Introduction to Complex Analysis

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Subsection 1

Complex Numbers and Their Properties

Complex Numbers

- The **imaginary unit** $i = \sqrt{-1}$ is defined by the property $i^2 = -1$.

Definition (Complex Number)

A **complex number** is any number of the form $z = a + ib$ where a and b are real numbers and i is the imaginary unit.

- The notations $a + ib$ and $a + bi$ are used interchangeably.
- The real number a in $z = a + ib$ is called the **real part** of z and the real number b is called the **imaginary part** of z.
- The real and imaginary parts of a complex number z are abbreviated $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, respectively.

  Example: If $z = 4 - 9i$, then $\mathrm{Re}(z) = 4$ and $\mathrm{Im}(z) = -9$.
- A real constant multiple of the imaginary unit is called a **pure imaginary number**.

  Example: $z = 6i$ is a pure imaginary number.

Equality of Complex Numbers

- Two complex numbers are equal if the corresponding real and imaginary parts are equal.

Definition (Equality)

Complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are **equal**, written $z_1 = z_2$, if $a_1 = a_2$ and $b_1 = b_2$.

- In terms of the symbols $\text{Re}(z)$ and $\text{Im}(z)$, we have

$$z_1 = z_2 \quad \text{if} \quad \text{Re}(z_1) = \text{Re}(z_2) \text{ and } \text{Im}(z_1) = \text{Im}(z_2).$$

- The totality of complex numbers or the **set of complex numbers** is usually denoted by the symbol $\mathbb{C}$.
- Because any real number a can be written as $z = a + 0i$, the set $\mathbb{R}$ of real numbers is a subset of $\mathbb{C}$.

Arithmetic Operations

- If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, the operations of addition, subtraction, multiplication and division are defined as follows:
  - **Addition**:
$$z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2).$$
  - **Subtraction**:
$$z_1 - z_2 = (a_1 + ib_1) - (a_2 + ib_2) = (a_1 - a_2) + i(b_1 - b_2).$$
  - **Multiplication**:
$$z_1 \cdot z_2 = (a_1 + ib_1)(a_2 + ib_2) = a_1 a_2 - b_1 b_2 + i(b_1 a_2 + a_1 b_2).$$
  - **Division**:
$$\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2} = \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i \frac{b_1 a_2 - a_1 b_2}{a_2^2 + b_2^2}.$$

Laws of Arithmetic

- The familiar commutative, associative, and distributive laws hold for complex numbers:
  - **Commutative laws**:

$$\begin{aligned} z_1 + z_2 &= z_2 + z_1 \\ z_1 z_2 &= z_2 z_1 \end{aligned}$$

  - **Associative laws**:

$$\begin{aligned} z_1 + (z_2 + z_3) &= (z_1 + z_2) + z_3 \\ z_1(z_2 z_3) &= (z_1 z_2) z_3 \end{aligned}$$

  - **Distributive law**:

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$$

- In view of these laws, there is no need to memorize the definitions of addition, subtraction, and multiplication.

How to Add, Subtract and Multiply

- Addition, Subtraction, and Multiplication can be performed as follows:
  (i) To add (subtract) two complex numbers, simply add (subtract) the corresponding real and imaginary parts.
  (ii) To multiply two complex numbers, use the distributive law and the fact that $i^2 = -1$.
- Example: If $z_1 = 2 + 4i$ and $z_2 = -3 + 8i$, find
  (a) $z_1 + z_2$; (b) $z_1 z_2$.
  (a) By adding real and imaginary parts, the sum of the two complex numbers z_1 and z_2 is

$$z_1 + z_2 = (2 + 4i) + (-3 + 8i) = (2 - 3) + (4 + 8)i = -1 + 12i.$$

  (b) By the distributive law and $i^2 = -1$, the product of z_1 and z_2 is

$$\begin{aligned} z_1 z_2 &= (2 + 4i)(-3 + 8i) = (2 + 4i)(-3) + (2 + 4i)(8i) \\ &= -6 - 12i + 16i + 32i^2 = (-6 - 32) + (16 - 12)i \\ &= -38 + 4i. \end{aligned}$$

Zero and Unity

- The **zero** in the complex number system is the number $0 + 0i$;
- The **unity** is $1 + 0i$.
- The zero and unity are denoted by 0 and 1, respectively.
- The zero is the **additive identity** in the complex number system: For any complex number $z = a + ib$,

$$z + 0 = (a + ib) + (0 + 0i) = a + ib = z.$$

- Similarly, the unity is the **multiplicative identity**: For any complex number $z = a + ib$, we have

$$z \cdot 1 = (a + ib)(1 + 0i) = a + ib = z.$$

Conjugates

Definition (Conjugate)

If z is a complex number, the number obtained by changing the sign of its imaginary part is called the **complex conjugate**, or simply **conjugate**, of z and is denoted by the symbol $\bar{z}$. In other words, if $z = a + ib$, then its conjugate is $\bar{z} = a - ib$.

- Example: If $z = 6 + 3i$, then $\bar{z} = 6 - 3i$. If $z = -5 - i$, then $\bar{z} = -5 + i$.
- If z is a real number, then $\bar{z} = z$.
- The conjugate of a sum and difference of two complex numbers is the sum and difference of the conjugates:

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2, \quad \overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2.$$

More Properties of Conjugates

- Moreover, we have the following three additional properties:
$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2, \quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, \quad \bar{\bar{z}} = z.$$
- The sum and product of a complex number z with its conjugate $\bar{z}$ is a real number:
$$\begin{array}{rcl} z + \bar{z} & = & (a + ib) + (a - ib) = 2a; \\ z\bar{z} & = & (a + ib)(a - ib) = a^2 - i^2 b^2 = a^2 + b^2. \end{array}$$
- The difference of a complex number z with its conjugate $\bar{z}$ is a pure imaginary number:
$$z - \bar{z} = (a + ib) - (a - ib) = 2ib.$$
- We obtain
$$\mathrm{Re}(z) = \frac{z + \bar{z}}{2}; \quad \mathrm{Im}(z) = \frac{z - \bar{z}}{2i}.$$

How to Divide

- To divide z_1 by z_2:
  - multiply the numerator and denominator of $\frac{z_1}{z_2}$ by the conjugate of z_2.

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \cdot \frac{\bar{z}_2}{\bar{z}_2} = \frac{z_1 \bar{z}_2}{z_2 \bar{z}_2};$$

  - Then use the fact that $z_2 \bar{z}_2$ is the sum of the squares of the real and imaginary parts of z_2.
- Example: If $z_1 = 2 - 3i$ and $z_2 = 4 + 6i$, find $\frac{z_1}{z_2}$.

