NOTES ON the HYDRAULICS OF HYDROPOWER PLANTS

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HYDROPOWER PLANTS: basic references

Main reference about hydropower topics:

- *Guide on how to develop a small hydropower plant, ESHA, 2004*
  (but MANY ERRORS IN THE EQUATIONS !!!)

Some genaral reference regarding hydraulics

- *F.H. White, "Fluid Mechanics", MacGraw-Hill Inc. USA*
- *V.L. Streeter and E.B. Wylie, Hydraulic Transients, McGraw-Hill Book Co., New York 1967*
- *V.T. Chow, Open Channel Hydraulics, McGraw-Hill Book Co., New York 1959*

Journals on the topic

- *Water Power and Dam construction*
- *Power Technology and Engineering (formerly Hydrotechnical Construction), Springer, DOI: 10.1007/BF02406938*

Organization

- *ESHA*
- *UNIPEDE (International Union of Producers and Distributors of Electricity). Pahes*

Web pages with interesting papers

- *http://web.me.com/bryanleyland/Site_3/Welcome.html*
- *Lessons learned from the design, construction and operation of hydroelectric facilities, ASCE - HYDROPOWER COMMITTEE - New-York, 1994*

Terminologia

Head (+ loss; gross +)	Carico, prevalenza
run of river plant	Impianto ad acqua fluente
dam	diga
weir	Traversa, stramazzo, soglia
intake	Opera di presa
power house	Centrale
forebay	Camera/vasca di carico
outlet	scarico
tailrace	canale di scarico
Trashrack (+ cleaner, +screen)	Griglia per l’intercettazione dei detriti galleggianti
generator	Generatore elettrico
turbine	turbina
pump	pump
multistage pump	Pompa multistadio
fishladder (fishpass)	scala a pesci
Flat gate	Paratoia piana
Sector gate	Paratoia a settore
Ecological flow	DMV – deflusso minimo vitale
spillway	sfioratore
flow-duration curve	curva di durata delle portate

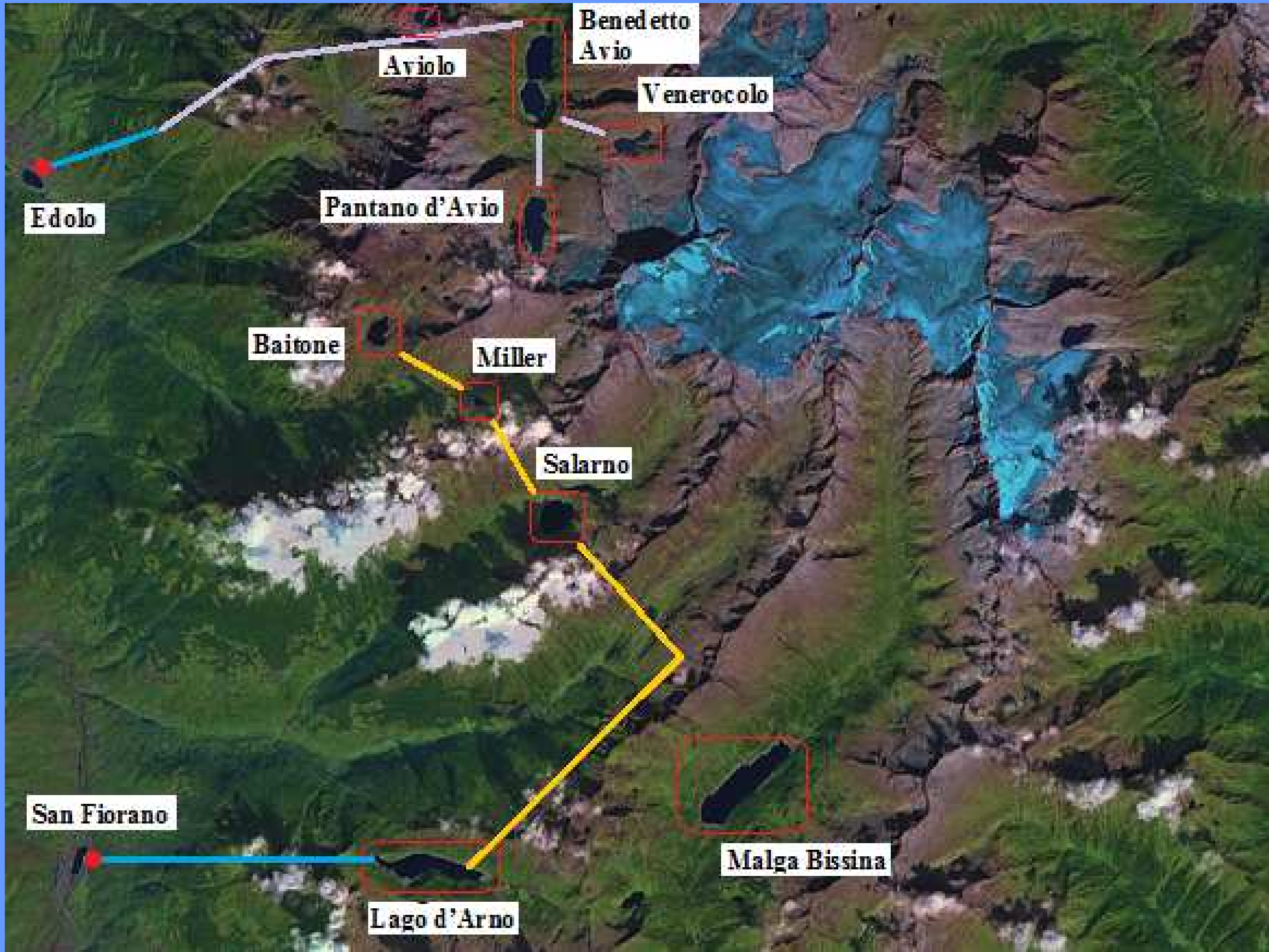
Terminologia

topographical survey	Rilievo topografico
Environmental Impact Assessment	Valutazione di Impatto ambientale
roughness	scabrezza
pipe	tubo
friction head loss	perdite di carico distribuite
Local head losses (entrances, bends, elbows, joints, racks, valves, sudden contractions or enlargements)	perdite di carico localizzate
spillway	sfioratore
Drop intake	Presa a trappola
Gross/Net head	Salto disponibile/Utile
Residual, reserved or compensation flow	Deflusso Minimo Vitale (DMV)

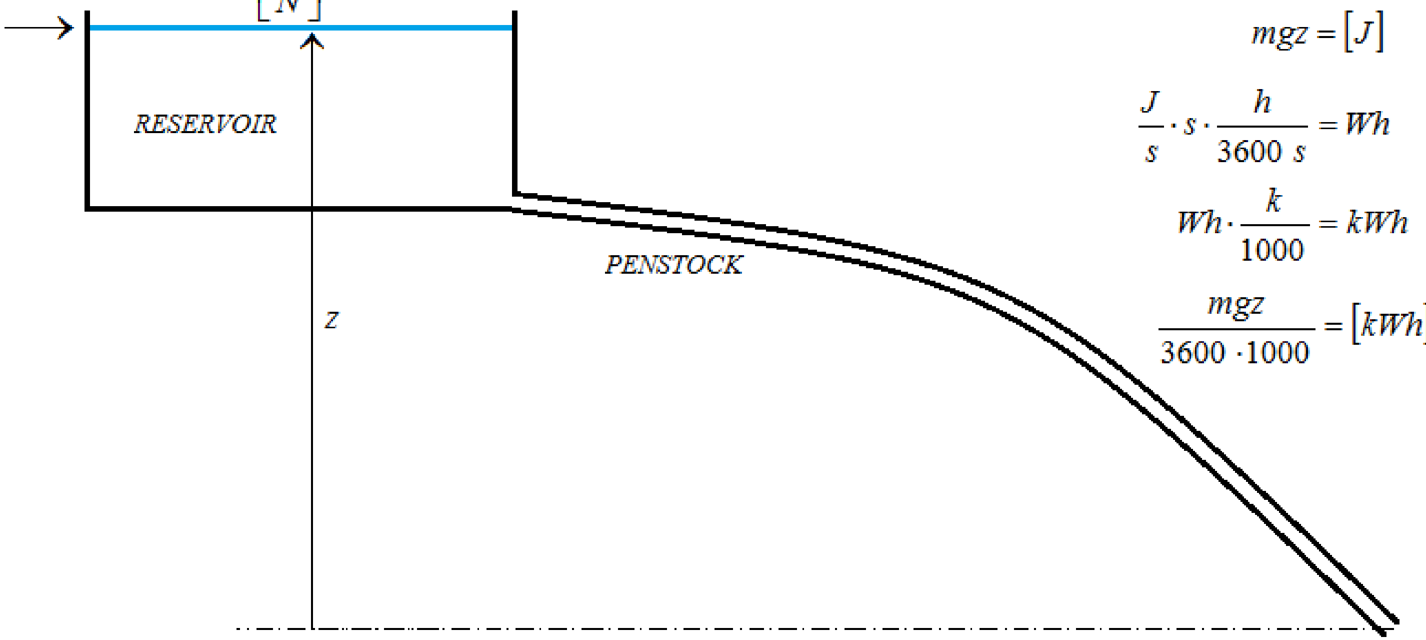
Terminologia Economica-Impiantistica

Firm Energy	Energia certa, producibile in una data parte della giornata da un impianto con il 95% di probabilità
Environmental Impact Assessment	Valutazione di Impatto ambientale
electrical network	Rete elettrica

