Introduction To *K*-theory and Some Applications*

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1. General Introduction and Overview

1.1 What is *K*-theory?

1.1.1 Roughly speaking, *K*-theory is the study of functors (bridges)

$$\underset{n\in \mathbb{Z}}{K_n} : \text{(Nice categories)} \to \text{(category of Abelian groups}$$

$$\mathcal{C} \to K_n\mathcal{C}$$

(See 2.4 (ii) for a formal definition of a functor).

Note: For $n \leq 0$, we have Negative *K*-theory
For $n \leq 2$, we have Classical *K*-theory
For $n \geq 3$, Higher *K*-theory

1.1.2 Some Historical Remarks

K-theory was so christened in 1957 by A. Grotherdieck who first studied $K_0(\mathcal{C})$ (then written $K(\mathcal{C})$) where for a scheme X, $\mathcal{C}$ is the category $\mathcal{P}(X)$ of locally free sheaves of O_X-modules. Because $K_0(\mathcal{C})$ classifies the isomorphism classes in $\mathcal{C}$ and he wanted the name of the theory to reflect 'class', he used the first letter 'K' in 'Klass' the German word meaning 'class'.

Next, M.F. Atiyah and F. Hirzebruch, in 1959 studied $K_0(\mathcal{C})$ where $\mathcal{C}$ is the category $\mathrm{Vect}_{\mathbf{C}}(X)$ of finite dimensional complex vector bundles over a compact space X yielding what became known as topological K-theory. It is usual to denote $K_0(\mathrm{Vect}_{\mathbf{C}}(X))$ by $KU(X)$ or $K^0_{\mathrm{top}}(X)$.

In 1962, R.G. Swan proved that for a compact space X, the category $\mathrm{Vect}_{\boldsymbol{C}}(X)$ is equivalent to the category $\mathsf{P}(C(X))$ of finitely generated projective modules over the ring $C(X)$ of complex valued functios on X.

i.e.,

$$\mathrm{Vect}_{\boldsymbol{C}}(X) \approx \mathsf{P}(C(X)).\ \text{So}\ K_0(\mathrm{Vect}_{\boldsymbol{C}}(X)) \approx K_0(\mathsf{P}(\boldsymbol{C}(X))).$$

Thereafter, H. Bass, R.G. Swan, etc. started replacing $C(X)$ by arbitrary rings A and studied $K_0(\mathsf{P}(A))$ for various rings A leading to the birth of Algebraic K-theory. Here $\mathsf{P}(A)$ denotes the category of finitely generated projective modules over any ring A. It is usual to denote $K_0(\mathsf{P}(A))$ by $K_0(A)$ for any ring A. $K_1(A)$ of a ring A was defined by H. Bass and $K_2(A)$ by J. Milnor. (see [3], [58] and [79]).

In 1970, D. Quillen came up with the definitions of all $K_n(\mathsf{C})$ for all $n \geq 0$ in such a way that $K_0(\mathsf{P}(A))$ coincides with $K_n(A) \quad \forall \quad n \geq 0$.

1.1.3 Some Features of $K_n(\mathsf{C})$

(1) $K_n(\mathsf{C})$ sometimes reflects the structure of objects of C.

For example,

(i) Let F be a field, G a finite group, $\mathbf{M}(FG)$ the category of finitely generated FG-modules. Then $K_0(\mathsf{M}(FG)) := G_0(FG)$ classifies representations of G in $\mathsf{P}(F)$ whose $\mathsf{P}(F)$ is the category of finite-dimensional vector spaces (see [42]),

(ii) $K_0(ZG)$ contains topological / geometric invariants. E.g., Swan-Well Invariants (see 2.7.1)

(iii) $K_i(ZG)$ contains Whitehead torsion – a topological invariant (see [3.2.3] or [57]).

(2) Each $K_n(\mathcal{C})$ yields a theory which could map or coincide with other theories.
For example,

(i) Galois, etale or Motivic cohomology theories (see [37]).
(ii) De Rham, cyclic cohomology (see [7] or [9, 10])
(iii) Representation theory, e.g., $K_0(\mathcal{M}(FG)) \approx G_0(FG)$ concides with Abelian group of characters of G (see [8, 42] or 2.3 vii).

(3) $K_n(\mathcal{C})$ satisfies various exact sequences connecting K_n, K_{n-1}, etc. For example, Localization sequences, Mayer-victories sequence, etc. These sequences are useful for computations (see [42] or [62]).

1.1.4 A Basic problem in this field is to understand and compute the Abelian groups $K_n(\mathsf{C})$ for various categories 'C'.

Two important examples of 'nice' categories are 'Abelian categories' and 'exact categories'. We now formally define these categories with copious examples and also develop notations for $K_n(\mathsf{C})$ for various C.

1.2 Abelian and Exact Categories – Definitions, Examples and Notations

1.2.1 A category consists of a class $\mathcal{C}$ of objects together with a set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ of morphisms from X to Y, for each ordered pair (X,Y) of objects of $\mathcal{C}$ such that

(1) For each triple (X,Y,Z) of objects of C, we have composition $\mathrm{Hom}_{\mathcal{C}}(Y,Z)\times \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}}(X,Z)$.

(2) Composition of morphisms is associative i.e., for composable morphisms f.g.h $g(hf)=(gh)f$

(3) There exists identity $1_X \in \mathrm{Hom}(X, X)$ such that if $g \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ and $h \in \mathrm{Hom}_{\mathsf{C}}(Z, X)$, $g 1_X = g$, and $1_X\ h = h$.

Examples:

(i) $\mathsf{S}p :=$ category of topological spaces, $ob(\mathsf{S}p) =$ topological spaces, $\mathrm{Hom}_{\mathsf{S}p}(X, Y) = \{\text{continuous maps } X \to Y\}$.

(ii) $\mathsf{G}p :=$ category of groups. $ob(\mathsf{G}p)$ are groups $\mathrm{Hom}_{\mathsf{G}p}(G, H) =$ groups homomorphisms $G \to G'$.

For more examples (see [55]).

1.2.2 Examples of Abelian Categories (for motivation)

(1) $\mathcal{A}b$ or $-\mathbf{M}\text{od} :=$ category of Abelian groups.

$ob\,(Ab) =$ Abelian groups

. Morphisms are Abelian group homomorphism.

(2) F a field; F-$\mathbf{M}\text{od} :=$ category of vector spaces over F.

$ob\,(F\text{-}\mathbf{M}\text{od}) :=$ vector spaces

Morphisms are linear transformation

(3) R a ring with identity.

$(R\text{-}\,\mathbf{M}\text{od}) :=$ category of R-modules

Morphisms are R-module homomorphisms.

1.2.3 Definitions of an Abelian Category

A category $\mathcal{A}$ is called an Abelian category if

(1) it is an Addictive category, that is:

  (a) There exists a zero object '0' in $\mathcal{A}$

  (b) Direct sum (= direct product) of any two objects of $\mathcal{A}$ exists in $\mathcal{A}$.

  (c) $\mathrm{Hom}_A(M, N)$ is an Abelian group such that composition distributes over addition.

(2) Every morphism in $\mathcal{A}$ has a kernel and a cokernel.

(3) For any morphism f, coker (ker f) = ker (coker f).

1.2.4 Note: A morphism $g: K \to M$ is called a kernel of a morphism $f: M \to N$ if for any morphism $h: P \to M$ with $f \cdot h = 0$, there exists a unique arrow $\kappa: P \to K$ such that $h = g \circ k$

$$\begin{tikzcd} K \arrow[r, "g"] & M \arrow[r, "f"] & N \\ & P \arrow[ul, dashed, "k"] \arrow[u, "h"'] & \end{tikzcd}$$

Equivalently: given an object P in A, we have an exact sequence

$$0 \to \hom_A(P, K) \xrightarrow{s_\kappa} \hom_A(P, M) \xrightarrow{f_p} \hom_A(P, N)$$

is exact.

Analogously, a morphism $g: N \to C$ is called a cokernel of $f: M \to N$ if for any $P \in 0b\ \mathsf{A}$

$$0 \to \hom_A(C, P) \to \hom_A(N, P) \xrightarrow{f^\wedge} \hom_A(M, P)$$

is exact.

Note: A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is said to be exact at B if $\ker(g) = \mathrm{Im}(f)$.

1.2.5 Definition of an Exact Category

An exact category is a small additive category $\mathcal{C}$ (embeddable in an Abelian category $\mathcal{A}$) together with a family $\mathcal{E}$ of short exact sequences $0 \to C' \to C \to C'' \to 0$ (I) such that

(i) $\mathcal{E}$ is the class of sequences in $\mathcal{C}$ that are exact in $\mathcal{A}$

(ii) $\mathcal{C}$ is closed under extensions i.e., for any exact sequence $0 \to C' \to C \to C'' \to 0$ in $\mathcal{A}$ with C', C'' in $\mathcal{C}$, we also have $C \in \mathcal{C}$.

Before giving a construction of K_n ($\mathcal{C}$) $n \geq 0$, we give some relevant examples of $\mathcal{C}$ and develop notations for K_n ($\mathcal{C}$).

1.2.6 Examples

1. An Abelian category is an exact category when it is considered together with a family of short exact sequences.

2. Let A be any ring with identity $C = \mathbf{P}(A)$ (resp. $\mathbf{M}(A)$) the category of finitely generated projective (resp. finitely generated) A-modules. Write $K_n(A)$ for $K_n(\mathbf{P}(A)$ and $G_n\ (A)$ for $K_n\ (\mathbf{M}(A))$ For $n \geq 0$, e.g.,

   (i) $A =$, , , .

   (ii) $A =$ integral domain, R.
   $A = F$ (a field, - could be quotient field of R)
   $A = D$ (a division ring)

   (iii) G any discrete group (could be finite)
   $A =$ G, RG, G, G, G (in the notation of (i) or (ii).
   - These are group-rings.

   (iv) G a finite group, ZG is an example of a Z-order in the semi-simple algebra QG.

(v) **Definition**

Let R be a Dedekind domain with quotient field F (e.g., $R = Z$ (resp. $\dot{Z}_p$), $F = Q$(resp $\dot{Q}_p$)

p a rational prime or more generally $\hat{R}_{\underline{p}}, \dot{F}_{\underline{p}}$ ($\underline{p}$ a prime ideal of R). An R-order Λ in semi-simple F-algebra Σ is a subring of Σ such that R is contained in the centre of Λ, Λ is a finitely generated R-module and

$$F \otimes_R \Lambda = \Sigma, \text{ (E.g., } \Lambda = ZG, \dot{Z}_p G, RG, \dot{R}_{\underline{p}} G \text{ G a finite group).}$$

(vi) Let A be a ring (with 1), α: $A \to A$ an automorphism of A, $A_\alpha(T) =$

A_α (t, t^{-1}) := α-twisted Laurent series ring over A (i.e., Additively $A_\alpha[T] = A[T]$, with multiplication given by (at^i) $\cdot$ (bt^i) $= a\ \alpha^{-1}(b)\ t^{i+j}$ for a, $b \in A$). Let $A_\alpha[t]$ be the subring of $A_\alpha(T)$ generated by A and t.

Note: If $\Lambda = RG$, $\Lambda_{\alpha}[T] = RV$ where $V = G \underset{\alpha}{\times} | T$ is a virtually infinite cyclic group and G is a finite group, α an automorphism of G and the action of the infinite cyclic group

$T = \langle t \rangle$ on G is given by $\alpha(g) = tgt^{-1}$ for all $g \in G$.

(3) X a compact topological space, $F =$ or , $\mathrm{Vect}_F(X) :=$ category of finite dimensional vector bundles on X. (See [2]). Write $K_n^F(X)$ for K_n (Vect_F (X).

Theorem (**Swan**): There exists an equivalence of categories $\mathrm{Vect}_{C}(X) \approx P(\quad X)$ where $\quad X$ is the ring of complex-valued functions on X. Hence

$$K^0(X) := K_n(\mathrm{Vect}_F(X) \approx K_n(\quad (\quad X)) = K_n(C(X)) \qquad \text{(I)}$$

Note: (I) gives the first connection between topological and Algebraic K-theory. (See [7])

Gelford-Naimark theorem says that any unital commutative C^*-algebra A has the form $A \approx C(X)$ for some compact space X. If A is a non-commulative $\mathrm{C}^{\wedge}$-algebra, then K-theory of A leads to "non-commutative geometry" in the sense that A could be conceived as ring of functions on a "non-commutative or quantum" space. Note that any not necessarily unital commutative C-algebra A has the form $C_0(X)$ where X is a locally compact space and $X^+ = X \quad \{p_\infty\}$, the one point compactification of X. When X is compact $C_0(X) = C(X)$.

Note that $C_0(X) = \left\{\alpha : X^+ \rightarrow \mathbf{C} \,\middle|\, \alpha \text{ continuous and } \alpha\left(C_0\right) = 0\right\}$.

(4) Let X be a scheme (e.g., an affine or projective algebraic variety). (See [8] or below). Let $\mathbf{P}(X)$ be the category of locally free sheaves of O_X-modules. Write $K_n(X)$ for $K_n(\mathbf{P}(X)$. Let $\mathbf{M}(X)$ be the category of coherent sheaves of O_X-modules. Write $G_n(X)$ for $K_n(\mathbf{M}(X)$. Note that if $X =$ Spec (A), A commutative ring we recover $K_n(A)$ and $G_n(A)$.

Recall (Definition of Affine and Projective Varieties)

(a) Let K be an algebraically closed field (e.g., or algebraic closure of a finite field. Can regard polynomials in $A = A_n = K[t_1,\ldots,t_n]$ as functions $f : K^n \to K$. An algebraic set in $K^n = \{x \in K^n \text{ satisfying } f_i(x) = 0 \quad 1 \le i \le r, f_i \in A\}$.

- If $S \subset A, V(S) = \{x \in K^n \mid f(x) = 0 \ \forall \ f \in S\}$ define closed sets for a topology (Zariski topology) on the affine space K^n, also denoted $\boldsymbol{A}^n(K)$.

  Note that $(V(S_1) \quad V(S_2)) = V(S_1 S_2)$

$\bigcap_{i \subset I} V(S_i) = V(\quad S_j), \ V(A) = \phi, \ V(\phi) = K^n)$.

- Also if $E \ \subset K^n, I(E) = \{f \in A \mid f(x) = 0 \quad \forall \quad x \in E\}$ is an ideal in A.
- Let $X \subset K^n$ be an algebraic set. A function $\varphi : X \to K$ is said to be regular if $\varphi = f \big|_X$ for some $f \in A$.
- The regular functions on A form a K-algebra $K[X]$ and $K[X] \cong A / \underline{a}$ where $\underline{a} = I(X)$.
- Call $(X_1 K[X])$ an affine algebraic variety where $K[X] = O_X(X)$.

(b) Let $V \in \mathsf{P}(K)$, $P(V) =$ set of lines (i.e., 1-dim subspaces) of V. Write $P_n(K)$ for $P\left(K^n\right)$. Elements of $P_n(K)$ are classes of (n + 1)-tangles $\left[x_0, x_1, \ldots, x_n\right]$ where $\left[x_0, \ldots, x_n\right] \cong [\lambda x_0, \ldots, \lambda x_n]$ if $\lambda \neq 0$ in K.

- If $S \subset K\left[t_0, \ldots, t_n\right]$ is a set of homogeneous polynomials $V(S) = \left\{x \in P_n(K) \middle| f(x) = 0 \quad \forall \quad f \in S\right\}$. The V(S) are closed sets for Zariski topology on $P_n(K)$.
- A projective algebraic variety X is a closed subspace of $P_n(K)$ together with its induced structure sheaf $O_X = O_{P_n}|_X$.

