rangle_J = 0.$$

- Next, we evaluate the current at

$$J_\Phi(y) = -\frac{\delta\Gamma[\Phi]}{\delta\Phi(y)}, \quad \langle\phi\rangle_{J_\Phi} = \Phi$$

  to give

$$\int_M dx\, \frac{\delta\Gamma[\Phi]}{\delta\Phi(x)}\langle f(\phi, \partial\phi)\rangle_{J_\Phi} = 0.$$

  Therefore, the effective action is invariant under the transformation

$$\Phi \to \Phi' = \Phi + \epsilon\langle f(\phi, \partial\phi)\rangle_{J_\Phi}$$

  which involves expectation values of the transformations, which can be rather complicated. This result is called a Slavnov–Taylor identity.

- When the symmetry is linear, i.e. $f$ is linear in the fields, we have

$$\langle f(\phi, \partial\phi)\rangle_{J_\Phi} = f(\Phi, \partial\Phi)$$

so the effective action is invariant under the transformation

$$\Phi \to \Phi' = \Phi + \epsilon f(\Phi, \partial\Phi).$$

Therefore, all symmetries of the classical action linear in the fields, under which the path integral measure is invariant, become symmetries of the quantum 1PI action. For example, this result ensures that a $\phi^3$ coupling cannot be generated in $\phi^4$ theory by $\mathbb{Z}_2$ symmetry, that $O(n)$ symmetry is preserved in the $O(n)$ model, and that Lorentz invariance is preserved.

# 9 Wilsonian Renormalization

## 9.1 Effective Actions

We now turn to the Wilsonian picture of quantum field theory.

**Note.** Consider a bead constrained to a steel hoop, in classical mechanics. The hoop itself has degrees of freedom, but they cost a huge amount of energy to excite. Then minimizing the action means that the hoop is in the ground state; plugging this back into the action gives a Lagrange multiplier than constrains the location of the bead. This is inherently a low-energy description, since the hoop can be excited by, e.g. smashing it with a sledgehammer.

In quantum field theory the situation is subtler, because we no longer have to sit at a minimum of the action. Instead, quantum fluctuations appear which involve degrees of freedom at arbitrarily high energies, e.g. in loop integrals. The content of renormalization is that it is still possible to calculate at low energies without knowing what those high-energy degrees of freedom are.

- We consider a scalar field theory with action

$$S_{\Lambda_0}[\varphi] = \int d^d x \, \frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi + \sum_i \Lambda_0^{d-d_i} g_{i0} \mathcal{O}_i(x).$$

  The subscript $\Lambda_0$ refers to the UV cutoff, which must be included to define the path integral. The $\mathcal{O}_i(x)$ are arbitrary local operators with dimensions $d_i > 0$, where the dimensions are found with ordinary dimensional analysis, and the $g_{i0}$ are defined so they are dimensionless.

- The partition function is

$$\mathcal{Z}_{\Lambda_0}(g_{i0}) = \int_{C^\infty(M)_{\le \Lambda_0}} \mathcal{D}\varphi \, e^{-S_{\Lambda_0}[\varphi]/\hbar}$$

  where $C^\infty(M)_{\le \Lambda_0}$ is the space of smooth functions on $M$ with momenta bounded by $\Lambda_0$. There are also IR divergences, which we can regulate by putting the system in a box.

- Since $C^\infty(M)_{\le \Lambda_0}$ is a vector space, we may split the modes as

$$\varphi(x) = \phi(x) + \chi(x), \quad \phi \in C^\infty(M)_{\le \Lambda}, \quad \chi \in C^\infty(M)_{(\Lambda, \Lambda_0]}, \quad \mathcal{D}\varphi = \mathcal{D}\phi \mathcal{D}\chi.$$

  Integrating over the high-energy modes $\chi$ gives the effective action

$$S_\Lambda^{\text{eff}}[\phi] = -\hbar \log \int_{C^\infty(M)_{(\Lambda, \Lambda_0]}} \mathcal{D}\chi \, \exp(-S_{\Lambda_0}[\phi + \chi]/\hbar).$$

  We can iterate this process, yielding a semigroup called the renormalization group. Note that the effective action $\mathcal{W}[J]$ is just a Wilsonian effective action with $\Lambda = 0$.

- Setting $\hbar = 1$, we may separate out the kinetic terms of the action as

$$S_{\Lambda_0}[\phi + \chi] = S^0[\phi] + S^0[\chi] + S^{\text{int}}_{\Lambda_0}[\phi, \chi]$$

  where there are no $\phi\chi$ mixing terms by momentum conservation. Therefore the interaction terms, which include mass terms, obey

$$\boxed{S^{\text{int}}_\Lambda[\phi] = -\log \int_{C^\infty(M)_{(\Lambda, \Lambda_0]}} \mathcal{D}\chi \, \exp(-S^0[\chi] - S^{\text{int}}_{\Lambda_0}[\phi, \chi])}$$

which yields effective couplings $g_i(\Lambda)$. Note that if we expand the right-hand side in powers of $S^{\text{int}}_{\Lambda_0}$, the even-power terms contribute with an extra minus sign, and the logarithm ensures we only sum over connected diagrams.

- By definition, we must have
$$\mathcal{Z}_\Lambda(g_i(\Lambda)) = \mathcal{Z}_{\Lambda_0}(g_{i0})$$
because both sides are the same integral; the left-hand side depends on $\Lambda$ through both the cutoff and the couplings. We can also include external currents to have partition functions $\mathcal{Z}[J]$. Then we have $\mathcal{Z}_\Lambda[J] = \mathcal{Z}_{\Lambda_0}[J]$ as long as $J$ has no Fourier components in $(\Lambda, \Lambda_0]$.

- Therefore, the total derivative with respect to $\log\Lambda$ is zero,
$$\frac{d\mathcal{Z}_\Lambda(g)}{d\log\Lambda} = \left(\frac{\partial}{\partial\log\Lambda}\bigg|_{g_i} + \frac{\partial g_i(\Lambda)}{\partial\log\Lambda}\frac{\partial}{\partial g_i}\bigg|_\Lambda\right)\mathcal{Z}_\Lambda(g) = 0.$$
This is an example of an RG equation, which we'll generically call a Callan-Symanzik equation.

- It will be useful to adjust our definitions to maintain canonical normalization of the kinetic term. With a generic initial action, the effective action will have the form
$$S^{\text{eff}}_\Lambda[\phi] = \int d^dx\,\frac{Z_\Lambda}{2}\partial^\mu\phi\partial_\mu\phi + \sum_i \Lambda^{d-d_i}Z_\Lambda^{n_i/2}g_i(\Lambda)\mathcal{O}_i(x)$$
where $n_i$ is the number of factors of $\phi$ in $\mathcal{O}_i(x)$, and we have redefined $g_i(\Lambda)$ to pull out factors of $Z_\Lambda$. We then define the renormalized field
$$\varphi = Z_\Lambda^{1/2}\phi$$
so that $S^{\text{eff}}_\Lambda[\varphi]$ has a canonically normalized kinetic term and dimensionless couplings $g_i(\Lambda)$.

- For convenience, we define the beta function
$$\beta_i(g_j(\Lambda)) = \frac{\partial g_i}{\partial\log\Lambda} = (d_i - d)g_i(\Lambda) + b_i^q(g_j).$$
where the first term is just from the variation of the explicit power of $\Lambda$, and the second represents the inherently quantum effect of integrating out high-energy modes. This term generically depends on all of the couplings; for example a $\phi^6$ vertex can renormalize a $\phi^4$ vertex by contracting two of the legs. It also depends on the field renormalization.

- We define the anomalous dimension $\gamma_\phi$ of the field $\phi$ by
$$\gamma_\phi = -\frac{1}{2}\frac{\partial\log Z_\Lambda}{\partial\log\Lambda}.$$
This is essentially what the beta function for the kinetic term would be, if we weren't fixing its normalization, and it appears in the RG flow of correlation functions. More generally, for multiple fields we would have a matrix of wavefunction renormalization factors, since the fields may mix as modes are integrated out. In practice, this matrix can be computed perturbatively, as we saw in the notes on Statistical Field Theory.

We now use our setup to compute correlation functions.

- We define the $n$-point correlator

$$\langle \phi(x_1) \dots \phi(x_n) \rangle = \frac{1}{\mathcal{Z}} \int_{C^\infty(M)_{\leq \Lambda}} \mathcal{D}\phi \, e^{-S_\Lambda^{\text{eff}}} \phi(x_1) \dots \phi_n(x_n).$$

  In terms of the canonically normalized field, this is

$$\langle \phi(x_1) \dots \phi(x_n) \rangle = Z_\Lambda^{-n/2} \langle \varphi(x_1) \dots \varphi(x_n) \rangle \equiv Z_\Lambda^{-n/2} \Gamma_\Lambda^{(n)}(x_1, \dots, x_n; g_i(\Lambda))$$

  where all other factors cancel out due to division by $\mathcal{Z}$.

- As long as the $\phi$ insertions involve modes with energy less than $\Lambda$, we can compute the $\phi$ correlator in either the original theory or the effective theory. Therefore we have

$$Z_{s\Lambda}^{-n/2} \Gamma_{s\Lambda}^{(n)}(x_1, \dots, x_n; g_i(s\Lambda)) = Z_\Lambda^{-n/2} \Gamma_\Lambda^{(n)}(x_1, \dots, x_n; g_i(\Lambda))$$

  for $s < 1$. Taking the differential gives the Callan-Symanzik equation

$$\frac{d}{d \log \Lambda} \Gamma_\Lambda^{(n)}(x_1, \dots, x_n; g_i(\Lambda)) = \left( \frac{\partial}{\partial \log \Lambda} + \beta_i \frac{\partial}{\partial g_i} + n\gamma_\phi \right) \Gamma_\Lambda^{(n)}(x_1, \dots, x_n; g_i(\Lambda)) = 0.$$

- It is also useful to consider an "autonomous" RG transformation, where the cutoff $\Lambda$ remains the same. To do this, we perform the change of variables

$$x' = sx, \quad \phi'(x') = s^{(2-d)/2} \phi(x)$$

  where the new field $\phi$ is chosen so that the kinetic term is invariant,

$$\int d^d x \, (\partial \phi(x))^2 = \int d^d x \, (\partial \phi'(x))^2 = \int d^d x' \, (\partial' \phi'(x'))^2.$$

  If $\phi(x)$ has UV cutoff $k = s\Lambda$, then $\phi'(x')$ has UV cutoff $k' = \Lambda$ where $k'$ is formally the Fourier conjugate to $x'$. Note that all of the *dimensionless* couplings are left invariant by this transformation. (Equivalently, everything dimensionful transforms with its engineering dimension.)

- At this point, the physical UV cutoff scale has not actually been changed; we have merely performed a change of coordinates. We next perform an active rescaling of the metric so that physical distances shrink by a factor of $s$. Then the new primed metric coordinates match the old unprimed metric coordinates.

- The composition of these steps (integrating out, changing to primed coordinates, actively rescaling the metric, renaming primed coordinates to unprimed) leaves $\Lambda$ and the metric invariant, but changes the dimensionless couplings from $g_i(\Lambda)$ to $g_i(s\Lambda)$. It is hence an autonomous RG transformation.

- It can be a bit puzzling to physically interpret the rescaling. In the nonrelativistic context, one might imagine pointing a camera at a physical sample. Then scaling the metric corresponds to "zooming out" the camera. In high energy physics, we would not include the scaling at all: we should not physically rescale the UV cutoff because it represents the scale of new physics. The calculational issues that would arise in the absence of rescaling are moot, because we do not use Wilsonian RG in practice for high energy physics anyway; we instead use "continuum RG". However, for the remainder of this section we will include the rescaling.

- Integrating out by a factor of $s$ and then rescaling by a factor of $s$ gives

$$\begin{aligned}\Gamma_{\Lambda}^{(n)}(x_1,\ldots,x_n;g_i(\Lambda)) &= \left(\frac{Z_\Lambda}{Z_{s\Lambda}}\right)^{n/2}\Gamma_{s\Lambda}^{(n)}(x_1,\ldots,x_n;g_i(s\Lambda)) \\ &= \left(s^{2-d}\frac{Z_\Lambda}{Z_{s\Lambda}}\right)^{n/2}\Gamma_{\Lambda}^{(n)}(sx_1,\ldots,sx_n;g_i(s\Lambda)).\end{aligned}$$

  Finally, replacing $x_i$ with $x_i/s$ gives

$$\Gamma_{\Lambda}^{(n)}(x_1/s,\ldots,x_n/s;g_i(\Lambda)) = \left(s^{2-d}\frac{Z_\Lambda}{Z_{s\Lambda}}\right)^{n/2}\Gamma_{\Lambda}^{(n)}(x_1,\ldots,x_n;g_i(s\Lambda)).$$

  For $s\to 0$, this says we can study the long-distance behavior of our theory by consider correlation functions at fixed separation, but using the couplings in the low-energy effective theory, which is intuitive.

- Taking the differential, every factor of the field scales as if the field had mass dimension

$$\Delta_\phi = (d-2)/2 + \gamma_\phi$$

  so the anomalous dimension is the difference between the scaling dimension and the naive classical dimension. Note that we pick up a minus sign here since mass dimension is inverse to length dimension.

## 9.2 RG Flow

Next, we consider the general picture of RG flow.

- A critical point is a point where all of the beta functions vanish, so the couplings are scale independent. A simple example is the Gaussian critical point, where $g_i^* = 0$. At a nontrivial fixed point, quantum effects exactly cancel the classical scaling; this usually requires strong coupling, since the classical terms are always $O(1)$.

- Now, the anomalous dimension $\gamma_\phi$ is only a function of the couplings, so at a critical point it is scale-invariant as well, $\gamma_\phi(g_i^*) = \gamma_\phi^*$. Then we have

$$\frac{\partial\Gamma_{\Lambda}^{(2)}(x,y)}{\partial\log\Lambda} = -2\gamma_\phi^*\Gamma_{\Lambda}^{(2)}(x,y)$$

  so that $\Gamma^{(2)}\propto\Lambda^{-2\gamma_\phi^*}$. By Lorentz invariance, correlation functions only depend on $|x-y|$. By classical dimensional analysis,

$$\langle\phi(x)\phi(y)\rangle = \Lambda^{d-2}G(\Lambda|x-y|,g_i^*).$$

  Combining these results together, we have

$$\Gamma_{\Lambda}^{(2)}(x,y;g_i^*) = \frac{\Lambda^{d-2}}{\Lambda^{2\Delta_\phi}}\frac{c(g_i^*)}{|x-y|^{2\Delta_\phi}}.$$

  This power-law behavior of correlation functions is typical at critical points; this is just a special case of our more general result above.

- Performing an RG step at a critical point leaves the partition function invariant, since it leaves the couplings invariant. On the other hand, an RG step is composed of (1) integrating out degrees of freedom, (2) changing to primed coordinates, and (3) rescaling the metric. The first two never change the partition function by definition. Hence at a critical point, the partition function is invariant under rescaling the metric alone, $g^{\mu\nu} \to e^{2\Omega} g^{\mu\nu}$. Now we have

$$0 = \delta g^{\mu\nu}(x) \frac{\delta \log \mathcal{Z}}{\delta g^{\mu\nu}(x)} = -\delta g^{\mu\nu}(x) \left\langle \frac{\delta S}{\delta g^{\mu\nu}(x)} \right\rangle \propto \delta g^{\mu\nu}(x) \langle T_{\mu\nu}(x) \rangle$$

  so at an RG critical point, the spacetime integral of $\langle T^{\mu}{}_{\mu} \rangle$ is zero.

- All known examples of Lorentz invariant, unitary QFTs that are scale invariant are actually invariant under conformal transformations $g^{\mu\nu} \to e^{2\Omega(x)} g^{\mu\nu}$. It is believed this is true in general; it has been proven in $d = 2$ and is an open question in high dimensions. Assuming this holds, at RG critical points $\langle T^{\mu}{}_{\mu}(x) \rangle = 0$.

We now linearize about a critical point.

- Near a critical point, at $g_i^* + \delta g_i$, the beta functions are

$$\beta_i = B_{ij} \delta g_j + O(\delta g^2).$$

  Let $\sigma_i$ be an eigenvector of $B_{ij}$ and let its eigenvalue be $\Delta_i - d$, so the coupling scales as

$$\sigma_i(\Lambda) = \left( \frac{\Lambda}{\Lambda_0} \right)^{\Delta_i - d} \sigma_i(\Lambda_0).$$

  The definition of $\Delta_i$ is slightly different from that of $\Delta_\phi$ because we are taking the couplings to be dimensionless but the field to be dimensionful.

- Classically, we would have $\Delta_i = d_i$, but more generally $\sigma_i$ has an anomalous dimension

$$\gamma_i = \Delta_i - d_i$$

  which is defined analogously to the anomalous dimension for $\phi$. For the Gaussian fixed point, all anomalous dimensions are zero, as we'll show below.

- It is a bit of an approximation to assign $\sigma_i$ a classical dimension, because in general the $\sigma_i$ will be complicated linear combinations of all the operators. However, when the corrections are 'weak' the off-diagonal elements of $B_{ij}$ are small compared to the classical diagonal terms, and the eigenvectors are close to the classical ones.

- Now imagine starting near a critical point and turning on the coupling to any operator with $\Delta_i > d$. Then the coupling becomes smaller as the scale $\Lambda$ is lowered, so we say the corresponding operator is irrelevant, as it does not affect long-distance physics; we just flow back to the critical point.

- More generally, the critical surface $\mathcal{C}$ is the set of points that flow back to the critical point under RG flow; then the irrelevant operators provide coordinates for $\mathcal{C}$ in the neighborhood of the critical point.

- Couplings with $\Delta_i < d$ instead grow as the scale is lowered, and are called relevant; RG flow instead drives us away from the critical surface. This flow may eventually terminate at a different critical point, or in exotic cases perform a limit cycle. Since each new field or derivative increases $\Delta_i$ classically, there are finitely many relevant operators for fixed $d$.
- In the classical approximation, and the convention where couplings have dimensions, irrelevant operators have couplings with negative mass dimension, and relevant operators have couplings with positive mass dimension. Then it is clear that a mass term is relevant.
- A marginal coupling is an RG eigenvector with $\Delta_i = d$, or equivalently about the Gaussian fixed point, $d_i = d$. In this case, we need to expand to second order about the fixed point, yielding a weak, logarithmic dependence on $\Lambda$. Depending on the sign of the next term, the coupling is marginally irrelevant or marginally relevant. Note that the kinetic term is exactly marginal by definition.

**Note.** Consider calculating the beta function to first order about the Gaussian fixed point. This is equivalent to considering diagrams with only one vertex, so renormalization only goes one way. For example, $\phi^6$ can renormalize $\phi^4$ but not vice versa. The matrix $B_{ij}$ is thus upper-triangular with the classical values on the diagonal, so all anomalous dimensions are zero. The only effect is that the eigenvectors are slightly tilted.

We must go to second order to figure out what classically marginal operators do. Note that in most practical computations in particle physics, we are implicitly expanding about the Gaussian fixed point. Also note that while the dimensionless couplings change, the dimensionful couplings are approximately constant.

**Note.** Given a term $\lambda_i \mathcal{O}_i$ in a weakly coupled field theory, its contribution to the action for a field configuration of energy $E \ll \Lambda$ is $\lambda_i E^{\Delta_i - d}$ by dimensional analysis. We don't use $\Lambda$ here because the field configuration can't know what $\Lambda$ is. To show this more quantitatively, consider a field configuration with lengthscale $L \sim 1/k$ and dimensionless amplitude $\hat{\phi} = \phi/k$. Then

$$\int dx\, (\partial\phi)^2 \sim \hat{\phi}^2, \quad \int dx\, m^2\phi^2 \sim \frac{m^2}{k^2}\hat{\phi}^2, \quad \int dx\, (\partial\phi)^p \phi^q \sim k^{2p+q-4}\hat{\phi}^{p+q}.$$

Assuming the kinetic term dominates, when we perform the path integral, configurations with $\hat{\phi} \sim 1$ dominate. Then the contribution of a term depends on $k$ just as our dimensional analysis suggests.

Then a dimensionless coupling $g_i$ has effect

$$\Delta S_i \sim g_i \left(\frac{E}{\Lambda}\right)^{\Delta_i - d}$$

where the kinetic term has $\Delta S \sim 1$. This measures the effect of the term on observable quantities derived from the action, such as cross sections. The tree-level RG evolution of $g_i$ simply states that $\Delta S_i$ is independent of $\Lambda$. Because loops are sensitive to $\Lambda$, anomalous dimensions appear at loop-level.

Relevant terms such as masses are more important at $E \ll \Lambda$, while all irrelevant terms become important as $E$ approaches $\Lambda$. For marginal terms, which are the majority of couplings we consider, the value of $g_i$ itself is a good estimate of its effect, so perturbation theory really does break down when $g_i$ is $O(1)$.

**Example.** For a scalar field $d = 4$ with $\mathbb{Z}_2$ symmetry near the gaussian fixed point, we have

$$\text{relevant: } 1, \phi^2, \quad \text{marginal: } \phi^4, (\partial\phi)^2$$

and everything else is irrelevant. The physical mass is independent of the cutoff, as one would expect. In $d = 3$, $\phi^4$ becomes relevant while $\phi^6$ becomes marginal. In $d = 2$, the field is dimensionless, so

$$\text{relevant: } 1, \phi^2, \phi^4, \ldots, \quad \text{marginal: } (\partial\phi)^2, \phi^2(\partial\phi)^2, \ldots.$$

That is, we get an infinite series of relevant terms, which is quite rare.

**Note.** The origin of universality. The theory space is generically infinite-dimensional. Now consider the finite-dimensional set of theories obtained by starting from an RG fixed point, turning on a relevant operator, and performing RG flow, creating a 'renormalized trajectory'. By the discussion above, every initial condition will approach a renormalized trajectory, so theories in the IR are described by a finite number of parameters.

If we continue RG flowing into the deep IR, we often end up at a trivial fixed point, like the "infinite temperature" fixed point in statistical field theory, since we integrate out all massive particles. Alternatively, we may end up at a nontrivial fixed point/CFT, such as the Wilson–Fisher fixed point, where the particles are massless; renormalization exactly cancels the tree-level mass.

We now consider a converse question, more relevant to particle physics: is it possible to fix a low-energy theory at scale $\Lambda$ while sending the cutoff $\Lambda_0$ to infinity?

- The simplest case is when $S_{\Lambda_0}$ is on the critical surface $\mathcal{C}$. Then taking $\Lambda_0 \to \infty$ gives $S_\Lambda^{\text{eff}}$ exactly at the critical point. Since $\mathcal{C}$ has finite codimension, we have to tune a finite number of parameters to do this, e.g. the temperature in a statistical field theory.

- Theories such as QCD and Yang–Mills are not CFTs, but instead have relevant and marginally relevant terms in their actions. In this case, to fix the low-energy theory we must modify the high-energy theory as $\Lambda_0$ varies, which is done perturbatively by adding counterterms. For example, we could parametrize the high-energy theory as

  $$S_{\Lambda_0}[\varphi] = S_\Lambda^{\text{eff}}[\varphi] + \hbar S_{\text{CT}}[\varphi, \Lambda_0]$$

  so that the counterterms vanish when $\Lambda = \Lambda_0$. The counterterms are tuned so that the limit

  $$e^{-S_\Lambda^{\text{eff}}[\phi]/\hbar} = \lim_{\Lambda_0 \to \infty} \int_{C^\infty(M)_{(\Lambda,\Lambda_0]}} \mathcal{D}\chi \, \exp\left( -\frac{S_\Lambda^{\text{eff}}[\phi + \chi]}{\hbar} - S_{\text{CT}}[\phi + \chi, \Lambda_0] \right)$$

  exists. We separate out the counterterms explicitly to organize perturbation theory. In perturbation theory, we work order by order in $\hbar$, canceling the (regularized) divergences produced by the bare action $S_\Lambda^{\text{eff}}$ by counterterms. The extra factor of $\hbar$ is because the one-loop divergences are canceled by tree-level counterterm diagrams, and so on.

- We need our low-energy theory to lie on a renormalized trajectory. Then the counterterms are chosen so that, as $\Lambda_0$ is increased, $S_{\Lambda_0}$ gets closer and closer to the critical point along the renormalized trajectory. Then there's one counterterm for each relevant operator, as we've seen perturbatively, and the continuum limit is the critical point.

- The existence of relevant operators leads to a fine-tuning problem. Suppose we have a very high, fixed cutoff $\Lambda_0 = \Lambda_{\text{Planck}}$. Then if the dimensionless coupling at scale $\Lambda$ is not extremely large, the dimensionless coupling at scale $\Lambda_0$ must be extremely small. This is 'unnatural', because we expect a fundamental theory, where the couplings can be calculated, to have $O(1)$ dimensionless couplings. Note there's no problem with having very large or very small dimensionless couplings in an *effective* theory.
- Marginally relevant operators aren't nearly as problematic as $\log(\Lambda_{\text{Planck}}/1\,\text{TeV}) \approx 40$ isn't huge, where 1 TeV is about the minimum possible cutoff $\Lambda$ for the Standard Model.
- In the case of a mass term, the only way to avoid unnaturalness would be to have $m \sim \Lambda_0$, i.e. we need particles so heavy we can't even produce them at $\Lambda$ in the first place.
- More generally, we could include finite coefficients for irrelevant operators in $S_{\Lambda_0}$. This affects the required tuning for the relevant operators, since the two will renormalize each other. However, the effects of the irrelevant operators will disappear exactly in the limit $\Lambda_0 \to \infty$. The point is that we only need one fine tuning for each relevant parameter.
- Note that generically, we should always think of every operator as being present. Even the points on a renormalized trajectory don't only have the relevant couplings turned on; instead these relevant couplings immediately turn on all others, so in general every coupling is nonzero, though the irrelevant operators remain small because they decay away as they're produced.
- As a result, the true coefficient of an irrelevant operator in the IR isn't nearly as small as naive scaling suggests. The real meaning of "irrelevant" is that the value of the coupling in the UV is irrelevant to the value in the IR, since the quick initial decay washes it out; irrelevant operators are still present in $S_\Lambda^{\text{eff}}$ even when $\Lambda_0 \to \infty$, but with small coefficients.

Now suppose that our low-energy effective theory contains an irrelevant operator with a large coefficient, which is often required to match experimental results.

- In this case, there is no hope of our theory lying on a renormalized trajectory. Since there are infinitely many irrelevant operators, which generically all renormalize each other, we now need an infinite number of counterterms and an infinite number of renormalization conditions.
- Moreover, the counterterm coefficients apparently diverge in the limit $\Lambda_0 \to \infty$, so we cannot take the continuum limit. Instead, we say the theory comes with a cutoff $\Lambda_{\text{cutoff}}$.
- Specifically, $\Lambda_{\text{cutoff}}$ is the scale where the dimensionless irrelevant couplings are $O(1)$. Now, the theory is perfectly predictive at energies $E \ll \Lambda_{\text{cutoff}}$, because the effects of more irrelevant terms are suppressed by more powers of $E/\Lambda_{\text{cutoff}}$, so we can truncate the Lagrangian after a few terms. On the other hand, perturbation theory breaks down as $E$ approaches $\Lambda_{\text{cutoff}}$. All of the terms become important at once, and we cannot determine all of the couplings with finite data, so the theory is not predictive.
- Generally, what must happen is that new physics takes over. This means that we switch to a new set of degrees of freedom (i.e. new fields, or even non-fields, such as in the Ising model) where the theory is either renormalizable or effective with a higher cutoff, at which point we can truncate the Lagrangian and do calculations again.

- Another possibility is a Landau pole, where a coupling diverges at finite energy, such as the marginally irrelevant coupling in QED. In practice this just means that new physics must take over before then. If it doesn't, then the theory is 'quantum trivial', meaning that the only way for it to have a continuum limit is for the coupling to be exactly zero.

- The usual exposition of the Landau pole uses a perturbatively computed beta function. This isn't valid since perturbation theory breaks down; instead we must establish a Landau pole nonperturbatively, through lattice simulations. Simulations of $\phi^4$ theory indicate a Landau pole indeed exists, so the theory is quantum trivial. On the other hand, simulations of QED indicate that we don't hit the Landau pole; instead chiral symmetry breaking occurs.

- The Standard Model is also suspected to be quantum trivial, through the marginally irrelevant quartic Higgs interaction. Of course, this isn't the main reason we suspect the Standard Model to be incomplete, or even a minor reason. A more compelling reason is the hierarchy problem: the Higgs mass is relevant, so as the usual logic goes, the Standard Model must be extended at the TeV scale to preserve naturalness. A critical evaluation of this argument is given here.

- A final possibility is that we simply flow to some unknown UV fixed point. This is the hypothesis of the asymptotically safe approach to quantum gravity. In this case, new physics need not take over, and our description can be valid up to arbitrarily high energies.

- All of these issues also occur in reverse for relevant couplings, such as in QCD, where the coupling would naively hit a Landau pole at $\Lambda_{\text{QCD}} \sim 1\,\text{GeV}$. In this case, confinement and chiral symmetry breaking occur; we must describe the dynamics in terms of hadrons rather than quarks and gluons.

**Note.** Relating our results to the old picture of perturbative renormalization.

- In the old picture, the counterterms are divergent order by order in $\hbar$. Here, they are perfectly finite. This is simply because limits don't commute with sums; for instance,

$$\lim_{x\to\infty} e^{-x} = 0$$

  but all the individual terms in the Taylor series for $e^{-x}$ diverge as $x \to \infty$. If we always maintain a finite cutoff and do the path integral exactly, there are never any divergences.

- Before, we thought of the regularization scale $\Lambda_0$ as an unphysical scale that had to be sent to infinity at the end of the calculation. Now, we think of it as a physical cutoff, beyond which might lie new physics. Taking $\Lambda_0$ to infinity is not essential, and not physical.

- Before, in renormalized perturbation theory, we thought of a bare Lagrangian as composed of a renormalized Lagrangian plus counterterms. Now, we identify the bare Lagrangian with the fundamental theory, at scale $\Lambda_0$, and the renormalized Lagrangian with the effective theory, at scale $\Lambda$.

- Before, we thought of irrelevant (nonrenormalizable) operators as dangerous and relevant (superrenormalizable) operators as benign. Now, we see that turning on an irrelevant operator changes nothing in the IR, while relevant operators present fine-tuning problems.

- Before, we identified a renormalizable theory as one that required only a finite number of counterterms to absorb the infinities. Now, we think of a renormalizable theory as one that admits a sensible continuum limit.

**Note.** Conventions differ between SFT and high energy QFT, for physical reasons. In SFT, we typically know the UV theory and want to find the IR behavior, so we keep $\Lambda$ the same by rescaling; the only thing that matters about the atomic cutoff is that it's very far away. In high energy physics, we know the IR behavior and are trying to find the UV behavior, so we keep the variation of $\Lambda$ explicit; this is what we've done above.

There is also another convention, which is whether to work with dimensionless couplings by pulling out factors of $\Lambda$, or to work with dimensionful couplings. For concreteness, consider $\mathcal{L} \supset m^2\phi^2$. In a free theory, integrating out UV degrees of freedom keeps the dimensionful coupling $m^2$ the same. However, one could also consider the dimensionless coupling $g = m^2/\Lambda^2$ (common in QFT), or stick with dimensionful couplings but keep $\Lambda$ the same by rescaling (common in SFT). These are essentially the same idea phrased in different ways. In both cases, the coupling will grow.

Philosophically, in SFT, the slogan is to "find something relevant". This is because we're interested in macroscopic behavior, and the atomic scale is so small that all effects of irrelevant operators are completely undetectable. The fact that relevant operators require fine tuning is acceptable because there is fine tuning in the lab, e.g. by tuning the temperature to a phase transition. In QFT, we already know the low-energy physics, and irrelevant operators are useful as a window into higher-energy physics. Relevant operators are a larger problem because there is nothing outside the universe that does the tuning.

**Note.** Before, we've argued that contributions to the action from various operators go as $g_i(E/\Lambda)^{\Delta_i - d}$. However, naive power counting can break down because of loops. For instance, in scalar field theory with a hard cutoff, a $\phi^6$ loop renormalizes the $\phi^4$ coupling at one-loop as

$$\delta g_4 \sim \frac{g_6}{\Lambda^2}\int^{\Lambda} \frac{đk}{k^2 - m^2} \sim g_6$$

which is too large. This occurs because only the external legs are fixed at energy $E$, while particles in the loop can go up to $\Lambda$. But if we regulate with a mass-independent scheme such as DR, we get a result compatible with power counting,

$$\delta g_4 \sim \frac{g_6\mu^{2\epsilon}}{\Lambda^2}\int \frac{đ^{4-\epsilon}k}{k^2 - m^2} \sim \frac{g_6}{\epsilon}\frac{m^2}{\Lambda^2} + g_6\frac{m^2}{\Lambda^2}\log\frac{m^2}{\mu^2} = O(m^2/\Lambda^2).$$

That is, DR removes the $\Lambda^2$ power divergence entirely; intuitively DR "only sees logarithmic divergences", which show up as $1/\epsilon$ terms that are subtracted out. Thus, if we don't use something like DR, it is difficult to self-consistently truncate the Lagrangian in an EFT. It is still possible, but one has to carefully keep track of counterterms, which can remove the contributions that violate naive power counting. (In this sense, it's the same as using a regulator that violates a symmetry: it's perfectly legal, but requires counterterms to be added with care.) For this reason, $\overline{\text{MS}}$ is used almost universally in practical EFT calculations.

**Note.** It is tempting to say that if a dimension-6 operator is measured with coefficient $1/\Lambda^2$, then new physics appears at scale $\Lambda$. This is deceptive, because operators come with UV couplings. What we really can conclude is that if nothing UV completes the theory, then the theory becomes strongly coupled at scale $\Lambda$. It is perfectly possible for new particles to appear at a lower scale. Indeed, there are many very small couplings in the SM, such as the electron Yukawa coupling, to which this reasoning applies.

## 9.3 Calculating RG Evolution

Next, we present an explicit, though somewhat impractical method for computing RG evolution.

- Concretely, we would like to evaluate

$$S_\Lambda^{\text{int}}[\phi] = -\log \int_{C^\infty(M)_{(\Lambda,\Lambda_0]}} \mathcal{D}\chi \, \exp(-S^0[\chi] - S_{\Lambda_0}^{\text{int}}[\phi+\chi])$$

  but this is hard in general, as the right-hand side is a general interacting path integral. The general approach is to expand $e^{-S_{\Lambda_0}^{\text{int}}[\phi+\chi]}$ as a series in $\chi$,

$$e^{-S_{\Lambda_0}^{\text{int}}[\phi+\chi]} = e^{-S_{\Lambda_0}^{\text{int}}[\phi]} + \int dx\, \chi(x) \frac{\delta}{\delta\phi(x)} e^{-S_{\Lambda_0}^{\text{int}}[\phi]} + \frac{1}{2}\int dxdy\, \chi(x)\chi(y) \frac{\delta}{\delta\phi(x)}\frac{\delta}{\delta\phi(y)} e^{-S_{\Lambda_0}^{\text{int}}[\phi]} + \dots.$$

  then perform the $\chi$ path integral. Note that we're Taylor expanding in position space, since eventually we want an action in position space.

- When we perform the $\chi$ path integral, we get factors of the position space propagator,

$$D_\Lambda(x,y) = \int_{\Lambda<|p|\leq\Lambda_0} đp \, \frac{e^{ip(x-y)}}{p^2+m^2}.$$

  Now the trick is to consider an infinitesimal RG step, $\Lambda = \Lambda_0 - \delta\Lambda$, so that

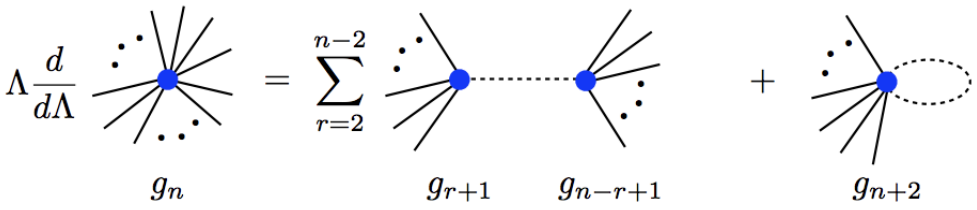
$$D_\Lambda(x,y) = \frac{1}{(2\pi)^d}\frac{\Lambda^{d-1}\delta\Lambda}{\Lambda^2+m^2}\int_{S^{d-1}} d\Omega\, e^{i\Lambda\hat{p}(x-y)}.$$

  This is an enormous improvement, because we only need keep track of terms linear in $\delta\Lambda$, which means we only want diagrams with a single $\chi$ propagator.

- Therefore, only the zeroth and second order terms in the Taylor expansion matter, and an explicit calculation yields Polchinski's equation,

$$-\frac{\partial S_\Lambda^{\text{int}}[\phi]}{\partial \log\Lambda} = \int dxdy \, \frac{\delta S_\Lambda^{\text{int}}}{\delta\phi(x)} D_\Lambda(x,y) \frac{\delta S_\Lambda^{\text{int}}}{\delta\phi(y)} - D_\Lambda(x,y)\frac{\delta^2 S_\Lambda^{\text{int}}}{\delta\phi(x)\delta\phi(y)}.$$

  More concretely, let $g_n$ be the coefficient of $\phi^n$. Then schematically we have




  which corresponds with the intuition of zooming out, shrinking $\chi$ propagators to points.

- Note that since the action is local, we must have $x = y$ in the second term to get a nonzero result. On the other hand, we can have $x \neq y$ in the first term, leading to non-local contributions, as the $\chi$ propagator falls off as $e^{-\sqrt{\Lambda^2+m^2}r}/r^{d-3}$. As usual, we simply Taylor expand to get a series of local terms with derivatives.

- Note that the variation of the action is

$$\frac{\delta S}{\delta\phi(x)} \equiv \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} + \dots$$

  so Polchinski's equation indeed accounts for the derivative terms.

- It's convenient to rewrite Polchinski's equation as

$$\frac{\partial}{\partial t} e^{-S^{\text{int}}[\phi]} = -\int dxdy\, D_\Lambda(x,y) \frac{\delta^2}{\delta\phi(x)\delta\phi(y)} e^{-S^{\text{int}}[\phi]}, \quad t = \log\Lambda$$

  which has the general form of a heat equation, with the 'Laplacian'

$$\Delta = \int dxdy\, D_\Lambda(x,y) \frac{\delta^2}{\delta\phi(x)\delta\phi(y)}.$$

  The eigenfunctions of the Laplacian grow or decay exponentially under RG flow; they are simply the RG eigenvectors we've seen earlier.

Next, we use the local potential approximation to explicitly compute RG evolution.

- We work in $d > 2$ near the Gaussian fixed point. Then besides the kinetic term, all operators involving derivatives are irrelevant, so we restrict attention to actions of the form

$$S_\Lambda^{\text{eff}}[\phi] = \int d^dx\, \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + V(\phi), \quad V(\phi) = \sum_k \Lambda^{d-k(d-2)} \frac{g_{2k}}{(2k)!}\phi^{2k}$$

  where for simplicity we've imposed $\mathbb{Z}_2$ symmetry. Neglecting the derivative interactions is called the local potential approximation, and works for slowly varying fields.

- Next, we set up our approximation somewhat differently. Before, we expanded $e^{-S_\Lambda^{\text{int}}[\phi+\chi]}$, yielding a Gaussian path integral for $\chi$. Now we instead write

$$S_\Lambda^{\text{eff}}[\phi+\chi] = S_\Lambda^{\text{eff}} + \int dx\, \frac{1}{2}(\partial\chi)^2 + \frac{1}{2}\chi^2 V''(\phi) + \dots$$

  so that we maintain a nontrivial action for $\chi$.

- We now consider an infinitesimal RG step $\delta\Lambda$ and work in momentum space. Then the momentum-space propagators are not small, but every loop picks up a factor of $\delta\Lambda$, so we only consider one-loop graphs. Note that there are no tree level graphs at all since the action is even in $\chi$, though we could also relax this assumption.

- We claim that one-loop graphs can only contain the $\chi^2$ vertex. To prove this rigorously, let there be $v_i$ copies of the $\chi^i$ vertex. Then Euler's identity gives

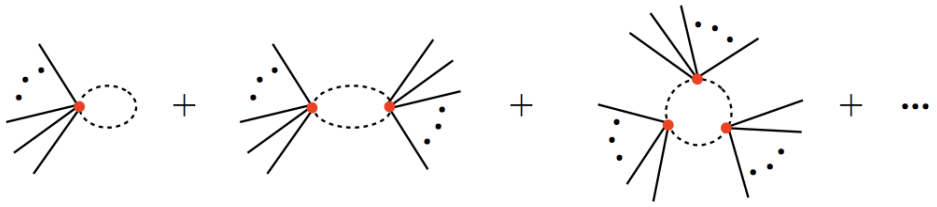
$$e - \sum_i v_i = \ell - 1$$

  where $\ell$ is the number of loops. Since there are no external lines,

$$2e = \sum_i i v_i, \quad \ell = 1 + \sum_i \frac{i-2}{2} v_i.$$

  Therefore only $v_2$ can be nonzero, and we recover a quadratic action for $\chi$, which is tractable.

- The types of diagrams that contribute are shown below.



Now, suppressing the position-dependence of $\phi$, the action for $\chi$ becomes

$$\begin{aligned} S^{(2)}[\chi,\phi] &= \int_{\Lambda-\delta\Lambda<|p|\leq\Lambda} đp\,\tilde{\chi}(p)\left(\frac{1}{2}p^2+\frac{1}{2}V''(\phi)\right)\tilde{\chi}(-p) \\ &= \frac{1}{2}\frac{\Lambda^{d-1}\delta\Lambda}{(2\pi)^d}(\Lambda^2+V''(\phi))\int_{S^{d-1}} d\Omega\,\tilde{\chi}(\Lambda\hat{p})\tilde{\chi}(-\Lambda\hat{p}). \end{aligned}$$

- Performing the Gaussian integration, we find

$$e^{-\delta\Lambda S^{\text{eff}}[\phi]} = \int \mathcal{D}\chi\, e^{-S^{(2)}[\chi,\phi]} = C\left(\frac{\pi}{\Lambda^2+V''(\phi)}\right)^{N/2}$$

where $N$ is the number of momentum modes. To regularize this, we put the theory in a box of side length $L$ and use periodic boundary conditions, giving

$$N = \frac{\text{vol}(S^{d-1})}{(2\pi)^d}\Lambda^{d-1}\delta\Lambda\, L^d.$$

We can thus rewrite our result as

$$\delta_\Lambda S^{\text{eff}}[\phi] = a\Lambda^{d-1}\delta\Lambda\int dx\,\log(\Lambda^2+V''(\phi(x)))$$

where we restored the position-dependence of $\phi$, turning a factor of $L^d$ into an integral over $x$.

- Finally, expanding the logarithm gives an infinite series of corrections,

$$\frac{dg_{2k}}{d\log\Lambda} = (k(d-2)-2)g_{2k} - a\Lambda^{k(d-2)}\frac{\partial^{2k}}{\partial\phi^{2k}}\log(\Lambda^2+V''(\phi))\Big|_{\phi=0}$$

where the first term is from the classical scaling. For example, the first two terms are

$$\frac{dg_2}{d\log\Lambda} = -2g_2 - \frac{ag_4}{1+g_2}, \quad \frac{dg_4}{d\log\Lambda} = (d-4)g_4 - \frac{ag_6}{a+g_2} + \frac{3ag_4^2}{(1+g_2)^2}.$$

For example, we see that in $d=4$, $g_4$ is marginally irrelevant.

- Looking back, we've traded one kind of complexity for another. In the local potential approximation, the beta functions are exact at one loop, but can contain arbitrarily high powers of the couplings. In Polchinski's equation, the beta functions are also exact at one loop, and are linear in the couplings, but we need to account for all derivative terms. One can think of the local potential approximation as solving for the derivative term couplings in terms of the non-derivative ones, and plugging them back into the beta functions to eliminate them.

Using our result, we can explicitly look at some simple RG flows.

- First, we consider the Gaussian fixed point. Linearizing the beta function about it gives

$$\beta_{2k} = (k(d-2)-d)g_{2k} - ag_{2k+2}.$$

  Thus as anticipated, there are no anomalous dimensions since the matrix $B$ in $\beta_i = B_{ij}g_j$ is upper-triangular, though the eigenvectors are mixed a bit.

