1 - SECOND-ORDER ACTIVE FILTERS

This section introduces circuits which have two zeros and two poles. The poles determine the natural frequencies of a circuit. These natural frequencies become time constants in the time-domain impulse response of circuit. The zeros determine the characteristics of the circuit in the frequency domain. For example, the zeros determine whether the circuit has a low-pass, bandpass, high-pass, bandstop, or an allpass behavior. The key difference between second-order and first-order circuits is that the roots of the second-order circuit can be complex whereas all roots of first-order circuits are constrained to the real axis.

It will be shown in this section that there is a significant difference between cascaded, first-order circuits and higher-order circuits such as second-order circuits. For example, assume that a circuit is to pass signals up to 10 kHz with a gain variation within 0 dB to -3 dB. Above 20 kHz the circuit must have a gain that is less than -20 dB. Fig. 1-1 shows this requirement. The magnitude response of the circuit must fall within the white areas and stay out of the shaded areas. In order to achieve this specification, four, first-order circuits are required. However, if we use second-order circuits which permit complex roots, we can satisfy the specification with one second-order circuit cascaded with one first-order circuit. The result will be the savings of one op amp and is due to the fact that we can make some of the poles complex.

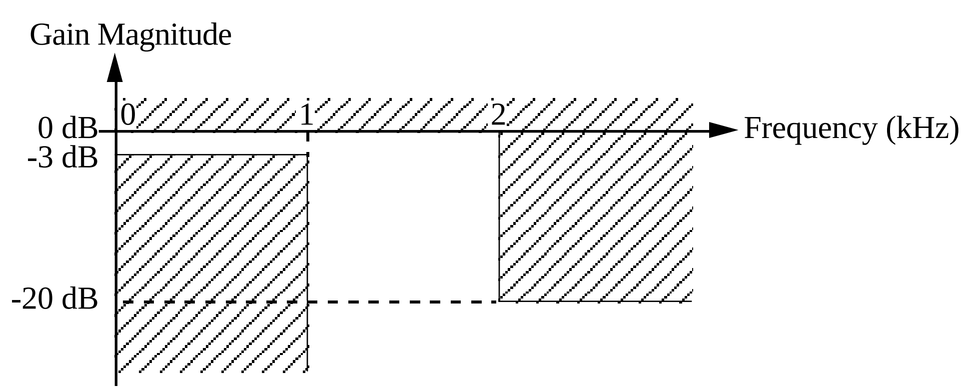

Figure 1-1 - Specification for a low-pass magnitude response in the frequency domain.

Second-Order, Passive, Low-Pass Filters

If we are willing to use resistors, inductances, and capacitors, then it is not necessary to use op amps to achieve a second-order response and complex roots. Let us consider the passive, second-order circuit of Fig. 1-2. Straight-forward analysis of this circuit using the complex frequency variable, *s*, gives

$$T(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{\frac{R/sC}{R+(1/sC)}}{sL+\frac{R/sC}{R+(1/sC)}} = \frac{\frac{R}{sC}}{sL\left(R+\frac{1}{sC}\right)+\frac{R}{sC}} = \frac{\frac{1}{LC}}{s^2+\frac{s}{RC}+\frac{1}{LC}} \quad . \tag{1-1}$$

We see that Eq. (1-1) has two poles at

$$p_1, p_2 = \frac{-1}{2RC} \pm \frac{1}{2}\sqrt{\left(\frac{1}{RC}\right)^2 - \frac{4}{LC}} \tag{1-2}$$

and two zeros at infinity. The poles will be complex if $(4/LC) > (1/RC)^2$.

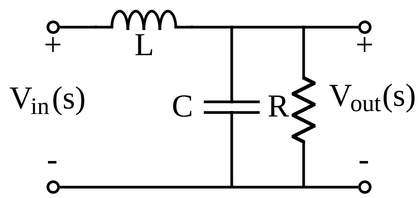

Figure 1-2 - Passive, RLC, low-pass filter.

The standard form of a second-order, low-pass filter is given as

$$T_{LP}(s) = \frac{T_{LP}(0)\omega_o^2}{s^2 + \left(\frac{\omega_o}{Q}\right)s + \omega_o^2} \tag{1-3}$$

where $T_{LP}(0)$ is the value of $T_{LP}(s)$ at dc, ω_o is the *pole frequency*, and Q is the *pole Q* or the *pole quality factor*. The *damping factor*, ζ, which may be better known to the reader, is given as

$$\zeta = \frac{1}{2Q} \quad . \tag{1-4}$$

The poles of Eq. (1-3) are

$$p_1,p_2 = \frac{-\omega_o}{2Q} \pm j\left(\frac{\omega_o}{2Q}\right)\sqrt{4Q^2-1} \quad . \tag{1-5}$$

The pole locations for the case where they are complex are shown on Fig. 1-3 and graphically illustrate the pole frequency and pole Q. Equating Eq. (1-1) with Eq. (1-3) gives $A_o = 1$, $\omega_o = 1/\sqrt{LC}$, and $Q = R/L$.

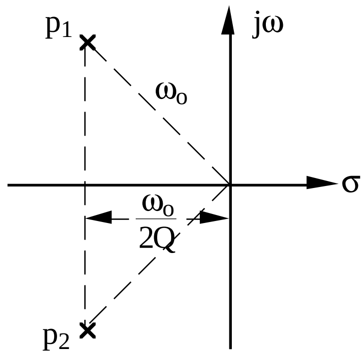

Figure 1-3 - Location of the poles of a second-order system in the complex frequency plane.

It is of interest to us to plot the locus of the poles, p_1 and p_2, as Q is varied from 0 to ∞. The resulting plot is called a *root locus* plot and is shown in Fig. 1-4. There are two loci on this plot, one corresponding to p_1 and the other to p_2. At Q=0, the poles are at 0

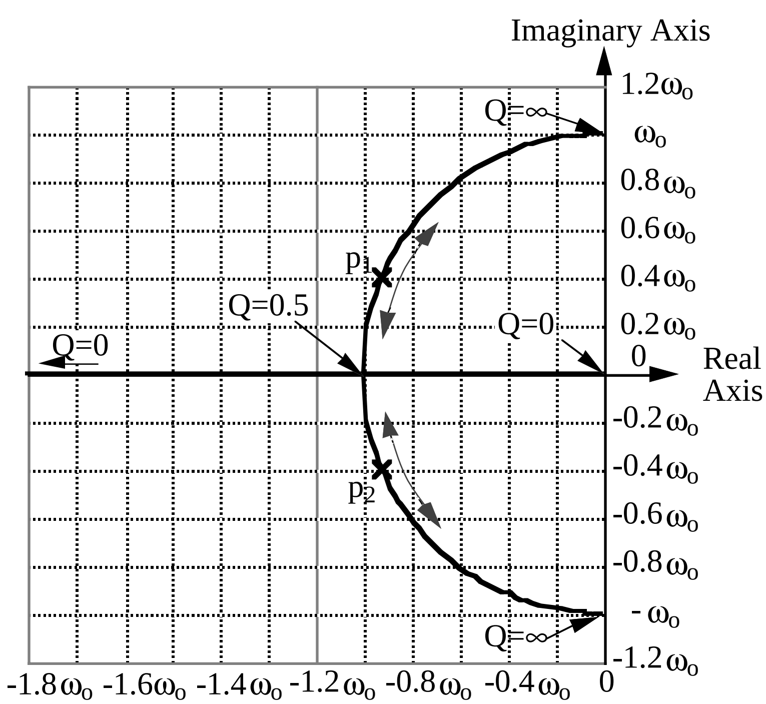

Figure 1-4 - Root-locus of the poles of Eq. (1-3) as Q is varied from 0 to ∞.

and ∞. As Q increases these poles move along the real axis towards $-\omega_o$. When Q=0.5, the two poles are identical and are at $-\omega_o$. As Q increases above 0.5, the poles leave the real axis and become complex. As Q increases further, one pole follows the upper quarter circle and the other the lower quarter circle. Finally, at $Q = \infty$, the poles are on the $j\omega$ axis at $\pm j1$.

Example 1-1 - Roots of a Passive RLC, Low-Pass Circuit

Find the roots of the passive RLC, low-pass circuit shown in Fig. 1-5.

Solution

First we must find the voltage transfer function. Using voltage division among the three series components results in

$$T(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{\frac{1}{sC}}{sL+R+\frac{1}{sC}} = \frac{\frac{1}{LC}}{s^2+\left(\frac{R}{L}\right)s+\frac{1}{LC}} = \frac{10^{12}}{s^2+141\text{x}10^4 s+10^{12}} .$$

Equating this transfer function to Eq. (1-3) gives $T_{LP}(0) = 1$, $\omega_o = 10^6$ rps, and $Q=1/\sqrt{2}$. Substituting these values into Eq. (1-5) gives

$$p_1,p_2 = -707{,}107 \pm j707{,}107 \text{ (rps).}$$

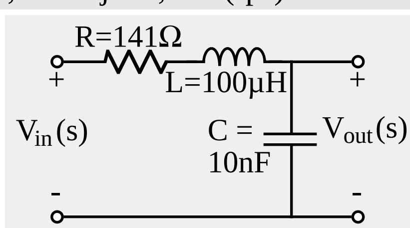

Figure 1-5 - A second-order low-pass RLC filter.

Standard, Second-Order, Low-Pass Transfer Function - Frequency Domain

The frequency response of the standard, second-order, low-pass transfer function can be normalized and plotted for general application. The normalization of Eq. (1-3) includes both amplitude and frequency and is defined as

$$T_{LPn}(s_n) = \frac{T_{LP}\left(\frac{s}{\omega_o}\right)}{|T_{LP}(0)|} = \frac{1}{s_n{}^2 + \frac{s_n}{Q} + 1} \qquad (1\text{-}6)$$

where

$$T_{LPn}(s) = \frac{T_{LP}(s)}{|T_{LP}(0)|} \tag{1-7}$$

and

$$s_n = \frac{s}{\omega_o} \quad . \tag{1-8}$$

The magnitude and phase response of the normalized, second-order, low-pass transfer function is shown in Fig. 1-6 where Q is a parameter. In this figure, we see that Q influences the frequency response in the vicinity of ω_o. If Q is greater than $\sqrt{2}$, then the normalized magnitude response has a peak value of