(5) Let G be an algebraic group over a field F, (a closed subgroup of $GL_n(F)$) e.g., $SL_n(F), O_n(F)$ and X a G-scheme, i.e., there exists an action $\theta : G \underset{F}{\times} X \to X$. Let $\mathbf{M}$(G,X) be the category of G-modules M over X. (i.e., M is a coherent O_X-module together with an isomorphism of $O_{G \underset{F}{\times} X}$ -module θ *(M) = p_2^* (M), with $p_2 : G \underset{F}{\times} X \to X$; satisfying some co-cycle conditions) (see [83]). Write G_n (G, X) for $K_n(\mathsf{M}(G, X))$.

- Let P(G,X) be the full subcategory of $\mathbf{M}$(G,X) consisting of locally free O_X-modules. Write $K_n(G,X)$ for $K_n(P(G,X))$. (see [43]).

(6) Let $\widetilde{G}$ be a semi-simple, connected, and simply connected algebraic group over a field F. $\overline{T} \subset \widetilde{G}$ a maximal G-split torus of $\widetilde{G}$, $\widetilde{P} \subset \widetilde{G}$ a parabolic subgroup of $\widetilde{G}$ containing the torus $\widetilde{T}$.

The factor variety $\widetilde{G}/\widetilde{F}$ is smooth and projective. Call **F** $= \widetilde{G}/\widetilde{P}$ a flag variety.

E.g.,

$$\widetilde{G} = SL_n \quad \widetilde{P} = \left\{ \begin{pmatrix} \underline{a} & \underline{b} \\ 0 & \underline{c} \end{pmatrix} \middle| \det \underline{a} \det \underline{c} = 1 \quad \underline{a} \in GL_n \quad \underline{c} \in GL_{n-k} \right\}.$$

Then $\mathsf{F} = \widetilde{G}/\widetilde{P}$ is the Grassmanian variety of k-dimensional linear subspaces of an n-dimensional vector space. Write $K_n(G, \mathsf{F})$ for $K_n(P(G, \mathsf{F}))$. (See [43])

6. Let F be a field and B a separable F-algebra, X a smooth projective variety equipped with the action of an affine algebraic group G over F. Let $\mathbf{V}B_G$ (X_1B) be the category of vector bundles on X equipped with left B-module structure. Write $K_n(X, B)$ for $K_n(\mathbf{V}B_G(X,B))$. In particular, in the notation of (5), we write $K_n(\mathbf{F}, B)$ for $K_n(\mathbf{V}B_G(\mathbf{F}, B))$. (See [43])

7. Let G be a finite group, S a G-set. Let $\underline{S}$ be a category defined by $ob\ \underline{S}$ = {elements of S); $\underline{S}$ (s.,t) = $\{(g,s) | g \in G, g s = t\}$. Let C be an exact category. [$\underline{S}$, C] the category of functors $\xi : \underline{S} \to C$ Then [$\underline{S}$, C] is also an exact category where a sequence

$0 \to \xi' \to \xi \to \xi'' \to 0$ is said to be exact in [S, C] if

$0 \to \xi'(s) \to \xi(s) \to \xi''(s) \to 0$ is exact in C. Write

$K_n^G(\underline{S},\mathsf{C})$ for $K_n([\underline{S},\mathsf{C}])$.

E.g., $\mathsf{C} = \mathsf{M}(A)$, A a commutative ring,

$S = G/H$, then $[G/H, \mathsf{M}(A))] = \mathsf{M}(AH)$.

- $\left[G/H, \mathsf{P}(A)\right] = \mathsf{P}_A(AH) =$ category of finitely generated AH-modules that are projective over A. (i.e., AH lattices)

$K_n\left(G/H, \mathsf{M}(A)\right) := G_n(AH)$.

If A is regular, then $G_n(A,H) \cong G_n(AH)$. (See [25])

2. $K_0(\mathsf{C})$, C an Exact Category: Definitions and Examples

2.1 Define the Grathendieck group $K_0(\mathsf{C})$ of an exact category C as the Abelian group generated by isomorphism classes (C) of C-objects subject to the relations $(C')+(C'')=(C)$ wherever $0 \to C' \to C \to C'' \to 0$ is exact in C.

2.2 Remarks

(i) $K_0(\mathsf{C}) \cong \mathsf{F}/\mathsf{R}$ where F is the free Abelian group on the isomorphism classes (C) of C -objects and R is the subgroup generated by all $(C')+(C'')-(C)$ for each short exact sequence $0 \to C' \to C \to C'' \to 0$ in C. Denote by $[C]$ the class of (C) in $K_0(\mathsf{C})$.

(ii) The construction in 2.1 satisfies a universal property. If $\chi\ C \to A$ is a map from C to an Abelian group A, given that $\chi(C)$ depends only on the isomorphism class of C and $\chi(C'') + \chi(C') = \chi(C)$ for any exact sequence $0 \to C' \to C \to C'' \to 0$, then there exists, a unique homomorphism $\chi' : K_0(C) \to A$ such that $\chi(C) = \chi'(C)$ for any C-object C.

(i) Let $F : C \to \mathsf{D}$ be an exact functor between two exact categories C, D (i.e., F is additive and takes short exact sequences in C to short exact sequences in D). Then F induces a group homomorphism $K_0(C) \to K_0(\mathsf{D})$.

(ii) Note that an Abelian category A is also an exact category and the definition of $K_0(\mathsf{A})$ is the same as in definition 2.1.

(i) If C is an exact category in which every s.e.s $0 \to C' \to C \to C'' \to 0$ splits. E.g., $\mathsf{P}(A), \mathrm{Vect}_{\mathbf{C}}(X)$, then $K_0(\mathsf{C})$ is the Abelian group on isomorphism classes of C-objects with relation $(C') + (C'') = (C' \oplus C')$. In this case, $(\mathsf{C}, \oplus)$ is an example of a "symmetric monoidal category" with one property that the isomorphism classes of objects of C form an Abelian monoid and $K_0(\mathsf{C})$ is then the 'group completion' or 'Grathendiuk group' of such a monoid (see [42], Chapter 1, 1.2, 1.3). In fact, this construction generalizes standard procedure of constructing integers from the natural numbers.

2.3 Examples

(i) If A is a field or division ring or a local ring or a principal ideal domain, then $K_0(A) \cong \boldsymbol{Z}$. This follows from the fact that every $P \in \mathsf{P}(A)$ is free (i.e., $P \cong A^s$ for some s) and moreover, A satisfies the invariant bases property i.e., $A^r \cong A^s \Rightarrow r = s$.

(ii) Let A be a (left) Noetherian ring (i.e., every left ideal is finitely generated). Then the category ($\mathsf{M}(\Lambda)$ of finitely generated (left)-A-modules is an exact category and we denote $K_0(\mathsf{M}(\Lambda))$ by $G_0(\Lambda)$. The inclusion functor $\mathsf{P}(\Lambda) \to \mathsf{M}(\Lambda)$ induces a map $K_0(A) \to G_0(\Lambda)$ called the Cartan map. For example, $\Lambda = RG$ (R a Dedekind domain, G a finite group) yields a Cartan map $K_0(RG) \to G_0(RG)$.

If Λ is left Artinian i.e., the left ideals of Λ satisfy descending chain condition, then $G_0(\Lambda)$ is free Abelian on $[S_1],\ldots,[S_r]$ where the $[S_i]$ are distinct classes of simple Λ-modules, while $K_0(\Lambda)$ is free Abelian on $[I_1],\ldots,[I_t]$ and tho I_i are distinct classes of indecomposable projective Λ-modules (see [8]). So, the map $K_0(A) \to G_0(\Lambda)$ gives matrix $(a_{i\,j})$ where $a_{i\,j}=$ the number of times S_j occurs in a composition series for I_i. This matrix is known as the Cartan matrix.

If Λ is left regular (i.e., every finitely generated left Λ-module has finite resolution by finitely generated projective left Λ-modules), then it is well known that the Cartan map is an isomorphism.

(iii) Recall also that a maximal R-order Γ in Σ is an order that is not contained in any other R-order. Note that Γ is regular. So, as in (ii) above, we have Cartan maps $K_0(\Gamma) \to G_0(\Gamma)$ and when Γ is a maximal order, we have $K_0(\Gamma) \cong G_0(\Gamma)$.

(i) Let R be a commutative ring with identity. Λ an R-algebra. Let $\mathsf{P}_R(\Lambda)$ be the category of left Λ-modules that are finitely generated and projective as R-modules (i.e., Λ-lattices). Then $\mathsf{P}_R(\Lambda)$ is an exact category and we write $G_0(R,\Lambda)$ for $K_0\left(\mathsf{P}_R(A)\right)$. If $\Lambda = RG$, G a finite group, we write $\mathsf{P}_R(G)$ for $\mathsf{P}_R(RG)$ and also write $G_0(R,G)$ for $G_0(R,RG)$. If $M, N \in \mathsf{P}_R(\Lambda)$, then, so is $\left(M \otimes_R N\right)$, and hence the multiplication given in $G_0(R,G)$ by $[M][N] = \left(M \otimes_R N\right)$ makes $G_0(R,G)$ a commutative ring with identity.

(v) If R is a commutative regular ring and Λ is an R-algebra that is finitely generated and projective as an R-modules (e.g., $\Lambda = RG$, G a finite group or R is a Dedekind domain with quotient field F, and Λ is an R-order in a semi-simple F-algebra), then $G_0 R, \Lambda) \cong G_0(\Lambda)$

(i) Let F be a field, G a finite group. A representation of G in $P(F)$ is a group homomorphism $p : G \to \mathrm{Aut}(V) \quad V \in \mathsf{P}(F)$. Call V a representation space for ρ. The dimension of V over F is called the degree of ρ.

Note:

- ρ determines a G-action on V i.e., $G \times V \to V \quad (g, v) \to \rho(g)v = gv$ and vice versa.
- Two representations $(V_1 \rho)$ and $(V_1' \rho')$ are said to be equivalent if there exists an F-isomomorphism $\beta : V \cong V'$ such that $\rho'(g) = \beta\rho(g)$

- There exists, 1 – 1 correspondence between representations of P in $\mathsf{P}(F)$ and FG-modules.
- Can write a representation of G in $\mathsf{P}(F)$ as a pair $(V_1\rho)$. $V \in \mathsf{P}(F)$ and $\rho : G \to \mathrm{Aut}(V)$.
- If C is any category and G a group. A representation of G in C (or a G-object in C) is a pair (X,ρ) $X \in \mathrm{ob}\,\mathsf{C}$, $\rho : G \to \mathrm{Aut}(X)$ a group-homomorphism.

The G-objects in C forms a category C_G where for $(X,\rho),(X',\rho')$, $\mathrm{mor}_{\mathsf{C}_G}(X,\rho),(X',\rho')$ is the set of all C-morphisms $\alpha : X \to X'$ such that for each $g \in G$, the diagram

$$\begin{tikzcd} X \arrow[r, "\rho_g"] \arrow[d, "\alpha"] & X \arrow[d, "\alpha"] \\ X' \arrow[r, "\rho'_g"] & X' \end{tikzcd} \quad \text{commutes}$$

(vii) Let G be a finite group, S a G-set, $\underline{S}$ the category associated to S, $\mathcal{C}$ an exact category, $[\underline{S}, \mathsf{C}]$ the category of covariant functors $\varsigma : \underline{S} \to \mathsf{C}$. We write ς_s for $\varsigma(s), s \in S$. Then, $[\underline{S}, \mathsf{C}]$ is an exact category where the sequence $0 \to \varsigma' \to \varsigma \to \varsigma'' \to 0$ in $[\underline{S}, \mathsf{C}]$ is defined to be exact if $0 \to \varsigma'_s \to \varsigma_s \to \varsigma''_s \to 0$ is exact in $\mathcal{C}$ for all $s \in S$. Denote by $K_0^G(S, \mathsf{C})$ the K_0 of $[\underline{S}, \mathsf{C}]$. Then $K_0^G(-, \mathsf{C}) : G\,\mathsf{Set} \to \mathsf{A}b$ is a functor called 'Mackey' functor. We also note the fact that $K_n^G(-, C), n \geq 0$ is also a 'Mackey' functor. (See [42])

If $\underline{S} = \underline{G/G}$, then $[\underline{G/G}, \mathsf{C}] \cong \mathsf{C}_G$ analogous constructions to the one above can be done for G, a profinite group, and compact Lie groups (see [42], [28], [35]).

Now if R is a commutative Noetherian ring with identity, we have $\left[\underline{G/G}, \mathsf{P}(R)\right] \cong \mathsf{P}(R)_G \cong \mathsf{P}_R(RG)$, and so, $K_0^G\left(\underline{G/G}, P(R)\right) \cong G_0(R,G) \cong G_0(RG)$. This provides an initial connection between K-theory of the group ring RG and Representation theory. As observed in (iv) above $G_0(R,G)$ is also a ring.

In particular, when $R = \mathbf{C}$, $\mathsf{P}(\mathbf{C}) = \mathsf{M}(\mathbf{C})$, and $K_0(\mathsf{P}(\mathbf{C})_G) \cong G_0(\mathbf{C}, G) = G_0(\mathbf{C}G)$ is the Abelian group of characters, $\chi : G \to \mathbf{C}$ (see [30]), as already observed in this paper.

If the exact category $\mathcal{C}$ has a pairing $\mathsf{C}\times\mathsf{C}\to\mathsf{C}$, which is naturally associative and commutative, and there exists $E\in\mathsf{C}$ such that $(E,M)=(M,E)=M$ for all $M\in\mathsf{C}$, then $K_0^G(-,\mathsf{C})$ is a Green functor and moreover, for all $n\geq 0$, $K_n^G(-,C)$ is a module over $K_0^G(-,C)$. (See [42])

2.4 K_0 of Schemes

(i) More Examples of Abelian Categories: Functor Categories and Sheaves

- Let $\mathbf{B}$ be a small category i.e., (*ob* $\mathbf{B}$ is a set), $\mathcal{A}$ an Abelian category. Then the category of functors $\mathbf{B}\to\mathcal{A}$ is also an Abelian category denoted by $\mathcal{A}^{\mathbf{B}}$.
  Note: $ob\,\mathcal{A}^{B}=\{\text{functors}:\mathbf{B}\to\mathcal{A})$
  Morphisms are natural transformations of functors.

- ***Recall.*** Let C, D be two categories. A covariant (resp. contravarient) functor from C to D is an assignment to each object $C \in ob(\mathsf{C})$ an object $F(C)$ in D as well as an assignment to each morphism $f, C \to C'$, a D-morphism $F(f): F(C) \to F(C')$ (resp. $F(C') \to F(C)$) such that

1. $F(1_C) = 1_{F(C)}$ for any $C \in \mathsf{C}$;
2. $F(gf) = (F(g)F(f)$ (resp. $F(gf) = F(f)\,F(s)$.

Example:

1. R a commutative ring, $F : R$-$\mathbf{M}$od $\to$ -$\mathbf{M}$od given by $F = \mathrm{Hom}_R(-, N)$

   N fixed in R-$\mathbf{M}$od. F is contravariant $F' = \mathrm{Hom}_R(M-)$ is covariant.

In fact $\mathrm{Hom}_R(-,-)$ is a bifunctor

$$R\text{-}\mathsf{M}od \times R\text{-}\mathsf{M}od \to \ \text{-}\mathbf{M}\mathrm{od}$$

$$(M, N) \to \mathrm{Hom}_R(M, N)$$

covariant in N and contravarient in M.

2. F : (Groups) $\to$ -mod

$$G \to G/[G,G]$$

is covariant – called Abelianization functor.

- Let F, F' be two functors - from C to D. A natural transformation from F to F' is an assignment to an object $C \in \mathsf{C}$ a D-morphisms $\eta_C : F(C) \to F(C)$ such that if $\alpha : C \to C'$ is a C-morphism, then the diagram

$$\begin{tikzcd} FC \arrow[r, "\eta_C"] \arrow[d, "F(\alpha)"] & F'C \arrow[d, "F'(\alpha)"] \\ FC' \arrow[r, "\eta_{C'}"] & F'C' \end{tikzcd}$$

commutes

- **Note:** A functor (roughly speaking) is a ‘bridge’ for crossing from one category to another.