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{2-3i}{4+6i} = \frac{2-3i}{4+6i} \cdot \frac{4-6i}{4-6i} = \frac{8-12i-12i+18i^2}{4^2+6^2} \\ &= \frac{-10-24i}{52} = -\frac{10}{52} - \frac{24}{52} i = -\frac{5}{26} - \frac{6}{13} i. \end{aligned}$$

Additive and Multiplicative Inverses

- In the complex number system, every number z has a unique **additive inverse**: The additive inverse of $z = a + ib$ is its negative, $-z$, where $-z = -a - ib$.

  For any complex number z, we have $z + (-z) = 0$.
- Similarly, every nonzero complex number z has a **multiplicative inverse**: For $z \neq 0$, there exists one and only one nonzero complex number z^{-1} such that $zz^{-1} = 1$. The multiplicative inverse z^{-1} is the same as the **reciprocal** $\frac{1}{z}$.
- Example: Find the reciprocal of $z = 2 - 3i$ and put the answer in the form $a + ib$.

$$\frac{1}{z} = \frac{1}{2-3i} = \frac{1}{2-3i} \cdot \frac{2+3i}{2+3i} = \frac{2+3i}{4+9} = \frac{2+3i}{13}.$$

$$\text{Therefore, } \frac{1}{z} = z^{-1} = \frac{2}{13} + \frac{3}{13}i.$$

Comparison with Real Analysis

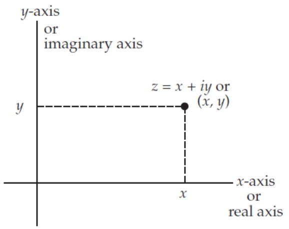
- Many of the properties of the real number system $\mathbb{R}$ hold in the complex number system $\mathbb{C}$, but there are some truly remarkable differences as well:
  (i) For example, the concept of order in the real number system does not carry over to the complex number system: We cannot compare two complex numbers $z_1 = a_1 + ib_1$, $b_1 \neq 0$, and $z_2 = a_2 + ib_2, b_2 \neq 0$, by means of inequalities.
  (ii) Some things that we take for granted as impossible in real analysis, such as $e^x = -2$ and $\sin x = 5$ when x is a real variable, are perfectly correct and ordinary in complex analysis when the symbol x is interpreted as a complex variable.

Subsection 2

Complex Plane

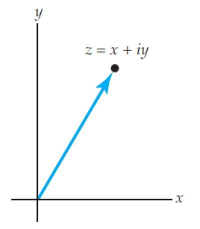
Complex Numbers and Points

- A complex number $z = x + iy$ is uniquely determined by an ordered pair of real numbers (x, y).
- The first and second entries of the ordered pairs correspond, in turn, to the real and imaginary parts of the complex number.
- Example: The ordered pair $(2, -3)$ corresponds to the complex number $z = 2 - 3i$. Conversely, $z = 2 - 3i$ determines the ordered pair $(2, -3)$. The numbers $7, i$ and $-5i$ are equivalent to $(7, 0), (0, 1), (0, -5)$ respectively.
- Because of the correspondence between a complex number $z = x + iy$ and one and only one point (x, y) in a coordinate plane, we shall use the terms complex number and point interchangeably.



Complex Numbers and Vectors: Modulus

- A complex number $z = x + iy$ can also be viewed as a two-dimensional position vector, i.e., a vector whose initial point is the origin and whose terminal point is the point (x, y).



Definition (Modulus of a Complex Number)

The **modulus** of a complex number $z = x + iy$, is the real number $|z| = \sqrt{x^2 + y^2}$.

- The modulus $|z|$ of a complex number z is also called the **absolute value** of z.
- Example: If $z = 2 - 3i$, then $|z| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$. If $z = -9i$, then $|-9i| = \sqrt{(-9)^2} = 9$.

Properties of the Modulus

- For any complex number $z = x + iy$, the product $z\bar{z}$ is the sum of the squares of the real and imaginary parts of z:

$$z\bar{z} = x^2 + y^2.$$

  This yields the relations:

$$|z|^2 = z\bar{z} \quad \text{and} \quad |z| = \sqrt{z\bar{z}}.$$

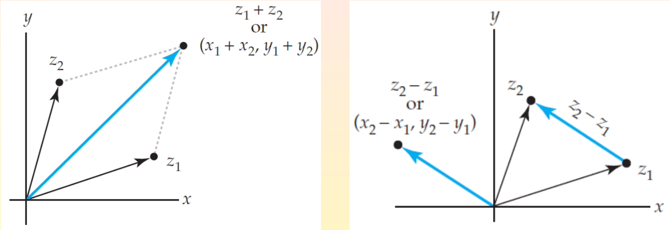
- The modulus of a complex number z has the additional properties:

$$|z_1 z_2| = |z_1||z_2| \quad \text{and} \quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$

  In particular, when $z_1 = z_2 = z$, we get $|z^2| = |z|^2$.

Addition and Subtraction Geometrically

- The addition of complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ takes the form $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$, i.e., it is simply the component definition of vector addition.



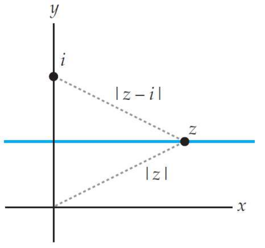
- The difference $z_2 - z_1$ can be drawn either starting from the terminal point of z_1 and ending at the terminal point of z_2, or as the position vector with terminal point $(x_2 - x_1, y_2 - y_1)$.
- Thus, the distance between $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is the same as the distance between the origin and $(x_2 - x_1, y_2 - y_1)$.

Sets of Points in the Complex Plane

- Example: Describe the set of points z in the complex plane that satisfy $|z| = |z - i|$.

  The given equation asserts that the distance from a point z to the origin equals the distance from z to the point i. Thus, the set of points z is a horizontal line:
  $|z| = |z - i| \Leftrightarrow \sqrt{x^2 + y^2} = \sqrt{x^2 + (y-1)^2} \Leftrightarrow x^2 + y^2 = x^2 + (y-1)^2 \Leftrightarrow x^2 + y^2 = x^2 + y^2 - 2y + 1.$

  Thus, $y = \frac{1}{2}$, which is an equation of a horizontal line. Complex numbers satisfying $|z| = |z - i|$ can be written as $z = x + \frac{1}{2} i$.



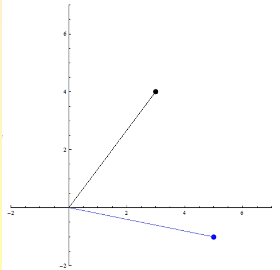
Comparing Moduli

- Since $|z|$ is a real number, we can compare the absolute values of two complex numbers.
- Example: If $z_1 = 3 + 4i$ and $z_2 = 5 - i$, then

$$|z_1| = \sqrt{25} = 5 \quad \text{and} \quad |z_2| = \sqrt{26}$$

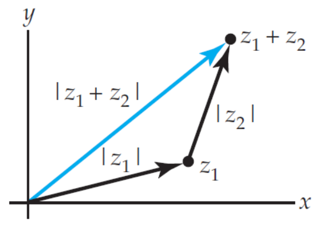
and, consequently, $|z_1| < |z_2|$.

