Fourier Analysis

Translation by Olof Staffans of the lecture notes

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by

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Contents

Chapter 0

Integration theory

This is a short summary of Lebesgue integration theory, which will be used in the course.

Fact 0.1. *Some subsets (="delmängder") $E \subset \mathbb{R} = (-\infty, \infty)$ are "measurable" (="mätbara") in the Lebesgue sense, others are not.*

General Assumption 0.2. *All the subsets E which we shall encounter in this course are measurable.*

Fact 0.3. *All measurable subsets $E \subset \mathbb{R}$ have a* measure *(="mått") $m(E)$, which in simple cases correspond to* "the total length" *of the set. E.g., the measure of the interval (a, b) is $b - a$ (and so is the measure of $[a, b]$ and $[a, b)$).*

Fact 0.4. *Some sets E have* measure zero*, i.e., $m(E) = 0$. True for example if E consists of finitely many (or countably many) points. ("måttet noll")*

The expression a.e. = "*almost everywhere*" (n.ö. = nästan överallt) means that something is true for all $x \in \mathbb{R}$, *except* for those x which belong to some set E with measure zero. For example, the function

$$f(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

is continuous almost everywhere. The expression $f_n(x) \to f(x)$ a.e. means that the measure of the set $x \in \mathbb{R}$ for which $f_n(x) \not\to f(x)$ is zero.

<u>Think</u>: "In all but finitely many points" (this is a simplification).

Notation 0.5. $\mathbb{R} = (-\infty, \infty)$, $\mathbb{C} =$ *complex plane.*

The set of Riemann integrable functions $f : I \mapsto \mathbb{C}$ ($I \subseteq \mathbb{R}$ is an interval) such that

$$\int_I |f(x)|^p dx < \infty, \; 1 \le p < \infty,$$

though much larger than the space $C(I)$ of continuous functions on I, is not big enough for our purposes. This defect can be remedied by the use of the Lebesgue integral instead of the Riemann integral. The Lebesgue integral is more complicated to define and develop than the Riemann integral, but as a tool it is easier to use as it has better properties. The main difference between the Riemann and the Lebesgue integral is that the former uses intervals and their lengths while the latter uses more general point sets and their measures.

Definition 0.6. A function $f : I \mapsto \mathbb{C}$ ($I \in \mathbb{R}$ is an interval) is **measurable** if there exists a sequence of *continuous* functions f_n so that

$$f_n(x) \to f(x) \text{ for almost all } x \in I$$

(i.e., the set of points $x \in I$ for which $f_n(x) \nrightarrow f(x)$ has measure zero).

General Assumption 0.7. *All the functions that we shall encounter in this course are measurable.*

Thus, the word "measurable" is understood throughout (when needed).

Definition 0.8. Let $1 \le p < \infty$, and $I \subset \mathbb{R}$ an interval. We write $f \in L^p(I)$ if (f is measurable and)

$$\int_I |f(x)|^p \mathrm{d}x < \infty.$$

We define the **norm** of f in $L^p(I)$ to be

$$\|f\|_{L^p(I)} = \left(\int_I |f(x)|^p \mathrm{d}x \right)^{1/p}.$$

Physical interpretation:

$$\boxed{p = 1} \quad \|f\|_{L^1(I)} = \int_I |f(x)| \mathrm{d}x$$

= "the total mass". "Probability density" if $f(x) \geq 0$, or a "size of the total population".

$$\boxed{p=2} \quad \|f\|_{L^2(I)} = \left(\int_I |f(x)|^2 \mathrm{d}x \right)^{1/2}$$

= "total energy" (e.g. in an electrical signal, such as alternating current).
These two cases are the two *important* ones (we ignore the rest). The third important case is $p = \infty$.

Definition 0.9. $f \in L^\infty(I)$ if (f is measurable and) there exists a number $M < \infty$ such that

$$|f(x)| < M \quad \text{a.e.}$$

The norm of f is

$$\|f\|_{L^\infty(I)} = \inf\{M : \ |f(x)| \leq M \text{ a.e.}\},$$

and it is denoted by

$$\|f\|_{L^\infty(I)} = \operatorname*{ess\,sup}_{x \in I} |f(x)|$$

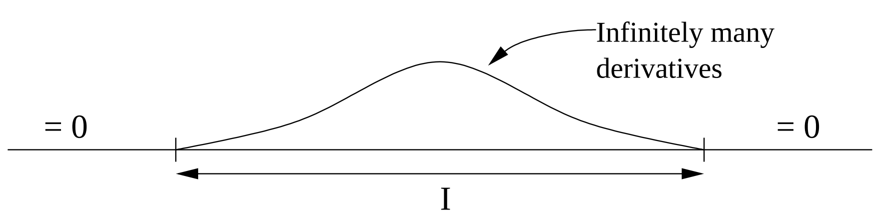
("essential supremum", "väsentligt supremum").

<u>Think</u>: $\|f\|_{L^\infty(I)}$ = "the largest value of f in I if we ignore a set of measure zero". For example:

$$f(x) = \begin{cases} 0, & x < 0 \\ 2, & x = 0 \\ 1, & x > 0 \end{cases}$$

$\Rightarrow \|f\|_{L^\infty(I)} = 1$.

Definition 0.10. $C_C^\infty(\mathbb{R}) = \mathfrak{D}$ = the set of (real or complex-valued) functions on $\mathbb{R}$ which can be differentiated as many times as you wish, and which vanish outside of a bounded interval (such functions do exist!). $C_C^\infty(I)$ = the same thing, but the function vanish outside of I.



Theorem 0.11. *Let $I \subset \mathbb{R}$ be an interval. Then $C_C^\infty(I)$ is dense in $L^p(I)$ for all p, $1 \le p < \infty$ (but not in $L^\infty(I)$). That is, for every $f \in L^p(I)$ it is possible to find a sequence $f_n \in C_C^\infty(I)$ so that*

$$\lim_{n\to\infty} \|f_n - f\|_{L^p(I)} = 0.$$

Proof. "Straightforward" (but takes a lot of work). $\square$

Theorem 0.12 (Fatou's lemma)**.** *Let $f_n(x) \ge 0$ and let $f_n(x) \to f(x)$ a.e. as $n \to \infty$. Then*

$$\int_I f(x)\,dx \le \lim_{n\to\infty} \int_I f_n(x)\,dx$$

(if the latter limit exists). Thus,

$$\int_I \left[\lim_{n\to\infty} f_n(x)\right] dx \le \lim_{n\to\infty} \int_I f_n(x)\,dx$$

if $f_n \ge 0$ ("f can have no more total mass than f_n, but it may have less"). Often we have equality, but not always.
Ex.

$$f_n(x) = \begin{cases} n, & 0 \le x \le 1/n \\ 0, & \textit{otherwise.} \end{cases}$$

Homework: Compute the limits above in this case.

Theorem 0.13 (Monotone Convergence Theorem)**.** *If*

$$0 \le f_1(x) \le f_2(x) \le \dots$$

and $f_n(x) \to f(x)$ a.e., then

$$\int_I f(x)\,dx = \lim_{n\to\infty} \int_I f_n(x)\,dx \quad (\le \infty).$$

Thus, for a positive increasing sequence we have

$$\int_I \left[\lim_{n\to\infty} f_n(x)\right] dx = \lim_{n\to\infty} \int_I f_n(x)\,dx$$

(the mass of the limit is the limit of the masses).

Theorem 0.14 (Lebesgue's dominated convergence theorem)**.** *(Extremely useful)*
If $f_n(x) \to f(x)$ a.e. and $|f_n(x)| \le g(x)$ a.e. and

$$\int_I g(x)\,dx < \infty \quad (\textit{i.e.}, g \in L^1(I)),$$

then

$$\int_I f(x)dx = \int_I \left[\lim_{n\to\infty} f_n(x)\right] dx = \lim_{n\to\infty} \int_I f_n(x)dx.$$

Theorem 0.15 (Fubini's theorem)**.** *(Very useful for multiple integrals).*
If f (is measurable and)

$$\int_I \int_J |f(x,y)|dy\ dx < \infty$$

then the double integral

$$\iint_{I\times J} f(x,y)dy\ dx$$

is well-defined, and equal to

$$\begin{aligned} &= \int_{x\in I} \left(\int_{y\in J} f(x,y)\,dy \right) dx \\ &= \int_{y\in J} \left(\int_{x\in I} f(x,y)\,dx \right) dy \end{aligned}$$

If $f \geq 0$, then all three integrals are well-defined, possibly $= \infty$, and if one of them is $< \infty$, then so are the others, and they are equal.

Note: These theorems are very useful, and often *easier to use* than the corresponding theorems based on the Rieman integral.

Theorem 0.16 (Integration by parts à la Lebesgue)**.** *Let $[a,b]$ be a finite interval, $u \in L^1([a,b]), v \in L^1([a,b])$,*

$$U(t) = U(a) + \int_a^t u(s)ds, \quad V(t) = V(a) + \int_a^t v(s)ds,\ t \in [a,b].$$

Then

$$\int_a^b u(t)V(t)dt = [U(t)V(t)]_a^b - \int_a^b U(t)v(t)dt.$$

Proof.

$$\begin{aligned} \int_a^b u(t)V(t) &= \int_a^b u(t) \int_a^t v(s)dsdt \\ &\stackrel{\text{Fubini}}{=} \big(U(b) - U(a)\big)V(a) + \int_a^b \left(\int_s^b u(t)dt \right) v(s)ds. \end{aligned}$$

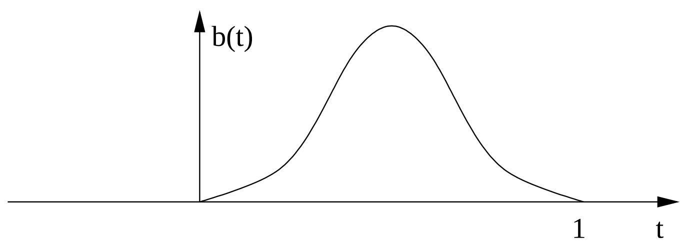
Since

$$\int_s^b u(t)dt = \Big(\int_a^b - \int_a^s\Big)u(t)dt = U(b) - U(a) - \int_a^s u(t)dt = U(b) - U(s),$$

we get

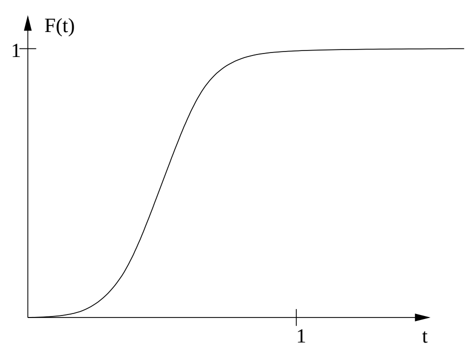
$$\begin{aligned}
\int_a^b u(t)V(t)dt &= \big(U(b)-U(a)\big)V(a) + \int_a^b \left(U(b)-U(s)\right)v(s)ds \\
&= \big(U(b)-U(a)\big)V(a) + U(b)\big(V(b)-V(a)\big) - \int_a^b U(s)v(s)ds \\
&= U(b)V(b) - U(a)V(a) - \int_a^b U(s)v(s)ds. \quad \square
\end{aligned}$$

Example 0.17. Sometimes we need test functions with special properties. Let us take a look how one can proceed.



$$b(t) = \begin{cases} e^{-\frac{1}{t(1-t)}} & ,0 < t < 1 \\ 0 & ,otherwise. \end{cases}$$

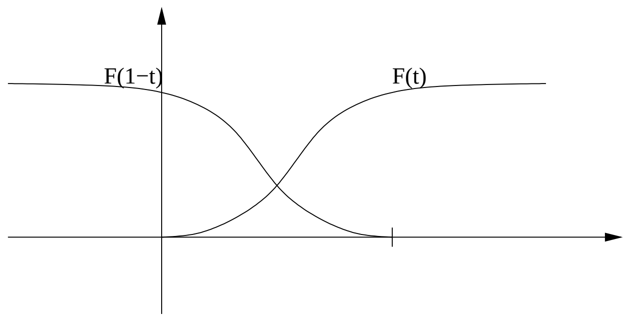
Then we can show that $b \in C^\infty(\mathbb{R})$, and b is a test function with compact support. Let $B(t) = \int_{-\infty}^t b(s)ds$ and norm it $F(t) = \frac{B(t)}{B(1)}$.



$$F(t) = \begin{cases} 0 & ,t \le 0 \\ 1 & ,t \ge 1 \\ \text{increase} & ,0 < t < 1. \end{cases}$$

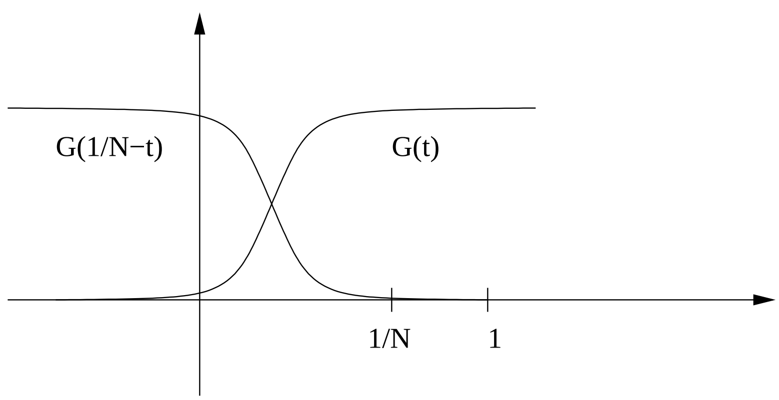
Further $F(t) + F(t-1) = 1,\ \forall t \in \mathbb{R}$, clearly true for $t \le 0$ and $t \ge 1$.
For $0 < t < 1$ we check the derivative

$$\frac{d}{dt}\left(F(t) - F(1-t)\right) = \frac{1}{B(1)}\left[B'(t) - B'(1-t)\right] = \frac{1}{B(1)}[b(t) - b(1-t)] = 0.$$



Let $G(t) = F(Nt)$. Then G increases from 0 to 1 on the interval $0 \leq t \leq \frac{1}{N}$.

$$G(t) + G(\frac{1}{N} - t) = F(Nt) - F(1 - Nt) = 1, \ \forall t \in \mathbb{R}.$$



Chapter 1

The Fourier Series of a Periodic Function

1.1 Introduction

Notation 1.1. *We use the letter $\mathbb{T}$ with a double meaning:*

a) $\mathbb{T} = [0, 1)$

b) In the notations $L^p(\mathbb{T})$, $C(\mathbb{T})$, $C^n(\mathbb{T})$ and $C^\infty(\mathbb{T})$ we use the letter $\mathbb{T}$ to imply that the functions are periodic *with* period 1, *i.e., $f(t+1) = f(t)$ for all $t \in \mathbb{R}$. In particular, in the continuous case we require $f(1) = f(0)$. Since the functions are periodic we know the whole function as soon as we know the values for $t \in [0, 1)$.*

Notation 1.2. $\|f\|_{L^p(\mathbb{T})} = \left(\int_0^1 |f(t)|^p dt\right)^{1/p}$, $1 \le p < \infty$. $\|f\|_{C(\mathbb{T})} = \max_{t \in T} |f(t)|$ *(f continuous).*

Definition 1.3. $f \in L^1(\mathbb{T})$ has the **Fourier coefficients**

$$\hat{f}(n) = \int_0^1 e^{-2\pi int} f(t) dt, \quad n \in \mathbb{Z},$$

where $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$. The sequence $\{\hat{f}(n)\}_{n \in \mathbb{Z}}$ is the (finite) **Fourier transform** of f.

Note:

$$\hat{f}(n) = \int_s^{s+1} e^{-2\pi int} f(t) dt \quad \forall s \in \mathbb{R},$$

since the function inside the integral is periodic with period 1.

<u>Note:</u> The Fourier transform of a *periodic function* is a *discrete sequence.*

Theorem 1.4.

i) $|\hat{f}(n)| \le \|f\|_{L^1(\mathbb{T})}, \quad \forall n \in \mathbb{Z}$

ii) $\lim_{n\to\pm\infty} \hat{f}(n) = 0.$

<u>Note:</u> ii) is called the Riemann–Lebesgue lemma.

Proof.

i) $|\hat{f}(n)| = |\int_0^1 e^{-2\pi int} f(t)dt| \le \int_0^1 |e^{-2\pi int} f(t)|dt = \int_0^1 |f(t)|dt = \|f\|_{L^1(\mathbb{T})}$ (by the triangle inequality for integrals).

ii) First consider the case where f is continuously differentiable, with $f(0) = f(1)$. Then integration by parts gives

$$\begin{aligned} \hat{f}(n) &= \int_0^1 e^{-2\pi int} f(t)dt \\ &= \frac{1}{-2\pi in}\left[e^{-2\pi int} f(t)\right]_0^1 + \frac{1}{2\pi in}\int_0^1 e^{-2\pi int} f'(t)dt \\ &= 0 + \frac{1}{2\pi in}\hat{f'}(n), \quad \text{so by i),} \end{aligned}$$

$$|\hat{f}(n)| = \frac{1}{2\pi n}|\hat{f'}(n)| \le \frac{1}{2\pi n}\int_0^1 |f'(s)|ds \to 0 \text{ as } n \to \infty.$$

In the general case, take $f \in L^1(\mathbb{T})$ and $\varepsilon > 0$. By Theorem 0.11 we can choose some g which is continuously differentiable with $g(0) = g(1) = 0$ so that

$$\|f - g\|_{L^1(\mathbb{T})} = \int_0^1 |f(t) - g(t)|dt \le \varepsilon/2.$$

By i),

$$\begin{aligned} |\hat{f}(n)| &= |\hat{f}(n) - \hat{g}(n) + \hat{g}(n)| \\ &\le |\hat{f}(n) - \hat{g}(n)| + |\hat{g}(n)| \\ &\le \|f - g\|_{L^1(\mathbb{T})} + |\hat{g}(n)| \\ &\le \epsilon/2 + |\hat{g}(n)|. \end{aligned}$$

By the first part of the proof, for n large enough, $|\hat{g}(n)| \le \varepsilon/2$, and so

$$|\hat{f}(n)| \le \varepsilon.$$

This shows that $|\hat{f}(n)| \to 0$ as $n \to \infty$. $\square$

Question 1.5. *If we know $\{\hat{f}(n)\}_{n=-\infty}^{\infty}$ then can we reconstruct $f(t)$?*
Answer: *is more or less "Yes".*

Definition 1.6. $C^n(\mathbb{T}) = n$ times continuously differentiable functions, periodic with period 1. (In particular, $f^{(k)}(1) = f^{(k)}(0)$ for $0 \le k \le n$.)

Theorem 1.7. *For all $f \in C^1(\mathbb{T})$ we have*

$$f(t) = \lim_{\substack{N\to\infty \\ M\to\infty}} \sum_{n=-M}^{N} \hat{f}(n) e^{2\pi i n t}, \quad t \in \mathbb{R}. \tag{1.1}$$

We shall see later that the convergence is actually uniform in t.

Proof. Step 1. We shift the argument of f by replacing $f(s)$ by $g(s) = f(s+t)$. Then

$$\hat{g}(n) = e^{2\pi i n t} \hat{f}(n),$$

and (1.1) becomes

$$f(t) = g(0) = \lim_{\substack{N\to\infty \\ M\to\infty}} \sum_{n=-M}^{N} \hat{f}(n) e^{2\pi i n t}.$$

Thus, it suffices to prove the case where $\boxed{t=0}$.
Step 2: If $g(s)$ is the constant function $g(s) \equiv g(0) = f(t)$, then (1.1) holds since $\hat{g}(0) = g(0)$ and $\hat{g}(n) = 0$ for $n \neq 0$ in this case. Replace $g(s)$ by

$$h(s) = g(s) - g(0).$$

Then h satisfies all the assumptions which g does, and in addition, $h(0) = 0$. Thus it suffices to prove the case where both $t = 0$ and $f(0) = 0$. For simplicity we write f instead of h, but we suppose below that $\boxed{t=0}$ and $\boxed{f(0)=0}$
Step 2: Define

$$g(s) = \begin{cases} \frac{f(s)}{e^{-2\pi i s}-1}, & s \neq \text{ integer } (=\text{“heltal”}) \\ \frac{i f'(0)}{2\pi}, & s = \text{integer}. \end{cases}$$

For $s = n =$ integer we have $e^{-2\pi i s} - 1 = 0$, and by l'Hospital's rule

$$\lim_{s\to n} g(s) = \lim_{s\to 0} \frac{f'(s)}{-2\pi i e^{-2\pi i s}} = \frac{f'(s)}{-2\pi i} = g(n)$$

(since $e^{-i2\pi n} = 1$). Thus g is continuous. We clearly have

$$f(s) = \left(e^{-2\pi i s} - 1\right) g(s), \tag{1.2}$$

so

$$\begin{aligned} \hat{f}(n) &= \int_{\mathbb{T}} e^{-2\pi i n s} f(s) ds \quad \text{(use (1.2))} \\ &= \int_{\mathbb{T}} e^{-2\pi i n s} (e^{-2\pi i s} - 1) g(s) ds \\ &= \int_{\mathbb{T}} e^{-2\pi i (n+1) s} g(s) ds - \int_{\mathbb{T}} e^{-2\pi i n s} g(s) ds \\ &= \hat{g}(n+1) - \hat{g}(n). \end{aligned}$$

Thus,

$$\sum_{n=-M}^{N} \hat{f}(n) = \hat{g}(N+1) - \hat{g}(-M) \to 0$$

by the Rieman–Lebesgue lemma (Theorem 1.4) $\square$

By working a little bit harder we get the following stronger version of Theorem 1.7:

Theorem 1.8. *Let $f \in L^1(\mathbb{T})$, $t_0 \in \mathbb{R}$, and suppose that*

$$\int_{t_0-1}^{t_0+1} \left| \frac{f(t) - f(t_0)}{t - t_0} \right| dt < \infty. \tag{1.3}$$

Then

$$f(t_0) = \lim_{\substack{N\to\infty \\ M\to\infty}} \sum_{n=-M}^{N} \hat{f}(n) e^{2\pi i n t_0} \quad t \in \mathbb{R}$$

Proof. We can repeat Steps 1 and 2 of the preceding proof to reduce the Theorem to the case where $t_0 = 0$ and $f(t_0) = 0$. In Step 3 we define the function g in the same way as before for $s \neq n$, but leave $g(s)$ undefined for $s = n$. Since $\lim_{s\to 0} s^{-1}(e^{-2\pi i s} - 1) = -2\pi i \neq 0$, the function g belongs to $L^1(\mathbb{T})$ if and only if condition (1.3) holds. The continuity of g was used only to ensure that $g \in L^1(\mathbb{T})$, and since $g \in L^1(\mathbb{T})$ already under the weaker assumption (1.3), the rest of the proof remains valid without any further changes. $\square$

Summary 1.9. *If $f \in L^1(\mathbb{T})$, then the Fourier transform $\{\hat{f}(n)\}_{n=-\infty}^{\infty}$ of f is well-defined, and $\hat{f}(n) \to 0$ as $n \to \infty$. If $f \in C^1(\mathbb{T})$, then we can reconstruct f from its Fourier transform through*

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n t} \left(= \lim_{\substack{N\to\infty \\ M\to\infty}} \sum_{n=-M}^{N} \hat{f}(n) e^{2\pi i n t} \right).$$

The same reconstruction formula remains valid under the weaker assumption of Theorem 1.8.

1.2 L^2-Theory ("Energy theory")

This theory is based on the fact that we can define an **inner product** (scalar product) in $L^2(\mathbb{T})$, namely

$$\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}dt, \quad f, g \in L^2(\mathbb{T}).$$

Scalar product means that for all $f, g, h \in L^2(\mathbb{T})$

i) $\langle f+g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$

ii) $\langle \lambda f, g \rangle = \lambda \langle f, g \rangle \quad \forall \lambda \in \mathbb{C}$

iii) $\langle g, f \rangle = \overline{\langle f, g \rangle}$ (complex conjugation)

iv) $\langle f, f \rangle \geq 0$, and $= 0$ only when $f \equiv 0$.

These are the same rules that we know from the scalar products in $\mathbb{C}^n$. In addition we have

$$\|f\|_{L^2(\mathbb{T})}^2 = \int_{\mathbb{T}} |f(t)|^2 dt = \int_{\mathbb{T}} f(t)\overline{f(t)}dt = \langle f, f \rangle.$$

This result can also be used to define the Fourier transform of a function $f \in L^2(\mathbb{T})$, since $L^2(\mathbb{T}) \subset L^1(\mathbb{T})$.

Lemma 1.10. *Every function $f \in L^2(\mathbb{T})$ also belongs to $L^1(\mathbb{T})$, and*

$$\|f\|_{L^1(\mathbb{T})} \leq \|f\|_{L^2(\mathbb{T})}.$$

PROOF. Interpret $\int_{\mathbb{T}} |f(t)| dt$ as the inner product of $|f(t)|$ and $g(t) \equiv 1$. By Schwartz inequality (see course on Analysis II),

$$|\langle f, g\rangle| = \int_{\mathbb{T}} |f(t)| \cdot 1 dt \leq \|f\|_{L^2} \cdot \|g\|_{L^2} = \|f\|_{L^2(\mathbb{T})} \int_{\mathbb{T}} 1^2 dt = \|f\|_{L^2(\mathbb{T})}.$$

Thus, $\|f\|_{L^1(\mathbb{T})} \leq \|f\|_{L^2(\mathbb{T})}$. Therefore:

$$\begin{aligned} f \in L^2(t) \quad &\Longrightarrow \quad \int_{\mathbb{T}} |f(t)| dt < \infty \\ &\Longrightarrow \quad \hat{f}(n) = \int_{\mathbb{T}} e^{-2\pi i n t} f(t) dt \ \text{ is defined for all } n. \end{aligned}$$

It is *not* true that $L^2(\mathbb{R}) \subset L^1(\mathbb{R})$. Counter example:

$$f(t) = \frac{1}{\sqrt{1+t^2}} \begin{cases} \in L^2(\mathbb{R}) \\ \notin L^1(\mathbb{R}) \\ \in C^\infty(\mathbb{R}) \end{cases}$$

(too large at ∞).

Notation 1.11. $e_n(t) = e^{2\pi i n t}$, $n \in \mathbb{Z}, t \in \mathbb{R}$.

Theorem 1.12 (Plancherel's Theorem)**.** *Let $f \in L^2(\mathbb{T})$. Then*

i) $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \int_0^1 |f(t)|^2 dt = \|f\|^2_{L^2(\mathbb{T})}$,

ii) $f = \sum_{n=-\infty}^{\infty} \hat{f}(n) e_n$ *in* $L^2(\mathbb{T})$ *(see explanation below).*

Note: This is a very central result in, e.g., signal processing.
Note: It follows from i) that the sum $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$ always converges if $f \in L^2(\mathbb{T})$
Note: i) says that

$$\begin{aligned} \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \quad &= \quad \text{the square of the total energy of the Fourier coefficients} \\ &= \quad \text{the square of the total energy of the original signal } f \\ &= \quad \int_{\mathbb{T}} |f(t)|^2 dt \end{aligned}$$

Note: Interpretation of ii): Define

$$f_{M,N} = \sum_{n=-M}^{N} \hat{f}(n) e_n = \sum_{n=-M}^{N} \hat{f}(n) e^{2\pi i n t}.$$

Then

$$\lim_{\substack{M\to\infty\\N\to\infty}} \|f - f_{M,N}\|^2 = 0 \Longleftrightarrow$$
$$\lim_{\substack{M\to\infty\\N\to\infty}} \int_0^1 |f(t) - f_{M,N}(t)|^2 dt = 0$$

($f_{M,N}(t)$ need not converge to $f(t)$ at every point, and not even almost everywhere).

The proof of Theorem 1.12 is based on some auxiliary results:

Theorem 1.13. *If $g_n \in L^2(\mathbb{T})$, $f_N = \sum_{n=0}^N g_n$, $g_n \perp g_m$, and $\sum_{n=0}^\infty \|g_n\|^2_{L^2(\mathbb{T})} < \infty$, then the limit*

$$f = \lim_{N\to\infty} \sum_{n=0}^N g_n$$

exists in L^2.

Proof. Course on "Analysis II" and course on "Hilbert Spaces". □

Interpretation: Every orthogonal sum with finite total energy converges.

Lemma 1.14. *Suppose that $\sum_{n=-\infty}^\infty |c(n)| < \infty$. Then the series*

$$\sum_{n=-\infty}^{\infty} c(n)e^{2\pi int}$$

converges uniformly to a continuous limit function $g(t)$.

Proof.

i) The series $\sum_{n=-\infty}^\infty c(n)e^{2\pi int}$ converges absolutely (since $|e^{2\pi int}| = 1$), so the limit

$$g(t) = \sum_{n=-\infty}^{\infty} c(n)e^{2\pi int}$$

exist for all $t \in \mathbb{R}$.

ii) The convergens is uniform, because the error

$$\begin{aligned}
\left|\sum_{n=-m}^{m} c(n)e^{2\pi int} - g(t)\right| &= \left|\sum_{|n|>m} c(n)e^{2\pi int}\right| \\
&\le \sum_{|n|>m} |c(n)e^{2\pi int}| \\
&= \sum_{|n|>m} |c(n)| \to 0 \text{ as } m \to \infty.
\end{aligned}$$

iii) If a sequence of <u>continuous</u> functions converge uniformly, then the limit is continuous (proof "Analysis II"). $\square$

PROOF OF THEOREM 1.12. (Outline)

$$\begin{aligned} 0 &\le \|f - f_{M,N}\|^2 = \langle f - f_{M,N}, f - f_{M,N}\rangle \\ &= \underbrace{\langle f, f\rangle}_{I} - \underbrace{\langle f_{M,N}, f\rangle}_{II} - \underbrace{\langle f, f_{M,N}\rangle}_{III} + \underbrace{\langle f_{M,N}, f_{M,N}\rangle}_{IV} \end{aligned}$$

$$\begin{aligned} I &= \langle f, f\rangle = \|f\|^2_{L^2(\mathbb{T})}. \\ II &= \langle \sum_{n=-M}^{N} \hat{f}(n)e_n, f\rangle = \sum_{n=-M}^{N} \hat{f}(n)\langle e_n, f\rangle \\ &= \sum_{n=-M}^{N} \hat{f}(n)\overline{\langle f, e_n\rangle} = \sum_{n=-M}^{N} \hat{f}(n)\overline{\hat{f}(n)} \\ &= \sum_{n=-M}^{N} |\hat{f}(n)|^2. \\ III &= \text{(the complex conjugate of } II) = II. \\ IV &= \Big\langle \sum_{n=-M}^{N} \hat{f}(n)e_n, \sum_{m=-M}^{N} \hat{f}(m)e_m \Big\rangle \\ &= \sum_{n=-M}^{N} \hat{f}(n)\overline{\hat{f}(m)}\underbrace{\langle e_n, e_m\rangle}_{\delta_n^m} \\ &= \sum_{n=-M}^{N} |\hat{f}(n)|^2 = II = III. \end{aligned}$$

Thus, adding $I - II - III + IV = I - II \ge 0$, i.e.,

$$\|f\|^2_{L^2(\mathbb{T})} - \sum_{n=-M}^{N} |\hat{f}(n)|^2 \ge 0.$$

This proves **Bessel's inequality**

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \le \|f\|^2_{L^2(\mathbb{T})}. \tag{1.4}$$

How do we get <u>equality</u>?

By Theorem 1.13, applied to the sums

$$\sum_{n=0}^{N} \hat{f}(n)e_n \text{ and } \sum_{n=-M}^{-1} \hat{f}(n)e_n,$$

the limit

$$g = \lim_{\substack{M\to\infty \\ N\to\infty}} f_{M,N} = \lim_{\substack{M\to\infty \\ N\to\infty}} \sum_{n=-M}^{N} \hat{f}(n) e_n \tag{1.5}$$

does exist. Why is $f = g$? (This means that the sequence e_n is *complete*!). This is (in principle) done in the following way

i) Argue as in the proof of Theorem 1.4 to show that if $f \in C^2(\mathbb{T})$, then $|\hat{f}(n)| \le 1/(2\pi n)^2 \|f''\|_{L^1}$ for $n \neq 0$. In particular, this means that $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$. By Lemma 1.14, the convergence in (1.5) is actually uniform, and by Theorem 1.7, the limit is equal to f. Uniform convergence implies convergence in $L^2(\mathbb{T})$, so even if we interpret (1.5) in the L^2-sense, the limit is still equal to f a.e. This proves that $f_{M,N} \to f$ in $L^2(\mathbb{T})$ if $f \in C^2(\mathbb{T})$.

ii) Approximate an arbitrary $f \in L^2(\mathbb{T})$ by a function $h \in C^2(\mathbb{T})$ so that $\|f - h\|_{L^2(\mathbb{T})} \le \varepsilon$.

iii) Use *i*) and *ii*) to show that $\|f - g\|_{L^2(\mathbb{T})} \le \varepsilon$, where g is the limit in (1.5). Since ε is arbitrary, we must have $g = f$. $\square$

Definition 1.15. Let $1 \le p < \infty$.

$$\ell^p(\mathbb{Z}) = \text{ set of all sequences } \{a_n\}_{n=-\infty}^{\infty} \text{ satisfying } \sum_{n=-\infty}^{\infty} |a_n|^p < \infty.$$

The *norm* of a sequence $a \in \ell^p(\mathbb{Z})$ is

$$\|a\|_{\ell^p(\mathbb{Z})} = \left(\sum_{n=-\infty}^{\infty} |a_n|^p \right)^{1/p}$$

Analogous to $L^p(I)$:

$$\begin{aligned} p = 1 \quad \|a\|_{\ell^1(\mathbb{Z})} &= \text{"total mass" (probability)}, \\ p = 2 \quad \|a\|_{\ell^2(\mathbb{Z})} &= \text{"total energy"}. \end{aligned}$$

In the case of $p = 2$ we also define an **inner product**

$$\langle a, b \rangle = \sum_{n=-\infty}^{\infty} a_n \overline{b_n}.$$

Definition 1.16. $\ell^\infty(\mathbb{Z})$ = set of all **bounded** sequences $\{a_n\}_{n=-\infty}^{\infty}$. The **norm** in $\ell^\infty(\mathbb{Z})$ is

$$\|a\|_{\ell^\infty(\mathbb{Z})} = \sup_{n\in\mathbb{Z}} |a_n|.$$

For details: See course in "Analysis II".

Definition 1.17. $c_0(\mathbb{Z})$ = the set of all sequences $\{a_n\}_{n=-\infty}^{\infty}$ satisfying $\lim_{n\to\pm\infty} a_n = 0$. We use the norm

$$\|a\|_{c_0(\mathbb{Z})} = \max_{n\in\mathbb{Z}} |a_n|$$

in $c_0(\mathbb{Z})$.

Note that $c_0(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z})$, and that

$$\|a\|_{c_0(\mathbb{Z})} = \|a\|_{\ell^\infty(\mathbb{Z})}$$

if $\{a\}_{n=-\infty}^{\infty} \in c_0(\mathbb{Z})$.

Theorem 1.18. *The Fourier transform maps* $L^2(\mathbb{T})$ one to one onto $\ell^2(\mathbb{Z})$, *and the Fourier inversion formula (see Theorem 1.12 ii) maps* $\ell^2(\mathbb{Z})$ one to one onto $L^2(\mathbb{T})$. *These two transforms* preserves *all* distances *and* scalar products.

PROOF. (Outline)

i) If $f \in L^2(\mathbb{T})$ then $\hat{f} \in \ell^2(\mathbb{Z})$. This follows from Theorem 1.12.

ii) If $\{a_n\}_{n=-\infty}^{\infty} \in \ell^2(\mathbb{Z})$, then the series

$$\sum_{n=-M}^{N} a_n e^{2\pi int}$$

converges to some limit function $f \in L^2(\mathbb{T})$. This follows from Theorem 1.13.

iii) If we compute the Fourier coefficients of f, then we find that $a_n = \hat{f}(n)$. Thus, $\{a_n\}_{n=-\infty}^{\infty}$ is the Fourier transform of f. This shows that the Fourier transform maps $L^2(\mathbb{T})$ *onto* $\ell^2(\mathbb{Z})$.

iv) Distances are preserved. If $f \in L^2(\mathbb{T})$, $g \in L^2(\mathbb{T})$, then by Theorem 1.12 i),

$$\|f - g\|_{L^2(\mathbb{T})} = \|\hat{f}(n) - \hat{g}(n)\|_{\ell^2(\mathbb{Z})},$$

i.e.,

$$\int_{\mathbb{T}} |f(t) - g(t)|^2 dt = \sum_{n=-\infty}^{\infty} |\hat{f}(n) - \hat{g}(n)|^2.$$

v) Inner products are preserved:

$$\begin{aligned}
\int_{\mathbb{T}} |f(t)-g(t)|^2 dt &= \langle f-g, f-g\rangle \\
&= \langle f,f\rangle - \langle f,g\rangle - \langle g,f\rangle + \langle g,g\rangle \\
&= \langle f,f\rangle - \langle f,g\rangle - \overline{\langle f,g\rangle} + \langle g,g\rangle \\
&= \langle f,f\rangle + \langle g,g\rangle - 2\Re Re\langle f,g\rangle.
\end{aligned}$$

In the same way,

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} |\hat{f}(n)-\hat{g}(n)|^2 &= \langle \hat{f}-\hat{g}, \hat{f}-\hat{g}\rangle \\
&= \langle \hat{f},\hat{f}\rangle + \langle \hat{g},\hat{g}\rangle - 2\Re\langle \hat{f},\hat{g}\rangle.
\end{aligned}$$

By iv), subtracting these two equations from each other we get

$$\Re\langle f,g\rangle = \Re\langle \hat{f},\hat{g}\rangle.$$

If we replace f by if, then

$$\begin{aligned}
\mathrm{Im}\langle f,g\rangle &= \mathrm{Re}\ i\langle f,g\rangle = \Re\langle if,g\rangle \\
&= \Re\langle i\hat{f},\hat{g}\rangle = \mathrm{Re}\ i\langle \hat{f},\hat{g}\rangle \\
&= \mathrm{Im}\langle \hat{f},\hat{g}\rangle.
\end{aligned}$$

Thus, $\langle f,g\rangle_{L^2(\mathbb{R})} = \langle \hat{f},\hat{g}\rangle_{\ell^2(\mathbb{Z})}$, or more explicitely,

$$\boxed{\int_{\mathbb{T}} f(t)\overline{g(t)}dt = \sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}.} \tag{1.6}$$

This is called **Parseval's identity**.