- In $d = 4$, we see that the mass term is relevant, while $\phi^6$ and higher couplings are irrelevant, as expected. To understand $g_4$, we expand to quadratic order for

$$\beta_4 = \frac{3}{16\pi^2}g_4^2$$

  so that the quartic interaction is marginally irrelevant; explicitly,

$$g_4(\Lambda) = \frac{16\pi^2}{3\log(\Lambda_0/\Lambda)}$$

  where $\Lambda_0$ is an integration constant, we need $\Lambda_0 > \Lambda$ for $g_4 > 0$, ensuring stability, and the theory has a Landau pole at $\Lambda_0 = \Lambda$.

- The fact that a dimensionless coupling implies an energy scale is known as dimensional transmutation. We've already seen this for irrelevant couplings, which give a natural cutoff for the effective field theory; marginal couplings have the scale appear in a logarithm. There's nothing puzzling about this, since all theories come with a natural energy scale, the cutoff $\Lambda$.

- We can find another critical point in $d = 4 - \epsilon$, the Wilson–Fisher fixed point, where

$$g_2 = -\frac{1}{6}\epsilon + O(\epsilon^2), \quad g_4 = \frac{1}{3a}\epsilon + O(\epsilon^2), \quad g_{2k} = O(\epsilon^2) \text{ for } k > 2.$$

  Staying near $d = 4$, where $g_4$ is marginal, allows us to find a nontrivial fixed point without requiring strong coupling.

- Linearizing about the fixed point gives

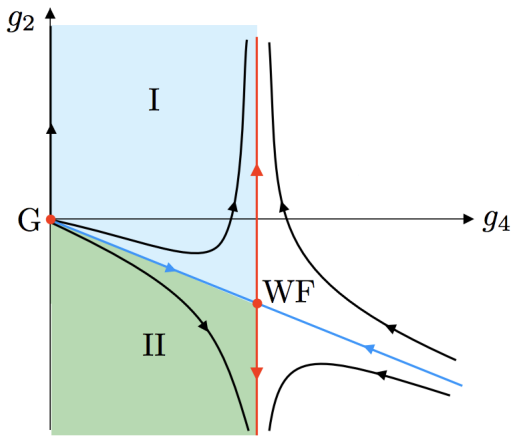
$$\frac{d}{d\log\Lambda}\begin{pmatrix}\delta g_2\\ \delta g_4\end{pmatrix} = \begin{pmatrix}\epsilon/3-2 & -a(1+\epsilon/6)\\ 0 & \epsilon\end{pmatrix}\begin{pmatrix}\delta g_2\\ \delta g_4\end{pmatrix}$$

  which has eigenvalues $\epsilon/3 - 2$ and $\epsilon$, so only one direction is relevant. Explicitly, we have

$$a = \frac{1}{(4\pi)^{d/2}}\frac{1}{\Gamma(d/2)}\bigg|_{d=4-\epsilon} = \frac{1}{16\pi^2} + \frac{\epsilon}{32\pi^2}(1-\gamma+\log 4\pi) + O(\epsilon^2).$$

  More sophisticated techniques can be used to show that the Wilson–Fisher fixed point indeed exists in $d = 3$ and $d = 2$.

- The RG flows in three dimensions are shown below.




Theories in region I flow to an interacting massive theory in the IR, and have the Gaussian theory as the continuum limit. Theories in region II are similar, but exhibit spontaneous symmetry breaking since $g_2 < 0$. Theories on the red line have the Wilson–Fisher fixed point as their continuum limit, while all others have no continuum limit at all.

- The situation in two dimensions in more complicated, as all of the operators $\phi^{2n}$ are marginal. It turns out there are infinitely many fixed points, where the $n^{\text{th}}$ fixed point can be reached from the Gaussian fixed point by turning on $\phi^{2(n+1)}$. The $n^{\text{th}}$ fixed point has $n$ relevant operators, which are essentially $\phi^2, \phi^4, \ldots, \phi^{2n}$. This result can be derived by conformal field theory.

## 9.4 Effective Field Theories

In this section, we give an array of examples of effective field theories, illustrating the general way of thinking about theories in the Wilsonian picture.

**Example.** In some cases, we can pick up extra symmetries just from the requirement of renormalizability; such symmetries are called "accidental" or "emergent". A familiar example is the emergence of rotational symmetry for a lattice spin system. As another example, consider electromagnetism with several fermions, where the most general renormalizable Lagrangian is

$$\mathcal{L} = \frac{1}{4e^2} Z_3 F^{\mu\nu} F_{\mu\nu} + (Z_L)_{ij} \overline{\psi}_{L_j} \not{D} \psi_{L_i} + (Z_R)_{ij} \overline{\psi}_{R_j} \not{D} \psi_{R_i} + M_{ij} \overline{\psi}_{L_i} \psi_{R_j} + \overline{M}_{ij} \overline{\psi}_{R_i} \psi_{L_j}.$$

Here, $Z_{L/R/3}$ are general wavefunction renormalizations. To have a real Lagrangian, we require $Z_{L/R}$ to be Hermitian. Note that we have no Majorana masses since the fields are charged.

We can dramatically simplify the Lagrangian by defining

$$\psi_L = S_L \psi'_L, \quad \psi_R = S_R \psi'_R$$

so the Lagrangian for the primed fields has

$$Z'_L = S_L^\dagger Z_L S_L, \quad Z'_R = S_R^\dagger Z_R S_R, \quad M' = S_L^\dagger M S_R.$$

Then it is possible to choose $S_L$ so that $Z'_L$ is the identity, and similarly choose $S_R$ so that $Z'_R$ is the identity, giving the usual kinetic terms. We can still redefine $S_{L/R}$ by unitaries, which we can use to make $M$ diagonal. Dropping the primes,

$$\mathcal{L} = \frac{1}{4e^2} Z_3 F^{\mu\nu} F_{\mu\nu} + \sum_i \overline{\psi}_i (\not{D} + m_i) \psi_i.$$

Then the number of $\psi_i$ particles is conserved, but this is an accidental symmetry; it could be broken by a nonrenormalizable term such as $Y_{ijkl}\overline{\psi}_i\gamma^\mu\psi_j\overline{\psi}_k\gamma_\mu\psi_\ell$. In the Standard Model, this is the origin of baryon and lepton number symmetry.

**Note.** More generally, it is believed that there are no continuous global symmetries in a quantum theory of gravity. Then from this perspective, all continuous symmetries we've found in particle physics are accidental! One piece of evidence in favor of this picture is Hawking radiation. The formation and evaporation of a black hole does not conserve baryon or lepton number, but it does conserve gauge quantum numbers, such as electric charge, since they may be measured by flux integrals at infinity. In string theory, it is a folk theorem that all exact symmetries, even discrete symmetries, are gauge symmetries.

**Example.** The Schrodinger field has kinetic term $\nabla^2\psi/2m$. Since the coefficient $1/2m$ has negative mass dimension, the operator is irrelevant. Hence upon renormalization, the Hamiltonian should have all terms allowed by symmetry, giving

$$H \sim \frac{p^2}{2m}\left(1 + a_1\frac{p^2}{m^2} + a_2\frac{p^4}{m^4} + \dots\right) + V(r)$$

where the factors of $m$ are put in by dimensional analysis. Note that here we are thinking of the renormalized theory which has no cutoff, $\Lambda \to \infty$, so the only mass scale is $m$.

This theory is perfectly predictive at energies where $p^2 \ll m^2$, despite the infinitely many unknown parameters $a_i$. But when $p^2 \sim m^2$, perturbation theory breaks down and all of the $a_i$ are important. The Schrodinger field can be UV-completed to either the Klein–Gordan or Dirac field, both of which are renormalizable.

**Example.** The classic example of an effective field theory is the 4-Fermi theory for $\beta$ decay,

$$\mathcal{L} \sim G_F\overline{\psi}_p\psi_n\overline{\psi}_e\psi_\nu$$

where we suppress gamma matrices and spinor indices. Similar operators can be used to describe the decay of the muon; this is used to determine

$$G_F = \left(\frac{1}{292.9\,\mathrm{GeV}}\right)^2$$

which indicates the energy where the theory breaks down; higher-dimension terms contribute in a series in $p^2G_F$. The specific values of these terms may be found by UV completing the 4-Fermi theory to the electroweak theory.

Sticking with only the dimension 6 term, the 4-Fermi theory is quite predictive. It can compute relations between decay rates, and angular distributions of decays. One can also include loops to get genuinely quantum predictions, which appear in cross sections as logarithmic dependence on the external momenta.

**Example.** Light scattering; or, why the sky is blue. We consider Rayleigh scattering of visible light off a nitrogen molecule, where the wavelength of the light is much larger than the size of the atom. We assume the photon energy is too weak to excite the molecule, so we can ignore its internal degrees of freedom and model it as a complex scalar field $\phi$. The molecule remains at rest since it is heavy, so we break Lorentz invariance, allowing us to use the four-velocity $u^\mu = (1, 0, 0, 0)$ in our

Lagrangian. Finally, the nitrogen molecule is electrically neutral, so the minimal coupling vanishes, $D_\mu\phi = \partial_\mu\phi$.

Given this setup, the kinetic term is $\pi^\dagger iv^\mu\partial_\mu\phi$, which shows that $\phi$ has mass dimension 3/2. Here, we can use relativistic dimensional analysis because the photon is relativistic, and the nitrogen is static. The effective interactions are

$$S^{\text{int}}[A,\phi] = \int dx\, \frac{g_1}{8\Lambda^3}\phi^2F^2 + \frac{g_2}{8\Lambda^3}\phi^2(u\cdot F)^2 + \dots$$

where the $g_i$ are dimensionless and the higher terms are suppressed by more powers of $\Lambda$. Now consider the tree-level contribution to $\phi + \gamma \to \phi + \gamma$ scattering. Since the field $F$ contains a derivative, the amplitude is proportional to $\omega^2$ and hence the cross section is proportional to $\omega^4$,

$$\sigma \sim g_2^2\frac{\omega^4}{\Lambda^6}.$$

Thus we get the frequency dependence of Rayleigh scattering almost for free! Note that when the scatterer is comparable in size to the wavelength, our effective field theory breaks down and we enter the regime of Mie scattering. It also breaks down once the light can excite the atom, which creates a Lorentzian scattering peak.

**Example.** Light propagation in an insulator. In vacuum, the unique gauge, Lorentz, and parity invariant action we can write for the photon is

$$S[A] = \frac{1}{4}\int dx\, \epsilon_0E^2 - \frac{1}{\mu_0}B^2.$$

Now we suppose the light propagates in a medium, whose degrees of freedom are modeled with additional fields $\Phi$. The Euler–Lagrange equations become

$$\partial^\mu F_{\mu\nu} = \mu_0 J_\nu, \quad J_\mu(x) = \frac{\delta S^{\text{int}}}{\delta A^\nu(x)}.$$

For example, in a conductor, we might ignore higher energy bands for the electrons, so $\Lambda$ would be the band gap. Then in an insulator, $\Lambda$ has the same interpretation, but since the Fermi surface lies in a band gap there are no relevant degrees of freedom at all! Then it would appear that $S^{\text{int}}$ must contain only irrelevant terms. But the insulator also picks out a preferred frame of reference, so we can write down non-Lorentz invariant terms,

$$S^{\text{int}}_\Lambda[A,\Phi] = \int dx\, \frac{1}{2}\left(\chi_eE^2 - \chi_mB^2 + \dots\right)$$

where the dimensionless couplings $\chi_e(\Lambda)$ and $\chi_m(\Lambda)$ are the susceptibilities. Then light can travel at a different speed, and this is the leading effect an insulator has. More generally, a crystalline insulator breaks rotational symmetry, so we can have

$$S^{\text{int}}_\Lambda[A,\Phi] = \int dx\, \frac{1}{2}\left((\chi_e)_{ij}E_iE_j - (\chi_m)_{ij}B_iB_j + \dots\right)$$

which leads to birefringence. At higher orders, we will find nonlinear terms such as $E^4/\Lambda^2$, yielding nonlinear optical effects.

**Note.** Quantum gravity may also be treated as an effective field theory. The action is

$$S[g] = \int dx\, \sqrt{-g} \left( \lambda + \frac{R}{16\pi G} \right)$$

where the dimensions are

$$[\lambda] = 4, \quad [g_{\mu\nu}] = 0, \quad [R] = [\partial\partial g] = 2, \quad [G] = -2.$$

In particular, one can see the metric is dimensionless because in flat spacetime its elements are simply $\delta$. Perturbing about the flat metric,

$$g = \delta + \sqrt{G} h$$

where $h$ is scaled to have canonically normalized kinetic terms. Higher-order terms in $h$ come with positive powers of $G$ and hence negative powers of $M_{\text{pl}}$, so they are all irrelevant. Their form can be obtained from taking the most general diffeomorphism invariant action,

$$S_\Lambda^{\text{eff}}[g] = \int dx\, \sqrt{-g} \left( c_0 \Lambda^4 + c_1 \Lambda^2 R + c_2 R^2 + c_3 R^{\mu\nu} R_{\mu\nu} + c_4 R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} + \ldots \right)$$

where the $c_i$ are dimensionless, and ordinary GR only uses $c_0$ and $c_1$. In the UV, either new degrees of freedom appear, such as strings, or the theory flows to a fixed point, a proposal known as asymptotic safety. (Note that "asymptotic freedom" as used above is just the case where this fixed point is the Gaussian fixed point.)

Of course, even though the interaction of gravity with matter is "irrelevant", it is very relevant to everyday life! This is because the gravitational interactions between the particles in macroscopic objects add up coherently, a fact which is not captured in our simple analysis above.

**Note.** Multiple kinds of renormalizability. So far we've been talking about Wilsonian renormalizability, which, near the Gaussian fixed point, simply corresponds to having no couplings with negative mass dimension. However, historically renormalizability was defined by demanding that all the divergences in a theory could be absorbed by redefinitions of the couplings already in the Lagrangian, which in turn must have a finite number of terms; we will call this counterterm renormalizability. In our picture, this means the Lagrangian must include all renormalizable terms consistent with symmetries; for instance, massless $\phi^4$ theory is not permitted. (Thus counterterm renormalizability is obliquely related to naturalness.) The main point is that it directly demands that *all* predictions up to arbitrary energy can be made given a finite number of parameters.

By contrast, in an effective field theory one requires more and more parameters to get an accurate prediction as one goes to higher energies. In practice, one fixes a desired precision, and then computes up to the order in $1/\Lambda$ required (or more generally, to the required order in the power counting parameter), neglecting all effects associated with higher-order terms. Then it is useful to say an EFT is renormalizable if it is renormalizable in the historic sense, at each order of the power counting expansion. (That is, divergences can appear associated with higher-order terms, but they can be simply neglected.) For example, $\phi^6$ theory is renormalizable at this sense, even though it induces a $\phi^8$ coupling, because the $\phi^8$ coupling is order $1/\Lambda^4$, while the $\phi^6$ coupling is only order $1/\Lambda^2$. EFTs are discussed further in the notes on the Standard Model.

**Note.** Counterterm renormalizability can be much more difficult to show than Wilsonian renormalizability, though the two are related.

- Consider the linear sigma model, a case with spontaneously broken global symmetry. Naively, there are more divergences than there are free parameters, so it looks like the theory is not renormalizable. But the divergences are still constrained by the broken $SO(n)$ symmetry, which is still present but nonlinearly realized, so "secret symmetry yields secret renormalizability".
- Consider a gauge theory. The gauge symmetry constrains the terms in the Lagrangian, but it does not manifestly constrain the divergences, because we need to gauge fix *before* quantization; that is, the divergences never "see" the gauge symmetry. Thus renormalizability is nontrivial, but can be proven using Ward identities. Using BRST symmetry, one can also prove Yang–Mills is indeed renormalizable.
- Now consider Yang–Mills with an explicit mass term $m^2 A^\mu A_\mu$. This makes the analysis of renormalizability even more difficult because the propagator no longer falls off at high momenta, so superficial degrees of divergence get worse. However, it turns out that it is renormalizable if and only if the gauge group is abelian.
- The Standard Model is more complicated, as it is a gauge theory with spontaneous symmetry breaking. 't Hooft proved its renormalizability in 1971 and we can perform this proof explicitly by working in $R_\xi$ gauge, introduced in the notes on the Standard Model.

These nontrivial facts should not be confused with Wilsonian renormalizability, which is easy to see in all these cases. Sometimes, counterterm and Wilsonian renormalizability are called perturbative and nonperturbative renormalizability. However, Wilsonian renormalizability does not imply counterterm renormalizability, e.g. in the case of quantum gravity if it were asymptotically safe. Here perturbative renormalizability fails because it expands about the Gaussian fixed point, while the nonperturbatively the theory is determined by a different fixed point in the UV.

# 10 Perturbative Renormalization

## 10.1 Power Counting

In this section, we explore the systematics of perturbative renormalization. We begin by estimating the degree of divergence of a Feynman diagram.

- We consider a scalar field theory with a $\phi^n$ interaction. Consider a connected diagram with $V$ such vertices, $I$ internal propagators, $E$ external lines, and $L$ loops. Euler's formula gives

  $$L = I - V + 1$$

  while counting the number of edges in two different ways gives

  $$nV = 2I + E$$

- The momentum integral will contain $L$ $\bar{d}^d k$ integrals and $I$ factors of $1/k^2$, so we define the superficial degree of divergence

  $$D = dL - 2I.$$

  We estimate $\mathcal{M} \sim \Lambda^D$, so diagrams with $D > 0$ are superficially divergent, diagrams with $D = 0$ are superficially log-divergent, and diagrams with $D < 0$ are superficially finite. Note that for particles with half-integer spin, propagators would instead contribute as $1/k$.

- Superficially divergent diagrams can actually be finite due to a symmetry. Moreover, superficially finite diagrams may diverge: the superficial estimate assumes that all of the loop momenta are of the same order $\Lambda$, but divergences can come from regions where only some of the loop momenta are large. One can show, rather tediously, that all such divergences come from superficially divergent subdiagrams.

- Rewriting $D$ in terms of $V$ and $E$, we have

  $$D = d - \left(d - n\frac{d-2}{2}\right) V - \frac{d-2}{2} E.$$

  Note that the quantity in brackets is simply the mass dimension of the $\phi^n$ coupling, while the coefficient of $E$ is simply the mass dimension of the field. This generalizes straightforwardly to multiply types of vertices and fields.

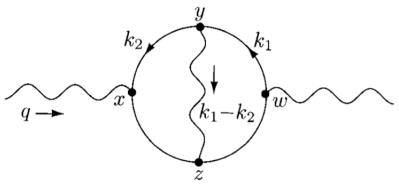
- We see in general that more external lines decreases $D$, while more internal vertices increases or decreases $D$ depending on the dimension of $V$. We say that a theory is (power-counting)

  - renormalizable if the number of superficially divergent amplitudes is finite, but superficial divergences appear at every order in perturbation theory,
  - superrenormalizable if the number of superficially divergent diagrams is finite,
  - nonrenormalizable if the number of superficially divergent amplitudes is infinite.

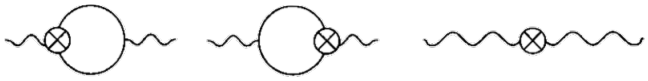
  In general, the couplings in a renormalizable or superrenormalizable theory all have nonnegative mass dimensions, while in a superrenormalizable theory the dimensions are all strictly positive. A single coupling with negative mass dimension renders a theory nonrenormalizable.

- Ignoring the issue of divergent subdiagrams, if a theory is (power-counting) renormalizable, then at every order in perturbation theory only a finite number of divergent diagrams appear, parametrized by a finite number of divergent constants.
- The BPHZ theorem ensures that divergent subdiagrams cannot change these conclusions; the divergences there are simply canceled by "counterterm subdiagrams", requiring no new counterterms to be introduced. The proof is by giving an explicit algorithm to perform this cancellation, and it is nontrivial because of the possibility of "overlapping divergences", where two divergent subdiagrams overlap, and hence their loop integrals can't be treated separately.

**Note.** A glimpse of higher-order renormalization. Consider the following two-loop diagram in QED.




This diagram contains two overlapping divergences. It is canceled by the counterterm diagrams shown below.



When $k_2$ is large, the points $x$, $y$, and $z$ are "close together", so the divergence is canceled by the first counterterm diagram. Similarly, the second counterterm diagrams accounts for the divergence where $k_1$ is large, while the third accounts for the divergence where both $k_1$ and $k_2$ are large. In DR, such a divergence appears as a double pole in $\epsilon$.

As an example, we apply power counting to QED.

- In QED, the interaction term is $e\overline{\psi}\not{A}\psi$ and the superficial degree of divergence of an amplitude with $E_\gamma$ external photons and $E_e$ external electrons is

$$D = 4 - E_\gamma - \frac{3}{2}E_e$$

  since $[e] = 0$. The theory is hence renormalizable.

- The amplitudes with $D \geq 0$, that don't vanish automatically by Lorentz invariance, are:
  - The vacuum energy with $D = 4$.
  - The photon tadpole with $D = 3$, and the three-photon amplitude with $D = 1$.
  - The four-photon amplitude with $D = 0$.
  - The photon propagator with $D = 2$, the electron propagator with $D = 1$, and the renormalized vertex with $D = 0$.

  Only the amplitudes in the final group are important, as we'll justify below.

- The vacuum energy is divergent, and we can absorb it with a vacuum energy counterterm. Since it has no observable consequences, we typically ignore it entirely by normal ordering.

- The one-photon and three-photon amplitudes all vanish by symmetry. QED is invariant under charge conjugation, $j^\mu \to -j^\mu$, which requires $A^\mu \to -A^\mu$. Therefore a correlation function of an odd number of photons vanishes, a result known as Furry's theorem. Diagrammatically, each diagram cancels against another with all of the electron loops reversed.

- The four-photon amplitude is finite by the Ward identity. To see this, let the amplitude be $A^{\mu\nu\rho\sigma}$. By the Ward identity, $k_\mu A^{\mu\nu\rho\sigma} = 0$, which implies that the amplitude must be proportional to something like $g^{\mu\nu}k^\sigma - g^{\mu\sigma}k^\nu$.

- To show this more explicitly, the divergent part of one diagram is

$$A^{\mu\nu\rho\sigma} \sim \int đk\, \frac{\mathrm{tr}(\not{k}\gamma^\mu \not{k}\gamma^\nu \not{k}\gamma^\rho \not{k}\gamma^\sigma)}{k^8}$$

  and there are five more diagrams corresponding to the different ways to attach the photons to the electron loop. Using spherical symmetry, we simplify the integral to

$$A^{\mu\nu\rho\sigma} \sim \int đk\, \frac{\mathrm{tr}(\gamma^{\alpha_1}\gamma^\mu\gamma^{\alpha_2}\gamma^\nu\gamma^{\alpha_3}\gamma^\rho\gamma^{\alpha_4}\gamma^\sigma)}{k^4}(\eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4} + \text{perms.})$$

  which can be simplified using the standard trace identities. Finally, symmetrizing over $\mu\nu\rho\sigma$ to account for the six diagrams yields zero.

- Pulling this factor out leaves an amplitude with $D = -1$, which is hence finite. One might naively think this means the amplitude is proportional to $1/\Lambda$ and hence vanishes, but the four-photon vertex is nonzero; it mediates light-by-light scattering. The point is that $D$ only gives us the UV dependence of the amplitude on $\Lambda$, while the finite contribution comes from the IR part of the loop integral.

- Finally, we arrive at the propagators. In our one-loop analysis of QED we found they were only logarithmically divergent, and this is indeed guaranteed by symmetries as we see below. For the degree of divergence, it suffices to look at the self energies.

- The electron self energy takes the form

$$A_0 + A_1\not{p} + A_2 p^2 + \dots$$

  and has mass dimension 1. Then the only place a linear divergence can appear is in the $A_0$ coefficient. Now, $A_0$ is an analytic function in the electron mass,

$$A_0 \sim \Lambda + m_e \log(\dots) + \frac{m_e^2}{\Lambda}\log(\dots) + \dots.$$

  When $m_e = 0$ the theory has an axial symmetry, by opposite rotations of the left and right-handed Weyl spinors; then a mass term cannot be generated by renormalization. Thus $A_0$ has no term independent of $m_e$ and the divergence is at most logarithmic in $\Lambda$.

- The photon self energy has mass dimension 2 and takes the form

$$\Pi^{\mu\nu} \sim B_0\eta_{\mu\nu} + B_2 q_\mu q_\nu + B_2'\eta_{\mu\nu}q^2 + \dots$$

  where quadratic divergences can come from $B_0$. But we saw that the Ward identities required

$$\Pi^{\mu\nu} = (\eta^{\mu\nu}q^2 - q^\mu q^\nu)\Pi(q^2)$$

  where $\Pi(q^2)$ is regular in $q$. Then we must have $B_0 = 0$.

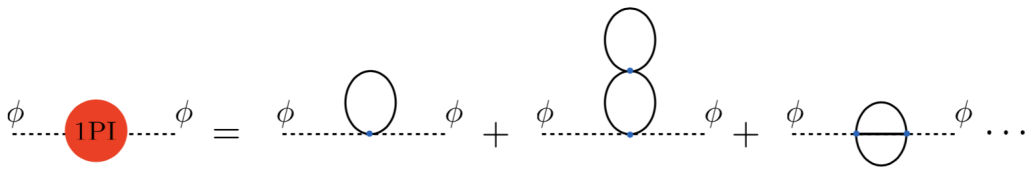
- These results ensure that the divergences of QED can be absorbed by renormalizing terms already present in the Lagrangian. For example, if $B_0$ had been nonzero, we would have had a quadratic divergence independent of $q$ in $\Pi^{\mu\nu}$. This could only be absorbed by adding a term $A_\mu A^\mu$ to the Lagrangian, which would break gauge invariance. Similarly, if the four-photon amplitude had been divergent, we would have had to add the term $(A_\mu A^\mu)^2$. Of course, these terms *would* appear, and be necessary, in QED with a massive "photon".
- The more general lesson here is that, in order to absorb all divergences, a renormalizable theory must contain *all* renormalizable terms that are allowed by symmetry, and no more.

**Note.** The logic above requires that the divergent part of an amplitude be written as a polynomial in external momenta, with divergent coefficients. To see this in general, note that we may simply differentiate any superficially divergent loop integral $D+1$ times with respect to external momenta to get a convergent integral (ignoring the issue of divergent subdiagrams), which has a finite result. Integrating $D+1$ times produces a finite piece plus the advertised divergent polynomial. The finite piece contains all the analytically nontrivial dependence on the external momenta, such as the logarithms.

## 10.2 Renormalization of $\phi^4$ Theory

In this section, we perform renormalization in $\phi^4$ theory with a hard cutoff. We will consider the renormalization of the propagator and the $\phi^4$ coupling. Equivalently, we are computing the quadratic and quartic terms in the 1PI effective action.

- We consider the action
$$S_{\Lambda_0}[\phi] = \int dx\, \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4$$
where we are working in four dimensions and Euclidean signature. We work perturbatively, which means we are near the Gaussian fixed point, so $\lambda$ is marginal.
- The exact propagator/connected two-point function in momentum space is
$$\Delta(k^2) = \int dx\, e^{ikx}\langle\phi(x)\phi(0)\rangle_c = \frac{1}{k^2+m^2-\Pi(k^2)}$$
which $\Pi(k^2)$ is the sum of 1PI diagrams, by the usual geometric series argument.
- The terms in $\Pi(k^2)$ up to two loops are shown below.




Here, a dotted line denotes an amputated propagator, i.e. they are not associated with $1/(p^2+m^2)$ factors. At one loop, the contribution is simply

$$\Pi(k^2) \supset -\frac{\lambda}{2}\int_{|p|\le\Lambda_0} \frac{đp}{p^2+m^2} = -\lambda\frac{\mathrm{Vol}(S^3)}{2(2\pi)^4}\int_0^{\Lambda_0}\frac{p^3\,dp}{p^2+m^2} = -\frac{\lambda}{2}\frac{m^2}{16\pi^2}\int_0^{\Lambda_0^2/m^2}\frac{u\,du}{1+u}.$$

Here, we used $\mathrm{Vol}(S^3) = 2\pi^2$ and let $u = p^2/m^2$. We see the appearance of the generic $1/16\pi^2$ "loop factor" that appears in four dimensions. Explicitly performing the integral gives

$$\Pi(k^2) \supset -\frac{\lambda}{32\pi^2}\left(\Lambda_0^2 - m^2 \log\left(1 + \frac{\Lambda_0^2}{m^2}\right)\right)$$

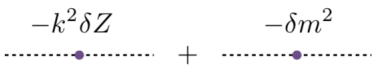
which has a quadratic and a logarithmic divergence.

- To "cancel" these divergences, we "add counterterms" to the action, subject to the same caveats as before. That is, we work with the action

$$S_{\Lambda_0}[\phi] + \hbar S^{\mathrm{CT}}[\phi, \Lambda_0], \quad S^{\mathrm{CT}}[\phi, \Lambda_0] = \int dx\, \frac{1}{2}\delta Z(\partial\phi)^2 + \frac{1}{2}\delta m^2 \phi^2 + \frac{1}{4!}\delta\lambda\phi^4.$$

  Note that this is rather different from what we did earlier, where we started from the low-energy effective action we wanted and added counterterms to get the high-energy action. At this point, the terms $S_{\Lambda_0}$ and $S^{\mathrm{CT}}$ have no physical meaning individually.

- At $O(\hbar)$, the counterterms contribute two diagrams,

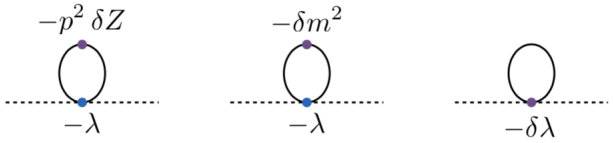



  so that

$$\Pi^{1\text{ loop}}(k^2) = -k^2\delta Z - \delta m^2 - \frac{\lambda}{32\pi^2}\left(\Lambda_0^2 - m^2 \log\left(1 + \frac{\Lambda_0^2}{m^2}\right)\right)$$

  and we must choose $\delta Z$ and $\delta m^2$ so the result is finite.

- At two loops, the corresponding counterterm diagrams have one loop, as shown.




- Since the counterterms guarantee we get finite results, we may take the continuum limit $\Lambda_0 \to \infty$ while fixing our action to match experiment. This is done by requiring the exact propagator have a pole at $p^2 = -m_{\mathrm{phys}}^2$ with residue one, where $m_{\mathrm{phys}}$ is the physical mass, which is measured by 'weighing' a stable particle, or more generally by looking for peaks in cross sections for processes involving the exchange of the particle.

- Quantitatively, this means we require

$$\Pi(-m_{\mathrm{phys}}^2) = m^2 - m_{\mathrm{phys}}^2, \quad \left.\frac{\partial\Pi}{\partial k^2}\right|_{k^2=-m_{\mathrm{phys}}^2} = 0.$$

  Note that these are only constraints on the total action; we are still free to pick how it is split into $S_{\Lambda_0}$ and $S^{\mathrm{CT}}$. In the on-shell scheme, we choose $S_{\Lambda_0}$ so that the parameter $m$ is equal to the physical mass, so

$$\Pi(-m^2) = 0, \quad \left.\frac{\partial\Pi}{\partial k^2}\right|_{k^2=-m^2} = 0.$$

  Hence, in the on-shell scheme $S_{\Lambda_0}$ does have a physical meaning, being roughly equivalent to the low-energy effective action encountered in Wilsonian renormalization. However, in other schemes $S_{\Lambda_0}$ has no direct interpretation.

- In our case, at one loop we have

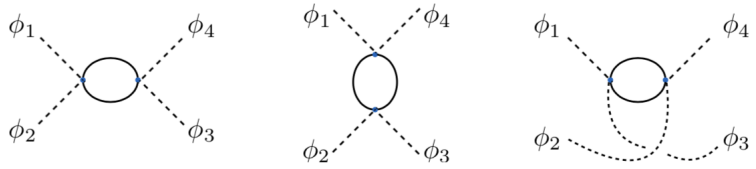
$$\delta Z = 0, \quad \delta m^2 = -\frac{\lambda}{32\pi^2}\left(\Lambda_0^2 - m^2 \log\left(1 + \frac{\Lambda_0^2}{m^2}\right)\right)$$

  so that the one loop correction exactly vanishes. This is rather special; the reason it happened is that the loop is a tadpole, which doesn't depend on the external momenta at all. The tadpole only renormalizes the mass, so its effects may be entirely canceled by a counterterm. One can reformulate perturbation theory so that all tadpoles are set to zero from the start, reflecting the fact that they don't affect the physics.

- More generally, a loop will generate an infinite series of terms, such as $\phi^2$, $(\partial\phi)^2$, and all higher derivatives, though the higher derivative terms will be suppressed by powers of $\Lambda_0$ rather than divergent. The on-shell scheme only fixes $\Pi$ and its derivative at a single point, so loop effects can be seen away from this point, e.g. in virtual particle exchange.

Next, we turn to the renormalization of the quartic coupling, which we take to mean the quartic part of the 1PI effective action.

- The one-loop corrections come from the graphs below.




  Letting the external momenta $k_i$ all flow into the diagram, the diagrams sum to

$$\frac{\lambda^2}{2}\int^{\Lambda_0} \frac{đp}{p^2+m^2}\left(\frac{1}{(p+k_1+k_2)^2+m^2} + \frac{1}{(p+k_1+k_4)^2+m^2} + \frac{1}{(p+k_1+k_3)^2+m^2}\right).$$

  Since the external momenta also flow through the loop, we generate an infinite series of derivative terms, as anticipated above.

- The contribution to the pure quartic coupling $\lambda$ comes from the momentum-independent part,

$$\frac{3\lambda^2}{2}\int^{\Lambda_0} \frac{đp}{(p^2+m^2)^2} = \frac{3\lambda^2}{32\pi^2}\left(\log\left(1+\frac{\Lambda_0^2}{m^2}\right) - \frac{\Lambda_0^2}{\Lambda_0^2+m^2}\right)$$

  which is logarithmically divergent. One possible choice for the counterterm is

$$\delta\lambda = \frac{3\lambda^2}{32\pi^2}\left(\log\frac{\Lambda_0^2}{m^2} - 1\right).$$

  The quartic term in the 1PI effective action is the negative of the sum of 1PI diagrams, so

$$\lambda_{\text{eff}} = \lambda - \frac{3\hbar\lambda^2}{32\pi^2}\left(\log\left(1+\frac{m^2}{\Lambda_0^2}\right) + \frac{m^2}{m^2+\Lambda_0^2}\right).$$

  This choice ensures that in the continuum limit, $\lambda_{\text{eff}} = \lambda$, in the spirit of the on-shell scheme.

- The contributions for terms with more derivatives come in a power series in $k^2/\Lambda_0^2$, and are hence finite, vanishing in the continuum limit $\Lambda_0 \to \infty$.

- The integrals here are essentially the same as in Wilsonian renormalization. There, we integrated loops over momenta in $[\Lambda, \Lambda_0]$, getting a renormalized coupling constant. Here, we are integrate over momenta in $[0, \Lambda_0]$ in essentially the same expressions. Hence in the limit $\Lambda_0 \to \infty$ the functional dependence is the same. This mean we know, e.g. that a marginal coupling gets logarithmically divergent corrections.

- While the above is true at one loop, matters are complicated at higher loops due to divergent subdiagrams, and we need the BPHZ theorem to ensure the divergences don't get worse than expected. But the Wilsonian intuition makes it clear that the BPHZ procedure must work.

- We now do a full calculation to see the derivative terms explicitly. Using Feynman's trick,

$$\frac{1}{(p+k_{12})^2+m^2}\frac{1}{p^2+m^2} = \int_0^1 \frac{dx}{[x((p+k_{12})^2+m^2)+(1-x)(p^2+m^2)]^2}$$

  so completing the square gives the loop integral

$$\frac{\lambda^2}{2}\int_0^1 dx \int \frac{đ\ell}{(\ell^2+M^2)^2}, \quad \ell = p - xk_{12}, \quad M^2 = m^2 + x(1-x)k_{12}^2.$$

  Note that integrating over the region $|p| \leq \Lambda_0$ is the same as integrating over the region $|\ell| \leq \Lambda_0$ up to error terms of order $|k_{12}|/\Lambda$, which vanish in the continuum limit $\Lambda_0 \to \infty$. For simplicity, we discard these terms now.

- Working in spherical coordinates, we get a factor of $\mathrm{Vol}(S^3)$ for

$$\pi^2 \int_0^1 dx \int_0^{\Lambda_0^2} \frac{\ell^2 d(\ell^2)}{(\ell^2+M^2)^2} = \pi^2 \int_0^1 dx \, \log\frac{\Lambda_0^2+M^2}{M^2} + \frac{M^2}{\Lambda_0^2+M^2} - 1$$

  Keeping only terms that survive in the continuum limit, our three loop diagrams yield

$$\frac{\lambda^2}{32\pi^2}\int_0^1 dx \, \log\frac{\Lambda_0^2}{m^2-x(1-x)s} + \log\frac{\Lambda_0^2}{m^2-x(1-x)t} + \log\frac{\Lambda_0^2}{m^2-x(1-x)u} - 3$$

  where the Mandelstam variables are

$$s = -(k_1+k_2)^2, \quad t = -(k_1+k_4)^2, \quad u = -(k_1+k_3)^2.$$

- Using the same choice of counterterm as in our rough calculation above, the quartic term in the effective action is

$$\mathcal{A}(k_i) = \lambda + \frac{\hbar\lambda^2}{32\pi^2}\int_0^1 dx \, \log\left(1 - \frac{x(1-x)s}{m^2}\right) + \log\left(1 - \frac{x(1-x)t}{m^2}\right) + \log\left(1 - \frac{x(1-x)u}{m^2}\right).$$

  Since this expression is in momentum space, expanding to get powers of the Mandelstam variables yields derivative terms. As expected, these terms correspond to no UV divergences, since they are irrelevant. But they are nonzero, because they are generated by the $\phi^4$ coupling.

- At this point, we can compute everything else using just tree-level diagrams. For example, the amplitude for $\phi^4$ scattering only contains one diagram, the quartic vertex, which is $-\delta^4\Gamma/\delta\phi^4 = \mathcal{A}(k_i)$. Similarly for $\phi^6$ scattering we have

so we only need the $\phi^2$, $\phi^4$, and $\phi^6$ terms in $\Gamma[\phi]$.

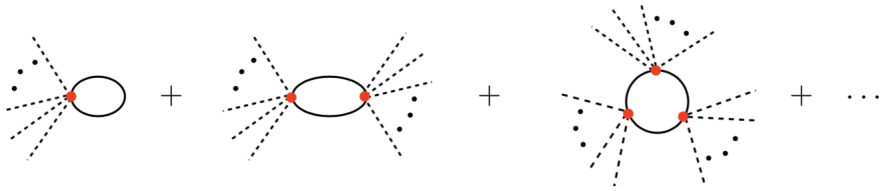
**Note.** A subtlety with getting a "pure coupling" is that with a hard cutoff, the effective action in momentum space is not analytic! Concretely, if the coupling is $f(k)$, then RG modifies it by an analytic function times step functions in the external momenta. This was not a problem here, but in general, computing $f(0)$ by taking zero external momenta will not give the same result as computing $f(k)$ for nonzero $k$, expanding in a Taylor series in $k$, and taking the constant term. The proliferation of step functions is one reason the hard cutoff is not practical.

Another fact we know from the Wilsonian picture is that irrelevant couplings in the high-energy theory should not affect anything at low energies. We can see this explicitly by including irrelevant couplings from the start.

- In this case, we take the action

$$S_{\Lambda_0}[\phi] = \int dx\, \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + V(\phi), \quad V(\phi) = \sum_{k\geq 2} g_{2k}\Lambda_0^{4-2k}\phi^{2k}.$$

  We would like to compute the $\phi^{2m}$ coupling in the 1PI effective action. At one loop, all such diagrams have the form below.



  This is merely the $O(\hbar)$ contribution to the effective action we've seen in the zero-dimensional case; it can also be found by expanding the functional determinant $\det(-\nabla^2 + m^2 + V''(\phi))^{-1/2}$.

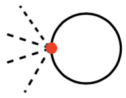
- We already know that the derivative terms are suppressed by powers of $k/\Lambda_0$, so we focus on the renormalization of the $\phi^{2m}$ coupling. A one-loop graph with $e$ propagators contributes

$$\int đp \prod_{j=1}^{e} \frac{1}{(p+K_j)^2+m^2} \sim \int \frac{đp}{p^{2e}} \sim \begin{cases} \Lambda_0^2 & e=1 \\ \log\Lambda_0 & e=2 \\ \text{finite but nonzero} & e>2. \end{cases}$$

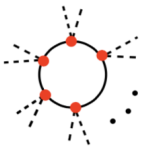
  For each vertex $\phi^{2k+2}$, we also pick up a factor of $\Lambda_0^{2(1-k)}$.

- Therefore, the only divergent one-loop diagrams are the mass and quartic vertex diagrams,

which we've already considered. Almost all other one-loop diagrams vanish in the continuum limit, but we find finite contributions from



which has no new effect, since we are already setting the $\phi^4$ coupling with a counterterm, and



where all of the vertices are $\phi^4$. That is, in the continuum limit the effective strength of the irrelevant interactions is totally fixed by the $\lambda\phi^4$ coupling, while the original high-energy irrelevant couplings play no role at all, just as in the Wilsonian picture.

**Note.** This result is very different from what we got by naive power counting, where we concluded that a single irrelevant coupling made every amplitude diverge. The difference is that there, we fixed the value of the dimensionful coupling. This corresponds to thinking about an effective field theory (e.g. the dimensionful masses should be fixed to match experiment), and indeed one cannot take the continuum limit of an effective field theory, and one should require infinitely many counterterms.

By contrast, here we've been thinking about $S_{\Lambda_0}$, the action of a fundamental theory. Then we've been fixing the dimensionless couplings, since they should all be presumably $O(1)$, and there is no issue with irrelevant couplings.

Finally, we repeat our calculations using DR and $\overline{\text{MS}}$.

- First, we consider the mass renormalization. We define the dimensionless coupling $g(\mu)$ by

$$\lambda = \mu^{4-d} g(\mu)$$

where $\mu$ is an arbitrary mass scale. Then the one-loop correction to $\Pi(k^2)$ is

$$\Pi(k^2) \supset -\frac{1}{2} g(\mu)\mu^{4-d} \int \frac{đp}{p^2+m^2} = -\frac{g(\mu)m^2}{2(4\pi)^{d/2}} \left(\frac{\mu}{m}\right)^{4-d} \Gamma\left(1-\frac{d}{2}\right)$$

by our usual DR formulas, with some extra signs due to the Euclidean signature.

- Setting $d = 4-\epsilon$ and expanding as usual, we find

$$\frac{g(\mu)m^2}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma + \log\left(\frac{4\pi\mu^2}{m^2}\right) - 1\right) + O(\epsilon)$$

as $\epsilon \to 0^+$. In MS we choose the counterterm

$$\delta m^2 = -\frac{g(\mu)m^2}{16\pi^2\epsilon}$$

to remove the divergence only, while in $\overline{\text{MS}}$ we choose

$$\delta m^2 = -\frac{g(\mu)m^2}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma + \log 4\pi\right), \quad \Pi_{1\,\text{loop}}(k^2) = \frac{g(\mu)m^2}{32\pi^2}\left(\log\frac{\mu^2}{m^2} - 1\right).$$

The one-loop mass correction now depends on $\mu$, but this is perfectly acceptable, as $\mu$ is part of the specification of the renormalization scheme. If we sum the corrections to all orders, $\mu$ must drop out, since we should just arrive back at the pole mass. Note that the pole mass is no longer equal to $m$, as we're not using the on-shell scheme.

- Next, we turn to the quartic coupling, where the 1PI diagrams are

$$\frac{g^2\mu^{4-d}}{2}\int \frac{d^dp}{(p^2+m^2)((p+k_1+k_2)^2+m^2)} + 2 \text{ other channels.}$$

We set the external momenta $k_i$ to zero to focus on the pure quartic coupling, giving

$$\frac{3g^2}{2(4\pi)^{d/2}}\left(\frac{\mu}{m}\right)^{4-d}\Gamma\left(2-\frac{d}{2}\right) = \frac{3g^2}{32\pi^2}\left(\frac{2}{\epsilon}-\gamma+\log\frac{4\pi\mu^2}{m^2}\right)+O(\epsilon).$$

Then in the $\overline{\text{MS}}$ scheme we have

$$\delta\lambda = \frac{3g^2}{32\pi^2}\left(\frac{2}{\epsilon}-\gamma+\log 4\pi\right), \quad g_{\text{eff}}(\mu) = g(\mu) - \frac{3\hbar g^2(\mu)}{32\pi^2}\log\frac{\mu^2}{m^2}+O(\hbar^2)$$

where we restored $\hbar$ to emphasize the one-loop effect is quantum.

