Algebraic Topology

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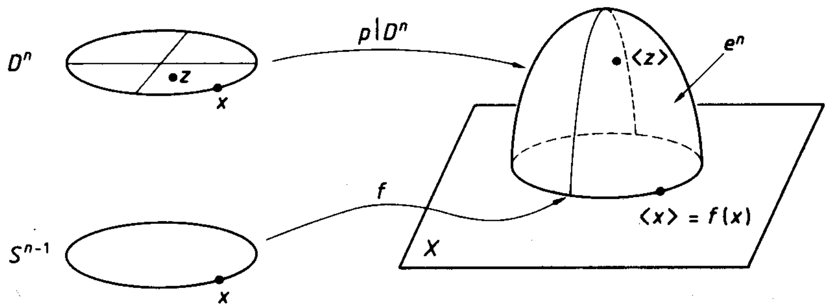


These lecture notes are inspired to a large extend by the book

R.Stöcker/H.Zieschang: Algebraische Topologie, B.G.Teubner, Stuttgart 1988

which I recommend for many of the topics I could not treat in this lecture course, in particular this concerns the homology of products [**7**, chapter 12], homology with coefficients [**7**, chapter 10], cohomology [**7**, chapter 13–15].

As always, I am very thankful for any feedback in the range from simple typing errors up to mathematical incomprehensibilities.

Vienna, 2000.08.01 Andreas Kriegl

Since Simon Hochgerner pointed out, that I forgot to treat the case $q = n - 1 - r$ for $r < n - 1$ in theorem 10.3, I adopted the proof appropriately.

Vienna, 2000.09.25 Andreas Kriegl

I translated chapter 1 from German to English, converted the whole source from amstex to latex and made some stylistic changes for my lecture course in this summer semester.

Vienna, 2006.02.17 Andreas Kriegl

I am thankfull for the lists of corrections which has been provided by Martin Heuschober and by Stefan Fürdös.

Vienna, 2008.01.30 Andreas Kriegl

Inhaltsverzeichnis

1 Building blocks and homeomorphy

For the first chapter I mainly listed the contents in form of short statements. For details please refer to the book.

Ball, sphere and cell

Problem of homeomorphy.
When is $X \cong Y$? Either we find a homeomorphism $f : X \to Y$, or a topological property, which hold for only one of X and Y, or we cannot decide this question.

1.1 Definition of basic building blocks. [7, 1.1.2]

1 $\mathbb{R}$ with the metric given by $d(x, y) := |x - y|$.

2 $I := [0, 1] := \{x \in \mathbb{R} : 0 \leq x \leq 1\}$, the unit interval.

3 $\mathbb{R}^n := \prod_n \mathbb{R} = \prod_{i \in n} \mathbb{R} = \prod_{i=0}^{n-1} \mathbb{R} = \{(x_i)_{i=0,\dots,n-1} : x_i \in \mathbb{R}\}$, with the product topology or, equivalently, with any of the equivalent metrics given by a norm on this vector space.

4 $I^n := \prod_n I = \{(x_i)_{i=0}^{n-1} : 0 \leq x_i \leq 1 \forall i\} = \{x \in \mathbb{R}^n : \|x - (\frac{1}{2}, \dots, \frac{1}{2})\|_\infty \leq 1\}$, the n-dimensional unit cube.

5 $\dot{I}^n := \partial_{\mathbb{R}^n} I^n = \{(x_i)_i \in I^n : \exists i : x_i \in \{0, 1\}\}$, the boundary of I^n in $\mathbb{R}^n$.

6 $D^n := \{x \in \mathbb{R}^n : \|x\|_2 := \sqrt{\sum_{i \in n}(x_i)^2} \leq 1\}$, the n-dimensional unit ball (with respect to the Euclidean norm).
A topological space X is called n-BALL iff $X \cong D^n$.

7 $\dot{D}^n := \partial_{\mathbb{R}^n} D^n = S^{n-1} := \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$, the $n - 1$-dimensional unit sphere.
A topological space X is called n-SPHERE iff $X \cong S^n$.

8 $\mathring{D}^n := \{x \in \mathbb{R}^n : \|x\|_2 < 1\}$, the interior of the n-dimensional unit ball.
A topological space X is called n-CELL iff $X \cong \mathring{D}^n$.

1.2 Definition. [7, 1.1.3] An AFFINE HOMEOMORPHISMS is a mapping of the form $x \mapsto A \cdot x + b$ with an invertible linear A and a fixed vector b.

Hence the ball in $\mathbb{R}^n$ with center b and radius r is homeomorphic to D^n and thus is an n-ball.

1.3 Example. [7, 1.1.4] $\mathring{D}^1 \cong \mathbb{R}$: Use the odd functions $t \mapsto \tan(\frac{\pi}{2}t)$, or $t \mapsto \frac{t}{1-t^2}$ with derivative $t \mapsto \frac{t^2+1}{(t^2-1)^2} > 0$, or $t \mapsto \frac{t}{1-|t|}$ with derivative $t \mapsto 1/(1 - |t|) > 0$ and inverse mapping $s \mapsto \frac{t}{1+|t|}$. Note, that a bijective function $f_1 : [0, 1) \to [0, +\infty)$ extends to an odd function $f : (-1, 1) \to \mathbb{R}$ by setting $f(x) := -f_1(-x)$ for $x < 0$. For $f_1(t) = \frac{t}{1-t}$ we have $f(t) = -\frac{-t}{1-(-t)} = \frac{t}{1-|t|}$ and for $f_1(t) = \frac{t}{1-t^2}$ we have $f(t) = -\frac{-t}{1-(-t)^2} = \frac{t}{1-t^2}$. Note that in both cases $f_1'(0) = \lim_{t \to 0+} f_1'(t) = 1$, hence f is a C^1 diffeomorphism. However, in the first case $\lim_{t \to 0+} f_1''(t) = 2$ and hence f is not C^2.

1.4 Example. [7, 1.1.5] $\mathring{D}^n \cong \mathbb{R}^n$: Use $f : x \mapsto \frac{x}{1-\|x\|} = \frac{x}{\|x\|} \cdot f_1(\|x\|)$ with $f_1(t) = \frac{t}{1-t}$ and directional derivative $f'(x)(v) = \frac{1}{1-\|x\|}\, v + \frac{\langle x|v\rangle}{(1-\|x\|)^2\|x\|}\, x \to v$ for $x \to 0$.

1.5 Corollary. [7, 1.1.6] *$\mathbb{R}^n$ is a cell; products of cells are cells, since $\mathbb{R}^n \times \mathbb{R}^m \cong \mathbb{R}^{n+m}$ by "associativity" of the product.*

1.6 Definition. A subset $A \subseteq \mathbb{R}^n$ ist called CONVEX, iff $x + t(y-x) \in A$ for $\forall x, y \in A$, $t \in [0,1]$.

1.7 Definition. A PAIR (X, A) of spaces is a topological space X together with a subspace $A \subseteq X$. A MAPPING $f : (X,A) \to (Y,B)$ of pairs is a continuous mapping $f : X \to Y$ with $f(A) \subseteq B$. A HOMEOMORPHISM $f : (X,A) \to (Y,B)$ of pairs is a mapping of pairs which is a homeomorphism $f : X \to Y$ and induces a homeomorphism $f|_A : A \to B$.

1.8 Definition. [7, 1.3.2] A mapping $f : (X,A) \to (Y,B)$ of pairs is called RELATIVE HOMEOMORPHISM, iff $f : X \setminus A \to Y \setminus B$ is a well-defined homeomorphism. A homeomorphism of pairs is a relative homeomorphism, but not conversely even if $f : X \to Y$ is a homeomorphism.

However, for X and Y compact any homeomorphism $f : X\setminus\{x_0\} \to Y\setminus\{y_0\}$ extends to a homeomorphism $\tilde{f} : (X, \{x_0\}) \to (Y, \{y_0\})$ of pairs, since $X \cong (X \setminus \{x_0\})_\infty$, cf. 1.35.

1.9 Example. [7, 1.1.15]

1 $\mathbb{R}^n \setminus \{0\} \cong S^{n-1} \times (0, +\infty) \cong S^{n-1} \times \mathbb{R}$ via $x \mapsto (\frac{1}{\|x\|}x, \|x\|)$, $e^t\, y \leftarrow\!\shortmid (y,t)$.

2 $D^n \setminus \{0\} \cong S^{n-1} \times (0,1] \cong S^{n-1} \times (\varepsilon, 1]$, via $(0,1] \cong (\varepsilon, 1]$ and (1).

1.10 Theorem. [7, 1.1.8] *$X \subseteq \mathbb{R}^n$ compact, convex, $\mathring{X} \neq \emptyset \Rightarrow (X, \dot{X}) \cong (D^n, S^{n-1})$.*
In particular, X is a ball, $\dot{X}$ is a sphere and $\mathring{X}$ is a cell.
If $X \subseteq \mathbb{R}^n$ is (bounded,) open and convex and not empty $\Rightarrow X$ is a cell.

Proof. W.l.o.g. let $0 \in \mathring{X}$ (translate X by $-x_0$ with $x_0 \in \mathring{X}$). The mapping $f : \dot{X} \ni x \mapsto \frac{1}{\|x\|}x \in S^{n-1}$ is bijective, since it keeps rays from 0 invariant and since for every $x \neq 0$ there is a $t > 0$ with $t\,x \in \dot{X}$ by the intermediate value theorem and this is unique, since $t\,x \in \mathring{X}$ for all $0 < t < t_0$ with $t_0\, x \in X$. Since $\dot{X}$ is compact it is a homeomorphism and by radial extension we get a homeomorphism

$$D^n \setminus \{0\} \cong S^{n-1} \times (0,1] \overset{f \times \mathrm{id}}{\cong} \dot{X} \times (0,1] \cong X \setminus \{0\},$$
$$x \mapsto \left(\frac{x}{\|x\|}, \|x\|\right) \mapsto \left(f^{-1}\left(\frac{x}{\|x\|}\right), \|x\|\right) \mapsto \|x\|\, f^{-1}\left(\frac{x}{\|x\|}\right)$$

which extends via $0 \mapsto 0$ to a homeomorphism of the 1-point compactifications and hence a homeomorphism of pairs $(D^n, S^{n-1}) \to (X, \dot{X})$.

The second part follows by considering $\overline{X}$, a compact convex set with non-empty interior X, since for $x \in \overline{X} \setminus X$ we have that $x = \lim_{t \to 1+} tx$ with $tx \notin X$ for $t > 1$ (if we assume $0 \in X$) and hence $x \in \dot{X}$.

That the boundedness condition can be dropped can be found for a much more general situation in [3, 16.21]. □

1.11 Corollary. [7, 1.1.9] *I^n is a ball and $\dot{I}^n$ is a sphere.*

1.12 Example. [7, 1.1.10] [7, 1.1.11] $D^p \times D^q$ is a ball, hence products of balls are balls, and $\partial(D^p \times D^q) = S^{p-1} \times D^q \cup D^p \times S^{q-1}$ is a sphere:
$D^p \times D^q$ is compact convex, and by exercise (1.1.1A) $\partial(A \times B) = \partial A \times B \cup A \times \partial B$. So by 1.10 the result follows.

1.13 Remark. [7, 1.1.12] 1.10 is wrong without convexity or compactness assumption: For compactness this is obvious. That, for example, a compact annulus is not a ball will follow from 2.19.

1.14 Example. [7, 1.1.13] $S^n = D^n_+ \cup D^n_-$, $D^n_+ \cap D^n_- = S^{n-1} \times \{0\} \cong S^{n-1}$, where $D^n_\pm := \{(x,t) \in S^n \subseteq \mathbb{R}^n \times \mathbb{R} : \pm t \geq 0\} \cong D^n$. The stereographic projection $S^n \setminus \{(0,\dots,0,1)\} \cong \mathbb{R}^n$ is given by $(x, x_n) \mapsto \frac{1}{1-x_n} x$.

1.15 Corollary. [7, 1.1.14] *$S^n \setminus \{*\}$ is a cell.*

1.16 Example. [7, 1.1.15.3] $D^n \setminus \{\dot{x}\} \cong \mathbb{R}^{n-1} \times [0, +\infty)$ for all $\dot{x} \in S^{n-1}$, via $\mathbb{R}^{n-1} \times [0,+\infty) \cong (S^{n-1} \setminus \{\dot{x}\}) \times (0,1] \cong D^n \setminus \{\dot{x}\}$, $(x,t) \mapsto x_0 + t(x - x_0)$.

1.17 Example. [7, 1.1.20] $S^n \not\cong \mathbb{R}^n$ and $D^n \not\cong \mathbb{R}^n$, since $\mathbb{R}^n$ is not compact.

None-homeomorphy of $X = S^1$ with I follows by counting components of $X \setminus \{*\}$.

1.18 Example. [7, 1.1.21] $S^1 \times S^1$ is called torus. It is embeddable into $\mathbb{R}^3$ by $(x,y) = (x_1, x_2; y_1, y_2) \mapsto ((R + r\, y_1)x, r\, y_2)$ with $0 < r < R$. This image is described by the equation $\{(x,y,z) : (\sqrt{x^2+y^2} - R)^2 + z^2 = r^2\}$. Furthermore, $S^1 \times S^1 \not\cong S^2$ by Jordan's curve theorem, since $(S^1 \times S^1) \setminus (S^1 \times \{1\})$ is connected.

1.19 Theorem (Invariance of a domain). [7, 1.1.16] *$\mathbb{R}^n \supseteq X \cong Y \subseteq \mathbb{R}^n$, X open in $\mathbb{R}^n$ $\Rightarrow$ Y open in $\mathbb{R}^n$.*

We will prove this hard theorem in 10.

1.20 Theorem (Invariance of dimension). [7, 1.1.17] *$m \neq n \Rightarrow \mathbb{R}^m \not\cong \mathbb{R}^n$, $S^m \not\cong S^n$, $D^m \not\cong D^n$.*

Proof. Let $m < n$.

Suppose $\mathbb{R}^n \cong \mathbb{R}^m$, then $\mathbb{R}^n \subseteq \mathbb{R}^n$ is open, but the image $\mathbb{R}^m \cong \mathbb{R}^m \times \{0\} \subseteq \mathbb{R}^n$ is not, a contradiction to 1.19.

$S^m \cong S^n \Rightarrow \mathbb{R}^m \cong S^m \setminus \{x\} \cong S^n \setminus \{y\} \cong \mathbb{R}^n \Rightarrow m = n$.

$f : D^m \cong D^n \Rightarrow \mathring{D}^n \cong f^{-1}(\mathring{D}^n) \subseteq D^m \subseteq \mathbb{R}^m \subset \mathbb{R}^n$ and $f^{-1}(\mathring{D}^n)$ is not open, a contradiction to 1.19. $\square$

1.21 Theorem (Invariance of boundary). [7, 1.1.18] *$f : D^n \to D^n$ homeomorphism $\Rightarrow$ $f : (D^n, S^{n-1}) \to (D^n, S^{n-1})$ homeomorphism of pairs.*

Proof. Let $\dot{x} \in \dot{D}^n$ with $y = f(\dot{x}) \notin \dot{D}^n$. Then $y \in \mathring{D}^n =: U$ and $f^{-1}(U)$ is homeomorphic to U but not open, since $x \in f^{-1}(U) \cap \dot{D}^n$. $\square$

1.22 Definition. [7, 1.1.19] Let X be an n-ball and $f : D^n \to X$ a homeomorphism. The BOUNDARY $\dot{X}$ of X ist defined as the image $f(\dot{D}^n)$. This definition makes sense by 1.21.

Quotient spaces

1.23 Definition. Quotient space. [7, 1.2.1] Cf. [2, 1.2.12]. Let $\sim$ be an equivalence relation on a topological space X. We denote the set of EQUIVALENCE CLASSES $[x]_\sim := \{y \in X : y \sim x\}$ by $X/\sim$. The QUOTIENT TOPOLOGY on $X/\sim$ is the final topology with respect to the mapping $\pi : X \to X/\sim$, $x \mapsto [x]_\sim$.

1.24 Proposition. [7, 1.2.2] *A subset $B \subseteq X/\sim$ is open/closed iff $\pi^{-1}(B)$ is open/closed. The quotient mapping π is continuous and surjective. It is open/closed iff for every open/closed $A \subseteq X$ the saturated hull $\pi^{-1}(\pi(A))$ is open/closed.*

The image of the closed subset $\{(x, y) : x\cdot y = 1, x, y > 0\} \subseteq \mathbb{R}^2$ under the projection $\mathrm{pr}_1 : \mathbb{R}_2 \to \mathbb{R}$ is not closed!

1.25 Definition. [7, 1.2.9] A mapping $f : X \to Y$ is called QUOTIENT MAPPING (or final), iff f is surjective continuous and satisfies one of the following conditions:

1 The induced mapping $X/\sim \to Y$ is a homeomorphism, where $x_1 \sim x_2 :\Leftrightarrow f(x_1) = f(x_2)$.

2 $B \subseteq Y$ is open (closed) if $f^{-1}(B)$ is it.

3 A mapping $g : Y \to Z$ is continuous iff $g \circ f$ is it.

(1⇒2) $X \to X/\sim$ has this property.
(2⇒3) $g^{-1}(W)$ open $\Leftrightarrow = (g \circ f)^{-1}(W) = f^{-1}(g^{-1}W)$ is open.
(3⇒1) $X/\sim \to Y$ is continuous by 1.27. $Y \to X/\sim$ is continuous by (3).

1.26 Example. [7, 1.2.3]

1 $I/\sim \cong S^1$, where $0 \sim 1$: The mapping $t \mapsto e^{2\pi i t}$, $I \to S^1$ factors to homeomorphism $I/\sim \to S^1$.

2 $I^2/\sim \cong S^1 \times I$, where $(0, t) \sim (1, t)$ for all t.

3 $I^2/\sim \cong S^1 \times S^1$, where $(t, 0) \sim (t, 1)$ and $(0, t) \sim (1, t)$ for all t.

1.27 Proposition. Universal property of $X/\sim$. [7, 1.2.11] [7, 1.2.6] [7, 1.2.5] *Let $f : X \to Y$ be continuous. Then f is compatible with the equivalence relation (i.e. $x \sim x' \Rightarrow f(x) = f(x')$) iff it factors to a continuous mapping $X/\sim \to Y$ over $\pi : X \to X/\sim$. Note that f is compatible with the equivalence relation iff the relation $f \circ \pi^{-1}$ is a mapping. The factorization $X/\sim \to Y$ is then given by $f \circ \pi^{-1}$.*

Proof.

$(z, y) \in f \circ \pi^{-1} \Leftrightarrow \exists x \in X : f(x) = y, \pi(x) = z$. Thus y is uniquely determined by z iff $\pi(x) = \pi(x') \Rightarrow f(x) = f(x')$. □

1.28 Proposition. [7, 1.2.4] *Functoriality of formation of quotients. Let $f : X \to Y$ be continuous and compatible with equivalence relations $\sim_X$ on X and $\sim_Y$ on Y. Then there is a unique induced continuous mapping $\tilde{f} : X/\sim_X \to Y/\sim_Y$.*

If f and f^{-1} are compatible with the equivalence relations and is a homeomorphism, then $\tilde{f}$ is a homeomorphism.

1.29 Proposition. [7, 1.2.7] [7, 1.2.12] *The restriction of a quotient-mapping to a closed/open saturated set is a quotient-mapping.*
Let $f : X \to Y$ be a quotient mapping, $B \subseteq Y$ open (or closed), $A := f^{-1}(B)$. Then $f|_A : A \to B$ is a quotient mapping.

For example, the restriction of $\pi : I \to I/\dot{I}$ to the open set $[0, 1)$ is not a quotient mapping.

Proof. Let $U \subseteq B$ with $(f|_A)^{-1}(U)$ open. Then $f^{-1}(U) = (f|_A)^{-1}(U)$ is open and hence $U \subseteq Y$ is open. □

1.30 Corollary. [7, 1.2.8] *A closed/open, $a \in A, x \in X, x \sim a \Rightarrow x = a$, $p : X \to Y$ quotient-mapping $\Rightarrow p|_A : A \to p(A) \subseteq Y$ is an embedding.*

Proof. $\Rightarrow A = p^{-1}(p(A)) \xRightarrow{\boxed{1.29}} p|_A; A \to p(A)$ is a quotient mapping and injective, hence a homeomorphism. □

1.31 Proposition. [7, 1.2.10] *Continuous surjective closed/open mappings are obviously quotient-mappings, but not conversely. Continuous surjective mappings from a compact to a T_2-space are quotient-mappings, since the image of closed subsets is compact hence closed. f, g quotient mapping $\Rightarrow g \circ f$ quotient mapping, by $\boxed{1.25.3}$. $g \circ f$ quotient mapping $\Rightarrow g$ quotient mapping, by $\boxed{1.25.3}$.*

1.32 Proposition. Theorem of Whitehead. [7, 1.2.13] *Let g be a quotient mapping and X locally compact. Then $X \times g$ is quotient mapping.*

For a proof and a counterexample for none locally compact X see [2, 2.2.9]:

Proof. Let $(x_0, z_0) \in W \subseteq X \times Z$ with open $f^{-1}(W) \subseteq X \times Y$, where $f := X \times g$ for $g : Y \to Z$. We choose $y_0 \in g^{-1}(z_0)$ and a compact $U \in \mathcal{U}(x_0)$ with $U \times \{y_0\} \subseteq f^{-1}(W)$. Since $f^{-1}(W)$ is saturated, $U \times g^{-1}(g(y)) \subseteq f^{-1}(W)$ provided $U \times \{y\} \subseteq f^{-1}(W)$. In particular, $U \times g^{-1}(z_0) \subseteq f^{-1}(W)$. Let $V := \{z \in Z : U \times g^{-1}(z) \subseteq f^{-1}(W)\}$. Then $(x_0, z_0) \in U \times V \subseteq W$ and V is open, since $g^{-1}(V) := \{y \in Y : U \times \{y\} \subseteq f^{-1}(W)\}$ is open. □

1.33 Corollary. [7, 1.2.14] *$f : X \to X'$, $g : Y \to Y'$ quotient mappings, X, Y' locally compact $\Rightarrow f \times g$ quotient mapping.*

Proof.

$$\begin{tikzcd}
X \times Y \arrow[r, "p \times Y"] \arrow[d, "X \times q"] & X' \times Y \arrow[d, "X' \times q"] \\
X \times Y' \arrow[r, "p \times Y'"] & X' \times Y'
\end{tikzcd}$$

□

Examples of quotient mappings

1.34 Proposition. Collapse of subspace. [7, 1.3.1] [7, 1.3.3] *$A \subseteq X$ closed $\Rightarrow$ $p : (X, A) \to (X/A, \{A\})$ is a relative homeomorphism. The functorial property for mappings of pairs is:*

$$\begin{tikzcd} (X,A) \arrow[r, "f"] \arrow[d] & (Y,B) \arrow[d] \\ (X/A, A/A) \arrow[r, dotted] & (Y/B, B/B) \end{tikzcd}$$

Proof. That $p : X \setminus A \to X/A \setminus A/A$ is a homeomorphism follows from 1.29. The functorial property follows from 1.27 $\square$

1.35 Example. [7, 1.3.4] $X/\emptyset \cong X$, $X/\{*\} \cong X$. $I/\dot{I} \cong S^1$, $X/A \cong (X \setminus A)_\infty$, provided X compact. In fact, X/A is compact, $X \setminus A$ is openly embedded into X/A and $X/A \setminus (X \setminus A)$ is the single point $A \in X/A$.

1.36 Example. [7, 1.3.5] $D^n \setminus S^{n-1} = \mathring{D}^n \cong \mathbb{R}^n$ and hence by 1.35 $D^n/S^{n-1} \cong (D^n \setminus S^{n-1})_\infty \cong (\mathbb{R}^n)_\infty \cong S^n$. Or, explicitly, $x \mapsto (\|x\|, \frac{x}{\|x\|}) \mapsto (\sin(\pi(1-t))\frac{x}{\|x\|}, \cos(\pi(1-t)))$.

1.37 Example. [7, 1.3.6] $X \times I$ is called cylinder over X. And $CX := (X \times I)/(X \times \{0\})$ is called the cone with base X. $C(S^n) \cong D^{n+1}$, via $(x,t) \mapsto t\,x$.

1.38 Example. [7, 1.3.7] Let (X_j, x_j) be pointed spaces. The 1-point union $\bigvee_{j\in J} X_j = \bigvee_{j \in J}(X_j, x_j)$ is $\bigsqcup_j X_j/\{x_j : j\}$. By 1.24 the projection $\pi : \bigsqcup_j X_j \to \bigvee_j X_j$ is a closed mapping.

1.39 Proposition. [7, 1.3.8] *X_i embeds into $\bigvee_j X_j$ and $\bigvee_j X_j$ is union of the images, which have pairwise as intersection the base point.*

Proof. That the composition $X_i \hookrightarrow \bigsqcup_j X_j \to \bigvee_j X_j$ is continuous and injective is clear. That it is an embedding follows, since by 1.38 the projection π is also a closed mapping. $\square$

1.40 Proposition. [7, 1.3.9] *Universal and functorial property of the 1-point-union:*

$$\begin{tikzcd} (X_i,x_i) \arrow[r, "f_i"] \arrow[d] & (Y,y) \\ \bigvee_j X_j \arrow[ur, dotted] & \end{tikzcd} \qquad \begin{tikzcd} (X_i,x_i) \arrow[r, "f_i"] \arrow[d] & (Y_i,y_i) \arrow[d] \\ \bigvee_j X_j \arrow[r, dotted] & \bigvee_j Y_j \end{tikzcd}$$

Proof. This follows from 1.28 and 1.27. $\square$

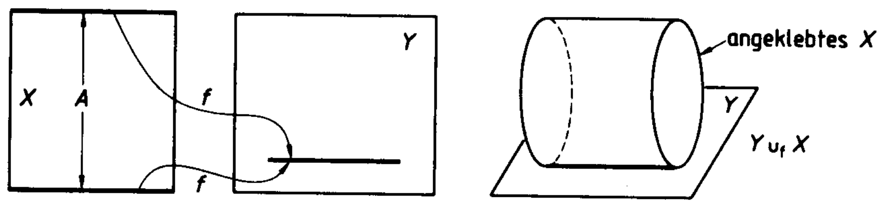
1.41 Proposition. [7, 1.3.10] *Embedding of $X_1 \vee \dots \vee X_n \hookrightarrow X_1 \times \ldots \times X_n$.*

Proof. Exercise (1.3.A1). $\square$

1.42 Example. [7, 1.3.11] 1.41 is wrong for infinite index sets: The open neighborhoods of the base point in $\bigvee_j X_j$ are given by $\bigvee_j U_j$, where U_j is an open neighborhood of the base point in X_j. Hence $\bigvee X_j$ is not first countable, whereas the product of countable many metrizable spaces X_j is first countable.

Also countable many circles in $\mathbb{R}^2$ which intersect only in a single point have as union in $\mathbb{R}^2$ not their one-point union, since a neighborhood of the single point contains almost all circle completely.

1.43 Definition. Gluing. [7, 1.3.12] $f : X \supseteq A \to Y$ with $A \subseteq X$ closed. $Y \cup_f X := Y \sqcup X/\sim$, where $a \sim f(a)$ for all $a \in A$, is called Y glued with X via f (oder along f).



1.44 Proposition. [7, 1.3.13] [7, 1.3.14] *$f : X \supseteq A \to Y$ with $A \subseteq X$ closed. $p|_Y : Y \to Y \cup_f X$ is a closed embedding. $p : (X, A) \to (Y \cup_f X, p(Y))$ is a relative homeomorphism.*

Proof. That $p|_Y : Y \to Y \cup_f X$ is continuous and injective is clear. Now let $B \subseteq Y$ be closed. Then $p^{-1}(p(B)) = B \sqcup f^{-1}(B)$ is closed and hence also $p(B)$.

That $p : X \setminus A \to Y \cup_f X \setminus Y$ is a homeomorphism follows from 1.29. □

1.45 Proposition. [7, 1.3.15] *Universal property of push-outs $Y \cup_f X$:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & Y \arrow[d] \arrow[ddr] & \\ X \arrow[r] \arrow[drr] & Y \cup_f X \arrow[dr, dotted] & \\ & & Z \end{tikzcd}$$

Proof. 1.27. □

1.46 Lemma. *Let $f_i : X_i \supseteq A_i \to Y$ be given, $X := X_1 \sqcup X_2$, $A := A_1 \sqcup A_2 \subseteq X$ and $f := f_1 \sqcup f_2 : X \supseteq A \to Y$. Then $Y \cup_f X \cong (Y \cup_{f_1} X_1) \cup_{f_2} X_2$.*

Proof.

$$\begin{tikzcd}
A_2 \arrow[dr, hook] \arrow[drrrr, "f_2"] \arrow[ddddd, hook] & & & & \\
& A \arrow[rrr, "f"] \arrow[ddd, dotted] & & & Y \arrow[ddl, "i_1"] \arrow[ddd, "i"] \\
& & A_1 \arrow[ul, hook] \arrow[urr, "f_1"] \arrow[d, hook] & & \\
& & X_1 \arrow[r, "p_1"] \arrow[dl, hook] & Y \cup_{f_1} X_1 \arrow[dd, "i_2"] \arrow[dr, dotted] & \\
& X \arrow[rrr, "p"] \arrow[drr, dotted] & & & Y \cup_f X \arrow[dl, dotted, shift left] \\
X_2 \arrow[ur, hook] \arrow[rrr, "p_2"] & & & (Y \cup_{f_1} X_1) \cup_{f_2} X_2 \arrow[ur, dotted, shift left] &
\end{tikzcd}$$

□

1.47 Example. [7, 1.3.16] $f : X \supseteq A \to Y = \{*\} \Rightarrow Y \cup_f X \cong X/A$, since X/A satisfies the universal property of the push-out.

$f : X \supseteq \{*\} \to Y \Rightarrow Y \cup_f X \cong X \vee Y$, by definition.

$f : X \supseteq A \to Y$ constant $\Rightarrow Y \cup_f X \cong X/A \vee Y$.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, hook] & \{*\} \arrow[r, hook] \arrow[d, hook] & Y \arrow[d, hook] \arrow[ddr] & \\
X \arrow[r] \arrow[drrr] & X/A \arrow[r] \arrow[drr, dotted] & Y \vee X/A \arrow[dr, dotted] & \\
& & & Z
\end{tikzcd}$$

1.48 Example. [7, 1.3.17] $f : X \supseteq A \to B \subseteq Y$ homeomorphism of closed subsets. $\Rightarrow Y \cup_f X = \pi(X) \cup \pi(Y)$ with $\pi(X) \cong X$, $\pi(Y) \cong Y$ and $\pi(X) \cap \pi(Y) \cong A \cong B$. Note however, that $Y \cup_f X$ depends not only on $X \supseteq A$ and $Y \supseteq B$ but also on the gluing map $f : A \to B$ as the example $X = I \times I = Y$ and $A = B = I \times \dot{I}$ with $\mathrm{id} \neq f : (x, 1) \mapsto (1 - x, 1)$, $(x, 0) \mapsto (x, 0)$ of a Möbius-strip versus a cylinder shows.

1.49 Proposition. [7, 1.3.18]

$$\begin{tikzcd}
X \arrow[d, "\cong"', "F"] & A \arrow[l, hook'] \arrow[r, "f"] \arrow[d, "\cong"', "F"] & Y \arrow[d, "G"', "\cong"] \\
X' & A' \arrow[l, hook'] \arrow[r, "f'"] & Y'
\end{tikzcd}$$

$\Rightarrow Y \cup_f X \cong Y' \cup_{f'} X'$.

Proof. By 1.45 we obtain a uniquely determined continuous map $G \cup F : Y \cup_f X \to Y' \cup_{f'} X'$ with $(G \cup F) \circ \pi|_X = \pi|_{X'} \circ F$ and $(G \cup F) \circ \pi|_Y = \pi|_{Y'} \circ F$. Since

$G^{-1} \circ f' = G^{-1} \circ f' \circ F \circ F|_A^{-1} = G^{-1} \circ G \circ f \circ F|_A^{-1} = f \circ F^{-1}|_{A'}$ we can similarly $G^{-1} \cup F^{-1} : Y' \cup_{f'} X' \to Y \cup_f X$. On X and Y (resp. X' and Y') they are inverse to each other, hence define a homeomorphism as required. □

1.50 Example. [7, 1.3.19]

(1) $Z = X \cup Y$ with X, Y closed. $\Rightarrow Z = Y \cup_{id} X$: The canonical mapping $Y \sqcup X \to Z$ induces a continuous bijective mapping $Y \cup_{\mathrm{id}} X \to Z$, which is closed and hence a homeomorphism, since $Y \sqcup X \to Z$ is closed.

(2) $Z = X \cup Y$ with X, Y closed, $A := X \cap Y$, $f : A \to A$ extendable to a homeomorphism of $X \Rightarrow Z \cong Y \cup_f X$: Apply 1.49 to

$$\begin{tikzcd}
X \arrow[d, "\tilde f", "\cong"'] & A \arrow[l, hook'] \arrow[r, "f"] \arrow[d, "f", "\cong"'] & A \arrow[r, hook] \arrow[d, "\mathrm{id}", "\cong"'] & Y \arrow[d, "\mathrm{id}", "\cong"'] \\
X & A \arrow[l, hook'] \arrow[r, "\mathrm{id}"] & A \arrow[r, hook] & Y
\end{tikzcd}$$

(3) $D^n \cup_f D^n$ for all homeomorphisms $f : S^{n-1} \to S^{n-1}$: We can extend f radially to a homeomorphism $\tilde f : D^n \to D^n$ by $\tilde f(x) = \|x\|\, f(\frac{x}{\|x\|})$ and can now apply (2).

(4) Gluing two identical cylinders $X \times I$ along any homeomorphism $f : X \times \{0\} \to X \times \{0\}$ yields again the cylinder $X \times I$: Since f extends to a homeomorphism $X \times I \to X \times I$, $(x,t) \mapsto (f(x), t)$ we may apply (2) to obtain $(X \times I) \cup_f (X \times I) = (X \times I) \cup_{\mathrm{id}} (X \times I) \cong X \times I$.

Manifolds

1.51 Definition. [7, 1.4.1] [7, 1.5.1] An m-dimensional MANIFOLD is a topological space X (which we will always require to be Hausdorff and second countable), for which each of its points $x \in X$ has a neighborhood A which is an n-ball, i.e. a homeomorphism $\varphi : A \to D^m$ (which we call chart at x) exists. A point $x \in X$ is called BOUNDARY POINT iff for some (and by 1.21 any) chart φ at x the point is mapped to $\varphi(x) \in S^{m-1}$. The set of all boundary points is called the BOUNDARY of X and denoted by $\dot X$ or ∂X. A manifold is called closed if it is compact and has empty boundary.

Let X be an m-manifold and $U \subseteq X$ open. Then U is an m-manifold as well and $\dot U = \dot X \cap U$:
If $x \in U \subseteq X$ is a boundary point of X, i.e. $\exists \varphi : A \xrightarrow{\cong} D^m$ with $\varphi(x) \in S^{m-1}$. Then $\varphi(U)$ is an open neighborhood of $\varphi(x)$ in D^m and hence contains a neighborhood B which is an m-ball. Then $\varphi : U \supseteq \varphi^{-1}(B) \cong B \subseteq D^m$ is the required chart for U, and $x \in \dot U$.
If $x \in U \subseteq X$ is not a boundary point of X, i.e. $\exists \varphi : A \xrightarrow{\cong} D^m$ with $\varphi(x) \in \mathring{D}^m$. Then $\varphi(U)$ is an open neighborhood of $\varphi(x)$ in D^m and hence contains a neighborhood B which is an m-ball. Then $\varphi : U \supseteq \varphi^{-1}(B) \cong B \subseteq D^m$ is the required chart for U, and $x \notin \dot U$.

1.52 Proposition. [7, 1.4.2] [7, 1.5.2] *Let $f : X \to Y$ be a homeomorphism between manifolds. Then $f(\dot X) = \dot Y$.*

Proof. Let $x \in X$ and $\varphi : A \cong D^m$ a chart at x. Then $\varphi \circ f^{-1} : f(A) \to D^m$ is a chart of Y at $f(x)$ and hence $x \in \dot{X} \Leftrightarrow f(x) \in \dot{Y}$. □

1.53 Proposition. [7, 1.4.3] [7, 1.5.3] *Let X be an m-manifold and $x \in \dot{X}$. Then there exists a neighborhood U of x in X with $(U, U \cap \partial X, x) \cong (D^{n-1} \times I, D^{n-1} \times \{0\}, (0,0))$.*

Proof. By assumption there exists a neighborhood A of x in X and a homeomorphism $\varphi : A \to D^m$ with $\varphi(x) \in S^{m-1}$. Choose an open neighborhood $W \subseteq A$ of x. Then $\dot{W} = \dot{X} \cap W$ and the manifold W is homeomorphic to $\varphi(U) \subseteq D^m$. Obviously $\varphi(W)$ contains a neighborhood B of $\varphi(x)$ homeomorphic to $D^{m-1} \times I$, where $S^{m-1} \cap B$ corresponds to $D^{m-1} \times \{0\}$. The set $U := \varphi^{-1}(B)$ is then the required neighborhood. □

1.54 Corollary. [7, 1.5.4] The boundary ∂X of a manifold is a manifold without boundary.

Proof. By 1.53 ∂X is locally homeomorphic to $D^{n-1} \times \{0\}$ and $x \in \partial X$ corresponds to $(0,0)$ thus is not in the boundary of ∂X. □

1.55 Proposition. [7, 1.5.7] *Let M be a m-dimensional and N an n-dimensional manifold. Then $M \times N$ is a $m+n$-dimensional manifold with boundary $\partial(M \times N) = \partial M \times N \cup M \times \partial N$.*

Proof. 1.12 □

1.56 Examples. [7, 1.4.4] Quadrics like hyperboloids ($\cong \mathbb{R}^2 \sqcup \mathbb{R}^2$ or $\cong S^1 \times \mathbb{R}$), paraboloids ($\cong \mathbb{R}^2$), the cylinder $S^1 \times \mathbb{R}$ are surfaces. Let X be a surface without boundary and $A \subseteq X$ be a discrete subset. Then $X \setminus A$ is also a surface without boundary. Let A be the set of a lines parallel to the coordinate axes through points with integer coordinates. Then the set $X = \{x \in \mathbb{R}^m : d(x, A) = 1/4\}$ is a surface without boundary.

1.57 Example. [7, 1.4.5] D^m is a manifold with boundary S^{m-1}, the halfspace $\mathbb{R}^{m-1} \times [0, +\infty)$ is a manifold with boundary $\mathbb{R}^{m-1} \times \{0\}$, For a manifold X without boundary (like S^1) the cylinder $X \times I$ is a manifold with boundary $X \times \{0, 1\}$.

1.58 Examples. [7, 1.5.8]

1 0-manifolds are discrete countable topological spaces.

2 The connected 1-manifolds are $\mathbb{R}$, S^1, I and $[0, +\infty)$.

3 The 2-manifolds are the surfaces.

4 $M \times N$ is a 3-manifold for M a 1-manifold and N a 2-manifold; e.g.: $S^2 \times \mathbb{R}$, $S^2 \times I$, $S^2 \times S^1$.

5 S^n, $\mathbb{R}^n$, $\mathbb{R}^{n-1} \times [0, +\infty)$, D^n are n-manifolds.

1.59 Example. Möbius strip. [7, 1.4.6] The Möbius-strip X is defined as $I \times I/\sim$, where $(x, 0) \cong (1-x, 1)$ for all x. Its boundary is $(I \times \dot{I})/\sim \cong S^1$ and hence it is not homeomorphic to the cylinder $S^1 \times I$.

An embedding of X into $\mathbb{R}^3$ is given by factoring $(\varphi, r) \mapsto ((2+(2r-1)\cos\pi\varphi)\cos 2\pi\varphi, 2+(2r-1)\cos\pi\varphi)\sin 2\pi\varphi, (2r-1)\sin\pi\varphi)$ over the quotient.

The Möbius-strip is not orientable which we will make precise later.

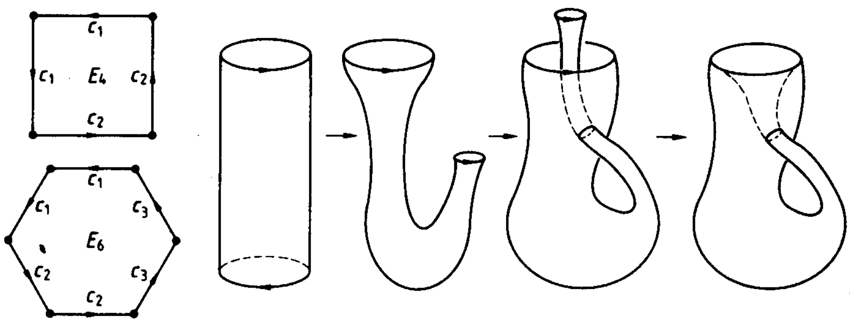
1.60 Proposition. [7, 1.4.7] [7, 1.5.5] *By cutting finitely many disjoint holes into a manifold one obtains a manifold whose boundary is the union of the boundary of X and the boundaries of the holes, in detail: Let X be an m-manifold and $f_i : D^m \to X$ embeddings with pairwise disjoint images. Let $\mathring{D}_i := \{f_i(x) : |x| < \frac{1}{2}\}$ and $S_i := \{f_i(x) : |x| = \frac{1}{2}\}$. Then $X \setminus \bigcup_{i=1}^n \mathring{D}_i$ is an m-manifold with boundary $\dot{X} \sqcup \bigsqcup_{i=1}^n S_i$.*

Proof. No point in $\{f_i(x) : |x| < 1\}$ is a boundary point of X hence the result follows. □

1.61 Proposition. [7, 1.4.8] [7, 1.5.6] *Let F and F' be two manifolds and R and R' components of the corresponding boundaries and $g : R \to R'$ a homeomorphism. Then $F' \cup_g F$ is a manifold in which F and F' are embedded as closed subsets with boundary $(\partial F \setminus R) \cup (\partial F' \setminus R')$.*

Proof. Let $A \cong D^m \times I$ and $A' \cong D^m \times I$ be neighborhoods of $x \in R$ and $g(x) \in R'$ with $\dot{F} \cap A = D^{m-1} \times \{0\}$ and $\dot{F}' \cap A' = D^{m-1} \times \{0\}$. W.l.o.g. we may assume that $g(\dot{F} \cap A) = \dot{F}' \cap A'$. The image of $A' \sqcup A$ in $F' \cup_g F$ is given by gluing $D^{m-1} \times I \cup D^{m-1} \times I$ along a homeomorphism $D^{m-1} \times \{0\} \to D^{m-1} \times \{0\}$ and hence by 1.50.3 is homeomorphic to $D^{m-1} \times I$ where x corresponds to $(0, 0)$. □

1.62 Example. [7, 1.4.9] $S^1 \times S^1$ can be obtained from two copies of $S^1 \times I$ that way. The same is true for Klein's bottle but with different gluing homeomorphism.



1.63 Example. Gluing a handle. [7, 1.4.10] [7, 1.5.8.7] Let X be a surface in which we cut two holes as in 1.60. The surface obtained from X by gluing a handle is $(X \setminus (\mathring{D}^2 \sqcup \mathring{D}^2)) \cup_f (S^1 \times I)$, where $f : S^1 \times I \supseteq S^1 \times \dot{I} \cong S^1 \sqcup S^1 \subseteq D^2 \sqcup D^2$.

More generally, one can glue handles $S^{n-1} \times I$ to n-manifolds.

1.64 Example. Connected sum. [7, 1.4.11] [7, 1.5.8.8] The connected sum of two surfaces X_1 and X_2 is given by cutting a whole into each of them and gluing along boundaries of the respective holes. $X_1 \sharp X_2 := (X_1 \setminus \mathring{D}^2) \cup_f (X_2 \setminus \mathring{D}^2)$, where $f : D^2 \supseteq S^1 \cong S^1 \subseteq D^2$.

More generally, one can define analogously the connected sum of n-manifolds. This however depends essentially on the gluing map.

1.65 Example. Doubling of a manifold with boundary. [7, 1.4.12] [7, 1.5.8.9] The doubling of a manifold is given by gluing two copied along their boundaries with the identity. $2X := X \cup_f X$, where $f = \mathrm{id} : \partial X \to \partial X$.

More generally, one can define the doubling of n-manifolds, e.g. $2D^n \cong S^n$.

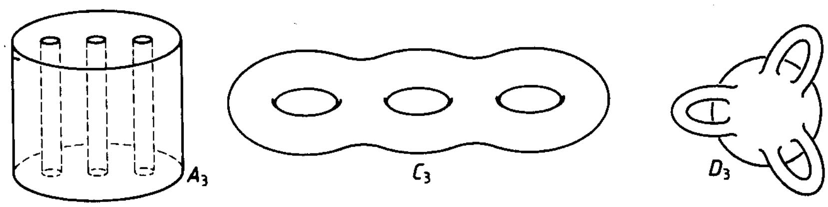
1.66 Example. [7, 1.4.13] The compact oriented surfaces can be described as:

1 boundary of a brezel $V_g := D^2_g \times I$ of genus g.

2 doubling $2D^2_g$.

3 connected sum of tori.

4 sphere with g handles.



1.67 Example. [7, 1.4.14] The compact oriented surface als quotient of an $4g$-polygon. By induction this surface is homeomorphic to those given in 1.66.

1.68 Example. [7, 1.4.15] [7, 1.5.13] The projective plane $\mathbb{P}^2$ as $(\mathbb{R}^3 \setminus \{0\})/\sim$ with $x \sim \lambda \cdot x$ für $\mathbb{R} \ni \lambda \neq 0$.

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Then the projective space is $\mathbb{P}^n_{\mathbb{K}} := (\mathbb{K}^{n+1} \setminus \{0\})/\sim$ where $x \sim \lambda x$ for $0 \neq \lambda \in \mathbb{K}$

1.69 Example. [7, 1.4.17] $\mathbb{P}^2 \cong D^2/\sim$ where $x \sim -x$ for all $x \in S^1$.
Consider a hemisphere $D^n_+ \subseteq S^n$. Then the quotient mapping $S^n \to \mathbb{P}^n$ restricts to a quotient mapping on the compact set D^n_+ with associated equivalence relation $x \sim -x$ on $S^{n-1} \subseteq D^n_+$.

1.70 Example. [7, 1.4.18] $\mathbb{P}^2$ als gluing a disk to a Möbius strip.
Consider the closed subsets $A := \{x \in S^2 : x_2 \leq 0, |x_3| \leq 1/2\}$ and $B = \{x \in S^3 : x_3 \geq 1/2\}$. The quotient mapping induces an homeomorphism on B, i.e. $\pi(B)$ is a 2-Ball. A is mapped to a Möbius-strip by 1.28 and 1.59. Since $\pi(B) \cup \pi(A) = \mathbb{P}^2$ and $\pi(B) \cap \pi(A) \cong S^1$ we are done.

1.71 Proposition. [7, 1.4.16] [7, 1.5.14] [7, 1.6.6] *$\mathbb{P}^n_{\mathbb{K}}$ is a dn-dimensional connected compact manifold, where $d := \dim_{\mathbb{R}} \mathbb{K}$. The mapping $p : S^{dn-1} \to \mathbb{P}^{n-1}_{\mathbb{K}}$, $x \mapsto [x]$ is a quotient mapping. In particular, $\mathbb{P}^1_{\mathbb{K}} \cong S^d$.*

Proof. Charts $\mathbb{K}^n \to \mathbb{P}^n_{\mathbb{K}}$, $(x^1, \dots, x^n) \mapsto [(x^1, \dots, x^i, 1, x^{i+1}, \dots, x^n)]$.
The restriction $\mathbb{K}^{n+1} \supseteq S^{d(n+1)-1} \to \mathbb{P}^n_{\mathbb{K}}$ is a quotient mapping since $\mathbb{K}^{n+1} \setminus \{0\} \to \mathbb{P}^n_{\mathbb{K}}$ is an open mapping. For $\mathbb{K} = \mathbb{R}$ it induces the equivalence relation $x \sim -x$. In particular $\mathbb{P}^n_{\mathbb{K}}$ is compact.
For $n = 1$ we have $\mathbb{P}^1_{\mathbb{K}} \setminus U_1 = \{[(0, 1)]\}$, therefore $\mathbb{P}^1_{\mathbb{K}} \cong \mathbb{K}_\infty \cong S^d$. □

1.72 Example. [7, 1.4.19] The none-oriented compact surfaces without boundary:

1 connected sum of projective planes.

2 sphere with glued Möbius strips.

1.86 Classification.

1.73 Proposition. [7, 1.4.20] *The none-orientable compact surfaces without boundary as quotient of a $2g$-polygon.*

Klein's bottle as sum of two Möbius strips.

1.74 Example. [7, 1.5.9] Union of filled tori $(D^2 \times S^1) \cup (S^1 \times D^2) = \partial(D^2 \times D^2) \cong \partial(D^4) \cong S^3$ by 1.12. Other point of view: $S^3 = D^3_+ \cup D^3_-$ and remove a filled cylinder from D_- and glue that to D_+ to obtain two tori. With respect to the stereographic projection the torus $\{(z_1, z_2) \in S^3 \subseteq \mathbb{C}^2 : |z_1| = r_1, |z_2| = r_2\}$ corresponds the torus with z-axes as axes and big radius $A := 1/r_1 \geq 1$ and small radius $a := \sqrt{A^2 - 1}$.

1.75 Example. [7, 1.5.10] Let $f : S^1 \times S^1 \to S^1 \times S^1$ be given by $f : (z, w) \mapsto (z^a w^b, z^c w^d)$, where $a, b, c, d \in \mathbb{Z}$ with $ad - bc = \pm 1$.

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "\begin{pmatrix} a & b \\ c & d \end{pmatrix}"] \arrow[d, two heads] & \mathbb{R}^2 \arrow[d, two heads] \\ S^1 \times S^1 \arrow[r, "f"] & S^1 \times S^1 \end{tikzcd}$$

A meridian $S^1 \times \{1\}$ on the torus is mapped to a curve $t \mapsto (e^{2\pi i a t}, e^{2\pi i c t})$ which winds a-times around the axes and c-times around the core.

$$M \begin{pmatrix} a & b \\ c & d \end{pmatrix} := (D^2 \times S^1) \cup_f (S^1 \times D^2).$$

In 1.88 we will see that M is often not homeomorphic to S^3.

1.76 Example. [7, 1.5.11] Cf. 1.61. By a Heegard decomposition of M one understands a representation of M by gluing two handle bodies of same genus along their boundary.

1.77 Example. [7, 1.5.12] Cf. 1.67 and 1.73. For relative prime $1 \leq q < p$ let the lens space be $L(\frac{q}{p}) := B^3/\sim$, where $(\varphi, \theta, 1) \sim (\varphi + 2\pi\frac{q}{p}, -\theta, 1)$ for $\theta \geq 0$ with respect to spherical coordinates. Thus the northern hemisphere is identified with the southern one after rotation by $2\pi\frac{q}{p}$. The interior of D^3 is mapped homeomorphically to a 3-cell in $L(\frac{q}{p})$. Also image of points in the open hemispheres have such neighborhoods (formed by one half in the one part inside the northern hemisphere and one inside the southern). Each p-points on the equator obtained by recursively turning by $2\pi\frac{q}{p}$ get identified. After squeezing D^3 a little in direction of the axes we may view a neighborhood of a point on the equator as a cylinder over a sector of a circle (a piece of cake) where the flat sides lie on the northern and southern hemisphere. In the quotient p many of these pieces are glued together along their flat sides thus obtaining again a 3-cell as neighborhood. We will come to this description again in 1.89.

Group actions and orbit spaces

1.78 Definition. [7, 1.7.3] Group action of a (topological) group G on a topological space X is a subgroup G of Homeo(X). The ORBIT SPACE is $X/G := X/\sim = \{Gx : x \in X\}$, where $x \sim y :\leftrightarrow \exists g \in G : y = g \cdot x$.

1.79 Examples. [7, 1.7.4]

1 S^1 acts on $\mathbb{C}$ by multiplication $\Rightarrow \mathbb{C}/S^1 \cong [0, +\infty)$.

2 $\mathbb{Z}$ acts on $\mathbb{R}$ by translation $(k, x) \mapsto k + x \Rightarrow \mathbb{R}/\mathbb{Z} \cong S^1$, $\mathbb{R}^2/\mathbb{Z} \cong S^1 \times \mathbb{R}$. ATTENTION: $\mathbb{R}/\mathbb{Z}$ has two meanings.

3 S^0 acts on S^n by reflection (scalar multiplication) $\Rightarrow S^n/S^0 \cong \mathbb{P}^n$.

1.80 Definition. [7, 1.7.5] G ACTS FREELY on X, when $gx \neq x$ for all x and $g \neq 1$.

1.81 Theorem. [7, 1.7.6] *Let G act strictly discontinuously on X, i.e. each $x \in X$ has a neighborhood U with $gU \cap U \neq \emptyset \Rightarrow g = \mathrm{id}$. In particular, this is the case, when G is finite and acts without fixed points If X is a closed m-manifold then so is X/G.*

Proof. $U \cong p(U)$ is the required neighborhood. □

1.82 Example. [7, 1.7.7] $1 \le q_1, \dots, q_k < p$ with q_i, p relative prime. $E_p := \{z : z^p = 1\} \cong \mathbb{Z}_p$ acts fixed point free on $S^{2k-1} \subseteq \mathbb{C}^k$ by $(z, (z_1, \dots, z_k)) \mapsto (z^{q_1} z_1, \dots, z^{q_k} z_k)$. The lens space $L_{2k-1}(p; q_1, \dots, q_k) := S^{2k-1}/E_p$ of type $(p; q_1, \dots, q_k)$ is a closed manifold of dimension $2k - 1$.

In particular, $L_3(p; q, 1) \cong L(\frac{q}{p})$: We may parametrize $S^3 \subseteq \mathbb{C}^2$ by $D^2 \times S^1 \to S^3$, $(z_1, z_2) \mapsto (z_1, \sqrt{1 - |z_1|^2}\, z_2)$ and the action of $E_3 = \langle a \rangle \cong \mathbb{Z}_p$, where $a = e^{2\pi i/p}$, lifts to the action given by $a \cdot (z_1, z_2) = (a^q z_1, a z_2)$. Only the points in $\{z_1\} \times S^1$ for $z_1 \in S^1$ get identified by p. A representative subset of S^3 for the action is given by $\{(z_1, z_2) \in S^3 : |\arg(z_2)| \le \frac{\pi}{p}\}$, whose preimage in $D^2 \times S^1$ is homeomorphic to $D^2 \times I$, and only points $(z_1, 0)$ and $(a^q z_1, 1)$ are in the same orbit. Thus the top $D^2 \times \{1\}$ and the bottom $D^2 \times \{0\}$ turned by a^q have to be identified in the orbit space and the generators $\{x_1\} \times I$ for $x_1 \in S^1$. Only the points in $\{z_1\} \times S^1$ for $z_1 \in S^1$ get identified by p. in the quotient. This gives the description of $L(\frac{q}{p})$ in 1.77.

Keep in mind, that only $p_i \mod q$ is relevant.

$L_3(p; q_1, q_2) \cong L_3(p; q_2, q_1)$ via the reflection $\mathbb{C} \times \mathbb{C} \supseteq S^3 \to S^3 \subseteq \mathbb{C} \times \mathbb{C}$, $(z_1, z_2) \mapsto (z_2, z_1)$.

For $q' \equiv -q \mod p$, we have $L_3(p; q, 1) \cong L_3(p; q', 1)$ by the bijection $g \mapsto \overline{g}$ on the group and the homeomorphism $(z_1, z_2) \mapsto (z_1, \overline{z_2})$ of S^3, via

$$
\begin{tikzcd}
(z_1, z_2) \arrow[rr] \arrow[dd, "g"] & & (z_1, \bar{z}_2) \arrow[d, "\bar{g}"] \\
& & (\bar{g}^{q'} z_1, \bar{g}\bar{z}_2) \\
(g^q z_1, g z_2) \arrow[r] & (g^q z_1, \bar{g}\bar{z}_2) \arrow[ur, Rightarrow] &
\end{tikzcd}
$$

For $qq' \equiv 1 \mod p$, we have $L_3(p; q, 1) \cong L_3(p; q', 1)$, since $L_3(p; q, 1) \cong L_3(p; q' q, q') = L_3(p; 1, q') \cong L_3(p; q', 1)$ via the group isomorphism $g \mapsto g^{q'}$.

