A First Course in Elementary Differential Equations

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Contents

1 Basic Terminology

In many models, we will have equations involving the derivatives of a dependent variable y with respect to one or more independent variables and are interested in discovering this function y. Such equations are referred to as **differential equations** (abbreviated DE). They arise in many applications such as population growth, decay of radioactive substance, the motion of a falling object, electrical network, and many more models that we will discuss throughout this book.

A First Source of Differential Equation: Vertical Motion of an Object

Suppose that an object initially at height y_0 is moving straight up or down with initial velocity v_0. Let $y(t)$ denote the distance of the object from the ground , $v(t)$ the object's velocity, and $a(t)$ the object's acceleration at time t. We assume y to be positive in the upward direction.

If air resistance is neglected, then by Newton's second law which states that the net force is equal to the product of mass and acceleration we have $ma(t) = -mg$. The negative sign on the right-hand of the equation is due to the fact that acceleration due to gravity is pointing downward. Using the fact that $a(t) = y''(t)$ and eliminating the mass, we obtain the equation

$$y'' = -g.$$

To find the velocity $v(t)$ we integrate for a first time and obtain

$$v(t) = -gt + C_1.$$

Since the initial velocity is v_0 then $C_1 = v(0) = v_0$ so that

$$v(t) = -gt + v_0.$$

Integrating for the second time we find the position function

$$y(t) = -\frac{1}{2}gt^2 + v_0 t + C_2.$$

Since y_0 is the initial height then $C_2 = y_0$ and so

$$y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0$$

Example 1.1
An object is dropped from the top of a cliff that is 144 feet about ground level.

(a) When will the object reach ground level?
(b) What is the velocity with which the object strikes the ground?

Solution.
(a) The motion of the object translates to the differential equation $y'' = -32$ with solution $y(t) = -16t^2 + 144$. The object reaches ground level when $y(t) = 0$ or $16t^2 = 144$. Solving for t we find $t = \sqrt{\frac{144}{16}} = 3\ sec$. The object will reach the ground 3 seconds after it is dropped from the tower.
(b) The object strikes the ground with velocity $v(3) = -32(3) = -96\ ft/sec$ ■

Problem 1.1
A ball is thrown straight up from ground level and reaches its greatest height after 5 seconds. Find the initial velocity of the ball and the value of its maximum height above ground level.

Basic Concepts of Differential Equations
We next discuss some basic notions of differential equations. There are two types of differential equations: ordinary and partial differential equations.
By an **ordinary differential equation** (abbreviated ODE) we mean an equation that involves an unknown function (the **dependent variable**) of a single variable, its **independent variable**, and one or more of its derivatives. The highest order derivative that appears in the equation is known as the **order** of the equation. Thus, an **nth order ordinary differential equation** is an equation of the form

$$y^{(n)} = f(t, y, y', \cdots, y^{(n-1)})$$

or

$$G(t, y, y', y'', \cdots, y^n) = 0.$$

A **first-order ordinary differential equation**, for example, takes the form $f(t, y(t), y'(t)) = 0$, and may alternatively be written as

$$y'(t) = g(t, y(t))$$

for all t in the interval of existence of y.
Similarly a **second-order ordinary differential equation** takes the form $f(t, y(t), y'(t), y''(t)) = 0$.

Example 1.2
Determine the order of each equation.

(a) $y' + 2ty = e^{-x^2}$
(b) $\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y(t) = 0$
(c) $y'' + 3ty' + 2y = \sin(5t)$.

Solution.
(a) This is a first order differential equation because the highest derivative is the first derivative.
(b) and (c) are second order differential equations since the highest derivative in each equation is the second order derivative ■

Problem 1.2
Find the order of the following differential equations.

(a) $ty'' + y = t^3$
(b) $y' + y^2 = 2$
(c) $\sin(y''') + 3t^2y = 6t$

Problem 1.3
What is the order of the differential equation?

(a) $y'(t) - 1 = 0$
(b) $y''(t) - 1 = 0$
(c) $y''(t) - 2ty(t) = 0$
(d) $y''(t)(y'(t))^{\frac{1}{2}} - \frac{t}{y(t)} = 0$

When a dependent function is a function of two or more independent variables then the derivatives are known as **partial derivatives**. An equation that involves a function of more than two independent variables and its partial derivatives is called **partial differential equation** (abbreviated PDE). For example, the **wave equation** is a partial differential equation of the form

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0.$$

In this course, when we use the term differential equation, we'll mean an ordinary differential equation.

Problem 1.4
In the equation
$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = x - 2y$$
identify the independent variable(s) and the dependent variable.

Problem 1.5
Classify the following equations as either ordinary or partial.

(a) $(y''')^4 + \frac{t^2}{(y')^2+4} = 0$

(b) $\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{y-x}{y+x}$

(c) $y'' - 4y = 0$

A **solution** of a differential equation is a function that satisfies the equation: When you substitute this function or its derivatives into the differential equation, you get a true mathematical statement.

Example 1.3
Show that the function $y = 100 + e^{-t}$ is a solution to the differential equation
$$y' = 100 - y.$$

Solution.
Indeed, finding the first order derivative of y we have $y' = -e^{-t}$. Also, $100 - y = 100 - (100 + e^{-t}) = -e^{-t}$. Thus, $y' = 100 - y$ so that $y = 100 + e^{-t}$ is a solution to the given DE.$\blacksquare$

Example 1.4 *(A Piecewise Defined Solution)*
Consider the differential equation $ty' - 4y = 0$ on the interval $(-\infty, \infty)$. Verify that the piecewise defined function
$$y = \begin{cases} -t^4, & t < 0 \\ t^4, & t \geq 0 \end{cases}$$
is a solution.

Solution.
For $t < 0$ we have $ty' - 4y = t(-t^4)' - 4(-t^4) = -4t^4 + 4t^4 = 0$. For $t \geq 0$ we have $ty' - 4y = t(t^4)' - 4t^4 = 4t^4 - 4t^4 = 0$. Thus, the given function is a solution $\blacksquare$

Solving a differential equation means finding all possible solutions of the equation.

Example 1.5
Solve the differential equation:

$$y'' = -2t.$$

Solution.
Integrating twice, all the solutions have the form

$$y(t) = -\frac{t^3}{3} + C_1 t + C_2 \blacksquare$$

Note that the function of the previous example defines all the solutions to the differential equation. Such a function will be referred to as the **general solution.** The constants C_1 and C_2 are called the **parameters.** Specific values of C_1 and C_2 determine what is called a **particular solution**. To find a particular solution additional conditions on the values of the function must be given. Such conditions are called **initial conditions**. A differential equation together with a set of initial conditions is called an **initial value problem** (abbreviated IVP).

Example 1.6
Consider the differential equation $y''(t) - 1 = 0$.
(a) Find the general solution of this equation.
(b) Find the solution that satisfies the initial conditions $y(1) = 1$ and $y'(1) = 4$.

Solution.
(a) Integrating twice we find the general solution

$$y(t) = \frac{t^2}{2} + C_1 t + C_2.$$

(b) Since $y'(t) = t + C_1$ and $y'(1) = 4$ then $4 = 1 + C_1$ so that $C_1 = 3$. Hence, $y(t) = \frac{t^2}{2} + 3t + C_2$. Now, since $y(1) = 1$ then $1 = \frac{1}{2} + 3 + C_2$. Solving for C_2 we find $C_2 = -\frac{5}{2}$. Hence, the solution to the IVP

$$\begin{cases} y''(t) - 1 = 0 \\ y'(1) = 4, y(1) = 1 \end{cases}$$

is

$$y(t) = \frac{t^2}{2} + 3t - \frac{5}{2} \blacksquare$$

The graph of a particular solution is called a **solution curve.** The function $y(t) = Ce^{-3t} + 2t + 1$ is the general solution to the differential equation $y' + 3y = 6t + 5$ (See Problem 1.20). A family of solution curves is shown in Figure 1.1. Notice for $C \neq 0$ the solution cruves have an oblique asymptote with equation $y(t) = 2t + 1$.

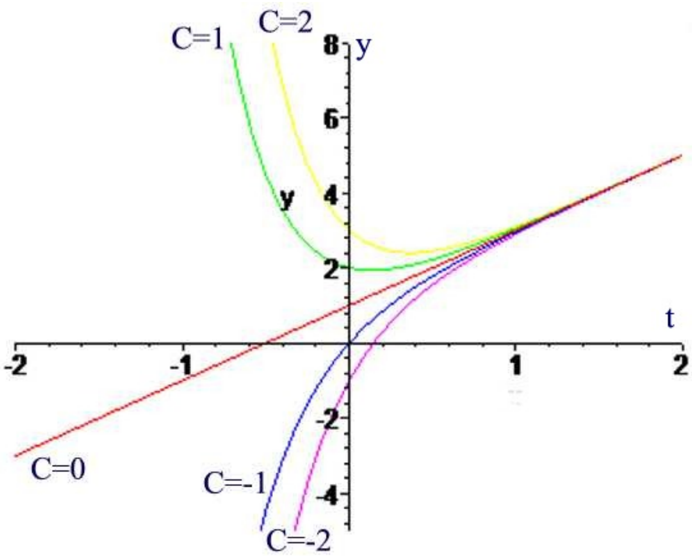


Figure 1.1

Sometimes a differential equation possesses a solution that cannot be obtained by assigning values to the parameters in a family of solutions. Such a solution is called a **singular solution.**

Example 1.7
We will show later on that the nonzero solutions to the differential equation $y' = ty^{\frac{1}{2}}$ are given by $y(t) = (\frac{t^2}{4} + C)^2$. Find the singular solution.

Solution.
The function $y(t) \equiv 0$ is a solution to the differential equation. This is a singular solution since it cannot be obtained from the family for any choice of the parameter C. The general solution consists of all the solutions of the form $y(t) = (\frac{t^2}{4} + C)^2$ together with the zero solution ■

Problem 1.6
Solve the equation $y'''(t) - 2 = 0$ by computing successive antiderivatives.

Problem 1.7
Solve the initial-value problem

$$\frac{dy}{dt} = 3y, \quad y(0) = 50.$$

What is the domain of the solution?

Problem 1.8
For what real value(s) of λ is $y = \cos \lambda t$ a solution of the equation $y'' + 9y = 0$?

Problem 1.9
For what value(s) of m is $y = e^{mt}$ a solution of the equation $y'' + 3y' + 2y = 0$?

Problem 1.10
Show that any function of the form $y(t) = C_1 \cos \omega t + C_2 \sin \omega t$ satisfies the differential equation

$$\frac{d^2y}{dt^2} + \omega y = 0.$$

Problem 1.11
Show that any function of the form $y(t) = C_1 \cos \omega t + C_2 \sin \omega t$ satisfies the differential equation

$$\frac{d^2y}{dt^2} + \omega^2 y = 0.$$

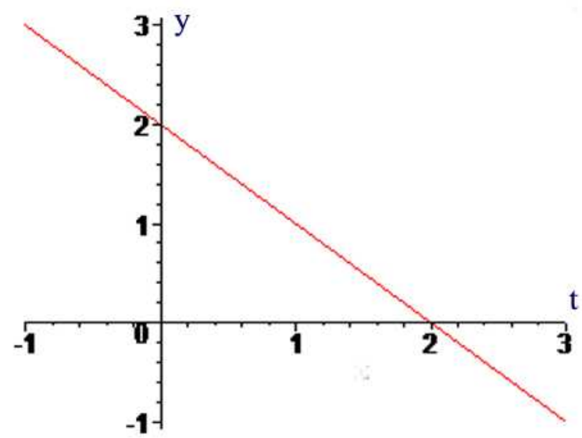
Problem 1.12
Suppose $y(t) = 2e^{-4t}$ is the solution to the initial value problem $y' + ky = 0, y(0) = y_0$. Find the values of k and y_0.

Problem 1.13
Consider $t > 0$. For what value(s) of the constant n, if any, is $y(t) = t^n$ a solution to the differential equation

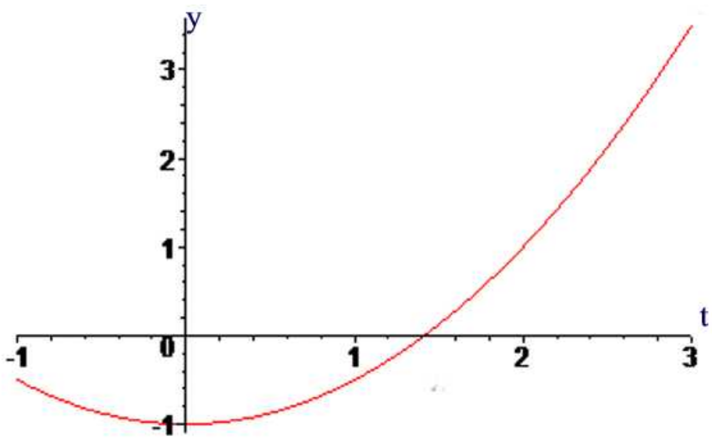
$$t^2y'' - 2ty' + 2y = 0?$$

Problem 1.14
(a) Show that $y(t) = C_1e^{2t} + C_2e^{-2t}$ is a solution of the differential equation $y'' - 4y = 0$, where C_1 and C_2 are arbitrary constants.
(b) Find the solution satisfying $y(0) = 2$ and $y'(0) = 0$.
(c) Find the solution satisfying $y(0) = 2$ and $\lim_{t\to\infty} y(t) = 0$

Problem 1.15
Suppose that the graph below is the particular solution to the initial value problem $y'(t) = m + 1, y(1) = y_0$. Determine the constants m and y_0 and then find the formula for $y(t)$.



Problem 1.16
Suppose that the graph below is the particular solution to the initial value problem $y'(t) = mt, y(t_0) = -1$. Determine the constants m and t_0 and then find the formula for $y(t)$.



Problem 1.17
Show that $y(t) = e^{2t}$ is not a solution to the differential equation $y'' + 4y = 0$.

Problem 1.18
At time $t = 0$ an object having mass m is released from rest at a height y_0 above the ground. Let g represent the constant gravitational acceleration. Derive an expression for the impact time (the time at which the object strikes the ground). What is the velocity with which the object strikes the ground?

Problem 1.19
At time $t = 0$, an object of mass m is released from rest at a height of 252 ft above the floor of an experimental chamber in which gravitational acceleration has been slightly modified. Assume (instead of the usual value of 32 ft$/sec^2$), that the acceleration has the form $32 - \epsilon \sin\left(\frac{\pi t}{4}\right)\ ft/sec^2$, where ϵ is a constant. In addition, assume that the projectile strikes the ground exactly 4 sec after release. Can this information be used to determine the constant ϵ? If so, determine ϵ.

Problem 1.20
Consider the initial-value problem

$$y' + 3y = 6t + 5, \quad y(0) = 3$$

(a) Show that $y = Ce^{-3t} + 2t + 1$ is a solution to the above differential equation.
(b) Find the value of C.

2 Qualitative Analysis: Direction Field of $y' = f(t, y)$

Solutions to differential equations can be given in one of the following forms:
• by an explicit formula: For example, the function $y = \sqrt{t^3 + 1}$ is an explicit solution to the initial value problem $2yy' = 3t^2, \ y(1) = \sqrt{2}$;
• by an implicit equation: The solution y to the equation $y' = -\frac{1+ye^{ty}}{1+te^{ty}}$ is defined implicitly by the equation $t + y + e^{ty} = 0$;
• by a power series representation. For example the general solution to the equation $(1 - t^2)y'' - 2ty' + 3y = 0$ is given by

$$y(t) = C_1 \left(1 - t^2 - \frac{1}{3}t^4 - \frac{1}{5}t^6 - \cdots\right) + C_2 t;$$

• numerically (Euler's and Runge-Kutta methods);
• graphically (direction fields, phase portraits, and phase lines).

Since explicit solutions of differential equations are often unobtainable, we explore methods of finding properties of solutions from the differential equation itself; the principal tool is the geometry of direction field.
A **direction field** (also known as **slope field**) consists of an array of short line segments in the ty-plane having the property that the line plotted at a point (t, y) has slope $f(t, y)$. Direction fields are basically used to visualize the family of solutions of a given differential equation without the need of solving the equation. Direction fields give **qualitative** information about solutions of ODEs.
In this section we use direction fields for solving initial value problems of the form

$$\left\{ \begin{array}{rcl} \frac{dy}{dt} & = & f(t, y) \\ y(t_0) & = & y_0 \end{array} \right.$$

In the special case where $f(t, y) = f(y)$, i.e. the independent variable t does not appear on the right side, the first order DE $\frac{dy}{dt} = f(y)$ is called **autonomous**.

Example 2.1
Find the direction field of the differential equation

$$\frac{dy}{dt} = 2t$$

What is the form of the general solution? Graph the particular solution going through $(0, -1)$.

Solution.
Figure 2.1 shows the slope field and the graph of the particular solution to the given DE passing through the point $(0, -1)$. The figure was plotted using the following MAPLE commands:
>with(plots):
>with(DEtools):
>slopeplot:= DEplot(diff(y(x),x)=2*x,y(x),x=-3..3,y=-3..3):
>g:=plot($x^2 - 1$, x=-3..3, y=-3..3, color=black):
>display([slopeplot,g]);

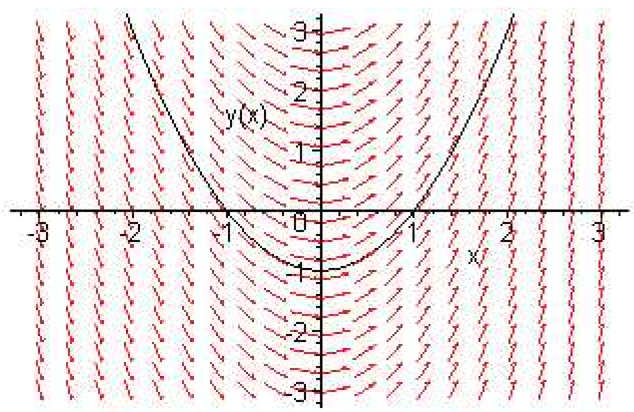


Figure 2.1

The solution curves look like parabolas. Thus, the general solution is given by the equation $y = t^2 + C$.■

Example 2.2
Using direction field, guess the form of the solution curves of the differential equation

$$\frac{dy}{dt} = -\frac{t}{y}.$$

Solution.
The direction fields (See Figure 2.2) is obtained by executing the following Maple commands

> with(plots):
> with (DEtools):

> slopeplot := DEplot($diff(y(x), x) = -x/y, y(x), x = -2..2, y = -2..2$):
> display(slopeplot);

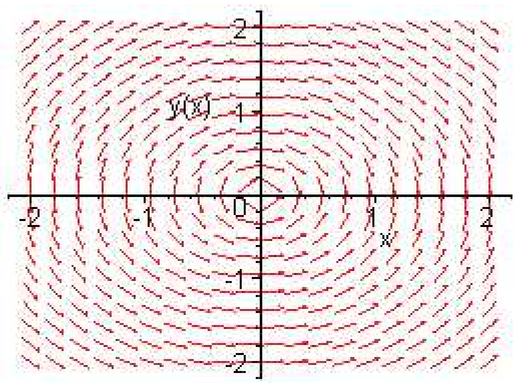


Figure 2.2

The solution curves look like circles centered at the origin. Thus, the general solution is given implicitly by the equation $t^2 + y^2 = C$ where C is a positive constant.■

The **phase portrait** of a differential equation is the family of graphs of the solutions of the equation. Thus, the family of all circles centered at the origin form the phase portrait of the differential equation of the previous example.

Remark 2.1
We point out here that even though one can draw solution curves, some do not have simple formula. For instance, the equation $\frac{dy}{dt} = y^2 - t$ does not have explicit solutions.

Problem 2.1
Sketch the direction field for the differential equation in the window $-5 \leq t \leq 5, -5 \leq y \leq 5$.

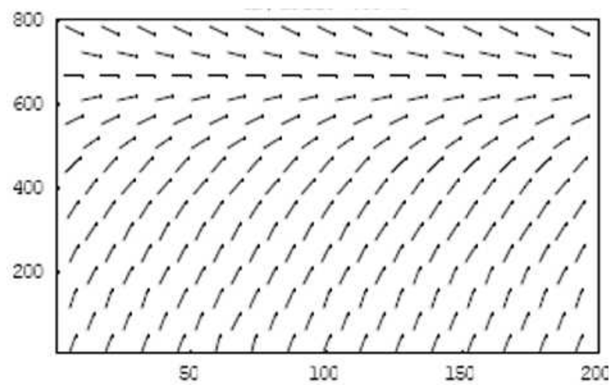
(a) $y' = y$ (b) $y' = t - y$.

Problem 2.2
Sketch solution curves to the differential equation

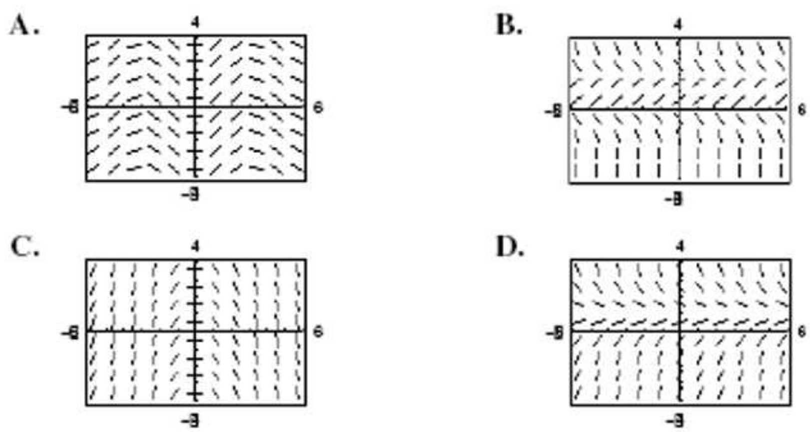
$$\frac{dy}{dt} = 20 - 0.03y$$

represented by the slope field below for the initial values

$$(t_0, y_0) = \{(0, 200), (0, 400), (0, 600), (0, 650), (0, 800)\}$$



Problem 2.3
Match each direction field with the equation that the slope field could represent. Each direction field is drawn in the portion of the ty-plane defined by $-6 \le t \le 6, -4 \le y \le 4$.
(a) $y' = -t$ (b) $y' = \sin t$ (c) $y' = 1 - y$ (d) $y' = y(2 - y)$



Problem 2.4
State whether or not the equation is autonomous.

(a) $y' = -t$ (b) $y' = \sin t$ (c) $y' = 1 - y$ (d) $y' = y(2 - y)$

The Method of Isoclines

An alternative scheme, useful for plotting direction fields by hand, is the **method of isoclines.** An **isocline** (which means "equal slope") of a differential equation is a curve in the ty-plane along which the slope is constant. For example, the isoclines of the equation $y' = f(t, y)$ are the **level curves** $f(t, y) = c$ of the function $f(t, y)$ in the $ty-$plane. The special isocline obtained by setting $c = 0$ is known as the **nullcline.**

To carry out the method of isoclines we first sketch the level curves $f(t, y) = c$ for various values of c. Then at representative points on these curves, we sketch short line segments each having the same slope c. This is illustrated in the next example.

Example 2.3

Use the method of isoclines to draw the direction field for the following differential equation

$$\frac{dy}{dt} = y - t.$$

Solution.

Here $f(t, y) = y - t$ so the isoclines $y - t = c$ consist of straight lines as shown in Figure 2.3.

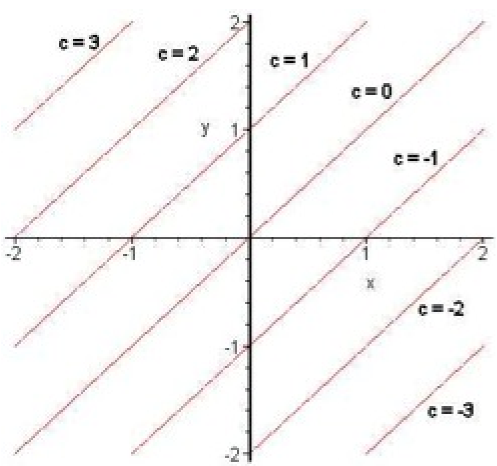


Figure 2.3

At selected points along an isocline of the form $y = t + c$ we draw short line segments each having slope c as shown in Figure 2.4.

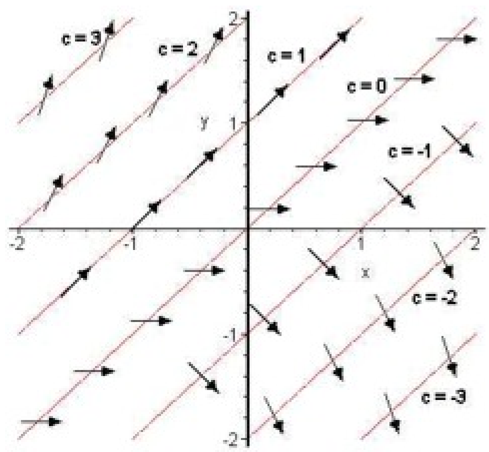


Figure 2.4

The direction field is shown in Figure 2.5.

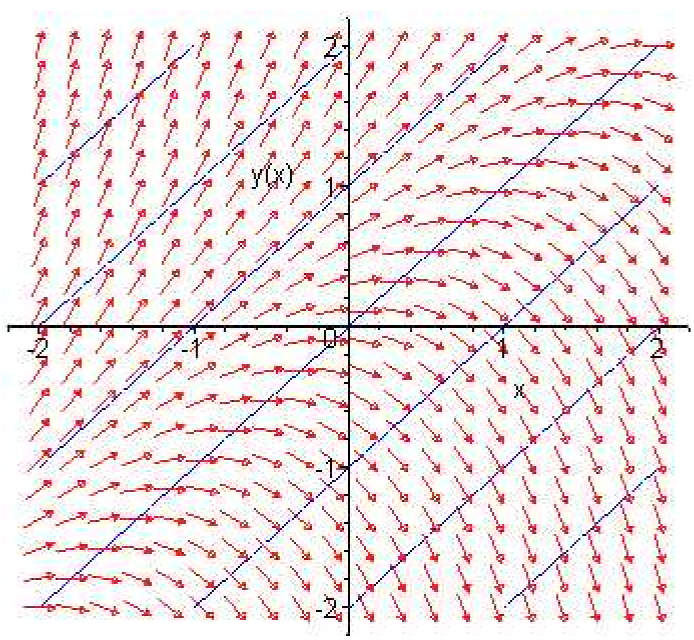


Figure 2.5

Problem 2.5
Find the equations of the isoclines for the DE $y' = \frac{2y}{t}$.

Equilibrium Solutions and Stability for Autonomous Equations
A physical system is often said to be in equilibrium if it doesn't change in time. We adopt this idea and say that a solution to a differential equation is an equilibrium solution if it is a constant function.
Thus, in a direction field of an autonomous equation **equilibrium** solutions are solution curves represented by horizontal lines. It follows that the equations of such solutions have the form $y(t) \equiv c$ where c is a constant. The following result tells us where to look for equilibrium solutions.

Theorem 2.1
The function $y(t) \equiv c$, where c is a constant, is an equilibrium solution to $y' = f(y)$ if and only if c is a root of $f(y) = 0$.

Proof.
Suppose that $y(t) \equiv c$, where c is a constant, is an equilibrium solution to $y' = f(y)$. Then, $f(y) = f(c) = y' = 0$ so that c is a solution to the equation $f(y) = 0$. Conversely, suppose that c is a constant satisfying $f(c) = 0$. The function $y(t) \equiv c$ satisfies $y' = f(y)$. That is, $y(t) \equiv c$ is an equilibrium solution $\blacksquare$

Example 2.4
Find the equilibrium solutions to the DE

$$\frac{dy}{dt} = 2y(1-y)$$

Solution.
The roots of $f(y) = 2y(1-y) = 0$ are $y = 0$ and $y = 1$. According to the previous theorem, the equilibrium solutions are $y(t) \equiv 0$ and $y(t) \equiv 1$. The direction field of the DE is shown in Figure 2.6 $\blacksquare$

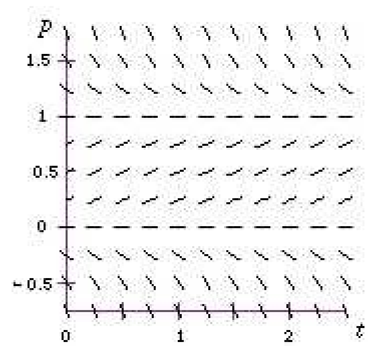


Figure 2.6

Remark 2.2
Equilibrium solutions can be defined for nonautonomous differential equations. For example, the function $y(t) \equiv 1$ is an equilibrium solution to the DE $y' = (1-y)t^2$ ■

The direction field of a given differential equation indicates that as t increases without bound, every solution either moves towards or moves away from an equilibrium solution.
If all nearby solutions move towards a certain equilibrium solution, then that equilibrium solution is called **asymptotically stable**, **stable**, or **attracting**. The solution $y = 1$ in Figure 2.6 is attracting. An equilibrium solution is called **unstable** or **repelling** when all nearby solutions move away from it. The solution $y = 0$ in Figure 2.6 is repelling.
In cases where solutions on one side of an equilibrium solution move towards the equilibrium solution and on the other side of the equilibrium solution move away from it we call the equilibrium solution **semi-stable.**
An equilibrium solution does not necessarily have to be either attracting or repelling. The next example illustrates this situation.

Example 2.5
Sketch the field direction of the differential equation

$$y' = 4y(1-y)^2$$

Show that $y = 1$ is neither stable or unstable.

Solution.
The direction field is shown in Figure 2.7.

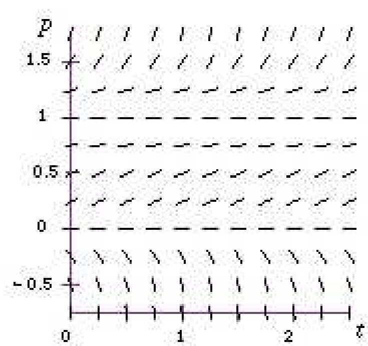


Figure 2.7

Note that the equilibrium solution $y(t) \equiv 1$ is neither stable or unstable. Nearby solutions that start below it are attracted upward towards it but nearby solutions that start above it are repelled upward and away from it ■
Another qualitative representation of a differential equation is the so-called phase line. A **phase line** consists of solid dots and arrows. The solid dots represent the equilibrium points and the arrows indicate the directions that solutions move as t increases. Figure 2.8 shows an example of a phase line.

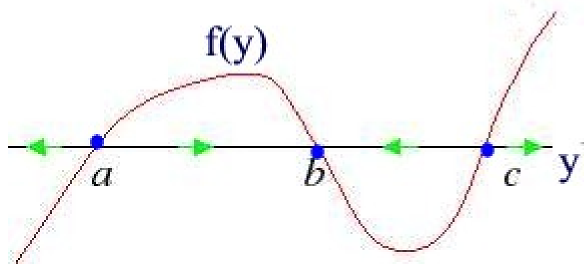


Figure 2.8

We see that the equilibrium b is stable, whereas the equilibria a and c are unstable.

Problem 2.6
Find all the equilibrium solutions of each of the autonomous differential equations below

(a) $y' = (y-1)(y-2)$
(b) $y' = (y-1)(y-2)^2$
(c) $y' = (y-1)(y-2)(y-3)$

Problem 2.7
Find an autonomous differential equation with an equilibrium solution at $y = 1$ and satisfying $y' < 0$ for $-\infty < y < 1$ and $1 < y < \infty$.

Problem 2.8
Find an autonomous differential equation with no equilibrium solutions and satisfying $y' > 0$.

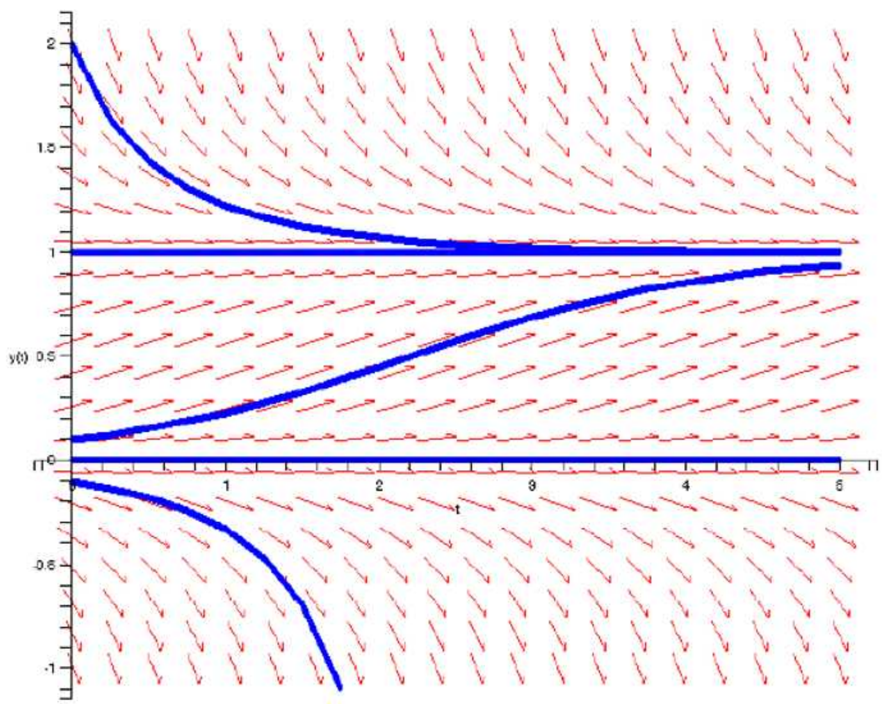
Problem 2.9
Find an autonomous differential equation with equilibrium solutions $y = \frac{n}{2}$, where n is an integer.

Problem 2.10
Find an autonomous differential equation with equilibrium solutions $y = 0$ and $y = 2$ and satisfying the properties $y' > 0$ for $0 < y < 2$; $y' < 0$ for $y < 0$ or $y > 2$.

Problem 2.11
Classify whether the equilibrium solutions are stable, unstable, or neither.

(a) $y' = 1 - y^2$
(b) $y' = (y + 1)^2$

Problem 2.12
Consider the direction field below. Classify the equilibrium points, as asymptotically stable, semi-stable, or unstable.



Problem 2.13
Sketch the direction field of the equation $y' = y^3$. Sketch the solution satisfying the condition $y(1) = -1$. What is the domain of this solution?

Problem 2.14
Find the equilibrium solutions and determine their stability

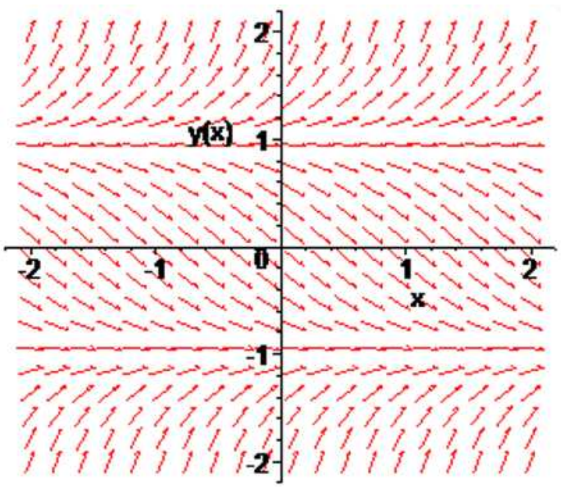
$$y' = y^2(y^2 - 1), \quad y(0) = y_0$$

Problem 2.15
Find the equilibrium solutions of the equation

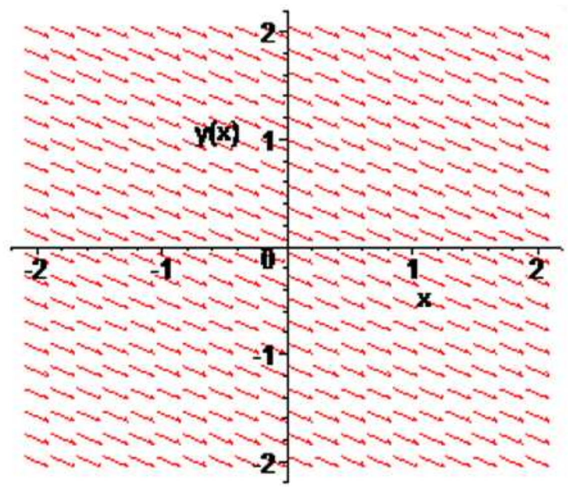
$$y' = y^2 - 4y$$

then decide whether they are stable or unstable. What is the long-time behavior if $y(0) = 5? y(0) = 4? y(0) = 3?$

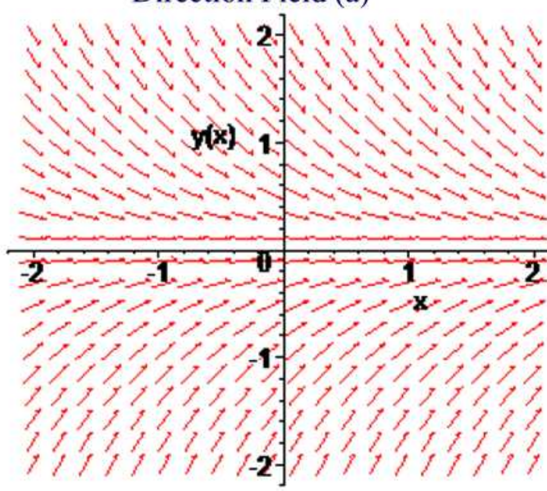
Problem 2.16
Consider the six direction fields shown. Match a direction field with each of the following differential equations.
(i) $y' = -y$ (ii) $y' = -t + 1$ (iii) $y' = y^2 - 1$ (iv) $y' = -\frac{1}{2}$ (v) $y' = y + t$
(vi) $y' = \frac{1}{y^2+1}$



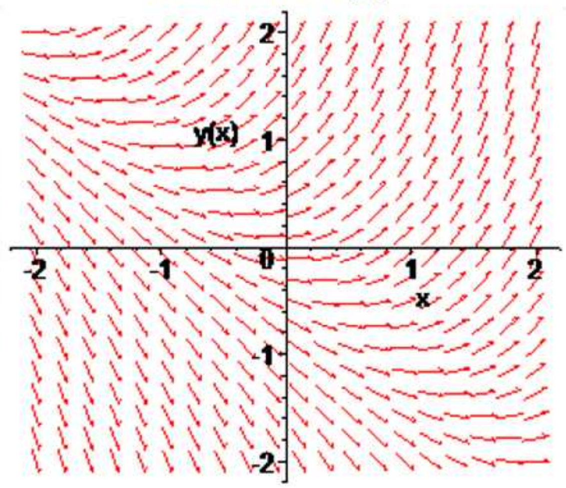
Direction Field (a)



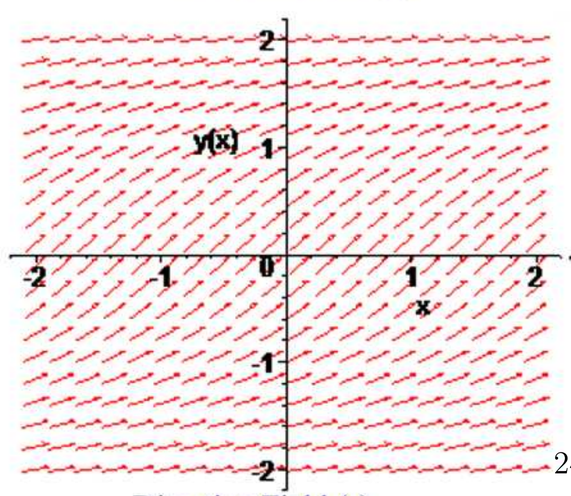
Direction Field (b)



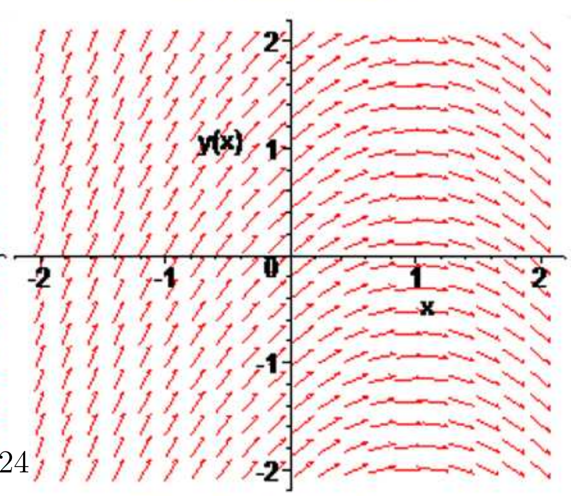
Direction Field (c)



Direction Field (d)



Direction Field (e)



Direction Field (f)

Problem 2.17
What is $\lim_{t\to\infty} y(t)$ for the initial-value problem

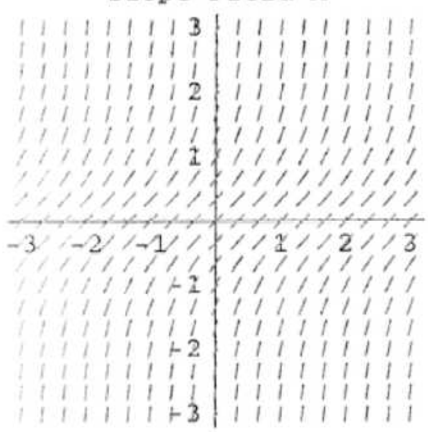
$$y' = \sin\left(y(t),\ y(0) = \frac{\pi}{2}\right.$$

Problem 2.18
Consider the following first order differential equations

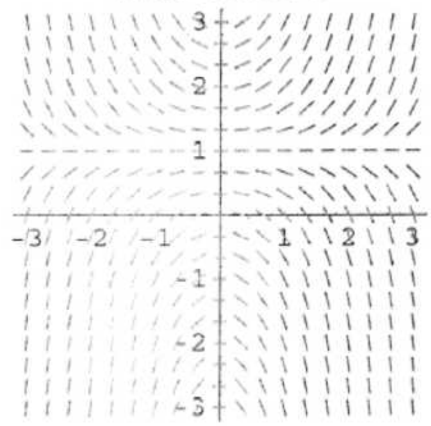
$$\begin{array}{llllllll} (a) & y' = ty + t & (b) & y' = y^2 + 1 & (c) & y' = ty - t & (d) & y' = \sin t \\ (e) & y' = y - t^2 & (f) & y' = \cos t & (g) & y' = y + t^2 & (h) & y' = 1 - y^2 \end{array}$$

Match the direction fields with their associated equations. Provide a brief justification for your choice.

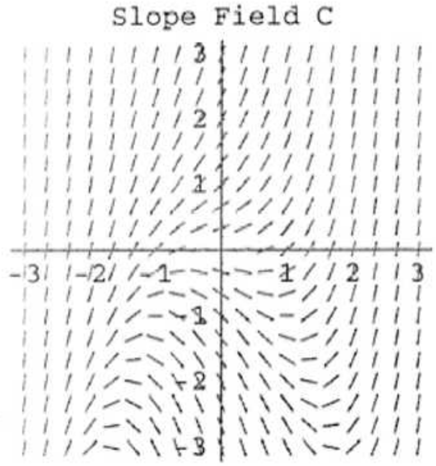
Slope Field A



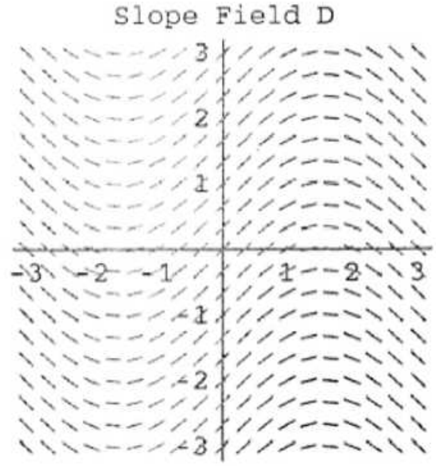
Slope Field B



Slope Field C



Slope Field D



Problem 2.19
The slope fields of $y' = 2 - y$ and $y' = \frac{t}{y}$ are shown in Figure 2.9(a) and Figure 2.9(b).
(a) On each slope field, sketch solution curves with initial conditions

$$\text{(i) } y(0) = 1 \quad \text{(ii) } y(1) = 0 \quad \text{(iii) } y(0) = 3$$

(b) For each solution curve, what can you say about the long run behavior of y? That is, does $\lim_{t\to\infty} y$ exist? If so, what is its value?

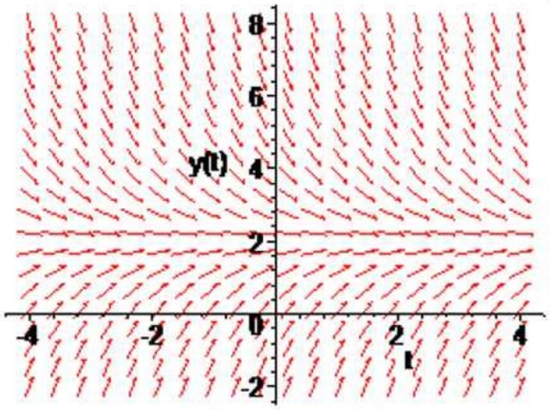


Figure 29(a)

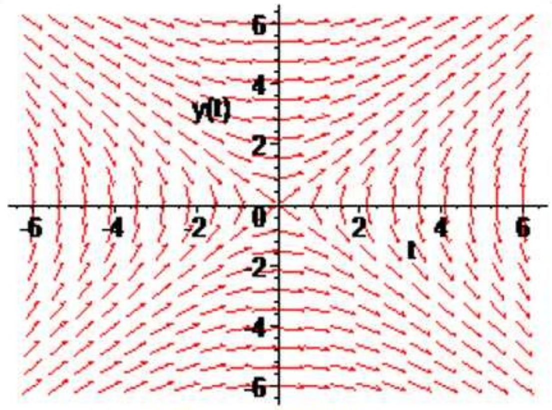


Figure 29(b)

Problem 2.20
The slope field for the equation $y' = t(y - 1)$ is shown in Figure 2.10.
(a) Sketch the solutions passing through the points

$$\text{(i) } (0,1) \quad \text{(ii) } (0,-1) \quad \text{(iii) } (0,0)$$

(b) From your sketch, write down the equation of the solution with $y(0) = 1$.
(c) Check your solution to part (b) by substituting it into the differential equation.