- If we work to all orders, the dependence of $g_{\text{eff}}(\mu)$ on $\mu$ must drop out, because it's just the $\phi^4$ coefficient of the effective action. Therefore its derivative is zero,

$$\frac{dg_{\text{eff}}}{d\log\mu} = 0.$$

This is a "continuum RG" equation, not to be confused with a Wilsonian RG equation.

- Differentiating our result and keeping only the lowest order terms, we have

$$\beta(g) = \frac{dg}{d\log\mu} = \frac{3\hbar g^2}{16\pi^2}+O(\hbar^2)$$

where the classical term, independent of $\hbar$, vanishes since the $\phi^4$ coupling is marginal. Therefore, at one loop the coupling is marginally irrelevant.

- Separating and integrating, we find

$$\frac{1}{g(\mu')} = \frac{1}{g(\mu)}+\frac{3\hbar}{16\pi^2}\log\frac{\mu}{\mu'}, \quad g(\mu) = \frac{16\pi^2}{3\hbar}\frac{1}{\log(\Lambda_{\phi^4}/\mu)}$$

at least at one-loop order. The fact that a dimensionless initial condition can be exchanged for a dimensionful scale, $\Lambda_{\phi^4}$, where the coupling diverges, is called dimensional transmutation.

**Note.** Interpreting the running coupling $g(\mu)$. The amplitude to all orders can't depend on $\mu$, but changing $\mu$ modifies each of the terms. Generically, we have a perturbation series in $g(\mu)\log(\mu^2/E^2)$ where $E$ is the energy scale of the process. Thus we can significantly improve the convergence of the series if we take $\mu \approx E$. In particular, when $\mu \approx E$ the sum of the series $g_{\text{eff}}$ is well-approximated by its first term $g(\mu)$. Thus $g(\mu)$ estimates the strength of interactions at energy scale $\mu$.

Previously, we used the on-shell scheme, where the coupling $e$ was fixed. Using this fixed coupling we computed the one-loop amplitude for $e^+e^- \to e^+e^-$ by exchange of a photon with momentum $q$, and found it depended on $q^2$, yielding a running coupling. While the particulars were different, in both cases the running coupling referred to the fact that the vertices in the 1PI effective action are momentum-dependent.

**Note.** The energy scale $\mu$ is not to be interpreted as a cutoff, but it does have a relationship. Note that DR integral measures take the form $\int \bar{d}^{4-\epsilon}p\,\mu^\epsilon$. For positive $\epsilon$, we have more momentum modes at energies less than $\mu$, and less at energies greater than $\mu$. DR removes a greater and greater fraction of the momentum modes as their energy increases, acting like a smooth cutoff.

Of course, there's more to DR than just that, because DR is defined in more subtle way that preserves symmetries, but this can lead to some useful intuition. Note that the density of momentum modes starts to be significantly changed when $(p/\mu)^\epsilon \sim 1 + \epsilon\log(p/\mu)$ is significantly greater than one, which corresponds to $\epsilon\log(p/\mu) \sim 1$. Treating this momentum scale as a UV cutoff, $p \sim \Lambda$, we have $1/\epsilon \sim \log(\Lambda/\mu)$. This is the reason $1/\epsilon$ poles in DR correspond to logarithmic divergences in cutoff regularization.

## 10.3 Renormalization of Quantum Electrodynamics

Finally, we revisit QED from a more sophisticated perspective.

- We work in Euclidean signature. In a theoretical context, we would define the action as

$$S[A,\psi] = \int d^dx\,\frac{1}{4e^2}F^{\mu\nu}F_{\mu\nu} + \overline{\psi}(\not{D}+m)\psi, \quad \not{D} = \not{\partial} + i\not{A}.$$

  Under these conventions we have

$$[e] = \frac{4-d}{2}, \quad [A] = 1$$

  in general dimensions, so the kinetic term is irrelevant when $d < 4$. Then the only relevant and marginal terms are topological ones, such as the Chern–Simons term, and this normalization is convenient for highlighting this fact: irrespective of normalization, the topological terms dominate the kinetic ones.

- However, when we do practical calculations for QED in $d = 4$, we would like the photon field to be canonically normalized. This is done by multiplying it by $e^{-1}$, giving

$$S[A,\psi] = \int d^dx\,\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \overline{\psi}(\not{\partial}+m)\psi + ie\overline{\psi}\not{A}\psi, \quad \not{D} = \not{\partial} + ie\not{A}$$

  so that the photon field now has the usual dimension $[A] = (d-2)/2$ and the kinetic term is marginal in all dimensions. There is no contradiction: while being marginal or relevant depends on the normalization, the Chern–Simons term is more relevant in all normalizations. We'll use this convention below and suppress the $d$ dependence.

- The Clifford algebra is modified since the metric is, giving

$$\{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu}, \quad (\gamma^\mu)^\dagger = -\gamma^\mu, \quad \operatorname{tr}\gamma^\rho\gamma^\sigma = 4\delta^{\rho\sigma}$$

  so that $\gamma^0$ is no longer distinguished. Then the boost generators

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu,\gamma^\nu]$$

  are Hermitian, reflecting the fact that $SO(d)$ is compact, and the action is real, and

$$\operatorname{tr}(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma) = 4(\delta^{\mu\rho}\delta^{\nu\sigma} - \delta^{\mu\nu}\delta^{\rho\sigma} + \delta^{\mu\sigma}\delta^{\nu\rho})$$

- To find the classical photon propagator, we expand the kinetic term in momentum space, for

$$\begin{aligned}\frac{1}{4}\int dx\, F^{\mu\nu}F_{\mu\nu} &= \frac{1}{4}\int dk\,(-i)(k^\mu \tilde{A}^\nu(-k) - k^\nu \tilde{A}^\mu(-k))i(k_\mu \tilde{A}_\nu(k) - k_\nu \tilde{A}_\mu(k)) \\ &= \frac{1}{2}\int dk\, k^2 \left(\delta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\tilde{A}_\mu(-k)\tilde{A}_\nu(k).\end{aligned}$$

  The propagator is the inverse of the kinetic term, and one possible inverse is

$$\Delta^0_{\mu\nu}(k) = \frac{1}{k^2}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right).$$

  The other possibilities are indexed by the $\xi$ parameter in $R_\xi$ gauge. Here we have chosen Landau/Lorenz gauge, $\partial^\mu A_\mu(x) = 0$ or equivalently $k^\mu \Delta^0_{\mu\nu}(k) = 0$.

- Next, we define the exact photon propagator

$$\Delta_{\mu\nu}(k) = \int dx\, e^{ikx}\langle A_\mu(x)A_\nu(0)\rangle$$

  and by the usual argument, denoting the sum of 1PI diagrams as $\Pi^\rho{}_\sigma$,

$$\Delta_{\mu\nu}(k) = \Delta^0_{\mu\nu}(k) + \Delta^0_{\mu\rho}(k)\Pi^\rho_\sigma(k)\Delta^{0\sigma}{}_\nu + \dots.$$

- Using the Ward identity, we can show that

$$\Pi_{\mu\nu}(k) = k^2\pi(k^2)P_{\mu\nu}, \quad P_{\mu\nu} = \left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right).$$

  The quantity in brackets is just an idempotent projection operator, $k^\mu P_{\mu\nu} = 0$, so the series sums to

$$\Delta_{\mu\nu}(k) = \frac{\Delta^0_{\mu\nu}(k)}{1 - \pi(k^2)}$$

- Since $\Delta_{\mu\nu}(k)$ is the inverse of the quadratic term in the 1PI effective action $S_{\text{eff}}$,

$$S^{(2)}_{\text{eff}}[\tilde{A}] = \frac{1}{2}\int dk\,(1 - \pi(k^2))k^2\left(\delta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\tilde{A}_\mu(-k)\tilde{A}_\nu(k).$$

  Then in position space, we get a wavefunction renormalization of $1 - \pi(0)$, along with an infinite series of derivative interactions suppressed by powers of the electron mass.

**Note.** Showing that the photon self-energy is transverse. By diagrammatics, we have

$$\langle j^\mu j^\nu\rangle(k) = \Pi^{\mu\nu}(k) + (\Pi\Delta\Pi)^{\mu\nu}(k)$$

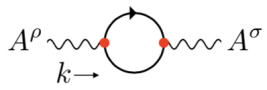
where $j^\mu = \overline{\psi}\gamma^\mu\psi$. Then the Ward identity ensures

$$k_\mu(\Pi + \Pi\Delta\Pi)^{\mu\nu} = 0.$$

Expanding order by order and using the fact that $\Pi$ starts at order $e$ gives $k_\mu\Pi^{\mu\nu} = 0$, as claimed. Note that in both this proof and our earlier proof, we've had to work perturbatively. Indeed, the statement does not hold in general; it fails in the case of spontaneous symmetry breaking.

Now we turn to the one-loop contribution to $\Pi^{\rho\sigma}(k)$.

- We use dimensional regularization, defining the dimensionless coupling $g$ by $e^2 = \mu^{4-d} g^2(\mu)$. The only diagram that contributes is




  where there is no symmetry factor, since the electron and positron are distinct.

- Applying the Feynman rules, we have

$$\Pi^{\rho\sigma}_{1\,\text{loop}}(k) = -\mu^{4-d}(ig)^2 \int đp \ \text{tr}\left(\frac{1}{i\not{p}+m}\gamma^\rho \frac{1}{i(\not{p}-\not{k})+m}\gamma^\sigma\right)$$

  where we picked up a minus sign from the fermion loop. As in the canonical formalism, we can understand this minus sign by going back to the level of contractions. When we apply Wick's theorem, we will always need to perform an odd number of Grassmann anticommutations to get the spinors all 'in order'. Equivalently, the path integral for fermions gives $\det(\not{D}+m)$ in the numerator rather than the denominator, and taking a logarithm gives the sign.

- Simplifying, the loop integral becomes

$$\int đp \frac{\text{tr}\left((-i\not{p}+m)\gamma^\rho(-i(\not{p}-\not{k})+m)\gamma^\sigma\right)}{(p^2+m^2)((p-k)^2+m^2))}.$$

  We begin by using the Feynman trick to complete the square, turning the denominator into

$$\int_0^1 \frac{dx}{((p^2+m^2)(1-x)+((p-k)^2+m^2)x)^2} = \int_0^1 \frac{dx}{((p-kx)^2+m^2+k^2x(1-x))^2}.$$

  We then shift $p \to p+kx$, noting that this also changes the numerator.

- Next, we apply the trace identities, giving

$$\Pi^{\rho\sigma}_{1\,\text{loop}}(k) = 4\mu^{4-d}g^2 \int đp \int_0^1 dx \, \frac{\text{numerator}}{(p^2+\Delta)^2}, \quad \Delta = m^2 + k^2x(1-x)$$

  where the numerator is

$$-(p+kx)^\rho(p-k(1-x))^\sigma + (p+kx)\cdot(p-k(1-x))\delta^{\rho\sigma} - (p+kx)^\sigma(p-k(1-x))^\rho + m^2\delta^{\rho\sigma}.$$

- We can dramatically simplify the numerator using symmetry. All terms odd in $p$ must vanish, while since the integral is isotropic we may replace

$$p^\mu p^\nu \to \frac{1}{d}\delta^{\mu\nu}p^2.$$

  We finally use two of our DR results to find

$$\Pi^{\rho\sigma}_{1\,\text{loop}} = (k^2\delta^{\rho\sigma} - k^\rho k^\sigma)\pi_{1\,\text{loop}}(k^2)$$

as required by the Ward identity, where

$$\pi_{1\,\text{loop}}(k^2) = -\frac{8g^2(\mu)\Gamma(2-d/2)}{(4\pi)^{d/2}} \int_0^1 dx\, x(1-x) \left(\frac{\mu^2}{\Delta}\right)^{2-d/2}.$$

Finally, setting $d = 4 - \epsilon$ yields

$$\pi_{1\,\text{loop}}(k^2) = -\frac{g^2(\mu)}{2\pi^2} \int_0^1 dx\, x(1-x) \left(\frac{2}{\epsilon} - \gamma + \log\frac{4\pi\mu^2}{\Delta}\right).$$

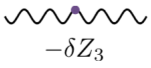
Next, we set the counterterms.

- As argued earlier, we require three counterterms for QED, which are

$$S^{\text{CT}}[A,\psi] = \int dx\, \delta Z_3 \frac{1}{4} F^{\mu\nu}F_{\mu\nu} + \delta Z_2\, \overline{\psi} \not{D} \psi + \delta m\, \overline{\psi}\psi.$$

  Here we've used $Z_1 = Z_2$ to combine the electron field strength renormalization and the vertex renormalization into one term. This guarantees the counterterm Lagrangian is gauge invariant, as it must be if our regulator preserved gauge invariance.

- In $\overline{\text{MS}}$ the $\delta Z_3$ counterterm contributes




  with

$$\delta Z_3 = -\frac{g^2(\mu)}{12\pi^2}\left(\frac{2}{\epsilon} - \gamma + \log 4\pi\right).$$

  Thus, the total one-loop contribution is

$$\pi_{1\,\text{loop}}(k^2) = \frac{g^2(\mu)}{2\pi^2} \int_0^1 dx\, x(1-x) \log\frac{m^2 + x(1-x)k^2}{\mu^2}.$$

- The logarithm produced here is typical for loop corrections. Moreover, it creates a branch cut for the photon propagator in the region $m^2 + x(1-x)k^2 \leq 0$. This is inaccessible for real Euclidean momenta, but in Lorentz signature with $k_0 = iE$ we have

$$x(1-x)(E^2 - \mathbf{k}^2) \geq m^2$$

  so the branch cut starts at $E = 2m$, the threshold for creating a real electron-positron pair.

- Now, reverting to our original normalization of the photon field, where the coefficient of the kinetic term is $1/e^2$, we find

$$\frac{1}{g_{\text{eff}}^2} = \frac{1-\pi(0)}{g^2(\mu)} = \frac{1}{g^2(\mu)} + \frac{\hbar}{12\pi^2}\log\frac{\mu^2}{m^2}.$$

  where we explicitly restore factors of $\hbar$. Note that there is no contribution from the vertex renormalization because, as we've shown, the covariant derivative $D_\mu = \partial_\mu + ieA_\mu$ is renormalized as a 'single piece' (i.e. $Z_1 = Z_2$) and hence is absorbed by a field renormalization of $\psi$ alone, without changing $e$.

- Setting $\partial g_{\text{eff}}/\partial \log \mu$ to zero gives

$$\beta(g) = \frac{\hbar g^3(\mu)}{12\pi^2} + O(\hbar^2), \quad g^2(\mu) = \frac{6\pi^2}{\hbar} \frac{1}{\log(\Lambda_{\text{QED}}/\mu)}, \quad \Lambda_{\text{QED}} \approx 10^{286}\,\text{GeV}$$

  where we set $\alpha(m_e) = g^2(m_e)/4\pi \approx 1/137$. That is, the QED coupling is marginally irrelevant and there is a Landau pole.

**Note.** How are our results here compatible with our earlier analysis of QED? In this section, we think in terms of the 1PI effective action. For simplicity, we set $m = 0$. Then

$$\mathcal{L} = \frac{1}{4} F_0^2 + \overline{\psi}_0 \partial\!\!\!/ \psi_0 + i e_0 \overline{\psi}_0 A\!\!\!/_0 \psi_0.$$

This is gauge invariant because the interactions are built from the covariant derivative $D = \partial + i e_0 A_0$. The effective Lagrangian in Fourier space takes the form

$$\mathcal{L}^{\text{eff}} \supset \frac{1}{4} f_3(q^2) |F_0^q|^2 + \frac{1}{Z_2} f_2(q^2) \overline{\psi}_0^q \partial\!\!\!/ \psi_0^q + i e_0 f_1(q,p) \overline{\psi}_0^{q+p} A\!\!\!/_0{}^q \psi_0^p.$$

Here, we're using an abbreviated notation where superscripts stand for momenta; we get this momentum dependence because the effective action has an infinite series of derivative terms. The parameters $Z_1$, $Z_2$, and $Z_3$ are defined by

$$\frac{1}{Z_3} = f_3(q^2)|_{q^2=0}, \quad \frac{1}{Z_2} = f_2(q^2)|_{q^2=m_e^2}, \quad \frac{1}{Z_1} = f_1(q,p)|_{q^2=0}$$

where $m_e$ is the pole mass of the electron, in this case zero. Therefore, at low momenta, we have

$$\mathcal{L}^{\text{eff}} \supset \frac{1}{4} \frac{1}{Z_3} F_0^2 + \frac{1}{Z_2} \overline{\psi}_0 \partial\!\!\!/ \psi_0 + \frac{i e_0}{Z_1} \overline{\psi}_0 A\!\!\!/_0 \psi_0.$$

We are thus motivated to define the renormalized fields and renormalized/physical coupling

$$A = \frac{A_0}{\sqrt{Z_3}}, \quad \psi = \frac{\psi_0}{\sqrt{Z_2}}, \quad e = \frac{Z_2}{Z_1} \sqrt{Z_3}\, e_0$$

which gives

$$\mathcal{L} = \frac{1}{4} Z_3 F^2 + Z_2 \overline{\psi} \partial\!\!\!/ \psi + i Z_1 e\, \overline{\psi} A\!\!\!/ \psi, \quad \mathcal{L}^{\text{eff}} \supset \frac{1}{4} F^2 + \overline{\psi} \partial\!\!\!/ \psi + i e \overline{\psi} A\!\!\!/ \psi.$$

Therefore, $e$ is a direct measure of the all-orders interaction strength at low momentum. In terms of these rescaled fields and couplings, the covariant derivative in $\mathcal{L}$ is $D = \partial + i(Z_1/Z_2) e A$. On the other hand, $\mathcal{L}^{\text{eff}}$ is built from $\partial + ieA$. Hence conceptually the result $Z_1 = Z_2$ tells us that the effective action is built from covariant derivatives in the same way as the original action.

We are now working solely in terms of physical variables, which is the essential point of renormalization. The last step is to split $\mathcal{L}$ into a finite and "counterterm" part. In renormalized perturbation theory, we set the finite part of $\mathcal{L}$ equal to $\mathcal{L}^{\text{eff}}$ evaluated at on-shell momenta, i.e. low momenta here. The running coupling arises because the coefficient of $\overline{\psi} A\!\!\!/ \psi$ in $\mathcal{L}^{\text{eff}}$ depends on momentum.

In $\overline{\text{MS}}$, we instead set the counterterm to have a simple form; then the parameters in the finite part of $\mathcal{L}$ are not physical parameters; instead we have an '$\overline{\text{MS}}$ mass' and '$\overline{\text{MS}}$ coupling' $e(\mu)$. But as argued earlier, $e(\mu)$ is physically meaningful because it approximates the coefficient of $\overline{\psi} A\!\!\!/ \psi$ in $\mathcal{L}^{\text{eff}}$ at momentum scale $\mu$.

**Note.** The $\overline{\text{MS}}$ scheme is a bit strange, since it does not have "decoupling". The coupling continues to run even when we are far below the mass of the electron, even though one would intuitively think the electron should have no effect at these energies. As another example, applying $\overline{\text{MS}}$ to the $SU(5)$ GUT, all three gauge couplings are equal at all energies, since $\overline{\text{MS}}$ knows nothing about the symmetry breaking. This is perfectly consistent, as the full perturbation series in principle sums to the same thing, but not physically transparent.

An alternative scheme that makes the decoupling explicit is to demand that the quantum corrections vanish at an arbitrary scale $k^2 = \mu^2$, which is the MOM/momentum space subtraction scheme. That is, we set

$$\delta Z_3 = -\frac{\hbar g^2(\mu)}{2\pi^2}\int_0^1 dx\, x(1-x)\left(\frac{2}{\epsilon} - \gamma + \log\frac{4\pi\mu^2}{m^2 + x(1-x)\mu^2}\right)$$

so the total one-loop contribution is

$$\pi(k^2) = \frac{\hbar g^2(\mu)}{2\pi^2}\int_0^1 dx\, x(1-x)\log\left(\frac{m^2 + x(1-x)k^2}{m^2 + x(1-x)\mu^2}\right) + O(\hbar^2).$$

Then the beta function is

$$\beta(\mu) = \frac{g^3(\mu)}{2\pi^2}\int_0^1 dx\, \frac{x^2(1-x)^2\mu^2}{m^2 + x(1-x)\mu^2}$$

which indeed goes to zero when $\mu \ll m$, freezing $\alpha \approx 1/137$. This is an example of the Appelquist–Carazzone decoupling theorem. In general, a mass-dependent (or "physical") scheme is one where the beta function depends explicitly on the renormalization scale $\mu$, and hence explicitly on the particle mass $m$. Decoupling only holds for mass-dependent schemes, which include this scheme and a hard cutoff, where $\Lambda$ plays the role of the renormalization scale.

The disadvantage of mass-dependent schemes is that perturbation theory is harder; we've seen that for a hard cutoff the loops are not small. Thus in practice, a modified version of $\overline{\text{MS}}$ is used, where the decoupling is put in by hand. We consider two theories, one with an electron, valid for $\mu > m_e$, and one with no electron, valid for $\mu < m_e$. We perform renormalization in both theories with $\overline{\text{MS}}$ or another mass-independent scheme and match physical quantities at $\mu = m_e$, such as $S$-matrix elements. At leading order, this means that $g(\mu)$ is continuous at $\mu = m_e$, and stops running below $m_e$. However, at higher orders, or if we choose $\mu \neq m_e$, the coupling jumps discontinuously at the matching. This is a "threshold correction", and must be computed carefully to get, e.g. gauge coupling unification to work out correctly.

This method is called "continuum EFT" in contrast to "Wilsonian EFT", since we never have a momentum cutoff. The idea is so common today that it is often used implicitly. For example, the QCD beta function is written in terms of $n_f$, the "number of light quarks". But in $\overline{\text{MS}}$ all of the quarks contribute in the same way to the beta function, regardless of mass. The standard $g(\mu)$ for QCD is actually using successive matchings, integrating out each heavy quark as we pass its mass.

**Note.** The first two terms of the beta function, corresponding to the one-loop and two-loop contributions, are independent of the subtraction scheme, provided that we maintain canonical normalization. Changing the normalization clearly adds terms to the beta function even at tree level; in this case that corresponds to setting $Z = 1$ at the physical mass pole. We did this for $\overline{\text{MS}}$ but not for our other scheme, which instead sets $Z = 1$ at $\mu$.

**Note.** In $\overline{\text{MS}}$, the mass is not the same as the pole mass, so what is its physical meaning? One can think of the mass term as like any other renormalized coupling. For example, the up quark

mass provides an extra two-point vertex in QCD scattering problems. The $\overline{\text{MS}}$ mass at scale $\mu$ approximates the overall all-order effect of this interaction term at tree level. (Note that there is nothing preventing $\overline{\text{MS}}$ mass terms from being negative!) The PDG gives most quark masses as $\overline{\text{MS}}$ masses (the universal choice for modern theory calculations), since there is no way to define a pole mass for them. But its top quark mass is a "Pythia" mass, meaning it's simply what one gets by fitting to Pythia, which has no direct link to any theoretical calculation. When speaking of the mass of any fundamental particle, we need to specify the scheme in which it's defined.

Finally, we prove that $Z_1 = Z_2$ using the Ward–Takahashi identity.

- Starting from the global $U(1)$ symmetry of QED,

$$\psi \to \psi' = e^{i\alpha\psi}, \quad j^\mu = i\overline{\psi}\gamma^\mu\psi$$

we may apply the Ward–Takahashi identity to find

$$\partial_\mu\langle j^\mu(x)\psi(x_1)\overline{\psi}(x_2)\rangle = -i\delta(x-x_1)\langle\psi(x_1)\overline{\psi}(x_2)\rangle + i\delta(x-x_2)\langle\psi(x_1)\overline{\psi}(x_2)\rangle.$$

Note that the presence of gauge fixing terms makes no difference here, because we're only considering global symmetry transformations. However, we will see an alternate derivation below that uses the local symmetry.

- We define the exact electron propagator by Fourier transforming the two-point function,

$$\langle\psi(k_1)\overline{\psi}(-k_2)\rangle \equiv \int dx_1\, dx_2\, e^{ik_1x_2}e^{-ik_2x_2}\langle\psi(x_1)\overline{\psi}(x_2)\rangle = \not\delta(k_1-k_2)S(k)$$

where we defined

$$S(k) = \int dx\, e^{ikx}\langle\psi(x)\overline{\psi}(0)\rangle = \frac{1}{i\not k + m - \Sigma(\not k)}$$

and $\Sigma(\not k)$ is the sum of 1PI diagrams.

- We define the exact vertex by

$$\langle j_\mu(k)\psi(k_1)\overline{\psi}(-k_2)\rangle \equiv \not\delta(p+k_1-k_2)S(k_1)\Gamma_\mu(k_1,k_2)S(k_2).$$

To understand this expression, consider computing the left-hand side diagrammatically. Here, it can be useful conceptually to draw a 'phantom' photon attached to $x$, though no such photon field actually appears in the correlator. The lowest-order term is $\gamma_\mu$, so indeed

$$\Gamma_\mu(k_1,k_2) = \gamma_\mu + \text{quantum corrections}.$$

At one-loop order, either the $\psi$ and $\overline{\psi}$ can emit and reabsorb a photon. This has nothing to do with the vertex, and is accordingly absorbed by the $S(k_1)$ and $S(k_2)$ factors. Recalling that 1PI and amputated diagrams are equivalent for three external fields, we end up computing the sum of 1PI diagrams with an external photon and two external fermions.

- Finally, taking the Fourier transform of the Ward–Takahashi identity,

$$(k_1-k_2)_\mu S(k_1)\Gamma^\mu(k_1,k_2)S(k_2) = iS(k_1) - iS(k_2)$$

which is equivalent to

$$(k_1-k_2)\Gamma^\mu(k_1,k_2) = iS^{-1}(k_2) - iS^{-1}(k_1).$$

This result relates the vertex to electron kinetic term in the 1PI effective action.

- To extract the desired result, differentiate with respect to $k_1$ and then take $k_1, k_2 \to k$ for

$$\Gamma_\mu(k,k) = -i\frac{\partial}{\partial k^\mu} S^{-1}(k) = \gamma_\mu + i\frac{\partial}{\partial k^\mu}\Sigma(\not{k}).$$

  Thus, at lowest order in $k$, the effective action looks like

$$S^{\text{eff}} \supset \overline{\psi}\left(\gamma_\mu + i\frac{\partial}{\partial k^\mu}\Sigma(\not{k})\right) D\psi$$

  so the covariant derivative $D\psi$ is renormalized as a single piece. Expanding our result in a series in $k$ shows that interactions must come in the form $\overline{\psi} D^p \psi$.

## 10.4 The Euler–Heisenberg Lagrangian

Finally, we sketch the derivation of the Euler–Heisenberg Lagrangian, which results from integrating out the electron. Remarkably, this result was first derived in 1936, long before renormalization was understood or Feynman diagrams were introduced.

- We expect we should be able to integrate out the electron because it appears quadratically in the QED action,

$$S[A,\psi] = \int dx\, \frac{1}{4} F^{\mu\nu}F_{\mu\nu} + \overline{\psi}(\not{D} + m)\psi.$$

  Since the electron is fermionic, we get a functional determinant in the numerator,

$$\Gamma_{\text{eff}}[A] = \frac{1}{4}\int dx\, F^{\mu\nu}F_{\mu\nu} - \hbar \log\det(\not{D} + m).$$

  Here, the determinants are in both spinor space and functional space.

- To understand the functional determinant better, note that we can expand

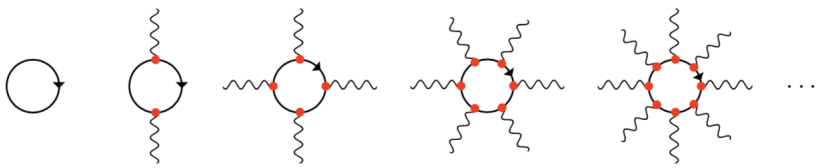
$$\log\det(\not{D} + m) = \log\det(\not{\partial} + m) + \operatorname{tr}\log(1 + ie(\not{\partial} + m)^{-1}\not{A})$$

  where we used $\log\det M = \operatorname{tr}\log M$. The first term comes from an electron loop vacuum bubble. The other term gives an infinite series in powers of $A$,

$$\operatorname{tr}\log(1 + ie(\not{\partial} + m)^{-1}\not{A}) = -\sum_{n=1}^{\infty}\frac{(-ie)^n}{n}\int\prod_{i=1}^{n} dx_i\, \operatorname{tr}(S(x_n,s_1)\not{A}(x_1)\dots S(x_{n-1},x_n)\not{A}(x_n))$$

  where the trace on the right is just a Dirac trace, and the functional trace is handled by the integration. All terms odd in $A$ vanish by Furry's theorem, and $S(x_i, x_{i+1})$ is the Dirac propagator, i.e. the inverse of the kinetic term.

- Thus the diagrams that contribute are



  so the effective action has interaction terms for any even number of photons. To make the calculation tractable, we will assume the electromagnetic field is constant while taking the trace, which is equivalent to neglecting derivative terms; from an effective field theory standpoint this is valid if we consider light with frequency much less than the electron mass.

- As a result, the first diagram contributes only to vacuum energy, the second renormalizes the photon field, and all the others produce new interactions, which mediate light-by-light scattering. Thus the Euler–Heisenberg Lagrangian is useful when considering high-intensity, low-frequency light.

- We now turn to the explicit evaluation of $\log\det(\not{D}+m)$. Since the trace of an odd number of gamma matrices vanishes, we must have

$$\operatorname{tr}\log(\not{D}+m)=\operatorname{tr}\log(-\not{D}+m)$$

  which gives

$$\operatorname{tr}\log(\not{D}+m)=\frac{1}{2}\left(\log\det(\not{D}+m)+\log\det(-\not{D}+m)\right)=\frac{1}{2}\log\det(-\not{D}^2+m^2).$$

- Next we can simplify $\not{D}^2$ as

$$\not{D}^2=\gamma^\mu\gamma^\nu D_\mu D_\nu=\left(\frac{1}{2}\{\gamma^\mu,\gamma^\nu\}+\frac{1}{2}[\gamma^\mu,\gamma^\nu]\right)D_\mu D_\nu=D^\mu D_\mu-eS^{\mu\nu}F_{\mu\nu}.$$

  The same manipulations yield the electron's magnetic moment in the Dirac equation.

- Therefore, we want to evaluate the functional

$$\begin{aligned}&\frac{1}{2}\operatorname{tr}\log(-D^2+eS^{\mu\nu}F_{\mu\nu}+m^2)\\&\qquad=\frac{1}{2}\operatorname{tr}\log\left(-((\partial+ieA)^2+m^2)\begin{pmatrix}1&0\\0&1\end{pmatrix}+e\begin{pmatrix}(\mathbf{B}+i\mathbf{E})\cdot\boldsymbol{\sigma}&0\\0&(\mathbf{B}-i\mathbf{E})\cdot\boldsymbol{\sigma}\end{pmatrix}\right)\end{aligned}$$

  where the $\boldsymbol{\sigma}$ are the usual Pauli matrices and the electromagnetic field is constant.

- Suppressing the spinor indices temporarily, we can perform the functional trace in position space, giving

$$\Gamma_{\text{eff}}[A]=\int dx\,\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\frac{1}{2}\langle x|\log(-D^2+eS^{\mu\nu}F_{\mu\nu}+m^2|x\rangle.$$

  To remove the logarithm, we use the asymptotic relation

$$\lim_{s_0\to0^+}\int_{s_0}^\infty e^{-sX}\frac{ds}{s}=-\log(s_0X)+\text{finite}$$

  to find

$$\frac{1}{2}\langle x|\log(-D^2+eS^{\mu\nu}F_{\mu\nu}+m^2)|x\rangle=\lim_{s_0\to0^+}\frac{1}{2}\int_{s_0}^\infty\frac{ds}{s}e^{-sm^2/2}\langle x|e^{-s(-D^2+eS^{\mu\nu}F_{\mu\nu})}|x\rangle.$$

  Thus the problem has been reduced to nonrelativistic quantum mechanics; we essentially have to find the energy eigenstates of the Pauli equation in position space. For example, in the case of a constant magnetic field, the solutions are Landau levels.

- A complicated calculation yields the effective Lagrangian

$$\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \lim_{s_0\to 0^+}\frac{e^2}{64\pi^2}\int_{s_0}^{\infty}\frac{ds}{s}e^{-sm^2}\frac{\operatorname{Re}\cosh(esX)}{\operatorname{Im}\cosh(esX)}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}, \quad X = \sqrt{\mathbf{E}^2 - \mathbf{B}^2 + \mathbf{E}\cdot\mathbf{B}}.$$

  Finally, we need to handle the UV divergences, which occur for small $s$. Expanding the integrand as a series in $es$, we find

$$\frac{\operatorname{Re}\cosh(esX)}{\operatorname{Im}\cosh(esX)}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = \frac{4}{e^2s^2} + \frac{2}{3}F^{\mu\nu}F_{\mu\nu} - \frac{e^2s^2}{45}\left((F^{\mu\nu}F_{\mu\nu})^2 + \frac{7}{4}(\widetilde{F}^{\mu\nu}F_{\mu\nu})^2\right) + O(e^4)$$

  and the first two terms are a vacuum energy and photon field renormalization, which we remove by minimal subtraction, and all other terms are finite. This yields the Euler–Heisenberg Lagrangian. It is fully nonperturbative in $e$, since the real expansion parameter is $E/m_e$.

**Note.** One might wonder what the most general terms in the series above look like. In particular, is it true that every term is built out of $F^2 \sim E^2 - B^2$ and $F\tilde{F} \sim \mathbf{E}\cdot\mathbf{B}$? One slick way to prove this is to consider the complex vector $\mathbf{E} + i\mathbf{B}$. Boosts correspond to imaginary rotations of this vector, so rotations and boosts together cover all possible rotations. But the only invariant of a vector with respect to rotation is its square, and

$$(\mathbf{E} + i\mathbf{B})\cdot(\mathbf{E} + i\mathbf{B}) = (E^2 - B^2) + 2i(\mathbf{E}\cdot\mathbf{B}).$$

Note that this proof only works in $d = 4$.

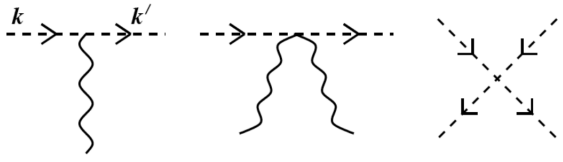
## 10.5 Scalar QED

As a final example, we'll consider the one-loop renormalization of scalar QED in $(+---)$ signature with $\overline{\text{MS}}$. Note that switching from MS to $\overline{\text{MS}}$ is equivalent to using $\mu^2_{\overline{\text{MS}}} = \mu^2_{\text{MS}} 4\pi e^{-\gamma}$ throughout, so if we define $\mu = \mu_{\overline{\text{MS}}}$, we won't need to add extra "macaroni and pie" to the counterterms. We will use a new technique, to be explained below, that can compute the beta functions from only the divergent parts of the counterterms, hence displaying the simplicity of the $\overline{\text{MS}}$ scheme.

- The scalar QED Lagrangian is

$$\mathcal{L} = |\partial_\mu\phi|^2 - m^2|\phi|^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\lambda}{4}|\phi|^4 - ieA_\mu(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) + e^2A_\mu A^\mu\phi^*\phi.$$

  Here we must include a $\phi^4$ interaction because it is consistent with the symmetries, or equivalently because it may be generated by the other interaction terms. The Feynman rules are:




$-ie(k+k')_\mu$ $\quad 2ie^2 g_{\mu\nu}$ $\quad -i\lambda$

$\dfrac{-ig_{\mu\nu}}{k^2+i\epsilon}$

$\dfrac{i}{k^2-m^2+i\epsilon}$

$\epsilon^\mu_{\lambda_i}(k), \epsilon^{\mu*}_{\lambda_i}(k)$ for incoming and outgoing photons respectively

However, for this analysis, we will work in Landau gauge, so the photon propagator comes with a factor of $P_{\mu\nu} = \eta_{\mu\nu} - q_\mu q_\nu/q^2$ rather than just $\eta_{\mu\nu}$. As we'll see, this sets many diagrams to zero, shortening the analysis.

- Relabeling all the fields above as bare fields with 0 subscripts, and defining physical fields as

$$\phi = \frac{\phi_0}{\sqrt{Z_2}}, \quad A = \frac{A_0}{\sqrt{Z_3}}, \quad e = \frac{Z_2}{Z_1}\sqrt{Z_3}e_0$$

we have

$$\mathcal{L} = Z_2|\partial_\mu\phi|^2 - Z_m m^2|\phi|^2 - \frac{Z_3}{4}F^{\mu\nu}F_{\mu\nu} \\ - Z_\lambda\frac{\lambda}{4}|\phi|^4 - iZ_1 eA_\mu(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) + Z_4 e^2 A_\mu A^\mu\phi^*\phi.$$

Here we have $Z_4 = Z_1^2/Z_2$, and we should have $Z_1 = Z_2$. In the calculations below we will not assume these results, but rather directly compute the $Z_i$ at one-loop order in DR and $\overline{\text{MS}}$ to demonstrate they hold.

- Defining $Z_i = 1 + \delta_i$, it is convenient to split the Lagrangian as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_{\text{ct}}$$

where

$$\mathcal{L}_0 = |\partial_\mu\phi|^2 - m^2|\phi|^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \quad \mathcal{L}_{\text{ct}} = \delta_2|\partial_\mu\phi|^2 - \delta_m m^2|\phi|^2 - \frac{\delta_3}{4}F^{\mu\nu}F_{\mu\nu}$$

and $\mathcal{L}_1$ contains the interactions. Below we will work to order $O(e^2)$ and $O(\lambda)$ in the $Z_i$.

First, we consider the photon propagator.

- As we've seen before, the exact photon propagator remains transverse in Landau gauge, as

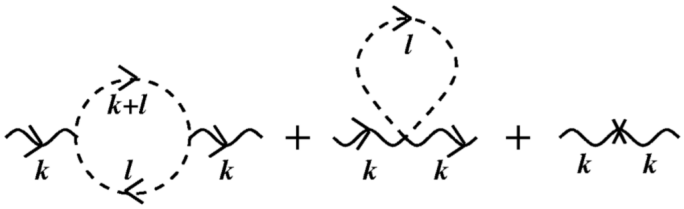
$$\Delta_{\mu\nu}(k) = \frac{\Delta^0_{\mu\nu}(k)}{1 - \pi(k^2)}$$

where $\pi(k^2)$ is determined by the sum of 1PI diagrams,

$$\Pi_{\mu\nu}(k^2) = k^2\pi(k^2)P_{\mu\nu}$$

which is transverse by the Ward identity.

- At order $e^2$, we have the contributing diagrams:



- Since $Z_i = 1 + O(e^2)$, we can replace $Z_1$ and $Z_4$ in the first two diagrams with just 1, giving

$$i\Pi^{\mu\nu}(k^2) = (-ie)^2 i^2 \int đ\ell \frac{(2\ell+k)^\mu (2\ell+k)^\nu}{(\ell+k)^2 - m^2)(\ell^2 - m^2)} + (2i)e^2 \eta^{\mu\nu} i \int \frac{đ\ell}{\ell^2 - m^2} - i(Z_3 - 1)(k^2\eta^{\mu\nu} - k^\mu k^\nu).$$

  The first two terms can be combined into

$$e^2 \int đ\ell \, \frac{N^{\mu\nu}}{((\ell+k)^2 - m^2)(\ell^2 - m^2)}, \quad N^{\mu\nu} = (2\ell+k)^\mu (2\ell+k)^\nu - 2((\ell+k)^2 - m^2)\eta^{\mu\nu}.$$

- Applying Feynman parameters as usual, this term becomes

$$e^2 \int_0^1 dx \int đq \, \frac{N^{\mu\nu}}{(q^2 - \Delta)^2}, \quad q = \ell + xk, \quad \Delta = -x(1-x)k^2 + m^2.$$

  Replacing $\ell$ with $q$ in $N^{\mu\nu}$, it simplifies to

$$N^{\mu\nu} \sim 4q^\mu q^\nu + (1-2x)^2 k^\mu k^\nu - 2(q^2 + (1-x)^2 k^2 - m^2)\eta^{\mu\nu}$$

  where we removed terms odd in $q$ as they integrate to zero. We replace $q^\mu q^\nu \to q^2 \eta^{\mu\nu}/D$ by isotropy, leaving two standard DR integrals.

- Performing these integrals with the standard formulas and then performing the $x$ integral gives

$$i\Pi^{\mu\nu}(k^2) = ie^2 \mu^\epsilon (k^\mu k^\nu - k^2 \eta^{\mu\nu}) \left( Z_3 - 1 + \frac{1}{3} \frac{1}{(4\pi)^{d/2}} \Gamma\left(\frac{4-d}{2}\right) + \text{finite} \right)$$

  where $\epsilon = 4 - d$, with the transverse structure required by the Ward identity. The finite parts are momentum-dependent and can be used to compute the running coupling, as in QED.

- In this section, we'll work with $\overline{\text{MS}}$, and hence can ignore the finite parts. Then we have

$$Z_3 = 1 - \frac{e^2}{24\pi^2} \frac{1}{\epsilon}$$

  at one loop. We will use this later to recover the running coupling by another way.

**Note.** In QED, we used the Ward–Takahashi identity associated with the global $U(1)$ symmetry to derive $Z_1 = Z_2$. For variety, we can do the same here using a somewhat different setup, essentially using the Schwinger–Dyson equation corresponding to a $U(1)$ gauge transformation. In this case, we must account for the gauge fixing term, which in $R_\xi$ gauge is

$$\mathcal{L} \supset \mathcal{L}_{\text{gf}} = -\frac{1}{2\xi} (\partial_\mu A^\mu)^2.$$

The bare photon propagator is now

$$\Delta^0_{\mu\nu}(k) = \frac{1}{k^2} \left( \eta_{\mu\nu} - (1-\xi) \frac{k_\mu k_\nu}{k^2} \right) = \frac{1}{k^2} \left( P_{\mu\nu}(k) + \xi \frac{k_\mu k_\nu}{k^2} \right).$$

Now, the sum of 1PI diagrams remains transverse, so the additional non-transverse piece is unaffected when we sum over 1PI diagrams, giving the exact propagator

$$\Delta_{\mu\nu}(k) = \frac{P_{\mu\nu}}{k^2(1-\pi(k^2))} + \xi\frac{k_\mu k_\nu}{k^2}.$$

The strategy we will use to derive our relations is to start with the generating functional with currents. Schematically, consider some local symmetry $\phi \to \phi' = \phi + \epsilon(x)\delta\phi$ under which the path integral measure is invariant. Then we have

$$Z[J] = \int \mathcal{D}\phi\, e^{i(S[\phi]+\phi J)} = \int \mathcal{D}\phi'\, e^{i(S[\phi']+\phi' J)} = \int \mathcal{D}\phi\, e^{i(S[\phi]+\phi J)} \left(1 + i\left(\delta S + \int dx\, J\delta\phi\right)\right).$$

Therefore, we must have

$$\int \mathcal{D}\phi\, e^{i(S[\phi]+\phi J)} \left(\delta S + \int dx\, J\delta\phi\right) = 0.$$

Integrating by parts to get an overall factor of $\epsilon(x)$, we can use the fact that $\epsilon(x)$ is arbitrary to peel it off and remove the $dx$ integral, leaving an correlation function identity. Since we are working with the generating functional, we can differentiate with respect to $J$ to yield identities for higher-point functions.