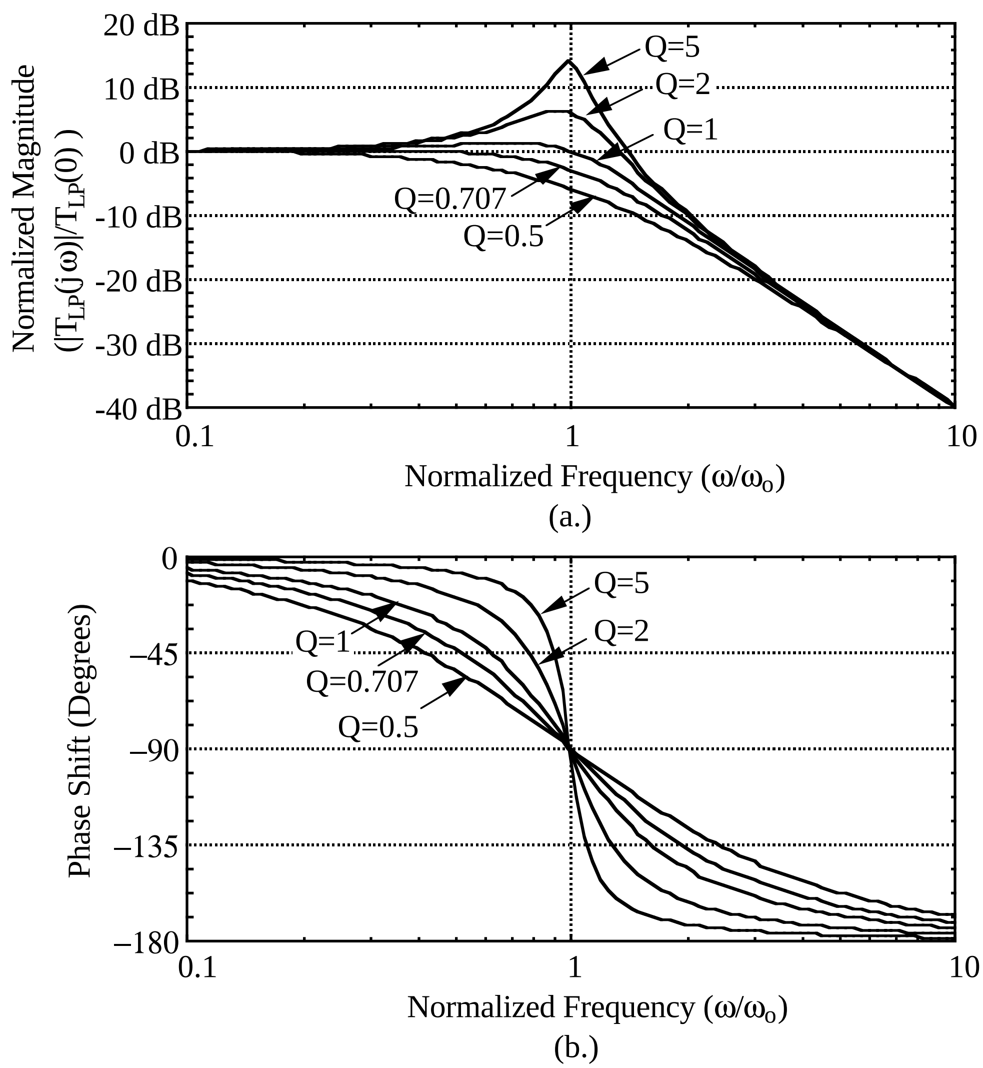

Figure 1-6 - (a.) Normalized magnitude and (b.) phase response of the standard second-order, low-pass transfer function with Q as a parameter.

$$|T_n(\omega_{max})| = \frac{Q}{\sqrt{1 - \frac{1}{4Q^2}}} \tag{1-9}$$

at a frequency of

$$\omega_{max} = \omega_o\sqrt{1 - \frac{1}{2Q^2}} \ . \tag{1-10}$$

Example 1-2 - Second-Order, Low-Pass Transfer Function

Find the pole locations and $|T(\omega_{max})|$ and ω_{max} of a second-order, low-pass transfer function if $\omega_o = 10^4$ rps and Q = 1.5.

Solution

From Eq. (1-5) we get

$$p_1,p_2 = \frac{-\omega_o}{2Q} \pm j\left(\frac{\omega_o}{2Q}\right)\sqrt{4Q^2-1} = -3{,}333 \pm j10{,}541 \text{ rps.}$$

Eqs. (1-9) and (1-10) give $|T(\omega_{max})|$ = 1.591 or 4.033 dB and ω_{max} = 8,819 rps.

Standard, Second-Order, Low-Pass Transfer Function - Step Response

The unit step response of the standard, second-order, low-pass transfer function can be found by multiplying Eq. (1-3) by 1/s to get

$$V_{out}(s) = \frac{T_{LP}(s)}{s} = \frac{T_{LP}(0)\omega_o^2}{s\left(s^2 + \left(\frac{\omega_o}{Q}\right)s + \omega_o^2\right)} = \frac{T_{LP}(0)\omega_o^2}{s(s+p_1)(s+p_2)} \ . \tag{1-11}$$

The solution of the step response depends on whether the poles p_1 and p_2 are real or complex which can be determined from Q or ζ. When Q > 0.5, the poles are complex and the step response of the second-order, low-pass transfer function is said to be underdamped. When Q = 0.5, the step response is critically damped. When Q < 0.5, the step response is overdamped.

The underdamped or critically damped solution ($Q \geq 0.5$) is of interest to us here. For purposes of notation simplicity, we shall use the damping factor ζ (=1/2Q) in place of the pole Q. Thus the poles of the standard, second-order transfer function when $\zeta \leq 1$ are

$$p_1, p_2 = -\zeta\omega_o \pm j\, \zeta\omega_o\sqrt{4Q^2-1} \ . \tag{1-12}$$

Substituting these roots into Eq. (1-11) and taking the inverse Laplace transform of $V_{out}(s)$ gives

$$L^{-1}[V_{out}(s)] = v_{out}(t) = T_{LP}(0)\left[1 - \frac{e^{-\zeta\omega_o t}}{\sqrt{1-\zeta^2}}\sin\left(\sqrt{1-\zeta^2}\,\omega_o t + \phi\right)\right] \qquad (1\text{-}13)$$

where

$$\phi = \tan^{-1}\left[\frac{\sqrt{1-\zeta^2}}{\zeta}\right] \quad . \qquad (1\text{–}14)$$

Eulers formula has been used to combine a sine and cosine having the same arguments into a single sinusoid with a phase shift of ϕ. Figure 1-7 shows the normalized step response of the standard, second-order, low-pass transfer function for ζ = 1, 0.707, 0.5, 0.25, and 0.1 which correspond to Q = 0.5, 0.707, 1, 2, and 5.

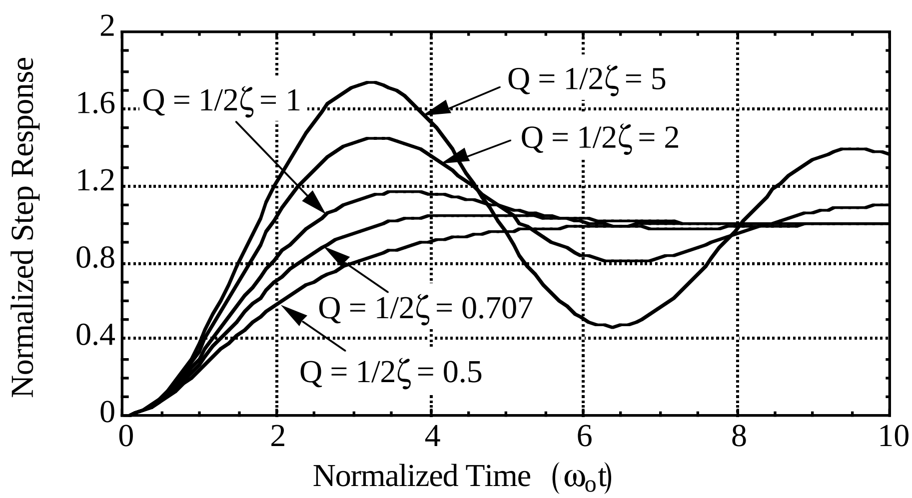

Figure 1-7 - Normalize step response of a standard, low-pass, second-order transfer function for Q ≥ 5 (underdamped).

We see from the normalized step response of Fig. 1-7 that for Q > 0.5, the output exceeds the final value of 1. This behavior is called overshoot. If the response has more than one oscillation (ring), the first oscillation is used because it is always the largest. If we differentiate Eq. (1-13) and set the result equal to zero, we will find that the peak value of the first oscillation occurs at

$$t_p = \frac{\pi}{\omega_o\sqrt{1-\zeta^2}} \quad .$$

Substituting this value into Eq. (1-13) gives

$$v_{out}(t_p) = 1 - \frac{e^{-\zeta\pi}}{\sqrt{1 - \zeta^2}} \quad . \qquad (1\text{-}16)$$

Fig. 1-8 helps to illustrate these results.

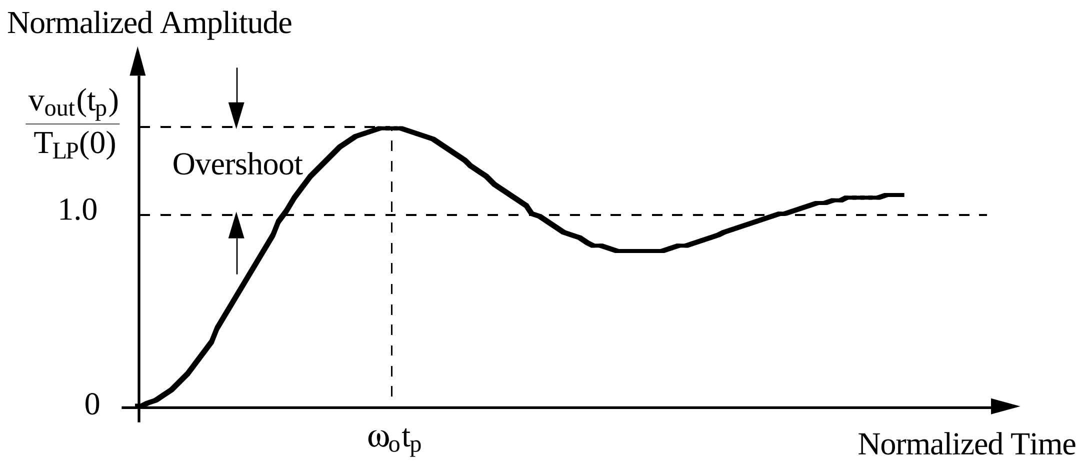

Figure 1-8 - Normalized step response for Q = 2.

From Fig. 1-8, we define overshoot is defined as

$$\text{Overshoot} = \frac{\text{Largest peak value - Final value}}{\text{Final value}} = \frac{\exp(-\pi\zeta)}{\sqrt{1-\zeta^2}} \quad . \qquad (1\text{-}17)$$

In general we want the step response of a second-order, low-pass circuit to approach its final value as quickly as possible. Therefore, high values of Q are undesirable because the oscillations of the step response take a long time to die out. Shortly, we shall show how to relate the overshoot of the step response of a feedback system to its stability. This will provide a quick method of examining stability of feedback circuits in the time domain.

Example 1-3 - Step Response of a Second-Order, Low-Pass Circuit

Find the t_p and the overshoot of the second-order, low-pass circuit of Ex. 1-2.

Solution

In Ex. 1-2, $\omega_o = 10^4$ rps and Q = 1.5. Q = 1.5 corresponds to $\zeta = 1/3$. Substituting these values in Eq. (1-15) gives $t_p = \frac{3.1416}{(10^4)(0.9428)} = 0.3332$ ms. The overshoot is found from Eq. (1-17) and is $\frac{\exp(-1.0472)}{0.9428} = 0.3722$. We typically multiply overshoot by 100 and express it as 37.22%.

How Does an Active-RC Filter Work?