1.83 Theorem. [7, 1.9.5] *$L(p, q) \cong L(p', q') \Leftrightarrow p = p'$ and ($q \equiv \pm q' \mod p$ or $qq' \equiv \pm 1 \mod p$).*

Proof. ($\Leftarrow$) We have shown this in 1.82. ($\Rightarrow$) is beyond the algebraic methods of this lecture. □

1.84 Definition. [7, 1.7.1] A TOPOLOGICAL GROUP is a topological space together with a group structure, s.d. $\mu : G \times G \to G$ and $\mathrm{inv} : G \to G$ are continuous.

1.85 Examples of topological groups. [7, 1.7.2]

1 $\mathbb{R}^n$ with addition.

2 $S^1 \subseteq \mathbb{C}$ and $S^3 \subseteq \mathbb{H}$ with multiplication.

3 $G \times H$ for topological groups G and H.

4 The general linear group $GL(n) := GL(n, \mathbb{R}) := \{A \in L(\mathbb{R}^n, \mathbb{R}^n) : \det(A) \neq 0\}$ with composition.

5 The orthogonal group $O(n) := \{A \in GL(n) : A^t \cdot A = \mathrm{id}\}$ and the (path-)connected component $SO(n) := \{T \in O(n) : \det(T) = 1\}$ of the identity in $O(n)$. As topological space $O(n) \cong SO(n) \times S^0$.

6 The special linear group $SL(n) := \{A \in GL(n) : \det(A) = 1\}$.

7 $GL(n, \mathbb{C}) := \{A \in L_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^n) : \det(C) = 1\}$.

8 The unitary group $U(n) := \{A \in GL(n, \mathbb{C}) : A^* \cdot A = \mathrm{id}\}$ with (path-)connected component $SU(n) := \{A \in U(n) : \det(A) = 1\}$. As topological space $U(n) \cong SU(n) \times S^1$.

9 In particular $SO(1) = SU(1) = \{*\}$, $SO(2) \cong U(1) \cong S^1$, $SU(2) \cong S^3$, $SO(3) \cong \mathbb{P}^3$.

The problem of homeomorphy

1.86 Theorem. [7, 1.9.1] *Each connected closed surface is homeomorphic to a surface $S^2 = F_0, S^1 \times S^1 = F_1, \ldots; \mathbb{P}^2 = N_1, N_2, \ldots$.*

Remark. For 3-manifolds one is far from a solution to the classification problem. For $n > 3$ there can be no algorithm. Orientation!

1.87 Theorem. [7, 1.9.2] *Each closed orientable 3-manifold admits a Heegard-decomposition.*

Hence in order to solve the classification problem one has to investigate only the homeomorphisms of closed oriented surfaces and determine gluing with which of them gives homeomorphic manifolds.

In the following example we study this for the homeomorphisms of the torus considered in 1.75.

1.88 Example. [7, 1.9.3] Let $M = M\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $M' = M\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ in $SL(2,\mathbb{Z})$, see 1.75. For $\alpha, \beta, \gamma, \delta \in S^0$ and $m, n \in \mathbb{Z}$ consider the homeomorphisms

$$\begin{aligned} F &: D^2 \times S^1 \to D^2 \times S^1, & (z,w) &\mapsto (z^\alpha w^m, w^\beta) \\ G &: S^1 \times D^2 \to S^1 \times D^2, & (z,w) &\mapsto (z^\gamma, z^n w^\delta) \end{aligned}$$

If

$$\begin{pmatrix} \gamma & 0 \\ n & \delta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} \alpha & m \\ 0 & \beta \end{pmatrix},$$

i.e.

$$\gamma a = a'\alpha, \quad \gamma b = a'm + b'\beta, \quad na + \delta c = c'\alpha, \quad nb + \delta d = c'm + d'\beta$$

then $(G|_{S^1\times S^1}) \circ f = f' \circ (F|_{S^1\times S^1})$ and thus $M \cong M'$ by 1.49.

$(a \le 0)$ $\alpha := -1$, $\beta := \gamma := \delta := 1$, $m := n := 0$
$\Rightarrow M\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cong M\begin{pmatrix} -a & -b \\ c & d \end{pmatrix}$, i.e. w.l.o.g. $a \ge 0$.

$(ad - bc < 0)$ $\alpha := \beta := \gamma := 1$, $\delta := -1$, $m := n := 0$
$\Rightarrow M\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cong M\begin{pmatrix} a & b \\ -c & -d \end{pmatrix}$, i.e. w.l.o.g. $ad - bc = 1$.

$(a = 0)$ $\Rightarrow bc = -1$. $\alpha := c$, $\beta := b$, $\gamma := 1$, $\delta := 1$, $n := 0$, $m := d$
$\Rightarrow M\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cong M\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cong (D^2 \cup_{\mathrm{id}} D^2) \times S^1 \cong S^2 \times S^1$.

$(a = 1)$ $\alpha := \delta := a$, $\beta := ad - bc$, $\gamma := 1$, $m := b$, $n := -c$
$\Rightarrow M\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cong M\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cong S^3$, by 1.74.

$(ad' - b'c = 1)$ $\Rightarrow a(d' - d) = c(b' - b)$ and by $ggT(a,c) = 1$, since $a\,d - b\,c = 1$ $\exists m$: $b = b' + m\,a$, $d = d + m\,c$. $\alpha := \beta := \gamma := \delta := 1$, $n := 0 \Rightarrow M\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cong M\begin{pmatrix} a & b' \\ c & d' \end{pmatrix} =: M(a,c)$.

$(c' := c - na)$ $\alpha := \beta := \gamma := \delta := -1$, $m := 0 \Rightarrow M(a,c) \cong M(a,c')$, i.e. w.l.o.g. $0 \le c < a$ (If $c = 0 \Rightarrow a = 1 \Rightarrow M(a,c) \cong S^3$).

Thus only the spaces $M(a.c)$ with $0 < c < a$ and $ggT(a,c) = 1$ remain.

1.89 Theorem (Heegard-decomposition via lens spaces). [7, 1.9.4] *For relative prime $1 \leq c < a$ we have $L(\frac{c}{a}) \cong M(a,c)$.*

Proof. We start with $L(\frac{c}{a}) = D^3/\sim$ and drill a cylindrical hole into D^3 and glue its top and bottom via $\sim$ to obtain a filled torus, where collections of a generators of the cylinder are glued to from a closed curve which winds c-times around the core of the torus (i.e. the axes of the cylinder) and a-times around the axes of the torus. The remaining D^3 with hole is cut into a sectors, each homeomorphic to a piece of a cake, which yield $\mathbb{D}^2 \times I$ after gluing the flat sides (which correspond to points on S^2) and groups of a generators of the cylindrical hole are glued to a circle $S^1 \times \{t\}$. After gluing the top and the by $2\pi\frac{1}{a}$ rotated bottom disc we obtain a second filled torus, where the groups of a generators of the cylinder form a meridian. In contrast the top circle of the cylindrical hole corresponds to a curve which winds a times around the axes and c times around the core. This is exactly the gluing procedure described in 1.75 for $M(a,c)$. $\square$

1.90 Definition. [7, 1.9.6] Two embeddings $f, g : X \to Y$ are called TOPOLOGICAL EQUIVALENT, if there exists a homeomorphism $h : Y \to Y$ with $g = h \circ f$. Each two embeddings $S^1 \to \mathbb{R}^2$ are by Schönflies's theorem equivalent.

1.91 Definition. [7, 1.9.7] A KNOT is an embedding $S^1 \to \mathbb{R}^3 \subseteq S^3$.

Remark. To each knot we may associated the complement of a tubular neighborhood in S^3. This is a compact connected 3-manifold with a torus as boundary.

By a result of [1] a knot is up to equivalence uniquely determined by the homotopy class of this manifold.

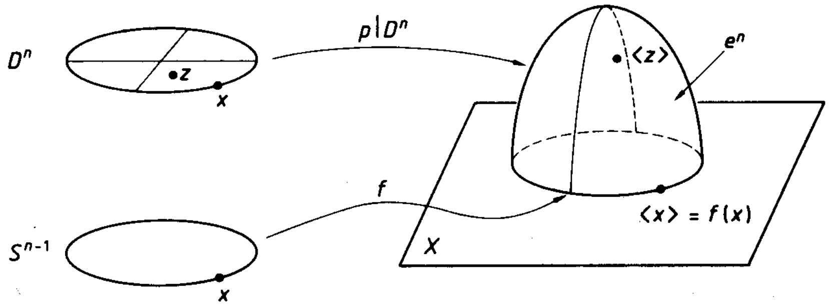
On the other hand, we may consider closed (orientable) surfaces in $\mathbb{R}^3$ of minimal genus which have the knot as boundary.

Gluing cells

1.92 Notation. [7, 1.6.1] $f : D^n \supseteq S^{n-1} \to X$. Consider $X \cup_f D^n$, $p : D^n \sqcup X \to X \cup_f D^n$, $e^n := p(\mathring{D}^n)$, $i := p|_X : X \hookrightarrow X \cup_f D^n =: X \cup e^n$.

By 1.44 $p : (D^n, S^{n-1}) \to (X \cup e^n, X)$ is a relative homeomorphism and $i : X \to X \cup e^n$ is a closed embedding.

For X T_2 also $X \cup e^n$ is T_2: Points in X can be separated in X by U_i and the sets $U_i \cup \{tx : 0 < t < 1, f(x) \in U_i\}$ separate them in $X \cup e^n$. When both points are in the open subset e_n, this is obvious. Otherwise one lies in e_n and the other in X, so a sphere in D^n separates them.



Conversely we have:

1.93 Proposition. [7, 1.6.2] *Let Z T_2, $X \subseteq Z$ closed and $F : (D^n, S^{n-1}) \to (Z, X)$ a relative homeomorphism. $\Rightarrow X \cup_f D^n \cong Z$, where $f := F|_{S^{n-1}}$, via $(F \sqcup i) \circ p^{-1}$.*

Proof. We consider

$$\begin{tikzcd}
S^{n-1} \arrow[rr, "f=F|_{S^{n-1}}"] \arrow[dd, hook] & & X \arrow[dl] \arrow[dd, hook, "j"] \\
& X \cup_f D^n \arrow[dr, dotted, "g"] & \\
D^n \arrow[ur, "p|_{D^n}"] \arrow[rr, "F"] & & Z
\end{tikzcd}$$

$j : X \hookrightarrow Z$ is closed and also F, since D^n is compact and Z is T_2. Thus g is closed and obviously bijective and continuous, thus a homeomorphism. □

1.94 Theorem. [7, 1.6.3] *Let $f : S^{n-1} \to X$ be continuous and surjective and X T_2 $\Rightarrow p|_{D^n} : D^n \to X \cup_f D^n$ is a quotient mapping.*

Proof. p is surjective, since f is. Since D^n is compact and $X \cup_f D^n$ is T_2, p is a quotient mapping. □

1.95 Example. [7, 1.6.4]

(1) $f : S^{n-1} \to \{*\} =: X \Rightarrow X \cup_f D^n \overset{\boxed{1.47}}{\cong} D^n/S^{n-1} \overset{\boxed{1.36}}{\cong} S^n$.

(2) $f : S^{n-1} \to X$ constant $\Rightarrow X \cup_f D^n \overset{\boxed{1.47}}{\cong} X \vee (D^n/S^{n-1}) \overset{\boxed{1.36}}{\cong} X \vee S^n$.

(3) $f = \mathrm{id} : S^{n-1} \to S^{n-1} =: X \Rightarrow X \cup_f D^n \cong D^n$ by 1.94.

(4) $f = \mathrm{incl} : S^{n-1} \hookrightarrow D^n =: X \Rightarrow X \cup_f D^n \cong S^n$ by 1.50.2.

1.96 Definition. [7, 1.6.5] We obtain an embedding $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$ via $\mathbb{K}^n \cong \mathbb{K}^n \times \{0\} \subseteq \mathbb{K}^{n+1}$. Let F and p be given by:

$$F : \mathbb{K}^n \supseteq D^{dn} \to \mathbb{P}^n_{\mathbb{K}}, \quad (x^1, \dots, x^n) \mapsto [(x^1, \dots, x^n, 1 - |x|)],$$
$$p : S^{dn-1} \to \mathbb{P}^n_{\mathbb{K}}, \quad (x^1, \dots, x^n) \mapsto [(x^1, \dots, x^n, 0)].$$

1.97 Proposition. [7, 1.6.7] *$F : (D^{dn}, S^{dn-1}) \to (\mathbb{P}^n_{\mathbb{K}}, \mathbb{P}^{n-1}_{\mathbb{K}})$ is a relative homeomorphism with $F|_{S^{dn-1}} = p$. Thus, by 1.93, $\mathbb{P}^n_{\mathbb{K}} = \mathbb{P}^{n-1}_{\mathbb{K}} \cup_p D^{dn}$ and furthermore, $\mathbb{P}^n_{\mathbb{K}} = D^{dn}/\sim$, where $x \sim -x$ for $x \in S^{dn-1}$.*

Proof. The charts $\mathbb{K}^n \cong U_{n+1} = \mathbb{P}^n_{\mathbb{K}} \setminus \mathbb{P}^{n-1}_{\mathbb{K}}$, $(x^1,\dots,x^n) \mapsto [(x^1,\dots,x^n,1)]$ where constructed in the proof of 1.71. The mapping $D^{dn} \setminus S^{dn-1} \to \mathbb{K}^n$, given by $x \mapsto (\frac{x_1}{1-|x|},\dots,\frac{x_n}{1-|x|})$, is a homeomorphism as in 1.4, and thus the composite F is a relative homeomorphism as well. Now use 1.93 and 1.94. □

1.98 Example. [7, 1.6.8]

$$\begin{aligned}\mathbb{P}^n_{\mathbb{R}} &\cong e^0 \cup e^1 \cup \dots \cup e^n\\ \mathbb{P}^n_{\mathbb{C}} &\cong e^0 \cup e^2 \cup \dots \cup e^{2n}\\ \mathbb{P}^n_{\mathbb{H}} &\cong e^0 \cup e^4 \cup \dots \cup e^{4n}\end{aligned}$$

.

1.99 Example. [7, 1.6.10] Let $g_n : S^1 \to S^1$, $z \mapsto z^n$. Then $S^1 \cup_{g_0} D^2 \cong S^1 \vee S^1$ by 1.95.2, $S^1 \cup_{g_1} D^2 \cong D^2$ by 1.95.3, $S^1 \cup_{g_2} D^2 \cong \mathbb{P}^2$ by 1.69, $S^1 \cup_{g_k} D^2 \cong S^1 \cup_{g_{-k}} D^2$ by conjugation $z \mapsto \bar{z}$.

1.100 Definition. [7, 1.6.9] Let $i^n_j : S^1 \hookrightarrow \bigvee_{k=1}^r S^1$, $z \mapsto z^n$ on the j^{th} summand S^1, furthermore, $B_k := \{\exp(\frac{2\pi i t}{m}) : k-1 \le t \le k\}$ an arc of length $\frac{2\pi}{m}$ and $f_k : B_k \to S^1$, $\exp(\frac{2\pi i t}{m}) \mapsto \exp(2\pi i(t-k+1))$. Finally, let $i^{n_1}_{j_1} \cdot \dots \cdot i^{n_m}_{j_m} : S^1 \to \bigvee^r S^1$ the mapping which coincides on B_k with $i^{n_k}_{j_k} \circ f_k$, i.e. one runs first n_1-times along the j_1-th summand S^1, etc.

1.101 Theorem. [7, 1.6.11] *Let $g \ge 1$ and $f := i_1 \cdot i_2 \cdot i_1^{-1} \cdot i_2^{-1} \cdot \dots \cdot i_{2g-1} \cdot i_{2g} \cdot i_{2g-1}^{-1} \cdot i_{2g}^{-1}$ resp. $f := i_1^2 \cdot i_2^2 \cdot \dots \cdot i_g^2$. Then $\bigvee^{2g} S^1 \cup_f D^2 \cong F_g$ and $\bigvee^g S^1 \cup_f D^2 \cong N_g$.*

Proof. 1.94 $\Rightarrow X_g := \bigvee S^1 \cup_f D^2 \cong D^2/\sim$ where $x \sim y$ for $x,y \in S^1 \Leftrightarrow f(x) = f(y)$. This is precisely the relation from 1.67, resp. 1.73. □

1.102 Definition. Gluing several cells. [7, 1.6.12] For continuous mappings $f_j : D^n \supseteq S^{n-1} \to X$ for $j \in J$ let

$$X \cup_{(f_j)_j} \bigcup_{j\in J} D^n := X \cup_{\bigsqcup_{j\in J} f_j} \bigsqcup_{j\in J} D^n.$$

1.103 Example. [7, 1.6.13]

(1) $f_j : S^{n-1} \to \{*\} \Rightarrow X \cup_{(f_j)_j} \bigcup_{j\in J} D^n \cong \bigvee_J S^n$: By 1.36 $\lambda : (D^n, S^{n-1}) \to (S^n, \{*\})$ is a relative homeomorphism and hence also $\bigsqcup_J \lambda = J \times \lambda : (J \times D^n, J \times S^{n-1}) \to (J \times S^n, J \times \{*\})$. By 1.32 the induced map $(J \times D^n)/(J \times S^{n-1}) \to (J \times S^n)/(J \times \{*\}) = \bigvee_j S^n$ is a quotient mapping, since J is locally compact as discrete space. Obviously this mapping is bijective, hence a homeomorphism.

(2) $X \cup_{(f_1,f_2)} (D^n \sqcup D^n) \cong (X \cup_{f_1} D^n) \cup_{f_2} D^n$, by 1.46.

(3) $f_j = \mathrm{id} : S^{n-1} \to S^{n-1} \Rightarrow S^{n-1} \cup_{(f_1,f_2)} (D^n \sqcup D^n) \overset{2}{\cong} (S^{n-1} \cup e^n) \cup e^n \overset{1.95.3}{\cong} D^n \cup e^n \overset{1.95.4}{\cong} S^n$.

Inductive limits

1.104 Definition. [7, 1.8.1] Let X be a set and $A_j \subseteq X$ topological spaces with $X = \bigcup_{j\in J} A_j$ and the trace topology on $A_j \cap A_k$ induced from A_j and from A_k should be identical and the intersection closed. The final topology on X induces on A_j the given topology, i.e. $A_j \hookrightarrow X$ is a closed embedding, since for each closed $B \subseteq A_j$ the set $B \cap A_k = B \cap A_j \cap A_k$ is closed in A_k, so B_k is closed in der final topology.

$p : \bigsqcup_j A_j \to X$ is a quotient mapping and we thus have the corresponding universal property.

1.105 Proposition. [7, 1.8.3] *Let $\mathcal{A}$ be a closed locally finite covering of X. Then X carries the final topology with respect to $\mathcal{A}$.*

Proof. See [2, 1.2.14.3]: Let $B \subseteq X$ be such that $B \cap A \subseteq A$ is closed. In order to show that $B \subseteq X$ is closed it suffices to prove that $\overline{\bigcup \mathcal{B}} = \bigcup_{B\in\mathcal{B}} \overline{B}$ for locally finite families $\mathcal{B}(:= \{B \cap A : A \in \mathcal{A}\})$. ($\supseteq$) is obvious. ($\subseteq$) Let $x \in \bigcup_{B\in\mathcal{B}} \overline{B}$ and U an open neighborhood of x with $\mathcal{B}_0 := \{B \in \mathcal{B} : B \cap U \neq \emptyset\}$ being finite. Then $x \notin \overline{\bigcup_{B\in\mathcal{B}\setminus\mathcal{B}_0} B}$ and

$$x \in \bigcup_{B\in\mathcal{B}} \overline{B} = \bigcup_{B\in\mathcal{B}_0} \overline{B} \cup \overline{\bigcup_{B\in\mathcal{B}\setminus\mathcal{B}_0} B}$$

thus $x \in \bigcup_{B\in\mathcal{B}_0} \overline{B} \subseteq \bigcup_{B\in\mathcal{B}} \overline{B}$. □

1.106 Example. [7, 1.8.4] In particular, this is valid for finite closed coverings.

1.107 Definition. [7, 1.8.5] Let A_n be an increasing sequence of topological spaces, where each A_n is a closed subspace in A_{n+1}. Then $A := \bigcup_{n\in\mathbb{N}} A_n$ with the final topology is called LIMIT of the sequence $(A_n)_n$ and one writes $A = \varinjlim_n A_n$.

1.108 Example. [7, 1.8.6] $\mathbb{R}^\infty := \varinjlim_n \mathbb{R}^n$, the space of finite sequences. Let $x \in \mathbb{R}^\infty$ with $\varepsilon_n > 0$. Then $\{y \in \mathbb{R}^\infty : |y_n - x_n| < \varepsilon_n \forall n\}$ is an open neighborhood of x in $\mathbb{R}^\infty$. Conversely, let $U \subseteq \mathbb{R}^\infty$ be an open set containing x. Then there exists an $\varepsilon_1 > 0$ with $K_1 := \{y_1 : |y_1 - x_1| \leq \varepsilon_1\} \subseteq U \cap \mathbb{R}^1$. Since $K_1 \subseteq \mathbb{R}^1 \subseteq \mathbb{R}^2$ is compact there exists by [2, 2.1.11] an $\varepsilon_2 > 0$ with $K_2 := \{(y_1, y_2) : y_1 \in K_1, |y_2 - x_2| \leq \varepsilon_2\} \subseteq U \cap \mathbb{R}^2$. Inductively we obtain ε_n with $\{y \in \mathbb{R}^\infty : |y_k - x_k| \leq \varepsilon_k \forall k\} = \bigcup_n K_n \subseteq U$. Thus the sets from above form a basis of the topology. The sets $\bigcup_n \{y \in \mathbb{R}^n : |y - x| < \varepsilon_n\}$ do not, since for $\varepsilon_n \searrow 0$ they contain none of the neighborhoods from above, since $(\frac{\delta_n}{2}, \dots, \frac{\delta_n}{2}, 0, \dots)$ is not contained therein.

1.109 Example. [7, 1.8.7]

1 $S^\infty := \varinjlim_n S^n$ is the set of unit vectors in $\mathbb{R}^\infty$.

2 $\mathbb{P}^\infty := \varinjlim_n \mathbb{P}^n$ is the space of lines through 0 in $\mathbb{R}^\infty$.

3 $O(\infty) := \varinjlim_n O(n)$, where $GL(n) \hookrightarrow GL(n+1)$ via $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$.

4 $SO(\infty) := \varinjlim_n SO(n)$

5 $U(\infty) := \varinjlim_n U(n)$

6 $SU(\infty) := \varinjlim_n SU(n)$

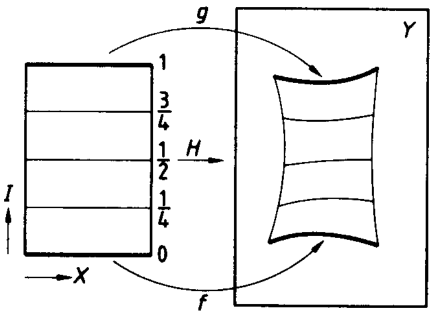
2 Homotopy

Homotopic mappings

2.1 Definition. [7, 2.1.1] A homotopy is a mapping $h : I \to C(X,Y)$, which is continuous as mapping $\hat{h} : I \times X \to Y$. Note that this implies, that $h : I \to C(X,Y)$ is continuous for the compact open topology (a version of the topology of uniform convergence for general topological spaces instead of uniform spaces Y) but not conversely.

Two mappings $h_j : X \to Y$ for $j \in \{0,1\}$ are called homotopic (we write $h_0 \sim h_1$) if there exists a homotopy $h : I \to C(X,Y)$ with $h(j) := h_j$ for $j \in \{0,1\}$, i.e. a continuous mapping $H : I \times X \to Y$ with and $H(x,j) = f_j(x)$ for all $x \in X$ and $j \in \{0,1\}$.

$$\begin{tikzcd} \{0,1\} \times X \arrow[r, "h_0 \sqcup h_1"] \arrow[d, hook] & Y \\ I \times X \arrow[ur, dotted, "H"'] & \end{tikzcd}$$



2.2 Lemma. [7, 2.1.2] *To be homotopic is an equivalence relation on $C(X,Y)$.*

2.3 Definition. [7, 2.1.5] The homotopy class $[f]$ of a mapping $g \in C(X,Y)$ is $[f] := \{g \in C(X,Y) : g \text{ is homotopic to } f\}$. Let $[X,Y] := \{[f] : f \in C(X,Y)\}$.

2.4 Lemma. [7, 2.1.3] *Homotopy is compatible with the composition.*

Proof. $h : I \to C(X,Y)$ a homotopy, $f : X' \to X$, $g : Y \to Y'$ continuous $\Rightarrow$ $C(f,g) \circ h := f^* \circ g_* \circ h : I \to C(X',Y')$ is a homotopy, since $(C(f,g) \circ h)\hat{} = g \circ \hat{h} \circ (f \times I)$ is continuous. $\square$

2.5 Definition. [7, 2.1.4] A mapping $f : X \to Y$ is called 0-homotopic iff it is homotopic to a constant mapping.
Any two constant mappings into Y are homotopic iff Y is path-connected. In fact a path $y : I \to I$ induces a homotopy $t \mapsto \mathrm{const}_y$.
X is called contractible, iff id_X is 0-homotopic.

2.6 Example. [7, 2.1.6]

(1) $[\{*\}, Y]$ is in bijection to the path-components of Y: Homotopy = Path.

(2) Star-shaped subsets $A \subseteq \mathbb{R}^n$ are contractible: scalar-multiplication.

(3) This is true in particular for $A = \mathbb{R}^n$ or a convex subset $A \subseteq \mathbb{R}^n$.

(4) For a contractible space X there need not exist a homotopy h which keeps x_0 fixed, see the comb 2.40.9. Contractible spaces are path-connected.

(5) A composition of a 0-homotopic mapping with any mapping is 0-homotopic by 2.4.

(6) If Y is contractible then any two mappings $f_j : X \to Y$ are homotopic, i.e. $[X, Y] := \{*\}$, by 2.4.

(7) Any continuous none-surjective mapping $f : X \to S^n$ is 0-homotopic: $S^n \setminus \{*\} \cong \mathbb{R}^n$ by 1.14, now use 2 and 6.

(8) If X is contractible and Y is path-connected then again any two mappings $f_j : X \to Y$ are homotopic, i.e. $[X, Y] = \{*\}$: 5 and 2.5.

(9) Any mapping $f : \mathbb{R}^n \to Y$ is 0-homotopic: 8 and 3.

2.7 Definition. [7, 2.1.7] A HOMOTOPY h relative $A \subseteq X$ is a homotopy $h : I \to C(X, Y)$ with $\mathrm{incl}^* \circ h : I \to C(X, Y) \to C(A, Y)$ constant. Two mappings $h_j : X \to Y$ are called homotopic relative $A \subseteq X$, iff there exists a homotopy $h : I \to C(X, Y)$ relative A with boundary values $h(j) = h_j$ for $j \in \{0, 1\}$.

2.8 Definition. [7, 2.1.8] A homotopy h of pairs (X, A) and (Y, B) is a homotopy $H : I \to C(X, Y)$ with $H(I)(A) \subseteq B$ Two mappings $f_j : (X, A) \to (Y, B)$ of pairs are called HOMOTOPIC, iff there exists a homotopy of pairs $H : I \to C(X, Y)$ with $H(j) = f_j$ and $H(t)(A) \subseteq B$. We denote with $[f]$ this homotopy class and with $[(X, A), (Y, B)]$ the set of all these classes.

2.9 Definition. [7, 2.1.10] A homotopy of pairs with $A = \{x_0\}$ and $B = \{y_0\}$ is called BASE-POINT PRESERVING HOMOTOPY. We have $f \simeq g : (X, \{x_0\}) \to (Y, \{y_0\})$ iff $f \simeq g$ relative $\{x_0\}$.

2.10 Example. [7, 2.1.9] Since I is contractible we have $[I, I] = \{0\}$. But $[(I, \dot{I}), (I, \dot{I})] = \{[\mathrm{id}], [t \mapsto 1 - t], [t \mapsto 0], [t \mapsto 1]\}$.

2.11 Lemma. [7, 2.1.11] *Let $p : X' \to X$ be a quotient mapping and let $h : I \to C(X, Y)$ be a mapping for which $p^* \circ h : I \to C(X', Y)$ is a homotopy. Then h is a homotopy.*

Proof. Note that for quotient-mappings p the induced injective mapping p^* is in general not an embedding (we may not find compact inverse images). However $\widehat{p^* \circ h} = \hat{h} \circ (I \times p)$ and $I \times p$ is a quotient-mapping by 1.32=[2, 2.2.9]. □

2.12 Lemma. [7, 2.1.12]

(1) *Let $p : X' \to X$ be a quotient mapping, $h : I \to C(X', Y)$ be a homotopy and $h_t \circ p^{-1} : X \to Y$ be well-defined for all t. Then this defines a homotopy $I \to C(X, Y)$ as well:* 2.11.

(2) *Let $h : I \to C(X, Y)$ be a homotopy compatible with equivalence relations $\sim$ on X and on Y, i.e. $x \sim x' \Rightarrow h(x, t) \sim h(x', t)$. Then h factors to a homotopy $I \to C(X/\sim, Y/\sim)$: Apply* 2.11 *to $(q_Y)_* \circ h : I \to C(X, Y/\sim)$.*

(3) *Let $f : X \supseteq A \to Y$ be a gluing map and $h : I \to C(X, Z)$ and $k : I \to C(Y, Z)$ be homotopies with $\mathrm{incl}^* \circ h = f^* \circ k$. Then they induce a homotopy $I \to C(Y \cup_f X, Z)$: Apply 2.11 to $p : Y \sqcup X \to Y \cup_f X$.*

(4) *Each homotopy $h : I \to C((X, A), (Y, B))$ of pairs induces a homotopy $I \to C(X/A, Y/B)$: 2.*

(5) *Homotopies $h^j : I \to C((X_j, x_j^0), (Y_j, y_j^0))$ induce a homotopy $\bigvee_j f^j : I \to C((\bigvee_j X_j, x^0), (\bigvee_j Y_j, y^0))$.*

2.13 Example. [7, 2.1.13]

(1) Let $h_t : (X, I) \to (X, I)$ be given by $h_t(x, s) := (x, ts)$. This induces a contraction of $CX := (X \times I)/(X \times \{0\})$.

(2) The contraction of $D^n = CS^{n-1}$ given by 1 does not leave S^{n-1} invariant, hence induces no contraction of D^n/S^{n-1}. We will see later, that $S^n \cong D^n/S^{n-1}$ is not contractible at all.

Homotopy classes for mappings of the circle

2.14 Definition. [7, 2.2.1] We consider the quotient mapping $p : \mathbb{R} \to S^1$, $t \mapsto \exp(2\pi i t)$ as well as its restriction $p|_I : I \to S^1$.

A mapping $\varphi : I \to \mathbb{R}$ factors to a well defined mapping $\overline{\varphi} := p \circ \varphi \circ p^{-1} : S^1 \to S^1$ iff $n := \varphi(1) - \varphi(0) \in \mathbb{Z}$.

$$\begin{tikzcd} I \arrow[r, "\varphi"] \arrow[d, "p"] & \mathbb{R} \arrow[d, "p"] \\ S^1 \arrow[r, dotted, "\overline{\varphi}"] & S^1 \end{tikzcd}$$

Conversely:

2.15 Lemma. [7, 2.2.2] *Let $f : S^1 \to S^1$ be continuous, then there exists a $\varphi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ with $f(s) = f(1) \cdot \hat{\varphi}(s)$.*

$$\begin{tikzcd} (\mathbb{R}, 0) \arrow[r, dotted, "\varphi"] \arrow[d, "p"] & (\mathbb{R}, 0) \arrow[d, "p"] \\ S^1 \arrow[r, "f"] & S^1 \end{tikzcd}$$

Proof. Replace f by $f(1)^{-1} \cdot f$, i.e. w.l.o.g. $f(1) = 1$. Let $h := f \circ p$. Then h is periodic and uniformly continuous. So chose $\delta > 0$ with $|t - t'| \le \delta \Rightarrow |h(t) - h(t')| < 2$. Let $t_j := j\,\delta$. The mapping $t \mapsto e^{2\pi i t}$ is a homeomorphism $(-\pi, \pi) \to S^1 \setminus \{-1\}$ let $\arg : S^1 \setminus \{-1\} \to (-\pi, \pi) \subseteq \mathbb{R}$ be its inverse. Then for $t_j \le t \le t_{j+1}$ let

$$\varphi(t) := \frac{1}{2\pi}\left(\arg \frac{h(t_1)}{h(t_0)} + \cdots + \arg \frac{h(t)}{h(t_j)}\right)$$

which gives the desired lifting. □

2.16 Definition. [7, 2.2.3] Let $f : S^1 \to S^1$ be continuous and φ as in 2.15, then $\operatorname{grad} f := \varphi(1) \in \mathbb{Z}$ is called MAPPING DEGREE of f.

2.17 Theorem. [7, 2.2.4] $\mathrm{grad} : [S^1, S^1] \cong \mathbb{Z}$.

(1) *The mapping $z \mapsto z^n$ has degree n.*

(2) *Two mappings are homotopic iff they have the same degree.*

Proof. $\boxed{1}$ follows since $\varphi(t) = n \cdot t$.

$\boxed{2}$ Let f be a homotopy $I \to C(S^1, S^1)$. Then there exists a lifting $\varphi : I \to C(\mathbb{R}, \mathbb{R})$ with $p(\varphi_t(s)) = f_t(1)^{-1} \cdot f_t(s)$. In particular $\varphi_t(1) \in p^{-1}(1) = \mathbb{Z}$ and hence is constant. So $\mathrm{grad}(f_0) = \varphi_0(1) = \varphi_1(1) = \mathrm{grad}(f_1)$.

Conversely, we define $\varphi : I \to C(\mathbb{R}, \mathbb{R})$ by $\varphi_t := (1-t)\varphi_0 + t\varphi_1$. Then this induces a homotopy $f : I \to C(S^1, S^1)$, since $\varphi_t(1) = \mathrm{grad}(f_0) = \mathrm{grad}(f_1) \in \mathbb{Z}$. □

2.18 Example. [7, 2.2.5]

(1) $\mathrm{grad}(\mathrm{id}) = 1$: $\mathrm{id} = g_1$; $f \sim 0 \Rightarrow \mathrm{grad}(f) = 0$: $f \sim g_0$; $\mathrm{grad}(z \mapsto \overline{z}) = -1$: g_{-1} by $\boxed{2.17}$.

(2) $\mathrm{grad}(f \circ g) = \mathrm{grad}(f) \cdot \mathrm{grad}(g)$: $g_n \circ g_m = g_{nm}$, now $\boxed{2.17}$.

(3) f homeomorphism $\Rightarrow \mathrm{grad}(f) \in \{\pm 1\}$: $\mathrm{grad}(f)$ is invertible.

(4) $\mathrm{incl} : S^1 \hookrightarrow \mathbb{C} \setminus \{0\}$ is not 0-homotopic, since id_{S^1} is not: $\mathrm{grad}(\mathrm{id}) = 1$ now $\boxed{2.17}$. We can use $[S^n, X]$ to detect holes in X.

(5) The two inclusions of $S^1 \hookrightarrow S^1 \times S^1$ are not homotopic: $\mathrm{pr}_1 \circ \mathrm{inc}_1 = \mathrm{id}$, $\mathrm{pr}_1 \circ \mathrm{inc}_2 \sim 0$.

2.19 Lemma. [7, 2.2.6] *S^1 is not contractible.*

Proof. $\mathrm{grad}(\mathrm{id}) = 1$. □

2.20 Definition. [7, 2.3.1] A subspace $A \subseteq X$ is called RETRACT iff there exists an $r : X \to A$ with $r|_A = \mathrm{id}_A$.

Being a retract is transitive, and retracts in Hausdorff spaces are closed ($A = \{x \in X : r(x) = x\}$)

2.21 Lemma. [7, 2.3.2] *A subspace $A \subseteq X$ is a retract of X iff every function $f : A \to Y$ can be extended to $\tilde{f} : X \to Y$.*
Let $A \subseteq X$ be closed. Then a function $f : A \to Y$ can be extended to X iff Y is a retract of $Y \cup_f X$.

Proof. We prove that id_A can be extended iff any $f : A \to Y$ can be extended. The extensions $\tilde{f}$ of $f : A \to Y$ correspond to retractions $r = \mathrm{id}_Y \cup \tilde{f}$ of $Y \subseteq Y \cup_f X$:

$$\begin{tikzcd}
A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, hook] \arrow[ddr, "f"'] & & A \arrow[ddl, "f"] \\
& X \arrow[ur, dotted, "r"'] \arrow[d, dotted, "\tilde{f}"] & \\
& Y &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, hook] & Y \arrow[d, hook] \arrow[ddr, "\mathrm{id}_Y"] & \\
X \arrow[r] \arrow[drr, dotted, "\tilde{f}"'] & Y \cup_f X \arrow[dr, dotted, "r"] & \\
& & Y
\end{tikzcd}$$
□

2.22 Lemma. [7, 2.2.7] *There is no retraction to $S^1 \hookrightarrow D^2$.*

Proof. Otherwise, let $r : D^2 \to S^1$ be a retraction to $\iota : S^1 \hookrightarrow D^2$. Then $\mathrm{id} = r \circ \iota \sim r \circ 0 = 0$, a contradiction. $\square$

2.23 Lemma. Brouwer's fixed point theorem. [7, 2.2.8] *Every continuous mapping $f : D^2 \to D^2$ has a fixed point.*

Proof. Assume $f(x) \neq x$ and let $r(x)$ the unique intersection point of the ray from $r(x)$ to x with S^1. Then r is a retraction, a contradiction to 2.22. $\square$

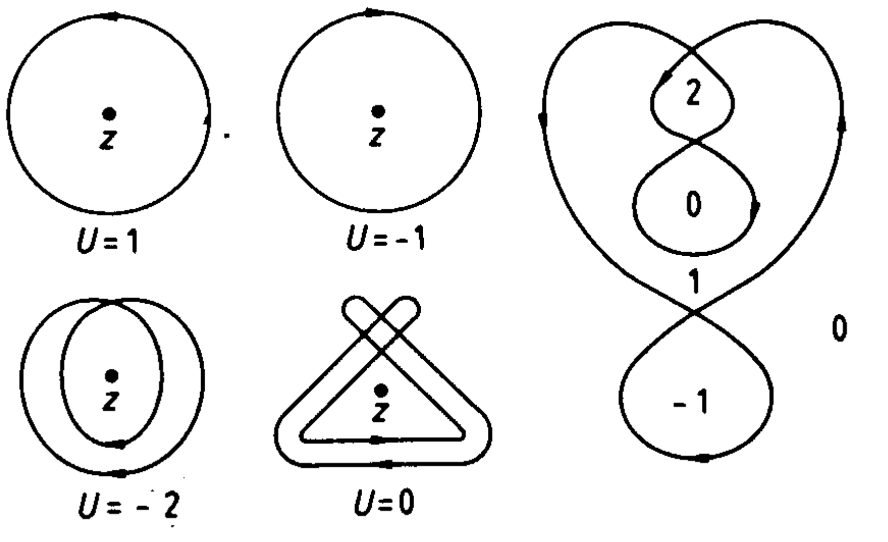
2.24 Lemma. Fundamental theorem of algebra. [7, 2.2.9] *Every none constant polynomial has a root.*

Proof. Let $p(x) = a_0 + \cdots + a_{n-1}x^{n-1} + x^n$ be a polynomial without root and $n \geq 1$, $s := |a_0| + \cdots + |a_{n-1}| + 1 \geq 1$ and $z \in S^1$. Then

$$\begin{aligned} |p(sz) - (sz)^n| &\leq |a_0| + s|a_1| + \cdots + s^{n-1}|a_{n-1}| \\ &\leq s^{n-1}(|a_0| + \cdots + |a_{n-1}|) < s^n = |(sz)^n|. \end{aligned}$$

Hence $0 \notin \overline{p(sz), (sz)^n}$. Thus $g_n : z \mapsto s^n z^n$ is homotopic to $z \mapsto p(sz)$ and consequently 0-homotopic. Hence $0 \sim g_n : zz^n 0$, a contradiction to 2.17. $\square$

2.25 Definition. [7, 2.2.10] The DEGREE of $f : S^1 \to \mathbb{R}^2$ with respect to $z_0 \notin f(S^1)$ is the degree of $x \mapsto \frac{f(x)-z_0}{|f(x)-z_0|}$ and will be denoted by $U(f, z_0)$ the TURNING (WINDING) NUMBER of f around z_0.



2.26 Lemma. [7, 2.2.11] *If z_0 and z_1 are in the same component of $\mathbb{C} \setminus f(S^1)$ then $U(f, z_0) = U(f, z_1)$.*

Proof. Let $t \mapsto z_t$ be a path in $\mathbb{C} \setminus f(S^1)$. Then $t \mapsto (x \mapsto \frac{f(x)-z_t}{|f(x)-z_t|})$ is a homotopy and hence $U(f, z_0) = U(f, z_1)$. $\square$

2.27 Lemma. [7, 2.2.12] *There is exactly one unbounded component of $\mathbb{C} \setminus f(S^1)$ and for z in this component we have $U(f, z) = 0$.*

Proof. For x' outside a sufficiently large disk containing $f(S^1)$ (this is connected and contained in the (unique) unbounded component) the mapping

$$t \mapsto \left(x \mapsto \frac{tf(x) - x'}{|tf(x) - x'|} \right)$$

is a homotopy showing that $x \mapsto \frac{f(x)-x'}{|f(x)-x'|}$ is 0-homotopic and hence $U(f, x') = 0$ and $U(f, \_)$ is zero on the unbounded component. $\square$

By Jordan's curve theorem there are exactly two components for an embedding $f : S^1 \to \mathbb{C}$. And one has $U(f, z) \in \{\pm 1\}$ for z in the bounded component.

2.28 Theorem of Borsuk and Ulam. [7, 2.2.13] *For every continuous mapping $f : S^2 \to \mathbb{R}^2$ there is a $z \in S^2$ with $f(z) = f(-z)$.*

Proof. Suppose $f(x) \neq f(-x)$. Consider $f_1 : S^2 \to S^1$, $x \mapsto \frac{f(x)-f(-x)}{|f(x)-f(-x)|}$ and $f_2 : D^2 \to S^1$, $x \mapsto f_1(x, \sqrt{1 - |x|^2})$. Then $g := f_2|_{S^1} \sim 0$ via f_2. Since f_1 is odd, so is g. Let φ be the lift of $g(1)^{-1}g$ and hence $\varphi(1) = \operatorname{grad}(g)$. For all t we have $g(\exp(2\pi i(t + \frac{1}{2}))) = g(-\exp(2\pi it)) = -g(\exp(2\pi it))$ and hence

$$\begin{aligned}\exp(2\pi i\varphi(t + \frac{1}{2})) &= g(1)^{-1}g(\exp(2\pi i(t + \frac{1}{2}))) = -g(1)^{-1}g(\exp(2\pi it)) \\ &= -\exp(2\pi i\varphi(t)) = \exp(2\pi i(\varphi(t) + \frac{1}{2})).\end{aligned}$$

Hence $k := \varphi(t + \frac{1}{2}) - \varphi(t) - \frac{1}{2} \in \mathbb{Z}$ and independent on t. For $t = 0$ we get $\varphi(\frac{1}{2}) = k + \frac{1}{2}$ and for $t = \frac{1}{2}$ we get $\operatorname{grad}(g) = \varphi(1) = \varphi(\frac{1}{2}) + \frac{1}{2} + k = 2k + 1 \neq 0$, a contradiction. $\square$

2.29 Ham-Sandwich-Theorem. [7, 2.2.14] *Let A_0, A_1, A_2 be 3 subsets of $\mathbb{R}^3$, such that the volume of the trace on each half-space is well-defined and continuous and such that for every $a \in S^2$ there exists a unique distance d_a such that the volume of A_0 on both sides of the plane with unit-normal a and distance d_a from zero are equal. Let furthermore $a \mapsto d_a$ be continuous. Then there is a plane which cuts A_0, A_1 and A_2 in exactly equal parts.*

Proof. Let $f : S^2 \to \mathbb{R}^2$ be the volume of A_1 and A_2 on the right side of the plane with unit-normal a and distance d_a. By 2.28 there exists a $b \in S^2$ with $f(b) = f(-b)$. Since $d_{-a} = -d_a$ we have that $f(-b)$ is the volume of A_1 and A_2 on the left side of this plane. $\square$

Retracts

2.30 Theorem. [7, 2.3.3] *A mapping $f : X \to Y$ is 0-homotopic iff there exists an extension $\tilde{f} : CX \to Y$ with $\tilde{f}|_X = f$*

Proof. We proof that homotopies $h : X \times I \to Y$ with constant h_0 correspond to extensions $\tilde{h}_1 : CX \to Y$ of h_1:

$$\begin{tikzcd}
& X \times \{1\} \arrow[r, equal] \arrow[dl, hook'] & X \arrow[dd, "f"] \\
X \times I \arrow[r, two heads] \arrow[drr, dotted, "h"] & CX \arrow[dr, dotted] & \\
& X \times \{0\} \arrow[ul, hook] \arrow[r, "\text{const}"'] & Y
\end{tikzcd} \qquad \square$$

2.31 Definition. [7, 2.3.4] A pair (X, A) is said to have the general HOMOTOPY EXTENSION PROPERTY (HEP) (equiv. a COFIBRATION) if A is closed in X and we have

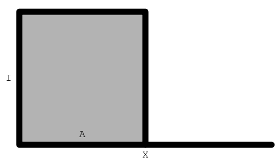
$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d, hook] & X \arrow[d, hook] \arrow[dr, "F_0"] & \\
A \times I \arrow[r, hook] \arrow[rr, bend right, "f"'] & X \times I \arrow[r, dotted, "F"] & Y
\end{tikzcd}$$

or, equivalently,

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d, "\check{f}"'] & X \arrow[d, "F_0"] \arrow[dl, dotted, "\check{F}"'] \\
C(I, Y) \arrow[r, "\text{ev}_0"] & Y
\end{tikzcd}$$

This is dual to the notion of FIBRATION (mappings with the homotopy lifting property):

$$\begin{tikzcd}
A & X \arrow[l, "\check{f}"'] \\
Y \times I \arrow[u, "h"] \arrow[ur, dotted, "H"] & Y \arrow[l, "\text{ins}_0"] \arrow[u, "H_0"']
\end{tikzcd}$$



2.32 Theorem. [7, 2.3.5] *(X, A) has HEP $\Leftrightarrow$ $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.*

Proof. (X, A) HEP $\Leftrightarrow$ any $f : W := X \times \{0\} \cup A \times I \to Y$ extends to $X \times I \overset{\boxed{2.21}}{\Longleftrightarrow}$ $W \subseteq X \times I$ is a retract.

2.33 Example. [7, 2.3.6]

(1) The pair (D^n, S^{n-1}) has the HEP: radial project from the axis at some point above the cylinder.

(2) If (X, A) has HEP then $(Y \cup_f X, Y)$ has HEP for any $f : A \to Y$.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, hook] & Y \arrow[r, "h"] \arrow[d, hook] & C(I, Z) \arrow[d, "\mathrm{ev}_0"] \\
X \arrow[r] \arrow[urr, dotted] & Y \cup_f X \arrow[r, "H_0"'] \arrow[ur, dotted] & Z
\end{tikzcd}$$

(3) If Y is obtained from X by gluing cells, then (Y, X) has HEP ($\Leftarrow$ (a), (b)).

2.34 Remark. [7, 2.3.7] Let (X, A) has HEP.

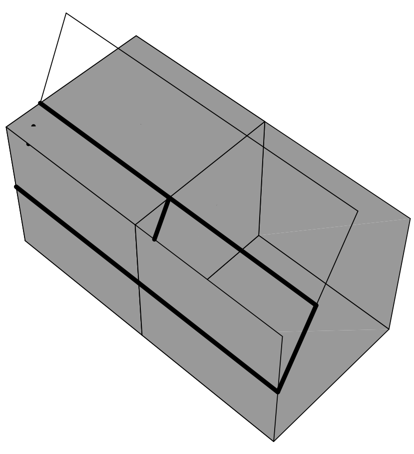
(1) If $f \sim g : A \to Y$ and f extends to X then so does g (by Definition of HEP).

(2) If $f : X \to Y$ is 0-homotopic on A, then there exists a mapping g homotopic to f, which is constant on A (Consider the homotopy on $I \times A$ and f on $\{0\} \times X$).

(3) If $A = \{x_0\}$ and Y is path-connected, then every mapping is homotopic to a base-point preserving one (Consider f on $\{0\}\times X$ and a path w on $I\times\{x_0\}$ between $f(x_0)$ and y_0).

2.35 Theorem. [7, 2.3.8] *If (X, A) has HEP, then so has $(X \times I, X \times \dot{I} \cup A \times I)$.*

Proof. We use 2.32 to show that $X\times I\times I$ has $W := X\times I\times\{0\}\cup(X\times\dot{I}\cup A\times I)\times I$ as retract. For this we consider planes E through the axis $X \times (1/2, 2)$. For planes intersecting the bottom $X \times I \times \{0\}$ we take the retraction r of the intersection $E\cap(X \times I \times I) \cong X \times I$ (via horizontal projection) onto the intersection $E\cap W \cong X \times \{0\} \cup A \times I$. For the other planes meeting the sides we take the retraction r of the intersection $E \cap (X \times I \times I) \cong X \times [0, t]$ (via vertical projection) onto the intersection $E\cap W \cong X\times\{0\}\cap A\times[0, t]$. For this we have to use that the retraction $r : (x, t) \mapsto (r_1(x, t), r_2(x, t))$ given by 2.32 can be chosen such that $r_2(x, t) \le t$, in fact replace $r_2(x, t)$ by $\min\{t, r_2(x, t)\}$. $\square$



Homotopy equivalences

2.36 Definition. [7, 2.4.1] A HOMOTOPY EQUIVALENCE is a mapping having up to homotopy an inverse. Two spaces are called HOMOTOPY EQUIVALENT if there exists a homotopy equivalence.

2.37 Definition. [7, 2.4.2] A continuous mapping between pairs is called HOMOTOPY EQUIVALENCE OF PAIRS, if there is a mapping of pairs in the opposite direction which is inverse up to homotopy of pairs.

2.38 Definition. [7, 2.4.3] A subspace $A \subseteq X$ is called DEFORMATION RETRACT (DR) iff there is a homotopy $h_t : X \to X$ with $h_0 = \mathrm{id}_X$ and $h_1 : X \to A \subseteq X$ a retraction. If h_t is a homotopy rel. A it is called STRICT DEFORMATION RETRACT.

2.39 Theorem. [7, 2.4.4] *Let (X, A) have the HEP. Then the following statements are equivalent:*

1 *$A \to X$ is a homotopy-equivalence;*

2 *A is a DR of X;*

3 *A is a SDR of X.*

Proof. ($3 \Rightarrow 2$) is obvious.

($2 \Rightarrow 1$) is always true. In fact let h_t be a deformation, with end value a retraction $r := h_1 : X \to A$. Then r is a homotopy inverse to $\iota : A \to X$, since $r \circ \iota = \mathrm{id}_A$ and $\iota \circ r = h_1 \sim h_0 = \mathrm{id}_X$.

($1 \Rightarrow 2$) Let g be a homotopy inverse to the inclusion $\iota : A \to X$. Since $g \circ \iota \sim \mathrm{id}_A$ and $g \circ \iota$ extends to $g : X \to A$ we conclude from 2.34.1 that $\mathrm{id}_A : A \to A$ has an extension $r : X \to A$, i.e. a retraction. On the other hand we conclude from $\iota \circ g \sim \mathrm{id}_X$ that $\mathrm{id}_X \sim \iota \circ g = r \circ \iota \circ g \sim r \circ \mathrm{id}_X = r$.

($2 \Rightarrow 3$) Let $h_t : X \to X$ be a deformation from id_X to a retraction $r : X \to A$. Let $F : X \times \dot{I} \cup A \times I \to X$ be given by $F(\_, 1) := r$ $F(x, t) = x$ elsewhere and let

$$H(x, s, t) := \begin{cases} h_{st}(r(x)) & \text{für } s = 1 \text{(the back side)} \\ h_{st}(x) & \text{elsewhere, i.e. for } x \in A \text{ or for } (s = 0 \text{ (the front side) or } t = 1 \text{ (the top side))} \end{cases}$$

because of $r(x) = x$ for $x \in A$ the definition coincides on the intersection. Then $F \sim H_1$ on $W := X \times \dot{I} \cup A \times I$. Since H_1 lives on $X \times I$, we can extend F to $X \times I$ by 2.34.1 and 2.35. This is the required deformation relative A. □

2.40 Example. [7, 2.4.5]

(1) X is homotopy-equivalent to a point iff it is contractible (in fact $\{*\} \subseteq X$ is a homotopy-equivalence iff $\mathrm{const}_* \sim \mathrm{id}_X$, i.e. X is contractible).

(2) Every set which is star-shaped with respect to some point, has this point as SDR.

(3) Composition of SDRs are SDRs.

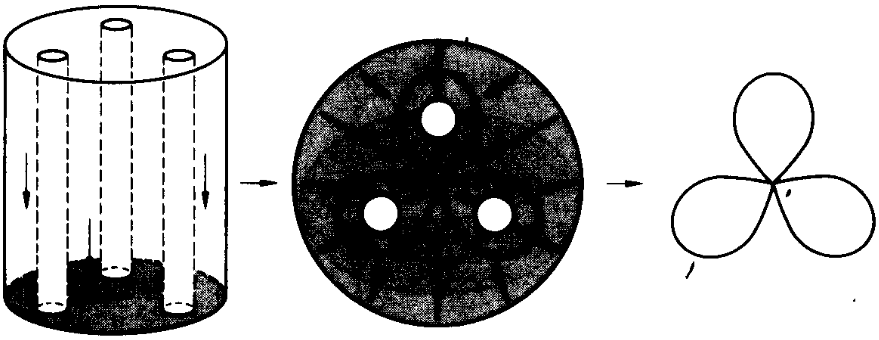
(4) If $\{y\}$ is a SDR of Y then so is $X \times \{y\}$ of $X \times Y$ and of $X \vee Y \subseteq X \times Y$.

(5) $X \times \{0\}$ is a SDR of $X \times I$ ($\Leftarrow$ 4, 2); $\{X\}$ is a SDR of CX.

(6) The complement of a k-dimensional affine subspace of $\mathbb{R}^n$ has an S^{n-k-1} as SDR ($\mathbb{R}^n \setminus \mathbb{R}^k = \mathbb{R}^k \times (\mathbb{R}^{n-k} \setminus \{0\}) \sim \{0\} \times S^{n-k-1}$ by (5) and (3)).

(7) The following spaces have S^1 as SDR: $X \times S^1$ for every strictly contractible X and the Möbius strip.

(8) Every handle-body of genus g has $S^1 \vee \cdots \vee S^1$ as SDR.



(9) An infinite comb has one tip as DR but not as SDR.

2.41 Proposition. [7, 2.4.6] *If A is a SDR of X and $f : A \to Y$ is continuous, then Y is a SDR in $Y \cup_f X$.*

Proof.

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, hook] & Y \arrow[d, hook] & \mathrm{id} \\ h_t & X \arrow[r] & Y \cup_f X & \tilde{h}_t \quad \square \end{tikzcd}$$

2.42 Corollary. [7, 2.4.7] *If Y is built from X by gluing simultaneously cells, then Y is a SDR in $Y \setminus P$, where P is given by picking in every cell a single point.*

Proof. Use 2.40.2. $\square$

2.43 Example. [7, 2.4.8] The pointed compact surfaces have $S^1 \vee \cdots \vee S^1$ as SDR.

Proof. By 1.101 they are $S^1 \vee \cdots \vee S^1 \cup_f (D^2 \setminus \{*\})$. Now use 2.41. $\square$

2.44 Theorem. [7, 2.4.9] *(X, A) has HEP iff $X \times \{0\} \cup A \times I$ is a SDR of $X \times I$.*

Proof. ($\Leftarrow$) follows from 2.32.

