Algebraic Topology;
MATHS 750 lecture notes

1 Some algebraic preliminaries

Definition 1.1 *A* group *is a set G together with a binary operation (thought of as multiplication, so we write ab for the result of applying this operation to (a, b)) such that the following conditions are satisfied:*

- $\forall a, b, c \in G, (ab)c = a(bc)$;
- $\exists 1 \in G$ *such that* $\forall a \in G, a1 = 1a = a$ *(1 is called the* identity*);*
- $\forall a \in G, \exists a' \in G$ *such that* $aa' = a'a = 1$ *(a' is called an* inverse *of a).*

If in addition

- $\forall a, b \in G, ab = ba$

then the groups is called abelian*.*

Of course every group contains at least one element. One can check that the inverse is unique.

Often the binary operation is thought of as addition, in which case it is more common to write $a + b$, and the identity is called 0 and the inverse $-a$. This is especially the case when the group is abelian.

Definition 1.2 *Given two groups G and H, a* homomorphism *from G to H is a function $\theta : G \to H$ such that $\forall a, b \in G$ we have $\theta(ab) = \theta(a)\theta(b)$. Given a homomorphism $\theta : G \to H$, the sets $Ker(\theta)$ and $Im(\theta)$ are defined by:*

$$Ker(\theta) = \{a \in G : \theta(a) = 1\}$$
$$Im(\theta) = \{\theta(a) \in H : a \in G\}.$$

Note that if θ is a homomorphism then $\theta(a^{-1}) = \theta(a)^{-1}$.

Example 1.3 *The trivial group is $\{1\}$ with the only possible operation. The trivial group is abelian.*

Example 1.4 *The next simplest group is $\mathbb{Z}_2 = \{0, 1\}$.*

This group is also abelian. Addition is defined by 1+1=0, all other sums being already specified by the identity axiom. Note then that $-1 = 1$.

We can generalise the example above to get a group with n elements for any positive integer n.

Example 1.5 *The abelian group $\mathbb{Z}_n$ consists of $\{0, 1, \ldots, n-1\}$, with addition defined using ordinary addition modulo n.*

Example 1.6 *The set of integers, $\mathbb{Z}$, is an abelian group under addition, as are the sets of rational numbers and real numbers.*

Example 1.7 *The group of permutations of $\{1, 2, \ldots, n\}$ is the set of all bijections of the set $\{1, 2, \ldots, n\}$.*

This group has $n!$ elements. When $n > 2$ this group is not abelian; for example if a cycles 1 to 2, 2 to 3 and 3 to 1 and b interchanges 1 and 2 and leaves 3 fixed then $ab \neq ba$ as $ab(1) = a(2) = 3$ whereas $ba(1) = b(2) = 1$.

Example 1.8 *Other easily visualised groups are the groups of symmetries of geometric figures in which the elements of the group are rigid motions which take the figure onto itself. For example rotations of an equilateral triangle through 120° and reflections of the triangle about an angle bisector (this group is the permutation group when $n = 3$).*

There are many infinite groups, both abelian and non-abelian.

Example 1.9 *If $\theta : G \to H$ is defined by $\theta(a) = 1$ for each a then θ is a homomorphism.*

There is no other homomorphism from $\mathbb{Z}_n$ to $\mathbb{Z}$.

Example 1.10 *The function $\theta : \mathbb{Z} \to \mathbb{Z}_n$ defined by $\theta(m) =$ the remainder obtained when m is divided by n, is a homomorphism.*

Example 1.11 *Let G be the positive reals with ordinary multiplication as the operation and H the reals with ordinary addition as the operation. Then the logarithm functions $log : G \to H$ are homomorphisms.*

Theorem 1.12 *A homomorphism $\theta : \mathbb{Z} \to G$, for any group G, is determined by $\theta(1)$. Indeed, if n is any positive integer then $\theta(n) = (\theta(1))^n$ or $n\theta(1)$ depending on whether the operation is multiplication or addition.*

This theorem is a bit like the theorem that allows us to specify a linear transformation of vector spaces by merely specifying what happens to elements of a basis.

Definition 1.13 *A homomorphism $\theta : G \to H$ is called a* monomorphism *(or a one-to-one homomorphism) provided that $\theta(a) = \theta(b) \implies a = b$ and an* epimorphism *(or an onto homomorphism) provided that $\forall b \in H$, $\exists a \in G$ such that $\theta(a) = b$. A homomorphism which is both a monomorphism and an epimorphism is called an* isomorphism. *If there is an isomorphism $\theta : G \to H$ then the two groups G and H are called* isomorphic, *denoted $G \approx H$.*

Theorem 1.14 *A homomorphism $\theta : G \to H$ is a monomorphism if and only if $Ker(\theta)$ contains only the identity.*

Definition 1.15 *Given two groups G and H, their* direct sum *is the group $G \oplus H$ defined as follows:*

$$G \oplus H = \{(a, b) : a \in G, b \in H\}; \quad (a, b)(c, d) = (ac, bd).$$

Although they both have 4 elements and are abelian, the two groups $\mathbb{Z}_4$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ are not isomorphic. In fact $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is isomorphic to the group $G = \{0, a, b, c\}$ with $a + a = b + b = c + c = 0$ and $a + b = c, b + c = a, c + a = b$, so has no element of *order* 4 (ie no element d with $d+d+d+d = 0$ but d, $d+d$, $d+d+d \neq 0$) whereas $\mathbb{Z}_4$ has two such elements, and it may be checked that any isomorphism between groups takes an element of order n to an element of the same order. In the group G in the previous sentence the elements a, b and c all have order 2.

Theorem 1.16 *If the composition of two homomorphisms $\theta : G \to H$ and $\varphi : H \to K$ is an isomorphism then θ is a monomorphism and φ is an epimorphism.*

Definition 1.17 *If we have a sequence of groups and homomorphisms linking them:*

$$\cdots \to G_{n+1} \xrightarrow{\theta_{n+1}} G_n \xrightarrow{\theta_n} G_{n-1} \to \cdots$$

then we say that the sequence is exact *at G_n provided that $Im(\theta_{n+1}) = Ker(\theta_n)$. We say that the sequence is* exact *provided that it is exact at each group.*

The following two theorems, and particularly the corollary, will be used over and over in the homology lectures.

Theorem 1.18 *If we have a sequence of groups and homomorphisms linking them which contains the following part which is exact at G:*

$$1 \to G \xrightarrow{\theta} H$$

then θ is a monomorphism.

Theorem 1.19 *If we have a sequence of groups and homomorphisms linking them which contains the following part which is exact at H:*

$$G \xrightarrow{\theta} H \to 1$$

then θ is an epimorphism.

Corollary 1.20 *If we have a sequence of groups and homomorphisms linking them which contains the following part which is exact at G and H:*

$$1 \to G \xrightarrow{\theta} H \to 1$$

then θ is an isomorphism.

Less can be said when there are 3 intermediate groups. We might hope that if the sequence

$$1 \to G \xrightarrow{\theta} H \xrightarrow{\varphi} K \to 1$$

is exact then $H \approx G \oplus K$, but this need not be so.

Example 1.21 *Define the exact sequence*

$$0 \to \mathbb{Z}_2 \xrightarrow{\theta} \mathbb{Z}_4 \xrightarrow{\varphi} \mathbb{Z}_2 \to 0$$

by $\theta(1) = 2$ *and* $\varphi(1) = 1$. *As noted above,* $\mathbb{Z}_4 \not\approx \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Frequently we will be considering diagrams of groups and homomorphisms, for example maybe something like:

$$\begin{tikzcd} G_1 \arrow[r, "\varphi"] \arrow[d, "\theta_1"] & G_2 \arrow[d, "\theta_2"] \\ H_1 \arrow[r, "\psi"] & H_2 \end{tikzcd} .$$

Definition 1.22 *The diagram is said to* commute *if* $\psi\theta_1 = \theta_2\varphi$. *A more general diagram* commutes *provided that for any pair of groups in the diagram, if there are two or more paths of homomorphisms leading from the first group to the second then the compositions of the homomorphisms along such paths are the same.*

2 The Fundamental Group

Definition 2.1 *Let* X *be a topological space and* $a \in X$. *A* loop *in* X based at a *is a continuous function* $\sigma : [0,1] \to X$ *such that* $\sigma(0) = \sigma(1) = a$. *The* product *of two loops* σ *and* τ *in* X *based at* a *is the loop* $\sigma * \tau$ *defined by* $\sigma * \tau(s) = \sigma(2s)$ *if* $s \leq 1/2$ *and* $\sigma * \tau(s) = \tau(2s-1)$ *if* $s \geq 1/2$. *The* reverse *of a loop* σ *is the loop* $\bar{\sigma}$ *defined by* $\bar{\sigma}(s) = \sigma(1-s)$. *Declare two loops* σ *and* τ *to be* homotopic, *denoted* $\sigma \sim \tau$ *if there exists a homotopy* $H : [0,1] \times [0,1] \to X$ *such that* $H(s,0) = \sigma(s)$, $H(s,1) = \tau(s)$ *and* $H(0,t) = H(1,t) = a$ *for each* $s, t \in [0,1]$. *Denote by* $\ddot{a}$ *the constant loop based at* a.

Lemma 2.2

1. *Let* $\sigma_1, \sigma_2, \tau_1, \tau_2 : [0,1] \to X$ *be four loops based at* a *such that* $\sigma_1 \sim \sigma_2$ *and* $\tau_1 \sim \tau_2$. *Then* $\sigma_1 * \tau_1 \sim \sigma_2 * \tau_2$.
2. *Let* $\sigma_1, \sigma_2, \sigma_3 : [0,1] \to X$ *be three loops based at* a. *Then* $(\sigma_1 * \sigma_2) * \sigma_3 \sim \sigma_1 * (\sigma_2 * \sigma_3)$.
3. *Let* $\sigma : [0,1] \to X$ *be a loop based at* a. *Then* $\sigma * \ddot{a} \sim \ddot{a} * \sigma \sim \sigma$.
4. *Let* $\sigma : [0,1] \to X$ *be a loop based at* a. *Then* $\sigma * \bar{\sigma} \sim \ddot{a}$.

Definition 2.3 *The* fundamental group *of* X *at* a *is the set of* $\sim$*-equivalence classes of loops in* X *based at* a *together with the group operation determined by the product* $*$. *This fundamental group is denoted by* $\pi(X,a)$, *or sometimes* $\pi_1(X,a)$.

By Lemma 2.2 the group operation is well-defined and the axioms for a group really are satisfied.

Example 2.4 *Suppose* X *consists of just a single point. Then* $\pi(X,a)$ *is trivial.*

Example 2.5 $\pi(\mathbb{R}, 0)$ *is trivial.*

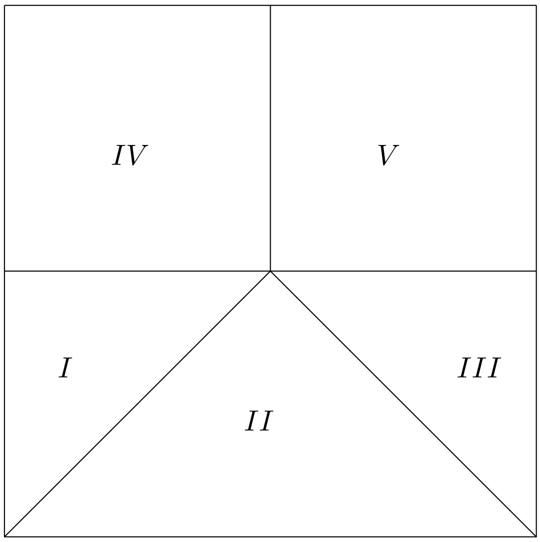
In fact these two examples can be further generalised as follows.