Theorem 1.19. *The Fourier transform maps $L^1(\mathbb{T})$ <u>into</u> $c_0(\mathbb{Z})$ (but not onto), and it is a <u>contraction</u>, i.e., the norm of the image is $\leq$ the norm of the original function.*

PROOF. This is a rewritten version of Theorem 1.4. Parts $i)$ and $ii)$ say that $\{\hat{f}(n)\}_{n=-\infty}^{\infty} \in c_0(\mathbb{Z})$, and part $i)$ says that $\|\hat{f}(n)\|_{c_0(\mathbb{Z})} \leq \|f\|_{L^1(\mathbb{T})}$. $\square$

The proof that there exist sequences in $c_0(\mathbb{Z})$ which are <u>not</u> the Fourier transform of some function $f \in L^1(\mathbb{T})$ is much more complicated.

1.3 Convolutions ("Faltningar")

Definition 1.20. The **convolution** ("faltningen") of two functions $f, g \in L^1(\mathbb{T})$ is

$$(f * g)(t) = \int_{\mathbb{T}} f(t-s)g(s)ds,$$

where $\int_{\mathbb{T}} = \int_{\alpha}^{\alpha+1}$ for all $\alpha \in \mathbb{R}$, since the function $s \mapsto f(t-s)g(s)$ is periodic.

<u>Note:</u> In this integral we need values of f and g outside of the interval $[0, 1)$, and therefore the periodicity of f and g is important.

Theorem 1.21. *If $f, g \in L^1(\mathbb{T})$, then $(f * g)(t)$ is defined almost everywhere, and $f * g \in L^1(\mathbb{T})$. Furthermore,*

$$\|f * g\|_{L^1(\mathbb{T})} \leq \|f\|_{L^1(\mathbb{T})} \|g\|_{L^1(\mathbb{T})} \tag{1.7}$$

Proof. (We ignore measurability)
We begin with (1.7)

$$\begin{aligned}
\|f * g\|_{L^1(\mathbb{T})} &= \int_{\mathbb{T}} |(f * g)(t)| \, dt \\
&= \int_{\mathbb{T}} \Big| \int_{\mathbb{T}} f(t-s)g(s) \, ds \Big| \, dt \\
&\overset{\triangle\text{-ineq.}}{\leq} \int_{t \in T} \int_{s \in T} |f(t-s)g(s)| \, ds \, dt \\
&\overset{\text{Fubini}}{=} \int_{s \in T} \left(\int_{t \in T} |f(t-s)| \, dt \right) |g(s)| \, ds \\
&\overset{\text{Put } v=t-s, dv=dt}{=} \int_{s \in T} \underbrace{\left(\int_{v \in T} |f(v)| \, dv \right)}_{=\|f\|_{L^1(\mathbb{T})}} |g(s)| \, ds \\
&= \|f\|_{L^1(\mathbb{T})} \int_{s \in T} |g(s)| \, ds = \|f\|_{L^1(\mathbb{T})} \|g\|_{L^1(\mathbb{T})}
\end{aligned}$$

This integral is finite. By Fubini's Theorem 0.15

$$\int_{\mathbb{T}} f(t-s)g(s)ds$$

is defined for almost all t. $\square$

Theorem 1.22. *For all $f, g \in L^1(\mathbb{T})$ we have*

$$(\widehat{f * g})(n) = \hat{f}(n)\hat{g}(n), \quad n \in \mathbb{Z}$$

PROOF. Homework.

Thus, the Fourier transform maps convolution onto pointwise multiplication.

Theorem 1.23. *If $k \in C^n(\mathbb{T})$ (n times continuously differentiable) and $f \in L^1(\mathbb{T})$, then $k * f \in C^n(\mathbb{T})$, and $(k * f)^{(m)}(t) = (k^{(m)} * f)(t)$ for all $m = 0, 1, 2, \ldots, n$.*

PROOF. (Outline) We have for all $h > 0$

$$\frac{1}{h}\left[(k * f)(t+h) - (k * f)(t)\right] = \frac{1}{h}\int_0^1 \left[k(t+h-s) - k(t-s)\right] f(s) ds.$$

By the mean value theorem,

$$k(t+h-s) = k(t-s) + hk'(\xi),$$

for some $\xi \in [t-s, t-s+h]$, and $\frac{1}{h}[k(t+h-s) - k(t-s)] = f(\xi) \to k'(t-s)$ as $h \to 0$, and $\left|\frac{1}{h}[k(t+h-s) - k(t-s)]\right| = |f'(\xi)| \leq M$, where $M = \sup_T |k'(s)|$. By the Lebesgue dominated convergence theorem (which is true also if we replace $n \to \infty$ by $h \to 0$)(take $g(x) = M|f(x)|$)

$$\lim_{h \to 0} \int_0^1 \frac{1}{h}[k(t+h-s) - k(t-s)]f(s)\ ds = \int_0^1 k'(t-s)f(s)\ ds,$$

so $k * f$ is differentiable, and $(k * f)' = k' * f$. By repeating this n times we find that $k * f$ is n times differentiable, and that $(k * f)^{(n)} = k^{(n)} * f$. We must still show that $k^{(n)} * f$ is continuous. This follows from the next lemma. □

Lemma 1.24. *If $k \in C(\mathbb{T})$ and $f \in L^1(\mathbb{T})$, then $k * f \in C(\mathbb{T})$.*

PROOF. By Lebesgue dominated convergence theorem (take $g(t) = 2\|k\|_{C(\mathbb{T})} f(t)$),

$$(k * f)(t+h) - (k * f)(t) = \int_0^1 [k(t+h-s) - k(t-s)]f(s)ds \to 0 \text{ as } h \to 0.$$

Corollary 1.25. *If $k \in C^1(\mathbb{T})$ and $f \in L^1(\mathbb{T})$, then for all $t \in \mathbb{R}$*

$$(k * f)(t) = \sum_{n=-\infty}^{\infty} e^{2\pi i n t} \hat{k}(n) \hat{f}(n).$$

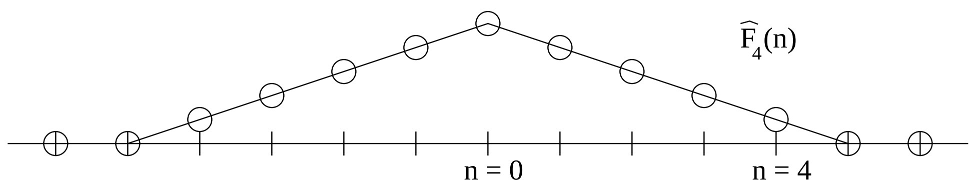
PROOF. Combine Theorems 1.7, 1.22 and 1.23.

Interpretation: This is a generalised inversion formula. If we choose $\hat{k}(n)$ so that

i) $\hat{k}(n) \approx 1$ for small $|n|$

ii) $\hat{k}(n) \approx 0$ for large $|n|$,

then we set a "filtered" approximation of f, where the "high frequences" (= high values of $|n|$) have been damped but the "low frequences" (= low values of $|n|$) remain. If we can take $\hat{k}(n) = 1$ for all n then this gives us back f itself, but this is impossible because of the Riemann-Lebesgue lemma.
Problem: Find a "good" function $k \in C^1(\mathbb{T})$ of this type.
Solution: **"The Fejer kernel"** is one possibility. Choose $\hat{k}(n)$ to be a "triangular function":



Fix $m = 0, 1, 2 \ldots$, and define

$$\hat{F}_m(n) = \begin{cases} \frac{m+1-|n|}{m+1} & , \quad |n| \leq m \\ 0 & , \quad |n| > m \end{cases}$$

($\neq 0$ in $2m + 1$ points.)
We get the corresponding time domain function $F_m(t)$ by using the invertion formula:

$$F_m(t) = \sum_{n=-m}^{m} \hat{F}_m(n) e^{2\pi i n t}.$$

Theorem 1.26. *The function $F_m(t)$ is explicitly given by*

$$F_m(t) = \frac{1}{m+1} \frac{\sin^2\left((m+1)\pi t\right)}{\sin^2(\pi t)}.$$

Proof. We are going to show that

$$\sum_{j=0}^{m} \sum_{n=-j}^{j} e^{2\pi i n t} = \left(\frac{\sin(\pi(m+1)t)}{\sin \pi t}\right)^2 \quad \text{when } t \neq 0.$$

Let $z = e^{2\pi it}$, $\overline{z} = e^{-2\pi it}$, for $t \neq n$, $n = 0, 1, 2, \ldots$. Also $z \neq 1$, and

$$\begin{aligned}
\sum_{n=-j}^{j} e^{2\pi int} &= \sum_{n=0}^{j} e^{2\pi int} + \sum_{n=1}^{j} e^{-2\pi int} = \sum_{n=0}^{j} z^n + \sum_{n=1}^{j} \overline{z}^n \\
&= \frac{1 - z^{j+1}}{1 - z} + \frac{\overline{z}(1 - \overline{z}^j)}{1 - \overline{z}} = \frac{1 - z^{j+1}}{1 - z} + \frac{\overbrace{z \cdot \overline{z}}^{=1}(1 - \overline{z}^j)}{z - \underbrace{z \cdot \overline{z}}_{=1}} \\
&= \frac{\overline{z}^j - z^{j+1}}{1 - z}.
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{j=0}^{m} \sum_{n=-j}^{j} e^{2\pi int} &= \sum_{j=0}^{m} \frac{\overline{z}^j - z^{j+1}}{1 - z} = \frac{1}{1 - z} \left(\sum_{j=0}^{m} \overline{z}^j - \sum_{j=0}^{m} z^{j+1} \right) \\
&= \frac{1}{1 - z} \left(\frac{1 - \overline{z}^{m+1}}{1 - \overline{z}} - z \left(\frac{1 - z^{m+1}}{1 - z} \right) \right) \\
&= \frac{1}{1 - z} \left[\frac{1 - \overline{z}^{m+1}}{1 - \overline{z}} - \frac{\overbrace{\overline{z} \cdot z}^{=1}(1 - z^{m+1})}{\overline{z}(1 - z)} \right] \\
&= \frac{1}{1 - z} \left[\frac{1 - \overline{z}^{m+1}}{1 - \overline{z}} - \frac{1 - z^{m+1}}{\overline{z} - 1} \right] \\
&= \frac{-\overline{z}^{m+1} + 2 - z^{m+1}}{|1 - z|^2}.
\end{aligned}$$

$\sin t = \frac{1}{2i}(e^{it} - e^{-it})$, $\cos t = \frac{1}{2}(e^{it} + e^{-it})$. Now

$$|1 - z| = |1 - e^{2\pi it}| = |e^{i\pi t}(e^{-i\pi t} - e^{i\pi t})| = |e^{-i\pi t} - e^{i\pi t}| = 2|\sin(\pi t)|$$

and

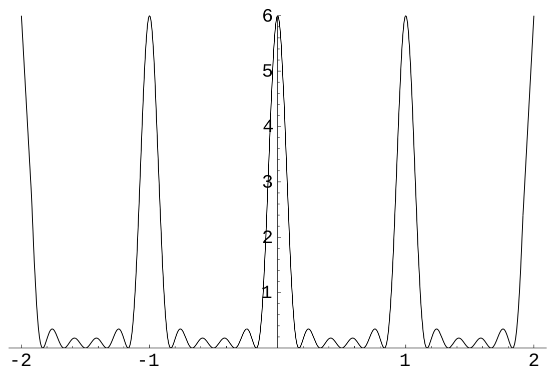
$$\begin{aligned}
z^{m+1} - 2 + \overline{z}^{m+1} &= e^{2\pi i(m+1)} - 2 + e^{-2\pi i(m+1)} \\
&= \left(e^{\pi i(m+1)} - e^{-\pi i(m+1)}\right)^2 = \left(2i \sin(\pi(m+1))\right)^2.
\end{aligned}$$

Hence

$$\sum_{j=0}^{m} \sum_{n=-j}^{j} e^{2\pi int} = \frac{4(\sin(\pi(m+1)))^2}{4(\sin(\pi t))^2} = \left(\frac{\sin(\pi(m+1))}{\sin(\pi t)}\right)^2$$

Note also that

$$\sum_{j=0}^{m} \sum_{n=-j}^{j} e^{2\pi int} = \sum_{n=-m}^{m} \sum_{j=|n|}^{m} e^{2\pi int} = \sum_{n=-m}^{m} (m + 1 - |n|) e^{2\pi int}.$$



Comment 1.27.

i) $F_m(t) \in C^\infty(\mathbb{T})$ *(infinitely many derivatives).*

ii) $F_m(t) \geq 0$.

iii) $\int_{\mathbb{T}} |F_m(t)|dt = \int_{\mathbb{T}} F_m(t)dt = \hat{F}_m(0) = 1$,

so the total mass of F_m *is 1.*

iv) For all $\delta, 0 < \delta < \frac{1}{2}$,

$$\lim_{m\to\infty} \int_\delta^{1-\delta} F_m(t)dt = 0,$$

i.e. the mass of F_m *gets concentrated to the integers* $t = 0, \pm 1, \pm 2 \ldots$ *as* $m \to \infty$.

Definition 1.28. A sequence of functions F_m with the properties i)-iv) above is called a (**periodic**) **approximate identity**. (Often i) is replaced by $F_m \in L^1(\mathbb{T})$.)

Theorem 1.29. *If* $f \in L^1(\mathbb{T})$, *then, as* $m \to \infty$,

i) $F_m * f \to f$ *in* $L^1(\mathbb{T})$, *and*

ii) $(F_m * f)(t) \to f(t)$ *for almost all* t.

Here i) means that $\int_{\mathbb{T}} |(F_m * f)(t) - f(t)|dt \to 0$ *as* $m \to \infty$

Proof. See page 27.
By combining Theorem 1.23 and Comment 1.27 we find that $F_m * f \in C^\infty(\mathbb{T})$. This combined with Theorem 1.29 gives us the following periodic version of Theorem 0.11:

Corollary 1.30. *For every $f \in L^1(\mathbb{T})$ and $\varepsilon > 0$ there is a function $g \in C^\infty(\mathbb{T})$ such that $\|g - f\|_{L^1(\mathbb{T})} \leq \varepsilon$.*

Proof. Choose $g = F_m * f$ where m is large enough. $\square$

To prove Theorem 1.29 we need a number of simpler results:

Lemma 1.31. *For all $f, g \in L^1(\mathbb{T})$ we have $f * g = g * f$*

Proof.

$$\begin{aligned}(f * g)(t) &= \int_{\mathbb{T}} f(t-s)g(s)ds \\ &\overset{t-s=v, ds=-dv}{=} \int_{\mathbb{T}} f(v)g(t-v)dv = (g * f)(t) \qquad \square\end{aligned}$$

We also need:

Theorem 1.32. *If $g \in C(\mathbb{T})$, then $F_m * g \to g$ uniformly as $m \to \infty$, i.e.*

$$\max_{t \in \mathbb{R}} |(F_m * g)(t) - g(t)| \to 0 \text{ as } m \to \infty.$$

Proof.

$$\begin{aligned}(F_m * g)(t) - g(t) &\overset{\text{Lemma 1.31}}{=} (g * F_m)(t) - g(t) \\ &\overset{\text{Comment 1.27}}{=} (g * F_m)(t) - g(t) \int_{\mathbb{T}} F_m(s)ds \\ &= \int_{\mathbb{T}} [g(t-s) - g(t)] F_m(s)ds.\end{aligned}$$

Since g is continuous and periodic, it is uniformly continuous, and given $\varepsilon > 0$ there is a $\delta > 0$ so that $|g(t-s) - g(t)| \leq \varepsilon$ if $|s| \leq \delta$. Split the intgral above into (choose the interval of integration to be $[-\frac{1}{2}, \frac{1}{2}]$)

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} [g(t-s) - g(t)] F_m(s)ds = \Big(\underbrace{\int_{-\frac{1}{2}}^{-\delta}}_{I} + \underbrace{\int_{-\delta}^{\delta}}_{II} + \underbrace{\int_{\delta}^{\frac{1}{2}}}_{III} \Big) [g(t-s) - g(t)] F_m(s)ds$$

Let $M = \sup_{t \in \mathbb{R}} |g(t)|$. Then $|g(t-s) - g(t)| \leq 2M$, and

$$\begin{aligned}|I + III| &\leq \left(\int_{-\frac{1}{2}}^{-\delta} + \int_{\delta}^{\frac{1}{2}} \right) 2M F_m(s)ds \\ &= 2M \int_{\delta}^{1-\delta} F_m(s)ds\end{aligned}$$

and by Comment 1.27iv) this goes to zero as $m \to \infty$. Therefore, we can choose m so large that

$$|I + III| \leq \varepsilon \quad (m \geq m_0, \text{ and } m_0 \text{ large.})$$

$$\begin{aligned} |II| &\leq \int_{-\delta}^{\delta} |g(t-s) - g(t)| F_m(s) ds \\ &\leq \varepsilon \int_{-\delta}^{\delta} F_m(s) ds \\ &\leq \varepsilon \int_{-\frac{1}{2}}^{\frac{1}{2}} F_m(s) ds = \varepsilon \end{aligned}$$

Thus, for $m \geq m_0$ we have

$$|(F_m * g)(t) - g(t)| \leq 2\varepsilon \quad (\text{for all } t).$$

Thus, $\lim_{m\to\infty} \sup_{t\in\mathbb{R}} |(F_m * g)(t) - g(t)| = 0$, i.e., $(F_m * g)(t) \to g(t)$ <u>uniformly</u> as $m \to \infty$. □

The proof of Theorem 1.29 also uses the following weaker version of Lemma 0.11:

Lemma 1.33. *For every $f \in L^1(\mathbb{T})$ and $\varepsilon > 0$ there is a function $g \in C(\mathbb{T})$ such that $\|f - g\|_{L^1(\mathbb{T})} \leq \varepsilon$.*

Proof. Course in Lebesgue integration theory. □

(We already used a stronger version of this lemma in the proof of Theorem 1.12.)

Proof of Theorem 1.29, part i): (The proof of part ii) is bypassed, typically proved in a course on integration theory.)

Let $\varepsilon > 0$, and choose some $g \in C(\mathbb{T})$ with $\|f - g\|_{L^1(\mathbb{T})} \leq \varepsilon$. Then

$$\begin{aligned} \|F_m * f - f\|_{L^1(\mathbb{T})} &\leq \|F_m * g - g + F_m * (f-g) - (f-g)\|_{L^1(t)} \\ &\leq \|F_m * g - g\|_{L^1(\mathbb{T})} + \|F_m * (f-g)\|_{L^1(\mathbb{T})} + \|(f-g)\|_{L^1(\mathbb{T})} \\ &\overset{\text{Thm 1.21}}{\leq} \|F_m * g - g\|_{L^1(\mathbb{T})} + \underbrace{(\|F_m\|_{L^1(\mathbb{T})} + 1)}_{=2} \underbrace{\|f - g\|_{L^1(\mathbb{T})}}_{\leq\varepsilon} \\ &= \|F_m * g - g\|_{L^1(\mathbb{T})} + 2\varepsilon. \end{aligned}$$

$$\begin{aligned}
\text{Now } \|F_m * g - g\|_{L^1(\mathbb{T})} &= \int_0^1 |(F_m * g(t) - g(t)|\, dt \\
&\le \int_0^1 \max_{s\in[0,1]} |(F_m * g(s) - g(s)|\, dt \\
&= \max_{s\in[0,1]} |(F_m * g(s) - g(s)| \cdot \underbrace{\int_0^1 dt}_{=1}.
\end{aligned}$$

By Theorem 1.32, this tends to zero as $m \to \infty$. Thus for large enough m,

$$\|F_m * f - f\|_{L^1(\mathbb{T})} \le 3\varepsilon,$$

so $F_m * f \to f$ in $L^1(\mathbb{T})$ as $m \to \infty$. $\square$

(Thus, we have "almost" proved Theorem 1.29 i): we have reduced it to a proof of Lemma 1.33 and other "standard properties of integrals".)

In the proof of Theorem 1.29 we used the "trivial" triangle inequality in $L^1(\mathbb{T})$:

$$\begin{aligned}
\|f + g\|_{L^1(\mathbb{T})} &= \int |f(t) + g(t)|\, dt \le \int |f(t)| + |g(t)|\, dt \\
&= \|f\|_{L^1(\mathbb{T})} + \|g\|_{L^1(\mathbb{T})}
\end{aligned}$$

Similar inequalities are true in all $L^p(\mathbb{T}), 1 \le p \le \infty$, and a more "sophisticated" version of the preceding proof gives:

Theorem 1.34. *If $1 \le p < \infty$ and $f \in L^p(\mathbb{T})$, then $F_m * f \to f$ in $L^p(\mathbb{T})$ as $m \to \infty$, and also pointwise a.e.*

PROOF. See Gripenberg.

Note: This is not true in $L^\infty(\mathbb{T})$. The correct "L^∞-version" is given in Theorem 1.32.

Corollary 1.35. *(Important!) If $f \in L^p(\mathbb{T}), 1 \le p < \infty$, or $f \in C^n(\mathbb{T})$, then*

$$\lim_{m\to\infty} \sum_{n=-m}^{m} \frac{m+1-|n|}{m+1} \hat{f}(n) e^{2\pi i n t} = f(t),$$

where the convergence is in the norm of L^p, and also pointwise a.e. In the case where $f \in C^n(\mathbb{T})$ we have uniform convergence, and the derivatives of order $\le n$ also converge uniformly.

Proof. By Corollary 1.25 and Comment 1.27,

$$\sum_{n=-m}^{m} \frac{m+1-|n|}{m+1}\hat{f}(n)e^{2\pi int} = (F_m * f)(t)$$

The rest follows from Theorems 1.34, 1.32, and 1.23, and Lemma 1.31. □

Interpretation: We improve the convergence of the sum

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi int}$$

by multiplying the coefficients by the "damping factors" $\frac{m+1-|n|}{m+1}$, $|n| \le m$. This particular method is called Césaro summability. (Other "summability" methods use other damping factors.)

Theorem 1.36. *(Important!) The Fourier coefficients $\hat{f}(n), n \in \mathbb{Z}$ of a function $f \in L^1(\mathbb{T})$* determine f uniquely *a.e., i.e., if $\hat{f}(n) = \hat{g}(n)$ for all n, then $f(t) = g(t)$ a.e.*

Proof. Suppose that $\hat{g}(n) = \hat{f}(n)$ for all n. Define $h(t) = f(t) - g(t)$. Then $\hat{h}(n) = \hat{f}(n) - \hat{g}(n) = 0, n \in \mathbb{Z}$. By Theorem 1.29,

$$h(t) = \lim_{m\to\infty} \sum_{n=-m}^{m} \frac{m+1-|n|}{m+1} \underbrace{\hat{h}(n)}_{=0} e^{2\pi int} = 0$$

in the "L^1-sense", i.e.

$$\|h\| = \int_0^1 |h(t)|\, dt = 0$$

This implies $h(t) = 0$ a.e., so $f(t) = g(t)$ a.e. □

Theorem 1.37. *Suppose that $f \in L^1(\mathbb{T})$ and that $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$. Then the series*

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi int}$$

converges uniformly to a continuous limit function $g(t)$, and $f(t) = g(t)$ a.e.

Proof. The uniform convergence follows from Lemma 1.14. We must have $f(t) = g(t)$ a.e. because of Theorems 1.29 and 1.36. □

The following theorem is much more surprising. It says that *not every* sequence $\{a_n\}_{n\in\mathbb{Z}}$ is the set of Fourier coefficients of some $f \in L^1(\mathbb{T})$.

Theorem 1.38. *Let $f \in L^1(\mathbb{T})$, $\hat{f}(n) \geq 0$ for $n \geq 0$, and $\hat{f}(-n) = -\hat{f}(n)$ (i.e. $\hat{f}(n)$ is an odd function). Then*

$$i) \sum_{n=1}^{\infty} \frac{1}{n} \hat{f}(n) < \infty$$

$$ii) \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} |\frac{1}{n} \hat{f}(n)| < \infty.$$

Proof. Second half easy: Since $\hat{f}$ is odd,

$$\begin{aligned} \sum_{\substack{n \neq 0 \\ n \in \mathbb{Z}}} |\frac{1}{n} \hat{f}(n)| &= \sum_{n>0} |\frac{1}{n} \hat{f}(n)| + \sum_{n<0} |\frac{1}{n} \hat{f}(-n)| \\ &= 2 \sum_{n=1}^{\infty} |\frac{1}{n} \hat{f}(n)| < \infty \qquad \text{if } i) \text{ holds.} \end{aligned}$$

i): Note that $\hat{f}(n) = -\hat{f}(-n)$ gives $\hat{f}(0) = 0$. Define $g(t) = \int_0^t f(s)ds$. Then $g(1) - g(0) = \int_0^1 f(s)ds = \hat{f}(0) = 0$, so that g is continuous. It is not difficult to show (=homework) that

$$\hat{g}(n) = \frac{1}{2\pi i n} \hat{f}(n), \; n \neq 0.$$

By Corollary 1.35,

$$\begin{aligned} g(0) &= \hat{g}(0) \underbrace{e^{2\pi i \cdot 0 \cdot 0}}_{=1} + \lim_{m \to \infty} \sum_{n=-m}^{m} \underbrace{\frac{m+1-|n|}{m+1}}_{\text{even}} \underbrace{\hat{g}(n)}_{\text{even}} \underbrace{e^{2\pi i n 0}}_{=1} \\ &= \hat{g}(0) + \frac{2}{2\pi i} \lim_{m \to \infty} \sum_{n=0}^{m} \frac{m+1-n}{m+1} \underbrace{\frac{\hat{f}(n)}{n}}_{\geq 0}. \end{aligned}$$

Thus

$$\lim_{m \to \infty} \sum_{n=1}^{m} \frac{m+1-n}{m+1} \frac{\hat{f}(n)}{n} = K = \text{a finite pos. number.}$$

In particular, for all finite M,

$$\sum_{n=1}^{M} \frac{\hat{f}(n)}{n} = \lim_{m \to \infty} \sum_{n=1}^{M} \frac{m+1-n}{m+1} \frac{\hat{f}(n)}{n} \leq K,$$

and so $\sum_{n=1}^{\infty} \frac{\hat{f}(n)}{n} \leq K < \infty$. $\square$

Theorem 1.39. *If $f \in C^k(\mathbb{T})$ and $g = f^{(k)}$, then $\hat{g}(n) = (2\pi i n)^k \hat{f}(n)$, $n \in \mathbb{Z}$.*

Proof. Homework.

Note: True under the weaker assumption that $f \in C^{k-1}(\mathbb{T})$, $g \in L^1(\mathbb{T})$, and $f^{k-1}(t) = f^{k-1}(0) + \int_0^t g(s)ds$.

1.4 Applications

1.4.1 Wirtinger's Inequality

Theorem 1.40 (Wirtinger's Inequality)**.** *Suppose that $f \in L^2(a,b)$, and that "f has a derivative in $L^2(a,b)$", i.e., suppose that*

$$f(t) = f(a) + \int_a^t g(s)ds$$

where $g \in L^2(a,b)$. In addition, suppose that $f(a) = f(b) = 0$. Then

$$\begin{aligned} \int_a^b |f(t)|^2 dt &\leq \left(\frac{b-a}{\pi}\right)^2 \int_a^b |g(t)|^2 dt \qquad (1.8) \\ \Bigg(&= \left(\frac{b-a}{\pi}\right)^2 \int_a^b |f'(t)|^2 dt \Bigg). \end{aligned}$$

Comment 1.41. *A function f which can be written in the form*

$$f(t) = f(a) + \int_a^t g(s)ds,$$

where $g \in L^1(a,b)$ is called absolutely continuous *on (a,b). This is the* "Lebesgue version of differentiability"*. See, for example, Rudin's "Real and Complex Analysis".*

Proof. i) First we reduce the interval (a,b) to $(0,1/2)$: Define

$$\begin{aligned} F(s) &= f(a + 2(b-a)s) \\ G(s) &= F'(s) = 2(b-a)g(a + 2(b-a)s). \end{aligned}$$

Then $F(0) = F(1/2) = 0$ and $F(t) = \int_0^t G(s)ds$. Change variable in the integral:

$$t = a + 2(b-a)s, \ dt = 2(b-a)ds,$$

and (1.8) becomes

$$\int_0^{1/2} |F(s)|^2 ds \leq \frac{1}{4\pi^2} \int_0^{1/2} |G(s)|^2 ds. \qquad (1.9)$$

We extend F and G to periodic functions, period one, so that F is odd and G is even: $F(-t) = -F(t)$ and $G(-t) = G(t)$ (first to the interval $(-1/2, 1/2)$ and then by periodicity to all of $\mathbb{R}$). The extended function F is continuous since $F(0) = F(1/2) = 0$. Then (1.9) becomes

$$\begin{aligned} \int_{\mathbb{T}} |F(s)|^2 ds &\leq \frac{1}{4\pi^2} \int_{\mathbb{T}} |G(s)|^2 ds \qquad \Leftrightarrow \\ \|F\|_{L^2(\mathbb{T})} &\leq \frac{1}{2\pi} \|G\|_{L^2(\mathbb{T})} \end{aligned}$$

By Parseval's identity, equation (1.6) on page 20, and Theorem 1.39 this is equivalent to

$$\sum_{n=-\infty}^{\infty} |\hat{F}(n)|^2 \leq \frac{1}{4\pi^2} \sum_{n=-\infty}^{\infty} |2\pi n \hat{F}(n)|^2. \tag{1.10}$$

Here

$$\hat{F}(0) = \int_{-1/2}^{1/2} F(s) ds = 0.$$

since F is odd, and for $n \neq 0$ we have $(2\pi n)^2 \geq 4\pi^2$. Thus (1.10) is true. $\square$

Note: The constant $\left(\frac{b-a}{\pi}\right)^2$ is the best possible: we get equality if we take $\hat{F}(1) \neq 0$, $\hat{F}(-1) = -\hat{F}(1)$, and all other $\hat{F}(n) = 0$. (Which function is this?)

1.4.2 Weierstrass Approximation Theorem

Theorem 1.42 (Weierstrass Approximation Theorem)**.** *Every continuous function on a closed interval* $[a, b]$ *can be* uniformly approximated *by a polynomial: For every* $\varepsilon > 0$ *there is a polynomial* P *so that*

$$\max_{t \in [a,b]} |P(t) - f(t)| \leq \varepsilon \tag{1.11}$$

Proof. First change the variable so that the interval becomes $[0, 1/2]$ (see previous page). Then extend f to an even function on $[-1/2, 1/2]$ (see previous page). Then extend f to a continuous 1-periodic function. By Corollary 1.35, the sequence

$$f_m(t) = \sum_{n=-m}^{m} \hat{F}_m(n) \hat{f}(n) e^{2\pi i n t}$$

(F_m = Fejer kernel) converges to f uniformly. Choose m so large that

$$|f_m(t) - f(t)| \leq \varepsilon/2$$

for all t. The function $f_m(t)$ is analytic, so by the course in analytic functions, the series

$$\sum_{k=0}^{\infty} \frac{f_m^{(k)}(0)}{k!} t^k$$

converges to $f_m(t)$, uniformly for $t \in [-1/2, 1/2]$. By taking N large enough we therefore have

$$|P_N(t) - f_m(t)| \leq \varepsilon/2 \text{ for } t \in [-1/2, 1/2],$$

where $P_N(t) = \sum_{k=0}^{N} \frac{f_m^{(k)}(0)}{k!} t^k$. This is a polynomial, and $|P_N(t) - f(t)| \leq \varepsilon$ for $t \in [-1/2, 1/2]$. Changing the variable t back to the original one we get a polynomial satisfying (1.11). $\square$

1.4.3 Solution of Differential Equations

There are many ways to use Fourier series to solve differential equations. We give only two examples.

Example 1.43. Solve the differential equation

$$y''(x) + \lambda y(x) = f(x), \quad 0 \leq x \leq 1, \tag{1.12}$$

with boundary conditions $y(0) = y(1)$, $y'(0) = y'(1)$. (These are *periodic* boundary conditions.) The function f is given, and $\lambda \in \mathbb{C}$ is a constant.

Solution. Extend y and f to all of $\mathbb{R}$ so that they become periodic, period 1. The equation + boundary conditions then give $y \in C^1(\mathbb{T})$. If we in addition assume that $f \in L^2(\mathbb{T})$, then (1.12) says that $y'' = f - \lambda y \in L^2(\mathbb{T})$ (i.e. f' is "absolutely continuous").

Assuming that $f \in C^1(\mathbb{T})$ and that f' is absolutely continuous we have by one of the homeworks

$$\widehat{(y'')}(n) = (2\pi i n)^2 \hat{y}(n),$$

so by transforming (1.12) we get

$$\begin{aligned} -4\pi^2 n^2 \hat{y}(n) + \lambda \hat{y}(n) &= \hat{f}(n), \quad n \in \mathbb{Z}, \text{ or} \\ (\lambda - 4\pi^2 n^2)\hat{y}(n) &= \hat{f}(n), \quad n \in \mathbb{Z}. \end{aligned} \tag{1.13}$$

Case A: $\lambda \neq 4\pi^2 n^2$ for all $n \in \mathbb{Z}$. Then (1.13) gives

$$\hat{y}(n) = \frac{\hat{f}(n)}{\lambda - 4\pi^2 n^2}.$$

The sequence on the right is in $\ell^1(\mathbb{Z})$, so $\hat{y}(n) \in \ell^1(\mathbb{Z})$. (i.e., $\sum |\hat{y}(n)| < \infty$). By Theorem 1.37,

$$y(t) = \sum_{n=-\infty}^{\infty} \underbrace{\frac{\hat{f}(n)}{\lambda - 4\pi^2 n^2}}_{=\hat{y}(n)} e^{2\pi i n t}, \quad t \in \mathbb{R}.$$

Thus, this is *the only possible* solution of (1.12).
How do we know that it is, indeed, a solution? Actually, we don't, but by working harder, and using the results from Chapter 0, it can be shown that $y \in C^1(\mathbb{T})$, and

$$y'(t) = \sum_{n=-\infty}^{\infty} 2\pi i n \hat{y}(n) e^{2\pi i n t},$$

where the sequence

$$2\pi i n \hat{y}(n) = \frac{2\pi i n \hat{y}(n)}{\lambda - 4\pi^2 n^2}$$

belongs to $\ell^1(\mathbb{Z})$ (both $\frac{2\pi i n}{\lambda - 4\pi^2 n^2}$ and $\hat{y}(n)$ belongs to $\ell^2(\mathbb{Z})$, and the product of two ℓ^2-sequences is an ℓ^1-sequence; see Analysis II). The sequence

$$(2\pi i n)^2 \hat{y}(n) = \frac{-4\pi^2 n^2}{\lambda - 4\pi^2 n^2} \hat{f}(n)$$

is an ℓ^2-sequence, and

$$\sum_{n=-\infty}^{\infty} \frac{-4\pi^2 n^2}{\lambda - 4\pi^2 n^2} \hat{f}(n) \to f''(t)$$

in the L^2-sense. Thus, $f \in C^1(\mathbb{T})$, f' is "absolutely continuous", and equation (1.12) *holds in the L^2-sense* (but not necessary everywhere). (It is called a mild solution of (1.12)).
Case B: $\lambda = 4\pi^2 k^2$ for some $k \in \mathbb{Z}$. Write

$$\lambda - 4\pi^2 n^2 = 4\pi^2 (k^2 - n^2) = 4\pi^2 (k - n)(k + n).$$

We get two additional *necessary conditions*: $\hat{f}(\pm k) = 0$. (If this condition is *not true* then the equation has *no solutions.*)
If $\hat{f}(k) = \hat{f}(-k) = 0$, then we get *infinitely many* solutions: Choose $\hat{y}(k)$ and $\hat{y}(-k)$ *arbitrarily*, and

$$\hat{y}(n) = \frac{\hat{f}(n)}{4\pi^2(k^2 - n^2)}, \quad n \neq \pm k.$$

Continue as in Case A.