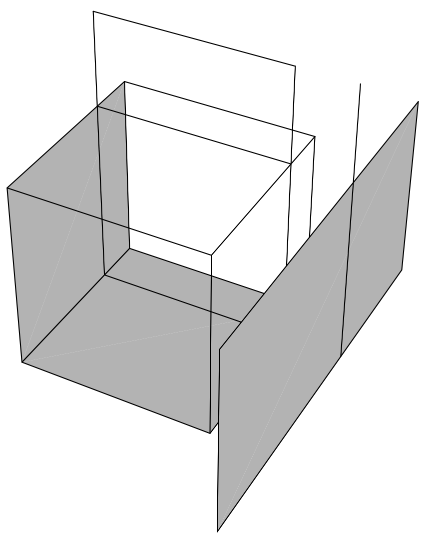
($\Rightarrow$) By 2.32 $W := X \times \{0\} \cup A \times I$ is a retract of $X \times I$. Let $r = (r_1, r_2)$ be the retraction. Then $h_t(x, s) := (r_1(x, ts), (1 - t)s + t\, r_2(x, s))$ is a homotopy between id and r rel. W: In fact $h_t(x, s) = (x, (1 - t)s + t\, s) = (x, s)$ for $(x, s) \in W$. $\square$

2.45 Definition. [7, 2.4.10] The MAPPING CYLINDER M_f of a mapping $f : X \to Y$ is given by $Y \cup_f (X \times I)$, where f is considered as mapping $X \times \{1\} \cong X \to Y$.

We have the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, hook, "i"'] & & Y \arrow[dl, hook, shift left] \\ & M_f \arrow[ur, "r", shift left] & \end{tikzcd}$$

where $f = r \circ i$ and i is a closed embedding with HEP and $Y \to M_f$ a SDR (along the generators $X \times I$) with retraction r. To see the HEP, use 2.32 to construct a retraction $M_f \times I \to M_f \times \{0\} \cup X \times I$ by projecting radially in the plane $\{x\} \times I \times I$ from $\{x\} \times \{1\} \times \{2\}$.



2.46 Corollary. [7, 2.4.12] *Two spaces are homotopy equivalent iff there exists a third one which contains both as SDRs.*

Proof. ($\Rightarrow$) Use the mapping cylinder as third space. Since f is a homotopy equivalence, so is $i : X \to M_f$ by 2.45 and by the HEP it is a SDR by 2.39.

($\Leftarrow$) Use that SDRs are always homotopy equivalences. $\square$

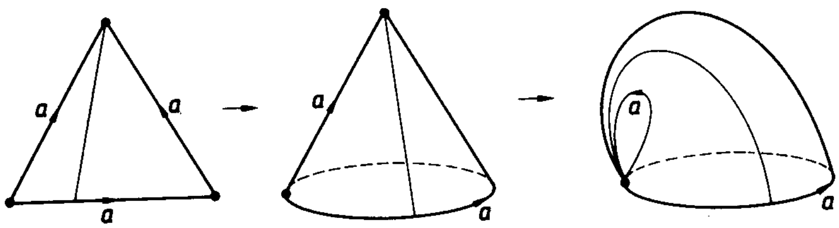
2.47 Proposition. [7, 2.4.13] *Assume (X, A) has HEP and $f_j : X \supseteq A \to Y$ are homotopic. Then $Y \cup_{f_0} X$ and $Y \cup_{f_1} X$ are homotopy equivalent rel. Y.*

Proof. Consider the homotopy $f : A \times I \to Y$ and the space $Z := Y \cup_f (X \times I)$. We show that $Y \cup_{f_j} X$ are SDRs of Z and hence are homotopy equivalent by 2.46:

$$
\begin{tikzcd}
A \arrow[rrr, "f_j"] \arrow[dr, hook] \arrow[dd, hook] & & & Y \arrow[dd] \\
& A \times I \arrow[urr, "f"'] \arrow[dd, hook] & & \\
X \arrow[rrr] \arrow[dr, hook] & & & Y \cup_{f_j} X \arrow[d, "\text{SDR}"', "\boxed{2.41}"] \\
& W \arrow[urr] \arrow[d, "\text{SDR}"', "\boxed{2.44}"] & & Y \cup_f X \times I \\
& X \times I \arrow[urr, "f"'] & &
\end{tikzcd}
$$

Where we use that the composite of two push-outs is a push-out, and if the composite of push-out and a commuting square is a push-out then so is the second square. □

2.48 Example. [7, 2.4.14] The dunce hat, i.e. a triangle with sides a, a, a^{-1} identified, is contractible: By 1.100, 1.94 and 2.47 we have $D \cong S^1 \cup_f e^2 \sim S^1 \cup_{\mathrm{id}} e^2 \cong D^2$.



2.49 Proposition. [7, 2.4.15] *Let A be contractible and let (X, A) have the HEP. Then the projection $X \to X/A$ is a homotopy equivalence.*

Proof. Consider

$$
\begin{tikzcd}
A \arrow[r, hook] \arrow[d, "f_t"'] & X \arrow[r, two heads] \arrow[d, "F_t"'] & X/A \arrow[d, "\tilde{F}_t"] \arrow[dl, "\tilde{R}"'] \\
A \arrow[r, hook] & X \arrow[r, two heads] & X/A
\end{tikzcd}
$$

Then $\tilde{R}$ given by factoring F_1 is the desired homotopy inverse to $X \to X/A$, since $F_0 = \mathrm{id}$ and $F_1(A) = \{*\}$. □

3 Simplicial Complexes

Basic concepts

3.1 Remark. [7, 3.1.1] A finite set of points $x_0, \dots, x_q$ in $\mathbb{R}^n$ is said to be in general position if one of the following equivalent conditions is satisfied:

1 The affine subspace $\{\sum_i \lambda_i x_i : \sum_i \lambda_i = 1\}$ generated by the x_i has dimension q;

2 No strict subset of $\{x_0, \dots, x_q\}$ generates the same affine subspace;

3 The vectors $x_i - x_0$ for $i > 0$ are linear independent;

4 The representation $\sum_i \lambda_i x_i$ with $\sum_i \lambda_i = 1$ is unique.

3.2 Definition. [7, 3.1.2] A SIMPLEX of dimension q (or short: a q-simplex) is the set

$$\sigma := \Big\{ \sum_i \lambda_i x_i : \sum_i \lambda_i = 1, \forall i : \lambda_i > 0 \Big\}$$

for points $\{x_0, \dots, x_q\}$ in general position. Its closure in $\mathbb{R}^n$ is the set

$$\bar\sigma := \Big\{ \sum_i \lambda_i x_i : \sum_i \lambda_i = 1, \forall i : \lambda_i \ge 0 \Big\}.$$

The points x_i are then called the VERTICES of σ. Remark that as extremal points of $\bar\sigma$ they are uniquely determined. The set $\dot\sigma := \bar\sigma \setminus \sigma$ is called boundary of σ.

3.3 Lemma. [7, 3.1.3] *Let σ be a q-simplex. Then* $(\bar\sigma, \dot\sigma) \cong (D^q, S^{q-1})$.

Proof. Use 1.10 for the affine subspace generated by σ. □

3.4 Definition. [7, 3.1.4] Let σ and τ be simplices in $\mathbb{R}^n$. Then σ is called FACE of τ ($\sigma \le \tau$) iff the vertices of σ form a subset of those of τ.

3.5 Remark. [7, 3.1.5]

(1) Every q-simplex has 2^{q+1} many faces: In fact this is the number of subsets of $\{x_0, \dots, x_q\}$

(2) The relation of being a face is transitive.

(3) The closure of a simplex σ is the disjoint union of all its faces $\overline{\sigma} = \bigcup_{\tau \le \sigma} \tau$: Remove all summands $\lambda_i x_i$ in $\sum_i \lambda_i x_i$ for which $\lambda_i = 0$.

3.6 Definition. [7, 3.1.6] A SIMPLICIAL COMPLEX K is a finite set of simplices in some $\mathbb{R}^n$ with the following properties:

1 $\sigma \in K, \tau \le \sigma \Rightarrow \tau \in K$.

2 $\sigma, \tau \in K, \sigma \ne \tau \Rightarrow \sigma \cap \tau = \emptyset$.

The 0-simplices are called vertices and the 1-simplices are called edges. The number $\max\{\dim \sigma : \sigma \in K\}$ is called dimension of K.

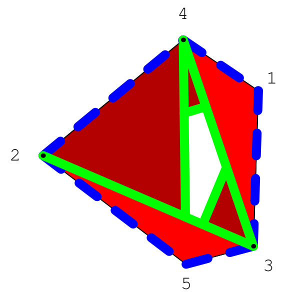
3.7 Definition. [7, 3.1.7] For a simplicial complex K the subspace $|K| := \bigcup_{\sigma \in K} \sigma$ is called the UNDERLYING TOPOLOGICAL SPACE. Every space which is the underlying space of a simplicial complex is called POLYHEDRA. A corresponding simplicial complex is called a TRIANGULATION of the space.

3.8 Remark. [7, 3.1.8] By 3.6 we have $|K| = \bigcup_{\sigma\in K} \bar{\sigma}$, and $\bar{\sigma}\cap\bar{\tau}$ is a either empty or the closure of a common face. Every polyhedra is compact and metrizable.

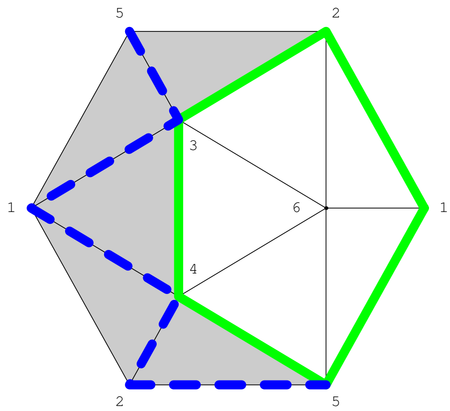
3.9 Remarks. [7, 3.1.9]

1 Regular polyhedra are triangulations of a 2-sphere.

2 There is a triangulation of the Möbius strip by 5 triangles.



3 =3.10 There is a (minimal) triangulation of the projective plane by 10 triangles.



1 One can show, that every compact surface, every compact 3-dimensional manifold and every compact differentiable manifold has a triangulation.

2 It is not known if every compact manifold has a triangulation.

3 Every ball and every sphere has a triangulation given by any n-Simplex with all its faces.

4 A countable union of circles tangent at some point is not a polyhedra, since it consists of infinite many 1-simplices.

3.11 Definition. [7, 3.1.10] For every $x \in |K|$ exists a unique simplex $\sigma \in K$ with $x \in \sigma$. It is called the CARRIER SIMPLEX of x and denoted $\operatorname{carr}_K(x)$.

3.12 Lemma. [7, 3.1.11] *Every point $x \in |K|$ has a unique representation $x = \sum_i \lambda_i x_i$, with $\sum_i \lambda_i = 1$ and $\lambda_i \geq 0$ and vertices x_i of K. The x_i with $\lambda_i > 0$ are just the vertices of the carrier simplex of x.*

3.13 Definition. [7, 3.1.12] A SUBCOMPLEX is a subset $L \subseteq K$, that is itself a simplicial complex. This is exactly the case if $\tau \leq \sigma \in L \Rightarrow \tau \in L$ since (2) is obvious.

3.14 Lemma. [7, 3.1.13] *A subset $L \subseteq K$ is a subcomplex iff $|L|$ is closed in $|K|$.*

Proof. ($\Rightarrow$) since $|L|$ is compact by 3.8.
($\Leftarrow$) $\tau \leq \sigma \in L \Rightarrow \tau \subseteq \bar{\sigma} \subseteq |L| \Rightarrow \tau \in L$. Use 3.5.3 and 3.8. $\square$

3.15 Definition. [7, 3.1.14] Two simplices σ and τ are called CONNECTIBLE in K iff there are simplices $\sigma_0 = \sigma, \dots, \sigma_r = \tau$ with $\bar{\sigma}_j \cap \bar{\sigma}_{j+1} \neq \emptyset$. The equivalence classes with respect to being connected are called the COMPONENTS of K. If there is only one component then K is called connected.

3.16 Lemma. [7, 3.1.15] *The components of K are subcomplexes and their underlying spaces are the path-components (connected components) of $|K|$.*

Proof. Since $\overline{\sigma}$ is a closed convex subset of some $\mathbb{R}^n$, it is path connected and hence the underlying subspace of a component is (path-)connected. Conversely, if two simplices σ and τ belong to the same component of the underlying space, then there is a curve c connecting σ with τ. This curve meets finitely many simplices $\sigma_0 = \sigma, \dots, \sigma_N = \tau$ and we may assume that it meets σ_i before σ_j for $i < j$. By induction we show that all $\overline{\sigma_i}$ belong to the same component of K. In fact if $\sigma_0, \dots, \sigma_{i-1}$ does so, then let $t_0 := \min\{t \in [0,1] : c(t) \in \overline{\sigma_i}\}$. Then $c(t) \in \bigcup_{j<i} \sigma_j$ for $t < t_0$ and hence $c(t_0) \in \bigcup_{i<j} \overline{\sigma_j} \cap \overline{\sigma_i}$. Thus $\overline{\sigma_i}$ is connected with $\overline{\sigma_j}$ for some $j < i$. $\square$

3.17 Definition. [7, 3.1.16] A mapping $\varphi : K \to L$ between simplicial complexes is called SIMPLICIAL MAPPING iff

1 φ maps vertices to vertices;

2 And if σ is generated by vertices $x_0, \dots, x_q$ then $\varphi(\sigma)$ is generated by the vertices $\varphi(x_0), \dots, \varphi(x_q)$, i.e. $\varphi(\langle x_0, \dots, x_q \rangle) = \langle \varphi(x_0), \dots, \varphi(x_q) \rangle$.

3.18 Lemma. [7, 3.1.17]

1 *A simplicial mapping is uniquely determined by its values on the vertices.*

2 *If $\sigma \leq \tau \in K$ then $\varphi(\sigma) \leq \varphi(\tau) \in L$.*

3 $\dim(\varphi\sigma) \leq \dim \sigma$.

Proof. This follows immediately, since $\varphi(\langle x_0, \dots, x_k \rangle) = \langle \{\varphi(x_i) : 0 \leq i \leq k\} \rangle$. $\square$

3.19 Definition. [7, 3.1.18] Let $\varphi : K \to L$ be simplicial. Then

$$|\varphi|\Big(\sum_i \lambda_i x_i\Big) := \sum_i \lambda_i \varphi(x_i) \text{ for } x_i \in K,\ \sum_i \lambda_i = 1 \text{ and } \lambda_i \geq 0$$

defines a continuous map from $|K| \to |L|$ (which is affine on every closed simplex $\bar\sigma$).

3.20 Remark. [7, 3.1.19] There are only finitely many simplicial mappings from K to L. For every simplicial map φ the map $|\varphi|$ is not dimension increasing.

3.21 Lemma. [7, 3.1.21]

1 *A map $\varphi : K \to L$ is an simplicial isomorphism (i.e. has an inverse, which is simplicial) iff it is simplicial and bijective.*

2 *For every simplicial isomorphism φ the mapping $|\varphi|$ is a homeomorphism.*

Proof. (1) We have to show that the inverse of a bijective simplicial mapping is simplicial.

Let y be a vertex of L and $\varphi(\sigma) = y$. We have to show that σ is a 0-simplex. Let $x_0, \dots, x_q$ be the vertices of σ. Then $\varphi(x_0), \dots, \varphi(x_q)$ generate the simplex $\varphi(\sigma) = y$ and hence have to be equal to the vertex y of y. Since φ is injective $q = 0$ and $\sigma = x_0$.

Now let $\tau = \varphi(\sigma)$ be a simplex in L with vertices $y_0, \dots, y_q$. Let $x_0, \dots, x_p$ be the vertices of σ. Since φ is simplicial and injective the images $\varphi(x_0), \dots, \varphi(x_p)$ are distinct and generate the simplex $\varphi(\sigma)$ hence are just the vertices of τ. Thus w.l.o.g. $p = q$ and $\varphi(x_j) = y_j$ for all j. □

Simplicial approximation

3.22 Definition. [7, 3.2.4] Let K and L be two simplicial complexes, $f : |K| \to |L|$ be continuous. Then a simplicial mapping $\varphi : K \to L$ is called SIMPLICIAL APPROXIMATION for f iff for all $x \in |K|$ we have $|\varphi|(x) \in \overline{\operatorname{carr}_L(f(x))}$, i.e. $f(x) \in \sigma \in L \Rightarrow |\varphi|(x) \in \bar\sigma$. This can be expressed shortly by $\forall \sigma \in L : |\varphi|(f^{-1}(\sigma)) \subseteq \bar\sigma$. In particular for every $x \in |K|$ there is a simplex $\sigma \in L$ with $f(x), |\varphi|(x) \in \bar\sigma$. Recall that $|\varphi|(\bar\sigma) = \overline{\varphi(\sigma)}$.

3.23 Lemma. [7, 3.2.5] *Let φ be a simplicial approximation of f, then $|\varphi| \sim f$.*

3.24 Example. [7, 3.2.6]

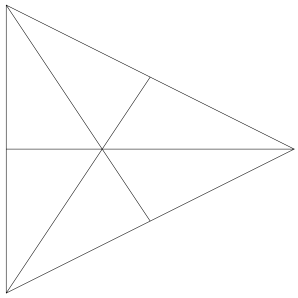
1 Let $X := |\dot\sigma^2|$. Then $X \cong S^1$. If $\varphi : K \to K$ is simplicial, then either φ is bijective or not surjective. So it has degree in $\{\pm 1, 0\}$. Every continuous map of absolute degree greater then 1 has no simplicial approximation.

- For $f : t \mapsto 4t(1-t)$ from $[0,1] \to [0,1]$ there is no simplicial approximation $\varphi : K \to K := \{\langle 0\rangle, \langle 1\rangle, \langle 0,1\rangle\}$, since any such must satisfy $\varphi(0) = \varphi(1) = \{0\}$, but $\varphi(1/2) \in \{1\}$.

In order to get simplicial approximations we have to refine the triangulation of $|K|$. This can be done with the following barycentric refinement.

3.25 Definition. [7, 3.2.1] The BARYCENTER $\hat\sigma$ of a q-simplex σ with vertices x_i is given by

$$\hat\sigma = \frac{1}{q+1} \sum_i x_i.$$



For every simplicial complex K the BARYCENTRIC REFINEMENT K' is given by all simplices having as vertices the barycenter of strictly increasing sequences of faces of a simplex in K, i.e.

$$K' := \{\langle \hat\sigma_0, \dots, \hat\sigma_q\rangle : \sigma_0 < \dots < \sigma_q \in K\}.$$

3.26 Theorem. [7, 3.2.2] *For every simplicial complex K the barycentric refinement K' is a simplicial complex of the same dimension d and the same underlying space but with $\max\{d(\sigma') : \sigma' \in K'\} \le \frac{d}{d+1}\max\{d(\sigma) : \sigma \in K\}$. Here $d(\sigma) := \max\{|x-y| : x, y \in \sigma\}$ denotes the diameter of σ.*

Proof. If $\sigma_0 < \dots < \sigma_q$, then the barycenter $\hat\sigma_0, \dots, \hat\sigma_q$ all lie in $\bar\sigma_q$ and are in general position: In fact, let $\sigma_i = \langle x_0, \dots, x_{n_i}\rangle$ and

$$x = \sum_{i=0}^{q} \lambda_i \hat\sigma_i = \sum_i \lambda_i \frac{1}{n_i+1}\sum_{j=0}^{n_i} x_j = \sum_j x_j \underbrace{\sum_{\substack{i\\ n_i \ge j}} \lambda_i \frac{1}{n_i+1}}_{=:\mu_j}.$$

Then $\mu_j \ge 0$ and

$$\sum_j \mu_j = \sum_j \sum_{\substack{i\\ n_i\ge j}} \lambda_i \frac{1}{n_i+1} = \sum_i \sum_{\substack{j\\ n_i \ge j}} \lambda_i \frac{1}{n_i+1} = \sum_i \lambda_i = 1.$$

Hence the μ_j are uniquely determined and so also the λ_i

We now show by induction on $q := \dim(\sigma)$ that for $\sigma \in K$ the set $\{\sigma' \in K' : \sigma' \subseteq \sigma\}$ is a disjoint partition of σ: For $(q = 0)$ this is obvious. For $(q > 0)$ and $x \in \sigma \setminus \{\hat\sigma\}$ the line through $\hat\sigma$ and x meets $\dot\sigma$ in some point y_x. By induction hypothesis $\exists \tau' \in K' : y_x \in \tau'$. Thus y_x is a convex combination of $\hat\tau_0, \dots, \hat\tau_j$ with $\tau_0 \le \dots \le \tau_j$. Hence x is a convex combination of $\hat\tau_0, \dots, \hat\tau_j, \hat\sigma$.

Now let x', y' be two vertices of some $\sigma' \in K'$, i.e. $x' = \frac{1}{r+1}(x_0 + \dots + x_r)$ and $y' = \frac{1}{s+1}(x_0 + \dots + x_s)$ with $r < s \le q$ for some simplex $\sigma = \langle x_0, \dots, x_q\rangle \in K$. Then

$$|x' - y'| \le \frac{1}{r+1}\sum_i |x_i - y'| \le \max\{|x_i - y'| : i\}$$

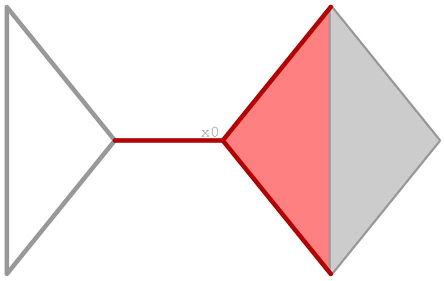
$$|x_i - y'| \le \frac{1}{s+1}\sum_{j\ne i} |x_i - x_j| \le \frac{s}{s+1} d(\sigma) \le \frac{d}{d+1} d(\sigma) \quad \square$$

3.27 Corollary. [7, 3.2.3] *For every simplicial complex K and every $\varepsilon > 0$ there is a iterated barycentric refinement $K^{(q)}$ with $d(\sigma) < \varepsilon$ for all $\sigma \in K^{(q)}$.*

Proof. $\left(\frac{d}{d+1}\right)^q \to 0$ for $q \to \infty$. $\square$

3.28 Definition. [7, 3.2.8] Let p be a vertex of K. Then the STAR of p in K is defined as

$$\mathrm{st}_K(p) := \bigcup_{\{p\} \le \sigma \in K} \sigma = \{x \in |K| : p \in \overline{\mathrm{carr}_K(x)}\}.$$



3.29 Lemma. [7, 3.2.9] *The family of stars form an open covering of $|K|$. For every open covering there is a refinement by the stars of some iterated barycentric refinement $K^{(q)}$ of K.*

Proof. Let $K_p := \{\sigma \in K : p \text{ is not vertex of } \sigma\}$. Then K_p is a subcomplex and hence $\mathrm{st_K(p)} = |K| \setminus |K_p|$ is open. If $\sigma \in K$ and p is a vertex of σ then obviously $\sigma \subseteq \mathrm{st}_K(p)$ and hence the stars form a covering.

By the Lebesgue-covering lemma, there is an $\varepsilon > 0$ such that every set of diameter less then ε is contained in some $U \in \mathcal{U}$. Choose by 3.26 a barycentric refinement $K^{(q)}$, such that $d(\sigma) < \frac{\varepsilon}{2}$ for all $\sigma \in K^{(q)}$. For every $x \in \mathrm{st}_{K^{(q)}}(p)$ we have $d(x,p) \le \max\{d(\sigma) : \sigma\} < \frac{\varepsilon}{2}$ hence $d(\mathrm{st}_{K^{(q)}}(p)) < \varepsilon$, and thus the stars form a refinement of $\mathcal{U}$. $\square$

3.30 Corollary. Simplicial Approximation. [7, 3.2.7] *For every continuous map $f : |K| \to |L|$ there a simplicial approximation $\varphi : K^{(q)} \to L$ of f for some iterated barycentric refinement $K^{(q)}$.*

Proof. Let q be chosen so large, that by 3.29 the stars of $K^{(q)}$ form a refinement of the covering $\{f^{-1}(\mathrm{st}_L(p)) : p \in L\}$. For sake of simplicity we write K instead of $K^{(q)}$. For every vertex $x \in K$ we may choose a vertex $\varphi(x) \in L$ with $f(\mathrm{st}_K(x)) \subseteq \mathrm{st}_L(\varphi(x))$. For $\sigma \in K$ with vertices $x_0, \dots, x_p$ define $\varphi(\sigma)$ to be the simplex generated by the $\varphi(x_i)$. We have to show that this simplex belongs to L. Let $x \in \sigma$ be any point in σ. Then $\sigma \subseteq \bigcap_i \mathrm{st}_K(x_i)$ and hence $f(x) \in f(\sigma) \subseteq f(\bigcap_i \mathrm{st}_K(x_i)) \subseteq \bigcap_i f(\mathrm{st}_K(x_i)) \subseteq \bigcap_i \mathrm{st}_L(\varphi(x_i))$. Thus $f(x) \in \mathrm{st}_L(\varphi(x_i))$, i.e. $\varphi(x_i) \le \tau := \mathrm{carr}_L(f(x)) \in L$, for all i. Hence $|\varphi|(x) \in \varphi(\sigma) = \langle \varphi(x_0), \dots, \varphi(x_p) \rangle \in L$ and φ is a simplicial approximation of f. $\square$

3.31 Corollary. [7, 3.2.10] *Let X and Y be polyhedra. Then $[X,Y]$ is countable.*

3.32 Remark. [7, 3.2.11]

We have a simplicial approximation $\chi : K' \to K$ of $\mathrm{id} : |K'| \to |K|$ by choosing for every vertex $\hat\sigma \in K'$ a vertex $\chi(\hat\sigma)$ of σ. Let $\hat\sigma_0, \dots, \hat\sigma_p$ be the vertices of some simplex $\sigma' \in K'$ with $\sigma_0 < \dots < \sigma_p$. Then the $\chi(\hat\sigma_j)$ are vertices of σ_p and hence they generate a face of $\sigma_p \in K$. Thus χ is a simplicial map. Let $x \in \sigma' \subseteq \sigma_p$. Then $|\chi|(x) \in \chi(\sigma') \subseteq \bar\sigma_p = \overline{\operatorname{carr}_K(x)}$, hence χ is a simplicial approximation of id.

Let σ be any q-simplex of K. Then there exists a unique simplex $\sigma' \subseteq \sigma$ which is mapped to σ and all other $\sigma' \subseteq \sigma$ are mapped to true faces of σ.

Proof. We use induction on q. For $q = 0$ this is obvious, since χ is the identity. If $q > 0$ and $x := \chi(\hat\sigma)$ let τ be the face of σ opposite to x. By induction hypothesis there is a unique $\tau' \subseteq \tau$ of K' which is mapped to τ. But then the simplex generated by τ' and $\hat\sigma$ is the unique simplex mapped to σ. In fact a simplex contained in σ with vertices $\hat\sigma_0, \dots, \hat\sigma_r$ that is mapped via χ to σ has to satisfy $\sigma_0 < \dots < \sigma_r \le \sigma$. Hence r is at most $\dim(\sigma)$, hence equal to $\dim(\sigma)$ and $\sigma_r = \sigma$. Since $\chi(\hat\sigma) = x$ we have that $\chi(\sigma_0), \dots, \chi(\sigma_{r-1})$ generate τ and thus τ' is the simplex with vertices $\chi(\sigma_0), \dots, \chi(\sigma_{r-1})$. □

Freeing by deformations

3.33 Proposition. [7, 3.3.2] *Let K be a simplicial complex and $n > \dim K$. Then every $f : |K| \to S^n$ is 0-homotopic. In particular, this is true for $K := \dot\sigma^{k+1}$ with $k < n$.*

Proof. By 3.30 we may take a simplicial approximation φ of $f : |K| \to |\dot\sigma^{n+1}|$ for some iterated barycentric subdivision. Then $|\varphi| : |K| \to |\dot\sigma^{n+1}|$ cannot be surjective ($n > \dim K$) and hence $f \sim |\varphi|$ is 0-homotopic since $\dot\sigma^{n+1} \setminus \{*\}$ is contractible. □

3.34 Theorem. Freeing of a point. [7, 3.3.3] *Let (K, L) be a simplicial pair and e^n be a n-cell with $n > \dim(K)$. Then every $f_0 : (|K|, |L|) \to (e^n, e^n \setminus \{0\})$ is homotopic relative $|L|$ to a mapping $f_1 : |K| \to e^n \setminus \{0\}$.*

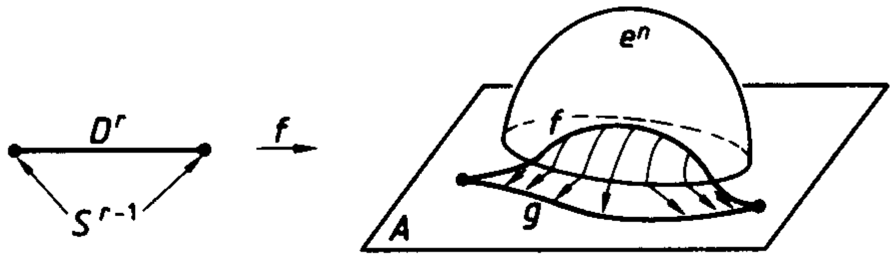
Proof. We first show this result for $(|K|, |L|) = (D^k, S^{k-1})$. By 2.40.6 we have $e^n \setminus \{0\} \sim S^{n-1}$. Hence $f_0|_{S^{k-1}} : S^{k-1} \to e^n \setminus \{0\}$ is 0-homotopic by 3.33. By 2.30 this homotopy gives an extension $f_1 : D^k = C(S^{k-1}) \to e^n \setminus \{0\}$. Consider a mapping $h : (D^k \times I)^\cdot \to e^n$ which is this extension on the top, and is f_0 on the bottom and on $S^{k-1} \times I$. Since e^n is contractible this mapping h is 0-homotopic by 2.6.6 and hence extends to $C((D^k \times I)^\cdot) \cong D^k \times I$ by 2.30. This extension is the desired homotopy.

For the general case we proceed by induction on the number of cells in $K \setminus L$. For $K = L$ the homotopy is constant f_0. So let $K \supset L$ and take $\sigma \in K \setminus L$ of maximal dimension. Then $M := K \setminus \{\sigma\}$ is a simplicial complex with $L \subseteq M$. Obviously $M \cup \bar\sigma = K$ and $M \cap \bar\sigma = \dot\sigma$. Consider the diagram

$$\begin{tikzcd}
 & {|K|} \arrow[d, dotted, "3\ h_t"] & \\
{|M|} \arrow[ur, hook] \arrow[r, dotted, "1"] & e^n & \bar\sigma \arrow[ul, hook'] \arrow[l, dotted, "2"'] \\
{|L|} \arrow[u, hook] & \dot\sigma \arrow[ul, hook'] \arrow[ur, hook] \arrow[u, dotted, "(1')"'] &
\end{tikzcd}$$

By induction hypothesis we have the required homotopy (1) relative $|L|$ on $|M|$. Since $(\bar\sigma,\dot\sigma)$ has HEP by 2.33.1, we may extend its restriction (1') to $\dot\sigma$ to a homotopy (2) rel. $\dot\sigma$. The union of these two homotopies gives a homotopy h_t indicated by arrow (3) which satisfies $h_0 = f$ and $h_1(|M|) \subseteq e^n \setminus \{0\}$. By the special case treated above, there is a homotopy g_t relative $\dot\sigma$ with $g_0 = h_1 : (\bar\sigma,\dot\sigma) \to (e^n, e^n \setminus \{0\})$ and $g_1(\bar\sigma) \subseteq e^n \setminus \{0\}$. Let $f_1 := h_1|_{|M|} \cup g_1|\bar\sigma$. Then $f_1 \sim f_0$ relative $|L|$ and $f_1(|K|) \subseteq e^n \setminus \{0\}$. □

3.35 Theorem. Freeing of a cell. [7, 3.3.4] *Let X be obtained from gluing an n-cell e^n to a subspace A and $k < n$. Then every $f : (D^k, S^{k-1}) \to (X, A)$ is homotopic relative S^{k-1} to a mapping $f_1 : D^k \to A$.*



Proof. We consider a triangulation K of D^k such that S^{k-1} corresponds to a subcomplex L. Then we consider the subcomplexes $K_1 := \{\sigma \in K : f(\bar\sigma) \subseteq e^n\} = \{\sigma \in K : \bar\sigma \subseteq f^{-1}(e^n)\}$ and $K_0 := \{\sigma \in K : \overline{\sigma} \subseteq f^{-1}(X \setminus \{0\})\}$. By passing to a appropriate iteration (again denoted K) of barycentric subdivisions, we may assume that $K = K_0 \cup K_1$ by 3.29. Now consider the diagram

$$\begin{tikzcd}
A \arrow[rr, hook] & & X & & & \\
& X\setminus\{0\} \arrow[ul, "r"'] \arrow[ur, hook] & & & e^n \arrow[ull, hook'] & \\
A \arrow[uu, "\mathrm{id}"] \arrow[ur, hook] & & D^k \arrow[uu, "f"'] \arrow[ul, dotted, "2"'] \arrow[d, equal] & e^n\setminus\{0\} \arrow[ull, hook'] \arrow[ur, hook] & & \{0\} \arrow[ul, hook'] \\
& & {|K|} & & & \\
S^{k-1} \arrow[uu, "f"] \arrow[uurr, hook] \arrow[d, equal] & {|K_0|} \arrow[uuu, "f"] \arrow[ur, hook] & & & {|K_1|} \arrow[ull, hook'] \arrow[uuu, "f"'] \arrow[uul, dotted, "1"'] & \\
{|L|} \arrow[ur, hook] & & & {|K_0\cap K_1|} \arrow[uuu, "f"] \arrow[ull, hook'] \arrow[ur, hook] & &
\end{tikzcd}$$

By 3.34 there exists a mapping (1) homotopic to $f|_{|K_1|}$ relative $|K_1 \cap K_0|$. Gluing the homotopy with the $f|_{|K_0|}$ gives a homotopy relative $|K_0|$ to a mapping (2). Composing with the retraction r (homotopic to id relative A) from 2.42 gives the desired mapping $f_1 : D^k \to A$ homotopic to f relative $|L|$. Remark that the triangle on top, as those above 1 and 2 commute only up to homotopy. □

4 CW-Spaces

Basics

4.1 Definition. [7, 4.1.3] A CW-complex is a partition $\mathcal{X}$ of a Hausdorff space into cells e, such that the following properties hold:

(C1) For every n-cell $e \in \mathcal{X}$ there exists a so-called characteristic map $\chi^e : D^n \to X$, which restricts to a homeomorphism from $(D^n)^o$ onto e and which maps S^{n-1} into the $n-1$-skeleton X^{n-1} of X, which is defined to be the union of all cells of dimension less than n in $\mathcal{X}$.

(C2) The closure $\bar{e}$ of every cell meets only finitely many cells.

(W) X carries the final topology with respect to $\bar{e}$ for all cells $e \in \mathcal{X}$.

A CW-space is a Hausdorff-space X, which admits a CW-complex $\mathcal{X}$ (which is called CW-decomposition of X).

Remark that if $\mathcal{X}$ is finite (X is then called finite CW-complex), then the conditions (C) and (W) are automatically satisfied.

If $X = X^n \neq X^{n-1}$ then the CW-complex is said to be of dimension n. If $X \neq X^n$ for all n, then it is said to be of infinite dimension.

Remark that, since the image $\chi(D^n)$ of the n-ball under a characteristic map is compact, it coincides with $\bar{e}$ and $\chi : D^n \to \bar{e}$ is a quotient mapping. So $\dot{e} := \bar{e} \setminus e = \chi(D^n)\setminus\chi((D^n)^o) \subseteq \chi(D^n\setminus(D^n)^o) = \chi(S^{n-1})$ and conversely $\chi(S^{n-1}) \subset \chi(D^n) \subseteq \bar{e}$ and $\chi(S^{n-1}) \subseteq X^{n-1} \subseteq X\setminus e$, thus $\dot{e} = \chi(S^{n-1})$ and χ is a relative homeomorphism $(D^n, S^{n-1}) \to (\bar{e}, \dot{e})$.

$$\begin{tikzcd}
(D^n)^o \arrow[r, hook] \arrow[d, two heads, "\cong"] & D^n \arrow[d, two heads, "\chi_e"] & S^{n-1} \arrow[l, hook'] \arrow[d, two heads] \\
e \arrow[r, hook] & \bar{e} & \dot{e} \arrow[l, hook']
\end{tikzcd}$$

4.2 Example. [7, 4.1.4] For every simplicial complex K the underlying space $|K|$ is a finite CW-complex, the cells being the simplices of K and the characteristic maps the inclusions $\bar{e} \subseteq |K|$.

The sphere S^n is a CW-complex with one 0-cell e^0 and one n-cell e^n, in particular **the boundary $\dot{e} = \bar{e} \setminus e$ of an n-cell, needn't be a sphere**.

The one point union of spheres is a CW-space with one 0-cell and for each sphere a cell of the same dimension.

$S^1 \vee S^2$ can be made in a different way into a CW-complex by taking a point $e^0 \in S^1$ different from the base point. Then $S^1 = e^0 \cup e^1$ and $S^1 \vee S^2 = e^0 \cup e^1 \cup e^2$. But **the boundary $\dot{e}^2$ of the two-cell is not a union of cells**.

The compact surfaces of genus g are all CW-complexes with one 0-cell and one 2-cell and $2g$ 1-cells (in the orientable case) and g 1-cells (in the non-orientable case), see 1.101.

The projective spaces P^n are CW-complexes with one cell of each dimension from 0 to n, see 1.98.

4.3 Definition. [7, 4.1.5] For a subset $\mathcal{Y}$ of a CW-decomposition $\mathcal{X}$ of a space X the underlying space $Y := \bigcup\{e : e \in \mathcal{Y}\}$ is called CW-SUBSPACE and $\mathcal{Y}$ is called CW-SUBCOMPLEX, iff $\mathcal{Y}$ is a CW-decomposition of Y with the trace topology.

In this situation (X, Y) is called CW-PAIR.

Let us first characterize finite CW-subcomplexes:

4.4 Lemma. *Let $\mathcal{Y}$ be a finite subset of a CW-decomposition $\mathcal{X}$ of a space X. Then $\mathcal{Y}$ forms a CW-subcomplex iff $Y := \bigcup\{e : e \in \mathcal{Y}\}$ is closed. Cf. 3.14.*

Proof. ($\Rightarrow$) If Y is a CW-subcomplex, then for every cell $e \in \mathcal{Y}$, there is a characteristic map $\chi : D^n \twoheadrightarrow \bar{e}^Y$. Hence $\bar{e}^Y$ is compact and thus coincides with the closure of e in X, so $Y = \bigcup\{\bar{e} : e \in \mathcal{Y}\}$ is closed.

($\Leftarrow$) Since Y is closed the characteristic maps for $e \in \mathcal{Y} \subseteq \mathcal{X}$ are also characteristic maps with respect to $\mathcal{Y}$. The other properties are obvious. □

4.5 Lemma. [7, 4.1.9] *Every compact subset of a CW-complex is contained in some finite subcomplex. In particular a CW-complex is compact iff it is finite.*

Proof. Let X be a CW-complex. We first show by induction on the dimension of X that the closure $\bar{e}$ of every cell is contained in a finite subcomplex. Assume this is true for all cells of dimension less than n and let e be an n-cell. By (C2) the boundary $\dot{e}$ meets only finitely many cells, each of dimension less than n. By induction hypotheses each of these cells is contained in some finite subcomplex X_i. Then union of these complexes is again a complex, by 4.4. If we add e itself to this complex, we get the desired finite complex.

Let now K be compact. For every $e \in \mathcal{X}$ with $e \cap K \neq \emptyset$ choose a point x_e in the intersection. Every subset $A \subseteq K_0 := \{x_e : e \cap K \neq \emptyset\}$ is closed, since it meets any $\bar{e}$ only in finitely many points. Hence K_0 is a discrete compact subset, and hence finite, i.e. K meets only finitely many cells. Since every $\bar{e}$ is contained in a finite subcomplex, we have that K is contained in the finite union of these subcomplexes.

The last statement of the lemma is now obvious. □

4.6 Corollary. *Every CW-complex carries the final topology with respect to its finite subcomplexes and also with respect to its skeletons.*

Proof. Since the closure $\bar{e}$ of every cell e is contained in a finite subcomplex and every finite subcomplex is contained in some skeleton X^n, these families are confinal to $\{\bar{e} : e \in \mathcal{X}\}$ and each of its spaces carries as subspace of X the final topology with respect to the inclusions of $\bar{e}$ by W. Hence they induce the same topology. (Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two families of mappings into a space X, and assume $\mathcal{F}_2$ is confinal to $\mathcal{F}_1$, i.e. for every $f_1 \in F_1$ there is some $f_2 \in \mathcal{F}_2$ and a map h such that $f_1 = f_2 \circ h$. Let X_j denote the space X with the final topology induced by $\mathcal{F}_j$. Then the identity from $X_1 \to X_2$ is continuous, since for every $f_1 \in \mathcal{F}_1$ we can write $\mathrm{id} \circ f_1 = f_2 \circ h$) □

Now we are able to extend 4.3 to infinite subcomplexes.

4.7 Proposition. *Let $\mathcal{X}$ a CW-decomposition of X and let $\mathcal{Y}$ be a subset of $\mathcal{X}$. Then the following statements are equivalent:*

1 *$\mathcal{Y}$ is a CW-decomposition of Y with the trace topology;*

2 *$Y := \bigcup\{e : e \in \mathcal{Y}\}$ is closed;*

3 *For every cell $e \in \mathcal{Y}$ we have $\bar{e} \subseteq Y$.*

Proof. (2⇒ 3) is obvious.
(1⇒ 3) follows, since the closure $\bar{e}^Y$ in Y is compact and hence equals $\bar{e}^X$.

For the converse directions we show that from (3) we obtain: If $A \subseteq Y$ has closed trace on $\bar{e} = \bar{e}^X = \bar{e}^Y$ (see 4.4) for each $e \in \mathcal{Y}$, then A is closed in X.
By 4.6 we have to show that the trace on every finite CW-subcomplex $\mathcal{X}_0 \subseteq \mathcal{X}$ is closed. Since there are only finitely many cells e_i in $\mathcal{X}_0 \cap \mathcal{Y}$ and for these $\bar{e}_i \subseteq X_0 \cap Y$ by (3) and 4.4, we get

$$X_0 \cap A = (X_0 \cap Y) \cap A = \Big(\bigcup \bar{e}_i\Big) \cap A = \bigcup(\bar{e}_i \cap A),$$

which is closed.

(3⇒ 2) by taking $A = Y$ in the previous claim.

(3⇒ 1) the previous claim shows the condition (W) for $\mathcal{Y}$. The other conditions for a CW-complex are obvious since $\bar{e}^X = \bar{e}^Y$. □

4.8 Corollary. [7, 4.1.6] *Intersections and unions of CW-complexes are CW-complexes. Connected components and topological disjoint unions of CW-complexes are CW-complexes. If $\mathcal{E} \subseteq \mathcal{X}$ is family of n-cells, then $X^{n-1} \cup \bigcup \mathcal{E}$ is a CW-complex. Each n-cell e is open in X^n.*

Proof. For intersections this follows from (1 ⇔ 2) in 4.7. For unions this follows from (1 ⇔ 3) in 4.7. That $X^{n-1} \cup \bigcup \mathcal{E}$ is a CW-complex follows also from (1 ⇔ 3) in 4.7. Since $X^n \setminus e = X^{n-1} \cup \bigcup\{e_1 \neq e : e_1 \text{ an } n\text{-cell in } X^n\}$ is a CW-complex, it is closed and hence e is open in X^n.

The statement on components and topological sums follows, since $\bar{e}$ is connected and by 4.7 (1⇔3). □

Constructions of CW-spaces

4.9 Proposition. [7, 4.2.9] *Let X and Y be two CW-complexes. Then $X \times Y$ with cells $e \times f$ for $e \in \mathcal{X}$ and $f \in \mathcal{Y}$ satisfies all properties of a CW-complex, with the possible exception of W. If X or Y is in addition locally compact, then $X \times Y$ is a CW-complex.*

Proof. Take the product of the characteristic maps in order to obtain a characteristic map for the product cell.

In order to get the property (W) we have to show that the map $\bigsqcup_{e,f} \bar{e} \times \bar{f} \to X \times Y$ is a quotient map. Since it can be rewritten as

$$\bigsqcup_e \bar{e} \times \bigsqcup_f \bar{f} \to \bigsqcup_e \bar{e} \times Y \to X \times Y$$

this follows from 1.33 using compactness of $\bar{e}$ and locally compactness of Y. □

4.10 Proposition. *Let (X, A) be a CW-pair. Then $A \cup X^n$ is obtained from $A \cup X^{n-1}$ by gluing all n-cells in $X^n \setminus A$ via the characteristic mappings.*

Proof. Let $\mathcal{E}$ be the set of all n-cells of $X \setminus A$ and let characteristic mappings $\chi^e : D^n \to \bar{e}$ for every $e \in \mathcal{E}$ be chosen. Let $\chi := \bigsqcup_{e\in\mathcal{E}} \chi_e : \bigsqcup_{e\in\mathcal{E}} D^n \to \bigcup_{e\in\mathcal{E}} \bar{e} \subseteq X^n$ and $f := \chi|_{\bigsqcup_e S^{n-1}}$. We have to show that

$$\begin{tikzcd}
 & & S^{n-1} \arrow[r, hook] \arrow[dll] & D^n \arrow[dll, "\chi_e"'] \arrow[ddll] \arrow[ddd, "g^e"] \\
\bigsqcup_e S^{n-1} \arrow[r, hook] \arrow[d, "f"'] & \bigsqcup_e D^n \arrow[d, "\chi"'] \arrow[ddrr] & & \\
A \cup X^{n-1} \arrow[r, hook] \arrow[drrr, "g^{n-1}"'] & A \cup X^n \arrow[drr, dotted, "g", "!"'] & & \\
 & & & Z
\end{tikzcd}$$

is a push-out. So let $g^{n-1} : A \cup X^{n-1} \to Z$ and $g^e : D^n \to Z$ be given, such that $g^{n-1} \circ \chi^e|_{S^{n-1}} = g^e|_{S^{n-1}}$. Then $g : A \cup X^n \to Z$ given by $g|_{A\cup X^{n-1}} = g^{n-1}$ and $g|_e = g^e \circ (\chi^e|_{(D^n)^o})^{-1}$ is the unique mapping making everything commutative. It is continuous, since on $\bar{e}$ it equals g^{n-1} if $e \subseteq A \cup X^{n-1}$ and composed with the quotient-mapping $\chi^e : D^n \to \bar{e}$ it equals g^e for the remaining e. □

Now we give an inductive description of CW-spaces.

4.11 Theorem. [7, 4.2.2] *A space X is a CW-complex iff there are spaces X^n, with X^0 discrete, X^n is formed from X^{n-1} by gluing n-cells and X is the limit of the X^n with respect to the natural inclusions $X^n \hookrightarrow X^{n+1}$.*

Proof. ($\Rightarrow$) We take X^n to be the n-skeleton. Then X carries the final topology with respect to the closed subspaces X^n and X^0 is discrete by 4.6. Taking $A := \emptyset$ in 4.10 we get that X^n can be obtained from X^{n-1} by gluing all the n-cells via their corresponding characteristic maps restricted to the boundary spheres.

($\Leftarrow$) We first show by induction that X^n is a CW-complex, with $n-1$-skeleton X^{n-1} and the cells which have been glued to X^{n-1} as n-cells.
For X^0 being a discrete space this is obvious. Since X^n is obtained from X^{n-1} by gluing n-cells

$$\begin{tikzcd}
\bigsqcup_e S^{n-1} \arrow[r, hook] \arrow[d, "f"'] & \bigsqcup_e D^n \arrow[d, "p"'] \\
X^{n-1} \arrow[r, hook] & X^n
\end{tikzcd}$$

we have that X^n is Hausdorff by 1.92 and is the disjoint union of the closed subspace X^{n-1}, which is a CW-complex by induction hypothesis, and the homeomorphic image $\bigcup_e e$ of $\bigsqcup_e D^n \setminus \bigsqcup_e S^{n-1} = \bigsqcup_e (D^n)^o$. As characteristic mappings for the n-cells e we may use $p|_{D^n}$, since it induces a homeomorphism $(D^n)^o \to e$ and it maps S^{n-1} to $f(S^{n-1}) \subseteq X^{n-1}$, which is compact and hence contained in a finite subcomplex of X^{n-1}. The remaining condition (W) follows, since X^n carries by construction the final topology with respect to X^{n-1} and $p : \bigsqcup D^n \to X^n$, and $\bigsqcup D^n$ carries the final topology with respect to the inclusion of the summands $D^n \subset \bigsqcup_e D^n$.

The inductive limit $X := \operatorname{inj\,lim}_n X^n$ now obviously satisfies all axioms of a CW-complex only Hausdorffness is to be checked. So let x, y be different points in X. We may assume that they lie in some X^n. So we find open disjoint neighborhoods

U^n and V^n in X^n. We construct open disjoint neighborhoods U^k and V^k in X^k with $k \geq n$ inductively. In fact, take $U^k := U^{k-1} \cup p\Big((p|_{\bigsqcup D^n \setminus \{0\}})^{-1}(U^{k-1})\Big)$. Then U^k is the image of the open and saturated set $U^{k-1} \sqcup \Big(p^{-1}(U^{k-1}) \setminus \bigsqcup\{0\}\Big)$ and hence open, and $U^k \cap X^{k-1} = U^{k-1}$. Proceeding the same way with V^k gives the required disjoint open sets $U := \bigcup U^k$ and $V := \bigcup V^k$. □

Example. In general gluing a CW-pair to a CW-space does not give a CW-space. Consider for example a surjective map $f : S^1 \to S^2$. Then the boundary $\dot{e} = S^2$ of $e := (D^2)^o$ is not contained in any 1-dimensional complex.

So we define

4.12 Definition. [7, 4.2.4] A continuous map $f : X \to Y$ between CW-complexes is called CELLULAR iff it maps X^n into Y^n for all n.

4.13 Lemma. *Let $f : X \supseteq A \to Y$ be given and let $Y' \subseteq Y$ and $X' \subseteq X$ be two closed subspaces, such that $f(A \cap X') \subseteq Y'$. Then the canonical mapping $Y' \cup_{f'} X' \to Y \cup_f X$ is a closed embedding, where $f' := f|A'$ with $A' := A \cap X'$.*

Proof. Consider the commutative diagram:

$$\begin{tikzcd}
A' \arrow[rr, hook] \arrow[dr, hook] \arrow[dd, "f'"'] & & X' \arrow[dr, hook] \arrow[dd, "p'"] & \\
& A \arrow[rr, hook] \arrow[dd, "f"] & & X \arrow[dd, "p"] \\
Y' \arrow[rr, hook] \arrow[dr, hook] & & Y' \cup_{f'} X' \arrow[dr, dashed] & \\
& Y \arrow[rr, hook] & & Y \cup_f X
\end{tikzcd}$$

The dashed arrow ι exists by the push-out property of the upper left square.

Since $Y' \cup_{f'} X' = Y' \sqcup (X' \setminus A')$ as sets, we get that ι is the inclusion $Y' \sqcup (X' \setminus A') \subseteq Y \sqcup (X \setminus X \cap A) = Y \sqcup (X \setminus A)$ and hence injective. Now let $B \subseteq Y' \cup_{f'} X'$ be closed, i.e. $B = B_1 \sqcup B_2$ with $B_1 \subseteq Y'$ closed and $B_2 \subseteq X' \setminus A'$ such that $p^{-1}(B) = B_2 \cup (f')^{-1}(B_1)$ is closed in X'. In order to show that $\iota(B) = B_1 \sqcup B_2 \subseteq Y' \cup (X' \setminus A') \subseteq Y \cup (X \setminus A)$ is closed we only have to show that $B_2 \cup f^{-1}(B_1)$ is closed in X, which follows from

$$B_2 \cup f^{-1}(B_1) = B_2 \cup ((f')^{-1}(B_1) \cup f^{-1}(B_1)) = (B_2 \cup (f')^{-1}(B_1)) \cup f^{-1}(B_1),$$

since $B_2 \cup (f')^{-1}(B_1) \subseteq X' \subseteq X$ is closed and $f^{-1}(B_1) \subseteq A \subseteq X$ is closed. □

4.14 Theorem. [7, 4.2.5] *Let (X, A) be a CW-pair and $f : A \to Y$ a cellular mapping into a CW-complex Y. Then $(Y \cup_f X, Y)$ is a CW-pair with the cells of Y and of $X \setminus A$ as cells.*

Proof. We consider the spaces $Z^n := Y^n \cup_{f_n} X^n$, where $f_n := f|_{A^n}$. Note that $A^n = A \cap X^n$. By 4.13 the Z^n form an increasing sequence of closed subspaces of the Hausdorff space $Z := Y \cup_f X$. Obviously Z^0 is discrete. So by 4.11 it remains to show that Z^n can be obtained from Z^{n-1} by gluing all n-cells of Y^n and of $X^n \setminus A^n$. For this we consider the following commutative diagram:

$$\begin{tikzcd}
 & \sqcup S^{n-1} \arrow[r, hook] \arrow[d] & \sqcup D^n \arrow[d] \arrow[dddr] & & \\
 & Y^{n-1} \arrow[r, hook] \arrow[dr, hook] & Y^n \arrow[drr, hook] \arrow[ddr, dotted, "1"] & & \\
A^{n-1} \arrow[ur, "f^{n-1}"] \arrow[r, hook] \arrow[ddr, hook] & A^n \arrow[ur, "f^n"] \arrow[ddr, hook] & Z^{n-1} \arrow[rr, hook] \arrow[dr] & & Z^n \arrow[dl, dotted, "4"] \\
 & & & W & \\
 & X^{n-1} \arrow[uur, hook] \arrow[r, hook] & A^n \cup X^{n-1} \arrow[ur, dotted, "2"] \arrow[rr, hook] & & X^n \arrow[uu] \arrow[ul, dotted, "3"'] \\
 & & \sqcup S^{n-1} \arrow[ul] \arrow[u] \arrow[rr, hook] & & \sqcup D^n \arrow[u] \arrow[uul]
\end{tikzcd}$$

The following spaces are push-outs of the arrows leading into them: Y^n, X^n, Z^{n-1}, Z^n and $A^n \cup X^{n-1}$. We have to show that Z^n is the push out of the inclusion $Z^{n-1} \to Z^n$ and the union of the two mappings $\sqcup D^n \to Y^n \to Z^n$ and $\sqcup D^n \to X^n \to Z^n$. So let mappings on all the D^n and on Z^{n-1} into a space W be given whose composites with the arrows from S^{n-1} into these spaces are the same. Then (1), (2), (3), and (4) exist uniquely by the push-out property of the corresponding domains Y^n, $A^n \cup X^{n-1}$, X^n and Z^n shown in 4.10. The map (4) is then the required unique mapping from $Z^n \to W$. □

4.15 Corollary. [7, 4.2.6] *Let (X, A) be a CW-pair with $A \neq \emptyset$. Then X/A is a CW-space with A as one 0-cell and the image of all cells in $X \setminus A$.*

Proof. $X/A = \{*\} \cup_f X$, where $f : A \to \{*\}$ is constant by 1.47, now apply 4.14. □

4.16 Corollary. [7, 4.2.8] *Let X be a CW-complex. Then X^n/X^{n-1} is a join of spheres of dimension n, for each n-cell one.*

Proof. By 4.15 X^n/X^{n-1} is a CW-space consisting of one 0-cell and all the n-cells of X. The characteristic mappings for the n-cells into the 0-cell have to be constant and hence $X^n/X^{n-1} \cong \bigvee_e S^n$ by 1.103.1. □

4.17 Corollary. [7, 4.2.7] *Let X_i be CW-spaces with base-point $x_i \in X_i^0$. Then the join $\bigvee_i X_i$ is a CW-space.*

Proof. $\bigvee_i X_i = (\sqcup_i X_i)/\{x_i : i\}$ is a CW-space by 4.8 and 4.14. □

Homotopy properties

4.18 Theorem. [7, 4.3.1] *Every CW-pair (X, A) has the HEP.*

Proof. The extension of the homotopy is constructed by induction on n for $A \cup X^n$. The induction step follows, since $(\bigsqcup D^n, \bigsqcup S^{n-1})$ has the HEP by 2.33.1 and gluing does not change it by 2.33.2. The union of these extensions is then the required homotopy because of property (W). □

4.19 Theorem. [7, 4.3.2] *For every CW-subcomplex A of a CW-space X we can find a open neighborhood $U(A)$ such that A is a SDR in $U(A)$. These neighborhoods can be chosen coherently, i.e. $U(A \cap B) = U(A) \cap U(B)$.*

Proof. For every n-cell e we have a radial deformation $h^e_t : X^{n-1} \cup (e \setminus \{0_e\}) \to X^{n-1} \cup (e \setminus 0_e)$ relative X^{n-1} with $h^e_0 = \mathrm{id}$ and $h^e_1 =: r^e : X^{n-1} \cup (e \setminus 0_e) \to X^{n-1}$ is a retraction, such that $r^e \circ h^e_t := r^e$.