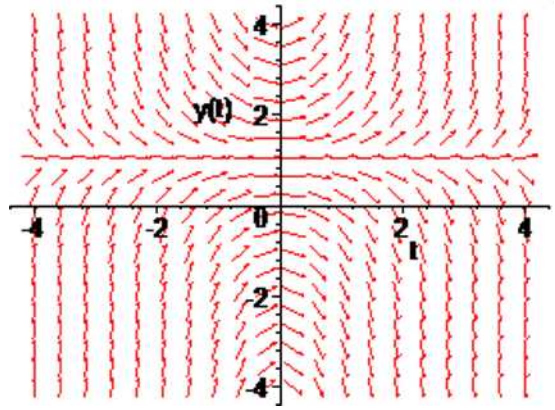
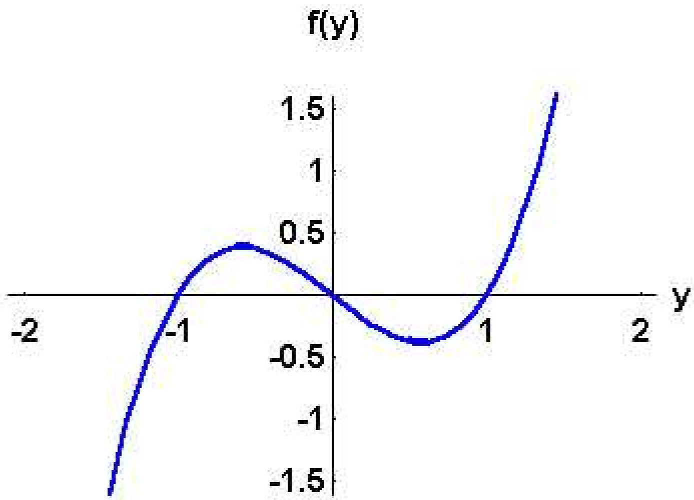


Figure 2.10

Problem 2.21
Consider the autonomous differential equation $\frac{dy}{dt} = f(y)$ where the graph of $f(y)$ is



(a) Sketch the phase line.
(b) Sketch the Slope Field of this differential equation.
(c) Sketch the graph of the solution to the IVP $y' = f(y),\ y(0) = \frac{1}{2}$. Find $\lim_{t\to\infty} y(t)$.
(d) Sketch the graph of the solution to the IVP $y' = f(y),\ y(0) = -\frac{1}{2}$. Find $\lim_{t\to\infty} y(t)$.

3 Existence and Uniqueness of Solutions to First Order Linear IVP

Before worrying about how to solve a differential equation, either analytically, qualitatively, or numerically, one should first resolve the basic issues of existence and uniqueness. First, does a solution exist? If, not, it makes no sense trying to find one. Second, is the solution uniquely determined by the initial conditions? Otherwise, the differential equation probably has little relevance for physical applications since we cannot use it as a predictive tool. Since differential equations inevitably have lots of solutions, the only way in which we can deduce uniqueness is by imposing suitable initial (or boundary) conditions.

Summarizing what just mentioned, the main important questions in the theory of differential equations are the following:

• When does a given initial value problem have a solution on some interval (a, b) containing t_0, where $y(t_0) = y_0$ is the initial condition?

• When is a solution of a given initial value problem unique?

• How large an interval containing t_0, is the existence of a unique solution guaranteed?

In this section we discuss the conditions needed to guarantee the existence of a unique solution to first order linear initial value problems. We start with the definition of a first order linear differential equation.

Any differential equation that can be written in the form

$$y' + p(t)y = g(t) \tag{1}$$

where $p(t)$ and $g(t)$ are functions with common domain $a < t < b$, is called a **first order linear differential equation.** The term linear is used because $f(t, y) = g(t) - p(t)y$ is linear in y. A DE that is not linear is called **nonlinear.**

In mathematics and physics, linear generally means "simple" and non-linear means "complicated". The theory for solving linear equations is very well developed because linear equations are simple enough to be solvable. Nonlinear equations can usually not be solved exactly and are the subject of much on-going research.

Now, we say that Equation (1) is **homogeneous** if $g(t) \equiv 0$ for all $a < t < b$. If there is a $a < t < b$ such that $g(t) \neq 0$ then Equation (1) is called **nonhomogeneous**.

Example 3.1
Classify each of the following first order differential equations as linear or nonlinear. If the equation is linear, decide whether it is homogeneous or nonhomogeneous.
(a) $\frac{dy}{dt} + \frac{y}{10} = ty$
(b) $t^2 - 3y^2 + 2ty\frac{dy}{dt} = 0$
(c) $t\frac{dy}{dt} = t^2 - 2y$
(d) $\frac{dy}{dt} = \frac{t-y}{t+y}$

Solution.
(a) Notice that the given equation can be written as $\frac{dy}{dt} + (\frac{1}{10} - t)y = 0$ which is a homogeneous first order linear DE.
(b) This is nonlinear because of the term y^2.
(c) This is a nonhomogeneous first order linear DE since the right-hand side is not identically zero on any interval.
(d) This is nonlinear because of the y in the denominator ■

Problem 3.1
Find $p(t)$ and y_0 so that the function $y(t) = 3e^{t^2}$ is the solution to the IVP $y' + p(t)y = 0, y(0) = y_0$.

First order linear differential equations possess important linearity or **superposition** properties.

Theorem 3.1
(a) If $y_1(t)$ and $y_2(t)$ are any two solutions of the homogeneous equation $y' + p(t)y = 0$ then for any constants c_1 and c_2 the linear combination $c_1y_1(t) + c_2y_2(t)$ is also a solution of the homogeneous equation.
(b) If $y_1(t)$ is a solution to the homogeneous equation $y' + p(t)y = 0$ and $y_2(t)$ is a solution to the nonhomogeneous equation $y' + p(t)y = g(t)$ then $Cy_1(t) + y_2(t)$ is also a solution to the nonhomogeneous equation, where C is an arbitrary constant.

Proof.
(a) Since $y_1(t)$ and $y_2(t)$ are solutions to the homogeneous equation then

$$(c_1y_1 + c_2y_2)' + p(t)(c_1y_1 + c_2y_2) = c_1(y_1' + p(t)y_1) + c_2(y_2' + p(t)y_2) = 0 + 0 = 0$$

(b) We have

$$(Cy_1 + y_2)' + p(t)(Cy_1 + y_2) = C(y_1' + p(t)y_1) + y_2' + p(t)y_2 = 0 + g(t) = g(t) \blacksquare$$

Remark 3.1
Part (a) of the previous theorem is not true in general for nonhomogeneous equations. For example, consider the equation $y' = 1$. Then $y_1(1) = t$ and $y_2(t) = t+1$ are both solutions to the DE. However, $y_1(t) + y_2(t) = 2t+1$ is not a solution since $(y_1 + y_2)' = 2 \neq 1$ ■

Next, we look at the conditions that guarantee the existence of a unique solution to the IVP

$$y' + p(t)y = g(t), y(t_0) = y_0 \tag{2}$$

Theorem 3.2
If $p(t)$ and $g(t)$ are continuous functions in the open interval $I = (a, b)$ and t_0 a point inside I then the IVP (2) has a unique solution $y(t)$ defined on I.

Proof.
The proof is very constructive and should not be ignored. The proof consists of two parts: the existence of a solution and uniqueness.
Existence: The technique we use is a well known technique for solving any first order linear DE known as the method of **integrating factor** which we will discuss in Section 5. Since $p(t)$ is continuous then by the Second Fundamental Theorem of Calculus the function

$$\int_{t_0}^{t} p(s)ds$$

is differentiable with derivative

$$\frac{d}{dt}\int_{t_0}^{t} p(s)ds = p(t), a < t < b$$

Let

$$I(t) = e^{\int_{t_0}^{t} p(s)ds}$$

From this, one can notice that Equation (1) can be written as

$$(I(t)y)' = I(t)g(t)$$

Integrating this last equation to obtain

$$I(s)y(s)|_{t_0}^{t} = \int_{t_0}^{t} I(s)g(s)ds$$

Thus,

$$I(t)y(t) - I(t_0)y(t_0) = \int_{t_0}^{t} I(s)g(s)ds$$

or

$$I(t)y(t) - y_0 = \int_{t_0}^{t} I(s)g(s)ds$$

Divide the last equation by $I(t)$ to obtain

$$y(t) = \frac{1}{e^{\int_{t_0}^{t} p(s)ds}} \int_{t_0}^{t} I(s)g(s)ds + \frac{y_0}{e^{\int_{t_0}^{t} p(s)ds}} \qquad (3)$$

Uniqueness: Suppose that $y_1(t)$ and $y_2(t)$ are two solutions of (2). Let $w(t) = y_1(t) - y_2(t)$ for any $a < t < b$. We will show that $w(t) \equiv 0$ for all $a < t < b$. First, we show that $w(t)$ satisfies the homogeneous equation

$$w' + p(t)w = 0 \qquad (4)$$

Indeed,

$$w' + p(t)w = (y_1' + p(t)y_1) - (y_2' + p(t)y_2) = g(t) - g(t) = 0.$$

Multiply Equation (4) by $e^{\int_{t_0}^{t} p(s)ds}$ to obtain

$$\left(e^{\int_{t_0}^{t} p(s)ds} w\right)' = 0$$

Now integrate both sides and then solve for $w(t)$ to obtain

$$w(t) = Ce^{-\int_{t_0}^{t} p(s)ds} \qquad (5)$$

But $w(t_0) = y_1(t_0) - y_2(t_0) = y_0 - y_0 = 0$ so that $C = 0$. Hence, $w(t) \equiv 0$ for all $a < t < b$ or $y_1(t) = y_2(t)$ for all $a < t < b$ ■

Example 3.2
If $p(t)$ is continuous on (a, b) and $a < t_0 < b$ then what is the unique solution to the IVP

$$y' + p(t)y = 0, y(t_0) = 0?$$

Solution.
Replace $g(t) \equiv 0$ and $y_0 = 0$ in Equation (3) to obtain

$$y(t) \equiv 0 \blacksquare$$

Example 3.3
Find the unique solution to the IVP

$$y' + \frac{1}{t\ln t}y = 9t^2, y(e) = 2e^3, \quad t > 0$$

Solution.
Let $I(t) = e^{\int_e^t \frac{1}{s\ln s}ds} = \ln t$. Then

$$(I(t)y)' = 9t^2 I(t)$$

Integrating both sides from e to t to obtain

$$I(t)y(t) - I(e)y(e) = 3t^3\ln t - t^3 - 2e^3.$$

Thus,

$$y(t) = 3t^3 - \frac{t^3}{\ln t} \blacksquare$$

Remark 3.2
The above theorem asserts that if the hypotheses are satisfied then a unique solution exists on an interval containing t_0. However, the solution might actually exist on a larger interval than what the theorem asserts. To be more precise, consider the IVP $ty' + 2y = 0,\ y(1) = 0$. If we apply Theorem 3.2, then a unique solution exists say on the interval $0 < t < \infty$ since this the interval containing 1 and where $p(t) = \frac{1}{t}$ is defined. Actually, the solution is $y(t) \equiv 0$. But one can easily see $y(t) \equiv 0$ is a solution for all $-\infty < t < \infty$. So our theorem asserts the existence of a local solution rather than a global one $\blacksquare$

Problem 3.2
For each of the initial conditions, determine the largest interval $a < t < b$ on which Theorem 3.2 guarantees the existence of a unique solution

$$y' + \frac{1}{t^2+1}y = \sin t$$

(a) $y(0) = \pi$ (b) $y(\pi) = 0$

Problem 3.3
For each of the initial conditions, determine the largest interval $a < t < b$ on which Theorem 3.2 guarantees the existence of a unique solution

$$y' + \frac{t}{t^2-4}y = \frac{e^t}{t-3}$$

(a) $y(5) = 2$ (b) $y(-\frac{3}{2}) = 1$ (c) $y(-6) = 2$

Problem 3.4
(a) For what value of the constant C and the exponent r is $y = Ct^r$ the solution of the IVP

$$2ty' - 6y = 0, y(-2) = 8?$$

(b) Determine the largest interval of the form $a < t < b$ on which Theorem 3.2 guarantees the existence of a unique solution.
(c) What is the actual interval of existence for the solution found in part (a)?

Problem 3.5
Solve the IVP

$$y' + 0.196y = 9.8,\ y(0) = 48$$

Problem 3.6
Solve the IVP

$$y' + \frac{2}{t}y = 4t, y(1) = 2$$

Problem 3.7
Let $w(t)$ be the unique solution to $w' + p(t)w = 0$ for all $a < t < b$ and $w(t_0) = w_0$. Show that either $w(t) \equiv 0$ for all $a < t < b$ or $w(t) \neq 0$ for all $a < t < b$ depending on whether $w_0 = 0$ or $w_0 \neq 0$. This result will be very useful when discussing Abel's Theorem (i.e., Theorem 16.3) in Section 16.

Problem 3.8
What information does the Existence and Uniqueness Theorem gives about the initial value problem $ty' = y + t^3 \cos t,\ y(1) = 1? y(-1) = 1$

Problem 3.9
Consider the following differential equation

$$(t-4)y' + 3y = \frac{1}{t^2+5t}$$

Without solving, find the interval over which a unique solution is guaranteed for each of the following initial conditions:
(a) $y(-3) = 4$ (b) $y(1.5) = -2$ (c) $y(-6) = 0$ (d) $y(4.1) = 3$

Problem 3.10
Without solving the initial value problem, $(t-1)y' + (\ln t)y = \frac{2}{t-3}, y(t_0) = y_0$, state whether or not a unique solution is guaranteed to exist for the $y(t_0) = y_0$ listed below. If a unique solution is guaranteed, find the largest interval for which the solution satisfies the differential equation and the initial condition.
(a) $y(2) = 4$ (b) $y(0) = 0$ (c) $y(4) = 2$

Problem 3.11
(a) State precisely the theorem (hypothesis and conclusion) for existence and uniqueness of a first order initial value problem.
(b) Consider the equation $y' + t^2y = e^{t^3}$ with initial conditions $y(t_0) = y_0$. Briefly discuss if this has a solution, if it is unique and why.

Problem 3.12
Is the differential equation linear or nonlinear? If the equation is linear, decide whether it is homogeneous or nonhomogeneous.
(a) $y' = ty^2$ (b) $y' = t^2y$ (c) $(\cos t)y' + e^ty = \sin t$ (d) $\frac{y'}{y} + t^3 = \sin t,\ y > 0$

Problem 3.13
Consider the initial value problem

$$y' + p(t)y = g(t), \quad y(3) = 1$$

Suppose that $p(t)$ and/or $g(t)$ have discontinuities at $t = -2, t = 0$, and $t = 5$ but are continuous for all other values of t. What is the largest interval (a, b) on which the exitence and uniqueness theorem is applied.

Problem 3.14
Determine α and y_0 so that the graph of the solution to the initial-value problem

$$y' + \alpha y = 0, \quad y(0) = y_0$$

passes through the points $(1, 4)$ and $(3, 1)$.

Problem 3.15
Match the following objects with the correct description. Every equation matches exactly one description.
(a) $y' = 3y - 5t$
(b) $\frac{\partial y}{t} = \frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 y}{\partial x^2}$
(c) $y' - y^2 = \sin t$
(d) $y' + 3y = 0$

(i) A partial differential equation
(ii) A homogeneous one-dimensional first order linear differential equation.
(iii) A nonlinear first order differential equation. (iv) An nonhomogenous first order linear differential equation

Problem 3.16
Consider the differential equation $y' = -t^2 y + \sin y$. What is the order of this equation? Is it linear or nonlinear?

Problem 3.17
Verify that $y(t) = e^{t^2} \int_0^t e^{-s^2} ds + e^{t^2}$ is a solution of the differential equation $y' - 2ty = 1$.

Problem 3.18
Consider the initial value problem

$$y' = -\frac{y}{t} + 2, \quad y(1) = 2$$

(a) Are the conditions of the Existence and Uniqueness theorem satisfied? Why or why not?
(b) Solve the IVP and state the domain of definition.

Problem 3.19
Solve the differential equation $y'' + y' = e^t$ as follows. Let $z = y' + y$, find a differential equation for z, and find the general solution. Then using this general value of z, find y by solving the differential equation $y' + y = z$.

Problem 3.20
Show that $y' = \frac{t+y}{t}$ is a linear first order nonhomogeneous equation.

4 Solving First Order Linear Homogeneous DE

In this section we are interested in finding the general solution to the first order linear homogeneous equation

$$y' + p(t)y = 0 \tag{6}$$

where $p(t)$ is continuous on the open interval $a < t < b$.
To find the general solution to Equation (43) we proceed as follows.

$$\begin{array}{ccc} y' & = & -p(t)y \\ y' + p(t)y & = & 0 \\ \left(e^{\int p(t)dt}y\right)' & = & 0 \\ \int \left(e^{\int p(t)dt}y\right)' dt & = & 0 \\ y(t) & = & Ce^{-\int p(t)dt} \end{array}$$

Hence, the function $y(t) = Ce^{-\int p(t)dt}$ is the **general solution** to Equation (43). Notice that when evaluating $\int p(t)dt$ the constant of integration will be ignored since it is included in the C as you have noticed from the above derivation of $y(t)$.

Example 4.1
Find the general solution of

$$y' + (\sin t)y = 0$$

Solution.
Since $p(t) = \sin t$ then $\int \sin t dt = -\cos t$. Thus, the general solution is $y(t) = Ce^{\cos t}$ ■

Remark 4.1
Instead of using indefinite integrals in the above process one can use definite integrals instead. For example, replace $\int p(t)dt$ by $\int_{t_0}^t p(s)ds$ for some fixed $a < t_0 < b$. Using definite integral is proven to be useful when $p(t)$ does not have an elementary function as an antiderivative. For example, when $p(t) = \frac{\sin t}{t}$ or $p(t) = \sin(t^2)$ ■

Solving First Order Linear Homogeneous DE with Initial Condition
Consider the IVP

$$y' + p(t)y = 0, y(t_0) = y_0$$

To solve this IVP, we first solve the differential equation without concern for the initial condition. We know that the function $y(t) = Ce^{-\int p(t)dt}$ is the general solution to the DE. Next, the constant C is determined by using the given initial condition. We illustrate this process in the next example.

Example 4.2
Solve the IVP

$$y' - 2\cos(2t)y = 0, y(\pi) = -2$$

Solution.
Since $p(t) = -2\cos(2t)$ then $\int -2\cos 2tdt = -\sin 2t$. Thus, the general solution to the DE is $y(t) = Ce^{\sin(2t)}$. Since $y(\pi) = -2$ then $C = -2$. Hence, the unique solution is given by $y(t) = -2e^{\sin(2t)}$ ■

Problem 4.1
Solve the IVP

$$y' = -2ty, \ y(1) = 1$$

Problem 4.2
Solve the IVP

$$y' + e^t y = 0, \ y(0) = 2$$

Problem 4.3
Consider the first order linear nonhomogeneous IVP

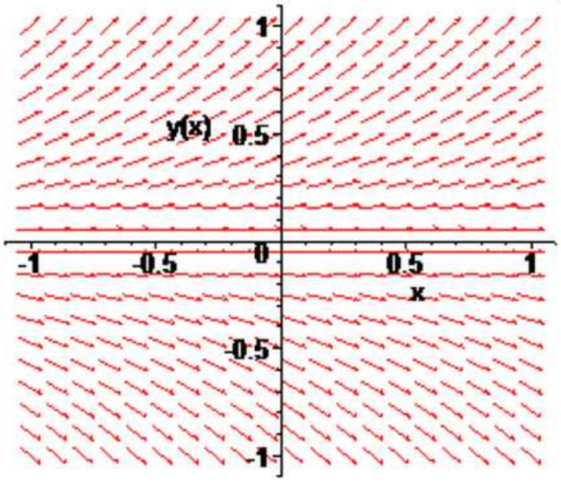
$$y' + p(t)y = \alpha p(t), \ y(t_0) = y_0$$

(a) Show that the IVP can be reduced to a first order linear homogeneous IVP by the change of variable $z(t) = y(t) - \alpha$.
(b) Solve this initial value problem for $z(t)$ and use the solution to determine $y(t)$.

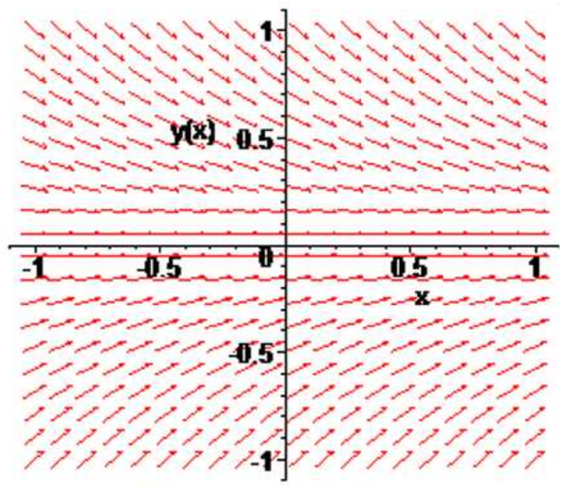
Problem 4.4
Apply the results of the previous problem to solve the IVP

$$y' + 2ty = 6t, \quad y(0) = 4$$

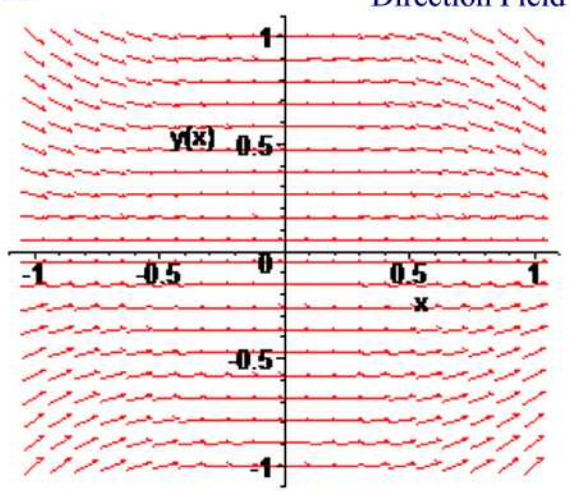
Problem 4.5
Consider the three direction fields shown below. Match each of the direction field with one of the following differential equations.
(a) $y' + y + 0$ (b) $y' + t^2y = 0$ (c) $y' - y = 0$



Direction Field 1



Direction Field 2



Direction Field 3

Problem 4.6
The unique solution to the IVP

$$ty' - \alpha y = 0, y(1) = y_0$$

goes through the points $(2, 1)$ and $(4, 4)$. Find the values of α and y_0.

Problem 4.7
The table below lists values of t and $\ln[y(t)]$ where $y(t)$ is the unique solution to the IVP

$$y' + t^n y = 0, \quad y(0) = y_0.$$

t	1	2	3	4
$\ln[y(t)]$	-0.25	-4.00	-20.25	-64.00

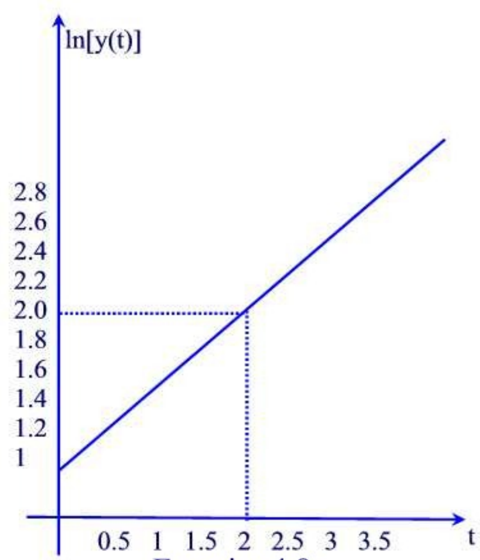
(a) Determine the values of n and y_0.
(b) What is $y(-1)$?

Problem 4.8
The figure below is the graph of $\ln[y(t)]$ versus $t, 0 \le t \le 4$, where $y(t)$ is the solution to the IVP

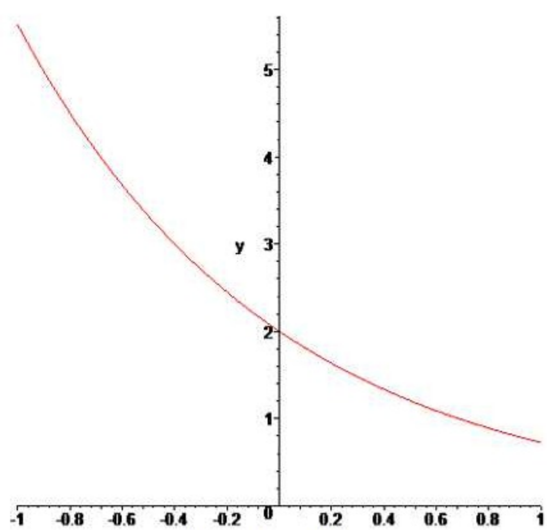
$$y' + p(t)y = 0, \quad y(0) = y_0.$$

Determine $p(t)$ and y_0

Problem 4.9
Given the initial value problem $y' + cy = 0, y(0) = y_0$. A portion of the graph of the solution is shown. Use the information contained in the graph to determine the constants c and y_0.

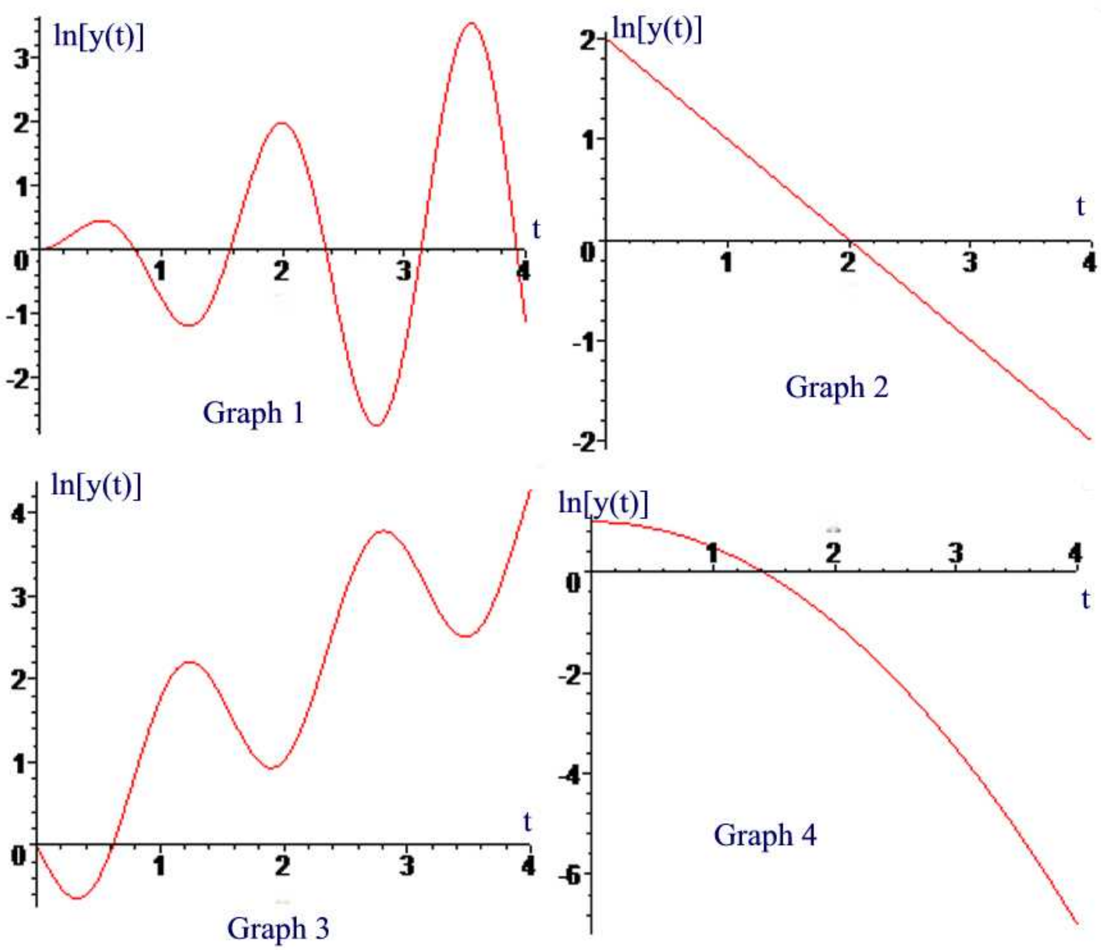


Exercise 4.8



Exercise 4.9

Problem 4.10
Given the four graphs of $\ln[y(t)]$ versus $0 \leq t \leq 4$, corresponding of the four differential equations (a)-(d). Match the graphs to the differential equations. For each match identify the initial condition, $y(0)$.
(a) $y' + y = 0$ (b) $y' - (\sin(4t) + 4t\cos(4t))y = 0$ (c) $y' + ty = 0$ (d) $y' - (1 - 4\cos(4t))y = 0$.



Problem 4.11
Consider the differential equation $y' + p(t)y = 0$. Find $p(t)$ so that $y = \frac{c}{t}$ is the general solution.

Problem 4.12
Consider the differential equation $y' + p(t)y = 0$. Find $p(t)$ so that $y = ct^3$ is the general solution.

Problem 4.13
Solve the initial-value problem: $y' - \frac{3}{t}y = 0,\ y(2) = 8$.

Problem 4.14
Solve the differential equation $y' - 2ty = 0$

Problem 4.15
Solve the initial-value problem $\frac{dP}{dt} - kP = 0, \quad P(0) = P_0$

Problem 4.16
Find the value of t so that $P(t) = \frac{P_0}{2}$ where $P(t)$ is the solution to the initial-value problem $\frac{dP}{dt} = -kP, k > 0, P(0) = P_0$

Problem 4.17
Find the function $f(t)$ that crosses the point $(0, 4)$ and whose slope satisfies $f'(t) = 2f(t)$.

Problem 4.18
Find the general solution to the differential equation $y'' - 2y' = 0$

Problem 4.19
Consider the differential equation: $y' = 3y - 2$
(a) Find the general solution y_h to the equation $y' = 3y$
(b) Show that $y_p = \frac{2}{3}$ is a solution to $y' = 3y - 2$
(c) Show that $y = y_h + y_p$ satisfies the given equation.
(d) Find the solution to the initial-value problem $y' = 3y - 2,\ y(0) = 2$

Problem 4.20
Consider the differential equation $y'' = 3y' - 2$
(a) Find the general solution y_h to the equation $y'' = 3y'$
(b) Show that $y_p = \frac{2}{3}t$ is a solution to $y'' = 3y' - 2$
(c) Show that $y = y_h + y_p$ satisfies the given equation.

5 Solving First Order Linear Non Homogeneous DE: The Method of Integrating Factor

In this section we discuss a technique for solving the first order linear nonhomogeneous equation

$$y' + p(t)y = g(t) \tag{7}$$

where $p(t)$ and $g(t)$ are continuous in $a < t < b$.
Now, since $p(t)$ is continuous then it has an antiderivative namely $\int p(t)dt$. Let $\mu(t) = e^{\int p(t)dt}$. Multiply Equation (44) by $\mu(t)$ and notice that the left hand side of the resulting equation is the derivative of a product. Indeed,

$$\frac{d}{dt}(\mu(t)y) = \mu(t)g(t).$$

Integrate both sides of the last equation with respect to t to obtain

$$\mu(t)y = \int \mu(t)g(t)dt + C$$

Hence,

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)g(t)dt + \frac{C}{\mu(t)}$$

or

$$y(t) = e^{-\int p(t)dt} \int e^{\int p(t)dt} g(t)dt + Ce^{-\int p(t)dt}$$

Notice that the second term of the previous expression is just the general solution for the homogeneous equation

$$y' + p(t)y = 0.$$

The first term is a solution to the nonhomogeneous equation as shown in the next example. Thus, the general solution to Equation (44) is the sum of a particular solution of the nonhomogeneous equation and the general solution of the homogeneous equation. This solution structure will appear again when discussing higher order linear equations and systems of linear equations.

Remark 5.1
1. Notice that multiplying Equation (44) by $\mu(t)$ was a key factor in the integration step discussed above. That's why $\mu(t)$ is known as the **integrating factor.**
2. The above argument remains valid if the indefinite integral is replaced by a definite integral with lower limit t_0 and an upper limit t, where t_0 is a point in the interval (a, b). (See Section 3) ■

Example 5.1
Show that $y_p = e^{-\int p(t)dt} \int e^{\int p(t)dt} g(t)dt$ satisfies Equation (44)

Solution.
We have

$$\begin{array}{rcc} y_p' + p(t)y_p & = & -p(t)e^{-\int p(t)dt} \int e^{\int p(t)dt} g(t)dt + e^{-\int p(t)dt} \cdot e^{\int p(t)dt} g(t) \\ & + & p(t)e^{-\int p(t)dt} \int e^{\int p(t)dt} g(t)dt \\ & = & g(t) \blacksquare \end{array}$$

Example 5.2
Solve the initial value problem

$$y' - \frac{y}{t} = 4t, \quad y(1) = 5$$

Solution.
By Theorem 3.2, the solution is defined on the interval $(0, \infty)$ since 1 belongs to that interval.
We have $p(t) = -\frac{1}{t}$ so that $\mu(t) = \frac{1}{t}$. Multiplying the given equation by the integrating factor and using the product rule we notice that

$$\left(\frac{1}{t}y\right)' = 4$$

Integrating with respect to t and then solving for y we find that the general solution is given by

$$y(t) = t \int 4dt + Ct = 4t^2 + Ct.$$

Since $y(1) = 5$ then $C = 1$ and hence the unique solution to the IVP is $y(t) = 4t^2 + t$ ■

Example 5.3
Find the general solution to the equation

$$y' + \frac{2}{t}y = \ln t,\ t > 0$$

Solution.
The integrating factor is $\mu(t) = e^{\int \frac{2}{t}dt} = t^2$. Multiplying the given equation by t^2 to obtain

$$(t^2y)' = t^2 \ln t$$

Integrating with respect to t we find

$$t^2y = \int t^2 \ln t dt + C$$

The integral on the right-hand side is evaluated using integration by parts with $u = \ln t, dv = t^2dt, du = \frac{dt}{t}, v = \frac{t^3}{3}$ obtaining

$$t^2y = \frac{t^3}{3}\ln t - \frac{t^3}{9} + C$$

Thus,

$$y = \frac{t}{3}\ln t - \frac{t}{9} + \frac{C}{t^2} \blacksquare$$

Problem 5.1
Solve the IVP: $y' + 2ty = t, \quad y(0) = 0$

Problem 5.2
Find the general solution: $y' + 3y = t + e^{-3t}$

Problem 5.3
Find the general solution: $y' + \frac{1}{t}y = 3\cos t,\ t > 0$

Problem 5.4
Find the general solution: $y' + 2y = \cos(3t)$.

Problem 5.5
Find the general solution: $y' + (\cos t)y = -3\cos t$.

Problem 5.6
Given that the solution to the IVP $ty' + 4y = \alpha t^2,\ y(1) = -\frac{1}{3}$ exists on the interval $-\infty < t < \infty$. What is the value of the constant α?

Problem 5.7
Suppose that $y(t) = Ce^{-2t} + t + 1$ is the general solution to the equation $y' + p(t)y = g(t)$. Determine the functions $p(t)$ and $g(t)$.

Problem 5.8
Suppose that $y(t) = -2e^{-t} + e^t + \sin t$ is the unique solution to the IVP $y' + y = g(t),\ y(0) = y_0$. Determine the constant y_0 and the function $g(t)$.

Problem 5.9
Find the value (if any) of the unique solution to the IVP $y' + (1 + \cos t)y = 1 + \cos t,\ y(0) = 3$ in the long run?

Case when either $p(t)$ or $g(t)$ has a jump discontinuity
Consider the IVP

$$y' + p(t)y = g(t), \quad y(t_0) = y_0, \quad a \le t_0 \le b$$

where either $p(t)$ or $g(t)$ has a jump discontinuity at $a < c < b$.
To solve this problem, we first solve the initial value problem on the interval $a \le t < c$ where both $p(t)$ and $g(t)$ are continuous. Theorem 3.2 asserts the existence of a unique solution $y_1(t)$ for $a \le t < c$. Also, $y_1(t)$ has a left-hand limit, i.e.,

$$\lim_{t \to c^-} y_1(t) = y_1(c^-)$$

Next, since $p(t)$ and $g(t)$ are continuous on $c \le t \le b$ then Theorem 3.2 asserts the existence of a unique solution $y_2(t)$ to the IVP

$$y' + p(t)y = g(t), y(c) = y_1(c^-)$$

The unique solution to the original IVP is then given by

$$y(t) = \begin{cases} y_1(t) & \text{if } a \le t < c \\ y_2(t) & \text{if } c \le t \le b \end{cases}$$

This solution is continuous on the interval $[a, b]$ but not differentiable at $t = c$. We will illustrate this in the next example.

Example 5.4
Find the solution to the IVP

$$y' + \frac{1}{t}y = g(t), \quad y(1) = 1$$

where

$$g(t) = \begin{cases} 3t & \text{if } 1 \le t \le 2 \\ 0 & \text{if } 2 < t \le 3 \end{cases}$$

The graph of $g(t)$ is given in Figure 5.1.

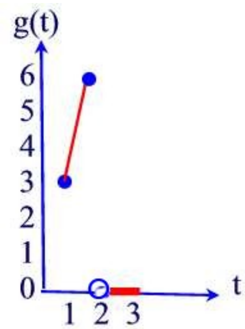


Figure 5.1

Solution.
First, we solve the IVP

$$y' + \frac{1}{t}y = 3t, \quad y(1) = 1, \quad 1 \le t \le 2$$

The integrating factor is $\mu(t) = t$ and the general solution is $y_1(t) = t^2 + \frac{C}{t}$. Since $y(1) = 1$ then $C = 0$. Hence, $y_1(t) = t^2$ and $y_1(2) = 4$.
Next, we solve the IVP

$$y' + \frac{1}{t}y = 0, \quad y(2) = 4, \quad 2 < t \le 3$$

The integrating factor is $\mu(t) = t$ and the general solution is $y_2(t) = \frac{C}{t}$. Since $y_2(2) = 4$ then $C = 8$. Thus,

$$y(t) = \begin{cases} t^2 & \text{if } 1 \le t \le 2 \\ \frac{8}{t} & \text{if } 2 < t \le 3 \end{cases} \blacksquare$$

The graph of $y(t)$ is given in Figure 5.2.

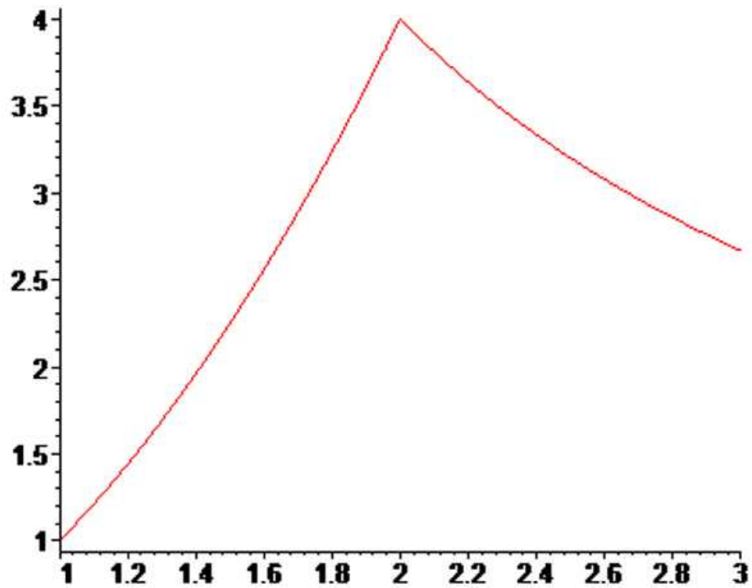


Figure 5.2

As you can see from the graph, $y(t)$ is continuous on $[1, 3]$ but not differentiable at $t = 2$ ■

Problem 5.10
Find the solution to the IVP

$$y' + p(t)y = 2, \quad y(0) = 1$$

where

$$p(t) = \begin{cases} 0 & \text{if } 0 \le t \le 1 \\ \frac{1}{t} & \text{if } 1 < t \le 2 \end{cases}$$

Problem 5.11
Find the solution to the IVP

$$y' + (\sin t)y = g(t), \quad y(0) = 3$$

where

$$g(t) = \begin{cases} \sin t & \text{if } 0 \le t \le \pi \\ -\sin t & \text{if } \pi < t \le 2\pi \end{cases}$$

Problem 5.12
Find the solution to the IVP

$$y' + y = g(t), \ t > 0, \quad y(0) = 3$$

where

$$g(t) = \begin{cases} 1 & \text{if } 0 \le t \le 1 \\ 0 & \text{if } t > 1 \end{cases}$$

Sketch an accurate graph of the solution, and discuss the long-term behavior of the solution. Is the solution differentiable on the interval $t > 0$? Explain your answer.

Problem 5.13
Find the solution to the IVP

$$y' + p(t)y = 0, \quad y(0) = 3$$

where

$$p(t) = \begin{cases} 2t - 1 & \text{if } 0 \le t \le 1 \\ 0 & \text{if } 1 < t \le 3 \\ -\frac{1}{t} & \text{if } 3 < t \le 4 \end{cases}$$

Problem 5.14
Solve $y' - \frac{1}{t}y = \sin t, \quad y(1) = 3$. Express your answer in terms of the **sine integral**, $Si(t) = \int_0^t \frac{\sin s}{s} ds$.

Problem 5.15
Solve the initial-value problem $ty' + 2y = t^2 - t + 1, \quad y(1) = \frac{1}{2}$

Problem 5.16
Solve the initial-value problem $y' + y = e^t y^2, y(0) = 1$ using the substitution $u(t) = \frac{1}{y(t}$

Problem 5.17
Show that if a and λ are positive constants, and b is any real number, then every solution of the equation

$$y' + ay = be^{-\lambda t}$$

has the property that $y \to 0$ as $t \to \infty$. Hint: Consider the cases $a = \lambda$ and $a \neq \lambda$ separately.

Problem 5.18
Solve the initial value problem $ty' = y + t, \quad y(1) = 7$

Problem 5.19
Solve the differential equation $y' = -ay + b$ without using the susbtitution $w = -ay + b$ where a and b are constants with $a \neq 0$ and $y(t) \neq \frac{b}{a}$.

Problem 5.20
Consider the following method of solving the equation

$$y' + p(t)y = g(t)$$

(a) Show that $y_h(t) = Ce^{-\int p(t)dt}$ is the general solution to the homogeneous equation $y' + p(t)y = 0$.
(b) Find a funcion $u(t)$ such that $y_p(t) = u(t)e^{-\int p(t)dt}$ is a solution to the nonhomogeneous equation.
This technique of finding a solution to the nonhomogeneous equation is known as the method of **variation of parameters**.

6 Modeling with First Order Linear Differential Equations

What is modeling? The process of representing a phenomenon mathematically, i.e. by means of a function or an equation, is referred to as **mathematical modeling**.
One of the mathematical modeling approach to problem solving consists of the following five steps:
1. Ask a question.
2. Set up a model.
3. Formulate the mathematical model.
4. Solve the mathematical model.
5. Answer the question.

We can summarise these steps of modeling into three stages: *formulation, solution, and application.* The formulation stage consists of steps 1 through 3. The solution stage consists of step 4, and the application stage consists of step 5. These stages are important in modeling; however, not all modeling will follow this exact pattern. This is just a guide to what modeling is about. Most mathematical models in the physical sciences, engineering, and economics require solving differential equations. In this and the next section we discuss few applications of first order linear differential equations.

Compound Interest
The term **compound interest** refers to a procedure for computing interest whereby the interest for a specified interest period is added to the original principal. The resulting sum becomes a new principal for the next interest period. The interest earned in the earlier interest periods earn interest in the future interest periods.
Suppose that you deposit P dollars into a saving account that pays annual interest r and the bank agrees to pay the interest at the end of each time period(usually expressed as a fraction of a year). If the number of periods in a year is n then we say that the interest is **compounded** n times per year (e.g.,'yearly'=1, 'quarterly'=4, 'monthly'=12, etc.). Thus, at the end of the first period the balance will be

$$B = P + \frac{r}{n}P = P\left(1 + \frac{r}{n}\right).$$

At the end of the second period the balance is given by

$$B = P\left(1+\frac{r}{n}\right) + \frac{r}{n}P\left(1+\frac{r}{n}\right) = P\left(1+\frac{r}{n}\right)^2.$$

Continuing in this fashion, we find that the balance at the end of the first year, i.e. after n periods, is

$$B = P\left(1+\frac{r}{n}\right)^n.$$

If the investment extends to another year then the balance would be given by

$$P\left(1+\frac{r}{n}\right)^{2n}.$$

For an investment of t years the balance is given by

$$B = P\left(1+\frac{r}{n}\right)^{nt}.$$

Since $\left(1+\frac{r}{n}\right)^{nt} = \left[\left(1+\frac{r}{n}\right)^n\right]^t$ then the function B can be written in the form $B(t) = Pa^t$ where $a = \left(1+\frac{r}{n}\right)^n$. That is, B is an exponential function.

Remark 6.1
Interest given by banks are known as **nominal rate** (e.g. "in name only"). When interest is compounded more frequently than once a year, the account effectively earns more than the nominal rate. Thus, we distinguish between nominal rate and **effective rate.** The effective annual rate tells how much interest the investment actually earns. The quantity $(1+\frac{r}{n})^n - 1$ is known as the **effective interest rate**.

Example 6.1
Translating a value to the future is referred to as **compounding**. What will be the maturity value of an investment of \$15,000 invested for four years at 9.5% compounded semi-annually?

Solution.
Using the formula for compound interest with $P = \$15,000, t = 4, n = 2$, and $r = .095$ we obtain

$$B = 15,000\left(1+\frac{0.095}{2}\right)^8 \approx \$21,743.20 \blacksquare$$

Problem 6.1
Translating a value to the present is referred to as **discounting.** We call $(1+\frac{r}{n})^{-nt}$ the **discount factor**. What principal invested today will amount to $\$8,000$ in 4 years if it is invested at 8% compounded quarterly?

Problem 6.2
What is the effective rate of interest corresponding to a nominal interest rate of 5% compounded quarterly?

Problem 6.3
Suppose you invested \$1200 on January 1 of this year in an account at an annual rate of 6%, compounded monthly.
1. Set up (write down) the equation that models this problem.
2. Determine your account balance after 5 years.

Continuous Compound Interest
When the compound formula is used over smaller time periods the interest becomes slightly larger and larger. That is, frequent compounding earns a higher effective rate, though the increase is small.
This suggests compounding more and more, or equivalently, finding the value of B in the long run. In Calculus, it can be shown that the expression $\left(1+\frac{r}{n}\right)^n$ approaches e^r as $n \to \infty$, where e (named after Euler) is a number whose value is $e = 2.71828\cdots$. The balance formula reduces to $B(t) = Pe^{rt}$. This formula is known as the **continuous compound formula**. In this case, the annual effective interest rate is found using the formula $e^r - 1$.

Remark 6.2
Notice that $B(t) = Pe^{rt}$ is the unique solution to the IVP

$$\frac{dB}{dt} = rB, B(0) = P$$

Example 6.2
Find the effective rate if \$1000 is deposited at 5% annual interest rate compounded continuously.

Solution.
The effective interest rate is $e^{0.05} - 1 \approx 0.05127 = 5.127\%$■

Example 6.3
An amount of \$3,000.00 is deposited in a bank paying an annual interest rate of 3 %, compounded continuously.
(a) Find the balance after 4 years.
(b) How long would it take for the money to double?