Reservoirs and Hydro Power exploitation in Valle Camonica



$$h = z \left[\frac{J}{N}\right]$$



$$mgz = [J]$$

$$\frac{J}{s} \cdot s \cdot \frac{h}{3600\ s} = Wh$$

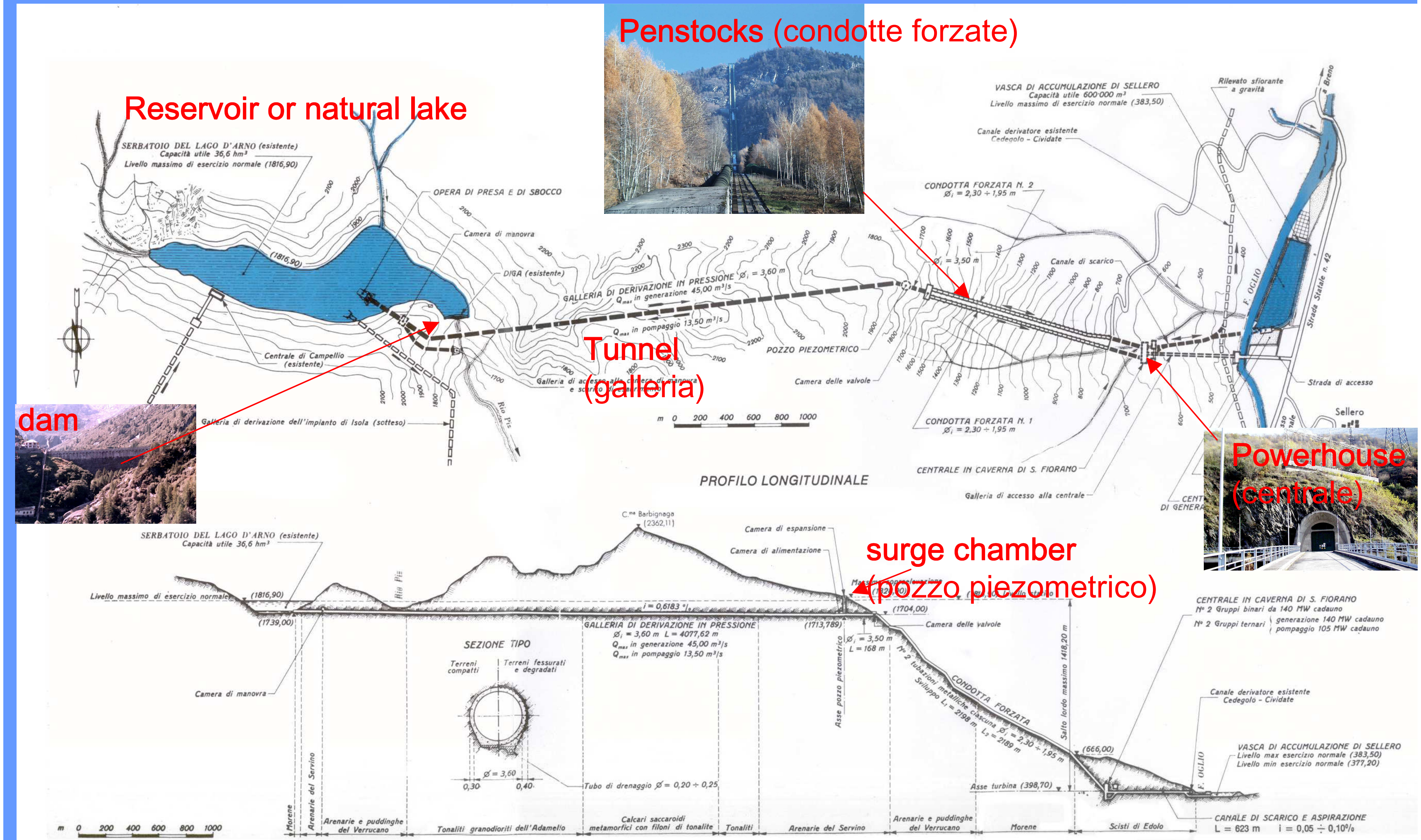
$$Wh \cdot \frac{k}{1000} = kWh$$

$$\frac{mgz}{3600 \cdot 1000} = [kWh]$$

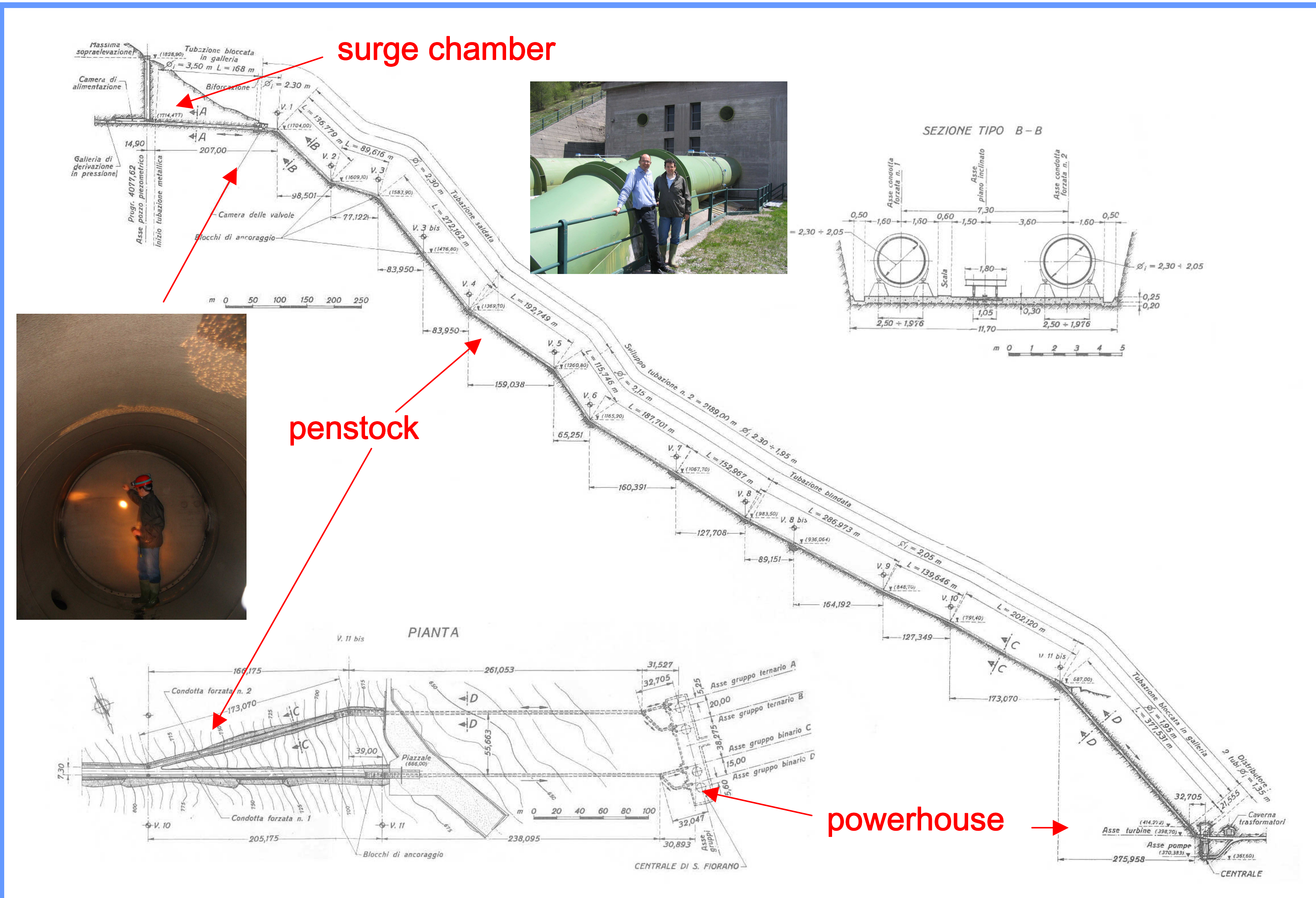
Energy stored within the reservoir in kWh