Now we construct inductively open sets $U^n(A)$ in $A \cup X^n$ by $U^{-1}(A) = A$ and $U^n(A) := U^{n-1}(A) \cup \bigcup_e (r^e)^{-1}(U^{n-1}(A))$, which satisfy $U^n(A) \cap (A \cup X^{n-1}) = U^{n-1}(A)$. The h^e_t can be joined to a deformation $U^n(A) \to U^n(A)$ relative $U^{n-1}(A)$ with $h^n_0 = \mathrm{id}$ and $h^n_1 =: r^n$ a retraction of $U^n(A) \to U^{n-1}(A)$.

By induction on n we construct homotopies $H^n_t : U^n \to U^n$, by

$$H^n_t = \begin{cases} \mathrm{id} & \text{for } t \le \frac{1}{n+1}, \\ h^n_s \text{for } \frac{1}{n+1} \le t \le \frac{1}{n} \text{ where } s := n(t(n+1)-1), \\ H^{n-1}_t \circ r^n \text{for } t \ge \frac{1}{n}. \end{cases}$$

Then $H^n_t|_{U^{n-1}} = H^{n-1}_t$, since $r_n|_{U^{n-1}} = \mathrm{id}$.

The union $U(A) := \bigcup U^n(A)$ is then the required open (by 4.6) neighborhood of A in X and the union of the H^n_t is the required deformation relative A. □

4.20 Corollary. [7, 4.3.3] *Every point x in a CW-complex X has a open neighborhood, of which it is a SDR.*

Proof. Let first e be an n-cell. Let $A := X^n$. By restricting the homotopy H_t from 4.19 to $r^{-1}(e)$, where $r : U(A) \to A$ denotes the retraction, we obtain that e is the SDR of a neighborhood. Since every point in a cell e is a SDR of the cell, we obtain the required result by transitivity. □

4.21 Theorem. Cellular Approximation. [7, 4.3.4] *For every continuous mapping $f_0 : X \to Y$ between CW-complexes there exists a homotopic cellular mapping. If $f_0|_A$ is cellular for some CW-subspace A, then the homotopy can be chosen to be relative A.*

Proof. Again we recursively extend the constant homotopy on A to a homotopy $h^n_t : A \cup X^n \to Y$ with h^n_1 being cellular.

For the induction step we use for each n-cell $e \in X \setminus A$ a characteristic mapping $\chi^e : D^n \to \bar{e}$. By induction hypothesis we get a mapping $\varphi_0 : (D^n \times 0) \cup (S^{n-1} \times I) \to Y$ given by $f_0 \circ \chi$ on the bottom and $h^{n-1}_t \circ \chi$ on the mantle $\dot{D}^n \times I$. Since this domain is a DR in $D^n \times I$ by 2.33.1 we can extend it to a mapping φ_0 on $D^n \times I$. The image of $D^n \times \{1\}$ is compact and hence contained in a finite CW-complex. Let $e^{n_1}, \dots, e^{n_r}$ be the cells of dimension greater than n of this complex. Then

$\varphi_0 : (D^n \times \{1\}, S^{n-1} \times \{1\}) \to (Y^n \cup e^{n_1} \cup \dots \cup e^{n_r}, Y^{n-1})$ is well defined. Applying 3.35 now r-times we can deform $\varphi_0|_{D^n \times \{1\}}$ successively relative $S^{n-1} \times \{1\}$ so, that its image avoids $e^{n_r}, \dots, e^{n_1}$. Let φ_t be the corresponding homotopy. We can extend φ_1 via φ_0 to a continuous mapping on the boundary of $D^n \times I$, which is homotopic to $\varphi_0|_{(D^n \times I)^\cdot}$ via $\varphi_t \cup h_t^{n-1} \circ \chi$. Since $(D^n \times I, (D^n \times I)^\cdot)$ is a CW-pair and hence has the HEP by 4.18 and since φ_0 extends to $D^n \times I$ so does φ_t, see 2.34.1. Now define $h_t^n|_e := \varphi_t \circ \chi_e^{-1}$. This is well-defined, since χ_e is a quotient mapping and $\varphi_t \circ \chi^{-1}|_{\dot e} = h_t^{n-1} \circ \chi \circ \chi^{-1}|_{\dot e} = h_t^{n-1}|_{\dot e}$. The union of the $h_t^n|_e$ gives the required h_t^n. $\square$

4.22 Corollary. [7, 4.3.5] *Let $f_0, f_1 : X \to Y$ be homotopic and cellular. Then there exists a homotopy $H : X \times I \to Y$ such that $H_t(X^n) \subseteq Y^{n+1}$.*

Remark that the inclusions of the endpoints in I are homotopic and cellular, but every homotopy has to map that point into the 1-skeleton.

Proof. Consider the CW-pair $(X \times I, X \times \dot I)$ and the given homotopy $f : X \times I \to Y$. Since by assumption its boundary value $f|_{X \times \dot I}$ is cellular,by 4.21 we can find another mapping $H : X \times I \to Y$, which is cellular and homotopic to f relative $X \times \dot I$. Thus H is the required homotopy, since for every n-cell e^n of X the image $H(e \times \{t\})$ is contained in $H(e^n \times e^1) \subseteq Y^{n+1}$. $\square$

5 Fundamental Group

Basic properties of the fundamental group

5.1 Definition. [7, 5.1.1] A path is a continuous mapping $u : I \to X$. The CONCATENATION $u_0 \cdot u_1$ of two paths u_0 and u_1 is defined by

$$(u_0 \cdot u_1)(t) := \begin{cases} u_0(2t) & \text{for } t \leq \frac{1}{2} \\ u_1(2t-1) & \text{for } t \geq \frac{1}{2} \end{cases}.$$

It is continuous provided $u_0(1) = u_1(0)$. The INVERSE PATH $u^{-1} : I \to X$ is given by $u^{-1}(t) := u(1-t)$.

Note that concatenation is not associative and the constant path is not a neutral element. The corresponding identities hold only up to reparametrizations.

5.2 Lemma. Reparametrization. [7, 5.1.5] *Let $u : I \to X$ be a path and $f : I \to I$ be the identity on $\dot{I}$. Then $u \sim u \circ f$ rel. $\dot{I}$.*

Proof. A homotopy is given by $h(t,s) := u(ts + (1-t)f(s))$, see 2.4. □

5.3 Lemma. [7, 5.1.6]

1 *Let u, v and w be paths with $u(1) = v(0)$ and $v(1) = w(0)$, then $(u \cdot v) \cdot w \sim u \cdot (v \cdot w)$ rel. $\dot{I}$.*

2 *Let u be path with $x := u(0)$, $y := u(1)$ then $\mathrm{const}_x \cdot u \sim u \sim u \cdot \mathrm{const}_y$ rel. $\dot{I}$.*

3 *Let u be a path with $x := u(0)$ and $y := u(1)$. Then $u \cdot u^{-1} \sim \mathrm{const}_x$ and $u^{-1} \cdot u \sim \mathrm{const}_y$ rel. $\dot{I}$.*

Proof. In (1) and (2) we only have to reparametrize. In (3) we consider the homotopy, which has constant value on each circle with center $(\frac{1}{2}, 0)$. □

5.4 Definition. [7, 5.1.7] Let (X, x_0) be a pointed space. Then the FUNDAMENTAL GROUP (or FIRST HOMOTOPY GROUP) is defined by

$$\pi_1(X, x_0) := [(I, \dot{I}), (X, x_0)] \cong [(S^1, \{1\}), (X, \{x_0\})],$$

where multiplication is given by $[u] \cdot [w] := [u \cdot w]$, the neutral element is $1_{x_0} := [\mathrm{const}_{x_0}]$ and the inverse to $[u]$ is $[u^{-1}]$. Both are well-defined by 5.3.

5.5 Lemma. [7, 5.1.8] *Let $u : I \to X$ be a path from x_0 to x_1. Then $\mathrm{conj}_{[u]} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ is a group isomorphism, where $\mathrm{conj}_{[u]} : [v] \mapsto [u^{-1}][v][u] := [u^{-1} \cdot v \cdot u]$.*

5.6 Lemma. [7, 5.1.10] *Let $f : (I^2)^{\cdot} \to X$ be continuous, and let $u_j(s) := f(s,j)$ and $v_j(t) := f(j,t)$ be its values on the 4 sides. Let $h : I^2 \to I^2$ map $(j,t) \mapsto (j,j)$, $(t,0) \mapsto (2t,0)$ for $t \leq \frac{1}{2}$, $(t,0) \mapsto (1, 2t-1)$ for $t \geq \frac{1}{2}$, $(t,1) \mapsto (0,2t)$ for $t \leq \frac{1}{2}$ and $(t,1) \mapsto (2t-1, 1)$ for $t \geq \frac{1}{2}$ and a homeomorphism on the interior, e.g.*

$$h(t,s) := \begin{cases} (1-2t)(0,0) + 2t\Big(s(0,1) + (1-s)(1,0)\Big) & \text{for } t \leq 1/2 \\ (2-2t)\Big(s(0,1) + (1-s)(1,0)\Big) + (2t-1)(1,1) & \text{for } t \geq 1/2 \end{cases}$$

Then the following statements are equivalent

1 *There exists a continuous extension of f to I^2;*

2 *f is 0-homotopic;*

3 *There exists a continuous extension of $f \circ h$ to I^2;*

4 *$u_o \cdot v_1 \sim v_0 \cdot u_1$ rel. $\dot{I}$.*

Proof. ($1 \Leftrightarrow 2$) was shown in 2.30.

($3 \Leftrightarrow 4$) $f \circ h : (I^2)^\cdot \to X$ is the boundary data for the homotopy required in (4).

($1 \Rightarrow 3$) Take $\widetilde{f \circ h} := \tilde{f} \circ h$.

($3 \Rightarrow 1$) Since $\widetilde{f \circ h}$ is constant on $h^{-1}(s,t)$ for all $(s,t) \in (I^2)^\cdot$, it factors over the quotient mapping h to a continuous extension $\tilde{f} : I^2 \to X$.

$$\begin{tikzcd}
(I^2)^\cdot \arrow[r, "h"] \arrow[d, hook] & (I^2)^\cdot \arrow[d, hook] \arrow[ddr, "f"] & \\
I^2 \arrow[r, "h"] \arrow[drr, "\widetilde{f\circ h}"'] & I^2 \arrow[dr, dotted, "\tilde{f}"] & \\
& & X
\end{tikzcd}$$ □

5.7 Corollary. *Let X be a topological group (monoid) then $\pi_1(X, 1)$ is abelian, where 1 denotes the neutral element.*

Proof. Consider the map $(t, s) \mapsto u(t) \cdot v(s)$. □

5.8 Proposition. [7, 5.1.12] *Let $V : \pi_1(X, x_0) = [(S^1, 1), (X, x_0)] \to [S^1, X]$ be the mapping forgetting the base-points. Then*

1. *$[u]$ is in the image of V iff $u(1)$ can be connected by a path with x_0.*
2. *V is surjective iff X is path connected.*
3. *$V(\alpha) = V(\beta)$ iff there exists a $\gamma \in \pi_1(X, x_0)$ with $\beta = \gamma^{-1} \cdot \alpha \cdot \gamma$.*
4. *V is injective iff $\pi_1(X, x_0)$ is abelian.*
5. *The kernel $V^{-1}([\mathrm{const}_{x_0}])$ of V is trivial.*

Warning: Since V is not a group-homomorphism, 5 does not contradict 4.

Proof. (1) $[u]$ is in the image of V if u is homotopic to a base point preserving closed path. The homotopy at the base-point gives a path connecting $u(1)$ with x_0. Conversely any path v from $c(1)$ to x_0 can be used to give a homotopy between u and the base point preserving path $v^{-1} \cdot u \cdot v$, cf. 2.34.3, since $(S^1, 1)$ has HEP by 4.18.

($1 \Rightarrow 2$) is obvious.

(3) Let $\alpha = [u]$ and $\beta = [v]$. Then $V(\alpha) = V(\beta)$ iff u is homotopic to v.
($\Rightarrow$) Let h be such a homotopy and $w(t) := h(1, t)$. Then by 5.6 ($1 \Rightarrow 4$) we have $w \cdot v \sim u \cdot w$ rel. $\dot{I}$, i.e. $\gamma \cdot \beta = \alpha \cdot \gamma$ and hence $\beta = \gamma^{-1} \cdot \alpha \cdot \gamma$ for $\gamma := [w]$.

($\Leftarrow$) Let $\beta = \gamma^{-1} \cdot \alpha \cdot \gamma$ and $\gamma = [w]$. Then $\gamma \cdot \beta = \alpha \cdot \gamma$ and hence $w \cdot u \sim v \cdot w$ rel. $\dot{I}$. Then by 5.6 (1$\Leftarrow$4) we have $u \sim v$, i.e. $V(\alpha) = V(\beta)$.

(3$\Rightarrow$ 4)
($\Rightarrow$) Let $\alpha, \gamma \in \pi_1(X)$ and $\beta := \gamma^{-1} \cdot \alpha \cdot \gamma$. By (3) we have $V(\alpha) = V(\beta)$ and since V is assumed to be injective we get $\alpha = \beta$, i.e. $\gamma \cdot \alpha = \alpha \cdot \gamma$.
($\Leftarrow$) Conversely, if $V(\alpha) = V(\beta)$, then by (3) there exists a γ with $\beta = \gamma^{-1} \cdot \alpha \cdot \gamma$ and $\gamma^{-1} \cdot \alpha \cdot \gamma = \alpha$, provided α and γ commute.

(3$\Rightarrow$ 5) Let $V(\alpha) = [\mathrm{const}_{x_0}] = V(\mathrm{const}_{x_0})$. By (3) there exists a γ with $\alpha = \gamma^{-1} \cdot [\mathrm{const}_{x_0}] \cdot \gamma = \gamma^{-1} \cdot \gamma = 1$. □

5.9 Corollary. [7, 5.1.13] *Let X be path connected. Then the following statements are equivalent:*

1 $\pi_1(X, x_0) \cong 1$ *for some (any) $x_0 \in X$, i.e. every $u : S^1 \to X$ is 0-homotopic rel. $\dot{I}$ by 5.3;*

2 $[S^1, X] = \{0\}$;

3 *Every $u : S^1 \to X$ is 0-homotopic, i.e. any two paths which agree on the endpoints are homotopic rel. $\dot{I}$.*

A path connected space satisfying these equivalent conditions is called SIMPLY CONNECTED.

Proof. (1$\Rightarrow$ 2) since $V : \pi^1(X, x_0) \to [S^1, X]$ is onto by 5.8.2.

(2$\Rightarrow$ 3) From 5.6 (2$\Rightarrow$ 4) with $v_j := \mathrm{const}_{x_j}$ follows $u \cdot \mathrm{const}_{x_0} \sim \mathrm{const}_{x_1} \cdot v$ rel. $\dot{I}$ and hence $u \sim v$ rel. $\dot{I}$ by 5.3.2.

(3$\Rightarrow$ 1) is obvious, since $u \sim \mathrm{const}_{x_0}$ □

5.10 Example. [7, 5.1.9] *Let X be a CW-complex without 1-cells, e.g. $X = S^n$ for $n > 1$. Then $\pi_1(X, x_0) = \{1\}$ for all $x_0 \in X^0$.*
In fact every $u : (I, \dot{I}) \to (X, x_0)$ is by 4.21 homotopic rel. $\dot{I}$ to a cellular mapping v, i.e. $v(I) \subseteq X^1 = X^0$, hence v is constant.

Note that such an X is path connected iff it has exactly one 0-cell.
($\Rightarrow$) Let x_0 and x_1 be two 0-cells and u be a path between them. By 4.21 u is homotopic to a cellular and hence constant path rel. $\dot{I}$, since X has no 1-cells. Thus $x_0 = x_1$.
($\Leftarrow$) Since cells are path-connected each point in X^n can be connected with some point in X^{n-1} and by induction with the unique point in X^0.

Corollary. *Let X be contractible, then X is simply connected.*

Proof. By 2.6.6 we get that $[S^1, X] = \{0\}$ provided X is contractible. □

5.11 Definition. [7, 5.1.15] Every $f : (X, x_0) \to (Y, y_0)$ induces a group homomorphism $\pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ given by $\pi_1(f)[u] := [f \circ u]$. Just use that $u \sim v \Rightarrow f \circ u \sim f \circ v$ and $f \circ (u \cdot v) = (f \circ u) \cdot (f \circ v)$ to get well-definedness and the homomorphic-property.

5.12 Corollary. [7, 5.1.16] *π_1 is a functor from the category of pointed topological spaces to that of groups, i.e. it preserves identities and commutativity of diagrams.*

Proof. trivial □

5.13 Proposition. [7, 5.1.18] *π_1 is homotopy invariant.*
More precisely: If $f \sim g$ rel. x_0 then $\pi_1(f) = \pi_1(g)$. If $f \sim g$ then $\pi_1(g) = \operatorname{conj}_{[u]} \circ \pi_1(f)$, where u is the path given by the homotopy at x_0. If $f : X \to Y$ is a homotopy equivalence then $\pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism.

Proof. If $f \sim g$ rel. x_0 and $[v] \in \pi_1(X, x_0)$ then $f \circ v \sim g \circ v$ rel. $\dot{I}$, i.e. $\pi_1(f)[v] = \pi_1(g)[v]$.

If h is a free homotopy from f to g, then $w(t) := h(x_0, t)$ defines a path from $f(x_0)$ to $g(x_0)$. And applying 5.6 (1⇒4) to $(s, t) \mapsto h(t, v(s))$ we get $(f \circ v) \cdot w \sim w \cdot (g \circ v)$ rel. $\dot{I}$, and hence $[f \circ v] \cdot [w] = [(f \circ v) \cdot w] = [w \cdot (g \circ v)] = [w] \cdot [g \circ v]$, i.e. $\pi_1(g)[v] = [g \circ v] = [w]^{-1} \cdot [f \circ v] \cdot [w] = [w]^{-1} \cdot \pi_1(f)[v] \cdot [w] = (\operatorname{conj}_{[w]} \circ \pi_1(f))([v])$.

Let now $f : X \to Y$ be a homotopy equivalence with homotopy inverse $g : Y \to X$. Then up to conjugation $\pi_1(f)$ and $\pi_1(g)$ are inverse to each other. □

The fundamental group of the circle

5.15 Proposition. [7, 5.2.2] *The composition $\deg \circ V : \pi_1(S^1, 1) \to [S^1, S^1] \to \mathbb{Z}$ is a group isomorphism.*

Proof. By 2.17 we have that deg is a bijection. By 5.8 V is surjective since S^1 is path-connected. By 5.7 and 5.8 it is also injective since S^1 is a topological group.

Remains to show that the composite is a group-homomorphism: Recall that $\deg([u])$ is given by evaluating a lifting $u : (S^1, 1) \to (S^1, 1)$ to a curve $\tilde{u} : \mathbb{R} \to \mathbb{R}$ with $\tilde{u}(0) = 0$ and $\exp(2\pi i \tilde{u}(t)) = u(\exp(2\pi i t))$ at 1. Given $u, v \in \pi_1(S^1, 1)$ with lifts $\tilde{u}$ and $\tilde{v}$, then the lift of $u \cdot v$ is given by

$$t \mapsto \begin{cases} \tilde{u}(2t) & \text{for } t \leq \frac{1}{2} \\ \tilde{u}(1) + \tilde{v}(2t - 1) & \text{for } t \geq \frac{1}{2}. \end{cases} \qquad \square$$

5.16 Corollary. [7, 5.2.4] *$\pi(X, x_0) \cong \mathbb{Z}$ for every space X which contains S^1 as DR. In particular this is true for $\mathbb{C} \setminus \{0\}$, the Möbius strip, a full torus and the complement of a line in $\mathbb{R}^3$.*

Constructions from group theory

5.17 Definition. [7, 5.3.1] We will denote with e the NEUTRAL ELEMENT in a given group.

A SUBGROUP of a group G is a subset $H \subseteq G$, which is with the restricted group operations itself a group, i.e. $h_1, h_2 \in H \Rightarrow h_1 h_2 \in H$, $h_1^{-1} \in H$, $1 \in H$.

The SUBGROUP $\langle X \rangle_{SG}$ *generated by a subset* $X \subseteq G$ is defined to be the smallest subgroup of G containing X, i.e.

$$\begin{aligned} \langle X \rangle_{SG} &= \bigcap \{H : X \subseteq H \leq G\} \\ &= \{x_1^{\varepsilon_1} \cdot \dots \cdot x_n^{\varepsilon_n} : x_j \in X, \varepsilon_j \in \{\pm 1\}\}. \end{aligned}$$

Given an equivalence relation $\sim$ on G we can form the quotient set $G/\sim$ and have the natural mapping $\pi : G \to G/\sim$. In order that $G/\sim$ carries a group structure, for which π is a homomorphism, i.e. $\pi(x \cdot y) = \pi(x) \cdot \pi(y)$, we need precisely that $\sim$ is a CONGRUENCE RELATION, i.e. $x_1 \sim x_2$, $y_1 \sim y_2 \Rightarrow x_1^{-1} \sim x_2^{-1}$, $x_1 \cdot y_1 \sim x_2 \sim y_2$.

Then $H := \{x : x \sim e\} = \pi^{-1}(e)$ is a normal subgroup (we write $H \lhd G$), i.e. is a subgroup such that $g \in G$, $h \in H \Rightarrow g^{-1}hg \in H$. And conversely, for normal subgroups $H \lhd G$ we have that $x \sim x \cdot h$ for all $x \in G$ and $h \in H$ defines a congruence relation $\sim$ and $G/H := G/\sim = \{gH : g \in G\}$. This shows, that normal subgroups are exactly the kernels of group homomorphisms. Every surjective group morphism $p : G \to G_1$ is up to an isomorphism $G \to G/\ker p$.

The NORMAL SUBGROUP $\langle X \rangle_{NG}$ *generated by a subset* $X \subseteq G$ is defined to be the smallest normal subgroup of G containing X, i.e.

$$\begin{aligned} \langle X \rangle_{NG} &= \bigcap \{H : X \subseteq H \lhd G\} \\ &= \{g_1^{-1} y_1 g_1 \cdot \dots \cdot g_n^{-1} y_n g_n : g_j \in G, y_j \in \langle X \rangle_{SG}\}. \end{aligned}$$

5.18 Definition. Let G_i be groups. Then the PRODUCT $\prod_i G_i$ of is defined to be the solution of the following universal problem:

$$\begin{tikzcd} G_i & & \prod_i G_i \arrow[ll, "\mathrm{pr}_i"'] \\ & H \arrow[ul, "f_i"] \arrow[ur, dotted, "(f_i)_i", "!"'] & \end{tikzcd}$$

A concrete realization of $\prod_i G_i$ is the cartesian product with the component-wise group operations.

Definition. Let G_i be groups. Then the COPRODUCT (FREE PRODUCT) $\coprod_i G_i$ of is defined to be the solution of the following universal problem:

$$\begin{tikzcd} G_i \arrow[rr, "\mathrm{inj}_i"] \arrow[dr, "f_i"'] & & \coprod_i G_i \arrow[dl, dotted, "(f_i)_i"', "!"] \\ & H & \end{tikzcd}$$

5.19 Remark. [7, 5.3.3] A concrete realization of $\coprod_i G_i$ is constructed as follows. Take the set X of all finite sequences of elements of the G_i. With concatenation of sequences X becomes a monoid, where the empty sequence is the neutral element. Every G_i is injectively mapped into X by mapping g to the sequence with the single entry g. However this injection is not multiplicative and X is not a group. So we consider the congruence relation generated by $(g, h) \sim (gh)$ if g, h belong to the same group and $(1) = \emptyset$ for the neutral element 1 of any group G_i. Then $X/\sim$ is a group and the composite $G_i \to X \to X/\sim$ is the required group homomorphism and this object satisfies the universal property of the coproduct.

In every equivalence class of $X/\sim$ we find a unique representative of the form $(g_1, \dots, g_n)$, with $g_j \in G_{i_j} \setminus \{1\}$ and $i_j \neq i_{j+1}$. Since $(g_1, \dots, g_n)$ is just the product of the images of $g_i \in G_i$ we may write this also as $g_1 \cdot \dots \cdot g_n$.

5.20 Definition. [7, 5.7.8] Let H, G_1, G_2 be groups and $f_j : H \to G_j$ group homomorphisms. Then the PUSH-OUT $G_1 \coprod_H G_2$ of (f_1, f_2) is a solution of the following universal problem:

$$\begin{tikzcd} H \arrow[r, "f_2"] \arrow[d, "f_1"'] & G_2 \arrow[d, "g_2"] \arrow[ddr, "k_2"] & \\ G_1 \arrow[r, "g_1"] \arrow[drr, "k_1"'] & G_1 \coprod_H G_2 \arrow[dr, dotted, "!"] & \\ & & K \end{tikzcd}$$

It can be constructed as follows:

$$G_1 \coprod_H G_2 := (G_1 \coprod G_2)/N,$$

where $N := \langle f_1(h) \cdot f_2(h)^{-1} : h \in H\rangle_{NT}$ and where g_j is given by composing the inclusion $G_j \to G_1 \coprod G_2$ with the natural quotient mapping $G_1 \coprod G_2 \to (G_1 \coprod G_2)/N$.

5.21 Definition. [7, 5.6.3] Let G be a group. Then the ABELIZATION ${}^{ab}G$ of G is an Abelian group being solution of the following universal problem:

$$\begin{tikzcd} G \arrow[rr, "\pi"] \arrow[dr, "f"'] & & {}^{ab}G \arrow[dl, dotted, "\tilde f", "!"'] \\ & A & \end{tikzcd}$$

where A is an arbitrary Abelian group.

A realization of ${}^{ab}G$ is given by G/G', where the COMMUTATOR SUBGROUP G' denotes the normalizer generated by all COMMUTATORS $[g,h] := ghg^{-1}h^{-1}$. Remark that $G' = \{[g_1,h_1] \cdot \dots \cdot [g_n,h_n] : g_j, h_j \in G\}$.

Remark. From general categorical results we conclude that the product (and more general limits) in the category of Abelian groups is the product (limit) formed in that of all groups. And abelization of a coproduct (more generally a colimit) is just the coproduct (colimit) of the abelizations formed in the category of Abelian groups.

5.22 Definition. [7, 5.3.7] Let G_i be abelian groups. Then the COPRODUCT (DIRECT SUM) ${}^{ab}\coprod_i G_i$ of is defined to be the solution of the following universal problem:

$$\begin{tikzcd} G_i \arrow[rr, "\mathrm{inj}_i"] \arrow[dr, "f_i"'] & & {}^{ab}\coprod_i G_i \arrow[dl, dotted, "(f_i)_i", "!"'] \\ & H & \end{tikzcd}$$

where H is an arbitrary Abelian group.

Remark. A concrete realization of ${}^{ab}\coprod_i G_i$ is given by those elements of $\prod_i G_i$, for which almost all coordinates are equal to the neutral element.

5.23 Definition. [7, 5.5.3] Let X be a set. Then the FREE GROUP $\mathcal{F}(X)$ is the universal solution to

$$\begin{tikzcd} X \arrow[rr, "\iota"] \arrow[dr, "f"'] & & \mathcal{F}(X) \arrow[dl, dotted, "\tilde{f}"', "!"] \\ & H & \end{tikzcd}$$

where the arrows starting at X are just mappings and $\tilde{f}$ is a group homomorphism.

5.24 Remark. [7, 5.5.2] One has $\mathcal{F}(X) \cong \mathcal{F}(\bigsqcup_{x\in X}\{x\}) \cong \coprod_{x\in X} F(\{x\})$ by a general categorical argument, and $\mathcal{F}(\{*\}) \cong \mathbb{Z}$, as is easily seen.

Definition. Let X be a set. Then the FREE ABELIAN GROUP ${}^{ab}\mathcal{F}(X)$ is the universal solution to

$$\begin{tikzcd} X \arrow[rr, "\iota_i"] \arrow[dr, "f"'] & & {}^{ab}\mathcal{F}(X) \arrow[dl, dotted, "\tilde{f}"', "!"] \\ & A & \end{tikzcd}$$

where the arrows starting at X are just mappings and $\tilde{f}$ is a group homomorphism.

Remark. By a general categorical argument we have ${}^{ab}(\mathcal{F}(X)) \cong {}^{ab}\mathcal{F}(X)$. And ${}^{ab}\mathcal{F}(X) \cong {}^{ab}\coprod_x \mathcal{F}(\{x\}) \cong {}^{ab}\coprod_x \mathbb{Z}$, which are just the finite sequences in $\mathbb{Z}^X$.

5.25 Definition. [7, 5.6.1] Given a set X and a subset $R \subseteq \mathcal{F}(X)$ we define

$$\langle X : R\rangle := \mathcal{F}(X)/\langle R\rangle_{NT}$$

to be the GROUP WITH GENERATORS X AND DEFINING RELATIONS R. If $\langle X : R\rangle \cong G$, then $\langle X : R\rangle$ is called REPRESENTATION of the group G.

5.26 Example.

$$\begin{aligned} \mathcal{F}(X) &:= \langle X : \emptyset\rangle \\ \mathbb{Z}_n &:= \langle x : x^n\rangle \end{aligned}$$

If $G_j = \langle X_j : R_j\rangle$, then $\coprod_j G_j = \langle \bigsqcup X_j : \bigcup_j R_j\rangle$ and ${}^{ab}G := \langle X : R \cup \{[x,y] : x, y \in X\}\rangle$

5.27 Remark. [7, 5.8.1] Obviously we have

$$\begin{aligned} \langle X : R\rangle &\cong \langle X : R \cup \{r'\}\rangle \\ \langle X : R\rangle &\cong \langle X \cup \{a\} : R \cup \{a^{-1}w\}\rangle \end{aligned}$$

provided $r' \in \langle R\rangle_{NT}$, $a \notin X$ and $w \in \mathcal{F}(X)$. These operations are called Tietze operations.

5.28 Theorem. [7, 5.8.2] *Two finite representations $\langle X : R\rangle$ and $\langle Y : S\rangle$ describe isomorphic groups iff there is a finite sequence of Tietze operations converting one description into the other.*

Proof. See [7]. □

Remark. The word problem for finitely presented groups is the problem to determine whether two elements $w, w' \in \mathcal{F}(X)$ define the same element of $\langle X : R\rangle$, or equivalently whether $w \in \langle R\rangle_{NT}$.

The isomorphy problem is to determine whether two finite group representations describe isomorphic groups.

Both problems have been shown to have no algorithmic solution.

Group descriptions of CW-spaces

5.29 Proposition. [7, 5.2.6] *For pointed spaces (X_i, x_i) we have the following isomorphism $\pi_1(\prod_i X_i, (x_i)_i) \cong \prod_i \pi_1(X_i, x_i)$.*

Proof. Obvious, since $[(Y, y), (\prod_i X_i, (x_i)_i)] \cong \prod_i [(Y, y), (X_i, x_i)]$, by composition with the coordinate projections, and since the multiplication of paths in $\prod_i X_i$ is given component-wise. □

5.30 Proposition. [7, 5.1.21] *Let X_0 be a path component of X and let $x_0 \in X_0$. Then the inclusion of $X_0 \subseteq X$ induces an isomorphism $\pi_1(X_0, x_0) \cong \pi_1(X, x_0)$.*

Proof. Since S^1 and $S^1 \times I$ is path connected, the paths and the homotopies have values in X_0. □

5.31 Proposition. *Let X_α be subspaces of X such that every compact set is contained in some X_α. And for any two of these subspaces there is a third one containing both. Let $x_0 \in X_\alpha$ for all α. Then $\pi_1(X, x_0)$ is the inductive limit of all $\pi_1(X_\alpha, x_0)$.*

Proof. Let G be any group and $f_\alpha : \pi_1(X_\alpha) \to G$ be group-homomorphisms, such that for every inclusion $i : X_\alpha \subseteq X_\beta$ we have $f_\beta \circ \pi_1(i) = f_\alpha$. We have to find a unique group-homomorphism $f : \pi_1(X) \to G$, which satisfies $f \circ \pi_1(i) = f_\alpha$ for all inclusions $i : X_\alpha \to X$. Since every closed curve w in X is contained in some X_α, we have to define $f([w]_X) := f_\alpha([w]_{X_\alpha})$. We only have to show that f is well-defined: So let $[w_1]_X = [w_2]_X$ for curves w_1 in X_{α_1} and $w_2 \in X_{\alpha_2}$. The image of the homotopy $w_1 \sim w_2$ is contained in some X_α, which we may assume to contain X_{α_1} and X_{α_2}. Thus $f_{\alpha_1}([w_1]_{X_{\alpha_1}}) = f_\alpha([w_1]_{X_\alpha}) = f_\alpha([w_2]_{X_\alpha}) = f_{\alpha_2}([w_2]_{X_{\alpha_2}})$. □

5.32 Definition. [7, 5.5.11] A CW-complex X with $X = X^1$ is called a GRAPH. A graph is called TREE if it is simple connected.

5.33 Theorem, Seifert van Kampen. [7, 5.3.11] *Let X be covered by two open path connected subsets U_1 and U_2 such that $U_1 \cap U_2$ is path connected. Let $x_0 \in U_1 \cap U_2$. Then*

$$\begin{tikzcd} \pi_1(U_1 \cap U_2, x_0) \arrow[r, "i^2_*"] \arrow[d, "i^1_*"] & \pi_1(U_2, x_0) \arrow[d, "j^2_*"'] \\ \pi_1(U_1, x_0) \arrow[r, "j^1_*"] & \pi_1(X, x_0) \end{tikzcd}$$

is a push-out, where all arrows are induced by the corresponding inclusions.

Proof. Let $G_j := \pi_1(U_j, x_0)$ für $j \in \{1, 2\}$, $G_0 := \pi_1(U_1 \cap U_2, x_0)$, $G := \pi_1(U_1 \cup U_2, x_0) = \pi_1(X, x_0)$ and $\bar{G} := (G_1 \coprod G_2)/N$ the push-out, where N is the normal subgroup generated by $\{i^1_*([u]) \cdot i^2_*([u])^{-1} : [u] \in G_0\}$. By the universal property of the push-out there exists a unique group-homomorphism $q : \bar{G} \to G$ and we only have to show that it is bijective.

Surjectivity: Let $[w] \in \pi_1(X)$. Take n sufficiently large such that for each $0 \le i < n$ we have $w[t_i, t_{i+1}] \subseteq U_{\varepsilon_i}$ for some $\varepsilon_i \in \{1, 2\}$ and $t_i := \frac{i}{n}$. Let w_j be the restriction of w to $[t_j, t_{j+1}]$ and let v_i be a path from x_0 to $w(t_i)$ in $U_{\varepsilon_i} \cap U_{\varepsilon_{i-1}}$. We may take v_0 and v_n to be constant x_0. Let $u_i := v_i \cdot w_i \cdot v_{i+1}^{-1}$. Then u_i is a closed path in U_{ε_i} and $w \sim u_0 \cdot \ldots \cdot u_{n-1}$ in X rel. $\dot{I}$. Hence

$$\begin{aligned}[w]_X &= [u_0]_X \cdot \ldots \cdot [u_{n-1}]_X = q([u_0]_{U_{\varepsilon_0}}) \cdot \ldots \cdot q([u_{n-1}]_{U_{\varepsilon_{n-1}}}) \\ &= q\Big([u_0]_{U_{\varepsilon_0}} \cdot \ldots \cdot [u_{n-1}]_{U_{\varepsilon_{n-1}}}\Big) \in G.\end{aligned}$$

Injectivity: Let $z \in \bar{G} = (G_1 \coprod G_2)/N$ with $q(z) = 1 = [\mathrm{const}_{x_0}] \in G$. Then we find closed paths u_i in U_{ε_i} for certain $\varepsilon_i \in \{1, 2\}$ with $z = [u_1]_{U_{\varepsilon_1}} \cdot \ldots \cdot [u_n]_{U_{\varepsilon_n}}$. Since

$$\begin{aligned}[\mathrm{const}_{x_0}]_X &= q(z) = q\Big([u_1]_{U_{\varepsilon_1}} \cdot \ldots \cdot [u_n]_{U_{\varepsilon_n}}\Big) \\ &= q([u_1]_{U_{\varepsilon_1}}) \cdot \ldots \cdot q([u_n]_{U_{\varepsilon_n}}) = [u_1]_X \cdot \ldots \cdot [u_n]_X = [u_1 \cdot \ldots \cdot u_n]_X\end{aligned}$$

there is a homotopy $H : I \times I \to X$ relative $\dot{I}$ between $u_1 \cdot \ldots \cdot u_n$ and const_{x_0}. We partition $I \times I$ into squares Q, such that $H(Q) \subseteq U_{\varepsilon_Q}$ for certain $\varepsilon_Q \in \{1, 2\}$. We may assume that the resulting partition on the bottom edge $I \times \{0\} \cong I$ is finer than $0 < \frac{1}{n} < \frac{2}{n} < \ldots < \frac{n}{n} = 1$. For every corner k of this partition we choose a curve v_k connecting x_0 with $H(k)$. If $H(k) \in U_j$ then we may assume that $v_k(I) \subseteq U_j$. If $H(k) = x_0$, we may assume that v_k is constant. For every edge c of such a square Q we define the curve closed curve $u_c := v_{c(0)} \cdot (H \circ c) \cdot v_{c(1)}^{-1}$ through x_0. Since u_c is contained in some U_j we may consider $[u_c]_{U_j}$ and its image $\bar{c} := q([u_c]_{U_j}) \in \bar{G}$. This is well defined, since if u_c is contained in $U_1 \cap U_2$ then $[u_c]_{U_1 \cap U_2}$ is mapped to $[u_c]_{U_j} \in G_j$ for $i \in \{1, 2\}$ and further on to the same element $\bar{c}$ in the push-out $\bar{G}$.

Let now Q be a fixed square with edges d, r, u, l:

$$\begin{tikzcd} \cdot \arrow[r, "u"] & \cdot \\ \cdot \arrow[u, "l"] \arrow[r, "d"'] & \cdot \arrow[u, "r"'] \end{tikzcd}$$

Then $d \cdot r \sim l \cdot u$ rel. $\dot{I}$ in Q, hence $u_d \cdot u_r \sim u_l \cdot u_u$ rel. $\dot{I}$ in U_{ε_Q}, i.e. $[u_d] \cdot [u_r] = [u_l] \cdot [u_u]$ in G_{ε_Q} and thus $\bar{d} \cdot \bar{r} = q([u_d]) \cdot q([u_r]) = q([u_l]) \cdot q([u_u]) = \bar{l} \cdot \bar{u}$ in $\bar{G}$. Multiplying in $\bar{G}$ all these equations resulting from one row, gives that the product corresponding to the top line equals in $\bar{G}$ that corresponding to the bottom line, since the inner vertical parts cancel, and those at the boundary are 1. Since the top row represents 1, we get that the same is true for the bottom one. But u_i is homotopic in U_{ε_i} rel. $\dot{I}$ to the concatenation of the corresponding u_c in the bottom row. Hence $[u_i]_{U_{\varepsilon_i}} = \prod_c [u_c]_{U_{\varepsilon_i}}$ in G_{ε_i} and thus $z = \prod_i [u_i]_{U_{\varepsilon_i}} = \prod_c [u_c]_{U_{\varepsilon_i}} = \prod_c \bar{c} = 1$ in $\bar{G}$. $\square$

5.34 Corollary. [7, 5.3.9] [7, 5.3.12] *Let $X = U_1 \cup U_2$ be as in 5.33.*

1 *If $U_1 \cap U_2$ is simply connected, then $\pi_1(U_1 \cup U_2) \cong \pi_1(U_1) \amalg \pi_1(U_2)$.*

2 *If U_1 and U_2 are simply connected, then $U_1 \cup U_2$ is simply connected.*

3 *If U_2 is simply connected, then $\mathrm{incl}_* : \pi_1(U_1) \to \pi_1(X)$ in the push-out square is an epimorphism and its kernel is generated by the image of $\mathrm{incl}_* : \pi_1(U_1 \cap U_2) \to \pi_1(U_1)$.*

4 *If U_2 and $U_1 \cap U_2$ are simply connected, then $\pi_1(U_1) \cong \pi_1(U_1 \cup U_2)$.*

In particular, $\pi_k(S^n) := [(S^k, 1), (S^n, 1)] = \{0\}$ for $k \geq 2$.

Proof.

5 In this situation $N = \{1\}$ and hence $G_1 \coprod G_2$ is the push-out.

6 Here $G_1 \coprod G_2 = \{1\} \coprod \{1\} = \{1\}$ and hence also the push-out.

7 In this situation $G_1 \coprod G_2 = G_1 \coprod \{1\} \cong G_1$ and N is the normal subgroup generated by the image of G_0 in G_1.

8 Here we have $N = \{1\}$ and hence the push-out is isomorphic to G_1.

□

5.35 Theorem. [7, 5.4.8] *Let a CW-complex X be the union of two connected CW-subcomplexes A and B. Let $x_0 \in A \cap B$ and $A \cap B$ be connected. Then π_1 maps the push-out square into push-out.*

Proof. By 4.19 we may choose open neighborhoods $U(A)$, $U(B)$ and $U(A \cap B) = U(A) \cap U(B)$ which contain A, B and $A \cap B$ as DRs and the deformation of $U(A)$ and of $U(B)$ coincide on $U(A) \cap U(B)$, so also $A \cup B$ is a DR of $U(A) \cup U(B)$. Then application of 5.33 and of 5.13 gives the result.

$$\begin{tikzcd}
 & U(A) \cap U(B) \arrow[rr, hook] \arrow[dd, hook] & & U(A) \arrow[dd, hook] \\
A \cap B \arrow[ur, hook, "\sim"] \arrow[rr, hook] \arrow[dd, hook] & & A \arrow[ur, hook, "\sim"] \arrow[dd, hook] & \\
 & U(B) \arrow[rr, hook] & & U(A) \cup U(B) \\
B \arrow[ur, hook, "\sim"] \arrow[rr, hook] & & A \cup B \arrow[ur, hook, "\sim"] &
\end{tikzcd}$$

□

5.36 Proposition. [7, 5.4.9] *Let A and B be (connected) CW-complexes. Then $\pi_1(A \vee B, x_0) \cong \pi_1(A) \coprod \pi_1(B)$.*

Proof. Since $A \cap B$ in $A \vee B$ is $\{x_0\}$ and hence simply connected this follows from 5.35 and 5.34. □

5.37 Example. We have $\pi_1(S^1 \vee S^1) \cong \mathbb{Z} \coprod \mathbb{Z}$. However, for spaces being not CW-spaces in general $\pi_1(A \vee B) \neq \pi_1(A) \coprod \pi_1(B)$: Take for example for A and B the subset of $\mathbb{R}^2$ formed by infinite many circles tangent at the base point. The closed curve which passed through all those circles alternatingly can not be expressed as finite product of words in $\pi_1(A)$ and $\pi_1(B)$.

5.38 Proposition. [7, 5.5.9] *Let X_j be a CW-complex with base-point $x_j \in X_j^0$. Then $\pi_1(\bigvee X_j) \cong \coprod_j \pi_1(X_j)$. In particular we have $\pi_1(\bigvee_J S^1) \cong \coprod_J \mathbb{Z} \cong \mathcal{F}(J)$, where the free generators of $\pi_1(\bigvee_j S^1)$ are just the inclusions $\mathrm{inj}_j : S^1 \to \bigvee_J S^1$.*

Proof. This follows by induction and by 5.31 from 5.36, since every compact subset is by 4.5 contained in a finite subcomplex of the CW-complex of $\bigvee_{j\in J} X_j$ given by 4.17. $\square$

5.39 Corollary. [7, 5.4.1] [7, 5.4.2] *Let Y be path connected with $y_0 \in Y$ and $f : S^{n-1} \to Y$ be continuous. Then the inclusion $Y \subseteq Y \cup_f e^n$ induces an isomorphism $\pi_1(Y, y_0) \to \pi_1(Y \cup_f e^n, y_0)$ if $n \geq 3$ and an epimorphism if $n = 2$. In the later case the kernel is the normal subgroup generated by $[v][f][v^{-1}]$, where v is a path from y_0 to $f(1)$.*

One could say that by gluing e^2 to Y the element $[f] \in \pi_1(Y)$ gets killed.

Proof. We take $U := Y \cup_f (e^n \setminus \{0\})$ and $V := e^n$.

$$\begin{tikzcd}
& U\cap V \arrow[rr, hook] \arrow[dd, hook] & & V \arrow[dd, hook] \\
S^{n-1} \arrow[ur, hook, "\sim"] \arrow[rr, hook] \arrow[dd, hook] & & \{*\} \arrow[ur, hook, "\sim"] \arrow[dd, hook] & \\
& U \arrow[rr, hook] & & U\cup V \\
Y \arrow[ur, hook, "\sim"] \arrow[rr, hook] & & X \arrow[ur, equal] &
\end{tikzcd}$$

Then V and $U \cap V = e^n \setminus \{0\} \sim S^{n-1}$ are simply connected, by 5.10. Thus the inclusion $U \subseteq Y \cup_f e^n$ induces an isomorphism by 5.34. Since Y is a DR of U by 2.42 the inclusion of $Y \to U$ induces an isomorphism by 5.13.

Now for $n = 2$. Again V is simply connected, but $U\cap V \sim S^1$ and hence $\pi_1(U\cap V, y_0)$ is the infinite cyclic group generated by the image of a circle of radius say $1/2$. This path is homotopic to $[v][f][v^{-1}]$ in $Y \cup_f e^2$, hence everything follows. $\square$

5.40 Example. [7, 5.4.4] We have $\pi_1(S^1 \cup_{z^n} e^2) \cong \mathbb{Z}_n$. In particular, for the projective plane $\pi_1(\mathbb{P}^2) = \pi_1(S^1 \cup_{z^2} e^2) \cong \mathbb{Z}_2$.

This can be easily visualized: by vertical homotopy the top semi-circle is homotopic to the inverse of the bottom one. So the whole circle is $\alpha \cdot \alpha \sim \alpha^{-1} \cdot \alpha \sim 1$.

Equally one may cut a small disk from D^2. The gluing of the boundary gives a Möbius strip and the generator α is just the middle line on the Möbius strip. Its square is homotopic to the boundary of the Möbius strip which is contractible on the small disk.

5.41 Corollary. [7, 5.4.3] *We have $\pi_1(Y \cup_{f_1} e^2 \cup \cdots \cup_{f_n} e^2) \cong \pi_1(Y)/N$, where N is the normal subgroup generated by $\mathrm{conj}_{[v_j]}[f_j]$, where $f_j : S^1 \to Y$ is the gluing map and v_j a path joining y_0 and $f_j(1)$ in Y.*

This is also valid for infinite many cells by 5.31.

Proof. By induction from 5.39. $\square$

5.42 Proposition. [7, 5.4.6] *Let X be a CW-complex and $x_0 \in X^0$. Then $X^1 \hookrightarrow X$ induces an epimorphism $\pi_1(X^1, x_0) \to \pi_1(X, x_0)$ and $X^2 \hookrightarrow X$ an isomorphism $\pi_1(X^2, x_0) \to \pi_1(X, x_0)$.*

Proof. This follows from 4.5 using 5.39 and 5.31.

$$\pi_1(X^1) \twoheadrightarrow \pi_1(X^2) \cong \pi_1(X^3) \cong \ldots \cong \operatorname{inj\,lim}_j \pi_1(X^j) = \pi_1(X). \qquad \square$$

5.43 Example. [7, 5.4.7] Since $P^n = P^2 \cup e^3 \cup \cdots \cup e^n$ we have $\pi_1(P^n) \cong \pi_1(P^2) \cong \mathbb{Z}_2$.

5.44 Lemma. [7, 5.5.12] *A connected graph is a tree iff it is contractible.*

Proof. ($\Rightarrow$) Let X^0 be the 0-skeleton of a tree X. And let $x_0 \in X^0$ be fixed. Every $x \in X^0$ can be connected by a path with x_0, which gives a homotopy $X^0 \to X$. By 4.18 it can be extended to a homotopy $h_t : X \to X$ with $h_0 = \mathrm{id}_X$ and $h_1(X^0) = \{x_0\}$. Let $e \subseteq X$ be a 1-cell with characteristic map $\chi_e : I \cong D^1 \to X$. Then $[h_1 \circ \chi_e] \in \pi_1(X, x_0) = \{1\}$, hence there is a homotopy $h_t^e : (I, \dot{I}) \to (X, \{x_0\})$ with $h_0^e = h_1 \circ \chi_e$ and $h_1^e(I) = \{x_0\}$. Let $k_t^e : X^0 \cup e \to X$ be defined by $k_t^e(X^0) = \{x_0\}$ and $k_t^e = h_t^e \circ \chi_e^{-1}$ on e. Taking the union of all k_t^e gives a homotopy $k_t : X^1 \to X$ between h_1 and a constant map. $\square$

5.45 Lemma. [7, 5.5.13] *Every connected graph X contain a maximal tree. Any maximal tree in X contains all vertices of X.*

Proof. Let $\mathcal{M}$ be the set of trees of X ordered by inclusion. Since the union of linear ordered subset of $\mathcal{M}$ is a tree (use 4.5), we get by Zorns lemma a maximal tree $Y \subseteq X$.

Let Y be a maximal tree and suppose that there is some $x_0 \in X^0 \setminus Y^0$. Let $w : I \to X$ be a path connecting x_0 and Y. Let t_1 be minimal in $w^{-1}(Y)$ and $t_0 < t_1$ be maximal in $w^{-1}(X^0 \setminus Y^0)$. Then $w([t_0, t_1])$ is the closure of a 1-cell e and $Y \cup \bar{e}$ is a larger tree, since Y is a SDR of $Y \cup \bar{e}$. $\square$

5.46 Proposition. [7, 5.5.14] *Let X be a connected graph and $x_0 \in X^0$. Let $Y \subseteq X$ be a maximal tree. For every 0-cell x choose a path v_x in Y connecting x_0 with x. And for every 1-cell $e \subseteq X^1 \setminus Y$ with characteristic mapping $\chi^e : I \cong D^1 \to X^1$ let $s(e) := [v_{\chi^e(0)}][\chi^e][v_{\chi^e(1)}]^{-1} \in \pi_1(X, x_0)$. Then $s : \mathcal{F}(\{e : e$ is 1-cell in $X^1 \setminus Y\}) \xrightarrow{\cong} \pi_1(X, x_0)$, $e \mapsto s(e)$, i.e. $\pi_1(X, x_0)$ is the free group generated by $\{s(e) : e$ is 1-cell in $X^1 \setminus Y\}$.*

Proof. The quotient X^1/Y is a CW-complex with only one vertex $y_0 = [Y]$ and $p : X^1 \to X^1/Y$ is a homotopy-equivalence by 2.49 since Y is contractible and (X^1, Y) has HEP by 4.18. By 4.15 $X^1/Y \cong \bigvee_e S^1$, where e runs through the 1-cells in $X^1 \setminus Y$, see also 4.16. Thus $\pi_1(X, x_0) \cong \pi_1(X^1/Y, y_0) = \mathcal{F}(\{e : e$ is 1-cell in $X^1 \setminus Y\})$ by 5.38. The inverse of this isomorphism is given by $e \mapsto [v_{\chi^e(0)} \cdot \chi^e \cdot v_{\chi^e(1)}^{-1}] = s(e)$. $\square$

5.47 Corollary. [7, 5.5.17] *Every connected CW-space is homotopy equivalent to a CW-complex with just one 0-cell.*

Proof. By 2.49 we have that $X \to X/Y$ with a maximal tree Y as constructed before is a homotopy equivalence. □

5.48 Corollary. [7, 5.5.16] *Let X be a connected graph with d_0 vertices and d_1 edges. Then $\pi_1(X)$ is a free group of $1 - d_0 + d_1$ generators.*

Proof. By induction we show that for all $1 \le n \le d_0$ there are trees $Y_n \subseteq X$ with n vertices and $n-1$ edges: Let Y_n be given and choose a point $x_0 \in X^0 \setminus Y_n$ and a path w connecting x_0 with Y_n. Then proceed as in the proof of 5.45 to find an edge $w([t_0, t_1])$ connecting a vertex outside Y_n with one in Y_n. Now $Y_{n+1} = Y_n \cup w([t_0, t_1])$ is the required tree with one more vertex and one more edge.

By 5.46 the result follows, since there are $d_1 - (d_0 - 1)$ many 1-cells not in Y_{d_1}. □

5.49 Theorem. [7, 5.6.4] *Let X be a CW-complex with maximal tree Y. Let generators $s(e^1)$ be constructed for every $e^1 \in X^1 \setminus Y^1$ as in 5.46. For every 2-cell $e^2 \in X^2$ define $r(e^2) := [u \cdot \chi_{e^2}|_{S^1} \cdot u^{-1}] \in \pi_1(X^1, x_0)$, where u is a path from x_0 to $\chi_{e^2}(1)$ in X^1 and $\chi_{e^2} : D^2 \to e^2$ a characteristic mapping. Then*

$$\pi_1(X, x_0) \cong \langle \{s(e^1) : e^1 \textit{ is 1-cell in } X^1 \setminus Y^1\} : \{r(e^2) : e^2 \in X^2\} \rangle.$$

Proof. By 5.42 the mapping $\pi_1(X^1, x_0) \to \pi_1(X, x_0)$ induced by $X^1 \to X$ is surjective and by 5.41 its kernel is the normal subgroup generated by $r(e^2) = [u \cdot \chi_{e^2}|_{S^1} \cdot u^{-1}] \in \pi_1(X^1) \cong \mathcal{F}(\{s(e^1) : e^1 \text{ is 1-cell in } X^1 \setminus Y^1\})$. □

5.50 Example. [7, 5.6.5] For every group $G = \langle S : R \rangle$ there is a 2-dimensional CW-complex X denoted $\mathrm{CW}(S : R)$ with $\pi_1(X) \cong G$.

Proof. Let $X^1 := \bigvee_S S^1$. Consider every $r \in R \subseteq \mathcal{F}(S) \cong \pi_1(X^1)$ as mapping $f_r : S^1 \to X^1$ and glue a 2-cell to X^1 via this mapping. I.e. $X = CW(S : R) := X^1 \bigcup_f \bigsqcup_R e^2$, where $f := \bigsqcup_{r \in R} f_r$. □

Remark that this construction depends on the choice of $w \in [w]$. However different choices give rise to homotopy equivalent spaces by 2.47

5.51 Proposition. [7, 5.8.6] *Every connected CW-complex of dimension less or equal to 2 is homotopy equivalent to $CW(S : R)$ for some representation $\langle S : R \rangle$ of its fundamental group.*

Proof. Choose a maximal tree $M \subseteq X^1$. Then by the proof of 5.46 we have that X is homotopy equivalent to X/M, which has as 1-skeleton $\bigvee_S S^1$. For every 2-cell e we choose a characteristic map χ^e. Thus $X/M = (\bigvee_S S^1) \cup_{\chi^e|_{S^1}} \bigcup_e D^2$. By 2.34.3 and 2.47 we can deform $\chi^e|_{S^1}$ to a base point preserving map $f^e : S^1 \to X^1$. Hence by 2.47 X/M is homotopy equivalent to $CW\langle S : \{f^e : e\} \rangle$. □

Remark. Note that this does not solve the isomorphy problem for 2-dimensional CW-complexes, since although two such spaces X and X' with isomorphic fundamental group are homotopy equivalent to $CW(S, R)$ and $CW(S, R')$ for representations $\langle S : R \rangle \cong \langle S' : R' \rangle$ of the homotopy group, the space $CW(S : R)$ and $CW(S' : R')$ need not be homotopy equivalent, e.g. $\pi_1(S^2) = \{1\} = \pi_1(\{*\})$ but S^2 is not homotopy equivalent to a point $\{*\}$.