A geometric interpretation of the last inequality is that the point $(3, 4)$ is closer to the origin than the point $(5, -1)$.



The Triangle Inequality

- Consider the triangle



The length of the side of the triangle corresponding to $z_1 + z_2$ cannot be longer than the sum of the lengths of the remaining two sides. In symbols

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

- From the identity $z_1 = z_1 + z_2 + (-z_2)$, we get $|z_1| = |z_1 + z_2 + (-z_2)| \leq |z_1 + z_2| + |-z_2| = |z_1 + z_2| + |z_2|$. Hence $|z_1 + z_2| \geq |z_1| - |z_2|$. Because $z_1 + z_2 = z_2 + z_1$, $|z_1 + z_2| = |z_2 + z_1| \geq |z_2| - |z_1| = -(|z_1| - |z_2|)$. Combined with the last result, this implies
$$|z_1 + z_2| \geq ||z_1| - |z_2||.$$

The Triangle Inequality: More Consequences

- We have shown that
$$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|.$$
- By replacing z_2 by $-z_2$, we get
$|z_1 + (-z_2)| \leq |z_1| + |(-z_2)| = |z_1| + |z_2|$, i.e.,
$$|z_1 - z_2| \leq |z_1| + |z_2|.$$
- Replacing z_2 by $-z_2$, we also find
$$|z_1 - z_2| \geq ||z_1| - |z_2||.$$
- The triangle inequality extends to any finite sum of complex numbers:
$$|z_1 + z_2 + z_3 + \cdots + z_n| \leq |z_1| + |z_2| + |z_3| + \cdots + |z_n|.$$

Establishing Upper Bounds

- Find an upper bound for $\left|\frac{-1}{z^4-5z+1}\right|$ if $|z|=2$.

  Since the absolute value of a quotient is the quotient of the absolute values and $|-1|=1$, $\left|\frac{-1}{z^4-5z+1}\right| = \frac{1}{|z^4-5z+1|}$. Thus, we want to find a positive real number M such that $\frac{1}{|z^4-5z+1|} \leq M$. To accomplish this task we want the denominator as small as possible. We have

  $$|z^4-5z+1| = |z^4-(5z-1)| \geq ||z^4|-|5z-1||.$$

  To make the difference in the last expression as small as possible, we want to make $|5z-1|$ as large as possible. We have

  $$|5z-1| \leq |5z|+|-1| = 5|z|+1.$$

  Using $|z|=2$,

  $$|z^4-5z+1| \geq ||z^4|-|5z-1|| \geq ||z|^4-(5|z|+1)| = ||z|^4-5|z|-1| = 5.$$

  Hence for $|z|=2$, we have $\frac{1}{|z^4-5z+1|} \leq \frac{1}{5}$.

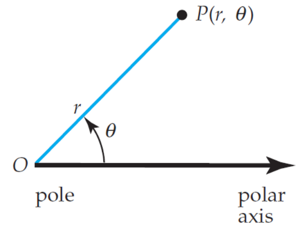
Subsection 3

Polar Form of Complex Numbers

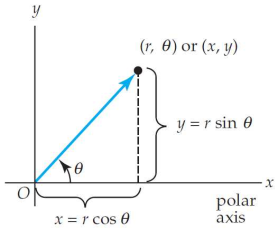
Polar Coordinates

- A point P in the plane whose rectangular coordinates are (x, y) can also be described in terms of polar coordinates.
- The polar coordinate system consists of
  - a point O called the **pole**;
  - the horizontal half-line emanating from the pole called the **polar axis**.
- If
  - r is the directed distance from the pole to P,
  - θ an angle (in radians) measured from the polar axis to the line OP,

  then the point P can be described by the ordered pair (r, θ), called the **polar coordinates** of P:



The Polar Form of a Complex Number

- 

  Suppose that a polar coordinate system is superimposed on the complex plane with
  - the pole O at the origin;
  - the polar axis coinciding with the positive x-axis.
- Then x, y, r and θ are related by $x = r\cos\theta$, $y = r\sin\theta$.
- These equations enable us to express a nonzero complex number $z = x + iy$ as

$$z = (r\cos\theta) + i(r\sin\theta) \quad \text{or} \quad z = r(\cos\theta + i\sin\theta).$$

  This is called the **polar form** or **polar representation** of the complex number z.

The Polar Form of a Complex Number

- In the polar form $z = r(\cos\theta + i\sin\theta)$, the coordinate r can be interpreted as the distance from the origin to the point (x, y).
- We adopt the convention that r is never negative so that we can take r to be the modulus of z: $r = |z|$.
- The angle θ of inclination of the vector z, always measured in radians from the positive real axis, is positive when measured counterclockwise and negative when measured clockwise.
- The angle θ is called an **argument** of z and is denoted by $\theta = \arg(z)$.
- An argument θ of a complex number must satisfy the equations

$$\cos\theta = \frac{x}{r} \quad \text{and} \quad \sin\theta = \frac{y}{r}.$$

- An argument of a complex number z is not unique since $\cos\theta$ and $\sin\theta$ are 2π-periodic.

Example: Expressing a Complex Number in Polar Form

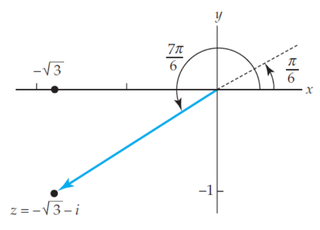
- Express $-\sqrt{3}-i$ in polar form.

  With $x=-\sqrt{3}$ and $y=-1$, we obtain

$$r=|z|=\sqrt{(-\sqrt{3})^2+(-1)^2}=2.$$

  Now $\frac{y}{x}=\frac{-1}{-\sqrt{3}}=\frac{1}{\sqrt{3}}$. We know that $\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}$.

  However, the point $(-\sqrt{3},-1)$ lies in the third quadrant, whence, we take the solution of $\tan\theta=\frac{-1}{-\sqrt{3}}=\frac{1}{\sqrt{3}}$ to be $\theta=\arg(z)=\frac{\pi}{6}+\pi=\frac{7\pi}{6}$.



  It follows that a polar form of the number is $z=2(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6})$.