- Any partially ordered set $(E,\leq)$ has the structure of a category where

$$ob(E) = \text{elements of } E$$

$$\hom_E(x,y) = \phi \text{ unless } x \leq y.$$

- Let X be a topological space, U the poset of open subsets of X. A contravariant functor $F : \mathsf{U} \to \mathsf{A}$ (A an Abelian category) is called a *presheaf* on X.

  Note: The presheaves on X form an Abelian category denoted by

  Presh (X).

A sheaf on X is a presheaf F satisfying:

If $\{U_i\}$ is an open covering of a subset $U \subset X$, then we have an exact sequence:

$$0 \to F(U) \to \Pi F(U_i) \rightrightarrows \prod_{i<j} F(U_i \cap U_j)$$

(i.e., if $f_i \in F(U_i)$ are such that f_i and f_j agree on $F(U_i \cap U_j)$, then there exists, a unique $f \in F(U)$ that maps to every f_i under $F(U) \to F(U_i)$.

Note: **S**$h(X)$ is also an Abelian category. (See [93] or [18])

(i) A ringed space (X, O_X) is a topological space X together with a sheaf O_X of rings on X.

(ii) An O_X-module is a sheaf M together with a sheaf morphism $O_X \times M \to M$ s.t for each $U \subset X$, $M(U)$ is a unitary $O_X(U)$-module.

(ii) Let R be a commutative ring with identity Spec(R) = {prime ideals of R}

A subset $Y \subset \operatorname{Spec}(R)$ is closed off

$$Y = V(I) = \{\underline{p} \in \operatorname{Spec}(R) \,\big|\, \underline{p} \supset I\}\text{, } I \text{ an ideal of } R.$$

One could view R as the ring of functions on Spec (R) and $V[I]$ as the set of points $y \in \operatorname{Spec}(R)$ at which all the functions in I vanish. If $f \in R$ is viewed as a function on Spec (R), its value at $y \in \operatorname{Spec}(R)$ is its image in the residue class field $k(y) := $ the field of fractions of R/y.

- If $X = \operatorname{Spec}(R)$, there exists a sheaf of rings O_X on X where $O_X(U) = S^{-1}R$ and $S = \{f \in R \,|\, \forall\ y \in U,\ f \notin y\}$ $O_X(X) = R$. Call the ringed space $\operatorname{Spec}(X, O_X)$ an affine scheme.

(iii) A scheme is a topological space X together with a sheaf of rings on X such that $X = U_i$, $\left(U_i \text{ open in } X\right)$ and $\left(U_i, O_X|U_i\right)$ is an affine scheme.

A morphism of schemes $f, X \to Y$ is a continuous map of the underlying topological space together with (for each open set $U \subset Y$) a ring homomorphism $f_U^* : O_Y(U_i) \to O_X(f^{-1}U)$ compatible with the restriction maps for each $V \subset U$. In addition, we require hat for $x \in f^{-1}(U)$ $g \in O_Y(U)$, if g vanishes on $f(x)$, then $f^*(y) \in O_X\left(f^{-1}U\right)$ vanishes at x.

Note: Say that $f \in O_Y(U)$ vanishes at a point $y \in U$ if given any affine neighbourhood W of y, the image of f in $O_W(U \cap W)$ lies in the prime ideal corresponding to y.

Recall: $k[X] = k[t_1, \ldots, t_n] / \underline{a}_X$. View $f \in k[X]$ as a function on the set of points of X.

(iv) A scheme X over Z is a morphism of schemes $X \to Z$
Let $X_1, Y,$ be schemes over Z

$$\begin{array}{ccc} X \underset{Z}{\times} Y & \rightarrow & Y \\ \downarrow & & \downarrow g \\ X & \xrightarrow[f]{} & Z \end{array} \quad (I) \text{ pull back}$$

$X \underset{Z}{\times} Y$ is the fibre product in the category of schemes over Z given by the diagram (I).

$X \underset{Z}{\times} Y$ is the fibre product in the category of schemes over Z given by the diagram (I).

- If $\quad X = \mathrm{Spec}(A), \quad Y = \mathrm{Spec}(B) \quad Z = \mathrm{Spec}(C)$

  $X \underset{Z}{\times} Y = \mathrm{Spec}\left(A \underset{C}{\otimes} B\right)$

- Put $A_X^n = Spec\left(Z\left[t_1, \ldots, t_n\right]\right)$

  $A_X^n = A_Z \underset{\mathrm{Spec}(Z)}{\times} X$

(v) Let X be a scheme. Define an algebraic bundle on X as a morphism of schemes $\pi : E \to X$ together with maps

$$s : E \underset{X}{\times} E \to E$$

$$\mu : A_Z^1 \underset{\mathrm{Spec}(Z)}{\times} E \to E$$

(satisfying axioms similar to those of a topological vector bundles) together with local triviality: i.e., there exists an open covering $X = \quad U_{\alpha}$ of X together with isomorphism

$$E\big|_{U_{\alpha}} \cong \pi^{-1}\left(U_{\alpha}\right) \cong \boldsymbol{A}^{n}$$

Recall that a topological vector bundle E over X consists of continuous maps $\pi : E \to X$ and $\boldsymbol{C} \times E \to E$ (scalar multiplicator), $\rho : E \underset{X}{\times} E \to E$ (addition) satisfying

(1) for $\lambda \in \boldsymbol{C}$, $v \in E$, $\pi(\lambda \cdot (v)) = \pi(v)$,

(2)
$$\begin{aligned} \pi(\rho(v, w)) &= \pi(v) \\ &= \pi(w) \end{aligned}$$

(3) If $E_x = \pi'(x)$, $\mu : \boldsymbol{C} \times E_x \to E_x$, $\sigma_x : E_x \times E_x \to E_x$ makes E_x into a complex vector space.

(vi) It is usual to view a vector bundle $\pi : E \to X$ via its sheaf of sections $\mathsf{E}(U) = \{\text{maps } s\text{: } X \to E \text{ s.t } \pi \circ s = id\}$ i.e., $\boldsymbol{E}$ is required to be a locally free sheaf of O_X-modules i.e., there exists an open cover $X = \; U_\alpha$ such that

$$\mathsf{E}\big|_{U_\alpha} \cong A_{U_\alpha}^{n_\alpha} \text{ for each } n_\alpha \in \boldsymbol{N}.$$

A morphism of bundles is just an O_X-linear map $f : \mathsf{E} \to \mathsf{F}$ i.e., for each open set $U \subset X$ we have an $O_X(U)$-linear map of modules $f\big|_U$

$\mathsf{E}(U) \to \mathsf{F}(U)$ s.t for $V \subset U$, the map $\rho_{vu}\, f\big|_U = f\big|_V \rho_{VU}$.

(i) If X is a scheme. Define $K_0(X) := K_0(\mathsf{P}(X))$.

If E is a vector bundle, $\boldsymbol{E}$ a locally free sheaf with

$[E]=[\mathsf{E}]\in K_0(X)$

$[\mathsf{E}]\cdot[\mathsf{F}]=\left[\mathsf{E}\otimes_{O_X}\mathsf{F}\right]$ (product in $K_0(X)$ where $\left(\mathsf{E}\otimes\mathsf{F}\right)(U)=\mathsf{E}(U)\otimes_{O_X}\mathsf{F}(U)$).

So $K_0(X)$ is a commutative ring.

- If $f:X\to Y$ is a morphism of schemes, there exists an exact functor $f^*:\mathsf{P}(Y)\to\mathsf{P}(X):\mathsf{E}\to f^*\mathsf{E}$

Note: If $U\subset X,\quad V\subset Y$, are affine open sets with $f(U)\subset V$, then $f^*:K_0(Y)\to K_0(X)$

So K_0 is a contravarient functor (schemes) $\longrightarrow$ (commutative rings)

(iii) If X is a smooth projective curve over a field k, (see [18]) then

$$K_0(X) \approx Z \oplus \mathrm{Pic}(X)$$
$$[E] \to rk(E) \oplus \left(\Lambda^{rk(E)} E\right)$$

where Pic(X) = group of isomomorphism classes of line bundles (i.e., variant bundles of rank 1) over X.

(iv) $K_0\left(\boldsymbol{P}_k^n\right) \cong \boldsymbol{Z}^{n+1}$

(v) If X is a regular scheme (i.e., any coherent sheaf of O_X-modules has a finite global resolution by locally free sheaves) then $K_0(X) \cong G_0(X)$.

2.5 Some Topological *K*-theory

2.5.1 Let X be a compact space.

Recall: $K_{\mathbf{C}}^0(X) := K_0(\mathrm{Vect}_{\mathbf{C}}(X)) \cong K_0(\mathbf{C}X)$. $K_{\mathbf{C}}^0(X)$ is also written $K_{tor}^0(X)$ or $KU(X)$.

$$K_{\mathbf{R}}^0(X) := K_0(\mathrm{Vect}_{\mathbf{R}}(X)).$$

Write $KO(X)$ for $K_0(\mathrm{Vect}_{\mathbf{R}}(X))$.

Note: $K_{tor}^0(X)$ as a generalized cohomology theory arises as homotopy groups of spectra. We now introduce the notion of spectra.

2.5.2 An Ω-spectrum $\underline{E}$ is a set of pointed spaces $\{E^0, E^1, \cdots\}$ each of which has the homotopy type of a CW-complex such that each map $E^i \to \Omega(E^{i+1})$ is a homotopy equivalence i.e., we have a 'sequence of homotopy equivalences $E^0 \cong \Omega E^1 \cong \Omega^2 E^2 \cong \cdot \cong \Omega^n E^n$.

2.5.3 Theorem (see [2]).

Let $\underline{E}$ be an Ω-spectrum. For any topological space $A \subset X$, put $h^n_{\underline{E}}(X, A) = [(X, A), E^n]$ $n \geq 0$.

Then $(X, A) \to h^*_{\underline{E}}(X, A)$ is a generalized cohomology theory, namely, it satisfies all of the Eulenberg-Steenrod axioms except that its value at a point $(*, \phi)$ may not be that of ordinary cohomology.

So,

(1) $h_E^*(-)$ is a functor (Topological pairs) $\rightarrow$ (Graded Abelian groups),

(2) For each $n \geq 0$, and each pair (X, A) of spaces, there exists, a functorial connecting homomorphism

$$\alpha : h_{\underline{E}}^n(A) \rightarrow h_{\underline{E}}^{n+1}(X, A)$$

(3) The connecting homomorphisms in (2) determine long exact sequence for every pair (X, A).

(4) $h_{\underline{E}}^n(-)$ satisfies excision i.e., for every pair (X, A) and every subspace $U \subset A$ s.t. $\overline{U} \subset \operatorname{Int}(A)$

$h_{\underline{E}}^*(X, A) \cong h_{\underline{E}}^m(X - U,\ A - U)$

Note: Above, $h_{\underline{E}}^\infty(X) := h_{\underline{E}}^*(X, \phi) = h_{\underline{E}}^*(X_+, *)$ where X_+ is the disjoint union of X and a point *.

2.5.4 $KO^*_{ton}(-), K^*_{ton}(-) = KU(-)$ are the generalized cohomology theories associated to the Ω-spectrum given by $BO\times\mathbf{Z}$ and $BU\times\mathbf{Z}$

i.e.,

$$K^{2j}_{ton}(X)=\left[X_{i}\ BU\times\mathbf{Z}\right]$$

$$K^{2j-1}_{ton}(X)=[X,U];K.$$

2.5.5 Bott Periology

1. $BO\times\mathbf{Z} \sim \Omega^8(BO\times\mathbf{Z})$

Moreover, the homotopy groups $\pi_i(BO\times\mathbf{Z})\cong KO^i$ are given by $\mathbf{Z}, \mathbf{Z}/2, \mathbf{Z}/2, 0, \mathbf{Z}, 0, 0, 0$ respectively for $i\equiv 0,1,2,3,4,5,6,7 \pmod 8$

2. $BU \times Z \sim \Omega^2(BU \times \boldsymbol{Z})$ and $\pi_i(BU \times \boldsymbol{Z}) = \begin{cases} Z \text{ if } i \text{ is even} \\ 0 \text{ if } i \text{ is odd} \end{cases}$.

3. For any topological space X, and any $i \geq 0$, we have a natural homomorphism

$$\beta : K_{top}^{-1}(X) \to K_{top}^{-i-2}(X)$$

Note:

For $i \in Z$, $K_{top}^{i}(X) = \begin{cases} K_{top}^{0}(X) \text{ for } i \text{ even} \\ K_{top}^{-1}(X) \text{ for } i \text{ odd} \end{cases}$,

Let $S^0 = (*,*) = *$.

Then

$$K_{\text{top}}^{n}(*) = \begin{cases} \boldsymbol{Z} \text{ if } n \text{ is even} \\ 0 \text{ if } n \text{ is odd} \end{cases}$$

$$K_{\text{top}}^{i}\left(S^{n}\right) = \begin{cases} \boldsymbol{Z} \text{ if } n+n \text{ is even} \\ 0 \text{ if } n \text{ is odd} \end{cases}$$

2.6 *K*-theory of C^*-algebras

2.6.1 A C^*-algebra is a Banach algebra satisfying $|a*a| = |a|^2$ for all $a \in A$. Let A be a C^*-algebra. Define $K_i^{\text{ton}}(A) := \pi_i\left(BGL(A)\right) = \pi_{i-1}\left(GL(A)\right) \cdot (GLA)$ is a topological group).

Note: $K_0(A) = K_0(P(A)) \approx K_0^{ton}(A) = \pi_0(GL(A))$.

$K_i(A) := GL(A)/GL_0(A)$ where $GL_0(A)$ of the connected component of the identity in $GL(A)$ …

Bott periodicity is also satisfied i.e., $K_n^{ton}(A) = K_{n+2}(A) \quad \forall \quad n \geq 0$ and so, the theory is $\boldsymbol{Z}_2$-graded having only $K_0^{ton}(A) = K_0(A)$ and $K_1^{ton}(A)$.

2.6.2 Example

1. Let G be a discrete groups, $\ell^2(G)$ the Hilbert space of square summable complex-valued functions on G, i.e., any element of $f \in \ell^2(G)$ can be written as

$$f = \sum_{g \in G} \lambda_g \, g, \lambda_g \in \boldsymbol{C}, \sum_{g \in G} (\lambda_g)^2 < \infty .$$

The group algebra $\boldsymbol{C}\,G$ is a subspace of $\ell^2(G)$. There exists a left regular representation λ_G of G on the space $\ell^2(G)$ given by

$$\lambda_G(g)\left(\sum_{h\in G}\lambda_h h\right)=\sum_{g\in G}\lambda_G gh$$

where $g\in G$ and

$$\sum\lambda_h\, h\in \ell^2 G.$$

This unitary representation extends linearly to $\mathbf{C}\ G$.

Now define reduced C*-algebra $C_r^* G$ of G by the image of $\lambda_G(\mathbf{C}G)$ in the C*-algebra of bounded operators on $\ell^2(G)$.

- If G is finite, the $C_r^{\alpha}(G)=\mathbf{C}\ G$ and $K_0(\mathbf{C}\ G)=R(G)$ the additive groups of representation ring of G.

(i) $K_0(\mathbf{C})=\mathbf{Z}, K_1(\mathbf{C})=\pi_G GL(\mathbf{C})=0$ such that $GL(\mathbf{C})$ is connected.

(ii) $H\,G=\mathbf{Z}/2,\ K_0\left(C_r^*(G)\right)\cong K_0(\mathbf{C})\oplus K_0(\mathbf{C})\cong \mathbf{Z}\oplus\mathbf{Z}$ since $C_r^n G\cong\mathbf{C}\ G=\mathbf{C}\oplus C$.

2.7 Some Applications of $K_0(\mathcal{C})$

2.7.1 Geometric and Topological Invarients

Let $R = \mathbf{Z}\pi_1(X)$, the integral grouping of the fundamental group of a space of the homotopy type of a *CW*-complex.