The following lemma shows exactly how the homotopy type might change while passing to other representations of the same group.

5.52 Lemma. [7, 5.8.7] *Let $\langle S' : R'\rangle$ be obtained by a Tietze process from $\langle S : R\rangle$. Then $CW(S' : R')$ is homotopy equivalent to $CW(S : R)$ or to $CW(S : R) \vee S^2$.*

This shows that $CW(G) = CW(S : R) := CW(S : R)$ would not be well-defined.

Proof. If $X = CW(S : R)$ and $Y = CW(S : R \cup \{r\})$ with $r \in \langle R\rangle_{NG}$. Then $Y = X \cup_f e^2$, where $f : S^1 \to \bigvee_S S^1 = X^1 \subseteq X$ is such that $[f] = r \in \pi_1(\bigvee_S S^1) = \mathcal{F}(S)$. Since $r \in \langle R\rangle_{NG}$, we have that $[f]_X = 1 \in \pi_1(X) = \pi_1(\bigvee_S S^1)/\langle R\rangle_{NG}$, hence $f \sim 0$ in X. Thus $Y = X \cup_f e^2 \sim X \cup_0 e^2 = X \vee S^2$ by 2.38.

If $X = CW(S : R)$ and $Y = CW(S \cup \{s\} : R \cup \{s^{-1}w\})$. Then $Y = (X \vee S^1) \cup_f e^2$, where $f : S^1 \to \bigvee_S S^1 = X^1$ is such that $[f] = s^{-1}w \in \mathcal{F}(S \cup \{s\})$. Let s is represented by the inclusion $S^1 \to X \vee S^1$ and w by $f_- := f|_{S^1_-} : S^1_- \to X^1$. Thus $Y = X \cup_{f_-} D^2$ and since $S^1_- \subseteq D^2$ is a SDR we have that X is also a SDR in Y, by 2.41. □

5.53 Example. [7, 5.7.1] The fundamental group of the orientable compact surface of genus $g \geq 1$ is

$$\langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g : [\alpha_1, \beta_1] \cdot \cdots \cdot [\alpha_g, \beta_g]\rangle.$$

That of the non-orientable compact surface of genus g is

$$\langle \alpha_1, \ldots, \alpha_g : \alpha_1^2 \cdot \cdots \cdot \alpha_g^2\rangle.$$

Proof. By 1.101 these surfaces are obtained by gluing one 2-cell to a join $\bigvee S^1$ of $2g$, respectively g, many S^1 and the gluing map is given by $i_1 \cdot i_2 \cdot i_1^{-1} \cdot i_2^{-1} \cdot \ldots$ and $i_1^2 \cdot \ldots \cdot i_g^2$, so the homotopy class of the characteristic mapping χ^e is $[\alpha_1, \beta_1] \cdot \cdots \cdot [\alpha_g, \beta_g]$ and $\alpha_1^2 \cdot \cdots \cdot \alpha_g^2$, respectively. □

5.54 Corollary. [7, 5.7.2] *None of the spaces in 5.53 are homotopy equivalent.*

Proof. The abelization of the fundamental groups are $\mathbb{Z}^{2g}$ and $\mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$. In fact

$$\begin{aligned}
&{}^{ab}\langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g : [\alpha_1, \beta_1] \cdot \ldots [\alpha_g, \beta_g]\rangle = \\
&\qquad = \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g : [\alpha_1, \beta_1] \cdot \ldots [\alpha_g, \beta_g], [\alpha_i, \alpha_j], [\beta_i, \beta_j], [\alpha_i, \beta_j]\rangle \\
&\qquad \overset{T1}{=} \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g : [\alpha_i, \alpha_j], [\beta_i, \beta_j], [\alpha_i, \beta_j]\rangle \\
&\qquad = {}^{ab}\langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g : \emptyset\rangle \\
&\qquad = {}^{ab}\mathcal{F}(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g) = \mathbb{Z}^{2g}
\end{aligned}$$

and

$$\begin{aligned}
&{}^{ab}\langle \alpha_1, \ldots, \alpha_g : \alpha_1^2 \cdot \cdots \cdot \alpha_g^2\rangle \\
&\qquad = {}^{ab}\langle \alpha_1, \ldots, \alpha_g : (\alpha_1 \cdot \cdots \cdot \alpha_g)^2\rangle \\
&\qquad \overset{T2}{=} {}^{ab}\langle \alpha_1, \ldots, \alpha_g, \alpha : \alpha^2, \alpha^{-1}\alpha_1 \ldots \alpha_g\rangle \\
&\qquad = {}^{ab}\langle \alpha_1, \ldots, \alpha_{g-1}, \alpha : \alpha^2\rangle \\
&\qquad = {}^{ab}(\langle \alpha_1, \ldots, \alpha_{g-1} : \emptyset\rangle \amalg \langle \alpha : \alpha^2\rangle) \\
&\qquad = \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2. \quad \square
\end{aligned}$$

Geometric interpretations are the following: S^2 is simply connected hence π_1 has no generator and no relation.

$S^1 \times S^1$ is a torus. The generators α and β of π_1 are given by $S^1 \times \{1\}$ and $\{1\} \times S^1$, which are a meridian and an equator in the 3-dimensional picture. This can be also seen by gluing the 4 edges of a square as $\alpha\beta\alpha^{-1}\beta^{-1}$. The relation $\alpha\beta = \beta\alpha$ is seen geometrically by taking as homotopy the closed curves given by running through some arc on the equator, then the meridian at that position and then the rest of the equator.

Now the oriented surface of genus g is obtained by cutting $2g$ holes into the sphere and gluing g cylinders to these wholes. Let x_0 be one point on the sphere not contained in the holes. The generators α_j are curves through x_0 along some generator $\{x\} \times I$ of the cylinder. The generators β_i are given by loops at around one boundary component $S^1 \times \{0\}$ of the cylinder. Then $\alpha_i\beta_i\alpha_i^{-1}$ describes the loop around the other component and $\alpha_i\beta_i\alpha_i^{-1}\beta_i^{-1}$ is a loop around both holes. The product of all these loops is a loop with all holes lying on one side and hence homotopic to a point.

We have seen the meaning of the generator α and the relation $\alpha^2 \sim 1$ on P^2 already.

The non-orientable surface of genus g is obtained from a sphere by cutting g holes and gluing g Möbius-strips. The generators α_j are just conjugates of the middle lines on the Möbius strips. Their squares are homotopic to the boundary circles. And hence the product of all α_i^2 is homotopic to a loop around all holes, which is in turn homotopic to a point.

This shows that beside the sphere, the torus and the projective plane these fundamental groups are not abelian.

6 Coverings

Basics

We pick up the method leading to the calculation $\pi_1(S^1) \cong \mathbb{Z}$ in 5.15. Basic ingredient was the lifting property of the mapping $t \mapsto \exp(2\pi i t)$, $\mathbb{R} \to S^1$, see 2.17. Its main property can be stated abstractly as follows:

6.1 Definition. Coverings. [7, 6.1.1] A COVERING MAP $p : Y \to X$ is a surjective continuous map, such that every $x \in X$ has an open neighborhood $U \subseteq X$ for which $p|_{p^{-1}(U)} : p^{-1}(U) \to U$ is up to an homeomorphism just the projection $\mathrm{pr} : \bigsqcup_J U \to U$ for some set $J \neq \emptyset$. I.e.

$$\begin{tikzcd} Y \arrow[dr, "p"'] & p^{-1}(U) \arrow[l, hook'] \arrow[dr, "p|_{p^{-1}(U)}"] & & \bigsqcup_J U \arrow[ll, "\cong"'] \arrow[dl, "\mathrm{pr}"] & U \arrow[l, hook', "\mathrm{inj}_j"'] \\ & X & U \arrow[l, hook'] & & \end{tikzcd}$$

The images of the summands U in $p^{-1}(U) \subseteq Y$ are called the LEAVES and U is called a TRIVIALISING NEIGHBORHOOD. The inverse images of points under p are called FIBERS. X is called BASE and Y TOTAL SPACE.

6.2 Example. Let $Y := \{(\sin(2\pi t), \cos(2\pi t), t) : t \in \mathbb{R}\} \cong \mathbb{R}$ and $p = \mathrm{pr}_{1,2} : Y \to S^1 \subseteq \mathbb{R}^2$. Then p is a covering map. Compare with 6.15 for $S^1 = \mathbb{R}/\mathbb{Z}$.

The map $z \mapsto z^n : S^1 \to S^1$ is an n-fold covering map. Compare with 6.15 for $S^1 = S^1/\mathbb{Z}_n$.

The map $S^n \to P^n$ is a two-fold covering map. Compare with 6.15 for $P^n = S^n/\mathbb{Z}_2$.

Let $p_1 : Y_1 \to X_1$ and $p_2 : Y_2 \to X_2$ be two covering maps, then so is $p_1 \times p_2 : Y_1 \times Y_2 \to X_1 \times X_2$. Examples: $\mathbb{R}^2 \to S^1 \times S^1$, $\mathbb{R}^2 \to \mathbb{R} \times S^1$ and $\mathbb{R} \times S^1 \to S^1 \times S^1$.

There is a twofold covering map from $I \times S^1$ to the Möbius strip. Compare with 6.15 for the action of $\mathbb{Z}_2$ on $[-1, 1] \times S^1$ given by $(t, \varphi) \mapsto (-t, \varphi + \pi)$.

The torus is a two fold covering of Klein's bottle. Compare with 6.15 for the action of $\mathbb{Z}_2$ on $S^1 \times S^1$ given by $(\varphi, \psi) \mapsto (-\varphi, \psi + \pi)$.

S^3 is a twofold covering of $SO(3)$. Compare with 6.15 for the action of S^3 on $S^3 \subset \mathbb{R}^4 = \mathbb{H}$ given by $x \mapsto (y \mapsto \mathrm{conj}_x(y))$. In particular $P^3 \cong SO(3)$.

$\mathbb{Z}_p$ acts transitively on S^{2k-1} and the orbit space is the lens space, so we get a covering $S^{2k-1} \to L(p; q_1, \dots, q_k)$.

Lemma. *Let X and Y be path connected and locally path connected and Hausdorff. Let Y be compact and $f : Y \to X$ be a local homeomorphism. Then f is a covering.*

Proof. Since f is a local homeomorphism, the inverse image of every point $x \in X$ is discrete and closed and hence finite since Y is compact.

Let us show next that f is surjective. In fact the image is open in X, since f is a local homeomorphism. It is closed, since Y is compact and X is Hausdorff. Since X is assumed to be (path-)connected it has to be all of X.

Let $x \in X$. For every $y \in f^{-1}(x)$ let U_y be a neighborhood, that is mapped homeomorphically onto some neighborhood of x. By taking the inverse images of the (finite) intersection of the corresponding neighborhoods of x we may assume that the image is the same neighborhood U for all $y \in U_y$. Hence $p : Y \to X$ is a covering. □

Example. Not every local homeomorphism is a covering map. Take for example the disjoint union $Y := \bigsqcup_{n\in\mathbb{N}}(-\infty, n)$ with the obvious projection to $X := \mathbb{R}$. The points $n \in \mathbb{N}$ have no trivialising neighborhoods. By taking the join of X with S^1 we may even assume that Y is path connected.

6.3 Lemma. [7, 6.1.3] *Let $p : Y \to X$ be a covering. Then*

1. *The fibers are discrete in Y.*
2. *Every open subset of a trivialising set is trivialising.*
3. *Let $A \subseteq X$. Then $p|_{p^{-1}(A)} : p^{-1}(A) \to A$ is a covering map.*
4. *If $B \subseteq Y$ is connected and $p(B) \subseteq U$ for some trivialising set U, then B is contained in some leave.*
5. *The projection is a surjective open local homeomorphism and hence a quotient mapping.*

Proof. (1) Points in the fiber are separated by the leaves.

(2) and (3) Take the restriction of the diagram above.

(4) B is covered by the leaves. Since each leave is open, so is the trace on B. Since B is connected only one leave may hit B, thus B is contained in this leave.

(5) Obviously the projection is a local homeomorphism. Hence it is open and a quotient mapping. □

Lifting properties

Let $p : (Y, y_0) \to (X, x_0)$ be a covering and $f : (Z, z_0) \to (X, x_0)$. Take a trivialising neighborhood U of x_0 and let $\tilde{U}$ be the leave of p over U which contains y_0. Then $(p|_{\tilde{U}}) : \tilde{U} \to U$ is a homeomorphism and hence $(p|_{\tilde{U}})^{-1} \circ f : Z \supseteq f^{-1}(U) \to \tilde{U} \subseteq Y$ is a continuous local lift of f.

Let $\tilde{f}$ be any continuous (local) lift of f with $\tilde{f}(z_0) = y_0$. Then $W := \tilde{f}^{-1}(\tilde{U})$ is a neighborhood of z_0 and $\tilde{f}(W) \subseteq \tilde{U}$, hence $f = p \circ \tilde{f}$ implies $(p|_{\tilde{U}})^{-1} \circ f = (p|_{\tilde{U}})^{-1} \circ p|_{\tilde{U}} \circ \tilde{f} = \tilde{f}$ on W. I.e. locally the lift of $\tilde{f}$ is unique.

We are now going to show that the lift is globally unique and we will determine when it exists.

6.4 Lemma. [7, 6.2.4] *Let $p : Y \to X$ be a covering map and $f : Z \to X$ be continuous, where Z is connected. Then any two lifts of f, which coincide in one point are equal.*

Proof. Let f^1, f^2 be two lifts of f. Then the set of points $\{z \in Z : f^1(z) = f^2(z)\}$ is clopen. In fact if U^j is the leave over U containing of $f^j(z)$, then $f^j = (p|_{U^j})^{-1} \circ f$ on the neighborhood $(f^1)^{-1}(U^1) \cap (f^2)^{-1}(U^2)$. Hence $f^1 = f^2$ locally around z iff $f^1(z) = f^2(z)$. □

6.5 Theorem. Lifting of curves. [7, 6.2.2] [7, 6.2.5] *Every path $w : I \to X$ has a unique lift ${}^y\tilde{w}$ with ${}^y\tilde{w}(0) = y$ for given $y \in p^{-1}(w(0))$. Paths homotopic relative their initial value have homotopic lifts.*

In particular we have an action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$ given by $[u] : y \mapsto {}^y\tilde{u}(1)$, i.e. the end-point of the lift of u, which starts bei y.

The total space Y is path connected iff X is path connected and this action is transitive (i.e. there exists a $y_0 \in p^{-1}(x_0)$ with $y_0 \cdot \pi_1(X, x_0) = p^{-1}(x_0)$, which is equivalent to: For all $y_1, y_2 \in p^{-1}(x_0)$ there exists a $g \in \pi_1(X, x_0)$ with $y_1 \cdot g = y_2$).

We will study this group-homomorphism $\pi_1(X, x_0) \to \mathrm{Bij}(p^{-1}(x_0))$ in 6.12 and in 6.17.

Proof. By 6.4 we have to show existence of a lift. By considering a path w as a homotopy being constant in the second factor, it is enough to show that homotopies $h : I \times I \to X$ can be lifted.

For this choose a partition of I^2 into squares $Q_{i,j}$, such that $h(Q_{i,j})$ is contained in a trivialising neighborhood $U_{i,j}$ of X. Now construct inductively a lift $\tilde{h}^1$ along $\bigcup_i Q_{i,1}$, by taking the leave $\tilde{U}_{i,1}$ over the trivialising neighborhood of $Q_{i,1}$ which contains the image under $\tilde{h}$ of the right bottom corner of $Q_{i-1,1}$ and hence also of the right side edge of $Q_{i-1,1}$ (by 6.3.3). Then $\tilde{h}|_{Q_{i,1}}$ can be defined as $(p|_{U_{i,1}})^{-1} \circ h|_{Q_{i,1}}$. Now proceed by induction in the same way to obtain lifts $\tilde{h}^j$ for all stripes $\bigcup_i Q_{i,j}$. By induction we can show that the lifts agree on the horizontal lines: In fact the image of h on a horizontal edge is contained in the intersection of the trivialising sets containing the image of the square above and below. And since the lifts $\tilde{h}^j$ and $\tilde{h}^{j-1}$ are contained in the respective leaves, and thus in the leave over the intersection, they have to be equal. We call the unique homotopy ${}^{y_0}\tilde{h}$.

Now suppose h is a homotopy rel. $\dot{I}$ between two paths w_0 and w_1 from x_0 to x_1 and let $y_0 \in p^{-1}(x_0)$. The homotopy ${}^{y_0}\tilde{h}$ has as boundary values lifts $\tilde{w}_0$ and $\tilde{w}_1$ with $\tilde{w}_0(0) = y_0$. Since $s \mapsto h(0, s)$ is a lift of the constant path x_0, it has to be constant, hence $\tilde{w}_1(0) = y_0$. So these are precisely the lifts of w_j. Since $s \mapsto h(1, s)$ is a lift of the constant path x_1, it is constant, i.e. $\tilde{h}$ is a homotopy rel. $\dot{I}$.

Composition law: The lift of ${}^{y_0}\widetilde{u \cdot w}$ is ${}^{y_0}\tilde{u} \cdot {}^{y_1}\tilde{v}$, where $y_1 := {}^{y_0}\tilde{u}(1)$.

The lifting property of 6.27 gives us a mapping $\pi_1(X, x_0)$ to the bijections of $p^{-1}(x_0)$ by setting $[u](y) := {}^y\tilde{u}(1)$. This is well defined, since curves u homotopic relative $\dot{I}$ have lifts ${}^y\tilde{u}$ homotopic relative $\dot{I}$ and hence have the same end point.

Moreover we have $[u \cdot v](y) = {}^y\widetilde{u \cdot v}(1) = ({}^y\tilde{u} \cdot {}^{y_1}\tilde{v})(1) = {}^{y_1}\tilde{v}(1) = [v](y_1) = [v]([u](y))$, where $y_1 = {}^y\tilde{u}(1) = [u](y)$. Hence, we consider this mapping as a **right** action, i.e. we write $y \cdot [u]$ for $[u](y)$. Then we have $y \cdot ([u] \cdot [v]) = (y \cdot [u]) \cdot [v]$.

Now the statement on path-connectedness:
If Y is path connected then so is the surjective image X. Furthermore a curve v connecting $y_1, y_2 \in p^{-1}(x_0)$ has a closed curve $u := p \circ v$ as image and $v = {}^{y_1}\tilde{u}$, so $y_1 \cdot [u] = y_2$.

Conversely let $y_1 \in Y$ be arbitrary. Since X is path connected we have a curve u connecting $p(y_1)$ with x_0. Its lift ${}^{y_1}\tilde{u}$ connects y_1 with $y := {}^{y_1}\tilde{u}(1) \in p^{-1}(x_0)$. Since $\pi_1(X, x_0)$ acts transitive on $p^{-1}(x_0)$ there is a $[u'] \in \pi_1(X, x_0)$ with $y \cdot [u'] = y_0$, i.e. the curve ${}^y\tilde{u}'$ connects y with y_0. $\square$

6.6 Proposition. [7, 6.3.5] *Let X be path-connected. Then the fibers of any covering $p : Y \to X$ can be mapped bijectively onto one another by lifting a curve connecting the foot points.*

Proof. Let $F_0 := p^{-1}(x_0)$, $F_1 := p^{-1}(x_1)$ and let u be a path from x_0 to x_1 then $y \mapsto {}^y\tilde{u}(1)$ defines a mapping $F_0 \to F_1$ and $y \mapsto {}^y\widetilde{u^{-1}}(1)$ a mapping $F_1 \to F_0$ and these mappings are inverse to each other, since the lift of the curve $u \cdot u^{-1} \sim 0$ is 0-homotopic rel. $\dot{I}$ and hence closed. $\square$

6.7 Lifting Theorem. [7, 6.2.6] *Let Z be path connected and locally path connected. Let $p : Y \to X$ be a covering and $f : Z \to X$ continuous. Let $x_0 \in X$, $y_0 \in Y$ and $z_0 \in Z$ be base points and all maps base point preserving. Then f has a base point preserving lift $\tilde{f}$ iff* $\operatorname{im}(\pi_1(f)) \subseteq \operatorname{im}(\pi_1(p))$.

Proof. ($\Rightarrow$) If $f = p \circ \tilde{f}$ then $\operatorname{im}(\pi_1(f)) = \operatorname{im}(\pi_1(p) \circ \pi_1(f)) \subseteq \operatorname{im}(\pi_1(p))$.

($\Leftarrow$) Let $z \in Z$ be arbitrary. Since Z is path-connected we may choose a path w from z_0 to z and take the lift ${}^{y_0}\widetilde{f \circ w}$ and define $\tilde{f}(z) := {}^{y_0}\widetilde{f \circ w}(1)$.

First we have to show that this definition is independent from the choice of w. So let w_j be two paths from z_0 to z. Then $f \circ (w_0 \cdot w_1^{-1}) = (f \circ w_0) \cdot (f \circ w_1)^{-1}$ is a closed path through x_0, hence by assumption there exists a closed path v through y_0 with $p \circ v \sim (f \circ w_0) \cdot (f \circ w_1)^{-1}$ rel. $\dot{I}$ and hence $(p \circ v) \cdot (f \circ w_1) \sim (f \circ w_0)$. Thus ${}^{y_0}\widetilde{f \circ w_0}(1) = {}^{y_0}\Big((p \circ v) \cdot (f \circ w_1)\Big)^{\sim}(1) = ({}^{y_0}\widetilde{p \circ v} \cdot {}^{y_0}\widetilde{f \circ w_1})(1) = {}^{y_0}\widetilde{f \circ w_1}(1)$.

Remains to show that $\tilde{f}$ is continuous. Let $z \in Z$ be fix and let $\tilde{U}$ be a small leaf over a trivialising neighborhood U of $f(z)$ containing $\tilde{f}(z)$. Let W be a path-connected neighborhood of z with $f(W) \subseteq U$ and let w be a path from z_0 to z. Then for every $z' \in W$ we can choose a path $w_{z'}$ in W from z to z'. Hence $\tilde{f}(z') = \widetilde{f \circ (w \cdot w_{z'})}^{y_0}(1) = ({}^{y_0}\widetilde{f \circ w} \cdot {}^{\tilde{f}(z)}\widetilde{f \circ w_{z'}})(1) = {}^{\tilde{f}(z)}\widetilde{f \circ w_{z'}}(1)$. But since $f \circ w_{z'}$ is contained in the trivialising neighborhood U and $\tilde{U}$ is the leaf over U containing the lift $\tilde{f}(z)$, we have that ${}^{\tilde{f}(z)}\widetilde{f \circ w_{z'}} = (p|_{\tilde{U}})^{-1} \circ f \circ w_{z'}$, and hence $\tilde{f}(z') = ((p|_{\tilde{U}})^{-1} \circ f)(z')$ and thus is continuous. $\square$

Thus it is important to determine the image of $\pi_1(p) : \pi_1(Y, y_0) \to \pi_1(X, x_0)$.

6.8 Proposition. [7, 6.3.1] *Let $p : (Y, y_0) \to (X, x_0)$ be a covering. Then the induced map $\pi_1(p) : \pi_1(Y, y_0) \to \pi_1(X, x_0)$ is injective and the image are formed by those classes in $\pi_1(X, x_0)$ for which (some) any representative w the lift ${}^{y_0}\tilde{w}$ is closed.*

So the image is formed by those $g \in \pi_1(X, x_0) =: G$ which act trivial on y_0, i.e. $y_0 \cdot g = y_0$. This is the so called isotropy subgroup *G_{y_0} of G at y_0.*

Proof. Injectivity: Let $[v] \in \pi_1(Y, y_0)$ be such that $1 = [p \circ v]$, i.e. $p \circ v \sim \mathrm{const}_{x_0}$. By 6.5 we may lift the homotopy. Since the lift of const_{x_0} is just const_{y_0} we have $[v] = 1$.

If some u has a closed lift v, then $\pi_1(p)[v] = [p \circ v] = [u]$, hence $[u] \in \operatorname{im}(\pi_1(p))$. Conversely let $[u] \in \operatorname{im}\pi_1(p)$. Then there exists a closed curve v through y_0 with $[p \circ v] = \pi_1(p)[v] = [u]$, hence $u \sim p \circ v$ rel. $\dot{I}$, and so ${}^{y_0}\tilde{u} \sim {}^{y_0}\widetilde{p \circ v} = v$ rel. $\dot{I}$, thus ${}^{y_0}\tilde{u}$ is closed as well. $\square$

Transitive actions

Since the action of the fundamental group on the typical fiber plays an important role we first study transitive actions in general. Note, that such actions of a group G on a set F can be considered as pendants of 1-dimensional vector-spaces F over a field G. Keeping this in mind makes the following lemma intuitively clear.

6.9 Lemma. Transitive Actions. [7, 6.3.3] *Let G act transitively on F (and F') from the right. An G-EQUIVARIANT MAPPING or G-HOMOMORPHISM φ is a mapping $\varphi : F \to F'$, which satisfies $\varphi(y \cdot g) = \varphi(y) \cdot g$. We write $\mathrm{Hom}_G(F, F')$ for the set of all G-homomorphisms $F \to F'$ and $\mathrm{Aut}_G(F)$ for the set of all G-equivariant isomorphisms $F \to F$. This is the commutant of the image of G in $\mathrm{bij}(F)$, or the set of all bijective G-homomorphisms of F. Then*

1. *We have $G_{y\cdot g} = g^{-1}G_y g$.*
2. *$\{G_y : y \in F\}$ is a conjugacy class of subgroups of G, i.e. an equivalence class of subgroups of H with respect to the relation of being conjugate.*
3. *Let H be a subgroup of G. Then the set $H\backslash G := \{Hg : g \in G\}$ of right classes carries a unique (transitive) right G-module structure, such that the canonical projection $\pi : G \to H\backslash G$, $g \mapsto Hg$ is G-equivariant.*
4. *For any conjugate subgroup $H' := g^{-1}Hg$ of H we have a G-equivariant bijection $H\backslash G \to H'\backslash G$ given by $Hg' \mapsto H'g^{-1}g'$.*
5. *For $y \in F$ the mapping $G \to F$ given by $g \mapsto y\cdot g$ factors to a G-isomorphism $G_y\backslash G \to F$.*
6. *For $\varphi \in \mathrm{Hom}_G(F, F')$ we have $G_y \subseteq G_{\varphi(y)}$. Conversely if $y_0 \in F$ and $y_1 \in F'$ satisfy $G_{y_0} \subseteq G_{y_1}$, then there is a unique $\varphi \in \mathrm{Hom}_G(F, F')$ with $\varphi(y_0) = y_1$.*
7. *Let $A \subseteq \mathrm{Aut}_G(F)$. Then $A = \mathrm{Aut}_G(F) \Leftrightarrow \forall y_0, y_1 : G_{y_0} = G_{y_1} \Rightarrow \exists \varphi \in A : \varphi(y_0) = y_1$. I.e. the only subgroup of $\mathrm{Aut}_G(F)$, which acts transitively on the points with the same isotropy-subgroup, is $\mathrm{Aut}_G(F)$ itself.*
8. *$\mathrm{Aut}_G(F) \cong \mathrm{Norm}_G(G_{y_0})/G_{y_0}$, where for a subgroup H of G one denotes with $\mathrm{Norm}_H(G) := \{g \in G : g^{-1}Hg \subseteq H\}$, the smallest subgroup of G, which contains H as normal subgroup.*
9. *$F \cong_G F' \Leftrightarrow \{G_y : y \in F\} = \{G_{y'} : y' \in F'\} \Leftrightarrow \{G_y : y \in F\} \cap \{G_{y'} : y' \in F'\} \neq \emptyset$.*

Proof. (1) We have $g^{-1}G_y g = G_{y\cdot g}$, since $h \in G_{y\cdot g} \Leftrightarrow y\cdot g\cdot h = y\cdot g \Leftrightarrow y\cdot(ghg^{-1}) = y$, i.e. $ghg^{-1} \in G_y$.

(2) From (1) we conclude that the subgroups G_y are conjugated to each other. Conversely if $H = g^{-1}G_y g = G_{y\cdot g}$, it is of this form.

(3) The only possible action on $H\backslash G$ such that π is G-equivariant is given by $Hg \cdot g' = \pi(g)\cdot g' := \pi(g\cdot g') = \pi(gg') = H(gg')$. That the so defined action makes sense, follows from $Hg_1 = Hg_2 \Rightarrow H(g_1 g) = (Hg_1)g = (Hg_2)g = H(g_2 g)$.

(4) The mapping is well defined, since $Hg_1 = Hg_2 \Leftrightarrow g_2 g_1^{-1} \in H \Leftrightarrow g^{-1}g_2 g_1^{-1} g \in g^{-1}Hg = H' \Leftrightarrow H'g^{-1}g_2 = H'g^{-1}g_1$. It is a G-equivariant, since $g' \mapsto g^{-1}g'$ from $G \to G$ is G-equivariant.

(5) Consider $G \to y \cdot G$ given by $g \mapsto y \cdot g$. This mapping has image $y \cdot G = F$, since G acts transitively. Furthermore g' and g have the same image $y \cdot g' = y \cdot g$ iff $g' g^{-1} \in G_y$.

(6) We have $G_y = \{g : y \cdot g = y\} \subseteq \{g : \varphi(y) \cdot g = \varphi(y \cdot g) = \varphi(y)\} = G_{\varphi(y)}$. Conversely let $G_{y_0} \subseteq G_{y_1}$ and $y \in F$. Since G acts transitively there exists a $g \in G$ with $y = y_0 \cdot g$. Define $\varphi(y) = \varphi(y_0 \cdot g) := \varphi(y_0) \cdot g = y_1 \cdot g$. This definition makes sense, since $y_0 \cdot g' = y_0 \cdot g$ implies $g' g^{-1} \in G_{y_0} \subseteq G_{y_1}$ and hence $y_1 \cdot g' = y_1 \cdot g$. By construction φ is G-equivariant.

(7) Remark that the right side just says that A acts transitively on $S := \{y : G_y = G_{y_0}\}$.
($\Rightarrow$) Let $G_{y_0} = G_{y_1}$. By (6) there exists a unique $\varphi \in \mathrm{Hom}_G(F, F')$ with $\varphi(y_0) = y_1$ and a unique $\varphi' \in \mathrm{Hom}_G(F', F)$ with $\varphi'(y_1) = y_0$. These are inverse to each other, since every element in $\mathrm{Aut}_G(F)$ is uniquely determined by its value on y_0 (use that G acts transitive).

($\Leftarrow$) Let $\varphi \in \mathrm{Aut}_G(F)$. Then by (6) $G_{y_0} = G_{\varphi(y_0)}$ hence there exists a $\psi \in A$ with $\psi(y_0) = \varphi(y_0)$. By transitivity of the action we have $\varphi = \psi \in A$.

(8) By (7) we know that in order that $y_0 \mapsto y_1 := y_0 \cdot g$ can be extended to a $\varphi \in \mathrm{Aut}_G(F)$ we need that $G_{y_0} = G_{y_1} = G_{y_0 \cdot g} = g^{-1} G_{y_0} g$. Thus we have shown that precisely the $g \in \mathrm{Norm}_G(G_{y_0})$ give rise to an (uniquely determined) $\varphi_g \in \mathrm{Aut}_G(F)$. I.e. $\mathrm{ev}_{y_0} : \mathrm{Aut}_G(F) \cong \{y : G_y = G_{y_0}\}$.

Let us next show that $g \mapsto \varphi_g$ is a group (anti-) homomorphism. So let $g_1, g_2 \in \mathrm{Norm}_G(G_{y_0})$. We have to show that $\varphi_{g_1} \circ \varphi_{g_2}$ maps y_0 to $y_0 \cdot g_1 \cdot g_2$. We have

$$\begin{aligned}(\varphi_{g_1} \circ \varphi_{g_2})(y_0) &= \varphi_{g_1}(\varphi_{g_2}(y_0)) = \varphi_{g_1}(y_0 \cdot g_2) \\ &= \varphi_{g_1}(y_0) \cdot g_2 = y_0 \cdot g_1 \cdot g_2.\end{aligned}$$

Remains to calculate the kernel of this homomorphisms, i.e. $\{g \in \mathrm{Norm}_G(G_{y_0}) : \varphi_g = \mathrm{id}\} = \{g : y_0 = y_0 \cdot g\} = G_{y_0}$.

(9) ($\Rightarrow$) Let $\varphi : F \to F'$ be a G-equivariant isomorphism. Then $G_y \subseteq G_{\varphi(y)} \subseteq G_{\varphi^{-1}(\varphi(y))} = G_y$ by (6).

($\Leftarrow$) By assumption there are $y \in F$ and $y' \in F'$ with $G_y = G_{y'}$ hence $F \cong_G G_y \backslash G = G_{y'} \backslash G \cong_G F'$. □

Corollary. *The following statements are equivalent:*

1a G_y *is normal for all* $y \in F$*;*

2a $G_y = G_{y'}$ *for all* $y, y' \in G$*;*

3a $\mathrm{Aut}_G(F)$ *acts transitive on* F*;*

4a *If* g *had a fixed point then it acts as identity.*

Proof. (2a$\Rightarrow$ 1a) is obvious, since $G_y = G_{y \cdot g} = g^{-1} \cdot G_y \cdot g$.

(1a$\Rightarrow$ 3a) If G_y is normal, then $\mathrm{Norm}_G(G_y) = G$ and hence $\mathrm{Aut}_G(F) = G/G_y$ which obviously acts transitive, since G does.

(3?a$\Rightarrow$ 4a) Let $y_0 \cdot g = y_0$ and $y \in F$. Since $\mathrm{Aut}_G(F)$ acts transitive there is an automorphism φ with $y = \varphi(y_0) = \varphi(y_0 \cdot g) = \varphi(y_0) \cdot g = y \cdot g$.

(4a$\Rightarrow$ 2a) Let $g \in G_y$, i.e. y is a fixed point of g. Hence g acts as identity, so $g \in G_{y'}$ for all $y' \in F$. □

Corollary. *We have a bijection between isomorphy classes of transitive right actions of G and conjugacy classes of subgroups of G.*

Proof. To each action on a space F we associate $\{G_y : y \in F\}$. This is well-defined by (2). By (9) this induces a well-defined injective mapping from the isomorphy classes to the conjugacy classes. This induced mapping is onto, since in each conjugacy class we can pick a subgroup H and form $F := H\backslash G$ with its canonical right action of G and isotropy subgroup H at $H \cdot e$. □

Corollary. *Let $p : Y \to X$ be a covering with path connected Y and $x_0 \in X$. The images $\pi_1(p)(\pi_1(X, y))$ for $y \in p^{-1}(x_0)$ form a conjugacy class of subgroups in $\pi_1(X, x_0)$.* □

This conjugacy class is called the CHARACTERISTIC CONJUGACY CLASS of the covering p.

6.10 Definition. Let $p : Y \to X$ and $p' : Y' \to X$ be two coverings. A HOMOMORPHISM f of coverings is a fiber preserving map $f : Y \to Y'$, i.e.

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "p"'] & & Y' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

We denote the set of all homomorphisms from $p : Y \to X$ to $p' : Y' \to X$ by $\mathrm{Hom}(p, p')$. Remark that a homomorphism f is nothing else but a lift of $p : Y \to X$ along $p' : Y' \to X$. The automorphisms f, i.e. invertible homomorphisms $p \to p$, are also called COVERING TRANSFORMATIONS or DECKTRANSFORMATIONS, and we write $\mathrm{Aut}(p)$ for the group formed by them.

Remark. For a covering $p : Y \to X$ and base-point $x_0 \in X$ we considered the fundamental group $G := \pi_1(X, x_0)$ of the base X acting by 6.5 transitively from the right on the typical fibre $F := p^{-1}(x_0)$.

This gives a functor from the category $\mathrm{Cov}_{\mathrm{pc}}(X)$ of homomorphisms of coverings with base X to the category $\mathrm{Act}_{\mathrm{tr}}(G)$ of G-equivariant mappings between transitive G-actions. In fact, for a homomorphism $h : p \to p'$ between coverings the restriction $f|_F : F \to F' := (p')^{-1}(x_0)$ is G-equivariant, since for $[u] \in G$ and $y \in F$ we have $f(y \cdot [u]) = f({}^y\tilde{u}(1)) = {}^{f(y)}\tilde{u}(1) = f(y) \cdot [u]$, because $f \circ {}^y\tilde{u}$ is the lift of u with start value $f(y)$.

One aim is to show that this functor is an equivalence of categories, i.e. there exists a functor in the opposite direction and the compositions of these two are up to natural isomorphisms the identity.

On the other hand, if we try to classify coverings $Y \to X$ with fixed Y we can consider the subgroup $A := \mathrm{Aut}(p)$ of the group $\mathrm{Homeo}(Y)$ of homeomorphisms. If f is a homomorphism between two coverings $p : Y \to X$ and $p' : Y \to X'$, i.e. $f : X \to X'$ is a continuous mapping with $f \circ p = p'$, then $A \subseteq A' := \mathrm{Aut}(p')$, since $h \in \mathrm{Aut}(p)$, i.e. $p \circ h = p$ implies $p' \circ h = f \circ p \circ h = f \circ p = p'$, i.e. $h \in \mathrm{Aut}(p')$. So this defines a functor from the category of homomorphisms between coverings with total space Y to the category of strictly discontinuously acting subgroups of $\mathrm{Homeo}(Y)$ given by the partial ordering '$\supseteq$'.

So the second aim will be to determine whether this is an equivalence of categories as well.

Let us first show that $\mathrm{Cov}_{\mathrm{pc}}(X) \to \mathrm{Act}_{\mathrm{tr}}(G)$ is FULL AND FAITHFUL, i.e.

6.11 Proposition. *Let X be locally path connected. Let $p : Y \to X$ and $p' : Y' \to X$ be two path-connected coverings with typical fibers $F := p^{-1}(x_0)$ and $F' := (p')^{-1}(x_0)$ and $G := \pi_1(X, x_0)$. Then $\mathrm{Hom}(p, p') \cong \mathrm{Hom}_G(F, F')$ via $\Phi \mapsto \Phi|_F$.*

Proof. The mapping $\Phi \mapsto \Phi|_F$ is injective, since $\Phi_1|_F = \Phi_2|_F$ implies $\Phi_1(y_0) = \Phi_2(y_0)$ and hence $\Phi_1 = \Phi_2$, by the uniqueness of lifts of p proved in 6.5.

Now the surjectivity: Let $\varphi : F \to F'$ be G-equivariant. As $\Phi : Y \to Y'$ we take the lift of $p : Y \to X$ which maps $y_0 \in F$ to $\varphi(y_0) \in F'$. This lift exists by 6.7, since $\pi_1(p)(Y, y_0) = G_{y_0} \subseteq G_{\varphi(y_0)} = \pi_1(p')(Y', \varphi(y_0))$ and Y is locally path-connected by 6.3.4.

Finally $\Phi|_F = \varphi$, since both are G-equivariant and coincide on y_0 by 6.9.7 □

6.12 Corollary. [7, 6.5.5] *Let Y be path connected and X be locally path connected. For any covering map $p : Y \to X$ we have*

$$\mathrm{Aut}(p) \cong \mathrm{Aut}_{\pi_1(X,x_0)}(p^{-1}(x_0)) \cong \mathrm{Norm}(\pi_1(p)(\pi_1(Y, y_0)))/\pi_1(p)(\pi_1(Y, y_0)).$$

The inverse of this isomorphism is given by mapping $[u] \in \mathrm{Norm}(\pi_1(p)(\pi_1(Y, y_0)))$ to the unique covering transformation f which maps y_0 to ${}^{y_0}\tilde{u}$.

Proof. Since the elements of Aut are just the isomorphisms of an object with itself, this follows directly from 6.11 and 6.9.9 □

6.13 Corollary. [7, 6.5.6] *If $p : Y \to X$ is a simple-connected covering with locally path connected base, then the group $\mathrm{Aut}(p)$ of covering transformations is isomorphic to the fundamental group $\pi_1(X, x_0)$.*

This can be used to calculate $\pi(X, x_0)$ by finding a covering $p : \tilde{X} \to X$ with simply connected total space $\tilde{X}$ (see 6.20) and then determine its automorphism group.

Proof. In this situation $\pi_1(Y, y_0) = \{1\}$ and hence $\mathrm{Norm}_G(G_{y_0}) = G$ and so we have $\mathrm{Aut}_G(F) \cong \mathrm{Norm}_G(G_{y_0})/\{1\} \cong G$. □

6.14 Lemma. *Let $p' : Y' \to X$ and $p : Y \to X$ be coverings of a locally path connected space X and $f : Y \to Y'$ be continuous and surjective with $p' \circ f = p$. Then f is a covering. If Y' is path-connected then f is automatically surjective.*

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "p"'] & & Y' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

Proof. Take a path connected trivialising set $U \subseteq X$. We show that every leaf $\tilde{V}$ of p over U is mapped by f into a leaf $\tilde{V}'$ of p' over U. In fact since the leafs are homeomorphic to U, they are path-connected as well, hence $f(\tilde{V})$ is completely contained in a leaf $\tilde{V}'$ of p' over $U = (p' \circ f)(\tilde{V})$. Thus $f^{-1}(\tilde{V}')$ is the topological disjoint union of certain leafs $\tilde{V}$ of p over U.

Let y_0 be a base point in Y and define $y_0' := f(y_0)$ and $x_0 := p'(y_0')$. If Y' is path connected and $y' \in Y'$ arbitrary, we can take a path c from y_0' to y'. Let ${}^{y_0}\widetilde{p' \circ c}$

be the lift with respect to p and y be its end point. Then $f \circ {}^{y_0}\widetilde{p' \circ c}$ is a lift of $p' \circ f \circ {}^{y_0}\widetilde{p' \circ c} = p \circ {}^{y_0}\widetilde{p' \circ c} = p' \circ c$, and hence equals c. Hence $f(y) = c(1) = y'$. $\square$

Coverings with fixed total space

Let us now tackle our second aim and try to invert the functor from the category of coverings with total space Y to that of subgroups of $\mathrm{Homeo}(Y)$.

So let A be any group acting on Y. Then we can consider the orbit space $Y/A := Y/\sim$, where $y \sim y' :\Leftrightarrow \exists g \in A$: $y' = g \cdot a$ with the quotient topology and the corresponding quotient mapping $\pi : Y \to Y/A$. Let us assume that this is a covering, i.e. for every $y \in Y$ there has to exist an open neighborhood $U \subseteq Y/A$ such that π^{-1} is a disjoint union of open subsets $\tilde{U}$ homeomorphic via π to U. So $U = \pi(\tilde{U})$ and $\pi^{-1}(U) = \pi^{-1}(\pi(\tilde{U})) = A(\tilde{U})$. Thus we would like that $g(\tilde{U}) \cap g'(\tilde{U}) = \emptyset$ for all $g \neq g'$. A group A satisfying this conditions is said to ACT STRICTLY DISCONTINUOUS on Y, i.e. every $y \in Y$ has a neighborhood V such that $g(V) \cap V = \emptyset$ for all $g \neq e$.

Obviously the action of $\mathrm{Aut}(p)$ of a covering $p : Y \to X$ has this property, since for f in $\mathrm{Aut}(p)$ we have that $f(\tilde{U}) \cap \tilde{U} \neq \emptyset$ implies that there exists some $y \in \tilde{U}$ with $f(y) \in \tilde{U}$. From $p(f(y)) = p(y)$ and since $p|_{\tilde{U}} : \tilde{U} \to U$ is injective we conclude that $f(y) = y$, but then $f = \mathrm{id}$ by 6.4.

6.15 Proposition. [7, 6.1.7] *Let G be a group acting strict discontinuous on a space Y. Then the canonical projection $\pi : Y \to Y/G$ is a covering. If Y is path connected and locally path connected then $\mathrm{Aut}(p) = G$.*

Warning: See below for an example, where Y/G is not Hausdorff.

Proof. We denote with $p : Y \to Y/G$ the quotient mapping. We have that $p(V)$ is trivialising with leafs ΦV for $\Phi \in G$.

Obviously every $\Phi \in G$ acts as covering transformation. Conversely let $\Phi \in \mathrm{Aut}(p)$. Then $p(y) = p(\Phi(y))$ and hence there is some $\Phi_y \in G$ with $\Phi_y \cdot y = \Phi(y)$. Since the two mappings Φ and Φ_y cover the identity and coincide on y they are equal by 6.4. $\square$

6.16 Example. [7, 5.7.5] See all the examples in 6.2. In particular we have $\mathbb{Z}$ as group of covering transformations of $\mathbb{R} \to S^1$ and $\mathbb{Z}_2$ as group of covering transformations of $S^n \to P^n$ and $\mathbb{Z}_n$ of $f_n : S^1 \to S^1$. Furthermore, the homotopy group of $M\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\mathbb{Z}_{|a|}$. and that of $L(p,q)$ is $\mathbb{Z}_p$.

Lemma. *Let G be a topological group and H a discrete subgroup. Then H acts strictly discontinuous on G by multiplication.*

Proof. Let V be a neighborhood of e in G such that $V \cap H = \{e\}$. Since $(x,y) \mapsto xy^{-1}$ is continuous $G \times G \to G$ we can find a neighborhood U of e with $U^{-1}U \subseteq V$. Then $hU \cap U \neq \emptyset$ implies the existence of $u \in U$ with $u' := hu \in U$. Hence $h = u'u^{-1} \in UU^{-1} \subseteq V$ and $h = e$. $\square$

Example. Consider the ordinary differential equation

$$\frac{dx}{dt} = \cos^2 x, \quad \frac{dy}{dt} = \sin x$$

Since this vector field is bounded, the solutions exist globally and we get a smooth function $\varphi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ associating to each $t \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^2$ the solution with value (x, y) at 0 at time t.
If the initial value satisfies $\cos^2 x = 0$ then the solution is $y(t) = y(0) + t \cdot \sin x$. Otherwise we have $\frac{dy}{dx} = \frac{\sin x}{\cos^2 x} = \frac{d}{dx}\frac{1}{\cos x}$, hence it has to be contained in $\{(y, x) : y(x) = \frac{1}{\cos x}\}$. Moreover the time it takes from $x = x_0$ to $x = x_1$ is given by $t(x_1) - t(x_0) = \int_{x_0}^{x_1} \frac{dt}{dx} = \int_{x_0}^{x_1} \frac{1}{\cos^2 x} dx = \tan x|_{x=x_0}^{x_1}$.

Remark that the quotient space $\mathbb{R}^2/\mathbb{R}$ is not Hausdorff (and $\mathbb{R}^2/\mathbb{Z}$ as well). It consists of a countable union $\bigsqcup_{\mathbb{Z}} \mathbb{R}$ of $\mathbb{R}'s$ together with the points $\pi/2 + \pi \cdot \mathbb{Z}$. A neighborhood basis of $\pi/2 + k\pi$ is given by end-interval of the two neighboring $\mathbb{R}'s$.

We may form the space $X := ([-\pi/2, \pi/2] \times \mathbb{R})/\sim$, where $(-\pi/2, -t) \sim (\pi/2, t)$. Since the action of $\mathbb{R}$ is compatible with this equivalence relation $\mathbb{R}$ acts fixed-point free on this Möbius strip X as well. The orbits of the discrete subgroup $\mathbb{Z} \subseteq \mathbb{R}$ are obviously closed subsets. However the action is not strictly discontinuous, since for any neighborhood of $[(\pi/2, 0)]$ some translate by $t \in \mathbb{Z}$ meets it again.

Note that for a group A acting strictly discontinuous on Y the covering $\pi : Y \to Y/A$ is normal in the following sense, since condition 6.17.3 is satisfied by the definition of Y/A.

6.17 Proposition. [7, 6.5.8] *A path-connected covering of a locally connected space X is called normal (or also regular), iff one of the following equivalent conditions are satisfied:*

1 *$\pi_1(p)(\pi_1(Y, y_0))$ is normal for (some) all y_0 in the fiber over x_0.*

2 *The conjugacy class of the covering consists of a single group.*

3 *The covering transformations act transitive on the fiber over x_0.*

4 *If one lift of a closed path through x_0 is closed, then so are all lifts.* □

In particular this is true if $\pi_1(X, x_0)$ is abelian or the covering is 2-fold or $\pi_1(Y, y_0) = \{1\}$.

Proof. This follows directly from the corollary in 6.9, since $\pi_1(p)(\pi_1(Y, y_0)) = G_y$ by 6.8, by $\mathrm{Aut}(p) \cong \mathrm{Aut}_G(F)$ by 6.12, and since the fixed points of $[u] \in G$ are exactly those $y \in F$ for which ${}^y\tilde{u}$ is closed. □

See 6.25 for examples of coverings which are not normal.

6.18 Theorem. [7, 6.5.3] *Let Y path-connected and locally path connected. Then we have an equivalence of categories $\mathrm{Cov}_{norm}(Y) \sim \mathrm{Act}_{str.dis.}(Y)$, where $\mathrm{Cov}_{norm}(Y)$ denotes the category of normal coverings with total space Y and $\mathrm{Act}_{str.dis.}(Y)$ the partially ordered set of strictly discontinuously acting subgroups of* $\mathrm{Homeo}(Y)$.

Proof. By what we have shown before we have functors $p \mapsto \mathrm{Aut}(p)$ and $(X \to X/A) \leftarrow A$. Remains to show that they are inverse to each other up to natural isomorphisms.

If we start with a group A, consider the covering $\pi : Y \to Y/A$, and its group $\mathrm{Aut}(\pi)$ of decktransformations, we obviously have $A \subseteq \mathrm{Aut}(\pi)$. The converse is true as well, since for $\Phi \in \mathrm{Aut}(\pi)$ and $y_0 \in F$ we have $\Phi(y_0) \in F$ and hence there exists a $g \in A$ with $g \cdot y_0 = \Phi(y_0)$. So the decktransformations Φ and g agree on y_0 and hence are identical being lifts of $\pi : Y \to Y/A$, i.e. $\Phi = g \in A$.

Conversely let $p : Y \to X$ be a normal covering and $A := \operatorname{Aut}(p)$. Then $\pi : Y \to Y/A$ is a covering and p factors over π to a homeomorphism $Y/\operatorname{Aut}(p) \cong X$.

Since every $\Phi \in \operatorname{Aut}(p)$ is fiber preserving, we have that p is constant on the $\operatorname{Aut}(p)$-orbits and hence factors to a surjective mapping $Y/\operatorname{Aut}(p) \to X$.

$$\begin{tikzcd} & Y \arrow[dl, "q"'] \arrow[dr, "p"] & \\ Y/\operatorname{Aut}(p) \arrow[rr, dotted, tail, two heads] & & X \end{tikzcd}$$

If $\operatorname{Aut}(p)$ acts transitive (i.e. p is normal) then this factorization is injective, since every two points in the same fiber are in the same orbit under $\operatorname{Aut}(p)$. Since $Y \to X$ and $Y \to Y/\operatorname{Aut}(p)$ are quotient mappings this bijection has to be a homeomorphism. $\square$

Coverings with fixed base space

Now we attack the first aim. It remains to show that the functor $\operatorname{Cov}_{\mathrm{pc}}(X) \to \operatorname{Act}_{\mathrm{tr}}(G)$ is surjective on objects up to isomorphisms.

So let a transitive right action of G on F be given. By 6.9.5 we have $F \cong_G G_y\backslash G$ for any isotropy group G_y with $y \in F$. In the particular case, where $G_y = \{1\}$ we have to find a covering $Y \to X$ with $\pi_1(Y, y) \cong G_y = \{1\}$, i.e. with simply connected total space Y.

6.19 Universal Covering.

For this we try to find "maximal" elements first. For transitive actions the maximal object is G with the right multiplication on itself by 6.9.7, since for every such action of G on some F we have a G-equivariant mapping $G \to F$.

The corresponding maximal covering $\pi : \tilde X \to X$ should thus have as fiber G and the action should be given by right multiplication. In particular we must have $G_y = \{1\}$ for all $y \in G$. Choose a base-point $y_0 \in \tilde X$ and let $x_0 := p(y_0)$. Since $G_{y_0} = \pi(p)\pi(\tilde X, y_0)$, we have that $\tilde X$ should be simply connected.

For every point $y \in \tilde X$ we find a path v_y from y_0 to y and since $\tilde X$ is simply connected the homotopy class $[v_y]$ rel. $\dot I$ is well defined. Thus we have a bijection $\tilde X \cong C((I, \{0\}), (\tilde X, y_0))/\underset{\mathrm{rel}.\dot I}{\sim}$. By the lifting property 6.5, this homotopy classes correspond bijectively to homotopy classes of paths starting at x_0, i.e.

$$\begin{tikzcd} \tilde X \arrow[r, "\cong"] \arrow[dr, "\pi"'] & C((I, \{0\}), (\tilde X, y_0))/\underset{\mathrm{rel}.\dot I}{\sim} \arrow[r, "\cong"] \arrow[d, "\pi\circ\mathrm{ev}_1"'] & C((I, \{0\}), (X, x_0))/\underset{\mathrm{rel}.\dot I}{\sim} \arrow[dl, "\mathrm{ev}_1"] \\ & X & \end{tikzcd}$$

Let U be a path-connected neighborhood of $x_1 \in X$. We calculate $p^{-1}(U)$. Note that $p^{-1}(x_1) = \{[v] : v \text{ is a path in } X \text{ from } x_0 \text{ to } x_1\}$.

$$\begin{aligned} p^{-1}(U) &= \{[w] : w(1) \in U\} \qquad \text{(now write } w = w \cdot u^{-1} \cdot u \text{ then)} \\ &= \{[v] \cdot [u] : v(0) = x_0, v(1) = x_1, u(0) = x_1, u(I) \subseteq U\} \\ &= \{[v] \cdot [u] : [v] \in p^{-1}(x_1), u(0) = x_1, u(I) \subseteq U\} \\ &= \bigcup_{[v] \in p^{-1}(x_1)} {}^{[v]}\tilde{U}, \end{aligned}$$

where ${}^{[v]}\tilde{U} := \{[v] \cdot [u] : u(0) = x_1, u(I) \subseteq U\}$ for $[v] \in p^{(-1)}(x_1)$. If U is path-connected, then $\pi|_{{}^{[v]}\tilde{U}} : {}^{[v]}\tilde{U} \to U$ is onto. In order that it is injective, we need that $u_0(1) = u_1(1) \Rightarrow [u_0] = [u_1]$, i.e. every closed curve in U through x_1 should be 0-homotopic in X. A space X which has a neighborhood of sets with this property is called SEMI-LOCALLY SIMPLY CONNECTED. Note that the closed curves are assumed to be local (i.e. contained in U), whereas the homotopy may leave U. Since any subset of such a set U has the same property, we get for a locally connected semi-locally simply connected space a neighborhood-basis of connected sets U with this property.

For a path-connected, locally path connected and semi-locally simply connected space X we thus define $\tilde{X}$ to be the set $C((I, \{0\}), (X, x_0))/ \underset{\text{rel.}\dot{I}}{\sim}$ and $\pi : \tilde{X} \to X$ by $\pi([u]) := u(1)$. Since for every U as above we want ${}^{[u]}\tilde{U}$ to be a leaf over U, we declare those sets to be open in $\tilde{X}$. In order that theses sets form the basis of a topology we have to take two such neighborhoods U_0 and U_1 and $y \in {}^{y_0}\tilde{U}_0 \cap {}^{y_1}\tilde{U}_1$. Then $\pi(y) \in U_0 \cap U_1$ and hence we can find such neighborhood $U \subseteq U_0 \cap U_1$ of $\pi(y)$. Then $y \in {}^{y}\tilde{U}$ and ${}^{y}\tilde{U} \subseteq \tilde{U}_0^{y_0} \cap \tilde{U}_1^{y_1}$.