The Principal Argument

- The symbol $\arg(z)$ represents a set of values, but the argument θ of a complex number that lies in the interval $-\pi < \theta \leq \pi$ is called the **principal value** of $\arg(z)$ or the **principal argument** of z.
- The principal argument of z is unique and is represented by the symbol $\mathrm{Arg}(z)$, that is,
$$-\pi < \mathrm{Arg}(z) \leq \pi.$$
- Example: If $z = i$, some values of $\arg(i)$ are $\frac{\pi}{2}, \frac{5\pi}{2}, -\frac{3\pi}{2}$, and so on. However, $\mathrm{Arg}(i) = \frac{\pi}{2}$.
Similarly, the argument of $-\sqrt{3} - i$ that lies in the interval $(-\pi, \pi)$, the principal argument of z, is $\mathrm{Arg}(z) = \frac{\pi}{6} - \pi = -\frac{5\pi}{6}$. Using $\mathrm{Arg}(z)$, we can express this complex number in the alternative polar form: $z = 2(\cos(-\frac{5\pi}{6}) + i \sin(-\frac{5\pi}{6}))$.
- In general, $\arg(z)$ and $\mathrm{Arg}(z)$ are related by
$$\arg(z) = \mathrm{Arg}(z) + 2\pi n, \ \ n = 0, \pm 1, \pm 2, \ldots.$$

Multiplying and Dividing in Polar Form

- Suppose $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, where θ_1 and θ_2 are any arguments of z_1 and z_2, respectively.
- Then

  $$z_1 z_2 = r_1 r_2[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$

  $$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)].$$
- From the addition formulas for the cosine and sine, we get

  $$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

  and

  $$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)].$$
- The lengths of $z_1 z_2$ and $\frac{z_1}{z_2}$ are the product of the lengths of z_1 and z_2 and the quotient of the lengths of z_1 and z_2, respectively.
- The arguments of $z_1 z_2$ and $\frac{z_1}{z_2}$ are given by $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ and $\arg(\frac{z_1}{z_2}) = \arg(z_1) - \arg(z_2)$.

Example of Multiplication and Division in Polar Form

- We have seen that for $z_1 = i$ and $z_2 = -\sqrt{3} - i$, $\text{Arg}(z_1) = \frac{\pi}{2}$ and $\text{Arg}(z_2) = -\frac{5\pi}{6}$, respectively. Thus, arguments for the product and quotient $z_1 z_2 = i(-\sqrt{3} - i) = 1 - \sqrt{3}i$ and $\frac{z_1}{z_2} = \frac{i}{-\sqrt{3}-i} = \frac{-1}{4} - \frac{\sqrt{3}}{4}i$ are:
$$\arg(z_1 z_2) = \frac{\pi}{2} + (-\frac{5\pi}{6}) = -\frac{\pi}{3}$$
and
$$\arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2} - (-\frac{5\pi}{6}) = \frac{4\pi}{3}.$$

Integer Powers of a Complex Number

- We can find integer powers of a complex number z from the multiplication and division formulas.
- If $z = r(\cos\theta + i\sin\theta)$, then
$$z^2 = r^2[\cos(\theta+\theta) + i\sin(\theta+\theta)] = r^2(\cos 2\theta + i\sin 2\theta).$$
- Since $z^3 = z^2 z$, we also get
$$z^3 = r^3(\cos 3\theta + i\sin 3\theta), \text{ and so on.}$$
- For negative powers, taking $\arg(1) = 0$,
$$\frac{1}{z^2} = z^{-2} = r^{-2}[\cos(-2\theta) + i\sin(-2\theta)].$$
- A general formula for the n-th power of z, for any integer n, is
$$z^n = r^n(\cos n\theta + i\sin n\theta).$$
- When $n = 0$, we get $z^0 = 1$.

Calculating the Power of a Complex Number

- Compute z^3 for $z = -\sqrt{3} - i$.

  A polar form of the given number is $z = 2[\cos(\frac{7\pi}{6}) + i\sin(\frac{7\pi}{6})]$. Using the previous formula, with $r = 2$, $\theta = \frac{7\pi}{6}$, and $n = 3$, we get

$$\begin{aligned} z^3 &= (-\sqrt{3} - i)^3 \\ &= 2^3 \left(\cos(3\frac{7\pi}{6}) + i\sin(3\frac{7\pi}{6})\right) \\ &= 8\left(\cos(\frac{7\pi}{2}) + i\sin(\frac{7\pi}{2})\right) \\ &= -8i, \end{aligned}$$

  since $\cos(\frac{7\pi}{2}) = 0$ and $\sin(\frac{7\pi}{2}) = -1$.

De Moivre's Formula

- When $z = \cos\theta + i\sin\theta$, we have $|z| = r = 1$, whence, we obtain **de Moivre's Formula**:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

- Example: If $z = \frac{\sqrt{3}}{2} + \frac{1}{2}i$, calculate z^3.

  Since $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and $\sin\frac{\pi}{6} = \frac{1}{2}$, we get:

$$\begin{array}{rcl} z^3 & = & (\frac{\sqrt{3}}{2} + \frac{1}{2}i)^3 \\ & = & (\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})^3 \\ & = & \cos(3\frac{\pi}{6}) + i\sin(3\frac{\pi}{6}) \\ & = & \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \\ & = & i. \end{array}$$

Some Remarks

(i) It is not true, in general, that $\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$ and $\mathrm{Arg}(\frac{z_1}{z_2}) = \mathrm{Arg}(z_1) - \mathrm{Arg}(z_2)$.

(ii) An argument can be assigned to any nonzero complex number z. However, for $z = 0$, $\arg(z)$ cannot be defined in any way that is meaningful.

(iii) If we take $\arg(z)$ from the interval $(-\pi, \pi)$, the relationship between a complex number z and its argument is single-valued; i.e., every nonzero complex number has precisely one angle in $(-\pi, \pi)$.
But there is nothing special about the interval $(-\pi, \pi)$.
For the interval $(-\pi, \pi)$, the negative real axis is analogous to a barrier that we agree not to cross (called a **branch cut**). If we use $(0, 2\pi)$ instead of $(-\pi, \pi)$, the branch cut is the positive real axis.

(iv) The "cosine i sine" part of the polar form of a complex number is sometimes abbreviated cis, i.e., $z = r(\cos\theta + i\sin\theta) = r\mathrm{cis}\theta$.

Subsection 4

Powers and Roots

n-th Complex Roots of a Complex Number

- Recall from algebra that -2 and 2 are said to be square roots of the number 4 because $(-2)^2 = 4$ and $(2)^2 = 4$.
- In other words, the two square roots of 4 are distinct solutions of the equation $w^2 = 4$.
- Similarly, $w = 3$ is a cube root of 27 since $w^3 = 3^3 = 27$.
- In general, we say that a number w is an ***n*-th root** of a nonzero complex number z if $w^n = z$, where n is a positive integer.
- Example: $w_1 = \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}i$ and $w_2 = -\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}i$ are the two square roots of the complex number $z = i$.
- We will demonstrate that there are exactly n solutions of the equation $w^n = z$.