In this context, we will use the gauge transformation

$$\delta\phi(x) = ie\alpha(x)\phi(x), \quad \delta A_\mu(x) = \partial_\mu\alpha(x).$$

In this case $\delta S$ is solely due to the gauge fixing term $\mathcal{L}_{\text{gf}}$,

$$\delta S = -\int dx\, \frac{1}{\xi}\alpha(x)\partial^2\partial_\mu A^\mu(x).$$

We also get three terms from the variations of the currents, so we have, suppressing $x$ arguments,

$$\int \mathcal{D}\phi\mathcal{D}\phi^*\mathcal{D}A_\mu\, e^{i(S[\phi,A_\mu]+\phi J^*+\phi^* J+J^\mu A_\mu)} \int dx \left(-\frac{1}{\xi}\alpha\partial^2\partial_\mu A^\mu\right) + J^*\delta\phi + J\delta\phi^* - iJ^\mu\delta A_\mu = 0.$$

Integrating by parts to get a factor of $\alpha(x)$ and using the fact that $\alpha(x)$ is arbitrary, we have

$$\left\langle -\frac{1}{\xi}\partial^2\partial_\mu A^\mu + ie\phi J^* - ie\phi^* J + i\partial_\mu J^\mu \right\rangle = 0.$$

Dividing by $Z[J^\mu, J^*, J]$ puts this in terms of the connected generating functional $W[J^\mu, J^*, J]$,

$$\frac{1}{\xi}\partial^2\partial_\mu\frac{\delta W}{\delta J_\mu(x)} - ieJ^*(x)\frac{\delta W}{\delta J^*(x)} + ieJ(x)\frac{\delta W}{\delta J(x)} + \partial_\mu J^\mu(x) = 0.$$

Acting with $(\delta/\delta J^*(y))(\delta/\delta J(z))$ on both sides and setting the currents to zero,

$$\frac{1}{\xi}\partial^2\partial_\mu\langle A_\mu(x)\phi(y)\phi(z)\rangle_c + ie\delta(y-x)\langle\phi(x)\phi^*(z)\rangle_c - ie\delta(z-x)\langle\phi^*(x)\phi(y)\rangle_c = 0.$$

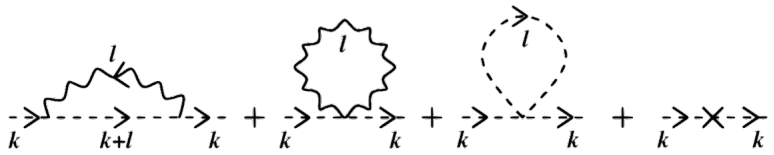
The last two terms are exact scalar propagators, while the first term is the product of the exact photon propagator, two exact scalar propagators, and the exact three-point vertex. However, upon a Fourier transform the derivatives on the photon propagator yield

$$\frac{1}{\xi}k^2 k^\mu\Delta_{\mu\nu}(k) = k^\mu.$$

This reduces the identity we have here to the Ward–Takahashi identity we saw earlier for QED, and the proof continues in the same way.

Next, we consider the scalar propagator, where Landau gauge will be somewhat useful.

- The scalar propagator $i\Pi_\phi(k^2)$ has five contributions:




  Here, the final diagram contains both the $\delta_2$ and $\delta_m$ counterterms.

- The second diagram is a "scaleless" integral, and using the DR formulas gives an answer of zero. But this is too quick: the DR formulas are not actually valid at all, because there does not exist *any* value of $d$ for which a scaleless integral converges (i.e. it is always either UV or IR divergent), so there are no values of $d$ to analytically continue from!

- A glib way to justify the result is to say "the DR answer has to depend on $\mu$, but there are no other mass scales in the integral, so what else could it be?" A bit more seriously, the integral can be IR regulated, in which case the DR formulas legitimately give

$$i\Pi_\phi(k^2) \supset (2Z_4 ie^2\eta^{\mu\nu})(-i)\int đ\ell \frac{P_{\mu\nu}}{\ell^2 - m_\gamma^2}.$$

  The resulting integral is proportional to $m_\gamma^2/\epsilon$ in DR, which vanishes in the limit $m_\gamma \to 0$.

- The first diagram contributes

$$i\Pi_\phi(k^2) \supset (-iZ_1 e)^2(-i)i\int đ\ell\, \frac{P_{\mu\nu}(\ell)(\ell+2k)^\mu(\ell+2k)^\nu}{\ell^2((\ell+k)^2 - m^2)}$$

  in Landau gauge, and since $\ell_\mu P^{\mu\nu} = 0$ we have

$$i\Pi_\phi(k^2) \supset -e^2\int đ\ell\, \frac{4(\ell^2k^2 - (k\cdot\ell)^2)}{\ell^4((\ell+k)^2 - m^2)}$$

  where we again set $Z_1 = 1$ since we only want the $O(e^2)$ terms.

- The denominator can be simplified using Feynman parameters,

$$\frac{1}{\ell^4((\ell+k)^2 - m^2)} = 2\int_0^1 dx_1 \int_0^{1-x_1} dx_2\, \frac{dx_1 dx_2}{((1-x_1-x_2)\ell^2 + x_2\ell^2 + x_1((\ell+k)^2 - m^2))^3}.$$

  The $x_2$ dependence drops out, so evaluating the $x_2$ integral and renaming $x_1$ to $x$,

$$\frac{1}{\ell^4((\ell+k)^2 - m^2)} = 2\int \frac{(1-x)\,dx}{(\ell^2 + 2x\ell\cdot k + xk^2 - xm^2)^3}.$$

  Using the usual shift $q = \ell + xk$ we have

$$i\Pi_\phi(k^2) \supset -8e^2\int_0^1 dx \int đq \frac{q^2k^2 - (q\cdot k)^2}{(q^2-\Delta)^3}, \quad \Delta = -x(1-x)k^2 + m^2.$$

  Applying isotropy and a standard DR formula we find

$$i\Pi_\phi(k^2) \supset -i\frac{3e^2}{8\pi^2}\frac{1}{\epsilon}\,k^2.$$

- Next, for the scalar loop contribution, we have

$$i\Pi_\phi(k^2) \supset (-i\lambda)(i)\int \frac{đ\ell}{\ell^2 - m^2} = \frac{i\lambda}{8\pi^2}\frac{1}{\epsilon}m^2$$

  by a standard DR formula.

- For comparison, the counterterms contribute

$$i\Pi_\phi(k^2) \supset i\delta_2 k^2 - i\delta_m m^2$$

  from which we conclude

$$Z_2 = 1 + \frac{3e^2}{8\pi^2}\frac{1}{\epsilon}, \quad Z_m = 1 + \frac{\lambda}{8\pi^2}\frac{1}{\epsilon}.$$

**Note.** A bit more about scaleless integrals in DR. The argument above is sketchy, because it requires a separate IR regulator; in principle DR can handle both UV and IR divergences by itself. Within pure DR, the justification for setting the integral to zero is the 't Hooft–Veltman conjecture, which simply states that "no inconsistencies" arise from doing this.

A bit more precisely, a proper treatment of DR would begin with a list of postulates that DR integrals must obey, just as we begin with postulates for Grassmann integration. Changing notation for clarity, these postulates are linearity, shift invariance, and the scaling

$$\int d^d\mathbf{p}\, f(\lambda\mathbf{p}) = \lambda^{-d}\int d^d\mathbf{p}\, f(\mathbf{p}).$$

It can be shown (e.g. in Collins) that these postulates essentially uniquely determine the DR prescription. The result is equivalent to how we've presented DR earlier: take the measure to be $d^d\mathbf{p} = p^{d-1}\Omega_d\, dp$, do the integral for the $d$ where it is defined, and analytically continue in $d$.

Now, using linearity, we can split the scaleless integral as

$$I = \int \frac{đ\ell}{\ell^2} = \int đ\ell\, \frac{\ell^2}{\ell^2(\ell^2 - m^2)} - \int đ\ell\, \frac{m^2}{\ell^2(\ell^2 - m^2)}$$

where $m$ is an arbitrary scale, thereby converting it into two ordinary DR integrals. Both integrals are convergent in an open region of the complex plane, so they may be analytically continued as usual, and the standard DR formulas apply as $d \to 4$. We find that as long as the $\epsilon$'s and $\mu$'s used in each are the same, the two results cancel, yielding the desired result. Further arguments can be used to show that, due to the DR postulates, $\int đ\ell\, (\ell^2)^\alpha = 0$ for any $\alpha$.

Of course, in this view the 't Hooft–Veltman conjecture simply reduces to the conjecture that these postulates do not lead to inconsistencies. (This is by no means guaranteed, as inconsistencies can arise when we try to account for spinors and the associated $\gamma^\mu$.) The "proof" for our purposes is simply experience. However, it turns out that it will eventually lead to trouble, in the form of "renormalons", i.e. poor convergence at large orders in perturbation theory, which are connected to the fact that the QFT perturbation series is asymptotic.

**Note.** There are more subtleties for integrals which are simultaneously UV and IR divergent, like

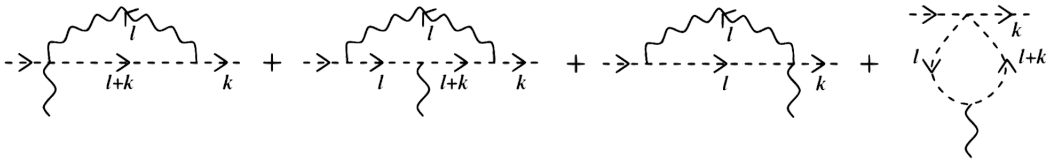
$$I = \int \frac{đ\ell}{\ell^4}.$$

Using the same trick as above, we can split this into a UV divergent and an IR divergent integral, which are each evaluated in DR with $(\epsilon_{\text{UV}}, \mu_{\text{UV}})$ and $(\epsilon_{\text{IR}}, \mu_{\text{IR}})$ respectively. The integrals converge

for $\epsilon_{\text{UV}} > 0$ and $\epsilon_{\text{IR}} < 0$ respectively, and the integral is zero when we take $\epsilon_{\text{UV}} = \epsilon_{\text{IR}}$. The subtlety here is that this integral cannot simply be ignored, because the counterterm, being a UV quantity, only can cancel the $1/\epsilon_{\text{UV}}$ pole; something else is required to cancel the $1/\epsilon_{\text{IR}}$ pole. That is, the UV and IR divergences are conceptually distinct. **(think more about this)**

Now we consider the vertices, where clever choices for the external momenta will greatly simplify the calculation in Landau gauge.

- First, we consider the one-loop corrections to the three-point vertex $V_3^\mu$.




  These are renormalized by $Z_1$. Since we only care about the divergence structure, we are free to choose any external momenta we want, so we choose the incoming $\phi$ to have zero momentum and the outgoing $\phi$ to have $k$. Then in the second and third diagrams, the left vertex yields a factor of $\ell_\mu$ which vanishes when contracted with the $P^{\mu\nu}(\ell)$ from the photon propagator in Landau gauge.

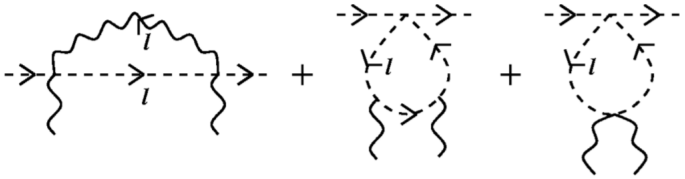
- The remaining terms take the form

$$\begin{aligned} iV_3^\mu(k,0) = ieZ_1 k^\mu + (iZ_1 e)(-2iZ_4 e^2\eta^{\mu\nu})(-i)^2 \int đ\ell\, \frac{P_{\nu\rho}(\ell)(\ell+2k)^\rho}{\ell^2((\ell+k)^2+m^2)} \\ + (-iZ_\lambda\lambda)(iZ_1 e)(-i)^2 \int đ\ell\, \frac{(2\ell+k)^\mu}{(\ell^2+m^2)((\ell+k)^2+m^2)}. \end{aligned}$$

  Upon Feynman parametrization, shifting the loop momentum, and integrating over $x$, the final term is proportional to $\ell^\mu$ and hence vanishes by symmetry. Evaluating the remaining integral by the usual methods gives

$$Z_1 = 1 + \frac{3e^2}{8\pi^2}\frac{1}{\epsilon}.$$

- Next, consider the four-point, scalar-scalar-photon-photon vertex. In this case, a convenient choice is to set all external momenta to zero, so that only five diagrams survive.




  For each of the first two diagrams, there is another diagram which swaps how the photon lines attach. Then we have

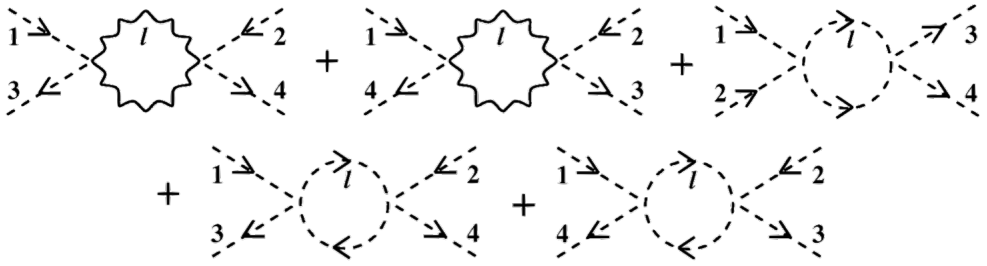
$$\begin{aligned} iV_4^{\mu\nu}(0,0,0) = -2iZ_4 e^2\eta^{\mu\nu} + (-2iZ_4 e^2)^2(-i)^2 \int đ\ell\, \frac{P^{\mu\nu}}{\ell^2(\ell^2+m^2)} + (\mu \leftrightarrow \nu) \\ + (iZ_1 e)^2(-iZ_\lambda\lambda)(-i)^3 \int đ\ell\, \frac{(2\ell)^\mu(2\ell)^\nu}{(\ell^2+m^2)^3} + (\mu \leftrightarrow \nu) \\ + (-iZ_\lambda\lambda)(-2iZ_4 e^2\eta^{\mu\nu})(-i)^2 \int đ\ell\, \frac{1}{(\ell^2+m^2)^2}. \end{aligned}$$

All of these may be evaluated in the usual ways to find

$$Z_4 = 1 + \frac{3e^2}{8\pi^2}\frac{1}{\epsilon}.$$

We've now shown $Z_1 = Z_2 = Z_4$, as expected and required by general considerations above.

- Finally, consider the four-scalar vertex. Again we may set all external momenta to zero, giving the remaining diagrams below.




Note that the first three have a symmetry factor of 2. The first two and the last three all have the same structure, giving

$$\begin{aligned} iV_4(0,0,0) = -iZ_\lambda\lambda &+ \left(\frac{1}{2}+\frac{1}{2}\right)(-2iZ_4e^2)^2(-i)^2\int đ\ell\,\frac{P^{\mu\nu}(\ell)P_{\mu\nu}(\ell)}{(\ell^2+m_\gamma^2)^2} \\ &+ \left(\frac{1}{2}+1+1\right)(-iZ_\lambda\lambda)^2(-i)^2\int đ\ell\,\frac{1}{(\ell^2+m^2)^2} \end{aligned}$$

which may be evaluated in the usual ways to find

$$Z_\lambda = 1 + \left(\frac{3e^4}{2\pi^2\lambda} + \frac{5\lambda}{16\pi^2}\right)\frac{1}{\epsilon}.$$

We now consider the running couplings.

- One of the best features of $\overline{\text{MS}}$ is that we can read off the beta functions directly from the $Z_i$, without ever having to compute the momentum-dependence of the vertices. We note that

$$e_0 = Z_3^{-1/2}\mu^{\epsilon/2}e, \quad \lambda_0 = Z_2^{-2}Z_\lambda\mu^\epsilon\lambda.$$

The $Z_i$ may be expanded in a series in the couplings $e^2$ and $\lambda$, and in $1/\epsilon$.

- The key result is that the $e_0$ and $\lambda_0$ are independent of $\mu$. Now note that

$$\log e_0 = \log(Z_3^{-1/2}) + \log e + \frac{\epsilon}{2}\log\mu + \ldots = \frac{e^2}{48\pi^2}\frac{1}{\epsilon} + \log e + \frac{\epsilon}{2}\log\mu + \ldots.$$

Differentiating with respect to $\log\mu$ on both sides and letting $\beta_e = de/d\log\mu$, we have

$$0 = \frac{\epsilon e}{2} + \beta + \frac{e^2}{24\pi^2}\frac{1}{\epsilon}\beta.$$

Solving for $\beta$ and taking the limit $e \to 0$ (crucially, before the limit $\epsilon \to 0$), we get

$$\beta = -\frac{\epsilon e}{2}\left(1 + \frac{e^2}{24\pi^2}\frac{1}{\epsilon}\right)^{-1} + \ldots = -\frac{\epsilon e}{2}\left(1 - \frac{e^2}{24\pi^2}\frac{1}{\epsilon}\right) + \ldots = \frac{e^3}{48\pi^2} + \ldots$$

which is the beta function in scalar QED.

- A slightly more complicated calculation along the same lines gives

$$\beta_\lambda = \frac{1}{16\pi^2}\left(5\lambda^2 - 6\lambda e^2 + 24e^4\right) + \dots$$

for the quartic coupling.

- For reference, the analogous factors for QED are

$$Z_1 = Z_2 = 1 - \frac{e^2}{8\pi^2\epsilon}, \quad Z_m = 1 - \frac{\delta_m}{m} = 1 - \frac{e^2}{2\pi^2\epsilon}, \quad Z_3 = 1 - \frac{e^2}{6\pi^2\epsilon}$$

where $Z_3$ immediately gives the QED beta function

$$\beta = \frac{e^3}{12\pi^2}$$

by the same argument as in scalar QED.

# 11 Non-Abelian Gauge Theory

## 11.1 Classical Yang–Mills

Before beginning, we review classical pure Yang–Mills theory to establish notation.

- We consider a Lie group $H$ with Lie algebra $\mathfrak{h}$, so every $h \in H$ can be written as

$$h = e^{ig\alpha}, \quad \alpha = \alpha_a T^a, \quad T^a \in \mathfrak{h}.$$

  Here the generators $T^a$ satisfy

$$[T^a, T^b] = i f^{ab}{}_c T^c.$$

  Note that conventions differ here, as one can flip the sign of $g$, $\alpha$, or $A_\mu$. In our current convention, also used by Peskin and Schroeder, the covariant derivative comes with a minus sign, $D \sim \partial - igA$, and gauge transformations come with plus signs.

- The Lie algebra contains a symmetric bilinear form called the Killing form,

$$\kappa^{ab} \equiv T^a \circ T^b$$

  which is invariant under the adjoint action,

$$hT^a h^{-1} \circ hT^b h^{-1} = T^a \circ T^b.$$

  Note that when we focus on the Lie algebra alone, we usually take the infinitesimal version of this equation by setting $h \approx 1 - ig\alpha$, for

$$[T^a, T^c] \circ T^b + T^a \circ [T^c, T^b] = 0$$

  as defined earlier.

- For an abstract Lie algebra, the Killing form is defined as we've seen earlier,

$$\kappa^{ab} = \mathrm{tr}[T^a, [T^b, \cdot]].$$

  On the other hand, for a matrix Lie group where the generators are traceless we have

$$T^a \circ T^b = \mathrm{tr}\, T^a T^b.$$

  We will normalize generators in the fundamental representation as

$$T^a \circ T^b = \frac{1}{2}\delta^{ab}$$

  and think of all generators below as being in the fundamental representation. There's no reason to do this at this point, but when we introduce physical matter, it will be in the fundamental representation, so we'll only need one set of generators.

- Under a gauge transformation $U(x) = e^{ig\alpha(x)}$ the gauge potential/connection transforms as

$$A_\mu \to U A_\mu U^{-1} - \frac{i}{g}(\partial_\mu U) U^{-1}$$

  where infinitesimally we have

$$A_\mu \to A_\mu + D_\mu \alpha, \quad D_\mu \alpha \equiv \partial_\mu \alpha - ig[A_\mu, \alpha]$$

  where we have defined the covariant derivative acting on a $\mathfrak{h}$-valued field. Note that $A_\mu$ doesn't transform in a representation of $H$, but the difference of two $A_\mu$'s transforms in the adjoint.

- It will be useful to write the above in components,

$$D_\mu \alpha^a = \partial_\mu \alpha^a + g A^b_\mu \alpha^c f^{abc}.$$

The same applies to any other quantity transforming in the adjoint.

- We define the curvature/field strength as

$$F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$$

as can be shown by acting with both sides on a "test object".

- The field strength transforms in the adjoint representation, so that

$$F_{\mu\nu} \to U F_{\mu\nu} U^{-1} = F_{\mu\nu} + ig[\alpha, F_{\mu\nu}] + O(\alpha^2).$$

Since the covariant derivative obeys the Jacobi identity, we have the Bianchi identity,

$$D_\mu F_{\alpha\beta} + D_\beta F_{\mu\alpha} + D_\alpha F_{\beta\mu} = 0.$$

- The pure Yang–Mills Lagrangian is

$$\mathcal{L} = -\frac{1}{2} \operatorname{tr} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}.$$

Note that this automatically contains terms cubic and quartic in $A$.

- To derive the equation of motion, it is useful to note that

$$\delta F_{\mu\nu} = D_\mu \delta A_\nu - D_\nu \delta A_\mu.$$

Then the variation of the action is proportional to

$$\operatorname{tr}(F_{\mu\nu}(\partial^\mu \delta A^\nu + ig[A^\mu, \delta A^\nu])) = -\operatorname{tr}(\delta A^\nu D^\mu F_{\mu\nu})$$

using the fact that $\delta A^\mu$ is in the adjoint, the cyclic property of the trace, and integration by parts. Since $\delta A^\mu$ is arbitrary, and the Killing form is nondegenerate, we have the equation of motion

$$D_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} - ig[A_\mu, F^{\mu\nu}] = 0.$$

- Matter fields in a representation $R$ will be introduced through the Lagrangian

$$\mathcal{L} \supset \overline{\psi}(i\not{D} - m)\psi, \quad (i\not{D} - m)\psi = 0, \quad \psi \to e^{ig\alpha}\psi = (1 + ig\alpha^a T^a_R)\psi + O(\alpha^2).$$

The covariant derivative acts on $\psi$ in whatever representation it transforms in,

$$D_\mu \psi_i = \partial_\mu \psi_i - ig A^a_\mu (T^a_R)_{ij} \psi_j$$

which leads to the interaction term

$$\mathcal{L} \supset g \overline{\psi}_i \gamma^\mu A^a_\mu (T^a_R)_{ij} \psi_j.$$

- The matter provides a current for the gauge fields, so that the equation of motion is

$$D_\mu F^{\mu\nu} = J^\nu, \quad J^{\mu a} = -g\overline{\psi}\gamma^\mu T_R^a \psi.$$

  By the transformation properties of $D_\mu F^{\mu\nu}$, we see that $J^\nu$ transforms in the adjoint. Acting with $D_\nu$ on both sides and using symmetry, we have $D_\mu J^\mu = 0$, so the current is "covariantly conserved" rather than conserved; this can also be shown with the equation of motion for $\psi$.

- In fact, there is no gauge-invariant conserved current in this theory. This result is a special case of the Weinberg–Witten theorem, which states that any theory with a global non-abelian symmetry under which massless spin 1 particles are charged does not admit a gauge-invariant conserved current.

- The second part of the Weinberg–Witten theorem implies that any theory with a conserved, Lorentz-covariant energy-momentum tensor cannot contain a massless particle of spin 2. This rules out scenarios where the graviton is a composite particle and space is fundamentally flat.

**Note.** The reason the classical Yang–Mills equations are not relevant to everyday life is best seen in the "theoretical" normalization, where the Yang–Mills action has a prefactor $1/g^2\hbar$. Then the classical limit is the weak coupling limit. However, in everyday life we're looking at the low energy limit, and since Yang–Mills is asymptotically free, this is precisely where the coupling is strong!

**Note.** The statements made above are straightforward to prove by expanding in components, but one can show them using differential forms, in a clean index-free notation. We define

$$A' = -igA$$

and hereafter work with $A'$, dropping the prime. We regard $A$ as as one-form and suppress explicit wedge products. Then the transformation of $A$ is

$$A \to UAU^{-1} - (dU)U^{-1} = UAU^{-1} + UdU^{-1}.$$

Taking the exterior derivative of both sides,

$$dA \to (dU)AU^{-1} + U(dA)U^{-1} - UAdU^{-1} + (dU)(dU^{-1})$$

where we picked up a minus sign from anticommuting $d$ and $A$. Similarly, we have

$$A^2 \to UA^2U^{-1} + UA(dU^{-1}) + U(dU^{-1})UAU^{-1} + U(dU^{-1})U(dU^{-1}).$$

This may be simplified by noting that $0 = d(UU^{-1}) = U(dU^{-1}) + (dU)U^{-1}$, giving

$$A^2 \to UA^2U^{-1} + UA(dU^{-1}) - (dU)AU^{-1} - (dU)(dU^{-1}).$$

Adding these expressions, we see that

$$F \to UFU^{-1}, \quad F = dA + A^2.$$

In this notation, the covariant derivative is

$$D = d + A, \quad F = D^2.$$

Furthermore, the Bianchi identity is simply $DF = 0$. To prove this, note that the $A$ in $D$ acts on $F$ by commutator, so

$$DF = dF + AF - FA = d(A^2) + AdA + A^3 - (dA)A - A^3 = 0$$

where we expanded $d(A^2) = (dA)A - AdA$. In this notation, the usual kinetic term is $\operatorname{tr}(F \star F)$, and the theta term is $\operatorname{tr} F^2$. Expanding the theta term, we have

$$\operatorname{tr} F^2 = \operatorname{tr}((dA)(dA) + A^2 dA + (dA)A^2 + A^4).$$

Due to the trace, the final term vanishes while the second and third term are in fact equal. This is easiest to see by expanding explicitly in components. For instance,

$$\operatorname{tr} A^4 = A^a_\mu A^b_\nu A^c_\rho A^d_\sigma \operatorname{tr}(T^a T^b T^c T^d) dx^\mu dx^\nu dx^\rho dx^\sigma$$

which vanishes because the trace is symmetric under cyclic permutations while the wedge product on the right is antisymmetric. In fact, we have

$$\operatorname{tr} F^2 = d \operatorname{tr}\left(AdA + \frac{2}{3}A^3\right).$$

where the quantity differentiated is the Chern–Simons current. This result will play a role in the theory of anomalies below.

## 11.2 Faddeev–Popov Quantization

First, we discuss some of the challenges associated with quantizing the theory.

- In canonical quantization, we again find that

  $$\Pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0$$

  as in QED, so it is a non-dynamical field which enforces the constraint

  $$D_i F^{0i} = 0$$

  which is the non-abelian generalization of Gauss's law.

- As in QED, we can enforce the constraints using Dirac brackets, as we did in Coulomb gauge. Alternatively, we can quantize naively and then restrict our Hilbert space to a 'physical' Hilbert space, as we did in the Gupta-Bleuler method. However, these methods are more involved for non-abelian gauge theory because the constraints are more complicated.

- Again as in QED, we still have a gauge symmetry. The gauge symmetry must always be fixed to define the quantum theory in the first place; for instance, with a gauge symmetry the propagator is not defined since the kinetic term is not invertible.

- Unlike for QED, we will first quantize using the path integral, which is much easier. To handle first-class constraints/gauge redundancies, we restrict the path integral to integrate over only one point in each gauge orbit. Note that in Yang–Mills, there are no second-class constraints, so we don't have to worry about them.

Next, we formally introduce the Faddeev–Popov method.

- Let $\mathcal{A}$ be the space of all gauge fields $A_\mu(x)$, and let $H$ be the space of gauge transformations. The physically inequivalent gauge field configurations live in the quotient space $\mathcal{A}/H$, where we identify
$$A_\mu \sim A_\mu^h, \quad A_\mu^h = hA_\mu h^{-1} + \frac{i}{g}(\partial_\mu h)h^{-1}.$$
Thus we want to define the path integral partition function as
$$\mathcal{Z} = \int_{\mathcal{A}/H} d\mu[A]\, e^{iS[A]}$$
but this is computationally inconvenient, since $\mathcal{A}/H$ is complicated. In the Faddeev–Popov method, we cleverly insert the identity to equate this path integral with another path integral over all of $\mathcal{A}$.

- Specifically, note that the naive path integral measure $\mathcal{D}A$ over $\mathcal{A}$ factors as
$$\int_{\mathcal{A}} \mathcal{D}A = \int_H d\mu[h] \int_{\mathcal{A}/H} d\mu[A].$$
Here, we will take $\mu[h]$ to be the Haar measure over $H$, which is shift-invariant,
$$\mu[g(x)h(x)] = \mu[h(x)].$$

- Now consider a Lie-algebra valued gauge fixing function $F$, such as $F(A) = \partial_\mu A^\mu$, so that $F(A) = 0$ once on every gauge orbit, say $F(A^{h_0}) = 0$. Then we have
$$1 = \int_H d\mu[H]\, \delta[F(A^h)] M(A^h), \quad M(A^h) = \det \frac{\partial F(A^h)}{\partial h}$$
which holds in analogy with the delta function identity
$$1 = \int dx\, \delta[f(x)] \left| \frac{\partial f}{\partial x} \right|.$$

- Inserting the identity, we have
$$\int_{\mathcal{A}} \mathcal{D}A = \int_H d\mu[h] \int_{\mathcal{A}} \mathcal{D}A\, \delta[F(A^h)]\, M(A^h).$$
Next, we make the nontrivial assumption that we can find a measure $\mathcal{D}A$ so that
$$\mathcal{D}A = \mathcal{D}A^h$$
which means the gauge symmetry is not anomalous. Upon relabeling, we thus have
$$\int_{\mathcal{A}} \mathcal{D}A = \left( \int_H d\mu[h] \right) \int_{\mathcal{A}} \mathcal{D}A\, \delta[F(A)]\, M(A).$$

- Equating this to our other expression for $\int_{\mathcal{A}} \mathcal{D}A$ and canceling a factor of the volume of $H$, we find that the partition function is

$$\mathcal{Z} = \int_{\mathcal{A}} \mathcal{D}A\, \delta[F(A)] M(A)\, e^{iS[A]}$$

  where we used the fact that the action was gauge-invariant.

- Next, we find a more explicit expression for $M$. Using the shift-invariance of the Haar measure we can bring the delta function peak to the identity,

$$1 = \int_H d\mu[h]\, \delta[F(\tilde{A}^h)] M(\tilde{A}^h), \quad \tilde{A} = A^{h_0}$$

  so that the peak is at $h(x) = e$. Expanding $h = e^{i\alpha}$, we get

$$1 = \int_{\mathfrak{h}} d\alpha\, \delta[F(\tilde{A}^\alpha)] M(\tilde{A}^\alpha), \quad M(\tilde{A}^\alpha) = \det \frac{\partial F(\tilde{A}^\alpha)}{\partial \alpha}$$

  for a suitably normalized Lie algebra measure $d\alpha$. Using the chain rule,

$$\frac{\partial F(\tilde{A}^\alpha)}{\partial \alpha} = \frac{\partial F(\tilde{A}^\alpha)}{\partial \tilde{A}^\alpha} \frac{\partial \tilde{A}^\alpha}{\partial \alpha} = \frac{\partial F(\tilde{A}^\alpha)}{\partial \tilde{A}^\alpha_\mu} D_\mu.$$

- Finally, using the gauge invariance of $\mathcal{D}A$ and $S[A]$, we have the partition function

$$\mathcal{Z} = \int_{\mathcal{A}} \mathcal{D}A\, \delta[F(A)]\, \det \Delta_{\text{FP}}\, e^{iS[A]}, \quad \Delta_{\text{FP}} = \frac{\partial F(A)}{\partial A_\mu} D_\mu.$$

  Note that the covariant $D_\mu$ acts on the Lie algebra, in the adjoint representation. Similarly, we may define expectation values of gauge invariant operators.

**Note.** In the absence of matter, the measure is, formally, automatically invariant under gauge transformations. Infinitesimally we have

$$A'^a_\mu = A^a_\mu + \partial_\mu \alpha^a + igf^{ade} A^d_\mu \alpha^e.$$

The Jacobian factor is

$$\det \frac{\partial A'^a_\mu}{\partial A^b_\nu} = \det \left( \delta^\mu_\nu \delta^a_b + ig\delta^\mu_\nu f^{abc} \alpha^c \right)$$

where the determinant is over both color and Lorentz indices. Using $\det(1+A) = 1 + \operatorname{tr}(A) + O(A^2)$,

$$\det \frac{\partial A'^a_\mu}{\partial A^b_\nu} = 1 + igf^{aac}\alpha^c = 1$$

by the antisymmetry of the structure constants, so we pick up no Jacobian at all.

**Note.** In more mathematical language, we begin with a principal $G$-bundle $P \to M$ and pick a 'base-point' connection $A_0$. Then any connection can be written as

$$A = A_0 + \delta A, \quad \delta A \in \mathcal{A}.$$

The space $\mathcal{A}$ is an affine space, i.e. a vector space without an origin, and there is a natural inner product on it, obtained by integration over the manifold along with contraction by the Killing form, giving a natural path integral measure.

We want to integrate over connections on the bundle, but note that $\mathcal{A}$ counts a connection and the very same connection with a different local trivialization as distinct. Hence we want to integrate over $\mathcal{A}/\mathcal{G}$ where $\mathcal{G}$ is the space of all gauge transformations; note this is much larger than $G$. The space $\mathcal{A}/\mathcal{G}$ is much more complicated than $\mathcal{A}$, and it is unclear how to define a measure on it.

The purpose of the Faddeev–Popov procedure is to write the desired path integral over $\mathcal{A}/\mathcal{G}$ in terms of a path integral over all of $\mathcal{A}$ but with a delta function, which is easier to handle. Many sources describe this procedure essentially in reverse, starting with a naive path integral over all of $\mathcal{A}$ and then "factoring out the volume of $\mathcal{G}$".

**Note.** Gauge fixing is more complicated in the non-abelian case. It's difficult to show that Lorenz gauge can even be attained. A worse problem is the Gribov ambiguity: generically gauge orbits intersect the gauge slice more than once. To see this, note that $\mathcal{A}$ is a principal bundle over $\mathcal{A}/\mathcal{G}$ with structure group $\mathcal{G}$, and a gauge slice is equivalent to a global section. Thus we require the bundle to be trivial,

$$\mathcal{A} \cong \mathcal{B} \times \mathcal{G}.$$

Since $\mathcal{A}$ is a vector space, it has trivial homotopy groups. However, it can be shown that $\mathcal{G}$ doesn't, giving the result. We won't worry about the 'Gribov copies' since we'll only work perturbatively.

At this point, we have a path integral with a $\mathcal{D}A$ measure, but it contains inconvenient extra factors. We now perform a few more tricks to absorb these terms into the action.

- First, we rewrite the delta function as an action contribution by introducing a new field, called the Nakanishi–Lautrup field. The simplest way to do this is

  $$\delta[F(A)] = \int \mathcal{D}B^a(x)\, e^{i\int dx\, B^a(x)F^a(A)}.$$

  Since $F(A)$ is Lie-algebra valued, $B(x) = B^a(x)T^a$ is as well and hence transforms in the adjoint; it is a bosonic scalar. The auxiliary field just acts like a Lagrange multiplier. Note that $B^aF^a$ is not gauge-invariant and we shouldn't expect it to be, because we are using the delta function to do gauge fixing.

- Next, we can simplify the determinant using the general formula

  $$\int d\theta^* d\theta\, e^{\theta_i^* M_{ij}\theta_j} = \det M.$$

  Since the Faddeev–Popov matrix acts on the Lie algebra, we have

  $$\det \Delta_{\mathrm{FP}} = \int \mathcal{D}\overline{c}\mathcal{D}c\, e^{-i\int dx\, \overline{c}^a(x)(\Delta_{\mathrm{FP}}c(x))^a}$$

  where $c$ and $\overline{c}$ are $\mathfrak{h}$-valued Grassmann fields, called Faddeev–Popov ghosts and anti-ghosts, which transform in the adjoint. Note that we have added a factor of $-i$ to $\Delta_{\mathrm{FP}}$. This is allowed since it merely multiplies $Z[J]$ by a phase and will ensure a canonical kinetic term for the ghosts in the gauge we'll use below.

- The ghosts are fermionic scalars, violating the spin-statistics theorem; this is allowed because the theorem assumes a positive-definite norm, but the ghost states do not have one. Their role is to cancel the unphysical polarizations of the Yang–Mills field: if the $A_\mu$ field has 4 polarizations, the ghosts heuristically have $-2$ because of their fermionic statistics, which cause them to contribute oppositely in loops.

- Now, the partition function is

$$\mathcal{Z} = \int \mathcal{D}[A, B, \overline{c}, c]\, e^{iS[A,B,\overline{c},c]}, \quad S = \int dx \left( -\frac{1}{4}(F^a_{\mu\nu})^2 + B^a F^a(A) + \overline{c}^a(\Delta_{\mathrm{FP}} c)^a \right)$$

  which is the desired form. However, we can make it even more convenient.

- Our gauge fixing condition was $F(A) = 0$, but we could also have required $F(A(x)) = f(x)$ for any $\mathfrak{h}$-valued function $f(x)$. We may also integrate over $f$ with a Gaussian weight,

$$\delta[F(A)] \to \int \mathcal{D}f\, \delta[F(A) - f] e^{-\frac{i}{2\xi}\int dx\, f^a(x) f^a(x)}.$$

  The point of this manipulation is that we do not need an auxiliary field; we may simply perform the integral over $f$, using the delta function, to get a contribution to the action quadratic in $F$,

$$\mathcal{Z} = \int \mathcal{D}[A, \overline{c}, c]\, e^{iS[A,\overline{c},c]}, \quad S = \int dx \left( -\frac{1}{4}(F^a_{\mu\nu})^2 - \frac{1}{2\xi} F^a F^a + \overline{c}^a(\Delta_{\mathrm{FP}} c)^a \right).$$

- Alternatively, we can express the delta function using the auxiliary field as before, then integrate over $f$. This gives

$$\mathcal{Z} = \int \mathcal{D}[A, B, \overline{c}, c]\, e^{iS[A,B,\overline{c},c]}, \quad S = \int dx \left( -\frac{1}{4}(F^a_{\mu\nu})^2 + B^a F^a + \frac{\xi}{2} B^a B^a + \overline{c}^a(\Delta_{\mathrm{FP}} c)^a \right).$$

  Both of these results are commonly used, and integrating out $B$ in the latter gives the former.

- Specializing to $F(A) = \partial_\mu A^\mu$, we have

$$\Delta_{\mathrm{FP}} = \partial^\mu D_\mu$$

  so the former action becomes

$$\mathcal{Z} = \int \mathcal{D}[A, \overline{c}, c]\, e^{iS[A,\overline{c},c]}, \quad S = \int dx \left( -\frac{1}{4}(F^a_{\mu\nu})^2 - \frac{1}{2\xi} F^a F^a + \overline{c}^a(-\partial^\mu D_\mu c)^a \right).$$

  This is known as $R_\xi$ gauge.

- We now revisit QED. In $R_\xi$ gauge, we recover the gauge-fixed QED action we had postulated earlier. The covariant derivative $D_\mu$ in the adjoint representation is simply $\partial_\mu$, giving a determinant factor of $\det(-\partial^2)$. This is independent of the gauge field, so we factor it out as a constant; this is why we didn't see ghosts in QED. However, we can encounter ghosts in QED if we use other gauges.

- The choice $\xi = 0$ is called Landau gauge, while $\xi = 1$ is called Feynman gauge; we often use it by default. In the path integral, it's conceptually clear the results do not depend on $\xi$, since $\xi$ just parametrizes how we do the gauge fixing.

**Example.** The axial gauge, $F(A) = n^\mu A_\mu$. In this case we have

$$\Delta_{\rm FP} = \frac{\partial F(A)}{\partial A_\mu} D_\mu = n^\mu D_\mu.$$

The ghosts automatically decouple, because when we use the delta function, the dependence on $\Delta_{\rm FP}$ on $A_\mu$ via $n^\mu A_\mu$ vanishes. Then $\det \Delta_{\rm FP}$ is simply a constant which can be ignored. The downside is that the gauge explicitly breaks Lorentz invariance.

Alternatively, we can perform the same integration over $f$ as above, giving

$$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 - \frac{1}{2\xi}(n^\mu A_\mu)^2 - \bar{c} n^\mu D_\mu c.$$

In this case it is straightforward, if a bit tedious, to invert the quadratic part of the Lagrangian to find the propagator for $A_\mu$, which gives

$$i\Delta^{\mu\nu}_{ab}(k) = \frac{i\delta_{ab}}{k^2 + i\epsilon}\left(\eta^{\mu\nu} - \frac{k^\mu n^\nu + n^\mu k^\nu}{k\cdot n} - \frac{\xi k^2 - n^2}{(k\cdot n)^2} k^\mu k^\nu\right).$$

Note that the interaction vertex between the ghosts and gauge boson contains the factor $n_\mu A^\mu$, and hence is proportional to

$$n_\mu \Delta^{\mu\nu}_{ab}(k) = -\frac{\xi}{k\cdot n}\delta_{ab} k^\nu.$$

Hence in the limit $\xi \to 0$ where the gauge condition is exactly imposed, all diagrams where a ghost attaches to a gluon vanish, so the ghosts decouple as we'd expect. A useful special case of this gauge is light cone gauge, the limit $\xi \to 0$ with $n^2 = 0$, giving

$$i\Delta^{\mu\nu}_{ab}(k) = \frac{i\delta_{ab}}{k^2 + i\epsilon}\left(\eta^{\mu\nu} - \frac{k^\mu n^\nu + n^\mu k^\nu}{k\cdot n}\right).$$

## 11.3 Canonical Quantization

Next, we proceed to the canonical quantization of Yang–Mills theory.

- Motivated by the path integral treatment above, we take the gauge-fixed Lagrangian

  $$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 - \partial^\mu B^a A^a_\mu + \frac{\xi}{2} B^a B^a + \partial^\mu \bar{c}^a D_\mu c^a$$

  where the derivatives only act on the field immediately to their right.

- The field momenta are

  $$\Pi^{a\mu}_A = \frac{\partial \mathcal{L}}{\partial \dot{A}^a_\mu} = F^{a\mu 0}, \quad \Pi^a_c = \frac{\partial \mathcal{L}}{\partial \dot{c}^a} = -\dot{\bar{c}}^a, \quad \Pi^a_B = \frac{\partial \mathcal{L}}{\partial \dot{B}^a} = -A^a_0, \quad \Pi^a_{\bar{c}} = \frac{\partial \mathcal{L}}{\partial \dot{\bar{c}}^a} = \dot{c}^a - g f^{abc} A^b_0 c^c.$$

  The conjugate momentum of $A^0$ still vanishes, but this is just because we have the auxiliary field $B$. We can eliminate $B$ by plugging in its equations of motion.

- Next, we perform canonical quantization, imposing the usual commutation relations

  $$[A_\mu(t,\mathbf{x}), \Pi_{\nu A}(t,\mathbf{y})] = i\eta_{\mu\nu}\delta(\mathbf{x}-\mathbf{y}).$$

Then by direct computation, we find for spacelike $j$ and $k$,

$$[A_j^a(t,\mathbf{x}), \dot{A}_k^b(t,\mathbf{y})] = i\delta^{ab}\delta_{kj}\delta(\mathbf{x}-\mathbf{y}).$$

Meanwhile, we have

$$[A_0^a(t,\mathbf{x}), B^b(t,\mathbf{y})] = i\delta^{ab}\delta(\mathbf{x}-\mathbf{y}).$$

- Since the ghosts were Grassmann variables, we impose the anti-commutation relations

$$\{c^a(t,\mathbf{x}), \Pi_c^b(t,\mathbf{y})\} = \{\overline{c}^a(t,\mathbf{x}), \Pi_{\overline{c}}^b(t,\mathbf{y})\} = i\delta^{ab}\delta(\mathbf{x}-\mathbf{y}).$$

We also define

$$c^{a\dagger} = c^a, \quad (\overline{c}^a)^\dagger = -\overline{c}^a$$

which ensures that $\mathcal{L} = \mathcal{L}^\dagger$, yielding a unitary $S$-matrix.

- At this point we would usually construct the Fock space, but Yang–Mills is an intrinsically interacting theory. Thus the spectrum will contain complicated bound states, e.g. hadrons and glueballs. On the other hand, to compute $S$-matrix elements it suffices to define 'in' and 'out' states via the LSZ reduction formula. These asymptotic states are free fields, which are fully renormalized but have gauge coupling $g = 0$.