Note that ${}^{[v_1]}\tilde{U} \cap {}^{[v_2]}\tilde{U} \neq \emptyset$ iff there exist curves u_i with $[v_1] \cdot [u_1] = [v_2] \cdot [u_2]$, where u_i are curves in U from x_1 to the same endpoint. Hence $[u_1] = [u_2] \in \pi_1(X, x_1)$ and thus $v_1 \sim v_2$ rel.$\dot{I}$, i.e. $[v_1] = [v_2]$. Obviously we have that $\pi|_{{}^{y}\tilde{U}} : {}^{y}\tilde{U} \to U$ is a homeomorphism, and hence $\pi : \tilde{X} \to X$ is a covering map.

Remark that for any path u starting at x_0 we have that $t \mapsto [u_t]$ is the lift with starting value $[\mathrm{const}_{x_0}] =: y_0$, where $u_t(s) := u(ts)$. Thus $\tilde{X}$ is path-connected, since for $[u] \in \tilde{X}$ the path $[u_t]$ connects $[\mathrm{const}_{x_0}]$ with $[u]$ in $\tilde{X}$.

Finally $\tilde{X}$ is simply connected. Let v be a closed curve in $\tilde{X}$ through y_0. Then $u := p \circ v$ is a closed curve trough x_0. Then $v(t) = [s \mapsto u(ts)]$, since both sides are lifts of u with starting point y_0. Hence $[\mathrm{const}_{x_0}] = [u_0] = v(0) = v(1) = [u_1] = [u]$. Since homotopies can be lifted, we have $\mathrm{const}_{y_0} \sim v$ rel. $\dot{I}$.

So we have proved:

6.20 Theorem. [7, 6.6.2] *Let X be path-connected, locally path-connected and semi-locally simply-connected. Then there exists a path-connected, simple-connected covering of X.*
Every simply connected path-connected covering of X covers any other path-connected covering.

Proof. The last statement follows, since we can lift the projection of any simple connected covering by 6.7 and the lift is a covering by 6.14. □

Corollary. *Let X be simply connected and locally path connected and $Y \to X$ be a path connected covering. Then p is a homeomorphism. In particular every simply connected open subset on the base space of a covering is a trivialising neighborhood.*

Proof. Since $\pi_1(X, x_0) = \{1\}$ acts transitively on $p^{-1}(x_0)$ the fiber has to be 1-pointed. □

Let us return to the question of surjectivity of $\mathrm{Cov}_{\mathrm{pc}}(X) \to \mathrm{Act}_{\mathrm{tr}}(G)$. Let $G \to F \cong G_y \backslash G$ be such an action. Since for any covering $p : Y \to X$ the universal covering $\pi : \tilde{X} \to X$ lifts by 6.7 to a mapping $f : \tilde{X} \to Y$ with $p \circ f = \pi$ we are interested in the following:

6.21

Now let $G = \pi_1(X, x_0)$ act transitively on $F \cong_G H\backslash G$, where $H := G_y < G$ is an isotropy subgroup. Let $\pi : \tilde{X} \to X$ be the universal covering, see 6.20. The associated action of G is then given by right multiplication and has as automorphism group $\mathrm{Aut}(\pi) = G$. Thus the subgroup H acts also strictly discontinuously on $\tilde{X}$ and hence $\tilde{X} \to \tilde{X}/H =: Y$ is a covering by 6.15. Furthermore the mapping $\pi : \tilde{X} \to X = \tilde{X}/G$ factors over $\tilde{X} \to Y$ to give some $p : Y \to X$ which is a covering by the following

6.22 Lemma. *Let X locally path connected and let $q : Z \to Y$ and $p : Y \to X$ be given.*

1 *If Y is path-connected, p and $p \circ q$ are coverings, then q is a covering.*

2 *If X is locally simply-connected, p and q are coverings, then $p \circ q$ is a covering.*

3 *If q and $p \circ q$ are coverings, then p is a covering.*

Proof. (1) is 6.14.

(2) Let p and q be coverings, with X locally simply connected. Then the leafs of p over a simply connected neighborhood U are again simply connected, hence are trivialising neighborhoods of q by the corollary in 6.20 and hence $(p \circ q)^{-1}(U) = q^{-1}(p^{-1}(U)) = q^{-1}(\bigsqcup_J V_j) = \bigsqcup_j q^{-1}(V_j)$ and $q^{-1}(V_j) \cong \bigsqcup_{J_j} V_j$. Thus $p \circ q$ is a covering as well.

(3) Let $p \circ q$ and q be coverings. We claim that p is a covering. Let U be path-connected and trivialising for $p \circ q$ and take a (necessarily open, since with X also Z and Y are locally path-connected) path component V of $p^{-1}(U)$. Let W be a path-component of $q^{-1}(V)$. Then W is contained in $q^{-1}(V) \subseteq q^{-1}(p^{-1}(U)) = (p \circ q)^{-1}(U)$ and hence is contained in some leaf $\tilde{U}$ of $p \circ q$ over U. Thus $p \circ q|_W : W \to X$ is an embedding, and hence also $q|_W : W \to V$. Since q is open and $V \subseteq p^{-1}U = q(q^{-1}(p^{-1}U)) = q((p \circ q)^{-1}U)$, we conclude from the path-connectedness of V that $q|_W \to V$ is onto and hence a homeomorphism. Thus $p|_V = p \circ q \circ (q|_W)^{-1}|_V$ is a homeomorphism. □

In the context of 6.21 it remains to show that the action corresponding to the covering $\tilde{X} \to \tilde{X}/H$ is isomorphic to H. The standard fiber of this covering is $H\backslash \pi^{-1}(x_0) = H\backslash G \cong F$. Furthermore, since the action of G on $G = \pi^{-1}(x_0)$ is given by right-multiplication, the same is true for the action on $p^{-1}(x_0) = H\backslash G$. Thus the action corresponding to Y is just the natural action of G on $H\backslash G$. Up to the isomorphism $H\backslash G \cong_G F$ we have thus found the desired covering of X, i.e. we have shown

6.23 Theorem. [7, 6.6.3] *Let X be path-connected, locally path-connected and semi-locally simply connected. Then we have an equivalence between the category of path-connected coverings of X and transitive actions of $G := \pi_1(X, x_0)$.*

$$\mathrm{Cov}_{pc}(X) \sim \mathrm{Act}_{tr}(G).$$

In particular we have a bijection between isomorphy classes of path-connected coverings of X with standard fiber F and conjugacy classes of transitive actions of $G := \pi_1(X, x_0)$ on F.

Proof. It is a general categorical result, that a full and faithful functor which is up to isomorphisms surjective on objects is an equivalence. In fact an inverse is given by selecting for every object in the image and inverse image up to an isomorphism and by the full and faithfulness this can be extended to a functor.

By 6.11 the functor is full and faithful and if a locally path connected space X has a simply connected covering. Then for every subgroup H of $G := \pi_1(X, x_0)$ there exists a covering $p : (Y, y_0) \to (X, x_0)$ with $H = \pi_1(p)\pi_1(Y, y_0)$ by 6.21.

In particular we have that the skeleton of the two categories are isomorphic. So we have a bijection between isomorphy-classes of path-connected coverings and isomorphy-classes of transitive actions, i.e. conjugacy classes of subgroups of $\pi_1(X, x_0)$. Remark that an isomorphisms of transitive actions is nothing else but a bijection of the representation spaces which intertwines the actions, i.e. $\rho(g, y) = \varphi^{-1}(\rho(g, \varphi(y)))$. If $F = F'$ this is just a bijection φ of F such that the two actions are conjugate to each other via φ. □

6.24 Corollary. [7, 6.3.4] *Two path-connected coverings of a locally path-connected space are equivalent, iff their conjugacy classes are the same.*

Proof. Use that $p \cong p' \Leftrightarrow F \cong_G F'$ by 6.11. □

Remark. We will now give an example that for two coverings $p : Y \to X$ and $p' : Y' \to X$ there may be more than one element in $\mathrm{Hom}(p, p')$ up to isomorphy.

By 6.23 it is enough to consider the corresponding question for transitive actions. For this we will consider subgroups $H < H' < G$ for which $\mathrm{Norm}_G(H) = H$ and $\mathrm{Norm}_G(H') = H'$ and for which a $g \notin H'$ exists with $gHg^{-1} \subseteq H'$.
In particular $\mathrm{Aut}_G(H) = \{1\}$ and $\mathrm{Aut}_G(H') = \{1\}$. Next we define two G-equivariant mappings $\varphi_j : G/H \to G/H'$. Such a mapping φ is uniquely determined by its value $\varphi(H) = H'g$ on H, since then $\varphi(Hg') = \varphi(H \cdot g') = H'g \cdot g' = H'(gg')$. This definition makes sense, since $Hg_1 = Hg_2$, i.e. $g_1 g_2^{-1} \in H \Rightarrow g g_1 g_2^{-1} g^{-1} \in gHg^{-1} \subseteq H'$, i.e. $H'gg_1 = H'gg_2$. So we may use the assumed g and e. These two mappings are not the same, since $g \notin H' \Rightarrow Hg \neq He$.
Remains to show that H, H', G and g can be found. So let F be finite, $G := \mathrm{Bij}(F)$ and let $\{F_j : j \in J\}$ be a partition of F in disjoint subsets of different non-zero cardinality. Then $H := \{\varphi \in G : \forall j \in J : \varphi(F_j) = F_j\}$ is a subgroup with $\mathrm{Norm}_G(H) = H$. In fact let $g \in G$ be such that $gHg^{-1} \subseteq H$ and assume there is some F_j with $g(F_j) \neq F_j$. Since $|g(F_j)| = |F_j|$ there have to exist $j \in J$ and $y_1, y_2 \in F_j$ such that $g(y_1)$ and $g(y_2)$ are in different sets F_{j_1} and F_{j_2}. Now take $h \in H$ given by exchanging y_1 and y_2. Then ghg^{-1} maps $g(y_1)$ to $g(y_2)$, and hence F_{j_1} is not invariant, so $ghg^{-1} \notin H$.

If $F = \{1,2,3,4,5,6,7,8,9,10\}$ and $F_1 = \{1\}$, $F_2 = \{2,3\}$, $F_3 = \{4,5,6\}$ and $F_4 = \{7,8,9,10\}$. Let H be given by the partition $\{F_1, F_2, F_3, F_4\}$ and H' be given by $\{F_1 \cup F_2, F_3 \cup F_4\}$ and let $g := (1,4)(2,5)(3,6) \notin H'$. Then $gHg^{-1} \subseteq H'$, since $g^{-1}(F_1 \cup F_2) = F_3$, $g^{-1}(F_3) = F_1 \cup F_2$ and $g^{-1}(F_4) = F_4$, hence $ghg^{-1}(F_3) = gh(F_1 \cup F_2) = g(F_1 \cup F_2) = F_3$, $ghg^{-1}(F_1 \cup F_2) = gh(F_3) = g(F_3) = F_1 \cup F_2$ and $ghg^{-1}(F_4) = F_4$.

Remark. See [4, p.181]: 2nd countability is inherited by the total space of a covering.

See [4, p.187]: Let $p : Y \to X$ and $p' : Y' \to X$ be two coverings. Then there may exists homomorphisms in $\mathrm{Hom}(p,p')$ and $\mathrm{Hom}(p',p)$ without $p \cong p'$.
In fact we can translate this to transitive actions. So we need subgroups $H < G$ and $H' < G$ which are not conjugate, but such that H is contained in some conjugate $g^{-1}H'g$ of H' and conversely. Then $H\backslash G \to (g^{-1}H'g)\backslash G \cong H'\backslash G$ is G-equivariant as is $H'\backslash G \to ((g')^{-1}Hg')\backslash G \cong H\backslash G$, but $H\backslash G$ is not isomorphic to $H'\backslash G$.
Remains to prove the existence of such groups. G can then be realized as fundamental group of a 2-dimensional CW-complex.

6.25 Example. [7, 6.7.3] Since every subgroup of an abelian group is normal and also any subgroup of index two, the simplest non-normal covering could best be found among the 3-fold coverings of $S^1 \vee S^1$. Thus we try to identify all these coverings and also those of the torus $S^1 \times S^1$ and Klein's bottle. For G we have in these cases $\langle \alpha, \beta : \emptyset \rangle$, $\langle \alpha, \beta : \alpha\beta = \beta\alpha \rangle$ and $\langle \alpha, \beta : \alpha^2\beta^2 = 1 \rangle$.

First we have to determine all transitive actions of $\langle \alpha, \beta : \emptyset \rangle$ on $\{0,1,2\}$, i.e. group-homomorphisms from from the free group with two generators a and b into that group of permutations of $\{1,2,3\}$. We write such permutations in cycle notation, i.e. these are

$$\{(1), (12), (13), (23), (123), (132)\}.$$

Where (1) has order 1, (123) and (132) have order 3 and the rest order 2. Up to symmetry we may assume that the image a of α has order less or equal to the image b of β.
If $\operatorname{ord} a = 1$, i.e. $a = (1)$ then $\operatorname{ord} b$ has to be 3 (otherwise the resulting action is not transitive) and the two possible choices are conjugate via (12).
If $\operatorname{ord} a = 2$, then $\operatorname{ord} b$ can be 2, but has to be different from a (otherwise not transitive) and any two choices $\{a, b\}$ and $\{a', b'\}$ are conjugate via the common element $c \in \{a, b\} \cap \{a', b'\}$, or b can have order 3, and again the choices of b are conjugate by a, and that of a are conjugate by b or b^{-1}.
If the order of a and b are 3, they can be either the same or different.

So we get representatives for all transitive actions

a	b	normal for $S^1 \vee S^1$	Klein	$S^1 \times S^1$
(1)	(123)	$+$	$-$	$+$
(123)	(1)	$+$	$-$	$+$
(12)	(13)	$-$	$+$	$-$
(12)	(123)	$-$	$-$	$-$
(123)	(12)	$-$	$-$	$-$
(123)	(123)	$+$	$-$	$+$
(123)	(132)	$+$	$+$	$+$

Remark that the action is normal iff every $g \in G$ acts either fixed-point free or is the identity. Thus at least the generators a and b have be of order 3 or 1. This excludes the 3 actions in the middle. Since the group generated by a and b is $\{(1), (123), (132)\}$ in all other cases, these are normal.

The last two columns are determined by checking $a^2b^2 = 1$ and $ab = ba$.

6.26 Example. [7, 6.1.5] There is a three-fold covering of $S^1 \vee S^1$, which is not normal.

Proof. Let $\{y_1, y_2, y_3\}$ be the fiber over x_0, let a and b denote parametrizations of the two factors S^1 in $S^1 \vee S^1$ and let a_1, a_2, a_3 be the leafs over a and b_1, b_2, b_3 be the leafs over b. Let b_i be from y_{i+1} to y_{i+2} (mod 3). Let a_1 be a closed path at y_1 and a_2 and a_3 connect y_2 and y_3 in opposite directions.

So a has closed as well as not closed lifts. □

6.27 Proposition. [7, 6.8.1] *Let $p : Y \to X$ be a covering. Then the following statements are true:*

1 *If X is a CW-complex then so is Y. The cells of Y are the path-components (leafs) of $p^{-1}(e)$ for all cells e of X.*

2 *If X is a manifold so is Y.*

3 *If X is a topological group, so is Y.*

Proof. (1) Let e be a cell of X. Since e is locally path-connected so is $p^{-1}(e)$ and every component of $p^{-1}(e)$ is homeomorphic to e via the projection, since the restriction of the projection is a covering map and e is simply connected. We may lift a characteristic map to a characteristic map of the lifted cell. And clearly the properties (C) and (W) are satisfied.

(2) We may take the chart domains to be trivialising sets in X. The leafs can then be used as chart domains of Y.

(3) The group structures $X \times X \to X$ and $X \to X$ can be lifted to mappings $Y \times Y \to Y$ and $Y \to Y$: In fact chose $1 \in p^{-1}(1)$. Then $\pi_1(\mu \circ (p \times p))([u_1], [u_2]) = [(p \circ u_1) \cdot (p \circ u_2)] = \pi_1(p)[u_1 \cdot u_2]$. Thus $\mu \circ (p \times p)$ has a unique lift to $\tilde{\mu} : Y \times Y \to Y$ by 5.7. Similarly $\pi_1(\nu \circ p)([u]) = [p \circ u]^{-1} = \pi_1(p)[u]^{-1}$. □

6.28 Theorem. [7, 6.9.1] *Every subgroup H of a free group G is free. If H has finite index k in G, then* $\mathrm{rank}(H) = (\mathrm{rank}(G) - 1) \cdot k + 1$. *In particular, there exist subgroups of any finite rank in the free group of rank 2.*

Proof. Let G be a free group and H a subgroup of G. Then G is the fundamental group of a join X of 1-spheres. Since X has a universal covering $\tilde{X} \to X$, there exists also a covering $Y \to X$ with isotropy subgroup H. By 6.27 Y is a graph, and hence its homotopy group H is a free group.

If H has finite index k in G, then $\mathrm{rank}(H) = (\mathrm{rank}(G) - 1) \cdot k + 1$, since by 5.48 Y has k-times as many cells of fixed dimension as X.

Let $G := \langle \{a, b\} : \emptyset \rangle$ and $k \geq 1$. Then there exists a unique surjective homomorphism $\varphi : G \to \mathbb{Z}_k$ with $\varphi(a) = 1$ and $\varphi(b) = 0$. Thus $H := \ker \varphi$ has index k in G and hence $\mathrm{rank}\, H = 1 + k$. □

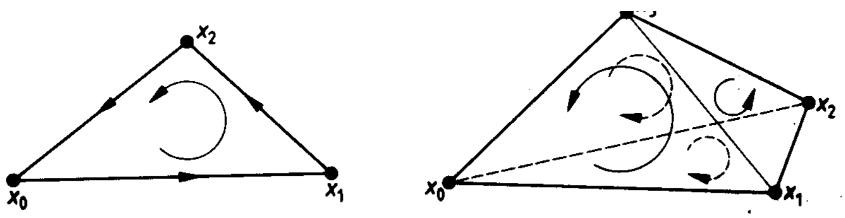
7 Simplicial Homology

Since it is difficult to calculate within non Abelian groups we try to associate abelian groups to a topological space. Certainly we could take ${}^{ab}\pi_1(X)$, but in order to calculate this we can hardly avoid the non-commutative group $\pi_1(X)$ as intermediate step. So we have to find a more direct approach. We start with the most explicitly describable spaces, i.e. the simplicial complexes K. By 3.30 there is to each closed curve $|\dot\Delta| = S^1 \to |K|$ a homotopic simplicial approximation c from some barycentric refinement of $\dot\Delta$ to K. Note that any barycentric refinement of $\dot\Delta$ is just a finite sequence of adjacent edges. If we want to get rid of commutativity we should consider c as formal linear combination $\sum_\sigma n_\sigma \cdot \sigma$ with integer coefficients n_σ of oriented edges σ in K (we dropped those edges with same start and end point). That c is a closed (and connected) curve corresponds to the assumption that every vertex occurs equally often as start and as end point. So we can associate to such a linear combination $\sum_\sigma n_\sigma \cdot \sigma$ a boundary $\partial \sum_\sigma n_\sigma \cdot \sigma := \sum_\sigma n_\sigma \cdot \partial\sigma$, where $\partial\sigma$ is just $x_1 - x_0$, when σ is the edge from x_0 to x_1. Thus $c := \sum_\sigma n_\sigma \cdot \sigma$ is closed iff $\partial c = 0$.

Next we should reformulate what it means that c is 0-homotopic, i.e. there exists an extension $\tilde c : |\Delta| = D^2 \to |K|$. Again by 3.30 we may assume that $\tilde c$ is simplicial from some barycentric refinement of Δ. The image of $\tilde c$ can be viewed as 2-chain, i.e. formal linear combination $\sum n_\sigma \cdot \sigma$ with integer coefficients n_σ of ordered 2-simplices σ of K. Note that an orientation of a triangle induces (or even is) a coherent orientation on the boundary edges. That $\tilde c$ is an extension of c means that the edges of these simplices, which do not belong to c, occur as often with one orientation as with the other. And those which do belong to c occur exactly that often more with that orientation than with the other. So we can define the boundary $\partial(\sum_\sigma n_\sigma \cdot \sigma)$ of a linear combination of 2-simplices as $\sum_\sigma n_\sigma \cdot \partial\sigma$, where $\partial\sigma = \langle x_0, x_1\rangle + \langle x_1, x_2\rangle + \langle x_2, x_0\rangle$, when σ is the triangle with vertices x_0, x_1, x_2 in that ordering. Then c is 0-homotopic iff there exists a 2-chain with boundary c. We call such a chain c exact or 0-homologue. The difference between closed and exact 1-chains is an obstruction to simply connectedness of $|K|$. At the same time this easily generalizes to k-chains:

Homology groups

7.1 Definition. [7, 7.1.1] [7, 7.1.4] An orientation of a q-simplex (with $q > 0$) is an equivalence class of linear orderings of the vertices, where two such orderings are equivalent iff they can be translated into each other by an even permutation. So if a q-simplex σ has vertices $x_0, \dots, x_q$ then an orientation is fixed by specifying an ordering $x_{\sigma(0)} < \dots < x_{\sigma(q)}$ and two such orderings σ and σ' describe the same oriented simplex iff $\mathrm{sign}(\sigma' \circ \sigma^{-1}) = +1$. We will denote the corresponding ordered simplex with $\langle x_{\sigma(0)}, \dots, x_{\sigma(q)}\rangle$. Let σ^{-1} denote the simplex with the same vertices as σ but the opposite orientation.



The q^{th}-chain group

$C_0(K) := {}^{ab}\langle\{\sigma : \sigma \text{ is 0-simplex in } K\}\rangle$

$C_q(K) := {}^{ab}\langle\{\sigma : \sigma \text{ is ordered } q\text{-simplex in } K\} : \{\sigma^{-1} + \sigma : \sigma \text{ is ordered } q\text{-simplex in } K\}\rangle$

is the free abelian group with all ordered q-simplices as generators modulo the relation $\sigma + \sigma^{-1} = 0$.

7.2 Lemma. [7, 7.1.5] *By picking an ordering of each simplex we get an unnatural isomorphism with the free abelian group with the unordered q-simplices as generators.*

Proof. We consider the map which associates to each ordered simplex either the unordered simplex, if the ordering is the selected one, or the negative of the unordered simplex, otherwise. This induces a surjective group-homomorphisms $O := {}^{\mathrm{ab}}\mathcal{F}(\{\sigma : \sigma \text{ is ordered } q\text{-simplex in} K\}) \to U := {}^{\mathrm{ab}}\mathcal{F}(\{\sigma : \sigma \text{ is unordered } q\text{-simplex in} K\})$. It factors over $C_q(K)$, since $\sigma + \sigma^{-1}$ is mapped to 0. The induced surjective homomorphism $C_q(X) \to U$ is injective, since $g := \sum n_\sigma \cdot \sigma \in O$ is mapped to $\sum \pm(n_\sigma - n_{\sigma^{-1}}) \cdot \sigma$ and this vanishes only if $n_\sigma = n_{\sigma^{-1}}$, i.e. if the image of g in $C_q(K)$ is 0. □

Note that

$$\begin{aligned}
\partial\langle x_0, x_1\rangle &= x_1 - x_0 = \langle\widehat{x_0}, x_1\rangle + \langle x_0, \widehat{x_1}\rangle^{-1};\\
\partial\langle x_0, x_1, x_2\rangle &= \langle x_0, x_1\rangle + \langle x_1, x_2\rangle + \langle x_2, x_0\rangle\\
&= \langle x_0, x_1, \widehat{x_2}\rangle + \langle\widehat{x_0}, x_1, x_2\rangle + \langle x_0, \widehat{x_1}, x_2\rangle^{-1}\\
&= \langle\widehat{x_0}, x_1, x_2\rangle + \langle x_0, \widehat{x_1}, x_2\rangle^{-1} + \langle x_0, x_1, \widehat{x_2}\rangle
\end{aligned}$$

Let σ be the tetrahedron with the natural orientation $x_0 < x_1 < x_2 < x_3$. Its faces should have orientation $\langle x_1, x_2, x_3\rangle$, $\langle x_0, x_2, x_3\rangle^{-1}$, $\langle x_0, x_1, x_3\rangle$ and $\langle x_0, x_1, x_2\rangle^{-1}$.

This leads to the generalized definition:

7.3 Definition. [7, 7.1.2] [7, 7.1.6] The ORDERING of the face σ' opposite to the vertex x_j in $\sigma = \langle x_0, \dots, x_q\rangle$ should be given by $\sigma' := \langle x_0, \dots, x_{j-1}, \widehat{x_j}, x_{j+1}, \dots, x_q\rangle^{(-1)^j}$. Let us show that this definition makes sense. So let σ be an even permutation of $\{0, \dots, q\}$ and σ' be the face opposite to j. Then $\langle x_0, \dots, x_q\rangle = \langle x_{\sigma(0)}, \dots, x_{\sigma(q)}\rangle$ and we have to show that

$$\langle x_0, \dots, x_{j-1}, \widehat{x_j}, x_{j+1}, \dots, x_q\rangle^{(-1)^j} = \langle x_{\sigma(0)}, \dots, \widehat{x_j}, \dots, x_{\sigma(q)}\rangle^{(-1)^i},$$

where i is the position of j in $\sigma(0), \dots, \sigma(q)$, i.e. $i = \sigma^{-1}(j)$. Without loss of generality let $i \le j$ (otherwise use σ^{-1}). Consider the permutations of $\{0, \dots, q\}$ given by the table

$$\begin{array}{ccccccccc}
0 & \dots & i-1 & i & \dots & j-1 & j & j+1 & \dots & q\\
0 & \dots & i-1 & i+1 & \dots & j & i & j+1 & \dots & q\\
\sigma(0) & \dots & \sigma(i-1) & \sigma(i+1) & \dots & \sigma(j) & \sigma(i) & \sigma(j+1) & \dots & \sigma(q)
\end{array}$$

The first one is the cyclic permutation $(i, i+1, \dots, j-1, j)$, hence has sign $(-1)^{j-i}$. The second one is σ and the composite leafs j invariant has sign $(-1)^{j-i} \cdot \operatorname{sign}\sigma = (-1)^{j-i}$ and induces the identity

$$\langle x_0, \dots, x_{j-1}, \widehat{x_j}, x_{j+1}, \dots, x_q\rangle = (-1)^{j-i}\langle x_{\sigma(0)}, \dots, \widehat{x_j}, \dots, x_{\sigma(q)}\rangle.$$

Now we define the BOUNDARY OF AN ORIENTED q-SIMPLEX $\sigma = \langle x_0, \dots, x_q\rangle$ (for $q > 0$) to be

$$\partial\sigma = \sum_{j=0}^{q} (-1)^j \langle x_0, \dots, x_{j-1}, \widehat{x_j}, x_{j+1}, \dots, x_q\rangle.$$

For $q \leq 0$ one puts $\partial\sigma = 0$. Extended by linearity we obtain a linear mapping $\partial := \partial_q : C_{q+1}(K) \to C_p(K)$.

7.4 Definition. [7, 7.1.7] With $Z_q(K) := \operatorname{Ker}(\partial_q)$ we denote the set of CLOSED q-CHAINS. With $B_q(K) := \operatorname{Im}(\partial_{q+1})$ we denote the set of EXACT (or 0-HOMOLOGOUS) q-chains. Two q chains are called HOMOLOGOUS iff their difference is exact.

7.5 Remark. [7, 7.1.8] If $q < 0$ then $C_q(K) := \{0\}$. If $q = 0$ then $C_q(K) = Z_q(K)$, If $q = \dim(K)$ then $B_q(K) = \{0\}$

7.6 Theorem. [7, 7.1.9] $0 = \partial^2 = \partial_{q-2} \circ \partial_{q-1}$ *and hence* $B_q \subseteq Z_q$.

Proof. Let $\sigma = \langle x_0, \dots, x_q\rangle$ with $q \geq 2$. Then

$$\begin{aligned}
\partial\partial\sigma &= \partial \sum_{j=0}^{q} (-1)^j \langle x_0, \dots, \widehat{x_j}, \dots, x_q\rangle \\
&= \sum_{j=0}^{q} (-1)^j \Bigg(\sum_{i=0}^{j-1} (-1)^i \langle x_0, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_q\rangle + \\
&\quad + \sum_{i=j+1}^{q} (-1)^{i-1} \langle x_0, \dots, \widehat{x_j}, \dots, \widehat{x_i}, \dots, x_q\rangle \Bigg) \\
&= \sum_{i<j} ((-1)^{i+j} - (-1)^{j+i}) \langle x_0, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_q\rangle \\
&= 0 \quad \square
\end{aligned}$$

7.7 Definition. [7, 8.3.1] A CHAIN COMPLEX is a family $(C_q)_{q\in\mathbb{Z}}$ of Abelian groups together with group-homomorphisms $\partial_q : C_q \to C_{q-1}$ which satisfy $\partial_q \circ \partial_{q+1} = 0$. We can consider also $C := \coprod_{q\in\mathbb{Z}} C_q$, which is a graded Abelian group and $\partial := \coprod_{q\in\mathbb{Z}} \partial_q$, which is a graded group homomorphism $C \to C$ of degree -1 and satisfies $\partial^2 = 0$.

For a chain complex (C, ∂) we define its HOMOLOGY $H(C, \partial) := \ker\partial / \operatorname{im}\partial$. This is a graded abelian group with $H(C, \partial) = \coprod_{k\in\mathbb{Z}} H_q(C, \partial)$, where $H_q(C, \partial) := \ker\partial_q / \operatorname{im}\partial_{q+1}$.

7.8 Definition. [7, 7.1.10] The factor group $H_q(K) := Z_q(K)/B_q(K)$ is called the q-th HOMOLOGY GROUP of K.

Examples and exact sequences

7.9 Example. [7, 7.2.1] We consider the following simplicial complex K formed by one triangle σ_2 with vertices x_0, x_1, x_2 and edges σ_1^0, σ_1^1, σ_1^2 one further point x_3 connected by 1-simplices σ_1^3 and σ_1^4 with x_1 and with x_2.

The generic chains are of the form

$$c_0 = \sum_i a_i\, x_i \qquad \text{with } a_i \in \mathbb{Z}$$
$$c_1 = \sum_i b_i\, \sigma_1^i \qquad \text{with } b_i \in \mathbb{Z}$$
$$c_2 = m\, \sigma_2$$

Since $\partial c_2 = m(\sigma_1^0 + \sigma_1^1 + \sigma_1^2) \neq 0$ for $m \neq 0$ the only closed 2-cycle is 0, hence $H_2(K) = 0$.

$$\partial c_1 = (b_1 - b_2)\, x_0 + (b_2 - b_0 + b_3)\, x_1 + (b_0 - b_1 - b_4)\, x_2 + (b_4 - b_3)\, x_3$$

vanishes, iff $b_1 = b_2$, $b_3 = b_4$ and $b_1 + b_3 = b_0$. So $Z_1(K)$ is formed by $c_1 = b_1(\sigma_1^0+\sigma_1^1+\sigma_1^2)+b_3(\sigma_1^0+\sigma_1^3+\sigma_1^4)$ and hence $z_1 := \sigma_1^0+\sigma_1^1+\sigma_1^2$ and $z_1' := \sigma_1^0+\sigma_1^3+\sigma_1^4$ form a basis with $\partial c_2 = m\, z_1$. So $B_1(K) = \{m\, z_1 : m \in \mathbb{Z}\}$ und $H_1(K) \cong \mathbb{Z}$.

For the determination of $H_0(K)$ see 7.12.

7.10 Remark. [7, 7.2.2] We have $H_q(K) = 0$ for $q < 0$ and $q > \dim K$. Furthermore $H_{\dim K}(K)$ is a free abelian group as subgroup of $C_{\dim K}(K)$.

7.11 Lemma. [7, 7.2.3] *If K has components $K_1, \ldots, K_m$ then $C_q(K) \cong \bigoplus_{j\le m} C_q(K_j)$ and $H_q(K) \cong \bigoplus_{j\le m} H_q(K_j)$.* □

7.12 Lemma. [7, 7.2.4] *$H_0(K)$ is a free abelian group. Generators are given by choosing in each component one point.*

Proof.

$$\begin{tikzcd} C_1(K) \arrow[r, two heads, "\partial"] & B_0(K) \arrow[r, hook] \arrow[d, equal] & Z_0(K) \arrow[r, two heads] \arrow[d, equal] & C_0(K) \arrow[d, dotted, "\cong"] \\ & \ker(\varepsilon) \arrow[r, hook] & C_0(K) \arrow[r, "\varepsilon"] & \mathbb{Z} \end{tikzcd}$$

Because of 7.11 we may assume that K is connected. Let $\varepsilon : C_0(K) \to \mathbb{Z}$ be the linear map given by $x \mapsto 1$ for all vertices $x \in K$. Obviously ε is surjective. Remains to show that its kernel is $B_0(K)$. Every two vertices x_0 and x_1 are homologous, since there is a 1-chain connecting x_0 with x_1. Thus $c := \sum_x n_x \cdot x$ is homologous to $(\sum_x n_x) \cdot x_0 = \varepsilon(c) \cdot x_0$ and hence $\operatorname{Ker}(\varepsilon) \subseteq B_0$. Conversely let $c = \partial(\sum_\sigma n_\sigma \cdot \sigma) = \sum_\sigma n_\sigma \cdot \partial\sigma$. Since $\varepsilon(\partial\langle x_0, x_1\rangle) = \varepsilon(x_1 - x_0) = 0$ we have the converse inclusion. □

7.13 Example. [7, 7.2.10] The homology of $X := S^1 \times I$. Note that $S^1 \times I \sim S^1$ and hence we would expect $H_2(X) = 0$ and $H_1(X) = {}^{ab}\pi_1(S^1) = \mathbb{Z}$. Let us show that this is in fact true. We consider the triangulation given by 6 triangles. We will show in a later section that the homology does not depend on the triangulation. We orient the triangles in the natural way.

$H_2(X)$: Let $z_2 = \sum_{\dim\sigma=2} n_\sigma \cdot \sigma \in Z_2(X)$, i.e. $\partial z_2 = 0$. Since those edges, which join the inner boundary with the outer one belong to exactly two 2-simplices, the coefficients of these two simplices have to be equal. So $n := n_\sigma$ is independent on σ. However $\partial(\sum_\sigma \sigma)$ is the difference of the outer boundary and the inner one, hence not zero, and so $z_2 = n(\sum_\sigma \sigma)$ is a cycle only if $n = 0$, i.e. $H_2(X) = \{0\}$.

$H_1(X)$: Let $[z_1] \in H_1(X)$, i.e. $z_1 = \sum_{\dim\sigma=2} n_\sigma \cdot \sigma \in Z_1(X)$ with $\partial z_1 = 0$. Since we may replace z_1 by a homologous chain, it is enough to consider linear combinations of a subset of edges, such that for each triangle at least 2 edges belong to this subset. In particular we can use the 6 interior edges. Since each vertex is a boundary point of exactly two of these edges the corresponding coefficients have to be equal (if we orient them coherently). Thus z_1 is homologous to a multiple of the sum c_1 of theses 6 edges. Hence $H_1(X)$ is generated by $[c_1]$. The only multiple of c_1, which is a boundary, is 0, since the boundary of $\sum_{\dim\sigma=2} n_\sigma \cdot \sigma$ contains $n_\sigma \cdot \sigma_1$, where σ_1 is the edge of σ, which is not an interior one.

7.14 Example. [7, 7.2.14] The homology of the projective plane $X := P^2$. We use the triangulation of P^2 by 10 triangles described in 3.10. And we take the obvious orientation of all triangles. Note however that on the "boundary edges" these orientations are not coherent.

$H^2(X)$: Let $z_2 = \sum_{\dim\sigma=2} n_\sigma \cdot \sigma \in Z_2(X)$, i.e. $\partial z_2 = 0$. Since those edges, which belong to the "interior" in the drawing belong to exactly two 2-simplices, the coefficient of these two simplices have to be equal. So $n := n_\sigma$ is independent on σ. However $\partial(\sum_\sigma \sigma)$ is twice the sum $a+b+c$ of three edges along which we have to glue, and hence is not zero. So $z_2 = n(\sum_\sigma \sigma)$ is a cycle only if $n = 0$, i.e. $H_2(X) = \{0\}$.

$H_1(X)$: Let $[z_1] \in H_1(X)$, i.e. $z_1 = \sum_{\dim\sigma=1} n_\sigma \cdot \sigma \in Z_1(X)$ with $\partial z_1 = 0$. Now we may replace z_1 by a homologous chain using all edges except the 3 inner most ones and the 3 edges normal to the "boundary". Now consider the vertices on the inner most triangle. Since for each such point exactly two of the remaining edges have it as a boundary point, they have to have the same coefficient, and hence may be replaced by the corresponding "boundary" parts. So z_1 is seen to be homologous to a sum of "boundary" edges. But another argument of the same kind shows that they must occur with the same coefficient. Hence $H_1(X)$ is generated by $a+b+c$. As we have show above $2(a+b+c)$ is the boundary of the sum over all triangles. Whereas $a+b+c$ is not a boundary of some 2-chain $\sum_\sigma n_\sigma \cdot \sigma$, since as before such a chain must have all coefficients equal to say n and hence its boundary is $2n(a+b+c)$. Thus $H_1(P^2) = \mathbb{Z}_2$, which is no big surprise, since $\pi_1(P^2) = \mathbb{Z}_2$.

7.15 Definition. [7, 8.2.1] A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of abelian groups is called EXACT at B iff $\ker g = \operatorname{im} f$. A finite (or infinite) sequence of groups C_q and group homomorphisms $f_q : C_{q+1} \to C_q$ is called exact if it is exact at all (but the end) points.

7.16 Remark. [7, 8.2.2] (1) A sequence $0 \to A \xrightarrow{f} B$ is exact iff f is injective.

(2) A sequence $A \xrightarrow{f} B \to 0$ is exact iff f is surjective.

(3) A sequence $0 \to A \xrightarrow{f} B \to 0$ is exact iff f is bijective.

(4) Let $A_{q+1} \xrightarrow{f_{q+1}} A_q \xrightarrow{f_q} A_{q-1} \xrightarrow{f_{q-1}} A_{q-2}$ be exact. Then the following statements are equivalent.

1 $f_q = 0$;

2 f_{q+1} is onto;

3 f_{q-1} is injective.

7.17 Lemma. *Let $0 \to C_0 \to C_1 \to \cdots \to C_n \to 0$ be an exact sequence of finitely generated free abelian groups. Then $\sum_{q=0}^n (-1)^q \operatorname{rank} C_q = 0$.*

Proof. For an $\mathbb{Z}$-linear map (i.e. abelian group homomorphism) f we have $\operatorname{rank}(\ker f)+\operatorname{rank}(\operatorname{im} f) = \operatorname{rank}(\operatorname{dom} f)$ by the pendent to the classical formula from linear algebra. Thus taking the alternating sum of all $\operatorname{rank}(\operatorname{dom} f_q)$ gives a telescoping one and hence evaluates to 0. □

7.18 Proposition. [7, 7.2.5] *Let K be a one dimensional connected simplicial complex. Then $H_1(K)$ is a free abelian group with $1-\alpha_0+\alpha_1$ many generators, where α_i are the number of i-simplices.*

Compare with 5.48.

Proof. Consider the sequence

$$\begin{tikzcd} 0 \arrow[r] & Z_1 \arrow[r, hook] \arrow[d, "\cong"'] & C_1 \arrow[r, "\partial"] & C_0 \arrow[r, two heads, "\varepsilon"] \arrow[dr, "\varepsilon"'] & H_0 \arrow[r] \arrow[d, "\cong"] & 0 \\ & H_1 & & & \mathbb{Z} & \end{tikzcd}$$

It is exact by definition and the vertical arrow is an isomorphism by 7.12 and hence we get by 7.17 the equation $0 = \operatorname{rank}(H_1) - \alpha_1 + \alpha_0 - 1$ □

7.19 Definition. [7, 7.2.6] Let K be a simplicial complex in $\mathbb{R}^n$. Let $p \in \mathbb{R}^n$ be not contained in the affine subspace generated by all $\sigma \in K$. Let $p\star\langle x_0,\dots,x_q\rangle := \langle p, x_0, \dots, x_q\rangle$. Let $p\star K := K \cup \{p\star\sigma : \sigma \in K\} \cup \{p\}$. It is called the CONE over K with vertex p and is obviously a simplicial complex. Note that we can extend $p\star(\_)$ to a linear mapping $C_q(K) \to C_q(p\star K)$.

7.20 Proposition. [7, 7.2.7] *He have $H_q(p\star K) = \{0\}$ for all $q \neq 0$.*

Proof. Let c be a q-chain of K. Then

$$\partial(p\star c) = c - \varepsilon(c)p \quad \text{if } q = 0$$
$$\partial(p\star c) = c - p\star\partial c \quad \text{if } q > 0$$

Note that this shows that any q-chain c (with $q > 0$) is homologous to $p\star\partial c$. We may assume $c = \langle x_0, \dots, x_q\rangle$. For $q = 0$ we have $\partial(p\star c) = \partial\langle p, x_0\rangle = x_0 - p = c - \varepsilon(c)p$. For $q > 0$ we get

$$\begin{aligned}\partial(p\star c) &= \partial\langle p, x_0, \dots, x_q\rangle \\ &= \langle x_0,\dots,x_q\rangle - \sum_{i=0}^{q}(-1)^i\langle p, x_0, \dots, \widehat{x_i}, \dots, x_q\rangle = c - p\star\partial c.\end{aligned}$$

Now let $c \in Z_q(p\star K)$. We have to show that it is a boundary. Clearly c is a combination of simplices of the form $\langle x_0,\dots,x_q\rangle$ and $\langle p, x_0, \dots, x_{q-1}\rangle$, i.e. $c = c_q + p\star c_{q-1}$ with $c_q \in C_q$ and $c_{q-1} \in C_{q-1}$. Hence $c = c_q + p\star c_{q-1} = \partial(p\star c_q) + p\star\partial c_q + p\star c_{q-1}$. So $p\star(\partial c_q + c_{q-1}) \in Z_q$. But by the equation above the boundary of such a cone vanishes only if $\partial c_q + c_{q-1} = 0$, hence c is a boundary. □

7.21 Corollary. [7, 7.2.8] *Let σ_n be an n-simplex and define $K(\sigma_n) := \{\tau : \tau \leq \sigma_n\}$. Then $K(\sigma_n)$ is a connected simplicial complex of dimension n with $|K(\sigma_n)|$ being an n-ball. We have $H_q(K(\sigma_n)) = 0$ for $q \neq 0$.*

Proof. Let $\sigma_n = \langle x_0,\dots,x_n\rangle$. Then $K(\sigma_n) = x_0\star K(\sigma_{n-1})$, where $\sigma_{n-1} = \langle x_1,\dots,x_n\rangle$. □

7.22 Proposition. [7, 7.2.9] *Let σ_{n+1} be an $n+1$-simplex and define $K(\dot\sigma_{n+1}) := \{\tau : \tau < \sigma_{n+1}\}$. Then $K(\dot\sigma_{n+1})$ is a connected simplicial complex of dimension n with $|K(\dot\sigma_{n+1})|$ being an n-sphere and we have*

$$H_q(K(\dot\sigma_{n+1})) \cong \begin{cases} \mathbb{Z} & \text{for } q \in \{0,n\} \\ 0 & \text{otherwise} \end{cases}$$

A generator of $H_n(K(\dot\sigma_{n+1}))$ is given by

$$\partial\sigma_{n+1} := \sum_{j=0}^{n+1} (-1)^j \langle x_0, \dots, \widehat{x_j}, \dots, x_{n+1}\rangle.$$

Proof. Let $K := K(\dot\sigma_{n+1})$ and $L := K(\sigma_{n+1})$. Then $L \setminus K = \{\sigma_{n+1}\}$ and we have

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(L) \arrow[r, "\partial"] & C_n(L) \arrow[r, "\partial"] \arrow[d, equal] & \cdots \arrow[r, "\partial"] & C_{q+1}(L) \arrow[r, "\partial"] \arrow[d, equal] & C_q(L) \arrow[r, "\partial"] \arrow[d, equal] & \cdots \\
& 0 \arrow[r, "\partial"] & C_n(K) \arrow[r, "\partial"] & \cdots \arrow[r, "\partial"] & C_{q+1}(K) \arrow[r, "\partial"] & C_q(K) \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

By 7.21 the top row is exact (for $q > 0$). Thus we have exactness for all $0 < q < n$. By exactness the arrow $\langle\sigma_{n+1}\rangle \cong C_{n+1}(L) \xrightarrow{\partial} C_n(L)$ is injective, and $H_n(K) = Z_n(K) = Z_n(L) = \partial(C_{n+1}(L)) \cong C_{n+1}(L) = \mathbb{Z}$. $\square$

We will show later that if $|K| \sim |L|$ then $H_q(K) \cong H_q(L)$ for all $q \in \mathbb{Z}$, hence it makes sense to speak about the homology groups of a polyhedra.

7.23 5'Lemma. [7, 8.2.3] *Let*

$$\begin{tikzcd}
A_1 \arrow[r, "\varphi_1"] \arrow[d, "f_1"', "\cong"] & A_2 \arrow[r, "\varphi_2"] \arrow[d, "f_2"', "\cong"] & A_3 \arrow[r, "\varphi_3"] \arrow[d, "f_3"'] & A_4 \arrow[r, "\varphi_4"] \arrow[d, "f_4"', "\cong"] & A_5 \arrow[d, "f_5"', "\cong"] \\
B_1 \arrow[r, "\psi_1"] & B_2 \arrow[r, "\psi_2"] & B_3 \arrow[r, "\psi_3"] & B_4 \arrow[r, "\psi_4"] & B_5
\end{tikzcd}$$

be a commutative diagram with exact horizontal rows. If all but the middle vertical arrow are isomorphisms so is the middle one.

Proof. (f_3 is injective)

$$\begin{aligned}
f_3 a_3 = 0 \Rightarrow 0 = \psi_3 f_3 a_3 = f_4 \varphi_3 a_3 \\
&\overset{f_4 \text{ inj.}}{\Rightarrow} \varphi_3 a_3 = 0 \\
&\overset{\text{exact at } A_3}{\Rightarrow} \exists a_2 : a_3 = \varphi_2 a_2 \\
&\Rightarrow 0 = f_3 a_3 = f_3 \varphi_2 a_2 = \psi_2 f_2 a_2 \\
&\overset{\text{exact at } B_2}{\Rightarrow} \exists b_1 : f_2 a_2 = \psi_1 b_1 \\
&\overset{f_1 \text{ surj.}}{\Rightarrow} \exists a_1 : b_1 = f_1 a_1 \\
&\Rightarrow f_2 a_2 = \psi_1 f_1 a_1 = f_2 \varphi_1 a_1 \\
&\overset{f_2 \text{ inj.}}{\Rightarrow} a_2 = \varphi_1 a_1 \\
&\overset{\text{exact at } A_2}{\Rightarrow} a_3 = \varphi_2 a_2 = \varphi_2 \varphi_1 a_1 = 0
\end{aligned}$$

$$\begin{tikzcd} a_1 \arrow[r, "\varphi_1"] \arrow[d, "f_1"', "\cong"] & a_2 \arrow[r, "\varphi_2"] \arrow[d, "f_2"', "\cong"] & a_3 \arrow[r, "\varphi_3"] \arrow[d, "f_3"'] & 0 \arrow[d, "f_4"', "\cong"] & \bullet \\ b_1 \arrow[r, "\psi_1"] & f_2(a_2) \arrow[r, "\psi_2"] & 0 \arrow[r, "\psi_3"] & 0 & \bullet \end{tikzcd}$$

(f_3 is onto)

$$\begin{aligned} b_3 &\overset{f_4 \text{ surj.}}{\Rightarrow} \exists a_4 : f_4 a_4 = \psi_3 b_3 \\ &\overset{\text{exact at } B_4}{\Rightarrow} f_5 \varphi_4 a_4 = \psi_4 f_4 a_4 = \psi_4 \psi_3 b_3 = 0 \\ &\overset{f_5 \text{ inj.}}{\Rightarrow} \varphi_4 a_4 = 0 \\ &\overset{\text{exact bei } A_4}{\Rightarrow} \exists a_3 : a_4 = \varphi_3 a_3 \\ &\Rightarrow \psi_3 f_3 a_3 = f_4 \varphi_3 a_3 = f_4 a_4 = \psi_3 b_3 \\ &\overset{\text{exact at } B_3}{\Rightarrow} \exists b_2 : b_3 - f_3 a_3 = \varphi_2 b_2 \\ &\overset{f_2 \text{ surj.}}{\Rightarrow} \exists a_2 : b_2 = f_2 a_2 \\ &\Rightarrow b_3 = f_3 a_3 + \psi_2 b_2 = f_3 a_3 + \psi_2 f_2 a_2 = f_3(a_3 + \varphi_2 a_2) \end{aligned}$$

$$\begin{tikzcd} \bullet & a_2 \arrow[r, "\varphi_2"] \arrow[d, "f_2"', "\cong"] & a_3 \arrow[r, "\varphi_3"] \arrow[d, "f_3"'] & a_4 \arrow[r, "\varphi_4"] \arrow[d, "f_4"', "\cong"] & \varphi_4 a_4 \arrow[d, "f_5"', "\cong"] \\ \bullet & b_2 \arrow[r, "\psi_2"] & b_3 \arrow[r, "\psi_3"] & \psi_3 b_3 \arrow[r, "\psi_4"] & 0 \end{tikzcd}$$ □

Remark. An exact sequence of the from

$$0 \to A \to B \to C \to 0$$

is called SHORT EXACT.

1 We have that the top line in the diagram

$$\begin{tikzcd} & A_{q+1} \arrow[r, "f_{q+1}"] \arrow[d, two heads] & A_q \arrow[r, "f_q"] \arrow[d, equal] & A_{q-1} & \\ 0 \arrow[r] & f_{q+1}(A_{q+1}) \arrow[r, hook] & A_q \arrow[r, two heads] & f_q(A_q) \arrow[r] \arrow[u, hook] & 0 \end{tikzcd}$$

is exact at A_q iff the bottom row is short exact.

2 Up to an isomorphism we have the following description of short exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "\cong"'] & B \arrow[r, "p"] \arrow[d, equal] & C \arrow[r] \arrow[d, "\cong"'] & 0 \\ & i(A) \arrow[r, hook] & B \arrow[r, two heads] & B/i(B) & \end{tikzcd}$$

3 The sequence $0 \to A \to A \oplus C \to C \to 0$ is short exact.

4 The sequence $0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}_m \to 0$ is short exact.

7.24 Lemma. [7, 8.2.4] *For a short exact sequence* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *the following statements are equivalent:*

1 *There is an isomorphism* $\varphi : A \oplus C \to B$ *with* $f(a) = \varphi(a,0)$ *and* $g(\varphi(a,c)) = c$*;*

2 g *has a right inverse* r*;*

3 f *has a left inverse.*

Under these equivalent conditions the sequence is called SPLITTING.

Proof. (1⇒ 2) The mapping $c \mapsto \varphi(0,c)$ is right inverse to g.

(2⇒ 1) Consider $\varphi : (a,c) \mapsto f(a) + r(c)$. By the 5'lemma it is the desired isomorphism

$$\begin{tikzcd}
 & A & & C \arrow[d] \arrow[dl, dotted] & \\
0 \arrow[r] & A \arrow[r, "f"] \arrow[u] & B \arrow[r, "g"] \arrow[ul, dotted] & C \arrow[r] & 0 \\
0 \arrow[r] \arrow[u, equal] & A \arrow[r] \arrow[u, equal] & A \oplus C \arrow[r] \arrow[u, "\varphi"] & C \arrow[r] \arrow[u, equal] & 0 \arrow[u, equal]
\end{tikzcd}$$

(1⇔3) is shown similarly. □

7.25 Example. [7, 8.2.5] The sequence $0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}_m \to 0$ does not split. In fact, every $a \in \mathbb{Z}_m$ has order $\operatorname{ord}(a) \le m < \infty$ but all $b \in \mathbb{Z}$ have order $\operatorname{ord}(b) = \infty$.

7.26 If C is free abelian than any short exact sequence $0 \to A \to B \to C \to 0$ splits.

If $0 \to A \to B \to C \to 0$ is exact and A and C is finitely generated then so is B. In fact, the generators of A together with inverse images of those of C generate B.

7.27 Definition. [7, 8.3.4] Let C and C' be two chain complexes. A CHAIN MAPPING is a family of homomorphisms $f_q : C_q \to C'_q$ which commutes with ∂, i.e. $\partial'_q \circ f_q = f_{q-1} \circ \partial_q$.

7.28 Definition. [7, 7.3.1] Let $\varphi : K \to L$ be a simplicial map between simplicial complexes. Define group homomorphisms $C_q(\varphi) : C_q(K) \to C_q(L)$ by $C_q(\varphi) := 0$ for $q \le 0$ or $q > \dim K$ and by $C_q(\varphi)(\langle x_0, \dots, x_q\rangle) := \langle \varphi(x_0), \dots, \varphi(x_q)\rangle$ if φ is injective on $\{x_0, \dots, x_q\}$ and $C_q(\varphi)(\langle x_0, \dots, x_q\rangle) := 0$ otherwise.

7.29 Proposition. [7, 7.3.2] *For every simplicial map* $\varphi : K \to L$ *the induced map* $(C_q(\varphi))_{q\in\mathbb{Z}}$ *is a chain mapping.*

Proof. We have to show that $\partial_q(C_q(\varphi)(\sigma)) = C_{q+1}(\varphi)(\partial_q \sigma)$ for every q-simplex $\sigma = \langle x_0, \dots, x_q\rangle$. If all vertices $\varphi(x_j)$ are distinct or are at least two pairs identical this is obvious. So we may assume that exactly two are the same. By reordering we may assume $\varphi(x_0) = \varphi(x_1)$. Then $C_q(\varphi)(\sigma) = 0$ and hence also $\partial C_q(\varphi)(\sigma)$. On the other hand $\partial\sigma = \langle x_1, \dots, x_q\rangle - \langle x_0, x_2, \dots, x_q\rangle + \sum_{j=2}^q (-1)^j \langle x_0, \dots, \widehat{x_j}, \dots, x_q\rangle$. The first two simplices have the same image under $C_{q-1}(\varphi)$. The sum is mapped to 0, since $\varphi(x_0) = \varphi(x_1)$. □

7.30 Lemma. [7, 8.3.5] *Any chain map* f *induces homomorphisms* $H_q(f) : H_q(C) \to H_q(C')$*. The chain maps form a category.*

Proof. Since $f \circ \partial = \partial \circ f$ we have that $f(Z_q) \subseteq Z'_q$ and $f(B_q) \subseteq B'_q$ and hence $H_q(f) : H_q(C) \to H_q(C')$ makes sense,

$$\begin{tikzcd}
0 \arrow[r] & B_q(C) \arrow[r] \arrow[d] & Z_q(C) \arrow[r] \arrow[d] & H_q(C) \arrow[r] \arrow[d, dotted] & 0 \\
0 \arrow[r] & B_q(C') \arrow[r] & Z_q(C') \arrow[r] & H_q(C') \arrow[r] & 0
\end{tikzcd}$$ □

7.31 Theorem. [7, 8.3.8] *Let $0 \to C' \xrightarrow{f} C \xrightarrow{g} C'' \to 0$ be a short exact sequence of chain mappings. Then we obtain a long exact sequence in homology:*

$$\dots \xrightarrow{\partial_*} H_q(C') \xrightarrow{H_q(f)} H_q(C) \xrightarrow{H_q(g)} H_q(C'') \xrightarrow{\partial_*} H_{q-1}(C') \xrightarrow{H_{q-1}(f)} \dots$$

In particular we can apply this to a chain complex C and a chain subcomplex C' and $C'' := C/C'$, since ∂ factors over $\partial'' : C'' \to C''$, via $\partial''(c + C') := \partial c + C'$.