An active-RC filter uses only resistors, capacitors, and amplifiers to achieve complex poles. If we do not use inductors, how can complex poles be achieved? Once more,the answer is feedback. When feedback is applied around a system containing real roots, the closed loop transfer function may contain complex roots. To illustrate how this occurs, assume that A(s) of a single-loop, negative feedback circuit (Fig. 4.3-2) can be written as

$$A(s) = \frac{A_o\omega_1\omega_2}{(s+\omega_1)(s+\omega_2)} \quad . \tag{1-18}$$

Therefore, the poles of A(s) are real and are located in the complex frequency plane at $-\omega_1$ and $-\omega_2$. Now assume that frequency independent negative feedback of $ß_o$ is placed around the amplifier. The closed-loop transfer function becomes

$$A_F(s) = \frac{A(s)}{1+ß_oA(s)} = \frac{A_o\omega_1\omega_2}{(s+\omega_1)(s+\omega_2) + A_oß_o\omega_1\omega_2}$$

$$= \frac{A_o\omega_1\omega_2}{s^2+(\omega_1+\omega_2)s+\omega_1\omega_2(1+A_oß_o)} = \frac{A_o\omega_1\omega_2}{(s+p_{f1})(s+p_{f2})} \quad . \tag{1-19}$$

The poles of the closed-loop transfer function, $A_F(s)$, are given as

$$p_{f1}, p_{f2} = -\frac{\omega_1+\omega_2}{2} \pm \frac{1}{2}\sqrt{(\omega_1+\omega_2)^2 - 4\omega_1\omega_2(1+A_oß_o)} \quad . \tag{1-20}$$

We can see that the closed-loop poles can become complex if $ß_o$ is large enough. The process by which feedback creates complex poles is illustrated by the root-locus of Fig. 1-9 for Eq. (1-22) where $\omega_2 = 9\omega_1$ and $A_oß_o$ is varied from 0 to ∞. We note that when $A_oß_o \geq 2$, the poles are complex.

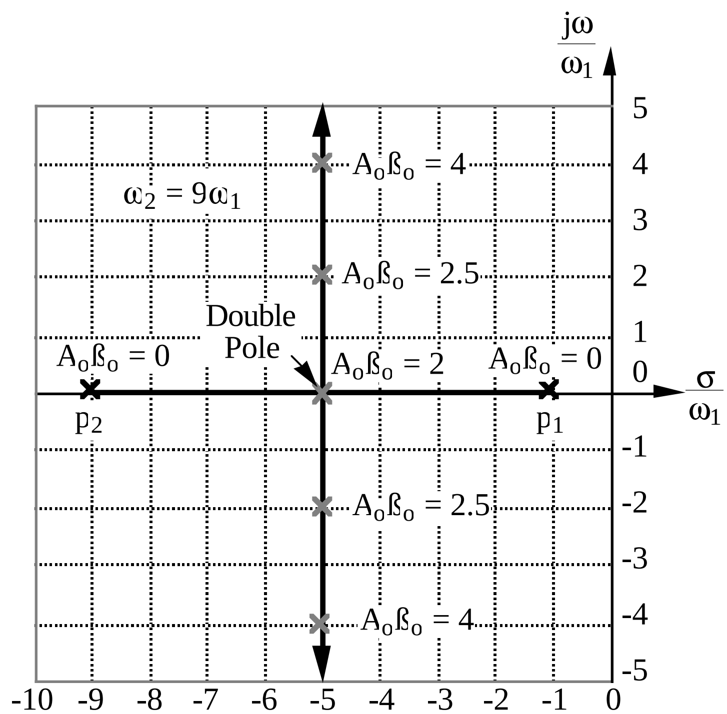

Figure 1-9 - Normalized root-locus of the poles of Eq. (1-19) when $A_o ß_o$ is varied from 0 to ∞ for the case where $\omega_2 = 9\omega_1$.

While the root-locus of Eq. (1-19) never crosses the jω axis, there are some cases where feedback will cause the locus of the closed-loop poles to cross the jω axis. When this occurs the feedback system is unstable. The root-locus of the poles is another way to examine the stability of a feedback system.

Example 1-4 - Illustration of Achieving Complex Poles using Negative Feedback

Suppose the A(s) part of a single-loop, negative feedback circuit is shown in Fig. 1-10. If negative feedback of $ß_o$ is applied around this amplifier, find the transfer function, $V_{out}(s)/V_{in}(s)$, and show whether or not complex poles can be obtained and under what conditions.

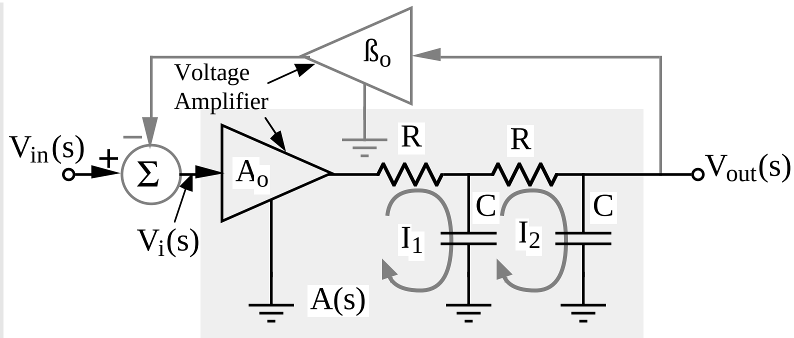

Figure 1-10 - Feedback circuit for Ex. 1-4.

Solution

First, we must find A(s). The two loop equations that describe the RC network are written as

$$A_oV_i(s) = \left(R+\frac{1}{sC}\right)I_1(s) - \left(\frac{1}{sC}\right)I_2(s)$$

and

$$0 = -\left(\frac{1}{sC}\right)I_1(s) + \left(R+\frac{2}{sC}\right)I_2(s) \ .$$

Solving these equations using Kramer's rule gives

$$I_2(s) = \frac{\begin{vmatrix} R+\frac{1}{sC} & A_oV_i(s) \\ \frac{-1}{sC} & 0 \end{vmatrix}}{\begin{vmatrix} R+\frac{1}{sC} & \frac{-1}{sC} \\ \frac{-1}{sC} & R+\frac{2}{sC} \end{vmatrix}} = \frac{\frac{A_oV_i(s)}{sC}}{\left(R+\frac{1}{sC}\right)\left(R+\frac{2}{sC}\right) - \left(\frac{1}{sC}\right)^2} \ .$$

Because $V_{out}(s) = I_2(s)/sC$, we can use the above to solve for $A(s) = V_{out}(s)/V_i(s)$ as

$$A(s) = \frac{V_{out}(s)}{V_i(s)} = \frac{\frac{A_o}{(sC)^2}}{R^2 + \frac{3Rs}{C} + \frac{1}{(sC)^2}} = \frac{\frac{A_o}{(RC)^2}}{s^2 + \frac{3s}{RC} + \frac{1}{(RC)^2}} \ .$$

Next, we substitute A(s) and $ß_o$ into Eq. (4.3-1) to get

$$A_F(s) = \frac{A(s)}{1+A(s)ß_o} = \frac{\frac{A_o}{(RC)^2}}{s^2 + \frac{3s}{RC} + \frac{1}{(RC)^2}(1+A_oß_o)} \ .$$

The poles of the closed-loop transfer function, $A_F(s)$, are given as

$$p_{f1},p_{f2} = \frac{-1.5}{RC} \pm \frac{1}{2}\sqrt{\frac{9}{(RC)^2} - \frac{4}{(RC)^2}(1+A_o\beta_o)}$$

which can be complex if $A_o\beta_o \geq 1.25$.

Time and Frequency Domain Perspective of Stability

We can now show a useful relationship between the step response and the stability of a feedback system based on the above results. The key to this relationship is to assume that we can approximate a third-order system by a second-order system. As a result, the relationship we will develop is useful for determining the degree of stability (i.e. phase margin) but not whether the circuit is stable or unstable.

Let us suppose that negative feedback has been used as illustrated above to create a second-order, low-pass transfer function having complex poles. The closed-loop transfer function can be written as

$$A_F(s) = T_{LP}(s) = \frac{T_{LP}(0)\omega_o^2}{s^2 + \left(\frac{\omega_o}{Q}\right)s + \omega_o^2} = \frac{A_F(0)\omega_o^2}{s^2 + (2\zeta\omega_o)s + \omega_o^2} \quad . \qquad (1\text{-}21)$$

Next, assume that ß is real (β_o) and multiply both side of Eq. (4.3-1) by β_o to get

$$A_F(s)\beta_o = \frac{A(s)\beta_o}{1+A(s)\beta_o} \quad . \qquad (1\text{-}22)$$

Solve for the quantity, $A(s)\beta_o$ of Eq. (4.2-22) to get

$$A(s)\beta_o = \frac{\beta_o}{\frac{1}{A_F(s)} - \beta_o} \quad . \qquad (1\text{-}23)$$

Finally, substitute Eq. (1-21) into Eq. (1-23) resulting in

$$A(s)\beta_o = \frac{\beta_o}{\frac{s^2+2\zeta\omega_o s + \omega_o^2}{A_F(0)\omega_o^2} - \beta_o} = \frac{A_F(0)\beta_o\omega_o^2}{s^2+2\zeta\omega_o s + \omega_o^2 - A_F(0)\beta_o\omega_o^2} \quad . \qquad (1\text{-}24)$$

When the loop gain is much greater than unity, we know that $A_F(0) \approx 1/\beta_o$. Therefore $A_F(0)\beta_o \approx 1$. Substituting this approximate relationship into Eq. (1-24) gives

$$A(s)\beta_o = \frac{A_F(0)\beta_o\omega_o^2}{s^2 + 2\zeta\omega_o s + \omega_o^2 - A_F(0)\beta_o\omega_o^2} \approx \frac{\omega_o^2}{s^2 + 2\zeta\omega_o s} \quad . \tag{1-25}$$

We have studied the stability properties of $A(s)\beta_o$ in Sec. 4.3 and know that ω_{0dB} occurs when $|A(j\omega)\beta_o| = 1$. Thus we can take the magnitude of Eq. (1-25) and set it equal to one to get

$$\omega_{0dB} = \omega_o\sqrt{\sqrt{4\zeta^4+1} - 2\zeta^2} \quad . \tag{1-26}$$

The phase shift of $A(j\omega)\beta_o$ can be expressed from Eq. (1-25) as

$$\mathrm{Arg}[A(j\omega_{0dB})\beta_o] = -\frac{\pi}{2} - \tan^{-1}\left(\frac{\omega_{0dB}}{2\zeta\omega_o}\right). \tag{1-27}$$

Substituting this value of phase shift of into the definition of phase margin below gives

$$\text{Phase Margin} = \pi - \left[\frac{\pi}{2} + \tan^{-1}\left(\frac{\omega_{0dB}}{2\zeta\omega_o}\right)\right] = \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega_{0dB}}{2\zeta\omega_o}\right) = \tan^{-1}\left(\frac{2\zeta\omega_o}{\omega_{0dB}}\right) \quad . \tag{1-28}$$

Substituting for ω_{0dB} in Eq. (1-28) gives

$$\text{Phase Margin} = \tan^{-1}\left[\frac{2\zeta}{\sqrt{\sqrt{4\zeta^4+1} - 2\zeta^2}}\right] = \cos^{-1}\left[\sqrt{4\zeta^4+1} - 2\zeta^2\right] \quad . \tag{1-29}$$

Eq. (1-29) gives the phase margin of a negative feedback system used to implement a second-order, low-pass transfer function in terms of the damping factor ζ. Previously, we related the peak overshoot to step response of a second-order, low-pass transfer function to the damping factor ζ in Eq. (1-17). Eqs. (1-17) and (1-29) allow us to relate the peak overshoot of the step response of a second-order, low-pass system to its phase margin. Figure 1-11 consist of a plot of Eqs. (1-17) and (1-29) as a function of the damping factor ζ. The dotted line shows how to use the figure. For example, suppose that we observed a 10% overshoot to the step response. We project horizontally to the overshoot curve to find a value of ζ. Next, project vertically to the phase margin curve. Finally, projecting horizontally gives the approximate equivalent phase margin. For this example, a 10%

overshoot corresponds to a ζ of approximately 0.59 (Q ≈ 0.85) which gives a phase margin of approximately 58°.