- Therefore, we instead consider the free Lagrangian

$$\mathcal{L}_0 = -\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 - \partial^\mu B^a A_\mu^a + \frac{\xi}{2}B^a B^a + \partial^\mu \overline{c}^a \partial_\mu c^a.$$

We set $\xi = 1$ and integrate out $B^a$ using the equation of motion $B^a = -\partial^\mu A_\mu^a$, giving

$$[A_\mu^a(t,\mathbf{x}), \dot{A}_\nu^b(t,\mathbf{y})] = -i\delta^{ab}\eta_{\mu\nu}\delta(\mathbf{x}-\mathbf{y})$$

just as we had in QED.

- The gauge field and ghost field have free mode expansions,

$$A_\mu^a(x) = \int \frac{đ\mathbf{k}}{\sqrt{2E_k}} \sum_\lambda \epsilon_\mu^a(\mathbf{k},\lambda)(a_\lambda^a(\mathbf{k})e^{-ikx} + a_\lambda^{a\dagger}(\mathbf{k})e^{ikx}), \quad \epsilon(\mathbf{k},\lambda)\cdot\epsilon(\mathbf{k},\lambda') = \eta_{\lambda\lambda'}$$

and

$$c^a(x) = \int \frac{d\mathbf{p}}{\sqrt{2E_p}}\, c^a(\mathbf{p})e^{-ipx} + c^{a\dagger}(\mathbf{p})e^{ipx}$$

with the commutation relations

$$[a_\lambda^{a\dagger}(\mathbf{k}), a_{\lambda'}^b(\mathbf{k}')] = \delta^{ab}\eta_{\lambda\lambda'}đ(\mathbf{k}-\mathbf{k}'), \quad \{c^a(\mathbf{p}), \overline{c}^{b\dagger}(\mathbf{p}')\} \propto \{\overline{c}^a(\mathbf{p}), c^{b\dagger}(\mathbf{p}')\} \propto \delta^{ab}\delta(\mathbf{p}-\mathbf{p}').$$

Thus we have negative norm and zero norm gluon states, as well as zero norm ghost states.

- The physical Hilbert space $H_{\text{phys}}$ must have a positive-definite norm. In addition, the $S$-matrix must be unitary when restricted to $H_{\text{phys}}$, i.e. the unphysical states must decouple. This is much more difficult to guarantee, but it follows if we define $H_{\text{phys}}$ using a symmetry of the full interacting theory, as in this case the $S$-matrix respects this symmetry by assumption.

- We cannot use gauge symmetry for this purpose, because we had to fix the gauge to quantize at all. In QED, we used the $U(1)$ global symmetry, which gave the Ward–Takahashi identities. But in a non-abelian gauge theory, this is complicated by the ghost and auxiliary fields. Instead, we will use the more subtle BRST symmetry.

## 11.4 BRST Symmetry

BRST symmetry is a fermionic global symmetry of the interacting, gauge-fixed Yang–Mills Lagrangian, which roughly corresponds to gauge symmetry when applied to the gauge field alone. It is theoretically useful because it allows us to recover some of the consequences of gauge symmetry, even though the Lagrangian is gauge-fixed.

- We begin with the gauge-fixed Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \frac{\xi}{2}B^aB^a + B^a\partial_\mu A^a_\mu + \overline{c}^a(-\partial^\mu D_\mu c)^a.$$

  We define a Grassmann odd operator $S$ so that

$$SA_\mu = -D_\mu c = -(\partial_\mu c + ig[A_\mu, c]), \quad Sc = \frac{i}{2}gf^{abc}c^bc^ct^a \equiv \frac{g}{2}[c,c], \quad S\overline{c} = -B, \quad SB = 0.$$

  Note that $Sc$ is nonzero since $c$ is Grassmann odd. Also note that $Sc$ is not just the conjugate of $S\overline{c}$. We regard $c$ and $\overline{c}$ as independent real Grassmann fields, not conjugates of each other.

- Note that $S$ obeys a graded Leibniz rule: if $c$ is Grassmann odd,

$$S(cA) = (Sc)A - c(SA).$$

  We now show $S^2\Phi = 0$ for any $\Phi \in \{A, c, \overline{c}, B\}$. This is obvious for $B$ and $\overline{c}$, and we have

$$S^2c \propto t^a f^{abc}f^{bde}c^cc^dc^e \propto t^a\left(\sum_{\text{cde}}^{\text{cyclic}} f^{abc}f^{bde}\right)c^cc^dc^e = 0$$

  by the Jacobi identity. The proof that $S^2A_\mu = 0$ is similar, but more complicated.

- Now, for a product of any two fields, we have

$$S^2(\Phi_1\Phi_2) = S(S\Phi_1\,\Phi_2 \pm \Phi_1 S\Phi_2) = \mp S\Phi_1\,S\Phi_2 \pm S\Phi_1\,S\Phi_2 = 0$$

  where we used $S^2\Phi_i = 0$ and the fact that $S$ flips the Grassmann parity. Similar logic applies for any product of fields, which implies $S$ is nilpotent, $S^2 = 0$.

- Next, we note the Lagrangian has the form

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} - S\left(\overline{c}^a\partial^\mu A^a_\mu + \frac{\xi}{2}\overline{c}^aB^a\right)$$

  as can be shown by direct expansion. Then $S\mathcal{L} = 0$, where the first term vanishes by gauge invariance and the second by nilpotence. We define the BRST symmetry transformation by

$$\delta_\epsilon\Phi = \epsilon S\Phi$$

  where $\epsilon$ is a constant independent Grassmann number, so $\delta_\epsilon\mathcal{L} = 0$. Note that the BRST transformation preserves the Grassmann parity, and obeys the Leibniz rule without grading.

- By Noether's theorem, we can construct the conserved current and charge

$$J^\mu = \sum_I \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_I)}\delta_\epsilon\Phi_I, \quad \partial_\mu J^\mu = 0, \quad Q = \int d\mathbf{x}\,J^0, \quad \dot{Q} = 0.$$

- Upon canonical quantization, one can show that we find a BRST charge operator $\hat{Q}^\dagger = \hat{Q}$ which generates the BRST transformation, by

$$[\hat{Q}, \hat{X}] = iS\hat{X}$$

where the bracket is a commutator or anticommutator depending on whether $\hat{X}$ is bosonic or fermionic. The conservation of $\hat{Q}$ is expressed as $[\hat{H}, \hat{Q}] = 0$, and the nilpotency of the BRST transformation implies

$$\hat{Q}^2 = 0.$$

- The continuous symmetries of the Yang–Mills Lagrangian are Lorentz invariance, global gauge invariance, BRST invariance, ghost number (a $U(1)$ assigning $+1$ charge to $c^a$ and $-1$ to $\overline{c}^a$), and anti-ghost translation invariance $\overline{c} \to \overline{c} + \chi$. In fact, the Lagrangian already contains all terms consistent with these symmetries whose coefficients have nonnegative mass dimension. Upon renormalization, BRST symmetry will ensure that the separate appearances of $g$ renormalize in the same way, providing identities analogous to $Z_1 = Z_2$ in QED.

- There are some minor variations we can consider. We can easily include matter fields, which, like the gauge field, transform as under a gauge transformation with gauge parameter $\alpha = -\epsilon c$,

$$\delta\psi_i = -ig\epsilon c^a t^a_{ij}\psi_j.$$

It is straightforward to show that $S^2\psi_i$ as well.

- We could also integrate out the $B$ field, in which case the only change is

$$S\overline{c} = \frac{1}{\xi}F^a = \frac{1}{\xi}\partial_\mu A^\mu$$

in $R_\xi$ gauge. However, to show that $S^2\overline{c} = 0$, we have to use the equation of motion for the ghost field. This is a general phenomenon: if we eliminate auxiliary fields, then often symmetries that held off-shell will only hold on-shell, reducing their power.

We now use the BRST operator to define a cohomology and physical Hilbert space.

- Since $\hat{Q}^2 = 0$, it defines a cohomology.
  - An element $|\psi\rangle \in \ker \hat{Q}$ is called $\hat{Q}$-closed.
  - An element $|\psi\rangle \in \operatorname{im} \hat{Q}$ is called $\hat{Q}$-exact. All exact states are closed.
  - For $|\psi\rangle$ exact, note that

$$\langle\psi|\psi\rangle = \langle\chi|\hat{Q}^\dagger\hat{Q}|\chi\rangle = \langle\chi|\hat{Q}^2|\chi\rangle = 0.$$

Then all exact states are null. More generally, all exact states are orthogonal to all closed states, and any two closed states differing by an exact state have the same norm. Physically, an exact state is gauge-equivalent to vacuum.

  - We define the $\hat{Q}$-cohomology by

$$\operatorname{cohom}(\hat{Q}) = \frac{\ker(\hat{Q})}{\operatorname{im}(\hat{Q})}.$$

Then there is a well-defined inner product on $\operatorname{cohom}(\hat{Q})$.

- Consider the variation of the time evolution operator

$$\hat{U} = \exp\left(i\int \Pi\dot{\phi} - \mathcal{L}\,dt\right), \quad \mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} - S\psi$$

under a change in the gauge fixing condition $F(A) = 0$. Only $S$ depends on $F$, so

$$\delta_F\langle\alpha|\hat{U}|\beta\rangle = \langle\alpha|i\delta_F S\psi|\beta\rangle = \langle\alpha|iS\delta_F\psi|\beta\rangle = \langle\alpha|\hat{Q}\delta_F\psi - \delta_F\psi\hat{Q}|\beta\rangle.$$

For this to vanish for all $\delta_F\psi$, we require $\hat{Q}|\beta\rangle = 0$. Since the time-evolution of physical states should not depend on the gauge choice, the physical Hilbert space must be BRST closed.

- Within this space, the BRST exact states have zero overlap with all other states and hence can never be measured. Thus we identify $H_{\text{phys}} = \text{cohom}(\hat{Q})$.

- We can explicitly find the physical Hilbert space for the free in/out states. The states are

$$|A^a_\mu(\mathbf{k})\rangle = \sum_\lambda \epsilon^a_\mu(\mathbf{k},\lambda)a^{a\dagger}_\lambda(\mathbf{k})|0\rangle, \quad |c^a(\mathbf{k})\rangle = c^{a\dagger}(\mathbf{k})|0\rangle, \quad |\overline{c}^a(\mathbf{k})\rangle = \overline{c}^{a\dagger}(\mathbf{k})|0\rangle$$

and we don't have separate $B$ states since $B \sim \partial A$. The BRST transformation is

$$SA_\mu = -\partial_\mu c, \quad Sc = 0, \quad S\overline{c} = -B = -\partial A, \quad SB = 0$$

where we work in Feynman gauge.

- Therefore, the BRST charge acts as

$$\hat{Q}|A^{\mu a}(\mathbf{k})\rangle = \alpha k^\mu|c^a(\mathbf{k})\rangle, \quad \hat{Q}|c^a(\mathbf{k})\rangle = 0, \quad \hat{Q}|\overline{c}^a(\mathbf{k})\rangle = \beta k^\mu|A^a_\mu(\mathbf{k})\rangle$$

where $\alpha, \beta \neq 0$. Since $\hat{Q}^2 = 0$, we have $k^2 = 0$. We now drop the $a$ index.

- The states that are BRST closed must have no $\overline{c}$ excitations, and moreover

$$|\xi\rangle = \xi_\mu|A^\mu(\mathbf{k})\rangle$$

is only closed if $\xi^\mu k_\mu = 0$, removing one of the unphysical polarizations.

- Note that $c$ excitations are BRST exact, as is $k^\mu|A_\mu(\mathbf{k})\rangle$. Therefore, the physical one-particle Hilbert space consists of states

$$|\psi\rangle = \xi^\mu|A_\mu(\mathbf{k})\rangle, \quad k^2 = 0, \quad \xi^\mu k_\mu = 0$$

where $\xi^\mu \sim \xi^\mu + k^\mu$. These are indeed the two degrees of freedom we want.

- Next, we check unitarity of the $S$-matrix. Since $\mathcal{L}^\dagger = \mathcal{L}$, the $S$-matrix is unitary on the entire Fock space. Letting $|\gamma\rangle$ be a basis,

$$\sum_\gamma \langle\alpha|S^\dagger|\gamma\rangle\langle\gamma|S|\beta\rangle = \langle\alpha|\beta\rangle.$$

The nontrivial thing to check is that it is unitary on $H_{\text{phys}}$,

$$\sum_{\chi_T}\langle\phi_T|S^\dagger|\chi_T\rangle\langle\chi_T|S|\psi_T\rangle = \langle\phi_T|\psi_T\rangle, \quad |\phi_T\rangle, |\psi_T\rangle \in H_{\text{phys}}$$

where $|\chi_T\rangle$ is a basis for $H_{\text{phys}}$.

- Since $\hat{Q}$ commutes with $\hat{H}$, it commutes with $S$, so

$$\hat{Q}S|\psi_T\rangle = S\hat{Q}|\psi_T\rangle = 0.$$

Therefore, $S$ maps $\ker \hat{Q}$ to itself. However, it can and does produce BRST exact states. These don't spoil unitarity because such states are orthogonal to BRST closed states. Explicitly,

$$\langle\phi_T|\psi_T\rangle = \sum_\gamma \langle\phi_T|S^\dagger|\gamma\rangle\langle\gamma|S|\psi_T\rangle = \sum_{\chi_T} \langle\phi_T|S^\dagger|\chi_T\rangle\langle\chi_T|S|\psi_T\rangle$$

as desired. Here, $\gamma$ ranges over physical states, BRST exact states, and states that are not BRST closed, and the latter two don't contribute by the arguments above.

- In the special case of QED in Lorenz gauge, the ghosts automatically decouple, and the constraints above reduce to the Gupta-Bleuler condition $\partial A^+|\psi\rangle = 0$. The difference in the non-abelian case is that such a constraint is not preserved by time evolution, as we can create BRST exact states.

- Accordingly, the Ward identity $k_\mu \mathcal{M}^\mu = 0$ does not hold in the non-abelian case, and this is related to the nonexistence of a conserved current. Instead, we use the BRST current to construct Slavnov–Taylor identities, which can be used to prove decoupling just like the Ward identities. The Slavnov–Taylor identities also ensure the gluon remains massless, as the Ward identities did for the photon.

- To see the role of the ghosts, note that the optical theorem relates a loop amplitude to a production amplitude squared, where the latter sums over physical external particles. Only two gauge boson polarizations are physical, but all four run around the loop. To maintain the optical theorem and hence unitarity, the ghosts also run around the loop with an extra factor of $-1$, canceling the unphysical polarizations. In the abelian case, the ghosts aren't necessary because the unphysical polarizations contribute nothing to loop amplitudes.

## 11.5 Perturbative Renormalization

Next, we show the non-abelian gauge theories are asymptotically free. We continue to follow the Peskin and Schroeder conventions.

- By the same reasoning as in QED, the gauge boson propagator is

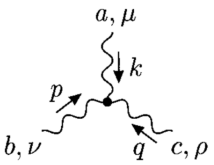
$$D_F^{\mu\nu ab}(p) = \frac{-i\delta^{ab}}{p^2 + i\epsilon}\left(\eta^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}\right).$$

The propagator is diagonal in color space. Incoming gauge bosons get $\epsilon_\mu^a(\mathbf{k}, \lambda)$ and outgoing gauge bosons get $\epsilon^{*a}_\mu(\mathbf{k}, \lambda)$.

- Next, in our convention the gauge boson interaction terms are

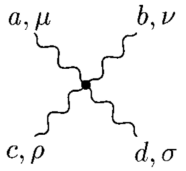
$$\mathcal{L} \supset g f^{abc}\eta^{\nu\rho}\partial^\mu A^a_\nu A^b_\mu A^c_\rho - \frac{g^2}{4} f^{abc} f^{ade}\eta^{\mu\rho}\eta^{\nu\sigma} A^b_\mu A^c_\nu A^d_\rho A^e_\sigma.$$

For the cubic Feynman rule, note that every interaction comes with an automatic factor of $i$, while the derivative contributes $-ip^\mu$ for incoming momentum $p$. There are six distinct contractions, yielding:



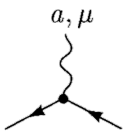

$$= \quad gf^{abc}\big[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu\big]$$

- As for the quartic interaction, there are 4! ways to perform the contractions. It turns out that they come in identical groups of 4, canceling the 1/4 factor and giving six terms:




$$= \quad -ig^2\big[f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho}) + f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\sigma}g^{\nu\rho}) + f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma})\big]$$

- Next, when we include matter fields in a representation $t^a$, they have the interaction



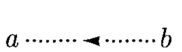

$$= \quad ig\gamma^\mu t^a$$

  where again our convention's sign is flipped. The propagator is the same as usual, with an extra $\delta_{ij}$ in flavor space.

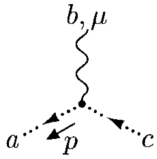
- Finally, we consider the ghost fields, where

$$\mathcal{L} \supset \overline{c}^a(-\partial^\mu D_\mu c)^a = \overline{c}^a(-\partial^2\delta^{ab})c^b + gf^{abc}\overline{c}^a\partial^\mu(A^b_\mu c^c).$$

  The resulting Feynman rules are shown below.




$$= \quad \frac{i\delta^{ab}}{p^2}$$




$$= \quad -gf^{abc}p^\mu$$

  The propagator is just the usual one, but it has a direction because the ghosts are fermionic; also note that a ghost loop contributes a factor of $-1$. To get the ghost vertex, we integrate by parts to put the derivative on $\overline{c}^a$. Also note that here, $p$ is outgoing rather than incoming.

Next, we briefly discuss the "group theory" factors that appear in amplitudes.

- In QED, we saw that the spinor-related parts of amplitudes could be constructed by following every fermion line, then writing the entries right-to-left. Similarly, in QCD, we get color factors by following the quark lines, with a generator $t^a$ every time a gluon attaches. The ordering is the same: reading matrix indices along the fermion line will give them ordered right-to-left.

- We also get color factors from gluons, since they themselves are colored. These yield products of structure factors, which are essentially matrix products in the adjoint representation, since $(t^a_A)^{bc} = if^{abc}$.

- We define the Dynkin index and quadratic Casimir as

$$\mathrm{tr}(t_R^a t_R^b) = T(R)\delta^{ab}, \quad t_R^a t_R^a = C(R)$$

with implicit summation over repeated indices and an identity in the second result. Conventionally, we normalize the generators so that

$$T(F) = \frac{1}{2}$$

which implies that

$$C(F) = \frac{N^2-1}{2N} = \frac{4}{3}, \quad T(A) = C(A) = N = 3.$$

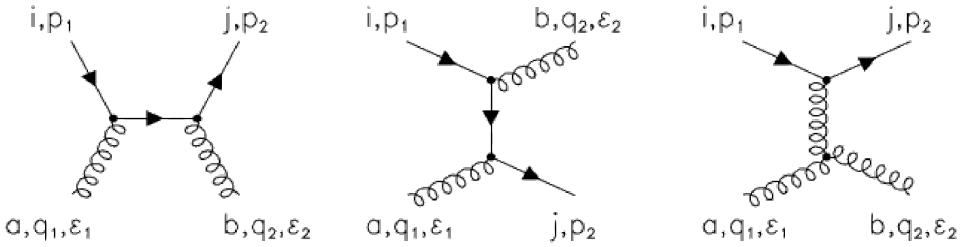
Here $F$ stands for the fundamental representation.

- We also often get factors of the dimensions of these representations,

$$d(F) = N = 3, \quad d(A) = N^2 - 1 = 8.$$

Further rules for computing these constants are given in the lecture notes on Group Theory, but these will suffice for our purposes.

**Example.** The process $qg \to qg$ at tree level. There are three diagrams:




which are in the $s$-channel, $u$-channel, and $t$-channel respectively. We'll focus on the color factors, since the kinematics are quite complicated. Setting the quark mass to zero for simplicity, we have

$$\mathcal{M}_s = g^2 \overline{u}_{s'}(p_2) \not{\epsilon}_2^* \frac{i(\not{p}_1 + \not{p}_2)}{s} \not{\epsilon}_1 u_s(p_1)\, t^b_{jk} t^a_{ki}$$

and

$$\mathcal{M}_u = g^2 \overline{u}_{s'}(p_2) \not{\epsilon}_1 \frac{i(\not{p}_1 - \not{q}_2)}{u} \not{\epsilon}_2^* u_s(p_1)\, t^a_{jk} t^b_{ki}$$

and

$$\mathcal{M}_t = g^2 \overline{u}_{s'}(p_2)\gamma^\mu u_s(p_1) \left(\frac{-i}{t}\right) f^{abc} t^c_{ji} \epsilon_2^{*\rho} \epsilon_1^\nu \left(\eta^{\nu\rho}(q_1+q_2)^\mu + \eta^{\rho\mu}(q_1 - 2q_2)^\nu + \eta^{\mu\nu}(q_2 - 2q_1)^\rho\right).$$

where repeated color indices are *not* summed. Upon squaring, and summing over final spins and averaging over initial spins, we get complicated expressions involving traces of up to eight gamma matrices, which can be simplified using the Clifford algebra. Ignoring this, we get, for example,

$$|\mathcal{M}_s|^2 \sim (t^b t^a)_{ji} (t^b t^a)^*_{ji} = (t^b t^a)_{ji} (t^b t^a)^\dagger_{ij} = (t^b t^a)_{ji} (t^a t^b)_{ij}.$$

Summing over final colors and averaging over initial colors, and keeping only the color factor,

$$|\mathcal{M}_s|^2 \sim \frac{1}{3}\frac{1}{8}\,\mathrm{tr}(t^b t^a t^a t^b) = \frac{1}{24} d(F) C(F)^2 = \frac{2}{9}.$$

As another example, we have

$$|\mathcal{M}_t|^2 \sim \frac{1}{24} f^{abc} f^{abd} \operatorname{tr}(t^c t^c) = \frac{1}{24} T(F) f^{abc} f^{abc} = \frac{1}{24} T(F) C(A) d(A) = \frac{1}{2}.$$

As a final example,

$$\mathcal{M}_t \mathcal{M}_s^* \sim \frac{1}{24} f^{abc} \operatorname{tr}(t^a t^b t^c) = \frac{1}{48} f^{abc} \operatorname{tr}(t^a [t^b, t^c]) = \frac{i}{48} f^{abc} f^{bcd} \operatorname{tr}(t^a t^d) = \frac{i}{48} C(A) d(A) T(F) = \frac{i}{4}.$$

Now we proceed to computing the beta function of Yang–Mills. **(todo)**

- 
- 
- 

# 12 Solitons

## 12.1 Kinks

We begin with a brief introduction to solitons.

- Solitons are stable, spatially localized smooth solutions of classical PDEs. We will focus on their description in classical field theory. Often, we will find that they "connect distinct vacua", giving them topological stability. The solitons have a topological charge that makes them distinct from the vacuum.
- To have soliton solutions, we need nonlinear equations, which can be achieved by nontrivial interactions between fields, or by self-interaction, as in $\phi^4$ theory.
- Solitons can be quantized, at which point we can interpret them as particles. For example, solitons carry energy and momentum. By Lorentz invariance, a stationary soliton with energy $E_0$ can be boosted to yield moving solitons, obeying

  $$E^2 = p^2 + m^2$$

  where the mass of the soliton is $m = E_0$. Solitons can also interact with each other like particles.
- However, solitons cannot be seen by perturbing about a vacuum state. Instead, we fix a soliton solution and treat it as a background for quantization. This is difficult, as we typically can't write down exact soliton solutions; we won't consider this subject too closely.
- Another feature that distinguishes solitons is that ordinary quantum particles have masses proportional to $\hbar$, since $E = \hbar\omega$, while the masses of solitons are independent of $\hbar$.
- We will focus on a few types of solitons, all in relativistic field theory.
  - In one spatial dimension, we have kink solutions.
  - In two spatial dimension, we have vortices. We will also investigate solitons in nonlinear $\sigma$ models.
  - In three spatial dimensions, we have Skyrmions.

  Nonrelativistic solitons include domain walls in ferromagnets and two-dimensional "baby" Skyrmions in exotic magnets. Solitons also appear in cosmology, where we can have domain walls, cosmic strings, and monopoles.
- Skyrmions are solitons in an effective field theory of interacting pions, representing the (fermionic) nucleons. They were pioneered by Skyrme, who had the strong philosophical opinion that all fermionic fields had to be emergent in this way. Though the skyrmion approach is not particularly accurate in real QCD, it was famously shown by Witten to be a good description in the large $N_c$ limit.
- Finally, we comment on quantization. Typically, for a field whose particles have mass $m$ and coupling $g$, solitons classically have energy and size

  $$E \sim \frac{m}{g}, \quad \ell \sim \frac{1}{m}$$

  in natural units; this will hold for our examples below.

- Then the Compton wavelength of a soliton is

$$\lambda \sim \frac{1}{E} \sim g\ell.$$

  Therefore, we don't expect quantization to significantly affect the soliton when the coupling is weak; we simply find perturbative corrections to $E$. The fact that $E$ diverges as $g \to 0$ also indicates the solitons cannot be seen in perturbation theory.

- We can also have some insight in the strong coupling regime. For example, the sine-Gordan theory and massive Thirring model are dual, with strong coupling in one mapping to weak coupling in the other, and solitons exchanging with elementary particles. Hence there is no intrinsic difference between an elementary particle and a soliton; all that matters is which viewpoint is more convenient for calculation in a particular regime.

Next, we introduce the basic kink solution.

- We work in $1+1$ dimensions with metric $\text{diag}(1,-1)$. The action is

$$S = \int d^2x \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - U(\phi) \right).$$

  The equation of motion is

$$\partial_\mu \partial^\mu \phi + \frac{dU}{d\phi} = 0$$

  which is called a nonlinear Klein-Gordan equation.

- For explicit calculations, it will be useful to work 'nonrelativistically'. We define $\dot{\phi} = \partial\phi/\partial t$ and $\phi' = \partial\phi/\partial x$ with

$$L = T - V, \quad T = \int \frac{1}{2}\dot{\phi}^2 \, dx, \quad V = \int \frac{1}{2}\phi'^2 + U(\phi) \, dx.$$

- We are interested in static solutions, which obey $\phi'' = dU/d\phi$. We choose the potential

$$U(\phi) = \frac{1}{2}(1-\phi^2)^2$$

  so two static solutions to the field equations are the vacua $\phi(x) = \pm 1$. There are soliton solutions that connect these vacua.

- There are plenty of other choices, such as

$$U(\phi) = 1 - \cos\phi$$

  which is called the sine-Gordan theory since the field equation contains $\sin\phi$. This theory is integrable, which means that we can find explicit exact solutions involving interacting solitons.

- Sticking with our original choice, the field equation is

$$\phi'' = -2(1-\phi^2)\phi.$$

This is a nonlinear second order differential equation, so instead we think in terms of energy. By Noether's theorem, we have a conserved energy

$$E = \int \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\phi'^2 + U(\phi)\, dx$$

which is simply $V$ for a static solution. For static solutions, the action is proportional to $V$ and is also extremized, which means that $E$ is minimized.

- We shift $U(\phi)$ so that it is nonnegative, and its minimum value is zero. Then finite energy solutions have fixed boundary conditions $U(\pm\infty)$. Now, we define the function $W$ by

$$U(\phi) = \frac{1}{2}\left(\frac{dW}{d\phi}\right)^2.$$

Now the energy is quadratic, so we may complete the square for

$$\begin{aligned} E &= \frac{1}{2}\int \phi'^2 + \left(\frac{dW}{d\phi}\right)^2 dx \\ &= \frac{1}{2}\int \left(\phi' \mp \frac{dW}{d\phi}\right)^2 dx \pm \int \frac{dW}{d\phi}\frac{d\phi}{dx}dx \\ &= \frac{1}{2}\int \left(\phi' \mp \frac{dW}{d\phi}\right)^2 dx \pm (W(\phi(\infty)) - W(\phi(-\infty))). \end{aligned}$$

- Since the second term is fixed, the energy is minimized if the first term vanishes, so

$$\phi' = \pm\frac{dW}{d\phi}$$

while the energy of the solution is

$$E = \pm(W(\phi(\infty)) - W(-\infty))$$

where we take the positive solution. Moreover, any soliton will have at least this energy; this is an example of a so-called Bogomolny bound.

- Note that the equation $\phi'' = dU/d\phi$ for static solutions is equivalent to that of a particle of unit mass and position $x(t)$ in the potential $-U$. This is just like the situation for instantons, and provides some intuition.

- In the case of the $\phi^4$ kink, we have

$$W = \phi - \frac{1}{3}\phi^3.$$

Fixing the boundary conditions $\phi(\infty) = 1$ and $\phi(-\infty) = -1$, we have $E = \pm 4/3$. Since the energy is positive, we take the plus sign, so $M = 4/3$ and

$$\phi' = \frac{dW}{d\phi} = 1 - \phi^2, \quad \phi(x) = \tanh(x - a)$$

where $a$ is a constant of integration, the position of the soliton. We thus call $a$ a modulus (or collective coordinate) of the solution, and the set of possible moduli is the moduli space; in this case it is $\mathbb{R}$.

- We can also have moving kink solutions, by simply performing a Lorentz transformation,

$$\phi(x,t) = \tanh\gamma(x - vt).$$

It is useful to focus on slowly moving solitons. Then

$$\phi(x,t) \approx \tanh(x - a(t)), \quad a(t) = vt$$

which looks like a static solution with a time-dependent modulus.

- We can write the action in terms of the time-varying modulus in this 'adiabatic' approximation. We have $\dot{\phi} = -\dot{a}\phi'$, so

$$T = \int \frac{1}{2}\dot{\phi}^2\,dx = \frac{1}{2}\dot{a}^2 \int \phi'^2\,dx = \frac{1}{2}M\dot{a}^2$$

where the final result can be simply written down using Lorentz invariance or computed by an ugly integral. Alternatively, the Bogomolny equation shows that the two terms in the $E$ integral above contribute equally, which means that the integral of $\phi'^2$ is just $M$.

- Meanwhile, $V$ doesn't depend on $\dot{a}$, so it is simply $4/3$, and we have the particle Lagrangian

$$L = \frac{1}{2}M\dot{a}^2 - \frac{4}{3}$$

This is an 'effective field theory' style approach; we are restricting our variational problem in field space to a 'valley' given by static solutions, parametrized by $a(t)$.

- The soliton Lagrangian has the equation of motion

$$M\ddot{a} = 0, \quad a(t) = vt + a_0$$

which is exactly what we found above. Geometrically, we can think of the equation of motion as the geodesic equation on the moduli space $\mathbb{R}$, where $M$ specifies the metric. This is trivial here, but generalizes to higher-dimensional systems and multi-soliton dynamics.

- In general, zero modes arise whenever a soliton breaks a symmetry in the Lagrangian; in this case the soliton breaks translational symmetry. Later we'll see more complex examples where a soliton breaks an internal symmetry, i.e. it carries a conserved charge.

## 12.2 Dynamics of Kinks

Next, we consider interactions of kinks.

- In our current theory, there aren't multi-kink solutions. However, we can ask how a solution consisting of a kink and anti-kink behaves.

- First, it's useful to look at conserved quantities. We have

$$T^\mu{}_\nu = \partial^\mu\phi\partial_\nu\phi - \delta^\mu_\nu\mathcal{L}.$$

The energy is the integral of $T^0{}_0$, giving the same result we found earlier. The momentum is

$$P = -\int T^0{}_1\,dx = -\int \dot{\phi}\phi'\,dx.$$

For our moving kink solution $\phi = \tanh(x - a(t))$ we have, by the same tricks as above,

$$P = M\dot{a}$$

which is what we would expect for a particle of mass $M$.

- Next, we would like to compute the interaction force between a widely separated kink and antikink, as only this regime is analytically tractable. Let the kink and antikink be located at $\pm a$, and let $b$ be a point in between them far away from each of them; we'll show the result doesn't depend sensitively on $b$. We define the momentum of the kink as

$$P = -\int_{-\infty}^{b} \dot{\phi}\phi' \, dx.$$

By conservation of energy-momentum we have

$$\partial_t T^0{}_1 + \partial_x T^1{}_1 = 0.$$

- The force on the kink is defined as

$$F = \dot{P} = -\int_{-\infty}^{b} \partial_t T^0{}_1 \, dx = \int_{-\infty}^{b} \partial_x T^1{}_1 \, dx = T^1{}_1(b) = \left(-\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\phi'^2 + U\right)_{x=b}$$

- To make further progress, we need to explicitly write down $\phi$. An approximate solution is

$$\phi(x) = \tanh(x + a) - \tanh(x - a) - 1 \equiv \phi_1 + \phi_2 - 1$$

as long as $a \gg 1$. We will assume that the kinks are initially at rest, $\dot{\phi} = 0$, though they will begin moving as they exert a force on each other.

- Next, at the point $b$, $\phi_2 - 1$ and hence $\phi_2'$ are small, so we expand to first order in them,

$$F = \left(-\frac{1}{2}\phi_1^2 + U(\phi_1) - \phi_1'\phi_2' + (\phi_2 - 1)\frac{dU}{d\phi}(\phi_1)\right)_{x=b}.$$

The first two terms cancel out by the Bogomolny equation; physically they must cancel because a kink cannot exert a force on itself.

- Next, the field equation gives $(dU/d\phi)(\phi_1) = \phi_1''$, so we have

$$F = (-\phi_1'\phi_2' + (\phi_2 - 1)\phi_1'')_{x=b}.$$

This is as far as we can go without using the explicit solution.

- To keep the calculations manageable we use the asymptotic form of tanh,

$$\phi_1(x) \sim 1 - 2e^{-2(x+a)}, \quad \phi_2 \sim 1 - 2e^{2(x-a)}$$

which is valid when $x$ is far from both $\pm a$. The factors of 2 here are called the 'amplitude of the tail'. Plugging this in and simplifying,

$$F = 32e^{-4a}$$

where the value of $b$ drops out.

- To understand this solution better, we restore parameters to find

$$F = 2m^2A^2e^{-ms}$$

  where $s = 2a$ is the separation, $A$ is the amplitude of the tail, and $m$ is the mass of the quanta in the theory. But this looks just like the result from tree-level exchange of quanta of mass $m$. Physically, applying perturbation theory makes sense because the interaction happens at $b$, where the field is close to vacuum.

- Evidently, the kink and antikink attract, eventually colliding. We have to resort to numerics to find what happens. Generally, if the kinks are moving slowly, they annihilate into 'meson radiation'. If they are moving very quickly, they can bounce off each other.

Next, we turn to kinks in sine-Gordan theory.

- In this case, the potential has minima at multiples of $2\pi$,

$$U(\phi) = 1 - \cos\phi, \quad \frac{dW}{d\phi} = 2\sin\frac{\phi}{2}.$$

  The Bogomolny equation is

$$\frac{d\phi}{dx} = 2\sin\frac{\phi}{2}, \quad \phi(x) = 4\tan^{-1}(e^{x-a})$$

  which describes a kink that interpolates between 0 and $2\pi$, with $M = 8$.

- However, there is *no* solution to the Bogomolny equations which interpolates between 0 and $4\pi$. The reason is that sine-Gordan kinks repel each other (since we know a kink and antikink attract), no matter how far apart they are. Thus the energy can be continually lowered by bringing the kinks further apart; there are no static multi-kink solutions.

- We can describe dynamical multi-kink solutions. One explicit example is

$$\phi(x,t) = 4\tan^{-1}\frac{v\sinh\gamma x}{\cosh\gamma vt}.$$

  Physically, the kinks move towards each other until $t = 0$, then bounce off each other, so evidently they repel. There is also a "breather" solution consisting of a kink and antikink bound together and oscillating.

- We might expect that a system of many kinks and antikinks, all with different velocities, will behave in a complicated way. Numerical simulations indicate that the result is simple; there is no energy loss due to "radiation". This is because the sine-Gordan theory is integrable.

- Since everything is periodic modulo $2\pi$, we can choose to physically identify $\phi \sim \phi + 2\pi$. Then we have $\phi : \mathbb{R} \to S^1$ and the boundary condition is $\phi(x) = 0$ at infinity.

- For a topological approach to classifying solitons, we can compactify $\mathbb{R}$ to get $\phi : S^1 \to S^1$, and solitons are classified by their winding number, i.e. their homotopy class in $\pi_1(S^1)$. We call this winding number the topological charge $Q$.

- We can also take a "physics" style approach. Define the topological current

$$j^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi = \frac{1}{2\pi}(\partial_x\phi, -\partial_t\phi), \quad \epsilon^{01} = 1.$$

  This current is conserved by the symmetry of mixed partial derivatives. This is remarkable because it is completely independent of the field equations, making its conservation topological rather than dynamical.

- The conserved charge associated with the current is

$$Q = \int j^0\,dx = \frac{1}{2\pi}\int \partial_x\phi\,dx$$

  which is simply the topological charge $Q$ as defined earlier.

- Finally, we can define $Q$ geometrically. The target space $S^1$ has a normalized volume form

$$\int_{S^1} \omega = 1$$

  where one example is $\omega = d\phi/2\pi$.

- Given a mapping $\phi : \mathbb{R} \to S^1$ we can pullback the volume form,

$$\phi^*\omega = \frac{1}{2\pi}\frac{d\phi}{dx}dx$$

  and we define the degree of the map to be

$$Q = \int_{\mathbb{R}} \phi^*\omega = \frac{1}{2\pi}\int \frac{d\phi}{dx}\,dx$$

  which agrees with the previous expressions.

Next, we briefly discuss quantization of the kink.

- At the simplest level, we can use the moduli space approximation,

$$L = \frac{1}{2}M\dot{a}^2, \quad H = \frac{P^2}{2M}$$

  and hence upon canonical quantization we get the ordinary Schrodinger equation for a free particle of mass $M$. Stationary states take the plane wave form

$$\psi(a) = e^{ika}, \quad P = \hbar k, \quad E = \frac{\hbar^2 k^2}{2M}.$$

  Note that the wavefunction is on the moduli space.

- More properly, we would quantize the field about a kink solution; this is similar to quantizing a field in a curved spacetime, in that we just generalize the usual plane-wave mode expansion. For example, one can form wavepackets far from the kink that don't feel it at all, while there are also modes that are "bound to the kink", and look like the kink's shape wobbling.

- The zero-point energies of these modes provide an infinite renormalization of the kink mass. This is acceptable, since we get a finite result when we subtract it against the zero-point energy of the vacuum solution. The remainder is a finite shift to the kink mass, which is indeed small when the coupling is weak.

- This computation will be complicated by the presence of zero-frequency modes. These correspond to moduli, and we can't ignore them; they remain important even when the coupling goes all the way to zero. Accounting for the moduli alone and ignoring all other terms in the Lagrangian is exactly what we did above.

- Another way to see the weak coupling requirement is to say that the soliton is essentially unchanged by quantization if there is a length scale $L$, much smaller than the kink size, where the size of the quantum fluctuations is small; this is the same requirement to treat the electromagnetic field classically. One can show explicitly this is equivalent to weak coupling.

- In the case of strong coupling, the soliton typically survives in the quantum theory, but it doesn't behave anything like the classical soliton.

## 12.3 Vortices

We will first attempt to find vortices in the simplest possible model, a complex scalar field in $2+1$ dimensions.

- We take the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^*(\partial^\mu \phi) - \frac{\lambda}{4}\left(|\phi|^2 - \frac{\mu^2}{\lambda}\right)^2$$

  where the potential is minimized for $|\phi| = v = \sqrt{\mu^2/\lambda}$.

- We expect solitons on topological grounds. Letting $\phi = \rho e^{i\alpha}$ and defining the winding number

$$N = \frac{1}{2\pi}\int_C d\boldsymbol{\ell} \cdot \nabla \alpha$$

  we find that $N$ takes integer values. On the other hand, it appears that $N$ must be zero if $\phi$ is non-singular since we can shrink the loop to zero. Thus for $N$ to be nonzero, there must be at least one point where $\phi$ vanishes. The solutions with nonzero $N = n$ are topologically stable with vorticity $n \in \mathbb{Z}$.

- To make further progress we must solve the field equations

$$\nabla^2 \phi - \lambda(|\phi|^2 - v^2)\phi = 0.$$

  We will try the rotationally symmetric ansatz $\phi(\mathbf{x}) = f(r)e^{i\theta}$. Also, if we demand symmetry by a reflection about the $x$-axis followed by complex conjugation, $f(r)$ must be real.

- There is a general reason the ansatz will work. Naively, taking an ansatz makes the equations of motion overdetermined, so we generically have no solutions at all. But suppose the action is invariant under a group of transformations $G$, and the ansatz has the most general possible form invariant under $G$. A variation of the ansatz can be decomposed into a part that is $G$-invariant

and a part that is not, where the second part averages to zero upon integration. Thus the only nontrivial equations of motion come from $G$-invariant variations, so the ansatz has enough parameters to generically yield a solution. A similar principle in quantum mechanics is that energy eigenstates have the symmetries of the Hamiltonian.

- However, we do need to look at variations that are not $G$-invariant to determine whether $G$-invariant solutions are stable, since physically there will be external influences that slightly break the symmetry.
- Now, plugging into the equations of motion gives a single equation for the single function $f$,

$$\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{f}{r^2} + \lambda(v^2 - f^2)f = 0$$

so we indeed get a solution, by the general reasoning above, with $f(0) = 0$ and $f(\infty) = v$.

- However, the solution has infinite energy, because

$$E \supset \int d^2x \, \frac{1}{2}|\phi|^2(\nabla\theta)^2 \sim \int \frac{dr}{r}$$

which diverges logarithmically. Such configurations could be physically relevant, i.e. we would have a finite energy per vortex if we had a finite density of vortices, but a single vortex is not physically meaningful.

It's possible to understand this result using general scaling reasoning.

- The energy of a finite-energy, static solution $\overline{\phi}$ can be written as $E = I_K + I_V$ where the gradient energy/"kinetic energy" $I_K$ is bilinear in the first derivatives of $\phi$ and the potential energy $I_V$ is an integral of $V(\phi)$.
- Now consider the scaled field $\phi(\mathbf{x}) = \overline{\phi}(\lambda\mathbf{x})$. In $D$ spatial dimensions,

$$E(\lambda) = \lambda^{2-D}I_K + \lambda^{-D}I_V.$$

Since $\overline{\phi}$ is a static solution, it locally minimizes the energy, so $dE/d\lambda = 0$ at $\lambda = 1$, giving

$$0 = (D-2)I_K + DI_V.$$

This places strong restrictions on solitons.

  - For $D = 1$, we get $I_K = I_V$, which is a generalization of a result we found earlier.
  - For $D = 2$, we require $I_V = 0$. This rules out non-singular solitons in our model above, since $\phi$ must vanish at some point, so we cannot have $|\phi| = v$ everywhere. Solitons can exist in more complicated models, such as the $O(3)$ sigma model. However, then $E(\lambda) = \lambda^{2-D}I_K$ is independent of $\lambda$, so the soliton is neutrally stable against expansion and contraction.
  - For $D \geq 3$, we require $I_K = I_V = 0$, so the only solution is the constant vacuum solution.

This result is a version of Derrick's theorem, and forbids simple solitons in $D > 1$.

- Solitons can evade Derrick's theorem in several ways. They can have infinite energy (as for our vortices), they can necessarily vary in time (which we won't consider), or they can be unstable or neutrally stable against compression, as we saw for $D = 2$. In this last case, however, this means we don't actually have stable solitons at all: when the soliton size shrinks close to the lattice cutoff, the topological obstruction fades away and the soliton can vanish entirely.

- Thus, in order to have stable finite-energy static solitons, we must add more structure to the theory. One example is an abelian Higgs model, where the energy is now

$$E = I_F + I_K + I_V, \quad I_F = \frac{1}{2}\int d^D x \ \mathrm{tr}\, F_{ij}^2, \quad I_K = \frac{1}{2}\int d^D x \, (D_j\phi)^*(D_j\phi).$$

  The new term $I_F$ transforms differently under scaling.

- We now consider the scaled fields

$$\phi(x) = \overline{\phi}(\lambda x), \quad A(x) = \lambda\overline{A}(\lambda x)$$

  where the factor of $\lambda$ is necessary to maintain the functional form of $I_K$. Then

$$E(\lambda) = \lambda^{4-D} I_F + \lambda^{2-D} I_K + \lambda^{-D} I_V$$

  which is stationary at $\lambda = 1$ if

$$(D-4)I_F + (D-2)I_K + DI_V = 0$$

  which allows soliton solutions in $D = 2$ and $D = 3$.

- Note that in pure Yang–Mills, we have only $I_F$, in which case a neutrally stable soliton exists only when $D = 4$. These solitons are just Yang–Mills instantons in spacetime dimension $d = 4$, after a Wick rotation to $D = 4$. The neutral stability indicates that instantons have any size.