Definition 2.6 *A space X is* contractible *provided that there is a map $C : X \times [0,1] \to X$ and a point $b \in X$ such that $C(x,0) = x$ and $C(x,1) = b$ for each $x \in X$. The function C is called a* contraction.

Proposition 2.7 *If X is contractible and $a \in X$ then $\pi(X, a)$ is trivial.*

Proof. The obvious thing to do is to apply the contraction to any loop based at a and in this way shrink the loop down to a. The problem is that during the contraction a may get moved about so that a shrinking loop may not always be a loop based at a.

Suppose that C is a contraction as in the definition. Define a new contraction $c : X \times [0,1] \to X$ by setting $c(x,t) = C(x, 2t)$ if $t \leq 1/2$ and $c(x,t) = b$ if $t \geq 1/2$. The difference is that c contracts X more quickly then sends all of X to b during the second half of the contraction. We don't really need to do this but it makes life a bit simpler. We now break the square $[0,1] \times [0,1]$ into 5 closed pieces on each of which we define part of a function $H : [0,1] \times [0,1] \to X$ so that H exhibits a homotopy of loops based at a from a given loop to the constant loop.



Suppose that $\sigma : [0,1] \to X$ is a loop based at a. We must exhibit a homotopy of loops which is based at a and which begins at σ and ends at the constant loop $\ddot{a}$. Set

$$H(s,t) = \begin{cases} c(a, 2s) & : \quad 0 \leq s \leq t \leq \frac{1}{2} \text{ ie in region I} \\ c(\sigma(\frac{s-t}{1-2t}), 2t) & : \quad 0 \leq t \leq s \leq 1-t \text{ ie in region II} \\ c(a, 2(1-s)) & : \quad \frac{1}{2} \leq 1-t \leq s \leq 1 \text{ ie in region III} \\ c(a, 2s(2-2t)) & : \quad s \leq \frac{1}{2} \text{ and } t \geq \frac{1}{2} \text{ ie in region IV} \\ c(a, 2(1-s)(2-2t)) & : \quad s \geq \frac{1}{2} \text{ and } t \geq \frac{1}{2} \text{ ie in region V} \end{cases}$$

Note that there is a problem with the definition in region II when $s = t = \frac{1}{2}$. However the function defined in this part is continuous at this point provided that we set it equal to b because of the fact that c sends all of the region $t \geq \frac{1}{2}$ to b.

The first thing we must check is that H is well-defined, for on each of the lines which form part of the boundary of two regions we have two possibly conflicting ways of defining H (and at $(\frac{1}{2}, \frac{1}{2})$ we have five!). There are five boundary segments and we will look at them individually;

- when $s = t \leq \frac{1}{2}$ we have $c(a, 2s) = c(a, 2t) = c(\sigma(\frac{s-t}{1-2t}), 2t)$;
- when $s + t = 1$ and $t \leq \frac{1}{2}$ we have $c(a, 2(1 - s) = c(a, 2t) = c(\sigma(\frac{s-t}{1-2t}), 2t)$;
- when $t = \frac{1}{2}$ and $s \leq \frac{1}{2}$ we have $c(a, 2s) = c(a, 2s(2 - 2t))$;
- when $t = \frac{1}{2}$ and $s \geq \frac{1}{2}$ we have $c(a, 2(1 - s)) = c(a, 2(1 - s)(2 - 2t))$;
- when $s = \frac{1}{2}$ and $t \geq \frac{1}{2}$ we have $c(a, 2s(2 - 2t)) = c(a, 2(1 - s)(2 - 2t))$.

It is easy to see that H is continuous in each of the separate regions. Thus by a standard theorem from point set topology H is continuous on $[0, 1] \times [0, 1]$.

Now $H(s, 0) = c(\sigma(s), 0) = \sigma(s)$, so that $H_0 = \sigma$; $H(s, 1) = c(a, 0) = a$, so that $H_1 = \ddot{a}$; $H(0, t) = H(1, t) = c(a, 0) = a$, so that each H_t really is a loop based at a. ■

Question 1 (Reasonable Question!) *Are there any spaces which have non-trivial fundamental group?*

Proposition 2.8 *Let X be a topological space, $a_0, a_1 \in X$ and $\rho : [0, 1] \to X$ be continuous so that $\rho(i) = a_i$ for $i = 0, 1$. Then ρ induces an isomorphism $\tilde{\rho} : \pi(X, a_0) \to \pi(X, a_1)$.*

Proof. Given a loop $\sigma : [0, 1] \to X$ based at a_0 we construct a loop $\bar{\rho} * \sigma * \rho$ based at a_1 much as in the definition of composition of two loops. ■

Definition 2.9 *If $f : X \to Y$ is a map with $f(a) = b$ then there is a natural homomorphism $f_* : \pi(X, a) \to \pi(Y, b)$ defined by setting $f_*([\sigma]) = [f\sigma]$. Here by $[\sigma]$ we mean the $\sim$-equivalence class of the loop σ.*

Proposition 2.10

1. *If $f : X \to X$ is the identity then so is $f_* : \pi(X, a) \to \pi(X, a)$.*
2. *If $f : X \to Y$ and $g : Y \to Z$ are such that $f(a) = b$ and $g(b) = c$ then $(gf)_* = g_* f_* : \pi(X, a) \to \pi(Z, c)$.*
3. *If $f, g : (X, a) \to (Y, b)$ are homotopic by a homotopy preserving the base point then $f_* = g_* : \pi(X, a) \to \pi(Y, b)$.*

3 Covering Projections

Definition 3.1 . *A* covering projection *is a continuous function $p : E \to X$ such that for each $x \in X$ there is an open subset V of X containing x such that $p^{-1}(V)$ is a disjoint union of open subsets of E each of which is mapped homeomorphically by p onto V. The set V is said to be* evenly covered.

Example 3.2 *Any homeomorphism is a covering projection.*

Example 3.3 *Consider* $\mathbb{S}^1$ *as the set of complex numbers of unit modulus and define* $e : \mathbb{R} \to \mathbb{S}^1$ *by* $e(t) = e^{2\pi it}$. *Then* e *is a covering projection.*

We may take the two open subsets $V_+ = \mathbb{S}^1 - \{-1\}$ and $V_- = \mathbb{S}^1 - \{1\}$ of $\mathbb{S}^1$ so that $p^{-1}(V_\pm)$ is a disjoint union of open intervals each of which is mapped homeomorphically onto $V_\pm$ by e.

Example 3.4 *Let* $\mathbb{S}^n$ *be the unit sphere in* $\mathbb{R}^{n+1}$, *ie* $\mathbb{S}^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^n : x_0^2 + \cdots + x_n^2 = 1\}$, *and define* $\sim$ *on* $\mathbb{S}^n$ *by* $x \sim y$ *if and only if* $x = \pm y$. *The quotient space is called* projective n-space *and denoted* $\mathbb{P}^n$ *and the quotient map* $\mathbb{S}^n \to \mathbb{P}^n$ *is a covering projection.*

Definition 3.5 . *Suppose that* $p : E \to X$ *is a covering projection and* $f : Y \to X$ *is a map. Then a map* $\hat{f} : Y \to E$ *is called a* lifting of f over p *provided that* $p\hat{f} = f$.

Theorem 3.6 (Unique Lifting Theorem) *Let* $p : E \to X$ *be a covering projection and* $f : Y \to X$ *a map. Suppose that* Y *is connected and that* $e_0 \in E$, $x_0 \in X$ *and* $y_0 \in Y$ *satisfy* $p(e_0) = f(y_0) = x_0$. *Then there is at most one map* $g : Y \to E$ *which lifts* f *over* p *such that* $g(y_0) = e_0$.

Proof. Suppose that we have two liftings $g_1, g_2 : Y \to E$ of f such that $g_1(y_0) = g_2(y_0) = e_0$. Let $A = \{y \in Y \ / \ g_1(y) = g_2(y)\}$. Then A is both open and closed in Y because $p : E \to X$ is a covering projection. Now $A \neq \varnothing$ so, because Y is connected, it follows that $A = Y$. ∎

Theorem 3.7 (Covering Homotopy Theorem) *Let* $p : E \to X$ *be a covering projection and suppose that* $e_0 \in E$ *and* $x_0 \in X$ *satisfy* $p(e_0) = x_0$. *Let* Y *be any topological space and* $y_0 \in Y$ *and suppose that* $f : Y \to X$ *satisfies* $f(y_0) = x_0$ *and has a lifting* $\hat{f}$ *satisfying* $\hat{f}(y_0) = e_0$. *Let* $F : Y \times [a, b] \to X$ *be a map so that* $F(y, a) = f(y)$ *for each* $y \in Y$. *Then there is a lifting* $\hat{F} : Y \times [a, b] \to E$ *of* F *with* $\hat{F}(y, a) = \hat{f}(y)$ *for each* $y \in Y$. *Furthermore if* $F(y_0, t) = x_0$ *for each* $t \in [a, b]$ *then* $\hat{F}(y_0, t) = e_0$ *for each* t.

Proof. **Case 1** Suppose that X itself is evenly covered. For each $y \in Y$, let E_y be the sheet containing $\hat{f}(y)$: thus $p|E_y : E_y \to X$ is a homeomorphism. Define $\hat{F}(y, t) = (p|E_y)^{-1}F(y, t)$. The function $\hat{F}$ is continuous, for if $y \in Y$ then $\hat{f}^{-1}(E_y)$ is a neighbourhood of y and if $z \in \hat{f}^{-1}(E_y)$ then $E_z = E_y$ so throughout $\hat{f}^{-1}(E_y) \times [a, b]$, we have $\hat{F} = (p|E_y)^{-1}F$, a composition of continuous functions. Continuity of $\hat{F}$ now follows.

Case 2 General case. For each $y \in Y$, there is an open neighbourhood N_y of y in Y and a partition $\langle a = t_0 < t_1 < \ldots < t_n = b\rangle$ (which may depend on y) such that $F(N_y \times [t_{i-1}, t_i])$ lies in an evenly covered open subset of X. By the first case, $F|N_y \times [t_0, t_1]$ lifts to $\hat{F}$. Inductively assume that $F|N_y \times [t_0, t_{i-1}]$ lifts to $\hat{F}$. Again by the first case we can lift $F|N_y \times [t_0, t_i]$ to $\hat{F}$. Thus F lifts over $N_y \times [a, b]$. Furthermore if $y_0 \in N_y$ and $F(y_0, t) = x_0$ for each t then $\hat{F}(y_0, t) = e_0$ for each t.