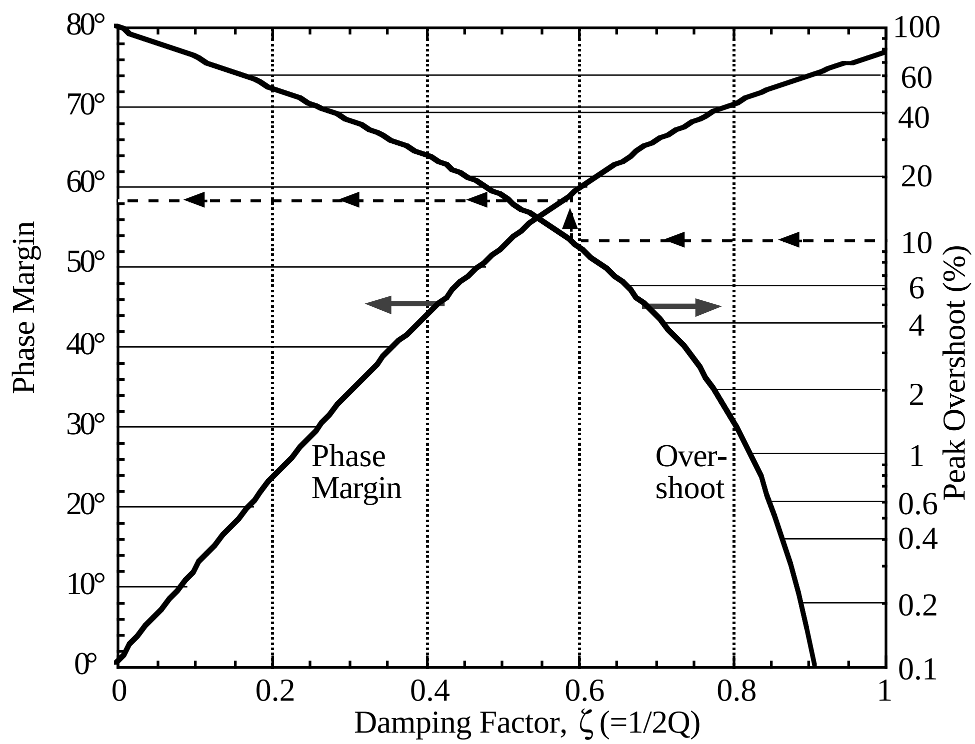

Figure 1-11 - Graphical illustration of the relationship between peak overshoot and phase margin of a negative feedback system.

Positive Feedback, Second-Order, Low-Pass, Active Filter†

The basic principle of active-RC filters is to use feedback to create complex poles which are necessary for efficient filter applications. The feedback can be either positive or negative as long as the circuit is not unstable. Fig. 1-12a shows one of the popular, second-order, low-pass filters which employs positive feedback. The voltage amplifier has a voltage gain of K and is assumed to have an infinite input resistance and a zero output resistance. This voltage amplifier can be realized by the noninverting voltage amplifier shown in Fig. 1-12b.

† This type of filter is called Sallen and Key after the inventors who published their work in the often referenced paper, R.P. Sallen and E.L. Key, "A Practical Method of Designing RC Active Filters," *IRE Trans. Circuit Theory*, vol. CT-2, March 1955, pp. 74-85.

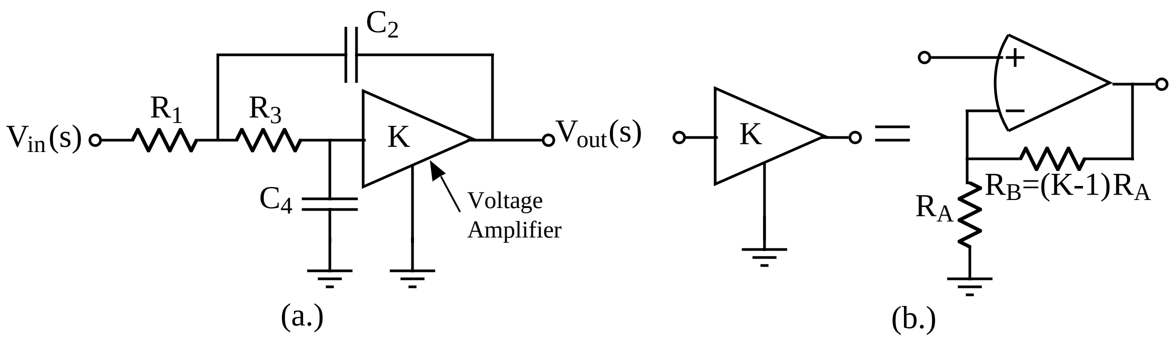

Figure 1-12 - (a.) A second-order, low-pass active filter using positive feedback. (b.) The realization of the voltage amplifier K by the noninverting op amp configuration.

The closed-loop, voltage transfer function of Fig. 1-12a can be found (see Prob. PA1-2) as

$$\frac{v_{out}(s)}{V_{in}(s)} = \frac{\dfrac{K}{R_1R_3C_2C_4}}{s^2 + s\left(\dfrac{1}{R_3C_4} + \dfrac{1}{R_1C_2} + \dfrac{1}{R_3C_2} - \dfrac{K}{R_3C_4}\right) + \dfrac{1}{R_1R_3C_2C_4}} \quad . \tag{1-30}$$

In order to use this result, we must be able to express the component values of Fig. 1-12a (R_1, R_3, C_2, C_4, and K) in terms of the parameters of the standard, second-order, low-pass transfer function ($T_{LP}(0)$, Q, and ω_o). These relationships are called *design equations* and are the key to designing a given active filter. When equating the coefficients of Eq. (1-30) to Eq. (1-3), three independent equations result. Unfortunately, there are 5 unknowns and therefore a unique solution does not exist. This circumstance happens often in active filter design. To solve this problem, the designer chooses as many additional constraints as necessary to obtain a unique set of design equations.

In order to achieve a unique set of design equations for Fig. 1-12, we need two more independent relationships. Let us choose these relationships as

$$R = R_1 = R_3 \tag{1-31}$$

and

$$C = C_2 = C_4 \quad . \tag{1-32}$$

Substituting these relationships into Eq. (1-30) gives

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{\dfrac{K}{(RC)^2}}{s^2 + \left(\dfrac{3-K}{RC}\right)s + \dfrac{1}{(RC)^2}} \quad . \tag{1-33}$$

Now, if we equate Eq. (1-33) to Eq. (1-3) we get three design equations which are

$$RC = \frac{1}{\omega_o} \tag{1-34}$$

$$K = 3 - \frac{1}{Q} \tag{1-35}$$

and

$$K = T_{LP}(0) \quad . \tag{1-36}$$

Unfortunately, our choice of equal resistors and equal capacitors resulted in two, rather than three, independent design equations. This means we cannot simultaneously satisfy a specification for Q and for $T_{LP}(0)$. However, this not a real disadvantages and the design equations are so simple that we shall call them the *equal-R*, *equal-C* design equations. Other design equations are given in the problems (see PR1-8 and PR1-9).

Example 1-5 - Application of the Equal-R, Equal-C Design Approach

Use the equal-R, equal-C design approach to design a second-order, low-pass filter using Fig. 1-12a if Q = 0.707 and f_o = 1 kHz. What is the value of $T_{LP}(0)$?

Solution

First, we must pick a value of R or C in order to use Eq. (1-34). Let us select C = 1μF. Therefore $R = \frac{1}{(6.2832\text{x}10^3)(10^{-6})} = 159.2\ \Omega$. This is probably too small for R so let us decrease our choice for C by 100 which gives $C = C_1 = C_2 = 0.01\mu F$ and $R = R_1 = R_2 = 15.92\ k\Omega$.

Next, we design K from Eq. (1-35). We see that K = 3 - 1.4142 = 1.5858. Now we have to design the resistors of Fig. 1-12b so that this gain is achieved. We see that $R_B = 0.5858R_A$. If we pick $R_A = 10\ k\Omega$, then $R_B = 5.858\ k\Omega$. We note that gains less than 1 or Q ≤ 0.5 cannot be achieved without modification to Fig. 1-12b (see Prob. PA1-3) and that Q ≤ 0.333 is impossible for Fig. 1-12a to realize.

The reason for the lower limit of Q of Fig. 1-12a noted in the above example can be seen by letting all passive components equal unity and plotting the poles of Eq. (1-30) as K varies from 0 to ∞. The result is shown in Fig. 1-13. We see that K = 1, that the poles are identical an on negative real axis at -1. If we use Fig. 1-12b to implement the voltage amplifier, this value of K will be the smallest possible corresponding to a Q = 0.5. For this reason and the fact that first-order circuits can realize poles on the negative real axis, we restrict Fig. 1-12a for the case of complex poles.

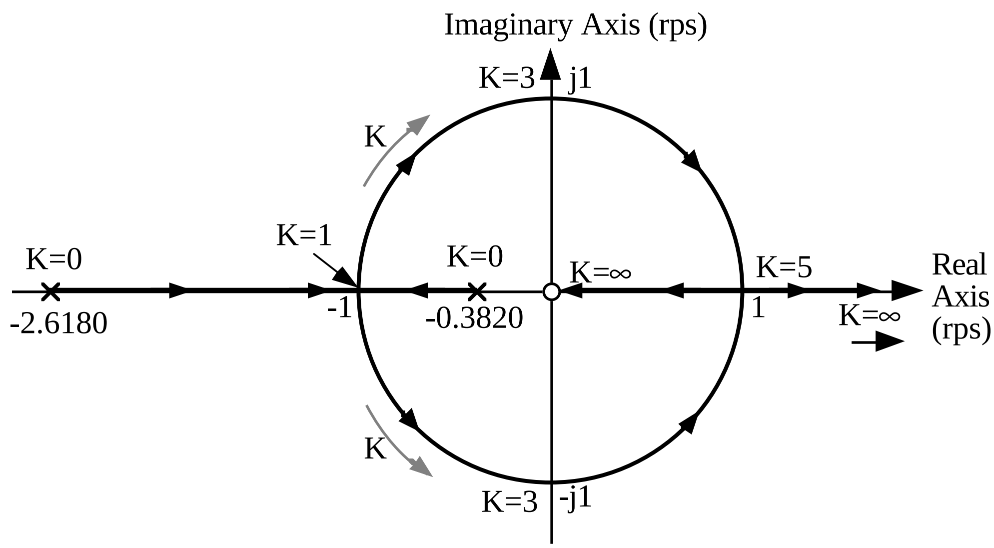

Figure 1-13 - Root locus of the poles of Fig. 1-12a as a function of K.