Solution.
Use the continuous compound interest formula, $B = Pe^{rt}$, with $P = 3000, r = 3/100 = 0.03, t = 4$.
(a) Therefore,

$$B(4) = 3000e^{0.03(4)} \approx \$3382.49$$

(b) Since the original investment is \$3,000.00, doubling means that the current balance is \$6,000.00. To find out how long it takes for this to happen (i.e. to find t), plug in $P = 3000, B = 6000$, and $r = 0.03$ in the continuous compound interest formula, and solve for t. Doing this, one gets,

$$\begin{array}{ccc} 3000e^{0.03t} & = & 6000 \\ e^{0.03t} & = & 2 \\ 0.03t & = & \ln 2 \\ t & = & \frac{\ln 2}{0.03} \approx 23.1 \; years \; \blacksquare \end{array}$$

Problem 6.4
Which is better: An account that pays 8% annual interest rate compounded quarterly or an account that pays 7.95% compounded continuously?

Problem 6.5
An amount of \$2,000.00 is deposited in a bank paying an annual interest rate of 2.85 %, compounded continuously.
(a) Find the balance after 3 years.
(b) How long would it take for the money to double?

Radioactive Decay
All materials are made of atoms. Radioactive atoms are unstable; that is, they have too much energy. When radioactive atoms release their extra energy, they are said to **decay.** All radioactive atoms decay. After releasing all their excess energy, the atoms become stable and are no longer radioactive. In order to understand this decaying process, we begin with a description of the atom. Atoms are made up of three subatomic particles: protons,

neutrons, and electrons. The protons and neutrons are packed together in the nucleus at the center of the atom (See Figure 6.1). The electrons orbit the nucleus. The number of protons in the nucleus determines what material (element) the atom is. For example, if the nucleus contains 8 protons, the atom is oxygen. If the nucleus contains 17 protons, the atom is chlorine.

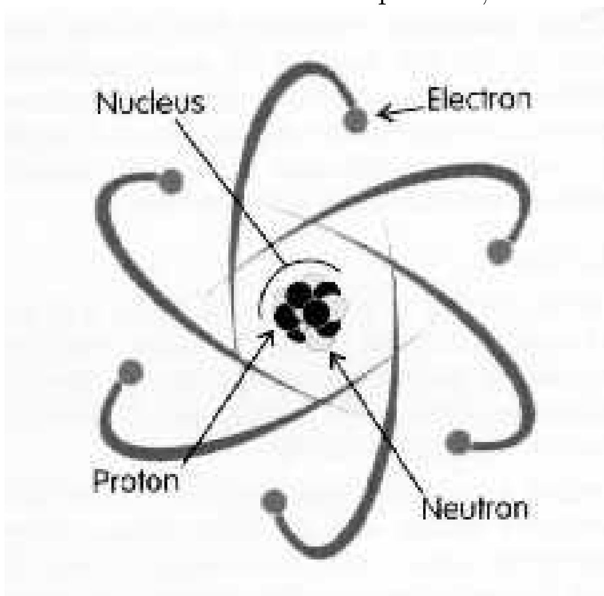


Figure 6.1

While all atoms of the same element have the same number of protons, it is possible for atoms of one element to have different numbers of neutrons. Atoms of the same element with different numbers of neutrons are called **isotopes.** For example, all atoms of the element carbon have 6 protons, but while most carbon atoms have 6 neutrons, some have 7 or 8. Isotopes are named by giving the name of the element followed by the sum of the neutrons and protons in the isotope's nucleus. So a carbon atom with 6 protons and 6 neutrons in its nucleus is called Carbon-12. The carbon atom with 8 neutrons is called Carbon-14.

When the nucleus of a radioactive isotope gives up its extra energy, that energy is called **ionizing radiation.** Ionizing radiation may take the form of alpha particles, beta particles, or gamma rays. The process of emitting the radiation is called **radioactive decay.** As the atoms decay, the rate of change of the mass of the radioactive isotope is proportional to the mass present. If $m(t)$ denotes the mass of radioactive isotope at time t then by the above statement we have

$$\frac{dm}{dt} = -km, k > 0$$

which is a first order linear differential equation with general solution $m(t) = Ce^{-kt}$. If m_0 denotes the initial mass then

$$m(t) = m_0 e^{-kt}$$

Half- Life: If $t_{\frac{1}{2}}$ is the time it takes for the radioactive substance to reduce to half its initial amount, then $m(t_{\frac{1}{2}}) = \frac{m(0)}{2}$. Solving for $t_{\frac{1}{2}}$ we find

$$\begin{array}{rcl} m(t_{\frac{1}{2}}) & = & \frac{m(0)}{2} \\ \\ m(0)e^{kt_{\frac{1}{2}}} & = & \frac{m(0)}{2} \\ \\ e^{kt_{\frac{1}{2}}} & = & \frac{1}{2} \\ \\ kt_{\frac{1}{2}} & = & -\ln 2 \\ \\ t_{\frac{1}{2}} & = & -\frac{\ln 2}{k} \end{array}$$

$t_{\frac{1}{2}}$ is called the **half-life** of the radioactive substance.

Example 6.4

The mass (in grams) of radioactive material in a sample is given by $m(t) = 100e^{-0.0017t}$, where t is measured in years. Find the half-life of this radioactive substance.

Solution.

The mass of the radioactive material is 100 grams at time $t = 0$. Therefore the half-life is the amount of time necessary for the sample to decay to 50 grams. So we can find the half-life by setting $m(t_{\frac{1}{2}})$ equal to 50 and solving for $t_{\frac{1}{2}}$.

$$\begin{array}{rcl} 100e^{-0.0017t_{\frac{1}{2}}} & = & 50 \\ e^{-0.0017t_{\frac{1}{2}}} & = & 0.5 \\ -0.0017t_{\frac{1}{2}} & = & \ln(0.5) \\ t_{\frac{1}{2}} & = & \frac{\ln(0.5)}{-0.0017} \approx 408 \ years \ \blacksquare \end{array}$$

Example 6.5
The half-life of Phosphorus is 14 days. Find an exponential model for this rate of decay and use it to determine the percentage of Phosphorus in substance which is left after 35 days.

Solution.
The model's equation is $m(t) = m_0e^{-kt}$. Since the half-life is 14 then $m_0e^{-14k} = \frac{m_0}{2}$. Solving for k we find $k \approx 0.0495105129$. Hence, $m(t) = m_0e^{-0.0495105129t}$. The percentage of amount remaining after 35 days is

$$\frac{m(35)}{m_0} = e^{-0.0495105129(35)} \approx 17.68\% \blacksquare$$

Problem 6.6
Carbon-14 is a radioactive isotope of carbon that has a half life of 5600 years. It is used extensively in dating organic material that is tens of thousands of years old. What fraction of the original amount of Carbon-14 in a sample would be present after 10,000 years?

Problem 6.7
In 1986 the Chernobyl nuclear power plant exploded, and scattered radioactive material over Europe. Of particular note were the two radioactive elements iodine-131 whose half-life is 8 days and cesium-137 whose half life is 30 years. Predict how much of this material would remain over time.

Problem 6.8
A team of archaeologists thinks they may have discovered Fred Flintsone's fossilized bowling ball. But they want to determine whether the fossil is authentic before they report their discovery to ABC's "Nightline." (Otherwise they run the risk of showing up on "Hard Copy" instead.) Fortunately, one of the scientists is a graduate of ATU's Math 3243, so he calls upon his experience as follows:
The radioactive substance (Carbon 14) has a half-life of 5730 years. By measuring the amount of Carbon present in a fossil, scientists can estimate how old the fossil is.
Analysis of the "Flinstone bowling ball" determines that 15% of the radioactive substance has already decayed. How old is the fossil ?

Problem 6.9
The half-life of Iodine-123 is about 13 hours. You begin with 50 grams of this substance. What is a formula for the amount of Iodine-123 remaining after t hours?

Population Models
We will examine the way that a simple differential equation arises when we study the phenomenon of population growth of species in a well-defined environment which we call a **colony**.
We will let $N(t)$ be the number of species in a population at time t. The population will change with time. Indeed the rate of change of N will be due to births or migration into the colony (that increase N) and deaths or migration out of the colony (that decrease it). By the "conservation of population law" we have

Rate of change of N = rate of pop. increase - rate of pop. decrease.

Now, let r_b and r_d be positive constants representing the birth and death rates per unit population. In general, for a given population, these would have certain numerical values that one could obtain by experiment, by observation, or by simple assumptions. Then $r_bN(t)$ represents the rate of population increase through births at time t. Similarly, $r_dN(t)$ represents the rate of population decrease through deaths at time t. Let $M(t)$ denote the migration rate at time t. Note that $M(t) > 0$ when the rate of immigration into the colony exceeds the exodus rate and $M(t) < 0$ otherwise. Thus, by the conservation of population law we arrive at the differential equation

$$\frac{dN}{dt} = r_bN - r_dN + M(t).$$

Letting $k = r_b - r_d$ the previous equation reduces to

$$\frac{dN}{dt} = kN + M(t).$$

We solve the previous equation when no migration exists, i.e., $M(t) = 0$. In this case, we have

$$\begin{array}{ccc}
N' & = & kN \\
N' - kN & = & 0 \\
(e^{-kt}N)' & = & 0 \\
\int_0^t (e^{-ks}N)'ds & = & 0 \\
e^{-kt}N(t) - N(0) & = & 0 \\
N(t) & = & N(0)e^{kt}
\end{array}$$

Note that the population will grow provided $k > 0$ which happens when $r_b - r_d > 0$ i.e. when more people were born then dead. In this case, we call k the **growth rate.** Similarly, if $k < 0$, or equivalently, $r_b < r_d$ then more people die on average than are born. k is called the **decay rate.**

Example 6.6
The population of Mexico city grows by 2.6% per year. In 1980 the population was 67.38 million. Find a formula for it.

Solution.
Let $N(t)$ be the population at time t [years] after 1980. To say that $N(t)$ grows by $k = 2\%$ per year means that

$$\frac{dN}{dt} = kN$$

where $k = 2\% = \frac{2}{100} = 0.02$. Thus, $N(t) = N(0)e^{0.02t}$. But $N(0) = 67.38$ so that

$$N(t) = 67.38e^{0.02t} \text{ } millions \blacksquare$$

Example 6.7
Suppose the population of a certain country was 56 million in 2000 and the natural rate of the growth of the population was 2% per year. Moreover, suppose $k(t)$ represents the net rate of growth of the population due to immigration and emigration t years after 2000.
(a) Let $y(t)$ be the population of the country t years after 2000. Write down the initial value problem involving y.
(b) Solve the equation if $k(t) = 0.04t$
(c) What does this model predict for the population of the country in the year 2010?
(d) When will the population of the country reach 100 million?

Solution.
(a) $y(t)$ satisfies the initial value problem

$$\frac{dy}{dt} = 0.02y + k(t), \quad y(0) = 56$$

(b) Rewriting the equation in part (a) as $y' - 0.02y = 0.04t$ so that $p(t) = -0.02$ and $g(t) = 0.04t$. Using the integrating factor method with $\mu(t) =$

$e^{-0.02t}$ we find

$$\begin{array}{rcl} y(t) & = & e^{0.02t}\int e^{-0.02t}(0.04t)dt + Ce^{0.02t} \\ & = & e^{0.02t}(0.04)\left(-\frac{t}{0.02}e^{-0.02t} - \frac{1}{(0.02)^2}e^{-0.02t}\right) + Ce^{0.02t} \\ & = & 2\left(-t - \frac{1}{0.02}\right) + Ce^{0.02t} \end{array}$$

Since $y(0) = 56$ then $C = 56 + 100 = 156$ so that

$$y(t) = 2\left(-t - \frac{1}{0.02}\right) + 156e^{0.02t}$$

(c) $y(10) = 2\left(-10 - \frac{1}{0.02}\right) + 156e^{0.02(10)} \approx 70.54$ *million*
(d) We set the equation $2\left(-t - \frac{1}{0.02}\right) + 156e^{0.02t} = 100$. Solving this equation for t using a graphing calculator we find $t \approx 23$ *years* ■

Problem 6.10
Statistics indicate that the world population since World War II has been growing at the rate of 1.9% per year. Further, United Nations records indicate that the world population in 1975 was (approximately) 4 billion. Assuming an exponential growth model.
(a) What will the population of the world be in the year 2000?
(b) When will the world population be 7 billion?

Problem 6.11
During the 1980s the population of a certain city went from 100,000 to 205,000. Populations by year are listed in the table below. $N(t)$ is the population (in thousands) at time t (in years).

Year	1980	1981	1982	1983	1984	1985	1986	1987	1988	1989
N(t)	100	108	117	127	138	149	162	175	190	205

(a) Use your calculator (i.e. exponential regression) to show that the population satisfies an equation of the form $N(t) = N(0)e^{kt}$.
(b) Use the model to predict the population of the city in 1994.
(c) According to our model, when will the population reach 300 thousand?

Problem 6.12
The population of fish in a pond is modeled by the differential equation

$$\frac{dN}{dt} = 480 - 4N$$

where time t is measured in years.
(a) Towards what number does the population of fish tend?
(b) If there are initially 10 fish in the pond, how long does it take for the number of fish to reach 90% of the eventual population?

Problem 6.13
The number of bacteria in a liquid culture is observed to grow at a rate proportional to the number of cells present. At the begining of the experiment there are 10,000 cells and after three hours there are 500,000. How many will there be after one day of growth if this unlimited growth continues? What is the doubling time of the bacteria, i.e. the amount of time it takes for the population to double in size.?

Problem 6.14
Bacteria is being cultured for the production of medication. Without harvesting the bacteria, the rate of change of the population is proportional to its current population, with a proportionality constant of 0.2 per hour. Also, the bacteria are being harvested at a rate of 1000 per hour. If there are initially 8000 bacteria in the culture, solve the initial value problem:

$$\frac{dN}{dt} = 0.2N - 1000, \; N(0) = 8000$$

for the number N of bacteria as a function of time and find the time it takes for the population to double its initial number.

Problem 6.15
A small lake supports a population of fish which, under normal circumstances, enjoys a natural birth process with birth rate $r > 0$. However, a fishing company has just discovered the lake and is now drawing fish out of the lake at a rate of h fish per day. A model capturing this situation is:

$$\frac{dP}{dt} = -h + rP, P(0) = P_0$$

(a) Find the equilibrium level P_e of fish in the lake.
(b) Find $P(t)$ explicitly (i.e. solve the initial value problem.)

Problem 6.16
The population of mosquitoes in a certain area increases at a rate proportional to the current population and, in the absence of other factors, the

population doubles each week. There are 200,000 mosquitoes in the area initially, and predators (birds, etc.) eat 20,000 mosquitoes per day. Determine the population of mosquitoes in the area at any time.

Problem 6.17
At the time of the 1990 census the city of Renton, WA had a population of 8000 people. The last (2000) census revealed that the population of Renton was 12000 people. The city planners do not wish to limit growth until the population reaches 18000. Assuming the rate of change of the population is proportional to the population, when will this occur?

Problem 6.18
If initially there are 50 grams of a radioactive substance and after 3 days there are only 10 grams remaining, what percentage of the original amount remains after 4 days?

Problem 6.19
The half-life of radioactive cobalt is 5.27 years. A sample of radioactive cobalt weighing 100 kilograms is buried in a nuclear waste storage facility. After 200 years, how much cobalt will remain in the sample? (Give the answer in exact form, involving a fractional power of 2.)

7 Additional Applications: Mixing Problems and Cooling Problems

In this section we discuss two additional problems modeled by first order linear differential equations: mixing problems and cooling problems.

Mixing Models

All mixing problems we consider here will involve a tank into which a certain mixture will be added at a certain input rate and the mixture will leave the system at a certain output rate. We shall always reserve $y = y(t)$ to denote the amount of substance in the tank at any given time t.

The differential equation involved here arises from the following natural relationship:

$$\frac{dy}{dt} = \textit{input rate} - \textit{output rate}.$$

The main assumption that we will be using here is that the concentration of the substance in the liquid is uniform throughout the tank. Clearly this will not be the case, but if we allow the concentration to vary depending on the location in the tank the problem becomes very difficult and will involve partial differential equations, which is not the focus of this course.

Consider a tank initially containing a volume V_0 of mixture (substance and liquid) of concentration c_0. Then the initial amount of the substance is given by $y_0 = c_0V_0$.

Suppose a mixture of concentration $c_i(t)$ flows into the tank at the volume rate $r_i(t)$. Then the substance is entering the tank at the rate $c_i(t)r_i$. Suppose that the well-mixed solution is pumped out of the tank at the volume rate $r_o(t)$. The concentration of this outflow is $\frac{y(t)}{V(t)}$ where $V(t)$ is the current volume of solution in the tank. Then clearly

$$\frac{dy}{dt} = c_i(t)r_i(t) - \frac{y(t)}{V(t)}r_o(t), \quad y(0) = y_0$$

and

$$\frac{dV}{dt} = r_i(t) - r_o(t).$$

Solving the last equation we find

$$V(t) = V_0 + \int_0^t (r_i(s) - r_o(s))ds.$$

Example 7.1
Consider a tank with volume 600 liters containing a salt solution with concentration of $\frac{1}{15}$ $kg/liter$. Suppose a solution with 1/5 kg/liter of salt flows into the tank at a rate of 25 liters/min. The solution in the tank is well-mixed. Solution flows out of the tank at a rate of 50 liters/min. If initially there is 40 kg of salt in the tank, how much salt will be in the tank as a function of time?

Solution.
Since the inflow rate is different from the outflow rate then the volume at any time t satisfies $\frac{dV}{dt} = 25-50 = -25$ $liters/min$ so that $V(t) = -25t+C$. But $V(0) = 600$ so that $C = 600$. Thus, $V(t) = -25t + 600$. If $y(t)$ is the amount of salt in the tank at any time t then

$$y' = \frac{1}{5} \times 25 - \frac{y}{600-25t} \times 50, \quad y(0) = 40$$

or

$$y' + \frac{2y}{24-t} = 5, \quad y(0) = 40.$$

The integrating factor is $\mu(t) = e^{\int \frac{2dt}{24-t}} = e^{-\ln(24-t)^2} = \frac{1}{(24-t)^2}$. Thus, the general solution is

$$y(t) = (24-t)^2 \int 5(24-t)^{-2} dt + C(24-t)^2 = 5(24-t) + C(24-t)^2$$

Since $y(0) = 40$ then $C = -\frac{5}{36}$. Thus,

$$y(t) = 5(24-t) - \frac{5}{36}(24-t)^2$$

The graph of $y(t)$ is shown in Figure 7.1 ■

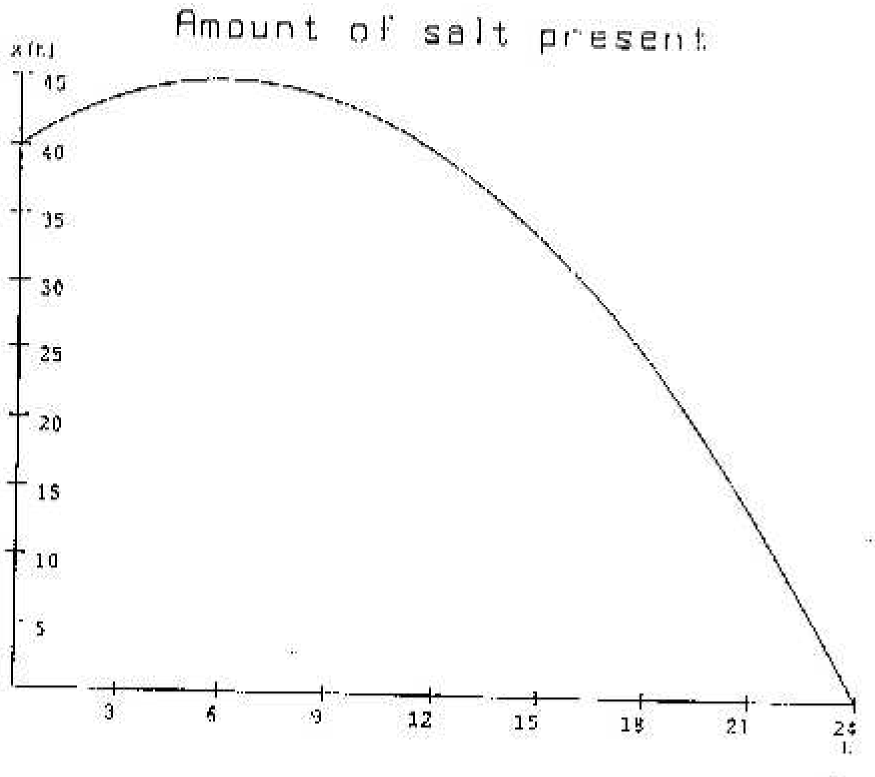


Figure 7.1

Problem 7.1
Consider a tank with volume 100 liters containing a salt solution. Suppose a solution with 2kg/liter of salt flows into the tank at a rate of 5 liters/min. The solution in the tank is well-mixed. Solution flows out of the tank at a rate of 5 liters/min. If initially there is 20 kg of salt in the tank, how much salt will be in the tank as a function of time?

Problem 7.2
A tank initially contains 50 gal of pure water. A solution containing 2 lb/gal of salt is pumped into the tank at 3 gal/min. The mixture is stirred constantly and flows out at the same rate of 3 gal/min.
(a) What initial-value problem is satisfied by the amount of salt $y(t)$ in the tank at time t?

(b) What is the actual amount of salt in the tank at time t?
(c) How much salt is in the tank at after 20 minutes?
(d) How much salt in in the tank after a long time?

Problem 7.3
Brine containing 1 lb/gal of salt is poured at 1 gal/min into a tank that initially contained 100 gal of fresh water. The stirred mixture is drained off at 2 gal/min.
(a) What initial value problem is satisfied by the amount of salt in it?
(b) What is the formula for this amount of salt?

Problem 7.4
Consider a large tank holding 1000 L of pure water into which a brine solution of salt begins to flow at a constant rate of 6 L/min. The solution inside the tank is kept well stirred, and is flowing out of the tank at a rate of 6 L/min. If the concentration of salt in the brine solution entering the tank is 0.1 Kg/L, determine when the concentration of salt will reach 0.05 Kg/L.

Problem 7.5
A tank containing chocolate milk initially contains a mixture of 460 gallons of milk and 40 gallons of chocolate syrup. Milk is added to the tank at the rate of 8 gallons per minute and syrup is added at a rate of 2 gallons per minute. At the same time, chocolate milk is withdrawn at the rate of 10 gallons per minute. Assuming perfect mixing of milk and syrup:
(a) Write up an initial value problem for the amount of syrup in the tank.
(b) Determine how much syrup will be in the tank over a long time.
(c) Determine how much syrup will be in the tank after 10 minutes.

Problem 7.6
A tank contains 100 L of water with 5kg of salt initially. An inlet pipe adds salt water with concentration of 2 kg/L at the constant rate of 10 L/min. The solution is well-stirred and is flowing out of the tank at the rate of 10 L/min. Give the IVP for the amount of salt $y(t)$ in the tank at time t. Solve the IVP and determine $y(2)$.

Problem 7.7
A tank initially contains 120 liters of pure water. A mixture containing a concentration of γ g/liter of salt enters the tank at the rate of 2 liters/min,

and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of γ for the amount of salt in the tank at any time t. Also find the limiting amount of salt in the tank at $t \to \infty$.

Problem 7.8
Consider a large tank holding 2,000 gallons of brine solution, initially containing 10 lbs of salt. At time t = 0, more brine solution begins to flow into the tank at the rate of 2 gal/min. The concentration of salt in the solution entering the tank is $3e^{-t}$ lbs/gal, i.e. varies in time. The solution inside the tank is well-stirred and is flowing out of the tank at the rate of 5 gal/min. Write down the initial value problem giving $y(t)$ = the amount of salt in the tank (in lbs.) at time t. Do not solve for $y(t)$.

Cooling and Heating Models
We are all aware of instances in which a coroner is required to determine the approximate time of death of a homicide victim. Knowing something about how fast the temperature of a human body cools down from $98.6°F$ to room temperature can be of significant aid in the coroner's conclusion. A law of physics useful in such cases is called **Newton's Law of Cooling.**
Newton's Law of Cooling states that the rate of change of the temperature of an object is proportional to the difference between its own temperature and the ambient temperature. If $H(t)$ denotes the temperature of the object at time t and S the temperature of the surrounding environment then $H(t)$ satisfies the following differential equation

$$H' = k(S - H), k > 0.$$

Using the method of integrating factor with $\mu(t) = e^{kt}$ we can find a formula for $H(t)$ as follows:

$$H(t) = e^{-kt}\int e^{kt}(kS)dt + Ce^{-kt} = S + Ce^{-kt}$$

Example 7.2
A boiling ($100°C$) solution is set on a table where room temperature is assumed to be constant at $20°C$. The solution cooled to $60°C$ after five minutes.

(a) Find a formula for the temperature (H) of the solution, t minutes after it is placed on the table.
(b) Determine how long it will take for the solution to cool to $22°C$.

Solution.
(a) We are asked to find an explicit formula for H in terms of t. We know this is a heating and cooling question so Newton's law of cooling tells us

$$\frac{dH}{dt} = k(S - H)$$

for some constant $k > 0$. So letting $S = 20$, we have:

$$\tfrac{dH}{dt} = k(20 - H)$$

Solving this equation using the method of integrating factor we find the solution

$$H(t) = Ce^{-kt} + 20$$

Since the initial temperature of the solution was $100°C$, we know that $H(0) = 100$, so the last line above gives:

$$100 = C + 20 \to C = 80.$$

So we now have:

$$H(t) = 80e^{-kt} + 20.$$

Now since $H(5) = 60$ then

$$60 = 80e^{-5k} + 20 \to e^{-5k} = \frac{1}{2} \to -5k = -\ln 2 \to k = \frac{\ln 2}{5} \approx 0.013863.$$

Hence,

$$H(t) = 80e^{-0.013863} + 20.$$

(b) We wish to find out what t is when H is 22. We use the formula we just found in part (a):

$$\begin{aligned}
80e^{-0.013863t} + 20 &= 22 \\
80e^{-0.013863t} &= 2 \\
e^{-0.013863t} &= \tfrac{1}{40} \\
-0.013863t &= -\ln 40 \\
t &= \tfrac{\ln 40}{0.013863} \approx 26.2 \ minutes \ \blacksquare
\end{aligned}$$

Problem 7.9
As part of his summer job at a restaurant, Jim learned to cook up a big pot of soup late at night, just before closing time, so that there would be plenty of soup to feed customers the next day. He also found out that, while refrigeration was essential to preserve the soup overnight, the soup was too hot to be put directly into the fridge when it was ready. (The soup had just boiled at $100°C$, and the fridge was not powerful enough to accomodate a big pot of soup if it was any warmer than $20°C$). Jim discovered that by cooling the pot in a sink full of cold water, (kept running, so that its temperature was roughly constant at $5°C$) and stirring occasionally, he could bring the temperature of the soup to $60°C$ in ten minutes. How long before closing time should the soup be ready so that Jim could put it in the fridge and leave on time ?

Problem 7.10 *(Determinating the Time of Death)*
Police arrive at the scene of a murder at 12 am. They immediately take and record the body's temperature, which is $90°F$, and thoroughly inspect the area. By the time they finish the inspection, it is 1:30 am. They again take the temperature of the body, which has dropped to $87°F$, and have it sent to the morgue. The temperature at the crime scene has remained steady at $82°F$. Determine the time of death.

Problem 7.11
Suppose you have just made a cup of tea with boiling water in a room where the temperature is $20°C$. Let $y(t)$ denote the temperature (in Celsius) of the tea at time t (in minutes).
(a) Write a differential equation that expresses Newton's Law of Cooling in this particular situation. What kind of differential equation is it?
(b) What is the initial condition?
(c) Substitute $u(t) = y(t) - 20$. What initial value problem does this new function $u(t)$ satisfy? What is the solution?
(d) Suppose it is known that the tea cools at a rate of $2°C$ per minute when its temperature is $70°C$. Write a formula for $y(t)$.
(e) What is the temperature of the tea a half an hour later?
(f) When will the tea have cooled to $37°C$?

Problem 7.12
Newton's Law of Heating is a corresponding principle which applies if an

object is being warmed rather than cooled. The same formulas apply except the constant of proportionality is positive in the warming case. Use Newton's Law of Heating to solve the following problem: A chicken is removed from the refrigerator at a temperature of $40°F$ and placed in an oven kept at the constant temperaturre of $350°F$. After 10 minutes the temperature of the chicken is $70°F$. The chicken is considered cooked when its temperature reaches $180°F$. How long must it remain in the oven?

Problem 7.13
A corpse is discovered at midnight and its body temperature is $84°F$. If the body temperature at death is $98°F$, the room temperature is constant at $66°F$, and the proportionality constant is .10 per hour, how many hours have passed since the time of death when the corpse is found?

Problem 7.14
A tank initially contains 100 gal of a salt-water solution containing $0.05 = \frac{1}{20}$ lb of salt for each gallon of water. At time $t = 0$, pure water [containing no salt] is poured into the tank at a flow rate of 2 gal per minute. Simultaneously, a drain is opened at the bottom of the tank that allows salt-water solution to leave the tank at a flow rate of 3 gal per minute. What will be the salt content in the tank when precisely 50 gal of salt solution remain?

Problem 7.15
A tank contains 200 gal of a 2 % solution of HCl. A 5 % solution of HCl is added at 5 gal/min. The well mixed solution is being drained at 5 gal/min. When does the concentration of HCl in the solution reach 4 %?

Problem 7.16
Suppose that the temperature of the cup of coffee obeys Newton's law of cooling. If the coffee has a temperature of $200°F$ when freshly poured, and one minute later has cooled to $190°F$ in a room at $70°F$, determine when the coffee reaches a temperature of $150°F$.

Problem 7.17
Suppose that at 1:00 pm one winter afternoon, there is a power failure at your condo in Nanaimo, and your heat does not work without electricity. When the power goes out, it is 68°F in your condo. At 10:00 pm, it is 57°F in your condo, and you notice it is 10°F outside (what a pity!).
(i) Assuming that the temperature, H, in your condo obeys Newton's Law

of Cooling, write the differential equation satisfied by H and then solve the intial-value problem.
(ii) Estimate the temperature in your condo when you get up at 7:00 am the next morning.

Problem 7.18
Johnny is in the basement watching over a tank with a capacity of 100 L. Originally, the tank is full of pure water. Water containing a salt at a concentration of 2 g/L is flowing into the tank at a rate of r L/minute, and the well mixed liquid in the tank is flowing out at the same rate.
(a) Write down and solve an initial value problem describing the quantity of salt in the mixture at time t in terms of r.
(b) If Johnny's mixture contains 10 g of salt after 50 minutes, what is r?

Problem 7.19
A brine tank holds 15000 gallons of continuously mixed liquid. Let $y(t)$ be the amount of salt (in pounds) in the tank at time t. Brine is flowing in and out at 150 gallons per hour, and the concentration of salt flowing is 1 pound per 10 gallons of water.
(a) Find the differential equation of $y(t)$ and find the solution assuming that there is no salt in the water at time t.
(b) What is the limiting amount of salt as $t \to \infty$?

Problem 7.20
A 10 gal. tank initially contains an effluent at a concentration of 1 lb/gal. Water with an increasing concentration given by $1 - e^{-t}$ lbs/gal of effluent flows into the tank at a rate of 5 gal/day and the mixture in the tank flows out at the same rate.
(a) Assuming that the salt distributes itself uniformly, construct a mathematical model of this flow process for the effluent content $y(t)$ of the tank.
(b) Solve the initial-value problem.
(c) What is the limiting value of the effluent content as $t \to \infty$?

8 Existence and Uniqueness of Solutions to the IVP $y' = f(t, y),\ y(t_0) = y_0$

When a mathematical model is constructed for physical systems, two reasonable demands are made. First, solutions should exist if the model is to be useful at all. Second, to work effectively in predicting the future behavior of the physical system, the model should produce only one solution for a particular set of initial conditions. Existence and uniqueness theorems help to meet these demands.
In this section we discuss the conditions that guarantee the existence of a unique solution to the initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0. \tag{8}$$

There are many ways to prove the existence of a solution to an ordinary differential equation. The simplest way is to find one explicitly. This is a good approach for separable or exact equations, or linear equations with constant coefficients. But unfortunately there are many equations that cannot be solved by elementary methods, so attempting to prove the existence of a solution with this approach is not at all practical. An alternative approach is to approximate a solution to an IVP by constructing a sequence of functions that converges uniformly to a solution. This is precisely the approach we will use for the proof of existence of a solution. This approach is due to Picard. Before introducing Picard's itrerations we remind the reader of the following Taylor series expansions:

$$\cos x = 1 - \tfrac{x^2}{2!} + \tfrac{x^4}{4!} - \cdots + (-1)^n \tfrac{x^{2n}}{(2n)!} + \cdots, \text{ for all } x$$

$$\sin x = x - \tfrac{x^3}{3!} + \tfrac{x^5}{5!} + \cdots + \tfrac{(-1)^n}{(2n+1)!} x^{2n+1} + \cdots, \text{ for all } x$$

$$e^x = 1 + \tfrac{x}{1!} + \tfrac{x^2}{2!} + \cdots + \tfrac{x^n}{n!} + \cdots, \text{ for all } x$$

$$\ln(1 + x) = x - \tfrac{x^2}{2} + \tfrac{x^3}{3} - \cdots + (-1)^{n-1} \tfrac{x^n}{n} + \cdots, \text{ for } -1 < x \leq 1.$$

$$\tfrac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots, \text{ for } -1 < x < 1$$

Next, we start by reformulating (8) as an equivalent integral equation. Integration of both sides of (8) yields

$$\int_{t_0}^{t} y'(s)ds = \int_{t_0}^{t} f(t, y(s))ds \tag{9}$$

Applying the Fundamental Theorem of Calculus to the left side of (9) yields

$$y(t) = y(t_0) + \int_{t_0}^{t} f(t, y(s))ds \tag{10}$$

Thus, a solution of (10) is also a solution to (8) and vice versa.

Picard Iterations
Picard's functions are defined recursively as follows:

$$\begin{array}{rcl} y_0(t) & \equiv & y_0 \\ y_1(t) & = & y_0 + \int_{t_0}^{t} f(t, y_0(s))ds \\ y_2(t) & = & y_0 + \int_{t_0}^{t} f(t, y_1(s))ds \\ \vdots & & \vdots \\ y_n(t) & = & y_0 + \int_{t_0}^{t} f(t, y_{n-1}(s))ds \end{array}$$

If this sequence of fucntions converges uniformly to a function $y(t)$ then this function is the solution to our initial value problem as we shall establish in the proof of Theorem 8.1 below.

Example 8.1
Use Picard iterations to find the solution to the IVP

$$y' = 2y, \quad y(0) = 1$$

Solution.
The IVP is equivalent to $y(t) = 1 + \int_0^t 2sds$. So the Picard iterates are

$$\begin{array}{rcl} y_0(t) & = & 1 \\ y_1(t) & = & 1 + 2t \\ y_2(t) & = & 1 + 2t + \frac{(2t)^2}{2!} \end{array}$$

and so on. It can be shown by induction on n that the nth iterate is given by

$$y_n(t) = 1 + 2t + \frac{(2t)^2}{2!} + \cdots + \frac{(2t)^n}{n!}$$

which is the nth Taylor polynomial for e^{2t}. Thus, $y_n(t) \longrightarrow e^{2t}$ as $n \to \infty$ for all values of t so that the solution to the initial value problem is $y(t) = e^{2t}$ ∎

Example 8.2
Consider the IVP
$$y' = 2t(1+y), \quad y(0) = 0$$
Find the Picard functions, $y_0, y_1, \cdots, y_n$. Show that $\lim_{n\to\infty} y_n(t) = e^{t^2} - 1$.

Solution.
We have

$$y_0(t) \equiv 0$$

$$y_1(t) = \int_0^t 2sds = t^2$$

$$y_2(t) = \int_0^t 2s(1+s^2)ds = t^2 + \frac{t^4}{2}$$

$$y_3(t) = \int_0^t 2s(1+s^2+\frac{s^4}{2})ds = t^2 + \frac{t^4}{2} + \frac{t^6}{6}$$

and inductively we have

$$y_n(t) = t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \cdots + \frac{t^{2n}}{n!}.$$

This is a convergent Taylor polynomial that converges uniformly to $e^{t^2} - 1$ ■

Problem 8.1
Use Picard iterations to find the solution to the IVP

$$y' = y - t, \quad y(0) = 2$$

The following result from advanced calculus is useful in proving the next theorem.

Theorem 8.1 *(Weierstrass M-Test)*
Assume $\{y_N(t)\}_{N=1}^{\infty}$ is a sequence of functions defined in an open interval $a < t < b$. Suppose that $\{M_N\}_{N=1}^{\infty}$ is a sequence of positive constants such that
$$|y_N(t)| \le M_N$$
for all $a < t < b$. If $\sum_{N=1}^{\infty} M_N$ is convergent then $\sum_{N=1}^{\infty} y_N$ converges uniformly for all $a < t < b$.

Next, we state the major result of this section.

Theorem 8.2
Suppose the functions $f(t, y)$ and $\frac{\partial f}{\partial y}(t, y)$ are continuous in the closed rectangle

$$R = \{f(t, y) : t_0 - a \leq t \leq t_0 + a, y_0 - b \leq y \leq y_0 + b\}$$

Let M be the maximum of $|f|$ on R. Then there exists a positive number $h = min\{a, \frac{b}{M}\}$ such that the initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0 \tag{11}$$

has a unique solution for t in the interval $J = [t_0 - h, t_0 + h]$.

Remark 8.1
Notice that M exists and is a finite number since f is continuous on a closed and bounded set.

Proof.
We will show that the Picard's iterations defined above converges uniformly to a function $y(t)$ for all t in J and that $y(t)$ satisfies the integral equation (10) and thus the IVP (8). We will prove this in a series of claims.

Claim 1: $(t, y_n(t))$ is in R for all t in J, i.e. $y_n(t)$ is well-defined.

Proof of Claim 1
The proof is by induction on n. For $n = 0$, we have (t, y_0) in R since $|t - t_0| \leq h \leq a$ for all t in J. Suppose the claim is true up to n. Then for all t in J we have

$$|y_{n+1}(t) - y_0| = \left| \int_{t_0}^{t} f(s, y_n(s)) ds \right| \leq \left| \int_{t_0}^{t} |f(s, y_n(s))| ds \right| \leq M|t - t_0| \leq Mh \leq b$$

Hence, $(t, y_{n+1}(t))$ is in R for all t in J.((The extra set of absolute value signs in the third term in this chain of inequalities is needed because if $t < t_0$ then $\int_{t_0}^{t} |f(s, y_n(s))| ds < 0$) $\blacksquare$

Claim 2: There is a constant $K > 0$ such that for $n = 0, 1, 2, \cdots$ we have

$$|f(t, y_{n+1})| - f(t, y_n)| \leq K|y_{n+1} - y_n|.$$

Proof of Claim 2
Since $\frac{\partial f}{\partial y}(t,y)$ is continuous in a closed and bounded set then the number $K = \max_{(t,y)\in R}\left|\frac{\partial f}{\partial y}(t,y)\right|$ exists and is finite. Since f and f_y are continuous on R then by Claim 1 and the Mean Value Theorem

$$f(t,y_{n+1}) - f(t,y_n) = \frac{\partial f}{\partial y}(t, y^*(n))(y_{n+1} - y_n)$$

where $|y^*(n) - y_0| \le b$. Thus,

$$|f(t,y_{n+1}) - f(t,y_n)| = |\frac{\partial f}{\partial y}(t, y^*(n))||y_{n+1} - y_n| \le K|y_{n+1} - y_n| \blacksquare$$

Claim 3: The functions $y_n(t)$ satisfy, for all t in J and all n, the inequality

$$|y_{n+1}(t) - y_n(t)| \le MK^n \frac{|t-t_0|^{n+1}}{(n+1)!}$$

Proof of Claim 3
We may assume that $t_0 \le t \le t_0 + h$. Similar argument applies for $t_0 - h \le t \le t_0$. The proof is by induction on n. For $n = 0$ we have

$$|y_1(t) - y_0| = |\int_{t_0}^t f(s,y_0)ds| \le |\int_{t_0}^t |f(s,y_0)|ds| \le M|t - t_0|$$

Suppose it is true up to n. Then using Claim 2 we have

$$\begin{array}{rcl}
|y_{n+1}(t) - y_n(t)| & = & \left|\int_{t_0}^t (f(s,y_n) - f(s,y_{n-1}))ds\right| \\
& \le & \int_{t_0}^t |f(s,y_n) - f(s,y_{n-1}|ds \\
& \le & K\int_{t_0}^t |y_n(s) - y_{n-1}(s)|ds \\
& \le & K\int_{t_0}^t MK^{n-1}\frac{|s-t_0|^n}{n!}ds \\
& = & K\int_{t_0}^t MK^{n-1}\frac{(s-t_0)^n}{n!}ds \\
& = & MK^n\frac{|t-t_0|^{n+1}}{(n+1)!} \\
& \le & MK^n\frac{h^{n+1}}{(n+1)!} \blacksquare
\end{array}$$

But

$$\sum_{n=0}^{\infty} MK^n \frac{h^{n+1}}{(n+1)!} = \frac{M}{K}(e^{Kh} - 1)$$

then by Weierstrass M-test we conclude that the series $\sum_{n=0}^{\infty}[y_{n+1}(t)-y_n(t)]$ converges uniformly for all $|t-t_0| \leq h$. But

$$y_n(t) = \sum_{k=0}^{n-1}[y_{k+1}(t) - y_k(t)] + y_0$$

Thus, the sequence $\{y_n\}$ converges uniformly to a function $y(t)$ for all $|t-t_0| \leq h$.

The function $y(t)$ is a continuous function (a uniform limit of a sequence of continuous function is continuous). Also we can interchange the order of taking limits and integration for such sequences. Therefore

$$\begin{array}{rcl} y(t) & = & \lim_{n\to\infty} y_n(t) \\ & = & y_0 + \lim_{n\to\infty} \int_{t_0}^t f(s, y_{n-1})ds \\ & = & y_0 + \int_{t_0}^t \lim_{n\to\infty} f(s, y_{n-1})ds \\ & = & y_0 + \int_{t_0}^t f(s,y)ds \end{array}$$

This shows that $y(t)$ is a solution to the integral equation (10) and therefore a solution to (8).

Uniqueness:

Suppose that $u_1(t)$ and $u_2(t)$ are two solutions to the IVP defined on J. Let $w(t) = u_1(t) - u_2(t)$. We will show that $w(t) \equiv 0$ for all t in J. First, notice that $w(t_0) = u_1(t_0) - u_2(t_0) = 0$ and

$$\int_{t_0}^t w'(s)ds = w(t) - w(t_0) = \int_{t_0}^t [f(s, u_1(s)) - f(s, u_2(s))]ds.$$

By Claim 2 we have

$$|w(t)| \leq K \int_{t_0}^t |u_1(s) - u_2(s)|ds$$

By letting $z(t) = \int_{t_0}^t |w(s)|ds \geq 0$, the previous inequality becomes $z'(t) \leq Kz(t)$. Furthermore, we have

$$\begin{array}{ccc} e^{-K(t-t_0)}z'(t) - e^{-K(t-t_0)}Kz(t) & \leq & 0 \\ \left(e^{-K(t-t_0)}z(t)\right)' & \leq & 0 \\ \int_{t_0}^t [e^{-K(s-t_0)}z(s)]'ds & \leq & 0 \\ e^{-K(t-t_0)}z(t) & \leq & z(t_0) = 0 \\ z(t) & \leq & 0 \end{array}$$

Thus, $0 \leq z(t) \leq 0$ for all t in J. This shows that $z(t) \equiv 0$ in J and therefore $z'(t) \equiv 0$ in J. Hence, $w(t) \equiv 0$ for all t in J ■

Remark 8.2
Although existence can be proved with no hypotheses on f beyond continuity, some assumption such as the continuity of $\frac{\partial f}{\partial y}$ is necessary for uniqueness. For example, the IVP

$$y' = y^{\frac{1}{3}}, \quad y(0) = 0$$

has as solutions the functions $y(t) \equiv 0$ and

$$y(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ (\frac{2}{3}t)^{\frac{3}{2}} & \text{if } t > 0 \end{cases}$$

Note that $\frac{\partial f}{\partial y}(t, y) = \frac{1}{3}y^{-\frac{2}{3}}$ is not continuous at $y = 0$ ■

Remark 8.3
The choice of h guarantees that the Picard iterates all lie in the rectangle R.

Example 8.3
Consider the differential equation

$$y' = \frac{y^{\frac{1}{3}}}{t(y-2)}$$

Does the existence theorem gaurantees the existence of a unique solution to the following IVPs: (a) $y(3) = 4$ (b) $y(0) = 7$ (c) $y(0) = 2$ (d) $y(1) = 2$

Solution.
The function $f(t, y) = \frac{y^{\frac{1}{3}}}{t(y-2)}$ is continuous for $t \neq 0$ and $y \neq 2$. The function

$$\frac{\partial f}{\partial y}(t, y) = \frac{-2 - 2y}{3t(y-2)^2 y^{\frac{2}{3}}}$$

is continuous for $t \neq 0$ and $y \neq 0, 2$. Thus, f and $\frac{\partial f}{\partial y}$ are continuous for $t \neq 0$ and $y \neq 0, 2$. Theorem 8.2 guarantees the existence of a unique solution for the initial value problem in (a) whereas there is no guarantee that there is a unique solution- or any solution- for the remaining IVPs ■

Theorem 8.3
Suppose that $f(t,y)$ and $\frac{\partial f}{\partial y}(t,y)$ are continuous on the open rectangle

$$R = \{(t,y) : a < t < b, c < y < d\}.$$

Then for any (t_0, y_0) in R the IVP

$$y' = f(t,y), \quad y(t_0) = y_0$$

has a unique solution defined on an interval of the form $[t_0 - h, t_0 + h] \subset [a,b]$ for some positive h.

Proof.
Since R is open then we can find a closed rectangle of the form

$$S = \{(t,y) : |t - t_0| \leq \alpha, |y - y_0| \leq \beta\}$$

containing (t_0, y_0) and contained in R. Now, the result follows from the previous theorem with R replaced by S ■

Problem 8.2
On what interval we expect unique solutions to

$$y' = \frac{y^2}{1-t^2}, \ y(0) = 0?$$

Problem 8.3
Consider the IVP

$$y' = \frac{1}{2}(-t + \sqrt{t^2 + 4y}), \ y(2) = -1.$$

(a) Show that $y(t) = 1 - t$ and $y(t) = -\frac{t^2}{4}$ are two solutions to the above IVP.
(b) Does this contradict Theorem 8.2?