Now it is claimed that if $y, y' \in Y$ then the two liftings we have just found on $N_y \times [a, b]$ and $N'_y \times [a, b]$ agree on their common domain. Indeed suppose that $z \in N_y \cap N'_y$. Then we have two liftings of $F|\{z\} \times [a, b]$ which agree at (z, a). Since $\{z\} \times [a, b]$ is connected, these two liftings agree by Theorem 3.6. Thus the function $\hat{F}$ is continuous. ∎

Corollary 3.8 (Path Lifting Theorem) *Let $p : E \to X$ be a covering projection and suppose that $e_0 \in E$ and $x_0 \in X$ satisfy $p(e_0) = x_0$. Let $\sigma : [0,1] \to X$ be a path with $\sigma(0) = x_0$. Then there is a unique path $\tau : [0,1] \to E$ with $\tau(0) = e_0$ and $p\tau = \sigma$.*

Theorem 3.9 (Map Lifting Criterion) *Let $p : E \to X$ be a covering projection, Y a connected space which is also locally path connected, and suppose that $e_0 \in E$, $x_0 \in X$ and $y_0 \in Y$ satisfy $p(e_0) = x_0$. Let $f : Y \to X$ be continuous with $f(y_0) = x_0$. Then there is a lifting $\hat{f} : Y \to E$ with $\hat{f}(y_0) = e_0$ if and only if $f_*\pi(Y, y_0) \subset p_*\pi(E, e_0)$.*

Proof. $\Rightarrow$: relatively straightforward.
$\Leftarrow$: Given $y \in Y$, let $\sigma : [0,1] \to Y$ be a path from y_0 to y. Then $f\sigma$ is a path in X from x_0 to $f(y)$. By the path lifting theorem we can lift $f\sigma$ to a path $\widehat{f\sigma} : [0,1] \to E$ from e_0 to some point $\widehat{f\sigma}(1)$: declare $\hat{f}(y) = \widehat{f\sigma}(1)$.

We must show that $f(y)$ is well-defined. Suppose that $\tau : [0,1] \to Y$ is another path from y_0 to y. Then $\sigma * \bar{\tau}$ is a loop in Y based at y_0, so by the homotopy group assumption there is a loop in E based at e_0 which is mapped by p onto a loop which is homotopic to the loop $f(\sigma * \bar{\tau}) = (f\sigma) * (\overline{f\tau})$. Thus if we lift the path $(f\sigma) * (\overline{f\tau})$ to a path in E starting at e_0 the result is a loop in E. Thus if $f\sigma$ and $f\tau$ are each lifted to paths in E starting at e_0 they must have the same terminal point, ie $\widehat{f\sigma}(1) = \widehat{f\tau}(1)$.

Finally we must show that $\hat{f}$ is continuous. Suppose that $y \in Y$, let $f(y) = e$ and let N be any open neighbourhood of e. Choose an open set $U \subset N$ containing e so that p takes U homeomorphically onto an open set $V \subset X$. Since V is open and f is continuous, it follows that $f^{-1}(V)$ is open. It is also clear that $y \in f^{-1}(V)$. Thus by local path connectedness of Y there is a path connected open neighbourhood W of y such that $W \subset f^{-1}(V)$. It is claimed that W is an open set containing y which is mapped by $\hat{f}$ into N. Suppose that $\eta \in W$: we will show that $\hat{f}(\eta) \in N$. Choose a path in Y from y_0 to y; by the path lifting theorem this lifts to a path in E from e_0 to $\hat{f}(y)$. Now use path connectedness of W to choose a path in W from y to η; this path is carried by f to a path in V and hence by $(p|U)^{-1}$ to a path in U beginning at e. Combining these two paths, from e_0 to e then from e to wherever gives a lifting of a path in Y from y_0 to η to a path in E from e_0 to what must be $\hat{f}(\eta)$. Thus $\hat{f}(\eta) \in U \subset N$ as required. ∎

4 Running Around in Circles

Throughout this section we are thinking of $\mathbb{S}^1$ as the set of complex numbers of unit modulus and we define $e : \mathbb{R} \to \mathbb{S}^1$ by $e(t) = e^{2\pi i t}$. The points $0 \in \mathbb{R}$ and $1 \in \mathbb{S}^1$ serve as base points and, of course, $e(0) = 1$. We also denote by $\mathbb{B}^2$ the set of complex numbers of modulus at most 1.

Theorem 4.1 $\pi(\mathbb{S}^1, 1) \approx \mathbb{Z}$.

Proof. We define an isomorphism $\theta : \pi(\mathbb{S}^1, 1) \to \mathbb{Z}$. Let $\sigma : [0,1] \to \mathbb{S}^1$ be a loop based at 1: thus $[\sigma]$ represents a typical element of $\pi(\mathbb{S}^1, 1)$. Use Corollary 3.8 to lift σ over e to $\hat{\sigma} : [0,1] \to \mathbb{R}$ so that $\hat{\sigma}(0) = 0$: by Theorem 3.6 this lift is unique. Because $e\hat{\sigma}(1) = 1$ it follows that $\hat{\sigma}(1) \in \mathbb{Z}$: we declare $\theta([\sigma]) = \hat{\sigma}(1)$.

We need to verify the following:

1. θ is well-defined, ie if $\sigma \sim \tau$ then $\hat{\sigma}(1) = \hat{\tau}(1)$.
2. θ is a homomorphism, ie if σ and τ are two loops in $\mathbb{S}^1$ based at 1 then $\widehat{\sigma * \tau}(1) = \hat{\sigma}(1) + \hat{\tau}(1)$.
3. θ is a monomorphism, ie if $\hat{\sigma}(1) = 0$ then $\sigma \sim \ddot{1}$.
4. θ is an epimorphism, ie if $n \in \mathbb{Z}$ then there is a loop σ based at 1 such that $\hat{\sigma}(1) = n$.

1. θ is well-defined. Suppose that σ and τ are two loops in $\mathbb{S}^1$ based at 1 such that $\sigma \sim \tau$. Then there is a homotopy $H : [0,1] \times [0,1] \to \mathbb{S}^1$ such that $H(s,0) = \sigma(s)$, $H(s,1) = \tau(s)$ and $H(0,t) = H(1,t) = 1$ for each $s,t \in [0,1]$. Now apply Theorem 3.7 with $f = \sigma$, $\hat{f} = \hat{\sigma}$ and $F = H$; write the lifting of H as $\hat{H}$. By Theorem 3.7 we conclude that $\hat{H}(s,1) = \hat{\tau}(s)$. Because $e\hat{H}(1,t) = 1$ it follows that $\hat{H}(1,t)$ is an integer which must be the same for all t because $[0,1]$ is connected. In particular $\hat{H}(1,0) = \hat{H}(1,1)$, ie $\hat{\sigma}(1) = \hat{\tau}(1)$.

2. θ is a homomorphism. Suppose that σ and τ are two loops in $\mathbb{S}^1$ based at 1. Define $\widehat{\sigma * \tau}$ by:

$$\widehat{\sigma * \tau}(s) = \begin{cases} \hat{\sigma}(2s) & \text{if } 0 \leq s \leq \frac{1}{2} \\ \hat{\tau}(2s-1) + \hat{\sigma}(1) & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases}$$

Then $\widehat{\sigma * \tau}(1) = \hat{\sigma}(1) + \hat{\tau}(1)$.

3. θ is a monomorphism. Suppose that $\hat{\sigma}(1) = 0$. Because $\mathbb{R}$ is contractible there is a homotopy $\hat{H} : [0,1] \times [0,1] \to \mathbb{R}$ such that $\hat{H}(s,0) = \hat{\sigma}(s)$ and $\hat{H}(s,1) = \hat{H}(0,t) = \hat{H}(1,t) = 0$ for each $s,t \in [0,1]$. Then $e\hat{H}$ is a homotopy from σ to $\ddot{1}$.

4. θ is an epimorphism. Suppose that $n \in \mathbb{Z}$. Define σ_n by $\sigma_n(s) = e^{2n\pi i s}$. Then $\widehat{\sigma_n}(1) = n$. ■

Corollary 4.2 *Let $i : \mathbb{S}^1 \to \mathbb{B}^2$ be the inclusion. Then there is no continuous function $r : \mathbb{B}^2 \to \mathbb{S}^1$ such that ri is the identity on $\mathbb{S}^1$.*

Proof. Suppose that there were. Consider the two commutative diagrams:

$$\begin{tikzcd} (\mathbb{S}^1,1) \arrow[rr, "i"'] \arrow[dr, "1"'] & & (\mathbb{B}^2,1) \arrow[dl, "r"] \\ & (\mathbb{S}^1,1) & \end{tikzcd} \qquad \begin{tikzcd} \pi(\mathbb{S}^1,1) \arrow[rr, "i_*"'] \arrow[dr, "1"'] & & \pi(\mathbb{B}^2,1) \arrow[dl, "r_*"] \\ & \pi(\mathbb{S}^1,1) & \end{tikzcd}$$

Now we know from Example 2.5 and Theorem 4.1 that the groups at the top left and bottom are $\mathbb{Z}$ while that on the top right is trivial. It is not possible for such a commutative diagram to exist, for example consider the fate of the element $1 \in \pi(\mathbb{S}^1,1)$: under 1 this goes to 1 but under i_* it must go to 0 and hence under r_*i_* must also go to 0. ■

Corollary 4.3 (Brouwer's Fixed Point Theorem) *Suppose that $f : \mathbb{B}^2 \to \mathbb{B}^2$ is continuous. Then $f(z) = z$ for some $z \in \mathbb{B}^2$.*

Proof. Suppose instead that for each $z \in \mathbb{B}^2$ we have $f(z) \neq z$. Define $r : \mathbb{B}^2 \to \mathbb{S}^1$ by letting $r(z)$ be that point obtained by drawing the straight line from $f(z)$ through z and extending as necessary until it reaches $\mathbb{S}^1$. If $z \in \mathbb{S}^1$ then $r(z) = z$ contrary to Corollary 4.2. ∎

Definition 4.4 *By a* circle *we mean a homeomorph of $\mathbb{S}^1$. Two circles $J, K \subset \mathbb{R}^3$ are* unlinked *if there is a continuous function $f : \mathbb{B}^2 \to \mathbb{R}^3 - K$ such that $f|\mathbb{S}^1$ is an embedding with $f(\mathbb{S}^1) = J$; otherwise J and K are* linked.

There is a lack of symmetry in the definition of unlinked; it may be proved that the variant of the definition obtained by interchanging the roles of J and K is equivalent to that given.

Proposition 4.5 *Let $J, K \subset \mathbb{R}^3$ be the following two circles: J is the circle of radius 1 in the plane $y = 0$ with centre $(1, 0, 0)$ and K is the circle of radius 1 in the plane $z = 0$ with centre $(0, 0, 0)$. Then J and K are linked.*

Proof. Suppose instead that J and K are unlinked, say $p : \mathbb{B}^2 \to \mathbb{R}^3 - K$ is a continuous function such that $p|\mathbb{S}^1$ is an embedding with $p(\mathbb{S}^1) = J$. Let

$$A = \{(x, 0, z) \in \mathbb{R}^3 \ / \ x \geq 0\}$$

and let $t : A - \{(1, 0, 0)\} \to J$ be radial projection from $(1, 0, 0)$: if $(x, y, z) \in J$ then $t(x, y, z) = (x, y, z)$. Next define $o : \mathbb{R}^3 \to \mathbb{R}^3$ by $o(x, y, z) = (\sqrt{x^2 + y^2}, 0, z)$. Note that if $(x, y, z) \in A$ then $o(x, y, z) = (x, y, z)$, and $o(\mathbb{R}^3 - K) \subset A - \{(1, 0, 0)\}$. Finally define $s : J \to \mathbb{S}^1$ by $s(x, y, z) = p^{-1}(x, y, z)$: as $p|\mathbb{S}^1$ is a homeomorphism onto $\mathbb{S}^1$, it follows that s is continuous.

Now consider

$$stop : \mathbb{B}^2 \to \mathbb{S}^1.$$

Then $stop(x, y) = (x, y)$ for all $(x, y) \in \mathbb{S}^1$, contradicting Corollary 4.2. ∎

Definition 4.6 *Suppose that $f : (\mathbb{S}^1, 1) \to (\mathbb{S}^1, 1)$ is continuous. Then the integer $f_*(1)$ is called the* degree *of f, denoted $d(f)$.*

Example 4.7 *The degree of any constant map is 0. The degree of the map $z \to z^n$ is n.*

Note that if $f, g : \mathbb{S}^1 \to \mathbb{S}^1$ are homotopic then they have the same degree.