Example 1.44. Same equation, but new boundary conditions: Interval is $[0, 1/2]$, and

$$y(0) = 0 = y(1/2).$$

Extend y and f to $[-1/2, 1/2]$ as odd functions

$$\begin{aligned} y(t) &= -y(-t), \quad -1/2 \leq t \leq 0 \\ f(t) &= -f(-t), \quad -1/2 \leq t \leq 0 \end{aligned}$$

and then make them periodic, period 1. Continue as before. This leads to a Fourier series with *odd* coefficients, which can be rewritten as a sinus-series.

Example 1.45. Same equation, interval $[0, 1/2]$, boundary conditions

$$y'(0) = 0 = y'(1/2).$$

Extend y and f to *even* functions, and continue as above. This leads to a solution with *even* coefficients $\hat{y}(n)$, and it can be rewritten as a cosinus-series.

1.4.4 Solution of Partial Differential Equations

See course on special functions.

Chapter 2

Fourier Integrals

2.1 L^1-Theory

Repetition: $\mathbb{R} = (-\infty, \infty)$,

$$f \in L^1(\mathbb{R}) \Leftrightarrow \int_{-\infty}^{\infty} |f(t)| dt < \infty \quad \text{(and } f \text{ measurable)}$$

$$f \in L^2(\mathbb{R}) \Leftrightarrow \int_{-\infty}^{\infty} |f(t)|^2 dt < \infty \quad \text{(and } f \text{ measurable)}$$

Definition 2.1. The Fourier transform of $f \in L^1(\mathbb{R})$ is given by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-2\pi i \omega t} f(t) dt, \ \omega \in \mathbb{R}$$

Comparison to chapter 1:

$$f \in L^1(\mathbb{T}) \Rightarrow \hat{f}(n) \text{ defined for all } n \in \mathbb{Z}$$

$$f \in L^1(\mathbb{R}) \Rightarrow \hat{f}(\omega) \text{ defined for all } \omega \in \mathbb{R}$$

Notation 2.2. *$C_0(\mathbb{R})$ = "continuous functions $f(t)$ satisfying $f(t) \to 0$ as $t \to \pm\infty$". The norm in C_0 is*

$$\|f\|_{C_0(\mathbb{R})} = \max_{t \in \mathbb{R}} |f(t)| \ (= \sup_{t \in \mathbb{R}} |f(t)|).$$

Compare this to $c_0(\mathbb{Z})$.

Theorem 2.3. *The Fourier transform $\mathcal{F}$ maps $L^1(\mathbb{R}) \to C_0(\mathbb{R})$, and it is a contraction, i.e., if $f \in L^1(\mathbb{R})$, then $\hat{f} \in C_0(\mathbb{R})$ and $\|\hat{f}\|_{C_0(\mathbb{R})} \le \|f\|_{L^1(\mathbb{R})}$, i.e.,* *i)* $\hat{f}$ *is continuous*

ii) $\hat{f}(\omega) \to 0$ *as* $\omega \to \pm\infty$

iii) $|\hat{f}(\omega)| \le \int_{-\infty}^{\infty} |f(t)|dt,\ \omega \in \mathbb{R}$.

Note: Part ii) is again the Riemann-Lesbesgue lemma.

PROOF. iii) "The same" as the proof of Theorem 1.4 i).
ii) "The same" as the proof of Theorem 1.4 ii), (replace n by ω, and prove this first in the special case where f is continuously differentiable and vanishes outside of some finite interval).
i) (The only "new" thing):

$$\begin{aligned}
|\hat{f}(\omega+h) - \hat{f}(\omega)| &= \Big|\int_{\mathbb{R}} \left(e^{-2\pi i(\omega+h)t} - e^{-2\pi i\omega t}\right) f(t)dt\Big| \\
&= \Big|\int_{\mathbb{R}} \left(e^{-2\pi iht} - 1\right) e^{-2\pi i\omega t} f(t)dt\Big| \\
&\overset{\triangle\text{-ineq.}}{\le} \int_{\mathbb{R}} |e^{-2\pi iht} - 1||f(t)|\, dt \to 0 \text{ as } h \to 0
\end{aligned}$$

(use Lesbesgue's dominated convergens Theorem, $e^{-2\pi iht} \to 1$ as $h \to 0$, and $|e^{-2\pi iht} - 1| \le 2$). $\square$

Question 2.4. *Is it possible to find a function* $f \in L^1(\mathbb{R})$ *whose Fourier transform is the same as the original function?*

Answer: Yes, there are many. See course on special functions. All functions which are eigenfunctions with eigenvalue 1 are mapped onto themselves.

Special case:

Example 2.5. If $h_0(t) = e^{-\pi t^2},\ t \in \mathbb{R}$, then $\hat{h}_0(\omega) = e^{-\pi\omega^2},\ \omega \in \mathbb{R}$

PROOF. See course on special functions.

Note: After rescaling, this becomes the normal (Gaussian) distribution function. This is no coincidence!

Another useful Fourier transform is:

Example 2.6. The **Fejer kernel** in $L^1(\mathbb{R})$ is

$$F(t) = \left(\frac{\sin(\pi t)}{\pi t}\right)^2.$$

The transform of this function is

$$\hat{F}(\omega) = \begin{cases} 1 - |\omega| & , \quad |\omega| \le 1, \\ 0 & , \quad \text{otherwise.} \end{cases}$$

PROOF. Direct computation. (Compare this to the <u>periodic</u> Fejer kernel on page 23.)

Theorem 2.7 (Basic rules). *Let* $f \in L^1(\mathbb{R}), \tau, \lambda \in \mathbb{R}$

$$\begin{array}{lll}
a) & g(t) = f(t-\tau) & \Rightarrow \hat{g}(\omega) = e^{-2\pi i\omega\tau}\hat{f}(\omega) \\
b) & g(t) = e^{2\pi i\tau t}f(t) & \Rightarrow \hat{g}(\omega) = \hat{f}(\omega - \tau) \\
c) & g(t) = f(-t) & \Rightarrow \hat{g}(\omega) = \hat{f}(-\omega) \\
d) & g(t) = \overline{f(t)} & \Rightarrow \hat{g}(\omega) = \overline{\hat{f}(-\omega)} \\
e) & g(t) = \lambda f(\lambda t) & \Rightarrow \hat{g}(\omega) = \hat{f}(\frac{\omega}{\lambda}) \quad (\lambda > 0) \\
f) & g \in L^1 \text{ and } h = f * g & \Rightarrow \hat{h}(\omega) = \hat{f}(\omega)\hat{g}(\omega) \\
g) & \left.\begin{array}{l} g(t) = -2\pi i t f(t) \\ \quad\quad \text{and } g \in L^1 \end{array}\right\} & \Rightarrow \left\{\begin{array}{l} \hat{f} \in C^1(\mathbb{R}), \quad \text{and} \\ \hat{f}'(\omega) = \hat{g}(\omega) \end{array}\right. \\
h) & \left.\begin{array}{l} f \text{ is ``absolutely continuous``} \\ \text{and } f' = g \in L^1(\mathbb{R}) \end{array}\right\} & \Rightarrow \hat{g}(\omega) = 2\pi i\omega\hat{f}(\omega).
\end{array}$$

PROOF. (a)-(e): Straightforward computation.
(g)-(h): Homework(?) (or later).
The *formal inversion* for Fourier integrals is

$$\begin{aligned}
\hat{f}(\omega) &= \int_{-\infty}^{\infty} e^{-2\pi i\omega t} f(t)dt \\
f(t) &\stackrel{?}{=} \int_{-\infty}^{\infty} e^{2\pi i\omega t}\hat{f}(\omega)d\omega
\end{aligned}$$

This is true in "some cases" in "some sense". To prove this we need some additional machinery.

Definition 2.8. Let $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$, where $1 \le p \le \infty$. Then we define

$$(f * g)(t) = \int_{\mathbb{R}} f(t-s)g(s)ds$$

for all those $t \in \mathbb{R}$ for which this integral converges absolutely, i.e.,

$$\int_{\mathbb{R}} |f(t-s)g(s)|ds < \infty.$$

Lemma 2.9. *With f and p as above, $f * g$ is defined a.e., $f * g \in L^p(\mathbb{R})$, and*

$$\|f * g\|_{L^p(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})} \|g\|_{L^p(\mathbb{R})}.$$

If $p = \infty$, then $f * g$ is defined everywhere and uniformly continuous.

Conclusion 2.10. *If $\|f\|_{L^1(\mathbb{R})} \leq 1$, then the mapping $g \mapsto f * g$ is a contraction from $L^p(\mathbb{R})$ to itself (same as in periodic case).*

Proof. $p = 1$: "same" proof as we gave on page 21.
$p = \infty$: Boundedness of $f * g$ easy. To prove continuity we approximate f by a function with compact support and show that $\|f(t) - f(t+h)\|_{L^1} \to 0$ as $h \to 0$.
$p \neq 1, \infty$: Significantly harder, case $p = 2$ found in Gasquet.

Notation 2.11. $\mathcal{BUC}(\mathbb{R})$ = *"all bounded and continuous functions on $\mathbb{R}$". We use the norm*

$$\|f\|_{\mathcal{BUC}(\mathbb{R})} = \sup_{t \in \mathbb{R}} |f(t)|.$$

Theorem 2.12 ("Approximate identity")**.** *Let $k \in L^1(\mathbb{R})$, $\hat{k}(0) = \int_{-\infty}^{\infty} k(t)dt = 1$, and define*

$$k_\lambda(t) = \lambda k(\lambda t), \quad t \in \mathbb{R},\ \lambda > 0.$$

If f belongs to one of the function spaces

a) $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$ (note: $p \neq \infty$),

b) $f \in C_0(\mathbb{R})$,

c) $f \in \mathcal{BUC}(\mathbb{R})$,

then $k_\lambda * f$ belongs to the same function space, and

$$k_\lambda * f \to f \quad \text{as } \lambda \to \infty$$

in the norm of the same function space, i.e.,

$$\|k_\lambda * f - f\|_{L^p(\mathbb{R})} \to 0 \text{ as } \lambda \to \infty \text{ if } f \in L^p(\mathbb{R})$$

$$\sup_{t \in \mathbb{R}} |(k_\lambda * f)(t) - f(t)| \to 0 \text{ as } \lambda \to \infty \begin{cases} \text{if} & f \in \mathcal{BUC}(\mathbb{R}), \\ \text{or} & f \in C_0(\mathbb{R}). \end{cases}$$

It also conveges a.e. if we assume that $\int_0^\infty (\sup_{s \geq |t|} |k(s)|)dt < \infty$.

PROOF. "The same" as the proofs of Theorems 1.29, 1.32 and 1.33. That is, the *computations* stay the same, but the bounds of integration change ($\mathbb{T} \to \mathbb{R}$), and the motivations change a little (but not much). $\square$

Example 2.13 (Standard choices of k)**.**

i) The *Gaussian kernel*

$$k(t) = e^{-\pi t^2}, \ \hat{k}(\omega) = e^{-\pi\omega^2}.$$

This function is C^∞ and nonnegative, so

$$\|k\|_{L^1} = \int_{\mathbb{R}} |k(t)|dt = \int_{\mathbb{R}} k(t)dt = \hat{k}(0) = 1.$$

ii) The *Fejer kernel*

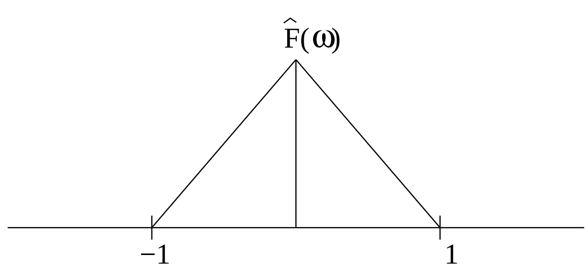
$$F(t) = \frac{\sin(\pi t)^2}{(\pi t)^2}.$$

It has the same advantages, and in addition

$$\hat{F}(\omega) = 0 \text{ for } |\omega| > 1.$$

The transform is a triangle:

$$\hat{F}(\omega) = \begin{cases} 1 - |\omega|, & |\omega| \le 1 \\ 0, & |\omega| > 1 \end{cases}$$



iii) $k(t) = e^{-2|t|}$ (or a rescaled version of this function. Here

$$\hat{k}(\omega) = \frac{1}{1 + (\pi\omega)^2}, \ \omega \in \mathbb{R}.$$

Same advantages (except C^∞)).

Comment 2.14. *According to Theorem 2.7 (e), $\hat{k}_\lambda(\omega) \to \hat{k}(0) = 1$ as $\lambda \to \infty$, for all $\omega \in \mathbb{R}$. All the kernels above are "low pass filters" (non causal). It is possible to use "one-sided" ("causal") filters instead (i.e., $k(t) = 0$ for $t < 0$). Substituting these into Theorem 2.12 we get "approximate identities", which "converge to a δ-distribution". Details later.*

Theorem 2.15. *If both $f \in L^1(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$, then the inversion formula*

$$f(t) = \int_{-\infty}^{\infty} e^{2\pi i\omega t}\hat{f}(\omega)d\omega \tag{2.1}$$

is valid for almost all $t \in \mathbb{R}$. By redefining f on a set of measure zero we can make it hold for all $t \in \mathbb{R}$ *(the right hand side of* (2.1) *is continuous).*

Proof. We approximate $\int_{\mathbb{R}} e^{2\pi i\omega t}\hat{f}(\omega)d\omega$ by

$$\begin{aligned}
&\textstyle\int_{\mathbb{R}} e^{2\pi i\omega t}e^{-\varepsilon^2\pi\omega^2}\hat{f}(\omega)d\omega && \text{(where } \varepsilon > 0 \text{ is small)}\\
&= \textstyle\int_{\mathbb{R}} e^{2\pi i\omega t-\varepsilon^2\pi\omega^2}\int_{\mathbb{R}} e^{-2\pi i\omega s}f(s)dsd\omega && \text{(Fubini)}\\
&= \textstyle\int_{s\in\mathbb{R}} f(s)\underbrace{\int_{\omega\in\mathbb{R}} e^{-2\pi i\omega(s-t)}\underbrace{e^{-\varepsilon^2\pi\omega^2}}_{k(\varepsilon\omega^2)}d\omega ds}_{(\star)} && \text{(Ex. 2.13 last page)}
\end{aligned}$$

($\star$) The Fourier transform of $k(\varepsilon\omega^2)$ at the point $s - t$. By Theorem 2.7 (e) this is equal to

$$= \frac{1}{\varepsilon}\hat{k}(\frac{s-t}{\varepsilon}) = \frac{1}{\varepsilon}\hat{k}(\frac{t-s}{\varepsilon})$$

(since $\hat{k}(\omega) = e^{-\pi\omega^2}$ is even).

The whole thing is

$$\int_{s\in\mathbb{R}} f(s)\frac{1}{\varepsilon}k\left(\frac{t-s}{\varepsilon}\right)ds = (f * k_{\frac{1}{\varepsilon}})(t) \to f \in L^1(\mathbb{R})$$

as $\varepsilon \to 0^+$ according to Theorem 2.12. Thus, for almost all $t \in \mathbb{R}$,

$$f(t) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}} e^{2\pi i\omega t}e^{-\varepsilon^2\pi\omega^2}\hat{f}(\omega)d\omega.$$

On the other hand, by the Lebesgue dominated convergence theorem, since

$$|e^{2\pi i\omega t}e^{-\varepsilon^2\pi\omega^2}\hat{f}(\omega)| \le |\hat{f}(\omega)| \in L^1(\mathbb{R}),$$

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}} e^{2\pi i\omega t}e^{-\varepsilon^2\pi\omega^2}\hat{f}(\omega)d\omega = \int_{\mathbb{R}} e^{2\pi i\omega t}\hat{f}(\omega)d\omega.$$

Thus, (2.1) holds a.e. The proof of the fact that

$$\int_{\mathbb{R}} e^{2\pi i\omega t}\hat{f}(\omega)d\omega \in C_0(\mathbb{R})$$

is the same as the proof of Theorem 2.3 (replace t by $-t$). $\square$

The *same proof* also gives us the following "approximate inversion formula":

Theorem 2.16. *Suppose that $k \in L^1(\mathbb{R})$, $\hat{k} \in L^1(\mathbb{R})$, and that*

$$\hat{k}(0) = \int_{\mathbb{R}} k(t)dt = 1.$$

If f belongs to one of the function spaces

a) $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$

b) $f \in C_0(\mathbb{R})$

c) $f \in \mathcal{BUC}(\mathbb{R})$

then

$$\int_{\mathbb{R}} e^{2\pi i\omega t}\hat{k}(\varepsilon\omega)\hat{f}(\omega)d\omega \to f(t)$$

in the norm of the given space (i.e., in L^p-norm, or in the sup-norm), and also a.e. if $\int_0^\infty (\sup_{s\geq|t|}|k(s)|)dt < \infty$.

Proof. Almost the same as the proof given above. If k is not even, then we end up with a convolution with the function $k_\varepsilon(t) = \frac{1}{\varepsilon}k(-t/\varepsilon)$ instead, but we can still apply Theorem 2.12 with $k(t)$ replaced by $k(-t)$. $\square$

Corollary 2.17. *The inversion in Theorem 2.15 can be interpreted as follows: If $f \in L^1(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$, then,*

$$\hat{\hat{f}}(t) = f(-t) \text{ a.e.}$$

Here $\hat{\hat{f}}(t)$ = the Fourier transform of $\hat{f}$ evaluated at the point t.

Proof. By Theorem 2.15,

$$f(t) = \underbrace{\int_{\mathbb{R}} e^{-2\pi i(-t)\omega}\hat{f}(\omega)d\omega}_{\text{Fourier transform of } \hat{f} \text{ at the point } (-t)} \quad \text{a.e.}$$

Corollary 2.18. *$\hat{\hat{\hat{\hat{f}}}}(t) = f(t)$ (If we repeat the Fourier transform 4 times, then we get back the original function). (True at least if $f \in L^1(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$.)*

As a prelude (=preludium) to the L^2-theory we still prove some additional results:

Lemma 2.19. *Let $f \in L^1(\mathbb{R})$ and $g \in L^1(\mathbb{R})$. Then*

$$\int_{\mathbb{R}} f(t)\hat{g}(t)dt = \int_{\mathbb{R}} \hat{f}(s)g(s)ds$$

Proof.

$$\begin{aligned}
\int_{\mathbb{R}} f(t)\hat{g}(t)dt &= \int_{t\in\mathbb{R}} f(t) \int_{s\in\mathbb{R}} e^{-2\pi its} g(s)dsdt \quad \text{(Fubini)} \\
&= \int_{s\in\mathbb{R}} \left(\int_{t\in\mathbb{R}} f(t)e^{-2\pi ist}dt \right) g(s)ds \\
&= \int_{s\in\mathbb{R}} \hat{f}(s)g(s)ds. \quad \square
\end{aligned}$$

Theorem 2.20. *Let $f \in L^1(\mathbb{R})$, $h \in L^1(\mathbb{R})$ and $\hat{h} \in L^1(\mathbb{R})$. Then*

$$\int_{\mathbb{R}} f(t)\overline{h(t)}dt = \int_{\mathbb{R}} \hat{f}(\omega)\overline{\hat{h}(\omega)}d\omega. \tag{2.2}$$

Specifically, if $f = h$, then ($f \in L^2(\mathbb{R})$ and)

$$\|f\|_{L^2(\mathbb{R})} = \|\hat{f}\|_{L^2(\mathbb{R})}. \tag{2.3}$$

Proof. Since $h(t) = \int_{\omega\in\mathbb{R}} e^{2\pi i\omega t}\hat{h}(\omega)d\omega$ we have

$$\begin{aligned}
\int_{\mathbb{R}} f(t)\overline{h(t)}dt &= \int_{t\in\mathbb{R}} f(t) \int_{\omega\in\mathbb{R}} e^{-2\pi i\omega t}\overline{\hat{h}(\omega)}d\omega dt \quad \text{(Fubini)} \\
&= \int_{s\in\mathbb{R}} \left(\int_{t\in\mathbb{R}} f(t)e^{-2\pi ist}dt \right) \overline{\hat{h}(\omega)}d\omega \\
&= \int_{\mathbb{R}} \hat{f}(\omega)\overline{\hat{h}(\omega)}d\omega. \quad \square
\end{aligned}$$

2.2 Rapidly Decaying Test Functions

("Snabbt avtagande testfunktioner").

Definition 2.21. $\mathcal{S}$ = the set of functions f with the following properties

i) $f \in C^\infty(\mathbb{R})$ (infinitely many times differentiable)

ii) $t^k f^{(n)}(t) \to 0$ as $t \to \pm\infty$ and this is true for *all*

$$k, n \in \mathbb{Z}_+ = \{0, 1, 2, 3, \dots\}.$$

Thus: Every derivative of $f \to 0$ at infinity faster than any negative power of t.
Note: There is *no natural norm* in this space (it is not a "Banach" space). However, it is possible to find a complete, shift-invariant metric on this space (it is a Frechet space).

Example 2.22. $f(t) = P(t)e^{-\pi t^2} \in \mathcal{S}$ for every *polynomial* $P(t)$. For example, the *Hermite functions* are of this type (see course in special functions).

Comment 2.23. *Gripenberg denotes* $\mathcal{S}$ *by* $C_{\downarrow}^{\infty}(\mathbb{R})$. *The functions in* $\mathcal{S}$ *are called* rapidly decaying test functions.

The main result of this section is

Theorem 2.24. $f \in \mathcal{S} \iff \hat{f} \in \mathcal{S}$

That is, both the Fourier transform and the inverse Fourier transform maps this class of functions onto itself. Before proving this we prove the following

Lemma 2.25. *We can replace condition (ii) in the definition of the class* $\mathcal{S}$ *by one of the conditions*

iii) $\int_{\mathbb{R}} |t^k f^{(n)}(t)| dt < \infty, \quad k, n \in \mathbb{Z}_+$ *or*

iv) $|\left(\frac{d}{dt}\right)^n t^k f(t)| \to 0$ *as* $t \to \pm\infty$, $k, n \in \mathbb{Z}_+$

without changing the class of functions $\mathcal{S}$.

Proof. If ii) holds, then for all $k, n \in \mathbb{Z}_+$,

$$\sup_{t \in \mathbb{R}} |(1 + t^2) t^k f^{(n)}(t)| < \infty$$

(replace k by $k + 2$ in ii). Thus, for some constant M,

$$|t^k f^{(n)}(t)| \le \frac{M}{1 + t^2} \implies \int_{\mathbb{R}} |t^k f^{(n)}(t)| dt < \infty.$$

Conversely, if iii) holds, then we can define $g(t) = t^{k+1} f^{(n)}(t)$ and get

$$g'(t) = \underbrace{(k+1) t^k f^{(n)}(t)}_{\in L^1} + \underbrace{t^{k+1} f^{(n+1)}(t)}_{\in L^1},$$

so $g' \in L^1(\mathbb{R})$, i.e.,

$$\int_{-\infty}^{\infty} |g'(t)| dt < \infty.$$

This implies

$$\begin{aligned} |g(t)| &\leq |g(0) + \int_0^t g'(s) ds| \\ &\leq |g(0)| + \int_0^t |g'(s)| ds \\ &\leq |g(0)| + \int_{-\infty}^{\infty} |g'(s)| ds = |g(0)| + \|g'\|_{L^1}, \end{aligned}$$

so g is bounded. Thus,

$$t^k f^{(n)}(t) = \frac{1}{t} g(t) \to 0 \text{ as } t \to \pm\infty.$$

The proof that *ii*) $\iff$ *iv*) is left as a homework. $\square$

Proof of Theorem 2.24. By Theorem 2.7, the Fourier transform of

$$(-2\pi i t)^k f^{(n)}(t) \text{ is } \left(\frac{d}{d\omega}\right)^k (2\pi i \omega)^n \hat{f}(\omega).$$

Therefore, if $f \in \mathcal{S}$, then condition iii) on the last page holds, and by Theorem 2.3, $\hat{f}$ satisfies the condition iv) on the last page. Thus $\hat{f} \in \mathcal{S}$. The same argument with $e^{-2\pi i \omega t}$ replaced by $e^{+2\pi i \omega t}$ shows that if $\hat{f} \in \mathcal{S}$, then the Fourier inverse transform of $\hat{f}$ (which is f) belongs to $\mathcal{S}$. $\square$

Note: Theorem 2.24 is the *basis* for the theory of Fourier transforms of *distributions*. More on this later.

2.3 L^2-Theory for Fourier Integrals

As we saw earlier in Lemma 1.10, $L^2(\mathbb{T}) \subset L^1(\mathbb{T})$. However, it is not true that $L^2(\mathbb{R}) \subset L^1(\mathbb{R})$. Counter example:

$$f(t) = \frac{1}{\sqrt{1+t^2}} \begin{cases} \in L^2(\mathbb{R}) \\ \notin L^1(\mathbb{R}) \\ \in C^\infty(\mathbb{R}) \end{cases}$$

(too large at ∞).

So how on earth should we define $\hat{f}(\omega)$ for $f \in L^2(\mathbb{R})$, if the integral

$$\int_{\mathbb{R}} e^{-2\pi i n t} f(t) dt$$

does not converge?

Recall: Lebesgue integral converges $\iff$ converges absolutely $\iff$

$$\int |e^{-2\pi int} f(t)| dt < \infty \iff f \in L^1(\mathbb{R}).$$

We are saved by Theorem 2.20. Notice, in particular, condition (2.3) in that theorem!

Definition 2.26 (L^2-Fourier transform)**.**

i) Approximate $f \in L^2(\mathbb{R})$ by a sequence $f_n \in \mathcal{S}$ which converges to f in $L^2(\mathbb{R})$. We do this e.g. by "smoothing" and "cutting" ("utjämning" och "klippning"): Let $k(t) = e^{-\pi t^2}$, define

$$k_n(t) = nk(nt), \quad \text{and}$$

$$f_n(t) = \underbrace{\underbrace{k\left(\frac{t}{n}\right)}_{\star} \underbrace{(k_n * f)(t)}_{\star\star}}_{\text{the product belongs to } \mathcal{S}}$$

($\star$) this tends to zero faster than any polynomial as $t \to \infty$.

($\star\star$) "smoothing" by an approximate identity, belongs to C^∞ and is bounded.

By Theorem 2.12 $k_n * f \to f$ in L^2 as $n \to \infty$. The functions $k\left(\frac{t}{n}\right)$ tend to $k(0) = 1$ at every point t as $n \to \infty$, and they are uniformly bounded by 1. By using the appropriate version of the Lesbesgue convergence we let $f_n \to f$ in $L^2(\mathbb{R})$ as $n \to \infty$.

ii) Since f_n converges in L^2, also $\hat{f}_n$ must converge to something in L^2. More about this in "Analysis II". This follows from Theorem 2.20. ($f_n \to f \Rightarrow f_n$ Cauchy sequence $\Rightarrow \hat{f}_n$ Cauchy seqence $\Rightarrow \hat{f}_n$ converges.)

iii) Call the limit to which f_n converges "The Fourier transform of f", and denote it $\hat{f}$.

Definition 2.27 (Inverse Fourier transform)**.** We do exactly as above, but replace $e^{-2\pi i\omega t}$ by $e^{+2\pi i\omega t}$.

Final conclusion:

Theorem 2.28. *The "extended" Fourier transform which we have defined above has the following properties: It maps $L^2(\mathbb{R})$ one-to-one onto $L^2(\mathbb{R})$, and if $\hat{f}$ is the Fourier transform of f, then f is the inverse Fourier transform of $\hat{f}$. Moreover, all norms, distances and inner products are preserved.*

Explanation:

i) "Normes preserved" means

$$\int_{\mathbb{R}} |f(t)|^2 dt = \int_{\mathbb{R}} |\hat{f}(\omega)|^2 d\omega,$$

or equivalently, $\|f\|_{L^2(\mathbb{R})} = \|\hat{f}\|_{L^2(\mathbb{R})}$.

ii) "Distances preserved" means

$$\|f - g\|_{L^2(\mathbb{R})} = \|\hat{f} - \hat{g}\|_{L^2(\mathbb{R})}$$

(apply i) with f replaced by $f - g$)

iii) "Inner product preserved" means

$$\int_{\mathbb{R}} f(t)\overline{g(t)} dt = \int_{\mathbb{R}} \hat{f}(\omega)\overline{\hat{g}(\omega)} d\omega,$$

which is often written as

$$\langle f, g\rangle_{L^2(\mathbb{R})} = \langle \hat{f}, \hat{g}\rangle_{L^2(\mathbb{R})}.$$

This was theory. How to do in practice?
One answer: We saw earlier that if $[a, b]$ is a finite interval, and if $f \in L^2[a, b] \Rightarrow f \in L^1[a, b]$, so for each $T > 0$, the integral

$$\hat{f}_T(\omega) = \int_{-T}^{T} e^{-2\pi i \omega t} f(t) dt$$

is defined for all $\omega \in \mathbb{R}$. We can try to let $T \to \infty$, and see what happens. (This resembles the theory for the inversion formula for the periodical L^2-theory.)

Theorem 2.29. *Suppose that $f \in L^2(\mathbb{R})$. Then*

$$\lim_{T\to\infty} \int_{-T}^{T} e^{-2\pi i \omega t} f(t) dt = \hat{f}(\omega)$$

in the L^2-sense as $T \to \infty$, and likewise

$$\lim_{T\to\infty} \int_{-T}^{T} e^{2\pi i \omega t} \hat{f}(\omega) d\omega = f(t)$$

in the L^2-sense.

Proof. Much too hard to be presented here. Another possibility: Use the Fejer kernel or the Gaussian kernel, or any other kernel, and define

$$\begin{aligned} \hat{f}(\omega) &= \lim_{n\to\infty} \int_{\mathbb{R}} e^{-2\pi i \omega t} k\left(\tfrac{t}{n}\right) f(t)dt, \\ f(t) &= \lim_{n\to\infty} \int_{\mathbb{R}} e^{+2\pi i \omega t} \hat{k}\left(\tfrac{\omega}{n}\right) \hat{f}(\omega)d\omega. \end{aligned}$$

We typically have the same type of convergence as we had in the Fourier inversion formula in the periodic case. (This is a well-developed part of mathematics, with lots of results available.) See Gripenberg's compendium for some additional results.

2.4 An Inversion Theorem

From time to time we need a better (= more useful) *inversion* theorem for the Fourier transform, so let us prove one here:

Theorem 2.30. *Suppose that $f \in L^1(\mathbb{R}) + L^2(\mathbb{R})$ (i.e., $f = f_1 + f_2$, where $f_1 \in L^1(\mathbb{R})$ and $f_2 \in L^2(\mathbb{R})$). Let $t_0 \in \mathbb{R}$, and suppose that*

$$\int_{t_0-1}^{t_0+1} \left| \frac{f(t) - f(t_0)}{t - t_0} \right| dt < \infty. \tag{2.4}$$

Then

$$f(t_0) = \lim_{\substack{S\to\infty \\ T\to\infty}} \int_{-S}^{T} e^{2\pi i \omega t_0} \hat{f}(\omega)d\omega, \tag{2.5}$$

where $\hat{f}(\omega) = \hat{f}_1(\omega) + \hat{f}_2(\omega)$.

Comment: Condition (2.4) is true if, for example, f is *differentiable* at the point t_0.

Proof. Step 1. First replace $f(t)$ by $g(t) = f(t + t_0)$. Then

$$\hat{g}(\omega) = e^{2\pi i \omega t_0} \hat{f}(\omega),$$

and (2.5) becomes

$$g(0) = \lim_{\substack{S\to\infty \\ T\to\infty}} \int_{-S}^{T} \hat{g}(\omega)d\omega,$$

and (2.4) becomes

$$\int_{-1}^{1} \left| \frac{g(t - t_0) - g(0)}{t - t_0} \right| dt < \infty.$$

Thus, it suffices to prove the case where $\boxed{t_0 = 0}$.

Step 2: We know that the theorem is true if $g(t) = e^{-\pi t^2}$ (See Example 2.5 and Theorem 2.15). Replace $g(t)$ by

$$h(t) = g(t) - g(0)e^{-\pi t^2}.$$

Then h satisfies all the assumptions which g does, and in addition, $h(0) = 0$. Thus it suffices to prove the case where both $(\star)\boxed{t_0 = 0}$ and $\boxed{f(0) = 0}$.
For simplicity we write f instead of h but assume $(\star)$. Then (2.4) and (2.5) simplify:

$$\int_{-1}^{1} \left|\frac{f(t)}{t}\right| dt \;<\; \infty, \tag{2.6}$$

$$\lim_{\substack{S\to\infty \\ T\to\infty}} \int_{-S}^{T} \hat{f}(\omega)d\omega \;=\; 0. \tag{2.7}$$

Step 3: If $f \in L^1(\mathbb{R})$, then we argue as follows. Define

$$g(t) = \frac{f(t)}{-2\pi i t}.$$

Then $g \in L^1(\mathbb{R})$. By Fubini's theorem,

$$\begin{aligned}
\int_{-S}^{T} \hat{f}(\omega)d\omega &= \int_{-S}^{T}\int_{-\infty}^{\infty} e^{-2\pi i\omega t} f(t)dt d\omega \\
&= \int_{-\infty}^{\infty}\int_{-S}^{T} e^{-2\pi i\omega t} d\omega f(t)dt \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{-2\pi i t} e^{-2\pi i\omega t}\right]_{-S}^{T} f(t)dt \\
&= \int_{-\infty}^{\infty} \left[e^{-2\pi i T t} - e^{-2\pi i(-S)t}\right] \frac{f(t)}{-2\pi i t} dt \\
&= \hat{g}(T) - \hat{g}(-S),
\end{aligned}$$

and this tends to zero as $T \to \infty$ and $S \to \infty$ (see Theorem 2.3). This proves (2.7).

Step 4: If instead $f \in L^2(\mathbb{R})$, then we use Parseval's identity

$$\int_{-\infty}^{\infty} f(t)\overline{h(t)}dt = \int_{-\infty}^{\infty} \hat{f}(\omega)\overline{\hat{h}(\omega)}d\omega$$

in a clever way: Choose

$$\hat{h}(\omega) = \begin{cases} 1, & -S \le t \le T, \\ 0, & \text{otherwise.} \end{cases}$$

Then the inverse Fourier transform $h(t)$ of $\hat{h}$ is

$$\begin{aligned} h(t) &= \int_{-S}^{T} e^{2\pi i\omega t} d\omega \\ &= \left[\frac{1}{2\pi it} e^{2\pi i\omega t}\right]_{-S}^{T} = \frac{1}{2\pi it}\left[e^{2\pi iTt} - e^{2\pi i(-S)t}\right] \end{aligned}$$

so Parseval's identity gives

$$\begin{aligned} \int_{-S}^{T} \hat{f}(\omega)d\omega &= \int_{-\infty}^{\infty} f(t)\frac{1}{-2\pi it}\left[e^{-2\pi iTt} - e^{-2\pi i(-S)t}\right] dt \\ &= \text{(with } g(t) \text{ as in Step 3)} \\ &= \int_{-\infty}^{\infty} \left[e^{-2\pi iTt} - e^{-2\pi i(S)t}\right] g(t)dt \\ &= \hat{g}(T) - \hat{g}(-S) \to 0 \text{ as } \begin{cases} T \to \infty, \\ S \to \infty. \end{cases} \end{aligned}$$

Step 5: If $f = f_1 + f_2$, where $f_1 \in L^1(\mathbb{R})$ and $f_2 \in L^2(\mathbb{R})$, then we apply Step 3 to f_1 and Step 4 to f_2, and get in both cases (2.7) with f replaced by f_1 and f_2. □

Note: This means that in "*most cases*" where f is continuous at t_0 we have

$$f(t_0) = \lim_{\substack{S\to\infty \\ T\to\infty}} \int_{-S}^{T} e^{2\pi i\omega t_0}\hat{f}(\omega)d\omega.$$

(continuous functions which do *not* satisfy (2.4) do exist, but they are difficult to find.) In some cases we can even use the inversion formula at a point where f is *discontinuous*.

Theorem 2.31. *Suppose that $f \in L^1(\mathbb{R}) + L^2(\mathbb{R})$. Let $t_0 \in \mathbb{R}$, and suppose that the two limits*

$$\begin{aligned} f(t_0+) &= \lim_{t\downarrow t_0} f(t) \\ f(t_0-) &= \lim_{t\uparrow t_0} f(t) \end{aligned}$$

exist, and that

$$\begin{aligned} \int_{t_0}^{t_0+1} \left|\frac{f(t)-f(t_0+)}{t-t_0}\right| dt &< \infty, \\ \int_{t_0-1}^{t_0} \left|\frac{f(t)-f(t_0-)}{t-t_0}\right| dt &< \infty. \end{aligned}$$

Then

$$\lim_{T\to\infty}\int_{-T}^{T} e^{2\pi i\omega t_0}\hat{f}(\omega)d\omega = \frac{1}{2}[f(t_0+)+f(t_0-)].$$

Note: Here we integrate $\int_{-T}^{T}$, not $\int_{-S}^{T}$, and the result is the *average* of the right and left hand limits.