For the given initial value problem in Problems 4 - 8,
(a) Rewrite the differential equation, if necessary, to obtain the form

$$y' = f(t,y), \quad y(t_0) = y_0.$$

Identify the function $f(t,y)$.
(b) Compute $\frac{\partial f}{\partial y}$. Determine where in the ty-plane both $f(t,y)$ and $\frac{\partial f}{\partial y}$ are continuous.
(c) Determine the largest open rectangle in the ty-plane that contains the point (t_0, y_0) and in which the hypotheses of Theorem 8.2 are satisfied.

Problem 8.4

$$3y' + 2t\cos y = 1, \quad y(\frac{\pi}{2}) = -1.$$

Problem 8.5

$$3ty' + 2\cos y = 1, \quad y(\frac{\pi}{2}) = -1.$$

Problem 8.6

$$2t + (1 + y^3)y' = 0, \quad y(1) = 1.$$

Problem 8.7

$$(y^2 - 9)y' + e^{-y} = t^2, \quad y(2) = 2$$

Problem 8.8

$$\cos yy' = 2 + \tan t, \quad y(0) = 0$$

Problem 8.9
Give an example of an initial value problem for which the open rectangle

$$R = \{(t,y) : 0 < t < 4, -1 < y < 2\}$$

represents the largest region in the ty-plane where the hypotheses of Theorem 8.2 are satisfied.

Problem 8.10
Consider the initial value problem: $t^2y' - y^2 = 0, \quad y(1) = 1.$

(a) Determine the largest open rectangle in the ty-plane, containing the point $(t_0, y_0) = (1,1)$, in which the hypotheses of Theorem 8.2 are satisfied.
(b) A solution of the initial value problem is $y(t) = t$. This solution exists on $-\infty < t < \infty$. Does this fact contradicts Theorem 8.2? Explain your answer.

Problem 8.11 *(Gronwall's Inequality)*
Let $u(t)$ and $h(t)$ be continuous functions defined on a closed interval $[a, b]$, with $h \geq 0$, let C be a non-negative constant, and suppose that

$$u(t) \leq C + \int_a^t u(s)h(s)ds$$

for all t in the interval. Show that

$$u(t) \leq Ce^{\int_a^t h(s)ds}$$

for all t in the interval.
Note in particular that if C = 0, then $u(t) \leq 0$ for all t.

Problem 8.12
Find the first three Picard iterates of the solution of the initial-value problem

$$y' = \cos t, \; y(0) = 0$$

and then try to find the nth Picard iterates.

Problem 8.13
Set up the Picard iteration technique to solve the initial value problem $y' = y^2, \; y(0) = 1$ and do the first three iterations.

Problem 8.14
Can we apply the basic existence and uniqueness theorem to the following problem ? Explain what (if anything) we can conclude, and why (or why not):

$$y' = \frac{y}{\sqrt{t}}, \quad y(0) = 2.$$

Problem 8.15
Consider the differential equation $y' = \frac{t-y}{t+y}$. For which of the following initial value conditions does Theorem 8.2 apply?
(a) $y(0) = 0$ (b) $y(1) = -1$ (c) $y(-1) = -1$

Problem 8.16
Does the initial value problem $y' = \frac{y}{t} + 2, \; y(0) = 1$ satisfy the conditions of Theorem 8.2?

Problem 8.17
Is it possible to find a function $f(t, y)$ that is continuous and has continuous partial derivatives such that the functions $y_1(t) = cost$ and $y_2(t) = 1 - \sin t$ are both solutions to the equation $y' = f(t, y)$ near $t = \frac{\pi}{2}$?

Problem 8.18
Does the initial value problem $y' = y \sin y + t,\ y(0) = -1$ satisfy the conditions of Theorem 8.2?

Problem 8.19
The condition of continuity of $f(t, y)$ in Theorem 8.2 can be replaced by the so-called Lipschitz continuous: A function $f(t, y)$ is said to be **Lipschitz continuous** in y on a closed interval $[a, b]$ if there is a positive constant k such that $|f(t, y_1) - f(t, y_2)| \leq k|y_1 - y_2|$ for all y_1, y_2 and $a \leq t \leq b$.
Show that the function $f(t, y) = 1 + t \sin ty$ is Lipschitz continuous in y for $0 \leq t \leq 2$. Hint: Use the Mean Value Theorem.

Problem 8.20
Find the region R of the ty-plane where both

$$f(t, y) = \frac{1}{\sqrt{y - \sin t}}$$

and $\frac{\partial f}{\partial y}(t, y)$ are continuous.

9 Separable Differential Equations

A first order differential equation is **separable** if it can be written with one variable only on the left and the other variable only on the right:

$$f(y)y' = g(t)$$

To solve this equation, we proceed as follows. Let $F(t)$ be an antiderivative of $f(t)$ and $G(t)$ be an antiderivative of $g(t)$. Then by the Chain Rule

$$\frac{d}{dt}F(y) = \frac{dF}{dy}\frac{dy}{dt} = f(y)y'$$

Thus,

$$f(y)y' - g(t) = \frac{d}{dt}F(y) - \frac{d}{dt}G(t) = \frac{d}{dt}[F(y) - G(t)] = 0$$

It follows that

$$F(y) - G(t) = C$$

which is equivalent to

$$\int f(y)y'dt = \int g(t)dt + C$$

As you can see, the result is generally an implicit equation involving a function of y and a function of t. It may or may not be possible to solve this to get y explicitly as a function of t. For an initial value problem, substitute the values of t and y by t_0 and y_0 to get the value of C.

Remark 9.1
If F is a differentiable function of y and y is a differentiable function of t and both F and y are given then the chain rule allows us to find $\frac{dF}{dt}$ given by

$$\frac{dF}{dt} = \frac{dF}{dy} \cdot \frac{dy}{dt}$$

For separable equations, we are given $f(y)y' = \frac{dF}{dt}$ and we are asked to find $F(y)$. This process is referred to as "reversing the chain rule."

Example 9.1
Solve the initial value problem $y' = 6ty^2, \quad y(1) = \frac{1}{25}$

Solution.
Since $f(t,y) = 6ty^2$ and $f_y(t,y) = 12ty$ are continuous in the rectangle

$$R = \{(t,y) : -\infty < t < \infty, \ -\infty < y < \infty\}$$

then by Theorem 8.2, the IVP has a unique solution on some interval containing $t = 1$.
Separating the variables and integrating both sides we obtain

$$\int \frac{y'}{y^2} dt = \int 6tdt$$

or

$$-\int \frac{d}{dt}\left(\frac{1}{y}\right) dt = \int 6tdt$$

Thus,

$$-\frac{1}{y(t)} = 3t^2 + C$$

Since $y(1) = \frac{1}{25}$ then $C = -28$. The unique solution to the IVP is then given explicitly by

$$y(t) = \frac{1}{28 - 3t^2}$$

The next question is the question of the interval of existence of this solution. Recall that there are two conditions that define an interval of validity. First, it must be a continuous interval with no breaks or holes in it. Second it must contain the value of the independent variable in the initial condition, $t = 1$ in this case.
There are three possible intervals where $y(t)$ is continuous:

$$-\infty < t < -\sqrt{\frac{28}{3}}, \quad -\sqrt{\frac{28}{3}} < t < \sqrt{\frac{28}{3}}, \quad t > \sqrt{\frac{28}{3}}$$

Only one of these will contain the value of t from the initial condition and so we can see that

$$-\sqrt{\frac{28}{3}} < t < \sqrt{\frac{28}{3}}$$

must be the interval of existence for this solution. Figure 9.1 shows the graph of the solution ■

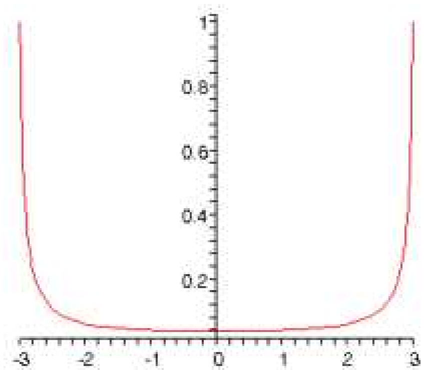


Figure 9.1

Example 9.2
Solve the IVP $yy' = 4\sin(2t), \quad y(0) = 1$

Solution.
This is a separable differential equation. Integrating both sides we find

$$\int \frac{d}{dt}\left(\frac{y^2}{2}\right) dt = 4\int \sin(2t)dt$$

Thus,

$$y^2 = -4\cos(2t) + C$$

Since $y(0) = 1$ then $C = 5$. Now, Solving explicitly for $y(t)$ we find

$$y(t) = \pm\sqrt{-4\cos t + 5}$$

Since $y(0) = 1$ then $y(t) = \sqrt{-4\cos t + 5}$. The interval of existence of the solution is the interval $-\infty < t < \infty$ ■

Example 9.3
Solve the initial value problem

$$y' = \sqrt{1 - y^2}, \quad y(0) = 0$$

Solution.
Separating the variables and then integrating we find

$$\int \frac{y'}{\sqrt{1-y^2}}dt = \int dt$$

or

$$\arcsin y = t + C$$

Since $y(0) = 0$ then $C = 0$ and consequently $y(t) = \sin t$ where $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$. Now, notice that $y(\frac{\pi}{2}) = 1$ and $y(t) = 1$ is the equilibrium solution. Similarly, $y(-\frac{\pi}{2}) = -1$ and $y(t) = -1$ is the equilibrium solution. This shows that the solution to the given IVP is

$$y(t) = \begin{cases} -1 & -\infty < t < -\frac{\pi}{2} \\ \sin t & -\frac{\pi}{2} \le t \le \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < t < \infty \end{cases} \blacksquare$$

The graph of this function is shown in Figure 9.2

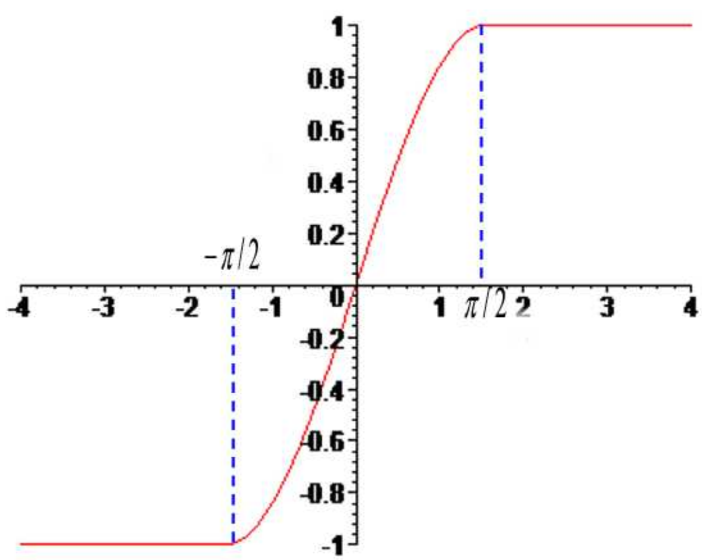


Figure 9.2

Problem 9.1
Solve the (separable) differential equation

$$y' = te^{t^2 - \ln y^2}.$$

Problem 9.2
Solve the (separable) differential equation

$$y' = \frac{t^2y - 4y}{t+2}$$

Problem 9.3
Solve the (separable) differential equation

$$ty' = 2(y-4)$$

Problem 9.4
Solve the (separable) differential equation

$$y' = 2y(2-y)$$

Problem 9.5
Solve the IVP

$$y' = \frac{4\sin(2t)}{y}, \quad y(0) = 1$$

Problem 9.6
Solve the IVP:

$$yy' = \sin t, \quad y(\frac{\pi}{2}) = -2$$

Problem 9.7
Solve the IVP:

$$y' + \frac{1}{y+1} = 0, \quad y(1) = 0.$$

Problem 9.8
Solve the IVP:

$$y' - ty^3 = 0, \quad y(0) = 2.$$

Problem 9.9
Solve the IVP:

$$y' = 1 + y^2, \quad y(\frac{\pi}{4}) = -1.$$

Problem 9.10
Solve the IVP:

$$y' = t - ty^2, \quad y(0) = \frac{1}{2}$$

Problem 9.11
Solve the IVP

$$(2y - \sin y)y' = \sin t - t, \quad y(0) = 0$$

Problem 9.12
For what values of the constants α, y_0, and integer n is the function $y(t) = (4+t)^{-\frac{1}{2}}$ a solution of the initial value problem?

$$y' + \alpha y^n = 0, \quad y(0) = y_0.$$

Problem 9.13
State an initial value problem, with initial condition imposed at $t_0 = 2$, having implicit solution $y^3 + t^2 + \sin y = 4$.

Problem 9.14
Consider the initial value problem

$$y' = 2y^2, \quad y(0) = y_0$$

For what value(s) of y_0 will the solution have a vertical asymptote at $t = 4$, where the t-interval of existence is $-\infty < t < 4$?

Problem 9.15
Consider the differential equation $y' = |y|$.
(a) Is this differential equation linear or nonlinear? Is the differentiable equation separable?
(b) A student solves the two initial value problems $y' = |y|, \ y(0) = 1$ and $y' = y, \ y(0) = 1$ and then graphs the two solution curves on the interval $-1 \le t \le 1$. Sketch the two graphs.
(c) The student next solves the two initial value problems $y' = |y|, \ y(0) = -1$ and $y' = y, \ y(0) = -1$. Sketch the solution curves.

Problem 9.16
Assume that $y \sin y - 3t + 3 = 0$ is an implicit solution of the initial value problem $y' = f(y), \quad y(1) = 0$. What is $f(y)$? What is an implicit solution to the initial value problem $y' = t^2 f(y), \quad y(1) = 0$?

Problem 9.17
Find all the solutions to the differential equation $y' = \frac{2ty}{1+t}$

Problem 9.18
Solve the initial-value problem $y' = \cos^2 y \cos^2 t, \quad y(0) = \frac{\pi}{4}$

Problem 9.19
Solve the initial-value problem $y' = e^{t+y},\ y(0) = 0$ and determine the interval on which the solution $y(t)$ is defined.

Problem 9.20
Solve the initial-value problem

$$y' = \frac{t^2}{e^{-y}} - \frac{e^y}{t^2}$$

(a) State the name of the method you are using.
(b) Find the solution which satisfies the condition $y(1) = 1$

10 Exact Differential Equations

We shall now present another technique for solving first order, non-linear, ordinary differential equations. This technique is a generalization of the one we used for separable equations.
We have seen that the solution procedure of separable equations consists of reversing the chain rule. This same procedure works for exact equations but this time the chain rule is for functions of two variables. We begin with brief review of partial derivatives.

Partial Derivatives
If $f(t,y)$ is a function of two variables t and y then the partial derivative $\frac{\partial f}{\partial t}$ of $f(t,y)$ is the derivative of $f(t,y)$ with respect to t, while pretending y is a constant. The partial derivative $\frac{\partial f}{\partial y}$ is the derivative of $f(t,y)$ with respect to y, while pretending t is constant. The precise definitions are

$$\frac{\partial f}{\partial t}(t,y) = \lim_{h\to 0} \frac{f(t+h,y)-f(t,y)}{h} \quad \text{and} \quad \frac{\partial f}{\partial y}(t,y) = \lim_{h\to 0} \frac{f(t,y+h)-f(t,y)}{h}$$

Example 10.1
Find $\frac{\partial f}{\partial t}$ and $\frac{\partial f}{\partial y}$ if $f(t,y) = t^4y^3 + t^5$.

Solution.
We have

$$\frac{\partial f}{\partial t}(t,y) = 4t^3y^3 + 5t^4 \quad \text{and} \quad \frac{\partial f}{\partial y}(t,y) = 3t^4y^2 \blacksquare$$

Problem 10.1
Find $\frac{\partial f}{\partial t}$ and $\frac{\partial f}{\partial y}$ if $f(t,y) = y \ln y - e^{-ty}$.

Problem 10.2
Find $\frac{\partial f}{\partial t}$ and $\frac{\partial f}{\partial y}$ if $f(t,y) = \ln ty + \frac{t^2+1}{y-5}$

The Extended Chain Rule
You recall the chain rule for functions of one variable: If u is differentiable at x and f is differentiable at $u(x)$ then the composite function $y = f(u(x))$ is also differentiable at x with derivative given by

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example 10.2
Find the derivative of the function $y = e^{\sqrt{x}}$.

Solution.
Let $u(x) = \sqrt{x}$ and $f(x) = e^x$. Then $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$ and $\frac{dy}{du} = e^u$. Hence,

$$\frac{dy}{dx} = e^u \frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{2\sqrt{x}} \blacksquare$$

The above chain rule can be extended to functions of two variables. Suppose that u and v are differentiable at t and f is a differentiable function of two variables. Then the function $z(t) = f(u(t), v(t))$ is differentiable at t with derivative

$$\frac{dz}{dt} = \frac{\partial f}{\partial u}\frac{du}{dt} + \frac{\partial f}{\partial v}\frac{dv}{dt}$$

Example 10.3
Let $z = f(u, v) = u^2 + 2u - uv + v^2$ where $u(t) = t^2 + 1$ and $v(t) = t^3 - t^2$. Find $\frac{dz}{dt}(2)$ in two different ways.

Solution.
First notice that $u(2) = 5$ and $v(2) = 4$. By using the extended chain rule we have

$$\begin{array}{rcl} \frac{dz}{dt} & = & \frac{\partial f}{\partial u}\frac{du}{dt} + \frac{\partial f}{\partial v}\frac{dv}{dt} \\ & = & (2u + 2 - v)(2t) + (2v - u)(3t^2 - 2t) \end{array}$$

Thus,

$$\frac{dz}{dt}(2) = (10 + 2 - 4)(4) + (8 - 5)(8) = 56$$

A different way for finding the derivative is to write z as only a function of t obtaining

$$z(t) = t^6 - 3t^5 + 3t^4 - t^3 + 5t^2 + 3$$

Finding the derivative of $z(t)$

$$z'(t) = 6t^5 - 15t^4 + 12t^3 - 3t^2 + 10t$$

Finally, $z'(2) = 56 \blacksquare$

Problem 10.3
Let $f(u, v) = 2u - 3uv$ where $u(t) = 2\cos t$ and $v(t) = 2\sin t$. Find $\frac{df}{dt}$.

Exact Differential Equations

The basic idea underlying separable equations is to reverse the chain rule for functions of one variable. The basic idea underlying exact equations is to reverse the extended chain rule. To this end, consider the differential equation

$$M(t,y) + N(t,y)\frac{dy}{dt} = 0 \tag{12}$$

Let $H(t,y)$ be a function satisfying the two conditions

$$\frac{\partial H}{\partial t}(t,y) = M(t,y) \;\; and \;\; \frac{\partial H}{\partial y}(t,y) = N(t,y) \tag{13}$$

Then Equation (12) can be written as

$$\frac{\partial H}{\partial t} + \frac{\partial H}{\partial y}\frac{dy}{dt} = 0 \tag{14}$$

By the extended chain rule, Equation (14) is the same as

$$\frac{d}{dt}H(t,y) = 0$$

Therefore, we obtain an implicitly defined solution given by

$$H(t,y) = C$$

An equation like (12) is called **exact** if there is a function $H(t,y)$ satisfying the conditions in (13).

Testing a Differentiable Equation for Exactness

The next question is the question of telling whether or not Equation (12) is exact. This is answered by the following theorem.

Theorem 10.1

Suppose that the functions $M(t,y)$ and $N(t,y)$ in (12) are continuous and have continuous first partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial t}$ in an open rectangle

$$R = \{(t,y) : a < t < b, \; c < y < d\}$$

Then (12) is exact in R if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

for all (t,y) in R.

Proof.
Let $M, N, \frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial t}$ be continuous functions in R.
Necessary condition:
Suppose, first, that (12) is exact. Then there is a function $H(t, y)$ satisfying conditions (13). But from multivariable calculus we know that

$$\frac{\partial^2 H(t,y)}{\partial y \partial t} = \frac{\partial^2 H(t,y)}{\partial t \partial y}$$

or

$$\frac{\partial}{\partial y}\left(\frac{\partial H(t,y)}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial H(t,y)}{\partial y}\right)$$

By (13) we see that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Notice that the equality of mixed partials is a consequence of the continuity of the first partial derivatives of $M(t, y)$ and $N(t, y)$.

Sufficient condition: Method for finding $H(t, y)$
Suppose that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

for all (t, y) in R. Let us find a function $H(t, y)$ satisfying (13). Indeed, since $\frac{\partial H(t,y)}{\partial t} = M(t, y)$ then

$$H(t,y) = \int M(t,y)dt + h(y) \tag{15}$$

Note that when integrating with respect to t the "constant of integration" is $h(y)$ since y is treated as a constant when the partial derivative with respect to t is computed.
Taking the derivative of this last equation we respect to y and using the fact that $\frac{\partial H(t,y)}{\partial y} = N(t, y)$ to obtain

$$\frac{\partial H(t,y)}{\partial y} = \frac{\partial}{\partial y}\int M(t,y)dt + \frac{dh(y)}{dy} = N(t,y)$$

This gives

$$\frac{dh(y)}{dy} = N(t,y) - \frac{\partial}{\partial y}\int M(t,y)dt \tag{16}$$

Notice that the right-hand side of this last equation is independent of t since

$$\begin{aligned} \frac{\partial}{\partial t}\left[N(t,y) - \frac{\partial}{\partial y}\int M(t,y)dt\right] &= \frac{\partial N}{\partial t} - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial t}\int M(t,y)dt\right) \\ &= \frac{\partial N}{\partial t} - \frac{\partial M}{\partial y} = 0 \end{aligned}$$

Finally, integrate (16) with respect to y and substitute the result in (15). The solution of the equation is $H(t,y) = C$. Since a function $H(t,y)$ satisfying (13) can be found then (12) is exact. This concludes a proof of the theorem ■

Remark 10.1
Every separable differential equation is exact. Indeed, since $g(t) + f(y)y' = 0$ then $\frac{\partial g}{\partial y} = 0$ and $\frac{\partial f}{\partial t} = 0$. However, not every exact equation is separable. For example, the differential equation $(2t+y) + (2y+t)y' = 0$ is exact since $\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial t} = 1$. This equation is clearly not separable ■

Example 10.4
Determine whether or not the equation is exact.

(a) $ty^2 + t + t^2yy' = 0$
(b) $y^2 + 1 + tyy' = 0$
(c) $\cos y + (y^2 + t\sin y)y' = 0$
(d) $\cos y + (y^2 - t\sin y)y' = 0$

Solution.
(a) Since $\frac{\partial(ty^2+t)}{\partial y} = 2ty$ and $\frac{\partial t^2y}{\partial t} = 2ty$ then the given equation is exact.
(b) Since $\frac{\partial(y^2+1)}{\partial y} = 2y$ and $\frac{\partial ty}{\partial t} = y$ then the given equation is not exact.
(c) Since $\frac{\partial \cos y}{\partial y} = -\sin y$ and $\frac{\partial(y^2+t\sin y)}{\partial t} = \sin y$ then the given equation is not exact.
(d) Since $\frac{\partial \cos y}{\partial y} = -\sin y$ and $\frac{\partial(y^2-t\sin y)}{\partial t} = -\sin y$ then the equation is exact ■

Example 10.5
Consider the initial value problem

$$t + y + (t+2y)y' = 0, \quad y(0) = 1$$

Show that the differential equation is exact and solve the IVP.

Solution.
We have $M(t,y) = t + y$ and $N(t,y) = t + 2y$. Since

$$\frac{\partial M(t,y)}{\partial y} = \frac{\partial N(t,y)}{\partial t} = 1$$

then by Theorem 10.1 the differential equation is exact. Thus,

$$H(t,y) = \int (t+y)dy = ty + \frac{y^2}{2} + c_1(t)$$

Hence,

$$t + 2y = \frac{\partial H(t,y)}{\partial t} = y + c_1'(t)$$

It follows that

$$c_1(t) = \int (t+y)dt = \frac{t^2}{2} + ty + C'$$

Hence,

$$2ty + \frac{y^2}{2} + \frac{t^2}{2} = C$$

Since $y(0) = 1$ then $C = \frac{1}{2}$. Thus, y satisfies the implicit equation

$$4ty + y^2 + t^2 = 1 \blacksquare$$

In Problems 4 - 8, determine whether the given differential equation is exact. If the equation is exact, find an implicit solution and (where possible) an explicit solution.

Problem 10.4

$$yy' + 3t^2 - 2 = 0, \quad y(-1) = -2.$$

Problem 10.5

$$y' = (3t^2 + 1)(y^2 + 1), \quad y(0) = 1$$

Problem 10.6

$$(6t + y^3)y' + 3t^2y = 0, \quad y(1) = 2.$$

Problem 10.7

$$(e^{t+y} + 2y)y' + (e^{t+y} + 3t^2) = 0, \quad y(0) = 0$$

Problem 10.8

$$(\sin(t+y) + y\cos(t+y) + t + y)y' + (y\cos(t+y) + y + t) = 0, \quad y(1) = -1$$

Problem 10.9
For what values of the constants $m, n,$ and α (if any) is the following differential equation exact?

$$t^m y^2 y' + \alpha t^3 y^n = 0$$

Problem 10.10
Assume that $N(t,y)y' + t^2 + y^2 \sin t = 0$ is an exact differential equation. Determine the general form of $N(t,y)$.

Problem 10.11
Assume that $t^3 y + e^t + y^2 = 5$ is an implicit solution to the differential equation

$$N(t,y)y' + M(t,y) = 0, \quad y(0) = y_0.$$

Determine possible functions $M(t,y), N(t,y)$, and the possible value(s) for y_0

Problem 10.12
Assume that $y = -t - \sqrt{4 - t^2}$ is an explicit solution of the following initial value problem

$$(y + at)y' + (ay + bt) = 0, \quad y(0) = y_0$$

Determine values for the constants a, b and y_0

Problem 10.13
Let k be a positive constant. Use the exactness criterion to determine whether or not the population equation $\frac{dP}{dt} = kP$ is exact. Do NOT try to solve the equation or carry out any furhter calculation.

Problem 10.14
Consider the differential equation $(2t+3) + (2y-2)y' = 0$. Determine whether this equation is exact or not. If it is, solve it.

Problem 10.15
Consider the differential equation $(ye^{2ty} + t) + bte^{2ty}y' = 0$. Determine for which value of b this equation is exact, and then solve it with this value of b.

Problem 10.16
Consider the differential equation $y + (2t - ye^y)y' = 0$. Check that this equation is not exact. Now multiply the equation by y. Check that the new equation is exact, and solve it.

Problem 10.17
(a) Consider the differential equation

$$y' + p(t)y = g(t)$$

with $p(t) \neq 0$. Show that this equation is not exact.
(b) Let $\mu(t) = e^{\int p(t)dt}$. Show that the equation

$$\mu(t)(y' + p(t)y) = \mu(t)g(t)$$

is exact and solve it.

Problem 10.18
Use the method of the previous problem to solve the linear, first-order equation $y' - \frac{y}{t} = 1$, with initial condition $y(1) = 7$. First, check that this equation is not exact. Next, find $\mu(t)$. Multiply the equation by $\mu(t)$ and check that the new equation is exact. Solve it, using the method of exact equations.

Problem 10.19
Put the following differential equation in the "Exact Differential Equation" form and find the general solution

$$y' = \frac{y^3 - 2ty}{t^2 - 3ty^2}$$

Problem 10.20
The following differential equations are exact. Solve them by that method.
(a) $(4t^3y + 4t + 4)y' = 8 - 4y - 6t^2y^2, \quad y(-1) = 1$
(b) $(6 - 4y + 16t) + (10y - 4t + 2)y' = 0, \quad y(1) = 2$

11 Substitution Techniques: Bernoulli and Ricatti Equations

A well-known nonlinear equation that reduces to a linear one with an appropriate substitution is the **Bernoulli equation** given by

$$y' + p(t)y = g(t)y^n \tag{17}$$

where n is an integer different from 0 and 1. Notice that for $n = 0$ or $n = 1$ the equation becomes linear.
To solve (17), first notice that $y(t) \equiv 0$ is the trivial solution. If $y(t) \not\equiv 0$ then (17) can be written as

$$\frac{y'}{y^n} + p(t)y^{1-n} = g(t) \tag{18}$$

Let $z = y^{1-n}$. Then by the chain rule of differentiation we have $z' = (1-n)y^{-n}y'$ and therefore $\frac{y'}{y^n} = \frac{1}{1-n}z'$. Thus, (17) reduces to

$$\frac{1}{1-n}z' + p(t)z = g(t)$$

which is a first order linear differential equation that can be solved by the technique of integrating factors. Once z is found then the desired solution is $y(t) = z^{\frac{1}{1-n}}$.

Example 11.1
Solve the Bernoulli equation

$$y' - \frac{1}{t}y = ty^2, \ t > 0$$

Solution.
Divide by y^2 and then let $z = y^{-1}$ to obtain

$$z' + \frac{1}{t}z = -t$$

The integrating factor is $\mu(t) = t$ and the general solution is

$$z(t) = \frac{1}{t}\int -t^2dt + Ct^{-1} = -\frac{t^2}{3} + Ct^{-1}$$

Thus, $y = \frac{1}{z} = \frac{1}{-\frac{t^2}{3}+Ct^{-1}}$ ■

Problem 11.1
Solve the Bernoulli equation

$$y' = \frac{t^2 + 3y^2}{2ty}, \quad t > 0$$

Problem 11.2
Find the general solution of $y' + ty = te^{-t^2}y^{-3}$

Problem 11.3
Solve the IVP $ty' + y = t^2y^2, \quad y(0.5) = 0.5$

Problem 11.4
Solve the IVP $y' - \frac{1}{t}y = -y^2, \quad y(1) = 1$

Problem 11.5
Solve the IVP $y' = y(1-y), \quad y(0) = \frac{1}{2}$

Problem 11.6
Solve the Bernoulli equation $y' + 3y = e^{3t}y^2$

Problem 11.7
Solve $y' + y = ty^4$

Problem 11.8
Solve the equation $y' = \sin(t+y)$ using the substitution $z = t + y$ and separable method.

Ricatti Equation
A differential equation is called a **Ricatti equation** if it can be written in the form:

$$y' = a(t)y^2 + b(t)y + c(t) \tag{19}$$

where a, b and c are functions of t. Clearly, this is a first order nonlinear and nonseparable differential equation. Ricatti and Bernoulli equations arise when we model logistic population and one-dimensional motion with air resistance . See Sections 12 and 13.
The solution of a Ricatti equation requires knowledge of a particular solution

to the ODE. To solve (19), we first find a particular solution y_1 to (19). Then we use the substitution $\frac{1}{z} = y - y_1$. Thus, $y = \frac{1}{z} + y_1$. Now, (19) reduces to

$$\begin{aligned}
-\frac{z'}{z^2} + y_1' &= a(t)(\frac{1}{z^2} + 2\frac{y_1}{z} + y_1^2) + b(t)(\frac{1}{z} + y_1) + c(t)\\
-\frac{z'}{z^2} + a(t)y_1^2 + b(t)y_1 + c(t) &= \frac{a(t)}{z^2} + \frac{2a(t)y_1}{z} + a(t)y_1^2 + \frac{b(t)}{z} + b(t)y_1 + c(t)\\
z' &= -a(t) - [2a(t)y_1 + b(t)]z
\end{aligned}$$

Thus, a Ricatti equation can be reduced to a linear equation

$$z' + [2a(t)y_1 + b(t)]z = -a(t) \tag{20}$$

that can be solved by the method of integrating factor. Once $z(t)$ is found then the solution to the original equation is $y(t) = \frac{1}{z(t)} + y_1(t)$. As the next example illustrates, in many cases a solution of a Ricatti equation cannot be expressed in terms of elementary functions.

Example 11.2
Solve : $y' = 2 - 2ty + y^2$ given that $y_1(t) = 2t$.

Solution.
We have $a(t) = 1, b(t) = -2t$, and $c(t) = 2$. Substituting in (20) to obtain

$$z' + 2tz = -1.$$

The integrating factor is $\mu(t) = e^{t^2}$ so that

$$\left(e^{t^2}z\right)' = -e^{t^2}$$

Now the integral $\int_{t_0}^t e^{s^2}ds$ cannot be expressed in terms of elementary functions. Thus we write

$$e^{t^2}z(t) = -\int_{t_0}^t e^{s^2}ds + e^{t_0^2}z(t_0)$$

Solving for z we find

$$z(t) = e^{-t^2}(e^{t_0^2}z(t_0) - \int_{t_0}^t e^{s^2}ds)$$

Finally,

$$y(t) = \frac{1}{e^{-t^2}(e^{t_0^2}z(t_0) - \int_{t_0}^t e^{s^2}ds)} + 2t \blacksquare$$

Problem 11.9
Solve the IVP: $y' = 2 + 2y + y^2, \quad y(0) = 0$ using the method of separation of variables.

Problem 11.10
Solve the differential equation $y' = 1 + t^2 - y^2$ given that $y_1(t) = t$ is a particular solution.

Problem 11.11
Solve the differential equation $y' = 5 - t^2 + 2ty - y^2$ given that $y_1(t) = t - 2$ is a particular solution.

Problem 11.12
Perform a change of variable that changes the Bernoulli equation $y' + y + y^2 = 0$ into a linear equation in the new variable. Do NOT try to solve the equation or proceed further than with any calculations.

Problem 11.13
Consider the equation

$$y' = \epsilon y - \sigma y^3, \ \epsilon > 0, \ \sigma > 0$$

(a) Use the Bernoulli transformation to change this nonlinear equation into a linear equation.
(b) Solve the resulting linear equation in part (a) and use the solution to find the solution of the given differential equation above.

Problem 11.14
Consider the differential equation

$$y' = f\left(\frac{y}{t}\right)$$

(a) Show that the substitution $z = \frac{y}{t}$ leads to a separable differential equation in z.
(b) Use the above method to solve the initial-value problem

$$y' = \frac{t+y}{t-y}, \quad y(1) = 0$$

Problem 11.15
Solve: $y' + \frac{y}{3} = e^t y^4$

Problem 11.16
Solve: $ty' + y = ty^3$

Problem 11.17
Solve: $y' + \frac{2}{t}y = -t^2y^2 \cos t$

Problem 11.18
Solve: $ty' + y = t^2y^2 \ln t$

Problem 11.19
Verify that $y_1(t) = 2$ is a particular solution to the Ricatti equation

$$y' = -2 - y + y^2,$$

and then find the general solution.

Problem 11.20
Verify that $y_1(t) = \frac{2}{t}$ is a particular solution to the Ricatti equation

$$y' = -\frac{4}{t^2} - \frac{1}{t}y + y^2,$$

and then find the general solution.

12 Applications of First Order Nonlinear Equations: The Logistic Population Model

One of the applications to first order linear differential equations that we discussed in Section 6 was Malthus population model described by $\frac{dP}{dt} = kP$ where $k = r_b - r_d$. The solution to this differential equation is $P(t) = P(0)e^{kt}$. As you can see, this model predicts either population growth without bound ($k > 0$) or inevitable extinction ($k < 0$). So basically the relative birth rate k is independent of the population size, i.e., constant.

Neither case is typically observed in reality, that is, what is actually observed differs substantially from what is predicted by the solution of the equation. What is often observed is that small populations often (though not always) increase in number (because resources are plentiful and the population should thrive and grow) while very large populations tend to decline in number (since resources become scarcer; for example, food availability decreases, waste products may accumulate and birth rates tend to decline while death rates tend to increase.) So the relative birth rate in Malthus' model should be replaced by a population-depedent relative birth rate.

In this section we consider a model that attempts to account for the effects mentioned previously. This model leads to a first order nonlinear differential equation.

The Logistic Model (Verhulst)

The realistic model that we consider is of the form

$$\frac{dP}{dt} = h(P)P$$

which is similar to Maltheus model except that now the growth rate $h(P)$ depends on the population size. We conjecture the following about $h(P)$:

• When P is small the population grows so that $h(P) > 0$.

• When P is large the population declines so that $h(P) < 0$.

The simplest way to implement this is by letting

$$h(P) = r - \alpha P$$

so that when the population is small then $h(P) \approx r > 0$ and when the population is large then $h(P) \approx -\alpha P < 0$. This then gives the following population equation known as the **logistic equation**

$$\frac{dP}{dt} = r\left(1 - \frac{P}{K}\right)P \tag{21}$$

where r and $K = \frac{r}{\alpha}$ are positive constants. Note that if $P(t) > K$ then $\frac{dP}{dt} < 0$ causing the population to decrease whereas if $0 < P(t) < K$ then $\frac{dP}{dt} > 0$ causing the population to increase. The constant K is called the **carrying capacity**. It represents the largest population that the environment can support. Note that the carrying capacity occurs at the equilibrium solution $P(t) = K$ so sometimes the carrying capacity is referred to as the **equilibrium value**. The phase portrait or the direction field looks like the one shown in Figure 12.1

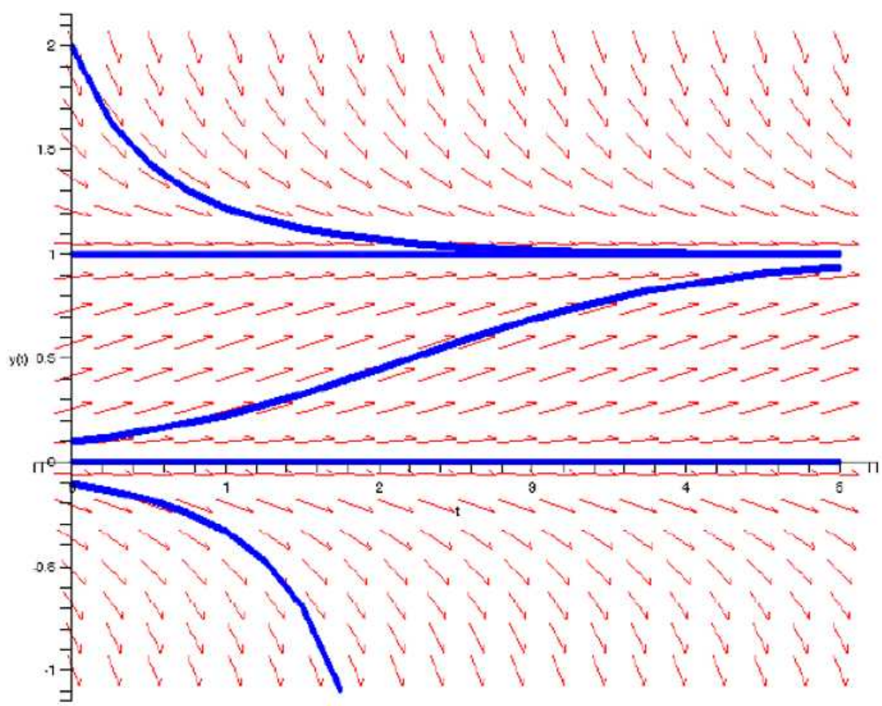


Figure 12.1

The curve below $P(t) = 0$ corresponds to negative initial population and do not have any physical significance.
Initial population between 0 and K grows almost exponentially at first.

Grows slow as P approach the limiting value K. As $t \to \infty$, $P \to K$.
Initial population larger than K decreases to K as $t \to \infty$. Physically the initial population is larger than the environment can support, and hence individuals die off.

Solving the Logistic Equation
The logistic equation (21) can be looked at as either a nonlinear separable equation or a Ricatti equation. The solution function is called **logistic function** and its graph is called the **logistic curve.** We will solve (21) by separating the variables. Indeed, separating the variables and using partial fractions we have the following:

$$\begin{array}{ccc}
\frac{dP}{dt} & = & r\left(1-\frac{P}{K}\right)P \\
\frac{P'}{\left(1-\frac{P}{K}\right)P} & = & r \\
\frac{P'}{P}-\frac{P'}{P-K} & = & r \\
\int \frac{P'}{P}dt - \int \frac{P'}{P-K}dt & = & rt+C \\
\ln\left|\frac{P}{P-K}\right| & = & rt+C \\
\frac{P}{P-K} & = & Ce^{rt} \\
P(t) & = & \frac{KCe^{rt}}{1-Ce^{rt}}
\end{array}$$

Since $P(0)$ is the initial population then $C = \frac{P(0)}{P(0)-K}$ so after substituting and simplifying the solution becomes

$$P(t) = \frac{KP(0)}{P(0)+(K-P(0))e^{-rt}} \tag{22}$$

Note that in the long run, $P(t)$ approaches K, that is,

$$\lim_{t\to\infty} P(t) = K.$$

Solving the Logistic Equation as a Ricatti Equation
An alternative method for solving (21) is to use the substitution $P(t) = \frac{1}{u}$.

Then by the chain rule we have

$$\frac{dP}{dt} = \frac{dP}{du}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{dt}$$

Rewriting the logistic equation in terms of u and t, and solving for u in terms of t we find

$$\begin{array}{ccc}
-\frac{1}{u^2}\frac{du}{dt} & = & \frac{r}{u}\left(1-\frac{1}{Ku}\right)\\
\frac{du}{dt} & = & -ru\left(1-\frac{1}{Ku}\right)\\
\frac{du}{dt} & = & -ru+\frac{r}{K} = r\left(\frac{1}{K}-u\right)\\
\int\frac{du}{\frac{1}{K}-u} & = & \int rdt\\
\ln|u-\frac{1}{K}| & = & -rt+C\\
u-\frac{1}{K} & = & Ce^{-rt}\\
u & = & \frac{1}{K}+Ce^{-kt}
\end{array}$$

Thus,

$$P(t) = \frac{K}{1+KCe^{-rt}}$$

Letting $t = 0$ to obtain

$$P(0) = \frac{K}{1+KC}$$

and solving for C we find $C = \frac{K-P(0)}{KP(0)}$. Substituting this in the last equation of $P(t)$ we find

$$P(t) = \frac{KP(0)}{P(0)+(K-P(0))e^{-rt}}$$

Example 12.1
Suppose a student carrying a flu virus returns to an isolated college campus of 1000 students. If it is determined that the rate at which the virus spreads is proportional not only to the number $P(t)$ of students infected but also to the number of students not infected. Determine the number of infected students after 6 days given that the number of infected students after 4 days is 50.

Solution.
We first must find a formula for $P(t)$ which is the solution to the IVP

$$\frac{dP}{dt} = r(1000-P)P, \quad P(0) = 1.$$

This equation can be rewritten in the form

$$\frac{dP}{dt} = 1000r(1 - \frac{P}{1000})P.$$

By (22) we find

$$P(t) = \frac{1000}{1 + 999e^{-1000rt}}$$

But $P(4) = 50$ so that

$$50 = \frac{1000}{1 + 999e^{-4000r}}$$

Solving this equation for r we find $r \approx 0.0009906$. Thus,

$$P(t) = \frac{1000}{1 + 999e^{-0.9906t}}$$

Finally,

$$P(6) = \frac{1000}{1 + 999e^{-0.9906(6)}} \approx 276 \textit{ students} \blacksquare$$

Problem 12.1
Find $\int \frac{dx}{(x-2)(3-x)}$

Problem 12.2
Find A and B so that $\frac{2x+3}{x^2-9} = \frac{A}{x+3} + \frac{B}{x-3}$

Problem 12.3
Write the partial fraction decomposition of $\frac{x+7}{x^2+x-6}$

Problem 12.4
An important feature of any logistic curve is related to its shape: *every logistic curve has a single inflection point which separates the curve into two equal regions of opposite concavity.* This inflection point is called the **point of diminishing returns.** Find the Coordinates of the Point of Diminishing Returns.

Problem 12.5
A population of roaches grows logistically in John's kitchen cabinet, feeding off 65 half-empty can of beef stew. There are 10 roaches initially, and the carrying capacity of the cabinet is $K = 10000$. The population reaches its maximum growth rate in 4 days. Determine the logistic equation for the growth of the population Find the number of roaches in the cabinet after 10 days.

Problem 12.6
The number of people $P(t)$ in a community who are exposed to a particular advertisement is governed by the logistic equation. Initially $P(0) = 500$, and it is observed that $P(1) = 1000$. If it is predicted that the limiting number of people in the community who will see the advertisement is 50,000, determine $P(t)$ at any time.

Problem 12.7
The population $P(t)$ at any time in a suburb of a large city is governed by the inititial value problem

$$\frac{dP}{dt} = (10^{-1} - 10^{-7}P)P, \quad P(0) = 5000$$

where t is measured in months. What is the limiting value of the population? At what time will the population be one-half of this limiting value?

Problem 12.8
Let $P(t)$ represent the population of a colony, in millions of individuals. Suppose the colony starts with 0.1 million individuals and evolves according to the equation

$$\frac{dP}{dt} = 0.1\left(1 - \frac{P}{3}\right)P$$

with time being measured in years. How long will it take the population to reach 90% of its equilibrium value?

Problem 12.9
Consider a population whose dynamics are described by the logistic equation with constant migration

$$\frac{dP}{dt} = r\left(1 - \frac{P}{K}\right)P + M,$$

where r, K, abd M are constants. Assume that K is a fixed positive constant and that we want to understand how the equilibrium solutions of this nonlinear autonomous equation depend upon the parameters r and M.
(a) Obtain the roots of the quadratic equation that define the equilibrium solution(s) of this differential equation. Note that for $M \neq 0$, the constants 0 and K are no longer equilibrium solutions. Does this make sense?

(b) For definiteness, set $K = 1$. Plot the equilibrium solutions obtained in (a) as functions of the ratio $\frac{M}{r}$. How many equilibrium populations exist for $\frac{M}{r} > 0$? How many exist for $-\frac{1}{4} < \frac{M}{r} \leq 0$?
(c) What happens when $\frac{M}{r} = -\frac{1}{4}$? What happens when $\frac{M}{r} < -\frac{K}{4} = -\frac{1}{4}$? Are these mathematical results consistent with what one would expect if migration rate out of the colony were sufficiently large relative to the colony's ability to gain size through reproduction?

Problem 12.10
Let $P(t)$ represent the number of individuals who, at time t, are infected with a certain disease. Let N denote the total number of individuals in the population. Assume that the spread of the disease can be modeled by the initial value problem

$$\frac{dP}{dt} = k(N - P)P, \quad P(0) = P_0$$

At time $t = 0$, when 100,000 members of the population of 500,000 are known to be infected, medical authorities intervene with medical treatment. As a consequence of this intervenetion, the rate factor k is no longer constant but varies with time as $k(t) = 2e^{-t} - 1$, where time t is measured in months and $k(t)$ represents the rate of infection per month per 100,000 individuals.
Initially as the effects of medical intervention begin to take hold, $k(t)$ remaind positive and the disease continues to spread. Eventually, however, the effects of medical treatment cause $k(t)$ to become negative and the number of infected individuals then decreases.
(a) Solve the appropriate initial value problem for the number of infected individuals, $P(t)$, at time t and plot the solution.
(b) From your plot, estimate the maximum number of individuals that are at any time infected with the disease.
(c) How long does it take before the number of infected individuals is reduced to 50,000?