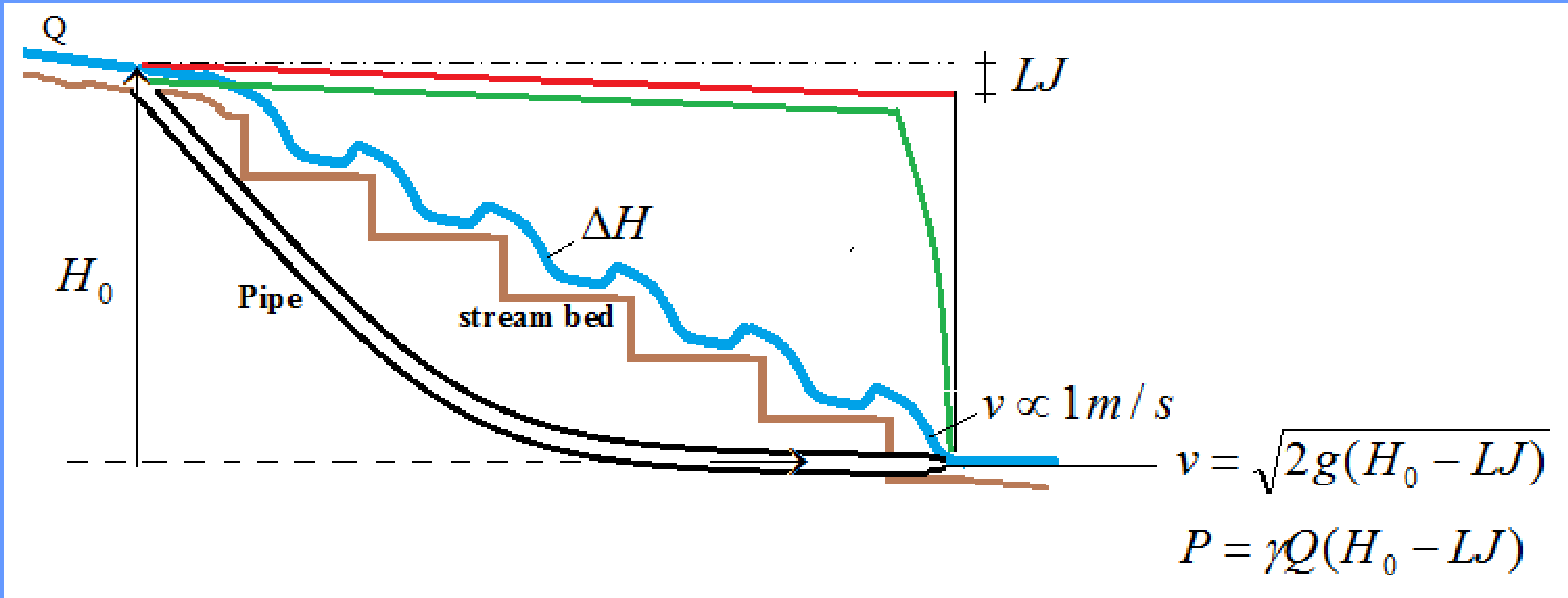
	S.Fiorano Power Plant					
1	Volume lago d'Arno	38000000	mc			
2	delta_z	1418	m			
3	Energia. Accumulata in KWh	146833900	kWh		(1)*ρ*g*(2)/3600	
4	En. Specifica Accumulata in KWh	3.86405	kWh/mc		(3)/(1)	
5	Prezzo KWh	0.13	euro			
6	Valore complessivo	19.088407	milioni euro		(3)*(5)	
7	Valore specifico	0.5023265	euro/mc		(6)/(1)	

How to retrieve this energy ?


Reservoir or natural lake
Penstocks (condotte forzate)
Tunnel (galleria)
dam
Powerhouse (centrale)
surge chamber (pozzo piezometrico)
SERBATOIO DEL LAGO D'ARNO (esistente)
Capacità utile 36,6 hm³
Livello massimo di esercizio normale (1816,90)
OPERA DI PRESA E DI SBOCCO
Camera di manovra
DIGA (esistente)
GALLERIA DI DERIVAZIONE IN PRESSIONE Ø = 3,60 m
Centrale di Campellio (esistente)
Galleria di derivazione dell'impianto di Isola (sotteso)
POZZO PIEZOMETRICO
Camera delle valvole
CONDOTTA FORZATA N. 2
CONDOTTA FORZATA N. 1
Canale di scarico
VASCA DI ACCUMULAZIONE DI SELLERO
Capacità utile 600 000 m³
Livello massimo di esercizio normale (383,50)
Canale derivatore esistente Cedegolo - Cividate
Rilevato sfiorante a gravità
F. OGLIO
Strada Statale n. 42
Strada di accesso
Sellero
CENTRALE IN CAVERNA DI S. FIORANO
Galleria di accesso alla centrale
PROFILO LONGITUDINALE
SEZIONE TIPO
Terreni compatti
Terreni fessurati e degradati
Tubo di drenaggio Ø = 0,20 ÷ 0,25
Camera di espansione
Camera di alimentazione
CONDOTTA FORZATA
Salto lordo massimo 1418,20 m
Asse turbina (398,70)
CENTRALE IN CAVERNA DI S. FIORANO
Nº 2 Gruppi binari da 140 MW cadauno
Nº 2 Gruppi ternari generazione 140 MW cadauno pompaggio 105 MW cadauno
VASCA DI ACCUMULAZIONE DI SELLERO
Livello max esercizio normale (383,50)
Livello min esercizio normale (377,20)
CANALE DI SCARICO E ASPIRAZIONE
L = 623 m i = 0,05 ÷ 0,10‰
Morene
Arenarie del Servino
Arenarie e puddinghe del Verrucano
Tonaliti granodioriti dell'Adamello
Calcari saccaroidi metamorfici con filoni di tonalite
Tonaliti
Scisti di Edolo

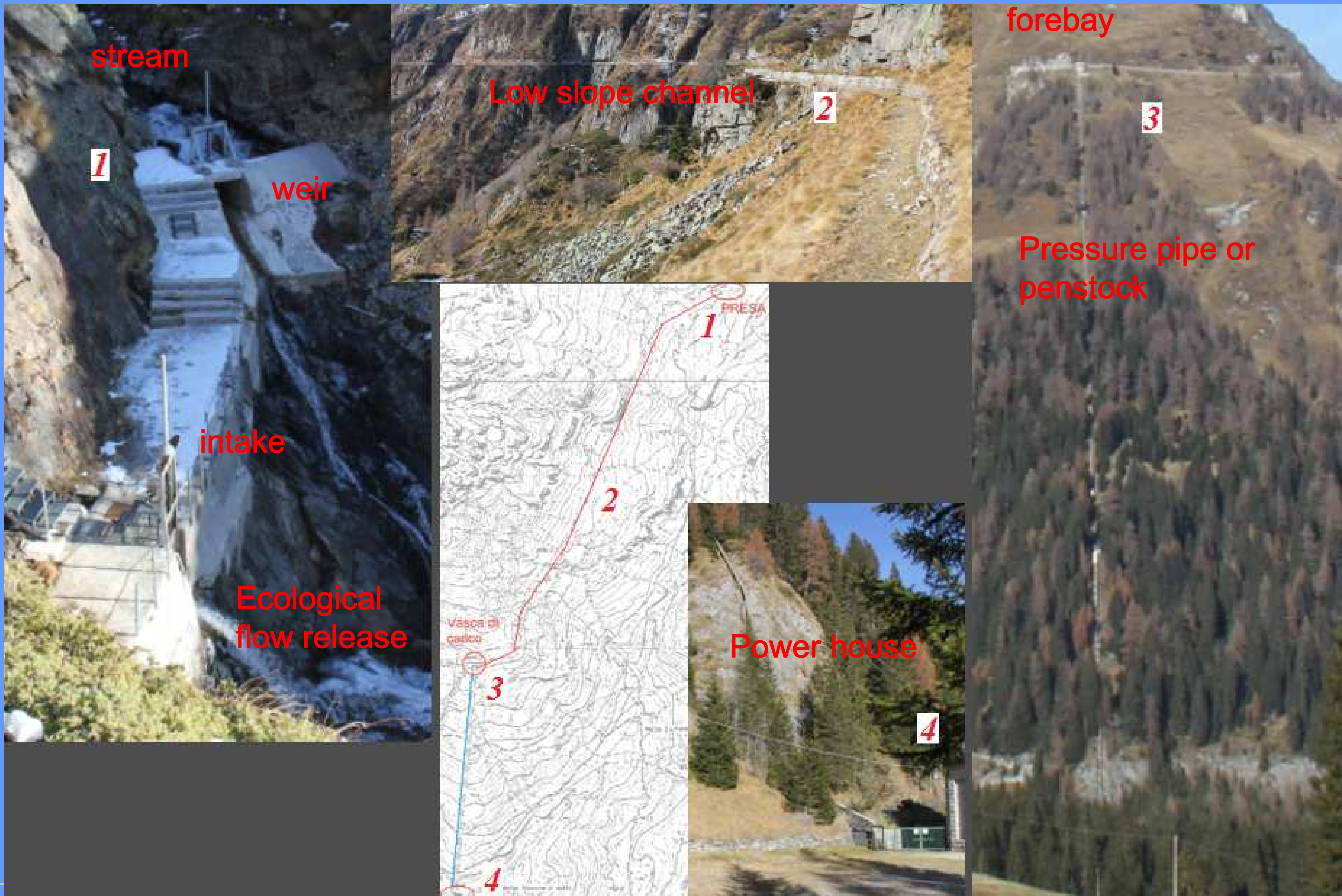
LARGE HYDROPOWER PLANTS: *pumped storage plant of San Fiorano*





How to retrieve this energy ?