Theorem (Wall) [87]

1. Let $C = (C_*, d)$ be a chain complex of projective *R*-modules that is homotopic to a chain complex of finite type of projective *R*-modules. Then $C = (C_*, d)$ is chain homotopic to a chain complex of finite type of free *R*-modules iff the Euler characteristics $\chi(C) = 0$ in $K_0(R)$.

Note: A bounded chain complex $C=\left(C_r,d\right)$ of R-modules is of finite type if all C_i are finitely generated. The Euler character of $C=\left(C_r,d\right)$ is given by $\chi(C)=\sum_{i=-\infty}^{\alpha}(-1)^r\left[C_i\right]\in K_0(R)$.

2. Computation of the group (SSP)

The calculation of $G_0(RG)$, G Abelian is connected to the calculation of the group (SSF) which houses obstructions constructed by Shub and Franks in their study of Morse-Smele diffeomorphisms.

3. Dynamical Systems

Dynamical systems can be classified by means of K_0 of C*-algebras.

2.7.2 Some other Miscellaneous Applications

1. Several classical problems in topology were solved via K-theory e.g., finding the number of independent vector fields on the n-space.

2. Index of Elliptic Operators

Let M be a closed manifold and $D: C^{\infty}(E) \to C^{\infty}(E)$ be an elliptic differential operator between two bundles E, F on M. Let $\widetilde{M} \to M$ be a normal covering of M with deck transformation group G (see [7]). Then, we can lift D to $\widetilde{M}$ and obtain an elliptic G-equivalent differential operators $\overline{D}: C^{\infty}\left(\widetilde{E}\right) \to C^{\infty}\left(\overline{F}\right)$ where $\overline{E}, \widetilde{F}$ are bundles on $\overline{M}$. Since the action is free, one can define an analytic index $\operatorname{ind}_G\left(\widetilde{D}\right)$ in $K_0\left(C_r^s G\right)$ (see [7]).

3. THE FUNCTORS K_1, K_2 - BRIEF REVIEW

We shall follow the historical development of the subject by briefly discussing k_1, K_2 of rings and their classical formulations.

3.1 K_1 of a Ring – Definition and Basic Properties

3.1.1 Let R be a ring with identity $GL_n(R)$ the group of invertible $n \times n$ matrices over R. Note that $GL_n(R) \subset GL_{n+1}(R)$ $A \to \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$.

Put $GL(R) = \varinjlim GL_n(R) = \bigcup_{n=1}^{\infty} GL_n(R)$.

Let $E_n(R)$ be the subgroup of $GL_n(R)$ generated by the elementary matrices, $e_{ij}(a)$ where

$e_{ij}(a)$ is the $n \times n$ matrix with 1's along the diagonal, a in the (i, j)-position with $i \neq j$ and zeros elsewhere. Put $E(R) = \varinjlim E_n(R)$.

3.1.2 Note that the matrices $e_{ij}(a)$ satisfy the following.

(i) $e_{ij}(a)\, e_{ij}(b) = e_{ij}(a+b) \quad \forall \quad a,b \in R$

(ii) $\left[e_{ij}(a), e_{jk}(b)\right] = e_{ik}(ab) \quad \forall \quad i \neq k,\ a,b \in R$

(iii) $\left[e_{ij}(a), e_{kl}(b)\right] = 1 \quad \forall \quad i \neq \ell,\ j \neq k$.

3.1.3 Whitehead Lemma

(i) $E(R) = [E(R), E(R)]$ i.e., $E(R)$ is perfect

(ii) $E(R) = [GL(R)], GL(R)]$.

3.1.4 Definition

$$K_1(R) := GL(R)\,/\,E(R) = GL(R)\,/[GL(R), GL(R)]$$

$$= H_1(GL(R))$$

3.1.5 Note that:

(i) K_1 is functorial in R i.e., $R \to R'$ is a ring homomorphism, we have $K_1(R) \to K_1(R')$

(ii) $K_1(R) \cong K_1(M_n(R))$ for any positive integer n and any ring R

(iii) $K_1(R) \cong K_1(\mathsf{P}(R))$.

3.1.6 If R is a commutative ring with identity, the determinant map $\det : GL_n(R) \to R^*$ commutes with $GL_n(R) \to GL_{n+1}(R)$ and hence defined a map $\det : GL(R) \to R^*$ which is surjective since given $a \in R^*$ there exists $A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ such that $\det(A) = a$.

- We also have an induced map

$$\det : GL(R)/[GL(R), GL(R)] \to R^*$$

i.e., $\det K_1(R) \to R^*$ that is split by a map

$$\alpha : R^{/*} \to K_1(R) : a \to \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

i.e., $\det \alpha = 1_R$. So $K_1(R) \cong R^* \oplus SK_1(R)$ where

$SK_1(R) := \ker(\det : K_1(R) \to R^*)$;

- Note that $SK_1(R) = SL(R)/E(R)$ where

$SL(R) = \varinjlim SL_n(R)$ and $SL_n(R) = \{x \in GL_n(R)/\det x = 1\}$.

3.1.6 Examples

(i) If R is a field $F, SK_1(F) = 0$ and $K_1(F) \cong F^*$

(ii) If R is a divisin ring $K_1(R) \cong R^*/[R^*, R^*]$.

3.1.7 Stability for K_1

Stability results are useful for reducing computations of $K_1(R)$ to computations of matrices of manageable size.

Definition: Let A be a ring with identity. An integer n is said to satisfy stable range condition (SR_n) for $GL(A)$ if whenever $r > n$, and $(a_1, a_2, \ldots, a_r)$ generates the unit ideal $\Sigma A a_i = A$, then there exists $b_1, b_2, \ldots, b_{r-1} \in A$ such that

$$(a_1 + a_r b, a_2 + a_r b_b, \cdots, a_{r-1} + a_r b_{r-1}) \text{ also}$$

generates the unit ideal i.e.,

$$\sum A(a_i + a_r b_i) = A$$

E.g., a semi-local ring (i.e., a ring with a finite number of maximal ideals satisfy SR_2).

3.1.8 Theorem

If SR_n is satisfied, then

(a) $GL_m(A)/E_m(A) \to GL(A)/E(A)$ is onto for $m \geq n$ and injective for all $m > n$.

(b) $E_m(A) \Delta GL_m(A)$ for $m \geq n_t 1$

(c) $GL_m(A)/E_m(A)$ is Abelian for $m > n$.

3.2 K_1, SK_1 of Orders and Group-rings

3.2.1 Let R be a Dedekind domain with quoted field F, Λ an R-order in a semi-simple F-algebra.

Put $SK_1(\Lambda) := \ker\left(K_1(\Lambda) \to K_1(\Sigma)\right)$.

Hence understanding $K_1(\Lambda)$ reduces to understanding $SK_1(\Lambda)$ and $K_1(\Sigma)$. Now $\Sigma = \Pi M_{n_i}\left(D_i\right)$. D_i a division ring.

- So $K_i(\Sigma) \cong \Pi K_1(D_i)$.
- One way of understanding $SK_1(\Lambda)$ is via reduced norm which generalizes the notion of determinant.

3.2.2 Let R be the ring of integers in a number field or p-adii field F. then there exists an extension E of F since that E is a splitting field of Σ i.e., $E \otimes_F \Sigma$ is a direct sum of metric algebras over E i.e.,

$$E \otimes_F \Sigma \cong \oplus M_{n_i}(E).$$

Let C be the centre of Σ.
If $a \in \Sigma, | \otimes a \in E \otimes_F \Sigma$ can be represented as a direct sum of matrices over E and so we have a map $nr : GL(\Sigma) \to C^*$.
If

$$\Sigma = \oplus \Sigma_i = \bigoplus_{i-1} M_{n_i}(E), \text{ and } C = \bigoplus_{i-1} C_i.$$

We could compute $nr(a)$ component-wise $v_{ia} GL(\Sigma_i) \to C_i$. Since C^n is Abelian, we have

$$nr : K_1(\Sigma) \to C^*.$$

- $SK_1(\Lambda) = \{x \in K_1(\Lambda) | nr(x) = 1\}$.

Hence we have access to $SK_1(RG)$ where G is any finite group.

3.2.3 Applications

1. **Whitehead Torsion**

J.H.C. Whitehead observed that if X is a topological space, with fundamental group $\pi_1(X) = G$, then the elementary row and column transformation of matrices over $\mathbb{Z}G$ have some topological meaning.

To enable him study homotopy between spaces, he introduce the group $\mathrm{Wh}(G) = K_1(\mathbf{Z}G)/w(\pm G)$ where w is the map $G \to GL_1(\mathbf{Z}G) \to GL(\mathbf{Z}G) \to K_1(\mathbf{Z}G)$ such that if $f : X \to Y$ is a homotopy equivalence, then there exists an invariant $\tau(f) \in \mathrm{Wh}(G)$ such that $\tau(f) = 0$ iff f is induced by elementary deformations transforming X to Y. The invariant $\tau(f)$ is called Whitehead torsion. (see [57])

- $K_1(\mathbf{Z}G) \cong (\pm 1) \times G^{ab} \times SK_1(\mathbf{Z}G)$ and so rank $K_1(\mathbf{Z}G) = \text{rank } \mathrm{Wh}(G)$ and $SK_1(\mathbf{Z}G)$ is the full torsion subgroup of Wh(G). So, computations of $\mathrm{Tor}\left(K_1(\mathbf{Z}G)\right)$ reduces to computation of $SK_1(\mathbf{Z}G)$.
  For information on computations of $SK_1(\mathbf{Z}G)$ (see [8], [60]).

3.3 K_2 of Rings and Fields

3.3.1 Let A be a ring with identity. The Stenberg group of order n $(n \geq 1)$ over A, denoted $St_n(A)$ is the group generated by $x_{lij}(a)$ $i \neq j,\ 1 \leq i,\ j \leq n,\ a \in A$, with relations

(i) $x_{ij}(a)\, x_{ij}(b) = x_{ij}(a+b)$

(ii) $\left[x_{ij}(a), x_{kl}(b)\right] = 1,\ \ j \neq k,\ \ i \neq \ell$

(iii) $\left[x_{ij}(a), x_{jk}(b)\right] = x_{ik}(ab),\ \ i, j, k$ distant

(iv) $\left[x_{ij}(a), x_{jk}(b)\right] = x_{ij}(-ba),\ \ j \neq k.$

Note: Since the generator $e_{ij}(a)$ of $E_n(A)$ satisfies relations (i) to (iv) above, we have a unique surjective homomorphism $\varphi_n : St_n(A) \to E_n(A)$ given by $\varphi_i\left(x_{ij}(a)\right) = e_{ij}(a)$.

Moreover the relations for $St_{n+l}(A)$ include those of $St_n(A)$ and so, there are maps $St_n(A) \to St_{n+l}(A)$. Then we have a conical map

$$St(A) \to E(A).$$

3.3.2 Define $K_2^M(A) := \ker \mathrm{St}(A) \to E(A)$.

3.3.3 Theorem: $K_2^M(A)$ is an Abelian group and is the centre of $St(A)$. Hence $St(A)$ is a central extension of $E(A)$.
i.e., we have a exact sequence
$1 \to K_2^M(A) \to St(A) \to E(A) \to 1.$

3.3.4 Definition: An exact sequence of groups of the form $1 \to A \to E \xrightarrow{\varphi} G \to 1$ is called a central extension of G by A if A is central in E. Write the extension as (E,φ). A central extension (E,φ) of G by A is said to be universal if for any other central extension (E',φ') of G, there is a unique morphism $(E,\varphi) \to (E',\varphi')$.

3.3.5 $St(A)$ is the universal central extension of $E(A)$. Hence there exists a natural isomorphism $K_2^M(A) \cong H_2(E(A), \boldsymbol{Z})$.

Note: The last statement follows from the fact that G (in this case, $E(A)$, the kernel of the universal central extension (E,φ) (in this case $(St(A),\varphi)$ is isomorphism to $H_2(G,\boldsymbol{Z})$ (in this case $H_2(E(A),\boldsymbol{Z})$.

3.3.6 Examples

(i) $K_2\boldsymbol{Z}$ is a cyclic group of order 2

(ii) $K_2(\boldsymbol{Z}(i)) = 1$, so is $K_2(\boldsymbol{Z}\sqrt{-7})$

(iii) $K_2(\boldsymbol{F}_q) = 1$ where $\boldsymbol{F}_q$ is a finite field with q elements

(i) If F is a field, $K_2(F[t]) \cong K_2(F)$ more generally $K_2(R[t]) \cong K_2(R)$ if R is a regular ring.

Note: $K_2^M(A) \cong K_2(\mathrm{P}(A)) = K_2(A)$.

3.3.7 Let A be a commutative ring with 1, $a \in A^{*}$. Put $x_{ij}(u)\, x_{ji}\left(-u^{-1}\right) x_{ij}(u)$.

Define $h_{ij}(u) = w_{ij}(u) w_{ij}(-)$.

For $u, v \in A^{r}$, one can easily check hat $\varphi([h_{12}(u), h_{13}(u)]) = 1$ and so, $[h_{12}(u), h_{13}(v)] \in K_2(A)$. One can also show that $[h_{12}(u), h_{13}(v)]$ is independent of $[h_{12}(u), h_{13}(v)]$ and call this the Stenberg symbol.

3.3.8 Theorem

Let A be a commutative ring with 1. The Stenberg symbol $\{\,,\}: A^{*} \times A \to K_2(A)$ is skew symmetric and bilinear i.e., $\{u, v\} = \{u, v\}^{-1}; \{u, u_2, v\} = \{u_1, v\}\, \{u_2, v\}$.

3.3.9 Theorem (Matsumoto)

Let F be a field. Then $K_2^M(F)$ is generated by $\{u,v),u,v\in F^*$ with relations

(i) $\left\{u\,u^1, v\right\} = \{u,v\}\{u^1, v\}$

(ii) $\left\{u, v\,v^1\right\} = \{u,v\}\{u, v^1\}$

(iii) $\left\{u, 1-u\right\} = 1$

i.e., $K_2^M(F)$ is the quotient of $F^* \otimes_{\mathbf{Z}} F^*$ by the subgroup generated by the elements $x \otimes (1-x),\, x \in F^*$.

3.4 Connections of K_2 with Brauer Groups of Fields and Galois Cohomology

3.4.1 Let F be a field and $Br(F)$ the Brauer group of F i.e., the group of stable isomorphism classes of central simple F-algebras with multiplication given by tensor product of algebras (see [7]).

A central simple F-algebra is said to be split by an extension E of F of $E \otimes A$ is E-isomorphic to $Mr(E)$ for some positive integer r.

It is well known that such E can be taken as some finite Galois extension of F.

Let $Br(F,E)$ be the group of stable isomorphism classes of E-split central simple F-algebras. Then $Br(F) := Br(F, F_s)$ where F_s is the separable closure of F.

3.4.3 For any $m > 0$, let μ_m be a group of m^{th} rods of 1, $G = \mathrm{Gil}(F_s)(F)$. Then we have a Kummer sequence of G-modules $0 \to \mu_m \to F_s^* \to 0$ from which we obtain an exact sequence of Galois cohomology groups

$$F^* \to F^* \to H^1(F, \mu_m) \to H^1(F, F_s^*) \to 0$$

where $H^1(F, F_s^*) = 0$ by Hilbert theorem 90 so, we obtain homomorphism

$$\chi_m : F^*/mF^* \cong F^* \otimes \boldsymbol{Z}/m \to H'(F, \mu_m).$$

Now, the composite

$$F^* \otimes_{\boldsymbol{Z}} F^* \to (F^* \otimes_{\boldsymbol{Z}} F^*) \otimes \boldsymbol{Z}/m \to H^1(F, \mu_m) \otimes H^1(F, \mu_m) \to H^2(F,$$

is given by $a \otimes b \to \chi_m(a) \quad \chi_m(b)$ (where is a cup product) which can be shown to be a Stanberg symbol inducing a homomorphism

$$g_{2,m} : K_2(F) \otimes \mathbf{Z} / m\mathbf{Z} \to H^2\left(F, \mu_m^{\otimes^2}\right) \tag{I}$$

we then have the following result

Theorem 3.4.4: Let F be a field, m an integer >0 such that the characteristic of F is prime to m. Then the map

$$g_{2,m} : K_2(F)/m\,K_2(P) \to H^2\left(F, \mu_m^{\otimes^2}\right)$$

is an isomorphism where $H^2\left(F, \mu_m^{\otimes^2}\right)$ can be identified with m torsion subgroup of $Br(F)$.