Problem 12.11
Consider a chemical reaction of the form $A + B \to C$, in which the rates of change of the two chemical reactants, A and B, are described by the following two differential equations

$$A' = -kAB, \quad B' = -kAB$$

where k is a positive constant. Assume that 5 moles of reactant A and 2 moles of reactant B are present at the beginning of the reaction.
(a) Show that the difference $A(t) - B(t)$ remains constant in time. What the value of this constant?
(b) Use the observation made in (a) to derive an initial value problem for reactant A.
(c) It was observed, after the reaction had progressed for 1 sec, that 4 moles of reactant A remained. How much of reactants A and B will be left after 4 sec of reaction time?

Problem 12.12
Suppose that a given population can be divided into two parts: those who have a given disease and can infect others, and those who do not have it but are susceptible. Let x be the proportion of susceptible individuals and y the proportion of the infectious individuals; then x + y = 1. Assume that the disease spreads by contact between sick and well members of the population, and that the rate of spread $\frac{dy}{dt}$ is proportional to the number of such contacts. Further, assume that members of both groups move about freely among each other, so that the number of contacts is proportional to the product of x and y. Since x = 1- y, we obtain the initial value problem

$$\frac{dy}{dt} = \alpha y(1-y), y(0) = y_0,$$

where α is a positive proportionality factor, and y_0 is the initial proportion of infectious individuals.
(a) Find the equilibrium points for the given differential equation, and determine whether each is stable or unstable. That is, do a complete qualitative analysis on the equation, complete with a graph of $\frac{dy}{dt}$ versus y, and a sketch of possible solutions in the ty-plane.
(b) Solve the initial value problem and verify that the conclusion you reached in part (a) are correct. Show that $y(t) \to 1$ as $t \to \infty$, which means that ultimately the disease spreads through the entire population.

Problem 12.13
Suppose that a population can be modeled by the logistic equation

$$\frac{dP}{dt} = 0.4P\left(1 - \frac{P}{3}\right)$$

Use qualitative techniques to describe the population over time.

Problem 12.14
Find the constants A and B so that

$$P(t) = \frac{e^{0.2t}}{A + Be^{0.2t}}$$

is the solution to the logistic model

$$\frac{dP}{dt} = 0.2P\left(1 - \frac{P}{200}\right), \quad P(0) = 150$$

Problem 12.15
A restricted access lake is stocked with 400 fish. It is estimated that the lake will be able to hold 10,000 fish. The number of fish tripled in the first year. Assuming that the fish population follows a logistic model and that 10,000 is the limiting population, find the length of time needed for the fish population to reach 5000.

Problem 12.16
Ten grizzly bears were introduced to a national park 10 years ago. There are 23 bears in the park at the present time. The park can support a maximum of 100 bears. Assuming a logistic growth model, when will the bear population reach 50?

Problem 12.17
Show that $P(t) = \frac{800}{1+15e^{-1.6t}}$ satisfies the differential equation

$$\frac{dP}{dt} = 0.002P(800 - P)$$

Problem 12.18
A population is observed to obey the logistic equation with eventual population 20,000. The initial population is 1000, and 8 hours later, the observed population is 1200. Find the reproductive rate r and the time required for the population to reach three quarters of its carrying capacity.

Problem 12.19
Let $P(t)$ be the population size for a bacteria colony at time t. The logistic model is that

$$\frac{dP}{dt} = kP(t)(M - P(t)),$$

where $k > 0$ and $M > 0$ are constants. Solve this equation when $k = 1$ and $M = 1000$ with $P(0) = 100$.

Problem 12.20
For the population model

$$P'(t) = 5P(t)(1000 - P(t))$$

with $P(0) = 100$ find the asymptotic population size $\lim_{t\to\infty} P(t)$.

13 Applications of First Order Nonlinear Equations: One-Dimensional Motion with Air Resistance

In Section 1 of this book, we discussed the motion of a free falling object, i.e., the falling of an object under the influence of gravity only - no air resistance or friction effects of any kind. This motion is described by Newton's second lawe given by F = mass × acceleration. The law results in a first order differential equation

$$m\frac{dv}{dt} = -mg$$

The negative sign on the right-hand of the equation is due to the fact that acceleration due to gravity is pointing downward whereas the displacement $y(t)$ is measured upward.

In this section, we shall examine in detail a more realistic model of the one-dimensional motion of an object where we include the effect of air resistance. Air resistance exists because air molecules collide into a falling body creating an upward force opposite gravity and thus reducing the fall of the object. We refer to such a force as the **drag** force. We consider two idealized models of drag force.

Model I: Drag Force is Proportional to Velocity (good for small, slowly falling objects)

If we assume that the drag force is proportional to velocity with positive constant of proportionality k then Newton's second law leads to the differential equation

$$m\frac{dv}{dt} = -mg - kv. \tag{23}$$

Here k depends on the properties of the falling object.

If the object is moving upward then the drag force is pointing downward and in this case $v > 0$ in (23). If the object is moving downward then the drag force is pointing upward and so $v < 0$ in (23).

Equation (23) is a first order linear nonhomogeneous equation that can be solved using the method of integrating factor. Rewriting (23) in the form

$$\frac{dv}{dt} + \frac{k}{m}v = -g$$

and letting

$$\mu(t) = e^{\int \frac{k}{m}dt} = e^{\frac{k}{m}t}$$

we have

$$\begin{array}{rcl}\left(e^{\frac{k}{m}t}v\right)' & = & -ge^{\frac{k}{m}t}\\ \int \left(e^{\frac{k}{m}t}v\right)' dt & = & -g\int e^{\frac{k}{m}t}dt\\ e^{\frac{k}{m}t}v & = & -\frac{mg}{k}e^{\frac{k}{m}t} + C\\ v(t) & = & -\frac{mg}{k} + Ce^{-\frac{k}{m}t}\end{array}$$

If v_0 is the initial velocity then $C = v_0 + \frac{mg}{k}$ and consequently

$$v(t) = -\frac{mg}{k} + (v_0 + \frac{mg}{k})e^{-\frac{k}{m}t} \tag{24}$$

Now, the equilibrium solution of (23) occurs when $v(t) = -\frac{mg}{k}$. At this velocity, the drag force and the gravitational force acting on the object (i.e., its weight) are equal and opposite side. This equilibrium velocity is referred to as the **terminal velocity** of the object. Thus, the terminal velocity of an object falling towards the ground, in non-vacuum, is the speed at which the gravitational force pulling it downwards is equal and opposite to the drag force pushing it upwards. At this speed, the object ceases to accelerate downwards and falls at constant speed.

Example 13.1
An object of mass 5 kg is released from rest 1000 m above the ground and allowed to fall freely under gravity. Assume that the force due to air resistance is proportional to the velocity of the object with proportionality constant $k = 50$ N-s/m. Determine the equation of motion of the object. When will the object strike the ground?

Solution.
Letting $v_0 = 0, m = 50, g = 9.8,$ and $k = 50$ in (24) we obtain

$$v(t) = y'(t) = -0.981 + 0.981e^{-10t}$$

Integrating with respect to t to obtain

$$y(t) = -0.981t - 0.0981e^{-10t} + C$$

But $y(0) = 0$ so that $C = 0.0981$. Hence, the equation of motion is

$$y(t) = -0.981t + 0.0981(1 - e^{-10t})$$

To find at what time the object hits the ground, we need to find T such that

$$y(T) = -1000.$$

That is, we must find T satisfying

$$-0.0981(10T) + 0.0981(1 - e^{-10T}) + 1000 = 0$$

or

$$0.0981(1 - 10T - e^{-10T}) + 1000 = 0.$$

Using a calculator we find $T \approx 1019.467\ s$. ■

Model II: Drag Force is Proportional to the Square of Velocity (more accurate for larger, more rapidly falling objects)
In this case, the model that represents the motion depends on the direction of the motion since $kv^2 \geq 0$. For an object moving upward the differential equation is given by

$$m\frac{dv}{dt} = -mg - kv^2, \quad v(t) \geq 0 \tag{25}$$

and for an object moving downward the differential equation is given by

$$m\frac{dv}{dt} = -mg + kv^2, \quad v(t) \leq 0 \tag{26}$$

In the case that an object is moving up and then down such as the motion of a projectile the model requires the use of both (25) and (26). For the upward dynamics, the motion is modeled by the initial value problem

$$m\frac{dv}{dt} = -mg - kv^2, \; v(0) = v_0, \quad v(t) \geq 0$$

The projectile will reach a highest point after some time t_m. After that point the projectile begins to fall and the motion is modeled by the initial value problem

$$m\frac{dv}{dt} = -mg + kv^2, \; v(t_m) = 0, \; v(t) \leq 0$$

Example 13.2
A projectile of mass m is shot upward from the origin with an initial velocity 300 ft/sec Assume that air resistance is proportional to the square of the velocity with $k = \frac{m}{2048}$.
(a) Find the velocity and position as a function of time
(b) Plot the position function.
(c) Find the time when maximum height is reached, the time when the projectile hits the ground, the maximum height, and the impact velocity, i.e., the velocity right before hiting the ground.

Solution.
In the upward motion we need to solve the initial value problem

$$mv' = -32m - \frac{m}{2048}v^2, \quad v(0) = 300$$

Separating the variables and integrating we find

$$\begin{array}{rcl}
\frac{v'}{1+\frac{v^2}{65536}} & = & -32 \\
\frac{v'}{1+\left(\frac{v}{256}\right)^2} & = & -32 \\
\int \frac{v'}{1+\left(\frac{v}{256}\right)^2} dt & = & \int -32 dt \\
256 \arctan\left(\frac{v}{256}\right) & = & -32t + C
\end{array}$$

But $v(0) = 300$ so that $C = 256 \arctan\left(\frac{75}{64}\right)$. Thus,

$$v(t) = 256 \tan\left[-\frac{1}{8}t + \arctan\left(\frac{75}{64}\right)\right]$$

To find the position function we integrate $v(t)$ with respect to t and find

$$\begin{array}{rcl}
y(t) & = & \int 256 \tan\left[-\frac{1}{8}t + \arctan\left(\frac{75}{64}\right)\right] dt \\
& = & 2048 \ln\left[\cos\left(-\frac{1}{8}t + \arctan\left(\frac{75}{64}\right)\right)\right] + C
\end{array}$$

But $y(0) = 0$ so that

$$C = -2048 \ln\left(\frac{64}{\sqrt{9721}}\right)$$

Hence

$$y(t) = 2048 \ln\left[\cos\left(-\frac{1}{8}t + \arctan\left(\frac{75}{64}\right)\right)\right] - 2048 \ln\left(\frac{64}{\sqrt{9721}}\right)$$

The highest point occur when $v(t) = 0$. That is,

$$256 \tan\left[-\frac{1}{8}t + \arctan\left(\frac{75}{64}\right)\right] = 0$$

Solving this equation for t we find

$$t_m = 6.91496 \; sec.$$

The graph of the ascent, valid for $0 \le t \le 6.91496$ is given in Figure 13.1.

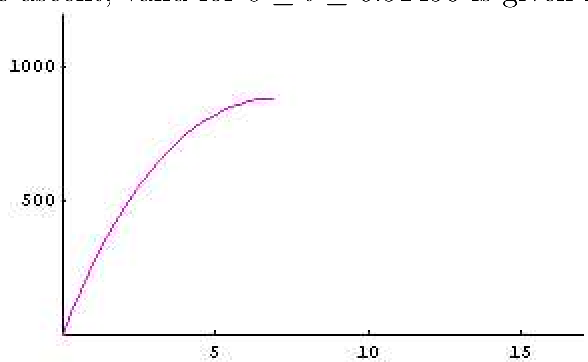


Figure 13.1

The maximum height is $y(6.91496) \approx 885.02 \; ft$.
Now, the initial value problem for the descent motion is given by

$$m\frac{dv}{dt} = -mg + \frac{m}{2048}v^2, \quad v(t_m) = 0$$

Solving this IVP we find

$$\begin{array}{rcl}
\frac{v'}{\frac{v^2}{65536}-1} & = & 32 \\
\frac{1}{2}\left(\frac{v'}{\left(\frac{v}{256}\right)-1} - \frac{v'}{\left(\frac{v}{256}\right)+1}\right) & = & 32 \\
128 \ln\left|\frac{v-256}{v+256}\right| & = & 32t + C \\
\frac{v-256}{v+256} & = & Ce^{\frac{1}{4}t}
\end{array}$$

But $v(6.91946) = 0$ so that $C = -e^{\frac{-1}{4}(6.91946)}$. Hence,

$$v(t) = 256\left(\frac{1 - e^{\frac{1}{4}(t-6.91496)}}{1 + e^{\frac{1}{4}(t-6.91496)}}\right)$$

This can be written in the form

$$v(t) = 256\left(-1 + \frac{2}{1 + e^{\frac{1}{4}(t-6.91496)}}\right)$$

Integrating this last equation with respect to t to obtain

$$y(t) = 256t - 3540.45952 - 2048\ln\left(1 + e^{\frac{1}{4}(t-6.91496)}\right) + C$$

Also, notice that

$$\begin{aligned}
-3540.45952 - 2048\ln\left(1 + e^{\frac{1}{4}(t-6.91496)}\right) &= -3540.45952 - 2048\ln e^{\frac{-6.91496}{4}} \\
&\quad -2048\ln\left(e^{\frac{6.91496}{4}} + e^{\frac{t}{4}}\right) \\
&\approx -2048\ln\left(5.63355 + e^{\frac{t}{4}}\right)
\end{aligned}$$

Hence,

$$y(t) = 256t - 2048\ln\left(5.63355 + e^{\frac{t}{4}}\right) + C$$

Since $y(6.91496) = 885.02$ then $C \approx 4074.82$. It follows that

$$y(t) = 256t - 2048\ln\left(5.63355 + e^{\frac{t}{4}}\right) + 4074.82$$

The plot for the descent is given in Figure 13.2.

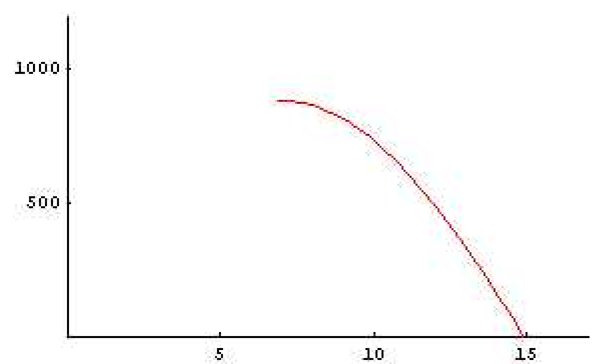


Figure 13.2

The projectile hits the ground at $t \approx 14.8977$ *sec.*
Combining the ascent and descent motion the graph of $y(t)$ is shown in Figure 13.3.

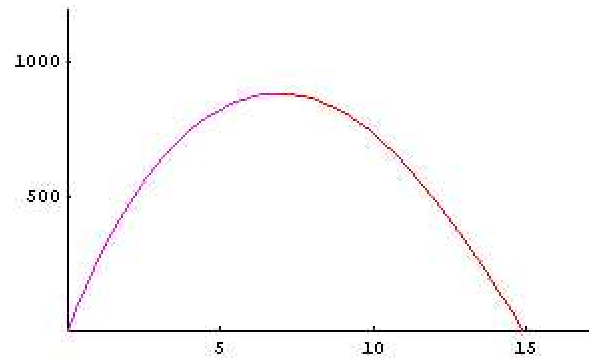


Figure 13.3

Finally, the terminal velocity is given by

$$\lim_{t\to\infty} 256\left(-1+\frac{2}{1+e^{\frac{1}{4}(t-6.91496)}}\right) = -256\ ft/sec$$

and the impact velocity is $v(14.8977) \approx -194.736\ ft/sec$ ■

Problem 13.1
A parachutist whose mass is 75 kg drops from a helicopter hovering 2000 m above ground, and falls towards the ground under the influence of gravity. Assume that the force due to air resistance is proportional to the velocity of the parachutist, with the proportionality constant $k = 30N - s/m$ when the chute is closed, and $k' = 90N - s/m$ when the chute is opened. If the chute doesn't open until the velocity of the parachutist reaches 20 m/s, after how many seconds will she reach the ground?

Problem 13.2
An object of mass m is dropped from a high altitude. How long will it take the object to achieve a velocity equal to one-half of its terminal velocity if the drag force is assumed proportional to the velocity?

Problem 13.3
An object of mass m is dropped from a high altitude. Assume the drag force is proportional to the square of the velocity with drag constant k. Find a formula for $v(t)$.

Problem 13.4
Assume that the action of a parachute can be modeled as a drag force proportional to the square of the velocity. What drag constant k of the parachute is needed for a 200 lb person to achieve a teminal velocity of 10 mph?

Problem 13.5
A drag chute must be deisgned to reduce the speed of 3000-lb dragster from 220 mph to 50 mph in 4 seconds. Assume that the drag force is proportional to the velocity.
(a) What value of the drag coefficient k is needed to accomplish this?
(b) How far will the dragster travel in the 4-sec interval?

Problem 13.6
A projectile of mass m is launched vertically upward from ground level at time $t = 0$ with initial velocity v_0 and is acted upon by gravity and air resistance. Assume the drag force is proportional to velocity, with drag coefficient k.
(a) Derive an expression for the time t_m when the projectile achieves its maximum height.
(b) Derive an expression for the maximum height.

Problem 13.7
A projectile is launched vertically upward from ground level with initial velocity v_0. Neglect air resistance. Show that the time it takes the projectile to reach its maximum height is equal to the time it takes to fall from this maximum height to the ground.

Problem 13.8
A 180-lb skydiver drops from a hot-air ballon. After 10 sec of free fall, a parachute is opened. The parachute immediately introduces a drag force proportional to the velocity. After an additional 4 sec, the parachutist reaches the ground. Assume that air resistance is negligible during free fall and that the parachute is designed so that a 200-lb person will reach a terminal velocity of 10 mph.
(a) What is the speed of the skydiver immediately before the parachute is opened?
(b)What is the parachutist impact velocity?
(c) At what altitude was the parachute opened?
(d) What is the ballon altitude?

Problem 13.9
A body of mass m is moving with velocity v in a gravity-free laboratory (i.e. outer space). It is known that the body experiences resistance in its flight proportional to the square root of its velocity. Consequently the motion of the body is governed by the initial-value problem

$$m\frac{dv}{dt} = -k\sqrt{v},\ v(0) = v_0$$

where k is a positive constant. Find $v(t)$. Does the body utimately come to rest? If so, when does this happen?

Problem 13.10
A mass m is thrown upward from ground level with initial velocity v_0. Assume that air resistance is proportional to velocity, the constant of proportionality being k. Show that the maximum height attained is

$$-\frac{m^2g}{k^2}\ln\left(1+\frac{kv_0}{mg}\right)+\frac{m}{k}\left(v_0+\frac{mg}{k}\right)\left(1-\frac{1}{\frac{k}{mg}v_0+1}\right)$$

Problem 13.11
A ball weighing 3/4 lb is thrown vertically upward from a point 6 ft above ground level with an initial velocity of 20ft/sec. As it rises it is acted upon by air resistance that is numerically equal to v/64 lbs where v is velocity (in ft/sec). How high will it rise?

Problem 13.12
A parachutist weighs 160 lbs (with chute). The chute is released immediately after the jump from a height of 1000 ft. The force due to air resistence is proportional to velocity and is given by $F_R = -8v$. Find the time of impact.

Problem 13.13
A parachutist weighs 100 Kg (with chute). The chute is released 30 seconds after the jump from a height of 2000 m. The force due to air resistance is defined by $F_R = -kv$ where $k = 15$ when the chute was closed and $k = 100$ when the chute was open. Find
(a) the distance and velocity function during the time the chute was closed (i.e., $0 \le t \le 30$ seconds).
(b) the distance and velocity function during the time the chute was open

(i.e., $t \geq 30$ seconds).
(c) the time of landing.
(d) the velocity of landing or the impact velocity.

Problem 13.14
Solve the equation
$$m\frac{dv}{dt} = -kv(t) - mg$$
with initial condition $v(0) = 0$ when $k = 0.1$ and $m = 1\ kg$.

Problem 13.15
A rocket is launched at time $t = 0$ and its engine provides a constant thrust for 10 seconds. During this time the burning of the rocket fuel constantly decreases the mass of the rocket. The problem is to determine the velocity $v(t)$ of the rocket at time t during this initial 10 second interval. Denote by $m(t)$ the mass of the rocket at time t and by U the constant upward thrust (force) provided by the engine. Applying Newton's Law gives
$$\frac{d}{dt}(m(t)v(t)) = U - kv(t) - m(t)g$$
where an air resistance term is included in addition to the gravitational and thrust terms. Find a fomrula for $v(t)$.

Problem 13.16
If $m(t) = 11 - t, U = 200$, and $k = 0$ the equation of motion of the rocket is
$$\frac{d}{dt}((11 - t)v(t)) = 200 - (11 - t)g.$$
Find $v(t)$ for $0 \leq t \leq 10$. Assume $v(0) = 0$. Make a graph of the velocity as a function of time.

Problem 13.17
If $m(t) = 11 - t, U = 200$, and $k = 2$ the equation of motion of the rocket is
$$\frac{d}{dt}((11 - t)v(t)) = 200 - 2v(t) - (11 - t)g.$$
Find $v(t)$ for $0 \leq t \leq 10$. Assume $v(0) = 0$. Make a graph of the velocity as a function of time.

Problem 13.18
Using (24), find the position function $y(t)$.

Problem 13.19
An arrow is shot upward from the origin with an initial velocity of 300 ft/sec. Assume that there is no air resistance and use the model

$$m\frac{dv}{dt} = -mg$$

Find the velocity and position as a function of time. Find the ascent time, the descent time, maximum height, and the impact velocity.

Problem 13.20
An arrow is shot upward from the origin with an initial velocity of 300 ft/sec. Assume that air resistance is proportional to the velocity, $F_R = 0.04mv$ and use the model

$$m\frac{dv}{dt} = -mg - kv$$

Find the velocity and position as a function of time, and plot the position function. Find the ascent time, the descent time, maximum height, and the impact velocity.

14 One-Dimensional Dynamics: Velocity as Function of Position

The computation is Example 13.2 was relatively complicated. The task of finding the maximum projectile height can be simplified by transforming the problem to one in which height rather than time is the independent variable. This transformation is the topic of this section.
We will assume that the motion of an object is either always increasing (i.e. $v(t) \geq 0$) or always decreasing (i.e. $v(t) \leq 0$) on the time interval of interest so that the displacement function $y(t)$ is invertible which allows us to write t in terms of y. In this case by applying the chain rule we can write

$$\frac{dv}{dt} = \frac{dv}{dy} \cdot \frac{dy}{dt} = v\frac{dv}{dy}$$

This change from $v(t)$ to $v(y)$ is useful when the net force acting on the object is a function of y and v and does not depend explicitly on t, i.e., we have

$$mv\frac{dv}{dy} = F(y, v)$$

Example 14.1
Find the maximum height in Example 13.2 by considering the velocity as a function of y.

Solution.
In the upward motion we need to solve the initial value problem

$$mv\frac{dv}{dy} = -32m - \frac{m}{2048}v^2, \quad v(0) = 300, y(0) = 0$$

This last equation is a Bernoulli equation since

$$\frac{dv}{dy} + \frac{v}{2048} = -32v^{-1}$$

By letting $w = v^2$ then $\frac{dw}{dy} = 2v\frac{dv}{dy}$ so that the last equation reduces to

$$\frac{1}{2}w' + \frac{w}{2048} = -32$$

or

$$w' + \frac{w}{1024} = -64$$

We solve this equation as follows

$$\begin{array}{lcl} e^{\frac{y}{1024}}\left(w' + \frac{w}{1024}\right) & = & -64e^{\frac{y}{1024}} \\ (e^{\frac{y}{1024}}w)' & = & -64e^{\frac{y}{1024}} \\ \int (e^{\frac{y}{1024}}w)'dy & = & \int -64e^{\frac{y}{1024}}dy \\ e^{\frac{y}{1024}}w & = & -65536e^{\frac{y}{1024}} + C \\ w(y) & = & -65536 + Ce^{-\frac{y}{1024}} \end{array}$$

But $w(0) = (v(0))^2 = 90000$ so that $C = 155536$. Thus,

$$w(y) = -65536 + 155536e^{-\frac{y}{1024}}$$

But $w = v^2$ so that

$$v(y) = \left[-65536 + 155536e^{-\frac{y}{1024}}\right]^{\frac{1}{2}}$$

This equation is valid for $0 \le y \le y_{max}$. The maximum height occurs when velocity is zero. That is

$$-65536 + 155536e^{-\frac{y}{1024}} = 0$$

Solving this equation for y we find

$$y_{max} = -1024 \ln\left(\frac{65536}{155536}\right) \approx 885.02\ ft \blacksquare$$

Newton's Law of Gravitation

The next example, an object falling through the atmosphere, shows that using position as the independent variable may convert a problem we cannot solve into one that we can solve. The example involves Newton's law of gravitation which states that any two objects exert a gravitational force of attraction on each other. The direction of the force is along the line joining the objects (See Figure 14.1). The magnitude of the force is proportional to the product of the gravitational masses of the objects, and inversely proportional to the square of the distance between them. That is

$$F_{12} = G\frac{m_1m_2}{r^2}$$

where $G \approx 6.673 \times 10^{-11} \frac{m^3}{kg \cdot s^2}$.

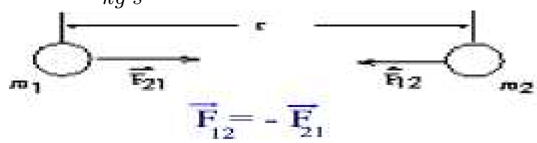


Figure 14.1

For an object of mass m falling to surface of the Earth the magnitude of the gravitational force becomes

$$F = G\frac{M_e m}{r^2} \tag{27}$$

where $M_e = 5.98 \times 10^{24}$ kg is the mass of the Earth and r is the distance of the object to the center of the Earth. Note that near the surface of the Earth we have

$$G\frac{M_e m}{R_e^2} = mg$$

where $R_e = 6.38 \times 10^6$ m is the radius of the Earth. Solving for g we find

$$g \approx 9.8 \ m/s^2$$

Example 14.2
Consider an object having mass $m = 100$ kg which is released from rest at an altitude of $h = 200$ km above the surface of the Earth. Find the velocity of the object right before hitting the ground. We assume no drag forces and considering only the force of gravitational attraction.

Solution.
By Newton's second law of motion we have

$$m\frac{dv}{dt} = -G\frac{M_e m}{r^2}.$$

But $v(t) = \frac{dr}{dt}$ so that

$$m\frac{d^2r}{dt^2} = -G\frac{M_e m}{r^2}.$$

If we regard v as a function of r then by the chain rule we arrive at the following IVP:

$$mv\frac{dv}{dr} = -G\frac{M_e m}{r^2}, \quad v(R_e + 200) = 0, \quad R_e < r < R_e + 200.$$

This is a separable nonlinear first order differential equation. Its solution is given by

$$\frac{v^2}{2} = \frac{GM_e}{r} + C$$

Since $v(R_e + 200) = 0$ then $C = -\frac{GM_e}{R_e+200}$. Hence, the implicit solution to the IVP is

$$\frac{v^2}{2} = GM_e \left[\frac{1}{r} - \frac{GM_e}{R_e + 200}\right]$$

But $r(t)$ is a decreasing function so that $\frac{dv}{dr} < 0$. This leads to the explicit solution

$$v(r) = -\sqrt{2GM_e \left[\frac{1}{r} - \frac{GM_e}{R_e + 200}\right]}$$

The impact velocity is then

$$v(R_e) = -\sqrt{2GM_e \left[\frac{1}{R_e} - \frac{GM_e}{R_e + 200}\right]} \approx -1952 \ m/s \ \blacksquare$$

In Problems 1 - 3, transform the equation into one having distance x as the independent variable. Determine the position x_f at which the object comes to rest.(If the object does not come to rest set $x_f = \infty$) Assume that $v = v_0$ when $x = 0$.

Problem 14.1

$$m\frac{dv}{dt} = -kx^2v$$

Problem 14.2

$$m\frac{dv}{dt} = -kxv^2$$

Problem 14.3

$$m\frac{dv}{dt} = \frac{kv}{1+x}$$

Problem 14.4
A boat having mass m is launched vertically with an initial velocity v_0. Assume the water exerts a drag force that is proportional to the square of the velocity. Determine the velocity of the boat when it is a distance d from the dock.

Problem 14.5
A block of mass m is pulled over a frictionless (smooth) surface by a cable having a constant tension F (See Figure 14.2). The block starts from rest at a horizontal distance D from the base of the pulley. Apply Newton's law of motion in the horizontal direction. What is the (horizontal) velocity of the block when $x = \frac{D}{3}$? (Assume the vertical component of the tensile force never exceeds the weight of the block.)

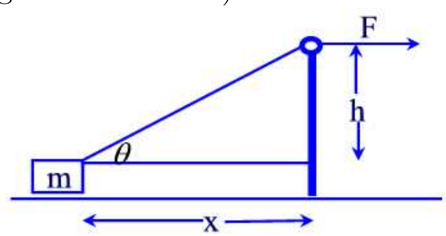


Figure 14.2

Problem 14.6
We need to design a ballistics chamber to deccelerate test projectiles fired into it. Assume the resistive force encountered by the projectile is proportional to the square of its velocity and neglect gravity. The coefficient k is given by $k(x) = k_0x$, where x_0 is a constant. If we use time as independent variable then Newton's second law of motion leads to the following differential equation

$$m\frac{dv}{dt} + k_0xv^2 = 0$$

(a) Adopt distance x as the indepndent variable and rewrite the above differential equation as a first order equation in terms of the new independent variable.
(b) Determine the value k_0 needed if the chamber is to reduce projectile velocity to 1% of its incoming value within d units of distance.

15 Second Order Linear Differential Equations: Existence and Uniqueness Results

To this point we have considered only first order differential equations. However, many of the most interesting differential equations involve second derivatives. Indeed, since acceleration is the second derivative of position, Newton's second law of motion, F = ma, is a second order differential equation. In this and the coming sections we turn our attention to linear second-order differential equations.

By a **second-order linear differential equation** we mean an equation of the form

$$y'' + p(t)y' + q(t)y = g(t)$$

If $g(t) \equiv 0$ we say that the equation is **homogeneous**. Otherwise the equation is **nonhomogeneous**. Initial-value problems for second-order linear differential equations require two initial-conditions. In this section we will consider the question of existence and uniqueness of solutions to the initial value problem

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \ y'(t_0) = y'_0 \tag{28}$$

Existence and uniqueness results similar to first-order equations exist for second-order equations as well. The following theorem tells us the conditions for the existence and uniqueness of solutions of a second order linear differential equation. Note how this theorem is analogous to the corresponding theorem for first order linear ODE's.

Theorem 15.1

If $p(t), q(t)$, and $g(t)$ are continuous functions over an interval $a < t < b$ containing t_0 then the initial value problem (28) has a unique solution in the interval (a,b).

Proof.

We provide a proof for the simple case when the coefficients are constants. In this case, one can apply a variant of the integrating factor applied to first order linear differential equations. So we assume that $p(t) \equiv C$ and $q(t) \equiv C'$ for all $a < t < b$.

Existence: The existence of a local solution is obtained here, as for all second order equations, by transforming the problem into a first order system.

This is done by introducing the variable $z = y'$. In this case, $z' = g(t) - Cz - C'y$. Thus, we can write the problem as a system:

$$\begin{bmatrix} y' \\ z' \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ C' & C \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$$

or in compact form

$$\mathbf{x}'(t) + \mathbf{A}\mathbf{x}(t) = \mathbf{b}(t)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ C' & C \end{bmatrix}, \qquad \mathbf{x}(t) = \begin{bmatrix} y \\ z \end{bmatrix}, \qquad \mathbf{b}(t) = \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$$

We solve this equation as if it were a scalar first-order linear differential equation, which we know how to solve. Multiply through with the correct integrating factor, integrate, and solve for $\mathbf{x}$. That is, we solve

$$\mathbf{x}'(t) + \mathbf{A}\mathbf{x}(t) = \mathbf{b}(t) \tag{29}$$

by multiplying with the integrating factor $e^{\int \mathbf{A} dt}$ where

$$\int \mathbf{A} dt = \begin{bmatrix} 0 & -t \\ C't & Ct \end{bmatrix}$$

and

$$e^{\int \mathbf{A} dt} = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\int \mathbf{A} dt \right]^n$$

Thus, we obtain

$$e^{\int \mathbf{A} dt}\mathbf{x}'(t) + e^{\int \mathbf{A} dt}\mathbf{A}\mathbf{x}(t) = e^{\int \mathbf{A} dt}\mathbf{b}(t)$$

which is

$$\left(e^{\int \mathbf{A} dt}\mathbf{x} \right)' = e^{\int \mathbf{A} dt}\mathbf{b}(t)$$

We integrate both sides with respect to t to get

$$e^{\int \mathbf{A} dt}\mathbf{x} = \int e^{\int \mathbf{A} dt}\mathbf{b}(t) dt + \mathbf{C}''$$

Finally we multiply by the inverse of the integrating factor, which of course is $e^{-\int \mathbf{A} dt}$, to get $\mathbf{x}$ alone,

$$\mathbf{x} = e^{-\int \mathbf{A} dt} \int e^{\int \mathbf{A} dt}\mathbf{b}(t) dt + e^{-\int \mathbf{A} dt}\mathbf{C}''$$

Note that the integration gives an integration constant, which is a vector, so that the general solution has a vector constant in it. That is to say, the general solution has two scalar constants in it. The initial conditions $y(t_0) = y_0$ and $y'(t_0) = y'_0$ determine these constants.

Uniqueness: Suppose that $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are two solutions to (29). Let $\mathbf{w}(t) = \mathbf{u}(t) - \mathbf{v}(t)$. Then by substitution into (29) we obtain

$$\mathbf{w}' + \mathbf{A}\mathbf{w} = \mathbf{0} \tag{30}$$

Multiplying through by $e^{\int \mathbf{A}dt}$ to obtain

$$\left(e^{\int \mathbf{A}dt}\mathbf{w}\right)' = \mathbf{0}$$

and then integrate to obtain

$$e^{\int \mathbf{A}dt}\mathbf{w} = \mathbf{D}$$

or

$$\mathbf{w}(t) = \mathbf{D}e^{-\int \mathbf{A}dt}$$

But $\mathbf{w}(t_0) = 0$ so that $\mathbf{D} = 0$. Hence, $\mathbf{w}(t) \equiv \mathbf{0}$ for all $a < t < b$ which is equivalent to $\mathbf{u}(t) = \mathbf{v}(t)$ for all $a < t < b$ ∎

Example 15.1
Use the method of integrating factor described in the above theorem to solve the initial value problem

$$y'' - y = 0, \quad y(0) = 1, \; y'(0) = 0$$

Solution.
In this problem, $p(t) = 0$ and $q(t) = -1$ so that

$$\mathbf{A} = \left[\begin{array}{cc} 0 & -1 \\ -1 & 0 \end{array}\right]$$

Hence,

$$-\int \mathbf{A}dt = \left[\begin{array}{cc} 0 & t \\ t & 0 \end{array}\right]$$

Now, one can easily see that for any nonnegative odd integer n we have

$$\left[-\int \mathbf{A}dt\right]^n = \left[\begin{array}{cc} 0 & t^n \\ t^n & 0 \end{array}\right]$$

and for nonnegative even integer n

$$\left[-\int \mathbf{A}dt\right]^n = \left[\begin{array}{cc} t^n & 0 \\ 0 & t^n \end{array}\right]$$

Thus,

$$e^{-\int \mathbf{A}dt} = \left[\begin{array}{cc} \sum_{n=0}^\infty \frac{t^{2n}}{(2n)!} & \sum_{n=0}^\infty \frac{t^{2n+1}}{(2n+1)!} \\ \sum_{n=0}^\infty \frac{t^{2n+1}}{(2n+1)!} & \sum_{n=0}^\infty \frac{t^{2n}}{(2n)!} \end{array}\right] = \left[\begin{array}{cc} \cosh t & \sinh t \\ \sinh t & \cosh t \end{array}\right]$$

Thus,

$$\mathbf{x}(t) = \left[\begin{array}{cc} \cosh t & \sinh t \\ \sinh t & \cosh t \end{array}\right]\left[\begin{array}{c} c_1 \\ c_2 \end{array}\right]$$

From this we obtain

$$y(t) = c_1 \cosh t + c_2 \sinh t$$

But $y(0) = 1$ so that $c_1 = 1$. Also, since $y'(0) = 0$ then $c_2 = 0$. Hence, the unique solution to the initial value problem is

$$y(t) = \cosh t \;\blacksquare$$

Remark 15.1
The approach used for the case of constant coefficients does not apply for the general case because, in general, one has

$$\frac{d}{dt}(e^{\int \mathbf{A}(t)dt}) \neq A(t)e^{\int \mathbf{A}(t)dt}$$

This is due to the fact that matrix multiplication is not commutative and so the power rule for differentiation does not apply. A proof for the general case of the above theorem is given in Section 27, Theorem 27.1.

Remark 15.2
The above theorem does not give the largest t-interval of exsitence. See Problem 15.3.

Example 15.2
Find the largest interval where

$$(t^2-1)y''+3ty'+(\cos t)y=e^t, \quad y(0)=4, \quad y'(0)=5$$

is guaranteed to have a unique solution.

Solution.
We first put it into standard form

$$y''+\frac{3t}{t^2-1}y'+\frac{\cos t}{t^2-1}y=\frac{e^t}{t^2-1}, \; y(0)=4, \; y'(0)=5$$

p, q, and g are all continuous except at $t=-1$ and $t=1$. The theorem tells us that there is a unique solution on (-1,1) since this interval contains 0 ■

In Problems 1 - 6, determine the largest t-interval on which Theorem 15.1 guarantees the existence of a unique solution.

Problem 15.1

$$y''+y'+3ty=\tan t, \; y(\pi)=1, \; y'(\pi)=-1$$

Problem 15.2

$$e^ty''+\tfrac{1}{t^2-1}y=\tfrac{4}{t}, y(-2)=1, \; y'(-2)=2$$

Problem 15.3

$$ty''+\tfrac{\sin 2t}{t^2-9}y'+2y=0, \; y(1)=0, y'(1)=1$$

Problem 15.4

$$ty''-(1+t)y'+y=t^2e^{2t}, y(-1)=0, y'(-1)=1$$

Problem 15.5

$$(\sin^2 t)y'' - (2\sin t\cos t)y' + (\cos^2 t + 1)y = \sin^3 t,\ y(\tfrac{\pi}{4}) = 0,\ y'(\tfrac{\pi}{4}) = \sqrt{2}$$

Problem 15.6

$$t^2y'' + ty' + y = \sec(\ln t),\ y(\tfrac{\pi}{3}) = 0,\ y'(\tfrac{\pi}{3}) = -1$$

In Problems 7 - 9, give an example of an initial value problem of the form (28) for which the given t-interval is the largest on which Theorem 15.1 guarantees a unique solution.

Problem 15.7

$$-\infty < t < \infty$$

Problem 15.8

$$3 < t < \infty$$

Problem 15.9

$$-1 < t < 5$$

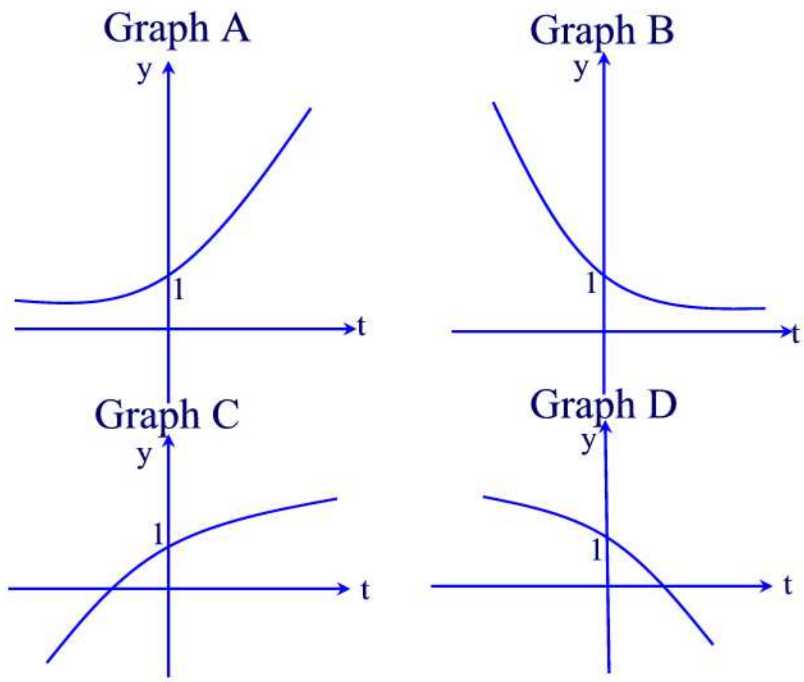
Problem 15.10
Consider the initial value problem $t^2y'' - ty' + y = 0,\ y(1) = 1,\ y'(1) = 1$.
(a) What is the largest interval on which Theorem 15.1 guarantees the existence of a unique solution?
(b) Show by direct substitution that the function $y(t) = t$ is the unique solution to this initial value problem. What is the interval on which this solution actually exists?
(c) Does this example contradict the assertion of Theorem 15.1? Explain.

Problem 15.11
Is there a solution $y(t)$ to the initial value problem

$$y'' + 2y' + \frac{1}{t-3}y = 0,\ y(1) = 1,\ y'(1) = 2$$

such that $\lim_{t\to 0^+} y(t) = \infty$?

Problem 15.12
Consider the graphs shown. Each graph displays a portion of the solution of one of the four initial value problems given. Match each graph with the appropriate initial value problem.
(a) $y'' + y = 2 - \sin t,\ y(0) = 1, y'(0) = -1$
(b) $y'' + y = -2t,\ y(0) = 1,\ y'(0) = -1$
(c) $y'' - y = t^2,\ y(0) = y'(0) = 1$
(d) $y'' - y = -2\cos t,\ y(0) = y'(0) = 1$



Problem 15.13
Determine the longest interval in which the initial-value problem

$$(t-3)y'' + ty' + (\ln|t|)y = 0,\ y(1) = 0,\ y'(1) = 1$$

is certain to have a unique solution.

Problem 15.14
The existence and uniqueness theorem tells us that the initial-value problem

$$y'' + t^2 y = 0, y(0) = y'(0) = 0$$

define exactly one function $y(t)$. Using only the existence and uniqueness theorem, show that this function has the additional property $y(-t) = y(t)$.

Problem 15.15
By introducing a new variable x, write $y'' - 2y + 1 = 0$ as a system of two first order linear equations of the form $\mathbf{x}' + \mathbf{A}\mathbf{X} = \mathbf{b}$

Problem 15.16
Write the differential equation $y'' + 4y' + 4y = 0$ as a first order system.

Problem 15.17
Using the substitutions $x_1 = y$ and $x_2 = y'$ write the differential equation $y'' + ky' + (t-1)y = 0$ as a first order system.

Problem 15.18
Consider the 2-by-2 matrix

$$\mathbf{A} = \left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right]$$

(a) Find $-\int \mathbf{A}(t)dt$
(b) Let $\mathbf{B} = -\int \mathbf{A}(t)dt$. Compute $\mathbf{B}^2, \mathbf{B}^3, \mathbf{B}^4, \mathbf{B}^5$.
(c) Show that

$$e^{\mathbf{B}} = \left[\begin{array}{cc} \sum_{n=0}^{\infty}(-1)^n\frac{(t)^{2n}}{(2n)!} & \sum_{n=0}^{\infty}(-1)^n\frac{t^{2n+1}}{(2n+1)!} \\ -\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n+1}}{(2n+1)!} & \sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{(2n)!} \end{array}\right] = \left[\begin{array}{cc} \cos t & \sin t \\ -\sin t & \cos t \end{array}\right]$$

Problem 15.19
Use the previous problem to solve the initial value problem

$$y'' + y = 0, \quad y(0) = 1, \; y'(0) = 0$$

Problem 15.20
Repeat the process of the previous two problems for solving the initial value problem

$$y'' - 2y' = 0, \; y(0) = 1, \; y'(0) = 2$$

16 The General Solution of Homogeneous Equations

In this section we discuss the structure of the general solution to the homogeneous second order linear differential equation

$$y'' + p(t)y' + q(t)y = 0 \qquad (31)$$

where $p(t)$ and $q(t)$ are continuous functions for $a < t < b$.
The first key property of (31) is its linear property also known as the **principle of superposition**.

Theorem 16.1 *(Principle of Superposition)*
If y_1 and y_2 are respective solutions of (31) then for any constants c_1 and c_2, the function $y = c_1y_1 + c_2y_2$ is also a solution to (31)

Proof.
To see why the linear property holds, we carry out the following computation for $y = c_1y_1 + c_2y_2$.

$$\begin{array}{rcl} y'' + p(t)y' + q(t)y &=& (c_1y_1 + c_2y_2)'' + p(t)(c_1y_1 + c_2y_2)' + q(t)(c_1y_1 + c_2y_2) \\ &=& c_1(y_1'' + p(t)y_1' + q(t)y_1) + c_2(y_2'' + p(t)y_2' + q(t)y_2) \\ &=& c_1 \cdot 0 + c_2 \cdot 0 = 0 \blacksquare \end{array}$$

The function $c_1y_1 + c_2y_2$ is called a **linear combination** of the functions y_1 and y_2.

Example 16.1
Write $y = 3\cos\left(2t + \frac{\pi}{4}\right)$ as a linear combination of $y_1 = \cos 2t$ and $y_2 = \sin 2t$.

Solution.
Using the identity

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

we arrive at

$$\begin{array}{rcl} 3\cos\left(2t + \frac{\pi}{4}\right) &=& 3\cos 2t\cos\frac{\pi}{4} - 3\sin 2t\sin\frac{\pi}{4} \\ &=& \left[3\cos\frac{\pi}{4}\right]\cos 2t + \left[-3\sin\frac{\pi}{4}\right]\sin 2t \\ &=& \left(\frac{3\sqrt{2}}{2}\right)\cos 2t + \left(-\frac{3\sqrt{2}}{2}\right)\sin 2t \blacksquare \end{array}$$

Theorem 16.1 states that if y_1 and y_2 are two given solutions of (31) then one can build many new solutions by taking a linear combination $y = c_1y_1 + c_2y_2$. However, this theorem does not say if every solution to (31)has to be written as a linear combination of y_1 and y_2. So our next interest is to find out if one can express every solution of (31) as a linear combination of two solutions of (31). If there are such solutions y_1 and y_2, we shall say that the set $\{y_1, y_2\}$ forms a **fundamental set of solutions** to (31).
It follows that if we know a fundamental set of solutions $\{y_1, y_2\}$ then we know the general solution to (31) which is given by

$$y(t) = c_1y_1(t) + c_2y_2(t)$$

Identifying Fundamental Sets
Given a particular homogeneous differential equation and two solutions of that differential equation. Is there a convenient way for checking whether or not these two solutions form a fundemental set of solutions? The answer is in the affirmative according to the following theorem.