Proof. As in the proof of Theorem 2.30 we may assume that

Step 1: $\boxed{t_0 = 0}$, (see Step 1 of that proof)

Step 2: $\boxed{f(t_0+)+f(t_0-)=0}$, (see Step 2 of that proof).

Step 3: The claim is true in the special case where

$$g(t) = \begin{cases} e^{-t}, & t>0,\\ -e^{t}, & t<0,\end{cases}$$

because $g(0+)=1,\ g(0-)=-1,\ g(0+)+g(0-)=0$, and

$$\int_{-T}^{T}\hat{g}(\omega)d\omega = 0 \quad \text{for all } T,$$

since f is odd $\implies \hat{g}$ is odd.

Step 4: Define $h(t)=f(t)-f(0+)\cdot g(t)$, where g is the function in Step 3. Then

$$\begin{aligned} h(0+) &= f(0+)-f(0+)=0 \quad \text{and}\\ h(0-) &= f(0-)-f(0+)(-1)=0, \quad \text{so}\end{aligned}$$

h is continuous. Now apply Theorem 2.30 to h. It gives

$$0 = h(0) = \lim_{T\to\infty}\int_{-T}^{T}\hat{h}(\omega)d\omega.$$

Since also

$$0 = f(0+)[g(0+)+g(0-)] = \lim_{T\to\infty}\int_{-T}^{T}\hat{g}(\omega)d\omega,$$

we therefore get

$$0 = f(0+)+f(0-) = \lim_{T\to\infty}\int_{-T}^{T}[\hat{h}(\omega)+\hat{g}(\omega)]d\omega = \lim_{T\to\infty}\int_{-T}^{T}\hat{f}(\omega)d\omega. \quad \square$$

Comment 2.32. *Theorems 2.30 and 2.31 also remain true if we replace*

$$\lim_{T\to\infty}\int_{-T}^{T} e^{2\pi i\omega t}\hat{f}(\omega)d\omega$$

by

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} e^{2\pi i\omega t}e^{-\pi(\varepsilon\omega)^2}\hat{f}(\omega)d\omega \tag{2.8}$$

(and other similar "summability" formulas). Compare this to Theorem 2.16. In the case of Theorem 2.31 it is important that the "cutoff kernel" ($= e^{-\pi(\varepsilon\omega)^2}$ in (2.8)) is *even*.

2.5 Applications

2.5.1 The Poisson Summation Formula

Suppose that $f \in L^1(\mathbb{R}) \cap C(\mathbb{R})$, that $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$ (i.e., $\hat{f} \in \ell^1(\mathbb{Z})$), and that $\sum_{n=-\infty}^{\infty} f(t+n)$ converges uniformly for all t in some interval $(-\delta, \delta)$. Then

$$\boxed{\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)} \tag{2.9}$$

Note: The uniform convergence of $\sum f(t+n)$ can be difficult to check. One possible way out is: If we define

$$\varepsilon_n = \sup_{-\delta<t<\delta} |f(t+n)|,$$

and if $\sum_{n=-\infty}^{\infty} \varepsilon_n < \infty$, then $\sum_{n=-\infty}^{\infty} f(t+n)$ converges (even absolutely), and the convergence is uniform in $(-\delta, \delta)$. The proof is roughly the same as what we did on page 29.

Proof of (2.9). We first construct a periodic function $g \in L^1(\mathbb{T})$ with the Fourier coefficients $\hat{f}(n)$:

$$\begin{aligned}
\hat{f}(n) &= \int_{-\infty}^{\infty} e^{-2\pi int} f(t)dt \\
&= \sum_{k=-\infty}^{\infty} \int_k^{k+1} e^{-2\pi int} f(t)dt \\
&\overset{t=k+s}{=} \sum_{k=-\infty}^{\infty} \int_0^1 e^{-2\pi ins} f(s+k)ds \\
&\overset{\text{Thm } 0.14}{=} \int_0^1 e^{-2\pi ins} \left(\sum_{k=-\infty}^{\infty} f(s+k) \right) ds \\
&= \hat{g}(n), \quad \text{where } g(t) = \sum_{n=-\infty}^{\infty} f(t+n).
\end{aligned}$$

(For this part of the proof it is enough to have $f \in L^1(\mathbb{R})$. The other conditions are needed later.)

So we have $\hat{g}(n) = \hat{f}(n)$. By the inversion formula for the periodic Fourier transform:

$$g(0) = \sum_{n=-\infty}^{\infty} e^{2\pi in0} \hat{g}(n) = \sum_{n=-\infty}^{\infty} \hat{g}(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n),$$

provided (=förutsatt) that we are allowed to use the Fourier inversion formula. This is allowed if $g \in C[-\delta, \delta]$ and $\hat{g} \in \ell^1(\mathbb{Z})$ (Theorem 1.37). This was part of our assumption.

In addition we need to know that the formula

$$g(t) = \sum_{n=-\infty}^{\infty} f(t+n)$$

holds at the point $t = 0$ (almost everywhere is no good, we need it in exactly this point). This is OK if $\sum_{n=-\infty}^{\infty} f(t+n)$ converges uniformly in $[-\delta, \delta]$ (this also implies that the limit function g is continuous).

Note: By working harder in the proof, Gripenberg is able to weaken some of the assumptions. There are also some counter-examples on how things can go wrong if you try to weaken the assumptions in the wrong way.

2.5.2 Is $\widehat{L^1(\mathbb{R})} = C_0(\mathbb{R})$?

That is, is every function $g \in C_0(\mathbb{R})$ the Fourier transform of a function $f \in L^1(\mathbb{R})$?

The answer is **no**, as the following counter-example shows. Take

$$g(\omega) = \begin{cases} \frac{\omega}{\ln 2} & , \quad |\omega| \le 1, \\ \frac{1}{\ln(1+\omega)} & , \quad \omega > 1, \\ -\frac{1}{\ln(1-\omega)} & , \quad \omega < -1. \end{cases}$$

Suppose that this would be the Fourier transform of a function $f \in L^1(\mathbb{R})$. As in the proof on the previous page, we define

$$h(t) = \sum_{n=-\infty}^{\infty} f(t+n).$$

Then (as we saw there), $h \in L^1(\mathbb{T})$, and $\hat{h}(n) = \hat{f}(n)$ for $n = 0, \pm 1, \pm 2, \ldots$. However, since $\sum_{n=1}^{\infty} \frac{1}{n} \hat{h}(n) = \infty$, this is not the Fourier sequence of any $h \in L^1(\mathbb{T})$ (by Theorem 1.38). Thus:

$$\boxed{\text{Not every } h \in C_0(\mathbb{R}) \text{ is the Fourier transform of some } f \in L^1(\mathbb{R}).}$$

But:

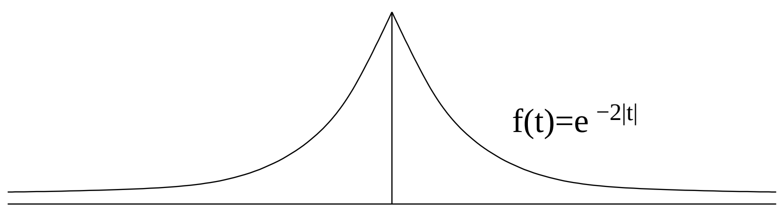
$$\begin{array}{llll} f \in L^1(\mathbb{R}) & \Rightarrow & \hat{f} \in C_0(\mathbb{R}) & (\text{ Page } 36) \\ f \in L^2(\mathbb{R}) & \Leftrightarrow & \hat{f} \in L^2(\mathbb{R}) & (\text{ Page } 47) \\ f \in \mathcal{S} & \Leftrightarrow & \hat{f} \in \mathcal{S} & (\text{ Page } 44) \end{array}$$

2.5.3 The Euler-MacLauren Summation Formula

Let $f \in C^\infty(\mathbb{R}^+)$ (where $\mathbb{R}^+ = [0,\infty)$), and suppose that

$$f^{(n)} \in L^1(\mathbb{R}^+)$$

for all $n \in \mathbb{Z}_+ = \{0,1,2,3\ldots\}$. We define $f(t)$ for $t<0$ so that $f(t)$ is **even**.
Warning: f is continuous at the origin, but f' may be discontinuous! For example, $f(t) = e^{-|2t|}$



We want to use Poisson summation formula. Is this allowed?
By Theorem 2.7, $\widehat{f^{(n)}} = (2\pi i\omega)^n \hat{f}(\omega)$, and $\hat{f}^{(n)}$ is bounded, so

$$\sup_{\omega\in\mathbb{R}} |(2\pi i\omega)^n||\hat{f}(\omega)| < \infty \text{ for all } n \;\Rightarrow \sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty.$$

By the note on page 52, also $\sum_{n=-\infty}^{\infty} f(t+n)$ converges uniformly in $(-1,1)$. By the Poisson summation formula:

$$\begin{aligned}
\sum_{n=0}^{\infty} f(n) &= \frac{1}{2}f(0) + \frac{1}{2}\sum_{n=-\infty}^{\infty} f(n)\\
&= \frac{1}{2}f(0) + \frac{1}{2}\sum_{n=-\infty}^{\infty} \hat{f}(n)\\
&= \frac{1}{2}f(0) + \frac{1}{2}\hat{f}(0) + \frac{1}{2}\sum_{n=1}^{\infty}\left[\hat{f}(n)+\hat{f}(-n)\right]\\
&= \frac{1}{2}f(0) + \frac{1}{2}\hat{f}(0) + \sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\underbrace{\frac{1}{2}\left(e^{2\pi int}+e^{-2\pi int}\right)}_{\cos(2\pi nt)} f(t)dt\\
&= \frac{1}{2}f(0) + \int_0^{\infty} f(t)dt + \sum_{n=1}^{\infty}\int_0^{\infty}\cos(2\pi nt)f(t)dt
\end{aligned}$$

Here we integrate by parts several times, always integrating the cosine-function and differentiating f. All the substitution terms containing **odd** derivatives of

f vanish since $\sin(2\pi nt) = 0$ for $t = 0$. See Gripenberg for details. The result looks something like

$$\sum_{n=0}^{\infty} f(n) = \int_0^{\infty} f(t)dt + \frac{1}{2}f(0) - \frac{1}{12}f'(0) + \frac{1}{720}f'''(0) - \frac{1}{30240}f^{(5)}(0) + \dots$$

2.5.4 Schwartz inequality

The Schwartz inequality will be used below. It says that

$$|\langle f, g\rangle| \leq \|f\|_{L^2}\|g\|_{L^2}$$

(true for all possible L^2-spaces, both $L^2(\mathbb{R})$ and $L^2(\mathbb{T})$ etc.)

2.5.5 Heisenberg's Uncertainty Principle

For all $f \in L^2(\mathbb{R})$, we have

$$\left(\int_{-\infty}^{\infty} t^2|f(t)|^2dt\right)\left(\int_{-\infty}^{\infty} \omega^2|\hat{f}(\omega)|^2d\omega\right) \geq \frac{1}{16\pi^2}\|f\|_{L^2(\mathbb{R})}^4$$

Interpretation: The more **concentrated** f is in the neighborhood of zero, the more **spread out** must $\hat{f}$ be, and conversely. (Here we must think that $\|f\|_{L^2(\mathbb{R})}$ is fixed, e.g. $\|f\|_{L^2(\mathbb{R})} = 1$.)
In quantum mechanics: The product of "time uncertainty" and "space uncertainty" cannot be less than a given fixed number.

PROOF. We begin with the case where $f \in \mathcal{S}$. Then

$$
\begin{aligned}
16\pi \int_{\mathbb{R}} |tf(t)|dt \int_{\mathbb{R}} |\omega \hat{f}(\omega)|d\omega \quad &= \quad 4\int_{\mathbb{R}} |tf(t)|dt \int_{\mathbb{R}} |f'(t)|dt
\end{aligned}
$$

($\widehat{f'(\omega)} = 2\pi i\omega \hat{f}(\omega)$ and Parseval's iden. holds). Now use Scwartz ineq.

$$
\begin{aligned}
&\geq \quad 4\left(\int_{\mathbb{R}} |tf(t)||f'(t)|dt\right) \\
&= \quad 4\left(\int_{\mathbb{R}} |t\overline{f(t)}||f'(t)|dt\right) \\
&\geq \quad 4\left(\int_{\mathbb{R}} Re[t\overline{f(t)}f'(t)]dt\right) \\
&= \quad 4\left(\int_{\mathbb{R}} t\left[\frac{1}{2}\left(\overline{f(t)}f'(t) + f(t)\overline{f'(t)}\right)\right] dt\right)^2 \\
&= \quad \int_{\mathbb{R}} t\frac{d}{dt}\underbrace{(f(t)\overline{f(t)})}_{=|f(t)|} dt \quad \text{(integrate by parts)} \\
&= \quad \left(\underbrace{[t|f(t)|]_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty} |f(t)|dt\right) \\
&= \quad \left(\int_{-\infty}^{\infty} |f(t)|dt\right)
\end{aligned}
$$

This proves the case where $f \in \mathcal{S}$. If $f \in L(\mathbb{R})$, but $f \in \mathcal{S}$, then we choose a sequence of functions $f_n \in \mathcal{S}$ so that

$$
\begin{aligned}
\int_{-\infty}^{\infty} |f_n(t)|dt \;&\to\; \int_{-\infty}^{\infty} |f(t)|dt \quad \text{and} \\
\int_{-\infty}^{\infty} |tf_n(t)|dt \;&\to\; \int_{-\infty}^{\infty} |tf(t)|dt \quad \text{and} \\
\int_{-\infty}^{\infty} |\omega \hat{f}_n(\omega)|d\omega \;&\to\; \int_{-\infty}^{\infty} |\omega \hat{f}(\omega)|d\omega
\end{aligned}
$$

(This can be done, not quite obvious). Since the inequality holds for each f_n, it must also hold for f.

2.5.6 Weierstrass' Non-Differentiable Function

Define $\sigma(t) = \sum_{k=0}^{\infty} a^k \cos(2\pi b^k t), \quad t \in \mathbb{R}$ where $0 < a < 1$ and ab≥ 1.

Lemma 2.33. *This sum defines a* ***continuous*** *function σ which is* ***not differentiable at any point****.*

PROOF. Convergence easy: At each t,

$$\sum_{k=0}^{\infty} |a^k \cos(2\pi b^k t)| \le \sum_{k=0}^{\infty} a^k = \frac{1}{1-a} < \infty,$$

and absolute convergence $\Rightarrow$ convergence. The convergence is even uniform: The error is

$$\Big|\sum_{k=K}^{\infty} a^k \cos(2\pi b^k t)\Big| \le \sum_{k=K}^{\infty} |a^k \cos(2\pi b^k t)| \le \sum_{k=K}^{\infty} a^k = \frac{a^K}{1-a} \to 0 \text{ as } K \to \infty$$

so by choosing K large enough we can make the error smaller than ε, and the same K works for all t.

By "Analysis II": If a sequence of continuous functions converges uniformly, then the limit function is continuous. Thus, σ *is continuous*.

Why is it *not differentiable*? At least does the formal derivative not converge: Formally we should have

$$\sigma'(t) = \sum_{k=0}^{\infty} a^k \cdot 2\pi b^k (-1) \sin(2\pi b^k t),$$

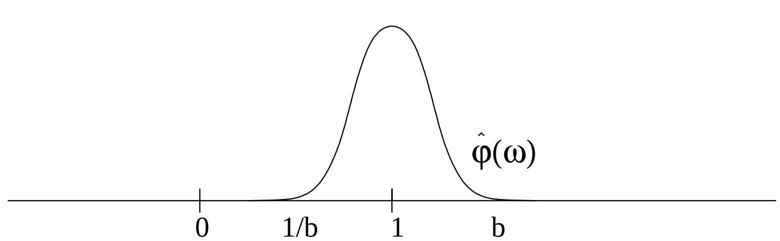
and the terms in this serie do not seem to go to zero (since $(ab)^k \ge 1$). (If a sum converges, then the terms must tend to zero.)

To prove that σ is not differentiable we cut the sum appropriatly: Choose some function $\varphi \in L^1(\mathbb{R})$ with the following properties:

i) $\hat{\varphi}(1) = 1$

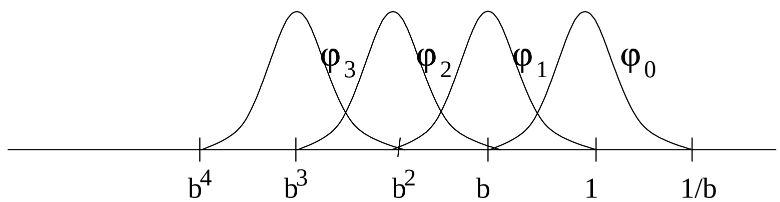
ii) $\hat{\varphi}(\omega) = 0$ for $\omega \le \frac{1}{b}$ and $\omega > b$

iii) $\int_{-\infty}^{\infty} |t\varphi(t)|dt < \infty$.



We can get such a function from the Fejer kernel: Take the square of the Fejer kernel ($\Rightarrow$ its Fourier transform is the convolution of $\hat{f}$ with itself), squeeze it (Theorem 2.7(e)), and shift it (Theorem 2.7(b)) so that it vanishes outside of

$(\frac{1}{b}, b)$, and $\hat{\varphi}(1) = 1$. (Sort of like approximate identity, but $\hat{\varphi}(1) = 1$ instead of $\hat{\varphi}(0) = 1$.)
Define $\varphi_j(t) = b^j\varphi(b^jt), \quad t \in \mathbb{R}$. Then $\hat{\varphi}_j(\omega) = \hat{\varphi}(\omega b^{-j})$, so $\hat{\varphi}(b^j) = 1$ and $\hat{\varphi}(\omega) = 0$ outside of the interval (b^{j-1}, b^{j+1}).



Put $f_j = \sigma * \varphi_j$. Then

$$\begin{aligned}
f_j(t) &= \int_{-\infty}^{\infty} \sigma(t-s)\varphi_j(s)ds \\
&= \int_{-\infty}^{\infty} \sum_{k=0}^{\infty} a^k \frac{1}{2}\left[e^{2\pi i b^k(t-s)} + e^{-2\pi i b^k(t-s)}\right]\varphi_j(s)ds \\
&\quad \text{(by the uniform convergence)} \\
&= \sum_{k=0}^{\infty} \frac{a^k}{2}\left[\underbrace{e^{2\pi i b^k t}\varphi_j(b^k)}_{=\delta_j^k} + \underbrace{e^{-2\pi i b^k t}\varphi_j(-b^k)}_{=0}\right] \\
&= \frac{1}{2}a^j e^{2\pi i b^k t}.
\end{aligned}$$

(Thus, this particular convolution picks out *just one* of the terms in the series.)
Suppose (to get a contradiction) that σ can be differentiated at some point $t \in \mathbb{R}$. Then the function

$$\eta(s) = \begin{cases} \frac{\sigma(t+s)-\sigma(t)}{s} - \sigma'(t) & , \quad s \neq 0 \\ 0 & , \quad s = 0 \end{cases}$$

is (uniformly) continuous and bounded, and $\eta(0) = 0$. Write this as

$$\sigma(t-s) = -s\eta(-s) + \sigma(t) - s\sigma'(t)$$

i.e.,

$$\begin{aligned}
f_j(t) &= \int_{\mathbb{R}} \sigma(t-s)\varphi_j(s)ds \\
&= \int_{\mathbb{R}} -s\eta(-s)\varphi_j(s)ds + \sigma(t)\underbrace{\int_{\mathbb{R}} \varphi_j(s)ds}_{=\hat{\varphi}_j(0)=0} - \sigma'(t)\underbrace{\int_{\mathbb{R}} s\varphi_j(s)ds}_{\frac{\hat{\varphi}_j'(0)}{-2\pi i}=0} \\
&= -\int_{\mathbb{R}} s\eta(-s)b^j\varphi(b^j s)ds \\
&\overset{b^j s=t}{=} -b^j\int_{\mathbb{R}} \underbrace{\eta\left(\frac{-s}{b^j}\right)}_{\to 0 \text{ pointwise}} \underbrace{s\varphi(s)ds}_{\in L^1} \\
&\to 0 \quad \text{by the Lesbesgue dominated convergence theorem as } j \to \infty.
\end{aligned}$$

Thus,

$$b^{-j}f_j(t) \to 0 \text{ as } j \to \infty \;\Leftrightarrow\; \frac{1}{2}\left(\frac{a}{b}\right)^j e^{2\pi i b^j t} \to 0 \text{ as } j \to \infty.$$

As $|e^{2\pi i b^j t}| = 1$, this is $\Leftrightarrow$ $\left(\frac{a}{b}\right)^j \to 0$ as $j \to \infty$. Impossible, since $\frac{a}{b} \geq 1$. Our assumption that σ is differentiable at the point t must be wrong $\Rightarrow$ $\sigma(t)$ is *not differentiable* in any point!

2.5.7 Differential Equations

Solve the differential equation

$$u''(t) + \lambda u(t) = f(t),\ t \in \mathbb{R} \tag{2.10}$$

where we require that $f \in L^2(\mathbb{R})$, $u \in L^2(\mathbb{R})$, $u \in C^1(\mathbb{R})$, $u' \in L^2(\mathbb{R})$ and that u' is of the form

$$u'(t) = u'(0) + \int_0^t v(s)ds,$$

where $v \in L^2(\mathbb{R})$ (that is, u' is "absolutely continuous" and its "generalized derivative" belongs to L^2).

The solution of this problem is based on the following lemmas:

Lemma 2.34. *Let $k = 1, 2, 3, \ldots$. Then the following conditions are equivalent:*

i) $u \in L^2(\mathbb{R}) \cap C^{k-1}(\mathbb{R})$, $u^{(k-1)}$ is "absolutely continuous" and the "generalized derivative of $u^{(k-1)}$" belongs to $L^2(\mathbb{R})$.

ii) $\hat{u} \in L^2(\mathbb{R})$ *and* $\int_{\mathbb{R}} |\omega^k \hat{u}(k)|^2 d\omega < \infty$.

PROOF. Similar to the proof of one of the homeworks, which says that the same result is true for L^2-Fourier series. (There ii) is replaced by $\sum |n\hat{f}(n)|^2 < \infty$.)

Lemma 2.35. *If u is as in Lemma 2.34, then*

$$\widehat{u^{(k)}}(\omega) = (2\pi i \omega)^k \hat{u}(\omega)$$

(compare this to Theorem 2.7(g)).

PROOF. Similar to the same homework.

<u>Solution</u>: By the two preceding lemmas, we can take Fourier transforms in (2.10), and get the equivalent equation

$$(2\pi i \omega)^2 \hat{u}(\omega) + \lambda \hat{u}(\omega) = \hat{f}(\omega),\ \omega \in \mathbb{R} \Leftrightarrow (\lambda - 4\pi^2\omega^2)\hat{u}(\omega) = \hat{f}(\omega),\ \omega \in \mathbb{R} \quad (2.11)$$

Two cases:

<u>Case 1</u>: $\lambda - 4\pi^2\omega^2 \neq 0$, for *all* $\omega \in \mathbb{R}$, i.e., λ must not be zero and not a positive number (negative is OK, complex is OK). Then

$$\hat{u}(\omega) = \frac{\hat{f}(\omega)}{\lambda - 4\pi^2\omega^2},\ \omega \in \mathbb{R}$$

so $u = k * f$, where $k =$ the inverse Fourier transform of

$$\hat{k}(\omega) = \frac{1}{\lambda - 4\pi^2\omega^2}.$$

This can be computed explicitly. It is called "Green's function" for this problem. Even without computing $k(t)$, we know that

- $k \in C_0(\mathbb{R})$ (since $\hat{k} \in L^1(\mathbb{R})$.)
- k has a generalized derivative in $L^2(\mathbb{R})$ (since $\int_{\mathbb{R}} |\omega \hat{k}(\omega)|^2 d\omega < \infty$.)
- k does not have a second generalized derivative in L^2 (since $\int_{\mathbb{R}} |\omega^2 \hat{k}(\omega)|^2 d\omega = \infty$.)

How to compute k? Start with a partial fraction expansion. Write

$$\lambda = \alpha^2 \quad \text{for some } \alpha \in \mathbb{C}$$

($\alpha =$ pure imaginary if $\lambda < 0$). Then

$$\begin{aligned} \frac{1}{\lambda - 4\pi^2\omega^2} &= \frac{1}{\alpha^2 - 4\pi^2\omega^2} = \frac{1}{\alpha - 2\pi\omega} \cdot \frac{1}{\alpha + 2\pi\omega} \\ &= \frac{A}{\alpha - 2\pi\omega} + \frac{B}{\alpha + 2\pi\omega} \\ &= \frac{A\alpha + 2\pi\omega A + B\alpha - 2\pi\omega B}{(\alpha - 2\pi\omega)(\alpha + 2\pi\omega)} \\ &\Rightarrow \left. \begin{array}{l} (A+B)\alpha = 1 \\ (A-B)2\pi\omega = 0 \end{array} \right\} \Rightarrow A = B = \frac{1}{2\alpha} \end{aligned}$$

Now we must still invert $\frac{1}{\alpha+2\pi\omega}$ and $\frac{1}{\alpha-2\pi\omega}$. This we do as follows:

Auxiliary result 1: Compute the transform of

$$f(t) = \begin{cases} e^{-zt} &, \quad t \ge 0, \\ 0 &, \quad t < 0, \end{cases}$$

where $\boxed{Re(z) > 0}$ ($\Rightarrow f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, but $f \notin C(\mathbb{R})$ because of the jump at the origin). Simply compute:

$$\begin{aligned} \hat{f}(\omega) &= \int_0^\infty e^{-2\pi i\omega t} e^{-zt} dt \\ &= \left[\frac{e^{-(z+2\pi i\omega)t}}{-(z + 2\pi i\omega)} \right]_0^\infty = \frac{1}{2\pi i\omega + z}. \end{aligned}$$

Auxiliary result 2: Compute the transform of

$$f(t) = \begin{cases} e^{zt} &, \quad t \le 0, \\ 0 &, \quad t > 0, \end{cases}$$

where $\boxed{Re(z) > 0}$ ($\Rightarrow f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, but $f \notin C(\mathbb{R})$)

$$\begin{aligned} \Rightarrow \hat{f}(\omega) &= \int_{-\infty}^0 e^{2\pi i\omega t} e^{zt} dt \\ &= \left[\frac{e^{(z-2\pi i\omega)t}}{(z - 2\pi i\omega)t} \right]_{-\infty}^0 = \frac{1}{z - 2\pi i\omega}. \end{aligned}$$

Back to the function k:

$$\begin{aligned} \hat{k}(\omega) &= \frac{1}{2\alpha} \left(\frac{1}{\alpha - 2\pi\omega} + \frac{1}{\alpha + 2\pi\omega} \right) \\ &= \frac{1}{2\alpha} \left(\frac{i}{i\alpha - 2\pi i\omega} + \frac{i}{i\alpha + 2\pi i\omega} \right). \end{aligned}$$

We defined α by requiring $\alpha^2 = \lambda$. Choose α so that $Im(\alpha) < 0$ (possible because α is not a positive real number).

$$\Rightarrow Re(i\alpha) > 0, \quad \text{and } \hat{k}(\omega) = \frac{1}{2\alpha}\left(\frac{i}{i\alpha - 2\pi i\omega} + \frac{i}{i\alpha + 2\pi i\omega}\right)$$

The auxiliary results 1 and 2 gives:

$$\boxed{k(t) = \begin{cases} \frac{i}{2\alpha}e^{-i\alpha t} & , \quad t \geq 0 \\ \frac{i}{2\alpha}e^{i\alpha t} & , \quad t < 0 \end{cases}}$$

and

$$\boxed{u(t) = (k * f)(t) = \int_{-\infty}^{\infty} k(t-s)f(s)ds}$$

Special case: $\lambda =$ negative number $= -a^2$, where $a > 0$. Take $\alpha = -ia$
$\Rightarrow i\alpha = i(-i)a = a$, and

$$k(t) = \begin{cases} -\frac{1}{2a}e^{-at} & , \quad t \geq 0 \\ -\frac{1}{2a}e^{at} & , \quad t < 0 \end{cases} \qquad \text{i.e.}$$

$$\boxed{k(t) = -\frac{1}{2a}e^{-|at|}, \ t \in \mathbb{R}}$$

Thus, the solution of the equation

$$u''(t) - a^2u(t) = f(t), \qquad t \in \mathbb{R},$$

where $a > 0$, is given by

$$\boxed{u = k * f} \qquad \text{where}$$

$$\boxed{k(t) = -\frac{1}{2a}e^{-a|t|}, \quad t \in \mathbb{R}}$$

This function k has many names, depending on the field of mathematics you are working in:

i) Green's function (PDE-people)

ii) Fundamental solution (PDE-people, Functional Analysis)

iii) Resolvent (Integral equations people)

<u>**Case 2**</u>: $\lambda = a^2 =$ a nonnegative number. Then

$$\hat{f}(\omega) = (a^2 - 4\pi^2\omega^2)\hat{u}(\omega) = (a - 2\pi\omega)(a + 2\pi\omega)\hat{u}(\omega).$$

As $\hat{u}(\omega) \in L^2(\mathbb{R})$ we get a necessary condition for the existence of a solution: If a solution exists then

$$\int_{\mathbb{R}} \left|\frac{\hat{f}(\omega)}{(a - 2\pi\omega)(a + 2\pi\omega)}\right|^2 d\omega < \infty. \tag{2.12}$$

(Since the denominator vanishes for $\omega = \pm\frac{a}{2\pi}$, this forces $\hat{f}$ to vanish at $\pm\frac{a}{2\pi}$, and to be "small" near these points.)
If the condition (2.12) holds, then we can continue the solution as before.

<u>Sideremark</u>: These results mean that this particular problem has no "eigenvalues" and no "eigenfunctions". Instead it has a "contionuous spectrum" consisting of the positive real line. (Ignore this comment!)

2.5.8 Heat equation

This equation:

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) &= \frac{\partial^2}{\partial x^2}u(t,x) + g(t,x), \begin{cases} t > 0 \\ x \in \mathbb{R} \end{cases} \\ u(0,x) &= f(x) \quad \text{(initial value)} \end{cases}$$

is solved in the same way. Rather than proving everything we proceed in a formal mannor (everything can be proved, but it takes a lot of time and energy.)
Transform the equation in the x-direction,

$$\hat{u}(t,\gamma) = \int_{\mathbb{R}} e^{-2\pi i\gamma x}u(t,x)dx.$$

<u>Assuming</u> that $\int_{\mathbb{R}} e^{-2\pi i\gamma x}\frac{\partial}{\partial t}u(t,x) = \frac{\partial}{\partial t}\int_{\mathbb{R}} e^{-2\pi i\gamma x}u(t,x)dx$ we get

$$\begin{cases} \frac{\partial}{\partial t}\hat{u}(t,\gamma) &= (2\pi i\gamma)^2\hat{u}(t,\gamma) + \hat{g}(t,\gamma) \\ \hat{u}(0,\gamma) &= \hat{f}(\gamma) \end{cases}$$

$$\Leftrightarrow$$

$$\begin{cases} \frac{\partial}{\partial t}\hat{u}(t,\gamma) &= -4\pi^2\gamma^2\hat{u}(t,\gamma) + \hat{g}(t,\gamma) \\ \hat{u}(0,\gamma) &= \hat{f}(\gamma) \end{cases}$$

We solve this by using the standard "variation of constants lemma":

$$\begin{aligned}\hat{u}(t,\gamma) &= \underbrace{\hat{f}(\gamma)e^{-4\pi^2\gamma^2 t}} + \underbrace{\int_0^t e^{-4\pi^2\gamma^2(t-s)}\hat{g}(s,\gamma)ds} \\ &= \quad \hat{u}_1(t,\gamma) \quad + \quad \hat{u}_2(t,\gamma)\end{aligned}$$

We can invert $e^{-4\pi^2\gamma^2 t} = e^{-\pi(2\sqrt{\pi t}\gamma)^2} = e^{-\pi(\gamma/\lambda)^2}$ where $\lambda = (2\sqrt{\pi t})^{-1}$: According to Theorem 2.7 and Example 2.5, this is the transform of

$$k(t,x) = \frac{1}{2\sqrt{\pi t}}e^{-\pi(\frac{x}{2\sqrt{\pi t}})^2} = \frac{1}{2\sqrt{\pi t}}e^{-\frac{x^2}{4t}}$$

We know that $\hat{f}(\gamma)\hat{k}(\gamma) = \widehat{k * f}(\gamma)$, so

$$\boxed{\begin{aligned}u_1(t,x) &= \textstyle\int_{-\infty}^{\infty}\frac{1}{2\sqrt{\pi t}}e^{-(x-y)^2/4t}f(y)dy, \\ &\quad \text{(By the same argument:} \\ &\quad s \text{ and } t-s \text{ are fixed when we transform.)} \\ u_2(t,x) &= \textstyle\int_0^t (k*g)(s)ds \\ &= \textstyle\int_0^t\int_{-\infty}^{\infty}\frac{1}{2\sqrt{\pi(t-s)}}e^{-(x-y)^2/4(t-s)}g(s,y)dyds, \\ u(t,x) &= u_1(t,x) + u_2(t,x)\end{aligned}}$$

The function

$$k(t,x) = \frac{1}{2\sqrt{\pi t}}e^{-\frac{x^2}{4t}}$$

is the *Green's function* or the *fundamental solution* of the heat equation on the real line $\mathbb{R} = (-\infty,\infty)$, or the *heat kernel*.

Note: To prove that this "solution" is indeed a solution we need to assume that

- all functions are in $L^2(\mathbb{R})$ with respect to x, i.e.,

$$\int_{-\infty}^{\infty}|u(t,x)|^2dx < \infty, \int_{-\infty}^{\infty}|g(t,x)|^2dx < \infty, \int_{-\infty}^{\infty}|f(x)|^2dx < \infty,$$

- some (weak) continuity assumptions with respect to t.

2.5.9 Wave equation

$$\begin{cases}\frac{\partial^2}{\partial t^2}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + k(t,x), & \begin{cases} t>0, \\ x\in\mathbb{R}.\end{cases} \\ u(0,x) = f(x), & x\in\mathbb{R} \\ \frac{\partial}{\partial t}u(0,x) = g(x), & x\in\mathbb{R}\end{cases}$$

Again we proceed *formally*. As above we get

$$\begin{cases} \frac{\partial^2}{\partial t^2}\hat{u}(t,\gamma) & = & -4\pi^2\gamma^2\hat{u}(t,\gamma) + \hat{k}(t,\gamma), \\ \hat{u}(0,\gamma) & = & \hat{f}(\gamma), \\ \frac{\partial}{\partial t}\hat{u}(0,\gamma) & = & \hat{g}(\gamma). \end{cases}$$

This can be solved by "the variation of constants formula", but to *simplify* the computations we assume that $k(t,x) \equiv 0$, i.e., $\hat{h}(t,\gamma) \equiv 0$. Then the solution is (check this!)

$$\hat{u}(t,\gamma) = \cos(2\pi\gamma t)\hat{f}(\gamma) + \frac{\sin(2\pi\gamma t)}{2\pi\gamma}\hat{g}(\gamma). \tag{2.13}$$

To invert the first term we use Theorem 2.7, and get

$$\frac{1}{2}[f(x+t) + f(x-t)].$$

The second term contains the "*Dirichlet kernel*", which is inverted as follows:
<u>Ex</u>. If

$$k(x) = \begin{cases} 1/2, & |t| \le 1 \\ 0, & \text{otherwise}, \end{cases}$$

then $\hat{k}(\omega) = \frac{1}{2\pi\omega}\sin(2\pi\omega)$.