Remark 3.4.5: J. Milnov defined 'higher Milnov K-groups' $K_n^M(F)\,(n \geq 1)$ fields as follows:

Definition

$$K_n^M(F) := \underbrace{F^* \otimes F^* \otimes \cdots \otimes F^*}_{n \text{ times}} \Big/ \{a_1 \otimes \cdots \otimes a_n \mid a_i + a_j = 1 \text{ for some } i \neq j, a_i \in$$

i.e., $K_n^M(F)$ is the quotient of $F^* \otimes F^* \cdots F^*$ (n times) by the subgroup generated by all $a_1 \otimes a_2 \otimes \cdots \otimes a_n$, $a_i \in F$ such that

$$a_i + a_j = 1.$$

Note: $\oplus_{n>0}^{\infty} K_n^M(F)$ is a ring.

Remarks 3.4.6: By generalizing the process outlined in 3.4.3, we obtain a map,

$$g_{n,m} : K_n^m(F) \big/ {}_m K_n^m(F) \to H^n\left(F, \mu_m^{\otimes^n}\right),$$

- It is a conjecture of Bloch-Kato hat $g_{n.m}$ is an isomorphism for all F, m, n.
- Theorem 3.4.4 above due to A. Merkurjev and A. Suslin, is the $g_{z,m}$ case of Bloch-Kato conjecture when m is prime to the characteristic of F.
- A Merkurjev proved that theorem 3.4.4 holds without any restriction of F with respect to m.
- It is also a conjecture of Milnor that $g_{n,z}$ is an isomomorphism. In 1996, V. Voevodsky proved that $g_{n,2^r}$ is an isomorphism for any r, leading to his being awarded a Fields medal.
- It is now believed that M. Rost and V. Voeodsky have now proved the Bloch-Kato conjective.

3.5 Applications

1. K_2 and Pseudo-isotopy

Let $R = \mathbf{Z}G$, G a group. For $u \in R^*$ put $w_{ij}(u) := x_{ij}(u) x_{ji}\left(-u^{-1}\right) x_{ij}(u)$. Let W_{ij} be the subgroup of $St(R)$ generated by all $w_{ij}(g), \quad g \in G$.

Now, let M be a smooth n-dimensional compact connected manifold without boundary. Two diffeomorphisms h_0, h_1 of M are said to be isotopic of they lie in the same path component of the diffeomorphism group. Say that h_0, h_1 are pseudo-isotopic if there is a diffeormorphism of the cylinder $M \times [0,1]$ restricted to h_0 on $M \times (0)$ and to h_1 on $M \times \{1\}$. Let $P(M)$ be the pseudo-isotopy space of M, i.e., the group of diffeomorphism L of $M \times [0,1]$ restricting to the identity on $M \times (0)$. Computation of $\pi_0\left(P\left(M^2\right)\right)$ helps to understand the differences between isotopes to and we have the following result due to A. Hatcher and J. Wagover.

Theorem: Let *M be* an n-dimensional $(n \geq 3)$ smooth compact manifold with boundary. Then there exists a subjective map

$$\pi_0\left(P(M) \to Wh_2\left(\pi_1(M)\right)\right)$$

where $\pi_1(M)$ is the fundamental group of *M*.

4. HIGHER ALGEBRAIC *K*-THEORY

4.1 The Plus Construction for $K_n(A)$

4.1.1 The plus construction of K_n of a ring *A* with identity makes use of the following theorem of Quillen.

Theorem 4.1.2: Let *X* be a connected CW-complex *N* a perfect normal subgroup of $\pi_1(X)$. Then there exists a CW-complex X^+ (depending on *N*) and a map $X \to X^+$ such that

(i) $i : \pi_1(X) \to \pi_1\left(X^+\right)$ is the quotient map $\pi_1(X) \to \pi_1\ X/N = \pi_1\left(X^+\right)$

(i) For any $\pi_1(X)/N$-module L, there is an isomorphism $i_a : H_a\left(X, i^* L\right) \to H_i\left(X^+, L\right)$ where $i^* L$ is L considered as a $\pi_1(X)$-module.

(ii) The space X^+ is universal in the sense that if Y is a CW-complex and $f : X \to Y$ is a map such that $f_* : \pi_1(X) \to \pi_1(Y)$ such that $f_\alpha(N) = 0$ then there exists a unique map $f^+, X^+ \to Y$ such that $f^+ i = f$.

Definition 4.1.3

Let A be a ring, $X = BGL(A)$ the classifying space of the group $GL(A)$, (a CW-complex characterized by the property that $\pi_1 BGL(A) = GL(A)$ and $\pi_i BGL(A) = 0$ for $i \neq 1$). Then $\pi_i BGL(A) = GL(A)$ contains $E(A)$ as a perfect normal subgroup. Hence, by theorem 4.1.2, there exists a $BGL(A)^+$. Define $K_n(A) = \pi_n\left(BL(A)^+\right)$.

Example/Remarks 4.1.4

(i) For $n = 1,2, K_n(A)$ as defined above can be identified with the classical definition.

(ii) $\pi_1 BGL(A)^H = GL(A)/E(A) = K_1(A)$.

(iii) $BE(A)^+$ is the universal covering space of $BGL(A)^+$ and so, we have

$$\pi_2 BGL(A)^+ \cong \pi_2\left(BE(A)^+\right) \cong H_2\left(BE(A)^+\right) \cong H_2\left(BE(A)\right)$$

$$\cong H_2\left(E(A)\right) \cong K_2\left(A\right).$$

(iv) $K_3(A) \cong H_3\left(St(A)\right)$ (see [42])

(v) If A is a finite ring, $K_n(A)$ is finite see [31] or [42]

(vi) For a finite field $\boldsymbol{F}_q$ with q elements

$$K_{2n}\left(\boldsymbol{F}_q\right) = 0 \text{ and } K_{2n-1}\left(\boldsymbol{F}_q\right) = \boldsymbol{Z}/\left(q_{n-1}\right).$$

4.2 Classifying Spaces and Simplical Objects

4.2.1 Definition

Let Δ be a category defined as follows: $ob(\Delta) := \{\underline{n} = \{0 < 1 < \cdots < n\}\}$
$\mathrm{Hom}_{\Delta}(\underline{m}, \underline{n}) = \{\text{monotone maps } f, \underline{m} \to \underline{n} \text{ i.e., } f(i \leq f(j) \text{ for } i < j\}$.

4.2.2 For any category A, a simplical object in A is a contravariant functor.
$X : \Delta \to \mathsf{A}$. Write X_n for $X(\underline{n})$
A cosimplical object in A is a covariant functor $X : \Delta \to \mathsf{A}$.

- Equivalently, one could define a simplical object in a category A as a set of objects $X_n (n \geq 0)$ in A and a set of morphisms $\delta_i : X_n \to X_{n-1}$ $(0 \leq i \leq n)$ called face maps as well as a set morphisms $s_i : X_n \to X_{n+1}$ $(0 \leq j \leq n)$ called degeneracies satisfying certain simplical identities (see [93]).

- The geometric n-simplex is the topological space $\hat{\Delta}^n = \{(x_0, x_1, \ldots, x_n) \in R^{n+1} \mid 0 \leq x_i \leq | \ \forall i \text{ and } \Sigma x_i = 1\}$

A functor $\hat{\Delta} : \Lambda \to \text{spaces} : \underline{n} \to \hat{\Delta}^n$ is a co-simplical space..

4.2.4 Definition: Let X_n be a simplical scl. The geometric realization of X_n, written $|X_n|$ is defined by

$$|X_n| = X \underset{\Delta}{\times} \Delta = \underset{n\geq 0}{} \left(X_n \times \hat{\Delta}_n\right) \Big/ \cong$$

where the equivalence relations $\cong$ is generated by $(x, \varphi_n(y)) \cong (\varphi^n(x), y)$ for any $x \in X_n$ $y \in Y_n$ and $\varphi : \underline{m} \to \underline{n} \in \Delta$ and where $X_n \times \Delta^n$ is given the product topology and x_n is considered as a discrete space.

4.2.5 Definition

Now let A be a small category. The Nerve of A, written $N\mathsf{A}$, is the simplical set whose n-simplices are diagrams

$$A_n = \left\{A_0 \xrightarrow{f_1} A_1 \longrightarrow \cdots \xrightarrow{f_n} A_n\right\}$$

where the A_i's are A-objects and the f_i are A-morphisms. The classifying space of A is defined as $|N\mathsf{A}|$ and denoted by $B\mathsf{A}$.

Remarks: $B\mathsf{A}$ is a CW-complex whose n-cells are in one-one correspondence with the diagrams A_n above.

4.2.6 Definition

Now let C be an exact category. We form a new category $Q\mathsf{C}$ such that $ob(Q\mathsf{C}) = ob\,\mathsf{C}$ and morphisms from M to P, say is an isomorphism class of diagrams $M \xleftarrow{j} N \xrightarrow{i} P$ where i an admissible monomorphism (or inflation) and j is an admissible *epi* morphism or deflation) in C i.e., i and j are part of some exact sequences $0 \longrightarrow N \xrightarrow{i} P \longrightarrow P' \to 0$ and $0 \longrightarrow N'' \xrightarrow{i} N \xrightarrow{j} M \to 0$, respectively.

Composition is also well defined (see [62]).

Definition 4.2.7: For $n \geq 0$, define

$$K_n(C) := \pi_{n+1}\left(BQ\mathsf{C}, 0\right) \quad n \geq 0.$$

Examples: Recall earlier examples.

(A) (1) $\mathsf{C} = \mathsf{P}(A), \quad K_n(\mathsf{C}) := K_n(A) \quad n \geq 0$

$\mathsf{C} = \mathsf{M}(A), \quad K_n(\mathsf{C}) = G_n(A) \quad n \geq 0$

Note that $K_n(\mathsf{P}(A)) \cong \pi_n\left(BGL\left(A^+\right)\right)$ for $n \geq 1$

We shall be interested in various rings A.

(i) $A = \boldsymbol{Z}, \boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{C}$

(ii) A = Integral domain R

(iii) $A = F$ (field possibly quotient field of R)

(iv) $A = \mathrm{D}$ a dunsion ring

(v) $A = \boldsymbol{Z}G, RG, \boldsymbol{Q}G, \boldsymbol{R}G, \boldsymbol{C}G$ (a finite group)

(vi) R = integers in a number field or p-adii field, A = RG, G finite group or more generally A = r-order Λ in a semi-simple F-algebra Σ

(vii) $A=\Lambda_\alpha(T)$ where Λ is as in (vi) When $A=RG$, $A=\Lambda_\alpha(T)=RV$ where $V=G\underset{\alpha}{\times}T$ is virtually cyclic group.

4.3 Some Sample Finiteness Results for $K_n(\mathcal{C})$- $(\mathcal{C}=\mathcal{P}(A), \mathcal{M}(A))$

4.3.1 Theorem

Let R be the ring of integers in a number field F, Λ any R-order in a semi-simple F-algebra Σ. Then,

(i) For all $n\geq 1, K_n(\Lambda), G_n(\Lambda)$ are finitely generated Abelian group
(Kuku, J. algebra 1984, AMS contemp. Math, 1986).

(ii) For all $n\geq 1, K_{2n}(\Lambda), G_{2n}(\Lambda)$ are finite Abelian groups, Kuku (K-theory 2005).

(iii) If F is totally real, then $G_{2m+2}(\Lambda)$ is also finite for all odd $m\geq 1$
(Algebras and Rep. Theory - to appear)

(i) For all $n \geq 1, G_{2n}\left(\Lambda_\alpha(T)\right)$ is a finitely generated Abelian group where $\Lambda_\alpha(T)$ is the twisted Laurent series ring over Λ. **(Kuku (2007): Algebras and Rep theory - to appear)**

(ii) There exists isomorphism $Q \otimes K_n\left(\Lambda_\alpha(T)\right) \approx Q \otimes G_n\left(\Lambda_\alpha(T)\right) \cong Q \otimes K_n\left(\sum_\alpha(T)\right) \; \forall \; n \geq$ **(Kuku (2007): Algebras and Rep. theory - to appear)**

(iii) If A is a finite ring, then $K_n(A)$, $G_n(A)$ are finite for all $n \geq 1$ **(Kuku AMS Cont. Mp. Math 1986).**

Note: Above results (i), (ii), (iii) apply to $\Lambda = RG$ (G a finite group) while (iv) and (v) apply to $\Lambda_\alpha(T) = (RG)_\alpha(T) = RV$ where $V = G \underset{\alpha}{\times} T$ is a virtually infinite cyclic group. (i) generalizes classical results known for n = 0,1 to higher dimensions.

4.3.2 K_n, SK_n of Orders and Group rings

Let R be a Dederkind domain (i.e., an integral domain in which every ideal is projective or equivalently R is Noetherian integrally closed and every prime ideal is maximal or equivalently every non-zero ideal $\underline{a}$ in R is invertible i.e., $\underline{a}\,\underline{a}^{-1} = R$ where $\underline{a}^{-1} = \{x \in F \mid x\underline{a} \subset R\}$. Let Λ be any R-order in a semi-simple F-algebra Σ. For $n \geq 0$, let $SK_n(\Lambda) := \ker\left(K_n(\Lambda)\right) \to K_n(\Sigma)$ and $SG_n(\Lambda) = \ker(G_n(\Lambda)) \to G_n(\Sigma) \cong K_n(\Sigma)$.

Note that for any regular ring R (e.g., Σ), $K_n(R) \cong G_n(R)$.

As observed earlier, when $\Lambda = RG$ (R integers in a number field, G a finite group), $SK_0(RG)$ $SK_1(RG)$ contain topological invariants – respectively, e.g., Swan in variants and Whitehead torsion). We have the following:

4.3.3 Theorem: (see Kuku Math. Zeit (1979) or Ku-Bk (2007).

Let p be a rational prime. F a p-adii field with ring of integers R, Γ a maximal R-order in a semi-simple F-algebra Σ, Then for all $n \geq 1$.

(a) $SK_{2n}(\Gamma) = 0$

(b) $SK_{2n-1}(\Gamma) = 0$ iff Σ is unified over its centre i.e., iff Σ is a direct product of matrix algebras over fields.

Note: Above result applies to $\Gamma = RG$ where $(|G|, p) = 1$.

4.3.4 Theorem: See Ku-Bk (2007) or Kuku (1984) J-algebra; Kuku (1986)
AMS Cont. Math; Kuku (2006) K-theory

Let R be the ring of integers in a number field F, Λ any R-order in a semi-simple F-algebra Σ. Then

(a) $SK_n(\Lambda), SG_n(\Lambda)$ are finite groups and $SG_{2n}(\Lambda) = 0$ for all $n \geq 1$

(b) $SK_n\left(\hat{\Lambda}_p\right)$, $SG_n\left(\hat{\Lambda}_p\right)$ are finite groups and

(c) If $\Lambda = \boldsymbol{Z}G$ where G is a finite p-group, then $SK_{2n-1}(\boldsymbol{Z}G)$, and $SK_{2n-1}\left(\hat{\boldsymbol{Z}}_{\boldsymbol{p}}G\right)$ are finite p-groups.