Theorem 16.2
Let $y_1(t)$ and $y_2(t)$ be two solutions to the homogeneous second order linear differential equation

$$y'' + p(t)y' + q(t)y = 0, a < t < b$$

where $p(t)$ and $q(t)$ are continuous in $a < t < b$. If there is a $a < t_0 < b$ such that

$$W(y_1(t_0), y_2(t_0)) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) \neq 0$$

then $\{y_1, y_2\}$ is a fundamental set of solutions. We call the function W the **Wronskian** function.

Proof.
We need to show that if $y(t)$ is a solution to (31) then we can write $y(t)$ as a linear combination of y_1 and y_2. That is

$$y(t) = c_1y_1(t) + c_2y_2(t).$$

So the problem reduces to finding the constants c_1 and c_2. These are found by solving the following linear system of two equations in the unknowns c_1 and c_2:

$$c_1y_1(t_0) + c_2y_2(t_0) = y(t_0)$$

$$c_1y_1'(t_0) + c_2y_2'(t_0) = y'(t_0)$$

By the method of elimination we find

$$c_1 = \frac{y(t_0)y_2'(t_0) - y'(t_0)y_2(t_0)}{W(y_1(t_0), y_2(t_0))}$$

and

$$c_2 = \frac{y'(t_0)y_1(t_0) - y(t_0)y_1'(t_0)}{W(y_1(t_0), y_2(t_0))}$$

Note that c_1 and c_2 exist since $W(y_1(t_0), y_2(t_0)) \neq 0$ ■

Example 16.2
Consider the differential equation

$$y'' + 4y = 0 \tag{32}$$

(a) Show that $y_1(t) = \cos 2t$ and $y_2(t) = \sin 2t$ are solutions to (32).
(b) Show that $\{\cos 2t, \sin 2t\}$ is a fundamental set of solutions.
(c) Write the solution $y(t) = 3\cos\left(2t + \frac{\pi}{4}\right)$ as a linear combination of y_1 and y_2.

Solution.
(a) A simple calculation shows

$$y_1'' + 4y_1 = -4\cos 2t + 4\cos 2t = 0$$

$$y_2'' + 4y_2 = -4\sin 2t + 4\sin 2t = 0$$

(b) For any t we have

$$W(y_1(t), y_2(t)) = \left| \begin{array}{cc} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{array} \right| = 2\cos^2 2t + 2\sin^2 2t = 2 \neq 0$$

Thus, $\{y_1, y_2\}$ is a fundamental set of solutions.
(c) Using the formulas for c_1 and c_2 with $t_0 = 0$ we find

$$\begin{array}{rcl} c_1 & = & \frac{y(0)y_2'(0) - y'(0)y_2(0)}{W(y_1(0), y_2(0))} \\ & = & \frac{6\cos\frac{\pi}{4}\cos 0 + 6\sin\frac{\pi}{4}\sin 0}{2} = \frac{3\sqrt{2}}{2} \end{array}$$

and

$$\begin{array}{rcl} c_2 & = & \frac{y'(0)y_1(0) - y(0)y_1'(0)}{W(y_1(0), y_2(0))} \\ & = & \frac{-6\sin\frac{\pi}{4}\cos 0 + 6\cos\frac{\pi}{4}\sin 0}{2} = -\frac{3\sqrt{2}}{2} \end{array}$$ ■

Theorem 16.2 says that if one can find $a < t_0 < b$ such that $W(y_1(t_0), y_2(t_0)) \neq 0$ then the set $\{y_1, y_2\}$ is a fundamental set of solutions. The following theorem shows that the condition $W(y_1(t_0), y_2(t_0)) \neq 0$ implies that $W(y_1(t), y_2(t)) \neq 0$ for all t in the interval (a, b). That is, the theorem tells us that we can choose our test point t_0 on the basis of convenience-any test point t_0 will do. That's why we choose $t_0 = 0$ in the previous example.

Theorem 16.3 *(Abel's)*
Let $y_1(t)$ and $y_2(t)$ be two solutions to the homogeneous second order linear differential equation

$$y'' + p(t)y' + q(t)y = 0, a < t < b$$

where $p(t)$ and $q(t)$ are continuous in $a < t < b$. If t_0 is any point in (a, b) then

$$W(y_1(t), y_2(t)) = W(y_1(t_0), y_2(t_0))e^{-\int_{t_0}^t p(s)ds}$$

Thus, if $W(y_1(t_0), y_2(t_0)) \neq 0$ then $W(y_1(t), y_2(t)) \neq 0$ for all $a < t < b$.

Proof.
Since $W(y_1(t), y_2(t)) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$ then $W'(y_1(t), y_2(t)) = y_1(t)y_2''(t) + y_1'(t)y_2'(t) - y_1'(t)y_2'(t) - y_1''(t)y_2'(t) = y_1(t)y_2''(t) - y_2(t)y_1''(t)$. But $y_1''(t) = -p(t)y_1'(t) - q(t)y_1(t)$ and $y_2''(t) = -p(t)y_2'(t) - q(t)y_2(t)$. Making these substitutions in the equation of $W'(y_1(t), y_2(t))$ we find

$$W' = y_1(t)(-p(t)y_2'(t) - q(t)y_2(t)) - y_2(t)(-p(t)y_1'(t) - q(t)y_1(t)) = -p(t)W$$

Solving this differential equation we find

$$\begin{array}{ccc}
W' & = & -p(t)W \\
W' + p(t)W & = & 0 \\
\left(e^{\int_{t_0}^t p(s)ds}W(y_1(s), y_2(s))\right)' & = & 0 \\
e^{\int_{t_0}^t p(s)ds}W(y_1(t), y_2(t)) - W(y_1(t_0), y_2(t_0)) & = & 0 \\
W(y_1(t), y_2(t)) & = & W(y_1(t_0), y_2(t_0))e^{-\int_{t_0}^t p(s)ds}
\end{array} \blacksquare$$

In the case y_1 and y_2 form a fundamental set of solutions then $W(y_1(t), y_2(t))$ is never zero in the interval $a < t < b$ as shown in the following theorem.

Theorem 16.4
Suppose that $\{y_1, y_2\}$ is a fundamental set of solutions to (31). Then $W(y_1(t), y_2(t)) \neq 0$ for all $a < t < b$.

Proof.
Let t_0 be any point in (a,b). By Theorem 15.1, there is a unique solution $y(t)$ to the initial value problem

$$y'' + p(t)y' + q(t)y = 0, \; y(t_0) = 1, \; y'(t_0) = 0$$

Since $\{y_1, y_2\}$ is a fundamental set then there exists unique constants c_1 and c_2 such that

$$\begin{array}{rcl} c_1y_1(t) + c_2y_2(t) & = & y(t) \\ c_1y_1'(t) + c_2y_2'(t) & = & y'(t) \end{array}$$

for all $a < t < b$. In particular for $t = t_0$ we obtain the system

$$\begin{array}{rcl} c_1y_1(t_0) + c_2y_2(t_0) & = & 1 \\ c_1y_1'(t_0) + c_2y_2'(t_0) & = & 0 \end{array}$$

This system has a unique solution (c_1, c_2) where c_1 and c_2 are found using the method of elimination

$$c_1 = \frac{y_2'(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)}$$

and

$$c_2 = \frac{-y_1'(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)}$$

But for c_1 and c_2 to exist we must have $W(y_1(t_0), y_2(t_0)) = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) \neq 0$. Since t_0 was arbitrary point (a,b) then $W(y_1(t), y_2(t)) \neq 0$ for all $a < t < b$ ∎

Combining Theorem 16.2 and Theorem 16.4 we obtain the following corollary characterizing a fundamental set of solutions.

Corollary 16.1
Let $y_1(t)$ and $y_2(t)$ be two solutions of (31). Let $W(y_1(t), y_2(t))$ denote the Wronskian of y_1 and y_2. Then $\{y_1, y_2\}$ is a fundamental set of solution if and only if $W(y_1(t), y_2(t)) \neq 0$ for all $a < t < b$.

Example 16.3
Consider the initial value problem

$$y'' - \frac{1}{t}y' - \frac{3}{t^2}y = 0, \; y(1) = 4, \; y'(1) = 8, \; 0 < t < \infty.$$

(a) Show that $y_1(t) = t^3$ and $y_2(t) = t^{-1}$ are solutions to the differential equation.
(b) Show that $\{y_1, y_2\}$ is a fundamental set of solutions to the differential equation.
(c) Solve the given initial value problem.

Solution.
(a) By substitution and simple calculation we find

$$y_1'' - \frac{1}{t}y_1' - \frac{3}{t^2}y_1 = 6t - \frac{1}{t}\cdot 3t^2 - \frac{3}{t^2}\cdot t^3 = 0$$
$$y_2'' - \frac{1}{t}y_2' - \frac{3}{t^2}y_2 = 2t^{-2} - \frac{1}{t}\cdot(-t^{-2}) - \frac{3}{t^2}\cdot t^{-1} = 0$$

(b) Finding the Wronskian at $t_0 = 1$ we see

$$W(y_1(1), y_2(1)) = \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} = -4 \neq 0$$

Thus, $\{y_1, y_2\}$ is a fundamental set of solution.
(c) The general solution to the differential equation has the form $y(t) = c_1y_1(t) + c_2y_2(t)$. The initial conditions yield the following linear system in the unknowns c_1 and c_2.

$$\begin{array}{rcl} c_1y_1(1) + c_2y_2(1) & = & 4 \\ c_1y_1'(1) + c_2y_2'(1) & = & 8 \end{array}$$

or

$$\begin{array}{rcl} c_1 + c_2 & = & 4 \\ 3c_1 - c_2 & = & 8 \end{array}$$

Solving this system by the method of elimination we find $c_1 = 3$ and $c_2 = 1$. Thus, $y(t) = 3t^3 + \frac{1}{4},\ 0 < t < \infty$ ■

In Problems 1-7, the t-interval of solution is $-\infty < t < \infty$ unless indicated otherwise.
(a) Determine whether the given functions are solutions to the differential equation.
(b) If both functions are solutions, calculate the Wronskian. Does this calculation show that the two functions form a fundamental set of solutions?
(c) If the two functions have been shown in (b) to form a fundamental set, construct the general solution and determine the unique solution satisfying the initial value problem.

Problem 16.1

$$y'' - 4y = 0,\ y_1(t) = e^{2t},\ y_2(t) = 2e^{-3t},\ y(0) = 1,\ y'(0) = -2$$

Problem 16.2

$$y'' + y = 0, y_1(t) = \sin t \cos t,\ y_2(t) = \sin t,\ y(\tfrac{\pi}{2}) = 1,\ y'(\tfrac{\pi}{2}) = 1$$

Problem 16.3

$$y'' - 4y' + 4y = 0,\ y_1(t) = e^{2t},\ y_2(t) = te^{2t},\ y(0) = 2,\ y'(0) = 0$$

Problem 16.4

$$ty'' + y' = 0,\ y_1(t) = \ln t,\ y_2(t) = \ln 3t,\ y(3) = 0,\ y'(3) = 3,\ 0 < t < \infty$$

Problem 16.5

$$t^2y'' - ty' - 3y = 0,\ y_1(t) = t^3,\ y_2(t) = -t^{-1},\ y(-1) = 0,\ y'(-1) = -2,\ \ -\infty < t < 0$$

Problem 16.6

$$y'' = 0,\ y_1(t) = t + 1, y_2(t) = -t + 2,\ y(1) = 4,\ y'(1) = -1$$

Problem 16.7

$$4y'' + 4y' + y = 0,\ y_1(t) = e^{\frac{t}{2}},\ y_2(t) = te^{\frac{t}{2}},\ y(1) = 1,\ y'(1) = 0$$

Problem 16.8
The functions $y_1(t) = t$ and $y_2(t) = t \ln t$ form a fundamental set of solutions to the differential equation

$$t^2y'' - ty' + y = 0, \quad 0 < t < \infty$$

(a) Show that $y(t) = 2t + t \ln 3t$ is a solution to the differential equation.
(b) Find c_1 and c_2 such that $y(t) = c_1y_1(t) + c_2y_2(t)$

Problem 16.9
The functions $y_1(t) = e^{3t}$ and $y_2(t) = e^{-3t}$ are known to be solutions of $y'' + \alpha y' + \beta y = 0$, where α and β are constants. Determine α and β.

Problem 16.10
The functions $y_1(t) = t$ and $y_2(t) = e^t$ are known to be solutions of $y'' + p(t)y' + q(t)y = 0$.
(a) Determine the functions $p(t)$ and $q(t)$.
(b) On what t-intervals are the functions $p(t)$ and $q(t)$ continuous?
(c) Compute the Wronskian of these two functions. On what t-intervals is the Wronskian nonzero?
(d) Are the observations in (b) and (c) consistent with Theorem 16.3?

Problem 16.11
It is known that two solutions of $y'' + ty' + 2y = 0$ has a Wronskian $W(t)$ that satisfies $W(1) = 4$. What is $W(2)$?

Problem 16.12
The pair of functions $\{y_1, y_2\}$ is known to form a fundamental set of solutions of $y'' + \alpha y' + \beta y = 0$, where α and β are constants. One solution is $y_1(t) = e^{2t}$, and the Wronskian formed by these two solutions is $W(t) = e^{-t}$. Determine the constants α and β.

Problem 16.13
The Wronskian of a pair of solutions of $y'' + p(t)y' + 3y = 0$ is $W(t) = e^{-t^2}$. What is the coefficient function $p(t)$?

Problem 16.14
Prove that if y_1 and y_2 have maxima or minima at the same point in an interval I, then they cannot be a fundamental set of solutions on that interval.

Problem 16.15
Without solving the equation, find the Wronskian of two solutions of Bessel's equation

$$t^2y'' + ty' + (t^2 - \mu^2)y = 0$$

Problem 16.16
If $W(y_1, y_2) = t^2e^t$ and $y_1(t) = t$ then find $y_2(t)$.

Problem 16.17
The functions t^2 and $1/t$ are solutions to a 2nd order, linear homogeneous ODE on $t > 0$. Verify whether or not the two solutions form a fundamental solution set.

Problem 16.18
Show that t^3 and t^4 can't both be solutions to a differential equation of the form $y'' + p(t)y' + q(t)y = 0$ where p and q are continuous functions defined on the real numbers.

Problem 16.19
Suppose that $t^2 + 1$ is the Wronskian of two solutions to the differential equation $y'' + p(t)y' + q(t)y = 0$. Find $p(t)$.

Problem 16.20
Suppose that $y_1(t) = t$ is a solution to the differential equation

$$t^2y'' - (t+2)ty' + (t+y)y = 0$$

Find a second solution y_2.

17 Existence of Many Fundamental Sets

Three questions are of importance about fundamental sets: Do they always exist? How many are there? How are different fundamental sets related? In this section we turn our attention to these questions.

Existence of Fundamental Sets of Solutions
For a given homogeneous equation, a fundamental set always exists according to the following theorem.

Theorem 17.1
Any homogeneous second order linear differential equation

$$y'' + p(t)y' + q(t)y = 0, a < t < b$$

where $p(t)$ and $q(t)$ are continuous in $a < t < b$ has a fundamental set of solutions $\{y_1, y_2\}$.

Proof.
Let t_0 be an arbitrary point in (a, b). Then by Theorem 15.1, there are unique solutions $y_1(t)$ and $y_2(t)$ on the interval (a, b) to the initial value problems

$$y'' + p(t)y' + q(t)y = 0,\ y(t_0) = 1,\ y'(t_0) = 0$$

$$y'' + p(t)y' + q(t)y = 0,\ y(t_0) = 0,\ y'(t_0) = 1$$

The fact that $\{y_1, y_2\}$ is a fundamental set of solutions follows from Theorem 16.2 since

$$W(t_0) = \left| \begin{array}{cc} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{array} \right| = \left| \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right| = 1 \neq 0 \blacksquare$$

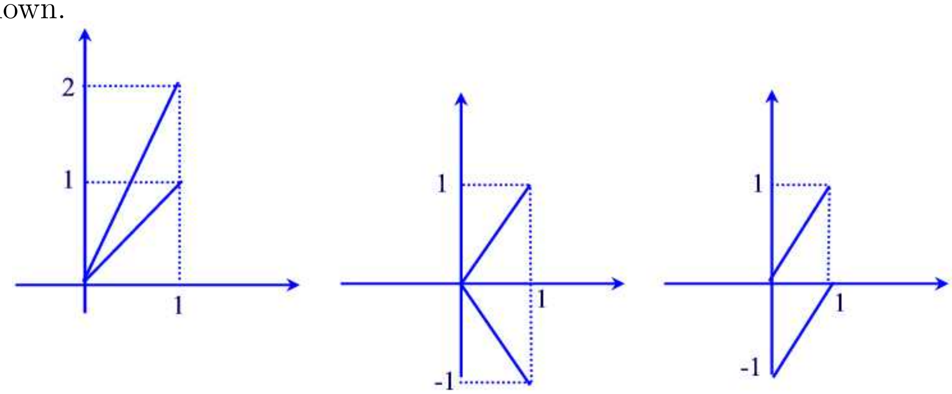
Fundamental Sets of Solutions and Linear Independence
In linear algebra, when two functions $f(t)$ and $g(t)$ are such that neither one can be a constant multiple of the other then they are said to be **linearly independent** functions. This is equivalent to saying that if $c_1 f(t) + c_2 g(t) = 0$ for all t where both f and g are defined then we must have $c_1 = c_2 = 0$. If either c_1 or c_2 is nonzero then this implies that one of the function is a constant multiple of the other function. In this situation the two functions are said to be **linearly dependent**. Loosely speaking, linearly independent functions are functions that are all "basically different". The above definition applies to any number of functions not just for two functions.

Example 17.1
(a) Show that the functions $y_1(t) = 2\sin^2 t$ and $y_2(t) = 1 - \cos^2 t$ are linearly dependent.
(b) Show that the functions $y_1(t) = t$ and $y_2(t) = -2$ are linearly independent.

Solution.
(a) Since $c_1y_1(t) + c_2y_2(t) = 0$ with $c_1 = 1$ and $c_2 = -2$ then the two functions are linearly dependent.
(b) Suppose that $c_1t + c_2(-2) = 0$ for all t. In particular, for $t = 0$ we see that $c_2 = 0$. Thus, $c_1t = 0$ for all t and for $t = 1$ we find $c_1 = 0$. Hence, y_1 and y_2 are linearly indepedent ■

Problem 17.1
Do the given functions form a linearly independent set on the indicated interval?
(a) $y_1(t) = 2,\ y_2(t) = t^2,\ \ -\infty < t < \infty$
(b) $y_1(t) = \ln t,\ y_2(t) = \ln t^2,\ 0 < t < \infty$
(c) $y_1(t) = 2,\ y_2(t) = t,\ y_3(t) = -t^2,\ \ -\infty < t < \infty$
(d) $y_1(t) = 2,\ y_2(t) = \sin^2 t,\ y_3(t) = 2\cos^2 t,\ \ -3 < t < 2$

Problem 17.2
Consider the graphs of the linear functions shown. In each case, determine if the functions form a linearly independent set of functions on the domain shown.



The following theorem asserts that a fundamental set of solutions to the second order linear differential equation

$$y'' + p(t)y' + q(t)y = 0,\ a < t < b$$

is linearly independent and vice versa any linearly independent pair of solution is a fundamental set.

Theorem 17.2
The set $\{y_1, y_2\}$ is a fundamental set of solutions to

$$y'' + p(t)y' + q(t)y = 0, \ a < t < b$$

where $p(t)$ and $q(t)$ are continuous on (a, b), if and only if the functions y_1 and y_2 are linearly independent.

Proof.
Suppose first that $\{y_1, y_2\}$ is a fundamental set of solutions. Then by Theorem 16.4 there is $a < t_0 < b$ such that $W(t_0) \neq 0$. Suppose that

$$c_1 y_1(t) + c_2 y_2(t) = 0$$

for all $a < t < b$. Differentiating the previous equation we find

$$c_1 y_1'(t) + c_2 y_2'(t) = 0$$

Thus, one finds c_1 and c_2 by solving the system

$$\begin{array}{rcl} c_1 y_1(t) + c_2 y_2(t) & = & 0 \\ c_1 y_1'(t) + c_2 y_2'(t) & = & 0 \end{array}$$

Solving this system by the method of elimination we find

$$c_1 = c_2 = \frac{0}{W(t_0)} = 0$$

Thus, $y_1(t)$ and $y_2(t)$ are linearly independent.
Conversely, suppose that $\{y_1, y_2\}$ is a linearly independent set. Suppose that $\{y_1, y_2\}$ is not a fundamental set of solutions. Then by Corollary 16.1, $W(t) = 0$ for all $a < t < b$. Choose any $a < t_0 < b$. Then $W(t_0) = 0$. But this says that the matrix

$$\left[\begin{array}{cc} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{array} \right]$$

is not invertible which means that there exist c_1 and c_2 not both zero such that

$$\begin{array}{rcl} c_1 y_1(t_0) + c_2 y_2(t_0) & = & 0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) & = & 0 \end{array}$$

Now, let $y(t) = c_1y_1(t) + c_2y_2(t)$ for all $a < t < b$. Then $y(t)$ is a solution to the differential equation and $y(t_0) = y'(t_0) = 0$. But the zero function also is a solution to the initial value problem. By the existence and uniqueness theorem (i.e, Theorem 15.1) we must have $c_1y_1(t) + c_2y_2(t) = 0$ for all $a < t < b$ with c_1 and c_2 not both equal to 0. But this means that y_1 and y_2 are linearly depedent which is a contradiction ∎

Remark 17.1
The fact that y_1 and y_2 are solutions is very critical in the above theorem. That is, if y_1 and y_2 are merely differentiable functions, then it is possible for them to be linearly indenpendent and yet have a vanishing Wronskian at some point in their common domain(See Problem 17.19).

Problem 17.3
Consider the differential equation $y'' + 2ty' + t^2y = 0$ on the interval $-\infty < t < \infty$. Assuming that $y_1(t)$ and $y_2(t)$ are two solutions satisfying the given initial conditions. Answer the following two questions.
(a) Do the solutions form a fundamental set?
(b) Do the two solutions form a linearly independent set of functions on $-\infty < t < \infty$?

(i) $y_1(1) = 2,\ y_1'(1) = 2,\ y_2(1) = -1,\ y_2'(1) = -1$
(ii) $y_1(-2) = 1,\ y_1'(-2) = 2,\ y_2(-2) = 0,\ y_2'(-2) = 1$
(iii) $y_1(3) = 0,\ y_1'(3) = 0,\ y_2(3) = 1,\ y_2'(3) = 2$

Problem 17.4
The property of linear dependence or independence depends not only upon the rule defining the functions but also on the domain. To illustrate this fact, show that the pair of functions, $f_1(t) = t,\ f_2(t) = |t|$, is linearly dependent on the interval $0 < t < \infty$ but is linearly independent on the interval $-\infty < t < \infty$.

Problem 17.5
Suppose that $\{f_1, f_2\}$ is a linearly indepedent set. Suppose that a function $f_3(t)$ can be expressed as a linear combination of f_1 and f_2 in two different ways,i.e., $f_3(t) = a_1f_1(t) + a_2f_2(t)$ and $f_3(t) = b_1f_1(t) + b_2f_2(t)$. Show that $a_1 = b_1$ and $a_2 = b_2$

Problem 17.6
Consider a set of functions containing the zero function. Can anything be said about whether they form a linearly dependent or linearly independent set? Explain.

Generating New Fundamental Sets from Old Ones
Next, we will show how to generate new fundamental sets from a given one and therefore establishing the fact that a homogeneous differenttial equation have many fundamental sets of solutions. We also show how different fundamental sets are related to each other. But first we start with the following theorem.

Theorem 17.3
Suppose that $\{y_1, y_2\}$ is a fundamental set of solutions to the homogeneous differential equation

$$y'' + p(t)y' + q(t)y = 0$$

where $p(t)$ and $q(t)$ are continuous functions for $a < t < b$. If $\overline{y}_1$ and $\overline{y}_2$ are any two solutions to the equation then one can write the matrix equation

$$\left[\begin{array}{c} \overline{y}_1(t) \\ \overline{y}_2(t) \end{array}\right] = \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}\right] \left[\begin{array}{c} y_1(t) \\ y_2(t) \end{array}\right] \tag{33}$$

Proof.
Since $\{y_1, y_2\}$ is a fundamental set of solutions then any solution to the equation is a linear combination of y_1 and y_2. Since $\overline{y}_1$ and $\overline{y}_1$ are solutions then we can find constants $a_{11}, a_{12}, a_{21}, a_{22}$ such that $\overline{y}_1(t) = a_{11}y_1(t) + a_{12}y_2(t)$ and $\overline{y}_2(t) = a_{21}y_1(t) + a_{22}y_2(t)$. But this is exactly (33). Note that the constants $\{a_{11}, a_{12}, a_{21}, a_{22}\}$ are unique since $\{y_1, y_2\}$ are linearly independent ∎

From this theorem we see that solutions can be generated by multiplying the matrix of fundamental sets by a 2×2 matrix of arbitrary numbers.

Example 17.2
Consider the differential equation

$$y'' - 4y = 0$$

(a) Show that $y_1(t) = e^{2t}$ and $y_2(t) = e^{-2t}$ are solutions to the equation.
(b) Show that $\{y_1, y_2\}$ is a fundamental set of solutions.

(c) Find solutions $y_3(t)$ and $y_4(t)$ satisfying the matrix equation

$$\begin{bmatrix} y_3(t) \\ y_4(t) \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

(d) Is $\{y_3, y_4\}$ a fundamental set of solutions?
(e) Find solutions $y_5(t)$ and $y_6(t)$ satisfying the matrix equation

$$\begin{bmatrix} y_5(t) \\ y_6(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

(f) Is $\{y_5, y_6\}$ a fundamental set of solutions?
(g) Compare the results in (d) and (f).

Solution.
(a) Since $y_1'' - 4y_1 = 4e^{2t} - 4e^{2t} = 0$ and $y_2'' - 4y_2 = 4e^{-2t} - 4e^{-2t} = 0$ then both $y_1(t)$ and $y_2(t)$ are solutions to the given differential equation.
(b) Since

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} e^{2t} & e^{-2t} \\ 2e^{2t} & -2e^{-2t} \end{vmatrix} = -4 \neq 0$$

then $\{y_1, y_2\}$ is a fundamental set of solution.
(c) Multiplying the right hand side matrices we find

$$\begin{bmatrix} y_3(t) \\ y_4(t) \end{bmatrix} = \begin{bmatrix} y_1(t) + \frac{1}{2}y_2(t) \\ 2y_1(t) + y_2(t) \end{bmatrix}$$

Thus, $y_3(t) = e^{2t} + \frac{1}{2}e^{-2t}$ and $y_4(t) = 2e^{2t} + e^{-2t}$
(d) Computing the Wronskian of y_3 and y_4 we find

$$W(t) = \begin{vmatrix} e^{2t} + \frac{1}{2}e^{-2t} & 2e^{2t} + e^{-2t} \\ 2e^{2t} - e^{-2t} & 4e^{2t} - 2e^{-2t} \end{vmatrix} = 0$$

for all t so that $\{y_3, y_4\}$ is not a fundamental set of solutions.
(e) Multiplying the right hand side matrices we find

$$\begin{bmatrix} y_5(t) \\ y_6(t) \end{bmatrix} = \begin{bmatrix} y_1(t) + 2y_2(t) \\ 3y_1(t) + 4y_2(t) \end{bmatrix}$$

Thus, $y_5(t) = e^{2t} + 2e^{-2t}$ and $y_6(t) = 3e^{2t} + 4e^{-2t}$
(f) Computing the Wronskian of y_5 and y_6 we find

$$W(t) = \left| \begin{array}{cc} e^{2t} + 2e^{-2t} & 3e^{2t} + 4e^{-2t} \\ 2e^{2t} - 4e^{-2t} & 6e^{2t} - 8e^{-2t} \end{array} \right| = -2(23 + 3e^{4t} + 24e^{-4t})$$

In particular, we see that $W(0) = -100 \neq 0$ so that $\{y_5, y_6\}$ is a fundamental set of solutions.
(g) The matrix in (b)

$$\left[\begin{array}{cc} 1 & \frac{1}{2} \\ 2 & 1 \end{array} \right]$$

is not invertible whereas the matrix in (f)

$$\left[\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array} \right]$$

is invertible ■

Theorem 17.4
$\{\overline{y}_1, \overline{y}_2\}$ is a fundamental set of solutions if and only if det(A)$\neq 0$ where A is the coefficient matrix

$$A = \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right]$$

Proof.
Since

$$\begin{array}{rcl} \overline{y}_1(t) & = & a_{11}y_1(t) + a_{12}y_2(t) \\ \overline{y}_2(t) & = & a_{21}y_1(t) + a_{22}y_2(t) \end{array}$$

Then

$$\begin{array}{rcl} \overline{y}_1{}'(t) & = & a_{11}y_1'(t) + a_{12}y_2'(t) \\ \overline{y}_2{}'(t) & = & a_{21}y_1'(t) + a_{22}y_2'(t) \end{array}$$

Thus, we can write

$$\left[\begin{array}{cc} \overline{y}_1 & \overline{y}_1{}' \\ \overline{y}_2 & \overline{y}_2{}' \end{array} \right] = \left[\begin{array}{cc} a_{11} & a_{21} \\ a_{12} & a_{22} \end{array} \right] \left[\begin{array}{cc} y_1 & y_1' \\ y_2 & y_2' \end{array} \right]$$

By taking the determinant of both sides and using the fact that the determinant of the product of two square matrices is the product of their determinant then we can write

$$\overline{W}(t) = det(A)W(t)$$

Since $W(t) \neq 0$ then $\overline{W}(t) \neq 0$ (i.e., $\{\overline{y}_1, \overline{y}_2\}$ is a fundamental set) if and only if $det(A) \neq 0$ ■

In Problems 7 - 9, answer the following questions.
(a) Show that $y_1(t)$ and $y_2(t)$ are solutions to the given differential equation.
(b) Determine the initial conditions satisfied by each function at the specified t_0.
(c) Determine whether the functions form a fundamental set on $-\infty < t < \infty$

Problem 17.7
$y'' - 4y = 0,\ y_1(t) = e^{2t},\ y_2(t) = e^{-2t},\ t_0 = 1.$

Problem 17.8
$y'' + 9y = 0,\ y_1(t) = \sin 3(t-1),\ y_2(t) = 2\cos 3(t-1),\ t_0 = 1.$

Problem 17.9
$y'' + 2y' - 3y = 0,\ y_1(t) = e^{-3t},\ y_2(t) = e^{-3(t-2)},\ t_0 = 2.$

In Problems 10 - 11, assume that $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions of $y'' + p(t)y' + q(t)y = 0$ on the t-interval of interest. Determine whether or not the functions $y_3(t)$ and $y_4(t)$, formed by the given linear combinations, also form a fundamental set of solutions on the same t-interval.

Problem 17.10
$y_3(t) = 2y_1(t) - y_2(t),\ y_4(t) = y_1(t) + y_2(t)$

Problem 17.11
$y_4(t) = 2y_1(t) - 2y_2(t),\ y_4(t) = y_1(t) - y_2(t)$

In Problems 12 - 13, the sets $\{y_1, y_2\}$ and $\{y_3, y_4\}$ are both fundamental sets of solutions for the given differential equation on the indicated interval. Find a constant 2×2 matrix

$$\left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}\right]$$

such that

$$\left[\begin{array}{c} y_3(t) \\ y_4(t) \end{array}\right] = \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}\right] \left[\begin{array}{c} y_1(t) \\ y_2(t) \end{array}\right]$$

Problem 17.12
$t^2y''-3ty'+3y=0,\ 0<t<\infty,\ y_1(t)=t,\ y_2(t)=t^3,\ y_3(t)=2t-t^3,\ y_4(t)=t^3+t$

Problem 17.13
$y''-4y'+4y=0,\ -\infty<t<\infty,\ y_1(t)=e^{2t},\ y_2(t)=te^{2t},\ y_3(t)=(2t-1)e^{2t},\ y_4(t)=(t-3)e^{2t}$

Problem 17.14
Verify whether the functions $f_1(t)=t^2,\ f_2(t)=2t^2-3t,\ f_3(t)=t$, and $f_4(t)=1$ are linearly independent. Do not use Wronskian to solve this problem.

Problem 17.15
(a) Compute the Wronskian of $y_1(t)=te^t$ and $y_2(t)=t^2e^t$
(a) Are they linearly independent on [0,1]? Explain your answer.

Problem 17.16
Determine if the following set of functions are linearly independent or linearly dependent,
(a) $y_1(t)=9\cos 2t$ and $y_2(t)=2\cos^2 t-2\sin^2 t$
(b) $y_1(t)=2t^2$ and $y_2(t)=t^4$

Problem 17.17
Without solving, determine the Wronskian of two solutions to the following differential equation.

$$t^4y''-2t^3y'-t^8y=0$$

Hint: Use Abel's Theorem

Problem 17.18
Without solving, determine the Wronskian of two solutions to the following differential equation.

$$y''-4ty'+\sin ty=0$$

Problem 17.19
Let $y_1(t)$ and $y_2(t)$ be any two differentiable functions on a closed interval $a\le t\le b$.
(a) Show that if $W(y_1(t),y_2(t))\neq 0$ for some $a\le t\le b$ then y_1 and y_2 are

linearly independent.
(b) Show that the two functions $y_1(t) = t^2$ and $y_2(t) = t|t|$ are linearly independent with zero Wronskian. Thus, a set of functions could be linearly independent on some interval and yet have a vanishing Wronskian.

Problem 17.20
Show that the two functions $y_1(t) = 1 - t$ and $y_2(t) = t^3$ cannot be both solutions to the differential equation

$$y'' + p(t)y' + q(t)y = 0$$

if $p(t)$ and $q(t)$ are continuous in $-1 \leq t \leq 5$.

18 Second Order Linear Homogeneous Equations with Constant Coefficients

In the previous two sections we established the structure of the general solution of a second order linear homogeneous differential equation. As we saw, the general solution is a linear combination of two solutions that form a fundamental set of solutions. In this and the next two sections we discuss methods for finding the fundamental set of solutions for second order homogeneous equations with constant coefficients, i.e., equations of the form

$$ay'' + by' + cy = 0 \tag{34}$$

where a, b and c are constants with $a \neq 0$.
Notice first that for $b = 0$ and $c \neq 0$ the function y'' is a constant multiple of y. So it makes sense to look for a function with such property. One such function is $y(t) = e^{rt}$. Substituting this function into (34) leads to

$$ay'' + by' + cy = ar^2e^{rt} + bre^{rt} + ce^{rt} = (ar^2 + br + c)e^{rt} = 0$$

Since $e^{rt} > 0$ for all t then the previous equation leads to

$$ar^2 + br + c = 0 \tag{35}$$

Thus, a function $y(t) = e^{rt}$ is a solution to (34) when r satisfies equation (35). We call (35) the **characteristic equation** for (34) and the polynomial $C(r) = ar^2 + br + c$ is called the **characteristic polynomial**.

Example 18.1
Solve: $y'' - 5y' - 6y = 0$.

Solution.
The characteristic polynomial for this equation is $C(r) = r^2 - 5r - 6 = (r-2)(r-3)$. Thus, the roots of the characteristic equation are $r = 2$ and $r = 3$. Since

$$W(t) = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} = e^{5t} \neq 0$$

then the functions $y_1(t) = e^{2t}$ and $y_2(t) = e^{3t}$ form a fundamental set of solutions. Hence, the general solution is given by $y(t) = c_1e^{2t} + c_2e^{3t}$ where c_1 and c_2 are arbitrary constants ■

We conclude from the previous example that the two distinct real solutions to the characteristic equation lead to the general solution. Does this result apply to any equation (34) whose characteristic equation has distinct solutions? The answer is in the affirmative. To see this, let r_1 and r_2 be the two distinct solutions to (35). Then

$$W(t) = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = r_2 e^{(r_1+r_2)t} - r_1 e^{(r_1+r_2)t} = (r_2 - r_1)e^{(r_1+r_2)t} \neq 0$$

since both $r_1 - r_2$ and $e^{(r_1+r_2)t}$ are not equal to 0. Hence, $e^{r_1 t}$ and $e^{r_2 t}$ form a fundamental set of solutions. As a result, the general solution of (34) is given by $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ where c_1 and c_2 are arbitrary constants.

Example 18.2
Solve the initial value problem

$$y'' - y' - 6y = 0,\ y(0) = 1,\ y'(0) = 2$$

Describe the behavior of the solution $y(t)$ as $t \to -\infty$ and $t \to \infty$.

Solution.
The characteristic polynomial is $C(r) = r^2 - r - 6 = (r-3)(r+2)$ so that the characteristic equation $r^2 - r - 6 = 0$ has the solutions $r_1 = 3$ and $r_2 = -2$. The general solution is then given by

$$y(t) = c_1 e^{3t} + c_2 e^{-2t}.$$

Taking the derivative to obtain

$$y'(t) = 3c_1 e^{3t} - 2c_2 e^{-2t}$$

The conditions $y(0) = 1$ and $y'(0) = 2$ lead to the system

$$\begin{array}{rcl} c_1 + c_2 & = & 1 \\ 3c_1 - 2c_2 & = & 2 \end{array}$$

Solving this system by the method of elimination we find $c_1 = \frac{4}{5}$ and $c_2 = \frac{1}{5}$. Hence, the unique solution to the initial value problem is

$$y(t) = \frac{1}{5}(4e^{3t} + e^{-2t})$$

As $t \to -\infty,\ e^{3t} \to 0$ and $e^{-2t} \to \infty$. Thus, $y(t) \to \infty$. Similarly, $y(t) \to \infty$ as $t \to \infty$ ■

Remark 18.1
In this section, we have only considered the case when (35) has two distinct solutions. In Section 19, we discuss the case of (35) having repeated solutions and in Section 20 we look at the complex solutions■

Problem 18.1
Solve the initial value problem

$$y'' + y' - 2y = 0,\ y(0) = 3,\ y'(0) = -3$$

Describe the behavior of the solution $y(t)$ as $t \to -\infty$ and $t \to \infty$.

Problem 18.2
Solve the initial value problem

$$y'' - 4y' + 3y = 0,\ y(0) = -1,\ y'(0) = 1$$

Describe the behavior of the solution $y(t)$ as $t \to -\infty$ and $t \to \infty$.

Problem 18.3
Solve the initial value problem

$$y'' - y = 0,\ y(0) = 1,\ y'(0) = -1$$

Describe the behavior of the solution $y(t)$ as $t \to -\infty$ and $t \to \infty$.

Problem 18.4
Solve the initial value problem

$$y'' + 5y' + 6y = 0,\ y(0) = 1,\ y'(0) = -1$$

Describe the behavior of the solution $y(t)$ as $t \to -\infty$ and $t \to \infty$.

Problem 18.5
Solve the initial value problem

$$y'' - 4y = 0,\ y(3) = 0,\ y'(3) = 0$$

Describe the behavior of the solution $y(t)$ as $t \to -\infty$ and $t \to \infty$.

Problem 18.6
Solve the initial value problem

$$2y'' - 3y' = 0,\ y(-2) = 3,\ y'(-2) = 0$$

Describe the behavior of the solution $y(t)$ as $t \to -\infty$ and $t \to \infty$.

Problem 18.7
Solve the initial value problem

$$y'' + 4y' + 2y = 0,\ y(0) = 0,\ y'(0) = 4$$

Describe the behavior of the solution $y(t)$ as $t \to -\infty$ and $t \to \infty$.

Problem 18.8
Solve the initial value problem

$$2y'' - y = 0,\ y(0) = -2,\ y'(0) = \sqrt{2}$$

Describe the behavior of the solution $y(t)$ as $t \to -\infty$ and $t \to \infty$.

Problem 18.9
Consider the initial value problem $y'' + \alpha y' + \beta y = 0,\ y(0) = 1,\ y'(0) = y_0'$, where α, β, and y_0' are constants. It is known that one solution of the differential equation is $y_1(t) = e^{-3t}$ and that the solution of the initial value problem satisfies $\lim_{t\to\infty} y(t) = 2$. Determine the constants α, β, and y_0'.

Problem 18.10
Consider the initial value problem $y'' + \alpha y' + \beta y = 0,\ y(0) = 3,\ y'(0) = 5$. The differential equation has a fundamental set of solutions $\{y_1, y_2\}$. It is known that $y_1(t) = e^{-t}$ and that the Wronskian formed by the two members of the fundamental set is $W(t) = 4e^{2t}$.
(a) Determine $y_2(t)$
(b) Determine the constants α and β.
(c) Solve the initial value problem.

Problem 18.11
Obtain the general solution to the differential equation $y''' - 5y'' + 6y' = 0$.

Problem 18.12
A particle of mass m moves along the x-axis and is acted upon by a drag force proportional to its velocity. The drag constant is denoted by k. If $x(t)$ represents the particle position at time t, Newton's law of motion leads to the differential equation $mx''(t) = -kx'(t)$.
(a) Obtain the general solution to this second order linear differential equation.
(b) Solve the initial value problem if $x(0) = x_0$ and $x'(0) = v_0$.
(c) What is $\lim_{t\to\infty} x(t)$?

Problem 18.13
Solve the initial-value problem $4y'' - y = 0, y(0) = 2, y'(0) = \beta$. Then find β so that the solution approaches zero as $t \to \infty$.

Problem 18.14
Find a homogeneous second-order linear ordinary differential equation whose general solution is $y(t) = c_1e^{2t} + c_2e^{-t}$.

Problem 18.15
Find the general solution of the differential equation $y'' - 3y' - 4y = 0$

Problem 18.16
Find the general solution of the differential equation $y'' + 4y' - 5y = 0$

Problem 18.17
Find the general solution of the differential equation $-3y'' + 2y' + y = 0$

Problem 18.18
Solve the initial-value problem: $y'' + 3y' - 4y = 0,\ y(0) = -1,\ y'(0) = 1$.

Problem 18.19
Solve the initial-value problem: $2y'' + 5y' - 3y = 0,\ y(0) = 2,\ y'(0) = 1$.

Problem 18.20
Show that if λ is a root of $a\lambda^3 + b\lambda^2 + c\lambda + d = 0$, then $e^{\lambda t}$ is a solution of $ay''' + by'' + cy' + dy = 0$.

19 Characteristic Equations with Repeated Roots

In this section we consider the question of the characteristic equation having a repeated real solution. This occurs when $b^2 - 4ac = 0$. The two equal roots are given by

$$r_1 = r_2 = -\frac{b}{2a}$$

The computation based on the trial form $y(t) = e^{rt}$ yields only one solution, namely

$$y_1(t) = e^{-\frac{b}{2a}t}$$

Since a fundamental set of solutions consists of two functions having a nonzero Wronskian then there must be another solution having a different functional form. The second solution follows from the following theorem.

Theorem 19.1
Suppose that $y_1(t)$ is a nontrivial solution to the differential equation

$$y'' + p(t)y' + q(t)y = 0 \tag{36}$$

Then any solution $y_2(t)$ can be written in the form

$$y_2(t) = C\left(\int \frac{e^{-\int p(t)dt}}{y_1^2(t)}dt\right) y_1(t) + C'y_1(t) \tag{37}$$

where C and C' are arbitrary constants.

Proof.
First, recall that the Wronskian $W(t)$ of any two solutions to (36) satisfies the differential equation $W' + p(t)W = 0$ so that $W(t) = Ce^{-\int p(t)dt}$. If y_2 is a solution to (36) then

$$\left(\frac{y_2}{y_1}\right)' = \frac{W(t)}{y_1^2(t)} = C\frac{e^{-\int p(t)dt}}{y_1^2(t)}$$

Integrating this last equation we find

$$y_2(t) = C\left(\int \frac{e^{-\int p(t)dt}}{y_1^2(t)}dt\right) y_1(t) + C'y_1(t)$$

where C and C' are arbitrary constant ■
Notice that the term $C'y_1(t)$ is simply a constant multiple of $y_1(t)$. Since the general solution of the differential equation (36) contains $y_1(t)$ multiplied by an arbitrary constant, we lose no generality by setting $C' = 0$. We can likewise take $C = 1$ since $y_2(t)$ will also be multiplied by an arbitrary constant in the general solution. With these simplification the second solution is

$$y_2(t) = \left(\int \frac{e^{-\int p(t)dt}}{y_1^2(t)} dt\right) y_1(t)$$

Now, for the equation

$$ay'' + by' + cy = 0 \tag{38}$$

we have $p(t) = \frac{b}{a}$. If $y_1(t) = e^{-\frac{b}{2a}t}$ then

$$y_2(t) = \left(\int \frac{e^{-\frac{b}{a}t}}{e^{-\frac{b}{a}t}} dt\right) e^{-\frac{b}{2a}t} = te^{-\frac{b}{2a}t}$$

Hence, the general solution to (38) is given by

$$y(t) = c_1 e^{-\frac{b}{2a}t} + c_2 te^{-\frac{b}{2a}t}$$

Example 19.1
Solve the initial value problem: $y'' + 2y' + y = 0,\ y(0) = 1,\ y_1'(0) = -1.$

Solution.
The characteristic equation $r^2 + 2r + 1 = 0$ has a repeated root: $r_1 = r_2 = -1$. Thus, the general solution is given by

$$y(t) = c_1 e^{-t} + c_2 te^{-t}.$$

The two conditions $y(0) = 1$ and $y'(0) = -1$ lead to $c_2 = 1$ and $c_1 = 0$. Hence, the unique solution is $y(t) = e^{-t}$ ■

Example 19.2
Consider the differential equation

$$t^2 y'' + 2ty' - 2y = 0, \quad 0 < t < \infty$$

Find the general solution given that $y_1(t) = t$ is a solution to the differential equation.

Solution.
Since $t > 0$ then we can rewrite the given equation in the form

$$y'' + \frac{2}{t}y' - \frac{2}{t^2}y = 0$$

In this case, $p(t) = \frac{2}{t}$ and

$$y_2(t) = \left(\int \frac{e^{-\int \frac{2}{t}dt}}{t^2}dt\right) t = \left(\int \frac{1}{t^4}dt\right) t = -\frac{1}{3t^2}.$$

Hence, the general solution is $y(t) = c_1 t + c_2 t^{-2}$ ■

In Problems 1 - 5 answer the following questions.

(a) Obtain the general solution of the differential equation.
(b) Impose the initial conditions to obtain the unique solution of the initial value problem.
(c) Describe the behavior of the solution as $t \to -\infty$ and $t \to \infty$.