Proof.

$$\hat{k}(\omega) = \frac{1}{2}\int_{-1}^{1} e^{-2\pi i\omega t}dt = \ldots = \frac{1}{2\pi\omega}\sin(\omega t).$$

Thus, the inverse Fourier transform of

$$\frac{\sin(2\pi\gamma)}{2\pi\gamma} \quad \text{is} \quad k(x) = \begin{cases} 1/2, & |x| \le 1, \\ 0, & |x| > 1, \end{cases}$$

(inverse transform = ordinary transform since the function is even), and the inverse Fourier transform (with respect to γ) of

$$\begin{aligned} \frac{\sin(2\pi\gamma t)}{2\pi\gamma} &= t\frac{\sin(2\pi\gamma t)}{2\pi\gamma t} \quad \text{is} \\ k(\frac{x}{t}) &= \begin{cases} 1/2, & |x| \le t, \\ 0, & |x| > t. \end{cases} \end{aligned}$$

This and Theorem 2.7(f), gives the inverse of the second term in (2.13): It is

$$\frac{1}{2}\int_{x-t}^{x+t} g(y)dy.$$

Conclusion: The solution of the wave equation with $h(t,x) \equiv 0$ seems to be

$$\boxed{u(t,x) = \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} g(y)dy,}$$

a formula known as *d'Alembert's formula.*

Interpretation: This is the sum of two waves: $u(t,x) = u^+(t,x) + u^-(t,x)$, where

$$u^+(t,x) = \frac{1}{2}f(x+t) + \frac{1}{2}G(x+t)$$

moves to the left with speed one, and

$$u^-(t,x) = \frac{1}{2}f(x-t) - \frac{1}{2}G(x-t)$$

moves to the right with speed one. Here

$$G(x) = \int_0^x g(y)dy, \quad x \in \mathbb{R}.$$

Chapter 3

Fourier Transforms of Distributions

Questions

1) How do we transform a function $f \notin L^1(\mathbb{R})$, $f \notin L^2(\mathbb{R})$, for example Weierstrass function
$$\sigma(t) = \sum_{k=0}^{\infty} a^k \cos(2\pi b^k t),$$
where $b \neq$ integer (if b is an integer, then σ is periodic and we can use Chapter I)?

2) Can we interpret both the periodic $\mathcal{F}$-transform (on $L^1(\mathbb{T})$) and the Fourier integral (on $L^1(\mathbb{R})$) as special cases of a "more general" Fourier transform?

3) How do you differentiate a discontinuous function?

The answer: Use "*distribution theory*", developed in France by Schwartz in 1950's.

3.1 What is a Measure?

We start with a simpler question: what is a "δ-function"? Typical definition:

$$\begin{cases} \delta(x) = 0, & x \neq 0 \\ \delta(0) = \infty \\ \int_{-\varepsilon}^{\varepsilon} \delta(x)dx = 1, & \text{(for } \varepsilon > 0). \end{cases}$$

We observe: *This is pure nonsense.* We observe that $\delta(x) = 0$ a.e., so $\int_{-\varepsilon}^{\varepsilon} \delta(x)dx = 0$.
Thus: $\boxed{\text{The } \delta\text{-function is not a function! What is it?}}$

Normally a δ-function is used in the following way: Suppose that f is continuous at the origin. Then

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\delta(x)dx &= \int_{-\infty}^{\infty} \underbrace{[f(x) - f(0)]}_{\substack{=0 \\ \text{when } x=0}} \underbrace{\delta(x)}_{\substack{=0 \\ \text{when } x\neq 0}} dx + f(0)\int_{-\infty}^{\infty} \delta(x)dx \\ &= f(0)\int_{-\infty}^{\infty} \delta(x)dx = f(0). \end{aligned}$$

This gives us a *new interpretation* of δ:

The δ-function is the "operator" which evaluates a continuous function at the point zero.

<u>Principle</u>: You feed a function $f(x)$ to δ, and δ gives you back the number $f(0)$ (forget about the integral formula).

Since the formal integral $\int_{-\infty}^{\infty} f(x)\delta(x)dx$ resembles an inner product, we often use the notation $\langle \delta, f\rangle$. Thus

$$\boxed{\langle \delta, f\rangle = f(0)}$$

Definition 3.1. The **δ-operator** is the (bounded linear) operator which maps $f \in C_0(\mathbb{R})$ into the number $f(0)$. Also called $\boxed{\text{Dirac's delta}}$.

This is a special case of *measure*:

Definition 3.2. A **measure** μ is a bounded linear operator which maps functions $f \in C_0(\mathbb{R})$ into the set of complex numbers $\mathbb{C}$ (or real). We denote this number by $\langle \mu, f\rangle$.

Example 3.3. The operator which maps $f \in C_0(\mathbb{R})$ into the number

$$f(0) + f(1) + \int_0^1 f(s)ds$$

is a measure.

Proof. Denote $\langle G, f\rangle = f(0) + f(1) + \int_0^1 f(s)ds$. Then

i) G maps $C_0(\mathbb{R}) \to \mathbb{C}$.

ii) G is *linear*:

$$\begin{aligned}\langle G, \lambda f + \mu g\rangle &= \lambda f(0) + \mu g(0) + \lambda f(1) + \mu g(1) \\ &\quad + \int_0^1 (\lambda f(s) + \mu g(s))ds \\ &= \lambda f(0) + \lambda f(1) + \int_0^1 \lambda f(s)ds \\ &\quad + \mu g(0) + \mu g(1) + \int_0^1 \mu g(s)ds \\ &= \lambda\langle G, f\rangle + \mu\langle G, g\rangle.\end{aligned}$$

iii) G is *continuous*: If $f_n \to f$ in $C_0(\mathbb{R})$, then $\max_{t\in\mathbb{R}}|f_n(t) - f(t)| \to 0$ as $n \to \infty$, so

$$f_n(0) \to f(0),\ f_n(1) \to f(1) \text{ and } \int_0^1 f_n(s)ds \to \int_0^1 f(s)ds,$$

so

$$\langle G, f_n\rangle \to \langle G, f\rangle \text{ as } n \to \infty.$$

Thus, G is a measure. □

Warning 3.4. *$\langle G, f\rangle$ is linear in f, not conjugate linear:*

$$\langle G, \lambda f\rangle = \lambda\langle G, f\rangle, \text{ and not } = \overline{\lambda}\langle G, f\rangle.$$

Alternative notation 3.5. *Instead of $\langle G, f\rangle$ many people write $G(f)$ or Gf (for example , Gripenberg).* $\boxed{\textit{See Gasquet}}$ *for more details.*

3.2 What is a Distribution?

Physisists often also use "the derivative of a δ-function", which is defined as

$$\langle \delta', f\rangle = -f'(0),$$

here $f'(0)$ = derivative of f at zero. This is *not a measure*: It is not defined for *all* $f \in C_0(\mathbb{R})$ (only for those that are differentiable at zero). It is linear, but it is *not continuous* (easy to prove). This is an example of a more general *distribution*.

Definition 3.6. A **tempered distribution** (=tempererad distribution) is a *continuous linear* operator from $\mathcal{S}$ to $\mathbb{C}$. We denote the set of such distributions by $\mathcal{S}'$. (The set $\mathcal{S}$ was defined in Section 2.2).

Theorem 3.7. *Every measure is a distribution.*

PROOF.

i) Maps $\mathcal{S}$ into $\mathbb{C}$, since $\mathcal{S} \subset C_0(\mathbb{R})$.

ii) Linearity is OK.

iii) Continuity is OK: If $f_n \to f$ in $\mathcal{S}$, then $f_n \to f$ in $C_0(\mathbb{R})$, so $\langle \mu, f_n \rangle \to \langle \mu, f \rangle$ (more details below!) $\square$

Example 3.8. Define $\langle \delta', \varphi \rangle = -\varphi'(0)$, $\varphi \in \mathcal{S}$. Then δ' is a tempered distribution

PROOF.

i) Maps $\mathcal{S} \to \mathbb{C}$? Yes!

ii) Linear? Yes!

iii) Continuous? Yes!

(See below for details!) $\square$

What does $\boxed{\varphi_n \to \varphi \text{ in } \mathcal{S}}$ mean?

Definition 3.9. $\varphi_n \to \varphi$ in $\mathcal{S}$ means the following: For all positive integers k, m,

$$t^k \varphi_n^{(m)}(t) \to t^k \varphi^{(m)}(t)$$

uniformly in t, i.e.,

$$\lim_{n\to\infty} \max_{t\in\mathbb{R}} |t^k(\varphi_n^{(m)}(t) - \varphi^{(m)}(t))| = 0.$$

Lemma 3.10. *If $\varphi_n \to \varphi$ in $\mathcal{S}$, then*

$$\varphi_n^{(m)} \to \varphi^{(m)} \text{ in } C_0(\mathbb{R})$$

for all $m = 0, 1, 2, \ldots$

PROOF. Obvious.

<u>Proof that δ' is continuous</u>: If $\varphi_n \to \varphi$ in $\mathcal{S}$, then $\max_{t\in\mathbb{R}} |\varphi_n'(t) - \varphi'(t)| \to 0$ as $n \to \infty$, so

$$\langle \delta', \varphi_n \rangle = -\varphi_n'(0) \to \varphi'(0) = \langle \delta', \varphi \rangle. \quad \square$$

3.3 How to Interpret a Function as a Distribution?

Lemma 3.11. *If $f \in L^1(\mathbb{R})$ then the operator which maps $\varphi \in \mathcal{S}$ into*

$$\langle F, \varphi \rangle = \int_{-\infty}^{\infty} f(s)\varphi(s)ds$$

is a continuous linear map from $\mathcal{S}$ to $\mathbb{C}$. (Thus, F is a tempered distribution*).*

Note: No complex conjugate on φ!
Note: F is even a measure.
Proof.

i) For every $\varphi \in \mathcal{S}$, the integral converges (absolutely), and defines a number in $\mathbb{C}$. Thus, F maps $\mathcal{S} \to \mathbb{C}$.

ii) *Linearity*: for all $\varphi, \psi \in \mathcal{S}$ and $\lambda, \mu \in \mathbb{C}$,

$$\begin{aligned} \langle F, \lambda\varphi + \mu\psi \rangle &= \int_{\mathbb{R}} f(s)[\lambda\varphi(s) + \mu\psi(s)]ds \\ &= \lambda \int_{\mathbb{R}} f(s)\varphi(s)ds + \mu \int_{\mathbb{R}} f(s)\psi(s)ds \\ &= \lambda\langle F, \varphi \rangle + \mu\langle F, \psi \rangle. \end{aligned}$$

iii) *Continuity*: If $\varphi_n \to \varphi$ in $\mathcal{S}$, then $\varphi_n \to \varphi$ in $C_0(\mathbb{R})$, and by Lebesgue's dominated convergence theorem,

$$\langle F, \varphi_n \rangle = \int_{\mathbb{R}} f(s)\varphi_n(s)ds \to \int_{\mathbb{R}} f(s)\varphi(s)ds = \langle F, \varphi \rangle. \quad \square$$

The *same proof* plus a little additional work proves:

Theorem 3.12. *If*

$$\int_{-\infty}^{\infty} \frac{|f(t)|}{1 + |t|^n} dt < \infty$$

for some $n = 0, 1, 2, \ldots$, then the formula

$$\langle F, \varphi \rangle = \int_{-\infty}^{\infty} f(s)\varphi(s)ds, \quad \varphi \in \mathcal{S},$$

defines a tempered distribution F.

Definition 3.13. We call the distribution F in Lemma 3.11 and Theorem 3.12 **the distribution induced by** f, and often write $\langle f, \varphi \rangle$ instead of $\langle F, \varphi \rangle$. Thus,

$$\boxed{\langle f, \varphi \rangle = \int_{-\infty}^{\infty} f(s)\varphi(s)ds, \quad \varphi \in \mathcal{S}.}$$

This is sort of like an inner product, but we *cannot change places* of f and φ: f is "*the distribution*" and φ is "*the test function*" in $\langle f, \varphi \rangle$.
Does "the distribution f" determine "the function f" uniquely? Yes!

Theorem 3.14. *Suppose that the two functions f_1 and f_2 satisfy*

$$\int_{\mathbb{R}} \frac{|f_i(t)|}{1+|t|^n} dt < \infty \quad (i = 1 \text{ or } i = 2),$$

and that they induce the same distribution, i.e., that

$$\int_{\mathbb{R}} f_1(t)\varphi(t)dt = \int_{\mathbb{R}} f_2(t)\varphi(t)dt, \quad \varphi \in \mathcal{S}.$$

Then $f_1(t) = f_2(t)$ almost everywhere.

Proof. Let $g = f_1 - f_2$. Then

$$\int_{\mathbb{R}} g(t)\varphi(t)dt = 0 \quad \text{for all } \varphi \in \mathcal{S} \iff$$
$$\int_{\mathbb{R}} \frac{g(t)}{(1+t^2)^{n/2}}(1+t^2)^{n/2}\varphi(t)dt = 0 \quad \forall \varphi \in \mathcal{S}.$$

Easy to show that $\underbrace{(1+t^2)^{n/2}\varphi(t)}_{\psi(t)} \in \mathcal{S} \iff \varphi \in \mathcal{S}$. If we define $h(t) = \frac{g(t)}{(1+t^2)^{n/2}}$, then $h \in L^1(\mathbb{R})$, and

$$\int_{-\infty}^{\infty} h(s)\psi(s)ds = 0 \quad \forall \psi \in \mathcal{S}.$$

If $\psi \in \mathcal{S}$ then also the function $s \mapsto \psi(t-s)$ belongs to $\mathcal{S}$, so

$$\int_{\mathbb{R}} h(s)\psi(t-s)ds = 0 \quad \begin{cases} \forall \psi \in \mathcal{S}, \\ \forall t \in \mathbb{R}. \end{cases} \tag{3.1}$$

Take $\psi_n(s) = ne^{-\pi(ns)^2}$. Then $\psi_n \in \mathcal{S}$, and by 3.1,

$$\psi_n * h \equiv 0.$$

On the other hand, by Theorem 2.12, $\psi_n * h \to h$ in $L^1(\mathbb{R})$ as $n \to \infty$, so this gives $h(t) = 0$ a.e. $\square$

Corollary 3.15. *If we know "the distribution f", then from this knowledge we can reconstruct $f(t)$ for* almost all *t.*

Proof. Use the same method as above. We know that $h(t) \in L^1(\mathbb{R})$, and that

$$(\psi_n * h)(t) \to h(t) = \frac{f(t)}{(1+t^2)^{n/2}}.$$

As soon as we know "the distribution f", we also know the values of

$$(\psi_n * h)(t) = \int_{-\infty}^{\infty} \frac{f(s)}{(1+s^2)^{n/2}}(1+s^2)^{n/2}\psi_n(t-s)ds$$

for all t. $\square$

3.4 Calculus with Distributions

(=Räkneregler)

3.16 (Addition)**.** *If f and g are two distributions, then $f+g$ is the distribution*

$$\langle f+g, \varphi\rangle = \langle f, \varphi\rangle + \langle g, \varphi\rangle, \quad \varphi \in \mathcal{S}.$$

(f and g distributions $\iff$ $f \in \mathcal{S}'$ and $g \in \mathcal{S}'$).

3.17 (Multiplication by a constant)**.** *If λ is a constant and $f \in \mathcal{S}'$, then λf is the distribution*

$$\langle \lambda f, \varphi\rangle = \lambda\langle f, \varphi\rangle, \quad \varphi \in \mathcal{S}.$$

3.18 (Multiplication by a test function)**.** *If $f \in \mathcal{S}'$ and $\eta \in \mathcal{S}$, then ηf is the distribution*

$$\langle \eta f, \varphi\rangle = \langle f, \eta\varphi\rangle \quad \varphi \in \mathcal{S}.$$

<u>Motivation</u>: If f would be induced by a function, then this would be the natural definition, because

$$\int_{\mathbb{R}} [\eta(s)f(s)]\varphi(s)ds = \int_{\mathbb{R}} f(s)[\eta(s)\varphi(s)]ds = \langle f, \eta\varphi\rangle.$$

Warning 3.19. *In general, you* cannot multiply two distributions. *For example,*

$$\boxed{\delta^2 = \delta\delta \text{ is nonsense}} \qquad \begin{array}{l}(\delta = \text{“}\delta\text{-function”}) \\ = \text{Dirac's delta}\end{array}$$

However, it is possible to multiply distributions by a larger class of "test functions":

Definition 3.20. By the class $C^\infty_{\text{pol}}(\mathbb{R})$ of *tempered test functions* we mean the following:

$$\psi \in C^\infty_{\text{pol}}(\mathbb{R}) \iff f \in C^\infty(\mathbb{R}),$$

and for every $k = 0, 1, 2, \ldots$ there are two numbers M and n so that

$$|\psi^{(k)}(t)| \leq M(1 + |t|^n), \quad t \in \mathbb{R}.$$

Thus, $f \in C^\infty_{\text{pol}}(\mathbb{R}) \iff f \in C^\infty(\mathbb{R})$, and every derivative of f grows at most as a polynomial as $t \to \infty$.

$$\text{Repetition:} \begin{cases} \mathcal{S} &= \text{"rapidly decaying test functions"} \\ \mathcal{S}' &= \text{"tempered distributions"} \\ C^\infty_{\text{pol}}(\mathbb{R}) &= \text{"tempered test functions".} \end{cases}$$

Example 3.21. Every *polynomial* belongs to C^∞_{pol}. So do the functions

$$\frac{1}{1+x^2}, \quad (1+x^2)^{\pm m} \quad (m \text{ need not be an integer})$$

Lemma 3.22. *If $\psi \in C^\infty_{pol}(\mathbb{R})$ and $\varphi \in \mathcal{S}$, then*

$$\psi\varphi \in \mathcal{S}.$$

Proof. Easy (special case used on page 72).

Definition 3.23. If $\psi \in C^\infty_{\text{pol}}(\mathbb{R})$ and $f \in \mathcal{S}'$, then ψf is the distribution

$$\boxed{\langle \psi f, \varphi \rangle = \langle f, \psi\varphi \rangle, \quad \varphi \in \mathcal{S}}$$

(O.K. since $\psi\varphi \in \mathcal{S}$).

Now to the $\boxed{\text{big surprise}}$: Every distribution has a *derivative*, which is another distribution!

Definition 3.24. Let $f \in \mathcal{S}'$. Then the **distribution derivative** of f is the distribution defined by

$$\boxed{\langle f', \varphi \rangle = -\langle f, \varphi' \rangle, \quad \varphi \in \mathcal{S}}$$

(This is O.K., because $\varphi \in \mathcal{S} \implies \varphi' \in \mathcal{S}$, so $-\langle f, \varphi' \rangle$ is defined).

<u>Motivation</u>: *If* f would be a function in $C^1(\mathbb{R})$ (not too big at ∞), then

$$\begin{aligned}
\langle f, \varphi' \rangle &= \int_{-\infty}^{\infty} f(s)\varphi'(s)ds \quad \text{(integrate by parts)} \\
&= \underbrace{[f(s)\varphi(s)]_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty} f'(s)\varphi(s)ds \\
&= -\langle f', \varphi \rangle. \quad \square
\end{aligned}$$

Example 3.25. Let

$$f(t) = \begin{cases} e^{-t}, & t \geq 0, \\ -e^{t}, & t < 0. \end{cases}$$

Interpret this as a distribution, and compute its distribution derivative.

<u>Solution</u>:

$$\begin{aligned}
\langle f', \varphi \rangle &= -\langle f, \varphi' \rangle = -\int_{-\infty}^{\infty} f(s)\varphi'(s)ds \\
&= \int_{-\infty}^{0} e^{s}\varphi'(s)ds - \int_{0}^{\infty} e^{-s}\varphi'(s)ds \\
&= [e^{s}\varphi(s)]_{-\infty}^{0} - \int_{-\infty}^{0} e^{s}\varphi(s)ds - \left[e^{-s}\varphi(s)\right]_{0}^{\infty} - \int_{0}^{\infty} e^{-s}\varphi(s)ds \\
&= 2\varphi(0) - \int_{-\infty}^{\infty} e^{-|s|}\varphi(s)ds.
\end{aligned}$$

Thus, $f' = 2\delta + h$, where h is the "function" $h(s) = -e^{-|s|}$, $s \in \mathbb{R}$, and $\delta =$ the Dirac delta (note that $h \in L^1(\mathbb{R}) \cap C(\mathbb{R})$).

Example 3.26. Compute the *second* derivative of the function in Example 3.25!

<u>Solution</u>: By definition, $\langle f'', \varphi \rangle = -\langle f', \varphi' \rangle$. Put $\varphi' = \psi$, and apply the rule $\langle f', \psi \rangle = -\langle f, \psi' \rangle$. This gives

$$\boxed{\langle f'', \varphi \rangle = \langle f, \varphi'' \rangle.}$$

By the preceding computation

$$\begin{aligned}
-\langle f, \varphi' \rangle &= -2\varphi'(0) - \int_{-\infty}^{\infty} e^{-|s|}\varphi'(s)ds \\
&= \text{(after an integration by parts)} \\
&= -2\varphi'(0) + \int_{-\infty}^{\infty} f(s)\varphi(s)ds
\end{aligned}$$

(f = original function). Thus,

$$\langle f'', \varphi \rangle = -2\varphi'(0) + \int_{-\infty}^{\infty} f(s)\varphi(s)ds.$$

Conclusion: In the *distribution sense*,

$$f'' = 2\delta' + f,$$

where $\langle \delta', \varphi \rangle = -\varphi'(0)$. This is the *distribution derivative* of *Dirac's delta*. In particular: f is a *distribution solution* of the differential equation

$$f'' - f = 2\delta'.$$

This has *something* to do with the differential equation on page 59. More about this later.

3.5 The Fourier Transform of a Distribution

Repetition: By Lemma 2.19, we have

$$\int_{-\infty}^{\infty} f(t)\hat{g}(t)dt = \int_{-\infty}^{\infty} \hat{f}(t)g(t)dt$$

if $f, g \in L^1(\mathbb{R})$. Take $g = \varphi \in \mathcal{S}$. Then $\hat{\varphi} \in \mathcal{S}$ (See Theorem 2.24), so we can interpret both f and $\hat{f}$ in the distribution sense and get

Definition 3.27. The Fourier transform of a distribution $f \in \mathcal{S}$ is the distribution defined by

$$\boxed{\langle \hat{f}, \varphi \rangle = \langle f, \hat{\varphi} \rangle, \quad \varphi \in \mathcal{S}.}$$

Possible, since $\varphi \in \mathcal{S} \iff \hat{\varphi} \in \mathcal{S}$.

Problem: Is this really a distribution? It is well-defined and linear, but is it continuous? To prove this we need to know that

$$\boxed{\varphi_n \to \varphi \text{ in } \mathcal{S} \iff \hat{\varphi}_n \to \hat{\varphi} \text{ in } \mathcal{S}.}$$

This is a true statement (see Gripenberg or Gasquet for a proof), and we get

Theorem 3.28. *The Fourier transform maps the class of tempered distributions onto itself:*

$$\boxed{f \in \mathcal{S}' \iff \hat{f} \in \mathcal{S}'.}$$

There is an obvious way of computing the inverse Fourier transform:

Theorem 3.29. *The inverse Fourier transform f of a distribution $\hat{f} \in \mathcal{S}'$ is given by*

$$\boxed{\langle f, \varphi \rangle = \langle \hat{f}, \psi \rangle, \quad \varphi \in \mathcal{S},}$$

where $\psi =$ the inverse Fourier transform of φ, i.e., $\psi(t) = \int_{-\infty}^{\infty} e^{2\pi i t\omega} \varphi(\omega) d\omega$.

Proof. If $\psi =$ the inverse Fourier transform of φ, then $\varphi = \hat{\psi}$ and the formula simply says that $\langle f, \hat{\psi} \rangle = \langle \hat{f}, \psi \rangle$. $\square$

3.6 The Fourier Transform of a Derivative

Problem 3.30. *Let $f \in \mathcal{S}'$. Then $f' \in \mathcal{S}'$. Find the Fourier transform of f'.*

Solution: Define $\eta(t) = 2\pi i t$, $t \in \mathbb{R}$. Then $\eta \in C^\infty_{\text{pol}}$, so we can multiply a tempered distribution by η. By various definitions (start with 3.27)

$$\begin{aligned}
\langle \widehat{(f')}, \varphi \rangle &= \langle f', \hat{\varphi} \rangle && \text{(use Definition 3.24)} \\
&= -\langle f, (\hat{\varphi})' \rangle && \text{(use Theorem 2.7(g))} \\
&= -\langle f, \hat{\psi} \rangle && \text{(where } \psi(s) = -2\pi i s \varphi(s)) \\
&= -\langle \hat{f}, \psi \rangle && \text{(by Definition 3.27)} \\
&= \langle \hat{f}, \eta\varphi \rangle && \text{(see Definition above of } \eta) \\
&= \langle \eta\hat{f}, \varphi \rangle && \text{(by Definition 3.23).}
\end{aligned}$$

Thus, $\boxed{\widehat{(f')} = \eta\hat{f}}$ where $\boxed{\eta(\omega) = 2\pi i\omega, \omega \in \mathbb{R}.}$
This proves one half of:

Theorem 3.31.

$$\boxed{\begin{aligned}
\widehat{(f')} &= (i2\pi\omega)\hat{f} \quad \textit{and} \\
(\widehat{-2\pi i t f}) &= (\hat{f})'
\end{aligned}}$$

More precisely, if we define $\eta(t) = 2\pi i t$, then $\eta \in C^\infty_{pol}$, and

$$\boxed{\widehat{(f')} = \eta\hat{f}, \ \widehat{(\eta f)} = -\hat{f}'.}$$

By repeating this result several times we get

Theorem 3.32.

$$\boxed{\begin{aligned}
\widehat{(f^{(k)})} &= (2\pi i\omega)^k \hat{f} \quad k \in \mathbb{Z}_+ \\
(\widehat{(-2\pi i t)^k f}) &= \hat{f}^{(k)}.
\end{aligned}}$$

Example 3.33. Compute the Fourier transform of

$$f(t) = \begin{cases} e^{-t}, & t > 0, \\ -e^{t}, & t < 0. \end{cases}$$

<u>Smart solution</u>: By the Examples 3.25 and 3.26.

$$f'' = 2\delta' + f \quad \text{(in the distribution sense)}.$$

Transform this:

$$[(2\pi i\omega)^2 - 1]\hat{f} = 2\widehat{(\delta')} = 2(2\pi i\omega)\hat{\delta}$$

(since δ' is the derivative of δ). Thus, we need $\hat{\delta}$:

$$\begin{aligned} \langle \hat{\delta}, \varphi \rangle &= \langle \delta, \hat{\varphi} \rangle = \hat{\varphi}(0) = \int_{\mathbb{R}} \varphi(s)ds \\ &= \int_{\mathbb{R}} 1 \cdot \varphi(s)ds = \int_{\mathbb{R}} f(s)\varphi(s)ds, \end{aligned}$$

where $f(s) \equiv 1$. Thus $\hat{\delta}$ is the distribution which is induced by the function $f(s) \equiv 1$, i.e., we may write $\boxed{\hat{\delta} \equiv 1}$.
Thus, $-(4\pi^2\omega^2 + 1)\hat{f} = 4\pi i\omega$, so $\hat{f}$ is induced by the function $\frac{4\pi i\omega}{-(1+4\pi^2\omega^2)}$. Thus,

$$\hat{f}(\omega) = \frac{4\pi i\omega}{-(1 + 4\pi^2\omega^2)}.$$

In particular:

Lemma 3.34.

$$\boxed{\begin{aligned} \hat{\delta}(\omega) &\equiv 1 \qquad \textit{and} \\ \hat{1} &= \delta. \end{aligned}}$$

(The Fourier transform of δ is the function $\equiv 1$, and the Fourier transform of the function $\equiv 1$ is the Dirac delta.)

Combining this with Theorem 3.32 we get

Lemma 3.35.

$$\boxed{\begin{aligned} \widehat{\delta^{(k)}} &= (2\pi i\omega)^k, \qquad k \in \mathbb{Z}_+ = 0, 1, 2, \ldots \\ \left[\widehat{(-2\pi it)^k}\right] &= \delta^{(k)} \end{aligned}}$$

3.7 Convolutions ("Faltningar")

It is *sometimes* (but not always) possible to define the *convolution* of two *distributions*. One possibility is the following: If $\varphi, \psi \in \mathcal{S}$, then we know that

$$\widehat{(\varphi * \psi)} = \hat{\varphi}\hat{\psi},$$

so we can define $\varphi * \psi$ to be the inverse Fourier transform of $\hat{\varphi}\hat{\psi}$. The same idea applies to distributions in *some* cases:

Definition 3.36. Let $f \in \mathcal{S}'$ and suppose that $g \in \mathcal{S}'$ happens to be such that $\hat{g} \in C^\infty_{\text{pol}}(\mathbb{R})$ (i.e., $\hat{g}$ is induced by a function in $C^\infty_{\text{pol}}(\mathbb{R})$, i.e., g is the inverse $\mathcal{F}$-transform of a function in C^∞_{pol}). Then we define

$$f * g = \text{the inverse Fourier transform of } \hat{f}\hat{g},$$

i.e. (cf. page 77):

$$\langle f * g, \varphi\rangle = \langle \hat{f}\hat{g}, \check{\varphi}\rangle$$

where $\check{\varphi}$ is the *inverse* Fourier transform of φ:

$$\check{\varphi}(t) = \int_{-\infty}^{\infty} e^{2\pi i\omega t}\varphi(\omega)d\omega.$$

This is possible since $\hat{g} \in C^\infty_{\text{pol}}$, so that $\hat{f}\hat{g} \in \mathcal{S}'$; see page 74

To get a direct interpretation (which does not involve Fourier transforms) we need two more definitions:

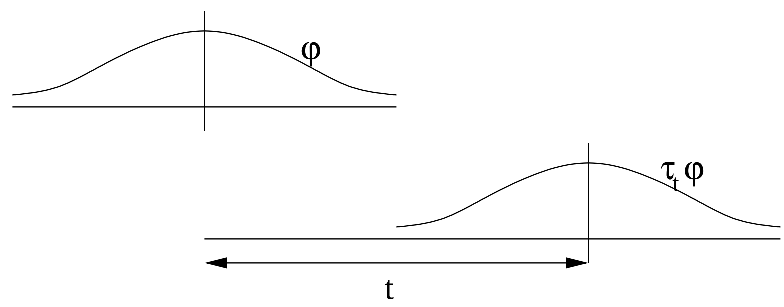
Definition 3.37. Let $t \in \mathbb{R}$, $f \in \mathcal{S}'$, $\varphi \in \mathcal{S}$. Then the **translations** $\tau_t f$and $\tau_t\varphi$ are given by

$$\boxed{\begin{array}{rcll} (\tau_t\varphi)(s) & = & \varphi(s-t), & s \in \mathbb{R} \\ \langle \tau_t f, \varphi\rangle & = & \langle f, \tau_{-t}\varphi\rangle & \end{array}}$$

<u>Motivation</u>: $\tau_t\varphi$ translates φ to the right by the amount t (if $t > 0$, to the left if $t < 0$).

For ordinary functions f we have

$$\begin{aligned} \int_{-\infty}^{\infty} (\tau_t f)(s)\varphi(s)ds &= \int_{-\infty}^{\infty} f(s-t)\varphi(s)ds \quad (s-t=v) \\ &= \int_{-\infty}^{\infty} f(v)\varphi(v+t)dv \\ &= \int_{-\infty}^{\infty} f(v)\tau_{-t}\varphi(v)dv, \end{aligned}$$

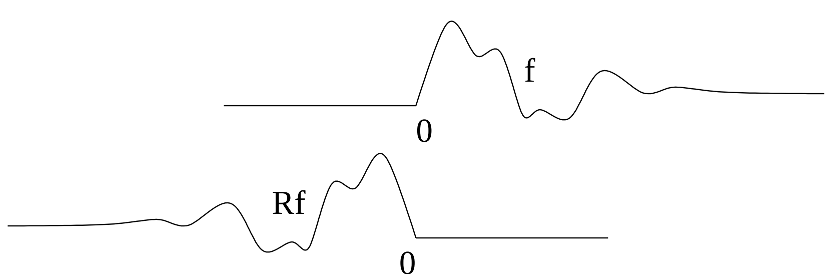


so the distribution definition coincides with the usual definition for functions interpreted as distributions.

Definition 3.38. The **reflection operator** R is defined by

$$\boxed{\begin{array}{rcll} (R\varphi)(s) & = & \varphi(-s), & \varphi \in \mathcal{S}, \\ \langle Rf, \varphi \rangle & = & \langle f, R\varphi \rangle, & f \in \mathcal{S}',\ \varphi \in \mathcal{S} \end{array}}$$

<u>Motivation</u>: Extra homework. If $f \in L^1(\mathbb{R})$ and $\eta \in \mathcal{S}$, then we can write $f * \varphi$



in the form

$$\begin{aligned} (f * \varphi)(t) &= \int_{\mathbb{R}} f(s)\eta(t-s)ds \\ &= \int_{\mathbb{R}} f(s)(R\eta)(s-t)ds \\ &= \int_{\mathbb{R}} f(s)(\tau_t R\eta)(s)ds, \end{aligned}$$

and we get an *alternative* formula for $f * \eta$ in this case.

Theorem 3.39. *If $f \in \mathcal{S}'$ and $\eta \in \mathcal{S}$, then $f * \eta$ as defined in Definition 3.36, is induced by the function*

$$t \mapsto \langle f, \tau_t R\eta \rangle,$$

and this function belongs to $C^\infty_{pol}(\mathbb{R})$.

We shall give a partial proof of this theorem (skipping the most complicated part). It is based on some auxiliary results which will be used later, too.

Lemma 3.40. *Let $\varphi \in \mathcal{S}$, and let*

$$\varphi_\varepsilon(t) = \frac{\varphi(t+\varepsilon) - \varphi(t)}{\varepsilon}, \quad t \in \mathbb{R}.$$

Then $\varphi_\varepsilon \to \varphi'$ in $\mathcal{S}$ as $\varepsilon \to 0$.

Proof. (Outline) Must show that

$$\lim_{\varepsilon\to 0} \sup_{t\in\mathbb{R}} |t|^k |\varphi_\varepsilon^{(m)}(t) - \varphi^{(m+1)}(t)| = 0$$

for all $t, m \in \mathbb{Z}_+$. By the mean value theorem,

$$\varphi^{(m)}(t+\varepsilon) = \varphi^{(m)}(t) + \varepsilon\varphi^{(m+1)}(\xi)$$

where $t < \xi < t + \varepsilon$ (if $\varepsilon > 0$). Thus

$$\begin{aligned} |\varphi_\varepsilon^{(m)}(t) - \varphi^{(m+1)}(t)| &= |\varphi^{(m+1)}(\xi) - \varphi^{(m+1)}(t)| \\ &= |\int_\xi^t \varphi^{(m+2)}(s)ds| \quad \left(\begin{array}{l} \text{where } t < \xi < t+\varepsilon \text{ if } \varepsilon > 0 \\ \text{or } t+\varepsilon < \xi < t \text{ if } \varepsilon < 0 \end{array}\right) \\ &\le \int_{t-|\varepsilon|}^{t+|\varepsilon|} |\varphi^{(m+2)}(s)|ds, \end{aligned}$$

and this multiplied by $|t|^k$ tends uniformly to zero as $\varepsilon \to 0$. (Here I am skipping a couple of lines). $\square$

Lemma 3.41. *For every $f \in \mathcal{S}'$ there exist two numbers $M > 0$ and $N \in \mathbb{Z}_+$ so that*

$$|\langle f, \varphi\rangle| \le M \max_{\substack{0\le j,k\le N \\ t\in\mathbb{R}}} |t^j \varphi^{(k)}(t)|. \tag{3.2}$$

Interpretation: Every $f \in \mathcal{S}'$ has a *finite order* (we need only derivatives $\varphi^{(k)}$ where $k \le N$) and a finite *polynomial growth rate* (we need only a finite power t^j with $j \le N$).