4.4 Higher Dimensional Class Groups of Orders and Group rings

Let R be the ring of integers in a number field F, Λ any R-order in a semi-simple F-algebra Σ. The higher class groups $Cl_n(\Lambda)$ of Λ are defined for all $n \geq 0$ by $C\ln(\Lambda) := \ker\left(SK_n(\Lambda)\right) \to \oplus SK_n\left(\hat{\Lambda}_1\right)$. Note that $Cl_n(\Lambda)$ coincides with the usual class group $Cl(\Lambda)$ of Λ which in turn generalizes the notion of class groups of integers in a number field. (see Ku-Bk (2007). For results on class groups of Λ (see Curtis/Reiner (1987) [8]).

Note also that computations of $Cl_1(\Lambda)$ which we already observed reduces to computation of Whitehead torsion (see Oliver (1988) [60]).

We now state known results for $Cl_n(\Lambda)\ n \geq 1$.

4.4.2 Theorem

Let R be the ring of integers in a number field F, Λ any R-order in a semi-simple F-algebra Σ. Then

(i) For all $n \geq 1$, $Cl_n(\Lambda)$ is a finite group **(see Ku-Bk (2007) or Kuku (1986) AMS Cont. Math.)**

(ii) For all $n \geq 1$, p-torsion in $Cl_{2n-1}(\Lambda)$ can occur only for primes p lying above prime ideals $\underline{p}$ at which $\hat{\Lambda}_{\underline{p}}$ is not maximal. Hence for any finite group G, for all $n \geq 1$, the only p-torsion possible in $Cl_{2n-1}(RG)$ is for those primes p dividing the order of G. **(see Kolster/Laubenbacher (1988) Math. Zeit).**

(iii) Let F be a number field with ring of integers R, Λ a hereditary R-order in a semi-simple F-algebra or and Eichler order in a quatermon algebra. Then the only p-torsion possible is for those primes p lying below the prime ideals $\underline{p}$ at which $\Lambda_{\underline{p}}$ is not maximal. **(see Ku-Bk (2007) or Guo/Kuku (2005) Comm. in Alg.).**

(i) Let S_n be a symmetric group of degree n. Then $Cl_{2n-1}(\mathbf{Z}S_2)$ is a finite z-torsion group **(see Kolster /Lauben bacher (1998) Math. Zeit).**

4.5 Higher *K*-theory of Schemes

4.5.1 Recall: If X is a scheme, we write $K_n(X)$ for $K_n(\mathcal{P}(X))$ and when X is a Neotherian scheme, we write $G_n(X)$ for $K_n(\mathcal{M}(X))$.

If G is an algebraic group over a field F, and X is a G-scheme, we write $K_n(G,X)$ for $K_n(\mathcal{P}(G,X))$ are $G_n(G,X)$ for $K_n(\mathcal{M}(G,X))$.

Note:

(a) If G is trival group $G_n(G,X)=G_n(X)$ and $K_n(G,X)=K_n(X)$.

(a) $G_n(G,-)$ is contravariant with respect to G-maps.

(b) $G_n(G,-)$ is covariant with respect to projective G-maps.

(c) $K_n(G,-)$ is contravariant with respect to any G-map.

(d) $G_n(-,X)$ is contravariant w.r.t. any group homomorphism.

(e) $K_n(-,X)$ is covariant w.r.t group homomorphisms. (see Thomason (1987) *K*-theory Proc. Princeton.

4.5.2 Recall: Let B be a finite dimensional separate F-algebra. X a smooth projective variety equipped with the action of an affine algebraic group G over F, γX the twisted form of X with respect to a cocycle $\gamma : \mathrm{Gal}\, F_{\mathrm{sep}}/F \to G(F_{\mathrm{sep}})$. Let $\boldsymbol{VB}_G(\boldsymbol{r}, \boldsymbol{B})$ be the category of vector bundle on γX equipped with left B-module structure. We write $K_n(\gamma X, B)$ for $K_n(VB_G(\gamma X, B))$. **(See Panin (1994) K-theory; Merurjer (preprint).**

We now have the following results.

4.5.3 Theorem: Kuku (2007) MPIM – Bonn, preprint

Let $\widetilde{G}$ be a semi-simple simply, connected and connected F-split algebraic group over a field F, $\widetilde{P}$ a parabolic subgroup of G, $\mathsf{F} = \widetilde{G}/\widetilde{P}$ the flag variety and $\gamma\mathsf{F}$ the twisted form of **F**, B a finite-dimensional separable F-algebra.

(a) Let F be a number field, then for all $n \geq 1$

(i) $K_{2n+1}(\gamma\mathsf{F}, B)$ is a finitely generated Abelian group;

(ii) $K_{2n}(\gamma\mathsf{F}, B)$ is a torsion group and has no non-trivial dunsible subgroups.

(b) Let F be a p-adii field, ℓ a rational prime such that $\ell \neq p$. Then for all $n \geq 1$ and any separate F-algebra B, $K_n(\gamma\mathsf{F}, B)_\ell$ is a finite group.

4.5.4 Theorem: (Kuku (2007) MPIM-Bonn (preprint))

Let V be a Brauer-Severi variety over a field F.

(a) If F is a number field, then $K_{2n+1}(V)$ is a finitely generated Abelian group for all $n \geq 1$.

(b) If F is a p-adii field, then for all $n \geq 1$, $K_n(V)_\ell$ is a finite group if ℓ is a prime $\neq p$.

4.6 Mod-*m* Higher K-theory of exact Categories, Schemes and Orders

4.6.1 Let X be an H-space, m a positive integer M_m^n an n-dimensional mod-m Moore space is the space obtained from S^{n-1} by attaching an n-cell via a map of degree m, **(See Ku-Bk (2007)** or **Niesendorfer 1980/ AMS Memoir).**

-). Write

$$\pi_n(X,\mathbf{Z}/m) \text{ for } [M_m^n,\ X] \quad n \geq 2$$

$$\pi_1(X,\mathbf{Z}/m) \text{ for } \pi_1(X)\otimes\mathbf{Z}/m\,.$$

The cofibration sequence

$$S^{n-1} \xrightarrow{m} S^{n-1} \xrightarrow{\beta} M_m^n \xrightarrow{\alpha} S^n \xrightarrow{m} S^n$$

yields an exact sequence

$$\pi_n(X) \xrightarrow{m} \pi_n(X) \xrightarrow{\beta} \pi_n(X,Z/m) \xrightarrow{\alpha} \pi_{n-1}(X) \xrightarrow{m} \pi_{n-1}(X)$$

and hence the following exact sequence

$$0 \to \pi_\ell(X)/m \to \pi_n(X,Z/m) \to \pi_{n-1}(X)[m] \to 0$$

where

$$\pi_{n-1}(X)[m] = \{x \in \pi_{n-1}(X) \mid mx = 0\}\,.$$

Example 4.6.2

(i) If $\mathcal{C}$ is an exact category, write $K_n(\mathcal{C},\mathbf{Z}/m)$ for $\pi_{n+1}(BQ\mathcal{C},\mathbf{Z}/m)$; $n \geq 1$ and write $K_0(\mathcal{C},\mathbf{Z}/m)$ for $K_0(\mathcal{C})\otimes\mathbf{Z}/m$.

(ii) If $\mathcal{C} = \mathcal{P}(A)$, a ring with 1, write $K_n(A,\mathbf{Z}/m)$ for $K_n(\mathcal{P}(A),\mathbf{Z}/m)$;

(iii) If X is a scheme, and $\mathsf{C} = \mathsf{P}(X)$, write $K_n(X, \mathbf{Z}/m)$ for $K_n(\mathsf{P}(X), \mathbf{Z}/m)$. Note that if $X = \mathrm{Spec}(A)$, A commutative, we recover $K_n(A, \mathbf{Z}/m)$.

(iv) Let A be a Noetherian ring. If $\mathsf{C} = \mathsf{M}(A)$, we write $G_n(A, \mathbf{Z}/m)$ for $K_n(\mathsf{M}(A), \mathbf{Z}/m)$.

(v) Let X be Noetherian scheme, $\mathsf{C} = \mathsf{M}(X)$. We write $G_n(X, \mathbf{Z}/m)$ for $K_n(\mathsf{M}(X), \mathbf{Z}/m)$. If $X = \mathrm{Spec}(A)$, we recover $G_n(A, \mathbf{Z}/m)$.

(vi) Let G be an Abelian group over a field F, X a G-scheme, $\mathsf{C} = \mathsf{M}(G, X)$. $G_n((G, X), \mathbf{Z}/m$ for $K_n(\mathsf{M}(G, X), \mathbf{Z}/m))$.

(vii) Let G be an algebraic group over a field F,X a G-scheme; $\mathsf{C} = \mathsf{P}(G, X)$. We write $K_n((G, X), \mathbf{Z}/m$ for $K_n(\mathsf{P}(G, X), \mathbf{Z}/m))$.

(viii) Let G be an algebraic group over a field F, X a G-scheme, B a finite dimensional separable F-algebra, ${}_rX$ the twisted form of X via a 1-cocycle r, $C = VB_G({}_rX, B)$. We write $K_n(({}_rX, B), \mathbf{Z}/m$ for $K_n(({}_rX, B), \mathbf{Z}/m))$.

4.6.2 Theorem: Kuku (2007) MPIM-Bonn Preprint

Let C, C' be exact categories and $f : \mathsf{C} \to \mathsf{C}'$ an exact factor which induces Abelian group homomorphism $f_0 : K_n(\mathsf{C}) \to K_n(\mathsf{C}')$ for each $n \geq 0$. Let ℓ be a rational prime

(a) Suppose that f_1 is injective (resp. surjective, resp. bijective), then so is $\bar{f}_1 : K_n(\mathsf{C}, \boldsymbol{Z}/m) \to K_n(\mathsf{C}', \boldsymbol{Z}/m)$;

(b) If f_α is split surjective (resp. split injective), then so is $\bar{f} : K_n(\mathsf{C}, \boldsymbol{Z}/m) \to K_n(\mathsf{C}', \boldsymbol{Z}/m)$.

4.7 Profinite Higher K-theory of Exact Categories, Schemes and Orders

4.7.1 Let C be an exact category, ℓ a rational prime, s a positive integer, put $M_{\ell^\infty}^{n+1} = \varinjlim M_{\ell^s}^{n+1}$. We define the profinite K-theory of C by $K_n^{pr}(\mathsf{C}, \hat{Z}_\ell) = [M_{\ell^\infty}^{n+1}, BQ\mathsf{C}]$. We also write $K_n(\mathsf{C}, \hat{\boldsymbol{Z}}_\ell)$ for $\varinjlim(\mathsf{C}, \boldsymbol{Z}/\ell^s)$.

Note: For all $n \geq 2$, we have an exact sequence

$$0 \to \varprojlim{}^1 K_{2n+1}(\mathsf{C}, \boldsymbol{Z}/\ell^s) \to K_n^{pr}(\mathsf{C}, \hat{\boldsymbol{Z}}_\ell) \to K_n(\mathsf{C}, \hat{\boldsymbol{Z}}_\ell) \to 0.$$

For more information on this construction, see Ku-Bk (2007), chapter 8 or [42].

Example 4.7.2

(i) Let $\mathsf{C} = \mathsf{P}(A)$, A a ring with 1. We write $K_n^{pr}\left(A, \hat{\boldsymbol{Z}}_\ell\right)$ for $K_n\left(\mathsf{P}(A), \hat{\boldsymbol{Z}}_\ell\right)$ and $K_n\left(\mathsf{P}(A), \hat{\boldsymbol{Z}}_\ell\right)$ for $K_n\left(\mathsf{P}(A), \hat{\boldsymbol{Z}}_\ell\right)$.

(ii) If X is a scheme and $\mathsf{C} = \mathsf{P}(X)$, we write $K_n^{pr}\left(X, \hat{\boldsymbol{Z}}_\ell\right)$ for $K_n^{pr}\left(\mathsf{P}(X), \hat{\boldsymbol{Z}}_\ell\right)$ and $K_n\left((X), \hat{\boldsymbol{Z}}_\ell\right)$ for $K_n\left(\mathsf{P}(X), \hat{\boldsymbol{Z}}_\ell\right)$.

(iii) Let $\mathsf{C} = \mathsf{M}(A)$, write $G_n^{pr}\left(A, \hat{\boldsymbol{Z}}_\ell\right)$ for $G_n^{pr}\left(\mathsf{M}(A), \hat{\boldsymbol{Z}}_\ell\right)$ and $G_n\left((A), \hat{\boldsymbol{Z}}_\ell\right)$ for $K_n\left(\mathsf{M}(A), \hat{\boldsymbol{Z}}_\ell\right)$.

(iv) If $\mathsf{C} = \mathsf{M}(X)$, X a scheme, write $G_n^{pr}\left(X, \hat{\boldsymbol{Z}}_\ell\right)$ for $K_n^{pr}\left(\mathsf{M}(X), \hat{\boldsymbol{Z}}_\ell\right)$ and $G_n\left(X, \hat{\boldsymbol{Z}}_\ell\right)$ for $K_n\left(\mathsf{M}(X), \hat{\boldsymbol{Z}}_\ell\right)$. If $X = \mathrm{Spec}(A)$ recover $G_n^{pr}\left(A, \hat{\boldsymbol{Z}}_\ell\right)$ and $G_n\left(A, \hat{\boldsymbol{Z}}_\ell\right)$.

(v) Let G be an algebraic group over a field F, X a G-scheme, $\mathsf{C} = \mathsf{M}(G,X)$. We write $G_n^{pr}\left((G,X),\hat{\boldsymbol{Z}}_\ell\right)$ for $G_n^{pr}\left(\mathsf{M}(G,X),\hat{\boldsymbol{Z}}_\ell\right)$.

(vi) Let G be an algebraic group over a field F, X a G-scheme, $\mathsf{C} = \mathsf{P}(G,X)$, we write $K_n^{pr}\left((G,X),\hat{\boldsymbol{Z}}_\ell\right)$ for $K_n^{pr}\left(\mathsf{P}(G,X),\hat{\boldsymbol{Z}}_\ell\right)$.

(vii) Let G be an algebraic group over a field F, X a G-scheme, γX the twisted form of X and B a finite-dimensional separable algebraic over F. If $\mathsf{C} = VB_G\left(({}_r X, B), \hat{\boldsymbol{Z}}_\ell\right)$, we write $K_n^{pr}\left(({}_r X,B),\hat{\boldsymbol{Z}}_\ell\right)$ for $K_n^{pr}\left(VB_G,({}_r X,B),\hat{\boldsymbol{Z}}_\ell\right)$

Theorem 4.7.3: Kuku (2007) MPIM –Bonn preprint

Let C,C' be exact categories and $f:\mathsf{C}\to\mathsf{C}'$ an exact factor which induces an Abelian group homomorphism $f_n, K_n(\mathsf{C})\to K_n(\mathsf{C}')$ for $n\geq 0$. Let ℓ be a rational prime, s a positive integer. Suppose that f_α is injective (resp. surjective; resp. bjective), then so is

$$f_\alpha : K_n^{pr}\left(\mathsf{C},\hat{\boldsymbol{Z}}_\ell\right)\to K_n^{pr}\left(\mathsf{C}',\hat{\boldsymbol{Z}}_\ell\right).$$

Theorem 4.7.4: Kuku (2007) MPIM-Bonn Preprint

Let F be a number field, $\widetilde{G}$ a semi-simple connected, simply connected split algebraic group over F, $\widetilde{P}$ a parabolic subgroup of $\widetilde{G}, \mathsf{F} = \widetilde{G}/\widetilde{P}$, γ a 1-cocycle : $\mathrm{Gal}(F_{\mathrm{sep}}/F) \to \widetilde{G}(F_{\mathrm{sep}})$, $\gamma\mathsf{F}$ the γ-twisted form of F, B a finite-dimensional separable F-algebra. Then for all $n \geq 1$,

(i) $K_{2n}^{pr}((\gamma\mathsf{F}, B), \hat{\boldsymbol{Z}}_{\ell})$ is an ℓ-complete Abelian group;

(ii) $\operatorname{div} K_n^{pr}((\mathsf{F}, B), \hat{Z}_{\ell}) = 0$.