Negative Feedback, Second-Order, Low-Pass, Active Filter

A second-order, low-pass active filter that uses negative feedback is shown in Fig. 1-14. It is necessary to have one more passive element in the feedback network compared to Fig. 1-12a in order to get sufficient open-loop phase shift to cause complex poles. The voltage transfer function of this circuit can be found as†

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{\dfrac{-1}{R_1R_3C_4C_5}}{s^2 + s\left(\dfrac{1}{R_1C_4} + \dfrac{1}{R_2C_4} + \dfrac{1}{R_3C_4}\right) + \dfrac{1}{R_2R_3C_4C_5}} \tag{1-37}$$

where we have assumed that the op amp is ideal.

† See for example, Sec. 5.1 of *Introduction to the Theory and Design of Active Filters*, L.P. Huelsman and P.E. Allen, McGraw-Hill Book Co., (1980).

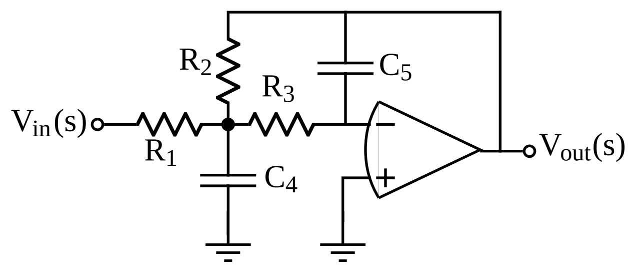

Figure 1-14 - A negative feedback realization of a second-order, low-pass active filter.

This circuit brings forth an important principle concerning the analysis of op amp circuits that should not go unnoticed. In general, *the more op amps and the less passive components in an op amp circuit, the easier is the analysis of the circuit*. The easiest type of op amp circuits to analyze are those where every passive element is connected between the following types of nodes: a node where the voltage is defined including ground, the inverting input terminal of the op amp with the potential on the noninverting input terminal known, or the output of the op amp. We see that the node common to R_1, R_2, R_3, and C_4 in Fig. 1-14 violates this principle. (Also the node common to R_1, C_2, and R_3 of Fig. 1-12a was in violation of this principle.)

If we equate Eq. (1-37) to Eq. (1-3) we get the following three equations,

$$\omega_o = \frac{1}{\sqrt{R_2R_3C_4C_5}} \tag{1-38}$$

$$\frac{1}{Q} = \sqrt{\frac{C_5}{C_4}}\left(\frac{\sqrt{R_2R_3}}{R_1} + \sqrt{\frac{R_3}{R_2}} + \sqrt{\frac{R_2}{R_3}}\right) \tag{1-39}$$

and

$$|T_{LP}(0)| = \frac{R_2}{R_1} \quad . \tag{1-40}$$

Again, we do not have a sufficient number of equations to be able to uniquely solve for all of the component values of the realization of Fig. 1-14. If we choose the constraints, $C_5 = C$ and $C_4 = 4Q^2(1 + |T_{LP}(0)|)C$, then a unique set of useful design equations result and are given as,

$$C_5 = C \tag{1-41}$$

$$C_4 = 4Q^2(1 + |T_{LP}(0)|)C \tag{1-42}$$

$$R_1 = \frac{1}{2|T_{LP}(0)|\omega_o QC} \tag{1-43}$$

$$R_2 = \frac{1}{2\omega_o QC} \tag{1-44}$$

and

$$R_3 = \frac{1}{2\omega_o QC(1+|T_{LP}(0)|)} \quad . \tag{1-45}$$

The following example will illustrate how to use these design equations for the second-order, low-pass realization of Fig. 1-14.

Example 1-6 - Design of A Negative Feedback, Second-Order, Low-Pass Active Filter

Use the negative feedback, second-order, low-pass active filter of Fig. 1-14 to design a low-pass filter having a dc gain of -1, $Q = 1/\sqrt{2}$, and f_o = 100 Hz.

Solution

Let us use the design equations given in Eqs. (1-41) through (1-45). Assume that $C_5 = C = 0.1\mu F$. Therefore, from Eq. (1-42) we get $C_4 = (8)(0.5)C = 0.4\ \mu F$. The resistors are designed using Eq. (1-43) for R_1 which gives

$$R_1 = \frac{\sqrt{2}}{(2)(1)(628.32)(10^{-7})} = 11.254\ k\Omega\ .$$

Eq. (1-44) gives

$$R_2 = \frac{\sqrt{2}}{(2)(628.32)(10^{-7})} = 11.254\ k\Omega\ .$$

Finally, Eq. (1-45) gives

$$R_3 = \frac{\sqrt{2}}{(2)(628.32)(2)(10^{-7})} = 5.627\ k\Omega\ .$$

One of the advantages of the negative feedback, second-order, low-pass active filter of Fig. 1-14 is that it can be used to achieve gains greater or less than unity independently of the values of ω_o or Q. Let us illustrate by the following example.

Example 1-7 - Design of A Second-Order, Low-Pass Filter with a DC Gain of -100

Repeat the Ex. 1-6 except let the gain at dc be -100.

Solution

Let us choose a smaller capacitor in anticipation that the resistances will be otherwise be smaller. Therefore, $C_5 = C = 0.01\mu F$. $C_4 = (4)(0.5)(101)(10^{-8}) = 2.02\ \mu F$. R_1 is found from Eq. (1-43) as

$$R_1 = \frac{\sqrt{2}}{(2)(100)(628.32)(10^{-8})} = 1.125\ k\Omega\ .$$

Eq. (1-44) gives

$$R_2 = \frac{\sqrt{2}}{(2)(628.32)(10^{-8})} = 112.54\ k\Omega\ .$$

Finally, Eq. (1-45) gives

$$R_3 = \frac{\sqrt{2}}{(2)(628.32)(101)(10^{-8})} = 1.114\ k\Omega\ .$$

Multiple Op Amp, Second-Order, Low-Pass, Active Filter

If one is willing to use more than one op amp, several useful second-order, low-pass realizations result. Figure 1-15 shows a two-op amp positive feedback circuit. This circuit has a lot of similarities to Fig. 1-12a except R_3 is isolated from R_1 and C_2 by the unity gain buffer, A1. This circuit has a unity dc voltage gain and is easy to analyze because all passive elements are connected to a voltage-defined node.

The output voltage of the first op amp, $V_{o1}(s)$, can be written in terms of $V_{out}(s)$ and $V_{in}(s)$ by superposition. The result is

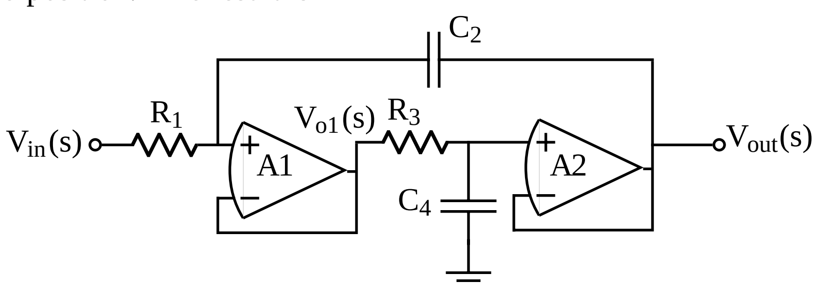

Figure 1-15 - A two-op amp, second-order, low-pass active filter realization.

$$V_{o1}(s) = \left(\frac{\frac{1}{sC_2}}{R_1+\frac{1}{sC_2}}\right)V_{in}(s) + \left(\frac{R_1}{R_1+\frac{1}{sC_2}}\right)V_{out}(s)$$

$$= \left(\frac{\frac{1}{R_1C_2}}{s+\frac{1}{R_1C_2}}\right)V_{in}(s) + \left(\frac{s}{s+\frac{1}{R_1C_2}}\right)V_{out}(s)\ . \qquad (1\text{-}46)$$

The output voltage, $V_{out}(s)$, can be expressed in terms of $V_{o1}(s)$ as

$$V_{out}(s) = \left(\frac{\frac{1}{sC_4}}{R_3+\frac{1}{sC_4}}\right)V_{o1}(s) = \left(\frac{\frac{1}{R_3C_4}}{s+\frac{1}{R_3C_4}}\right)V_{o1}(s)\ . \tag{1-47}$$

Now, substituting Eq. (1-47) into Eq. (1-46) and simplifying results in

$$V_{out}(s) = \left(\frac{\frac{1}{R_3C_4}}{s+\frac{1}{R_3C_4}}\right)\left[\left(\frac{\frac{1}{R_1C_2}}{s+\frac{1}{R_1C_2}}\right)V_{in}(s) + \left(\frac{s}{s+\frac{1}{R_1C_2}}\right)V_{out}(s)\right] \tag{1-48}$$

or

$$V_{out}(s)\left[\left(s + \frac{1}{R_1C_2}\right)\left(s + \frac{1}{R_3C_4}\right) - \frac{s}{R_3C_4}\right] = \frac{V_{in}(s)}{R_1R_3C_2C_4}\ . \tag{1-49}$$

Solving for the voltage transfer function gives the desired result which is

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{\frac{1}{R_1R_3C_2C_4}}{s^2 + \frac{s}{R_1C_2} + \frac{1}{R_1R_3C_2C_4}}\ . \tag{1-50}$$

When we equate Eq. (1-50) to Eq. (1-3) we get

$$\omega_o = \frac{1}{R_1R_3C_2C_4}\ ,\ Q = \sqrt{\frac{R_1C_2}{R_3C_4}}\ ,\ \text{and}\ \ T_{LP}(0) = 1\ . \tag{1-51}$$

In order to develop a set of design equations, we assume that $C_4 = mC_2 = mC$ and $R_3 = nR_1 = nR$. Using these constraints in Eq. (1-51) results in

$$m = \frac{Q}{\omega_o RC} \tag{1-52}$$

$$n = \frac{1}{\omega_o QRC} \tag{1-53}$$

$$C_4 = mC \tag{1-54}$$

and

$$R_3 = nR \tag{1-55}$$

where $R_1 = R$ and $C_2 = C$ are chosen arbitrarily and $T_{LP}(0)$ is always unity.

<u>Example 1-8 - Design of A Two-Op Amp, Second-order, Low-Pass Active Filter</u>

Use Fig. 1-15 to realize the filter of Ex. 1-6 if the gain is +1.

<u>Solution</u>

Let us arbitrarily pick $C_2 = C = 0.1\mu F$ and $R_1 = R = 10\ k\Omega$. Eq. (1-52) gives $m = \dfrac{1}{\sqrt{2}(628.32)(10^4)(10^{-7})} = 1.1254$. Eq. (1-53) gives $n = \dfrac{\sqrt{2}}{(628.32)(10^4)(10^{-7})} = 2.2508$. Thus Eqs. (1-54) and (1-55) give $C_4 = (1.1254)(0.1\ \mu F) = 0.11254\ \mu F$ and $R_3 = (2.2508)(10\ k\Omega) = 22.508\ k\Omega$.