Roots of a Complex Number

- Suppose $z = r(\cos\theta + i\sin\theta)$ and $w = \rho(\cos\phi + i\sin\phi)$ are polar forms of the complex numbers z and w.
- $w^n = z$ becomes $\rho^n(\cos n\phi + i\sin n\phi) = r(\cos\theta + i\sin\theta)$.
- We can conclude that $\rho^n = r$ and $\cos n\phi + i\sin n\phi = \cos\theta + i\sin\theta$.
- Let $\rho = \sqrt[n]{r}$ be the unique positive n-th root of the real number $r > 0$.
- The definition of equality of two complex numbers implies that $\cos n\phi = \cos\theta$ and $\sin n\phi = \sin\theta$. Thus, the arguments θ and ϕ are related by $n\phi = \theta + 2k\pi$, where k is an integer, i.e., $\phi = \frac{\theta+2k\pi}{n}$.
- As k takes on the successive integer values $k = 0, 1, 2, \ldots, n-1$, we obtain n distinct n-th roots of z.
- These roots have the same modulus $\sqrt[n]{r}$ but different arguments.
- The n nth roots of a nonzero complex number $z = r(\cos\theta + i\sin\theta)$ are given by

$$w_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right], k = 0, 1, \ldots, n-1.$$

Example: Finding Cube Roots

- Find the three cube roots of $z = i$.

  We are solving $w^3 = i$. With $r = 1$, $\theta = \arg(i) = \frac{\pi}{2}$, a polar form of the given number is given by $z = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$. From the previous work, with $n = 3$, we then obtain

$$w_k = \sqrt[3]{1}(\cos\frac{\frac{\pi}{2} + 2k\pi}{3} + i\sin\frac{\frac{\pi}{2} + 2k\pi}{3}), k = 0, 1, 2.$$

  Hence the three roots are,

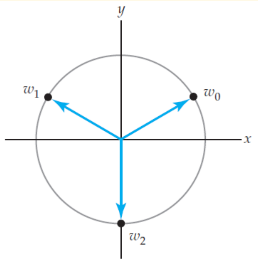
$$\begin{aligned} k &= 0, \quad w_0 = \cos\tfrac{\pi}{6} + i\sin\tfrac{\pi}{6} = \tfrac{\sqrt{3}}{2} + \tfrac{1}{2}i; \\ k &= 1, \quad w_1 = \cos\tfrac{5\pi}{6} + i\sin\tfrac{5\pi}{6} = -\tfrac{\sqrt{3}}{2} + \tfrac{1}{2}i; \\ k &= 2, \quad w_2 = \cos\tfrac{3\pi}{2} + i\sin\tfrac{3\pi}{2} = -i. \end{aligned}$$

The Principal n-th Root

- The symbol $\arg(z)$ really stands for a set of arguments for a complex number z.
- Similarly, $z^{1/n}$ is n-valued and represents the set of n n-th roots w_k of z.
- The unique root of a complex number z obtained by using the principal value of $\arg(z)$, with $k = 0$, is referred to as the **principal n-th root** of w.
- Example: Since $\mathrm{Arg}(i) = \frac{\pi}{2}$ and $w_k = \sqrt[3]{1}(\cos \frac{\frac{\pi}{2}+2k\pi}{3} + i \sin \frac{\frac{\pi}{2}+2k\pi}{3})$, $k = 0, 1, 2$,
$$w_0 = \frac{\sqrt{3}}{2} + \frac{1}{2} i$$
is the principal cube root of i.
- The choice of $\mathrm{Arg}(z)$ and $k = 0$ guarantees that when z is a positive real number r, the principal n-th root is $\sqrt[n]{r}$.

Geometry of the n Complex n-th Roots

- Since the roots have the same modulus, the n n-th roots of a nonzero complex number z lie on a circle of radius $\sqrt[n]{r}$ centered at the origin in the complex plane.
- Since the difference between the arguments of any two successive roots w_k and w_{k+1} is $\frac{2\pi}{n}$, the n nth roots of z are equally spaced on this circle, beginning with the root whose argument is $\frac{\theta}{n}$.
- To illustrate, look at the three cube roots of i:



$$w_0 = \frac{\sqrt{3}}{2} + \frac{1}{2}i;$$
$$w_1 = -\frac{\sqrt{3}}{2} + \frac{1}{2}i;$$
$$w_2 = -i.$$

Example: Fourth Roots of a Complex Number

- The four fourth roots of $z = 1 + i$.

  $r = \sqrt{2}$ and $\theta = \arg(z) = \frac{\pi}{4}$. From our formula, with $n = 4$, we obtain

$$w_k = \sqrt[8]{2}\left[\cos\left(\frac{\frac{\pi}{4} + 2k\pi}{4}\right) + i\sin\left(\frac{\frac{\pi}{4} + 2k\pi}{4}\right)\right], k = 0, 1, 2, 3.$$

  We calculate

$$\begin{aligned}
k = 0, \quad & w_0 = \sqrt[8]{2}(\cos\tfrac{\pi}{16} + i\sin\tfrac{\pi}{16}); \\
k = 1, \quad & w_1 = \sqrt[8]{2}(\cos\tfrac{9\pi}{16} + i\sin\tfrac{9\pi}{16}); \\
k = 2, \quad & w_2 = \sqrt[8]{2}(\cos\tfrac{17\pi}{16} + i\sin\tfrac{17\pi}{16}); \\
k = 3, \quad & w_3 = \sqrt[8]{2}(\cos\tfrac{25\pi}{16} + i\sin\tfrac{25\pi}{16}).
\end{aligned}$$

Remarks on Complex Roots

(i) The complex number system is closed under the operation of extracting roots. This means that for any $z \in \mathbb{C}$, $z^{1/n}$ is also in $\mathbb{C}$. The real number system does not possess a similar closure property since, if x is in $\mathbb{R}$, $x^{1/n}$ is not necessarily in $\mathbb{R}$.

(ii) Geometrically, the n nth roots of a complex number z can also be interpreted as the vertices of a regular polygon with n sides that is inscribed within a circle of radius $\sqrt[n]{r}$ centered at the origin.

(iii) When m and n are positive integers with no common factors, then we may define a rational power of z, i.e., $z^{m/n}$: It can be shown that the set of values $(z^{1/n})^m$ is the same as the set of values $(z^m)^{1/n}$. This set of n common values is defined to be $z^{m/n}$.