Lemma 4.8 *Suppose that $f : (\mathbb{S}^1, 1) \to (\mathbb{S}^1, 1)$ is continuous. Let $\widehat{fe}$ be a lifting of $fe : [0, 1] \to \mathbb{S}^1$ given by Theorem 3.8. Then $d(f) = \widehat{fe}(1) - \widehat{fe}(0)$.*

Proposition 4.9 *Suppose that $f : \mathbb{B}^2 \to \mathbb{R}^2$ is such that $f(\mathbb{S}^1) \subset \mathbb{S}^1$ and $f|\mathbb{S}^1$ has degree $k \neq 0$. Then $\mathbb{B}^2 \subset f(\mathbb{B}^2)$.*

Proof. Suppose not, say $a \in \mathbb{B}^2 - f(\mathbb{B}^2)$. As $f|\mathbb{S}^1$ has non-zero degree it follows that $f(\mathbb{S}^1) = \mathbb{S}^1$, so that a is in the interior of $\mathbb{B}^2$. Define $\rho : \mathbb{R}^2 - \{a\} \to \mathbb{S}^1$ by letting ρ project points outside $\mathbb{S}^1$ radially towards the origin and points inside $\mathbb{S}^1$ away from a. Let $i : \mathbb{S}^1 \to \mathbb{B}^2$ denote the inclusion.

Consider the composition $\mathbb{S}^1 \xrightarrow{i} \mathbb{B}^2 \xrightarrow{f} \mathbb{R}^2 - \{a\} \xrightarrow{\rho} \mathbb{S}^1$. This composition is just $f|\mathbb{S}^1$ so it has degree k. Now apply the fundamental group operator to get:

$$\pi(\mathbb{S}^1) \xrightarrow{i_*} \pi(\mathbb{B}^2) \xrightarrow{f_*} \pi(\mathbb{R}^2 - \{a\}) \xrightarrow{\rho_*} \pi(\mathbb{S}^1).$$

The composition is multiplication by k. However this is impossible as the composition factors through the trivial group $\pi(\mathbb{B}^2)$. ∎

Theorem 4.10 (Fundamental theorem of algebra) *Let $P(z)$ be a polynomial of positive degree with complex coefficients. Then there is a complex number ζ with $P(\zeta) = 0$.*

Proof. Let $k > 0$ be the degree of the polynomial $P(z)$. We may assume that the coefficient of z^k in $P(z)$ is 1, so that

$$P(z) = a_0 + a_1 z + \cdots + a_{k-1} z^{k-1} + z^k.$$

Define $F_k : \mathbb{C} \to \mathbb{C}$ by $F_k(z) = z^k$, and define $h : \mathbb{C} \times [0, 1] \to \mathbb{C}$ by

$$h(z, t) = tF_k(z) + (1 - t)P(z) = z^k + (1 - t) \sum_{i=0}^{k-1} a_i z^i \quad \forall z \in \mathbb{C}.$$

Then $h_0 = P,\ h_1 = F_k$ and $z^k = h(z, t) - (1 - t) \sum_{i=0}^{k-1} a_i z^i$,
so that $|z^k| \le |h(z, t)| + \sum_{i=0}^{k-1} |a_i| \times |z|^i$,
and hence $|h(z, t)| \ge |z|^k - \sum_{i=0}^{k-1} |a_i| \times |z|^i$.

Let $M = 1 + \sum_{i=0}^{k-1} |a_i|$. Now $\forall t \in [0, 1]$ and $\forall z \in \mathbb{C}$, if $|z| \ge M$ then $|z| \ge 1$ so $|z|^i \le |z|^{k-1}$ for $i < k$, and hence

$$|h(z, t)| \ge |z|^k - \sum_{i=0}^{k-1} |a_i| \times |z|^{k-1} = |z|^{k-1}[|z| - \sum_{i=0}^{k-1} |a_i|] \ge |z| - \sum_{i=0}^{k-1} |a_i| \ge 1.$$

Thus we may define $H : \mathbb{S}^1 \times [0, 1] \to \mathbb{S}^1$ by $H(z, t) = \frac{h(Mz,t)}{|h(Mz,t)|}$. Note that $H_1(z) = z^k$ so H_1, and hence (by Proposition 2.10) H_0, has degree k.

We now show that P has a root in the disc $M\mathbb{B}^2$. Indeed, suppose that there is no $z \in M\mathbb{B}^2$ such that $P(z) = 0$. Then the degree k function $H_0 : \mathbb{S}^1 \to \mathbb{S}^1$ of the last paragraph extends over all of $\mathbb{B}^2$ by defining $\widehat{H_0}(z) = \frac{P(Mz)}{|P(Mz)|}$. Furthermore this extended function maps all of $\mathbb{B}^2$ into $\mathbb{S}^1$, contrary to Proposition 4.9.

It follows that the polynomial function defined by $P(z) = a_0 + a_1 z + \cdots + a_{k-1} z^{k-1} + z^k$ has at least one root within $1 + \sum_{i=0}^{k-1} |a_i|$ of 0. ∎

For the following we need the $(n - 1)$-sphere. This is defined to be the set $\mathbb{S}^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \ / \ x_1^2 + \ldots x_n^2 = 1\}$.

Definition 4.11 *A* vector field *on the* $(n-1)$*-sphere is a continuous function* $v : \mathbb{S}^{n-1} \to \mathbb{R}^n$ *such that for each* $x \in \mathbb{S}^{n-1}$, *we have* $x \bullet v(x) = 0$ *(here the dot,* $\bullet$*, denotes the usual scalar product). The vector field* v *is* non-zero *provided that* $v(x) \neq 0$ *for each* x.

The condition $x \bullet v(x) = 0$ ensures that $v(x)$ is tangent to $\mathbb{S}^{n-1}$ when located at x. Using this one can formulate the notion of a vector field on any smooth manifold in euclidean space.

Definition 4.12 *A collection* $\{v_1, \dots, v_m\}$ *of vector fields on* $\mathbb{S}^{n-1}$ *is called* linearly independent *provided that for each* x, *the vectors* $\{v_1(x), \dots, v_m(x)\}$ *are linearly independent.*

Example 4.13 *On an odd dimensional sphere we can define a non-zero vector field by*

$$v(x_1, \dots, x_{2k}) = (-x_2, x_1, -x_4, x_3, \dots, -x_{2k}, x_{2k-1}).$$

Note that this vector field when applied to $\mathbb{S}^1$ may be expressed in terms of complex numbers by $v(z) = iz$. This same expression is valid in quaternions and Cayley numbers. Furthermore, we may obtain other, linearly independent, vector fields on $\mathbb{S}^3$ and $\mathbb{S}^7$ by replacing i by j or k etc. In this way we obtain 3 linearly independent vector fields on $\mathbb{S}^3$ and 7 on $\mathbb{S}^7$. Note that this is the maximum number; more generally, we cannot find more than $n-1$ linearly independent vector fields on $\mathbb{S}^{n-1}$.

Theorem 4.14 *There is no non-zero vector field on* $\mathbb{S}^2$.

Proof. Suppose $v : \mathbb{S}^2 \to \mathbb{R}^3$ is a vector field which is non-zero. We will obtain a contradiction. We may assume that $|v(x)| = 1$ for each x, for if not then we may replace v by the field which takes x to $v(x)/|v(x)|$.

Next suppose that $f : \mathbb{S}^1 \to \mathbb{S}^2$ is differentiable with $f'(z) \neq 0$ for each z. We will only need the case where f carries $\mathbb{S}^1$ diffeomorphically onto a line of constant latitude but will consider the more general case initially. Define $\bar{f} : \mathbb{S}^1 \to \mathbb{S}^1$ as follows: given $z \in \mathbb{S}^1$, let $T_z : \mathbb{R}^3 \to \mathbb{R}^3$ be the orientation-preserving orthogonal transformation which carries $f(z)$ to $(0,0,1)$ and $f'(z)$ to $(a,0,0)$ (for some $a > 0$). Since $vf(z)$ is orthogonal to $f(z)$ then $T_z vf(z)$ is orthogonal to $(0,0,1)$ so lies in $\mathbb{R}^2$. Moreover, since $vf(z)$ has unit length so has $T_z vf(z)$, so $T_z vf(z) \in \mathbb{S}^1$: set $\bar{f}(z) = T_z vf(z)$. The function $\bar{f}$ is continuous so its degree $d(\bar{f})$ is defined.

Consider now the particular case $f_r : \mathbb{S}^1 \to \mathbb{S}^2$, for $r \in (-1,1)$, defined by $f_r(x,y) = (x\sqrt{1-r^2}, y\sqrt{1-r^2}, r)$. It is easily checked that the parameter r provides a homotopy between any two of the corresponding functions $\bar{f}_r$ and hence all of the degrees $d(\bar{f}_r)$ are the same.

By continuity of v, there is an r near 1 and there is an s near -1 so that for each $z \in \mathbb{S}^1$, $vf_r(z)$ is within 1 of $v(0,0,1)$ and $vf_s(z)$ is within 1 of $v(0,0,-1)$: all of the vectors $vf_r(z)$ point about in the same direction as $v(0,0,1)$ and all of the vectors $vf_s(z)$ point about in the same direction as $v(0,0,-1)$. Looking at the sphere from the outside, f_r wraps $\mathbb{S}^1$ around a line of latitude in the anticlockwise direction so $d(\bar{f}_r) = -1$, and f_s wraps $\mathbb{S}^1$ around a line of latitude in the clockwise direction so $d(\bar{f}_s) = 1$. This contradicts $d(\bar{f}_r) = d(\bar{f}_s)$. ∎

Theorem 4.14 also holds for any even-dimensional sphere, ie any vector field on $\mathbb{S}^{2k}$ is zero. A natural question to ask is how many linearly independent vector fields are supported by an odd-dimensional sphere. We have already seen that there is at least 1, and there cannot possibly be more than $2k-1$ on $\mathbb{S}^{2k-1}$. The question was settled in 1962. In fact only on $\mathbb{S}^1$, $\mathbb{S}^3$ and $\mathbb{S}^7$ (and $\mathbb{S}^0$!!) is the maximum number the dimension of the sphere. If we write $n = 2^{4\alpha+\beta}(2\gamma+1)$, with α, β and γ integers and $0 \leq \beta < 4$, then the maximum number of linearly independent vector fields supported by $\mathbb{S}^{n-1}$ is $2^\beta + 8\alpha - 1$. Just as we used the complex, quaternionic and Cayley structures giving $\mathbb{R}^2$, $\mathbb{R}^4$ and $\mathbb{R}^8$ as (not necessarily commutative, not necessarily associative) algebras over $\mathbb{R}$ to obtain the maximum number of linearly independent vector fields over $\mathbb{S}^1$, $\mathbb{S}^3$ and $\mathbb{S}^7$, so can we deduce that there are no such algebras in other dimensions (except 1!).

5 The Homology Axioms

Definition 5.1 *A* topological pair *is a pair (X, A) consisting of a topological space X and a subspace A. A map of pairs $f : (X, A) \to (Y, B)$ is a continuous function $f : X \to Y$ such that $f(A) \subset B$.*

The topological pair $(X, \varnothing)$ will be abbreviated to X. The *identity* map is the map $\mathbf{1}_X : (X, A) \to (X, A)$ given by $\mathbf{1}_X(x) = x$; where no confusion will arise we will denote $\mathbf{1}_X$ by $\mathbf{1}$. We use $i : A \to X$ and $j : X \to (X, A)$ to denote the inclusion maps.