Theorem 4.7.5: Kuku (2007 – MPIM-Bonn Preprint

Let p be a rational prime, F a p-adii field, $\widetilde{G}$ a semi-simple connected and simply connected split algebraic group over F, $\widetilde{P}$ a parabolic subgroup of $\widetilde{G}$, $\overline{\mathsf{F}} = \widetilde{G}/\widetilde{P}$ the flag variety, γ a 1-cocycle $\mathrm{Gal}(F_{\mathrm{sep}}/F) \to G(F_{\mathrm{sep}})$, $\gamma\mathsf{F}$ the γ-twisted form of $\mathbf{F}$, B a finite-dimensional separable F-algebra, ℓ a rational prime such that $\ell \neq p$. Then for all $n \geq 2$.

(i) $K_n^{pr}\left((\gamma\mathsf{F}, B)\,\hat{\boldsymbol{Z}}_\ell\right)$ is an ℓ-complete profinite Abelian group.

(ii) $K_n^{pr}\left((\gamma\mathsf{F}, B)\,\hat{\boldsymbol{Z}}_\ell\right) = K_n\left((\gamma\mathsf{F}, B)\,\hat{\boldsymbol{Z}}_\ell\right)$,

(iii) The map $\varphi : K_n(\gamma\mathsf{F}, B) \to K_n^{pr}\left((\gamma\mathsf{F}, B), \hat{\boldsymbol{Z}}_\ell\right)$ induces isomomorphiss

- $K_n(\gamma\mathsf{F}, B), [\ell] \cong K_n^{pr}\left((\gamma\mathsf{F}, B), \hat{\boldsymbol{Z}}_\ell\right), \left[\ell^s\right]$
- $K_n(\gamma\mathsf{F}, B), / \ell^s \cong K_n^{pr}\left((\gamma\mathsf{F}, B), \hat{\boldsymbol{Z}}_\ell\right) / \ell^s$.

(iv) Kernel and cokernel of $K_n({}_r\mathsf{F}, B) \to K_n^{pr}\left(({}_r\mathsf{F}, B), \hat{\boldsymbol{Z}}_\ell\right)$ are uniquely ℓ-divisible.

(v) div $K_n^{pr}\left(({}_r\mathsf{F}, B), \hat{\boldsymbol{Z}}_\ell\right) = 0$ for $n \geq 2$.

5. Equivariant Higher K-theory Together with Relative Generalizations

In this section, we exploit representation theoretic techniques (especially induction theory) to define and study equivarient higher K-theory and their relative generalizatins. Induction theory has always aimed at computing various invariants of a group G in terms of corresponding invariants of subgroups of G. For lack of time and space, we discuss here finite group actions and note that analogous results exist for pro-finite group and compact lie group actions (see Ku-Bk (2007) chapter 9 –13).

5.1 Equivariant Higher *K*-theory for Exact Categories for Finite Group Actions

5.1.1 Definition

Let $\mathcal{B}$ be a category with finite sums final object and finite pullbacks (and hence finite products) e.g., category G-set of (finite) G-$\mathcal{S}$ets, where G is a finite groups, $\mathcal{D}$ an Abelian category (e.g., R-$\mathbf{M}$od)

A pair of functors $\left(M_\alpha, M^\alpha\right) : \mathsf{B} \to \mathsf{D}$ is called a Marchey functor if

(i) $M_\alpha : \mathsf{B} \to \mathsf{D}$ is covariant, $M^* : \mathsf{B} \to \mathsf{D}$ contravariant and
$M_\alpha(X) = M^\alpha(X) = M(X) \quad \forall \quad X \in ob\,\mathsf{B}$.

(ii) For any pull-back diagram

$$\begin{tikzcd} A' \arrow[r, "p_2"] \arrow[d, "p_1"] & A_2 \arrow[d, "p_2"] \\ A_1 \arrow[r, "f_1"] & A \end{tikzcd}$$

in B, the diagram

$$\begin{tikzcd} M(A') \arrow[r, "f_{2*}"] & M(A_2) \\ M(A_1) \arrow[r, "f_{1\alpha}"] \arrow[u, "p_1^\alpha"] & M(A) \arrow[u, "p_2^+"] \end{tikzcd}$$

commutes

(iii) M^α transforms finite coproducts in B into finite products in D i.e., the embeddings $X_i \to \coprod_{i=1} X_i$ induces an isomomorphism

$$M\left(X_i \coprod X_2 \cdots \coprod X_n\right) \cong M\left(X_1\right) \times \cdots \times M\left(X_n\right).$$

5.1.2 Note that (ii) above is an axiomatization of the Mackey subgroup theorem in classical representation theory (Put $\mathcal{B}$ = G-$\mathcal{S}$et, $A_1 = G/H$; $A_2 = G/H'$ $G/H \times G/H'$ can be identified with the set $D(H, H') = \{HgH' | g \in G\}$ of double cosets of H and H' in G. (see [8] for a statement of Mackey subgroup theorem).

5.1.3 We shall concentrate on exact categories in this section but observe that analogous theories exist for symmetric monoidal and Wildhanser category **(see Ku-Bk (2007) chapters 9, 10, 13).**

So, let $\mathcal{C}$ be an exact category, S a G-set, G a finite group, $\underline{S}$ the translation category of S. Recall that the category $[\underline{S}, \mathrm{C}]$ of covariant functors from $\underline{S}$ to $\mathcal{C}$ is also an exact category where a sequence $0 \to S' \to S \to S'' \to 0$ in $[\underline{S}, \mathrm{C}]$ is said to be exact if $0 \to S'(S) \to S(S) \to S''(S) \to 0$ is exact in $\mathcal{C}$.

5.1.3 Definition

Let $K_n^G(S,\mathsf{C})$ be the n^{th} algebraic K-group associated with the exact category $[\underline{S},\mathsf{C}]$ with respect to fibre-wise exact sequences.

Theorem 5.1.4

$K_n^G(-,\mathsf{C}): \mathrm{G}\mathsf{Set} \to \boldsymbol{Z}\text{-}\mathsf{M}od$ is a Mackey functor.
(For proof see Ku-Bk (2007) or Dress/Kuku Comm. in Alg. (1981).

5.1.5 Note: We want to turn $K_n^G(-,\mathsf{C})$ into a 'Green' functor and see that for suitable category C, $K_n^G(-,\mathsf{C})$ is a module over $K_n^G(-,\mathsf{C})$. We first define these notions of 'Green' functor and modules over 'Green' functors.

5.1.6 Definition

A Green functor $\mathsf{G} : \mathsf{B} \to R\text{-}\mathsf{M}od$ is a Mackey functor together with a pairing $\mathsf{G} \times \mathsf{G} \to \mathsf{G}$ such that for any B-object X, the R-bilinear map $\mathsf{G}(X) \to \mathsf{G}(X)$ makes $\mathsf{G}(X)$ into an R-algebra with a unit $1 \in \mathsf{G}(X)$ such that for any morphism $f : X \to Y$, we have $f^*\left(1_{G(Y)}\right) = 1_{G(X)}$.

A left (resp. right) G-module is a Mackey functor $M : \mathsf{B} \to R\text{-}\mathsf{M}od$ together with a pairing $\mathsf{G} \times \mathsf{M} \to \mathsf{M}$ (resp. $\mathsf{M} \times \mathsf{G} \to \mathsf{M}$) such that for any B-object X, $\mathbf{M}(X)$ becomes a left (resp. right) unitary $\mathbf{G}(X)$-module we shall refer to left G-modules just as G-modules.

5.1.7 Definition

Let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ be exact categories. An exact pairing (,). $\mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_3$ given by $(X_1, X_2) \to (X_1 \circ X_2)$ is a covariant functor such that

$$\begin{aligned}
&\mathrm{Hom}[(X_1, X_2), (X_1', X_2')] \\
&= \mathrm{Hom}(X_1, X_1') \times \mathrm{Hom}(X_2, X_2') \to \mathrm{Hom}(X_1 \circ X_2), (X_1' \circ X_2')
\end{aligned}$$

is bi-additive and bi-exact **(see Ku-Bk (2007) or [87]).**

5.1.8 Theorem
(for Proof see Ku-Bk (200) or Dress/Kuku. Comm. in Alg.

(1981)

Let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ be exact categories and $\mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_3$ an exact pairing of exact categories, S a G-$\mathcal{S}$et. Then the pairing induces a pairing $[\underline{S}, \mathcal{C}_1] \times [\underline{S}, \mathcal{C}_2] \to [\underline{S}, \mathcal{C}_3]$ and hence a pairing $K_n^G(S, \mathcal{C}_1) \times K_n^G(S, \mathcal{C}_2) \to K_n^G(S, \mathcal{C}_3)$.

Suppose that C is an exact category such that the pairing $\mathsf{C} \times \mathsf{C} \to \mathsf{C}$ is naturally associative and commutative and there exists $E \in \mathsf{C}$ such that $[E \circ N] = [N \circ E] = [N] \quad \forall \quad N \in \mathsf{C}$. Then $K_n^G(-,\mathsf{C})$ is a Green functor and $K_n^G(-,\mathsf{C})$ is a unitary $K_n^G(-,\mathsf{C})$-module.

5.1.9 Definition/Remarks

If $M : G\mathcal{S}\mathrm{et} \to \boldsymbol{Z}\text{-}\mathbf{M}\mathrm{od}$ is any Mackey functor, X a G-set, define a Mackey functor $M_X : \mathrm{G}\mathcal{S}\mathrm{et} \to \boldsymbol{Z}\text{-}\mathsf{M}od$ by $M_X(Y) = M(X \times Y)$. The projection map $pr : X \times Y \to Y$ defines a natural transformation $\theta_X : M_X \to M$ where $\theta_X(Y) = pr_1 \;\; M(X \times Y \to M(Y))$. M is said to be X-projective if θ_X is split surjective i.e., there exists a national transformation $\varphi : M \to M_X$ such that $O_X \varphi = id_M$.

Now define a defect base D_M of M by $D_M = \{H \leq G | X^H \neq \phi\}$ where X is a G-set (called defect set of M) such that M is Y-projective iff there is a G-map f, $X \to Y$ **(See Ku-Bk (2007) Prop. 9.1.1).**

If M is a module over a Green functor G, then M is X-projective iff G is X-projective iff the induction map $\mathsf{G}(X) \to \mathsf{G}(G/G)$ is surjective **(see Ku-Bk. Theorem 9.3.1).**

- In general, proving induction results reduce to determining G-sets X for where $\mathsf{G}(X) \to \mathsf{G}(G/G)$ is surjective and this in turn reduces to computing D_G (see Ku-Bk 9.6.1).

Hence one could apply induction techniques to obtain results on higher K-groups $K_n^G(-,\mathsf{C})$ which are modules over Green functors $K_n^G(-,\mathsf{C})$.

5.2 Relative Equivalent Higher Algebraic k-theory

Definition 5.2.1 Let S, T be G-$\mathcal{S}$ets. Then the projection $S\times T \xrightarrow{\varphi} S$ gives rise to a functor $\underline{S\times T} \xrightarrow{\varphi} \underline{S}$. Suppose that C is an exact category. If $\varsigma \in [\underline{S},\mathsf{C}]$, we write ς' for $\varsigma\circ\varphi : \underline{S\times T} \xrightarrow{\varphi} \underline{S} \xrightarrow{\varsigma} \underline{\mathsf{C}}$. Then a sequence $\varsigma_1 \to \varsigma_2 \to \varsigma 3$ of functors in $[\underline{S},\mathsf{C}]$ is said to be T-exact if the sequence $\varsigma_1' \to \varsigma_2' \to \varsigma_3'$ of restricted functors $\underline{S}\times\underline{T} \xrightarrow{\varphi} \underline{S} \xrightarrow{\varsigma} \mathsf{C}$ is split exact. If $\varphi : S_2 \to S_1$ is a G-map, and $\varsigma_1 \to \varsigma_2 \to \varsigma 3$ is a T-exact sequence in $[\underline{S},\mathsf{C}]$, and we put $\hat{\varsigma}_i = \varphi\circ\varsigma_i$, then $\hat{\varsigma}_1 \to \hat{\varsigma}_2 \to \hat{\varsigma}_3$ is T-exact in $[\underline{S}_1,\mathsf{C}]$. Let $K_n^G(S,\mathsf{C},T)$ be the nth algebraic K-group associated to the exact category $[\underline{S},\mathsf{C}]$ with respect to T-exact sequence.

Remarks: The use of the restriction functors $\varsigma', \hat{\varsigma}$ in 5.2.1 constitute a special case of the following general situation. Let ς be an exact category and B, B' any small categories. We define exactress in $[\mathsf{B},\mathsf{C}]$ relative to some covariant functor $\delta : \mathsf{B}' \to \mathsf{B}$.

Thus a sequence $\varsigma_1 \to \varsigma_2 \to \varsigma_3$ of functors in $[\mathsf{B},\mathsf{C}]$ is said to be exact relative to $\delta:\mathsf{B}' \to \mathsf{B}$ if it is exact fibrewise and if the sequence $\varsigma_1' \to \varsigma_2' \to \varsigma_3'$ of restricted functors $\varsigma_1' := \varsigma_i \circ \delta' : \mathsf{B}' \xrightarrow{\delta} \mathsf{B} \xrightarrow{\varsigma} \mathsf{C}$ is split exact. Let $K_n^G(S,\mathsf{C},T)$ be the nth algebraic K-group associated to the exact category $[\mathsf{S},\mathsf{C}]$ w.r.t exact sequences.

5.2.3 Definition

Let S, T be G-$\mathcal{S}ets$. A functor $\varsigma \in [\underline{S},\mathsf{C}]$ is said to be T-projective if any T-exact sequence $\varsigma_1 \to \varsigma_2 \to \varsigma$ is exact. Let $[\underline{S},\mathsf{C}]_T$ be the additive category of T-projective functors in $[\underline{S},\mathsf{C}]$ considered as an exact category with respect to split exact sequences. Note that the restriction functor associated to $S_1 \xrightarrow{\psi} S_2$ carries T-projective functors $\varsigma \in \left[\underline{S}_2,\mathsf{C}\right]$ into T-projective functors $\varsigma \circ \psi \in \left[\underline{S}_2,\mathsf{C}\right]$. Define $P_n^G(S,C,T)$ as the nth algebraic K-group associated to the exact category $[\underline{S},\mathsf{C}]_T$, with respect to split exact sequences.

5.2.3 Theorem

$K_n^G(-,C,T)$ and $P_n^G(-,C,T)$ are Mackey functors from *GSet* to *Ab* for all $n \geq 0$. If the pairing $C \times C \to C$ is naturally associative and commutative and contains a natural unit, then $K_n^G(-,C,T): GSet \to Ab$ is a Green functor, and $K_n^G(-,C,T)$ and $P_n^G(-,C,T)$ are $K_0^G(-,C,T)$-modules.

Also, the induction functor $\psi_*:[\underline{S}_1,\mathsf{C}] \to [\underline{S}_2,\mathsf{C}]$ associated to $\psi: S_1 \to S_2$ preserves *T*-exact sequences and *T*-projective functors and hence induces homomorphism
$K_n^G(\psi,C,T)_*: K_n^G(S_1,C,T) \to K_n^G(S_2,C,T)$ and
$P_n^G(\psi,C,T)_*: P_n^G(S_1,C,T) \to P_n^G(S_2,C,T)$, thus making $K_n^G(-,C,T)$ and $P_n^G(S_1,C,T)$ covariant functors. Other properties of Mackey functors can be easily verified.