Tow-Thomas (Resonator) Realization

A second, multiple op amp, second-order, low-pass active filter realization is shown in Fig. 1-16. This circuit will be called the Tow-Thomas filter and consists of a damped inverting integrator, cascaded with another undamped integrator, and an inverter with feedback applied around the entire structure. If it weren't for R_4, the feedback loop of this circuit would be unstable. An advantage of this circuit, not found in any of the previous realizations, is that it offers independent tuning of the pole Q and the pole frequency.

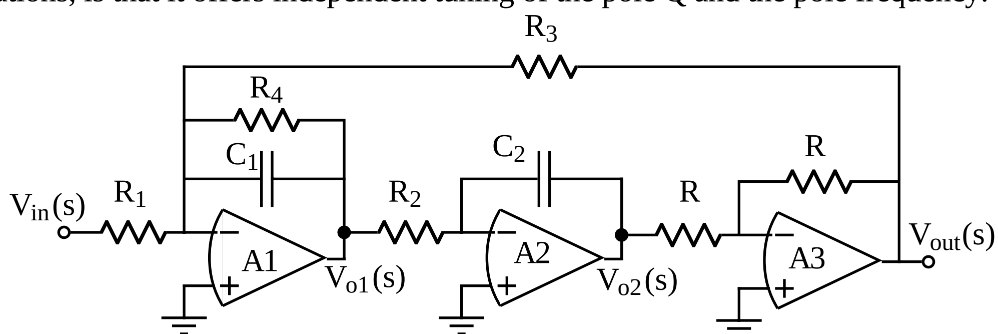

Figure 1-16 - The Tow-Thomas, second-order, low-pass active filter realization.

The number of op amps in Fig. 1-16 insure that the analysis of this circuit will be straight-forward. The output of op amp A1 can be expressed by superposition between $V_{in}(s)$ and $V_{out}(s)$ as

$$V_{o1}(s) = \left(\frac{\frac{1}{R_4C_1}}{s + \frac{1}{R_4C_1}}\right)\left[-\left(\frac{R_4}{R_1}\right)V_{in}(s) - \left(\frac{R_4}{R_3}\right)V_{out}(s)\right] \quad . \tag{1-56}$$

By inspection we can write that

$$V_{out}(s) = \frac{V_{o1}(s)}{sR_2C_2} \quad . \tag{1-57}$$

Substituting Eq. (1-57) into Eq. (1-56) yields

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{\frac{-1}{R_1R_2C_1C_2}}{s^2 + \frac{s}{R_4C_1} + \frac{1}{R_2R_3C_1C_2}} \quad . \tag{1-58}$$

Equating Eq. (1-58) to Eq. (1-3) gives

$$\omega_o = \frac{1}{\sqrt{R_2R_3C_1C_2}} \ , \ \frac{1}{Q} = \frac{1}{R_4}\sqrt{\frac{R_2R_3C_2}{C_1}} \ , \text{ and } \ |T_{LP}(0)| = \frac{R_3}{R_1} \quad . \tag{1-59}$$

If we let $R_2 = R_3 = R$ and $C_1 = C_2 = C$, then the design equations become

$$R = \frac{1}{\omega_o C} \quad \text{or} \quad C = \frac{1}{\omega_o R} \tag{1-60}$$

$$R_4 = QR \tag{1-61}$$

and

$$R_1 = \frac{R_3}{|T_{LP}(0)|} = \frac{R}{|T_{LP}(0)|} \quad . \tag{1-62}$$

We note that a bandpass realization is also available at V_{o2} and that this realization does not have the negative sign of Eq. (1-58).

Example 1-9 - Design of the Tow-Thomas, Second-Order, Low-Pass Active Filter

Apply the circuit of Fig. 1-16 to Ex. 1-7.

Solution

The specifications of the filter of Ex. 1-7 were a dc gain of -100, ω_o of 200π rps, and a Q of 0.707. Choosing C = 0.1µF gives $R_2 = R_3 = R = \frac{1}{(628.32)(10^{-7})} = 15.915\ \text{k}\Omega$ using Eq. (1-60). Eq. (1-61) gives $R_4 = (0.707)(15.915\ \text{k}\Omega) = 11.254\ \text{k}\Omega$. Finally, Eq. (1-62) gives $R_1 = (15.195\ \text{k}\Omega)/(100) = 151.95$. It might be worthwhile to go back and choose C = 0.01µF in order to raise all resistors by a value of 10.

High-Pass, Second-Order, Active Filters

A second useful second-order transfer function is the high-pass transfer function. This transfer function is exactly like the second-order, low-pass transfer function except that both zeros are at the origin of the complex frequency plane. Therefore, we can write the standard, second-order, high-pass transfer function as

$$T_{HP}(s) = \frac{T_{HP}(\infty)s^2}{s^2 + \left(\frac{\omega_o}{Q}\right)s + \omega_o^2} \tag{1-63}$$

where $T_{HP}(\infty)$ is equal to $T_{HP}(s)$ at $\omega = \infty$. The poles of the second-order high-pass transfer function are given by Eq. (1-5) and illustrated by Fig. 1-3.

We can normalize Eq. (1-63) as we did for $T_{LP}(s)$ to get

$$T_{HPn}(s_n) = \frac{T_{HP}\left(\frac{s}{\omega_o}\right)}{|T_{HP}(\infty)|} = \frac{s_n^2}{s_n^2 + \frac{s_n}{Q} + 1} \tag{1-64}$$

where

$$T_{HPn}(s) = \frac{T_{HP}(s)}{|T_{HP}(\infty)|} \tag{1-65}$$

and

$$s_n = \frac{s}{\omega_o} \quad . \tag{1-66}$$

The normalized frequency response of the standard, second-order, high-pass transfer function is shown in Fig. 1-17. We note that the slopes of the magnitudes as frequency becomes much greater or much less than ω_o is ±20 dB/dec. rather than -40 dB/dec. of the low-pass, second-order transfer function. This is because one pole is causing the high-frequency roll-off while the other pole is causing the low-frequency roll-off.

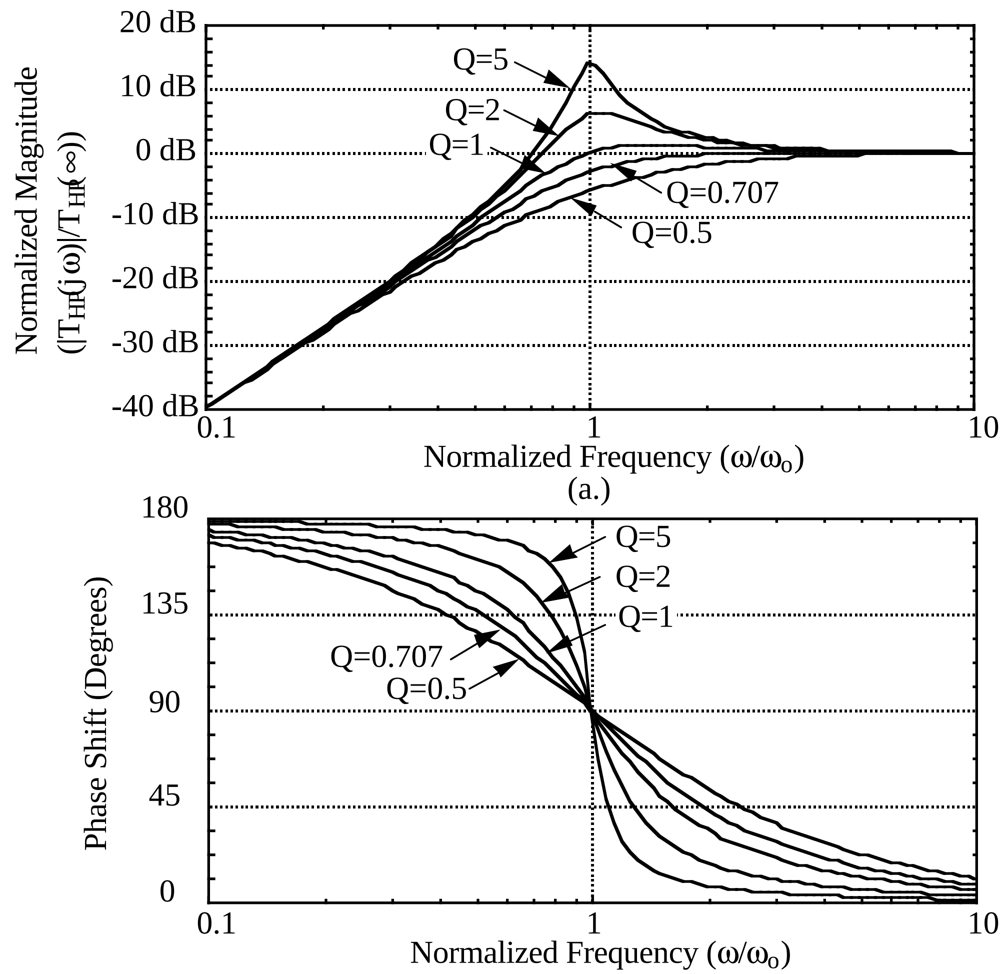

Figure 1-17 - (a.) Normalized magnitude and (b.) phase response of the standard, second-order, high-pass transfer function with Q as a parameter.

A realization for a second-order, high-pass transfer function can be derived from Fig. 1-12a and Fig. 1-14 by simply by replacing pertinent resistors with capacitors and the capacitors with resistors. The resulting second-order, high-pass positive feedback and negative feedback realizations are shown in Fig. 1-18. Example 1-10 will illustrate their application to high-pass filter design.

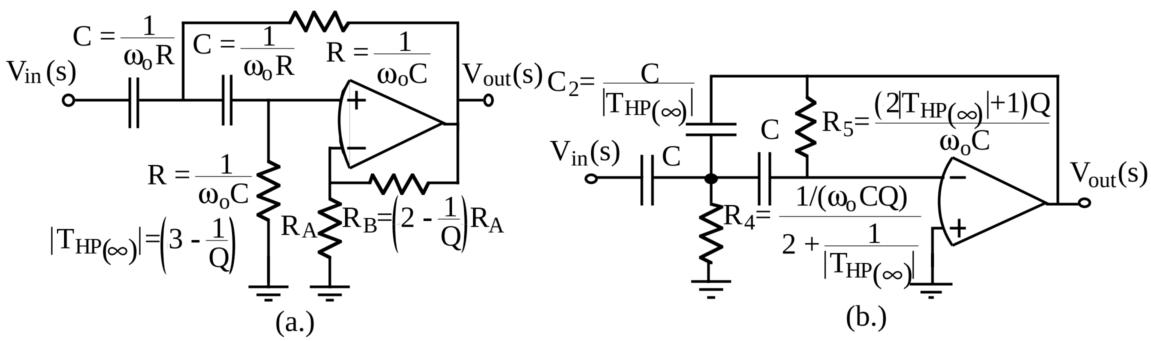

Figure 1-18 - (a.) Positive feedback and (b.) negative feedback, second-order, high-pass active filter realization.

Example 1-10 - Design of High-Pass Positive Feedback and Negative Feedback Filters

Use the realizations of Fig. 1-18 to realize a second-order, high-pass filter having $|T_{HP}(\infty)| = 1$,f_o = 10 kHz, and a Q = 1.