We will consider a class of topological pairs and maps of these pairs (technically a *category* of topological pairs).

Definition 5.2 *A* homology theory *assigns*

- *an abelian group $H_q(X, A)$ to each topological pair (X, A) and each $q \in \mathbb{Z}$;*
- *a homomorphism $f_* : H_q(X, A) \to H_q(Y, B)$ to each map $f : (X, A) \to (Y, B)$ of pairs and each $q \in \mathbb{Z}$; and*
- *a homomorphism $\partial : H_q(X, A) \to H_{q-1}(A)$, called the* boundary*, to each topological pair (X, A) and each $q \in \mathbb{Z}$*

such that the following seven axioms are satisfied:

Axiom 1 (Identity) $\mathbf{1}_* = \mathbf{1}$*;*

Axiom 2 (Composition) $(gf)_* = g_* f_*$ *whenever gf is defined;*

Axiom 3 (Commutativity) $\partial f_* = (f|A)_* \partial$ *when $f : (X, A) \to (Y, B)$;*

Axiom 4 (Exactness) *The sequence*

$$\cdots \to H_q(A) \xrightarrow{i_*} H_q(X) \xrightarrow{j_*} H_q(X, A) \xrightarrow{\partial} H_{q-1}(A) \to \cdots$$

of homomorphisms is exact, that is, the image of any one homomorphism is the kernel of the next;

Axiom 5 (Homotopy) *If f is homotopic to g then $f_* = g_*$;*

Axiom 6 (Excision) *If $U \subset X$ is open with $\overline{U} \subset \mathring{A}$ and $e : (X - U, A - U) \to (X, A)$ is the inclusion then $e_* : H_q(X - U, A - U) \to H_q(X, A)$ is an isomorphism for each q (the map e is called an* excision*);*

Axiom 7 (Dimension) *The group $H_q(\{0\})$ is trivial for each $q \neq 0$.*

The group $H_0(\{0\})$ is called the *coefficient group* and will be denoted by G. Much of the time we may take $G = \mathbb{Z}$.

Example 5.3 *Let $H_q(X, A)$ be the trivial group for each (X, A). Then all of the axioms are satisfied. Of course there is little profit in studying this homology theory!*

Example 5.4 . *If we restrict our attention to topological pairs consisting of simplicial complexes and let $H_q(X, A)$ be the simplicial homology group for each (X, A) and define the homomorphisms appropriately, then all of the axioms are satisfied.*

There are other examples of non-trivial homology theories, for example *singular homology theory* which is defined on the category of all topological pairs and maps, but we will not have time to describe any of them. Instead we will assume that such theories exist and draw some conclusions.

6 Immediate Consequences of the Homology Axioms

Theorem 6.1 *For each space X and each $q \in \mathbb{Z}$ the group $H_q(X, X)$ is trivial.*

Proof: Apply Axiom 4 to the pair (X, X), noting that Axiom 1 tells us that the homomorphisms $\mathbf{1}_* : H_q(X) \to H_q(X)$ are always the identity. ∎

Theorem 6.2 *If $f : (X, A) \to (Y, B)$ is a homotopy equivalence then $f_* : H_q(X, A) \to H_q(Y, B)$ is an isomorphism.*

Proof: Let $g : (Y, B) \to (X, A)$ be a homotopy inverse of f. Then by Axiom 5, gf is homotopic to $\mathbf{1}_X$ and fg is homotopic to $\mathbf{1}_Y$, hence by Axioms 1 and 2 we have that g_*f_* and f_*g_* are the respective identities. It follows that f_* and g_* are both isomorphisms. ∎

Corollary 6.3 *If X is homotopy equivalent to Y then the groups $H_q(X)$ and $H_q(Y)$ are isomorphic.*

Corollary 6.4 *If X is contractible then $H_0(X)$ is isomorphic to G and $H_q(X)$ is trivial for $q \neq 0$.*

Definition 6.5 *A subspace A of a space X is called a* retract *of X provided that there is a map $r : X \to A$ such that $ri = \mathbf{1}_A$.*

Theorem 6.6 *If A is a retract of X then i_* is a monomorphism, j_* is an epimorphism and ∂ is trivial. Moreover,*

$$H_q(X) \approx H_q(A) \oplus H_q(X, A).$$

Corollary 6.7 *If x_0 is any point of X then*

$$H_0(X) \approx G \oplus H_0(X, x_0) \text{ and } H_q(X) \approx H_q(X, x_0) \text{ if } q \neq 0.$$

Theorem 6.8 *If N is a finite discrete space with n points then $H_q(N)$ is trivial if $q \neq 0$ and $H_0(N)$ is a direct sum of n copies of G.*

Proof: We use induction on n, the result being true for $n = 1$ by Corollary 6.3 and Axiom 7.

Now suppose the result true for $n - 1$ and let $N = \{x_1, \ldots, x_n\}$. Set $A = \{x_n\}$. Then by Corollary 6.7, $H_0(N) \approx G \oplus H_0(N, A)$ and $H_q(N) \approx H_q(N, A)$ if $q \neq 0$. Thus it suffices to show that $H_q(N, A) \approx H_q(N - A)$. As $A \subset N$ is open and $\overline{A} \subset \mathring{A}$, by Axiom 6, $H_q(N - A, A - A) \to H_q(N, A)$ is an isomorphism, ie $H_q(N, A) \approx H_q(N - A)$.

Definition 6.9 *A space X is* deformable *into a subspace A provided that there is a homotopy $h_t : X \to X$ such that $h_0 = \mathbf{1}$ and $h_1(X) \subset A$.*

Theorem 6.10 *If X is deformable into A then i_* is an epimorphism, j_* is trivial and ∂ is a monomorphism. Moreover,*

$$H_q(A) \approx H_q(X) \oplus H_{q+1}(X, A).$$

Corollary 6.11 *If X is contractible and $A \subset X$ then $H_0(A) \approx G \oplus H_1(X, A)$ and $H_q(A) \approx H_{q+1}(X, A)$ if $q \neq 0$.*

Theorem 6.12 *If $U \subset X$ is open with $U \subset A$, and $V \subset X$ is open with $\overline{V} \subset U$ and the inclusion $(X - U, A - U) \to (X - V, A - V)$ is a homotopy equivalence then $e_* : H_q(X - U, A - U) \to H_q(X, A)$ is an isomorphism.*

7 Reduced Homology Groups

Except as noted at the end, throughout this section we assume that $A, B \neq \varnothing$. We will simplify notation by writing 0 and (0,0) for $\{0\}$ and $(\{0\},\{0\})$ respectively.

Definition 7.1 *Let $f : (X, A) \to (0, 0)$, $g : X \to 0$ and $h : A \to 0$ denote the unique maps. Then the* reduced homology groups *$\tilde{H}_q(X, A), \tilde{H}_q(X)$ and $\tilde{H}_q(A)$ are, respectively, the kernels of*

$$f_* : H_q(X, A) \to H_q(0, 0), g_* : H_q(X) \to H_q(0) \text{ and } h_* : H_q(A) \to H_q(0).$$

We have the following commutative diagram in which the homomorphisms in the top row are restrictions of those directly below:

$$\begin{tikzcd}
\cdots \arrow[r] & \tilde{H}_q(A) \arrow[r, "\tilde{\imath}_*"] \arrow[d] & \tilde{H}_q(X) \arrow[r, "\tilde{\jmath}_*"] \arrow[d] & \tilde{H}_q(X, A) \arrow[r, "\tilde{\partial}"] \arrow[d] & \tilde{H}_{q-1}(A) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_q(A) \arrow[r, "i_*"] \arrow[d, "h_*"] & H_q(X) \arrow[r, "j_*"] \arrow[d, "g_*"] & H_q(X, A) \arrow[r, "\partial"] \arrow[d, "f_*"] & H_{q-1}(A) \arrow[r] \arrow[d, "h_*"] & \cdots \\
\cdots \arrow[r] & H_q(0) \arrow[r] & H_q(0) \arrow[r] & H_q(0, 0) \arrow[r] & H_{q-1}(0) \arrow[r] & \cdots
\end{tikzcd}$$

Proposition 7.2 *In the diagram above the images of $\tilde{\imath}_*, \tilde{\jmath}_*$ and $\tilde{\partial}$ lie in $\tilde{H}_q(X)$, $\tilde{H}_q(X, A)$ and $\tilde{H}_{q-1}(A)$ respectively.*

Theorem 7.3 *For all q we have $\tilde{H}_q(X, A) = H_q(X, A)$; for all $q \neq 0$ we have $\tilde{H}_q(X) = H_q(X)$ and $\tilde{H}_q(A) = H_q(A)$ and we have $H_0(X) \approx G \oplus \tilde{H}_0(X)$ and $H_0(A) \approx G \oplus \tilde{H}_0(A)$.*

Proof: As $H_q(0, 0)$ is trivial for all q and $H_q(0)$ is trivial for all $q \neq 0$, it follows that $\tilde{H}_q(X, A)$ for all q and $\tilde{H}_q(X)$ and $\tilde{H}_q(A)$ for $q \neq 0$ are as claimed.

Let $k : 0 \to X$ be any map. Then $gk : 0 \to 0$ is the identity so $g_*k_* = \mathbf{1}$. Define θ :Im$(k_*)\oplus$Ker$(g_*) \to H_0(X)$ by $\theta(a, b) = a - b$.

θ is a monomorphism, for if $\theta(a, b) = 0$ then $a = b$ so $g_*(a) = g_*(b) = 0$ as $b \in$Ker(g_*). As $a \in$Im(k_*) and $g_*k_* = \mathbf{1}$, it follows that $a = 0$ and hence $b = 0$.

θ is an epimorphism, for if $x \in H_0(X)$ then $g_*(k_*g_*(x) - x) = 0$ so

$$(k_*g_*(x), k_*g_*(x) - x) \in \mathrm{Im}(k_*) \oplus \mathrm{Ker}(g_*).$$

Further $\theta(k_*g_*(x), k_*g_*(x) - x) = x$.

Thus θ is an isomorphism, so $H_0(X) \approx$Im$(k_*)\oplus$Ker(g_*).

As k_* is a monomorphism, Im$(k_*) \approx G$. By definition, Ker$(g_*) = \tilde{H}_0(X)$. Thus $H_0(X) \approx G \oplus \tilde{H}_0(X)$. Similarly for A. ■

Theorem 7.4 *The sequence*

$$... \to \tilde{H}_q(A) \xrightarrow{\tilde{\imath}_*} \tilde{H}_q(X) \xrightarrow{\tilde{\jmath}_*} \tilde{H}_q(X, A) \xrightarrow{\partial} \tilde{H}_{q-1}(A) \to ...$$

is exact.

Proof. As the only places where the reduced sequence differs from the original exact sequence is from $\tilde{H}_1(X, A)$ to $\tilde{H}_0(X, A)$, it suffices to verify the exactness at these four groups.

At $\tilde{H}_1(X, A)$: Im$(\tilde{\jmath}_*)$ =Im(j_*) =Ker(∂) =Ker$(\tilde{\partial})$.

At $\tilde{H}_0(A)$: Clearly Im$(\tilde{\partial})$ $\subset$Ker$(\tilde{\imath}_*)$. Conversely if $a \in$Ker$(\tilde{\imath}_*)$ then $i_*(a) = 0$ so there is $y \in H_1(X, A)$ with $\partial(y) = a$. As $\tilde{H}_1(X, A) = H_1(X, A)$ it follows that $y \in \tilde{H}_1(X, A)$ and $\tilde{\partial}(y) = a$.