Proof. Consider

$$\begin{tikzcd}
0 \arrow[r] & C'_q \arrow[r, "f"] \arrow[d, "\partial"'] & C_q \arrow[r, "g"] \arrow[d, "\partial"'] & C''_q \arrow[r] \arrow[d, "\partial"'] & 0 \\
0 \arrow[r] & C'_{q-1} \arrow[r, "f"] & C_{q-1} \arrow[r, "g"] & C''_{q-1} \arrow[r] & 0
\end{tikzcd}$$

Let $\partial_*[z''] := [(f^{-1} \circ \partial \circ g^{-1})(z'')]$ for $z'' \in C''$ with $\partial z'' = 0$.

We first show that it is possible to choose elements in the corresponding inverse images and then we will show that the resulting class does not depend on any of the choices.
So let $z''_q \in C''_q$ be a cycle, i.e. $\partial z''_q = 0$. Since g is onto we find $x_q \in C_q$ with $g x_q = z''_q$. Since $g \partial x_q = \partial g x_q = \partial z''_q = 0$, we find $x'_{q-1} \in C'_{q-1}$ with $f x'_{q-1} = \partial x_q$. And hence $x'_{q-1} \in f^{-1} \partial g^{-1} z''_q$. Furthermore $f \partial x'_{q-1} = \partial f x'_{q-1} = \partial \partial x_q = 0$. Since f is injective we get $\partial x'_{q-1} = 0$ and hence we may form the class $\partial_*[z''_q] := [x'_{q-1}]$.

$$\begin{tikzcd}
 & x_q \arrow[r, "g"] \arrow[d, "\partial"'] & z''_q \arrow[d, "\partial"'] \\
x'_{q-1} \arrow[r, "f"] \arrow[d, "\partial"'] & \partial x_q \arrow[r, "g"] \arrow[d, "\partial"'] & 0 \\
\partial x'_{q-1} \arrow[r, "f"] & 0 &
\end{tikzcd}$$

Now the independency from all choices, So let $[z''_q] = [\bar{z}''_q]$, i.e. $\exists x''_{q+1} : \partial x''_{q+1} = z''_q - \bar{z}''_q$. Choose $x_q, \bar{x}_q \in C_q$ as before, so that $g x_q = x''_q$ and $g \bar{x}_q = \bar{x}''_q$. Also as before choose $x'_{q-1}, \bar{x}'_{q-1} \in C'_{q-1}$ with $f x'_{q-1} = \partial x_q$ and $f \bar{x}'_{q-1} = \partial \bar{x}_q$. We have to show that $[x'_{q-1}] = [\bar{x}'_{q-1}]$. So choose $x_{q+1} \in C_{q+1}$ with $g x_{q+1} = x''_{q+1}$. Then $g \partial x_{q+1} = \partial g x_{q+1} = \partial x''_{q+1} = z''_q - \bar{z}''_q = g(x_q - \bar{x}_q)$, hence there exists an $x'_q \in C_q$ with $f x'_q = \partial x_{q+1} - x_q + \bar{x}_q$. And $f \partial x'_q = \partial f x'_q = \partial(\partial x_{q+1} - x_q + \bar{x}_q) = 0 - \partial x_q + \partial \bar{x}_q = -f(x'_{q-1} - \bar{x}'_{q-1})$. Since f is injective we have $x'_{q-1} = \bar{x}'_{q-1} + \partial x'_q$, i.e. $[x'_{q-1}] = [\bar{x}'_{q-1}]$.

Exactness at $H_q(C')$:
($\subseteq$) $f_* \partial_*[z''] = [f f^{-1} \partial g^{-1} z''] = [\partial g^{-1} z''] = 0$.

($\supseteq$) Let $\partial z' = 0$ and $0 = f_*[z'] = [fz']$, i.e. $\exists x$: $\partial x = fz'$. Then $x'' := gx$ satisfies $\partial x'' = \partial gx = g\partial x = gfz' = 0$ and $\partial_*[x''] = [f^{-1}\partial g^{-1}gx] = [f^{-1}\partial x] = [z']$.

Exactness at $H_q(C)$:
($\subseteq$) since $g \circ f = 0$.
($\supseteq$) Let $\partial z = 0$ with $0 = g_*[z] = [gz]$, i.e. $\exists x''$: $\partial x'' = gz$. Then $\exists x$: $gx = x''$. Hence $gz = \partial x'' = \partial gx = g\partial x \Rightarrow \exists x'$: $fx' = z - \partial x \Rightarrow \partial fx' = \partial(z - \partial x) = 0$ and $f_*[x'] = [fx'] = [z - \partial x] = [z]$.

Exactness at $H_q(C'')$:
($\subseteq$) We have $\partial_* g_*[z] = [f^{-1}\partial g^{-1}gz] = [f^{-1}\partial z] = [f^{-1}0] = 0$.
($\supseteq$) Let $\partial z'' = 0$ and $0 = \partial_*[z'']$, i.e. $\exists x'$: $\partial x' = z'$, where $z' \in f^{-1}\partial g^{-1}z''$, i.e. $\exists x$: $gx = z''$ and $fz' = \partial x$. Then $\partial(x - fx') = fz' - f(\partial x') = 0$ and $g(x - fx') = z'' - 0$, i.e. $g_*[x - fx'] = [z'']$. □

Relative homology

7.32 Definition. [7, 7.4.1] Let $K_0 \subseteq K$ be a simplicial subcomplex. Then $C(K_0)$ is a chain subcomplex of $C(K)$ and hence we may form the chain complex $C(K, K_0)$ given by $C_q(K, K_0) := C_q(K)/C_q(K_0)$. Note that we can identify this so-called RELATIVE HOMOLOGY GROUP with the free abelian group generated by all q-simplices in $K \setminus K_0$. The boundary operator is given by taking the boundary of $\sum_\sigma k_\sigma \cdot \sigma$ in $C(K)$, but deleting all summands of simplices in $C(K_0)$. Using the short exact sequence $0 \to C(K_0) \to C(K) \to C(K, K_0) \to 0$ we get a long exact sequence in homology and we denote the q-th homology group of $C(K, K_0)$ by $H_q(K, K_0)$ and call it the relative homology of K with respect to K_0.

7.33 Remark. [7, 7.4.2]

1 If $K_0 = K$ then $C_q(K, K_0) = C_q(\emptyset) = \{0\}$ and hence $H_q(K, K) = \{0\}$.

2 If $K_0 = \emptyset$ then $C_q(K, K_0) = C_q(K)$ and hence $H_q(K, \emptyset) = H_q(K)$.

3 If K is connected and $K \supseteq K_0 \neq \emptyset$, then $H_0(K, K_0) = \{0\}$. In fact let $z \in C_0(K, K_0)$, i.e. $z = \sum_{x \in K \setminus K_0} k_x \cdot x$. Let $x_0 \in K_0$ be chosen fixed. Since K is connected we find for every $x \in K$ a 1-chain c with boundary $\partial c = x - x_0$, hence $z \sim \varepsilon(z) \cdot x_0 = 0$ in $C_0(K, K_0)$.

4 Note that in 7.22 we calculated the relative chain complex $C_q(K, L)$, where $K := K(\sigma_n)$ and $L := K(\dot\sigma_n)$ and obtained $C_q(K, L) = \{0\}$ for $q \neq n$ and $C_n(K, L) = \langle \sigma_n \rangle \cong \mathbb{Z}$. Hence $H_q(K, L) \cong \{0\}$ for $q \neq 0$ and $H_n(K, L) \cong \mathbb{Z}$.

7.34 Example. [7, 7.4.7] Let M be the Möbius strip with boundary ∂M. We have a triangulation of M in 5 triangles as in 3.10. Since ∂M is a 1-sphere it has homology $H_1(\partial M) \cong \mathbb{Z}$, where a generator r is given by the 1-cycle formed by the 5-edges of the boundary.

Furthermore $H_1(M) \cong \mathbb{Z}$, where a generator is given by the sum m of the remaining edges. In fact every triangle has two of these edges. So for every cycle the coefficient of these two edges has to be equal.

If a combination of triangles has a multiple of m as boundary, their coefficients have to be 0.

Now consider the following fragment of the long exact homology sequence:

$$\begin{tikzcd}
H_1(\partial M) \arrow[r] \arrow[d, equal] & H_1(M) \arrow[r] \arrow[d, equal] & H_1(M, \partial M) \arrow[r] & H_0(\partial M) \arrow[r] \arrow[d, equal] & H_0(M) \arrow[d, equal] \\
\langle r \rangle & \langle m \rangle & & \langle x_0 \rangle & \langle x_0 \rangle
\end{tikzcd}$$

Since $H_0(\partial M) \cong \mathbb{Z} \cong H_0(M)$, where a generator is given by any point x_0 in $\partial M \subseteq M$, we have that the rightmost arrow is a bijection, so the one to the left is 0 and hence the previous one is onto. Remains to calculate the image of $\langle [r] \rangle = H_1(\partial M) \to H_1(M) = \langle [m] \rangle$. For this we consider the sum over all triangles. It has boundary $2m - r$ and hence $[r]$ is mapped to $2[m]$. Thus we have $H_1(M, \partial M) \cong \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$.

7.35 Proposition. [7, 8.3.11] *Let (C, C') and (D, D') be pairs of chain complexes, $C'' = C/C'$, $D'' = D/D'$ and $f : (C, C') \to (D, D')$ be a chain mapping of pairs. This induces a homomorphism which intertwines with the long exact homology sequences.*

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_*"] & H_q(C') \arrow[r, "H_q(i)"] \arrow[d, "f_*"'] & H_q(C) \arrow[r, "H_q(p)"] \arrow[d, "f_*"'] & H_q(C'') \arrow[r, "\partial_*"] \arrow[d, "f_*"'] & H_{q-1}(C') \arrow[r, "H_{q-1}(g)"] \arrow[d, "f_*"'] & \cdots \\
\cdots \arrow[r, "\partial_*"] & H_q(D') \arrow[r, "H_q(j)"] & H_q(D) \arrow[r, "H_q(q)"] & H_q(D'') \arrow[r, "\partial_*"] & H_{q-1}(D') \arrow[r, "H_{q-1}(g)"] & \cdots
\end{tikzcd}$$

Proof.

$$\begin{tikzcd}
 & x \arrow[r, "p"] \arrow[d] \arrow[ddr, "f"] & z'' \arrow[ddr, "f"] & \\
z' \arrow[r, "i"] \arrow[ddr, "f"] & \partial x \arrow[ddr, "f"] & & \\
 & & fx \arrow[r, "q"] \arrow[d] & fz'' \\
 & fz' \arrow[r, "j"] & \partial fx &
\end{tikzcd}$$

The commutativity of all but the rectangle involving ∂_* is obvious. For the remaining one let $z'' \in C''$ be a cycle. We have to show that $\partial_* f_*[z''] = f_* \partial_*[z'']$. So let $z' \in i^{-1}\partial p^{-1} z''$, i.e. $iz' = \partial x$ for some x with $px = z''$. Then $f_*\partial_*[z''] = [fz']$ and we have to show that $j(fz') \in \partial q^{-1} fz''$, which follows from $jfz' = fiz' = f\partial x = \partial fx$ and $q(fx) = f(px) = fz''$. □

7.36 Corollary. [7, 7.4.6] *Proposition 7.35 applies in particular to a simplicial mapping $\varphi : (K, K_0) \to (L, L_0)$ of pairs.*

7.37 Excision Theorem. [7, 7.4.9] *Let K be the union of two subcomplexes K_0 and K_1. Then $(K_1, K_0 \cap K_1) \to (K, K_0)$ induces an isomorphism $H(K_1, K_0 \cap K_1) \to H(K_1 \cup K_0, K_0)$.*

Proof. Note that we have

$$K_1 \setminus (K_0 \cap K_1) = K_1 \setminus K_0 = (K_0 \cup K_1) \setminus K_0$$

and also

$$\begin{tikzcd} 0 \arrow[r] & C_q(K_0 \cap K_1) \arrow[r, "i_1"] \arrow[d, "i_2"'] & C_q(K_1) \arrow[r] \arrow[d, "j_1"'] & C_q(K_1, K_0 \cap K_1) \arrow[r, "\cong"] \arrow[d, dotted, "\cong"] & \langle K_1 \setminus (K_0 \cap K_1)\rangle \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & C_q(K_0) \arrow[r, "j_2"] & C_q(K_0 \cup K_1) \arrow[r] & C_q(K_0 \cup K_1, K_0) \arrow[r, "\cong"] & \langle (K_0 \cup K_1) \setminus K_0\rangle \arrow[r] & 0 \end{tikzcd}$$

This is gives even an isomorphism on the level of chain complexes, as follows from the commutativity of the diagram. □

If $U := K \setminus K_1 = K_0 \setminus (K_0 \cap K_1)$ then $K_1 = K \setminus U$ and $K_0 \cap K_1 = K_0 \setminus U$, hence the isomorphism reads $H(K \setminus U, K_0 \setminus U) \cong H(K, K_0)$. Conversely, if (K, K_0) is a pair of simplicial complexes and $U \subseteq K_0$ is such that $K_1 := K \setminus U$ is a simplicial complex, then we get:

7.38 Corollary. [7, 7.4.8] *Let $K_0 \subseteq K$ be a pair of simplicial complexes. Assume there exists a set $U \subseteq K_0$ such that $\forall \tau \in U$, $\tau < \sigma \Rightarrow \sigma \in U$. Then $K_1 := K \setminus U$ and $K_0 \cap K_1 := K_0 \setminus U$ are simplicial complexes and $H(K, K_0) \cong H(K \setminus U, K_0 \setminus U)$.* □

8 Singular Homology

Basics

8.1 Definition. [7, 9.1.1] The STANDARD (CLOSED!) q-SIMPLEX Δ_q is the simplex spanned by the standard unit vectors $e_j \in \mathbb{R}^{q+1}$ for $0 \le j \le q$. So

$$\Delta_q := \Big\{(\lambda_0, \dots, \lambda_q) : 0 \le \lambda_j \le 1 : \sum_j \lambda_j = 1\Big\}.$$

8.2 Definition. [7, 9.1.2] For $q \ge 1$ and $0 \le j \le q$ let the FACE-MAP $\delta^j_{q-1} : \Delta_{q-1} \to \Delta_q$ be the unique affine map, which maps e_i to e_i for $i < j$ and to e_{i+1} for $i > j$, i.e.

$$e_0, \dots, e_{q-1} \mapsto e_0, \dots, \widehat{e_j}, \dots, e_q.$$

8.3 Lemma. [7, 9.1.3] *For $q \ge 2$ and $0 \le k < j \le q$ we have $\delta^j_{q-1} \circ \delta^k_{q-2} = \delta^k_{q-1} \circ \delta^{j-1}_{q-2}$.* □

Proof. The mapping on the left side has the following effect on the edges:

$$e_0, \dots, e_{q-2} \mapsto e_0, \dots, \widehat{e_k}, \dots, e_{q-1} \mapsto e_0, \dots \widehat{e_k}, \dots, \widehat{e_j}, \dots, e_q$$

And on the right side:

$$e_0, \dots, e_{q-2} \mapsto e_0, \dots, \widehat{e_{j-1}}, \dots, e_{q-1} \mapsto e_0, \dots \widehat{e_k}, \dots, \widehat{e_j}, \dots, e_q \quad \square$$

8.4 Definition. [7, 9.1.4] Let X be a topological space. A SINGULAR q-SIMPLEX is a continuous map $\sigma : \Delta_q \to X$. The q-th SINGULAR CHAIN GROUP $S_q(X)$ is the free abelian group generated by all singular q-simplices, i.e.

$$S_q(X) := {}^{\mathrm{ab}}\mathcal{F}(C(\Delta_q, X))$$

Its elements are called SINGULAR q-CHAINS. The boundary operator ∂ is the linear extension of

$$\partial : \sigma \mapsto \sum_{j=0}^{q} (-1)^j \sigma \circ \delta^j.$$

By $\boxed{8.3}$ the groups $S_q(X)$ together with ∂ from a chain complex $S(X)$:

$$\begin{aligned}
\partial\partial\sigma &= \partial\Big(\sum_{j=0}^{q} (-1)^j \sigma \circ \delta^j\Big) \\
&= \sum_{j=0}^{q} (-1)^j \sum_{k=0}^{q-1} (-1)^k \sigma \circ \delta^j \circ \delta^k \\
&= \sum_{0 \le k < j \le q} (-1)^{j+k} \sigma \circ \delta^j \circ \delta^k + \sum_{0 \le j \le k < q} (-1)^{j+k} \sigma \circ \delta^j \circ \delta^k \\
&\overset{\boxed{8.3}}{=\!=} \sum_{0 \le k < j \le q} (-1)^{j+k} \sigma \circ \delta^k \circ \delta^{j-1} + \sum_{0 \le j < k \le q} (-1)^{j+k-1} \sigma \circ \delta^j \circ \delta^{k-1} \\
&= \sum_{0 \le k < j \le q} (-1)^{j+k} \sigma \circ \delta^k \circ \delta^{j-1} + \sum_{0 \le k < j \le q} (-1)^{k+j-1} \sigma \circ \delta^k \circ \delta^{j-1} = 0. \quad \square
\end{aligned}$$

The q-th singular homology group $H_q(X)$ is defined to be $H_q(S(X))$. The elements of $B_q(X) := B_q(S(X))$ are called (singular) q-boundaries and those of $Z_q(X) := Z_q(S(X))$ are called (singular) q-cycles.

Remark that singular 0-simplices can be identified with the points in X and singular 1-simplices with paths in X.

8.5 Definition. [7, 9.1.6] [7, 9.1.8] [7, 9.1.9] Let $f : X \to Y$ be continuous. Then f induces a chain-mapping $f_* := S(f) : S(X) \to S(Y)$ and hence group-homomorphisms $f_* := H_q(f) : H_q(X) \to H_q(Y)$.

$$\partial\Big(S(f)(\sigma)\Big) = \partial(f\circ\sigma) = \sum_{j=0}^{q}(-1)^j f\circ\sigma\circ\delta^j = S(f)\Big(\sum_{j=0}^{q}(-1)^j\sigma\circ\delta^j\Big) = S(f)\Big(\partial(\sigma)\Big).$$

So H_q is a functor from continuous maps between topological spaces into group homomorphisms between abelian groups.

8.6 Remark. [7, 9.1.7] The identity $\mathrm{id}_{\Delta_q} : \Delta_q \to \Delta_q$ is a singular q-simplex of Δ_q, which we will denote again by Δ_q. If σ is a singular q-simplex in X, then $S(\sigma)(\Delta_q) = \sigma$. We will make use of this several times in order to construct natural transformations, by defining them first for the standard simplex, see 8.22 and 8.30.

8.7 Theorem. [7, 9.1.10] *Let $X = \{*\}$ be a single point. Then $H_q(X) = \{0\}$ for $q \neq 0$ and $H_0(X) = S_0(X) \cong \mathbb{Z}$.*

A space X is called acyclic iff it is path-connected and $H_q(X) = \{0\}$ for $q \neq 0$.

Proof. The only singular q-simplex is the constant mapping. Its boundary is $\partial\sigma_q = (\sum_{i=0}^{q}(-1)^j)\sigma_{q-1}$. For even $q > 0$ we have that $Z_q(X) = \{0\}$ for odd q we have that $B_q(X) = Z_q(X)$, hence in both cases $H_q(X) = \{0\}$. For $q = 0$ we have $B_0(X) = \{0\}$ and $Z_0(X) = S_0(X) \cong \mathbb{Z}$. □

8.8 Corollary. [7, 9.1.11] *Let $f : X \to Y$ be constant then $H_q(f) = 0$ for $q \neq 0$.*

Proof. Obvious, since f factors over a single point. □

8.9 Proposition. [7, 9.1.12] *Let X_j be the path components of X. Then the inclusions of $X_j \to X$ induce an isomorphism $\bigoplus_j H_q(X_j) \to H_q(X)$; cf. 7.11.*

Proof. This is follows as 7.11: Let σ be a singular simplex of X. Then σ is completely contained in some X_j, hence the induced map $\bigoplus_j S_q(X_j) \to S_q(X)$ is onto. Conversely this linear map is injective, since the chains in the various X_j have disjoint images. Thus we have a bijection $\bigoplus_j S_q(X_j) \cong S_q(X)$, which induces an isomorphism of homology groups. □

8.10 Proposition. [7, 9.1.13] *Let X be a topological space. Then $H_0(X)$ is a free abelian group with generators given by choosing one point in each path-component; cf. 7.12.*

Proof. Because of 8.9 we may assume that X is path-connected. The mapping $\varepsilon : S_0(X) \to \mathbb{Z}$, $\sum_\sigma n_\sigma\cdot\sigma \mapsto \sum_\sigma n_\sigma$ is onto and as in 7.12 its kernel is just $B_0(X)$, so ε induces an isomorphism $H_0(X) \cong \mathbb{Z}$; cf. 7.12. □

8.11 Corollary. [7, 9.1.14] *Let X and Y be path-connected. Then every continuous mapping $f : X \to Y$ induces an isomorphism $H_0(f) : H_0(X) \to H_0(Y)$.*

Proof. Obvious since the generator is mapped to a generator. $\square$

8.12 Definition. [7, 9.1.15] Let $A \subseteq \mathbb{R}^n$ be convex and $p \in A$ be fixed. For a singular q-simplex $\sigma : \Delta_q \to A$ we define the CONE $p \star \sigma : \Delta_{q+1} \to A$ by

$$(p \star \sigma)((1-t)e^0 + t\delta^0(x)) := (1-t)p + t\sigma(x) \text{ for } t \in [0,1] \text{ and } x \in \Delta_q.$$

For a q-chain $c = \sum_\sigma n_\sigma \cdot \sigma$ we extend it by linearity

$$p \star c := \sum_\sigma n_\sigma \cdot (p \star \sigma)$$

and obtain a homomorphism $S_q(A) \to S_{q+1}(A)$; cf. 7.19.

8.13 Lemma. [7, 9.1.16] *Let $c \in S_q(A)$ then*

$$\partial(p \star c) = c - \varepsilon(c)\, p \text{ for } q = 0,$$
$$\partial(p \star c) = c - p \star \partial c \text{ for } q > 0,$$

where $\varepsilon\Big(\sum_x n_x \cdot x\Big) = \sum_x n_x$; cf. 7.20.

Proof. It is enough to show this for singular simplices $c = \sigma_q$. For $q = 0$ we have that $p \star \sigma : \Delta_1 \to X$ is a path from p to σ hence $\partial(p \star \sigma) = \sigma - p = \sigma - \varepsilon(\sigma)p$. For $q > 0$ we have $(p \star \sigma) \circ \delta^0 = \sigma$ and $(p \star \sigma) \circ \delta^i = p \star (\sigma \circ \delta^{i-1})$ for $i > 0$, hence $\partial(p \star \sigma) = \sigma - p \star \partial\sigma$. $\square$

8.14 Corollary. [7, 9.1.18] *Let $A \subseteq \mathbb{R}^n$ be convex. Then A is acyclic; cf. 7.20 & 7.21.*

Proof. Let $p \in A$ and z be a q-cycle for $q > 0$. Then $z = \partial(p \star z)$ by 8.13 and hence $Z_q(A) = B_q(A)$, i.e. $H_q(A) \cong \{0\}$. $\square$

Relative homology

8.15 Definition. [7, 9.2.1] Let (X, A) be a pair of spaces. Then we get a pair of chain complexes $(S(X), S(A))$ and hence a short exact sequence

$$0 \to S(A) \to S(X) \to S(X, A) \to 0,$$

where $S_q(X, A) := S_q(X)/S_q(A)$. Its elements are called RELATIVE SINGULAR q-CHAINS. But **unlike 7.32 we can not identify them with formal linear combinations of simplices in $X \setminus A$.**

8.16 Remark. [7, 9.2.3] But as in 7.32 we get a long exact sequence in homology

$$\dots \to H_q(A) \to H_q(X) \to H_q(X, A) \to H_{q-1}(A) \to \dots,$$

where $H_q(X, A) := H_q(S(X, A))$. Note that $z \in S_q(X)$ with $\partial z \in S_q(A)$ describe the classes $[z + S_q(A)] \in H_q(X, A)$.
For a continuous mapping of pairs $(A, X) \to (B, Y)$ we get a homology ladder as in 7.35.

8.17 Remark. [7, 9.2.2] As in 7.33 we get

1 $H_q(X,X) = \{0\}$,

2 $H_q(X,\emptyset) \cong H_q(X)$, and

3 $H_0(X,A) = \{0\}$ for path connected X and $A \neq \emptyset$.

8.18 Remark. [7, 9.2.4] Using the long exact homology sequence

$$\dots H_{q+1}(X,A) \to H_q(A) \to H_q(X) \to H_q(X,A) \to H_{q-1}(A) \to \dots,$$

we obtain:

1. Let $A \subseteq X$ be such that $H_q(A) \to H_q(X)$ is injective for all q. Then we get short exact sequences $0 \to H_q(A) \to H_q(X) \to H_q(X,A) \to 0$, where the right most arrow is 0, since the next one in the exact homology sequence of the pair is assumed to be injective.
2. Let $A \subseteq X$ be a retract. Then by functorality $H_q(A) \to H_q(X)$ is a retract and hence by (1) we have (splitting) short exact sequences.
3. Let $x_0 \in X$. The constant mapping $X \to \{x_0\}$ is a retraction, hence $H_q(X) \cong H_q(\{x_0\}) \oplus H_q(X,\{x_0\})$ by 2. By 8.7 we have that $H_q(\{x_0\}) = \{0\}$ for $q \neq 0$ and $H_0(\{x_0\}) = \mathbb{Z}$, hence $H_q(X,\{x_0\}) \cong H_q(X)$ for $q > 0$ and $0 \to \mathbb{Z} \to H_0(X) \to H_0(X,\{x_0\}) \to 0$ is splitting exact.
4. Let $f : (X,A) \to (Y,B)$ be such that $f_* : H_q(A) \to H_q(B)$ and $f_* : H_q(X) \to H_q(Y)$ are isomorphisms for all q. Then the same is true for $f_* : H_q(X,A) \to H_q(Y,B)$ by the 5'Lemma.

8.19 Theorem. Exact Homology Sequence of a Triple. [7, 9.2.5] *Let $B \subseteq A \subseteq X$. Then we get a long exact homology sequence*

$$\dots \to H_{q+1}(X,A) \xrightarrow{\partial_*} H_q(A,B) \to H_q(X,B) \to H_q(X,A) \to \dots$$

The boundary operator ∂_* can also be described by $[z]_{(X,A)} \mapsto [\partial z]_{(A,B)}$ for $z \in S_q(X)$ or as composition $H_{q+1}(X,A) \xrightarrow{\partial_*} H_q(A) \to H_q(A,B)$.

Proof. We have a short sequence

$$0 \to S(A,B) \to S(X,B) \to S(X,A) \to 0.$$

given by

$$\begin{tikzcd}
& S(B) \arrow[r, equal] \arrow[d] & S(B) \arrow[r, hook] \arrow[d] & S(A) \arrow[d] & \\
& S(A) \arrow[r, hook] \arrow[d] & S(X) \arrow[r, equal] \arrow[d] & S(X) \arrow[d] & \\
0 \arrow[r] & S(A,B) \arrow[r, hook] \arrow[d] & S(X,B) \arrow[r, two heads] \arrow[d] & S(X,A) \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

Hence the bottom row is exact at $S(X,A)$ and at $S(A,B)$: In fact for $\dot a \in S(A,B)$ let the image in $S(X,B)$ be 0. Then $a = b \in S(B)$ and hence $\dot a = 0$ in $S(A,B)$.

$$\begin{tikzcd}
b \arrow[r, equal] \arrow[d, mapsto, "4."] & \exists b \arrow[d, mapsto, "3."] & \bullet \\
\exists a \arrow[r, mapsto, "2."] \arrow[d, mapsto, "1."] & a \arrow[d, mapsto, "2."] & \bullet \\
\dot a \arrow[r, mapsto, "0."] & 0 & \bullet
\end{tikzcd}$$

It is also exact at $S(X,B)$, since for $\dot x \in S(X,B)$ which is mapped to 0 in $S(X,A)$ the image $x \in S(X)$ is an $a \in S(A)$ and hence satisfies $\dot a$ is mapped to $\dot x$.

$$\begin{tikzcd}
\bullet & \bullet & \exists a \arrow[d, mapsto, "3."] \\
a \arrow[r, mapsto, "4."] \arrow[d, mapsto, "5."] & \exists x \arrow[r, equal] \arrow[d, mapsto, "1."] & x \arrow[d, mapsto, "2."] \\
\dot a \arrow[r, mapsto, "5."] & \dot x \arrow[r, mapsto, "0."] & 0
\end{tikzcd}$$

So this short exact sequence induces a long exact sequence in homology. The boundary operator maps by construction 7.31 the class $[z + S(A)]$ with $\partial z \in S(A)$ to $[\partial z + S(B)]$. This is precisely the image of value of the boundary operator $[\partial z]$ for the pair (X,A) under the natural map $H(A) \to H(A,B)$. $\square$

8.20 Exercise. Let X be path-connected and $A \subseteq X$. Then $H_0(X,A) = 0$.

Homotopy theorem

We are now going to prove that homotopic mappings induce the same mappings in homology. For this we consider first as homotopy which is as free and as natural as possible, i.e. the homotopy given by $\mathrm{inj}_t : X \to X \times I$, $x \mapsto (x,t)$. We have to show that inj_0 and inj_1 induce the same mapping in homology. So the images of a cycle should differ only by a boundary. Let $\sigma : \Delta_q \to X$ be a singular simplex. Then we may consider the cylinder $\sigma(\Delta_q) \times I$ over $\sigma(\Delta_q)$. It seems clear, that we can triangulate $\Delta_q \times I$. The image of the corresponding chain c_{q+1} under $\sigma \times I$ gives then a $q+1$-chain in $X \times I$, whose boundary consists of the parts $\sigma \times \{1\} = \mathrm{inj}_1 \circ \sigma$ and $\sigma \times \{0\} = \mathrm{inj}_0 \circ \sigma$ and a triangulation of $(\sigma \times I)_* \partial c_q$. Note that it would have been easier here, if we had defined the singular homology by using squares instead of triangles, since it is not so clear how to describe an explicit triangulation of $\Delta_q \times I$, in fact we will show the existence of c_{q+1} by induction in the following lemma.

We make use of the following

8.21 Definition. [7, 8.4.6] Let $R, S : \mathcal{X} \to \mathcal{Y}$ be two functors. A natural transformation $\varphi : R \to S$ is a family $\varphi_X : R(X) \to S(X)$ of $\mathcal{Y}$-morphisms for every object $X \in \mathcal{X}$ such that for every $\mathcal{X}$-morphism $f : X \to X'$ the following diagram commutes:

$$\begin{tikzcd}
R(X) \arrow[r, "\varphi_X"] \arrow[d, "R(f)"'] & S(X) \arrow[d, "S(f)"] \\
R(X') \arrow[r, "\varphi_{X'}"'] & S(X')
\end{tikzcd}$$

8.22 Lemma. [7, 9.3.7] *Let $\varphi_0, \varphi_1 : S(\_) \to S(\_ \times I)$ be two natural transformations and assume furthermore that $H_0(\varphi_0) = H_0(\varphi_1) : H_0(\{*\}) \to H_0(\{*\} \times I)$. Then φ_0 and φ_1 are chain homotopic, i.e. there exists natural homomorphisms $\mathcal{Z} = (\mathcal{Z}_q)_q$ with $\mathcal{Z}_q : S_q(X) \to S_{q+1}(X \times I)$ and $\partial \mathcal{Z}_q + \mathcal{Z}_{q-1}\partial = \varphi_1 - \varphi_0$ on $S_q(X)$.*

Proof. We construct $\mathcal{Z}_q$ by induction on q:
For $q < 0$ let $\mathcal{Z}_q := 0$. Now let $\mathcal{Z}_j$ for all $j < q$ be already constructed. Consider the natural transformation $\varphi := \varphi_1 - \varphi_0$. We first treat the case $X := \Delta_q$. So we have to find $\mathcal{Z}_q(\Delta_q) =: c_{q+1} \in S_q(\Delta_q \times I)$ with $\partial c_{q+1} = \varphi\Delta_q - \mathcal{Z}_{q-1}\partial\Delta_q$. For $q = 0$ this follows from the assumption $[\varphi(\Delta_0)] = 0 \in H_0(\Delta_0 \times I)$. For $q > 0$ we can use that $S(\Delta_q \times I)$ is acyclic (since $\Delta_q \times I$ is a convex subset of $\mathbb{R}^{q+2}$) by 8.14. So we only have to show that the right side is a cycle. In fact by induction hypothesis (applied to $\partial\Delta_q$) we have

$$\partial\Big(\varphi\Delta_q - \mathcal{Z}_{q-1}\partial\Delta_q\Big) = \varphi\partial\Delta_q - \Big(\varphi - \mathcal{Z}_{q-2}\Big)\partial(\partial\Delta_q) = \varphi\partial\Delta_q - \Big(\varphi\partial\Delta_q - \mathcal{Z}_{q-2}\partial\partial\Delta_q\Big) = 0.$$

Now we extend $\mathcal{Z}_q : S_q(X) \to S_{q+1}(X \times I)$ by naturality to the case of a general X: I.e. for $\sigma : \Delta_q \to X$ we define $\mathcal{Z}_q(\sigma) := S_{q+1}(\sigma \times I)(c_{q+1})$.

Then $\mathcal{Z}_q$ is in fact natural, since $S_q(f \times I)\mathcal{Z}_q(\sigma) = S_{q+1}(f \times I)S_{q+1}(\sigma \times I)c_{q+1}$ and $\mathcal{Z}_q S_q(f)(\sigma) = \mathcal{Z}_q(f\sigma) = S_q(f\sigma \times I)c_{q+1}$ and $(f \times I) \circ (\sigma \times I) = (f \circ \sigma) \times I$.

$$\begin{tikzcd}
S_q(Y) \arrow[r, "\mathcal{Z}_q"] & S_q(Y \times I) \\
S_q(X) \arrow[r, "\mathcal{Z}_q"] \arrow[u, "f_*"] & S_q(X \times I) \arrow[u, "(f\times I)_*"'] \\
S_q(\Delta_q) \arrow[r, "\mathcal{Z}_q"] \arrow[u, "\sigma_*"] & S_q(\Delta_q \times I) \arrow[u, "(\sigma\times I)_*"']
\end{tikzcd}$$

Furthermore $\mathcal{Z}_q$ is also a chain-homotopy, since

$$\begin{aligned}
\partial\mathcal{Z}_q(\sigma) &= \partial S_q(\sigma \times I)(c_{q+1}) = S_q(\sigma \times I)\partial c_{q+1} = S_q(\sigma \times I)(\varphi\Delta_q - \mathcal{Z}_{q-1}\partial\Delta_q) \\
&= \varphi S_q(\sigma)(\Delta_q) - \mathcal{Z}_{q-1}\partial S_q(\sigma)(\Delta_q) = \varphi(\sigma) - \mathcal{Z}_{q-1}\partial(\sigma). \quad \square
\end{aligned}$$

8.23 Definition. [7, 8.3.12] [7, 8.3.15] Two chain mappings $\varphi, \psi : C \to C'$ are called (CHAIN) HOMOTOPIC and we write $\psi \sim \varphi$ if there are homomorphisms $\mathcal{Z} : C_q \to C'_{q+1}$ such that $\psi - \varphi = \partial\mathcal{Z} + \mathcal{Z}\partial$.

8.24 Proposition. [7, 8.3.13] *Let $\varphi \sim \psi : C \to C'$ then $H(\varphi) = H(\psi) : H(C) \to H(C')$.*

Proof. Let $[c] \in H(C)$, i.e. $\partial c = 0$ then $H(\psi)[c] - H(\varphi)[c] = [(\psi - \varphi)c] = [\mathcal{Z}\partial c + \partial\mathcal{Z}c] = [\partial\mathcal{Z}c] = 0$. $\square$

8.25 Proposition. [7, 8.3.14] *Chain homotopies are compatible with compositions.*

Proof. Clearly, for $\varphi \sim \psi$ we have $\chi \circ \varphi \sim \chi \circ \psi$ and $\varphi \circ \chi \sim \psi \circ \chi$ and being homotopic is transitive. $\square$

8.26 Theorem. [7, 9.3.1] *Let $f \sim g : (X, A) \to (Y, B)$. Then $f_* = g_* : H_q(X, A) \to H_q(Y, B)$.*

Proof. By 8.22 we have that the chain mappings induced by the inclusions $\mathrm{inj}_j : X \to X \times I$ are chain homotopic to each other for $j \in \{0,1\}$ by a chain homotopy $\mathcal{Z}$. Let h be a homotopy between f and g, i.e. $f = h \circ \mathrm{inj}_0$ and $g = h \circ \mathrm{inj}_1$. By 8.25 we have $S(f) \sim S(g) : S(X) \to S(Y)$ and also the restrictions $S(f) \sim S(g) : S(A) \to S(B)$, since the constructed homotopy is natural. Thus $S(f) \sim S(g) : S(X,A) \to S(X,B)$. By 8.24 we have that $H(f) = H(g) : H(X,A) \to H(X,B)$. □

8.27 Corollary. [7, 9.3.2] *Let $f \sim g : X \to Y$. Then $f_* = g_* : H_q(X) \to H_q(Y)$.*

Proof. Obvious, since $H_q(X, \emptyset) \cong H_q(X)$. □

8.28 Corollary. [7, 9.3.3] *Let $f : X \to Y$ be a homotopy equivalence. Then $f_* : H_q(X) \to H_q(Y)$ is an isomorphism for all q. In particular all contractible spaces are acyclic.*

Proof. Obvious by functoriality and 8.27 since an inverse g up to homotopy induces an inverse $H(g)$ of $H(f)$. □

8.29 Corollary. [7, 9.3.4] [7, 9.3.5] [7, 9.3.6]

1 *Let $A \subseteq X$ be a DR. Then $H_q(A) \to H_q(X)$ is an isomorphism and hence $H_q(X,A) = \{0\}$ for all q.*

- *Let $B \subseteq A \subseteq X$ and A be a DR of X. Then $H_q(A,B) \to H_q(X,B)$ is an isomorphism.*
- *Let $B \subseteq A \subseteq X$ and B be a DR of A. Then $H_q(X,B) \to H_q(X,A)$ is an isomorphism.*

Proof. The first part follows as special case from 8.28 and from the long exact homology sequence of a pair, 8.15. The other two cases then follow by using the long exact homology sequence of a triple, 8.19. □

Excision theorem

In order to prove the general excision theorem we need barycentric refinement for singular simplices, since a singular simplex in X need neither be contained in $S(U)$ nor in $S(V)$ for a given covering $\{U,V\}$ of X.

8.30 Definition. [7, 9.4.1] For the standard q-simplex Δ_q we define the barycentric chain $B(\Delta_q) \in S_q(\Delta_q)$ recursively by

$$B(\Delta_0) := \Delta_0 = \{e_0\}$$

$$B(\Delta_q) := \widehat{\Delta_q} \star \sum_{j=0}^{q} (-1)^j S(\delta^j)(B(\Delta_{q-1})) \text{ for } q \geq 1,$$

where $\widehat{\Delta_q}$ denotes the barycenter $\frac{1}{q+1}\sum_{j=0}^{q} e^j$.

Now we define in a natural way $B(\sigma) := B\Big(S(\sigma)(\Delta_q)\Big) = S(\sigma)B(\Delta_q)$ for $\sigma : \Delta_q \to X$ and extend it linearly to $B : S_q(X) \to S_q(X)$ by setting $B\Big(\sum_\sigma n_\sigma \cdot \sigma\Big) := \sum_\sigma n_\sigma B(\sigma)$.

8.31 Proposition. [7, 9.4.2] *The barycentric refinement is a natural chain mapping $B : S(\_) \to S(\_)$ with $B \sim \mathrm{id}$.*

Proof. Let us first show naturality: So let $f : X \to Y$ be continuous. Then

$$(f_*B)\sigma = (f_*\sigma_*B)\Delta_q = (f \circ \sigma)_* B\Delta_q = B(f \circ \sigma) = Bf_*\sigma.$$

Next we prove that it is a chain mapping, i.e. $\partial B = B\partial$. On $S_q(X)$ with $q \le 0$ this is obvious. Note that the formula for $B(\Delta_q)$ can be rewritten as

$$B\Delta_q = \widehat{\Delta_q} \star B\partial\Delta_q.$$

so we use induction for $q > 0$:

$$\begin{aligned}\partial B\sigma &= \partial\sigma_* B\Delta_q = \sigma_*\partial B\Delta_q = \sigma_*\partial\Big(\widehat{\Delta_q} \star B\partial\Delta_q\Big)\\ &\overset{\boxed{8.13}}{=\!=} \sigma_*\Big(B\partial\Delta_q - \widehat{\Delta_q} \star \partial B\partial\Delta_q\Big) \overset{\text{I.Hyp.}}{=\!=} B\sigma_*\partial(\Delta_q) - \sigma_*\Big(\widehat{\Delta_q} \star B\partial\partial(\Delta_q)\Big)\\ &= B\partial\sigma_*(\Delta_q) - 0 = B\partial\sigma.\end{aligned}$$

Finally we prove the existence of a chain homotopy $\mathrm{id} \sim B : S \to S$. Let $i : X \to X \times I$ be given by $x \mapsto (x, 0)$ and $p : X \times I \to X$ given by $(x, t) \mapsto x$ then $S(p) \circ S(i) = \mathrm{id}$. By 8.22 we have a chain homotopy $S(i) \sim S(i) \circ B$. Composing with $S(p)$ gives a chain homotopy $\mathrm{id} = S(p) \circ S(i) \sim S(p) \circ S(i) \circ B = B$ by 8.25. □

8.32 Corollary. [7, 9.4.3] *Let $Y \subseteq X$ then $B_* = \mathrm{id} : H(X, Y) \to H(X, Y)$.*

By iteration we get the corresponding results for $B^r := \overbrace{B \circ \cdots \circ B}^{r \text{ times}}$.

Proof. Let $\alpha \in H_q(X, Y)$ be given, i.e. $\alpha = [z + S_q(Y)]$ for a $z \in S_q(X)$ with $\partial z \in S_{q-1}(Y)$. By 8.31 $B \sim \mathrm{id}$. Let $\mathcal{Z}$ be a corresponding natural chain homotopy, then $Bz - z = \partial\mathcal{Z}_q z - \mathcal{Z}_{q-1}\partial z \in \partial\mathcal{Z}_q z + S_q(Y)$, i.e. Bz is homologous to z relative Y and, furthermore, $\partial Bz \in \partial z + 0 + \partial\mathcal{Z}_{q-1}\partial z \in S_{q-1}(Y)$, so Bz is a cycle relative Y, i.e. $\alpha = [z + S_q(Y)] = [Bz + S_q(Y)] = B(\alpha)$. □

8.33 Lemma. [7, 9.4.4] *Let X be the union of two open subsets U and V. Then for every $c \in S_q(X)$ there is an $r > 0$ with $B^r c \in S_q(U) + S_q(V) \subseteq S_q(X)$.*

Proof. It is enough to show this for c being a singular simplex $\sigma : \Delta_q \to X$. The sets $\sigma^{-1}(U)$ and $\sigma^{-1}(V)$ form an open covering of Δ_q. Let λ be the Lebesgue number for this covering, i.e. all subsets of Δ_q of diameter less than λ belong to one of the two sets. Since $B^r(\Delta_q)$ is a finite linear combination of singular simplices, whose image are closed simplices of the r-th barycentric refinement of $K := \{\tau : \tau \le \Delta_q\}$, we have by 3.27 that each summand of $B^r(\Delta_q)$ has image in $\sigma^{-1}(U)$ or in $\sigma^{-1}(V)$. Hence $B^r(\sigma) = B^r S(\sigma)(\Delta_q) = S(\sigma)B^r(\Delta_q)$ is a combination of terms in $S_q(U)$ and in $S_q(V)$. □

8.34 Excision theorem. [7, 9.4.5]
*Let $X_j \subseteq X$ for $j \in \{0, 1\}$ such that the interiors X_j^o cover X.
Then the inclusion $i_* : (X_2, X_2 \cap X_1) \to (X_2 \cup X_1, X_1)$ induces isomorphisms $H_q(X_2, X_2 \cap X_1) \to H_q(X_2 \cup X_1, X_1)$ for all q.
In particular this applies to $X_1 := Y \subseteq X$ and $X_2 := X \setminus Z$ for subsets Z and Y satisfying $\bar{Z} \subseteq Y^o$ and so gives isomorphisms $H_q(X \setminus Z, Y \setminus Z) \to H_q(X, Y)$.*

Proof. We have to show that $i_*: H_q(X_2, X_2\cap X_1) \to H_q(X_2\cup X_1, X_1)$ is bijective.

i_* is onto: Let $\beta \in H_q(X_2\cup X_1, X_1)$, i.e. $\beta = [z + S_q(X_1)]$ for some $z \in S_q(X)$ with $\partial z \in S_q(X_1)$. By 8.33 there exists an $r > 0$ and $u_j \in S_q(X_j^o)$ such that $z \sim B^r z = u_1 + u_2 \sim u_2$ relative X_1 by 8.32. We have $\partial u_2 \in S_{q-1}(X_2)$ and $\partial u_2 = \partial B^r z - \partial u_1 = B^r \partial z - \partial u_1 \in S_{q-1}(X_1)$, hence $\partial u_2 \in S_{q-1}(X_1\cap X_2)$. So $\alpha := [u_2 + S_q(X_2\cap X_1)] \in H_q(X_2, X_2\cap X_1)$ and it is mapped by i_* to β.

i_* is injective: Let $\alpha \in H_q(X_2, X_2\cap X_1)$ be such that $i_*\alpha = 0$. Then $\alpha = [x_2 + S_q(X_2\cap X_1)]$ for some $x_2 \in S_q(X_2)$. Since $i_*(\alpha) = 0$ we have a $(q+1)$-chain c in X and a q-chain x_1 in X_1 with $\partial c = x_2 + x_1$. Again by 8.33 there is an $r > 0$ such that $B^r c = u_1 + u_2$ with $u_j \in S_q(X_j^o)$. Hence $\partial u_1 + \partial u_2 = \partial B^r c = B^r \partial c = B^r(x_2 + x_1)$. So $a := B^r x_2 - \partial u_2 = \partial u_1 - B^r x_1$ is a chain in $X_1 \cap X_2$ and $x_2 \sim B^r x_2 = \partial u_2 + a \sim 0$ relative X_1 by 8.32, i.e. $\alpha = [x_2 + S_q(X_2\cap X_1)] = [\partial u_2 + a + S_q(X_2\cap X_1)] = 0$.

The alternate description is valid, since the interiors of $X_1 := Y$ and $X_2 := X\setminus Z$ cover X iff $Y^o = X_1^o \supseteq X\setminus X_2^o = X\setminus(X\setminus \bar Z) = \bar Z$. Obviously $Y\setminus Z = X_1\cap X_2$. □

8.35 Corollary. [7, 9.4.6] [7, 9.4.7] *Let (X, A) be a CW-pair. Then the quotient map $p : (X, A) \to (X/A, A/A)$ induces an isomorphism in homology for all q and hence $H_q(X, A) \cong H_q(X/A)$ for all $q \neq 0$.*

Proof. By 4.19 we have an open neighborhood U of A in X, of which A is a SDR. Let $p : X \to X/A =: Y$ be the quotient mapping and let $V := p(U) \subseteq X/A =: Y$ and $y := A/A \in X/A$. Since U is saturated also $V \subseteq Y$ is open and $p(A) = \{y\}$ a SDR in V. Now consider

$$\begin{tikzcd} H_q(X,A) \arrow[r, "\cong"', "8.29"] \arrow[d, "p_*"'] & H_q(X,U) \arrow[d, "p_*"'] & H_q(X\setminus A, U\setminus A) \arrow[l, "\cong"', "8.34"] \arrow[d, "p_*", "\cong\ 1.34"'] \\ H_q(Y,\{y\}) \arrow[r, "\cong"', "8.29"] & H_q(Y,V) & H_q(Y\setminus\{y\}, V\setminus\{y\}) \arrow[l, "\cong"', "8.34"] \end{tikzcd}$$

By 1.34 we have that $p : (X, A) \to (Y, \{y\})$ is a relative homeomorphism, so the vertical arrow on the right side is induced by an isomorphism of pairs and hence is an isomorphism. The horizontal arrows on the right side are isomorphisms by the excision theorem 8.34. Hence the vertical arrow in the middle is an isomorphism. By 8.29 the horizontal arrows on the left are isomorphisms, hence also the vertical arrow on the left.

By 8.18.3 we have finally that $H_q(Y, \{y\}) \cong H_q(Y)$ for $q > 0$. □

8.36 Corollary. [7, 9.4.7] *We have $H_q(X, A) \cong H_q(X/A)$ for all $q \neq 0$ in the situation of 8.35.*

Proof. □

8.37 Proposition. [7, 9.4.8] *Let $f : (X, A) \to (Y, B)$ be a relative homeomorphism of CW-pairs. Assume furthermore that $X \setminus A$ contains only finitely many cells or $f : X \to Y$ is a quotient mapping. Then $f_* : H_q(X, A) \to H_q(Y, B)$ is an isomorphism for all q.*

Proof. By 1.34 we have an induced continuous bijective mapping $X/A \to Y/B$ making the following diagrams commute:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & B \arrow[d, hook] \\ X \arrow[r, "f"] \arrow[d, "p"] & Y \arrow[d, "q"] \\ X/A \arrow[r, "f", "\cong"'] & Y/B \end{tikzcd} \qquad \begin{tikzcd} H_q(X,A) \arrow[r, "f_*"] \arrow[d, hook, "p_*", "8.35"'] & H_q(Y,B) \arrow[d, hook, "q_*", "8.35"'] \\ H_q(X/A, A/A) \arrow[r, "8.18", "\cong"'] & H_q(Y/B, B/B) \end{tikzcd}$$

That this bijection is a homeomorphism follows in case $X \setminus A$ has only finitely many cells since then X/A is compact, and in the case where $f : X \to Y$ is a quotient map then so is $X \to Y \to Y/B$ and hence also $X/A \to Y/B$. Both X/A and Y/B are CW-complexes by 4.15 and by 8.18.2 the bottom arrow on the right is an isomorphism. By 8.35 the vertical arrows on the right are isomorphisms as well, so the same has to be true for the top one. □

8.38 Proposition. [7, 9.4.9] *Let X_j be CW-complexes which 0-cells $x_j \in X_j$ as base-points. Then we have natural isomorphisms ${}^{ab}\coprod_j H_q(X_j) \to H_q(\bigvee_j X_j)$ for $q \neq 0$.*

Proof. We have $\bigvee_j X_j := \bigsqcup_j X_j/A$ where $A := \{x_j : j \in J\}$. Hence $H_q(A) = 0$ for $q \neq 0$ and so

$$H_q\Big(\bigvee_j X_j\Big) \overset{8.36}{\cong} H_q\Big(\bigsqcup_j X_j, A\Big) \overset{A \text{ ``acyclic''}}{\cong} H_q\Big(\bigsqcup_j X_j\Big) \overset{8.9}{\cong} {}^{ab}\coprod_j H_q(X_j). \quad □$$

8.39 Proposition. [7, 9.4.10] *Let $X = X_1 \cup X_2$, where $X_j \subseteq X$ is open. Then there is a long exact sequence (the so called* Mayer-Vietoris sequence*)*

$$\cdots \to H_q(X_1 \cap X_2) \to H_q(X_1) \oplus H_q(X_2) \to H_q(X) \to H_{q-1}(X_1 \cap X_2) \to \ldots.$$

Proof. Let $S := S(X)$, $S_1 := S(X_1) \subseteq S(X)$ and $S_2 := S(X_2) \subseteq S(X)$. Then $S(X_1 \cap X_2) = S_1 \cap S_2$. Let $S_1 + S_2$ be the chain complex which has the subgroup of S generated by S_1 and S_2 in every dimension. We claim that the following short sequence

$$0 \to S_1/(S_1 \cap S_2) \to S/S_2 \to S/(S_1 + S_2) \to 0$$

is exact. In fact by the first isomorphy theorem we have $S_1/(S_1 \cap S_2) \cong (S_1 + S_2)/S_2$ and hence the inclusion $S_1 + S_2 \subseteq S$ induces an injection $S_1/(S_1 \cap S_2) \to S/S_2$. The quotient of it is by the second isomorphy theorem $(S/S_2)/((S_1 + S_2)/S_2) \cong S/(S_1 + S_2)$, which proves the claim. By 8.34 we have that the inclusion $(S_1, S_1 \cap S_2) \hookrightarrow (S, S_2)$ induces an isomorphism $H(S_1/(S_1 \cap S_2)) =: H(X_1, X_1 \cap X_2) \to H(X_1 \cup X_2, X_2) := H(S/S_2)$. Hence the long exact homology sequence 7.31 gives $H(S/(S_1 + S_2)) = 0$.

If we consider now the short exact sequence

$$0 \to S_1 + S_2 \to S \to S/(S_1 + S_2) \to 0$$

we deduce from the long exact homology sequence that $H(S_1 + S_2) \to H(S)$ is an isomorphism.

Now consider the sequence

$$0 \to S_1 \cap S_2 \to S_1 \oplus S_2 \to S_1 + S_2 \to 0,$$

where the inclusion is given by $c \mapsto (c, -c)$ and the projection by $(c_1, c_2) \mapsto c_1 + c_2$. This is obviously short exact, since (c_1, c_2) is mapped to 0 iff $c_1 + c_2 = 0$, i.e. $c := c_1 = -c_2 \in S_1 \cap S_2$ is mapped to (c_1, c_2).

$$\begin{tikzcd}
S_1 \cap S_2 \arrow[r, hook] \arrow[d, hook] & S_2 \arrow[r, equal] \arrow[d, hook] & S_2 \arrow[d, hook] & \\
S_1 \arrow[r, hook] \arrow[d] & S_1 + S_2 \arrow[r, hook] \arrow[d] & S \arrow[r, two heads] \arrow[d] & S/(S_1+S_2) \arrow[d, equal] \\
S_1/S_1 \cap S_2 \arrow[r, equal] & (S_1+S_2)/S_2 \arrow[r, hook] & S/S_2 \arrow[r, two heads] & S/(S_1+S_2)
\end{tikzcd}$$

So we get a long exact sequence in homology, where we may replace $H(S_1 + S_2)$ by $H(S) =: H(X)$ by what we said above. Note that the boundary operator is given by $[z] \mapsto [\partial z_1]$, where $B^r z = z_1 + z_2$. □

8.40 Remark. [7, 9.4.12]

(1) It is enough to assume in 8.39 that there are neighborhoods of X_1 and X_2 which have X_1 and X_2 and their intersection has $X_1 \cap X_2$ as DRs. In particular this applies to CW-subspaces X_i of a CW-complex X by 4.19.

(2) Let $X_1 \cap X_2$ be acyclic. Then the Mayer-Vietoris sequence gives $H_q(X) \cong H_q(X_1) \oplus H_q(X_2)$ for $q \neq 0$. In fact only the case $q = 1$ needs some argument: We have the exact sequence

$$\begin{tikzcd}
0 = H_1(X_1 \cap X_2) \arrow[r] & H_1(X_1) \oplus H_1(X_2) \arrow[r] & H_1(X) \arrow[dll] & \\
\mathbb{Z} = H_0(X_1 \cap X_2) \arrow[r] & H_0(X_1) \oplus H_0(X_2) \arrow[r] & H_0(X) \arrow[r] & 0
\end{tikzcd}$$

and the mapping $H_0(X_1 \cap X_2) \to H_0(X_1) \oplus H_0(X_2)$ is injective, since the generator is mapped to a generator of $H_0(X_1)$ and of $H_0(X_2)$.