Solution

If we pick C = 1 nF of Fig. 1-18a, then the resistors are $R_2 = R_4 = R$ = 15.915 kΩ. Because Q = 1, $R_B = R_A$. Let us choose $R_A = R_B$ = 10 kΩ. We note that $|T_{HP}(\infty)| = (4/\sqrt{2})-1 = 1.8284$ which does not meet the specification. In order to meet the specification, we can add a resistive attenuator at the output consisting of a 8.284 kΩ and 10 kΩ resistor. The only disadvantage of this solution is that the output resistance of the filter is now 10 kΩ. If this is unacceptable, then the use of an op amp buffer will solve the problem.

For the negative feedback realization of Fig. 1-18b, we again select C = 1 nF. Because $|T_{HP}(\infty)|$ = 1, C_2 = C = 1nF also. R_4 = 15.915 kΩ/3 = 5.305 kΩ. R_5 = (2)(2)15.915 kΩ or 63.662 kΩ.

Figure 1-19 shows how to modify the second-order Tow-Thomas circuit to realize a second-order, high-pass transfer function. Both noninverting and inverting realizations are possible in Fig. 1-19.

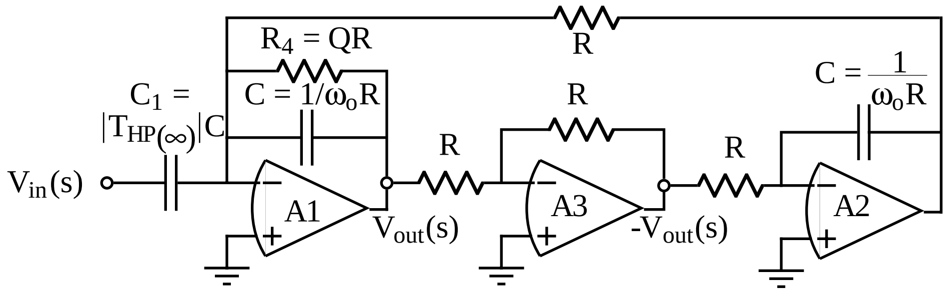

Figure 1-19 - The Tow-Thomas second-order, high-pass active filter realization.

Example 1-11 - Design of A High-Pass Filter using the Tow-Thomas of Fig. 1-19

Repeat Ex. 1-10 using the Tow-Thomas second-order, high-pass active filter realization of Fig. 1-19.

Solution

Picking R = 10 kΩ, the formulas on Fig. 1-19 give $C = 1/(2\pi x10^4)(10^4) = 1.5915$ nF, $C_1 = C = 1.5915$ nF, and $R_4 = R = 10$ kΩ.

Bandpass, Second-Order, Active Filters

Another useful second-order transfer function is the bandpass transfer function. This transfer function is exactly like the second-order, low-pass transfer function except that one zero is at the origin of the complex frequency plane and the other zero is at infinity. Therefore, we can write the standard, second-order, bandpass transfer function as

$$T_{BP}(s) = \frac{T_{BP}(\omega_o)\left(\frac{\omega_o}{Q}\right) s}{s^2 + \left(\frac{\omega_o}{Q}\right) s + \omega_o^2} \tag{1-67}$$

where $T_{BP}(\omega_o)$ is equal to $T_{BP}(s)$ at $\omega = \omega_o$. Fig. 1-3 also holds for the poles of $T_{BP}(s)$.

We can normalize Eq. (1-63) as we did for $T_{LP}(s)$ and $T_{HP}(s)$ to get

$$T_{BPn}(s_n) = \frac{T_{BP}\left(\frac{s}{\omega_o}\right)}{|T_{BP}(\omega_o)|} = \frac{\frac{s_n}{Q}}{s_n^2 + \frac{s_n}{Q} + 1} \tag{1-68}$$

where

$$T_{BPn}(s) = \frac{T_{BP}(s)}{|T_{BP}(\omega_o)|} \tag{1-69}$$

and

$$s_n = \frac{s}{\omega_o} \quad . \tag{1-70}$$

The normalized frequency response of the standard, second-order, bandpass transfer function is shown in Fig. 1-20. We note that the slopes of the magnitudes as frequency

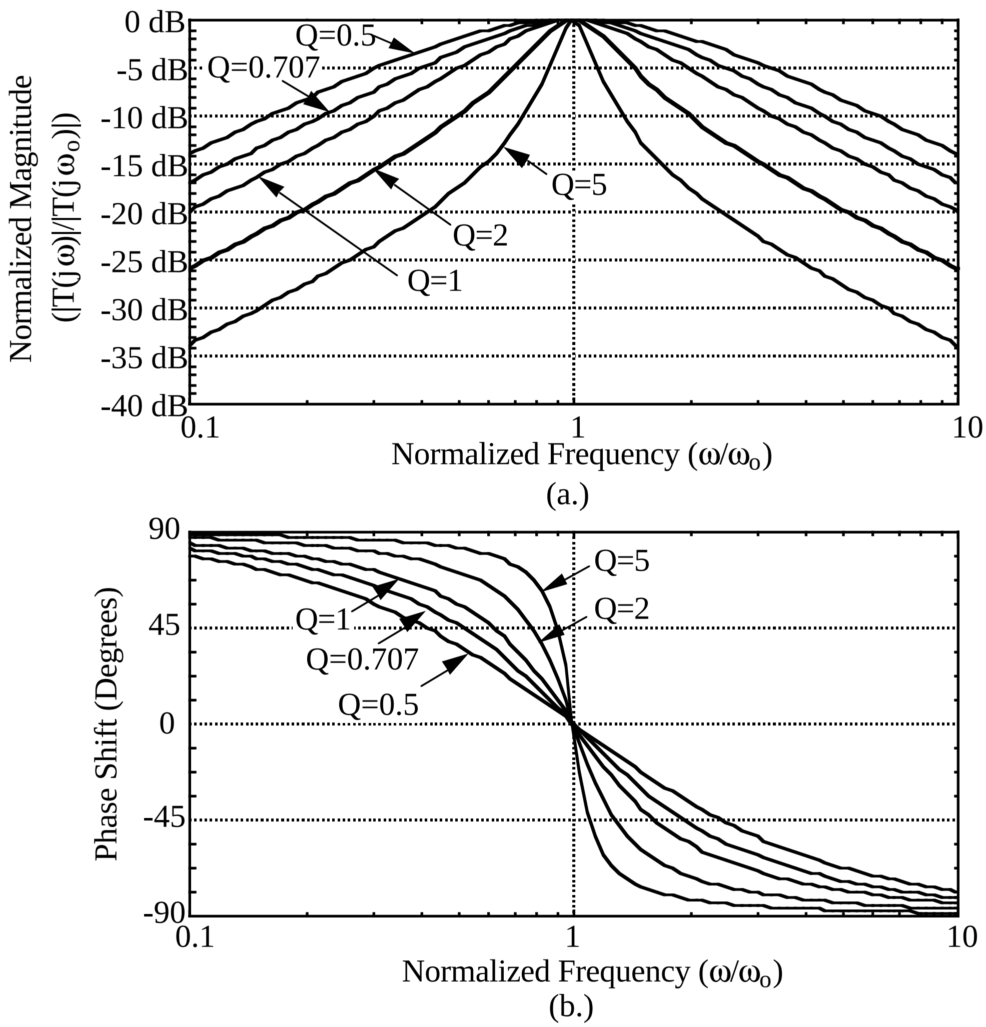

Figure 1-20 - (a.) Normalized magnitude and (b.) phase response of the standard second-order, bandpass transfer function with Q as a parameter.

becomes much greater or much less than ω_o is ±20 dB/dec. rather than -40 dB/dec. of the low-pass, second-order transfer function. This is because one pole is causing the high-frequency roll-off while the other pole is causing the low-frequency roll-off.

Fig. 1-21a shows a positive feedback, second-order, bandpass realization using the concepts of Fig. 1-12a. Fig. 1-21b illustrates a second-order, bandpass filter using the negative feedback approach of Fig. 1-14. In order to save space and summarize the results briefly, a set of design equations are shown on the circuits.

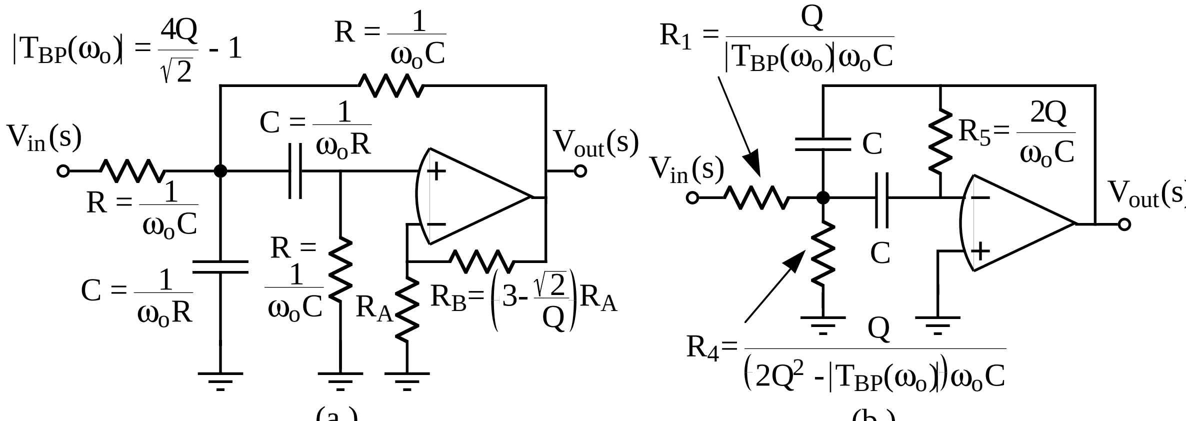

Figure 1-21 - (a.) Positive feedback and (b.) negative feedback, second-order, bandpass active filter realizations.

Example 1-12 - Application of the Second-Order, Bandpass Filters of Fig. 1-21

Use the realizations of Fig. 1-21 to realize a second-order, bandpass filter having $|T_{BP}(\omega_o)| = 1$,$f_o = 10$ kHz, and a Q = 1.

Solution

Let us pick C = 1 nF for the realization of Fig. 1-21a. Thus R becomes $1/(2\pi x10^4)(10^{-9}) =$ 15.915 kΩ. If we pick $R_A = 10$ kΩ, then $R_B = (3 - \sqrt{2})10$ kΩ = 12.858 kΩ. The gain at $s = j\omega_o$ is 2 and must be reduced using the same approach as proposed in Ex. 1-10.

For the infinite gain realization of Fig. 1-21b, we again pick C = 1 nF. $R_1 = 1/(2\pi x10^4)(10^{-9}) =$ 15.915 kΩ. R_2 and R_3 are both equal to R_1 in this case.

It turns out that the output of A1 of the second-order, low-pass circuit, designated as $V_{o1}(s)$, of Fig. 1-16 realizes a second-order, bandpass filter. In order to increase the versatility of the bandpass realization, we interchange the integrator (A2) and the inverter (A3) to get a positive or negative bandpass response. This modification is shown in Fig. 1-22.