SMALL HYDROPOWER PLANTS: Run of River High Head scheme - Gaver



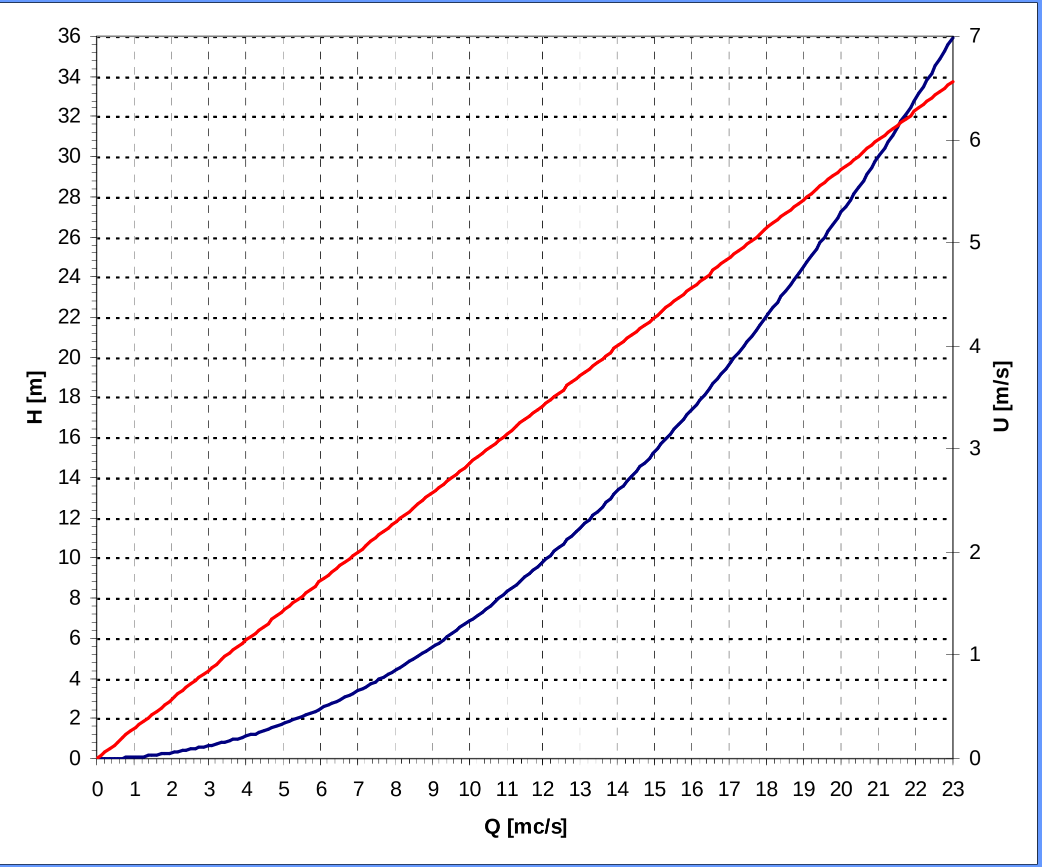
HYDROPOWER PLANTS: *Low Head scheme at the end of a channel- Prevalle*

Pressure Flow: exercise 1

Calculate, using the Moody chart, the friction loss in a 2.112 m diameter welded steel pipe along a length of 2189 m, conveying a flow of 22.5 m^3/s at 20 °C (Data taken from S. Fiorano plant, max. Q per penstock)

$D := 2.112$ m;
$Q := 22.5$ m^3/s;
$L := 2189$ m;
$U := Q/A(d) = 6.42$ m/s;
$Re := U^*d/\nu = 13430041$;
$\varepsilon = 0.6^*0.001$ m; (Manning's n = 0.012sm$^{-1/3}$);
$J = 0.015$;
$\Delta H := J^*L = 32.3$ m; $\Delta H/1418 = 0.023$

If $Q = 6.75$, $\Delta H := J^*L = 2.93$ m; $\Delta H/1418 = 0.0021$

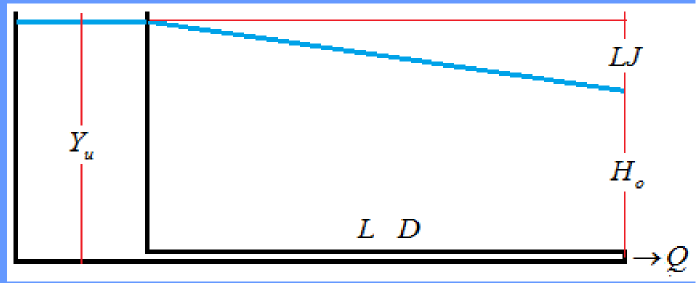


Manning coefficient n for several commercial pipes

Kind of pipe	n
Welded steel	0.012
Polyethylene (PE)	0.009
PVC	0.009
Asbestos cement	0.011
Ductile iron	0.015
Cast iron	0.014
Wood-stave (new)	0.012
Concrete (steel forms smooth finish)	0.014

Optimizing the production of a power plant with reservoir: when D is given, which Q will maximize the power ?

$$\left.\begin{aligned} P &= \gamma Q H_o \\ H_o &= Y_u - LJ = Y_u - \lambda \frac{Q^2 L}{2g\bar{D}A^2} \end{aligned}\right\} P = \gamma Q\left(Y_u - \lambda \frac{Q^2 L}{2g\bar{D}A^2}\right)$$



$$\frac{dP}{dQ} = 0 \quad \rightarrow \quad Y_u = 3\frac{\lambda Q^2 L}{2g\bar{D}A^2}; \quad LJ = \frac{Y_u}{3}$$

Depending on the type of turbine, H_0 can be totally converted into kinetic energy or partitioned between pressure related energy and kinetic energy

e.g;

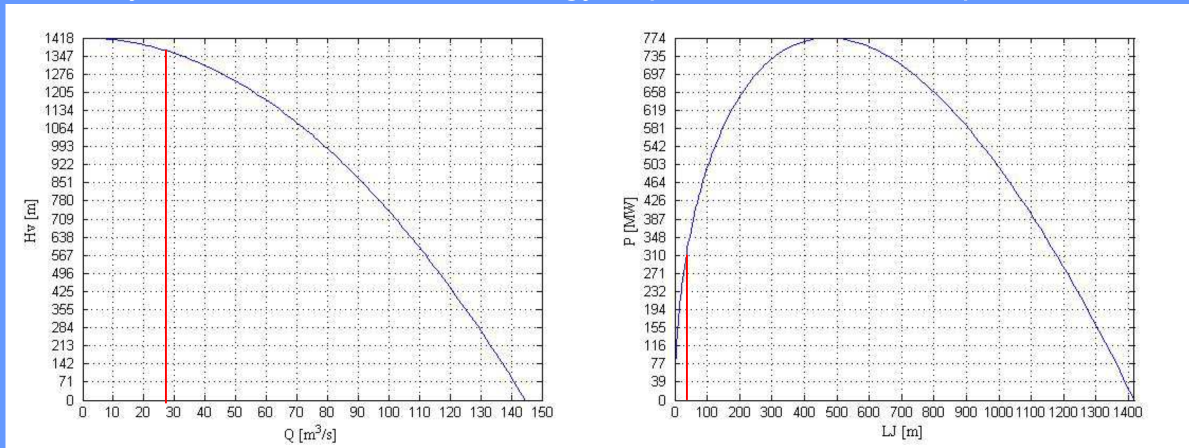
D := 2.112 m;

L := 2189 m;

*ε = 0.6*0.001 m;*

Y_u=1418 m H_V= Y_u-LJ

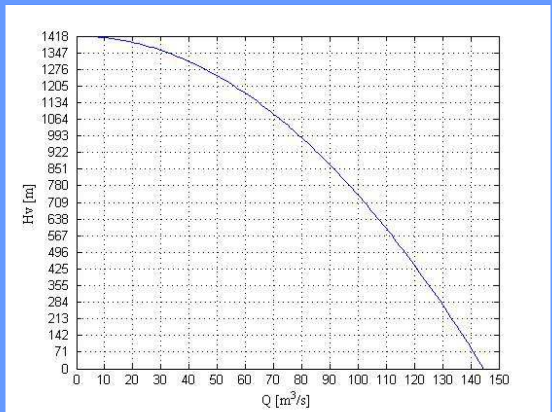
*(MIND: in S.Fiorano, the turbine max output is MW 4x140 = 560, with Q =22.5*2 cm/s)*

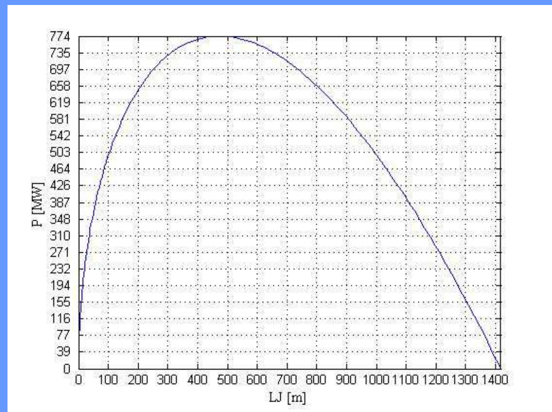


HYDROPOWER PLANTS: *steady state*

Optimizing the production of a power plant with reservoir: when D is given, which Q will maximize the power ?

Hydraulics of each S.Fiorano penstock - situation for actual maximum Q													
	D	**ε**	**ε/D**	**A**	**L**	**L/D**	**Q**	**U**	**Re**	**λ**	**alt.cin**	**J**	**JL**
2	2.13	0.001	0.000469484	3.563272928	2198	1031.925	22.5	6.314	13316548	0.01651	2.0329	0.0158	34.644
Power transferred for each penstock					***305.22***	**MW**							
Hydraulics of each S.Fiorano penstock when power is maximized													
deltaZ		1418											
	D	**ε**	**ε/D**	**A**	**L**	**L/D**	**Q**	**U**	**Re**	**λ**	**alt.cin**	**J**	**JL**
1	2.13	0.001	0.000469484	3.563272928	2198	1031.925	83.2	23.35	49253530	0.01647	27.811	0.215	472.67
Power transferred for each penstock					***771.44***	**MW**							
Time to empty Lake Arno reservoir with the 2 options													
Volume		38000000		mc									
t1		234.5679012		h									
t2		63.4195091		h									
Overall energy produced with the 2 options													
E_t1		143188		MWh									
E_t2		97849		MWh									