We will next investigate vortices in the abelian Higgs model in $2 + 1$ dimensions.

- We have a complex scalar field $\phi$ and a $U(1)$ gauge potential $a_\mu$ with

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu\phi)^*(D^\mu\phi) - \frac{\lambda}{4}\left(|\phi|^2 - \frac{\mu^2}{\lambda}\right)^2, \quad D_\mu\phi = \partial_\mu\phi + ieA_\mu\phi.$$

  The gauge transformations take the form

$$\phi(x) \to e^{ie\Lambda(x)}\phi(x), \quad A_\mu(x) \to A_\mu(x) - \partial_\mu\Lambda(x).$$

  Under such a gauge transformation, the winding number transforms as

$$N \to N + \frac{e}{2\pi}\int_C d\boldsymbol{\ell}\cdot\nabla\Lambda = N$$

  as long as $\Lambda$ is nonsingular and single-valued, so our topological argument still works.

- Taking spatial components with $\phi = \rho e^{i\alpha}$, we find

$$\mathbf{D}\phi = e^{i\alpha}(\nabla\rho + i\rho(\nabla\alpha - e\mathbf{A}))$$

which means the gradient energy is zero at large distances if

$$\mathbf{A} = \frac{1}{e}\nabla\alpha.$$

Hence the energy does not diverge as it did above.

- Evaluating the winding number by a loop at large $r$, for a solution with vorticity $n$,

$$n = \frac{1}{2\pi}\int_C d\boldsymbol{\ell}\cdot\nabla\alpha = \frac{e}{2\pi}\int_C d\boldsymbol{\ell}\cdot\mathbf{A} = \frac{e}{2\pi}\int d^2x\, B, \quad \Phi = \frac{2\pi n}{e}$$

so we find a quantized magnetic flux. Since the field is pure gauge at large $r$, the flux is localized at small $r$.

- We now consider a static vortex solution with $n = 1$. We work in the gauge $A_0 = 0$ and take the ansatz

$$\phi(\mathbf{x}) = ve^{i\theta}f(evr), \quad A_j(\mathbf{x}) = \epsilon_{jk}\hat{x}^k\frac{a(evr)}{er}$$

where $f$ and $a$ are real. This is the most general ansatz with the same symmetries we used for the global vortex case. The boundary conditions are

$$f(0) = a(0) = 0, \quad f(\infty) = a(\infty) = 1$$

which ensure the vortex is nonsingular at the origin, and has finite energy.

- There is no closed-form analytic solution, but the form of the solution is intuitive. When $|\phi| \neq 0$, the gauge field acquires a mass, so it is energetically costly to have a magnetic field. Hence it is localized near the origin, where $|\phi|$ is small. Since the magnetic energy depends on $B^2$, it favors a larger vortex core, but the potential for $\phi$ favors a smaller vortex core. The relative strength of these two effects is determined by the ratio $\lambda/e^2$.

- Also note that the same vortex solution can be written in other gauges, though it will no longer appear rotationally symmetric.

Next, we consider multiple vortices.

- We may also consider the interaction of vortices. Two vortices will interact through both the $\phi$ field and the $A_\mu$ field. Since the former is spin zero, it mediates a universally attractive force, while the latter makes like-charged vortices repel. Hence at large distances, the interaction is determined by the field with the slower falloff; the vortices attract if and only if $m_\phi < m_A$, or equivalently $\lambda/e^2 < 1/2$.

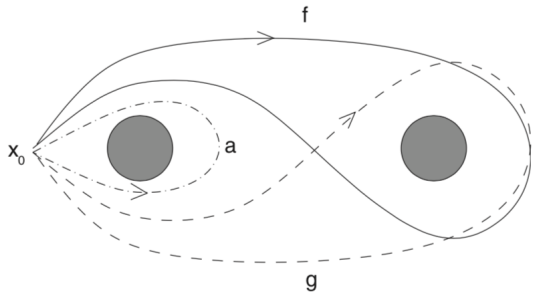
- It's harder to say what will happen at smaller distances, since the problem becomes nonlinear, but numerical simulations indicate that the sign of the force only depends on whether $m_\phi/m_A$ is greater or less than one. When they are equal, vortices don't interact at all, and there are static solutions with multiple vortices.

- Otherwise, such static solutions don't exist, but we can still find a static solution consisting of a single vortex with $n > 1$. However, when the force between vortices is repulsive, such a vortex is not energetically stable.
- This result is important for superconductivity, which is described by the same theory, but in three dimensions; our vortices are replaced with vortex lines. Consider forcing a magnetic flux through a superconductor by applying a field; it will then be energetically favorable to form vortices.
- In a type I superconductor, the superconductivity is lost at a relatively low external field, while a type II superconductor can persist up to a much higher field in a mixed state where the flux is confined to separated vortex lines. Physically, type I superconductors have $\lambda/e^2 < 1/2$, so the vortex lines attract, combining into extended regions where the superconductor breaks down. In a type II superconductor they repel, forming a lattice.

## 12.4 Vortices and Homotopy

Next, we formalize our statements with homotopy theory.

- We consider a theory with symmetry group $G$ broken down to $H$, so the vacuum manifold is $\mathcal{M} = G/H$. In two spatial dimensions, we consider a loop at spatial infinity (or at least well-separated from the vortex cores), which yields a loop in field space. Then naively, if this loop is not homotopic to the trivial loop, it must contain a soliton.
- However, there is a subtlety in this standard argument. When we consider homotopy of loops, we deform the loop while fixing a base point. However, when we consider the stability of solitons, we allow arbitrary variations of the field; there is no point on the loop where the field value is fixed. We are considering loops up to "free homotopy".
- As a specific example, consider the setup shown below.




  The loops $f$ and $g$ are not homotopic, but they are related by $g \sim afa^{-1}$. Hence they are freely homotopic. More generally, the set of loops up to free homotopy is the set of conjugacy classes of $\pi_1(\mathcal{M})$.
- Note that the set of conjugacy classes is not a group! Given multiple vortices, there are multiple paths that go around all of them, whose corresponding loops lie in different conjugacy classes.
- We must also account for gauge invariance. One can show that the freedom due to gauge transformations keeps loops within the same free homotopy class, so we don't have to further modify our conclusions.

- Below, we will focus on the abelian case, where these subtleties do not appear, as all conjugacy classes have one element. In this case, the quantum numbers of vortices simply add.

**Example.** In the models above, the symmetry group was $U(1)$ and completely broken, so $\mathcal{M} = U(1)$, and $\pi_1(\mathcal{M})$ indexes the vortices.

**Example.** Consider $SO(n)$ broken to $SO(n-1)$, which occurs when a scalar field transforming in the vector representation of $SO(n)$ acquires a vev. Then $\mathcal{M} = S^{n-1}$ and for $N \geq 3$, $\pi_1(\mathcal{M}) = 0$, so there are no topologically stable vortices.

**Example.** Consider an $SO(3)$ gauge theory with two scalar fields transforming in the vector representation, with the potential

$$V(\boldsymbol{\phi}, \boldsymbol{\chi}) = \frac{\lambda_\phi}{4}(\boldsymbol{\phi}^2 - v_\phi^2)^2 + \frac{\lambda_\chi}{4}(\boldsymbol{\chi}^2 - v_\chi^2)^2 + g(\boldsymbol{\phi} \cdot \boldsymbol{\chi})^2.$$

If $g$ is negative, the vevs are parallel, and the symmetry is broken to $U(1)$. If $g$ is positive they must be orthogonal, so the $SO(3)$ symmetry is completely broken, so $\mathcal{M} = SO(3)$ and $\pi_1(\mathcal{M}) = \mathbb{Z}_2$. Hence we have $\mathbb{Z}_2$ vortices, where a combination of two is topologically trivial. A simple ansatz is

$$\boldsymbol{\phi} = v_\phi(0, 0, 1), \quad \boldsymbol{\chi} = v_\chi f(r)(\cos\theta, \sin\theta, 0), \quad \mathbf{A}_j = \epsilon_{jk}\hat{x}^k \frac{a(r)}{r}(0, 0, 1)$$

where the latter two are just an embedding of the $U(1)$ gauged vortex.

**Note.** Consider the case $v_\phi \gg v_\chi$ above. Then if we integrate out the $\boldsymbol{\phi}$ field, we are left with a $U(1)$ gauge theory, and a combination of two vortices is topologically stable, even though it isn't in the original $SO(3)$ gauge theory. This is due to energetics. The deformation that takes two $\mathbb{Z}_2$ vortices to the untwisted configuration must involve a non-constant $\boldsymbol{\phi}$ field at intermediate stages, but if $v_\phi \gg v_\chi$ this has a very large energy cost. The lesson here is that topological stability is not absolute; it can be 'effective' like everything else in field theory.

**Note.** There is some freedom in the gauge groups. In the previous example, we could have taken gauge group $SU(2)$, in which case a $\mathbb{Z}_2$ symmetry would remain when $g > 0$. We get the same vacuum manifold, as $\mathcal{M} = SU(2)/\mathbb{Z}_2 = SO(3)$.

**Example.** The electroweak sector of the Standard Model has

$$G = SU(2) \times U(1), \quad H = U(1), \quad \mathcal{M} = S^3$$

because the Higgs vev can take any value with the same norm. Since $\pi_1(S^3) = 0$, there are no topological vortices. On the other hand, consider a theory with a local $U(1)$ symmetry but only a global $SU(2)$ symmetry. Then a vortex configuration with

$$\phi = \begin{pmatrix} 0 \\ f(r)e^{i\theta} \end{pmatrix}$$

is indeed topologically stable, as unwinding it would cost an infinite amount of $SU(2)$ gradient energy. Such a vortex is called a semilocal vortex. As before, we must have $f(0) = 0$ to avoid a singularity.

However, the semilocal vortex has a different source of instability. Recall that we can use a local $SU(2)$ symmetry to set the upper component of $\phi$ to zero everywhere, without loss of generality. With a global $SU(2)$ symmetry, we must consider the more general possibility

$$\phi = \begin{pmatrix} g(r) \\ f(r)e^{i\theta} \end{pmatrix}.$$

The scalar potential is minimized if $f^2 + g^2 = v^2$, so a vortex may be unstable to the formation of larger and larger regions where $f(r) = 0$, as long as $g(r) = v$. By the topological arguments, the vorticity must still remain at spatial infinity, but it will become unobservable at any finite radius. This instability occurs when the $U(1)$ gauge coupling is small compared to the scalar self-coupling.

**Example.** Alice strings. Consider an $SO(3)$ gauge theory with a Higgs field in the 5, i.e. an traceless symmetric $3 \times 3$ matrix transforming as

$$\phi \to R\phi R^T.$$

Also suppose that $V(\phi)$ is minimized when two of its eigenvalues are equal, so

$$\phi = R\,\mathrm{diag}(a, a, -2a)R^T = a(1 - 3\mathbf{e}\mathbf{e}^T), \quad \mathbf{e} = R\mathbf{e}_0.$$

Then the vacuum manifold is $S^2/\mathbb{Z}_2$ with fundamental group $\mathbb{Z}_2$. To find the unbroken gauge group, note that the vacuum remains invariant under rotations about the $\mathbf{e}$ axis, but it is also invariant under a $\pi$ rotation about the $\mathbf{e}'$ axis, where $\mathbf{e}'$ is an arbitrary vector perpendicular to $\mathbf{e}$. These operations do not commute; instead the group is

$$U(1) \rtimes \mathbb{Z}_2 = \mathrm{Pin}(2).$$

Of course, these results are consistent with the general principle $\mathcal{M} = G/H$.

Now consider a vortex solution described by $\mathbf{e}(\theta) = R(\theta)\mathbf{e}_0$ at large distances. If $R(\theta)$ traverses a nontrivial path through $S^2/\mathbb{Z}_2$, then the vortex is topologically stable. If it is present, it is not possible to define the charge of a particle under the unbroken $U(1)$ subgroup unambiguously, because it is generated by rotations about $\mathbf{e}$, and following $\mathbf{e}$ around a circle flips the sign. In order to remove the ambiguity, one may perform a gauge transformation so that $\mathbf{e}$ is constant everywhere, except for a "branch cut" running out to infinity, across which it flips. Then we may say the sign of the charge flips upon crossing the branch cut. In three dimensions, the vortices become strings, and the branch cuts become surfaces. The location of the surface is not gauge invariant, but its existence is, leading to the term "Alice strings".

Now consider two charges and an Alice string that closes on itself to make a circular loop. We consider two particles with the same charge starting next to each other; this is a physical statement, as it means that, e.g. they cannot annihilate. If we transport one of them in a circle that goes through the loop, it will come back with the opposite charge, and hence the two particles can annihilate; hence this ambiguity in the charge has a real physical effect. We can define the total charge of the system of the two charges and string at all times by a flux integral at infinity, and this must be conserved, so charge must have been transferred to the string. But since the surface associated with the string is arbitrary, it is meaningless to ask when the transfer occurred; hence we have a system where the total charge is defined but cannot be unambiguously localized!

**Example.** A non-abelian fundamental group. Consider the same theory as above, with a potential that is minimized when the eigenvalues of $\phi$ are all distinct,

$$\phi = R\,\mathrm{diag}(a_1, a_2, a_3)R^T, \quad a_1 \neq a_2 \neq a_3.$$

This vacuum is invariant under only the identity and rotations by $\pi$ about the $x$, $y$, and $z$ axes, which form the Klein four group. To find $\pi(\mathcal{M})$, it is more convenient to use the universal cover $G = SU(2)$. The unbroken group $H$ has eight elements, $\{\pm I, \pm\sigma_x, \pm\sigma_y, \pm\sigma_z\}$ forming the quaternion group. Since $G$ is simply connected,

$$\pi_1(G/H) = \pi_0(H)$$

so the fundamental group is non-abelian.

## 12.5 Quantizing Vortices (TODO)

# 13 Monopoles (TODO)

We begin by reviewing topological defects in general, in the SM and GUT theories.

- For a theory with vacuum manifold $\mathcal{M} = G/H$, the topological defects are classified by $\pi_0(\mathcal{M})$ (domain walls), $\pi_1(\mathcal{M})$ (vortices/strings), $\pi_2(\mathcal{M})$ (monopoles), and $\pi_3(\mathcal{M})$ (textures).
- The behavior of these objects changes dramatically if there is gauge symmetry, as in the SM. We call these defects with gauge symmetry "local", in contrast to "global" textures.
- Only local strings and monopoles are interesting as cosmological defects, because global strings and monopoles have infinite energy. However, collections of global strings or monopoles can have finite energy and play a role in other contexts, such as the Kosterlitz–Thouless transition.
- The global/local distinction is not relevant for domain walls; one could add a discrete gauge symmetry, but this doesn't do much because it doesn't produce a gauge field. Local textures are a bit trivial because the field lies in the vacuum manifold at all points, so the energy vanishes. Textures are instead interpreted as other vacuum states. **(right? anything else?)**
- In the SM, the relevant groups are $G = SU(2)_L \times U(1)_Y$ and $H = U(1)_A$, and $\mathcal{M} \cong S^3$. (This is easier to see by considering the possible states of the Higgs doublet, which takes values in $\mathbb{C}^2$.) As a result, the only possible defects are local textures, which are not interesting.
- Using results from the notes on Geometry and Topology, for the $SU(5)$ GUT, $G$ is connected and simply connected and $H = SU(3)_C \times U(1)_A$, so

$$\pi_1(G/H) = \pi_0(H) = 0, \quad \pi_2(G/H) = \pi_1(H) = \mathbb{Z}.$$

  This yields monopoles, but not other topological defects. But for more complex GUTs, symmetry breaking generally occurs in multiple stages, and generically produces cosmic strings.
- Cosmologically, there are strong constraints on domain walls and (local) monopoles coming from high-scale physics; both generically contain far too much energy and must be inflated away if they appear. Cosmic (local) strings are acceptable, but CMB measurements show they cannot play a dominant role in structure formation. Global textures were once hypothesized to play a role in structure formation, but this has also been ruled out.

# 14 Anomalies

## 14.1 Pion Decay

We begin with an overview of anomalies.

- A anomaly is a symmetry of a classical theory that is not present in the quantum theory. Specifically, anomalies mean the Ward–Takahashi identities will no longer hold. They arise because the theory cannot be regularized without breaking the symmetry, or equivalently because the path integral measure cannot be chosen to be invariant.
- Anomalies can apply to either gauge or global symmetries. A gauge anomaly is dangerous because it destroys the Ward identities, and hence prevents the decoupling of unphysical states, leading to violation of unitarity. Thus we must arrange our theories so that all gauge anomalies cancel. Such a theory is called 'anomaly free'.
- Global anomalies are not dangerous, and are instead ubiquitous. For example, the $U(1)_B$ baryon number symmetry is anomalous, and this allows for nonconservation of baryon number, which is required for baryogenesis.
- We will see that anomalies are infrared effects, resulting from massless particles in the spectrum. This leads to the idea of anomaly matching, which relates the spectrum of massless particles above and below a phase transition. In renormalizable gauge theories, anomalies arise exclusively from chiral fermions.
- More specifically, consider a theory with gauge group $G$ and global symmetry group $\tilde{G}$, with currents $j^\mu$ and $\tilde{j}^\mu$. It will turn out that anomalies can be computed by considering correlators of three currents, each of which may be from $G$ or $\tilde{G}$, at one-loop order with a massless particle in the loop.
- For reasons that will become clear later, we will write, e.g. the anomaly resulting from three $j^\mu$'s as $\mathrm{tr}(GGG)$. As a result, there are four important cases.
  - $\mathrm{tr}(GGG)$ corresponds to an anomalous gauge symmetry, which is dangerous for the reasons explained above. The Ward–Takahashi identities correspond roughly to $\partial_\mu j^\mu \sim F\tilde{F}$.
  - $\mathrm{tr}(GG\tilde{G})$ and $\mathrm{tr}(G\tilde{G}\tilde{G})$ may correspond to *either* an anomalous gauge symmetry or an anomalous global symmetry, depending on how we regularize the theory; for consistency we always choose the latter. This results in a non-conserved global current, $\partial_\mu \tilde{j}^\mu \sim F\tilde{F}$. For historical reasons, this is called a chiral anomaly.
  - Note that many anomalies of this mixed form automatically vanish, because $\mathrm{tr}(T_a) = 0$ for any generator of a semi-simple Lie group.
  - $\mathrm{tr}(\tilde{G}\tilde{G}\tilde{G})$ corresponds to a 't Hooft anomaly. These do not cause nonconservation of the current, $\partial_\mu \tilde{j}^\mu \sim 0$, but they are an obstruction to gauging $\tilde{G}$. It is these anomalies that are used in anomaly matching.
  - There are also linear/mixed/gravitational anomalies of the form $\mathrm{tr}(G\,\mathrm{grav}^2)$ or $\mathrm{tr}(\tilde{G}\,\mathrm{grav}^2)$ where two of the external legs are gravitons. These cause nonconservation of a current in the presence of spacetime curvature. This can be understood in terms of the previous anomalies by working with a vierbein, in which case coupling to gravity is like coupling to an $SO(3,1)$ gauge field.

- Since $F\tilde{F}$ is a total derivative term, one might think that an anomalous global symmetry still has a conserved *charge*. However, this is not true if we account for instantons, the subject of the next chapter, which can occur in non-abelian gauge theories. For gravitational anomalies, there are also corresponding gravitational instantons.
- Finally, quantum field theories generically break scale invariance. In this case the anomaly is called the trace anomaly, since scale invariance makes the energy-momentum tensor traceless, and it is proportional to the $\beta$-function. Conformal field theories are trace anomaly free.
- Historically, anomalies were first encountered in the computation of the decay rate for $\pi^0 \to \gamma\gamma$, where the chiral anomaly has an effect. In this context, it is also called an ABJ anomaly or the axial anomaly.
- The anomalies we have described above are sometimes called "perturbative" anomalies, in the sense that they can be computed using perturbative techniques. There are also more subtle anomalies, such as Witten's global $SU(2)$ anomaly (not to be confused with the anomalies of global symmetries we've considered above), which states that an $SU(2)$ gauge theory with an odd number of fermions in the fundamental representation is inconsistent.

First, we consider the decay $\pi^0 \to \gamma\gamma$ in the context of QCD with the up and down quark.

- An overview of the symmetries of the theory is given in the notes on the Standard Model. As a quick summary, the symmetry group is

  $$\tilde{G} = U(1)_L \times U(1)_R \times SU(2)_L \times SU(2)_R.$$

  This is spontaneously broken to

  $$\tilde{G} = U(1)_V \times SU(2)_V$$

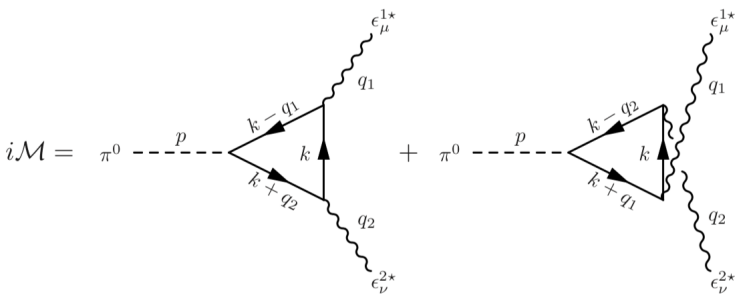
  by the formation of the quark condensate. The $U(1)_V$ symmetry is the total quark number, while the symmetry $U(1)_{\text{EM}} \subseteq SU(2)_V$ is gauged. The spontaneously broken symmetries $U(1)_A$ and $SU(2)_A$ correspond to the $\eta'$ and the pions, respectively.
- However, we should also account for the quark masses. Equal masses for the up and down quark break the symmetry group explicitly to

  $$\tilde{G} = U(1)_V \times SU(2)_V$$

  independent of the quark condensate. In this context $SU(2)_V$ is known as isospin. It is broken by the up-down quark mass difference, which we neglect.
- Now, suppose the $\pi^0$ couples to the proton $\psi$ by the interaction

  $$\mathcal{L}_{\text{int}} = i\lambda\pi\overline{\psi}\gamma^5\psi$$

  and it decays to photons by a proton loop, where the proton has mass $m$.
- The two contributing diagrams are shown below.




The diagrams may be computed in the usual way. The amplitude is superficially divergent, of the form $\int d^4k/k^3$, but the combination of the two terms turns out to be finite, with

$$\mathcal{M} = \frac{\lambda e^2}{4\pi^2 m}\epsilon^{\mu\nu\rho\sigma}\epsilon_\mu^{1*}\epsilon_\nu^{2*}q_1^\rho q_2^\sigma, \quad \Gamma = \frac{\alpha^2}{64\pi^3}\frac{\lambda^2 m_\pi^3}{m^2}.$$

- To make the result more quantitative, we need to find the value of $\lambda$. In the context of chiral perturbation theory, this coupling is related at tree level to the nucleon mass by $\lambda = m/f_\pi$, so

$$\Gamma = \frac{\alpha^2}{64\pi^3}\frac{m_\pi^3}{F_\pi^2}.$$

- Alternatively, the computation can be done by current algebra. The key fact is that the neutral pion is a Goldstone boson associated with the spontaneous breaking of $SU(2)_A$. The three pions satisfy

$$\langle\Omega|j_\mu^{5a}(x)|\pi^a(p)\rangle = ie^{ipx}F_\pi p_\mu, \quad j_\mu^{5a} = \overline{\Psi}\tau^a\gamma_\mu\gamma^5\Psi$$

  where $\Psi$ can be taken to be either $(p, n)$ or $(u, d)$. The neutral pion corresponds to $a = 3$.

- Now, if we had used $\Psi = (u, d)$ instead, there would be contributions from both up quark loops and down quark loops, of which there are $N$ colors, giving an overall factor of

$$N\left((2/3)^2 - (1/3)^2\right) = \frac{N}{3}$$

  in the amplitude, where the minus sign comes from the negative isospin of the down quark, and we get $q^2$ factors from the two photon vertices. Hence the pion decay rate was an early test that $N = 3$ in QCD.

- One might complain that higher-order effects in QCD should be significant. The above result is only accurate because the chiral anomaly is exact at one loop, as we'll see below.

- The fact that the spinor $\psi$ in the loop has a mass means that it violates $U(1)_A$ axial symmetry,

$$j^{\mu 5} = \overline{\psi}\gamma^\mu\gamma^5\psi, \quad \partial_\mu j^{\mu 5} = 2im\overline{\psi}\gamma^5\psi.$$

  This basic fact holds whether we think of the loop as containing nucleons or quarks.

- Now, the calculation we did above essentially says that

$$\langle A|\overline{\psi}\gamma^5\psi|A\rangle = i\frac{e^2}{32\pi^2}\frac{1}{m}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

where $F_{\mu\nu}$ is a background electromagnetic field. But then we have

$$\langle A|\partial_\mu j^{\mu 5}|A\rangle = -\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

which means that the axial current is not conserved in a background electromagnetic field, even when we take the mass $m$ to zero! Hence our simple calculation shows us something strange is going on with the axial symmetry – an anomaly is present.

- The reason the chiral anomaly is important here is that, if one reasoned from chiral symmetry alone, the decay rate should be much smaller. Naively we could have a term in the effective Lagrangian of the form

$$\mathcal{L}_{\text{eff}} = \frac{e^2}{8\pi^2 f_\pi}\pi^0\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

where the $f_\pi$ is by dimensional analysis, and the extra $1/4\pi^2$ is because we are dealing with a loop effect. This yields the same result that we found above.

- This is surprising since the pion is a Goldstone boson of $SU(2)_A$, and hence should not have a non-derivative coupling. We know that $SU(2)_A$ is not an exact symmetry, so such a coupling may arise due to the breaking of $SU(2)_A$ by the quark masses. But it must be proportional to $m_u + m_d \propto m_\pi^2$, and hence smaller by a factor of $m_\pi^2/f_\pi^2$, giving a result that is too small.

- The resolution comes from the chiral anomaly, which in this case shows that there is a violation of $SU(2)_A$ symmetry independent of the quark masses. We know that under an $SU(2)_A$ transformation with $a = 3$, the $\pi^0$ field shifts, $\delta\pi^0 = \theta F_\pi$. When the quarks are integrated out,

$$\mathcal{L}_{\text{eff}} \supset -\frac{e^2}{16\pi^2 f_\pi}\pi^0\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

almost exactly as expected by the naive argument. This is the vertex in the effective theory that replaces the triangle diagrams.

- In the language we used above, the $\pi^0$ decay rate is due to an $SU(2)_A U(1)_{\text{EM}}^2$ anomaly. There is also a $U(1)_A U(1)_{\text{EM}}^2$ anomaly. However, what distinguishes $U(1)_A$ is that it is anomalous even in pure QCD, by a $U(1)_A SU(3)_C^2$ anomaly. This has a large effect on the mass of the $\eta'$ because QCD is strongly coupled, resolving the $U(1)_A$ problem.

## 14.2 Triangle Diagrams

Next, we compute the anomaly directly, starting with a massless Dirac fermion. In this case we have the global symmetry $U(1)_A \times U(1)_V$.

- We consider the correlation function $\langle j^{\alpha 5}(x)j^\mu(y)j^\nu(z)\rangle$. Under an axial transformation we have $\delta j^\mu = 0$, so the Ward–Takahashi identity is simply

$$\partial_\alpha\langle j^{\alpha 5}(x)j^\mu(y)j^\nu(z)\rangle = 0.$$

This correlator is closely related to the pion decay amplitude, but without the external lines or the coupling constants.

- Defining

$$iM_5^{\alpha\mu\nu}(p,q_1,q_2)\not{p}(p-q_1-q_2)=\int dxdydz\, e^{ipx}e^{iq_1y}e^{iq_2z}\langle j^{\alpha5}(x)j^{\mu}(y)j^{\nu}(z)\rangle$$

  and applying the Feynman rules, we find at one loop

$$M_5^{\alpha\mu\nu}=\int đk\ \mathrm{tr}\left(\gamma^\mu\frac{1}{\not{k}}\gamma^\nu\frac{1}{\not{k}+\not{q}_2}\gamma^\alpha\gamma^5\frac{i}{\not{k}-\not{q}_1}+\begin{pmatrix}\mu\leftrightarrow\nu\\1\leftrightarrow2\end{pmatrix}\right).$$

- Using standard identities, we may show

$$p_\alpha M_5^{\alpha\mu\nu}=4i\epsilon^{\mu\nu\rho\sigma}\int đk\left(\frac{k^\rho q_2^\sigma}{k^2(k+q_2)^2}+\frac{k^\rho q_1^\sigma}{k^2(k-q_1)^2}\right)+\begin{pmatrix}\mu\leftrightarrow\nu\\1\leftrightarrow2\end{pmatrix}.$$

  By Lorentz invariance the integrals must be proportional to $q_1^\rho q_1^\sigma$ and $q_2^\rho q_2^\sigma$ respectively, and hence vanish when contracted with the Levi–Civita symbol. So naively it looks like the Ward–Takahashi identity is obeyed.

- Similarly, by contracting with $q_\mu^1$, we find

$$q_\mu^1 M_5^{\alpha\mu\nu}=-4i\epsilon^{\mu\nu\rho\sigma}\int đk\left(\frac{(k-q_1)^\rho(k+q_2)^\sigma}{(k-q_1)^2(k+q_2)^2}-\frac{(k-q_2)^\rho(k+q_1)^\sigma}{(k-q_2)^2(k+q_1)^2}\right).$$

  If we shift $k\to k+q_1$ in the first term and $k\to k+q_2$ in the second, the integral vanishes identically, so naively it looks like the vector current is conserved as well.

- However, this calculations are deceptive. If an integral is regularized, the integration variable can always be shifted, but we haven't done any regularization here. In fact, our standard regulators don't work, because DR has trouble with $\gamma^5$ and PV explicitly breaks chiral symmetry by introducing a massive fermion. One can still make sense of linearly divergent integrals, but their value changes upon a shift.

- To see this, consider the integral

$$\Delta(a)=\int dx\, f(x+a)-f(x)$$

  where the integration bounds are $\pm\infty$, and $f(x)$ goes to a constant at infinity. Naively the integral is zero by shifting the first term, but by Taylor expanding we have

$$\Delta(a)=\int dx\, af'(x)+\text{higher derivatives}=a(f(\infty)-f(-\infty))$$

  where the higher derivatives don't contribute since $f$ goes to a constant at infinity. This result holds regardless of whether or how we regulate $f(x)$ at infinity.

- By similar reasoning, one can show that for the linearly divergent integral

$$\Delta^\alpha(a^\mu)=\int đk\,(F^\alpha(k+a)-F^\alpha(k)),\quad \lim_{k_E\to\infty}F^\alpha(k_E)=A\frac{k_E^\alpha}{k_E^4}$$

  where $k_E$ is a Euclidean momentum, we have

$$\Delta^\alpha(a^\mu)=\frac{i}{32\pi^2}Aa^\alpha$$

  where the factor of $i$ comes from Wick rotation.

- Returning to the vector current calculation above, the quadratic divergences cancel, while for the linear divergence gives
$$q_\mu^1 M_5^{\alpha\mu\nu} = \frac{1}{4\pi^2}\epsilon^{\alpha\nu\rho\sigma} q_1^\rho q_2^\sigma.$$
Hence it appears the vector current is not conserved. The resolution is that the definitions of the loop momenta $k$ are independent between the two contributing diagrams. In other words, by choosing the $k$'s we did above, we had implicitly fixed a regularization scheme for the correlator.

- We may choose the vector current to be conserved by shifting $k \to k + q_1$ in the first diagram, which fixes $k \to k + q_2$ in the second by Bose symmetry. Then the linear divergences cancel exactly, but one can show that
$$p_\alpha M_5^{\alpha\mu\nu} = \frac{1}{4\pi^2}\epsilon^{\mu\nu\rho\sigma} q_1^\rho q_2^\sigma.$$
Taking the most general possible shift $k \to k + b_1 q_1 + b_2 q_2$ it can be shown that it is impossible to preserve both the vector and axial symmetry; we choose to preserve the former.

- To compare with what we found earlier, we can relate correlation functions to matrix elements by LSZ reduction. The form of the LSZ reduction formula for photons is
$$\langle f|i\rangle = i\epsilon^\mu \int dx\, e^{-ikx}(-\partial^2)\ldots\langle A_\mu(x)\ldots\rangle.$$
By the Schwinger–Dyson equation associated with the classical equation of motion $-\partial^2 A_\mu = j_\mu$,
$$\langle f|i\rangle = i\epsilon^\mu \int dx\, e^{-ikx}\ldots\langle j_\mu(x)\ldots\rangle + \text{contact terms}$$
where the latter do not contribute to $S$-matrix elements and hence don't matter here.

- Applying the LSZ reduction formula, we have
$$\langle q_1, q_2|j^{\alpha 5}(p)|0\rangle = (ig)^2\epsilon_\mu\epsilon'_\nu(iM^{\alpha\mu\nu}(p, q_1, q_2)\not{\delta}(p - q_1 - q_2)).$$
Therefore, contracting both sides with $p_\alpha$, we have
$$\langle q_1, q_2|\partial_\alpha j^{\alpha 5}(x)|0\rangle = -\frac{g^2}{2\pi^2}\epsilon^{\mu\nu\rho\sigma} q_{1\rho} q_{2\sigma}\epsilon_\mu\epsilon'_\nu e^{-i(q_1+q_2)z} + O(g^4)$$
where the higher-order terms vanish by the one-loop exactness of the anomaly. This is consistent with the operator equation
$$\partial_\mu j^{\mu 5} = -\frac{g^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}F_{\rho\sigma}.$$

**Note.** We haven't proven the above operator equation, only that a certain matrix element of both sides matches. Later, we will use the path integral to give a more general proof, which establishes that the two sides are equal when placed inside arbitrary time-ordered correlation functions.

The reason we hesitate to work with the actual operators is because they are difficult to define: quantum fields are operator-valued distributions and hence two fields at the same point can't be multiplied, but $j^{\mu 5}$ contains products of this form. In this case, we can "regularize" by point-splitting, separating the two $\psi$ fields in $j^{\mu 5}$ and connecting them with a Wilson line,
$$j^{\mu 5} = \lim_{\epsilon\to 0} \overline{\psi}(x + \epsilon/2)\gamma^\mu\gamma^5 \exp\left(-ie\int_{x-\epsilon/2}^{x+\epsilon/2} dz^\mu\, A_\mu(z)\right)\psi(x - \epsilon/2).$$

Taking the limit carefully proves the desired operator equation. This somewhat old-fashioned approach is given in detail in chapter 19 of Peskin and Schroeder. In modern high-energy theory, many simply circumvent this issue by working exclusively with the path integral. In this language, an "operator" is just defined as any expression that can go under a path integral, and an "operator equation" is defined to be any statement that holds underneath a path integral up to contact terms.

**Note.** As stated above, dimensional regularization has problems with $\gamma^5$. So far, we've mostly treated dimensional regularization as a series of ad hoc rules. That is, we assume that the trace obeys the usual properties (linearity, cyclicity), and further assume that

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \quad \{\gamma_5, \gamma^\mu\} = 0, \quad \gamma_5^2 = 1.$$

These properties can be used to derive others; for instance,

$$d \operatorname{tr} \gamma_5 = \operatorname{tr}(\gamma_5 \gamma^\mu \gamma_\mu) = -\operatorname{tr}(\gamma_5 \gamma_\mu \gamma^\mu) = -d \operatorname{tr} \gamma_5$$

which implies that $\operatorname{tr} \gamma_5 = 0$ for all $d \neq 0$. If $\operatorname{tr} \gamma_5$ is to be meromorphic in $d$, then $\operatorname{tr} \gamma_5 = 0$ for all $d$.

If $\gamma^5$ really could satisfy all these properties simultaneously, the axial anomaly would vanish. However, other results would come out incorrect. For instance, it can be shown that

$$(4-d) \operatorname{tr}(\gamma_5 \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) = 0$$

which this trace to be zero for all $d$, including $d = 4$, making many amplitudes vanish identically, and giving completely wrong physical results.

To avoid problems of this sort, we need to give a more constructive definition of dimensional regularization. For example, Collins defines integrals in non-integer dimensions axiomatically in chapter 4. One consistent way to define $\gamma_5$ is to take $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ in all dimensions. This formally means that $\gamma^5$ anticommutes with 4 of the gamma matrices, and commutes with $d-4$ of them, breaking Lorentz invariance. Practically, this means that to evaluate loop integrals, we must split the loop momentum as $\ell = \ell_\parallel + \ell_\perp$, where the two pieces have 4 and $d-4$ components, respectively, and treat the latter piece like a typical dimensional regularization integral in $d-4$ dimensions. This is used to derive the axial anomaly in chapter 19 of Peskin and Schroeder.

There are other methods for defining $\gamma_5$, which are described in chapter 13 of Collins and this paper. For instance, one can proceed naively if one only is interested in amplitudes with an even number of $\gamma_5$'s. This is used in chapter 21 of Peskin and Schroeder to handle one-loop computations in the Standard Model.

Now we reflect on our result.

- We see that we may choose which symmetry to preserve; we could also choose to preserve neither. If we give the Dirac fermion a charge, we must choose the vector symmetry to be preserved to avoid a gauge anomaly. If we give the fermion a mass, then axial symmetry is automatically broken and we may regulate with PV, so none of the subtleties above apply.

- The reason we considered this correlator was that taking more currents would yield divergences weaker than a linear one. Also, we needed one or three axial currents, because we need an odd number of $\gamma^5$ matrices (otherwise they can cancel out and we can use DR) and at least one vector current to check vector current conservation. However, we could also derive the axial anomaly by considering a correlator of three axial currents.

- In general, anomalies in four dimensions are studied with triangle diagrams, while anomalies in $2n$ dimensions are studied with $(n+1)$-gons. Moreover, we don't need to look at higher-loop diagrams since the chiral anomaly is one-loop exact. **(why not 2 currents?)**

- In general, for QED with any number of Dirac fermions, the gauge anomaly can be made to vanish by similar manipulations. However, for a single left-chiral Weyl fermion, this is not the case. The photon couples to the current

$$j_L^\mu = \overline{\psi}\gamma^\mu P_L \psi.$$

  The factors of $P_L$ can be manipulated to show that

$$\langle j_L^\alpha j_L^\mu j_L^\nu \rangle = \frac{1}{2}\left(\langle j^\alpha j^\mu j^\nu \rangle - \langle j_5^\alpha j^\mu j^\nu \rangle\right).$$

  The former vanishes when contracted with any of the momenta, as we may use DR there. Now we can have $p_\alpha \langle j_5^\alpha j^\mu j^\nu \rangle = 0$ or $q_{1\mu}\langle J_5^\alpha j^\mu j^\nu \rangle = 0$, but not both. Then the Ward–Takahashi identity for $j_L^\mu$ cannot be satisfied, and QED with a single Weyl fermion is inconsistent.

- Now suppose we have a left-chiral and right-chiral Weyl fermion with different charges. In this case the gauge boson $A_\mu$ couples to

$$j_{LR}^\mu = Q_L \overline{\psi}\gamma^\mu P_L \psi + Q_R \overline{\psi}\gamma^\mu P_R \psi.$$

  Since the mass is zero, there is no mixing between left-chiral and right-chiral fermions, so the contributions just add, giving

$$\langle j_{LR}^\alpha j_{LR}^\mu j_{LR}^\nu \rangle = Q_L^3 \langle j_L^\alpha j_L^\mu j_L^\nu \rangle + Q_R^3 \langle j_R^\alpha j_R^\mu j_R^\nu \rangle \supset \frac{1}{2}(Q_R^3 - Q_L^3) M_5^{\alpha\mu\nu}.$$

  Hence the theory is consistent if $Q_L = Q_R$.

- Another way to understand the relative minus sign is to convert the right-chiral Weyl fermions to left-chiral Weyl fermions, which flips their charge; then the total anomaly is just proportional to $\sum_i Q_i^3$. In the case of a non-chiral gauge theory, every $Q_i$ is paired with another of the opposite sign, which guarantees the sum is zero.

- In general, we only need to worry about vector symmetries being anomalous if the gauge theory is chiral. And chiral fermions in four dimensions must be massless, so we only need the low-energy spectrum; the chiral anomaly is an infrared effect. This allows us to check gauge anomaly cancellation in the Standard Model below. (However, we shouldn't forget that axial symmetries can be anomalous even in non-chiral theories, as we saw in $\pi^0$ decay.)

- By generalizing the computation of the triangle diagram to a non-abelian gauge symmetry, for a massless Dirac fermion in a representation $R$, we have

$$\partial_\mu j^{\mu 5} = -\frac{g^2}{16\pi^2} T(R) \epsilon^{\mu\nu\rho\sigma} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial_\rho A_\sigma^a - \partial_\sigma A_\rho^a) + O(g^3)$$

  which suggests the result

$$\partial_\mu j^{\mu 5} = -\frac{g^2}{16\pi^2} T(R) \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a = -\frac{g^2}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} \operatorname{tr} F_{\mu\nu} F_{\rho\sigma}$$

  by gauge invariance; it can properly be derived by computing square and pentagon diagrams.

- By the LSZ logic above, this anomaly will also affect Ward–Takahashi identities for correlation functions of the form $\langle j^5 jjj\rangle$ and $\langle j^5 jjjj\rangle$, since the term $F\tilde{F}$ can also annihilate three or four gauge bosons.
- Massive fermions don't contribute to the chiral anomaly, because there we can simply use PV regularization. Hence the chiral anomaly is an infrared effect. Note that the (historically prior) calculation we presented above can obscure this, since it makes it look like anomalies are due to loop divergences.

## 14.3 Anomalies from the Path Integral

Next, we see how anomalies can arise from the path integral.

- We consider the quantity

$$Z(A) = \int \mathcal{D}\Psi\mathcal{D}\overline{\Psi} e^{iS(A)}, \quad S(A) = \int dx\, \overline{\Psi}(i\not{D})\Psi, \quad D_\mu = \partial_\mu - igA_\mu$$

for a fixed background gauge field $A$, under which the Dirac fermion $\Psi$ transforms in a representation $R$. Formally, we have

$$Z(A) = \det(i\not{D})$$

but this expression must be regularized.

- Now consider a spacetime-dependent axial $U(1)$ transformation,

$$\Psi'(x) = e^{-i\alpha(x)\gamma_5}\Psi(x), \quad \overline{\Psi}'(x) = \overline{\Psi}(x)e^{-i\alpha(x)\gamma_5}.$$

If the measure were invariant, we would have

$$Z(A) = \int \mathcal{D}\Psi'\mathcal{D}\overline{\Psi'} \exp\left(i\int dx\, \overline{\Psi'}(i\not{D})\Psi'\right) = \int \mathcal{D}\Psi\mathcal{D}\overline{\Psi} \exp\left(i\int dx\, \overline{\Psi}(i\not{D})\Psi + j_A^\mu(x)\partial_\mu\alpha(x)\right)$$

where we used invariance of the measure in the second step. Integrating by parts and setting this equal to $Z(A)$, we have $\langle\partial_\mu j_A^\mu\rangle = 0$. If we had other operators present, we would have this result up to contact terms, which is precisely the Ward–Takahashi identity.

- Now we consider the Jacobian of the transformation more closely. We have

$$J(x,y) = \delta(x-y)e^{-i\alpha(x)\gamma_5}, \quad \mathcal{D}\Psi'\mathcal{D}\overline{\Psi'} = (\det J)^{-2}\mathcal{D}\Psi\mathcal{D}\overline{\Psi}$$

where the negative power is due to Grassmann variables, and

$$(\det J)^{-2} = \exp\left(-2\,\mathrm{tr}\log J\right) = \exp\left(2i\int dx\,\alpha(x)\,\mathrm{tr}(\delta(x-x)\gamma_5)\right)$$

where the trace is over spin and group indices.

- Again, this result is meaningless without regularization, as the delta function is infinite and the trace vanishes. Since we need to regularize $Z(A)$ in any case, we should use $\not{D}$, so we replace

$$\delta(x-y) \to e^{-(i\not{D}_x)^2/M^2}\delta(x-y) = \int đk\, e^{-(i\not{D}_x)^2/M^2} e^{ik(x-y)}$$

for a mass $M$ that will be taken to infinity. This replaces the delta function with a Gaussian in a gauge-invariant way, as we'll see below. Other cutoff functions would also work.