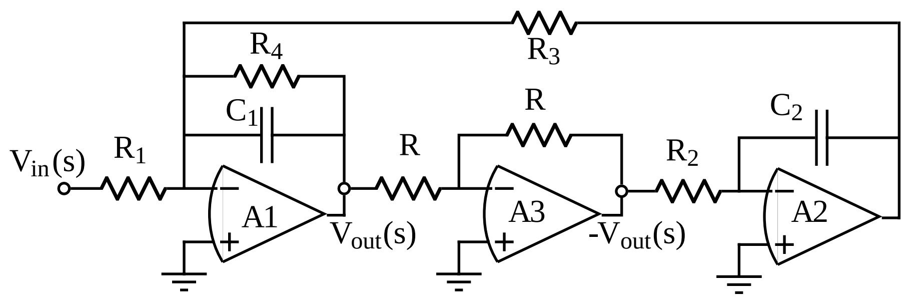

Figure 1-22 - The Tow-Thomas, second-order, bandpass active filter realization.

We can show that the transfer function of Fig. 1-22 is given by

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{\frac{-s}{R_1C_1}}{s^2 + \frac{s}{R_4C_1} + \frac{1}{R_2R_3C_1C_2}} \quad . \tag{1-71}$$

The relationships of Eq. (1-59) hold for Fig. 1-22 except that

$$|T_{BP}(\omega_o)| = \frac{R_4}{R_1} \quad . \tag{1-72}$$

The design equations given in Eqs. (1-60) and (1-61) are valid along with

$$R_1 = \frac{R_4}{|T_{BP}(\omega_o)|} = \frac{QR}{|T_{BP}(\omega_o)|} \quad . \tag{1-73}$$

Example 1-13 - Design of a Second-Order, Bandpass Filter using the Tow-Thomas Filter

Repeat Ex. 1-12 for Fig. 1-22.

Solution

Selecting C = 1 nF results in $R_2 = R_3 = 1/(2\pi x10^4)(10^{-9}) = 15.915$ kΩ from Eq. (1-60). Eq. (4.6-61) gives that $R_4 = R = 15.915$ kΩ. The unity value of $|T_{BP}(\omega_o)|$ and Q substituted into Eq. (1-73) gives $R_1 = 15.915$ kΩ.

Other Types of Second-Order Transfer Functions

There are two other types of second-order transfer functions filters which we have not covered here. They are the band-stop and allpass. These transfer functions have the same poles as the previous ones. However, the zeros of the band-stop transfer function are on the jω axis while the zeros of the allpass transfer function are quadrantally symmetrical to the poles (they are mirror images of the poles in the right-half plane). Both of these transfer functions can be implemented by a second-order biquadratic transfer function whose transfer function is given as

$$T_{BQ}(s) = \frac{K\left(s^2 \pm \left(\frac{\omega_{oz}}{Q_z}\right)s + \omega_{oz}^2\right)}{s^2 + \left(\frac{\omega_{op}}{Q_p}\right)s + \omega_{op}^2} \tag{1-74}$$

where K is a constant, ω_{oz} is the zero frequency, Q_z the zero-Q, ω_{op} is the pole frequency, and Q_p the pole-Q.

A realization of Eq. (1-74) is called a biquad. While there are many one and two amplifier realizations of the biquad we shall consider only a modification of the Tow-Thomas circuit shown in Fig. 1-23. This circuit is identical to Fig. 1-16 except the two resistors associated with the inverter stage (A3) have been designated as R_5 and R_6 and the input is applied to all three stages through R_7, R_8, and the parallel combination of C_3 and R_1.

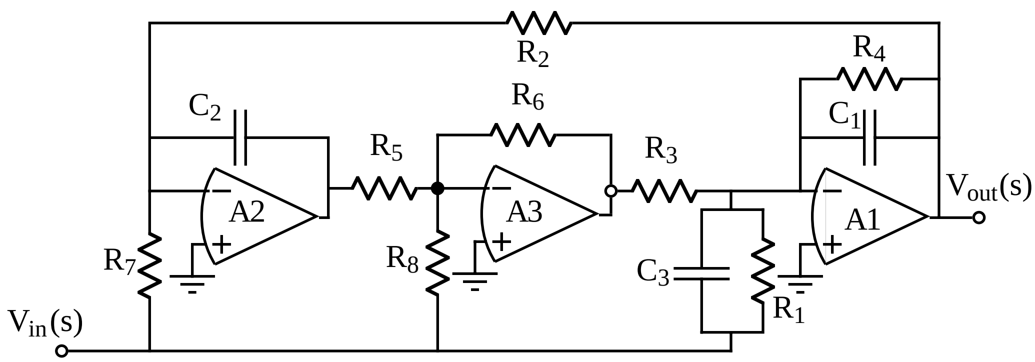

Figure 1-23 - Tow-Thomas biquad realization.

An analysis of Fig. 1-23 similar to that of Fig. 1-16 gives the transfer function as

$$T_{BQ}(s) = \frac{V_{out}(s)}{V_{in}(s)} = -\left(\frac{C_3}{C_1}\right)\left(\frac{s^2 + s\left(\frac{1}{R_1} - \frac{R_6}{R_3R_8}\right)\frac{1}{C_3} + \frac{R_6}{R_3R_5R_7C_2C_3}}{s^2 + \frac{s}{R_4C_1} + \frac{R_6}{R_2R_3R_5C_1C_2}}\right). \tag{1-75}$$

We can equate Eq. (1-75) to Eq. (1-74) to get

$$\omega_z = \sqrt{\frac{R_6}{R_3R_5R_7C_2C_3}} \quad \text{and} \quad \frac{1}{Q_z} = \left(\frac{1}{R_1} - \frac{R_6}{R_3R_8}\right)\sqrt{\frac{R_3R_5R_7C_2}{R_6C_3}} \tag{1-76}$$

and

$$\omega_p = \sqrt{\frac{R_6}{R_2R_3R_5C_1C_2}} \quad \text{and} \quad \frac{1}{Q_p} = \frac{1}{R_4}\sqrt{\frac{R_2R_3R_5C_2}{R_6C_1}} \quad . \tag{1-77}$$

We see that the Tow-Thomas biquad is capable of realizing all of the second-order filters we have studied so far. Table 1-1 shows the design equations for each of five different second-order filters.

Type of Second-Order Transfer Function	Element Constraints	Tuning Elements
Low-pass	$R_1 = R_8 = \infty$ $C_3 = 0$	R_2, R_4, R_7
High-pass	$R_1 = R_7 = R_8 = \infty$	R_2, R_4 and C_3*
Bandpass	R_1 or $R_8 = \infty$ (depending upon sign) $R_7 = \infty$ $C_3 = 0$	R_2, R_4, and R_1 or R_8, whichever is not infinite
Right-half plane zeros	$R_1 = \infty$	R_2, R_4, R_7, R_8, and C_3*
jω axis zeros	$R_1 = R_8 = \infty$	R_2, R_4, R_7, and C_3*

*Note that C_3 may be fixed if the passband gain is a free parameter.

Table 1-1 - Design and tuning relationships for the biquadratic circuit of Fig. 1-23.

Example 1-14 - Biquadratic, Transfer Function Design using the Tow-Thomas Biquad

A normalized, low-pass filter has the following transfer function

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{-(s^2 + 4)}{s^2 + \sqrt{2}\, s + 1} \quad .$$

Use the Tow-Thomas biquad of Fig. 1-23 to realize this transfer function and denormalize the frequency by 10^3 and the impedance by 10^5.

Solution

The roots of the numerator are ±j2 which corresponds to the jω axis case. Therefore, we let R_1 and R_8 be infinity. Let us also choose $R_5 = R_6$. Eq. (1-75) reduces to

$$\frac{V_{out}(s)}{V_{in}(s)} = -\left(\frac{C_3}{C_1}\right)\left(\frac{s^2 + \frac{1}{R_3R_7C_2C_3}}{s^2 + \frac{s}{R_4C_1} + \frac{1}{R_2R_3C_1C_2}}\right).$$

Next, we choose $R_2 = R_3 = R_5 = R_6 = 1\Omega$ and $C_1 = C_2 = C_3 = 1$ F. This results in $R_4 = 1/\sqrt{2} = 0.7071\ \Omega$ and $R_7 = 1/4 = 0.25\ \Omega$. Denormalizing the frequency by 10^3 and the impedance by 10^5 gives $R_2 = R_3 = R_5 = R_6 = 100$ kΩ, $C_1 = C_2 = C_3 = 10$ nF, $R_4 = 70.71$ kΩ, and $R_7 = 25$ kΩ.

Tuning Active Filters

After the second-order active filter has been designed, the next step is the actual implementation of the circuit. Unfortunately, the values of resistors and capacitors generally do not have the accuracy required in the examples illustrated in this section. As a result, the final step in implementing a second-order active filter is tuning. Tuning is the process of adjusting the passive component values so that the desired frequency performance is achieved. Tuning requires the ability to vary the components which can be done by component replacement or a variable component such as a potentiometer.

A general tuning procedure for most second-order active filters is outlined below. It is based on the magnitude of the frequency response of a low-pass filter. The filter parameters are assumed to be the pole frequency, f_o, the pole Q, Q, and the gain, T(0).

1.) The component(s) which set(s) the parameter f_o is(are) tuned by adjusting the magnitude of the filter response to be T(0)/10 (or T(0) (dB) - 20dB) at $10f_o$.

2.) The component(s) which set(s) the parameter T(0) is(are) tuned by adjusting the magnitude to T(0) at $f_o/10$.

3.) The component(s) which set(s) the parameter Q is(are) tuned by adjusting the magnitude of the peak (if there is one) to the value given by Fig. 1-6a. If there is no peaking, then adjust so that the magnitude at f_o is correct (i.e. -3dB for Q = 0707).

The tuning procedure should follow in the order of steps 1 through 3 and may be repeated if necessary. One could also use the phase shift to help in the tuning of the filter. The concept of the above tuning procedure is easily adaptable to other types of second-order filters.

Summary

Various types of op amp realizations for second-order transfer functions have been introduced in this section. The complexity of these realizations is such that one must be aware of the stability principles presented in the previous section. In fact, the phase margin of some of the realizations becomes smaller the higher the pole-Q.

We have seen that the poles for all of the second-order realizations are essentially the same and are determined by the pole frequency, ω_o, and the pole-Q, Q. It is the location of the zeros that determines the performance of the second-order transfer function.

We have also shown complex poles can be realized by amplifiers and only resistors and capacitors. In addition, we showed how the step response of a second-order, low-pass transfer function could be related to the phase margin of a negative feedback system.

Specific topics of importance include:

- Complex poles allow the design of filter with fewer op amps and passive components
- Feedback permits circuits containing only resistors, capacitors, and amplifiers to achieve complex poles
- If the step response of a feedback system has more than three rings (oscillations), the phase margin is poor
- The design equations for an active filter permit the unique design of all components given ω_o, Q (or ζ), and $|T_{LP}(0)|$, $|T_{BP}(j\omega_o)|$, or $|T_{HP}(\infty)|$.
- The simplest circuits to analyze are those that have all passive components connected between a defined potential (including ground), the negative input terminal of an ideal op amp with the potential at the positive input terminal defined, or the output terminal of an ideal op amp.