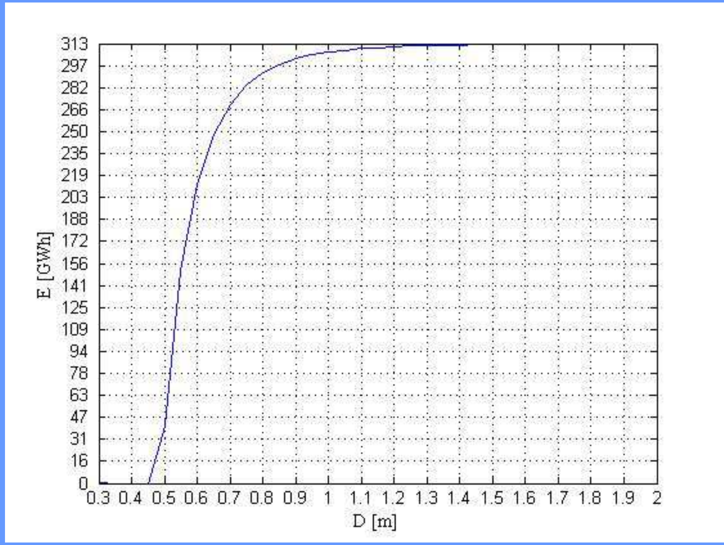




Optimizing the production of a run of river power plant: when Q is given, which D will maximize the power ?
Without an economic constraint, the trivial solution to this question is D = ∞

$$\left.\begin{aligned} P &= \gamma \bar{Q} H_o \\ H_o &= Y_u - LJ = Y_u - \lambda \frac{\bar{Q}^2}{2gDA^2} \end{aligned}\right\} P = \gamma Q\left(Y_u - \lambda \frac{\bar{Q}^2}{2gDA^2}\right)$$

$$\frac{dP}{dD} = 0 \quad \rightarrow \quad P = \gamma \bar{Q} Y_u; \qquad D = \infty$$



*L := 2189 m; ε = 0.6*0.001 m; Y_u=1418 m Q = 22.5 cm/s; yearly production for 1000 hours of peak functioning as a function of D*

However, the cost per unit length of a pipe grows with D.
For simplicity's sake we can make the hypothesis

$$C = L\left(\omega_0 + \omega_1 D^2\right)$$

Other lower constraints for D comes from the water hammer theory, as we shall see

HYDROPOWER PLANTS: steady state

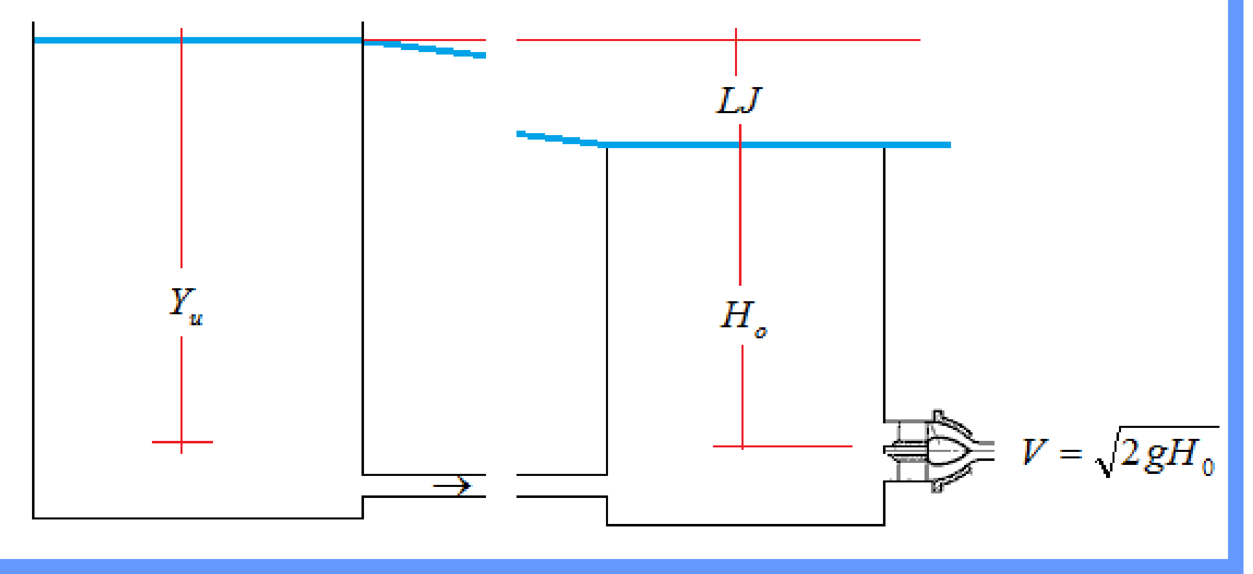
Depending on the type of turbine, H_0 can be totally converted into kinetic energy or partitioned between pressure related energy and kinetic energy.

In a Pelton turbine, used in most high head run of river plants, H0 is totally converted into kinetic energy. In the range of Q variation, V is slightly affected by the adjustable noozle regulation.

Considering S. Fiorano,
Y_u=1418 m
H_0= Y_u -JL = 1385.7 m

$$V_{\max} = \sqrt{2gY_u} = 166.8\ m/s$$

$$V_{\min} = \sqrt{2gH_0} = 164.8\ m/s$$

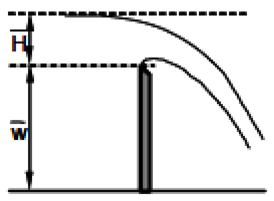
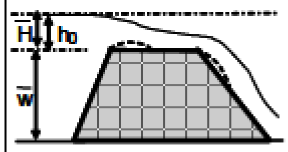
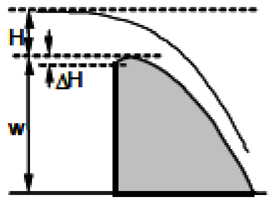


- Depending on the availability of the resource and the purpose of the exploitation, different water capturing structures may be present

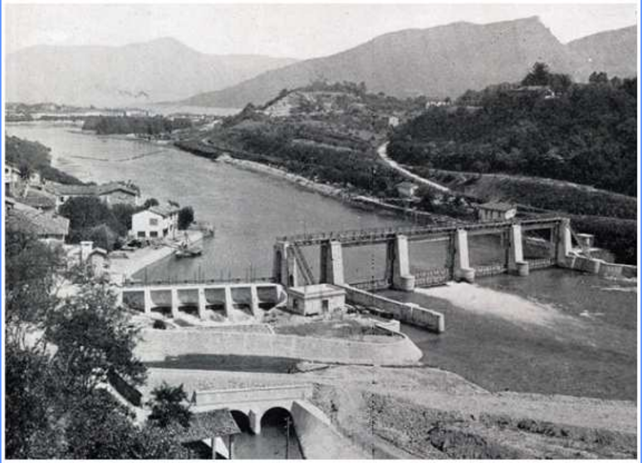
Type of structure	*example*	*purpose*
Dam	(e.g. San Fiorano)	Long-medium term storage
Water retention pond		
Weir + Water retention pond	(e.g., Saviore)	Short term storage
Weir		Improving water diversion, e.g, by rising water level at the intake

- In all this cases spillways (sfioratori, luci) are present in order to controll overflow (dams) and guarantee unaltered tailwater levels during floods.
- The weir in itself can be regarded as a spillway so that the two terms are often interchangeable.
- Often spillways have mobile devices by which they can actively control the water level upstream.
- In such a case they are gated spillways, where the gate can be flat, sector or radial. In this cases a first temptative discharge relationship can be provided by theoretical scheme but, with the exception of the simplest cases, is should more effectively be determined experimentally or on physical models

Weir / Spillways

Type	Design	Discharge relationship	Characteristics
Sharp-crested weir		$Q = b\cdot\overline{C}_d\cdot\overline{H}^{3/2}\cdot\sqrt{2g}$ $\overline{C}_d = 0.42$	Simple design Cost effective
Broad-crested weir		$Q = b\cdot\overline{C}_d\cdot\overline{H}^{3/2}\cdot\sqrt{2g}$ $\overline{C}_{d,mean} = 0.42$	Simple design, underpressures on crest Cost effective
Ogee weir		$Q = b\cdot C_{dD}\cdot H^{3/2}\cdot\sqrt{2g}$ $C_{dD} = 0.494$ (for $H = H_D$)	Highest discharge Costly design

Sarnico dam with flat gates as spillways



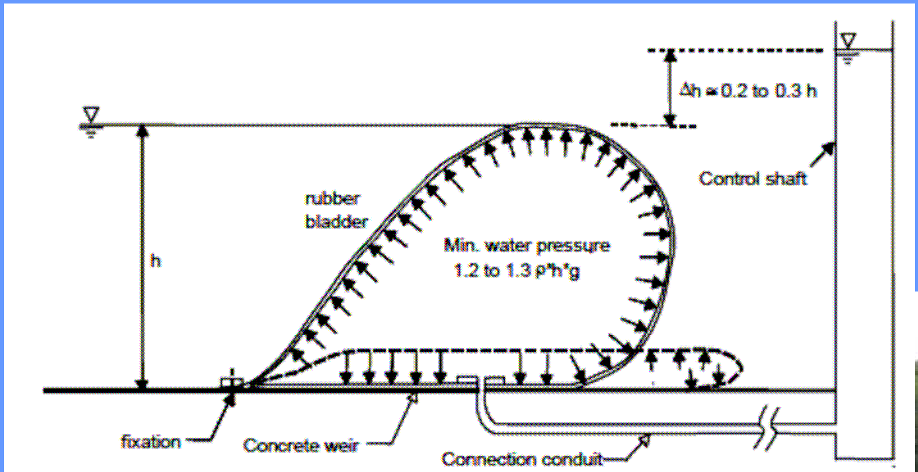
Gavardo: spillways with flat gates partly open to release ecological flow

HYDROPOWER PLANTS: flashboards and inflatable weir

Flashboards: *Extremely simple dynamic regulation with respect to the original weir crest, in order to increase water depth on the intake*

To be removed by hands during floods (a very difficult task)

On the right: flashboards used in Gavardo to prevent backwater during floods



Inflatable weir: *a simple and economic proxy, particularly when the water raise is limited and the span considerable.*