Proof. Assume to get a contradiction that (3.2) is false. Then for all $n \in \mathbb{Z}_+$, there is a function $\varphi_n \in \mathcal{S}$ so that

$$|\langle f, \varphi_n\rangle| \ge n \max_{\substack{0\le j,k\le n \\ t\in\mathbb{R}}} |t^j \varphi_n^{(k)}(t)|.$$

Multiply φ_n by a constant to make $\langle f, \varphi_n \rangle = 1$. Then

$$\max_{\substack{0 \le j,k \le n \\ t \in \mathbb{R}}} |t^j \varphi_n^{(k)}(t)| \le \frac{1}{n} \to 0 \text{ as } n \to \infty,$$

so $\varphi_n \to 0$ in $\mathcal{S}$ as $n \to \infty$. As f is continuous, this implies that $\langle f, \varphi_n \rangle \to 0$ as $n \to \infty$. This contradicts the assumption $\langle f, \varphi_n \rangle = 1$. Thus, (3.2) cannot be false. $\square$

Theorem 3.42. *Define $\varphi(t) = \langle f, \tau_t R\eta \rangle$. Then $\varphi \in C^\infty_{pol}$, and for all $n \in \mathbb{Z}_+$,*

$$\varphi^{(n)}(t) = \langle f^{(n)}, \tau_t R\eta \rangle = \langle f, \tau_t R\eta^{(n)} \rangle.$$

Note: As soon as we have proved Theorem 3.39, we may write this as

$$(f * \eta)^{(n)} = f^{(n)} * \eta = f * \eta^{(n)}.$$

Thus, to differentiate $f * \eta$ it suffices to differentiate either f or η (but not both). The derivatives may also be distributed between f and η:

$$(f * \eta)^{(n)} = f^{(k)} * \eta^{(n-k)}, \quad 0 \le k \le n.$$

Motivation: A formal differentiation of

$$\begin{aligned} (f * \varphi)(t) &= \int_{\mathbb{R}} f(t-s)\varphi(s)ds \quad \text{gives} \\ (f * \varphi)' &= \int_{\mathbb{R}} f'(t-s)\varphi(s)ds = f' * \varphi, \end{aligned}$$

and a formal differentiation of

$$\begin{aligned} (f * \varphi)(t) &= \int_{\mathbb{R}} f(s)\varphi(t-s)ds \quad \text{gives} \\ (f * \varphi)' &= \int_{\mathbb{R}} f(s)\varphi'(t-s)ds = f * \varphi'. \end{aligned}$$

Proof of Theorem 3.42.

i) $\frac{1}{\varepsilon}[\varphi(t+\varepsilon) - \varphi(t)] = \langle f, \frac{1}{\varepsilon}(\tau_{t+\varepsilon} R\eta - \tau_t R\eta) \rangle$. Here

$$\begin{aligned} \frac{1}{\varepsilon}(\tau_{t+\varepsilon} R\eta - \tau_t R\eta)(s) &= \frac{1}{\varepsilon}[(R\eta)(s-t-\varepsilon) - R\eta(s-t)] \\ &= \frac{1}{\varepsilon}[\eta(t+\varepsilon-s) - \eta(t-s)] \quad \text{(by Lemma 3.40)} \\ &\to \eta'(t-s) = (R\eta')(s-t) = \tau_t R\eta'. \end{aligned}$$

Thus, the following limit exists:

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[\varphi(t+\varepsilon)-\varphi(t)] = \langle f, \tau_t R\eta'\rangle.$$

Repeating the same argument n times we find that φ is n times differentiable, and that

$$\varphi^{(n)} = \langle f, \tau_t R\eta^{(n)}\rangle$$

(or written differently, $(f*\eta)^{(n)} = f*\eta^{(n)}$.)

ii) A direct computation shows: If we put

$$\psi(s) = \eta(t-s) = (R\eta)(s-t) = (\tau_t R\eta)(s),$$

then $\psi'(s) = -\eta'(t-s) = -\tau_t R\eta'$. Thus $\langle f, \tau_t R\eta'\rangle = -\langle f, \psi'\rangle = \langle f', \psi\rangle = \langle f', \tau_t R\eta\rangle$ (by the definition of distributed derivative). Thus, $\varphi' = \langle f, \tau_t R\eta'\rangle = \langle f', \tau_t R\eta\rangle$ (or written differently, $f*\eta' = f'*\eta$). Repeating this n times we get

$$f*\eta^{(n)} = f^{(n)}*\eta.$$

iii) The estimate which shows that $\varphi\in C^\infty_{\text{pol}}$: By Lemma 3.41,

$$\begin{aligned}
|\varphi^{(n)}(t)| &= |\langle f^{(n)}, \tau_t R\eta\rangle| \\
&\le M \max_{\substack{0\le j,k\le N\\ s\in\mathbb{R}}} |s^j(\tau_t R\eta)^{(k)}(s)| \quad (\psi \text{ as above}) \\
&= M \max_{\substack{0\le j,k\le N\\ s\in\mathbb{R}}} |s^j\eta^{(k)}(t-s)| \quad (t-s=v) \\
&= M \max_{\substack{0\le j,k\le N\\ v\in\mathbb{R}}} |(t-v)^j\eta^{(k)}(s)| \\
&\le \text{a polynomial in } |t|. \quad \square
\end{aligned}$$

To prove Theorem 3.39 it suffices to prove the following lemma (if two distributions have the same Fourier transform, then they are equal):

Lemma 3.43. *Define $\varphi(t) = \langle f, \tau_t R\eta\rangle$. Then $\hat\varphi = \hat f\hat\eta$.*

Proof. (Outline) By the distribution definition of $\hat\varphi$:

$$\langle\hat\varphi, \psi\rangle = \langle\varphi, \hat\psi\rangle \quad \text{for all } \psi\in\mathcal{S}.$$

We compute this:

$$
\begin{aligned}
\langle \varphi, \hat{\psi} \rangle &= \int_{-\infty}^{\infty} \underbrace{\varphi(s)}_{\substack{\text{function} \\ \text{in } C_{\text{pol}}^{\infty}}} \hat{\psi}(s) ds \\
&= \int_{-\infty}^{\infty} \langle f, \tau_s R\eta \rangle \hat{\psi}(s) ds \\
&= \text{(this step is \textit{too difficult}: To show that we may move} \\
&\quad \text{the integral to the other side of } f \text{ requires more theory} \\
&\quad \text{then we have time to present)} \\
&= \langle f, \int_{-\infty}^{\infty} \tau_s R\eta \hat{\varphi}(s) ds \rangle = (\star)
\end{aligned}
$$

Here $\tau_s R\eta$ is the function

$$
(\tau_s R\eta)(t) = (R\eta)(t-s) = \eta(s-t) = (\tau_t \eta)(s),
$$

so the integral is

$$
\begin{aligned}
\int_{-\infty}^{\infty} \eta(s-t) \hat{\psi}(s) ds &= \int_{-\infty}^{\infty} (\tau_t \eta)(s) \hat{\psi}(s) ds \quad \text{(see page 43)} \\
&= \int_{-\infty}^{\infty} \widehat{(\tau_t \eta)}(s) \psi(s) ds \quad \text{(see page 38)} \\
&= \underbrace{\int_{-\infty}^{\infty} e^{-2\pi i t s} \hat{\eta}(s) \psi(s) ds}_{\mathcal{F}\text{-transform of } \hat{\eta}\psi} \\
(\star) = \langle f, \widehat{\hat{\eta}\psi} \rangle &= \langle \hat{f}, \hat{\eta}\psi \rangle \\
&= \langle \hat{f}\hat{\eta}, \psi \rangle. \quad \text{Thus, } \hat{\varphi} = \hat{f}\hat{\eta}. \quad \square
\end{aligned}
$$

Using this result it is easy to prove:

Theorem 3.44. *Let $f \in \mathcal{S}'$, $\varphi, \psi \in \mathcal{S}$. Then*

$$
\underbrace{\underbrace{(f * \varphi)}_{in\ C_{pol}^{\infty}} * \underbrace{\psi}_{in\ \mathcal{S}}}_{in\ C_{pol}^{\infty}} = \underbrace{\underbrace{f}_{in\ \mathcal{S}'} * \underbrace{(\varphi * \psi)}_{in\ \mathcal{S}}}_{in\ C_{pol}^{\infty}}
$$

Proof. Take the Fourier transforms:

$$
\underbrace{\underbrace{(f * \varphi)}_{\substack{\downarrow \\ \hat{f}\hat{\varphi}}} * \underbrace{\psi}_{\substack{\downarrow \\ \hat{\psi}}}}_{(\hat{f}\hat{\varphi})\hat{\psi}} = \underbrace{\underbrace{f}_{\substack{\downarrow \\ \hat{f}}} * \underbrace{(\varphi * \psi)}_{\substack{\downarrow \\ \hat{\varphi}\hat{\psi}}}}_{\hat{f}(\hat{\varphi}\hat{\psi})}.
$$

The transforms are the same, hence so are the original distributions (note that both $(f * \varphi) * \psi$ and $f * (\varphi * \psi)$ are in C^∞_{pol} so we are allowed to take distribution Fourier transforms).

3.8 Convergence in $\mathcal{S}'$

We define convergence in $\mathcal{S}'$ by means of test functions in $\mathcal{S}$. (This is a special case of "weak" or "weak*"-convergence).

Definition 3.45. $f_n \to f$ in $\mathcal{S}'$ means that

$$\boxed{\langle f_n, \varphi\rangle \to \langle f, \varphi\rangle \quad \text{for all } \varphi \in \mathcal{S}.}$$

Lemma 3.46. *Let $\eta \in \mathcal{S}$ with $\hat{\eta}(0) = 1$, and define $\eta_\lambda(t) = \lambda\eta(\lambda t)$, $t \in \mathbb{R}$, $\lambda > 0$. Then, for all $\varphi \in \mathcal{S}$,*

$$\boxed{\eta_\lambda * \varphi \to \varphi \textit{ in } \mathcal{S} \textit{ as } \lambda \to \infty.}$$

<u>Note</u>: We had this type of "δ-sequences" also in the L^1-theory on page 36.

Proof. (Outline.) The Fourier transform is continuous $\mathcal{S} \to \mathcal{S}$ (which we have not proved, but it is true). Therefore

$$\begin{aligned} \eta_\lambda * \varphi \to \varphi \text{ in } \mathcal{S} \quad &\iff \quad \widehat{\eta_\lambda * \varphi} \to \hat{\varphi} \text{ in } \mathcal{S} \\ &\iff \quad \hat{\eta}_\lambda \hat{\varphi} \to \hat{\varphi} \text{ in } \mathcal{S} \\ &\iff \quad \hat{\eta}(\omega/\lambda)\hat{\varphi}(\omega) \to \hat{\varphi}(\omega) \text{ in } \mathcal{S} \text{ as } \lambda \to \infty. \end{aligned}$$

Thus, we must show that

$$\sup_{\omega \in \mathbb{R}} \left| \omega^k \left(\frac{d}{d\omega}\right)^j [\hat{\eta}(\omega/\lambda) - 1]\hat{\varphi}(\omega) \right| \to 0 \text{ as } \lambda \to \infty.$$

This is a "straightforward" mechanical computation (which does take some time). $\square$

Theorem 3.47. *Define η_λ as in Lemma 3.46. Then*

$$\eta_\lambda \to \delta \textit{ in } \mathcal{S}' \textit{ as } \lambda \to \infty.$$

<u>Comment</u>: This is the reason for the name "δ-sequence".

Proof. The claim (="påstående") is that for all $\varphi \in \mathcal{S}$,

$$\int_R \eta_\lambda(t)\varphi(t)dt \to \langle \delta, \varphi\rangle = \varphi(0) \text{ as } \lambda \to \infty.$$

(Or equivalently, $\int_R \lambda\eta(\lambda t)\varphi(t)dt \to \varphi(0)$ as $\lambda \to \infty$). Rewrite this as

$$\int_{\mathbb{R}} \eta_\lambda(t)(R\varphi)(-t)dt = (\eta_\lambda * R\varphi)(0),$$

and by Lemma 3.46, this tends to $(R\varphi)(0) = \varphi(0)$ as $\lambda \to \infty$. Thus,

$$\langle \eta_\lambda, \varphi \rangle \to \langle \delta, \varphi \rangle \quad \text{for all } \varphi \in \mathcal{S} \text{ as } \lambda \to \infty,$$

so $\eta_\lambda \to \delta$ in $\mathcal{S}'$. $\square$

Theorem 3.48. *Define η_λ as in Lemma 3.46. Then, for all $f \in \mathcal{S}'$, we have*

$$\eta_\lambda * f \to f \text{ in } \mathcal{S}' \text{ as } \lambda \to \infty.$$

Proof. The claim is that

$$\langle \eta_\lambda * f, \varphi \rangle \to \langle f, \varphi \rangle \quad \text{for all } \varphi \in \mathcal{S}.$$

Replace φ with the reflected

$$\begin{aligned}
\psi = R\varphi \quad &\Longrightarrow \quad \langle \eta_\lambda * f, R\psi \rangle \to \langle f, R\psi \rangle \text{ for all } \varphi \in \mathcal{S} \\
&\Longleftrightarrow \quad \text{(by Thm 3.39) } ((\eta_\lambda * f) * \psi)(0) \to (f * \psi)(0) \text{ (use Thm 3.44)} \\
&\Longleftrightarrow \quad f * (\eta_\lambda * \psi)(0) \to (f * \psi)(0) \text{ (use Thm 3.39)} \\
&\Longleftrightarrow \quad \langle f, R(\eta_\lambda * \psi) \rangle \to \langle f, R\psi \rangle.
\end{aligned}$$

This is true because f is continuous and $\eta_\lambda * \psi \to \psi$ in $\mathcal{S}$, according to Lemma 3.46.

There is a *General Rule* about *distributions*:

Metatheorem: All *reasonable* claims about *distribution convergence* are true.

Problem: What is "reasonable"?

Among others, the following results are reasonable:

Theorem 3.49. *All the operations on distributions and test functions which we have defined are continuous. Thus, if*

$$\begin{aligned}
&f_n \to f \text{ in } \mathcal{S}',\ g_n \to g \text{ in } \mathcal{S}', \\
&\psi_n \to \psi \text{ in } C^\infty_{pol} \text{ (which we have not defined!)}, \\
&\varphi_n \to \varphi \text{ in } \mathcal{S}, \\
&\lambda_n \to \lambda \text{ in } \mathbb{C}, \text{ then, among others,}
\end{aligned}$$

i) $f_n + g_n \to f + g$ *in* $\mathcal{S}'$

ii) $\lambda_n f_n \to \lambda f$ *in* $\mathcal{S}'$

iii) $\psi_n f_n \to \psi f$ *in* $\mathcal{S}'$

iv) $\check{\psi}_n * f_n \to \check{\psi} * f$ *in* $\mathcal{S}'$ ($\check{\psi}$ *=inverse* $\mathcal{F}$*-transform of* ψ)

v) $\varphi_n * f_n \to \varphi * f$ *in* C^∞_{pol}

vi) $f'_n \to f'$ *in* $\mathcal{S}'$

vii) $\hat{f}_n \to \hat{f}$ *in* $\mathcal{S}'$ *etc.*

PROOF. "Easy" but long.

3.9 Distribution Solutions of ODE:s

Example 3.50. Find the function $u \in L^2(\mathbb{R}_+) \cap C^1(\mathbb{R}_+)$ with an "absolutely continuous" derivative u' which satisfies the equation

$$\begin{cases} u''(x) - u(x) &= f(x), \quad x > 0, \\ u(0) &= 1. \end{cases}$$

Here $f \in L^2(\mathbb{R}_+)$ is given.

SOLUTION. Let v be the solution of homework 22. Then

$$\begin{cases} v''(x) - v(x) &= f(x), \quad x > 0, \\ v(0) &= 0. \end{cases} \tag{3.3}$$

Define $w = u - v$. Then w is a solution of

$$\begin{cases} w''(x) - w(x) &= 0, \quad x \geq 0, \\ w(0) &= 1. \end{cases} \tag{3.4}$$

In addition we require $w \in L^2(\mathbb{R}_+)$.

ELEMENTARY SOLUTION. The characteristic equation is

$$\lambda^2 - 1 = 0, \quad \text{roots } \lambda = \pm 1,$$

general solution

$$w(x) = c_1 e^x + c_2 e^{-x}.$$

The condition $w(x) \in L^2(\mathbb{R}_+)$ forces $c_1 = 0$. The condition $w(0) = 1$ gives $w(0) = c_2 e^0 = c_2 = 1$. Thus: $w(x) = e^{-x}$, $x \geq 0$.
Original solution: $u(x) = e^{-x} + v(x)$, where v is a solution of homework 22, i.e.,

$$u(x) = e^{-x} + \frac{1}{2}e^{-x}\int_0^\infty e^{-y} f(y)dy - \frac{1}{2}\int_0^\infty e^{-|x-y|} f(y)dy.$$

Distribution solution. Make w an *even* function, and differentiate: we denote the *distribution* derivatives by $w^{(1)}$ and $w^{(2)}$. Then

$$\begin{aligned} w^{(1)} &= w' \quad \text{(since } w \text{ is continuous at zero)} \\ w^{(2)} &= w'' + \underbrace{2w'(0)}_{\substack{\text{due to jump} \\ \text{discontinuity} \\ \text{at zero in } w'}} \delta_0 \quad \text{(Dirac delta at the point zero)} \end{aligned}$$

The problem says: $w'' = w$, so

$$\begin{aligned} & w^{(2)} - w = 2w'(0)\delta_0. \quad \text{Transform:} \\ & ((2\pi i\gamma)^2 - 1)\hat{w}(\gamma) = 2w'(0) \quad (\text{since } \hat{\delta}_0 \equiv 1) \\ & \Longrightarrow \hat{w}(\gamma) = \tfrac{2w'(0)}{1+4\pi^2\gamma^2}, \end{aligned}$$

whose inverse transform is $-w'(0)e^{-|x|}$ (see page 62). We are only interested in values $x \geq 0$ so

$$w(x) = -w'(0)e^{-x}, \quad x > 0.$$

The condition $w(0) = 1$ gives $-w'(0) = 1$, so

$$w(x) = e^{-x}, \quad x \geq 0.$$

Example 3.51. Solve the equation

$$\begin{cases} u''(x) - u(x) &= f(x), \quad x > 0, \\ u'(0) &= a, \end{cases}$$

where a =given constant, $f(x)$ given function.

Many different ways exist to attack this problem:
Method 1. Split u in two parts: $u = v + w$, where

$$\begin{cases} v''(x) - v(x) &= f(x), \quad x > 0 \\ v'(0) &= 0, \end{cases}$$

and

$$\begin{cases} w''(x) - w(x) = 0, & x > 0 \\ w'(0) = a, & \end{cases}$$

We can solve the first equation by making an *even* extension of v. The second equation can be solved as above.

METHOD 2. Make an *even* extension of u and transform. Let $u^{(1)}$ and $u^{(2)}$ be the distribution derivatives of u. Then as above,

$$\begin{aligned} u^{(1)} &= u' \quad (u \text{ is continuous}) \\ u^{(2)} &= u'' + 2\underbrace{u'(0)}_{=a}\delta_0 \quad (u' \text{ discontinuous}) \end{aligned}$$

By the equation: $u'' = u + f$, so

$$u^{(2)} - u = 2a\delta_0 + f$$

Transform this:

$$\begin{aligned} &[(2\pi i\gamma)^2 - 1]\hat{u} = 2a + \hat{f}, \quad \text{so} \\ &\hat{u} = \tfrac{-2a}{1+4\pi^2\gamma^2} - \tfrac{\hat{f}}{1+4\pi^2\gamma^2} \end{aligned}$$

Invert:

$$u(x) = -ae^{-|x|} - \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|}f(y)dy.$$

Since f is *even*, this becomes for $x > 0$:

$$u(x) = -ae^{-x} - \frac{1}{2}e^{-x}\int_0^{\infty} e^{-y}f(y)dy - \frac{1}{2}\int_0^{\infty} e^{-|x-y|}f(y)dy.$$

METHOD 3. The method to make u and f even or odd works, but it is a "dirty trick" which has to be memorized. A simpler method is to define $u(t) \equiv 0$ and $f(t) \equiv 0$ for $t < 0$, and to continue as above. We shall return to this method in connection with the *Laplace transform*.

Partial Differential Equations are solved in a similar manner. The computations become slightly more complicated, and the *motivations* become much more complicated. For example, we can replace all the functions in the examples on page 63 and 64 by distributions, and the results "stay the same".

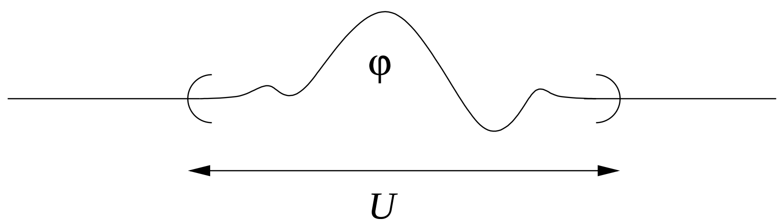
3.10 The Support and Spectrum of a Distribution

"Support" = "the piece of the real line on which the distribution stands"

Definition 3.52. The **support** of a continuous function φ is the closure (="slutna höljet") of the set $\{x \in \mathbb{R} : \ \varphi(x) \neq 0\}$.

Note: The set $\{x \in \mathbb{R} : \ \varphi(x) \neq 0\}$ is open, but the support contains, in addition, the boundary points of this set.

Definition 3.53. Let $f \in \mathcal{S}'$ and let $\mathcal{U} \subset \mathbb{R}$ be an open set. Then f **vanishes on** $\mathcal{U}$ (="försvinner på $\mathcal{U}$") if $\langle f, \varphi \rangle = 0$ for all test functions $\varphi \in \mathcal{S}$ whose support is contained in $\mathcal{U}$.



Interpretation: f has "no mass in $\mathcal{U}$", "no action on $\mathcal{U}$".

Example 3.54. δ vanishes on $(0, \infty)$ and on $(-\infty, 0)$. Likewise vanishes $\delta^{(k)}$ ($k \in \mathbb{Z}_+ = 0, 1, 2, \ldots$) on $(-\infty, 0) \cup (0, \infty)$.

Proof. Obvious.

Example 3.55. The function

$$f(t) = \begin{cases} 1 - |t|, & |t| \leq 1, \\ 0, & |t| > 1, \end{cases}$$

vanishes on $(-\infty, -1)$ and on $(1, \infty)$. The support of this function is $[-1, 1]$ (note that the end points are *included*).

Definition 3.56. Let $f \in \mathcal{S}'$. Then the **support** of f is the complement of the largest set on which f vanishes. Thus,

$$\operatorname{supp}(f) = M \Leftrightarrow \begin{cases} M \text{ is closed, } f \text{ vanishes on } \mathbb{R} \setminus M, \text{ and} \\ f \text{ does not vanish on any open set } \Omega \\ \text{which is strictly bigger than } \mathbb{R} \setminus M. \end{cases}$$

Example 3.57. The support of the distribution $\delta_a^{(k)}$ is the single point $\{a\}$. Here $k \in \mathbb{Z}_+$, and δ_a is point evaluation at a:

$$\langle \delta_a, \varphi \rangle = \varphi(a).$$

Definition 3.58. The **spectrum** of a distribution $f \in \mathcal{S}'$ is the **support of** $\hat{f}$.

Lemma 3.59. *If $M \subset \mathbb{R}$ is closed, then* $\operatorname{supp}(f) \subset M$ *if and only if f vanishes on $\mathbb{R} \setminus M$.*

PROOF. Easy. □

Example 3.60. Interpret $f(t) = t^n$ as a distribution. Then $\hat{f} = \frac{1}{(-2\pi i)^n}\delta^{(n)}$, as we saw on page 78. Thus the support of $\hat{f}$ is $\{0\}$, so the spectrum of f is $\{0\}$.

By adding such functions we get:

Theorem 3.61. *The spectrum of the function $f(t) \equiv 0$ is* empty. *The spectrum of every other polynomial is the single point $\{0\}$.*

PROOF. $f(t) \equiv 0 \iff$ spectrum is empty follows from definition. The other half is proved above. □
The *converse* is true, but much harder to prove:

Theorem 3.62. *If $f \in \mathcal{S}'$ and if the spectrum of f is $\{0\}$, then f is a polynomial $(\not\equiv 0)$.*

This follows from the following theorem by taking Fourier transforms:

Theorem 3.63. *If the support of f is one single point $\{a\}$ then f can be written as a* finite *sum*

$$f = \sum_{k=0}^{n} a_n \delta_a^{(k)}.$$

PROOF. Too difficult to include. See e.g., Rudin's "Functional Analysis".
Possible homework: Show that

Theorem 3.64. *The spectrum of f is $\{a\} \iff f(t) = e^{2\pi i a t}P(t)$, where P is a polynomial, $P \not\equiv 0$.*

Theorem 3.65. *Suppose that $f \in \mathcal{S}'$ has a bounded support, i.e., f vanishes on $(-\infty, -T)$ and on (T, ∞) for some $T > 0$ ($\iff$ $\operatorname{supp}(f) \subset [-T, T]$). Then $\hat{f}$ can be interpreted as a function, namely as*

$$\hat{f}(\omega) = \langle f, \eta(t)e^{-2\pi i \omega t}\rangle,$$

where $\eta \in \mathcal{S}$ is an arbitrary function satifying $\eta(t) \equiv 1$ for $t \in [-T-1, T+1]$ (or, more generally, for $t \in \mathcal{U}$ where $\mathcal{U}$ is an open set containing $\operatorname{supp}(f)$*). Moreover, $\hat{f} \in C^\infty_{pol}(\mathbb{R})$.*

Proof. (Not quite complete)

Step 1. Define

$$\psi(\omega) = \langle f, \eta(t)e^{-2\pi i\omega t}\rangle,$$

where η is as above. If we choose two *different* η_1 and η_2, then $\eta_1(t) - \eta_2(t) = 0$ is an open set $\mathcal{U}$ containing supp(f). Since f vanishes on $\mathbb{R}\backslash\mathcal{U}$, we have

$$\langle f, \eta_1(t)e^{-2\pi i\omega t}\rangle = \langle f, \eta_2(t)e^{-2\pi i\omega t}\rangle,$$

so $\psi(\omega)$ *does not depend* on *how we choose* η.

Step 2. For simplicity, choose $\eta(t)$ so that $\eta(t) \equiv 0$ for $|t| > T+1$ (where T as in the theorem). A "simple" but boring computation shows that

$$\frac{1}{\varepsilon}[e^{-2\pi i(\omega+\varepsilon)t} - e^{-2\pi i\omega t}]\eta(t) \to \frac{\partial}{\partial\omega}e^{-2\pi i\omega t}\eta(t) = -2\pi i t e^{-2\pi i\omega t}\eta(t)$$

in $\mathcal{S}$ as $\varepsilon \to 0$ (all derivatives converge uniformly on $[-T-1, T+1]$, and everything is $\equiv 0$ outside this interval). Since we have convergence in $\mathcal{S}$, also the following limit exists:

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(\psi(\omega+\varepsilon) - \psi(\omega)) &= \psi'(\omega)\\ &= \lim_{\varepsilon\to 0}\langle f, \frac{1}{\varepsilon}(e^{-2\pi i(\omega+\varepsilon)t} - e^{-2\pi i\omega t})\eta(t)\rangle\\ &= \langle f, -2\pi i t e^{-2\pi i\omega t}\eta(t)\rangle.\end{aligned}$$

Repeating the same computation with η replaced by $(-2\pi i t)\eta(t)$, etc., we find that ψ is infinitely many times differentiable, and that

$$\psi^{(k)}(\omega) = \langle f, (-2\pi i t)^k e^{-2\pi i\omega t}\eta(t)\rangle, \quad k \in \mathbb{Z}_+. \tag{3.5}$$

Step 3. Show that the derivatives grow at most polynomially. By Lemma 3.41, we have

$$|\langle f, \varphi\rangle| \le M \max_{\substack{0\le t,l\le N\\ t\in\mathbb{R}}} |t^j\varphi^{(l)}(t)|.$$

Apply this to (3.5) $\Longrightarrow$

$$|\psi^{(k)}(\omega)| \le M \max_{\substack{0\le j,l\le N\\ t\in\mathbb{R}}} \left|t^j\left(\frac{d}{dt}\right)^l (-2\pi i t)^k e^{-2\pi i\omega t}\eta(t)\right|.$$

The derivative $l = 0$ gives a constant independent of ω.

The derivative $l = 1$ gives a constant times $|\omega|$.

The derivative $l = 2$ gives a constant times $|\omega|^2$, etc.

Thus, $|\psi^{(k)}(\omega)| \leq$ const. $x[1 + |w|^N]$, so $\psi \in C^\infty_{\text{pol}}$.

Step 4. Show that $\psi = \hat{f}$. That is, show that

$$\int_{\mathbb{R}} \psi(\omega)\varphi(\omega)d\omega = \langle \hat{f}, \varphi\rangle (= \langle f, \hat{\varphi}\rangle).$$

Here we need the same "advanced" step as on page 83:

$$\begin{aligned}
\int_{\mathbb{R}} \psi(\omega)\varphi(\omega)d\omega &= \int_{\mathbb{R}} \langle f, e^{-2\pi i\omega t}\eta(t)\varphi(\omega)\rangle d\omega \\
&= (\text{why??}) = \langle f, \int_{\mathbb{R}} e^{-2\pi i\omega t}\eta(t)\varphi(\omega)d\omega\rangle \\
&= \langle f, \eta(t)\hat{\varphi}(t)\rangle \quad \begin{pmatrix} \text{since } \eta(t) \equiv 1 \text{ in a} \\ \text{neighborhood of } \operatorname{supp}(f) \end{pmatrix} \\
&= \langle f, \hat{\varphi}\rangle.
\end{aligned}$$

A *very short* explanation of why "why??" is permitted: Replace the integral by a Riemann sum, which converges in $\mathcal{S}$, i.e., approximate

$$\int_{\mathbb{R}} e^{-2\pi i\omega t}\varphi(\omega)d\omega = \lim_{n\to\infty} \sum_{k=-\infty}^{\infty} e^{-2\pi i\omega_k t}\varphi(\omega_k)\frac{1}{n},$$

where $\omega_k = k/n$.

3.11 Trigonometric Polynomials

Definition 3.66. A **trigonometric polynomial** is a sum of the form

$$\psi(t) = \sum_{j=1}^{m} c_j e^{2\pi i\omega_j t}.$$

The numbers ω_j are called the **frequencies** of ψ.

Theorem 3.67. *If we interpret ψ as a polynomial then the* spectrum *of ψ is $\{\omega_1, \omega_2, \ldots, \omega_m\}$, i.e., the spectrum consists of the* frequencies *of the polynomial.*

Proof. Follows from homework 27, since $\operatorname{supp}(\delta_{\omega_j}) = \{\omega_j\}$. □

Example 3.68. Find the spectrum of the Weierstrass function

$$\sigma(t) = \sum_{k=0}^{\infty} a^k \cos(2\pi b^k t),$$

where $0 < a < 1$, $ab \geq 1$.

To solve this we need the following lemma

Lemma 3.69. *Let $0 < a < 1$, $b > 0$. Then*

$$\sum_{k=0}^{N} a^k \cos(2\pi b^k t) \to \sum_{k=0}^{\infty} a^k \cos(2\pi b^k t)$$

in $\mathcal{S}'$ as $N \to \infty$.

Proof. Easy. Must show that for all $\varphi \in \mathcal{S}$,

$$\int_{\mathbb{R}} \left(\sum_{k=0}^{N} - \sum_{k=0}^{\infty} \right) a^k \cos(2\pi b^k t)\varphi(t)dt \to 0 \text{ as } N \to \infty.$$

This is true because

$$\begin{aligned} \int_{\mathbb{R}} \sum_{k=N+1}^{\infty} |a^k \cos(2\pi b^k t)\varphi(t)|dt &\le \int_{\mathbb{R}} \sum_{k=N+1}^{\infty} a^k |\varphi(t)|dt \\ &\le \sum_{k=N+1}^{\infty} a^k \int_{-\infty}^{\infty} |\varphi(t)|dt \\ &= \frac{a^{N+1}}{1-a} \int_{-\infty}^{\infty} |\varphi(t)|dt \to 0 \text{ as } N \to \infty. \end{aligned}$$

<u>Solution of 3.68</u>: Since $\sum_{k=0}^{N} \to \sum_{k=0}^{\infty}$ in $\mathcal{S}'$, also the Fourier transforms converge in $\mathcal{S}'$, so to find $\hat{\sigma}$ it is enough to find the transform of $\sum_{k=0}^{N} a^k \cos(2\pi b^k t)$ and to let $N \to \infty$. This transform is

$$\delta_0 + \frac{1}{2}[a(\delta_b + \delta_{-b}) + a^2(\delta_{-b^2} + \delta_{b^2}) + \ldots + a^N(\delta_{-b^N} + \delta_{b^N})].$$

Thus,

$$\hat{\sigma} = \delta_0 + \frac{1}{2} \sum_{n=1}^{\infty} a^n(\delta_{-b^n} + \delta_{b^n}),$$

where the sum converges in $\mathcal{S}'$, and the support of this is $\{0, \pm b, \pm b^2, \pm b^3, \ldots\}$, which is also the spectrum of σ.

Example 3.70. Let f be *periodic* with period 1, and suppose that $f \in L^1(\mathbb{T})$, i.e., $\int_0^1 |f(t)|dt < \infty$. Find the Fourier transform and the spectrum of f.

<u>Solution</u>: (Outline) The inversion formula for the periodic transform says that

$$f = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n t}.$$

Working as on page 86 (but a little bit harder) we find that the sum converges in $\mathcal{S}'$, so we are allowed to take transforms:

$$\hat{f} = \sum_{n=-\infty}^{\infty} \hat{f}(n)\delta_n \qquad \text{(converges still in } \mathcal{S}'\text{)}.$$

Thus, the spectrum of f is $\{n \in \mathbb{N} : \hat{f}(n) \neq 0\}$. Compare this to the theory of Fourier series.

General Conclusion 3.71. *The* distribution Fourier transform *contains* all *the other Fourier transforms in this course. A "universal transform".*

3.12 Singular differential equations

Definition 3.72. A linear differential equation of the type

$$\sum_{k=0}^{n} a_k u^{(k)} = f \tag{3.6}$$

is **regular** if it has *exactly one solution* $u \in \mathcal{S}'$ for every $f \in \mathcal{S}'$. Otherwise it is **singular**.

Thus: Singular means: For some $f \in \mathcal{S}'$ it has either no solution or more than one solution.

Example 3.73. The equation $u' = f$. Taking $f = 0$ we get many different solutions, namely u =constant (different constants give different solutions). Thus, this equation is singular.

Example 3.74. We saw earlier on page 59-63 that if we work with L^2-functions instead of distributions, then the equation

$$u'' + \lambda u = f$$

is singular iff $\lambda > 0$. The same result is true for distributions:

Theorem 3.75. *The equation* (3.6) *is regular*

$$\Leftrightarrow \sum_{k=0}^{n} a_k (2\pi i \omega)^k \neq 0 \quad \textit{for all } \omega \in \mathbb{R}.$$

Before proving this, let us define

Definition 3.76. The function $D(\omega) = \sum_{k=0}^{n} a^k (2\pi i\omega)^k$ is called the **symbol** of (3.6)

Thus: Singular $\Leftrightarrow$ the symbol vanishes for some $\omega \in \mathbb{R}$.
PROOF OF THEOREM 3.75. Part 1: Suppose that $D(\omega) \neq 0$ for all $\omega \in \mathbb{R}$. Transform (3.6):

$$\sum_{k=0}^{n} a_k (2\pi i\omega)^k \hat{u} = \hat{f} \quad \Leftrightarrow \quad D(\omega)\hat{u} = \hat{f}.$$

If $D(\omega) \neq 0$ for all ω, then $\frac{1}{D(\omega)} \in C^\infty_{pol}$, so we can multiply by $\frac{1}{D(\omega)}$:

$$\hat{u} = \frac{1}{D(\omega)}\hat{f} \quad \Leftrightarrow \quad u = K * f$$

where K is the inverse distribution Fourier transform of $\frac{1}{D(\omega)}$. Therefore, (3.6) has exactly one solution $u \in \mathcal{S}'$ for every $f \in \mathcal{S}'$.
Part 2: Suppose that $D(a) = 0$ for some $a \in \mathbb{R}$. Then

$$\langle D\delta_a, \varphi\rangle = \langle \delta_a, D\varphi\rangle = D(a)\varphi(a) = 0.$$

This is true for all $\varphi \in \mathcal{S}$, so $D\delta_a$ is the zero distribution: $D\delta_a = 0$.

$$\Leftrightarrow \sum_{k=0}^{n} a_k (2\pi i\omega)^k \delta_a = 0.$$

Let v be the inverse transform of δ_a, i.e.,

$$v(t) = e^{2\pi i a t} \;\Rightarrow\; \sum_{k=0}^{n} a_k v^{(k)} = 0.$$

Thus, v is one solution of (3.6) with $f \equiv 0$. Another solution is $v \equiv 0$. Thus, (3.6) has at least two different solutions $\Rightarrow$ the equation is singular. $\square$

Definition 3.77. If (3.6) is regular, then we call K =inverse transform of $\frac{1}{D(\omega)}$ the **Green's function** of (3.6). (Not defined for singular problems.)

How many solutions does a singular equation have? Find them all! (Solution later!)

Example 3.78. If $f \in C(\mathbb{R})$ (and $|f(t)| \leq M(1+|t|^k)$ for some M and k), then the equation

$$u' = f$$

has at least the solutions

$$u(x) = \int_0^x f(x)dx + \text{constant}$$

Does it have more solutions?
Answer: No! Why?
Suppose that $u' = f$ and $v' = f \Rightarrow u' - v' = 0$. Transform this $\Rightarrow (2\pi i\omega)(\hat{u} - \hat{v}) = 0$.
Let φ be a test function which vanishes in some interval $[-\varepsilon, \varepsilon]$($\Leftrightarrow$ the support of φ is in $(-\infty, -\varepsilon] \cup [\varepsilon, \infty)$). Then

$$\psi(\omega) = \frac{\varphi(\omega)}{2\pi i\omega}$$

is also a test function (it is $\equiv 0$ in $[-\varepsilon, \varepsilon]$), since $(2\pi i\omega)(\hat{u} - \hat{v}) = 0$ we get

$$\begin{aligned} 0 &= \langle (2\pi i\omega)(\hat{u} - \hat{v}), \psi \rangle \\ &= \langle \hat{u} - \hat{v}, 2\pi i\omega\psi(\omega) \rangle = \langle \hat{u} - \hat{v}, \varphi \rangle. \end{aligned}$$

Thus, $\langle \hat{u} - \hat{v}, \varphi \rangle = 0$ when $\text{supp}(\varphi) \subset (-\infty, -\varepsilon] \cup [\varepsilon, \infty)$, so by definition, $\text{supp}(\hat{u} - \hat{v}) \subset \{0\}$. By theorem 3.63, $\hat{u} - \hat{v}$ is a polynomial. The only polynomial whose derivative is zero is the constant function, so $u - v$ is a constant. $\square$

A more sophisticated version of the same argument proves the following theorem:

Theorem 3.79. *Suppose that the equation*

$$\sum_{k=0}^{n} a_k u^{(k)} = f \tag{3.7}$$

is singular, and suppose that the symbol $D(\omega)$ has exactly r simple *zeros $\omega_1, \omega_2, \ldots, \omega_r$. If the equation (3.7) has a solution v, then every other solution $u \in \mathcal{S}'$ of (3.7) is of the form*

$$u = v + \sum_{j=1}^{r} b_j e^{2\pi i\omega_j t},$$

where the coefficients b_j can be chosen arbitrarily.