- If one is unsatisfied, one can also define the path integral more explicitly, using the same method we use for instantons below. That is, we expand $\Psi$ in a basis of suitably orthonormalized eigenfunctions of $\not{D}$,

$$\Psi(x) = \sum_n a_n \phi_n(x), \quad i\not{D}\phi_n = \lambda_n \phi_n$$

and define the measure by $\mathcal{D}\Psi = \prod_n da_n$, with a similar definition for $\mathcal{D}\overline{\Psi}$. One can then derive the Jacobian above more carefully, getting

$$\operatorname{tr}(\delta(x-x)\gamma_5) \to \sum_n \overline{\phi}_n(x)\gamma^5\phi_n(x)$$

which is equally divergent without regularization, but less manifestly nonsensical. One can then regulate the sum by adding a factor of $e^{-\lambda_n^2/M^2}$, then switch to a plane wave basis, recovering the expression above.

- Next, moving the derivative through the exponential with the identity

$$f(\partial)e^{ikx} = e^{ikx}f(\partial + ik)$$

we have

$$\delta(x-y) \to \int đk\, e^{ik(x-y)} e^{-(i\not{D}-\not{k})^2/M^2}.$$

Expanding the exponential, we have

$$-(i\not{D}-\not{k})^2 = -\not{k}^2 + i\{\not{k}, \not{D}\} + \not{D}^2 = -k^2 + 2ik\cdot D + D^2 + \frac{1}{2}[\gamma^\mu, \gamma^\nu]D_\mu D_\nu$$

where we used standard gamma matrix identities and $\gamma^\mu\gamma^\nu = (\{\gamma^\mu, \gamma^\nu\} + [\gamma^\mu, \gamma^\nu])/2$. Finally, we replace $D_\mu D_\nu$ with $[D_\mu, D_\nu]/2 = igF_{\mu\nu}/2$.

- Finally, rescaling $k$ by $M$, we have

$$\delta(x-y) \to M^4 \int đk\, e^{iMk(x-y)} e^{-k^2} e^{2ik\cdot D/M + D^2/M^2 + igF_{\mu\nu}[\gamma^\mu,\gamma^\nu]/4M^2}.$$

Therefore, we have

$$\operatorname{tr}\delta(x-x)\gamma_5 \to M^4 \int đk\, e^{-k^2} \operatorname{tr} e^{2ik\cdot D/M + D^2/M^2 + igF_{\mu\nu}[\gamma^\mu,\gamma^\nu]/4M^2}\gamma_5.$$

- We now expand the exponential in powers of $1/M$, where only terms up to $M^{-4}$ survive the $M \to \infty$ limit. The only way to get a nonzero trace is to have four gamma matrices, so we get

$$\operatorname{tr}\delta(x-x)\gamma_5 \to \int đk\, e^{-k^2} \frac{1}{2}\left(\frac{ig}{4M^2}\right)^2 \operatorname{tr} F_{\mu\nu}F_{\rho\sigma}[\gamma^\mu,\gamma^\nu][\gamma^\rho,\gamma^\sigma]\gamma_5 = \frac{ig^2}{2}\epsilon^{\mu\nu\rho\sigma}\operatorname{tr}(F_{\mu\nu}F_{\rho\sigma})\int đk\, e^{-k^2}$$

using standard identities, where the remaining trace is over the group. The Gaussian integral is

$$\int đk\, e^{-k^2} = \frac{i}{(4\pi)^2}$$

where the factor of $i$ is from Wick rotation, and the denominator is the usual loop factor.

- Putting everything together, we have

$$(\det J)^{-2} = \exp\left(-\frac{ig^2}{16\pi^2}\int dx\,\alpha(x)\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\,F_{\mu\nu}F_{\rho\sigma}\right).$$

  Thus, the invariance of $Z(A)$ implies that

$$\left\langle \partial_\mu j_A^\mu + \frac{g^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\,F_{\mu\nu}F_{\rho\sigma}\right\rangle = 0$$

  with the same holding up to contact terms in correlators, as desired.

Now we reflect on the result.

- Taking $\alpha(x)$ to be constant, we have additionally shown that a chiral redefinition of a fermion field can induce extra terms in the Lagrangian, which is important for understanding the strong CP problem.
- The Adler–Bardeen theorem states that the anomaly is one-loop exact. It can be proven by a careful study of Feynman diagrams, or by noting that we never expanded in $g$ above. **(right?)**
- By direct computation, we see the divergence of the current is a total derivative,

$$\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\,F_{\mu\nu}F_{\rho\sigma} = \partial_\mu\left(4\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}(A_\nu\partial_\rho A_\sigma - \frac{2}{3}igA_\nu A_\rho A_\sigma)\right)$$

  where the quantity in brackets is called a Chern–Simons current. In particular, given an anomalous symmetry, we can construct an alternative current that is conserved. However, note that this current is not gauge-invariant.
- Note that the total value of the axial charge cannot change for topologically trivial configurations. Such changes must be mediated via instantons, but instantons don't exist for $U(1)$ gauge fields, so the axial charge is conserved even if the current locally isn't.
- A better proof for the one-loop exactness of the anomaly uses topology. Imagine higher-order corrections to the anomaly. The most general answer has the form

$$\partial_\mu j_A^\mu = f(g)g^2\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\,F_{\mu\nu}F_{\rho\sigma}$$

  by dimensional analysis and parity invariance. Integrating over spacetime,

$$\Delta Q_A = f(g)\int dx\,g^2\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\,F_{\mu\nu}F_{\rho\sigma}.$$

  Assuming the gauge group is non-abelian, both the integral and the left-hand side are integers, so $f(g)$ cannot depend on $g$, and hence is one-loop exact. (This proof doesn't work in the abelian case because both sides vanish identically.)
- A similar derivation can be used for any even dimension $d = 2n$, giving

$$\partial_\mu j^{\mu 5} = (-1)^{n+1}\frac{2g^n}{n!(4\pi)^n}\epsilon^{\mu_1\ldots\mu_{2n}}F_{\mu_1\mu_2}\ldots F_{\mu_{2n-1}\mu_{2n}}.$$

  Note that in $d = 2$, anomalies only arise from abelian gauge groups, as otherwise $\mathrm{tr}\,F_{\mu\nu} = 0$.

**Note.** The meaning of an anomaly for a global symmetry can be described directly in terms of the path integral as follows. Suppose $j^\mu$ is the current for a global symmetry. We can formally couple the symmetry to a background gauge field, which in the simplest case can be done by adding $j^\mu A_\mu$ to the Lagrangian for a background field $A_\mu$, giving a partition function $Z[A]$. The symmetry can be gauged if we add in the Yang–Mills action for $A_\mu$ and integrate $Z[A]$ over gauge-inequivalent $A_\mu$. This cannot be done if the global symmetry is anomalous, which occurs precisely when $Z[A]$ is not gauge-invariant.

## 14.4 Anomalies in the Standard Model

Next, we show gauge anomaly cancellation in the SM, introducing non-abelian anomalies.

- In the SM, we have the three currents $j_\mu^C$, $j_\mu^L$ and $j_\mu^Y$, which couple to $SU(3)_C$, $SU(2)_L$ and $U(1)_Y$ respectively. (For brevity, we'll drop the subscripts.) Gauge anomaly cancellation requires

$$\partial_\mu \langle j_\mu^i j_\nu^j j_\rho^k \rangle = 0$$

  for all combinations of $i$, $j$, and $k$.

- In the case where all of the currents are $j_\mu^Y$, we have the $U(1)^3$ anomaly

$$\partial_\mu j_Y^\mu = \left( \sum_{\text{left}} Y_i^3 - \sum_{\text{right}} Y_i^3 \right) \frac{g^2}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} B_{\mu\nu} B_{\rho\sigma}$$

  where $B_{\mu\nu}$ is the $U(1)$ field strength and $g$ is the associated coupling.

- For the anomaly to vanish, we need

$$0 = (2Y_L^3 - Y_e^3 - Y_\nu^3) + 3(2Y_Q^3 - Y_u^3 - Y_d^3)$$

  where the 3 accounts for color charge, the 2 accounts for the two members in a doublet, and we've added a right-handed neutrino for later.

- Plugging in the numbers

$$Y_L = -\frac{1}{2}, \quad Y_e = -1, \quad Y_\nu = 0, \quad Y_Q = \frac{1}{6}, \quad Y_u = \frac{2}{3}, \quad Y_d = -\frac{1}{3}$$

  we find the anomaly vanishes for each generation, but not for the quarks or leptons individually.

- Next, consider triangle diagrams with three of the same non-abelian current. Here a fermion $\psi$ in a representation $R$ contributes to the current as

$$j_\mu^a = \overline{\psi}_i (T_R^a)_{ij} \gamma^\mu T_j.$$

  Therefore, the two diagrams pick up factors of $\operatorname{tr}(T_R^a T_R^b T_R^c)$ and $\operatorname{tr}(T_R^a T_R^c T_R^b)$ respectively.

- By substituting $T_R^a T_R^b = ([T_R^a, T_R^b] + \{T_R^a, T_R^b\})/2$, we have

$$\operatorname{tr}(T_R^a T_R^b T_R^c) = \frac{i}{2} T_R f^{abc} + \frac{1}{4} d_R^{abc}, \quad d_R^{abc} = 2 \operatorname{tr}(T_R^a \{T_R^b, T_R^c\}).$$

The quantity $d_R^{abc}$ is a totally symmetric rank 3 tensor. In the case of $SU(n)$, there is only one such tensor, so we have

$$d_{abc}^R = A(R) d_{abc}$$

where $d_{abc}$ is defined with the generators in the fundamental representation, and $A(R)$ is called the anomaly coefficient of the representation $R$.

- The term proportional to $f^{abc}$ contributes through the difference of the diagrams and is UV divergent. However, it does not contribute to the anomaly; instead it merely renormalizes the three gauge boson vertex.
- The term proportional to $d_{abc}^R$ contributes to the anomaly by

$$\partial_\mu j^{\mu a}(x) = \left( \sum_{\text{left}} A(R_i) - \sum_{\text{right}} A(R_i) \right) \frac{g^2}{128\pi^2} d^{abc} \epsilon^{\mu\nu\rho\sigma} F^b_{\mu\nu} F^b_{\rho\sigma}$$

where $F_{\mu\nu}$ is the associated gauge field strength and $g$ is the coupling. The overall factor is fixed by $U(1)$, where we have $T^a = 1$ and hence $d^{abc} = 4$.

- Since $T^a_{\overline{R}} = -(T^a_R)^T$, the anomaly coefficient obeys

$$A(R) = -A(\overline{R}).$$

Since the representations of $SU(2)$ are all pseudoreal, there is no $SU(2)^3$ anomaly because the anomaly coefficients all vanish. Moreover, there is no $SU(3)^3$ anomaly because QCD is non-chiral. The anomaly coefficient obeys the useful identities

$$A(R_1 \oplus R_2) = A(R_1) + A(R_2), \quad A(R_1 \otimes R_2) = A(R_1) d(R_2) + d(R_1) A(R_2)$$

so we can compute anomaly coefficients by building up from the fundamental.

Now we consider the mixed anomalies involving different currents. In this case, the trace is taken over matrices which are tensor products of the matrices associated with each group factor.

- An $SU(n)U(1)^2$ anomaly would be proportional to $\operatorname{tr}(T_R^a\{1,1\}) \propto \operatorname{tr} T_R^a = 0$, so they automatically vanish. Similarly, any anomaly with exactly one factor of $SU(2)$ or $SU(3)$ vanishes. Therefore, the only cases to check are $SU(3)^2U(1)$ and $SU(2)^2U(1)$.
- One has to be careful when applying this point: not all $SU(n)U(1)^2$ anomalies vanish, only those where $SU(n) \times U(1)$ is a symmetry. For example, the $SU(2)_A U(1)^2$ anomaly does not vanish, and this is precisely what accounts for the decay $\pi^0 \to \gamma\gamma$.
- The $SU(3)^2U(1)$ anomaly only receives contributions from quarks, of the form

$$Y \operatorname{tr}(\{T^a, T^b\}) \propto \delta^{ab}$$

which means the overall contribution is

$$\delta^{ab}(6Y_Q - 3Y_u - 3Y_d) = 0$$

where the right-handed quarks contribute negatively as usual.

- The $SU(2)^2U(1)$ anomaly receives contributions from left-handed fields only, giving
$$\delta^{ab}(2Y_L + 6Y_Q) = 0.$$
- The final constraint is from the gravitational anomaly, which comes from diagrams with two external gravitons and one external gauge boson. We find
$$\partial^\mu j^a_\mu \propto \mathrm{tr}(T^a_R)\epsilon^{\mu\nu\rho\sigma}R_{\mu\nu\alpha\beta}R_{\rho\sigma}{}^{\alpha\beta}.$$
Since the generators of $SU(n)$ are traceless, the only constraint is from $\mathrm{grav}^2U(1)$, which gives
$$\sum_{\text{left}} Y_i - \sum_{\text{right}} Y_i = 0.$$
- In summary, we have four constraints on six hypercharges. There are two two-parameter families of allowed hypercharges, one of which is ruled out by demanding $Y_Q \neq 0$. The other is
$$Y_L = -\frac{a}{2} - b, \quad Y_e = -a - b, \quad Y_\nu = -b, \quad Y_Q = \frac{a}{6} + \frac{b}{3}, \quad Y_u = \frac{2a}{3} + \frac{b}{3}, \quad Y_d = -\frac{a}{3} + \frac{b}{3}.$$
Note that we may also swap $Y_d$ and $Y_u$, and $Y_e$ and $Y_\nu$, since the constraints don't distinguish between them; there is also an arbitrary hypercharge normalization we've neglected. The coupling $a$ corresponds to hypercharge as it is in the SM, while $b$ corresponds to $B - L$.
- No matter what the values of $a$ and $b$ are, we must have $Y_L + 3Y_Q = 0$ exactly, indicating that the electron and proton have exactly opposite electric charges. However, for general $b \neq 0$, the neutron is not electrically neutral, and charge is not quantized.
- If there is no right-handed neutrino, or if it is Majorana, then we must have $Y_\nu = 0$, implying $b = 0$ and therefore fixing the SM hypercharges up to scaling. On the other hand, if the opposite is true, then it is possible to extend the Standard Model by gauging both $U(1)_Y$ as in the usual Standard Model, and $U(1)_{B-L}$.

For more about anomaly coefficients, see the notes on Group Theory. Next, we consider global anomalies in the Standard Model.

- Recall the $U(1)_A$ axial symmetry considered earlier. We cannot compute an anomaly from a $U(1)_A^3$ triangle diagram, because there is nothing $\partial_\mu j^\mu_A$ can be equal to. Instead the anomaly is due to $U(1)_A U(1)_V^2$, since the latter is associated with the electromagnetic gauge group.
- By similar reasoning, all anomalies for a global symmetry $G$ come from $GH^2$ triangle diagrams, where $H$ is one of the gauge groups of the SM. We must use $H^2$ since the trace of a single factor is zero.
- Two important global symmetries are baryon number and lepton number. Now, we have anomalies due to $SU(2)^2U(1)_B$ and $SU(2)^2U(1)_L$, giving
$$\partial_\mu j^{\mu B} = \partial_\mu j^{\mu L} \supset n_g \frac{g^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}W^a_{\mu\nu}W^a_{\rho\sigma}$$
where $n_g = 3$ is the number of generations. But if there is a sterile neutrino, $U(1)_{B-L}$ is nonanomalous, as we saw earlier, so it can be gauged. Such gauge bosons are common in grand unified theories and cause baryon number violation. Note that in neutral atoms, such a gauge boson would couple to neutron number. If gauged $U(1)_{B-L}$ is unbroken, it would need an exceptionally small coupling. If it is broken, the gauge boson mass is at least 3.5 TeV.

- The right-hand side is a total derivative, which means that baryon and lepton number are conserved perturbatively. However, there are $SU(2)$ instantons for which the right-hand side integrates to a nonzero value over spacetime; these mediate violation of baryon and lepton number. The rate for this process is extremely small, but at high temperatures, it can be much higher because the transitions are caused by thermal fluctuations rather than quantum effects. In this case, we say the process is mediated by sphalerons.

- There are also anomalies due to $U(1)_Y^2 U(1)_B$ and $U(1)_Y^2 U(1)_L$. However, since there are no $U(1)$ instantons, such anomalies cannot lead to global nonconservation of baryon or lepton number, so they are less important.

- Next, we consider theta terms. The theta term of $U(1)_Y$ has no physical effects, because there are no $U(1)$ instantons. The theta term of $SU(2)$ may be rotated away by the $SU(2)^2 U(1)_B$ anomaly. There is also an $SU(2)^2 U(1)_L$ anomaly, which matches the $U(1)_B$ anomaly so that $U(1)_{B-L}$ is nonanomalous.

- However, the theta term of $SU(3)$ cannot be removed. The reason that we could easily remove the theta term of $SU(2)$ was because of the chiral structure of the weak force; rotating all quarks contributes to the anomaly because $SU(2)$ only couples to left-handed quarks.

- For $SU(3)$, we may try $SU(3)^2 U(1)_A$ since $U(1)_A$ is chiral, but $U(1)_A$ is not a symmetry of the SM because of the quark masses. Instead, such an axial transformation changes the phases of the quark Yukawa couplings, and the invariant quantity is

  $$\overline{\theta} = \theta - \arg\det M$$

  where $M$ is the quark mass matrix. The fact that $\overline{\theta}$ is observed to be zero within experimental error is the strong CP problem, as explained further in the notes on the Standard Model.

**Note.** A bit more about baryon and lepton number violation. In order to see what an $SU(2)_L$ instanton does, we note there are 12 $SU(2)_L$ doublets in the SM (3 families of left-handed leptons, and $3 \times 3$ families and colors of left-handed quarks). The number of corresponding particles changes by 1 in each, so for a unit instanton we have

$$\Delta L_e = \Delta L_\mu = \Delta L_\tau = 1, \quad \Delta B = 3.$$

Of course such a process must still conserve energy, electric charge, and color charge. Also note that instantons can't make isolated protons decay, because baryon number is violated in multiples of 3.

Finally, we describe the technique of anomaly matching.

- As an example, consider pure QCD with three massless quark flavors, where the global symmetry is $G = SU(3)_L \times SU(3)_R \times U(1)_V$. There are no $SU(3)^2 G$ anomalies, which ensures $G$ remains a symmetry in a background gauge field, though there are $G^3$ anomalies.

- We now imagine gauging the symmetry $G$ with an arbitrarily weak coupling. The $G^3$ anomalies now render the theory inconsistent, but they can be removed by adding spectator fermions. They are only coupled to the existing particles through the gauge field, and can be effectively decoupled by taking the gauge coupling arbitrarily small.

- Now consider the low-energy theory where quarks are confined. Since anomalies are infrared effects, gauge anomaly cancellation must still hold. This must be due to massless particles in the spectrum, which could be either Goldstone bosons or massless hadrons.

- Suppose the latter occurs, so $SU(3)_L \times SU(3)_R$ is not spontaneously broken, and consider the $SU(3)_L^3$ anomalies. In the deconfined phase, the left-handed quarks each contribute $A(\text{fund}) = 1$, and there are three colors, so the spectator fermions contribute an anomaly coefficient of $-3$.

- Now, for the anomalies to be canceled in the confined phase, color singlet fermions constructed from quarks must provide a total anomaly coefficient of 3. We have

$$3 \times 3 \times 3 = 10 + 8 + 8 + 1$$

  and the 8 cannot contribute since it is a real representation. To evaluate $A(10)$, note

$$A(6) = A(3 \times 3) = A(\overline{3}) = 3A(3) + 3A(3) - A(\overline{3}) = 7$$

  and

$$A(10) = A(6 \times 3) - A(8) = 3A(6) + 6A(3) - A(8) = 27.$$

  Hence a 10 of baryons can only contribute in multiples of 27. Thus, for consistency, pure QCD with three flavors must have spontaneous chiral symmetry breaking!

- In real QCD, chiral symmetry breaking indeed occurs, and the anomalies are transferred over to the Goldstone bosons. For example, anomaly matching allows us to conclude there must be a $U(1)_{\pi^0}U(1)_V^2$ anomaly, where $U(1)_{\pi^0} \subseteq SU(2)_A$ is associated with the $\pi^0$. Indeed, this was our original motivating example.

# 15 Instantons

## 15.1 Quantum Mechanics

An instanton is a classical solution of the equations of motion with finite, non-zero action. They can be used to describe transitions between vacuum states in a semiclassical approximation.

- First, consider $\phi^4$ theory,
$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - g^2\phi^4.$$
In the classical case, we are free to scale the Lagrangian. Letting $\phi' = g\phi$, we have
$$\mathcal{L} = \frac{1}{g^2}\left(\frac{1}{2}\partial_\mu\phi'\partial^\mu\phi' - \frac{1}{2}m^2\phi'^2 - \phi'^4\right)$$
which means that $g$ is not a physical parameter.

- In the quantum case, the scale of the Lagrangian is set by $\hbar$, so $g$ is important; the physically relevant quantity is
$$\frac{\mathcal{L}}{\hbar} = \frac{1}{g^2\hbar}\left(\frac{1}{2}\partial_\mu\phi'\partial^\mu\phi' + \dots\right).$$
Then the relevant dimensionless parameter is $g^2\hbar$, and an expansion in $\hbar$ is the same as an expansion in $g$, as we've seen before.

- Our approximations will be semiclassical and hence require weak coupling. However, they will extract nonperturbative information essentially by the WKB approximation, which says that the probability of barrier penetration for a particle of unit mass is
$$|T(E)| = \exp\left(-\frac{1}{\hbar}\int_{x_1}^{x_2} dx\,\sqrt{2(V-E)}\right)(1+O(\hbar))$$
which is zero to all orders in $\hbar$.

- We can't imagine barrier penetration as a classical process, because the velocity would be imaginary in the classically forbidden region. But this is exactly what we would expect if we worked in Euclidean (imaginary) time. This motivates us to consider Euclidean solutions.

- Specifically, Euclidean solutions are precisely those which maximize the WKB probability of barrier penetration. To see this, consider the system
$$L = \int \frac{1}{2}\dot{q}^2 - V(\mathbf{q})$$
and consider a tunneling path $\mathbf{q}(s)$ between minima, where $ds^2 = d\mathbf{q}^2$, $\mathbf{q}(0) = \mathbf{q}_0$, $\mathbf{q}(s_f) = \mathbf{q}_f$. The tunneling amplitude is approximately $e^{-B/2}$ where
$$B = 2\int_0^{s_f} ds\,\sqrt{2(V(\mathbf{q}(s)) - E)}.$$

- In the case where the motion is between maxima, and hence classically allowed, Jacobi's principle tells us these paths correspond to stationary points of the action. By similar reasoning, for tunneling between minima, the paths minimize the Euclidean action

$$S_E = \int_{\tau_0}^{\tau_f} d\tau \left( \frac{1}{2} \left( \frac{d\mathbf{q}}{d\tau} \right)^2 + V(\mathbf{q}) \right)$$

  which has the potential negated.

- The equation of motion is

$$\frac{d^2 q_i}{d\tau^2} = \frac{\partial V}{\partial q_i}$$

  and the particle is at rest at the start and end of the path $\overline{\mathbf{q}}(\tau)$, so

$$\frac{1}{2} \left( \frac{d\overline{\mathbf{q}}}{d\tau} \right)^2 = V(\overline{\mathbf{q}}) - E, \quad E = V(\overline{\mathbf{q}}_0).$$

- Explicitly, the Euclidean action of the path is

$$S_E[\overline{\mathbf{q}}] = \int_{\tau_0}^{\tau_f} d\tau \, 2(V(\overline{\mathbf{q}}) - V(\mathbf{q}_0)) + \int_{\tau_0}^{\tau_f} d\tau \, V(\mathbf{q}_0)$$

  and the first term is simply $B/2$ upon a change of variables, so

$$\frac{B}{2} = S_E(\overline{\mathbf{q}}) - S_E(\mathbf{q}_0)$$

  where $\mathbf{q}_0$ is the trivial path $\mathbf{q}_0(\tau) = \mathbf{q}_0$.

- Such a Euclidean solution is called an 'instanton' because it is centered around some point of Euclidean time. However, this is not physically related to real time; at this point it's simply some parameter related to $s$. Originally, instantons were called pseudoparticles.

- We may also consider tunneling out of a metastable minimum. In this case, the WKB amplitude goes across the potential barrier, but the Euclidean solution goes out and 'bounces' back in, doubling the action. Hence we have

$$B = S_E(\overline{\mathbf{q}}) - S_E(\mathbf{q}_0).$$

  Physically, the amplitude for a particle in the minimum $\mathbf{q}_0$ to stay there falls exponentially in time, with the exponent being proportional to $e^{-B/2}$. Alternatively, one can say the state peaked in $\mathbf{q}_0$ has complex energy. This could be made more precise by starting with a system where $\mathbf{q}_0$ is a true minimum and analytically continuing to the system considered here.

- It is clear here that there are many possible escape paths, and we've just found the most probable one. The other reasonably likely escape paths are simply perturbed about the most probable one, and their effect on $B$ can be found perturbatively.

Next, we'll turn to the path integral, which is better at calculating the subleading corrections.

- Setting $m = \hbar = 1$ and restricting to one dimension,

$$\langle x_f | e^{-HT} | x_i \rangle = \int_{x(-T/2)=x_i}^{x(T/2)=x_f} \mathcal{D}x\, e^{-S_E}, \quad H = \frac{p^2}{2} + V(x)$$

  where $S_E$ is the Euclidean action as above. Expanding in energy eigenstates, $H|n\rangle = E_n|n\rangle$,

$$\langle x_f | e^{-HT} | x_i \rangle = \sum_n e^{-E_n T} \langle x_f | n \rangle \langle n | x_i \rangle$$

  so for large $T$, we may compute the energy and wavefunction of the lowest energy states by evaluating the path integral.

- To define the measure, we choose a set of orthonormal functions $x_n(\tau)$ vanishing at the boundaries and choose $\overline{x}$ obeying the boundary conditions; this is equivalent to the usual $\mathcal{D}x$ measure up to a constant Jacobian. Then we define the measure by

$$x(\tau) = \overline{x}(\tau) + \sum_n c_n x_n(\tau), \quad \mathcal{D}x \equiv N \prod_n \frac{dc_n}{\sqrt{2\pi}}$$

  where $N$ is a divergent normalization factor.

- In the semiclassical limit, we expand around stationary points of $S$. Suppose $\overline{x}$ is the only one; it obeys the classical equations of motion in the potential $-V(x)$. We let the basis functions $x_n(t)$ be eigenfunctions of the second variational derivative of $S$ at $\overline{x}$,

$$-\frac{d^2 x_n}{dt^2} + V''(\overline{x}) x_n = \lambda_n x_n.$$

  Expanding about the stationary point, the path integral is a Gaussian, so

$$\langle x_f | e^{-HT} | x_i \rangle = N e^{-S(\overline{x})} \prod_n \lambda_n^{-1/2} (1 + O(\hbar)) \approx N e^{-S(\overline{x})} \det(-\partial_\tau^2 + V''(\overline{x}))^{-1/2}.$$

  Higher order terms here come from the higher-order variation of the action with respect to $c_n$.

- This result is a generalization of the usual functional determinants seen in field theory. In that case, we usually only explicitly perform path integrals when the action is quadratic, in which case the functional determinant $\det \delta^2 S|_{\overline{x}}$ is independent of $\overline{x}$ and diagonalized by plane waves.

**Example.** A trivial example. Let $x_i = x_f = 0$ and consider a potential with a minimum at $x = 0$ and $V(0) = 0$. Then the only solution for $\overline{x}$ is the constant $\overline{x} = 0$, where $S = 0$, so

$$\langle 0 | e^{-HT} | 0 \rangle = N \det(-\partial_t^2 + \omega^2)^{-1/2} (1 + O(\hbar)), \quad \omega = V''(0).$$

To evaluate the functional determinant, note the eigenvalues are

$$\lambda_n = \frac{\pi n^2}{T^2} + \omega^2$$

so we have

$$N \det(-\partial_\tau^2 + \omega^2)^{-1/2} = N \prod_n \left( \frac{\pi^2}{n^2} T^2 \right)^{-1/2} \prod_n \left( 1 + \frac{\omega^2 T^2}{\pi^2 n^2} \right)^{1/2}.$$

The constant $N$ is fixed by matching with the result for the free particle, $\omega = 0$, giving

$$N \det(-\partial_\tau^2 + \omega^2)^{-1/2} = \frac{1}{\sqrt{2\pi T}} \prod_n \left(1 + \frac{\omega^2 T^2}{\pi^2 n^2}\right)^{1/2} = \frac{1}{\sqrt{2\pi T}} \left(\frac{\sinh(\omega T)}{\omega T}\right)^{-1/2}.$$

Taking the limit $T \to \infty$ and neglecting exponentially small terms, and restoring $\hbar$,

$$N \det(-\partial_\tau^2 + \omega^2)^{-1/2} = \sqrt{\frac{\omega}{\pi\hbar}} e^{-\omega T/2}$$

which implies that the ground state energy is $\hbar\omega/2 + O(\hbar^2)$, and

$$|\langle x = 0 | n = 0\rangle|^2 = \sqrt{\frac{\omega}{\pi\hbar}}(1 + O(\hbar)).$$

These are indeed the correct semiclassical results.

Next, we consider the extended example of the double well.

- Consider an even potential with minima at $\pm a$ with $V(a) = 0$ and $V''(a) = \omega^2$. We will compute $\langle a|e^{-HT}|a\rangle$ and $\langle a|e^{-HT}|-a\rangle$ semiclassically. As before, we have classical solutions that stay motionless at $\pm a$, but we also have the 'instanton' solution that moves from $-a$ to $a$, and the 'anti-instanton' that moves from $a$ to $-a$.

- Since we'll take $T$ to infinity anyway, we take this limit now. Then the instanton solutions have zero energy, and
$$\frac{dx}{d\tau} = \sqrt{2V}, \quad \tau = \tau_1 + \int_0^x \frac{dx'}{\sqrt{2V}}.$$
Such a solution is called an instanton centered at $\tau_1$. They are like solitons, but localized in Euclidean time rather than space. The action of an instanton is
$$S_0 = \int dt\, \frac{1}{2}(dx/dt)^2 + V = \int dt\, (dx/dt)^2 = \int_{-a}^{a} dx\, \sqrt{2V}.$$

- For large times, when $x$ approaches $a$, the equation of motion is
$$\frac{dx}{dt} = \omega(a - x)$$
which is exponential decay, so instantons have size on the order of $1/\omega$. Then in the $T \to \infty$ limit, we must account for multi-instanton solutions, where the instanton separation is $O(T)$, since they are approximate solutions to the equation of motion.

- Consider a solution with $n$ widely separated instantons, centered at $t_1, \ldots, t_n$, with
$$T/2 > t_1 > \ldots > t_N > -T/2.$$
The action of this solution is $nS_0$. The functional determinant is
$$N \det(-\partial_t^2 + V''(\overline{x}))^{-1/2} = \sqrt{\frac{\omega}{\pi}} e^{-\omega T/2} K^n$$
where the factor $K$ accounts for how each instanton modifies the functional determinant. We may multiply by $K^n$ because the effect of each instanton on the spectrum of fluctuations is localized about each one, and they are widely separated.

- Integrating over the instanton locations gives a factor of $T^n/n!$, so we have

$$\langle a|e^{-HT}|a\rangle = \sqrt{\frac{\omega}{\pi}}e^{-\omega T/2}\sum_{n\,\text{even}}\frac{(Ke^{-S_0}T)^n}{n!}$$

  and carrying out the sums gives

$$\langle \pm a|e^{-HT}|a\rangle = \sqrt{\frac{\omega}{\pi}}e^{-\omega T/2}\frac{\exp(Ke^{-S_0}T)\mp\exp(-Ke^{-S_0}T)}{2}.$$

- Thus, we have two low-lying energy eigenstates, with energies

$$E_\pm = \frac{1}{2}\omega \pm Ke^{-S_0}.$$

  These are the two expected energy eigenstates, consisting of even and odd combinations of the ground states of the two wells; the degeneracy is broken by barrier tunneling. Unlike the WKB approximation, the path integral allows us to calculate the preexponential factor $K$, and if we worked to more accuracy, we could find higher-order corrections.

- Above, we have assumed we were working with an 'instanton gas', i.e. that the instantons were too widely separated to affect each other. To check self-consistency, note that the largest term in the series $\sum_n x^n/n!$ comes when $x \sim n$, so

$$n \sim KTe^{-S_0}.$$

  If the instanton has characteristic size $\delta\tau$, we require

$$(\delta\tau)Ke^{-S_0} \ll 1.$$

  On dimensional grounds $K \sim 1/\delta\tau$, so we just need $S_0 \gg \hbar$, which we've already assumed.

- To compute the value of $K$ more carefully, formally we would have

$$K = \left|\frac{\det(-\partial_\tau^2+\omega^2)}{\det(-\partial_\tau^2+V''(\overline{x}))}\right|^{1/2}.$$

  However, this is not well-defined because the functional determinant in the denominator has a zero eigenvalue. This zero mode results because the instanton breaks time translation invariance. Explicitly, it is $x_1 = S_0^{-1/2}d\overline{x}/dt$.

- Making the zero mode into a collective coordinate and evaluating its contribution explicitly,

$$K = \left(\frac{S_0}{2\pi}\right)^{1/2}\left|\frac{\det(-\partial_\tau^2+\omega^2)}{\det'(-\partial_\tau^2+V''(\overline{x}))}\right|^{1/2}$$

  where the prime denotes exclusion of the zero mode. The remaining determinants can be computed using standard methods.

**Example.** Consider a periodic potential with minima at the integers. By similar reasoning,

$$\langle j_+|e^{-HT}|j_-\rangle = \left(\frac{\omega}{\pi}\right)^{1/2}e^{-\omega T/2}\sum_{n,\overline{n}}\frac{1}{n!\,\overline{n}!}(Ke^{-S_0}T)^{n+\overline{n}}\delta_{n-\overline{n}-j_++j_-}$$

where $n$ is the number of instantons and $n'$ is the number of anti-instantons. Using the identity

$$\delta_{ab} = \int_0^{2\pi} d\theta \, \frac{e^{i\theta(a-b)}}{2\pi}$$

we have

$$\langle j_+|e^{-HT}|j_-\rangle = \left(\frac{\omega}{\pi}\right)^{1/2} e^{-\omega T/2} \int_0^{2\pi} \frac{d\theta}{2\pi} \, e^{i(j_- - j_+)\theta} \exp\left(2KT\cos\theta e^{-S_0}\right).$$

Then the Hamiltonian is diagonalized by states of the form $|\theta\rangle$ where

$$\langle\theta|j\rangle = e^{ij\theta}, \quad E(\theta) = \frac{1}{2}\omega + 2Ke^{-S_0}\cos\theta.$$

These $|\theta\rangle$ states are analogous to the "$\theta$-vacua" we will encounter in gauge theories.

**Example.** Decay from a metastable vacuum. We consider a local minimum at $x = a$ with $V(a) = 0$ and wish to compute the rate of decay. Naively, the instanton solutions $\overline{x}(\tau)$ are bounces, and we can sum over any number of bounces, giving the answer

$$\langle a|e^{-HT}|b\rangle = \sqrt{\frac{\omega}{\pi}} e^{-\omega T/2} \sum_n \frac{(Ke^{-S_0}T)^n}{n!} = \sqrt{\frac{\omega}{\pi}} e^{-\omega T/2} \exp\left(Ke^{-S_0}T\right)$$

where as before,

$$K = \left(\frac{S_0}{2\pi}\right)^{1/2} \left|\frac{\det(-\partial_\tau^2 + \omega^2)}{\det'(-\partial_\tau^2 + V''(\overline{x}))}\right|^{1/2}.$$

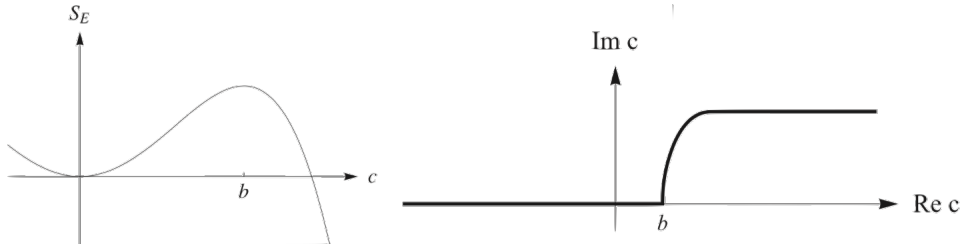
We expect the amplitude to decay exponentially, or equivalently for the state in the well to have a complex energy. Hence taking the logarithm, at large times we have

$$E_0 = \frac{1}{2}\omega - Ke^{-S_0}.$$

We get the desired imaginary part because there is a mode with a negative eigenvalue, which renders $K$ imaginary. To see this, note that the equation for the eigenfunctions $x_n$ has the form of the Schrodinger equation, and hence the mode with lowest eigenvalue should have no nodes. However, since the instanton solution bounces, the zero mode has one node, and hence there is one lower mode. This implies that the instanton solution is merely a saddle, not a minimum, of the potential.

To understand this, suppose the 'bounce' occurs at $x = b$, and consider the family of solutions that bounce at $x = c$. Then $S_E(c)$ has a local minimum at $c = 0$, corresponding to the trivial solution, and an extremum at $c = b$ corresponding to the instanton; hence it must be a maximum.

The reason that the bounce can be a local minimum of the barrier penetration integral $B$ but not a local minimum of $S_E$ is that they have different boundary conditions. In our earlier calculations, both were restricted over paths that went from one minimum to the other. But in this case, we're just going from one minimum back to itself, and $B$ has the additional constraint that the paths must bounce at $x = c$. (However, note that we must always consider minima of $B$, since otherwise our approximation will give a completely wrong result.)



To integrate over this mode, we must essentially perform the integral

$$J = \int_{-\infty}^{\infty} \frac{dc}{\sqrt{2\pi}} e^{-S(c)}.$$

This integral seems to diverge. To get a sensible result, we deform the contour as shown above, which gives a finite result. In the semiclassical limit, the only contribution to the integral comes from the vertical region near $c = b$, where we may apply the steepest descent approximation to get

$$\operatorname{Im} J = \frac{1}{2} e^{-S(b)} |S''(b)|^{-1/2}$$

where the factor of $1/2$ is because we only have half of the Gaussian peak; this is the factor of $1/2$ we anticipated on intuitive grounds earlier. Hence

$$K = \frac{i}{2} \left( \frac{S_0}{2\pi} \right)^{1/2} \left| \frac{\det(-\partial_\tau^2 + \omega^2)}{\det'(-\partial_\tau^2 + V''(\overline{x}))} \right|^{1/2}$$

and the decay rate is

$$\Gamma = -2 \operatorname{Im} E_0 = \left( \frac{S_0}{2\pi} \right) \left| \frac{\det(-\partial_\tau^2 + \omega^2)}{\det'(-\partial_\tau^2 + V''(\overline{x}))} \right|^{1/2} e^{-S_0}.$$

**Note.** The ideas above generalize directly to field theory. For the instantons to have finite $B$, the deviation of the fields from their initial value must be localized in space as well as time, resulting in additional zero modes; integrating over them gives a factor of the volume of the space. There will also be additional UV divergences that must be renormalized.

It's subtle why tunneling should be permitted in field theory at all. Naively, there are infinitely many degrees of freedom that all have to tunnel at once, giving an infinite suppression; this is exactly why spontaneous symmetry breaking is possible in quantum field theory but not in quantum mechanics. The first caveat is that in $d = 1 + 1$, the tunneling can occur by the formation and separation of domain walls, which incurs a finite cost; this corresponds to the Mermin–Wagner theorem. The second is gauge symmetry, which allows instantons to tunnel between vacua while being locally gauge-equivalent to zero on almost all of space, avoiding an infinite energy cost.

**Note.** The above formalism, based on analytic continuation to a Euclidean potential, is almost universally used in quantum field theory, but physically opaque. The instanton trajectories we sum over in Euclidean time bear no resemblance to the tunneling events occurring in physical time. (As they say, "you can't eat an instanton".) For a more physical approach based on the Minkowski path integral, see this paper and this paper.

## 15.2 Yang–Mills Vacua

We will work primarily in $A_0 = 0$ gauge, and begin with some classical subtleties.

- We use conventions where

  $$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu, A_\nu], \quad D_\mu = \partial_\mu - ieA_\mu$$

  and normalize generators so that $\operatorname{tr}(T^a T^b) = \delta^{ab}/2$.

- In the gauge $A_0 = 0$, the Lagrangian is

$$\mathcal{L} = \operatorname{tr} \dot{A}_j^2 - \frac{1}{2} \operatorname{tr} F_{ij}^2, \quad F_{0i} = \dot{A}_i.$$

  The dynamical variables are the $A_i$, and the conjugate momenta are

$$\Pi^j = F^{j0}.$$

  Note that we are working in the non-abelian case, but we suppress group indices.

- The classical equations of motion are

$$D_\mu F^{\mu j} = 0.$$

  By comparison, if we had not fixed the gauge, it would have been $D_\mu F^{\mu\nu} = 0$. Hence we are missing the Gauss's law constraint

$$C(\mathbf{x}) = D_\mu F^{\mu 0}(\mathbf{x}) = D_j \Pi^j(\mathbf{x}) = 0.$$

  However, the equations of motion do imply $\dot{C}(\mathbf{x}) = 0$.

- Physically, we have the freedom to make time-independent gauge transformations. If we view them as genuine symmetries, then they yield conserved charges, which correspond to the conservation of $C(\mathbf{x})$. Specifically, for a gauge transformation generated by $\Lambda(\mathbf{x})$ which vanishes at spatial infinity, the Noether charge is

$$C_\Lambda = 2 \int d\mathbf{x} \ \operatorname{tr} D_j \Lambda(\mathbf{x}) \Pi^j(\mathbf{x}) = -2 \int d\mathbf{x} \ \operatorname{tr} \Lambda(\mathbf{x}) D_j \Pi^j(\mathbf{x}) = -2 \int d\mathbf{x} \ \operatorname{tr} \Lambda(\mathbf{x}) C(\mathbf{x}).$$

  By Noether's theorem, $C_\Lambda$ is conserved for any $\Lambda(\mathbf{x})$, so $C(\mathbf{x})$ is. Hence we can just restrict to configurations where $C(\mathbf{x}) = 0$.

- The Lagrangian is in the standard $T - V$ form with quadratic kinetic energy, so going to Euclidean signature is straightforward, with

$$\mathcal{L}_E = \operatorname{tr}(\partial_4 A_j)^2 + \frac{1}{2} \operatorname{tr} F_{ij}^2, \quad x_4 = \tau.$$

  Conventionally, we take $\epsilon^{0123} = \epsilon^{123} = \epsilon^{1234} = 1$.

- Finally, we may include a field $A_4(x)$ which acts as a Lagrange multiplier that enforces Gauss's Law, in which case the Euclidean Lagrangian becomes

$$\mathcal{L}_E = \frac{1}{2} \operatorname{tr} F_{rs}^2, \quad r, s \in \{1, 2, 3, 4\}$$

  along with the boundary conditions $F_{j4} = 0$ on the temporal boundaries. This procedure, of first eliminating $A_0$, then rotating to Euclidean spacetime, then introducing $A_4$, is a bit longer than the naive approach but avoids some subtleties.

In $A_0 = 0$ gauge, there are topologically distinct vacua.

- Vacuum configurations have $F_{\mu\nu} = 0$, which implies in our gauge that

$$A_j = \frac{i}{g} G^{-1} \partial_j G$$

where $G(\mathbf{x})$ is an element of the gauge group, which we take to be $SU(2)$.

- We assume that $G$ tends to a constant $G_\infty$ as $r \to \infty$, to be justified later, giving a map $S^3 \to SU(2)$. The homotopy classes of such maps are classified by $\pi_3(SU(2)) = \mathbb{Z}$.

- To see why homotopy classes are relevant, we need to distinguish between 'large' and 'small' gauge transformations. Gauge transformations map $G(\mathbf{x})$ to $\Lambda(\mathbf{x})G(\mathbf{x})$. A small gauge transformation is one that can be built from a series of infinitesimal gauge transformations, with gauge functions $\Lambda(\mathbf{x})$ that vanish at spatial infinity. Large gauge transformations instead connect vacua in different homotopy classes.

- Physically, small gauge transformations must be 'do nothing' transformations, but it is ambiguous whether large gauge transformations are; this is a choice that affects the theory. For our purposes, it won't matter; we'll choose them to not be 'do nothing' transformations for clarity.