At $\tilde{H}_0(X)$: Clearly Im$(\tilde{\imath}_*)$ $\subset$Ker$(\tilde{\jmath}_*)$. Conversely if $x \in$Ker$(\tilde{\jmath}_*)$ then there is $a \in H_0(A)$ such that $i_*(a) = x$. Now $h_*(a) = g_*(i_*(a)) = g_*(x) = 0$ as $x \in$Ker(g_*). Thus $a \in$Ker$(h_*) = \tilde{H}_0(A)$ so $\tilde{\imath}_*(a) = x$.

At $\tilde{H}_0(X, A)$: Clearly Im$(\tilde{\jmath}_*)$ $\subset$Ker$(\tilde{\partial})$. Conversely if $y \in$Ker$(\tilde{\partial})$ =Ker(∂) =Im(j_*) then there is $x \in H_0(X)$ with $j_*(x) = y$. Choose any map $k : 0 \to A$. Note that $g_*i_*k_*g_*(x) = g_*(x)$ as $gik = \mathbf{1}$, so $x - i_*k_*g_*(x) \in$Ker$(g_*) = \tilde{H}_0(X)$. Furthermore, $\tilde{\jmath}_*(x - i_*k_*g_*(x)) = j_*(x) - j_*i_*k_*g_*(x) = j_*(x) = y$ as $j_*i_* = 0$. ■

Proposition 7.5 *If $f : (X, A) \to (Y, B)$ is a map then f naturally induces homomorphisms*

$$f_* : \tilde{H}_q(X, A) \to \tilde{H}_q(Y, B), g_* : \tilde{H}_q(X) \to \tilde{H}_q(Y) \text{ and } h_* : \tilde{H}_q(A) \to \tilde{H}_q(B).$$

Proof. The only cases needing consideration are $g_* : \tilde{H}_0(X) \to \tilde{H}_0(Y)$ and $h_* : \tilde{H}_0(A) \to \tilde{H}_0(B)$, and only the first of these is treated.

If $x \in \tilde{H}_0(X)$ then $g_*(x) \in \tilde{H}_0(Y)$ by commutativity of the diagram

$$\begin{tikzcd}
H_0(X) \arrow[d, "g_*"] \arrow[dr] & \\
 & H_0(0), \\
H_0(Y) \arrow[ur] &
\end{tikzcd}$$

so $g_* : \tilde{H}_0(X) \to \tilde{H}_0(Y)$ is merely the restriction of $g_* : H_0(X) \to H_0(Y)$. ∎

In the following propositions we relax the condition that $A, B \neq \varnothing$.

Proposition 7.6 $\mathbf{1} : (X, A) \to (X, A)$ *induces* $\mathbf{1} : \tilde{H}_q(X, A) \to \tilde{H}_q(X, A)$.

Proposition 7.7 *If* $f : (X, A) \to (Y, B)$ *and* $g : (Y, B) \to (Z, C)$ *are maps then* $(gf)_* = g_* f_* : \tilde{H}_q(X, A) \to \tilde{H}_q(Z, C)$.

Proposition 7.8 *If* f *is homotopic to* g *then* $f_* = g_* : \tilde{H}_q(X, A) \to \tilde{H}_q(Y, B)$

Proposition 7.9 *If* X *is homotopy equivalent to* Y *then* $\tilde{H}_q(X) \approx \tilde{H}_q(Y)$.

Proposition 7.10 *If* X *is contractible then for each* $q \in \mathbb{Z}$, $\tilde{H}_q(X)$ *is trivial.*

8 Homology Groups of Spheres

Let

$\mathbb{S}^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : \sum x_i^2 = 1\}$,

$\mathbb{S}^n_+ = \{(x_0, \ldots, x_n) \in \mathbb{S}^n : x_n \geq 0\}$,

$\mathbb{S}^n_- = \{(x_0, \ldots, x_n) \in \mathbb{S}^n : x_n \leq 0\}$,

We can consider $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ by identifying $(x_1, \ldots, x_n)$ with $(x_1, \ldots, x_n, 0)$. Then $\mathbb{S}^{n-1} = \mathbb{S}^n_+ \cap \mathbb{S}^n_-$.

For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we denote $\sum x_i^2$ by $\|x\|$.

Theorem 8.1 *The excision map* $(\mathbb{S}^n_-, \mathbb{S}^{n-1}) \to (\mathbb{S}^n, \mathbb{S}^n_+)$ *induces isomorphisms*

$$H_q(\mathbb{S}^n_-, \mathbb{S}^{n-1}) \approx H_q(\mathbb{S}^n, \mathbb{S}^n_+) \text{ and } \tilde{H}_q(\mathbb{S}^n_-, \mathbb{S}^{n-1}) \approx \tilde{H}_q(\mathbb{S}^n, \mathbb{S}^n_+).$$

Proof. The two isomorphisms are the same. To get them apply Theorem 6.12 with $X = \mathbb{S}^n, A = \mathbb{S}^n_+, U = \mathbb{S}^n - \mathbb{S}^n_-$ and $V = \{(x_0, ...x_n) \in \mathbb{S}^n : x_n > 0.5\}$. ∎

Theorem 8.2 *For each* $n \geq 0$ *we have* $\tilde{H}_q(\mathbb{S}^n) \approx G$ *if* $q = n$ *and* $\tilde{H}_q(\mathbb{S}^n)$ *is trivial if* $q \neq n$.

Proof. Consider the following diagram consisting of parts of the reduced exact sequences of $(\mathbb{S}^n, \mathbb{S}^n_+)$ and $(\mathbb{S}^n_-, \mathbb{S}^{n-1})$, and the excision of Theorem 8.1:

$$\begin{tikzcd}
\tilde{H}_q(\mathbb{S}^n_+) \arrow[r] & \tilde{H}_q(\mathbb{S}^n) \arrow[r] & \tilde{H}_q(\mathbb{S}^n, \mathbb{S}^n_+) \arrow[r] & \tilde{H}_{q-1}(\mathbb{S}^n_+) & \\
 & \tilde{H}_q(\mathbb{S}^n_-) \arrow[r] & \tilde{H}_q(\mathbb{S}^n_-, \mathbb{S}^{n-1}) \arrow[r] \arrow[u] & \tilde{H}_{q-1}(\mathbb{S}^{n-1}) \arrow[r] & \tilde{H}_{q-1}(\mathbb{S}^n_-)
\end{tikzcd}$$

In the top row, as $\mathbb{S}^n_+$ is contractible, by Proposition 7.10, $\tilde{H}_q(\mathbb{S}^n_+)$ and $\tilde{H}_{q-1}(\mathbb{S}^n_+)$ are trivial so $\tilde{H}_q(\mathbb{S}^n) \approx \tilde{H}_q(\mathbb{S}^n, \mathbb{S}^n_+)$. By contractibility of $\mathbb{S}^n_-$ the bottom row gives $\tilde{H}_q(\mathbb{S}^n_-, \mathbb{S}^{n-1}) \approx \tilde{H}_{q-1}(\mathbb{S}^{n-1})$. Thus $\tilde{H}_q(\mathbb{S}^n) \approx \tilde{H}_{q-1}(\mathbb{S}^{n-1})$, and hence by induction, $\tilde{H}_q(\mathbb{S}^n) \approx \tilde{H}_{q-n}(\mathbb{S}^0)$.

By Theorems 6.8 and 7.3, we obtain the claimed result. ■

Corollary 8.3 *For each n, $H_q(S^n)$ is trivial if $n \neq q \neq 0$, is isomorphic to G if $n \neq q = 0$ or $n = q \neq 0$ and is isomorphic to $G \oplus G$ if $n = q = 0$.*

For the first time we need to assume that there is a non-trivial homology theory. This assumption will remain in place from now on, though it is not essential for every result described.

Theorem 8.4 *There is no continuous $r : \mathbb{B}^n \to \mathbb{S}^{n-1}$ with $r|\mathbb{S}^{n-1} = \mathbf{1}$, where $\mathbb{B}^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$.*

Proof. Suppose there were. Then $ri = \mathbf{1}$, where $i : \mathbb{S}^{n-1} \to \mathbb{B}^n$ is the inclusion. Thus $(ri)_* = \mathbf{1}$. However this gives a commutative diagram:

$$\begin{tikzcd} \tilde{H}_{n-1}(\mathbb{S}^{n-1}) \arrow[r, "i_*"] \arrow[dr, "\mathbf{1}"'] & \tilde{H}_{n-1}(\mathbb{B}^n) \arrow[d, "r_*"] \\ & \tilde{H}_{n-1}(\mathbb{S}^{n-1}) \end{tikzcd} \quad \text{ie} \quad \begin{tikzcd} G \arrow[r] \arrow[dr, "\mathbf{1}"'] & 0 \arrow[d] \\ & G \end{tikzcd}$$

by Theorem 8.2 and Proposition 7.10. Provided that G is non-trivial, the last diagram is impossible. Hence if there is a homology theory with non-trivial coefficient group, such a map r cannot exist. ■

Theorem 8.5 (Brouwer's Fixed Point Theorem) *Let $f : \mathbb{B}^n \to \mathbb{B}^n$ be a map. Then there is $x \in \mathbb{B}^n$ with $f(x) = x$.*

Proof. Suppose instead that for each $x \in \mathbb{B}^n$ we have $f(x) \neq x$. Then $r : \mathbb{B}^n \to \mathbb{S}^{n-1}$ may be defined as follows: given $x \in \mathbb{B}^n$, let $r(x)$ be that point of $\mathbb{S}^{n-1}$ obtained by extending the line segment from $f(x)$ through x until it meets $\mathbb{S}^{n-1}$. Then r is continuous and $r|\mathbb{S}^{n-1} = \mathbf{1}$, contrary to Theorem 8.4. ■

Definition 8.6 *Suppose that X is a topological space. If $f : X \to X$ is a map for which there is a point $x \in X$ such that $f(x) = x$ then x is called a* fixed point *of f. If every continuous function $f : X \to X$ has a fixed point then X is said to have the* fixed point property*.*

Example 8.7 *Theorem 8.5 tells us that $\mathbb{B}^n$ has the fixed point property. On the other hand neither $\mathbb{S}^n$ nor $\mathbb{R}^n$ has the fixed point property, the antipodal map $\alpha : \mathbb{S}^n \to \mathbb{S}^n$ defined by $\alpha(x) = -x$ and any nontrivial translation of $\mathbb{R}^n$ both not having a fixed point.*

It is clear that if two topological spaces are homeomorphic and one of them has the fixed point property then so has the other.