Compare this to example 3.78: The symbol of the equation $u' = f$ is $2\pi i\omega$, which has a simple zero at zero.

Comment 3.80. *The equation* (3.7) always *has a distribution solution, for all* $f \in \mathcal{S}'$*. This is proved for the equation* $u' = f$ *in [GW99, p. 277], and this can be extended to the general case.*

Comment 3.81. *A zero* ω_j *of order* $r \geq 2$ *of* D *gives rise to terms of the type* $P(t)e^{2\pi i\omega_j t}$*, where* $P(t)$ *is a polynomial of degree* $\leq r - 1$*.*

Chapter 4

The Fourier Transform of a Sequence (Discrete Time)

From our earlier results we very quickly get a Fourier transform theory for *sequences* $\{a_n\}_{n=-\infty}^{\infty}$. We interpret this sequence as the distribution

$$\sum_{n=-\infty}^{\infty} a_n \delta_n \qquad (\delta_n = \text{Dirac's delta at the point } n)$$

For example, this converges in $\mathcal{S}'$ if

$$|a_n| \le M(1+|n|^N) \quad \text{for some } M, N$$

and the Fourier transform is:

$$\sum_{n=-\infty}^{\infty} a_n e^{-2\pi i \omega n} = \sum_{k=-\infty}^{\infty} a_{-k} e^{2\pi i \omega k}$$

which also converges in $\mathcal{S}'$. This transform is *identical* to the *inverse* transform discussed in Chapter 1 (periodic function!), except for the fact that we replace i by $-i$ (or equivalently, replace n by $-n$). Therefore:

Theorem 4.1. *All the results listed in Chapter 1 can be applied to the theory of Fourier transforms of* sequences*, provided that we* intercharge *the Fourier transform and the inverse Fourier transform.*

Notation 4.2. *To simplify the notations we write the original sequence as $f(n)$, $n \in \mathbb{Z}$, and denote the Fourier transform as $\hat{f}$. Then* $\boxed{\hat{f} \text{ is periodic}}$ *(function or* *distribution, depending on the size of* $|f(n)|$ *as* $n \to \infty$*), and*

$$\boxed{\hat{f}(\omega) = \sum_{n=-\infty}^{\infty} f(n) e^{-2\pi i \omega n}.}$$

From Chapter 1 we can give e.g., the following results:

Theorem 4.3.

i) $f \in \ell^2(\mathbb{Z}) \Leftrightarrow \hat{f} \in L^2(\mathbb{T})$,

ii) $f \in \ell^1(\mathbb{Z}) \Rightarrow \hat{f} \in C(\mathbb{T})$ *(converse false),*

iii) $\widehat{(fg)} = \hat{f} * \hat{g}$ *if e.g.* $\begin{cases} \hat{f} \in L^1(\mathbb{T}) \\ \hat{g} \in L^1(\mathbb{T}) \end{cases}$ *or* $\begin{cases} f \in \ell^2(\mathbb{Z}) \\ g \in \ell^2(\mathbb{Z}) \end{cases}$

iv) Etc.

We can also define discrete convolutions:

Definition 4.4. $(f * g)(n) = \sum_{k=-\infty}^{\infty} f(n-k) g(k)$.

This is defined whenever the sum converges absolutely. For example, if $f(k) \neq 0$ only for finitely many k or if

$$\begin{aligned} &f \in \ell^1(\mathbb{Z}), g \in \ell^\infty(\mathbb{Z}), \quad \text{or if} \\ &f \in \ell^2(\mathbb{Z}), g \in \ell^2(\mathbb{Z}), \quad \text{etc.} \end{aligned}$$

Lemma 4.5.

i) $f \in \ell^1(\mathbb{Z}), g \in L^p(\mathbb{Z}),\ 1 \le p \le \infty,\ \Rightarrow\ f * g \in \ell^p(\mathbb{Z})$

ii) $f \in \ell^1(\mathbb{Z}), g \in c_0(\mathbb{Z})\ \Rightarrow\ f * g \in c_0(\mathbb{Z})$.

Proof. "Same" as in Chapter 1 (replace all integrals by sums).

Theorem 4.6. *If* $f \in \ell^1(\mathbb{Z})$ *and* $g \in \ell^1(\mathbb{Z})$*, then*

$$\widehat{(f * g)}(\omega) = \hat{f}(\omega) \hat{g}(\omega).$$

Also true if e.g. $f \in \ell^2(\mathbb{Z})$ *and* $g \in \ell^2(\mathbb{Z})$*.*

Proof. ℓ^1-case: "Same" as proof of Theorem 1.21 (replace integrals by sums). In the ℓ^2-case we first approximate by an ℓ^1-sequence, use the ℓ^1-theory, and pass to the limit.

Notation 4.7. *Especially in the engineering literature, but also in mathematical literature, one often makes a change of variable: we have*

$$\begin{aligned}\hat{f}(\omega) &= \sum_{n=-\infty}^{\infty} f(n)e^{-2\pi i\omega n} = \sum_{n=-\infty}^{\infty} f(n)\left(e^{-2\pi i\omega}\right)^n \\ &= \sum_{n=-\infty}^{\infty} f(n)z^{-n},\end{aligned}$$

where $z = e^{2\pi i\omega}$.

Definition 4.8. Engineers define $F(z) = \sum_{n=-\infty}^{\infty} f(n)z^{-n}$ as the (bilateral) (="dubbelsidig") Z-transformation of f.

Definition 4.9. Most mathematicians define $F(z) = \sum_{n=-\infty}^{\infty} f(n)z^{n}$ instead.

Note: If $f(n) = 0$ for $n < 0$ we get the onesided (=unilateral) transform

$$F(z) = \sum_{n=0}^{\infty} f(n)z^{-n} \quad \left(\text{or } \sum_{n=0}^{\infty} f(n)z^{n}\right).$$

Note: The Z-transform is reduced to the Fourier transform by a *change of variable*

$$\boxed{z = e^{2\pi i\omega}}, \quad \text{so } \boxed{\omega \in [0,1] \Leftrightarrow |z| = 1}$$

Thus, z takes values *on the unit circle.* In the case of one-sided sequences we can also allow $|z| > 1$ (engineers) or $|z| < 1$ (mathematicians) and get *power series* like those studied in the theory of *analytic functions.*

$$\boxed{\text{All Fourier transform results apply}}$$

Chapter 5

The Discrete Fourier Transform

We have studied four types of Fourier transforms:

i) Periodic functions on $\mathbb{R} \Rightarrow \hat{f}$ defined on $\mathbb{Z}$.

ii) Non-periodic functions on $\mathbb{R} \Rightarrow \hat{f}$ defined on $\mathbb{R}$.

iii) Distributions on $\mathbb{R} \Rightarrow \hat{f}$ defined on $\mathbb{R}$.

iv) Sequences defined on $\mathbb{Z} \Rightarrow \hat{f}$ periodic on $\mathbb{R}$.

The final addition comes now:

v) f *a periodic sequence* (on $\mathbb{Z}$) $\Rightarrow \hat{f}$ *a periodic sequence.*

5.1 Definitions

Definition 5.1. $\Pi_N = \{$all periodic sequences $F(m)$ with period N, i.e., $F(m+N) = F(m)\}$.

Note: These are in principle defined for all $n \in \mathbb{Z}$, but the periodicity means that it is enough to know $F(0), F(1), \ldots, F(N-1)$ to know the whole sequence (or any other set of N consecutive (= på varandra följande) values).

Definition 5.2. The **Fourier transform** of a sequence $F \in \Pi_N$ is given by

$$\hat{F}(m) = \frac{1}{N} \sum_{k=0}^{N-1} e^{-\frac{2\pi imk}{N}} F(k), \quad m \in \mathbb{Z}.$$

Warning 5.3. *Some people replace the constant $\frac{1}{N}$ in front of the sum by $\frac{1}{\sqrt{N}}$ or omit it completely. (This affects the inversion formula.)*

Lemma 5.4. *$\hat{F}$ is* periodic *with the same period N as F.*

Proof.

$$\begin{aligned}\hat{F}(m+N) &= \frac{1}{N}\sum_{\text{one period}} e^{-\frac{2\pi i(m+N)k}{N}} F(k)\\ &= \frac{1}{N}\sum_{\text{one period}} \underbrace{e^{-2\pi ik}}_{=1} e^{-\frac{2\pi imk}{N}} F(k)\\ &= \hat{F}(m). \quad \square\end{aligned}$$

Thus, $\boxed{F \in \Pi_N \Rightarrow \hat{F} \in \Pi_N}$.

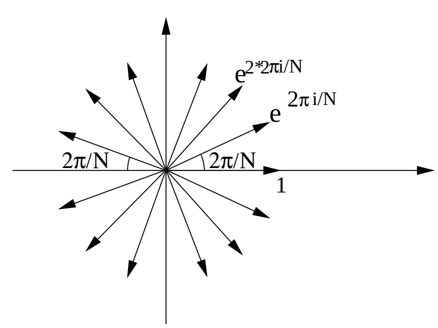
Theorem 5.5. *F can be reconstructed from $\hat{F}$ by the inversion formula*

$$F(k) = \sum_{m=0}^{N-1} e^{\frac{2\pi imk}{N}} \hat{F}(m).$$

Note: No $\frac{1}{N}$ in front here.
Note: Matlab puts the $\frac{1}{N}$ in front of the inversion formula instead!
Proof.

$$\sum_m e^{\frac{2\pi imk}{N}} \frac{1}{N}\sum_l e^{-\frac{2\pi iml}{N}} F(l) = \frac{1}{N}\sum_{l=0}^{N-1} F(l) \underbrace{\sum_{m=0}^{N-1} e^{\frac{2\pi im(k-l)}{N}}}_{=\begin{cases} N, & \text{if } l = k \\ 0, & \text{if } l \neq k\end{cases}} = F(k)$$

We know that $\left(e^{\frac{2\pi i}{N}}\right)^N = 1$, so $e^{\frac{2\pi i}{N}}$ is the N:th root of 1:



We add N numbers, whose absolute value is one, and who point symmetrically in all the different directions indicated above. For symmetry reasons, the sum must be zero (except when $l = k$). (You always jump an angle $\frac{2\pi(k-l)}{N}$ for each turn, and go $k - l$ times around before you are done.)

Definition 5.6. The **convolution** $F * G$ of two sequences in Π_N are defined by

$$(F * G)(m) = \sum_{\text{one period}} F(m-k)G(k)$$

(Note: Some indeces get out of the interval $[0, N-1]$. You must use the *periodicity* of F and G to get the corresponding values of $F(m-k)G(k)$.).

Definition 5.7. The (ordinary) **product** $F \cdot G$ is defined by

$$(F \cdot G)(m) = F(m)G(m), \ m \in \mathbb{Z}.$$

Theorem 5.8. *$(\widehat{F \cdot G}) = \hat{F} * \hat{G}$ and $(\widehat{F * G}) = N\hat{F} \cdot \hat{G}$ (note the extra factor N).*

Proof. Easy. (Homework?)

Definition 5.9. $(RF)(n) = F(-n)$ (**reflection** operator).

As before: The inverse transform = the ususal transform plus reflection:

Theorem 5.10. *$\check{F} = N(\widehat{RF})$ (note the extra factor N), where $\hat{}$ = Fourier transform and $\check{}$ = Inverse Fourier transform.*

Proof. Easy. We could have avoided the factor N by a different scaling (but then it shows up in other places instead).

5.2 FFT=the Fast Fourier Transform

Question 5.11. *How many flops do we need to compute the Fourier transform of $F \in \Pi_N$?*

FLOP=FLoating Point Operation={multiplication or addition or combination of both}.

1 Megaflop = 1 million flops/second (10^6)
1 Gigaflop = 1 billion flops/second (10^9)

(Used as speed measures of computers.)

Task 5.12. *Compute* $\hat{F}(m) = \frac{1}{N}\sum_{k=0}^{N-1} e^{-\frac{2\pi imk}{N}} F(k)$ *with the* minimum amount of flops *(=quickly).*

Good Idea 5.13. *Compute the coefficients* $\left(e^{-\frac{2\pi i}{N}}\right)^k = \omega^k$ only once, *and store them in a table. Since* $\omega^{k+N} = \omega^k$, *we have* $e^{-\frac{2\pi imk}{N}} = \omega^{mk} = \omega^r$ *where* $r =$ *remainder when we divide* mk *by* N. *Thus, only* N *numbers need to be stored.*
<u>Thus</u>: We can ignore the number of flops needed to compute the coefficients $e^{-\frac{2\pi imk}{N}}$ *(done in advance).*

Trivial Solution 5.14. *If we count multiplication and addition separetely, then we need to compute* N *coefficients (as* $m = 0, 1, \ldots, N-1$*), and each coefficient requires* N *muliplications and* $N-1$ *additions. This totals*

$$N(2N-1) = 2N^2 - N \approx \boxed{2N^2 \textit{ flops}}.$$

This is too much.

Brilliant Idea 5.15. *Regroup (=omgruppera) the terms, using the* symmetry. *Start by doing* even *coefficients and* odd *coefficients separetely:*
Suppose for simplicity that N *is* even. *Then, for* even m, *(put* $N = 2n$*)*

$$\begin{aligned}
\hat{F}(2m) &= \frac{1}{N}\sum_{k=0}^{N-1} \omega^{2mk} F(k) \\
&= \frac{1}{N}\Big[\sum_{k=0}^{n-1} \omega^{2mk} F(k) + \underbrace{\sum_{k=n}^{2n-1} \omega^{2mk} F(k)}_{\textit{Replace } k \textit{ by } k+n}\Big] \\
&= \frac{1}{N}\left[\sum_{k=0}^{n-1} \omega^{2mk} F(k) + \omega^{2m(k+n)} F(k+n)\right] \\
&= \frac{2}{N}\sum_{k=0}^{n-1} e^{-\frac{2\pi imk}{(N/2)}} \frac{1}{2}\left[F(k) + F(k+n)\right].
\end{aligned}$$

This is a new *discrete time periodic Fourier transform of the sequence* $G(k) = \frac{1}{2}[F(k) + F(n+k)]$ *with* $\boxed{\textit{period } n = \frac{N}{2}}$.
A similar computation (see Gripenberg) shows that the odd *coefficients can be computed from*

$$\hat{F}(2m+1) = \frac{1}{n}\sum_{k=0}^{n-1} e^{-\frac{2\pi imk}{n}} H(k),$$

where $H(k) = \frac{1}{2}e^{-\frac{i\pi k}{n}}\left[F(k) - F(k+n)\right]$*. Thus, instead of* one *transform of order* N *we get* two *transforms of order* $n = \frac{N}{2}$*.*

Number of flops: *Computing the new transforms by brute force (as in 5.14 on page 105) we need the following flops:*

Even: $n(2n-1) = \frac{N^2}{2} - \frac{N}{2} + n$ *additions* $= \frac{N^2}{2}$ *flops.*

Odd: *The numbers* $e^{-\frac{i\pi k}{n}} = e^{-\frac{2i\pi k}{N}}$ *are found in the table already computed.*

We essentially again need the same amount, namely $\frac{N^2}{2} + \frac{N}{2}$ *(n extra multiplications).*

Total: $\frac{N^2}{2} + \frac{N^2}{2} + \frac{N}{2} = N^2 + \frac{N}{2} \approx N^2$*. Thus, this* approximately halfed *the number of needed flops.*

Repeat 5.16. *Divide the new smaller transforms into two halfs, and again, and again. This is possible if* $N = 2^k$ *for some integer* k*, e.g.,* $N = 1024 = 2^{10}$*.*

Final conclusion: After some smaller adjustments we get down to

$$\frac{3}{2}2^k k \text{ flops.}$$

Here $N = 2^k$, so $k = \log_2 N$, and we get

Theorem 5.17. *The* Fast Fourier Transform *with* radius 2 *outlined above needs approximately* $\frac{3}{2}N\log_2 N$ *flops.*

This is much smaller than $2N^2 - N$ for large N. For example $N = 2^{10} = 1024$ gives

$$\frac{3}{2}N\log_2 N \approx 15000 \ll 2000000 = 2N^2 - N.$$

Definition 5.18. Fast Fourier transform with

$$\begin{cases} \text{radius } 2: & \text{split into 2 parts at each step} & N = 2^k \\ \text{radius } 3: & \text{split into 3 parts at each step} & N = 3^k \\ \text{radius } m: & \text{split into } m \text{ parts at each step} & N = m^k \end{cases}$$

Note: Based on *symmetries*. "The same" computations repeat themselves, so by combining them in a clever way we can do it quicker.

Note: The FFT is *so fast* that it caused a minor revolution to many branches of numerical analysis. It made it possible to compute Fourier transforms in practice.

Rest of this chapter: How to use the FFT to compute the *other* transforms discussed earlier.

5.3 Computation of the Fourier Coefficients of a Periodic Function

Problem 5.19. *Let $f \in C(\mathbb{T})$. Compute*

$$\hat{f}(k) = \int_0^1 e^{-2\pi i k t} f(t)dt$$

as efficiently as possible.

Solution: Turn f into a periodic sequence and use FFT!

Conversion 5.20. *Choose some $N \in \mathbb{Z}$, and put*

$$F(m) = f(\frac{m}{N}), \quad m \in \mathbb{Z}$$

(equidistant "sampling"). The periodicity of f makes F periodic with period N. Thus, $F \in \Pi_N$.

Theorem 5.21 (Error estimte)**.** *If $f \in C(\mathbb{T})$ and $\hat{f} \in \ell^1(\mathbb{Z})$ (i.e., $\sum|\hat{f}(k)| < \infty$), then*

$$\hat{F}(m) - \hat{f}(m) = \sum_{k\neq 0} \hat{f}(m + kN).$$

Proof. By the inversion formula, for all t,

$$f(t) = \sum_{j\in\mathbb{Z}} e^{2\pi i j t} \hat{f}(j).$$

Put $t_k = \frac{k}{N} \Rightarrow$

$$f(t_k) = F(k) = \sum_{j\in\mathbb{Z}} e^{\frac{2\pi i k j}{N}} \hat{f}(j)$$

(this series converges uniformly by Lemma 1.14). By the definition of $\hat{F}$:

$$\begin{aligned}
\hat{F}(m) &= \frac{1}{N}\sum_{k=0}^{N-1} e^{-\frac{2\pi i m k}{N}} F(k) \\
&= \frac{1}{N}\sum_{j\in\mathbb{Z}} \hat{f}(j) \qquad \underbrace{\sum_{k=0}^{N-1} e^{\frac{2\pi i (j-m)k}{N}}}_{=\begin{cases} N, & \text{if } \frac{j-m}{N} = \text{integer} \\ 0, & \text{if } \frac{j-m}{N} \neq \text{integer}\end{cases}} \\
&= \sum_{l\in\mathbb{Z}} \hat{f}(m + Nl).
\end{aligned}$$

Take away the term $\hat{f}(m)$ $(l = 0)$ to get

$$\hat{F}(m) = \hat{f}(m) + \sum_{l \neq 0} \hat{f}(m + Nl).$$

Note: If N is "large" and if $\hat{f}(m) \to 0$ "quickly" as $m \to \infty$, then the error

$$\sum_{l \neq 0} \hat{f}(m + Nl) \approx 0.$$

First Method 5.22. *Put*

i) $\hat{f}(m) \approx \hat{F}(m)$ *if* $|m| < \frac{N}{2}$

ii) $\hat{f}(m) \approx \frac{1}{2}\hat{F}(m)$ *if* $|m| = \frac{N}{2}$ *(*N *even)*

iii) $\hat{f}(m) \approx 0$ *if* $|m| > \frac{N}{2}$.

Here ii) is not important. *We could use* $\hat{f}(\frac{N}{2}) = 0$ *or* $\hat{f}(\frac{N}{2}) = \hat{F}(m)$ *instead. Here*

$$\hat{F}(m) = \frac{1}{N} \sum_{k=0}^{N-1} e^{-\frac{2\pi i m k}{N}} F(k).$$

Notation 5.23. *Let us denote (note the extra star)*

$$\sum_{|k| \leq N/2}^{*} a_k = \sum_{|k| \leq N/2} a_k$$

= the usual sum of a_k *if* N *odd (then we have exactly* N *terms), and*

$\sum_{|k| \leq N/2}^{*} a_k =$ *a sum where the first and last terms have been divided by two (these are the same if the sequence is periodic with period* N*, there is "one term too many" in this case).*

First Method 5.24 (Error)**.** *The first method gives the error:*

i) $|m| < \frac{N}{2}$ *gives the error*

$$|\hat{f}(m) - \hat{F}(m)| \leq \sum_{k \neq 0} |\hat{f}(m + kN)|$$

ii) $|m| = \frac{N}{2}$ *gives the error*

$$|\hat{f}(m) - \frac{1}{2}\hat{F}(m)|$$

iii) $|m| > \frac{N}{2}$ *gives the error* $|\hat{f}(m)|$.

we can simplify *this into the following crude (="grov") estimate:*

$$\boxed{\sup_{m\in\mathbb{Z}}|\hat{f}(m) - \hat{F}(m)| \leq \sum_{|m|\geq N/2}^{*} |\hat{f}(m)|} \tag{5.1}$$

(because this sum is $\geq$ *the actual error).*

First Method 5.25 (Drawbacks)**.**

1° *Large error.*

2° *Inaccurate error estimate* (5.1).

3° *The error estimate based on* $\hat{f}$ *and not on* f.

We need a better method.

Second Method 5.26 (General Principle)**.**

1° *Evaluate* t *at the points* $t_k = \frac{k}{N}$ *(as before),* $F(k) = f(t_k)$

2° *Use the sequence* F *to construct a new function* $P \in C(T)$ *which "approximates"* f.

3° *Compute the Fourier coefficients of* P.

4° *Approximate* $\hat{f}(n)$ *by* $\hat{P}(n)$.

For this to succeed we must choose P *in a smart way. The final result will be quite simple, but for later use we shall derive* P *from some "basic principles".*

Choice of P **5.27.** *Clearly* P *depends on* F. *To simplify the computations we require* P *to satisfy (write* $P = P(F)$*)*

A) P *is linear:* $P(\lambda F + \mu G) = \lambda P(F) + \mu P(G)$

B) P *is translation invariant: If we translate* F*, then* $P(F)$ *is translated by the same amount: If we denote*

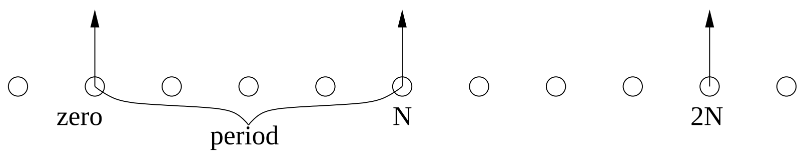
$$\begin{aligned}(\tau_j F)(m) &= F(m-j), \text{ then} \\ P(\tau_j F) &= \tau_{j/N} P(F)\end{aligned}$$

(j *discrete steps* $\iff$ *a time difference of* j/N*).*

This leads to simple computations: We want to compute $\hat{P}(m)$ (which we use as approximations of $\hat{f}(m)$) Define a δ-sequence:

$$D(n) = \begin{cases} 1, & \text{for } n = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

Then



$$(\tau_k D)(n) = \begin{cases} 1, & \text{if } n = k + jN, \ j \in \mathbb{Z} \\ 0, & \text{otherwise,} \end{cases}$$

so

$$[F(k)\tau_k D](n) = \begin{cases} F(k), & n = k + jN \\ 0, & \text{otherwise.} \end{cases}$$

and so

$$\boxed{F = \sum_{k=0}^{N-1} F(k)\tau_k D}$$

Therefore, the principles A) and B) give

$$\begin{aligned} P(F) &= \sum_{k=0}^{N-1} F(k) P(\tau_k D) \\ &= \sum_{k=0}^{N-1} F(k) \tau_{k/N} P(D), \end{aligned}$$

$\boxed{\text{Where } P(D) \text{ is the approximation of } D = \text{“unit pulse at time zero” } D.}$

We denote this function by p. Let us transform $P(F)$:

$$\begin{aligned}
(\widehat{P(F)})(m) &= \int_0^1 \sum_{k=0}^{N+1} F(k)(\tau_{k/N}p)(s)e^{-2\pi ism}ds \\
&= \sum_{k=0}^{N+1} F(k) \int_0^1 e^{-2\pi ism} p(s-\frac{k}{N})ds \quad (s-\frac{k}{N}=t) \\
&= \sum_{k=0}^{N+1} F(k) \int_{\substack{\text{one}\\ \text{period}}} e^{-2\pi im(t+\frac{k}{N})}p(t)dt \\
&= \sum_{k=0}^{N+1} F(k)e^{-\frac{-2\pi imk}{N}} \underbrace{\int_{\substack{\text{one}\\ \text{period}}} e^{-2\pi imt}p(t)dt}_{\hat{p}(m)} \\
&= \hat{p}(m) \underbrace{\sum_{k=0}^{N+1} F(k)e^{-\frac{2\pi imk}{N}}}_{=N\hat{F}(m)} \\
&= N\hat{p}(m)\hat{F}(m).
\end{aligned}$$

We can get rid of the factor N by replacing p by Np. This is our approximation of the "pulse of size N at zero"

$$\begin{cases} N, & n = 0 + jN \\ 0, & \text{otherwise.} \end{cases}$$

Second Method 5.28. *Construct F as in the First Method, and compute $\hat{F}$. Then the approximation of $\hat{f}(m)$ is*

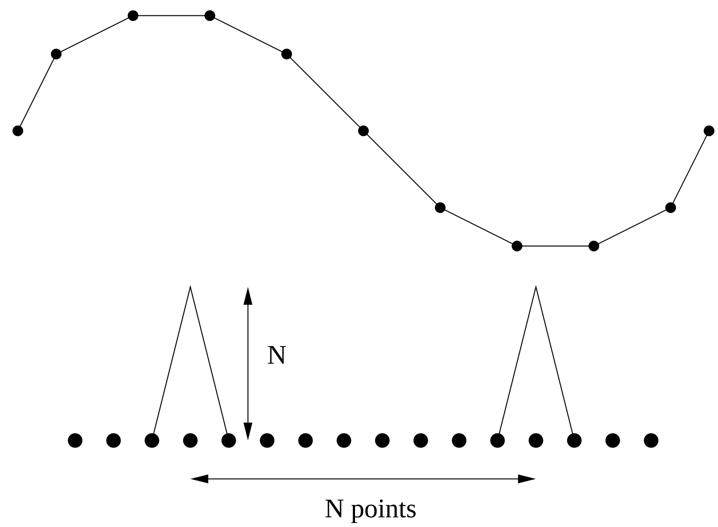
$$\hat{f}(m) \approx \hat{F}(m)\hat{p}(m),$$

where $\hat{p}$ is the Fourier transform of the function that we get when we apply our approximation procedure to the sequence

$$ND(n) = \begin{cases} N, & n = 0(+jN) \\ 0, & \textit{otherwise.} \end{cases}$$

Note: The complicated proof of this simple method will pay off in a second!

Approximation Method 5.29. *Use any type of translation-invariant interpolation method, for example* splines. *The simplest possible method is* linear interpolation*: If we interpolate the pulse ND in this way we get Thus,*



$$p(t) = \begin{cases} N(1-N|t|), & |t| \le \frac{1}{N} \\ 0, & \frac{1}{N} \le |t| \le 1 - \frac{1}{N} \end{cases}$$
(periodic extension)

This is a periodic version of the kernel. A direct computation gives

$$\hat{p}(m) = \left(\frac{\sin(\pi m/N)}{\pi m/N}\right)^2.$$

We get the following interesting theorem:

Theorem 5.30. *If we first discretize f, i.e. we replace f by the sequence $F(k) = f(k/N)$, the compute $\hat{F}(m)$, and finally multiply $\hat{F}(m)$ by*

$$\hat{p}(m) = \left(\frac{\sin(\pi m/N)}{\pi m/N}\right)^2.$$

then we get the Fourier coefficients for the function which we get from f by linear interpolation at the points $t_k = k/N$.

(This corresponds to the computation of the Fourier integral $\int_0^1 e^{-2\pi i m t} f(t)dt$ by using the trapetsoidal rule. Other integration methods have similar interpretations.)

5.4 Trigonometric Interpolation

Problem 5.31. *Construct a good method to approximate a periodic function $f \in C(T)$ by a* trigonometric polynomial

$$\sum_{m=-N}^{N} a_m e^{2\pi imt}$$

(a finite *sum, resembles inverse Fourier transformation).*

Useful for numerical computation etc.

Note: The earlier "Second Method" gave us a *linear interpolation*, not trigonometric approximation.

Note: This trigonometric polynomial has only finitely many Fourier coefficients $\neq 0$ (namely a_m, $|m| \leq N$).

Actually, the "First Method" gave us a trigonometric polynomial. There we had

$$\begin{cases} \hat{f}(m) \approx \hat{F}(m) & \text{for } |m| < \frac{N}{2}, \\ \hat{f}(m) \approx \frac{1}{2}\hat{F}(m) & \text{for } |m| = \frac{N}{2}, \\ \hat{f}(m) \approx 0 & \text{for } |m| > \frac{N}{2}. \end{cases}$$

By inverting this sequence we get a trigonometric approximation of f: $f(t) \approx g(t)$, where

$$g(t) = \sum_{|m|\leq N/2}^{*} \hat{F}(m) e^{2\pi imt}. \tag{5.2}$$

We have *two* different *errors*:

i) $\hat{f}(m)$ is replaced by $\hat{F}(m) = \frac{1}{N}\sum_{k=0}^{N-1} f(\frac{k}{N}) e^{\frac{2\pi ikm}{N}}$,

ii) The inverse series was truncated to N terms.

Strange fact: These two errors (partially) *cancel* each other.

Theorem 5.32. *The function g defined in* (5.2) *satisfies*

$$g(\frac{k}{N}) = f(\frac{k}{N}), \quad n \in \mathbb{Z},$$

i.e., g interpolates *f at the points t_k (which were used to construct first F and then g).*

Proof. We defined $F(k) = f(\frac{k}{N})$, and

$$\hat{F}(m) = \sum_{|k|\le N/2}^{*} F(k)e^{-\frac{2\pi imk}{N}}.$$

By the inversion formula on page 103,

$$\begin{aligned} g(\frac{k}{N}) &= \sum_{|m|\le N/2}^{*} \hat{F}(m)e^{\frac{2\pi imk}{N}} \text{ (use periodicity)} \\ &= \sum_{m=0}^{N-1} \hat{F}(m)e^{\frac{2\pi imk}{N}} \\ &= F(k) = f(\frac{k}{N}) \quad \square \end{aligned}$$

<u>Error estimate</u>: How large is $|f(t)-g(t)|$ between the mesh points $t_k = \frac{k}{N}$ (where the error is zero)? We get an estimate from the computation in the last section. Suppose that $\hat{f} \in \ell^1(\mathbb{Z})$ and $f \in C(T)$ so that the inversion formula holds for all t (see Theorem 1.37). Then

$$f(t) = \sum_{m\in\mathbb{Z}} \hat{f}(m)e^{2\pi imt}, \quad \text{and}$$

$$\begin{aligned} g(t) &= \sum_{|m|\le N/2}^{*} \hat{F}(m)e^{2\pi imt} \quad \text{(Theorem 5.21)} \\ &= \sum_{|m|\le N/2}^{*} \left[\hat{f}(m) + \sum_{k\neq 0} \hat{f}(m+kN)\right] e^{2\pi imt} \\ &= f(t) - \sum_{|m|\ge N/2}^{*} \hat{f}(m)e^{2\pi imt} + \sum_{|m|\le N/2}^{*} \sum_{k\neq 0} \hat{f}(m+kN)e^{2\pi imt}. \end{aligned}$$

Thus

$$|g(t)-f(t)| \le \sum_{|m|\ge N/2}^{*} |\hat{f}(m)| + \underbrace{\sum_{|m|\le N/2}^{*} \sum_{k\neq 0} |\hat{f}(m+kM)|}_{=\sum^{*}_{|l|\ge N/2}|\hat{f}(l)|} = 2 \sum_{|m|\ge N/2}^{*} |\hat{f}(m)|$$

(take $l = m + kN$, every $|l| > \frac{N}{2}$ appears one time, no $|l| < \frac{N}{2}$ appears, and $|l| = \frac{N}{2}$ two times).

This leads to the following theorem:

Theorem 5.33. *If* $\sum_{m=-\infty}^{\infty}|\hat{f}(m)| < \infty$, *then*

$$|g(t) - f(t)| \le 2 \sum_{|m|\ge N/2}^{*} |\hat{f}(m)|,$$

where

$$\begin{aligned} g(t) &= \textstyle\sum_{|m|\le N/2}^{*} \hat{F}(m)e^{2\pi imt}, \quad \textit{and} \\ \hat{F}(m) &= \tfrac{1}{N}\textstyle\sum_{|m|\le N/2}^{*} e^{-\frac{2\pi imk}{N}} f(\tfrac{k}{N}). \end{aligned}$$

This is nice if $\hat{f}(m) \to 0$ *rapidly as* $m \to \infty$. *Better accuracy by increasing* N.

5.5 Generating Functions

Definition 5.34. The **generating function** of the sequence $J_n(x)$ is the function

$$f(x,z) = \sum_n J_n(x)z^n,$$

where the sum over $n \in \mathbb{Z}$ or over $n \in \mathbb{Z}_+$, depending on for which values of n the functions $J_n(x)$ are defined.

Note: We did this in the course on *special functions*. E.g., if J_n = Bessel's function of order n, then

$$f(x,z) = e^{\frac{x}{2}(z-1/2)}.$$

Note: For a fixed value of x, this is the "mathematician's version" of the Z-transform described on page 101.
Make a change of variable:

$$\begin{aligned} z = e^{2\pi it} \Rightarrow f(x, e^{2\pi it}) &= \sum_{n\in\mathbb{Z}} J_n(x)(e^{2\pi it})^n \\ &= \sum_{n\in\mathbb{Z}} J_n(x)e^{2\pi int}, \end{aligned}$$

Comparing this to the inversion formula in Chapter 1 we get

Theorem 5.35. *For a fixed* x, *the* n*:th Fourier coefficient of the function* $t \mapsto f(x, e^{2\pi it})$ *is equal to* $J_n(x)$.

Thus, we can *compute* $J_n(x)$ by the method descibed in Section 5.3 to compute the coefficients $a_n = J_n(x)$ (x = fixed, n varies):

1) Discretize $F(k) = f(x, e^{\frac{2\pi ik}{N}})$

2) $\hat{F}(m) = \frac{1}{N}\sum^*_{|k|\le N/2} e^{-\frac{2\pi imk}{N}} F(k)$

3) $\hat{F}(m) - J_n(x) = \sum_{k\neq 0} a_{m+kN}$, (Theorem 5.21)

where $a_{m+kN} = J_{m+kN}(x)$.

5.6 One-Sided Sequences

So far we have been talking about *periodic* sequences (in Π_N). Instead one often wants to discuss

A) *Finite sequences* $A(0), A(1), \ldots, A(N-1)$ or

B) *One-sided sequences* $A(n), n \in \mathbb{Z}_+ = \{0, 1, 2 \ldots\}$

Note: 5.36. *A* finite *sequence is a special csse of a one-sided sequence: put $A(n) = 0$ for $n \ge N$.*

Note: 5.37. *A one-sided sequence is a special case of a two-sided sequence: put $A(n) = 0$ for $n < 0$.*

Problem: These extended sequences are *not periodic.* $\Rightarrow$ We cannot use the Fast Fourier Transform directly.

Notation 5.38. $\mathbb{C}^{\mathbb{Z}_+} = \{$*all complex valued sequences* $A(n), n \in \mathbb{Z}_+\}$

Definition 5.39. The **convolution** of two sequences $A, B \in \mathbb{C}^{\mathbb{Z}_+}$ is

$$(A * B)(m) = \sum_{k=0}^{m} A(m-k)B(k),\ m \in \mathbb{Z}_+$$

Note: The summation boundaries are the natural ones if we think that $A(k) = B(k) = 0$ for $k < 0$.

Notation 5.40.

$$A_{|n}(k) = \begin{cases} A(k), & 0 \le k < n \\ 0, & k \ge n. \end{cases}$$

Thus, this restricts *the sequence $A(k)$ to the n first terms.*

Lemma 5.41. $(A * B)_{|n} = (A_{|n} * B_{|n})_{|n}$

Proof. Easy.