Subsection 5

Sets of Points in the Complex Plane

Circles

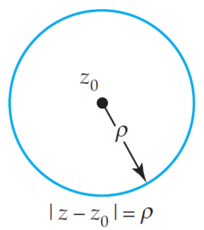
- Suppose $z_0 = x_0 + iy_0$.
- The distance between the points $z = x + iy$ and $z_0 = x_0 + iy_0$ is

$$|z - z_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2}.$$

- Thus, the points $z = x + iy$ that satisfy the equation

$$|z - z_0| = \rho, \rho > 0,$$

lie on a circle of radius ρ centered at the point z_0.



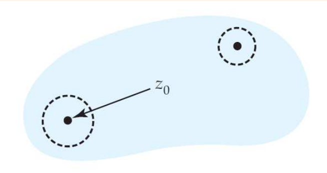
- Example:
  (a) $|z| = 1$ is an equation of a unit circle centered at the origin.
  (b) By rewriting $|z - 1 + 3i| = 5$ as $|z - (1 - 3i)| = 5$, we see that the equation describes a circle of radius 5 centered at the point $z_0 = 1 - 3i$.

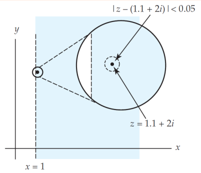
Disks and Neighborhoods

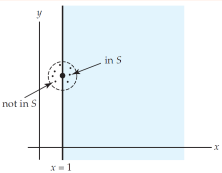
- The points z that satisfy the inequality $|z - z_0| \leq \rho$ can be either on the circle $|z - z_0| = \rho$ or within the circle.
- We say that the set of points defined by $|z - z_0| \leq \rho$ is a **disk** of radius ρ centered at z_0.
- The points z that satisfy the strict inequality $|z - z_0| < \rho$ lie within, and not on, a circle of radius ρ centered at the point z_0. This set is called a **neighborhood** of z_0.
- Occasionally, we will need to use a neighborhood of z_0 that also excludes z_0. Such a neighborhood is defined by the simultaneous inequality $0 < |z - z_0| < \rho$ and called a **deleted neighborhood** of z_0.
- Example: $|z| < 1$ defines a neighborhood of the origin, whereas $0 < |z| < 1$ defines a deleted neighborhood of the origin; $|z - 3 + 4i| < 0.01$ defines a neighborhood of $3 - 4i$, whereas the inequality $0 < |z - 3 + 4i| < 0.01$ defines a deleted neighborhood of $3 - 4i$.

Open Sets

- A point z_0 is called an **interior point** of a set S of the complex plane if there exists some neighborhood of z_0 that lies entirely within S.
- If every point z of a set S is an interior point, then S is said to be an **open set**.

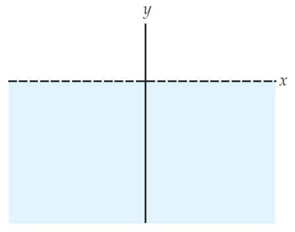




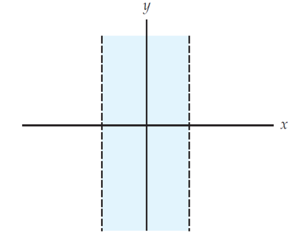


- Example: The inequality $\text{Re}(z) > 1$ defines a right half-plane, which is an open set. All complex numbers $z = x + iy$ for which $x > 1$ are in this set. E.g., if we choose $z_0 = 1.1 + 2i$, then a neighborhood of z_0 lying entirely in the set is defined by $|z - (1.1 + 2i)| < 0.05$.
- Example: The set S of points in the complex plane defined by $\text{Re}(z) \geq 1$ is not open because every neighborhood of a point lying on the line $x = 1$ must contain points in S and points not in S.

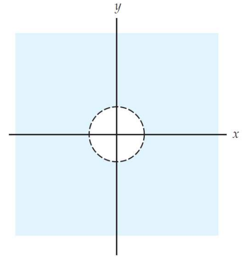
Additional Examples of Open Sets



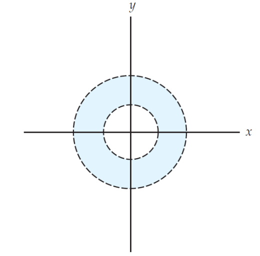
(a) Im(z) < 0; lower half-plane



(b) $-1 <$ Re(z) < 1; infinite vertical strip



(c) $|z| > 1$; exterior of unit circle



(d) $1 < |z| < 2$; interior of circular ring

Boundary and Exterior Points

- If every neighborhood of a point z_0 of a set S contains at least one point of S and at least one point not in S, then z_0 is said to be a **boundary point** of S.

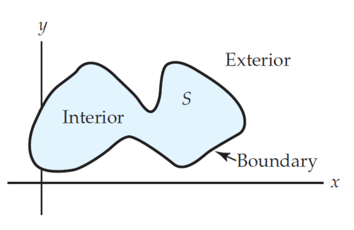
  Example: For the set of points defined by $\mathrm{Re}(z) \geq 1$, the points on the vertical line $x = 1$ are boundary points.

  Example: The points that lie on the circle $|z - i| = 2$ are boundary points for the disk $|z - i| \leq 2$ as well as for the neighborhood $|z - i| < 2$ of $z = i$.
- The collection of boundary points of S is called the **boundary** of S.

  Example: The circle $|z - i| = 2$ is the boundary for both the disk $|z - i| \leq 2$ and the neighborhood $|z - i| < 2$ of $z = i$.
- A point z that is neither an interior point nor a boundary point of a set S is said to be an **exterior point** of S, i.e., z_0 is an exterior point of a set S if there exists some neighborhood of z_0 that contains no points of S.

Interior, Boundary and Exterior Points

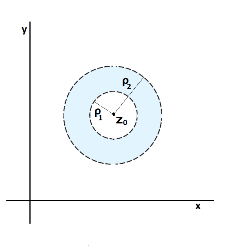
- Typical set S with interior, boundary, and exterior.



- An open set S can be as simple as the complex plane with a single point z_0 deleted.
  - The boundary of this "punctured plane" is z_0;
  - The only candidate for an exterior point is z_0. However, S has no exterior points since no neighborhood of z_0 lies entirely outside the punctured plane.

Annulus

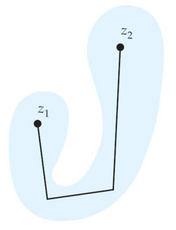
- The set S_1 of points satisfying the inequality $\rho_1 < |z - z_0|$ lie exterior to the circle of radius ρ_1 centered at z_0.
- The set S_2 of points satisfying $|z - z_0| < \rho_2$ lie interior to the circle of radius ρ_2 centered at z_0.
- Thus, if $0 < \rho_1 < \rho_2$, the set of points satisfying the simultaneous inequality $\rho_1 < |z - z_0| < \rho_2$ is the intersection of the sets S_1 and S_2. This intersection is an open circular ring centered at z_0, called an **open circular annulus**.



- By allowing $\rho_1 = 0$, we obtain a deleted neighborhood of z_0.