Theorem 8.8 *Suppose that A is a real $n \times n$ matrix all of whose entries are positive. Then A has a positive real eigenvalue.*

Proof. We will also denote by $A : \mathbb{R}^n \to \mathbb{R}^n$ the linear transformation determined by A using the usual basis of $\mathbb{R}^n$. The function A is continuous. Let $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) : x_1, \ldots x_n \geq 0\}$ and $B = \mathbb{S}^{n-1} \cap \mathbb{R}^n_+$. As B is homeomorphic to $\mathbb{B}^{n-1}$ it has the fixed point property. Since all of the entries of A are positive, we have $A(\mathbb{R}^n_+) \subset \mathbb{R}^n_+$. Moreover if $x \in \mathbb{R}^n_+ - \{0\}$ then $A(x) \neq 0$ so we may define $f : B \to B$ by $f(x) = A(x)/\|A(x)\|$. Then f is well-defined and continuous. Let $u \in B$ be a fixed point of f. Then $f(u) = u$ so $A(u) = \|A(u)\|u$, ie $A(u)$ is a real multiple of u, so u is an eigenvector of A and, since $u \neq 0$, $\|A(u)\|$ is an eigenvalue of A. ■

Corollary 8.9 *If $m \neq n$ then $\mathbb{S}^m$ and $\mathbb{S}^n$ are not homotopy equivalent.*

Theorem 8.10 *If $m \neq n$ then $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic.*

Proof. Suppose that $m \neq n$ yet $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$; say $h : \mathbb{R}^m \to \mathbb{R}^n$ is a homeomorphism. We may assume that $h(0) = 0$. Define $f : \mathbb{S}^{m-1} \to \mathbb{S}^{n-1}$ by $f(x) = h(x)/\|h(x)\|$. Then f is a homotopy equivalence, contrary to Corollary 8.9.■

9 Degrees of Spherical Maps again

We will now assume that there is a homology theory in which the coefficient group is the additive group of integers, $\mathbb{Z}$. Note that in that case by Theorem 8.2 we have $\tilde{H}_n(\mathbb{S}^n) \approx \mathbb{Z}$.

Definition 9.1 *Suppose that $f : \mathbb{S}^n \to \mathbb{S}^n$ is continuous. Then $f_* : \tilde{H}_n(\mathbb{S}^n) \to \tilde{H}_n(S^n)$ is multiplication by some integer: this integer is called the* degree *of f and is denoted deg$f$.*

It is clear that if f is homotopic to g then deg$f$=deg$g$. The converse also holds but we will not prove it.

For $n = 1$, this alternative definition of degree is equivalent to that given earlier for a map $\mathbb{S}^1 \to \mathbb{S}^1$.

Proposition 9.2 *Suppose that $f, g : \mathbb{S}^n \to \mathbb{S}^n$ are two maps. Then*

$$\deg(fg) = \deg(gf) = \deg f \deg g.$$

Proposition 9.3 *Let $\rho : \mathbb{S}^n \to \mathbb{S}^n$ be the reflection defined by*

$$\rho(x_0, \ldots, x_n) = (-x_0, x_1, \ldots, x_n).$$

Then ρ has degree -1.

Proof. The following diagram, which includes isomorphisms used in the proof of Theorem 8.2, commutes:

$$\begin{tikzcd}
\tilde{H}_n(\mathbb{S}^n) \arrow[r] \arrow[d, "\rho_*"] & \tilde{H}_n(\mathbb{S}^n, \mathbb{S}^n_+) \arrow[d, "\rho_*"] & \tilde{H}_n(\mathbb{S}^n_-, \mathbb{S}^{n-1}) \arrow[l] \arrow[r] \arrow[d, "\rho_*"] & \tilde{H}_{n-1}(\mathbb{S}^{n-1}) \arrow[d, "\rho_*"] \\
\tilde{H}_n(\mathbb{S}^n) \arrow[r] & \tilde{H}_n(\mathbb{S}^n, \mathbb{S}^n_+) & \tilde{H}_n(\mathbb{S}^n_-, \mathbb{S}^{n-1}) \arrow[l] \arrow[r] & \tilde{H}_{n-1}(S^{n-1})
\end{tikzcd}$$

Thus $\rho_* : \tilde{H}_n(\mathbb{S}^n) \to \tilde{H}_n(\mathbb{S}^n)$ is multiplication by -1 if and only if $\rho_* : \tilde{H}_{n-1}(\mathbb{S}^{n-1}) \to \tilde{H}_{n-1}(\mathbb{S}^{n-1})$ is multiplication by -1. Thus by induction it suffices to prove the result for $n = 0$.

Let $i^+ : \{1\} \to \mathbb{S}^0$, $j^+ : \mathbb{S}^0 \to (\mathbb{S}^0, \{1\})$, $i^- : \{-1\} \to \mathbb{S}^0$ and $e : \{-1\} \to (\mathbb{S}^0, \{1\})$ be the inclusions, and $r : \mathbb{S}^0 \to \{1\}$ the retraction.

Note that e induces an isomorphism $e_* : H_0(\{-1\}) \to H_0(\mathbb{S}^0, \{1\})$. As $ri^+ = \mathbf{1}$, it follows that $i^+_* : H_0(\{1\}) \to H_0(\mathbb{S}^0)$ is a monomorphism. Furthermore, $\rho i^+ r i^- = i^-$ and $\rho i^- = i^+ r i^-$. By definition, $\tilde{H}_0(\mathbb{S}^0)$ =Ker$(r_*) \subset H_0(\mathbb{S}^0)$.

Define $\theta : H_0(\{1\}) \oplus H_0(\{-1\}) \to H_0(\mathbb{S}^0)$ by $\theta(a, b) = i^+_*(a) + i^-_*(b)$. Then θ is an isomorphism. Indeed, θ is a monomorphism because if $\theta(a, b) = 0$ then $j^+_*\theta(a, b) = 0$, from which $e_*(b) = j^+_* i^-_*(b) = 0$ so $b = 0$ and hence, since i^+_* is a monomorphism, $a = 0$. Also θ is an epimorphism for if $x \in H_0(\mathbb{S}^0)$ then $b = e_*^{-1} j^+_*(x) \in H_0(\{-1\})$ and $j^+_*(x - i^-_*(b)) = 0$ so there is $a \in H_0(\{1\})$ with $i^+_*(a) = x - i^-_*(b)$: then $\theta(a, b) = x$.

Note that $\theta(a, b) \in$Ker(r_*) if and only if $r_*\theta(a, b) = 0$ if and only if $r_* i^+_*(a) + r_* i^-_*(b) = 0$. As $ri^+ = \mathbf{1}$, it follows that $\theta(a, b) \in \tilde{H}_0(\mathbb{S}^0)$ if and only if $a = -r_* i^-_*(b)$. Now
$\rho_*\theta(-r_* i^-_*(b), b) = \rho_* i^+_*(-r_* i^-_*(b)) + \rho_* i^-_*(b) = -(\rho i^+ r i^-)_*(b) + (\rho i^-)_*(b)$
$= -i^-_*(b) + i^+_* r_* i^-_*(b) = -\theta(-r_* i^-_*(b), b)$.

Thus ρ_* is multiplication by -1. ∎

Theorem 9.4 *Let $g : \mathbb{S}^n \to \mathbb{S}^n$ be an orthogonal transformation. Then g has degree* det$g$.

Proof. Either det$g = 1$ or det$g = -1$.
Case I. Suppose that det$g = 1$. It is claimed that $g \simeq \mathbf{1}$, so that $g_* = \mathbf{1}_* = \mathbf{1}$, which is multiplication by 1.

If $n = 0$ then $g = \mathbf{1}$ so the claim is true in this case.

Assume the claim is true for $n - 1 \geq 0$. Let P be a plane in $\mathbb{R}^{n+1}$ containing $N = (0, \dots, 0, 1)$, $g(N)$ and $(0, \dots, 0)$: P is unique if these points are not collinear. Let $P^\perp$ be the orthogonal complement of P in $\mathbb{R}^{n+1}$. Let $h : \mathbb{S}^n \to \mathbb{S}^n$ be the rotation leaving $P^\perp$ pointwise fixed and sending $g(N)$ to N. Then h is homotopic to $\mathbf{1}$ and det$h = 1$. Consider hg: now $hg(N) = N$, and $\mathbb{R}^n$ is the hyperplane in $\mathbb{R}^{n+1}$ through the origin perpendicular to N so $hg(\mathbb{R}^n) = \mathbb{R}^n$, and hence $hg(\mathbb{S}^{n-1}) = \mathbb{S}^{n-1}$. Note that $f = hg|\mathbb{S}^{n-1}$ is an orthogonal transformation, with determinant 1. Thus by inductive assumption $f \simeq \mathbf{1}$ so $g \simeq hg \simeq \mathbf{1}$.
Case II. Suppose that det$g = -1$. Let ρ be as in Proposition 9.3. Then ρ is an orthogonal transformation and det$\rho = -1$, so det$(\rho g) = 1$. Thus by Case I $\rho g \simeq \mathbf{1}$, so $\rho_* g_* = \mathbf{1}$ and so by Proposition 9.3, g_* is multiplication by -1. ∎

Corollary 9.5 *Let $\alpha : \mathbb{S}^n \to \mathbb{S}^n$ be the antipodal map. Then α has degree $(-1)^{n+1}$.*

Definition 9.6 *A* vector field *on $\mathbb{S}^n$ is a continuous function $v : \mathbb{S}^n \to \mathbb{R}^{n+1}$ such that for each $x \in \mathbb{S}^n$, $v(x) \neq 0$ and $v(x)$ is orthogonal to x.*

Theorem 9.7 *$\mathbb{S}^n$ has a vector field if and only if n is odd.*

Proof. If $\mathbb{S}^n$ has a vector field, say $v : \mathbb{S}^n \to \mathbb{R}^{n+1}$, then we may assume that $\|v(x)\| = 1$, dividing $v(x)$ by $\|v(x)\|$ if necessary to achieve this. Define $F : \mathbb{S}^n \times [0, 1] \to \mathbb{S}^n$ by

$F(x,t) = x\cos\pi t + v(x)\sin\pi t$. Then F is a homotopy from $\mathbf{1}$ to α. Thus by Corollary 9.5, $(-1)^{n+1} = 1$ and hence n is odd.

Conversely if n is odd, say $n = 2k-1$, define the vector field $v : \mathbb{S}^{2k-1} \to \mathbb{R}^{2k}$ by $v(x_1, x_2, \ldots, x_{2k}) = (x_2, -x_1, \ldots, x_{2k}, -x_{2k-1})$. ∎

Definition 9.8 *Let $f : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ be a continuous function. The* suspension *of f, denoted $\bar{f} : \mathbb{S}^n \to \mathbb{S}^n$, is defined by*

$$\bar{f}(x_0, \ldots, x_n) = ((1 - \|x_n\|^2)f(x_0, \ldots, x_{n-1}), x_n).$$

Note that $\bar{f}$ is continuous.

Proposition 9.9 *Let $f : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ be continuous. Then* $\deg\bar{f} = \deg f$.

Proof. This follows from commutativity of the following diagram, where the horizontal arrows are isomorphisms obtained in Theorem 8.2:

$$\begin{tikzcd}
\tilde{H}_q(\mathbb{S}^{n-1}) \arrow[r] \arrow[d, "f_*"] & \tilde{H}_{q+1}(\mathbb{S}^n) \arrow[d, "\bar{f}_*"] \\
\tilde{H}_q(\mathbb{S}^{n-1}) \arrow[r] & \tilde{H}_{q+1}(\mathbb{S}^n)\ .
\end{tikzcd}$$

∎

Theorem 9.10 *For each $n, k \in \mathbb{Z}$ with $n > 0$ there is a map $f : \mathbb{S}^n \to \mathbb{S}^n$ of degree k.*

Proof. Given k, the map f_k of Example 4.7 has degree k. Then $\overline{f_k} : \mathbb{S}^2 \to \mathbb{S}^2$ and by Proposition 9.9, $\deg\overline{f_k} = \deg f_k = k$. Note then that $\overline{\overline{f_k}} : \mathbb{S}^3 \to \mathbb{S}^3$ has degree k and so on. ∎

Theorem 9.11 *If $f : \mathbb{S}^n \to \mathbb{S}^n$ is a map with non-zero degree then f is surjective.*

Proof. Suppose that there is $y \in \mathbb{S}^n - f(\mathbb{S}^n)$. Since $\mathbb{S}^n - \{y\}$ is homeomorphic to $\mathbb{R}^n$ it is contractible. Thus f is homotopic to a constant map and hence has degree 0. ∎

10 Constructing Singular Homology Theory

Let $\Delta^q = \{(x_0, \ldots, x_n) \in \mathbb{R}^{q+1} \ / \ \sum_{i=0}^{q} x_i = 1 \text{ and } x_i \geq 0 \text{ for each } i\}$. Δ^q is called the *q-simplex*. Consider the points $v_0, \ldots, v_q \in \mathbb{R}^{q+1}$, where all coordinates of v_i are 0 except the $i+1^{\text{st}}$, which is 1. Then Δ^q is the convex hull of $v_0, \ldots, v_q$, so we may write $\Delta^q = \{\sum_{i=0}^{q} t_i v_i \ / \ t_i \geq 0 \text{ for each } i \text{ and } \sum_{i=0}^{q} t_i = 1\}$.