Observe that for any $G\mathsf{S}et\ T$, the pairing $[\underline{S}_1, \mathsf{C}] \times [\underline{S}_2, \mathsf{C}] \to [\underline{S}_3, \mathsf{C}]$ takes T-exact sequences into T-exact sequences, and so, if $[\underline{S}_i, \mathsf{C}],\ i = 1,2$ are considered as exact categories with respect to T-exact sequences, then we have a pairing $K_0^G(\underline{S}, C_1, T) \times K_n^G(\underline{S}, C_2, T) \to K_n^G(\underline{S}, C_3, T)$. Also if ς_3 is T-projective, so is $\langle \varsigma_1, \varsigma_2 \rangle$. Hence, if $[\underline{S}, \mathsf{C}_1]$ is considered as an exact category with respect to T-exact sequences, we have an induced pairing $K_0^G(\underline{S}, C_1, T) \times P_n^G(\underline{S}, C_2, T) \to P_n^G(\underline{S}, C_3, T)$. Now, if we put $C_1 = C_2 = C_3 = C$ such that the pairing $C \times C \to C$ is naturally associative and commutative and C has a natural unit, then, as in theorem 5.1.8 $K_0^G(-, C, T)$ is a Green functor and it is clear from the above that $K_n^G(-, C, T)$ and $P_n^G(-, C, T)$ are $K_0^G(-, C, T)$-modules.

5.2.4 Remarks

(i) In the notation of theorem 5.2.3, we have the following natural transformation of functors: $P_n^G(-,\mathsf{C},T) \to K_n^G(-,\mathsf{C},T) \to K_n^G(-,\mathsf{C})$, where T is any G-set, G a finite group, and C an exact category. Note that the first map is the 'Cartan' map.

(ii) If there exists a G-map $T_2 \to T_1$, we also have the following natural transformations $P_n^G(-,\mathsf{C},T_2) \to P_n^G(-,\mathsf{C},T_1)$ and $K_n^G(-,\mathsf{C},T_1) \to K_n^G(-,\mathsf{C},T_2)$ since, in this case, any T_1-exact sequence is T_2-exact.

5.3 Interpretation in Terms of Group-rings

In this subsection, we discuss how to interpret the theories in previous sections in terms of group-rings.

5.3.1 Recall that any G-set S can be written as a finite as a finite disjoint union of transitive G-sets, each of which is isomorphic to a quotient set G/H for some subgroup H of G. Since Mackey functors, by definition, take finite disjoint unions into finite direct sums, it will be enough to consider exact categories $[G/H, \mathsf{C}]$ where C is an exact category.

For any ring A, let $\mathsf{M}(A)$ be the category of finitely generated A-modules and $\mathsf{P}(A)$ the category of finitely generated projective A-modules. Recall from … that if G is a finite group, H a subgroup of G, A a commutative ring, then there exists and equivalence of exact categories $[\underline{G/H}, \mathsf{M}(A)] \to \mathsf{M}(AH)$. Under this experience, $[\underline{G/H}, \mathsf{P}(A)]$ is identified with the category of finitely generated A-projective left AH-modules, i.e., $[G/H, \mathsf{P}(A)] \cong \mathsf{P}_A(AH)$.

We now observe that a sequence of functors $\varsigma_1 \to \varsigma_2 \to \varsigma_3 \in [G/H, \mathsf{M}(A)]$ or $[G/H, \mathsf{P}(A)]$ is exact if the corresponding sequence $\varsigma_1(H) \to \varsigma_2(H) \to \varsigma_3(H)$ of AH-modules is exact.

Remarks 5.3.2

(i) It follows that for every $n \geq 0, K_n^G[G/H, \mathsf{P}(A)]$ can be identified with the nth algebraic K-group of the category of finitely generated A-projective AH-modules while $K_n^G[G/H, \mathsf{P}(A)] = G_n(AH)$ if A is Noetherian. It is well known that $K_n^G[G/H, \mathsf{P}(A)] = K_n^G[G/H, \mathsf{M}(A)]$ is an isomorphism when A is regular.

(i) Let $\varphi : G/H_1 \to G/H_2$ be a G-map for $H_1 \leq H_2 \leq G$. We may restrict ourselves to the case $H_2 = G$, and so, we have $\varphi^* \left[\underline{G/G}, \mathsf{M}(A)\right] \to \left[\underline{G/H}, \mathsf{M}(A)\right]$ corresponding to the restriction functor $\mathsf{M}(AG) \to \mathsf{M}(AH)$, while $\varphi_* : \left[\underline{G/H}, \mathsf{M}(A)\right] \to \left[\underline{G/G}, \mathsf{M}(A)\right]$ corresponds to the induction functor $\mathsf{M}(AH) \to \mathsf{M}(AG)$ given by $N \to AG \otimes_{AN} N$. Similar situations hold for functor categories involving $\mathsf{P}(A)$. So, we have corresponding restriction and induction homomorphisms for the respective K-groups.

(ii) If $\mathsf{C} = \mathsf{P}(A)$ and A is commutative, then the tensor product defines a naturally associative and commutative pairing $\mathsf{P}(A) \times \mathsf{P}(A) \to \mathsf{P}(A)$ with a natural unit, and so, $K_n^G(-, \mathsf{P}(A))$ are $K_0^G(-, \mathsf{P}(A))$-modules.

5.3.3 We now interpret the relative situation. So let T be a G-set. Note that a sequence $\varsigma_1 \to \varsigma_2 \to \varsigma_3$ of functors in $\left[\underline{G/H}, \mathsf{M}(A)\right]$ or $\left[\underline{G/H}, \mathsf{P}(A)\right]$ is said to be T-exact if $\varsigma_1(H) \to \varsigma_2(H) \to \varsigma_3(H)$ is AH'-split exac for all $H' \leq H$ such that $T^{H'} \neq \varnothing$ where $T^{H'} \to \left\{t \in T' \middle| \; gt = t \quad \forall\, g \in H'\right\}$. In particular, the sequence of G/H-exact (resp. G/G-exact) if an only if the corresponding sequence of AH-modules (resp. A/G-modules) is split exact. If ε is the trivial subgroup of G, it is G/ε-exact if it is split exact as a sequence of A-modules.

So, $K_n^G\left(G/H, \mathsf{P}(A), T\right)$ (resp. $K_n^G\left(G/H, \mathsf{M}(A), T\right)$ is the nth algebraic K-group of the category of finitely generated A-projective AH-modules (resp. category of finitely generated AH-modules) with respect to exact sequences that split when restricted to the various subgroups H' of H such that $T^{H'} \neq \varnothing$ with respect to exact sequences. In particular, $K_n^G\left(G/H, \mathsf{P}(A), G/\varepsilon\right) = K_n(AH)$. If A is commutative, then $K_n^G\left(-, \mathsf{P}(A), T\right)$ is a Green functor, and $K_n^G\left(-, \mathsf{P}(A), T\right)$ and $P_n^G\left(-, \mathsf{P}(A), T\right)$ are $K_0^G\left(-, \mathsf{P}(A), T\right)$-modules.

Now, let us interpret the map, associated to G-maps $S_1 \to S_2$. We may specialize to maps $\varphi : G/H_1 \to G/H_2$ for $H_1 \le H_2 \le G$, and for convenience we may restrict ourselves to the case $H_2 = G$, which we write $H_1 = H$. In this case, $\varphi^* : \left[\underline{G/G}, \mathsf{M}(A)\right] \to \left[\underline{G/H}, \mathsf{M}(A)\right]$ corresponds to the restriction of AG-modules to AH-modules, and $\varphi_* : \left[\underline{G/H}, \mathsf{M}(A)\right]$ corresponds to the induction of AH-modules to AG-modules.

Since any G-set S can be written as a disjoint union of transitive G-sets isomorphic to some coset-set G/H, and since all the above K-functors satisfy the additiveity condition, the above identification extend to K-groups, defined on an arbitrary G-set S.

5.4 Some Applications

5.4.1 We are now in position to draw various conclusions just by quoting well-established induction theorems concerning $K_0^G(-,\mathsf{P}(A))$ and $K_0^G(-,\mathsf{P}(A),T)$, and more generally $R\otimes_Z K_0^G(-,\mathsf{P}(A))$ and $R\otimes_Z K_0^G(-,\mathsf{P}(A),T)$ for R, a subring of Q, or just any commutative ring (see …) Since any exact sequence in $\mathsf{P}(A)$ is split exact, we have a canonical identification $K_0^G(-,\mathsf{P}(A),T)=K_0^G(-,\mathsf{P}(A),G/\varepsilon)$ (ε the trivial subgroup of G) and thus may direct our attention to the relative case only.

So, let T be a G-set. For p a prime and q a prime or 0, let $\mathsf{D}(p,T,q)$ denote the set of subgroups $H\le G$ such that the smallest normal subgroup H_1 of H with a q-factor group has a normal Sylow-subgroup H_2 with $T^{H_2}\neq\varnothing$ and a cyclic factor group H_1/H_2. Let H_q denote the set of subgroups $H\le G$, which are q-hyperelementary, i.e., have a cyclic normal subgroup with a q-factor group (or are cyclic for $q=0$).

For A and R being commutative rings, let $\mathsf{D}(A,T,R)$ denote the union of all $\mathsf{D}(p,T,q)$ with $pA \neq A$ and $qR \neq R$, and let H_R denote the set of all H_q with $qR \neq R$. Then , it has been proved (see [11], [44]) $R \otimes_Z K_0^G\left(-,\mathsf{P}(A),T\right)$ is S-projective for some G-set S if $S^H \neq \varnothing \quad \forall\, H \in \mathsf{D}(A,T,R) \quad H_R$. Moreover, if A is a field of characteristic $p \neq 0$, then $K_0^G\left(-,\mathsf{P}(A),T\right)$ is S-projective already if $S^H \neq \varnothing \quad \forall\, H \in \mathsf{D}(A,T,R)$. (Also see Ku-Bk).

5.4.2 Among the many possible applications of these results, we discuss just one special case. Let $A = k$ be a field of characteristic $p \neq 0$, let $R = \mathbf{Z}\left(\frac{1}{p}\right)$, and let $S = \cup_{H \in \mathsf{D}(k,T,R)} G/H$. Then, $R \otimes_Z K_n^G\left(-, \mathsf{P}(k), T\right)$ are S-projective. Moreover, the Cartan map $K_n^G\left(-, \mathsf{P}(k), T\right) \to K_n^G\left(-, \mathsf{P}(k), T\right)$ is an isomorphism for any G-set S for which the Sylow-p-subgroups H of the stabilizers of the elements in X have a non-empty fixed point set $T^H \in T$, since in this case T-exact sequences over X are split exact and thus all functors $\varsigma : \underline{X} \to \mathsf{P}(k)$ are T-projective, i.e., $\left[X, \mathsf{P}(k)\right]_\tau \quad \left[X, \mathsf{P}(k)\right]$ is an isomomorphism if $\left[X, \mathsf{P}(A)\right]$ is taken to be exact with respect to T-exact and thus split exact sequences. This implies in particular that for G-sets X, the Cartan map

$$P_n^G\left(X \times S, \mathsf{P}(k), T\right) \to K_n^G\left(X \times S, \mathsf{P}(k), T\right)$$

is an isomorphism since any stabilizer group of an element in $X \times S$ is a subgroup of a stabilizer group of an element in S, and thus, by the very definition of S and $D\left(k, T, \mathbf{Z}\left(\frac{1}{p}\right)\right)$, has a Sylow-$p$-subgroup H with $T^H \neq \varnothing$. This finally implies that $P_n^G(-, \mathsf{P}(k), T)s \to K_n^G(-, \mathsf{P}(k), T)s$ is an isomorphism. So, by the general theory of Mackey funcors,

$$\mathbf{Z}\left(\frac{1}{p}\right) \otimes P_n^G(-, \mathsf{P}(k)T) \to \mathbf{Z}\left(\frac{1}{p}\right) \otimes K_0^G(-, \mathsf{P}(k)T)$$

is an isomorphism. The special case $(T = G/\varepsilon)$ $P_n^G(-, \mathsf{P}(k), G/\varepsilon)$, just the K-theory of finitely generated projective kG-modules and $K_n^G(-, \mathsf{P}(k), G/\varepsilon)$ the K-theory of finitely generated kG-modules with respect to exact sequences. Thus we have proved the following.

Theorem 5.4.3

Let k be a filed of characteristics p, G a finite group. Then, for all $n \geq 0$, the Cartan map $K_n(kG) \to G_n(kG)$ induces isomorphisms

$$\mathbf{Z}\left(\frac{1}{p}\right) \otimes K_n(kG) \to \mathbf{Z}\left(\frac{1}{p}\right) \otimes G_n(kG).$$

Here are some applications of theorem 5.4.3. These applications are due to A.O. Kuku (see [42]).

Theorem 5.4.4

Let p be a rational prime, k a field of characteristic p, G a finite group. Then for all $n \geq 1$.

(i) $K_{2n}(kG)$ is a finite p-group.

(ii) The Cartan homomorphism $\varphi_{2n-1} : K_{2n-1}(kG) \to G_{2n-1}(kG)$ is surjective, and $\ker \varphi_{2n-1}$ is the Sylow-p-subgroup of $K_{2n-1}(kG)$.

Corollary 5.4.5

Let k be a field of characteristic p, C a finite $E1$ category. Then, for all $n \geq 0$, the Cartan homomorphism $K_n(k\mathsf{C}) \to G_n(k\mathsf{C})$ induces isomorphism

$$\mathbf{Z}\left(\frac{1}{p}\right) \otimes K_n(k\mathsf{C}) \cong \mathbf{Z}\left(\frac{1}{p}\right) \otimes G_n(k\mathsf{C}).$$

Corollary 5.4.6

Let R be the ring of integers in a number field F, m a prime ideal of R lying over a rational prime p. then for all, $n \geq 1$,

(a) the Cartan map $K_n((R/m)\mathsf{C}) \to G_n((R/m)\mathsf{C})$ is surjective;

(b) $K_{2n}((R/m)\mathsf{C})$ is a finite p-group

Finally, with the identification of Mackey functors: $G\mathsf{S}et \to \mathsf{A}b$ with Green's G-functors $\underline{\delta}G \to \mathsf{A}b$ as in [42] and above interpretations of our equivariant theory in terms of grouprings, we now have, from the forgoing, the following result, which says that higher algebraic K-groups are hyperelementary computable. First, we define this concept.

Definition 5.4.7

Let G be a finite group, U a collection of subgroups of G closed under subgroups and isomorphic images, A a commutative ring with identity. Then a Mackey functor $M : \delta G \to A\text{-}\mathsf{M}\mathrm{od}$ is said to be U-compatible if the restriction maps $M(G) \to \prod_{H \in \mathsf{U}} M(H)$ induces an isomorphism $M(G) \cong \varprojlim_{H \in \mathsf{U}} M(H)$ where $\varprojlim_{H \in \mathsf{U}}$ is the subgroup of all $(x) \in \prod_{H \in \mathsf{U}} M(H)$ such that for any $H, H' \in \mathsf{U}$ and $g \in G$ with $gH'g^{-} \subseteq H$, $\varphi : H' \to H$ given by $h \to ghg^{-1}$, then $M(\varphi)(x_H) = x_H$.

Now, if A is a commutative ring with identity, $M:\delta G \to \mathbf{Z}\text{-}\mathsf{M}\mathrm{od}$ a Mackey functor, then $A \otimes M(H)$. Now, let P be a set of rational primes, $\mathbf{Z}_{\mathsf{P}} = Z\left[\frac{1}{q} \;\middle|\; q \notin \mathsf{P}\right], C(G)$ the collection of all cyclic subgroups of $G, h_P C(G)$ the collection of all P-hyperelementary subgroups of G, i.e.,

$$h_P C(G) = \{H \le G \mid \exists H' \le H, H' \in (G), H/H' \text{ a } p\text{-group for some } p \in \mathsf{F}$$

Then we have the following theorem,

Theorem 5.4.7

Let R be a Dedekind ring, G a finite group, M any of the Green modules $K_n(k-1), G_n(k-1)\ SK_n(k-1),\ SG_n(R-1),\ Cl_n(R-1)$ over $G_0(R-1)$ then $\mathbf{Z}_{\mathsf{P}} \otimes M$ is $h_P(C(G))$-computable.

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