Notation 5.42. *$A = 0_n$ means that $A(k) = 0$ for $0 \le k < n-1$, i.e., $A_{|n} = 0$.*

Lemma 5.43. *If $A = 0_n$ and $B = 0_m$, then $A * B = 0_{n+m}$.*

Proof. Easy.

Computation of $A * B$ 5.44 (One-sided convolution)**.**

1) Choose a number $N \ge 2n$ (often a power of 2).

2) Define

$$F(k) = \begin{cases} A(k), & 0 \le k < n, \\ 0, & n \le k < N, \end{cases}$$

and extend F to be periodic, period N.

3) Define

$$G(k) = \begin{cases} B(k), & 0 \le k < n, \\ 0, & n \le k < N, \end{cases}$$

periodic extension: $G(k+N) = G(k)$.

Then, for all m, $0 \le m < n$,

$$\begin{aligned} \underbrace{(F * G)(m)}_{\text{periodic convolution}} &= \sum_{k=0}^{N-1} F(m-k)G(k) \\ &= \sum_{k=0}^{m} F(m-k)G(k) \\ &= \sum_{k=0}^{m} A(m-k)G(k) = \underbrace{(A * B)(m)}_{\text{one-sided convolution}} \end{aligned}$$

Note: Important that $N \ge 2n$.
Thus, this way we have computed the n first coefficients of $(A * B)$.

Theorem 5.45. *The method described below allows us to compute $(A * B)_{|n}$ (=the first n coefficients of $A * B$) with a number of FLOP:s which is*

$$C \cdot n \log_2 n, \text{ where } C \text{ is a constant.}$$

Method: 1)-3) same as above

4) Use FFT to compute

$$\hat{F} \cdot \hat{G} (= N(\widehat{F * G})).$$

5) Use the inverse FFT to compute

$$F * G = \frac{1}{N}(\hat{F} \cdot \hat{G})^{\vee}$$

Then $(A * B)_{|n} = (F * G)_{|n}$.

Note: A "naive" computation of $A * B_{|n}$ requires $C_1 \cdot n^2$ FLOPs, where C_1 is another constant.

Note: Use "naive" method if n small. Use "FFT-inverse FFT" if n is large.

Note: The rest of this chapter *applies* one-sided convolutions to different situations. In all cases the method described in Theorem 5.45 can be used to compute these.

5.7 The Polynomial Interpretation of a Finite Sequence

Problem 5.46. *Compute the product of two polynomials:*

$$p(x) = \sum_{k=0}^{n} a_k x^k \qquad q(x) = \sum_{l=0}^{m} b_l x^l.$$

Solution: Define $a_k = 0$ for $k > 0$ and $b_l = 0$ for $l > m$. Then

$$\begin{aligned} p(x)q(x) &= \underbrace{\left(\sum_{k=0}^{\infty} a_k x^k\right)\left(\sum_{l=0}^{\infty} b_l x^l\right)}_{\text{sums are actually finite}} \\ &= \sum_{k,l} a_k b_l x^{k+l} \quad (k+l=j,\ k=j-l) \\ &= \sum_j x^j \sum_{l=0}^{j} a_{j-l} b_l = \sum_{j=0}^{m+n} c_j x^j, \end{aligned}$$

where $c_j = \sum_{l=0}^{j} a_{j-l} b_l$. This gives

Theorem 5.47.

i) Multiplication *of two polynomials corresponds to a* convolution *of their* coefficients*: If*

$$p(x) = \sum_{k=0}^{n} a_k x^k, \quad q(x) = \sum_{l=0}^{m} b_l x^l,$$

then $p(x)q(x) = \sum_{j=0}^{m+n} c_j x^j$*, where* $c = a * b$*.*

ii) Addition *of two polynomials corresponds to* addition *of the coefficients:*

$$p(x) + q(x) = \sum c_j x^j, \quad \textit{where } c_j = a_j + b_j.$$

iii) Multiplication *of a polynomial* by *a complex* constant *corresponds to multiplication of the coefficients by the same constant.*

Operation	Polynomial	Coefficients
Addition	$p(x) + q(x)$	$\{a_k + b_k\}_{k=0}^{\max\{m,n\}}$
Multiplication by $\lambda \in \mathbb{C}$	$\lambda p(x)$	$\{\lambda a_k\}_{k=0}^{n}$
Multiplication	$p(x)q(x)$	$(a * b)(k)$

Thus there is a *one-to-one correspondence* between

$$\text{polynomials} \iff \text{finite sequences},$$

where the operations correspond as described above. This is used in all symbolic computer computations of polynomials.

Note: Two different conventions are in common use:

A) first coefficient is a_0 (= lowest order),

B) first coefficient is a_n (= highest order).

5.8 Formal Power Series and Analytic Functions

Next we extend "polynomials" so that they may contain *infinitely many* terms.

Definition 5.48. A **Formal Power Series** (FPS) is a sum of the type

$$\sum_{k=0}^{\infty} A(k) x^k$$

which *need not converge* for any $x \neq 0$. (If it does converge, then it defines an analytic function in the region of convergence.)

Example 5.49. $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges for all x (and the sum is e^x).

Example 5.50. $\sum_{k=0}^{\infty} x^k$ converges for $|x| < 1$ (and the sum is $\frac{1}{1-x}$).

Example 5.51. $\sum_{k=0}^{\infty} k!x^k$ converges for *no* $x \neq 0$.

All of these are formal power series (and the first two are "ordinary" power series).

Calculus with FPS 5.52. *We borrow the calculus rules from the polynomials:*

i) We add *two FPS:s by adding the coefficients:*

$$\left[\sum_{k=0}^{\infty} A(k)x^k\right] + \left[\sum_{k=0}^{\infty} B(k)x^k\right] = \sum_{k=0}^{\infty}[A(k) + B(k)]x^k.$$

ii) We multiply *a FPS* by a constant λ *by multiplying each coefficients by* λ*:*

$$\lambda \sum_{k=0}^{\infty} A(k)x^k = \sum_{k=0}^{\infty}[\lambda A(k)]x^k.$$

iii) We multiply *two FPS:s with each other by taking the* convolution *of the coefficients:*

$$\left[\sum_{k=0}^{\infty} A(k)x^k\right]\left[\sum_{k=0}^{\infty} B(k)x^k\right] = \sum_{k=0}^{\infty} C(k)x^k,$$

where $C = A * B$*.*

Notation 5.53. *We denote* $\tilde{A}(x) = \sum_{k=0}^{\infty} A(k)x^k$*.*

Conclusion 5.54. *There is a* one-to-one *correspondence between all* Formal Power Series *and all* one-sided sequences *(bounded or not). We denoted these by* $\mathbb{C}^{\mathbb{Z}_+}$ *on page 116.*

Comment 5.55. *In the sequence (="fortsättningen") we operate with FPS:s. These power series* often converge*, and then they define analytic functions, but this fact is not used anywhere in the proofs.*

5.9 Inversion of (Formal) Power Series

Problem 5.56. *Given a (formal) power series* $\tilde{A}(x) = \sum A(k)x^k$*, find the* inverse *formal power series* $\tilde{B}(x) = \sum B(k)x^k$*.*

Thus, we want to find $\tilde{B}(x)$ so that

$$\tilde{A}(x)\tilde{B}(x) = 1, \quad \text{i.e.,}$$
$$\left[\sum_{k=0}^{\infty} A(k)x^k\right]\left[\sum_{l=0}^{\infty} B(l)x^l\right] = 1 + 0x + 0x^2 + \dots$$

Notation 5.57. $\delta_0 = \{1, 0, 0, \dots\}$ = *the sequence whose power series is* $\{1 + 0x + 0x^2 + \dots\}$*. This series converges, and the sum is* $\equiv 1$*. More generally:*

$$\begin{aligned}\delta_k &= \{0, 0, 0, \dots, 0, 1, 0, 0, \dots\} \\ &= 0 + 0x + 0x^2 + \dots + 0x^{k-1} + 1x^k + 0x^{k+1} + 0x^{k+2} + \dots \\ &= x^k\end{aligned}$$

Power Series	*Sequence*
δ_k	x^k

SOLUTION. We know that $A*B = \delta_0$, or equivalently, $\tilde{A}(x)\tilde{B}(x) = 1$. Explicitly,

$$\begin{aligned}\tilde{A}(x)\tilde{B}(x) &= A(0)B(0) \quad (\text{times } x^0) && (5.3)\\ &\quad +[A(0)B(1) + A(1)B(0)]x && (5.4)\\ &\quad +[A(0)B(2) + A(1)B(1) + A(2)B(0)]x^2 && (5.5)\\ &\quad +\dots && (5.6)\end{aligned}$$

From this we can solve:

i) $A(0)B(0) = 1 \implies A(0) \neq 0$ and $B(0) = \frac{1}{A(0)}$.

ii) $A(0)B(1) + A(1)B(0) = 0 \implies B(1) = -\frac{A(1)B(0)}{A(0)}$ (always possible)

iii) $A(0)B(2) + A(1)B(1) + A(2)B(0) = 1 \implies B(2) = -\frac{1}{A(0)}[A(1)B(1) + A(2)B(0)]$, etc.

we get a theorem:

Theorem 5.58. *The FPS* $\tilde{A}(x)$ *can be inverted if and only if* $A(0) \neq 0$*. The inverse series* $[A(x)]^{-1}$ *is obtained* recursively *by the procedure described above.*

Recursive means:

i) Solve $B(0)$

ii) Solve $B(1)$ using $B(0)$

iii) Solve $B(2)$ using $B(1)$ and $B(0)$

iv) Solve $B(3)$ using $B(2)$, $B(1)$ and $B(0)$, etc.

This is *Hard Work*. For example

$$\begin{aligned}
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \\
\tan(x) &= \frac{\sin(x)}{\cos(x)} = \underbrace{\sin(x)\frac{1}{\cos(x)}}_{\text{convolution}} = ???
\end{aligned}$$

Hard Work means: Number of FLOPS is a *constant times* N^2. Better method: Use FFT.

Theorem 5.59. *Let $A(0) \neq 0$. and let $\tilde{B}(x)$ be the inverse of $\tilde{A}(x)$. Then, for every $k \geq 1$,*

$$B_{|2k} = (B_{|k} * (2\delta_0 - A * B_{|k}))_{|2k} \tag{5.7}$$

Proof. See Gripenberg.
Usage: First compute $B_{|1} = \{\frac{1}{A(0)}, 0, 0, 0, \dots\}$
Then $B_{|2} = \{B(0), B(1), 0, 0, 0, \dots\}$ (use (5.7))
Then $B_{|4} = \{B(0), B(1), B(2), B(3), 0, , 0, 0, \dots\}$
Then $B_{|8} = \{8 \text{ terms} \neq 0\}$ etc.
Use the method on page 117 for the convolutions. (Useful only if you need *lots* of coefficients).

5.10 Multidimensional FFT

Especially in image processing we also need the discrete Fourier transform in *several* dimensions. Let $d = \{1, 2, 3, \dots\}$ be the "*space dimension*". Put $\Pi_N^d = \{$ sequences $x(k_1, k_2, \dots, k_d)$ which are N-periodic in each variable separately$\}$.

Definition 5.60. The d-dimensional Fourier transform is obtained by transforming d successive (="efter varandra") "ordinary" Fourier transformations, one for each variable.

Lemma 5.61. *The d-dimensional Fourier transform is given by*

$$\hat{x}(m_1, m_2, \ldots, m_d) = \frac{1}{N^d} \sum_{k_1} \sum_{k_2} \cdots \sum_{k_d} e^{\frac{-2\pi i (k_1 m_1 + k_2 m_2 + \ldots + k_d m_d)}{N}} x(k_1, k_2, \ldots, k_d).$$

PROOF. Easy.

All 1-dimensional results generalize easy to the d-dimensional case.

Notation 5.62. *We call* $\underline{k} = (k_1, k_2, \ldots, k_d)$ *and* $\underline{m} = (m_1, m_2, \ldots, m_d)$ multi-indeces *(=pluralis av "multi-index"), and put*

$$\underline{k} \cdot \underline{m} = k_1 m_1 + k_2 m_2 + \ldots + k_d m_d$$

(=the "inner product" of $\underline{k}$ *and* $\underline{m}$*).*

Lemma 5.63.

$$\begin{aligned} \hat{x}(\underline{m}) &= \frac{1}{N^d} \sum_{\underline{k}} e^{-\frac{2\pi i \underline{m} \cdot \underline{k}}{N}} x(\underline{k}), \\ x(\underline{k}) &= \sum_{\underline{m}} e^{\frac{2\pi i \underline{m} \cdot \underline{k}}{N}} \hat{x}(\underline{m}). \end{aligned}$$

Definition 5.64.

$$\begin{aligned} (F \cdot G)(\underline{\mathrm{m}}) &= F(\underline{\mathrm{m}}) G(\underline{\mathrm{m}}) \\ (F * G)(\underline{\mathrm{m}}) &= \sum_{\underline{\mathrm{k}}} F(\underline{\mathrm{m}} - \underline{\mathrm{k}}) G(\underline{\mathrm{k}}), \end{aligned}$$

where all the components of $\underline{\mathrm{m}}$ and $\underline{\mathrm{k}}$ run over one period.

Theorem 5.65.

$$\begin{aligned} (F \cdot G)\hat{} &= \hat{F} * \hat{G}, \\ (F * G)\hat{} &= N^d \hat{F} \cdot \hat{G}. \end{aligned}$$

PROOF. Follows from Theorem 5.8.

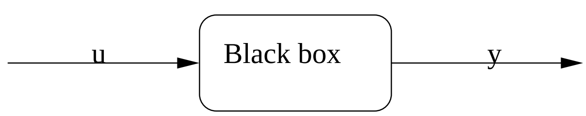
<u>In practice</u>: Either use *one* multi-dimensional, or use d one-dimensional transforms (not much difference, multi-dimensional a little faster).

Chapter 6

The Laplace Transform

6.1 General Remarks

Example 6.1. We send in a "signal" u into an "amplifier", and get an "output signal" y:



Under quite general assumptions it can be shown that

$$y(t) = (K * u)(t) = \int_{-\infty}^{t} K(t-s)u(s)ds,$$

i.e., the output is the convolution (="faltningen") of u with the "inpulse response" K.

Terminology 6.2. *"Impulse response" (=pulssvar) since $y = K$ if $u =$ a delta distribution.*

Causality 6.3. *The upper bound in the integral is t, i.e., $(K*u)(t)$ depends only on* past *values of u, and not on* future *values. This is called* causality*.*

If, in addition $u(t) = 0$ for $t < 0$, then $y(t) = 0$ for $t < 0$, and

$$y(t) = \int_{0}^{t} K(t-s)u(s)ds,$$

which is a *one-sided convolution.*

Classification 6.4. *Approximately: The Laplace-transform is the* Fourier transform *applied to* one-sided *signals (defined on $\mathbb{R}^+$). In addition there is a change of variable which rotate the complax plane.*

6.2 The Standard Laplace Transform

Definition 6.5. Suppose that $\int_0^\infty e^{-\sigma t}|f(t)|dt < \infty$ for some $\sigma \in \mathbb{R}$. Then we define the **Laplace transform** $\tilde{f}(s)$ of f by

$$\tilde{f}(s) = \int_0^\infty e^{-st} f(t) dt, \quad \Re(s) \geq \sigma.$$

Lemma 6.6. *The integral above converges absolutely for all $s \in \mathbb{C}$ with $\Re(s) \geq \sigma$ (i.e., $\tilde{f}(s)$ is well-defined for such s).*

Proof. Write $s = \alpha + i\beta$. Then

$$\begin{aligned} |e^{-st} f(t)| &= |e^{-\alpha t} e^{i\beta t} f(t)| \\ &= e^{-\alpha t}|f(t)| \\ &\leq e^{-\sigma t}|f(t)|, \text{ so} \\ \int_0^\infty |e^{-st} f(t)| dt &\leq \int_0^\infty e^{-\sigma t}|f(t)| dt < \infty. \qquad \square \end{aligned}$$

Theorem 6.7. *$\tilde{f}(s)$ is* analytic *in the open half-plane Re(s) $> \sigma$, i.e., $\tilde{f}(s)$ has a* complex derivative *with respect to s.*

Proof. (Outline)

$$\begin{aligned} \frac{\tilde{f}(z) - \tilde{f}(s)}{z - s} &= \int_0^\infty \frac{e^{-zt} - e^{-st}}{z - s} f(t) dt \\ &= \int_0^\infty \frac{e^{-(z-s)t} - 1}{z - s} e^{-st} f(t) dt \quad (\text{put } z - s = h) \\ &= \int_0^\infty \underbrace{\frac{1}{h}[e^{-ht} - 1]}_{\to -t \text{ as } h \to 0} e^{-st} f(t) dt \end{aligned}$$

As Re$(s) > \sigma$ we find that $\int_0^\infty |te^{-st} f(t)|d < \infty$ and a "short" computation (about $\frac{1}{2}$ page) shows that the Lebesgue dominated convergence theorem can be applied (show that $|\frac{1}{h}(e^{-ht} - 1)| \leq \text{const.} \cdot t \cdot e^{\alpha t}$, where $\alpha = \frac{1}{2}[\sigma + \Re(s)]$ (this is true for some small enough h), and then show that $\int_0^\infty te^{\alpha t}|e^{-st} f(t)|dt < \infty$). Thus, $\frac{d}{ds}\tilde{f}(s)$ exists, and

$$\boxed{\frac{d}{ds}\tilde{f}(s) = -\int_0^\infty e^{-st} t f(t) dt, \quad \Re(s) > \sigma}$$

Corollary 6.8. *$\frac{d}{ds}\tilde{f}(s)$ is the Laplace transform of $g(t) = -tf(t)$, and this Laplace transform converges (at least) in the half-plane Re(s) $> \sigma$.*

Theorem 6.9. *$\tilde{f}(s)$ is* bounded *in the half-plane $\Re(s) \geq \sigma$.*

Proof. (cf. proof of Lemma 6.6)

$$\begin{aligned} |\tilde{f}(s)| &= |\int_0^\infty e^{-st} f(t)dt| \leq \int_0^\infty |e^{-st} f(t)|dt \\ &= \int_0^\infty e^{-(\Re s)t} |f(t)|dt \leq \int_0^\infty e^{-\sigma t}|f(t)|dt < \infty. \end{aligned}$$

Definition 6.10. A **bounded analytic** function on the half-plane $\mathrm{Re}(s) > \sigma$ is called a H^∞**-function** (over this half-plane).

Theorem 6.11. *If f is absolutely continuous and $\int_0^\infty e^{-\sigma t}|g(t)|dt < \infty$ (i.e., $f(t) = f(0) + \int_0^t g(s)ds$, where $\int_0^\infty e^{-\sigma t}|g(t)|dt < \infty$), then*

$$(\tilde{f'})(s) = s\tilde{f}(s) - f(0), \quad \Re(s) > \sigma.$$

Proof. Integration by parts (a la Lebesgue) gives

$$\begin{aligned} \underbrace{\lim_{T\to\infty} \int_0^T e^{-st} f(t)\, dt}_{=\tilde{f}(s)} &= \lim_{T\to\infty} \left([\frac{e^{-st}}{-s} f(t)]_0^T + \frac{1}{s}\int_0^\infty e^{-st} f'(t)dt \right) \\ &= \frac{1}{s} f(0) + \frac{1}{s}\tilde{f}'(s), \quad \text{so} \\ (\tilde{f'})(s) &= s\tilde{f}(s) - f(0). \quad \square \end{aligned}$$

6.3 The Connection with the Fourier Transform

Let $\mathrm{Re}(s) > \sigma$, and make a change of variable:

$$\begin{aligned} &\int_0^\infty e^{-st} f(t)dt \quad (t = 2\pi v; dt = 2\pi dv) \\ &= \int_0^\infty e^{-2\pi sv} f(2\pi v) 2\pi dv \quad (s = \alpha + i\omega) \\ &= \int_0^\infty e^{-2\pi i\omega v} e^{-2\pi\alpha v} f(2\pi v) 2\pi dv \quad (\text{put } f(t) = 0 \text{ for } t < 0) \\ &= \int_{-\infty}^\infty e^{-2\pi i\omega t} g(t)dt, \end{aligned}$$

where

$$g(t) = \begin{cases} 2\pi e^{-2\pi\alpha t} f(2\pi t) & , \ t \geq 0 \\ 0 & , \ t < 0. \end{cases} \tag{6.1}$$

Thus, we got

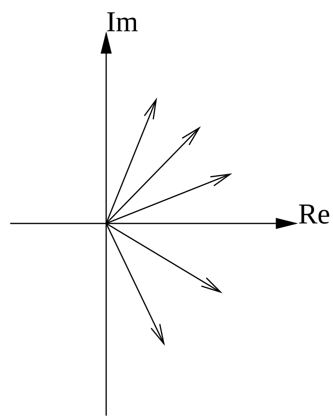
Theorem 6.12. *On the line Re*$(s) = \alpha$ *(which is a line parallell with the imaginary axis* $-\infty < \omega < \infty$*)* $\tilde{f}(s)$ *coincides (=sammanfaller med) with the Fourier transform of the function* g *defined in* (6.1).

Thus, modulo a change of variable, the Laplace transform is the Fourier transform of a function vanishing for $t < 0$. From Theorem 6.12 and the theory about Fourier transforms of functions in $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ we can derive a number of results. For example:

Theorem 6.13. *(Compare to Theorem 2.3, page 36) If* $f \in L^1(\mathbb{R}^+)$ *(i.e.,* $\int_0^\infty |f(t)|dt < \infty$*), then*

$$\lim_{\substack{|s|\to\infty \\ \Re(s)\geq 0}} |\tilde{f}(s)| = 0$$

(where $s \to \infty$ *in the half plane Re*$(s) > 0$ *in an arbitrary manner)*



Combining Theorem 6.12 with one of the theorems about the inversion of the Fourier integral we get formulas of the type

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{2\pi i\omega t}\tilde{f}(\alpha + i\omega)d\omega = \begin{cases} e^{-2\pi\alpha t}f(t), & t > 0, \\ 0, & t < 0. \end{cases}$$

This is often written as a complex line integral: We integrate along the line $\mathrm{Re}(s) = \alpha$, and replace $2\pi t \to t$ and multiply the formulas by $e^{2\pi\alpha t}$ to get $(s = \alpha + i\omega,\ ds = id\omega)$

$$\begin{aligned} f(t) &= \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{st}\tilde{f}(s)ds \qquad (6.2) \\ &= \frac{1}{2\pi i}\int_{\omega=-\infty}^{\infty} e^{(\alpha+i\omega)t}\tilde{f}(\alpha + i\omega)id\omega \end{aligned}$$

Warning 6.14. *This integral* seldom converges absolutely*. If it does converge absolutely, then (See Theorem 2.3 with the Fourier theorem replaced by the inverse Fourier theorem) the function*

$$g(t) = \begin{cases} 2\pi e^{-2\pi\alpha t} f(t), & t \geq 0, \\ 0, & t < 0 \end{cases}$$

must be continuous. *In other words:*

Lemma 6.15. *If the integral* (6.2) *converges absolutely, then f must be* continuous *and satisfy $f(0) = 0$.*

Therefore, the inversion theorems given in Theorem 2.30 and Theorem 2.31 are much more useful. They give (under the assumptions given there)

$$\frac{1}{2}[f(t+) + f(t-)] = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{\alpha - iT}^{\alpha + iT} e^{st} \tilde{f}(s) ds$$

(and we interpret $f(t) = 0$ for $t < 0$). By Theorem 6.11, if f is absolutely continuous and $f' \in L^1(\mathbb{R}^+)$, then (use also Theorem 6.13)

$$\tilde{f}(s) = \frac{1}{s}[(\tilde{f'})(s) + f(0)],$$

where $(\tilde{f'})(s) \to 0$ as $|s| \to \infty$, $\Re(s) \geq 0$. Thus, for large values of ω, $\tilde{f}(\alpha + i\omega) \approx \frac{f(0)}{i\omega}$, so the *convergence is slow* in general. Apart from the space H^∞ (see page 126) (over the half plane) another much used space (especially in Control theory) is H^2.

Theorem 6.16. *If $f \in L^2(\mathbb{R}^+)$, then the Laplace transform $\tilde{f}$ of f is analytic in the half-plane $\Re(s) > 0$, and it satisfy, in addition*

$$\sup_{\alpha > 0} \int_{-\infty}^{\infty} |\tilde{f}(\alpha + i\omega)|^2 d\omega < \infty,$$

i.e., there is a constant M so that

$$\int_{-\infty}^{\infty} |\tilde{f}(\alpha + i\omega)|^2 d\omega \leq M \quad (\textit{for all } \alpha > 0).$$

Proof. By Theorem 6.12 and the L^2-theory for Fourier integrals (see Section 2.3),

$$\begin{aligned}
\int_{-\infty}^{\infty} |\tilde{f}(\alpha + i\omega)|^2 d\omega &= \int_0^{\infty} |2\pi e^{-2\pi\alpha t} f(2\pi t)|^2 dt \quad (2\pi t = v) \\
&= 2\pi \int_0^{\infty} |e^{-\alpha v} f(v)|^2 dv \\
&\leq 2\pi \int_0^{\infty} |f(v)|^2 dv = 2\pi \|f\|_{L^2(0,\infty)}. \quad \square
\end{aligned}$$

Converesly:

Theorem 6.17. *If φ is analytic in $\Re(s) > 0$, and φ satisfies*

$$\sup_{\alpha>0} \int_{-\infty}^{\infty} |\varphi(\alpha + i\omega)|^2 d\omega < \infty, \tag{6.3}$$

then φ is the Laplace transform of a function $f \in L^2(\mathbb{R}^+)$.

Proof. Not too difficult (but rather long).

Definition 6.18. An **H^2-function** over the half-plane $\Re(s) > 0$ is a function φ which is analytic and satisfies (6.3).

6.4 The Laplace Transform of a Distribution

Let $f \in \mathcal{S}'$ (tempered distribution), and suppose that the *support* of f is contained in $[0, \infty) = \mathbb{R}^+$ (i.e., f vanishes on $(-\infty, 0)$). Then we can define the Laplace transform of f in two ways:

i) Make a change of variables as on page 126 and use the Fourier transform theory.

ii) Define $\tilde{f}(s)$ as f applied to the "test function" e^{-st}, $t > 0$. (Warning: this is *not* a test function!)

Both methods lead to the same result, but the second method is actually simpler. If $\Re(s) > 0$, then $t \mapsto e^{-st}$ behaves like a test function on $[0, \infty)$ but not on $(-\infty, 0)$. However, f is supported on $[0, \infty)$, so it does not matter how e^{-st} behaves for $t < 0$. More precisely, we take an arbitrary "cut off" function $\eta \in C_{\text{pol}}^{\infty}$ satisfying

$$\begin{cases} \eta(t) \equiv 1 & \text{for } t \geq -1, \\ \eta(t) \equiv 0 & \text{for } t \leq -2. \end{cases}$$

Then $\eta(t)e^{-st} = e^{-st}$ for $t \in [-1, \infty)$, and since f is supported on $[0, \infty)$ we can replace e^{-st} by $\eta(t)e^{-st}$ to get

Definition 6.19. If $f \in \mathcal{S}'$ vanishes on $(-\infty, 0)$, then we define the Laplace transform $\tilde{f}(s)$ of f by

$$\tilde{f}(s) = \langle f, \eta(t)e^{-st} \rangle, \quad \Re(s) > 0.$$

(Compare this to what we did on page 84).
Note: In the same way we can define the Laplace transform of a distribution that is not necessarily tempered, but which becomes tempered after multiplication by $e^{-\sigma t}$ for some $\sigma > 0$. In this case the Laplace transform will be defined in the half-plane $\Re s > \sigma$.

Theorem 6.20. *If f vanishes on $(-\infty, 0)$, then $\tilde{f}$ is* analytic *on the half-plane $\Re s > 0$.*

Proof omitted.
Note: $\tilde{f}$ need not be bounded. For example, if $f = \delta'$, then

$$\begin{aligned}\widetilde{(\delta')}(s) &= \langle \delta', \eta(t)e^{-st}\rangle = -\langle \delta, \eta(t)e^{-st}\rangle \\ &= \frac{d}{dt}e^{-st}{}_{|t=0} = -s.\end{aligned}$$

(which is unbounded). On the other hand

$$\tilde{\delta}(s) = \langle \delta, \eta(t)e^{-st}\rangle = e^{-st}{}_{|t=0} = 1.$$

Theorem 6.21. *If $f \in \mathcal{S}'$ vanishes on $(-\infty, 0)$, then*

$$\left.\begin{aligned} &i) \quad [\widetilde{tf(t)}](s) &&= -[\tilde{f}(s)]' \\ &ii) \quad \widetilde{f'(s)} &&= s\tilde{f}(s)\end{aligned}\right\} \Re(s) > 0$$

Proof. Easy (homework?)

Warning 6.22. *You can apply this distribution transform also to* functions*, but remember to put $f(t) = 0$ for $t < 0$. This automatically leads to a δ-term in the distribution derivative of f: after we define $f(t) = 0$ for $t < 0$, the distribution derivative of f is*

$$\underbrace{f(0)\delta_0}_{\textit{dervatives of jump at zero}} + \underbrace{f'(t)}_{\textit{usual derivative}}$$

6.5 Discrete Time: Z-transform

This is a short continuation of the theory on page 101.
In discrete time we also run into one-sided convolutions (as we have seen), and it is possible to compute these by the FFT. From a mathematical point of view the Z-tranform is often simpler than the Fourier transform.

Definition 6.23. The Z-transform of a sequence $\{f(n)\}_{n=0}^{\infty}$ is given by

$$\tilde{f}(z) = \sum_{n=0}^{\infty} f(n) z^{-n},$$

for all these $z \in \mathbb{C}$ for which the series converges absolutely.

Lemma 6.24.

i) There is a number $\rho \in [0, \infty]$ so that $\tilde{f}(z)$ converges for $|z| > \rho$ and $\tilde{f}(z)$ diverges for $|z| < \rho$.

ii) $\tilde{f}$ is analytic *for $|z| > \rho$.*

Proof. Course on analytic functions.

As we noticed on page 101, the Z-transform can be converted to the discrete time Fourier transform by a simple change of variable.

6.6 Using Laguerra Functions and FFT to Compute Laplace Transforms

We start by recalling some results from the course in special functions:

Definition 6.25. The **Laguerra polynomials** $\mathcal{L}_m$ are given by

$$\mathcal{L}_m(t) = \frac{1}{m!} e^t \Big(\frac{d}{dt}\Big)^m (t^m e^{-t}), \quad m \geq 0,$$

and the **Laguerra functions** ℓ_m are given by

$$\ell_,(t) = \frac{1}{m!} e^{\frac{t}{2}} \Big(\frac{d}{dt}\Big)^m (t^m e^{-t}), \quad m \geq 0.$$

Note that $\ell_m(t) = e^{-\frac{t}{2}} \mathcal{L}_m(t)$.

Lemma 6.26. *The Laguerra polynomials can be computed recusively from the formula*

$$(m+1)\mathcal{L}_{m+1}(t) + (t - 2m - 1)\mathcal{L}_m(t) + m\mathcal{L}_{m-1}(t) = 0,$$

with starting values $\mathcal{L}_{-1} \equiv 0$ and $\mathcal{L}_1 \equiv 1$.

We saw that the sequence $\{\ell_m\}_{m=0}^{\infty}$ is an ortonormal sequence in $L^2(\mathbb{R}^+)$, so that if we define, for some $f \in L^2(\mathbb{R}^+)$,

$$f_m = \int_0^\infty f(t)\ell_m(t)dt,$$

then

$$f(t) = \sum_{m=0}^{\infty} f_m \ell_m(t) \quad \text{(in the } L^2\text{-sense).} \tag{6.4}$$

Taking Laplace transforms in this equation we get

$$\tilde{f}(s) = \sum_{m=0}^{\infty} f_m \tilde{\ell}_m(s).$$

Lemma 6.27.

i) $\tilde{\ell}_m(s) = \frac{(s-1/2)^m}{(s+1/2)^{m+1}}$,

ii) $\tilde{f}(s) = \sum_{m=0}^{\infty} f_m \frac{(s-1/2)^m}{(s+1/2)^{m+1}}$, *where* $f_m = \int_0^\infty f(t)\ell_m(t)dt$.

Proof. Course on special functions.

The same method can be used to compute *inverse Laplace transforms*, and this gives a possibility to use FFT to compute the coefficients $\{f_m\}_{m=0}^{\infty}$ if we know $\tilde{f}(s)$. The argument goes as follows.

Suppose for simplicity that $f \in L^1(\mathbb{R})$, so that $\tilde{f}(s)$ is defined and bounded on $\mathbb{C}_+ = \{s \in \mathbb{C} | Re\ s > 0\}$. We want to expand $\tilde{f}(s)$ into a series of the type

$$\tilde{f}(s) = \sum_{m=0}^{\infty} f_m \frac{(s-1/2)^m}{(s+1/2)^{m+1}}. \tag{6.5}$$

Once we know the cofficients f_m we can recover $f(t)$ from formula (6.4). To find the coefficients f_m we map the right half-plane $\mathbb{C}_+$ into the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We define

$$z = \frac{s-1/2}{s+1/2} \iff sz + \frac{1}{2}z = s - \frac{1}{2} \iff$$
$$s = \frac{1}{2}\frac{1+z}{1-z} \quad \text{and} \quad s + s + 1/2 = \frac{1}{2}(1 + \frac{1+z}{1-z}) = \frac{1}{1-z}, \text{ so}$$
$$\frac{1}{s+1/2} = 1 - z$$

Lemma 6.28.

i) $\Re(s) > 0 \iff |z| < 1$ *item[ii)]*$\Re(s) = 0 \iff |z| = 1$

iii) $s = 1/2 \iff z = 0$

iv) $s = \infty \iff z = 1$

v) $s = 0 \iff z = -1$

vi) $s = -1/2 \iff z = \infty$

Proof. Easy.

Conclusion: The function $\tilde{f}(\frac{1}{2}\frac{1+z}{1-z})$ is analytic *inside* the *unit disc* $\mathbb{D}$, (and bounded if $\tilde{f}$ is bounded on $\mathbb{C}_+$).

Making the same change of variable as in (6.5) we get

$$\frac{1}{1-z}\tilde{f}(\frac{1}{2}\frac{1+z}{1-z}) = \sum_{m=0}^{\infty} f_m z^m.$$

Let us define

$$\boxed{g(z) = \frac{1}{1-z}\tilde{f}(\frac{1}{2}\frac{1+z}{1-z}), \quad |z| < 1.}$$

Then

$$\boxed{g(z) = \sum_{m=0}^{\infty} f_m z^m,}$$

so $g(z)$ is the "mathematical" version of the Z-transform of the sequence $\{f_m\}_{m=0}^{\infty}$ (in the control theory of the Z-transform we replace z^m by z^{-m}).

If we know $\tilde{f}(s)$, then we know $g(z)$, and we can use FFT to compute the coefficients f_m: Make a change of variable: Put $\alpha_N = e^{2\pi i/N}$. Then

$$g(\alpha_N^k) = \sum_{m=0}^{\infty} f_m \alpha_N^{mk} = \sum_{m=0}^{\infty} f_m e^{2\pi imk/N} \approx \sum_{m=0}^{N} f_m e^{2\pi imk/N}$$

(if N is large enough). This is the *inverse discrete Fourier transform* of a periodic extension of the sequence $\{f_m\}_{m=0}^{N-1}$. Thus, $f_m \approx$ the discrete transformation of the sequence $\{g(\alpha_N^k)\}_{k=0}^{N-1}$. We put

$$\boxed{G(k) = g(\alpha_N^k) = \frac{1}{1-\alpha_N^k}\tilde{f}(\frac{1}{2}\frac{1+\alpha_N^k}{1-\alpha_N^k}),}$$

and get $f_m \approx \hat{G}(m)$, which can be computed with the FFT.

<u>Error estimate</u>: We know that $f_m = \hat{g}(m)$ (see page 115) and that $\hat{g}(m) = 0$ for $m < 0$. By the error estimate on page 108 we get

$$|\hat{G}(m) - f_m| = \sum_{k \neq 0} |f_{m+kN}|$$

(where we put $f_m = 0$ for $m < 0$).

Bibliography

[Gri04] Gustaf Gripenberg, *Fourier-muunnosten perusteet (lecture notes)*, http://math.tkk.fi/teaching/fourier/, 2004.

[GW99] C. Gasquet and P. Witomski, *Fourier analysis and applications*, Texts in Applied Mathematics, vol. 30, Springer-Verlag, New York, 1999, Filtering, numerical computation, wavelets, Translated from the French and with a preface by R. Ryan. MR MR1657104 (99h:42003)

[Wal02] David F. Walnut, *An introduction to wavelet analysis*, Applied and Numerical Harmonic Analysis, Birkhäuser Boston Inc., Boston, MA, 2002. MR MR1854350 (2002f:42039)