Connected Sets and Domains

- If any pair of points z_1 and z_2 in a set S can be connected by a polygonal line that consists of a finite number of line segments joined end to end that lies entirely in the set, then the set S is said to be **connected**.



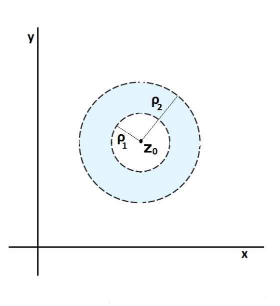
- An open connected set is called a **domain**.
  Example: The set of numbers z satisfying $\text{Re}(z) \neq 4$ is an open set but is not connected: it is not possible to join points on either side of the vertical line $x = 4$ by a polygonal line without leaving the set.
  Example: A neighborhood of a point z_0 is a connected set.

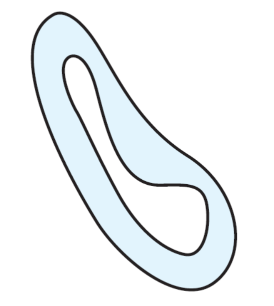
Region

- A **region** is a set of points in the complex plane with all, some, or none of its boundary points.
    - Since an open set does not contain any boundary points, it is automatically a region.
    - A region that contains all its boundary points is said to be **closed**.
- Example: The disk defined by $|z - z_0| \leq \rho$ is an example of a closed region and is referred to as a **closed disk**.
- Example: A neighborhood of a point z_0 defined by $|z - z_0| < \rho$ is an open set or an open region and is said to be an **open disk**.
- If the center z_0 is deleted from either a closed disk or an open disk, the regions defined by $0 < |z - z_0| \leq \rho$ or $0 < |z - z_0| < \rho$ are called **punctured disks**. A punctured open disk is the same as a deleted neighborhood of z_0.
- A region can be neither open nor closed.
  Example: The annular region defined by $1 \leq |z - 5| < 3$ contains only some of its boundary points, and so it is neither open nor closed.

General Annular Regions

- We have defined a circular annular region given by $\rho_1 < |z - z_0| < \rho_2$.



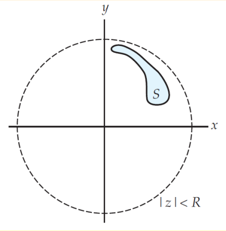


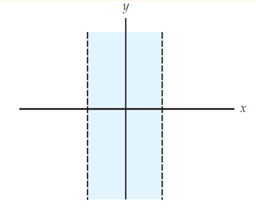
- In a more general interpretation, an **annulus** or **annular region** may have the appearance shown on the right.

Bounded Sets

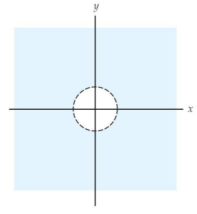
- We say that a set S in the complex plane is **bounded** if there exists a real number $R > 0$ such that $|z| < R$ every z in S, i.e., S is bounded if it can be completely enclosed within some neighborhood of the origin.

  Example: The set S shown below is bounded because it is contained entirely within the dashed circular neighborhood of the origin.





(b) $-1 < \mathrm{Re}(z) < 1$; infinite vertical strip



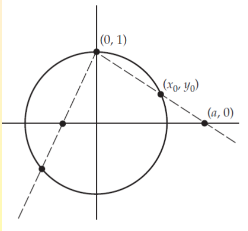
(c) $|z| > 1$; exterior of unit circle

- A set is **unbounded** if it is not bounded.

  Example: The sets on the rightmost figures above are unbounded.

Extended Real Number System

- On the real line, we have exactly two directions and we represent the notions of "increasing without bound" and "decreasing without bound" symbolically by $x \to +\infty$ $x \to -\infty$, respectively.
- We can avoid $\pm\infty$ by dealing with an "ideal point" called the **point at infinity**, which is denoted simply by ∞.
- We identify any real number a with a point (x_0, y_0):

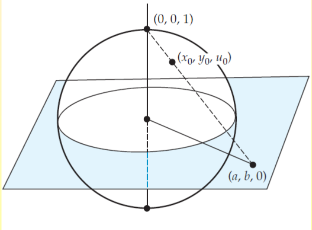


The farther the point $(a, 0)$ is from the origin, the nearer (x_0, y_0) is to $(0, 1)$. The only point on the circle that does not correspond to a real number a is $(0, 1)$. We identify $(0, 1)$ with ∞.

- The set consisting of the real numbers $\mathbb{R}$ adjoined with ∞ is called the **extended real-number system.**

Extended Complex Number System

- Since $\mathbb{C}$ is not ordered, the notions of z either "increasing" or "decreasing" have no meaning.
- By increasing the modulus $|z|$ of a complex number z, the number moves farther from the origin.
- In complex analysis, only the notion of ∞ is used because we can extend the complex number system $\mathbb{C}$ in a manner analogous to that just described for the real number system $\mathbb{R}$.
- We associate a complex number with a point on a unit sphere called the **Riemann sphere**:



Because the point $(0,0,1)$ corresponds to no number z in the plane, we correspond it with ∞. The system consisting of $\mathbb{C}$ adjoined with the "ideal point" ∞ is called the **extended complex-number system**.

Subsection 6

Applications

Complex Roots of Quadratic Equations

- Consider the quadratic equation
$$ax^2 + bx + c = 0,$$
where the coefficients $a \neq 0$, b and c are real.
- Completion of the square in x yields the quadratic formula:
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$
- When $D = b^2 - 4ac < 0$, the roots of the equation are complex.
- Example: The two roots of $x^2 - 2x + 10 = 0$ are
$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(10)}}{2(1)} = \frac{2 \pm \sqrt{-36}}{2}.$$
$\sqrt{-36} = \sqrt{36}\sqrt{-1} = 6i$. Therefore, the complex roots of the equation are
$$z_1 = 1 + 3i, \quad z_2 = 1 - 3i.$$

The Quadratic Formula for Complex Coefficients

- The quadratic formula is perfectly valid when the coefficients $a \neq 0$, b and c of a quadratic polynomial equation
$$az^2 + bz + c = 0$$
are complex numbers.
- Although the formula can be obtained in exactly the same manner, we choose to write the result as
$$z = \frac{-b + (b^2 - 4ac)^{1/2}}{2a}.$$
- When $D = b^2 - 4ac \neq 0$, the symbol $(b^2 - 4ac)^{1/2}$ represents the set of two square roots of the complex number $b^2 - 4ac$.
- Thus, the formula gives two complex solutions.
- In the sequel to keep notation clear, we reserve the use of the symbol $\sqrt{\ }$ to real numbers where $\sqrt{a}$ denotes the nonnegative root of the real number $a \geq 0$.