(3) Let X_1 and X_2 be acyclic, then we have $H_q(X_1 \cap X_2) \cong H_{q+1}(X)$ for $q > 0$ and furthermore $H_1(X)$ is free abelian and

$$\begin{tikzcd}
H_1(X_1) \oplus H_1(X_2) \arrow[r] \arrow[d, equal] & H_1(X) \arrow[r] & H_0(X_1 \cap X_2) \arrow[r] \arrow[d, equal] & H_0(X_1) \oplus H_0(X_2) \arrow[r] \arrow[d, equal] & H_0(X) \arrow[r] \arrow[d, equal] & 0 \\
0 & & \mathbb{Z}^k & \mathbb{Z}^2 & \mathbb{Z} &
\end{tikzcd}$$

gives $H_1(X) \cong \mathbb{Z}^{k-1}$ via the rank formula $\operatorname{rank}(\ker f) + \operatorname{rank}(\operatorname{im} f) = \operatorname{rank}(\operatorname{dom} f)$, where we used that $X = X_1 \cup X_2$ is connected being the union of two connected sets.

(4) Consider the covering $S^n = D^n_+ \cup D^n_-$. By (1) we get a long exact Mayer-Vietoris sequence. And since D^n_+ and D^n_- are contractible, they are acyclic. So $H_q(S^n) \cong H_{q-1}(D^n_+ \cap D^n_-) = H_{q-1}(S^{n-1})$ for $q, n > 0$. Inductively we hence

get $H_q(S^n) \cong H_{q-n}(S^0) = \{0\}$ for $q > n$, since S^0 is discrete and $H_q(S^n) \cong H_1(S^{n-q+1}) = \{0\}$ for $0 < q < n$, since

$$\begin{tikzcd} 0 \arrow[r] & H_1(S^{n-q+1}) \arrow[r] & H_0(S^{n-q}) \arrow[r] \arrow[d, equal] & H_0(D^{n-q}_+) \oplus H_0(D^{n-q}_-) \arrow[r] \arrow[d, equal] & H_0(S^{n-q+1}) \arrow[r] \arrow[d, equal] & 0 \\ & & \mathbb{Z} & \mathbb{Z}\oplus\mathbb{Z} & \mathbb{Z} & \end{tikzcd}$$

and $H_n(S^n) \cong H_1(S^1) \cong \mathbb{Z}$, since

$$\begin{tikzcd} 0 \arrow[r] & H_1(S^1) \arrow[r] & H_0(S^0) \arrow[r] \arrow[d, equal] & H_0(D^0_+) \oplus H_0(D^0_-) \arrow[r] \arrow[d, equal] & H_0(S^1) \arrow[r] \arrow[d, equal] & 0 \\ & & \mathbb{Z}\oplus\mathbb{Z} & \mathbb{Z}\oplus\mathbb{Z} & \mathbb{Z} & \end{tikzcd}$$

Homology of balls, spheres and their complements

8.41 Proposition. [7, 9.5.1] *Let $n \geq 0$ then*

$$H_q(\Delta_n, \dot\Delta_n) \cong \begin{cases} \mathbb{Z} & \text{for } q = n \\ 0 & \text{otherwise} \end{cases}$$

The generator in $H_n(\Delta_n, \dot\Delta_n)$ will be denoted $[\Delta_n]$ and is given by the relative homology class of the singular simplex $\mathrm{id}_{\Delta_n} : \Delta_n \to \Delta_n$.

Proof. We proof this by induction on n.

$(n = 0)$ $H_q(\Delta_0, \dot\Delta_0) = H_q(\{1\}, \emptyset) \overset{8.16}{=\!=} H_q(\{*\})$.

$(n > 0)$ We consider Δ_{n-1} as face opposite to e_n in Δ_n and let $A_n := \dot\Delta_n \setminus \Delta_{n-1}$. Since A_n is a DR of Δ_n, we conclude from the homology-sequence 8.19 of the triple $A_n \subseteq \dot\Delta_n \subseteq \Delta_n$ that $H_q(\Delta_n, \dot\Delta_n) \cong H_{q-1}(\dot\Delta_n, A_n)$. Since $\Delta_{n-1} \setminus \dot\Delta_{n-1} - \dot\Delta_n \setminus A_n$ we get from 8.37 or 8.36 that the inclusion induces an isomorphism $H_{q-1}(\Delta_{n-1}, \dot\Delta_{n-1}) \cong H_{q-1}(\dot\Delta_n, A_n)$. Hence $H_q(\Delta_n, \dot\Delta_n) \cong H_{q-1}(\Delta_{n-1}, \dot\Delta_{n-1})$ and by recursion we finally arrive in case $q \geq n$ at $H_{q-n}(\Delta_0, \dot\Delta_0)$ which we calculated above, and in case $q < n$ at $H_0(\Delta_{n-q}, \dot\Delta_{n-q}) = 0$ by 8.16, since Δ_{n-q} is connected and $\dot\Delta_{n-q} \neq \emptyset$.

Let $[\Delta_n]$ denote the relative homology class in $H_n(\Delta_n, \dot\Delta_n)$ of $\mathrm{id}_{\Delta_n} : \Delta_n \to \Delta_n$. Then its image in $H_{n-1}(\dot\Delta_n, A_n)$ ist given by $[\partial\,\mathrm{id}_{\Delta_n} + S_n(A_n)]$ which equals the image $[\mathrm{id}_{\Delta_{n-1}} + s_n(A_n)]$ of $[\Delta_{n-1}] \in H_{n-1}(\Delta_{n-1}, ;\Delta_{n-1})$. Obviously $[\Delta_0]$ is the generator of $H_0(\Delta_0, \dot\Delta_0) = H_0(\{1\})$. □

8.42 Corollary. [7, 9.5.2] *For $n \geq 0$ we have*

$$H_q(D^n, S^{n-1}) \cong \begin{cases} \mathbb{Z} & \text{for } q = n \\ 0 & \text{otherwise} \end{cases}$$

We denote the canonical generator by $[D^n]$. It is given by th relative homology class of a homeomorphism $\Delta_n \to D^n$. □

8.43 Corollary. [7, 9.5.3] *For $n > 0$ we have*

$$H_q(S^n) \cong \begin{cases} \mathbb{Z} & \text{for } q = n \text{ or } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

We denote the canonical generator by $[S^n]$. It is given by $[S^n] = \partial_*([D^n]) = [\partial D^n]$.

So this gives a different proof from 8.40.4

Proof. Consider the homology sequence of the pair $S^n \subseteq D^{n+1}$:

$$\begin{array}{ccccccc} H_{q+1}(D^{n+1}) & \longrightarrow & H_{q+1}(D^{n+1}, S^n) & \xrightarrow{\cong} & H_q(S^n) & \longrightarrow & H_q(D^{n+1}) \\ \| & & & & & & \|\,{\scriptstyle 8.14} \\ 0 & & & & & & 0 \end{array}$$

□

8.44 Corollary. [7, 9.5.6] *We have $H_q(\bigvee_j S^n) = 0$ for $q \notin \{0, n\}$ and $H_n(\bigvee_j S^n) \cong {}^{ab}\coprod_j \mathbb{Z}$. By 8.38 the generators are $(\mathrm{inj}_j)_*[S^n]$.* □

1.20 Proposition. *Let $m \neq n$ then $\mathbb{R}^m \ncong \mathbb{R}^n$ and $S^m \ncong S^n$.*

We have proved this by applying 1.19 the theorem of the invariance of domains.

Proof of 1.20 for $\mathbb{R}^n$ and S^n. Let $m \neq n$ and $m > 0$. Then $H_m(S^m) \cong \mathbb{Z}$ but $H_m(S^n) = \{0\}$, so $S^m \not\sim S^n$. Assume $\mathbb{R}^n \cong \mathbb{R}^m$ then $S^{m-1} \sim \mathbb{R}^m \setminus \{0\} \cong \mathbb{R}^n \setminus \{0\} \sim S^{n-1}$, hence $m = n$. □

9.1 Proposition. [7, 11.1.1] *S^n is not contractible and is not a retract in D^{n+1}*

Proof. Since $H_n(S^n) \cong \mathbb{Z} \ncong \{0\} = H_n(\{*\})$ the first statement is clear. And the second follows, since retracts of contractible spaces are contractible. In fact let $h_t : X \to X$ be a contraction and let $i : A \to X$ have a left inverse $p : X \to A$. Then $p \circ h_t \circ i : A \to A$ is a contraction of A. □

9.2 Corollary. Brouwers fixed point theorem. [7, 11.1.2] *Every continuous map $f : D^n \to D^n$ has a fixed point.*

Proof. Otherwise we can define a retraction as in 2.23. □

10.1 Proposition. [7, 11.7.1] *Let $B \subseteq S^n$ be a ball. Then $S^n \setminus B$ is acyclic.*

Proof. Induction on $r := \dim B$.
$(r = 0)$ Then B is a point and hence $S^n \setminus B \cong \mathbb{R}^n$ is contractible and thus acyclic.
$(r + 1)$ Let $z \in Z_q(S^n \setminus B)$ for $q > 0$ and $z := x - y \in Z_0(S^n \setminus B)$ for $q = 0$ with $x, y \in S^n \setminus B$. We have to show that $\exists b \in S_{q+1}(S^n \setminus B)$ with $\partial b = z$.

Consider a homeomorphisms $f : I^{r+1} = I^r \times I \cong B$. Then $B_t := f(I^r \times \{t\})$ is an r-ball. Thus by induction hypothesis there are $b_t \in S_{q+1}(S^n \setminus B_t)$ with $\partial b_t = z$ considered as element in $S_q(S^n \setminus B_t) \leftarrow S_q(S^n \setminus B)$. Since the image of b_t is disjoint to B_t, we can choose an open neighborhood V_t of t such that $I^r \times V_t \subseteq f^{-1}(S^n \setminus \mathrm{Im}(b_t))$. Using compactness we find a partition of $0 = t_0 < t_1 < \cdots < t_N = 1$ of I into finitely many intervals $I_j := [t_{j-1}, t_j]$ such that for each j there exists a t with $I_j \subset V_t$. Let $b_j := b_t \in S_{q+1}(Y_j)$ where Y_j is the open subset $S^n \setminus f(I^r \times I_j)$. Now let $X_j := \bigcap_{i<j} Y_i$. Then $X_j \cup Y_j = \bigcap_{i<j} Y_i \cup Y_j = Y_{j-1} \cup Y_j = S^n \setminus f(I^r \times \{t_j\})$ and $X_j \cap Y_j = X_{j+1}$.

We now show by induction on j that $[z] = 0$ in $H_q(X_j)$. For $(j = 1)$ nothing is to be shown, since $X_1 = S^n$. For $(j + 1)$ we apply the Mayer-Vietoris sequence 8.39 to the open sets X_j and Y_j:

$$\begin{array}{ccccc} H_{q+1}(\overbrace{X_j \cup Y_j}^{S^n \setminus f(I^r \times \{t_j\})}) & \longrightarrow & H_q(\overbrace{X_j \cap Y_j}^{X_{j+1}}) & \longrightarrow & H_q(X_j) \oplus H_q(Y_j) \\ {\scriptstyle ind.}\Big\| & & & & \\ 0 & & & & \end{array}$$

The image of $[z] \in H_q(X_{j+1})$ in $H_q(X_j) \oplus H_q(Y_j)$ is zero, since the first component is $[z] = 0 \in H_q(X_j)$ by induction hypothesis, and the second component $[z] = [\partial b_j] = 0 \in H_q(Y_j)$. Since the space on the left side is zero, the arrow on the right is injective we get that $[z] = 0 \in H_q(X_{j+1})$.

Since $X_j = S^n \setminus B$ finally, we are done. □

10.2 Theorem. [7, 11.7.4] *Let $S \subseteq S^n$ be an r-sphere with $0 \le r \le n-1$ and $n \ge 2$. Then*

$$H_q(S^n \setminus S) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{for } q = 0 \text{ and } r = n-1 \\ \mathbb{Z} & \text{for } q \in \{0, n-1-r\} \text{ and } r < n-1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Induction on r.
$(r = 0)$ Then $S \cong \{-1, +1\}$ and $S^n \setminus S \sim S^{n-1}$, so the result follows from 8.40 or 8.43.
$(r > 0)$ We have $S^r = D^r_- \cup D^r_+$ and $B_\pm := f(D^r_\pm)$ are r-balls and $S' := f(S^{r-1})$ an $(r-1)$-sphere. By 10.1 $S^n \setminus B_\pm$ are acyclic and since $S^n \setminus S' = (S^n \setminus B_+) \cup (S^n \setminus B_-)$ and $S^n \setminus S = (S^n \setminus B_+) \cap (S^n \setminus B_-)$ we get by 8.40.3 that $H_q(S^n \setminus S) \cong H_{q+1}(S^n \setminus S')$ for $q > 0$ and $H_0(S^n \setminus S) \cong H_1(S^n \setminus S') \oplus \mathbb{Z}$. By recursion we finally arrive at $H_{q+r}(S^n \setminus \{\pm 1\}) = H_{q+r}(S^{n-1})$, which be treated before.

10.3 Proposition. [7, 11.7.2] [7, 11.7.5] *Let $n \ge 2$. If $B \subseteq \mathbb{R}^n$ is a ball, then*

$$H_q(\mathbb{R}^n \setminus B) = \begin{cases} \mathbb{Z} & \text{for } q \in \{0, n-1\} \\ 0 & \text{otherwise.} \end{cases}$$

If $S \subseteq \mathbb{R}^n$ is an r-sphere with $0 \le r \le n-1$, then

$$H_q(\mathbb{R}^n \setminus S) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{for } (q = n-1, r = 0) \text{ or } (q = 0, r = n-1) \\ \mathbb{Z} & \text{for } (q = n-1, r \ne 0) \text{ or } (q \in \{0, n-1-r\}, r \ne n-1) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $A \subseteq \mathbb{R}^n \cong S^n \setminus \{P_+\} \subset S^n$ be compact. The long exact homology sequence 8.16 of the pair $(S^n \setminus A, \mathbb{R}^n \setminus A)$ gives

$$\to H_{q+1}(S^n \setminus A, \mathbb{R}^n \setminus A) \xrightarrow{\partial_*} H_q(\mathbb{R}^n \setminus A) \to H_q(S^n \setminus A) \to H_q(S^n \setminus A, \mathbb{R}^n \setminus A) \to$$

By the excision theorem 8.34 applied to $A \subseteq \mathbb{R}^n \subseteq S^n$ we get $H_q(S^n \setminus A, \mathbb{R}^n \setminus A) \cong H_q(S^n, \mathbb{R}^n)$, which is isomorphic by 8.29 to $H_q(S^n, \{*\})$, since $\mathbb{R}^n$ is contractible.

This homology equals for $q > 0$ by 8.18.3 that of $H_q(S^n)$ and is 0 for $q = 0$ by 8.20 since S^n is path-connected. So

$$H_q(S^n, \{*\}) = \begin{cases} \mathbb{Z} & \text{for } q = n \\ 0 & \text{otherwise} \end{cases}$$

The long exact sequence from above thus is

$$\to H_{q+1}(S^n, \{*\}) \xrightarrow{\partial_*} H_q(\mathbb{R}^n \setminus A) \to H_q(S^n \setminus A) \to H_q(S^n, \{*\}) \to$$

In particular, $H_q(\mathbb{R}^n \setminus A) \cong H_q(S^n \setminus A)$ for $q \notin \{n-1, n\}$ and near $q = n-1$ it is for A a sphere or ball:

$$0 \to H_n(\mathbb{R}^n \setminus A) \to 0 \to \mathbb{Z} \to H_{n-1}(\mathbb{R}^n \setminus A) \to H_{n-1}(S^n \setminus A) \to 0,$$

This gives $H_n(\mathbb{R}^n \setminus A) = 0$ and $H_{n-1}(\mathbb{R}^n \setminus A) \cong \mathbb{Z} \oplus H_{n-1}(S^n \setminus A)$, from which the claimed result follows. □

10.4 Corollary (Jordan's separation theorem generalized). [7, 11.7.6] [7, 11.7.7] *Let $X \in \{\mathbb{R}^n, S^n\}$. For any r-sphere S with $r < n-1$ we have that $X \setminus S$ is connected (We cannot cut X into two pieces along such a sphere).*
If S is an $n-1$-sphere then $X \setminus S$ has two components, both of which have S as boundary. If $X = S^n$ then the components are acyclic.

Proof. For spheres of dimension $r < n-1$ the result follows from 10.2 and 10.3 since $H_0(X \setminus S) \cong \mathbb{Z}$ in these cases.

If S is a sphere of dimension $n-1$, then $H_0(X \setminus S) \cong \mathbb{Z}^2$ by 10.2 and 10.3. Hence $X \setminus S$ has two components, say U and V.

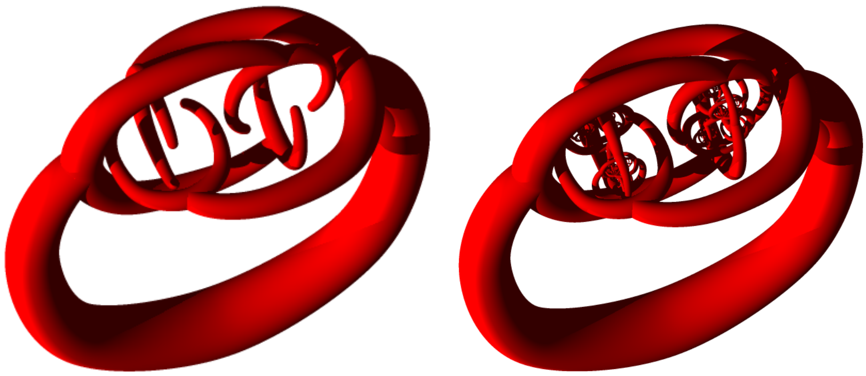
That for $X = S^n$ the components are acyclic follows from $H_q(U) \oplus H_q(V) \cong H_q(X \setminus S) = \{0\}$ for $q \neq 0$.

($\dot U \subseteq S$) In fact $\dot U \cap U = \emptyset$, since U is open and thus $\dot U = \overline{U} \setminus U^o = \overline{U} \setminus U$. From $U \subseteq {\sim}V$ we get $\overline{U} \subseteq \overline{{\sim}V} = {\sim}V$ since V is open and hence $U \cap V = \emptyset$. So $\dot U = \bar U \setminus U \subseteq (X \setminus V) \setminus U = X \setminus (U \cup V) = S$.

($S \subseteq \dot U$) Let $x \in S$ and W be a neighborhood of $x \in X$. Choose $n-1$-balls B and B' with $S = B \cup B'$ and such that $x \in B \subseteq W$. Let c be a path in $\mathbb{R}^n$ from U to V, which avoids $B' \subseteq S$ (this is possible by 10.3 since $X \setminus B'$ is path connected). Let $t_0 := \sup\{t : c(t) \in U\}$. Hence $y := c(t_0) \in \bar U \setminus U = \dot U \subseteq S = B \cup B'$. Hence $y \in B \subseteq W$ and so $W \cap \dot U$ contains y and is not empty, hence $x \in \dot U$. □

10.5 Remark. [7, 11.7.8] For dimension 2 we have Schönflies's theorem (see [5, §9]), i.e. for every Jordan curve in S^2, i.e. injective continuous mapping $c : S^1 \to S^2$ there exists a homeomorphism $f : S^2 \cong S^2$ with $f|_{S^1} = c$. Thus up to a homeomorphism a Jordan-curve looks like the equator $S^1 \subseteq S^2$.

In dimension greater than 2, Alexanders horned sphere is a counterexample: One component of the complement is not simply connected. This gives at the same time an example of an open subset of S^3, which is homologically trivial (i.e. acyclic) but not homotopy-theoretical ($\pi_1(U) \neq 0$).



1.19 **Corollary (Invariance of the domain).** *Let $X, Y \subseteq \mathbb{R}^n$ be homeomorphic. If X is open then so is Y.*

Proof. Take $x \in X$ and $y := f(x) \in Y$. By assumption there is a ball $B := \{z : |z - x| \le r\} \subseteq X$. Let $S := \partial B$. Then $\mathbb{R}^n \setminus f(S) = (\mathbb{R}^n \setminus f(B)) \cup (f(B) \setminus f(S))$. The first part is connected by 10.1 and the second one coincides with $f(B \setminus S) \cong B \setminus S = \overset{o}{D}{}^n$ and hence is connected as well. Thus they are the path components of $\mathbb{R}^n \setminus f(S)$ and hence open. Since the set on the right side is a neighborhood of y in Y, we have that Y is open. $\square$

10.6 Exercise. Let $\nu : S^1 \to S^1 \vee S^1$ be the closed path which first runs once through one and then through the other factorbe as in 5.14. Then $\nu_* : H_1(S^1) \to H_1(S^1 \vee S^1)$ is given by $\pi_1(\{S^1\}_+) = (i_1)_*(\{S^1\}_+) + (i_2)_*(\{S^1\}_+)$.

Cellular homology

10.7 Proposition. [7, 9.6.1] *Let X be a CW-complex. Then $H_p(X^q, X^{q-1}) = 0$ for $p \neq q$.*

Proof. For $q < 0$ this is clear. For $q = 0$ we have $H_p(X^q, X^{q-1}) = H_p(X^0, \emptyset) = 0$ by 8.15, 8.7 and 8.9.

So let $q > 0$. For $p = 0$ we have $H_0(X^{q-1}) \twoheadrightarrow H_0(X^q) \xrightarrow{0} H_0(X^q, X^{q-1}) \to 0$ and the first mapping is onto (since X^q has less components). So the next arrow is 0.

Now let $p \neq 0$. By 8.36 we have $H_p(X^q, X^{q-1}) \cong H_p(X^q/X^{q-1})$ and so the result follows from 8.44, since $X^q/X^{q-1} \cong \bigvee S^q$. $\square$

10.8 Corollary. [7, 9.6.2] *The inclusions induce an epimorphism $H_q(X^q) \to H_q(X)$ and an isomorphism $H_q(X^{q+1}) \to H_q(X)$.*

Proof. By 10.7 and

$$H_{p+1}(X^q, X^{q-1}) \to H_p(X^{q-1}) \to H_p(X^q) \to H_p(X^q, X^{q-1})$$

the first arrow in sequence

$$H_q(X^q) \to H_q(X^{q+1}) \to \cdots \to H_q(X^p) \to \cdots \to H_q(X)$$

is onto and the others are isomorphisms. So we have the result for finite CW-complexes. In the general case we use that every singular simplex lies in some

X^p, hence all mappings are surjective. Similar one shows injectivity, since $[z] = 0 \in H_q(X)$ implies $z = \partial c$ for some $c \in S_q(X) = \bigcup_j S_q(X^j)$, hence $[z] = 0 \in H_q(X^j)$. $\square$

10.9 Corollary. [7, 9.6.3] *Let X be a CW-space without q-cells. Then $H_q(X) = 0$. In particular $H_q(X) = 0$ for $q > \dim X$.*

Proof. From the homology sequence

$$H_{q+1}(X^p, X^{p-1}) \to H_q(X^{p-1}) \to H_q(X^p) \to H_q(X^p, X^{p-1})$$

for $q > p$ and $\boxed{10.7}$ we deduce $H_q(X^{q-1}) \cong \ldots \cong H_q(X^{-1}) = 0$. By assumption $X^q = X^{q-1}$ and hence $H_q(X^q, X^{q-1}) = 0$ so we get the surjectivity of $H_q(X^{q-1}) \to H_q(X^q)$ and thus $H_q(X^q) = 0$ as well. Now the result follows since by $\boxed{10.8}$ $H_q(X^q) \twoheadrightarrow H_q(X)$ is onto. $\square$

10.10 Definition. [7, 9.6.4] The q-th CELLULAR CHAIN GROUP of a CW-complex X is defined as

$$C_q(X) := H_q(X^q, X^{q-1}),$$

and its elements are called CELLULAR q-CHAINS. For every q-cell e in X with characteristic map $\chi^e : (D^q, S^{q-1}) \to (X^q, X^{q-1})$ we define a so-called orientation $\chi^e_*([D^q]) \in C_q(X)$ as the image of $\chi^e_* : H_q(D^q, S^{q-1}) \cong \mathbb{Z} \to H_q(X^q, X^{q-1})$, where $[D^q]$ denotes the generator in $H_q(D^q, S^{q-1})$ induced from a homeomorphism $\Delta^q \to D^q$.

For every cell there are exactly two orientations, which differ only by their sign. And $C_q(X)$ is a free abelian group generated by a selection of orientations for each q-cell.

Proof. Let χ_1 and χ_2 be two characteristic mappings for e. We can consider them as rel. homeomorphisms $\chi_j : (D^q, S^{q-1}) \to (X^{q-1} \cup e, X^{q-1})$. By $\boxed{8.37}$ these factorizations induce isomorphisms. Hence $H_q(\chi_1)[D^q] = \pm H_q(\chi_2)[D^q]$, since the generator $[D^q]$ has to be mapped to a generator of $H_q(D^q, S^{q-1})$, and the only ones are $\pm[D^q]$.

Obviously $C_0(X) = H_0(X^0)$ is free abelian.

For $q > 0$ the projection $p : (X^q, X^{q-1}) \to (Y, \{y_0\}) := (X^q/X^{q-1}, X^{q-1}/X^{q-1})$ induces by $\boxed{8.35}$ an isomorphism $p_* : C_q(X) \to H_q(Y, y_0)$. Since Y is a join of q-spheres we have that $p_*\chi^e_*[D^q]$ form a basis in the free abelian group $H_q(Y, y_0)$, as follows from $\boxed{8.44}$. In fact consider the following commutative diagram:

$$\begin{tikzcd}
(D^q, S^{q-1}) \arrow[r, "\chi^e"] \arrow[d, "h"'] & (X^q, X^{q-1}) \arrow[d, "p"] \\
(S^q, \{*\}) \arrow[r, "\boxed{8.44}"] & (X^q/X^{q-1}, \{*\}),
\end{tikzcd}$$

where the vertical arrows are rel. homeomorphisms and hence induce isomorphisms in homology

$$\begin{tikzcd} H_q(D^q, S^{q-1}) \arrow[r, "\chi^e_*", "\cong"'] \arrow[d, "h_*"', "\cong"] & H_q(X^q, X^{q-1}) \arrow[d, "p_*"', "\cong"] \\ H_q(S^q) \arrow[r] \arrow[d, equal] & H_q(\bigvee S^q) \arrow[d, equal] \\ \mathbb{Z} & {}^{\mathrm{ab}}\coprod \mathbb{Z} \end{tikzcd}$$

and the bottom arrow maps the generator $[S^q] \in H_q(S^q) \cong H_q(S^q, \{*\})$ to one of the generators in $H_q(X^q/X^{q-1}) \cong H_q(X^q/X^{q-1}, \{*\})$. $\square$

10.11 Definition. [7, 9.6.6] Using the long exact sequences for the pairs (X^{q+1}, X^q) and (X^q, X^{q-1}) we have

$$\begin{tikzcd} & C_{q+1}(X) \arrow[d, equal] & & & \\ \cdots \arrow[r] & H_{q+1}(X^{q+1}, X^q) \arrow[r, "\partial_*"] & H_q(X^q) \arrow[r] \arrow[d, equal] & H_q(X^{q+1}) \arrow[r] & \cdots \\ \cdots \arrow[r] & H_q(X^{q-1}) \arrow[r] & H_q(X^q) \arrow[r, hook, "j_*"] & H_q(X^q, X^{q-1}) \arrow[r] \arrow[d, equal] & \cdots \\ & & & C_q(X) & \end{tikzcd}$$

Let $\partial := j_* \circ \partial_* : C_{q+1}(X) \to C_q(X)$. We have $\partial^2 = 0$ by the exactness of the second sequence at $H_q(X^q, X^{q-1})$ and thus we obtain a chain complex. Its homology $H(C(X))$ is called CELLULAR HOMOLOGY of the CW-complex X.
For any $q+1$-cell e with characteristic map χ^e we get

$$\partial(\chi^e_*[D^{q+1}]) = j_*\partial_*\chi^e_*[D^{q+1}] = j_*(\chi^e|_{S^q})_*\partial_*[D^{q+1}] = j_*(\chi^e|_{S^q})_*[\partial D^{q+1}] = j_*(\chi^e|_{S^q})_*[S^q],$$

by the homology ladder

$$\begin{tikzcd} H_{q+1}(D^{q+1}, S^q) \arrow[r, "\chi^e_*"] \arrow[d, "\partial_*"'] & H_{q+1}(X^{q+1}, X^q) \arrow[r, equal] \arrow[d, "\partial_*"'] & C_{q+1}(X) \arrow[d, "\partial"'] \\ H_q(S^q) \arrow[r, "(\chi^e|S^q)_*"] & H_q(X^q) \arrow[r, "j_*"] & C_q(X). \end{tikzcd}$$

Singular versus cellular homology

10.12 Proposition. [7, 9.6.9] [7, 9.6.11] *The homomorphism $j_* : H_q(X^q) \to H_q(X^q, X^{q-1})$ is injective and maps onto the q-th cellular cycles. The map $i_* : H_q(X^q) \to H_q(X)$ is onto and its kernel is mapped by j_* onto the q-th cellular boundaries.*

Thus one obtains isomorphisms

$$j_* : H_q(C(X)) \xrightarrow{\cong} H_q(X),$$

which are natural for cellular mappings.

Proof. From the exact sequence $0 \overset{\boxed{10.9}}{=\!=} H_q(X^{q-1}) \to H_q(X^q) \xrightarrow{j_*} H_q(X^q, X^{q-1}) =: C_q(X)$ we deduce that j_* is injective and hence $\mathrm{Ker}(\partial) = \mathrm{Ker}(j_*\partial_*) = \mathrm{Ker}(\partial_*) = \mathrm{Im}(j_*)$.

From the exact homology sequence of the pair (X, X^{q+1})

$$\begin{tikzcd} H_{q+1}(X^{q+1}) \arrow[r, two heads, "\boxed{10.8}"'] & H_{q+1}(X) \arrow[r, "0"] & H_{q+1}(X, X^{q+1}) \arrow[r, "0"] \arrow[d, equal] & H_q(X^{q+1}) \arrow[r, "\cong", "\boxed{10.8}"'] & H_q(X) \\ & & 0 & & \end{tikzcd}$$

we get $H_{q+1}(X, X^{q+1}) = 0$. By the exact homology sequence $\boxed{8.19}$ for the triple $X^q \subseteq X^{q+1} \subseteq X$

$$\begin{tikzcd} H_{q+1}(X^{q+1}, X^q) \arrow[r, two heads] & H_{q+1}(X, X^q) \arrow[r] & H_{q+1}(X, X^{q+1}) \arrow[d, equal] \\ & & 0 \end{tikzcd}$$

we get that $H_{q+1}(X^{q+1}, X^q) \to H_{q+1}(X, X^q)$ is onto. The q-th cellular boundary is the image of the top row in

$$\begin{tikzcd} H_{q+1}(X^{q+1}, X^q) \arrow[r, "\partial_*"] \arrow[d] & H_q(X^q) \arrow[r, "j_*"] \arrow[d, equal] & H_q(X^q, X^{q-1}) \\ H_{q+1}(X, X^q) \arrow[r, "\partial_*"] & H_q(X^q) \arrow[r, two heads, "i_*", "\boxed{10.8}"'] & H_q(X) \end{tikzcd}$$

Since the rectangle commutes by naturality of ∂_* and since $\mathrm{Im}\,\partial_* = \mathrm{Ker}\,i_*$ we get

$$\mathrm{Im}(\partial) = \mathrm{Im}(j_*\partial_*) = j_*(\mathrm{Im}\,\partial_*) = j_*(\mathrm{Ker}\,i_*).$$

Hence the q-th cellular boundaries are the image of $\mathrm{Ker}\,i_*$ under j_*. Now we get the desired natural isomorphism

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker}\,i_* \arrow[r, hook] \arrow[ddd, "\cong", "j_*"'] & H_q(X^q) \arrow[rr, two heads, "i_*"] \arrow[ddd, "\cong", "j_*"'] \arrow[dr, "j_*"] & & H_q(X) \arrow[r] \arrow[ddd, dotted, "\cong", "j_*"'] & 0 \\ & & & H_q(X^q, X^{q-1}) \arrow[d, equal] & & \\ & & & C_q(X) & & \\ 0 \arrow[r] & \mathrm{Im}\,\partial_{q+1} \arrow[r, hook] & \mathrm{Ker}\,\partial_q \arrow[rr, two heads] \arrow[ur, hook] & & H_q(C(X)) \arrow[r] & 0 \end{tikzcd}$$ □

10.13 Proposition. [7, 9.6.10] *For $q \geq 1$ we have that in the short exact sequence*

$$0 \to \mathrm{Ker}(i_*) \to H_q(X^q) \xrightarrow{i_*} H_q(X) \to 0$$

$H_q(X^q)$ is free abelian and $\mathrm{Ker}(i_*)$ is generated by $H_q(\chi^e)[S^q]$, where $\chi^e : S^q \to X^q$ is a chosen gluing map for any $q+1$-cell e in X.

Proof.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} i_* \arrow[r, hook] \arrow[d, equal] & H_q(X^q) \arrow[r, two heads] \arrow[dd, equal] & H_q(X) \arrow[r] & 0 \\
& \operatorname{Bild} \partial_* \arrow[dr, hook] & & & \\
\cdots \arrow[r] & C_{q+1}(X) \arrow[r, "\partial_*"'] \arrow[u, "\partial_*"] & H_q(X^q) \arrow[r] & H_q(X^{q+1}) \arrow[r] \arrow[uu, "\cong", "\boxed{10.8}"'] & \cdots
\end{tikzcd}$$

By 10.12 we have that $H_q(X^q) \cong \operatorname{Ker} \partial_q \subseteq C_q(X)$ and hence is free abelian. Furthermore $H_q(X^{q+1}) \cong H_q(X)$ by 10.8, and hence the kernel of $i_* : H_q(X^q) \to H_q(X)$ equals the kernel of $H_q(X^q) \to H_q(X^{q+1}) \cong H_q(X)$, and equals by the homology sequence of the pair (X, X^q) the image of $\partial_* : C_{q+1}(X) := H_{q+1}(X^{q+1}, X^q) \to H_q(X^q)$. By 10.10 we have that $C_{q+1}(X)$ is the free abelian group generated by $\chi^e_*[D^{q+1}]$, where $\chi^e : (D^{q+1}, S^q) \to (X^{q+1}, X^q)$ are chosen characteristic maps for all $q+1$-cells e in X. By 10.11 (see 8.15) we have that $\partial_*(\chi^e_*[D^{q+1}]) = [\partial \chi^e[D^{q+1}]] = \chi^e_*[S^q]$. $\square$

10.14 Proposition. [7, 9.9.10] *For the projective spaces we have*

$$H_q(P^n(\mathbb{C})) \cong \begin{cases} \mathbb{Z} & \text{for } q = 0, 2, \dots, 2n \\ 0 & \text{otherwise} \end{cases}$$

and

$$H_q(P^n(\mathbb{H})) \cong \begin{cases} \mathbb{Z} & \text{for } q = 0, 4, \dots, 4n \\ 0 & \text{otherwise} \end{cases}$$

Proof. Since there are no-cells in all but the dimensions divisible by 2 (or 4) the boundary operator of the cellular homology is 0 (since either domain or codomain is zero) and hence the homology coincides with the cellular chain complex. $\square$

Simplicial versus singular homology

We are now going to show that the singular homology of a singular complex K is naturally isomorphic to the homology of the associated CW-space $|K|$. The idea behind this isomorphism is very easy: To a given simplex $\sigma = \langle x_0, \dots, x_q \rangle \in K$ one associates the affine singular simplex $\bar\sigma : \Delta_q \to |K|$, which maps $e_j \to x_j$ for all $0 \le j \le q$. We will show that this induces a map $H_q(K) \to H_q(|K|)$, $[\sigma] \mapsto [\sigma]$. In order that it is well defined we have to show that an even permutation of the vertices does not change the homology class of $\bar\sigma$. We do this in the following

10.15 Lemma. [7, 9.7.1] *Let τ be a permutation of $\{0, \dots, q\}$. Then τ induces a affine mapping $\tau : (\Delta_q, \dot\Delta_q) \to (\Delta_q, \dot\Delta_q)$, with $H_q(\tau)[\Delta_q] = \operatorname{sign}(\tau)[\Delta_q] \in H_q(\Delta_q, \dot\Delta_q)$.*

Proof. Since any permutation is a product of transpositions, we may assume that τ is a transposition, say $(0,1)$. Let an affine $\sigma : \Delta_{q+1} \to \Delta_q$ be defined by $e_0; e_1, e_2 \dots \mapsto e_1; e_0, e_1, \dots$. The boundary of this singular $q+1$-simplex in Δ_q is $\partial\sigma = \sigma \circ \delta^0 + \sum_{i \notin \{0,2\}} (-1)^i \sigma \circ \delta^i + \sigma \circ \delta^2 = \mathrm{id}_{\Delta_q} + c + \tau$ for $c := \sum_{i \notin \{0,2\}} (-1)^i \sigma \circ \delta^i \in S_q(\dot\Delta_q)$. Hence $H(\tau)[\Delta_q] = -[\tau] = -[\Delta_q]$. $\square$

Although this lemma shows that the mapping $H_q(K) \to H_q(|K|)$ is well-defined, it is not so obvious to show that it will be an isomorphism, since there are a lot more singular simplices in $|K|$ then just the simplices of K. So we will make a little detour via the cellular homology.

10.16 Definition. [7, 9.7.2] Let $\sigma = \langle x_0, \dots, x_q \rangle$ be an oriented q-simplex in a simplicial complex K. This induces an affine mapping $\bar\sigma : (\Delta_q, \dot\Delta_q) \to (|K|^q, |K|^{q-1})$, which can be considered as characteristic mapping for $\sigma \subseteq |K|$. Note however that $\bar\sigma$ depends on the chosen ordering of the vertices. Hence we get a mapping

$$\Phi : C_q(K) \to C_q(|K|) = H_q(|K|^q, |K|^{q-1}), \quad \Phi(\sigma) := \bar\sigma_*[\Delta_q] = [\bar\sigma].$$

This is well-defined (i.e. depends no longer on the ordering but only on the orientation) by 10.15. Note that we used an identification 7.2 of $C_q(K)$ with the free abelian group generated by the simplices with some fixed orientation.

10.17 Theorem. [7, 9.7.3] *The mapping Φ defines a natural isomorphism $C(\_) \to C(|\_|)$.*

Proof. That $\Phi_K : C(K) \to C(|K|)$ is an isomorphism is clear, since the free generators σ 7.2 are mapped to the free generators $[\bar\sigma]$ 10.10.

It is natural for simplicial mappings $\psi : K \to L$. In fact take a simplex $\sigma = \langle x_0, \dots, x_q \rangle \in K$. If ψ is injective on the vertices x_j of σ, then

$$\Phi\psi\sigma = \Phi\langle\psi(x_0), \dots, \psi(x_q)\rangle = [\overline{\langle\psi(x_0), \dots, \psi(x_q)\rangle}] = [|\psi| \circ \bar\sigma] = |\psi|_*\Phi\sigma.$$

In the other case $\psi\sigma = 0$, hence $\Phi\psi\sigma = 0$ and $|\psi|_*\Phi\sigma = |\psi|_*[\bar\sigma] = [|\psi| \circ \bar\sigma]$, but $|\psi| \circ \bar\sigma$ has values in $|L|^{q-1}$, hence $[|\psi| \circ \bar\sigma] = 0 \in H_q(|L|^q, |L|^{q-1})$.

Let us show that it is a chain mapping. For $\sigma = \langle x_0, \dots, x_q \rangle$ we have

$$\partial\Phi\sigma = j_*\partial_*[\bar\sigma] = j_*[\partial\bar\sigma] = [\partial\bar\sigma] = \Big[\sum_j (-1)^j \bar\sigma \circ \delta^j\Big] \quad \text{and}$$
$$\Phi\partial\sigma = \Phi\Big(\sum_j (-1)^j \langle x_0, \dots, \widehat{x_j}, \dots, x_q\rangle\Big) = \Big[\sum_j (-1)^j \bar\sigma \circ \delta^j\Big]$$

So $\partial\Phi = \Phi\partial$. □

10.18 Corollary. [7, 9.7.4] *Let K be a simplicial complex. Then we have natural isomorphisms $H_q(K) \xrightarrow{\Phi_*} H_q(C(|K|)) \to H_q(|K|)$, from the simplicial to the cellular and further on to the singular homology.*

Proof. This follows by composing the isomorphisms in 10.17 and 10.12. □

Let us now come back to the description of the isomorphism $H(K) \cong H(|K|)$ indicated in the introduction to this section.

10.19 Proposition. [7, 9.7.7] *The isomorphism $H(K) \cong H(|K|)$ between simplicial and singular homology can be described as follows: Choose a linear ordering of the vertices of K, and then map a simplex $\sigma = \langle x_0, \dots, x_q \rangle$ with $x_0 < \dots < x_q$ to $\bar\sigma$, which is just σ considered as map $\Delta_q \to |K|$.*

Proof. We consider the following commutative diagram

$$\begin{tikzcd}
 & H_q(|K|^q) \arrow[r, two heads, "i_*"] \arrow[dl, "j_*"'] \arrow[dd, "j_*"] & H_q(|K|) \arrow[dd, "\cong"] \\
H_q(|K|^q, |K|^{q-1}) \arrow[d, equal] & & \\
C_q(|K|) & \operatorname{Ker} \partial_q \arrow[ul, hook] \arrow[r, two heads] & H_q(C(|K|)) \\
C_q(K) \arrow[u, "\Phi", "\cong"'] & Z_q(K) \arrow[l, hook'] \arrow[u, "\Phi", "\cong"'] \arrow[r, two heads] & H_q(K) \arrow[u, "\Phi_*", "\cong"']
\end{tikzcd}$$

and take $\alpha \in H_q(K)$. It can be represented by a simplicial cycle $z := \sum_\sigma n_\sigma \sigma \in Z_q(K) \subseteq C_q(K)$. On the other hand we can consider the singular q-chain $\bar{z} := \sum_\sigma n_\sigma \bar{\sigma} \in S_q(|K|)$. It is a cycle, since $\partial \bar{z} = \sum_\sigma n_\sigma \partial \bar{\sigma} \stackrel{!}{=} \sum_\sigma n_\sigma \overline{\partial \sigma} = \overline{\partial(\sum_\sigma n_\sigma \sigma)} = \overline{\partial z} = \bar{0} = 0$. Since the image of $\bar{\sigma}$ is the closure of the simplex σ it is contained in the q-skeleton $|K|^q$, and hence we may consider $\beta := [\bar{z}] \in H_q(|K|^q)$. Note that $j_*(\beta) = j_*[\sum_\sigma n_\sigma \bar{\sigma}] = \sum_\sigma n_\sigma \bar{\sigma}_*[\Delta_q] = \sum_\sigma n_\sigma \Phi(\sigma) = \Phi(z)$. Thus the composition of isomorphisms $H_q(K) \xrightarrow{\Phi_*} H_q(C(|K|)) \to H_q(|K|)$ maps $\alpha = [z] \mapsto [\Phi(z)] \mapsto i_* j_*^{-1}(\Phi(z)) = i_*[\beta] = [\bar{z}]$. $\square$

Fundamental group versus first homology group

10.20 Proposition. [7, 9.8.1] *We have a natural homomorphism $h_1 : \pi_1(X, x_0) \to H_1(X)$ given by $[\varphi] \mapsto \varphi_*[S^1] = [\varphi]$. For the last equality we consider $\varphi : (S^1, 1) \to (X, x_0)$ either as singular chain $\dot{\Delta}_2 \cong S^1 \to X$ or as singular simplex $\Delta_1 \to S^1 \to X$.*
If X is path-connected then this homomorphism is surjective and its kernel is just the commutator subgroup. Thus $H_1(X)$ is just the abelization of $\pi_1(X, x_0)$.

Proof. That h is natural is clear. Let us show that it is a homomorphism. So let two closed curves $\varphi, \psi : (S^1, 1) \to (X, x_0)$ be given.

We need a formula for the concatenation of paths considered as mappings $(S^1, 1) \to (X, x_0)$: The corresponding paths $I \to S^1$ are obtained by composing with $t \mapsto e^{2\pi i t}$, hence $\varphi \cdot \psi$ is given by $(\varphi, \psi) \circ \nu : (S^1, 1) \to (S^1, 1) \vee (S^1, 1) \to (X, x_0)$, where $\nu : S^1 \to S^1 \vee S^1$ is given by $t \mapsto (e^{2\pi i 2t}, 1) \in S^1 \vee S^1 \subseteq S^1 \times S^1$ for $2t \leq 1$ and $t \mapsto (1, e^{2\pi i(2t-1)}) \in S^1 \vee S^1$ for $2t \geq 1$.

We also need a formula for $\nu_* : H_1(S^1) \to H_1(S^1 \vee S^1)$: Consider the relative homeomorphism $\sigma : (\Delta_1, \dot{\Delta}_1) \to (S^1, 1)$ given by $(1-t)e_0 + te_1 \mapsto e^{2\pi i t}$. It induces an isomorphism $H_1(\Delta_1, \dot{\Delta}_1) \to H_1(S^1, 1) \cong H_1(S^1)$, which maps the generator $[\Delta_1]$ to $[S^1]$. Now take the barycentric refinement $B\sigma$ of σ. We have $\nu_*[S^1] = \nu_*[\sigma] = \nu_*[B\sigma] = [\mathrm{inj}_1 \circ \sigma] + [\mathrm{inj}_2 \circ \sigma] = [S^1] \oplus [S^1] \in H_1(S^1 \vee S^1) = H_1(S^1) \oplus H_1(S^1)$.

Thus we have

$$\begin{aligned}
h_1([\varphi][\psi]) &\overset{\boxed{5.14}}{=} h_1([(\varphi,\psi)\circ\nu]) = ((\varphi,\psi)\circ\nu)_*[S^1] = (\varphi,\psi)_*\nu_*[S^1]\\
&\overset{\boxed{10.6}}{=} (\varphi,\psi)_*\Big([S^1]\oplus[S^1]\Big) = \varphi_*[S^1] + \psi_*[S^1]\\
&= h_1[\varphi] + h_1[\psi].
\end{aligned}$$

Although the theorem is valid for arbitrary path-connected topological spaces, see [6, IV.3.8], we give the proof only for connected CW-complexes X. Since π_1 and H_1 do not depend on cells of dimension greater then 2 by 5.42 and 10.9, we may assume $\dim X \leq 2$. The theorem is invariant under homotopy equivalences, hence we may assume by 5.47 and 2.49 that X has exactly one 0-cell and that this cell is x_0. So X^1 is a one point union of 1-cells and X is obtained by gluing 2-cells e via maps $f^e : S^1 \to X^1$. By 2.34 and 2.47 we may assume that $f^e(1) = x_0$. Now consider the commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \pi_1(X^1,x_0)' \arrow[r] \arrow[d, hook] & \pi_1(X,x_0)' \arrow[d, hook] & \\
0 \arrow[r] & N \arrow[r, hook] \arrow[d, "h_1|_N"'] & \pi_1(X^1,x_0) \arrow[r, two heads, "i_*"] \arrow[d, "h_1"'] & \pi_1(X,x_0) \arrow[r] \arrow[d, "h_1"'] & 0\\
0 \arrow[r] & U \arrow[r, hook] \arrow[d] & H_1(X^1) \arrow[r, two heads, "i_*"] \arrow[d] & H_1(X) \arrow[r] & 0\\
 & 0 & 0 & &
\end{tikzcd}$$

By 5.49 the top i_* is onto and its kernel N is the normal subgroup generated by the $[f^e]$. By 10.13 the bottom i_* is onto and its kernel U is the subgroup generated by $(f^e)_*[S^1]$. By the remark after 5.24 we know that the abelization of a free group is the free abelian group and by 5.38 and 8.44 the two spaces in the middle are free resp. free abelian, with the corresponding generators. So we have that the result is true for X^1. Furthermore $h_1(N) = U$, since the generators of N are mapped to those of U. By diagram chasing the general result follows: Let $G := \pi_1(X^1, x_0)$. The map $h_1 : \pi_1(X, x_0) \to H_1(X)$ is obviously surjective and its kernel is given by all gN for which $h_1(gN) = h_1(g)U = 0$, i.e. $h_1(g) \in U$. By surjectivity of $h_1 : N \to U$ we have an $n \in N$ with $h_1(n) = h_1(g)$, i.e. $gn^{-1} \in G'$. So $gN \in (G/N)'$. The converse inclusion is clear, since $H_1(X)$ is abelian. □

10.21 Corollary. [7, 9.8.2] *For the closed orientable surface X of genus g we have $H_1(X) \cong \mathbb{Z}^{2g}$ for the non-orientable one we have $H_1(X) \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$ for the projective spaces we have $H_1(P^n) \cong \mathbb{Z}_2$ for $2 \leq n \leq \infty$.*

Proof. Use the formulas given in 5.53, 5.54 and 5.43. □

10.23 Proposition. [7, 9.9.2] *Let $f : S^1 \to S^1$ be continuous of degree k. Then $f_* : H_1(S^1) \to H_1(S^1)$ is given by $[S^1] \mapsto k \cdot [S^1]$.*

Proof. We know that f acts by multiplication in homotopy and using the naturality of h_1 gives the same result for homology.

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\cong"] \arrow[d, "k\cdot"'] & \pi_1(S^1,1) \arrow[r, "\cong", "h_1"'] \arrow[d, "\pi_1(f)"'] & H_1(S^1) \arrow[d, "H_1(f)"'] \\
\mathbb{Z} \arrow[r, "\cong"] & \pi_1(S^1,1) \arrow[r, "\cong", "h_1"'] & H_1(S^1)
\end{tikzcd}$$

For a direct proof see [7, 9.5.5] and 2.17. $\square$

10.24 Proposition. [7, 9.9.9] *We have for the homology of the closed orientable surface of genus g:*

$$H_q(X) \cong \begin{cases} \mathbb{Z} & \text{for } q = 0, 2 \\ \mathbb{Z}^{2g} & \text{for } q = 1 \\ 0 & \text{otherwise} \end{cases}$$

and for the non-orientable ones:

$$H_q(X) \cong \begin{cases} \mathbb{Z} & \text{for } q = 0 \\ \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2 & \text{for } q = 1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. We calculate the cellular homology. Recall that in both cases X can be described as the CW-complex obtained by gluing one 2-cell e to a join of circles S^1 along a map $f : S^1 \to \bigvee^k S^1$ of the form $i_{j_1}^{n_1} \cdot \dots \cdot i_{j_m}^{n_m}$. Thus the non-vanishing cellular chain groups are $C_0(X) \cong \mathbb{Z}$, $C_1(X) \cong \mathbb{R}^k$ and $C_2(X) \cong \mathbb{Z}$ with generators given by the base-point 1, the 1-cells e_j^1 and the 2-cell e^2. As in the proof of 10.20 and using 10.23 one shows that $f_*[S^1] = n_1 \cdot [\bar{e}_{j_1}^1] + \dots + n_m \cdot [\bar{e}_{j_m}^1]$. Hence $\partial_2(e^2) = j_*(\chi^e|_{S^1})[S^1] = n_1 e_{j_1}^1 + \dots + n_m e_{j_m}^1$, whereas $\partial_1 = 0$.
In case of an oriented closed surface X of genus g we thus have $\partial_2 e^2 = e_1^1 + e_2^1 - e_1^1 - e_2^1 + \dots = 0$, hence $H_q(X) = H_q(C(X)) = C_q(X)$ is as claimed.
In case of a non-orientable surfaces X of genus g we have $\partial_2 e^2 = \partial(\chi_*^{e^2}[D^2]) \overset{10.11}{=\!=\!=} j_* f_*[S^1] = 2e_1^1 + \dots + 2e_g^1$, which shows that $H_2(X) = \operatorname{Ker} \partial_2 = \{0\}$ and $H_1(X) = \operatorname{Ker} \partial_1 / \operatorname{Ker} \partial_2 = \mathbb{Z}^g / 2\mathbb{Z}(e_1^1 + \dots + e_g^1) = {}^{ab}\langle e_1^1, \dots, e_g^1 : 2(e_1^1 + \dots + e_g^1) = 0\rangle = {}^{ab}\langle e_1^1, \dots, e_{g-1}^1, x := e_1^1 + \dots + e_g^1 : 2x = 0\rangle = \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$. $\square$

10.25 Proposition. [7, 9.9.14] *For the projective spaces we have*

$$H_q(P^n(\mathbb{R})) \cong \begin{cases} \mathbb{Z} & \text{for } q = 0 \text{ or } q = n \equiv 1 \mod 2 \\ \mathbb{Z}_2 & \text{for } 0 < q < n \text{ with } q \equiv 1 \mod 2 \\ 0 & \text{otherwise} \end{cases}$$

Proof. The idea is to consider a CW-decomposition of S^n compatible with the equivalence relation $x \sim -x$, which gives $P^n = S^n/\sim$. For this we consider the spheres $S^0 \subset S^1 \subset \dots \subset S^n$ and the cells $\{x \in S^q : \pm x_{q+1} > 0\}$ with characteristic map $f_\pm^q : x \mapsto (x, \pm\sqrt{1 - |x|^2})$. They form a cell decomposition of S^n and hence $e_\pm^q := (f_\pm^q)_*[D^q]$ is a basis in $C_q(S^n)$. We have the reflection $r : D^q \to D^q$, $x \mapsto -x$ and may consider it as mapping $r : (S^q, S^{q-1}) \to (S^q, S^{q-1})$ to obtain an homomorphism $r_* : C_q(S^n) \to C_q(S^n)$.

We claim $r_* e_+^q = (-1)^q e_-^q$: Note that $r_*[D^q] = (-1)^q [D^q]$ which is obvious for $q = 1$ and follows by induction for $q \geq 2$. Since $r \circ f_+^q = f_-^q \circ r$ we thus get $r_* e_+^q = r_*(f_+^q)_*[D^q] = (f_-^q)_* r_*[D^q] = (-1)^q e_-^q$.

Next we claim that $\partial e_+^{q+1} = \pm(e_+^q - e_-^q) = \partial e_-^{q+1}$: Since $f_\pm^{q+1}|S^q = \mathrm{id}$ we get $\partial e_\pm^{q+1} = j_*[S^q]$ using 10.11. Now consider

$$\begin{array}{ccccccccc}
H_q(S^{q-1}) & \longrightarrow & H_q(S^q) & \xrightarrow{j_*} & H_q(S^q, S^{q-1}) & \xrightarrow{\partial_*} & H_{q-1}(S^{q-1}) & \longrightarrow & H_{q-1}(S^q) \\
\| & & \| & & \| & & \| & & \| \\
0 & & \mathbb{Z} & & {}^{ab}\langle\{e_\pm^q\}\rangle & & \mathbb{Z} & & 0
\end{array}$$

So $\partial_* \neq 0$ since it is onto and in particular applied to the generators $e_\pm^q$ we have $\partial_* e_+^q = \partial_* e_-^q \neq 0$. So $\operatorname{Ker} \partial_* \cong \mathbb{Z}$ is generated by $e_+^q - e_-^q$, but it coincides with the image of j_* and hence is generated by $j_*[S^q]$. Thus $j_*[S^q] = \pm(e_+^q - e_-^q)$.

Now P^n is a CW-complex with cells $e^n = p(e_\pm^q)$ and with characteristic mappings $p \circ f_+^q : D^q \to P^q$. Hence the generators of $C_q(P^n)$ are given by $(p \circ f_+^q)_*[D^q]$ again denoted e^q. Since $p \circ r = p$ we have by the first claim that $p_*(e_-^q) = (-1)^q p_*(r_* e_+^q) = (-1)^q p_*(e_+^q) = (-1)^q e^q$. For $0 < q \leq n$ we get by the second claim that

$$\begin{aligned}
\partial e^q = \partial p_*(e_+^q) = p_* \partial(e_+^q) &= \pm p_*(e_+^{q-1} - e_-^{q-1}) \\
= \pm(1 - (-1)^{q-1}) e^{q-1} &= \begin{cases} 0 & \text{for odd } q \\ \pm 2e^{q-1} & \text{for even } q \end{cases}.
\end{aligned}$$

For even q with $0 < q \leq n$ we have no non-trivial cycle in $C_q(P^n)$, since $\partial e^q = \pm 2e^{q-1}$. For odd $0 < q \leq n$ we have that e^q is a cycle and $2e^q = \pm \partial e^{q+1}$ is a boundary for $q < n$. So the claimed homology follows. □

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