- We can explicitly compute the homotopy class by the integral

$$N[G] = \frac{1}{24\pi^2} \epsilon^{ijk} \int d\mathbf{x} \ \mathrm{tr}\, G^{-1} \partial_i G G^{-1} \partial_j G G^{-1} \partial_k G$$

which is normalized so that if $G$ is taken to be in the fundamental representation of $SU(2)$, with $T_a = \sigma_a/2$, then $N[G]$ is the usual winding number. In the case of gauge group $SU(2)$ only, it is also the Brouwer degree of the mapping $G$.

- To see this, note that near a point where $G$ is equal to $G_0$,

$$G(\mathbf{x}) = G_0 \exp(i\sigma_a \Lambda_a(\mathbf{x})) \approx G_0(I + i\sigma_a \Lambda_a(\mathbf{x})).$$

Plugging this in, we find the spatial integral of

$$-\frac{i}{24\pi^2} \epsilon^{ijk} \mathrm{tr}(\sigma_a \sigma_b \sigma_c) \partial_i \Lambda_a \partial_j \Lambda_b \partial_k \Lambda_c = \frac{1}{12\pi^2} \epsilon^{ijk} \epsilon^{abc} \partial_i \Lambda_a \partial_j \Lambda_b \partial_k \Lambda_c = \frac{1}{2\pi^2} \epsilon^{ijk} \partial_i \Lambda_1 \partial_j \Lambda_2 \partial_k \Lambda_3.$$

However, this is precisely the Jacobian factor for the transformation from the spatial coordinates $x_j$ to the group coordinates $\Lambda_a$, and $2\pi^2$ is simply the area of $S^3$.

- An example of a configuration with unit winding number is

$$g_1(x) = \exp(i\hat{r}_a \sigma_a f(r))$$

where $f(r)$ is any monotonic function with $f(0) = -\pi$ and $f(\infty) = 0$.

- Next, we have the useful identity

$$N[G_1 G_2] = N[G_1] + N[G_2].$$

Here the error term is

$$\frac{1}{8\pi^2} \epsilon^{ijk} \int d\mathbf{x} \ \mathrm{tr}\left( G_1^{-1} \partial_i G_1 \partial_j G_2 G_2^{-2} \partial_k G_2 G_2^{-1} + \partial_i G_2 G_2^{-1} G_1^{-1} \partial_j G_1 G_1^{-1} \partial_k G_1 \right)$$

and it can be shown to vanish using the product rule, the antisymmetry of $\epsilon^{ijk}$, and the fact that the $G_i$ tend to a constant at infinity.

- Also note that for an infinitesimal gauge transformation, the winding number is the integral of

$$\frac{1}{2\pi^2}\epsilon^{ijk}\partial_i\Lambda_1\partial_j\Lambda_2\partial_k\Lambda_3 = \frac{1}{2\pi^2}\epsilon^{ijk}\partial_i(\Lambda_1\partial_j\Lambda_2\partial_k\Lambda_3)$$

  which is a total derivative, so the winding number is zero. Hence $N[G]$ is invariant under continuous deformations/small gauge transformations.

- Next, we define the current

$$j^\mu_A = \frac{g^2}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\left(A_\nu\partial_\rho A_\sigma - \frac{2ig}{3}A_\nu A_\rho A_\sigma\right) = \frac{g^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\left(A_\nu F_{\rho\sigma} + \frac{2ig}{3}A_\nu A_\rho A_\sigma\right).$$

  The current is not gauge invariant, but it has a gauge-invariant divergence,

$$\partial_\mu j^\mu_A = \frac{g^2}{16\pi^2}\,\mathrm{tr}\,F_{\mu\nu}\tilde{F}^{\mu\nu}, \quad \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}.$$

  To verify this equation, note that the term quartic in $A_\mu$ on the right-hand side drops out by contraction with the Levi–Civita symbol.

- The charge associated with this current is

$$Q_A = \int d\mathbf{x}\, j^0_A = \frac{g^2}{16\pi^2}\int d\mathbf{x}\,\epsilon^{ijk}\,\mathrm{tr}\left(A_iF_{jk} + \frac{2ig}{3}A_iA_jA_k\right).$$

  Therefore, for a vacuum configuration we have

$$Q_A = N[G].$$

Next, we consider tunneling between these vacua.

- Consider a sequence of finite energy configurations $A_\mu(\mathbf{x}, t)$ interpolating between two vacuum configurations with winding numbers $N_1$ and $N_2$. We have

$$\int dx\,\partial_\mu j^\mu_A = \int dx\,(\partial_0 j^0_A + \partial_i j^i_A)$$

  which can be reduced to integrals over the bounding surface, conventionally depicted as a cylinder. The first term yields $N_2 - N_1$, while the second is a surface integral over $r = \infty$, whose only contribution in $A_0 = 0$ gauge is from the $A_jF_{0k}$ term. However, for finite energy configurations $F_{0k}$ must fall off faster than $1/r^{3/2}$, so $A_j$ must fall off faster than $1/r^{1/2}$, and hence this term vanishes in the limit $r \to \infty$. Thus,

$$\Delta N = \frac{g^2}{16\pi^2}\int dx\ \mathrm{tr}\,F_{\mu\nu}\tilde{F}^{\mu\nu}.$$

- Note that our derivation above is agnostic over whether $t$ is real or Euclidean time. In the context of Euclidean time, the most important vacuum tunneling solutions will be instantons. They connect vacua with $N$ differing by 1, and hence obey

$$\int dx\ \mathrm{tr}\,F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{16\pi^2}{g^2}.$$

- In the quantum theory, the state can be regarded as a wavefunctional of classical field configurations. The $C_\Lambda$ generate small gauge transformations, and the Gauss's law constraint means $C_\Lambda = 0$. Hence the wavefunctional must be constant on classical vacua related by small gauge transformations, in the same way that a wavefunction with $L_z = 0$ is independent of $\theta$. From this point on we ignore these states.

- Next, we have vacua $|n\rangle$ related by large gauge transformations, indexed by an integer $n$. These are not energy eigenstates, as tunneling between the vacua can occur.

- We define an operator $T$ by
$$T|n\rangle = |n+1\rangle$$
and note that by gauge invariant, $T$ commutes with the Hamiltonian. Then the energy eigenstates diagonalize $T$. They are the $\theta$-vacua,
$$|\theta\rangle = \sum_n e^{-in\theta}|n\rangle.$$
The $\theta$-vacua cannot evolve into each other, and this property extends to Fock spaces built upon these vacua. Hence we could simply regard $\theta$ as a constant of nature.

- When we perform the path integral, we must sum over gauge field configurations with all possible winding numbers. The net effect is that living in $|\theta\rangle$ is equivalent to ignoring such configurations, at the cost of adding the term
$$\Delta\mathcal{L} = \frac{\theta g^2}{16\pi^2}\,\mathrm{tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu}$$
to the Lagrangian. This is a total derivative, but has nonperturbative effects.

**Note.** The conclusions above depend on the gauge we're choosing. For example, in the axial gauge
$$A_3 = 0, \quad A_2(x,y,0,t) = 0, \quad A_1(x,0,0,t) = 0, \quad A_0(0,0,0,t) = 0$$
the gauge fixing is complete; there is a one-to-one relationship between $F_{\mu\nu}$ and $A_\mu$ and hence one classical vacuum, where $A_\mu = 0$. Intuitively, this is because the first condition leaves only $z$-independent gauge transformations; the second condition fixes $z$ and leaves only $z$-independent and $y$-independent gauge transformations, and so on, leaving only global gauge transformations.

In this gauge, the winding number of the vacuum is always zero. Instanton configurations still exist, connecting this vacuum to itself. However, in the surface integral of $n_\mu j^\mu_A$ considered above, only the sides of the cylinder contribute rather than the caps; the 'charge' created by an instanton flows out to spatial infinity. In this picture, the $\theta$ term is not a consequence of the $\theta$-vacuum, but simply a parameter in the Lagrangian. (This viewpoint also holds in $A_0 = 0$ gauge, so the theta term would be the sum of a bare, Lagrangian term and the effective term from living in a $\theta$ vacuum.) The fundamental difference between this gauge and the previous one is that this one counts the large gauge transformations as 'do nothing' transformations. However, the $\theta$ term has the same consequences in both pictures.

A simple analogy is given by the Lagrangian
$$L = \frac{1}{2}B\dot{\alpha}^2 - K(1-\cos\alpha).$$

For $\alpha \in \mathbb{R}$, it describes a particle of mass $B$ moving in a periodic potential, and the energy eigenstates are the analogues of the $\theta$-vacua. However, we can also regard $\alpha$ as an angle, in which case the Lagrangian describes a pendulum with moment of inertia $B$. The analogue of the $\theta$ term is

$$\Delta L = \frac{\theta}{2\pi}\dot{\alpha}$$

and appears in either case.

## 15.3 Yang–Mills Instantons

Next, we take a closer look at the instanton solutions. First, we take a detour into topology.

- We compactify Euclidean spacetime to $S^4$ and consider $G$-bundles over $S^4$. We may cover $S^4$ with two patches, which overlap on the "equator" $S^3$. Here the gauge fields are related by

  $$A_p^{\mathbf{II}} = U A_p^{\mathbf{I}} U^{-1} - \frac{i}{g}(\partial_p U)U^{-1}.$$

  We would like to classify all topologically distinct bundles.

- First, we consider the simpler situation of a $U(1)$-bundle over $S^2$, where

  $$A_p^{\mathbf{II}} - A_p^{\mathbf{I}} = -\partial_p \Lambda$$

  on the overlap region. We define

  $$I_1 = \frac{1}{4\pi}\int d^2x\, \epsilon^{pq} F_{pq}$$

  where the $x^p$ are coordinates on $S^2$. This quantity is manifestly gauge invariant.

- Next, we claim $I_1$ is a topological invariant, i.e. it does not change under a smooth deformation

  $$A_p^{\mathbf{I}} \to A_p^{\mathbf{I}} + \delta A_p^{\mathbf{I}}, \quad A_p^{\mathbf{II}} \to A_p^{\mathbf{II}} + \delta A_p^{\mathbf{II}}.$$

  In particular, it is continuous from one patch to the next, i.e. on the overlap region

  $$\delta A_p^{\mathbf{II}} = \delta A_p^{\mathbf{I}}.$$

  Now the integrand is a total derivative, and

  $$\delta I_1 = \frac{1}{4\pi}\int d^2x\, \epsilon^{pq}\delta F_{pq} = \frac{1}{2\pi}\int d^2x\, \partial_p(\epsilon^{pq}\delta A_q) = 0$$

  since $S^2$ has no boundary.

- Similarly, the value of $I_1$ itself is

  $$I_1 = \frac{1}{2\pi}\int d^2x\, \partial_p(\epsilon^{pq} A_q) \equiv \int d^2x\, \partial_p j^p.$$

  However, it is incorrect to simply set this to zero, because the current varies from patch to patch; it is not gauge invariant. Instead, we split it into integrals over the two hemispheres, yielding two integrals over the equator $\Sigma$,

  $$I_1 = \frac{1}{2\pi}\int_\Sigma d\ell_p\, \epsilon^{pq}(A_q^{\mathbf{I}} - A_q^{\mathbf{II}}) = \frac{1}{2\pi}\int_\Sigma d\ell_p\, \epsilon^{pq}\partial_q \Lambda$$

which is simply the winding number of the map $\Lambda : \Sigma \to U(1)$. Hence we have related $I_1$ to the winding number, a topological invariant of the fibre bundle. The integrand of $I_1$ is called the first Chern class/form and $I_1$ itself is called the first Chern number; the Chern class/form is only *locally* the derivative of the Chern–Simons 1-form.

- In four dimensions, we have the second Chern number

$$I_2 = \frac{g^2}{32\pi^2} \int d^4x \, \epsilon^{pqrs} \operatorname{tr} F_{pq} F_{rs}.$$

Again, this is a topological invariant, but now we have

$$\delta A_p^{\mathbf{II}} = U \delta A_p^{\mathbf{I}} U^{-1}.$$

Under a deformation we have

$$\delta I_2 = \frac{g^2}{16\pi^2} \int d^4 \, \epsilon^{pqrs} \operatorname{tr} F_{pq} \delta F_{rs} = \frac{g^2}{8\pi^2} \int d^4x \, \epsilon^{pqrs} \operatorname{tr} F_{pq} D_r \delta A_s$$

where the covariant derivative is taken with respect to the unperturbed potential. Now

$$\epsilon^{pqrs} \operatorname{tr} F_{pq} D_r \delta A_s = \epsilon^{pqrs} \partial_r (\operatorname{tr} F_{pq} \delta A_s) - \epsilon^{pqrs} \operatorname{tr}(D_r F_{pq}) \delta A_s$$

and the second term vanishes by the Bianchi identity. The first term is gauge invariant, so by the same argument as before $\delta I_2 = 0$ since $S^4$ has no boundary.

- Next, to relate $I_2$ to the topology of the bundle, we have shown above that

$$\frac{g^2}{32\pi^2} \epsilon^{pqrs} \operatorname{tr} F_{pq} F_{rs} = \partial_p j_A^p.$$

The current $j_A^p$ is the Chern–Simons 3-form, and is not gauge invariant. Hence by the same argument as above, we have

$$I_2 = \int_\Sigma dS_p \, (j_A^{\mathbf{I}p} - j_A^{\mathbf{II}p})$$

where $\Sigma$ is the 'equator' with topology $S^3$.

- An explicit computation shows that

$$j_A^{\mathbf{I}p} - j_A^{\mathbf{II}p} = \frac{1}{24\pi^2} \epsilon^{pqrs} \operatorname{tr} U^{-1} \partial_q U U^{-1} \partial_r U U^{-1} \partial_s U - \frac{ig}{8\pi^2} \epsilon^{pqrs} \partial_q (\operatorname{tr} U^{-1} \partial_q U A_s).$$

The second term, while not gauge invariant, is non-singular everywhere on $S^3$. That is, it can be computed using only the gauge field on a single patch. Then since $S^3$ has no boundary, the integral of this term vanishes, leaving

$$I_2 = \frac{1}{24\pi^2} \epsilon^{pqrs} \int_\Sigma dS_p \, \operatorname{tr} U^{-1} \partial_q U U^{-1} \partial_r U U^{-1} \partial_s U.$$

As shown above, for $G = SU(2)$ is is simply the winding number that counts the number of times $SU(2)$ is covered as one goes over $\Sigma$. Hence the bundles are classified by $\pi_3(S^3)$.

- More generally, they are classified by $\pi_3(G)$. Bott's theorem states that any continuous mapping from $S^3$ into a simple Lie group $G$ can be continuously deformed into a mapping into an $SU(2)$ subgroup of $G$. Hence our analysis holds unchanged for more general gauge groups. For a general compact Lie group, the $U(1)$ factors don't matter, while each other factor gives an independent winding number.

Finally, we return to the tricky business of justifying the boundary conditions.

- Mathematically, a gauge transformation is a fiber-preserving automorphism $P \to P$, and a large gauge transformation is one which is not connected to the identity. In particular, all gauge transformations preserve transition functions and hence the instanton number. They simply are basis changes for the fibers, which change the description of the connection.

- The transition functions between two patches are sometimes called gauge transformations, but they are completely different objects; they characterize the structure of the bundle itself and are fixed prior to the introduction of the connection.

- Now, the vacua and instantons were classified in different ways. The vacua are invariant under spatial gauge transformations; whether or not they are invariant under large spatial gauge transformations depends on the convention. The instantons, counted by the Chern number, are invariant under arbitrary continuous deformations of the gauge field, which include gauge transformations as a subset, as well as under large gauge transformations because the Chern form is gauge invariant. While instantons correspond to nontrivial fiber bundles over $S^4$, vacua are all trivial fiber bundles over $S^3$.

- In the case of instantons, one can shrink one of the patches to a very small region, which makes the transition functions singular. Naively, without the machinery of bundles, this looks like a singularity in the gauge field; one example is the Dirac string for the magnetic monopole.

- As we'll see below, we can write down instanton gauge fields on $\mathbb{R}^4$, but they will always be singular in at least one point. If this is the point at infinity, the instanton can described as the 'winding of the gauge field at long distances', allowing a sensible description without bundles. Alternatively, in the bundle language, the winding number of instanton is encoded in the transition function between this gauge field and a trivial gauge field in a topologically trivial patch containing the point at infinity.

- A more common approach in practice, which we will use below, is to take the gauge field to be constant at $x \to \infty$ but singular at $x = 0$. This is equally valid, and in the bundle language the winding would be encoded in the transition function between the gauge field in an infinitesimal patch containing the origin, and a patch containing everything else.

- The compactification to $S^4$ is a mathematical tool rather than a physical requirement. The physical content of an instanton is that it is a local minimum of the Euclidean action. Compactification of $S^4$ is only necessary to get topologically nontrivial fibre bundles, where the instanton number is additionally invariant under *all* continuous deformations, but this requirement has nothing to do with the physical business of computing tunneling rates: since instantons have finite sizes, modifying boundary conditions at infinity doesn't affect the Euclidean action.

- Note that there are two definitions of 'topological' going on here. Both the Chern and Chern–Simons forms are topological terms in the sense that they don't depend on the metric, but

only the Chern form is invariant under arbitrary deformations of the gauge field. These two types of terms yield topological information about the spacetime manifold and bundles over it, respectively. In the special case where the bundle is the tangent bundle, we can use it to get topological information about the manifold; the Gauss–Bonnet theorem is one example.

- Instantons are sometimes mistakenly called "vacua". The confusion arises from the fact that one may take an instanton background and perturb around it to compute subleading corrections, just like one does for a vacuum.
- It is simple to justify the condition $A_\mu(\mathbf{x}) \to \text{const}$ in the physical gauge, since this is required by the uniqueness of the vacuum $A_\mu(\mathbf{x}) = 0$.
- The situation is more subtle in $A_0 = 0$ gauge. If we do not take this condition, the quantity $N[G]$ will not be an integer, intuitively because there is a singular winding 'near the point at infinity'. On the other hand, the difference of $N[G]$ for two vacua will always be an integer, because it's simply the instanton number. Hence by performing a large gauge transformation we can always set $N[G]$ to an integer, and instantons will preserve this. Hence we can really just take $A_\mu(\mathbf{x})$ to be constant without loss of generality.

We now find explicit instanton solutions on $\mathbb{R}^4$ using the 't Hooft symbols.

- The trick is to use the relation between $\mathfrak{so}(4)$ and $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Define the matrices

$$e_p = \begin{cases} i\tau_p & p = 1,2,3 \\ I & p = 4 \end{cases}, \quad e_p^\dagger = \begin{cases} -i\tau_p & p = 1,2,3 \\ I & p = 4 \end{cases}$$

  where the $\tau_p$ are the usual Pauli matrices. Then for an $SO(4)$ vector $V_p$, define

$$V = e_p^\dagger V_p = \begin{pmatrix} V_4 - iV_3 & -V_2 - iV_1 \\ V_2 - iV_1 & V_4 + iV_3 \end{pmatrix}, \quad \det V = V_p V_p.$$

  Then $V$ is a unitary matrix times a multiple of the identity. We map the other way by

$$V_p = \frac{1}{2} \operatorname{tr} V e_p.$$

- Rotations of $V_p$ can be implemented on $V$ by

$$V \to U_L^{-1} V U_R, \quad U_L, U_R \in SU(2)$$

  where we need $U_L/U_R$ to be unitary to preserve the form of $V$ and special unitary to preserve the determinant. Since the transformation $U_L = U_R = -I$ does nothing, we conclude

$$SO(4) = (SU(2) \times SU(2))/\mathbb{Z}_2.$$

- To compute the effect of an infinitesimal transformation, let

$$U_L = e^{-i\boldsymbol{\omega}_L \cdot \boldsymbol{\tau}}, \quad U_R = e^{-i\boldsymbol{\omega}_R \cdot \boldsymbol{\tau}}.$$

  Then infinitesimally we have

$$\delta V = i(\boldsymbol{\omega}_L \cdot \boldsymbol{\tau})V - iV(\boldsymbol{\omega}_R \cdot \boldsymbol{\tau}).$$

- By some messy but direct calculation, we can show this implies

$$\delta V_p = (\omega_L^j \overline{\eta}_{pq}^j + \omega_R^j \eta_{pq}^j) V_q$$

and the numbers $\eta_{pq}^j$ are best written in terms of the matrices

$$\eta_{pq} = \eta_{pq}^j \tau_j = -i(e_p e_q^\dagger - \delta_{pq} I), \quad \overline{\eta}_{pq} = \overline{\eta}_{pq}^j \tau_j = -i(e_p^\dagger e_q - \delta_{pq} I).$$

- These matrices are antisymmetric, with the nonzero $\eta_{pq}^j$ and $\overline{\eta}_{pq}^j$ fixed by

$$\eta_{ij}^k = \overline{\eta}_{ij}^k = \epsilon_{ijk}, \quad \eta_{i4}^k = -\overline{\eta}_{i4}^k = \delta_{ik}.$$

This implies that $\eta_{pq}$ is self-dual and $\overline{\eta}_{pq}$ is anti-self-dual,

$$\eta_{pq} = \frac{1}{2}\epsilon_{pqrs}\eta_{rs}, \quad \overline{\eta}_{pq} = -\frac{1}{2}\epsilon_{pqrs}\overline{\eta}_{rs}.$$

Above, we have seen that $\overline{\eta}_{pq}$ determines the effect of $U_L$ while $\eta_{pq}$ determines the effect of $U_R$. This is because an antisymmetric rank two $SO(4)$ tensor transforms as $(1,0) \oplus (0,1)$ under $SU(2) \times SU(2)$. These two terms each transform trivially under one of the $SU(2)$ factors, and are self-dual and anti-self-dual.

- Finally, we will use the identities

$$\eta_{pq}\eta_{qr} = -2i\eta_{pr} - 3\delta_{pr} I, \quad \overline{\eta}_{pq}\overline{\eta}_{qr} = -2i\overline{\eta}_{pr} - 3\delta_{pr} I.$$

Next, we explicitly construct the unit instanton. It is convenient to proceed without gauge fixing.

- We can bound the action by noting that $F_{rs}^2 = \tilde{F}_{rs}^2$, so

$$\int d^4x \, \frac{1}{2} \operatorname{tr} F_{rs}^2 = \int d^4x \, \frac{1}{4} \operatorname{tr}(F_{rs}^2 + \tilde{F}_{rs}^2) = \int d^4x \, \frac{1}{4} \operatorname{tr}(F_{rs} - \tilde{F}_{rs})^2 + \int d^4x \, \frac{1}{2} \operatorname{tr} F_{rs}\tilde{F}_{rs} \geq \frac{8\pi^2}{g^2} k$$

where $k$ is the instanton number. The unit instanton has $k = 1$ and locally minimizes the action, which means its action is $8\pi^2/g^2$ and the field strength is self-dual. In the case of $k < 0$, we may flip a sign above to conclude that anti-instantons have anti-self-dual field strength.

- The result above is a Bogomolny bound, and it is useful because imposing self-duality requires solving only a first-order differential equation, while naively we would have to solve a second-order differential equation.

- We take the ansatz

$$A_p = \eta_{pq} x_q f(x^2)$$

with the corresponding field strength

$$F_{pq} = 2\eta_{qp} f + 2(\eta_{qr} x_r x_p - \eta_{pr} x_r x_q) f' - ig[\eta_{pr}\eta_{qs}] x_r x_s f^2, \quad f' = \frac{df}{d(x^2)}.$$

Requiring self-duality imposes constraints on $f(x^2)$.

- By rotational invariance, it suffices to examine the fields along the positive $x_4$ axis, where

$$F_{ij} = -2\eta_{ij} f - ig[\eta_{i4}, \eta_{j4}] x^2 f^2, \quad F_{k4} = -2\eta_{k4} f - 2\eta_{k4} x^2 f'$$

and the self-equality condition $F_{ij} = \epsilon_{ijk} F_{k4}$ reduces to

$$f' = -gf^2, \quad f = \frac{1}{g} \frac{1}{x^2 + \lambda^2}.$$

- Therefore, the unit instanton solution, also called the BPST instanton, is

$$A_p = \frac{1}{g} \frac{\eta_{pq} x_q}{x^2 + \lambda^2}, \quad A_p^j = \frac{2}{g} \frac{\eta_{pq}^j x_q}{x^2 + \lambda^2}.$$

The parameter $\lambda$ gives the characteristic size of the instanton; all values of $\lambda$ are allowed by the scale invariance of classical Yang–Mills. The anti-self-dual anti-instanton has the same form, with $\eta_{pq}$ replaced with $\overline{\eta}_{pq}$. The field strength is

$$F_{pq} = -\frac{2}{g} \frac{\eta_{pq} \lambda^2}{(x^2 + \lambda^2)^2}.$$

Note that this falls off as $1/x^4$ even though $A_p$ falls off as $1/x$. This is because the leading large-distance component of $A_p$ is pure gauge.

- Alternatively, by taking a gauge transformations, we have

$$A_p' = \frac{1}{g} \frac{\overline{\eta}_{pq} x_q \lambda^2}{x^2 (x^2 + \lambda^2)}.$$

In terms of bundles over $S^4$, the original gauge field is singular at the point at infinity, while this gauge field is singular at the origin. Such a singularity is necessary, since the fiber bundle is nontrivial.

- Yet another way to write the unit instanton is to let

$$G = \frac{x_4 + i\mathbf{x} \cdot \boldsymbol{\tau}}{|x|}, \quad A_p = \frac{x^2}{x^2 + \lambda^2} \left( -\frac{i}{g} \right) (\partial_p G) G^{-1}.$$

This works because $G$ wraps around $SU(2)$ once in an $S^3$ slice at constant $|x|$. Multiple instantons can be found by taking powers of $G$. Finally, one can explicitly transform this to $A_0 = 0$ gauge, giving explicit tunneling between vacua with different winding.

## 15.4 Physical Consequences

Finally, we investigate the physical consequence of instantons. First we connect them to anomalies.

- Consider a theory of $N_f$ massless fermions. The $U(1)_A$ symmetry is anomalous by

$$\partial_\mu j_5^\mu = \frac{N_f g^2}{8\pi^2} \operatorname{tr} F_{\mu\nu} \tilde{F}^{\mu\nu}.$$

We can form a divergenceless current

$$J_5^\mu = j_5^\mu - 2N_f j_A^\mu$$

where $j_A^\mu$ is as defined above; this current is not gauge invariant.

- In the $A_0 = 0$ gauge, the instanton corresponds to tunneling between vacua of different winding number; moreover this current vanishes at spatial infinity. Hence we have a conserved charge

$$Q_5 = n_R - n_L - 2N_f n$$

where $n$ is the winding number of the gauge field. Hence a change in winding number must be accompanied by a change in fermion chirality,

$$\Delta(n_R - n_L) = 2N_f \Delta n.$$

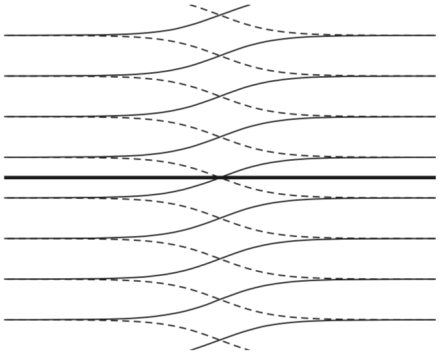
- To understand exactly how this happens, we consider the Dirac equation. The Hamiltonian for a single particle is

$$H = -i\alpha^j D_j, \quad \alpha^j = \gamma^0\gamma^j = \begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix}$$

in a particular representation of the $\gamma^\mu$ matrices. The four-component fermions naturally split into a pair of two-component Weyl fermions, where

$$H_R = -i\sigma^j D_j, \quad H_L = i\sigma^j D_j = -H_R.$$

- Now we work in the Dirac sea picture. Both $H_L$ and $H_R$ have both positive and negative energy eigenstates, and we suppose all the negative energy states are filled. During the tunneling process, $A_\mu$ is not in a vacuum state, so the fermion spectrum changes; however it must be the same in both the initial and final states. However, the spectrum can 'flow' as shown below.



Here the heavy line is the zero of energy. If the change is slow, the adiabatic theorem holds, so that the initial Dirac sea turns into a Dirac sea plus a right-chiral particle and a left-chiral negative energy hole, corresponding to a right-chiral positive energy antiparticle.

- Even if the adiabatic theorem does not hold, the Hamiltonian only couples fermion fields of the same chirality, so any transitions between states do not affect the change in net chirality. Similarly, the initial state need not be the vacuum state.

- The fact that the instanton number is generally equal to the number of energy levels that cross zero can be proven with an index theorem, which we won't go into here.

- Now consider the physical gauge. In this case, the vacua have been gauge transformed to all have the same winding number. However the effect of the instanton of course cannot be gauged away; the spectral flow occurs as before. Then the charge $Q_5$ is not conserved, which appears paradoxical. The resolution is that, as we've seen already, the current $j_A^\mu$ does not vanish at spatial infinity in this gauge; instead it yields a finite surface integral.

- The chirality-violating processes associated with instantons can be represented by a nonlocal effective Lagrangian density,

$$\mathcal{L}_{\text{eff}} = Ce^{-8\pi^2/g^2(\lambda)} \prod_{s=1}^{N_f} (\overline{\psi}_R^s \omega)(\overline{\omega}\psi_L^s)$$

  where we must integrate over the positions of the fermion fields, $\omega$ is a fixed Dirac spinor transforming under the fundamental representation of the gauge group which depends on the gauge orientation of the instanton, and $C$ is obtained from the one-loop corrections to the instanton. The Hermitian conjugate of this term accounts for the anti-instanton, and one must also integrate over instanton positions and scales. One may also integrate over gauge orientations of $\omega$ to get a gauge-invariant result.

- To solve the $U(1)$ problem of QCD, note that if we define

$$\mathcal{M}_{rs} = (\overline{q}_{Rr}\omega)(\overline{\omega}q_{Ls})$$

  then the anticommutativity of the fermion fields implies

$$\prod_{s=1}^{N_f} (\overline{q}_{Rs}\omega)(\overline{\omega}q_{Ls}) = \frac{1}{N_f!} \det \mathcal{M}.$$

  The chiral transformations of the quark fields are

$$q_L \to U_L q_L, \quad q_R \to U_R q_R, \quad \mathcal{M} \to U_R^\dagger \mathcal{M} U_L$$

  under which $\det \mathcal{M}$ is not invariant for a $U(1)_A$ transformation. Hence $U(1)_A$ is explicitly broken when one accounts for instanton effects, and so is not a spontaneously broken symmetry.

- Computing the coefficient $C$ amounts to evaluating functional determinants as we did for quantum mechanical tunneling, to find how the action varies as the instanton solution is deformed. We must also integrate over collective coordinates. The troublesome coordinate is the instanton size; we find a divergence due to large instantons, where the gauge coupling is strong. Hence unless we have a natural cutoff on the instanton size, it is difficult to estimate $C$. We only know that in QCD, instanton effects are significant.

**Note.** However, the $U(1)_A$ problem is a bit more confusing when one thinks in terms of $\theta$-vacua instead of Lagrangian terms. The charge associated with $U(1)_A$ generates $\theta$ translations. Hence it appears the $U(1)_A$ symmetry is spontaneously broken by the choice of a $\theta$ vacuum, and hence should yield a PNBG.

Formally, any explicitly broken symmetry can be thought of as a spontaneously broken symmetry by enlarging the Hilbert space (i.e. treating the symmetry breaking parameter as a spurion), but it will not correspond to a physical excitation. The key property that distinguishes $U(1)_A$ breaking from the genuine spontaneous rotational symmetry breaking of a ferromagnet is that the magnetization $\mathbf{M}(x)$ is a local field which can have local excitations, while $\theta$ is a global property of the state, so it doesn't make sense to promote it to $\theta(x)$. We dwell on this point because often the introduction of the axion is oversimplified to saying that "axions result from promoting $\theta$ to a field". As shown in the notes on the Standard Model, one has to work a bit harder than that!

**Note.** A very similar phenomenon occurs in the simpler context of QED with a massless fermion in $d = 2$, where we can do the calculations simply and explicitly. In this case, the particles are either right-movers or left-movers, with the numbers $n_R$ and $n_L$ separately conserved. The axial charge, proportional to $n_R - n_L$, is violated by the anomaly,

$$\Delta(n_R - n_L) = \int d^2x \, \frac{e}{2\pi} \epsilon^{\mu\nu} F_{\mu\nu}.$$

To relate this to spectral flow, suppose we adiabatically turn on a constant $A^1$ field, and for simplicity compactify space to length $L$. The Hamiltonian for a Weyl field is

$$H = \int dx \, \psi^\dagger (-i\alpha^1 D_a)\psi, \quad \alpha = \gamma^0\gamma^1 = \gamma^5.$$

Splitting this into right-moving and left-moving components,

$$H = \int dx \, \left( -i\psi_+^\dagger (\partial_1 - ieA^1)\psi_+ + i\psi_-^\dagger (\partial_1 - ieA^1)\psi_- \right).$$

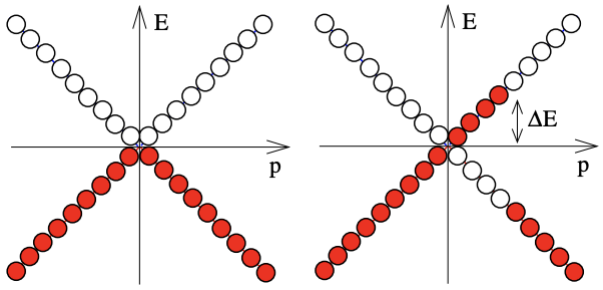
The eigenfunctions of the covariant derivative the form $e^{ik_n x}$ with $k = 2\pi n/L$, which implies the corresponding single-particle eigenstates of $H$ have the energies

$$E_n^+ = k_n - eA^1, \quad E_n = -(k_n - eA^1).$$

Suppose we shift $A^1$ by $\Delta A^1 = 2\pi/eL$. Then spectral flow occurs in exactly the same manner as we saw above for $d = 4$, changing $n_R - n_L$ by $-2$. Correspondingly, we have

$$\int d^2x \, \frac{e}{2\pi} \epsilon^{\mu\nu} F_{\mu\nu} = \int dt \, dx \, \frac{e}{\pi} \partial_0 A_1 = -2$$

as expected. We can understand the spectral flow in a very concrete way: turning on the $A_1$ field corresponds to giving an equal impulse $e\Delta A^1$ to each particle. This affects the Dirac sea as follows:




The impulse creates right-moving particles and left-moving antiparticles, violating axial charge conservation.

This derivation didn't need instantons: we just postulated a background field configuration, nonzero over all space. Instantons are necessary to get a nontrivial result if we also require the fields to vanish quickly at infinity, i.e. they are the only contributions that matter in the infinite volume limit, where constant field configurations like this one would have infinite action.

**Note.** As with all anomaly effects, the result depends on the UV regularization. This is because the spectral flow affects energy levels all the way to positive and negative infinity. For example, if we had a hard cutoff, then spectral flow would never change the charges; in the above example, for example, the creation of low-energy right-moving particles would be compensated by the creation of high-energy right-moving antiparticles. But we cannot just use this regulator because, as we've seen before, it would ultimately lead to an anomaly in the vector current, rendering the theory inconsistent.

Next, we investigate baryon number violation.

- In the $SU(2) \times U(1)$ electroweak theory there are also instantons. The Higgs acquires its usual vev at infinity but vanishes at the center of the instanton. The Higgs vev hence breaks the classical scale invariance, by favoring a smaller instanton size. As a result the integration over instanton size is finite, peaking around $\lambda \sim 1/\langle\phi\rangle$.

- Each electroweak instanton leads to a violation of baryon and lepton number by

$$\Delta B = \Delta L = 3$$

  where the factor of 3 is from the three generations. The effective Lagrangian is nonlocal at the electroweak scale and hence, at longer distances, one can approximate the terms as local. For example, an instanton could mediate the scattering process $p + n \to \overline{p} + e^+ + e^+ + \overline{\nu}_e$.

- The rate of the associated processes is suppressed by

$$\left(e^{-8\pi^2/g^2}\right)^2 = e^{-16\pi^2 \sin^2\theta_W/e^2} \sim 10^{-161}$$

  where we evaluated $e$ at the $Z$ mass, since the typical instanton size is set by the electroweak scale. The universe contains about $10^{78}$ protons and has an age of about $10^{10}$ years, or $10^{40}$ times a typical strong interaction time of $10^{-23}$ seconds. Hence baryon number violation by instantons appears to be completely negligible.

- However, one can also travel between different vacua by thermal fluctuations, rather than quantum tunneling. For every path connecting two vacua in spacetime, there is a point of highest energy; the minimum of all such maxima determines the thermal fluctuation rate. Such a solution is called a sphaleron, and it heuristically looks like the "midpoint" of the unit instanton solution. In particular, it has half-integer winding number. Sphalerons shouldn't be confused with instantons: instantons are spacetime configurations that are minima of the Euclidean action, while sphalerons are spatial configurations that are saddle points of the energy.

- Since a sphaleron is a critical point of the potential, it corresponds to a static solution (in $A_0 = 0$ gauge) of the field equations. Since it is a saddle point, it is unstable, leading to its name; sphaleron is etymologically related to "slippery".

- The size of the sphaleron is set by the Higgs vev, leading to the estimate

$$E_{\text{sph}} \sim \frac{4\pi v}{g} \sim 10\,\text{GeV}$$

  where $g$ is the $SU(2)$ gauge coupling. We then estimate $\Gamma \sim e^{-E_{\text{sph}}/T}$, though a more accurate calculation would give $\Gamma \sim e^{-F_{\text{sph}}/T}$ where the free energy $F_{\text{sph}}$ accounts for finite temperature corrections to the effective potential, which change the expectation value of the Higgs field.

- More quantitatively, the rate per spacetime volume goes as

$$\Gamma \sim \begin{cases} T^4 e^{-F_{\text{sph}}/T} & T \lesssim F_{\text{sph}} \\ T^4 \alpha_w^4 & T \gtrsim F_{\text{sph}} \end{cases}$$

  where the $T^4$ is by dimensional analysis, and the casework occurs because electroweak symmetry is unbroken for $T \gtrsim F_{\text{sph}}$.

- In any case, it is clear that at temperatures near or above that of the electroweak phase transition, baryon number violation can be significant. Depending on the model, this can be an obstacle to baryogenesis, diluting existing baryon number asymmetry, or the reason for baryogenesis itself.

Next, we discuss the QCD $\theta$ term.

- As we've motivated above, the Yang–Mills Lagrangian contains the $\theta$ term

$$\Delta\mathcal{L} = \frac{\theta g^2}{16\pi^2}\operatorname{tr} F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{\theta g^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\operatorname{tr} F_{\mu\nu}F_{\rho\sigma}.$$

  Such a term does not contribute classically but can in quantum mechanics.

- As a simple analogue, consider a particle on a ring, at angular coordinate $\alpha$, and

$$L = \frac{1}{2}B\dot{\alpha}^2 + \frac{\theta}{2\pi}\dot{\alpha}.$$

  The conjugate momentum to $\alpha$ is

$$J = B\dot{\alpha} + \frac{\theta}{2\pi}$$

  is quantized and takes on integer values, so for $J = n$,

$$E_n = \frac{1}{2B}\left(n - \frac{\theta}{2\pi}\right)^2.$$

  Hence a total derivative term affects the spectrum, though $\theta$ only matters up to multiples of $2\pi$. For example, when $\theta = 0$ or $\theta = \pi$, the energy levels are doubly degenerate; in both cases this is a consequence of time reversal symmetry.

- Now add a potential energy $K(1 - \cos\alpha)$ where $K \gg 1/B$, so we may think of the particle as a rigid pendulum. In this case, the $\theta$ dependence only comes from tunneling processes, so the $\theta$-dependence of the energy levels is exponentially suppressed.

- As for the physical consequences of the $\theta$ term, we cannot consider the $\theta$-dependence of the vacuum energy, since $\theta$ is fixed. (However, such dependence is important when there is an axion.) The $\theta$ term also cannot explain observed CP violating processes, such as kaon decay, because neutron EDM requirements mean $\theta$ must be extremely small. In fact, no effects of the $\theta$ term have ever been observed. This is the strong CP problem.

Finally, we describe the Witten effect.

- The Witten effect links the $\theta$ term to the electric charge of magnetic monopoles. Here we must distinguish between two senses of the electric charge: the quantity that is coupled to electric fields which can be measured by their Coulomb tail, and the conserved Noether charge $Q_n$ associated with phase rotations.

- Now consider a global $U(1)$ transformation composed with a gauge transformation,

$$\delta\mathbf{A}_\mu = \frac{1}{e}\mathbf{D}_\mu(\boldsymbol{\phi}/|\boldsymbol{\phi}|), \quad \delta\boldsymbol{\phi} = 0$$

  in a theory with gauge coupling $e$ where $SU(2)$ is broken to $U(1)$ by a triplet, and the bold refers to $SU(2)$ indices.

- The Noether charge is
$$Q_n = \int d\mathbf{x}\, \delta\mathbf{A}_j \cdot \mathbf{\Pi}^j$$
where the conjugate momentum $\mathbf{\Pi}^j$ is
$$\mathbf{\Pi}_j = \frac{\partial \mathcal{L}}{\partial(\partial_0 \mathbf{A}_j)} = \mathbf{F}^{j0}$$
which implies that when $\theta = 0$,
$$eQ_n = \int d\mathbf{x}\, \mathbf{D}_j \hat{\boldsymbol{\phi}} \cdot \mathbf{F}^{j0} = \int d\mathbf{x}\, \left( \partial_j (\hat{\boldsymbol{\phi}} \cdot \mathbf{F}^{j0}) - \hat{\boldsymbol{\phi}} \cdot \mathbf{D}_j \mathbf{F}^{j0} \right).$$
The field equation $\mathbf{D}_j \mathbf{F}^{j0} = \boldsymbol{\phi} \times \mathbf{D}^0 \boldsymbol{\phi}$ shows the second term vanishes, so
$$eQ_n = \int d^2 S_j\, \hat{\boldsymbol{\phi}} \cdot \mathbf{F}^{j0} = Q_E$$
where electric charge is defined by the unbroken $U(1)$) subgroup. Since the Noether charge $Q_n$ is conjugate to a periodic value, it is an integer, so the charge is quantized.

- Adding the $\theta$ term changes the conjugate momenta to
$$\mathbf{\Pi}^j = \mathbf{F}^{j0} - \frac{\theta e^2}{16\pi^2} \epsilon^{jkl} \mathbf{F}_{kl}$$
and repeating the steps above, using the Bianchi identity $\epsilon^{jkl} \mathbf{D}_j \mathbf{F}_{kl} = 0$ gives
$$eQ_n = \int d^2 S_j\, \hat{\boldsymbol{\phi}} \cdot \mathbf{F}^{j0} - \frac{\theta e^2}{16\pi^2} \epsilon^{jkl} \hat{\boldsymbol{\phi}} \cdot \mathbf{F}_{kl} = Q_E - \frac{e\theta}{2\pi} \left( \frac{eQ_M}{4\pi} \right).$$

- However, the Noether charge remains quantized to integer values. Thus, a monopole with magnetic charge $Q_M = 4\pi/e$ must have electric charge
$$Q_E = ne + \frac{e\theta}{2\pi}$$
for some integer $n$. Note that the result is periodic in $\theta$ as required.

- This can be understood by looking at the monopole from a distance, where it looks like a point charge. The Lagrangian is that of ordinary electromagnetism with the additional term
$$\Delta\mathcal{L} = -\frac{\theta e^2}{8\pi^2} \mathbf{E} \cdot \mathbf{B}$$
so that the abelian Gauss's law becomes
$$\nabla \cdot \mathbf{E} - \frac{\theta e^2}{8\pi^2} \nabla \cdot \mathbf{B} = \rho$$
where $\rho$ is a purely electric source, which vanishes for a monopole. Hence magnetic charge must be accompanied by electric charge.

- This doesn't contradict the fact that the $\theta$ term does not affect the equations of motion, because for trivial gauge bundles $\nabla \cdot \mathbf{B}$ vanishes identically. It can only be nonzero if the bundle is nontrivial, in which case the $\theta$ term, which is still a total derivative, cannot be converted to a vanishing surface integral at infinity; instead we get a contribution at the overlap of the patches.

- Note that the Witten effect is totally independent of instantons; it only relies on there being a $\theta$ term in the Lagrangian. This term is thought of as "caused" by instantons for historical reasons.