The i^{th} *face* of Δ^q is the $(q-1)$-simplex $\Delta_i^{q-1} = \{\sum_{i=0}^{q} t_j v_j \in \Delta^q \ / \ t_i = 0\}$. Define the *face map* $F_i^q : \Delta^{q-1} \to \Delta^q$ by

$$F_i^q(v_j) = \begin{cases} v_j & \text{if } j < i \\ v_{j+1} & \text{if } j \geq i \end{cases},$$

and extend F_i^q linearly. Observe that for $i < j$ we have $F_j F_i = F_i F_{j-1} : \Delta^{q-1} \to \Delta^{q+1}$.

Let X be a topological space. For each $q \geq 0$ denote by $S_q(X)$ the free abelian group generated by $\{\sigma : \Delta^q \to X \; / \; \sigma \text{ is continuous}\}$. A continuous function $\sigma : \Delta^q \to X$ is called a *singular q-simplex.* A typical element of $S_q(X)$ may be thought of as a formal sum $\sum_{i=1}^n a_i \sigma_i$, where $a_i \in \mathbb{Z}$ and σ_i is a singular q-simplex: such a sum is called a *q-chain.* Note that when $q < 0$ there are no singular q-simplexes so $S_q(X)$ is the trivial group in that case.

Define the *face homomorphism* $V_i^q : S_q(X) \to S_{q-1}(X)$ by $V_i^q(\sigma) = \sigma F_i^q$. For this to have meaning when $q = 0$ we require V_i^q to be the trivial homomorphism. Observe that $V_i^q V_j^{q+1} = V_{j-1}^q V_i^{q+1} : S_{q+1}(X) \to S_{q-1}(X)$, again with $i < j$. The *boundary homomorphism* $\partial_q : S_q(X) \to S_{q-1}(X)$ is defined by $\partial_q = \sum_{i=0}^q (-1)^i V_i^q$.

Lemma 10.1 *For each $q \in \mathbb{Z}$ we have $\partial_q \partial_{q+1} = 0$.*

Let $Z_q(X) = \operatorname{Ker}(\partial_q)$ and $B_q(X) = \operatorname{Im}(\partial_{q+1})$. The elements of $Z_q(X)$ are called *cycles* while those of $B_q(X)$ are called *boundaries.* From the Lemma we have $B_q(X) \subset Z_q(X)$. As each is an abelian group we may take the quotient $H_q(X) = Z_q(X)/B_q(X)$. The abelian group $H_q(X)$ is called the q^{th} *homology group* of X.

If X and Y are two spaces and $f : X \to Y$ is continuous then there is induced a homomorphism $f_\sharp : S_q(X) \to S_q(Y)$ defined by $f_\sharp(\sigma) = \sigma f$. This homomorphism in turn induces a homomorphism $f_* : H_q(X) \to H_q(Y)$.

Now suppose that (X, A) is a topological pair. Then the homomorphism $i_\sharp : S_q(A) \to S_q(X)$ induced by the inclusion $i : A \to X$ is a monomorphism. Declare $S_q(X, A) = S_q(X)/i_\sharp(S_q(A))$. When $A = \varnothing$ this reduces to $S_q(X)$. We may define a homomorphism $\partial_q : S_q(X, A) \to S_{q-1}(X, A)$ by $\partial_q([c]) = [\partial_q(c)]$. As $\partial_q \partial_{q+1} = 0$, we may define $Z_q(X, A) = \operatorname{Ker}(\partial_q)$, $B_q(X, A) = \operatorname{Im}(\partial_{q+1})$ and $H_q(X, A) = Z_q(X, A)/B_q(X, A)$, much as before.

If $f : (X, A) \to (Y, B)$ is a map of pairs then f induces a homomorphism $f_* : H_q(X, A) \to H_q(Y, B)$.

In the next diagram we have two interlocking sequences of groups forming an infinite sequence of short exact sequences. Beginning with a chain $c \in Z_{q+1}(X, A)$ we can pull this back to a chain $b \in S_{q+1}(X)$, map it to $\partial(b) \in S_q(X)$, and use exactness of that row to find $a \in S_q(A)$. One can show that this a is a cycle so determines a homology class $[a] \in H_q(A)$. Furthermore if $c' \in Z_{q+1}(X, A)$ and $[c] = [c']$ then $[a] = [a']$ too. Linearity is guaranteed so that the assignment of $[a]$ to $[c]$ gives a homomorphism $\partial_* : H_{q+1}(X, A) \to H_q(A)$.

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & S_{q+1}(A) \arrow[r] \arrow[d] & S_{q+1}(X) \arrow[r] \arrow[d] & S_{q+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_q(A) \arrow[r] \arrow[d] & S_q(X) \arrow[r] \arrow[d] & S_q(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_{q-1}(A) \arrow[r] \arrow[d] & S_{q-1}(X) \arrow[r] \arrow[d] & S_{q-1}(X,A) \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

Now we have an exact sequence

$$\ldots \to H_q(A) \xrightarrow{i_*} H_q(X) \xrightarrow{j_*} H_q(X,A) \xrightarrow{\partial_*} H_{q-1}(A) \to \ldots.$$

Exactness of this sequence involves quite a lot of diagram chasing using the double sequence above.

The proofs that the Homotopy and Excision Axioms are satisfied is quite complicated. The essential idea for the latter involves subdivision of a singular simplex. For a given singular simplex we chop the standard simplex into small enough pieces, each affinely homeomorphic to the standard simplex, in such a way that the image of each small simplex meeting U lies in A: this comes from the open cover $\{\mathring{A}, X - \overline{U}\}$. We are then able to drop off those small singular simplices which meet $\overline{U}$ because they lie inside the subgroup $i_\sharp(S_q(A))$.

The proof that the Dimension Axiom is satisfied is simple. Let 0 denote the one-point space. For each $q \geq 0$ there is only one singular simplex $\sigma_q : \Delta^q \to 0$, the constant map. Thus $S_q(0) \approx \mathbb{Z}$. Further $V_i^q = V_j^q$, so we have $\partial_q = 0$ when q is odd (because there are $q+1$ summands) while ∂_q is the identity when $q > 0$ is even. Of course for $q \leq 0$ we must have $\partial_q = 0$ too. So we get:

$$\begin{array}{ccccccccccccccc}
\cdots & \longrightarrow & S_4(0) & \longrightarrow & S_3(0) & \longrightarrow & S_2(0) & \longrightarrow & S_1(0) & \longrightarrow & S_0(0) & \longrightarrow & S_{-1}(0) & \longrightarrow & \cdots \\
\cdots & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{1} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{1} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{0} & 0 & \xrightarrow{0} & \cdots
\end{array}$$

Inspecting this sequence we see that for $q > 0$ and odd $B_q(0) = Z_q(0) \approx \mathbb{Z}$, for $q < 0$ or for $q > 0$ and even $B_q(0) = Z_q(0) = 0$, while for $q = 0$ we have $Z_0(0) \approx \mathbb{Z}$ but $B_0(0) = 0$.

Exercises

1. Suppose that the sequence $0 \to G \xrightarrow{\theta} H \xrightarrow{\varphi} K \to 0$ of abelian groups and homomorphisms is exact and that there is a homomorphism $\alpha : H \to G$ such that $\alpha\theta : G \to G$ is the identity. Prove that $H \approx G \oplus K$.

2. Suppose that the sequence $0 \to G \xrightarrow{\theta} H \xrightarrow{\varphi} K \to 0$ of abelian groups and homomorphisms is exact and that there is a homomorphism $\beta : K \to H$ such that $\varphi\beta : K \to K$ is the identity. Prove that $H \approx G \oplus K$.

3. Prove Lemma 2.2.
4. Prove Proposition 2.10.
5. Suppose that $p : E \to X$ and $q : X \to Y$ are both covering projections and that for each $y \in Y$ the set $q^{-1}(y)$ is finite. Prove that $qp : E \to Y$ is also a covering projection. Can the assumption that $q^{-1}(y)$ is finite be omitted?
6. Suppose that X is a path connected space. Then X is called *simply connected* provided that $\pi(X, a)$ is trivial for any choice of $a \in X$. Suppose that $X = U \cup V$ where

   - each of U and V is open;
   - each of U and V is simply connected;
   - $U \cap V$ is non-empty and path connected.

   Prove that X is simply connected.

   [Hint: Let $\sigma : [0, 1] \to X$ be a loop based at $a \in U \cap V$: it needs to be shown that σ is homotopic to the constant loop based at a though loops based at a. Show that there is a partition $\{0 = t_0 < t_1 < \ldots < t_n = 1\}$ of $[0, 1]$ such that for each $i = 1, \ldots, n$ the short path $\sigma([t_{i-1}, t_i]$ is a subset of either U or V. For each such i choose a path τ_i in $U \cap V$ from a to $\sigma(t_i)$. Use simple connectedness of U and V to show that each loop $\tau_{i-1} * \sigma|[t_{i-1}, t_i] * \overline{\tau_i}$ is homotopic to the constant loop based at a.]
7. Prove that $\mathbb{S}^n$ is simply connected if $n > 1$.
8. Let $\mathbb{S}^n$ be the unit sphere in $\mathbb{R}^{n+1}$ and define $\sim$ on $\mathbb{S}^n$ by $x \sim y$ if and only if $x = \pm y$. Then the quotient space is $\mathbb{P}^n$; denote the quotient map by $q : \mathbb{S}^n \to \mathbb{P}^n$. Prove that q is a covering projection.
9. By following the ideas of the proof of Theorem 4.1 prove that $\pi(\mathbb{P}^n) \approx \mathbb{Z}_2$ when $n > 1$.
10. Prove Theorem 6.6.
11. Prove Theorem 6.10.
12. Prove Theorem 6.12.
13. Let R_n be the n petalled rose, which is the following quotient space. Take the subspace C_n of $\mathbb{R}^2$ consisting of the n disjoint circles

    $$\{(x, y) \in \mathbb{R}^2 \ / \ (x - 3i)^2 + (y - 1)^2 = 1 \text{ for some } i = 1, \ldots, n\},$$

    define $\sim$ on C_n by $(x, y) \sim (\xi, \eta)$ if either $(x, y) = (\xi, \eta)$ or $y = \eta = 0$, and declare R_n to be the quotient space $C_n/\sim$. For example R_3 is homeomorphic to the subspace $\{(r, \theta) \ / \ r = \sin 3\theta\}$ (using polar coordinates).

    Prove that

    $$H_q(R_n) \approx \begin{cases} \mathbb{Z} & \text{if } q = 0 \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z} & \text{if } q = 1 \\ 0 & \text{if } q > 1. \end{cases}$$