Using the Quadratic Formula

- Solve the quadratic equation $z^2 + (1-i)z - 3i = 0$.
  Apply the quadratic formulas, with $a = 1$, $b = 1 - i$ and $c = -3i$:
  $$z = \frac{-(1-i) + [(1-i)^2 - 4(-3i)]^{1/2}}{2} = \frac{1}{2}[-1 + i + (10i)^{1/2}].$$
  To compute $(10i)^{1/2}$ we rewrite in polar form with $r = 10$, $\theta = \frac{\pi}{2}$, and use
  $$w_k = \sqrt{r}(\cos\frac{\theta + 2k\pi}{2} + i\sin\frac{\theta + 2k\pi}{2}),\ k = 0, 1.$$
  Thus, the two square roots of $10i$ are:
  $w_0 = \sqrt{10}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) = \sqrt{10}(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i) = \sqrt{5} + \sqrt{5}i$ and
  $w_1 = \sqrt{10}(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}) = \sqrt{10}(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i) = -\sqrt{5} - \sqrt{5}i$.
  Going back to the quadratic formula, we obtain
  $$z_1 = \frac{1}{2}[-1 + i + (\sqrt{5} + \sqrt{5}i)], \quad z_2 = \frac{1}{2}[-1 + i + (-\sqrt{5} - \sqrt{5}i)],$$
  or $z_1 = \frac{1}{2}(\sqrt{5} - 1) + \frac{1}{2}(\sqrt{5} + 1)i$, $z_2 = -\frac{1}{2}(\sqrt{5} + 1) - \frac{1}{2}(\sqrt{5} - 1)i$.

Factoring a Quadratic Polynomial

- By finding all the roots of a polynomial equation we can factor the polynomial completely.
- If z_1 and z_2 are the roots of $az^2 + bz + c = 0$, then $az^2 + bz + c$ factors as
$$az^2 + bz + c = a(z - z_1)(z - z_2).$$
- Example: We found that the quadratic equation $x^2 - 2x + 10 = 0$ has roots $z_1 = 1 + 3i$ and $z_2 = 1 - 3i$. Thus, the polynomial $x^2 - 2x + 10$ factors as
$x^2 - 2x + 10 = [x - (1 + 3i)][x - (1 - 3i)] = (x - 1 - 3i)(x - 1 + 3i).$
- Example: Similarly, $z^2 + (1 - i)z - 3i = (z - z_1)(z - z_2) =$
$[z - \frac{1}{2}(\sqrt{5} - 1) - \frac{1}{2}(\sqrt{5} + 1)i][z + \frac{1}{2}(\sqrt{5} + 1) + \frac{1}{2}(\sqrt{5} - 1)i].$

Differential Equations: The Auxiliary Equation

- The first step in solving a linear second-order ordinary differential equation $ay'' + by' + cy = f(x)$ with real coefficients a, b and c is to solve the **associated homogeneous equation** $ay'' + by' + cy = 0$.
- The latter equation possesses solutions of the form $y = e^{mx}$.
- To see this, we substitute $y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2e^{mx}$ into $ay'' + by' + cy = 0$:
  $ay'' + by' + cy = am^2e^{mx} + bme^{mx} + ce^{mx} = e^{mx}(am^2 + bm + c) = 0.$
- From $e^{mx}(am^2 + bm + c) = 0$, we see that $y = e^{mx}$ is a solution of the homogeneous equation whenever m is root of the polynomial equation $am^2 + bm + c = 0$.
- This equation is known as the **auxiliary equation**.

Differential Equations: Complex Roots of the Auxiliary

- When the coefficients of a polynomial equation are real, the complex roots of the equation must always appear in conjugate pairs.
- Thus, if the auxiliary equation possesses complex roots $\alpha + i\beta$, $\alpha - i\beta$, $\beta > 0$, then two solutions of $ay'' + by' + cy = 0$ are complex exponential functions $y = e^{(\alpha+i\beta)x}$ and $y = e^{(\alpha-i\beta)x}$.
- In order to obtain real solutions of the differential equation, we use Eulers formula $$e^{i\theta} = \cos\theta + i\sin\theta, \ \theta \text{ real}.$$
- We obtain $e^{(\alpha+i\beta)x} = e^{\alpha x}e^{i\beta x} = e^{\alpha x}(\cos\beta x + i\sin\beta x)$ and $e^{(\alpha-i\beta)x} = e^{\alpha x}e^{-i\beta x} = e^{\alpha x}(\cos\beta x - i\sin\beta x)$.
- Since the differential equation is homogeneous, the linear combinations $y_1 = \frac{1}{2}(e^{(\alpha+i\beta)x} + e^{(\alpha-i\beta)x})$, $y_2 = \frac{1}{2i}(e^{(\alpha+i\beta)x} - e^{(\alpha-i\beta)x})$ are also solutions.
- These expressions are real functions $$y_1 = e^{\alpha x}\cos\beta x \quad \text{and} \quad y_2 = e^{\alpha x}\sin\beta x.$$

Solving a Differential Equation

- Solve the differential equation $y'' + 2y' + 2y = 0$.
  We apply the quadratic formula to the auxiliary equation
  $$m^2 + 2m + 2 = 0.$$
  We obtain the complex roots $m_1 = -1 + i$ and $m_2 = \overline{m_1} = -1 - i$. With the identifications $\alpha = -1$ and $\beta = 1$, the preceding formulas give the two solutions
  $$y_1 = e^{-x} \cos x \quad \text{and} \quad y_2 = e^{-x} \sin x.$$
- The general solution of a homogeneous linear n-th-order differential equations consists of a linear combination of n linearly independent solutions.
- Thus, the general solution of the given second-order differential equation is
  $$y = c_1 y_1 + c_2 y_2 = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x,$$
  where c_1 and c_2 are arbitrary constants.

Exponential Form of a Complex Number

- In general, the complex exponential e^z is the complex number defined by
$$e^z = e^{x+iy} = e^x(\cos y + i \sin y).$$
- The definition can be used to show that the familiar law of exponents $e^{z_1} e^{z_2} = e^{z_1+z_2}$ holds for complex numbers.
- This justifies the results presented on differential equations.
- Euler's formula is a special case of this definition.
- Euler's formula provides a notational convenience for several concepts considered earlier in this chapter, e.g., the polar form of z
$$z = r(\cos\theta + i \sin\theta)$$
can now be written compactly as $z = re^{i\theta}$. This convenient form is called the **exponential form** of a complex number z.
- Example: $i = e^{\pi i/2}$ and $1 + i = \sqrt{2}e^{\pi i/4}$.
- Finally, the formula for the n nth roots of a complex number becomes
$$z^{1/n} = \sqrt[n]{r}e^{i(\theta+2k\pi)/n}, \quad k = 0, 1, 2, \dots, n-1.$$