Problem 19.1
$9y'' - 6y' + y = 0,\ y(3) = -2,\ y'(3) = -\frac{5}{3}$

Problem 19.2
$25y'' + 20y' + 4y = 0,\ y(5) = 4e^{-2},\ y'(5) = -\frac{3}{5}e^{-2}$

Problem 19.3
$y'' - 4y' + 4y = 0,\ y(1) = -4,\ y'(1) = 0$

Problem 19.4
$y'' + 2\sqrt{2}y' + y = 0,\ y(0) = 1,\ y'(0) = 0$

Problem 19.5
$3y'' + 2\sqrt{3}y' + y = 0,\ y(0) = 2\sqrt{3},\ y'(0) = 3$

In Problems 6 - 9, one solution, $y_1(t)$, of the differential equation is given.
(a) Find a second solution of the form $y_2(t) = u(t)y_1(t)$.
(b) Compute the Wronskian formed by the solutions $y_1(t)$ and $y_2(t)$. On what intervals is the Wronskian continuous and nonzero?
(c) Rewrite the differential equation in the form $y'' + p(t)y' + q(t)y = 0$. On what interval(s) are both $p(t)$ and $q(t)$ continuous? How does this observation compare with the interval(s) determined in part (b)?

Problem 19.6
$ty'' - (2t+1)y' + (t+1)y = 0, y_1(t) = e^t$

Problem 19.7
$y'' - (2\cot t)y' + (1 + 2\cot^2 t)y = 0,\ y_1(t) = \sin t$

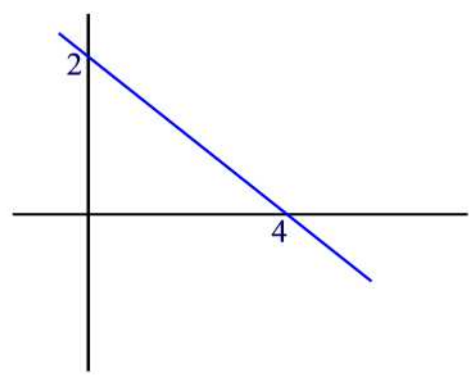
Problem 19.8
$y'' + 4ty' + (2 + 4t^2)y = 0,\ y_1(t) = e^{-t^2}$

Problem 19.9
$y'' - \left(2 + \frac{n-1}{t}\right) y' + \left(1 + \frac{n-1}{t}\right) y = 0$, where n is a positive integer, $y_1(t) = e^t$.

Problem 19.10
The graph of a solution $y(t)$ of the differential equation $4y'' + 4y' + y = 0$ passes through the points $(1, e^{-\frac{1}{2}})$ and $(2, 0)$. Determine $y(0)$ and $y'(0)$.

Problem 19.11
Find a second order linear differential equation whose general solution is given by $y(t) = c_1 e^{-3t} + c_2 te^{-3t}$.

Problem 19.12
The graph shown below is the solution of $y'' - 2\alpha y' + \alpha^2 y = 0, y(0) = y_0,\ y'(0) = y_0$. Determine the constants α, y_0, and y_0' as well as the solution $y(t)$.



Problem 19.13
Show that if λ is a double root of $at^3 + bt^2 + ct + d = 0$, then $te^{\lambda t}$ is also a solution of $ay''' + by'' + cy' + dy = 0$.

Problem 19.14
Find the general solution of $y'' - 6y' + 9y = 0$.

Problem 19.15
Find the general solution of $4y'' - 4y' + y = 0$

Problem 19.16
Solve the initial-value problem: $y'' + y' + \frac{y}{4} = 0,\ y(0) = 2,\ y'(0) = 0$.

Problem 19.17
Consider the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

Let y_1 be a solution of the corresponding homogeneous equation. Let $y = uy_1$ and show that y is a solution of the nonhomogeneous if u is a solution of

$$y_1u'' + [2y_1' + py_1]u' = g$$

The latter equation is a first-order linear equation for u'.

Problem 19.18
Given that $y_1(t) = t^2$ is a solution of

$$t^2y'' - 3ty' + 4y = 0,\ t > 0$$

find the general solution.

Problem 19.19
Let $y_1(t)$ be a nonzero solution of the third-order homogeneous linear ODE

$$y''' + p(t)y'' + q(t)y' + r(t)y = 0$$

Use the substitution $y = uy_1$ to reduce the problem to a second-order linear equation.

Problem 19.20
The following problem indicates a second way for finding the second root. It is known as the **method of reduction of order.** Consider the differential equation $y'' + p(t)y' + q(t)y = 0$ having one solution $y_1(t)$.

(a) If $y_2(t) = u(t)y_1(t)$ is a solution then show that the differential equation satisfied by $u(t)$ is given by

$$y_1 u'' + (2y_1' + py_1)u' = 0$$

(b) Use the substitution $v = u'$ to reduce the equation in part(a) into a first order linear differential equation in v.
(c) Solve the equation in part(b) for v.
(d) Find $u(t)$ and then $y_2(t)$

20 Characteristic Equations with Complex Roots

In this section we solve the linear second order homogeneous differential equation with constant coefficients

$$ay'' + by' + cy = 0, \ a \neq 0 \tag{39}$$

when the roots of the characteristic equation

$$ar^2 + br + c = 0 \tag{40}$$

are complex numbers. This occurs when $b^2 - 4ac < 0$. In this case, the complex roots of equation (40) are given by

$$r_{1,2} = \frac{-b \pm i\sqrt{4ac - b^2}}{2a}$$

where $i = \sqrt{-1}$. We will write

$$r_{1,2} = \alpha \pm i\beta$$

where $\alpha = -\frac{b}{2a}$ and $\beta = \frac{\sqrt{4ac-b^2}}{2a}$. Like before, we would like to conclude that the functions

$$c_1 e^{(\alpha+i\beta)x} \text{ and } c_1 e^{(\alpha-i\beta)x}$$

are solutions to (39). These are complex solutions, we would like to have real solutions to the original real differential equation. This requires the use of the so-called the **complex exponential function** which we introduce and discuss some of its properties.
For any complex number $z = \alpha + i\beta$ we define the Euler's function

$$e^z = e^\alpha(\cos\beta + i\sin\beta)$$

The exponential function satisfies the usual laws of exponentials such as

$$e^z e^w = e^{z+w}$$

To see this, we let $z = \alpha_1 + i\beta_1$ and $w = \alpha_2 + i\beta_2$. Then

$$\begin{array}{rcl} e^z e^w & = & e^{\alpha_1}(\cos\beta_1 + i\sin\beta_1)e^{\alpha_2}(\cos\beta_2 + i\sin\beta_2) \\ & = & e^{\alpha_1+\alpha_2}\left[(\cos\beta_1\cos\beta_2 - \sin\beta_1\sin\beta_2) + i(\sin\beta_1\cos\beta_2 + \cos\beta_1\sin\beta_2)\right] \\ & = & e^{\alpha_1+\alpha_2}(\cos(\beta_1+\beta_2) + i\sin(\beta_1+\beta_2)) \\ & = & e^{z+w} \end{array}$$

From the above rule we can write

$$(e^z)^n = e^z \cdot e^z \cdots e^z = e^{z+z+\cdots+z} = e^{nz}$$

where n is a positive integer.

Problem 20.1
For any $z = \alpha + i\beta$ we define the conjugate of z to be the complex number $\overline{z} = \alpha - i\beta$. show that $\alpha = \frac{1}{2}(z + \overline{z})$ and $\beta = \frac{1}{2i}(z - \overline{z})$.

Problem 20.2
Write each of the complex numbers in the form $\alpha + i\beta$, where α and β are real numbers.

1. $2e^{i\frac{\pi}{3}}$
2. $(2 - i)e^{i\frac{3\pi}{2}}$
3. $(\sqrt{2}e^{i\frac{\pi}{6}})^4$.

Problem 20.3
Write each functions in the form $Ae^{\alpha t}\cos\beta t + iB\sin\beta t$, where α, β, A, and B are real numbers.

1. $2e^{i\sqrt{2}t}$
2. $-\frac{1}{2}e^{2t+i(t+\pi)}$
3. $(\sqrt{3}e^{(1+i)t})^3$

It follows from the above discussion that the complex solutions to the differential equation are linear combinations of $e^{\alpha t}\cos\beta t$ and $e^{\alpha t}\cos\beta t$. Now letting $y_1(t) = e^{\alpha t}\cos\beta t$ and $y_2(t) = e^{\alpha t}\sin\beta t$ we find

$$\begin{array}{lcc}
ay_1'' + by_1' + cy & = & a(\alpha^2 e^{\alpha t}\cos\beta t - \beta^2 e^{\alpha t}\cos\beta t - 2\alpha\beta e^{\alpha t}\sin\beta t) \\
 & + & b(\alpha e^{\alpha t}\cos\beta t - \beta e^{\alpha t}\sin\beta t) + ce^{\alpha t}\cos\beta t \\
 & = & e^{\alpha t}\cos\beta t(a(\alpha^2 - \beta^2) + b\alpha + c) - e^{\alpha t}\sin\beta t(2a\alpha\beta + b\beta) \\
 & = & e^{\alpha t}\cos\beta t\left(a\left(\frac{b^2}{4a^2} - \frac{4ac-b^2}{4a^2}\right) + b\left(\frac{-b}{2a}\right) + c\right) \\
 & - & e^{\alpha t}\sin\beta t\left(2a\left(-\frac{b}{2a}\frac{\sqrt{4ac-b^2}}{2a}\right) + \frac{b\sqrt{4ac-b^2}}{2a}\right) \\
 & = & 0
\end{array}$$

Thus, $y_1(t) = e^{\alpha t}\cos\beta t$ is a solution to equation (39). Similarly, we show that $y_2(t) = e^{\alpha t}\sin\beta t$ is a solution to equation (39). Moreover,

$$W(t) = \begin{vmatrix} e^{\alpha t}\cos\beta t & e^{\alpha t}\sin\beta t \\ \alpha e^{\alpha t}\cos\beta t - \beta e^{\alpha t}\sin\beta t & \alpha e^{\alpha t}\sin\beta t + \beta e^{\alpha t}\cos\beta t \end{vmatrix} = \beta e^{2\alpha t} \neq 0$$

Hence, $\{y_1, y_2\}$ is a fundamental set of solutions to equation (39) so that the general solution is given by

$$y(t) = e^{\alpha t}(c_1\cos\beta t + c_2\sin\beta t)$$

where c_1 and c_2 are real numbers.

Example 20.1
Solve: $y'' + 2y' + 5y = 0$.

Solution.
The characteristic equation $r^2 + 2r + 5 = 0$ has complex roots $r_{1,2} = -1 \pm 2i$. The general solution is

$$y(t) = e^{-x}(c_1\cos 2x + c_2\sin 2x) \blacksquare$$

Example 20.2
Solve the initial value problem

$$y'' - 10y' + 29y = 0,\ y(0) = 1,\ y'(0) = 3$$

Solution.
The characteristic equation $r^2 - 10r + 29 = 0$ has the complex roots $r_{1,2} = 5 \pm 2i$. Thus, the general solution is given by the expression

$$y(t) = e^{5t}(c_1\cos 2t + c_2\sin 2t)$$

Finding y' we obtain

$$y'(t) = e^{5t}[(5c_1 + 2c_2)\cos 2t + (5c_2 - 2c_1)\sin 2t]$$

The initial conditions yield $c_1 = 1$ and $c_2 = -1$. Thus, the unique solution to the initial value problem is

$$y(t) = e^{5t}(\cos 2t - \sin 2t) \blacksquare$$

Next, we consider the question of representing the general solution $y(t) = e^{\alpha t}(c_1\cos\beta t + c_2\sin\beta t)$ in the form $y(t) = Ke^{\alpha t}\cos(\beta t - \delta)$, where $0 \leq \delta < 2\pi$. For this, we let $P(c_1, c_2)$ be a coordinate point in the plane and let δ be the angle between the t-axis and ray $\overrightarrow{OP}$. See Figure 20.1. Then

$$\cos\delta = \frac{c_1}{K} \text{ and } \sin\delta = \frac{c_2}{K}$$

where $K = \sqrt{c_1^2 + c_2^2}$. Then in terms of K and δ we can write

$$\begin{aligned} c_1 \cos\omega t + c_2 \sin\omega t &= K\left(\frac{c_1}{K}\cos\beta t + \frac{c_2}{K}\sin\beta t\right) \\ &= K(\cos\delta\cos\beta t + \sin\delta\sin\beta t) = K\cos(\beta t - \delta). \end{aligned}$$

It follows that

$$y(t) = Ke^{\alpha t}\cos(\beta t - \delta)$$

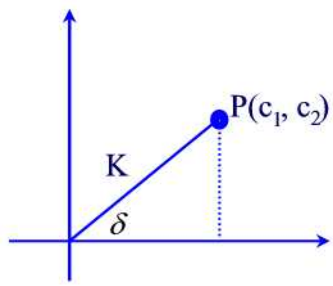


Figure 20.1

We call $Ke^{\alpha t}$ the **amplitude** of the oscillations. This means that the graph of $y(t)$ is bounded by the graphs of $\pm Ke^{\alpha t}$. The angle δ is the **phase angle** or **phase shift**. The term "phase shift" reflects the fact that we obtain the graph of $\cos(\beta t - \delta)$ by shifting the graph of $\cos\beta t$ to the right by an amount $t = \frac{\delta}{\beta}$.

Example 20.3
Put the solution of the initial value problem

$$y'' - 2y' + 17y = 0,\ y(0) = -4,\ y'(0) = 8$$

in the form $y(t) = Ke^{\alpha t}\cos(\beta t - \delta)$.

Solution.
The characterisitc equation $r^2 - 2r + 17 = 0$ has the complex roots $r_{1,3} = 1 \pm 4i$. Thus, the general solution to the differential equation is

$$y(t) = e^t(c_1\cos 4t + c_2\sin 4t)$$

Since $y(0) = -4$ then $c_1 = -4$. Since $y'(t) = e^t(c_1 \cos 4t + c_2 \sin 4t) + e^t(4c_2 \cos 4t - 4c_1 \sin 4t)$ and $y'(0) = 8$ then $8 = -4 + 4c_2$ so that $c_2 = 3$. Thus, the unique solution to the initial value problem is

$$y(t) = e^t(3 \sin 4t - 4 \cos 4t).$$

Now, $K = \sqrt{9 + 16} = \sqrt{25} = 5$. Thus, $\tan \delta = -\frac{3}{4}$ so that $\delta = \arctan\left(-\frac{3}{4}\right)$. Hence,

$$y(t) = 5e^t \cos\left(4t + \left(\arctan \frac{3}{4}\right)\right))$$

The graph of $y(t)$ together with the envelope containing it is shown in Figure 20.2.

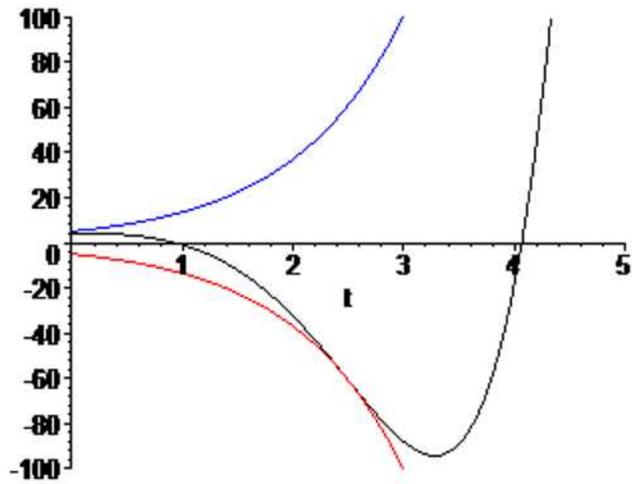


Figure 20.2

In Problems 4 - 8
(a) Determine the roots of the characteristic equation.
(b) Obtain the general solution as a linear combination of real-valued solutions.
(c) Impose the initial conditions and solve the initial value problem.

Problem 20.4
$y'' + 2y' + 2y = 0, \; y(0) = 3, \; y'(0) = -1$

Problem 20.5
$2y'' - 2y' + y = 0, \; y(-\pi) = 1, \; y'(-\pi) = -1$

Problem 20.6
$y'' + 4y' + 5y = 0,\ y(\frac{\pi}{2}) = \frac{1}{2},\ y'(\frac{\pi}{2}) = -2$

Problem 20.7
$y'' + 4\pi^2 y = 0,\ y(1) = 2,\ y'(1) = 1$

Problem 20.8
$9y'' + \pi^2 y = 0,\ y(3) = 2,\ y'(3) = -\pi$

In Problems 9 - 10, the function $y(t)$ is a solution of the initial value problem $y'' + ay' + by = 0,\ y(t_0) = y_0, y'(t_0) = y_0'$, where the point t_0 is specified. Determine the constants a, b, y_0, and y_0'.

Problem 20.9
$y(t) = 2\sin 2t + \cos 2t,\ t_0 = \frac{\pi}{4}$

Problem 20.10
$y(t) = e^{t-\frac{\pi}{6}}\cos 2t - e^{t-\frac{\pi}{6}}\sin 2t,\ t_0 = \frac{\pi}{6}$

In Problems 11 - 13, rewrite the function $y(t)$ in the form $y(t) = Ke^{\alpha t}\cos\beta t - \delta)$, where $0 \leq \delta < 2\pi$. Use this representation to sketch a graph of the given function, on a domain sufficiently large to display its main features.

Problem 20.11
$y(t) = \sin t + \cos t$

Problem 20.12
$y(t) = e^t\cos t + \sqrt{3}e^t\sin t$

Problem 20.13
$y(t) = e^{-2t}\cos 2t - e^{-2t}\sin 2t$

Problem 20.14
Consider the differential equation $y'' + ay' + 9y = 0$, where a is a real number. Suppose that we know the Wronskian of a fundamental set of solutions of this differential equation is constant: $W(t) = 1$ for all real numbers t. Find the general solution of this differential equation.

Problem 20.15
Rewrite $2\cos 7t - 11\sin 7t$ in phase-angle form. Give the exact function (so your answer will involve the inverse tangent function)

Problem 20.16
Find a homogeneous linear ordinary differential equation whose general solution is $y(t) = c_1 e^{2t} \cos(3t) + c_2 e^{2t} \sin(3t)$.

Problem 20.17
Rewrite $y(t) = 5e^{(5-2i)t} - 3e^{(5+2i)t}$, without complex exponents, using sines and cosines. What ODE of the form $ay'' + by' + cy = 0$, has y as a solution?

Problem 20.18
Consider the function $y(t) = 3\cos 2t - 4\sin 2t$. Find a second order linear IVP that y satisfies.

Problem 20.19
An equation of the form

$$t^2 y'' + \alpha t y' + \beta y = 0, \ t > 0$$

where α and β are real constants is called an **Euler equation.** Show that the substitution $u(t) = \ln t$ transforms Euler equation into an equation with constant coefficients.

Problem 20.20
Use the result of the previous problem to solve the differential equation $t^2 y'' + ty' + y = 0$.

21 Applications of Homogeneous Second Order Linear Differential Equations: Unforced Mechanical Vibrations

Second-order homogeneous linear differential equations have a variety of applications in science and engineering. In this section we explore one of them: the unforced or free mechanical vibration of a mass-spring system. The case of forced vibrations will be the topic of Section 24.
Consider a spring of length L hanging vertically. If we attach an object of mass m to the free end of the spring then the spring stretches to a new resting position or equilibrium position. Let Y represent the distance the spring stretches to achieve this new position. Then by Hooke's law the spring stretches until the restoring force F_R exactly counteracts the object's weight, i.e., we have

$$mg + F_R = mg - kY = 0$$

It follows from this equation that $k = \frac{mg}{Y}$.
In this section, we consider the motion of an object with mass at the end of a spring that is either vertical (as in Figure 21.1(a)) or horizontal on a level surface (as in Figure 21.1(b)). In the discussion below we will consider vertical motion.

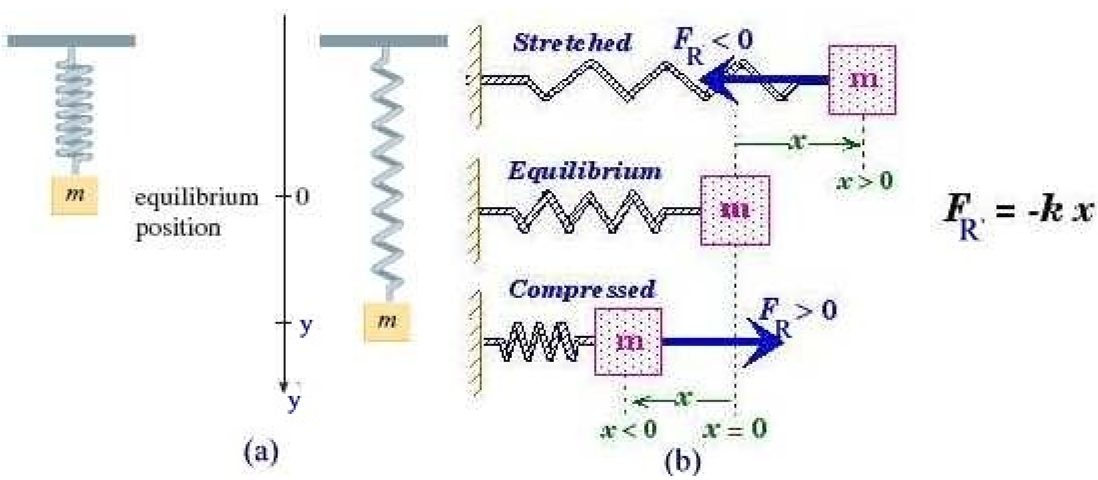


Figure 21.1

We consider two forces applied to the motion of the spring:
The restoring force of the spring: Hooke'ss Law states that if the spring

is stretched (or compressed) y units from its natural length, then it exerts a force that is proportional to y:

$$F_R = \text{restoring force} = -k(Y + y)$$

where k is a positive constant (called the **spring constant**). The negative sign indicates that the spring force is a restoring force, i.e., the force F_R always acts in the opposite direction from the direction in which the system is displaced. In the SI system, the unit of F_R is the Newton (N), that of k is the Newton per meter, and the unit for displacement y is the meter. The value of k depends on the stiffness of the spring. For large k the spring is stiff whereas for small k the spring is soft.

Damping force: We assume a damping mechanism is attached and surpresses the vibrating motion of the mass-spring system. An example is the damping force supplied by a shock absorber in a car or a bicycle.

We assume that the damping force is proportional to the velocity of the mass and acts in the direction opposite to the motion. (This has been confirmed, at least approximately, by some physical experiments.) Thus

$$F_D = \text{damping force} = -\gamma \tfrac{dy}{dt}$$

where γ is a positive constant, called the **damping constant.** Again the negative sign is present because the damping force acts to oppose the motion.

Now, by Newton's Second Law of motion we have

$$m\frac{d^2y}{dt^2} = F_D + F_R = mg - kY - ky - \gamma\frac{dy}{dt} = -ky - \gamma\frac{dy}{dt}$$

since $mg - kY = 0$. Thus,

$$m\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + ky = 0 \tag{41}$$

Equation (41) is a homogeneous second order linear differential equation with characteristic equation

$$mr^2 + \gamma r + k = 0 \tag{42}$$

and roots

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

We consider three cases depending on the sign of $\gamma^2 - 4mk$.

Case 1: $\gamma^2 - 4mk > 0$(Overdamping)
In this case we have two distinct real roots. The general solution is then given by

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Since the constants $\gamma, m,$ and k are all positive then $\sqrt{\gamma^2 - 4mk} < \gamma$. Thus, $-\gamma + \sqrt{\gamma^2 - 4mk} < 0$. So both r_1 and r_2 are negative numbers and this implies that $y(t) \to 0$ as $t \to \infty$. Typical graphs of $y(t)$ are shown in Figure 21.2. Notice that oscillations do not occur. (Its possible for the mass to pass through the equilibrium position once, but only once.) This is because $\gamma^2 > 4mk$ means that there is a strong damping force compared with a weak spring or small mass.

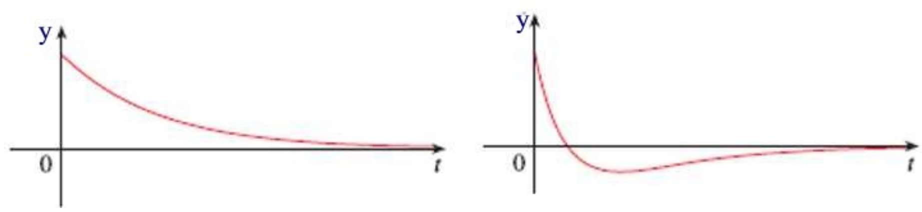


Figure 21.2

Case 2: $\gamma^2 - 4mk = 0$ (Critical Damping)
In this case, the roots r_1 and r_2 are both equal to $-\frac{\gamma}{2m}$ and the general solution to (41) is given by

$$y(t) = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$$

Since $e^{r_1 t} \to 0$ as $t \to \infty$ and

$$\lim_{t\to\infty} te^{r_1 t} = \lim_{t\to\infty} \frac{t}{e^{-r_1 t}} = \lim_{t\to\infty} \frac{1}{-r_1 e^{-r_1 t}} = 0$$

then $y(t) \to 0$ as $t \to \infty$. Here damping is also sufficiently strong to surpress oscillatory vibrations of the system. Typical graphs are similar to the ones in Figure 21.2.

Case 3: $\gamma^2 - 4mk < 0$(Underdamping)

Here the roots are complex conjugates $r_{1,2} = \alpha + i\beta$ where $\alpha = -\frac{\gamma}{2m}$ and $\beta = \frac{\sqrt{4mk-\gamma^2}}{2m}$. The general solution is given by

$$y(t) = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t)$$

In this case, damping here is too weak to surpress the vibrations. Note that $y(t) \to 0$ as $t \to \infty$, that is, the motion decays to 0 as time increases. A typical graph is shown in Figure 21.3.

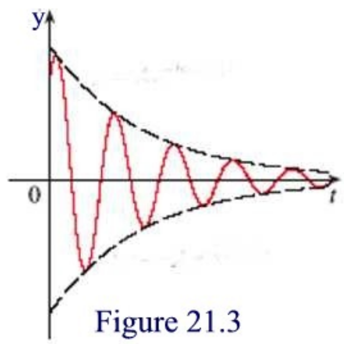


Figure 21.3

Example 21.1
A mass-spring system consists of a mass of 2 kg and a spring with natural length 0.5 m. A force of 25.6 N is required to maintain it stretched to a length of 0.7 m. Suppose the system is attached to a damping mechanism with $\gamma = 40$. Find the position of the mass at any time t if the spring is stretched from the equilibrium position with an initial velocity of 0.6 m/s.

Solution.
Given that $F_R = 25.6$. Thus, $k(0.2) = 25.6$ so that $k = 128\ N/m$. The motion of the system is described by the differential equation

$$2\frac{dy^2}{dt} + 40\frac{dy}{dt} + 128y = 0$$

0r

$$\frac{dy^2}{dt} + 20\frac{dy}{dt} + 64y = 0$$

The associated characteristic equation $r^2 + 20r + 64 = 0$ has roots $r_1 = -4$ and $r_2 = -16$. The displacement function is then given by

$$y(t) = c_1 e^{-4t} + c_2 e^{-16t}$$

Since $y(0) = 0$ then $c_1 + c_2 = 0$. Also, $y'(0) = 0.6$ so that $-4c_1 - 16c_2 = 0.6$ or $c_1 + 4c_2 = -0.15$. Solving for c_1 and c_2 we find $c_1 = 0.05$ and $c_2 = -0.05$. Therefore,

$$y(t) = 0.05(e^{-4t} - e^{-16t}) \blacksquare$$

Problem 21.1
A 10-kg mass, when attached to the end of a spring hanging vertically, stretches the spring 30 mm. Assume the mass is then pulled down another 70 mm and released (with no initial velocity).
(a) Determine the spring constant k.
(b) State the initial value problem (giving numerical values for all the constants) for $y(t)$, where $y(t)$ denotes the displacement (in meters) of the mass from its equilibrium rest position. Assuming that y is measured positive in the downward direction.
(c) Solve the initial value problem formulated in part (b).

Problem 21.2
A 20-kg mass was initially at rest, attached to the end of a vertically hanging spring. When given an initial velocity of 2 m/s from its equilibrium rest position, the mass was observed to attain a maximum displacement of 0.2 m from its equilibrium position. What is the value of the spring constant k?

Problem 21.3
A spring-mass-dashpot system consists of a 10-kg mass attached to a spring with spring constant $k = 100\ N/m$; the dashpot has damping constant $\gamma = 7\ kg/s$. At time $t = 0$, the system is set into motion by pulling the mass down 0.5 m from its equilibrium rest position while simultaneously giving it an initial downward velocity of 1 m/s.
(a) State the initial value problem to be solved for $y(t)$, the displacement from equilibrium (in meters) measured positive in the downward direction. Give numerical values to all constants involved.
(b) Solve the initial value problem. What is $\lim_{t\to\infty} y(t)$? Explain why your answer for this limit makes sense from a physical perspective.

Problem 21.4
A spring and dashpot system is to be designed for a 32-lb weight so that the overall system is critically damped.
(a) How must the damping constant γ and spring constant k be related?

(b) Assume the system is to be designed so that the mass, when given an initial velocity of 4 ft/sec from its rest position, will have a maximum displacement of 6 in. What values of damping constant γ and constant k are required?

Problem 21.5
A mass-spring-dashpot system can be modeled by the second order equation

$$my'' + ky' + \gamma y = 0$$

where m is the mass, k is the spring constant and γ is the damping coefficient. A certain system of this type with $m = 1$ can also be modeled by the first order system

$$\begin{bmatrix} y \\ y' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -5 & -4 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix}$$

What is the spring constant in this system? What is the damping coefficient?

Problem 21.6
Consider the mass-spring-dashpot system satisfying the differential equation

$$y'' + 2y' + 5y = 0$$

Is this system overdamped, critically damped, or underdamped?

Problem 21.7
Consider a mass-spring-dashpot system for which $m = 1, \gamma = 6$, and $k = 13$.
(a) Find the general solution of the corresponding second order differential equation that describes the displacement function.
(b) Is the system over-damped, under-damped, or critically damped?

Problem 21.8
A mass of 100 g stretches a spring 5 cm. If the mass is set in motion from equilibrium with a downward velocity of 10 cm/sec and there is no air resistance, then when does the mass return to equilibrium position for the first time?

Problem 21.9
A mass weighing 8 lb stretches a spring 1.5 in. The mass is attached to a damper with coefficient γ. Determine γ so the system is critically damped.

22 The Structure of the General Solution of Linear Nonhomogeneous Equations

In this section we consider the question of finding the general solution to the differential equation

$$y'' + p(t)y' + q(t)y = g(t) \qquad (43)$$

where $p(t), q(t)$ and $g(t)$ are continuous functions for $a < t < b$. The following theorem provides the structure of the general solution to equation (43).

Theorem 22.1
Let $\{y_1(t), y_2(t)\}$ be a fundamental set of solutions to the homogeneous equation $y'' + p(t)y' + q(t)y = 0$ and $y_p(t)$ be a particular solution of the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t), \quad a < t < b$$

The general solution of the nonhomogeneous equation is given by

$$y(t) = y_p(t) + c_1y_1(t) + c_2y_2(t)$$

for constants c_1 and c_2.

Proof.
Let $y(t)$ be any solution to equation (43). Since $y_p(t)$ is also a solution then

$$\begin{array}{rcl}(y-y_p)'' + p(t)(y-y_p)' + q(t)(y-y_p) & = & (y'' + p(t)y' + q(t)y) \\ & - & (y_p'' + p(t)y_p' + q(t)y_p) \\ & = & g(t) - g(t) = 0\end{array}$$

Therefore $y - y_p$ is a solution to the homogeneous equation. But $\{y_1, y_2\}$ is a fundamental set of solutions to the homogeneous equation so that there exist unique constants c_1 and c_2 such that $y(t) - y_p(t) = c_1y_1(t) + c_2y_2(t)$. Hence,

$$y(t) = y_p(t) + c_1y_1(t) + c_2y_2(t) \blacksquare$$

It follows from the above theorem that finding the general solution to nonhomogeneous equations consists of three steps:
1. Find the general solution of the associated homogeneous equation $y'' + p(t)y' + q(t)y = 0$.
2. Find a single solution of the original equation $y'' + p(t)y' + q(t)y = g(t)$
3. Add together the solutions found in steps 1 and 2.

Example 22.1
Use the fact that $y_p(t) = 3t - 1$ to find the unique solution to the initial value problem

$$y'' - 2y' - 3y = -9t - 3, \; y(0) = 1, \; y'(0) = 3$$

Solution.
Since we are given y_p then we need to find the general solution of the homogeneous equation $y'' - 2y' - 3y = 0$. The associated characteristic equation $r^2 - 2r - 3 = 0$ has roots $r_1 = -1$ and $r_2 = 3$. Hence, the general solution to the differential equation is

$$y(t) = c_1 e^{-t} + c_2 e^{3t} + 3t - 1$$

The derivative of this function is given by $y'(t) = -c_1 e^{-t} + 3c_2 e^{3t} + 3$. The condition $y(0) = 1$ leads to $c_1 + c_2 = 2$. The condition $y'(0) = 3$ leads to $-c_1 + 3c_2 = 0$. Solving for c_1 and c_2 we find $c_1 = \frac{3}{2}$ and $c_2 = \frac{1}{2}$. The unique solution is given by

$$y(t) = \frac{3}{2} e^{-t} + \frac{1}{2} e^{3t} + 3t - 1 \blacksquare$$

Note that $y_1(t) = 3t - 1$ and $y_2(t) = e^{3t} + 3t - 1$ both are particular solutions to the given differential equation. However, the sum $u(t) = y_1(t) + y_2(t) = e^{3t} + 6t - 2$ is not a solution since

$$u'' - 2u' - 3u = -18t - 6 \neq -9t - 3.$$

This shows that the superposition of solutions is valid only for homogeneous equations and not true in general for nonhomogeneous equations. However, we can have a property of superposition of nonhomogeneous if one is adding two solutions of two different nonhomogeneous equations. More precisely, we have

Theorem 22.2
Let $y_1(t)$ be a solution of $y'' + p(t)y' + q(t)y = g_1(t)$ and $y_2(t)$ a solution of $y'' + p(t)y' + q(t)y = g_2(t)$. Then for any constants c_1 and c_2 the function $Y(t) = c_1 y_1(t) + c_2 y_2(t)$ is a solution of the equation

$$y'' + p(t)y' + q(t)y = c_1 g_1(t) + c_2 g_2(t)$$

Proof.
We have

$$\begin{aligned} Y''+p(t)Y'+q(t)Y &= c_1y_1''+c_2y_2''+p(t)c_1y_1'+p(t)c_2y_2'+q(t)c_1y_1+q(t)c_2y_2 \\ &= c_1(y_1''+p(t)y_1'+q(t)y_1)+c_2(y_2''+p(t)y_2'+q(t)y_2) \\ &= c_1g_1(t)+c_2g_2(t) \blacksquare \end{aligned}$$

Example 22.2
The functions $u_1(t)$ and $u_2(t)$ are solutions to the following differential equations

$$\begin{aligned} u_1''+p(t)u_1'+q(t)u_1 &= 2e^{-t}-t-1 \\ u_2''+p(t)u_2'+q(t)u_2 &= 3t \end{aligned}$$

Use the functions u_1 and u_2 to construct a particular solution of the differential equation

$$u''+p(t)u'+q(t)u = 4e^{-t}-2$$

Solution.
The left-hand side of the given equation can be written as $4e^{-2}-2 = 2(2e^{-t}-t-1)+\frac{2}{3}(3t)$ so that by the previous theorem, the function $u(t) = 2u_1(t)+\frac{2}{3}u_2(t)$ is the required particular solution $\blacksquare$

In Problems 1- 7, answer the following three questions.
(a) Verify that the given function, $y_p(t)$, is a particular solution of the differential equations.
(b) Determine the general solution,y_h, of the homogeneous equation.
(c) Find the general solution to the differential equation and impose the initial conditions to obtain the unique solution of the initial value problem.

Problem 22.1
$y''-y'-2y = 4e^{-t},\ y(0)=0,\ y'(0)=0,\ \ y_p(t) = -\frac{4}{3}te^{-t}$

Problem 22.2
$y''-2y'-3y = e^{2t},\ y(0)=1,\ y'(0)=0,\ y_p(t) = -\frac{1}{3}e^{2t}$

Problem 22.3
$y''-y'-2y = 10,\ y(-1)=0,\ y'(-1)=1,\ y_p(t) = -5$

Problem 22.4
$y''+y' = 2e^{-t},\ y(0)=2,\ y'(0)=2,\ y_p(t) = -2te^{-t}$

Problem 22.5
$y'' + 4y = 10e^{t-\pi},\ y(\pi) = 2,\ y'(\pi) = 0,\ y_p(t) = 2e^{t-\pi}$

Problem 22.6
$y'' - 2y' + 2y = 5\sin t,\ y(\frac{\pi}{2}) = 1,\ y'(\frac{\pi}{2}) = 0,\ y_p(t) = 2\cos t + \sin t$

Problem 22.7
$y'' - 2y' + y = t^2 + 4 + 2\sin t,\ y(0) = 1,\ y'(0) = 3,\ y_p(t) = t^2 + 4t + 10 + \cos t$

The functions $u_1(t), u_2(t)$, and $u_3(t)$ are solutions to the following differential equations

$$\begin{array}{rcl} u_1'' + p(t)u_1' + q(t)u_1 & = & 2e^{-t} - t - 1 \\ u_2'' + p(t)u_2' + q(t)u_2 & = & 3t \\ u_3'' + p(t)u_3' + q(t)u_3 & = & 2e^t + 1 \end{array}$$

In Problems 8 - 9, use the functions $u_1, u_2(t)$ and u_3 to construct a particular solution of the differential equation

Problem 22.8
$u'' + p(t)u' + q(t)u = e^t + 2t + \frac{1}{2}$

Problem 22.9
$u'' + p(t)u' + q(t)u = \frac{e^t + e^{-t}}{2}$

In Problems 10 - 13, determine the function $g(t)$

Problem 22.10
$y'' - 2y' - 3y = g(t), \quad y_p(t) = 3e^{5t}$

Problem 22.11
$y'' - 2y' = g(t), \quad y_p(t) = 3t + \sqrt{t},\ t > 0$

Problem 22.12
$y'' + y' = g(t), \quad y_p(t) = \ln(1 + t),\ t > -1$

Problem 22.13
$y'' + 2y' + y = g(t), \quad y_p(t) = t - 2$

In Problems 14 - 16, the general solution of the nonhomogeneous differential equation $y'' + \alpha y' + \beta y = g(t)$ is given, where c_1 and c_2 are arbitrary constants. Determine the constants α and β and the function $g(t)$.

Problem 22.14
$y(t) = c_1 e^t + c_2 e^{2t} + 2t^{-2t}$

Problem 22.15
$y(t) = c_1 e^t + c_2 t e^t + t^2 e^t$

Problem 22.16
$y(t) = c_1 \sin 2t + c_2 \cos 2t - 1 + \sin t$

Problem 22.17
Given that the function $\frac{e^t}{5}$ satisfies the differential equation $y'' + 4y = e^t$, write a general solution of the differential equation $y'' + 4y = e^t$.

Problem 22.18
Find the general solution to the differential equation

$$y^{(4)} + 9y'' = 24 + 108t^2$$

given a particular solution $y_p(t) = \cos 3t + \sin 3t + t^4$

Problem 22.19
Show that the general solution of the third-order linear ODE $y''' + p(t)y'' + q(t)y' + r(t)y = g(t)$ is of the form $y = y_p + y_h$, where y_p is a particular solution, and y_h is the general solution of the corresponding homogeneous equation.

23 The Method of Undetermined Coefficients

In Section 22 we found out that the general solution to the nonhomogeneous differential equation

$$y'' + p(t)y' + q(t)y = g(t), \quad a < t < b \tag{44}$$

has the structure

$$y(t) = c_1y_1(t) + c_2y_2(t) + y_p(t)$$

where $y_p(t)$ is a particular solution to the nonhomogeneous equation. We will write $y(t) = y_h(t) + y_p(t)$ where $y_h(t) = c_1y_1(t) + c_2y_2(t)$.
In this and the next section we discuss methods for determining $y_p(t)$. The techinque we discuss in this section is known as the **method of undetermined coefficients.**
This method is limited in scope; it applies only to the special case of (44), where $p(t)$ and $q(t)$ are constants and $g(t)$ has some special form. The idea behind the method of undetermined coefficients is to look for $y_p(t)$ which is of a form like that of $g(t)$. This is possible only for special functions $g(t)$, but these special cases arise quite frequently in applications.
We will assume that $g(t)$ being simple means it is some combination of terms like $e^{rt}, \cos(kt), \sin(kt)$, and polynomials $a_nt^n + a_{n-1}t^{n-1} + \cdots a_1t + a_0$. (Note that if both cosine and sine terms are present, if they have the same argument kt they can be treated as one. But if they have different arguments they must be treated separately, each resulting in a combination of sine and cosine terms in y_p.) Based on those terms we will put together a candidate y_p that has some constants in it we need to solve for: Those are the undetermined coefficients this method is named for.
We start with the case where $g(t)$ is an exponential function.

Example 23.1
Find the general solution of the nonhomogeneous equation

$$y'' - 2y' - 3y = 36e^{5t}$$

Solution.
For a guessing function we will try $y_p(t) = Ae^{5t}$ where A is a constant to be determined. Inserting this into the given equation we arrive at

$$25Ae^{5t} - 10Ae^{5t} - 3Ae^{5t} = 36e^{5t}$$

Simplifying this last equation we find $12Ae^{5t} = 36e^{5t}$. Solving for A we find $A = 3$. Thus, $y_p(t) = 3e^{5t}$ is a particular solution to the differential equation. Next, the characteristic equation $r^2 - 2r - 3 = 0$ has the roots $r_1 = -1$ and $r_2 = 3$. Hence, the general solution to the differential equation is $y(t) = c_1e^{-t} + c_2e^{3t} + 3e^{5t}$ ■

Why our guess did work? The idea is simply that if y is an exponential, then so is y' and y'', and so if both y and g are exponentials, then all terms in the equation are exponentials and we can hope to obtain a solution by setting coefficients equal to each other.

Example 23.2
Find the general solution of the nonhomogeneous equation

$$y'' - y' - 2y = 4e^{-t}$$

Solution.
Let's try and proceed as in the previous example. Our choice of a particular solution is $y_p(t) = Ae^{-t}$. Substituting this into the differential equation leads to $0Ae^{-t} = 4e^{-t}$. Thus, A does not exist. Why did the procedure of the previous example fail here? The reason is that the function e^{-t} that appears in $g(t)$ is part of the general solution of the homogeneous equation $y_h(t) = c_1e^{-t} + c_2e^{2t}$. That is e^{-t} is a solution to the homogeneous equation. A correct form for the particular solution would be $y_p(t) = Ate^{-t}$. If we insert this into the differential equation we end up with $-3Ae^{-t} = 4e^{-t}$. Solving for A we find $A = -\frac{4}{3}$. Thus, $y_p(t) = -\frac{4}{3}te^{-t}$ and the general solution to the differential equation is $y(t) = c_1e^{-t} + c_2e^{2t} - \frac{4}{3}te^{-t}$ ■

The previous example illustrates the needs to first find the general solution $y_h(t)$ of the homogeneous equation before guessing the trial solution. The trial function must be modified if portions of $g(t)$ or its derivatives are present in $y_h(t)$.

Example 23.3
Find the general solution of the nonhomogeneous equation

$$y'' + 2y' + y = 2e^{-t}$$

Solution.
The characteristic equation is $r^2+2r+1=0$ with double roots $r_1=r_2=-1$. Thus, $y_h(t)=c_1e^{-t}+c_2te^{-t}$. Since $g(t)$ has the function e^{-t} which appears in the expression of $y_h(t)$ then a trial function of the form $y_p(t)=Ae^{-t}$ will fail to work. Choosing $y_p(t)=Ate^{-t}$ will also lead to a failure since te^{-t} is part appears in $y_h(t)$. Thus, a proper guess is $y_p(t)=At^2e^{-t}$. Findind derivatives up to order 2 we find $y_p'(t)=2Ate^{-t}-At^2e^{-t}$ and $y_p''(t)=2Ae^{-t}-4Ate^{-t}+At^2e^{-t}$. Substituting this in the original equation and collecting like terms we find

$$2Ae^{-t}=2e^{-t}$$

Solving for A we find $A=1$ so that $y_p(t)=t^2e^{-t}$. Hence, the general solution is given by

$$y(t)=c_1e^{-t}+c_2te^{-t}+t^2e^{-t} \blacksquare$$

It follows from the previous two examples that when guessing for y_p make sure that none of the functions in either $g(t)$ or y_p (or their derivatives) appears in $y_h(t)$.

Next, we consider the case of $g(t)$ being a polynomial.

Example 23.4
Find the general solution of

$$y''+4y'-2y=2t^2-3t+6$$

Solution.
We first solve the homogeneous equation. The characteristic equation $r^2+4r-2=0$ has the roots $r_1=-2-\sqrt{6}$ and $r_2=-2+\sqrt{6}$. Thus,

$$y_h(t)=c_1e^{(-2-\sqrt{6})t}+c_2e^{(-2+\sqrt{6})t}$$

Since $g(t)$ is a quadratic function then we are going to try $y_p(t)=At^2+Bt+C$. Inserting this into the differential equation leads to

$$-2At^2+(8A-2b)t+(2A+4B-2C)=2t^2-3t+6$$

Equating coefficients of like powers of t we find $A=-1, B=-\frac{5}{2}$, and $C=-9$. Thus a particular solution is

$$y_p(t)=-t^2-\frac{5}{2}t-9$$

The general solution of the given equation is

$$y(t) = y_h(t) + y_p(t) = c_1 e^{(-2-\sqrt{6})t} + c_2 e^{(-2+\sqrt{6})t} - t^2 - \frac{5}{2}t - 9 \blacksquare$$

Next, we consider the case when $g(t)$ is either a sine or a cosine function

Example 23.5
Find the general solution of

$$y'' - y' + y = 2\sin 3t$$

Solution.
The characteristic equation $r^2 - r + 1 = 0$ has roots $r_1 = \frac{1}{2} - i\frac{\sqrt{3}}{2}$ and $r_2 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. Thus, the general solution to the homogeneous equation is

$$y_h(t) = e^{\frac{1}{2}t}(c_1 \cos \frac{\sqrt{3}}{2}t + c_2 \sin \frac{\sqrt{3}}{2}t)$$

Our guess for the particular solution is $y_p(t) = A\cos 3t + B\sin 3t$. Inserting this into the given differential equation leads to

$$(-8A - 3B)\cos 3t + (3A - 8B)\sin 3t = 2\sin 3t$$

Setting $-8A - 3B = 0$ and $3A - 8B = 2$ and solving for A and B we find $A = \frac{6}{73}$ and $B = -\frac{16}{73}$. Thus, a particular solution is

$$y_p(t) = \frac{6}{73}\cos 3t - \frac{16}{73}\sin 3t.$$

The general solution to the differential equation is

$$y(t) = y_h(t) + y_p(t) = e^{\frac{1}{2}t}(c_1 \cos \frac{\sqrt{3}}{2}t + c_2 \sin \frac{\sqrt{3}}{2}t) + \frac{6}{73}\cos 3t - \frac{16}{73}\sin 3t \blacksquare$$

The following example illustrates the use of Theorem 22.2

Example 23.6
Find the general solution of

$$y'' - 2y' - 3y = 4t - 5 + 6te^{2t}$$

Solution.
The characteristic equation of the homogeneous equation is $r^2 - 2r - 3 = 0$ with roots $r_1 = -1$ and $r_2 = 3$. Thus,

$$y_h(t) = c_1 e^{-t} + c_2 t e^{3t}$$

By Theorem 22.2, a guess for the particular solution is $y_p(t) = At + B + Cte^{2t} + De^{2t}$. Inserting this into the differential equation leads to

$$-3At - 2A - 3B - 3Cte^{2t} + (2C - D)e^{2t} = 4t - 5 + 6te^{2t}$$

From this identity we obtain $-3A = 4$ so that $A = -\frac{4}{3}$. Also, $-2A - 3B = -5$ so that $B = \frac{23}{9}$. Since $-3C = 6$ then $C = -2$. From $2C - 3D = 0$ we find $D = -\frac{4}{3}$. It follows that

$$y(t) = c_1 e^{-t} + c_2 t e^{3t} - \frac{4}{3}t + \frac{23}{9} - \left(2t + \frac{4}{3}\right) e^{2t} \blacksquare$$

In the following table we list examples of $g(t)$ along with the corresponding form of the particular solution.

Form of $g(t)$	Form of $y_p(t)$
$a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$	$t^r[A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0$
$[a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0]e^{\alpha t}$	$t^r[A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0]e^{\alpha t}$
$[a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0]\cos \alpha t$	$t^r[(A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0)\cos \alpha t$
or	$+(B_n t^n + B_{n-1} t^{n-1} + \cdots + B_1 t + B_0)\sin \alpha t]$
$[a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0]\sin \alpha t$	
$e^{\alpha t}[a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0]\sin \beta t$	$t^r[(A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0)e^{\alpha t}\cos \beta t$
or	$+(B_n t^n + B_{n-1} t^{n-1} + \cdots + B_1 t + B_0)e^{\alpha t}\sin \beta t]$
$e^{\alpha t}[a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0]\cos \beta t$	

The number r is chosen to be the smallest nonnegative integer such that no term in the assumed form is a solution of the homogeneous equation $ay'' + by' + cy = 0$. The value of r will be 0, 1, or 2.

Example 23.7
List an appropriate form for a particular solution of
(a) $y'' + 4y = t^2 e^{3t}$
(b) $y'' + 4y = te^{2t}\cos t$
(c) $y'' + 4y = 2t^2 + 5\sin 2t + e^{3t}$
(d) $y'' + 4y = t^2 \cos 2t$

Solution.
The general solution to the homogeneous equation is $y_h(t) = c_1 \cos t + c_2 \sin t$.
(a) For $g(t) = t^2 e^{3t}$, an appropriate particular solution has the form $y_p(t) = t^r(A_2t^2 + A_1t + A_0)e^{3t}$. We take $r = 0$ since no term in the assumed form for y_p is present in the expression of $y_h(t)$. Thus

$$y_p(t) = (A_2t^2 + A_1t + A_0)e^{3t}$$

(b) An appropriate form is

$$y_p(t) = t^r[(A_1t + A_0)e^{2t}\cos t + (B_1t + B_0)e^{2t}\sin t]$$

We take $r = 0$ since no term in the assumed form for y_p is present in the expression of $y_h(t)$. Thus

$$y_p(t) = (A_1t + A_0)e^{2t}\cos t + (B_1t + B_0)e^{2t}\sin t$$

(c)

$$y_p(t) = A_2t^2 + A_1t + A_0 + B_0t\cos 2t + C_0t\sin 2t + D_0e^{3t}$$

(d)

$$y_p(t) = t(A_2t^2 + A_1t + A_0)\cos 2t + t(B_2t^2 + B_1t + B_0)\cos 2t \ \blacksquare$$

Problem 23.1
For the given differential equation
(a) Determine the general solution to the homogeneous equation
(b) Use the method of undetermined coefficients to find a particular solution.
(c) Form the general solution.

1. $y'' - 4y = \sin 2t$
2. $y'' + y = e^t \sin t$
3. $y'' - 4y' + 4y = 8 + \sin 2t$
4. $2y'' - 5y' + 2y = te^t$
5. $y'' + y' = 6t^2$
6. $y'' + y' = \cos t$
7. $y'' + 4y' + 5y = 5t + e^{-t}$

Problem 23.2
For the given differential equation

(a) Determine the general solution to the homogeneous equation
(b) List the form of particular solution prescribed by the method of undetermined coefficients; you need not evaluate the constants in the assumed form.

1. $y'' - 2y' - 3y = 2e^{-t}\cos t + t^2 + te^{3t}$
2. $y'' - y' = t^2(2 + e^t)$
3. $y'' - y = \frac{1}{2}(e^t + e^{-t} + e^{2t} - e^{-2t})$
4. $y'' + 4y = \frac{3}{2}e^{2t} - \frac{1}{2}e^{-2t} + 1$

Problem 23.3
Consider the differential equation $y'' + \alpha y' + \beta y = g(t)$. The general solution to the homogeneous equation and $g(t)$ are given. Determine α and β and then find the general solution of the differential equation.

1. $y_h(t) = c_1 + c_2e^{-t},\ g(t) = t$
2. $y_c(t) = c_1\cos t + c_2\sin t,\ g(t) = t + \sin 2t$

Problem 23.4
Consider the differential equation $y'' + \alpha y' + \beta y = g(t)$. The nonhomogeneous term $g(t)$ and the form of the particular solution prescribed by the method of undetermined coefficients are given. Determine α and β.

1. $g(t) = t + e^{3t},\quad y_p(t) = A_1t^2 + A_0t + B_0te^{3t}$
2. $g(t) = -e^t + \sin 2t + e^t\sin 2t,\quad y_p(t) = A_0e^t + B_0t\cos 2t + C_0t\sin 2t + D_0e^t\cos 2t + E_0e^t\sin 2t$

Problem 23.5
Find the form of the particular solution $y_p(t)$ for the following differential equation but do NOT solve for the coefficients

$$y'' + 3y' + 2y = e^t(t^2 + 1)\sin 2t + 3e^{-t}\cos t + 4e^t$$

Problem 23.6
A mass of 100 g is attached to a spring of length 50 cm. It is stretched 10cm by the addition of mass. It is then pulled 10 cm downwards and let go. Determine the subsequent motion, ignoring friction.

Problem 23.7
Express the solution to the initial-value problem

$$y'' + 4y = 5\sin 3t,\ y(0) = y'(0) = 0$$

as a sum of two oscillations.

Problem 23.8
Consider the following equation for $y(t)$:

$$y'' + 4y' + 5y = 2t$$

a) Find a fundamental set of solutions to the corresponding homogeneous equation.
b) Construct a particular solution.
c) Give the general solution.

Problem 23.9
a) Find two fundamental solutions to the equation $y'' + 5y' + 6y = 0$ and compute their Wronskian.
b) Find all solutions to the equation $y'' + 5y' + 6y = \sin t$.

Problem 23.10
Consider the differential equation $y'' - 4y' - 12y = g(t)$. For the $g(t)$ listed below, provide the correct initial guess for the particular solution, y_p, when using the Method of Undetermined Coefficients. (DO NOT SOLVE FOR THE COEFFICIENTS.)
(a) $g(t) = 2t^3 - t + 3$
(b) $g(t) = 12e^{-4t} \sin 2t$
(c) $g(t) = 7e^{8t} - e^t$

Problem 23.11
Consider the differential equation $y'' + 4y' + 3y = 2e^{2t}$.
(a) Determine the homogeneous solution.
(b) Compute a particular solution.
(c) Determine the general solution for the equation.
(d) Find the solution to the initial value problem when $y(1) = 0$ and $y'(1) = 1$.

Problem 23.12
(a) Find all solutions to the differential equation $y'' - 3y' + 2y = 60e^{7t}$
(b) Find all solutions to the differential equation $y'' - 2y' + y = t$
(c) Find all solutions to the differential equation $y'' + y = t^2$

Problem 23.13
For the following equations, write down the form of the particular solution, using the method of undetermined coefficients. You do not have to find the value of the coefficients.
(a) $y'' + y = te^{-t} + \cos t$
(b) $y'' + y = 10t^5 - t^3 + 23t^2 - t - 17)e^t \cos 6t$

Problem 23.14
Use the method of undetermined coefficients to find the general solution of the equation
$$y'' - 3y' - 4y = 8t + 2\sin t + 5e^{-t}$$

Problem 23.15
Use the method of undetermined coefficients to find the exact solution of the initial value problem
$$y'' + 2y' + 2y = 4\cos 3t,$$
with initial conditions $y(0) = -1$ and $y'(0) = 2$.

Problem 23.16
Given
$$y'' - 3y' + 2y = 6e^{-3t} + \sin 2t$$
Find general solution to the given equation using method of undetermined coefficients.

Problem 23.17
Using the method of undetermined coefficients find a particular solution 0f
$$y'' - 9y = te^{3t}$$

Problem 23.18
Verify that e^t and $(1 + t)$ are solutions of the homogeneous equation corresponding to
$$ty'' - (1 + t)y' + y = t^2e^{2t}$$
and use this to find the general solution.

Problem 23.19
Find the solution of the given initial value problem:
$$y'' - 2y' = e^{2t} + t^2 - 1,\ y(0) = \frac{1}{8},\ y'(0) = 0$$

Problem 23.20
Use the method of undetermined coefficients to solve: $y'' - 2y' + y = t^3 \cos 2t$

24 The Method of Variation of Parameters

In this section we discuss a second method for finding a particular solution to a nonhomogeneous differential equation

$$y'' + p(t)y' + q(t)y = g(t), \quad a < t < b \tag{45}$$

This method has no prior conditions to be satisfied by either $p(t), q(t)$, or $g(t)$. Therefore, it may sound more general than the method of undetermined coefficients. We will see that this method depends on integration while the previous one is purely algebraic which, for some at least, is an advantage.
To use this method, we first find the general solution to the homogeneous equation

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

Then we replace the parameters c_1 and c_2 by two functions $u_1(t)$ and $u_2(t)$ to be determined. From this the method got its name. Thus obtaining

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t).$$

Observe that if u_1 and u_2 are constant functions then the above y is just the homogeneous solution to the differential equation.
In order to determine the two functions one has to impose two constraints. Finding the derivative of y_p we obtain

$$y_p' = (y_1'u_1 + y_2'u_2) + (y_1u_1' + y_2u_2')$$

Finding the second derivative to obtain

$$y_p'' = y_1''u_1 + y_1'u_1' + y_2''u_2 + y_2'u_2' + (y_1u_1' + y_2u_2')'$$

Since it is up to us to choose u_1 and u_2 we decide to do that in such a way to make our computation simple. One way to achieving that is to impose the condition

$$y_1u_1' + y_2u_2' = 0 \tag{46}$$

Under such a constraint y_p' and y_p'' are simplified to

$$y_p' = y_1'u_1 + y_2'u_2$$

and

$$y_p'' = y_1''u_1 + y_1'u_1' + y_2''u_2 + y_2'u_2'$$

In particular, y_p'' does not involve u_1'' and u_2''.
Inserting y_p, y_p', and y_p'' into equation (45) to obtain

$$[y_1''u_1 + y_1'u_1' + y_2''u_2 + y_2'u_2'] + p(t)(y_1'u_1 + y_2'u_2) + q(t)(u_1y_1 + u_2y_2) = g(t)$$

Rearranging terms,

$$[y_1'' + p(t)y_1' + q(t)y_1]u_1 + [y_2'' + p(t)y_2' + q(t)y_2]u_2 + [u_1'y_1' + u_2'y_2'] = g(t)$$

Since y_1 and y_2 are solutions to the homogeneous equation then the previous equation yields our second constraint

$$u_1'y_1' + u_2'y_2' = g(t) \tag{47}$$

Combining equation (46) and (47) into the matrix form

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$$

Since $\{y_1, y_2\}$ is a fundamental set then the determinent of the coefficient matrix is nonzero so that one can find unique u_1' and u_2'. These functions are given by

$$u_1'(t) = -\tfrac{y_2(t)g(t)}{W(t)} \text{ and } u_2'(t) = \tfrac{y_1(t)g(t)}{W(t)}$$

Computing antiderivatives to obtain

$$u_1(t) = \int -\tfrac{y_2(t)g(t)}{W(t)}dt \text{ and } u_2(t) = \int \tfrac{y_1(t)g(t)}{W(t)}dt$$

Example 24.1
Find the general solution of

$$y'' - y' - 2y = 2e^{-t}$$

using the method of variation of parameters.

Solution.
The characteristic equation $r^2 - r - 2 = 0$ has roots $r_1 = -1$ and $r_2 = 2$. Thus, $y_1(t) = e^{-t}, y_2(t) = e^{2t}$ and $W(t) = 3e^t$. Thus,

$$u_1(t) = -\int \frac{e^{2t} \cdot 2e^{-t}}{3e^t} dt = \frac{2}{3}t$$

and

$$u_2(t) = \int \frac{e^{-t} \cdot 2e^{-t}}{3e^t} dt = -\frac{2}{9}e^{-3t}$$

Thus,

$$y_p(t) = \frac{2}{3}te^{-t} - \frac{2}{9}e^{-t}$$

The general solution is then given by

$$y(t) = c_1 e^{-t} + c_2 e^{2t} + \frac{2}{3}te^{-t} - \frac{2}{9}e^{-t} \blacksquare$$

Example 24.2
Find the general solution to $(2t-1)y'' - 4ty' + 4y = (2t-1)^2 e^{-t}$ if $y_1(t) = t$ and $y_2(t) = e^{2t}$ form a fundamental set of solutions to the equation.

Solution.
First we rewrite the equation in standard form

$$y'' - \frac{4t}{2t-1}y' + \frac{4}{2t-1}y = (2t-1)e^{-t}$$

Since $W(t) = (2t-1)e^{2t}$ then

$$u_1(t) = -\int \frac{e^{2t} \cdot (2t-1)e^{-t}}{(2t-1)e^{2t}} dt = e^{-t}$$

and

$$u_2(t) = \int \frac{t \cdot (2t-1)e^{-t}}{(2t-1)e^{2t}} dt = -\frac{1}{3}te^{-3t} - \frac{1}{9}e^{-3t}$$

Thus,

$$y_p(t) = te^{-t} - \frac{1}{3}te^{-t} - \frac{1}{9}e^{-t} = \frac{2}{3}te^{-t} - \frac{1}{9}e^{-t}$$

The general solution is

$$y(t) = c_1 t + c_2 e^{2t} + \frac{2}{3}te^{-t} - \frac{1}{9}e^{-t} \blacksquare$$

Problem 24.1
Solve $y'' + y = \sec t$ by variation of parameters.

Problem 24.2
Solve $y'' - y = e^t$ by undetermined coefficients and by variation of parameters. Explain any differences in the answers.

Problem 24.3
Solve the following 2nd order equation using the variation of parameter method:

$$y''4y = t^2 + 8\cos 2t$$

Problem 24.4
Find a particular solution by the variation of parameters to the equation

$$y'' + 2y' + y = e^{-t}\ln t$$

Problem 24.5
Use the variation of parameters to find a particular solution, and then check your answers by using the method of undetermined coefficients that we learned in Section 23.

$$y'' - 6y' + 8y = \cos 2t$$

Can you comment on the relative advantages of the two methods?

Problem 24.6
Solve the following initial value problem by using variation of parameters:

$$y'' + 2y' - 3y = te^t, +y(0) = -\frac{1}{64},\ y'(0) = \frac{59}{64}$$

Problem 24.7
(a) Verify that $\{e^{\sqrt{t}}, e^{-\sqrt{t}}\}$ is a fundamental set for the equation

$$4ty'' + 2y' - y = 0$$

on the interval $(0, \infty)$. You may assume that the given functions are solutions to the equation.
(b) Use the method of variation of parameters to find one solution to the equation

$$4ty'' + 2y' - y = 4\sqrt{t}e^{\sqrt{t}}.$$

Problem 24.8
Use the method of variation of parameters to find the general solution to the equation
$$y'' + y = \sin t$$

Problem 24.9
Consider the differential equation
$$t^2y'' + 3ty' - 3y = 0,\ t > 0$$
(a) Determine r so that $y = t^r$ is a solution.
(b) Use (a) to find a fundamental set of solutions.
(c) Use the method of variaion of parameters for finding a particular solution to
$$t^2y'' + 3ty' - 3y = \frac{1}{t^3},\ t > 0$$

Problem 24.10
Use the method of variation of parameters to find the general solution to the D. E.
$$y'' + y = \sin^2 t$$

Problem 24.11
Consider the differential equation
$$t^2y'' - 3ty' + 4y = t^2 \ln t, \quad t > 0$$
(a) Find a solution of the form $y = t^r$ to the homogeneous equation.
(b) Use part (a) to find a fundamental set of solution.
(c) Use the method of variation of parameters to find a particular solution to the nonhomogeneous problem.

Problem 24.12
Find the general solution to the differential equation $y'' + y' = \ln t,\ t > 0$

Problem 24.13
Find the general solution of
$$y'' + y = \frac{1}{2 + \sin t}$$

Problem 24.14
Find a particular solution of

$$t^2y'' - 2ty' + 2y = t^3, \ t > 0$$

Problem 24.15
Consider the homogeneous differential equation $y''+p(t)y'+q(t)y = g(t)$. Let $\{y_1, y_2\}$ be a fundamental set of solutions for the corresponding homogeneous equation and let $W(t)$ be the Wronskian of this fundamental set. Show that the particular solution that vanishes at $t = t_0$ is given by

$$y_p(t) = \int_{t_0}^{t} [y_2(t)y_1(\lambda) - y_1(t)y_2(\lambda)] \frac{g(\lambda)}{W(\lambda)} d\lambda.$$

In Problems 16 - 18, the given expression is the solution of the initial value problem

$$y'' + \alpha y' + \beta y = g(t), \ y(0) = y_0, \ y'(0) = y_0'$$

Determine α, β, y_0, and y_0'

Problem 24.16

$$y(t) = \frac{1}{2} \int_0^t \sin(2(t - \lambda))g(\lambda)d\lambda$$

Problem 24.17

$$y(t) = t + \int_0^t (t - \lambda)g(\lambda)d\lambda$$

Problem 24.18

$$y(t) = e^{-t} + \int_0^t \frac{e^{t-\lambda} - e^{-(t-\lambda)}}{2} g(\lambda)d\lambda$$

Problem 24.19
Did you ever wonder what would happen if the method of variation of parameters were applied to a first order linear equation? Let's figure it out. Start with a general first-order linear equation $y' + p(t)y = g(t)$. Suppose that $y_1(t)$ is some nonzero solution of the associated homogeneous equation

$y' + p(t)y = 0$. Vary it to get a trial solution for the nonhomogeneous equation, that is, write $y(t) = u(t)y_1(t)$.
(a) Calculate $y'(t)$
(b) Put the expressions for $y(t)$ and $y'(t)$ into the nonhomogeneous equation $y' + p(t)y = g(t)$. Simplify it using the fact that $y_1(t)$ is a solution to the associated homogeneous equation, and solve for $u'(t)$. Use this expression to obtain a formula giving $y(t)$ as an integral whose integrand involves $g(t)$.
(c) Now, let's see why this is nothing new. Use an integrating factor to find a solution $y_1(t)$ for $y' + p(t)y = 0$
(d) Put the expression for $y_1(t)$ into the formula you obtained using variation of parameters and simplify. Not surprisingly, the resulting formula is exactly the one that results when one solves $y' + p(t)y = g(t)$ using an integrating factor, although you need not check this.

25 Applications of Nonhomogeneous Second Order Linear Differential Equations: Forced Mechanical Vibrations

In this section we consider two applications of nonhomogeneous second order linear differential equations: forced mechanical vibrations and electrical circuits.

Forced Vibrations
In Section 22, we considered a mass-spring-dashpot system where the motion is affected by two forces: the spring restoring force and the damping force. In addition to these forces, suppose that there is an external applied force $F_a(t)$ affecting the motion of the spring. Then Newton's second law of motion gives

$$\begin{aligned} m\tfrac{d^2y}{dt^2} &= \text{restoring force} + \text{damping force} + \text{external force} \\ &= -ky - \gamma\tfrac{dy}{dt} + F_a(t) \end{aligned}$$

Thus, the motion of the spring is now being governed by the following nonhomogeneous second order linear differential equation

$$m\frac{d^2y}{dt^2} + ky + \gamma\frac{dy}{dt} = F_a(t) \tag{48}$$

If there is no dumping then equation (48) reduces to

$$y'' + \omega_0^2 y = f_a(t) \tag{49}$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ and $f_a(t) = \frac{F(t)}{m}$. We call ω_0 the **natural frequency** of the system.
A commonly used applied force is a periodic varying force function

$$f_a(t) = F\cos\omega t$$

For simplicity, we assume that the system is initially at rest so that we have the following initial value problem

$$y'' + \omega_0^2 y = F\cos\omega t, \quad y(0) = 0, \; y'(0) = 0 \tag{50}$$

The general solution to the associated homogeneous equation is given by

$$y_h(t) = c_1\cos\omega_0 t + c_2\sin\omega_0 t$$

To find a particular solution we consider the cases whether $\omega \neq \omega_0$ and $\omega = \omega_0$.

Case 1: $\omega \neq \omega_0$
In this case, $F\cos\omega t$ is not a solution to the homogeneous equation. Using the method of undetermined coefficients, an appropriate trial function is

$$y_p(t) = A\sin\omega t + B\cos\omega t$$

Substituting this into the nonhomogeneous equation we find

$$A(\omega_0^2 - \omega^2)\sin\omega t + B(\omega_0^2 - \omega^2)\cos\omega t = F\cos\omega t$$

Thus, $A = 0$ and $B = \frac{F}{\omega_0^2 - \omega^2}$ so that

$$y_p(t) = \frac{F}{\omega_0^2 - \omega^2}\cos\omega t$$

Hence, the general solution is

$$y(t) = c_1\cos\omega_0 t + c_2\sin\omega_0 t + \frac{F}{\omega_0^2 - \omega^2}\cos\omega t$$

Since $y(0) = 0$ then $c_1 = \frac{F}{\omega^2 - \omega_0^2}$. To find c_2 we need to use the condition $y'(0) = 0$. But

$$y'(t) = -c_1\omega_0\sin\omega_0 t + c_2\omega_0\cos\omega_0 t - \frac{F\omega}{\omega_0^2 - \omega^2}\sin\omega_0 t$$

Thus, $c_2 = 0$. It follows that the unique solution to the initial value problem (50) is given by

$$y(t) = \frac{F}{\omega^2 - \omega_0^2}(\cos\omega_0 t - \cos\omega t) \tag{51}$$

Utilizing a trigonometric identity of cosine the last equality can be written as

$$y(t) = \frac{2F}{\omega^2 - \omega_0^2}\sin\left(\frac{\omega - \omega_0}{2}t\right)\sin\left(\frac{\omega + \omega_0}{2}t\right)$$

When $\omega \approx \omega_0$ the factor $\sin\left(\frac{\omega+\omega_0}{2}t\right)$ oscillates much more rapidly than $\sin\left(\frac{\omega-\omega_0}{2}t\right)$. Therefore, $y(t)$ is a product of a slowly varying amplitude and a rapidly varying oscillating. The physical phenomenon of beats refers to the periodic cancellation of sound at a slow frequency. Figure 25.1 shows a rapidly varying oscillation $y(t) = 2\sin 4t \sin 40t$ and the two slowly varying envelopes $y_1(t) = 2\sin 4t$ and $y_2(t) = -2\sin 4t$.

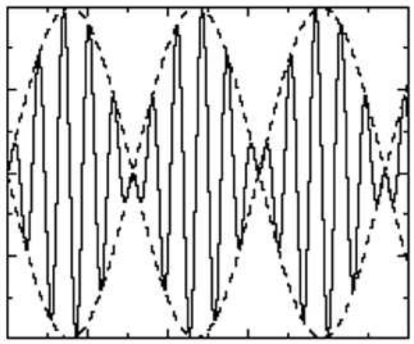


Figure 25.1

Case 2: $\omega = \omega_0$
In this case the applied frequency reinforces the natural frequency and the result is vibrations of large amplitude. This is the phenomenon of **resonance.** Since $F\cos\omega t$ is a solution to the homogeneous equation then an appropriate trial function is

$$y_p(t) = At\cos\omega_0 t + Bt\sin\omega_0 t$$

Substituting this into the nonhomogeneous equation we find

$$2B\omega_0\cos\omega_0 t - 2A\omega_0\sin\omega_0 t = F\cos\omega_0 t$$

Thus, $A = 0$ and $B = \frac{F}{2\omega_0}$. Hence,

$$y_p(t) = \frac{F}{2\omega_0}t\sin\omega_0 t$$

and the general solution is

$$y(t) = c_1\cos\omega_0 t + c_2\sin\omega_0 t + \frac{F}{2\omega_0}t\sin\omega_0 t$$

Since $y(0) = 0$ then $c_2 = 0$. Thus, $y(t) = c_1\cos\omega_0 t + \frac{F}{2\omega_0}t\sin\omega_0 t$. Finding the derivative we obtain $y'(t) = -c_1\omega_0\sin\omega_0 t + \frac{F}{2\omega_0}\sin\omega_0 t + \frac{F}{2}t\cos\omega_0 t$. The condition $y'(0) = 0$ implies that $c_1 = 0$. Therefore the unique solution to (50) is given by

$$y(t) = \frac{F}{2\omega_0} t \sin \omega_0 t \tag{52}$$

Presence of Damping When $\omega = \omega_0$
In this case, the initial value problem is

$$y'' + 2\delta y' + \omega_0^2 y = F \cos \omega_0 t, \quad y(0) = 0, \quad y'(0) = 0$$

where $\delta = \frac{\gamma}{2m}$. We assume that $\omega_0^2 > \delta^2$. Then the general solution to the homogeneous equation is given by

$$y_h(t) = c_1 e^{-\delta t} \cos((\sqrt{\omega_0^2 - \delta^2})t) + c_2 e^{-\delta t} \sin((\sqrt{\omega_0^2 - \delta^2})t)$$

An appropriate trial function for the particular solution is

$$y_p(t) = A \cos \omega_0 t + B \sin \omega_0 t$$

Substituting this into the differential equation we find

$$-2\delta A\omega_0 \sin \omega_0 t + 2\delta B\omega_0 \cos \omega_0 t = F \cos \omega_0 t$$

Thus, $A = 0$ and $B = \frac{F}{2\delta\omega_0}$. Therefore, the general solution is

$$y(t) = c_1 e^{-\delta t} \cos((\sqrt{\omega_0^2 - \delta^2})t) + c_2 e^{-\delta t} \sin((\sqrt{\omega_0^2 - \delta^2})t) + \frac{F}{2\delta\omega_0} \sin \omega_0 t$$

Since $y(0) = 0$ then $c_1 = 0$. Since $y'(t) = -\delta c_2 e^{-\delta t} \sin((\sqrt{\omega_0^2 - \delta^2})t) + c_2 e^{-\delta t}\sqrt{\omega_0^2 - \delta^2} \cos((\sqrt{\omega_0^2 - \delta^2})t) + \frac{F}{2\delta} \cos \omega_0 t$ and $y'(0) = 0$ then $c_2\sqrt{\omega_0^2 - \delta^2} + \frac{F}{2\delta} = 0$ and solving for c_2 we find

$$c_2 = -\frac{F}{2\delta\sqrt{\omega_0^2 - \delta^2}}$$

Therefore,

$$y(t) = \frac{F}{2\delta}\left[\frac{\sin \omega_0 t}{\omega_0} - \frac{e^{-\delta t}}{\sqrt{\omega_0^2 - \delta^2}} \sin((\sqrt{\omega_0^2 - \delta^2})t)\right] \tag{53}$$

Let's find out what happens to the previous expression of $y(t)$ as $\delta \to 0^+$. But first, we rewrite $y(t)$ in the form

$$y(t) = \frac{F}{2}\left[\frac{\sqrt{\omega_0^2 - \delta^2}\sin\omega_0 t - \omega_0 e^{-\delta t}\sin((\sqrt{\omega_0^2-\delta^2})t)}{\delta\omega_0\sqrt{\omega_0^2-\delta^2}}\right]$$

Notice that as $\delta \to 0^+$ the limit of $y(t)$ is of the form $\frac{0}{0}$ so we can apply L'Hopital's Rule to the limit. Letting $N(\delta) = \sqrt{\omega_0^2 - \delta^2}\sin\omega_0 t - \omega_0 e^{-\delta t}\sin((\sqrt{\omega_0^2-\delta^2})t)$ and $D(N) = \delta\omega_0\sqrt{\omega_0^2-\delta^2}$ we find

$$\begin{array}{rcl} \frac{dN}{d\delta} & = & \frac{1}{2}(\omega_0^2-\delta^2)^{-\frac{1}{2}}(-2\delta)\sin\omega_0 t + \omega_0 t e^{-\delta t}\sin((\sqrt{\omega_0^2-\delta^2})t) \\ - & & \omega_0 e^{-\delta t}\cos((\sqrt{\omega_0^2-\delta^2})t)\left(\frac{1}{2}\right)(\omega_0^2-\delta^2)^{-\frac{1}{2}}(-2\delta t) \end{array}$$

Thus, as $\delta \to 0^+$, $\frac{dN}{d\delta} \to 0 + \omega_0 t\sin\omega_0 t + 0 = \omega_0 t\sin\omega_0 t$..
Similarly, we find

$$\frac{dD}{d\delta} = \omega_0\sqrt{\omega_0^2-\delta^2} + \delta\omega_0\left(\frac{1}{2}\right)(\omega_0^2-\delta^2)^{-\frac{1}{2}}(-2\delta)$$

and $\frac{dD}{d\delta} \to \omega_0^2$ as $\delta \to 0^+$. From the above discussion we arrive at

$$\lim_{\delta\to 0^+} y(t) = \frac{Ft\sin\omega_0 t}{2\omega_0}$$

which is (52).

Presence of Damping When $\omega \neq \omega_0$
In this case, the initial value problem is

$$y'' + 2\delta y' + \omega_0^2 y = F\cos\omega t, \quad y(0) = 0, \quad y'(0) = 0$$

where $\delta = \frac{\gamma}{2m}$. We assume that $\omega_0^2 > \delta^2$. Then the general solution to the homogeneous equation is given by

$$y_h(t) = c_1 e^{-\delta t}\cos((\sqrt{\omega_0^2-\delta^2})t) + c_2 e^{-\delta t}\sin((\sqrt{\omega_0^2-\delta^2})t)$$

An appropriate trial function for the particular solution is

$$y_p(t) = A\cos\omega t + B\sin\omega t$$

Substituting this into the differential equation we find

$$[A(\omega_0^2-\omega^2)+2\delta\omega B]\cos\omega t+[B(\omega_0^2-\omega^2)-2\delta\omega A]\sin\omega t=F\cos\omega t$$

Solving for A and B we find

$$A=\frac{(\omega_0^2-\omega^2)F}{(\omega_0^2-\omega^2)^2+4\delta^2\omega^2}$$

and

$$B=\frac{2\delta\omega F}{(\omega_0^2-\omega^2)^2+4\delta^2\omega^2}$$

Therefore, the general solution is

$$\begin{array}{rcl} y(t) & = & c_1e^{-\delta t}\cos\left((\sqrt{\omega_0^2-\delta^2})t\right)+c_2e^{-\delta t}\sin\left((\sqrt{\omega_0^2-\delta^2})t\right)\\ & + & \frac{(\omega_0^2-\omega^2)F}{(\omega_0^2-\omega^2)^2+4\delta^2\omega^2}\cos\omega t+\frac{2\delta\omega F}{(\omega_0^2-\omega^2)^2+4\delta^2\omega^2}\sin\omega t \end{array}$$

Since $y(0)=0$ then

$$c_1=-\frac{(\omega_0^2-\omega^2)F}{(\omega_0^2-\omega^2)^2+4\delta^2\omega^2}$$

Since

$$\begin{array}{rcl} y'(t) & = & -\delta c_1e^{-\delta t}\cos\left((\sqrt{\omega_0^2-\delta^2})t\right)-c_1\sqrt{\omega_0^2-\delta^2}e^{-\delta t}\sin\left((\sqrt{\omega_0^2-\delta^2})t\right)\\ & - & \delta c_2e^{-\delta t}\sin\left((\sqrt{\omega_0^2-\delta^2})t\right)+c_2e^{-\delta t}\sqrt{\omega_0^2-\delta^2}\cos\left((\sqrt{\omega_0^2-\delta^2})t\right)\\ & - & \frac{(\omega_0^2-\omega^2)F}{(\omega_0^2-\omega^2)^2+4\delta^2\omega^2}\omega\sin\omega t+\frac{2\delta\omega F}{(\omega_0^2-\omega^2)^2+4\delta^2\omega^2}\omega\cos\omega t \end{array}$$

and $y'(0)=0$ then $-\delta c_1+c_2\sqrt{\omega_0^2-\delta^2}+\frac{2\delta\omega^2F}{(\omega_0^2-\omega^2)^2+4\delta^2\omega^2}=0$ and solving for c_2 we find

$$c_2=-\frac{(\omega_0^2+\omega^2)F}{4\delta\omega^2\sqrt{\omega_0^2-\delta^2}}$$

Therefore,

$$\begin{array}{rcl} y(t) & = & \frac{F}{(\omega_0^2-\omega^2)^2+4\delta^2\omega^2}[(\omega_0^2-\omega^2)\cos\omega t+2\delta\omega\sin\omega t]\\ & - & \frac{Fe^{-\delta t}}{(\omega_0^2-\omega^2)^2+4\delta^2\omega^2}\left[(\omega_0^2-\omega^2)\cos\left((\sqrt{\omega_0^2-\delta^2})t\right)\right.\\ & & \left.+\frac{(\omega_0^2+\omega^2)\delta}{\sqrt{\omega_0^2-\delta^2}}\sin\left((\sqrt{\omega_0^2-\delta^2})t\right)\right] \end{array}$$

Notice that for fixed $t > 0$, as $\omega \to \omega_0$ the previous expressions reduces to the expression given in (53). Also, notice that for fixed ω we have

$$\begin{array}{rcl}\lim_{\delta\to 0^+} y(t) & = & \frac{F}{\omega_0-\omega}\cos\omega t - \frac{F}{\omega_0^2-\omega^2}\cos\omega_0 t\\ & = & \frac{F}{\omega_0^2-\omega^2}[\cos\omega t - \cos\omega_0 t = \frac{F}{\omega^2-\omega_0^2}[\cos\omega_0 t - \cos\omega t]\end{array}$$

which is (54).

Electrical Circuits

Consider an electric circuit where a resistor, a capacitor and an inductor are connected in series with a battery or a generator. See Figure 25.2.

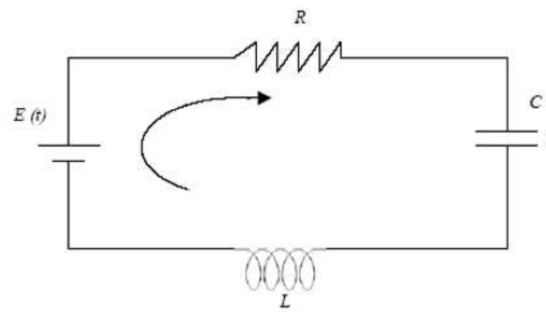


Figure 25.2

When the switch is closed, an instantaneous current will flow. If $Q(t)$ and $I(t)$ are respectively the charge on the capacitor and the current in the circuit at any instant t, then Kirchhoffs Voltage Law gives

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t), \tag{54}$$

where the inductance L, the resistance R and the capacitance C are all assumed to be constants, but the electromotive force $E(t)$ may depend on time.

Since the current flowing in a circuit must be equal to the instantaneous rate of change of charges on the capacitor, we have $I = \frac{dQ}{dt}$. As a result, the above circuit equation may be re-written as

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

which is a nonhomogeneous differential equation with constant coefficients.

Example 25.1
If an inductor of 0.5 henry is connected in series with a resistor of 6 ohms, a capacitor of 0.02 farad and a generator with electromotive force equals to $24\sin(10t)$ volts, then what is $q(t)$ at any instant t?

Solution.
The circuit equation is reduced to

$$\frac{d^2Q}{dt^2} + +12\frac{dQ}{dt} + 100Q = 48\sin 10t.$$

The characteristic equation is

$$r^2 + 12r + 100 = 0.$$

Solving this quadratic equation we find $r_{1,2} = -6 \pm 8i$. Thus,

$$Q_h(t) = e^{-6t}(c_1\cos 8t + c_2\sin 8t).$$

To find a particular solution we use the method of undetermined coefficients. So we let

$$Q_p(t) = A\cos 10t + B\sin 10t.$$

Substituting this equation into the above differential equation and then solving for A and B we find $A = 0$ and $B = -\frac{2}{5}$. Thus,

$$Q_p(t) = -\frac{2}{5}\sin 10t$$

and the general solution is

$$Q(t) = e^{-6t}(c_1\cos 8t + c_2\sin 8t) - \frac{2}{5}\sin 10t\blacksquare$$

Example 25.2
Show that the the current flow I satisfies the second order differential equation

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = E'(t) \qquad (55)$$

Solution.
Differentiating the equation

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t)$$

and using the fact that $I = \frac{dQ}{dt}$ to obtain the desired equation.$\blacksquare$

For an AC voltage we have $V(0) = 0$ and $V(t) = E_0 \sin \omega t$ so that Equation 55 becomes

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = E_0\omega \cos \omega t \tag{56}$$

Let us try and find the general solution to (56). First, we look for a particular solution of (56) of the form $I(t) = A \sin(\omega t + \phi)$ with the amplitude A and phase ϕ to be determined. Any such particular solution must obey

$$\begin{array}{rcl} -L\omega^2 A \sin(\omega t + \phi) + R\omega A \cos(\omega t + \phi) & + & \frac{1}{C}A \sin(\omega t + \phi) = \omega E_0 \cos \omega t \\ & = & \omega E_0 \cos(\omega t + \phi - \phi) \end{array}$$

and and hence

$$\begin{array}{rcl} \left(\frac{1}{C} - L\omega^2\right) A \sin(\omega t + \phi) + R\omega A \cos(\omega t + \phi) & = & \omega E_0 \cos \phi \cos(\omega t + \phi) \\ & + & \omega E_0 \sin \phi \sin(\omega t + \phi) \end{array}$$

Matching coefficients of $\sin(\omega t + \phi)$ and $\cos(\omega t + \phi)$ on the left and right hand sides gives

$$\left(\frac{1}{C} - L\omega^2\right) A = \omega E_0 \sin \phi \tag{57}$$

$$R\omega A = E_0 \omega \cos \phi \tag{58}$$

To find ϕ take the ratio $\frac{(57)}{(58)}$ to obtain

$$\tan \phi = \frac{\frac{1}{C} - L\omega^2}{R\omega}$$

and taking inverse tangent we find

$$\phi = \arctan\left(\frac{1}{RC\omega} - \frac{L\omega}{R}\right) \tag{59}$$

To find A we use the identity $\sin^2\phi + \sin^2\phi = 1$ to obtain

$$\sqrt{\left(\frac{1}{C} - L\omega^2\right)^2 A^2 + R^2\omega^2 A^2} = E_0\omega$$

so that

$$A = \frac{E_0\omega}{\sqrt{\left(\frac{1}{C} - L\omega^2\right)^2 + R^2\omega^2}} \tag{60}$$

Example 25.3
Show that if $R^2 \neq \frac{4L}{C}$ then the two roots to the characteristic equation are

$$r_{1,2} = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L}$$

Solution.
The characteristic equation is

$$Lr^2 + Rr + \frac{1}{C} = 0$$

Since $R^2 \neq \frac{4L}{C}$ then this equation has two distinct real solutions given by

$$r_{1,2} = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L} \blacksquare$$

It follows that the general solution to (55) is given by

$$i(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + A\sin(\omega t + \phi)$$

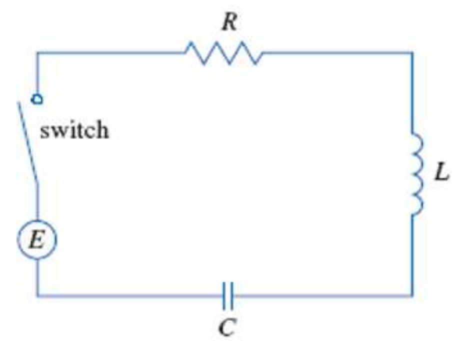
where r_1 and r_2 are given by the previous example and A and ϕ are given by (59) and (60).

Remark 25.1
Comparing Equations (48) and (54), we see that mathematically they are identical. This suggests the analogies given in the following chart between

physical situations that, at first glance, are very different.

Spring system		Electric circuit	
x	displacement	Q	charge
dx/dt	velocity	$I = dQ/dt$	current
m	mass	L	inductance
c	damping constant	R	resistance
k	spring constant	$1/C$	elastance
$F(t)$	external force	$E(t)$	electromotive force

Problem 25.1
Find the charge and current at time in the circuit below if $R = 40\Omega, L = 1\ H, C = 16 \times 10^{-4}\ F$, and $E(t) = 100\cos 10t$ and the initial charge and current are both zero.



Problem 25.2
A series circuit consists of a resistor with $R = 20\Omega$, an inductor with $L = 1\ H$, a capacitor with $C = 0.005\ F$, and a 12-V battery. If the initial charge and current are both 0, find the charge and current at time t.

Problem 25.3
The battery in previous problem is replaced by a generator producing a voltage of $E(t) = 12\sin 10t$. Find the charge at time t.

Problem 25.4
A series circuit contains a resistor with $R = 24\ \Omega$, an inductor with $L = 2\ H$, a capacitor with $C = 0.005\ F$, and a 12-V battery. The initial charge is $Q = 0.001\ C$ and the initial current is 0.
(a) Find the charge and current at time t.
(b) Graph the charge and current functions.

Problem 25.5

A vibrating spring with damping is modeled by the differential equation

$$y'' + 2y' + 4y = 0.$$

1. Find the general solution to the equation. Show each step of the process.
2. Is the solution under damped, over damped or critically damped?
3. Suppose that the damping were changed, keeping the mass and the spring the same, until the system became critically damped. Write the differential equation which models this critically damped system. Do not solve.
4. What is the steady state (long time) solution to

$$y'' + 2y' + 4y = \cos(2t).?$$

Problem 25.6

A vertical spring with a spring constant equal to 108 lb/ft has a 96 lb weight attached to it. A dashpot (or a shock absorber) with a damping coefficient c = 36 lb-sec/ft is attached to the weight. Suppose that a downward force of $f(t) = 72\cos 6t$ is applied to the weight. If the weight is released from rest at the equilibrium position at time $t = 0$ (a) show that the differential equation governing the displacement is $y(t)$

$$y'' + 12y' + 36y = 24\cos 6t$$

where g = 32 ft/sec is used .

(b) Find the solution satisfying the equation established in Part (a) and the given initial conditions.

Problem 25.7

A six Newton weight is attached to the lower end of a coil spring suspended from the ceiling, the spring constant of the spring being 27 Newtons per meter. The weight comes to rest in its equilibrium position, and beginning at t = 0 and external force given by $F(t) = 12\cos(20t)$ is applied to the system. Determine the resulting displacement as a function of time, assuming damping is negligible.

Problem 25.8

An inductor of 5 henries is connected in series with a capacitor of 1/180 farads, a resistor of 60 ohms and a voltage-supply given by $E(t) = 12\cos 6t$

in volts. Suppose that both the charge Q and the current I are zero initially.
(a) Show that the differential equation governing the charge $Q(t)$ is

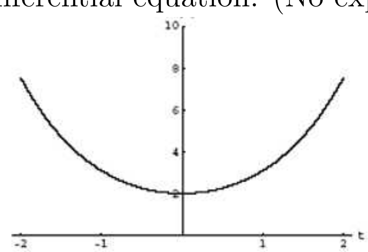
$$Q'' + 12Q' + 36Q = 24\cos 6t$$

(b) Find the charge $Q(t)$ satisfying the equation of Part (a) and the given initial conditions.

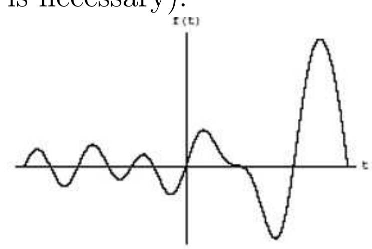
Problem 25.9
An inductor of 4 H is connected in series with a capacitor of 0.25 F and a resistor of 10 , without supplied voltage. Suppose that at t=0, there is a charge of 1/3 coulomb on the capacitor but no current.
(a) Write down the differential equation for the charge, Q(t), and the initial conditions.
(b) Find the charge as a function of time t.

Problem 25.10
Below we have five differential equations and five graphs. Next to each differential equation write the number of the graph that represents a solution to that differential equation. (No explanation is necessary).

I



II



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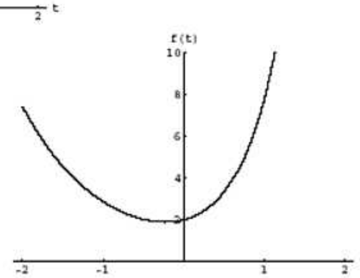

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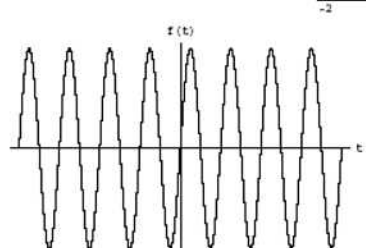
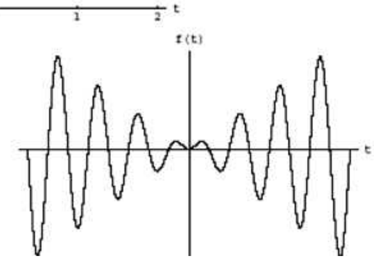


V



(a) $y' + y = 3e^{2t}$ has a solution represented by graph:
(b) $y'' - y = 0$ has a solution represented by graph:
(c) $y'' + y = 0$ has a solution represented by graph:
(d) $16y'' - 8y' + 17y = -16\cos 2t - 47\sin 2t$ has a solution represented by graph:
(e) $y'' + y = 2\cos t$ has a solution represented by graph:

Problem 25.11
Write a paragraph describing the similarities between the equations governing mechanical vis-a-vis electrical vibrations

Problem 25.12
Consider the IVP, $y'' + by' + 9y = \sin \omega t,\ y(0) = 0,\ y'(0) = 0$. For what values of b and ω is the solution periodic? For what values are there frequency beats? Solve the system in the resonant case and sketch the solution.