A 2 m high weir, 30 m wide, can be deflated in less than 1/2 hour that is a short time but not too short

Purpose: *to divert the required amount of water into a power canal or into a penstock*
Constraint: *minimum possible head losses and environmental impact. Sometimes a fish diversion systems is required; always an ecological release flow device. geological, hydraulic, structural and economic considerations must be taken into account.*
Challenges: *handling debris and sediment transport. Minimize the effects of ice formation*
Position: *The best disposition of the intake is with the screen at right angles to the spillway so that the flow pushes away the debris. The intake should not be located in an area of still water, in order to prevent the risk of debris and sediment build up.*
Additional equipment:*the intake should be equipped with a trashrack and a settling basin where the flow velocity is reduced, to remove all particles over 0.2 mm (when possible). There must be a sluicing system to flush the deposited silt, sand, gravel and pebbles with a minimum of water loss; and a spillway to divert the excess water.*

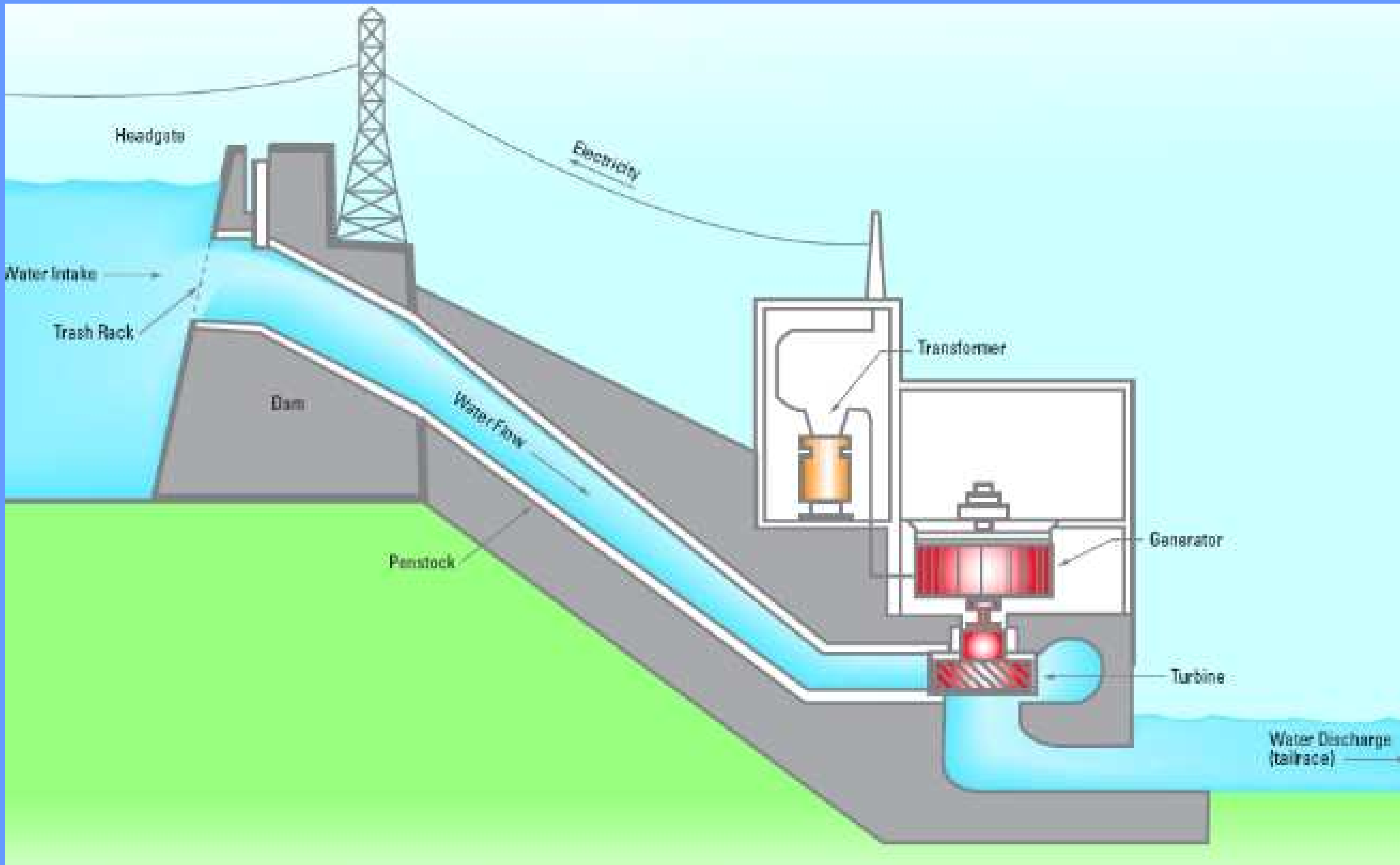
Types of intakes

Power intake: *supplies water directly to the turbine via a penstock. These intakes are usually encountered in lakes and reservoirs, i.e.,, where sediment and debris are non a problem*
Conveyance intake *(lateral, frontal and drop intakes) The intake supplies water to other waterways (power canal, flume, tunnel, etc.) that usually end in a power intake.These are most frequently encountered along waterways and generally transfer the water as free surface flow.*

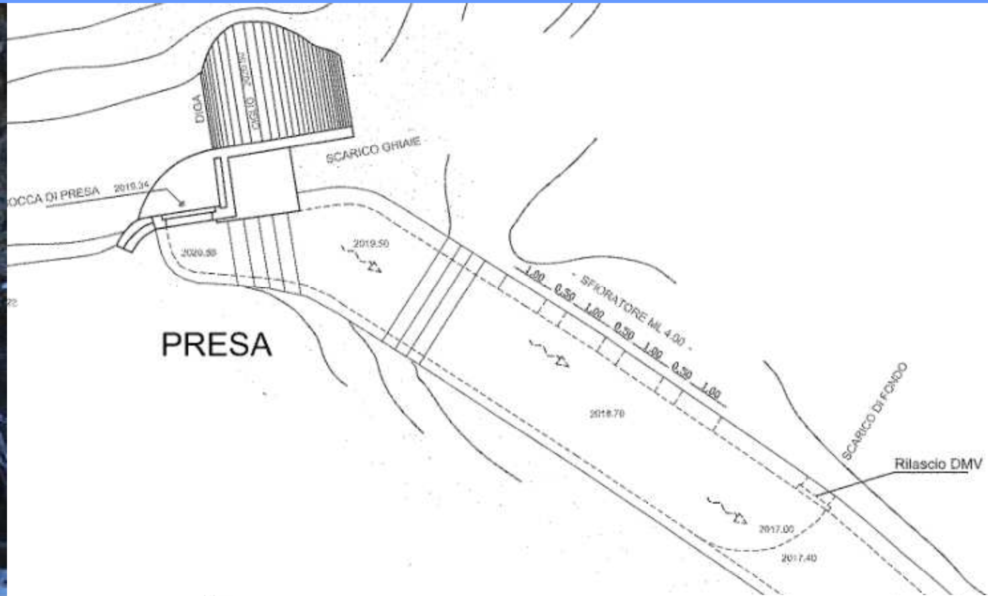
Power intake:

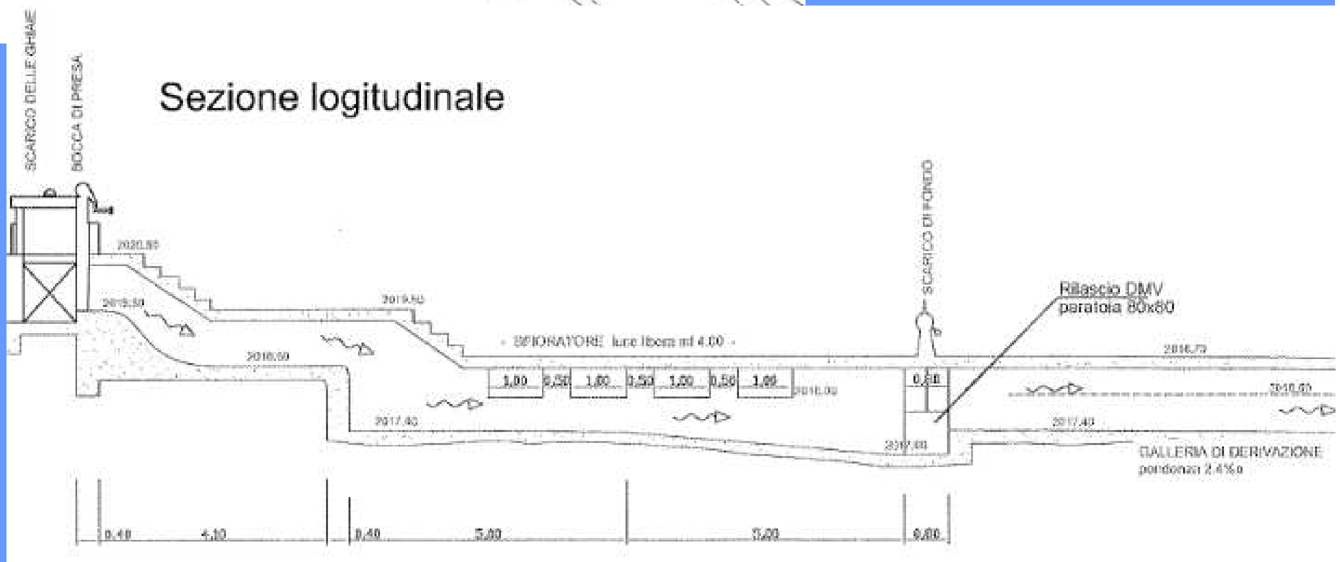
Only for lakes and reservoir

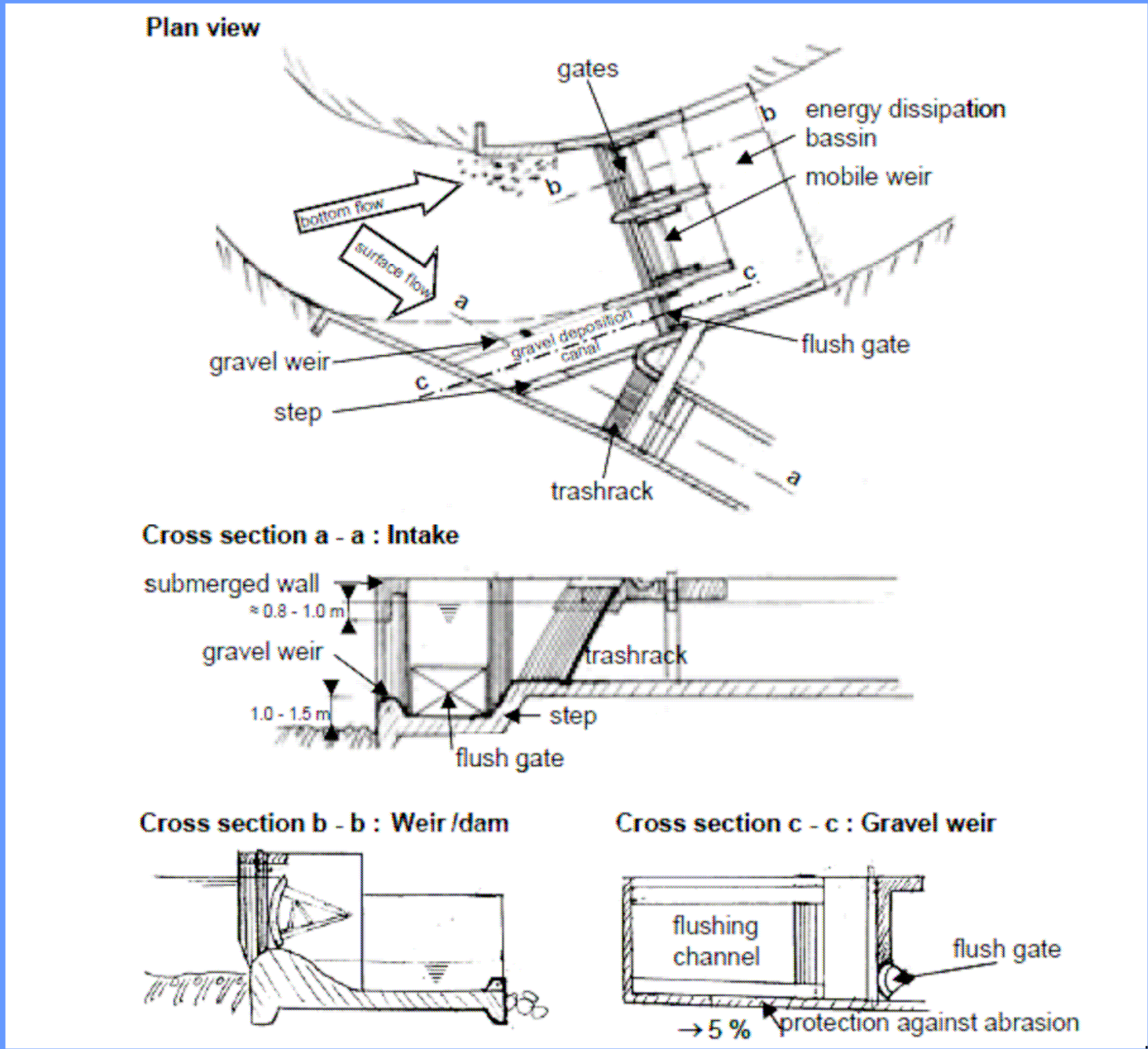
It is important to prevent the risk of vortex formation at their entrance and thus the formation of air pockets within the flow



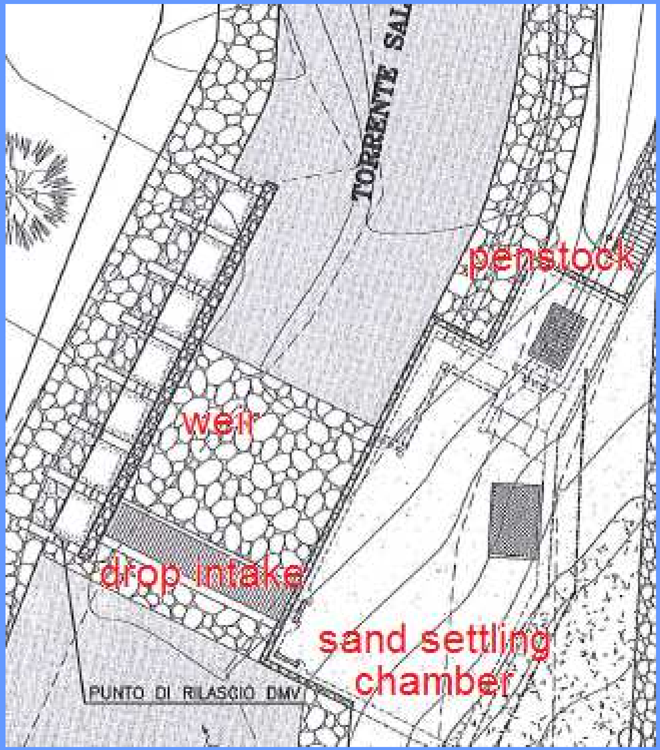
Lateral intake







Lateral intake



For a discussion of different design formulas, look for paper by Drobir et al. on the website

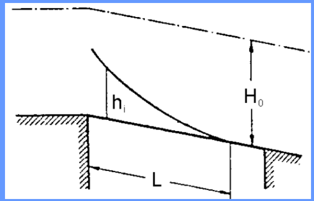
Drop (tyrolean) intake

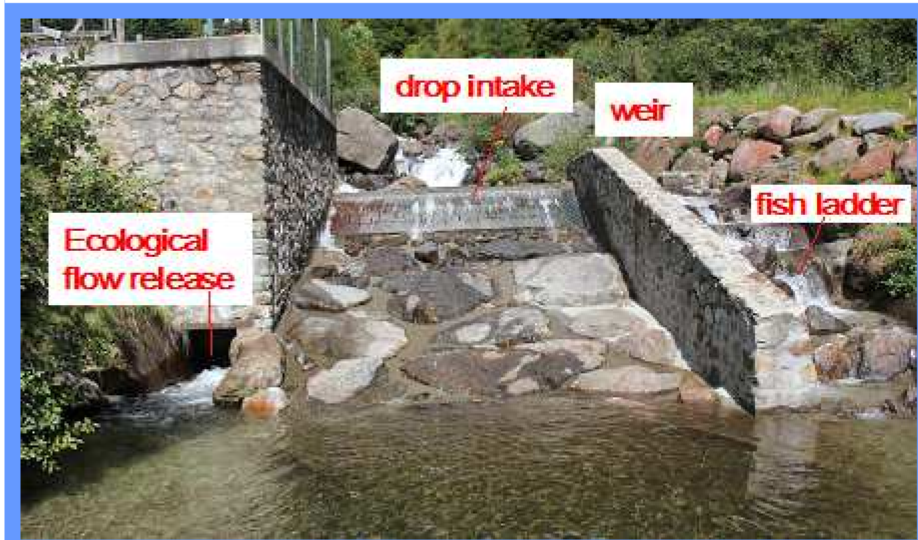
Design parameter: wetted rack length, as a function of the specific discharge, inclination of the rack and width of the bar-gaps

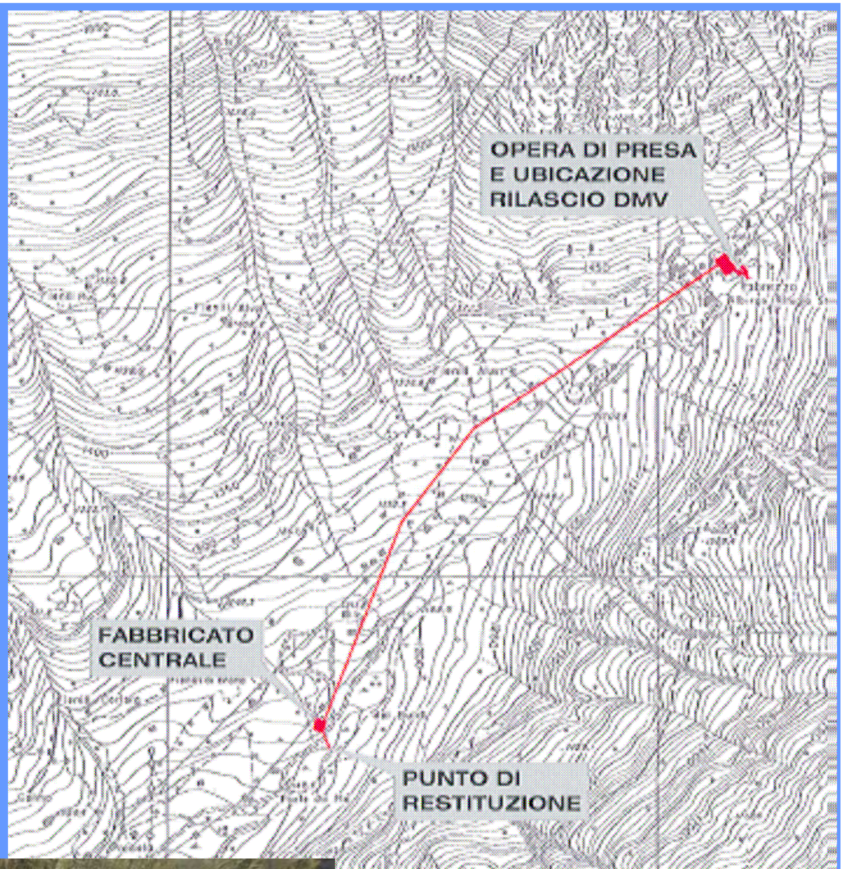
Noseda suggested to compute the wetted rack length on a constant energy head basis:

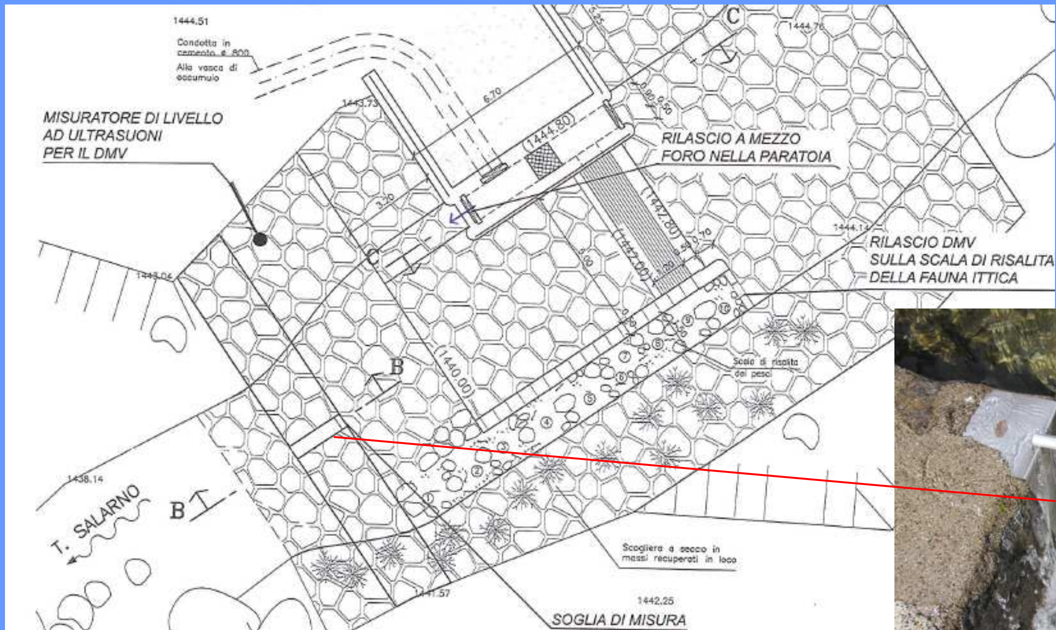
$$L = 1.185 \frac{H_0}{m\mu}$$

μ is the discharge coefficient of the rack which is dependent on the shape of the cross section of the bar, m is the void ratio .This equation neglects the inclination of the rack and thus it is only valid for horizontal racks.




drop intake
weir
fish ladder
Ecological
flow release


OPERA DI PRESA
E UBICAZIONE
RILASCIO DMV
FABBRICATO
CENTRALE
PUNTO DI
RESTITUZIONE


MISURATORE DI LIVELLO
AD ULTRASUONI
PER IL DMV
RILASCIO A MEZZO
FORO NELLA PARATOIA
RILASCIO DMV
SULLA SCALA DI RISALITA
DELLA FAUNA ITTICA
T. SALARNO
SOGLIA DI MISURA

HYDROPOWER PLANTS: Fish ladders

Saviore dell'Adamello (1)

Saviore dell'Adamello (2)

Upstream pond (forebay - vasca di carico)

Electricity prices at peak hours can be substantially higher than in off-peak hours, hence the interest in providing an extended pound, big enough to store the water necessary to operate, at maximum during peak hours. To evaluate this volume:

Q_R = river flow (m3/s); Q_r = available river flow = river flow – Q_{DMV} (m3/s);
Q_P = flow needed to operate in peak hours (m3/s)
Q_{OP} = flow needed to operate in off-peak hours (m3/s)
t_P = daily peak hours
t_{OP} = daily off-peak hours (24 - tP)

The volume for a daily pond is given by V:

$$V = 3600 t_p (Q_p - Q_r)$$

Provided that this volume must be refilled in off-peak hours, we have the constraint:

$$3600 t_{op} (Q_r - Q_{op}) \geq 3600 t_p (Q_p - Q_r)$$

that implies

$$Q_p \leq \left(1 + \frac{t_{op}}{t_p}\right) Q_r - \frac{t_{op}}{t_p} Q_{op}$$

HYDROPOWER PLANTS: *trash rack*

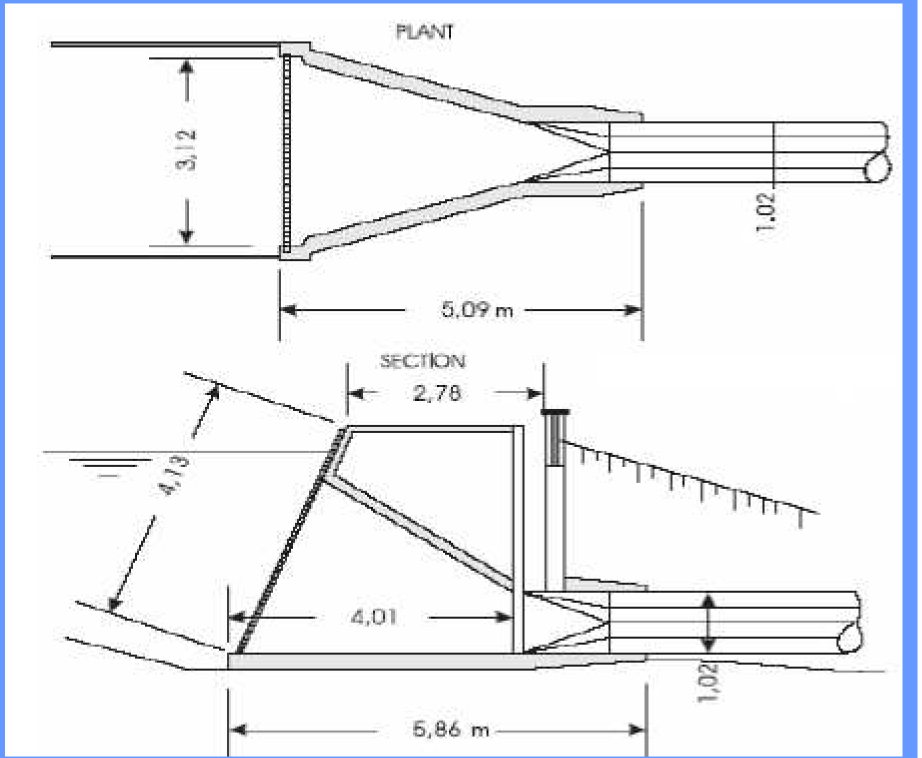
A screen is nearly always required at the entrance of both pressure pipes and intakes to avoid the entrance of floating debris. The flow of water through the rack also gives rise to a head loss. Though usually small, it must be included into the calculation.

The maximum possible spacing between the bars is generally specified by the turbine manufacturers. Typical values are 20-30 mm for Pelton turbines, 40-50 mm for Francis turbines and 80-100 mm for Kaplan turbines.

Head losses

In the design of small hydro plants care must be taken to minimise head losses because they can be of huge importance to the feasibility of the project. For large plants, although small in relative terms, can be economically very significative. Accordingly in any case care must be taken in order to:

- minimise flow separation
- distribute flow uniformly on the cross section
- suppress vortex generation
- choose an appropriate trashrack design



Rule of thumb: at the trashrack $0.8 < U < 1$ m/s (this dictates the dimensions of the rectangular section upstream).
In the penstock $3 < U < 5$ m/s (this dictates the penstock diameter).

For a well designed low head intake structure like the one on the right the head loss can be as low as

$$\Delta H = 0.19 \frac{U^2}{2g}$$

where U is the velocity in the penstock

HYDROPOWER PLANTS: … and Vortex formation

It is very important to prevent air entrance into the penstock.

In order to prevent this from happening, a mimimum submergence h_t at the penstock entrance must be guaranteed. According to Knauss:

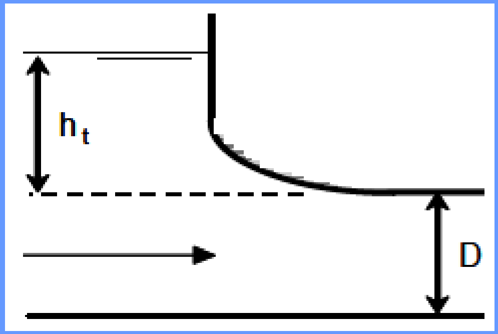
$$h_t \geq D\,[1+2{,}3U(g\,D)^{0{,}5}]$$

Where D is the penstock diameter in m, U the velocity within it and g the acceleration due to gravity.

Alternatively

$h_t \geq 1{,}474\, U^{0{,}48} D^{0{,}76}$ *[Rohan]*

$h_t \geq c\, U\, D^{0{,}50}$ *[Gordon]*



Where c is a factor that takes into account the simmetry of the approaching flow

c=0,7245 for asymettric flow

c=0,5434 for symmetric flow

For instance: D= 600 mm; Q= 0,75 m3/s; U =2,66 m/s:

Knauss	2,57 m
Rohan	2,00 m
Gordon	1,75 m

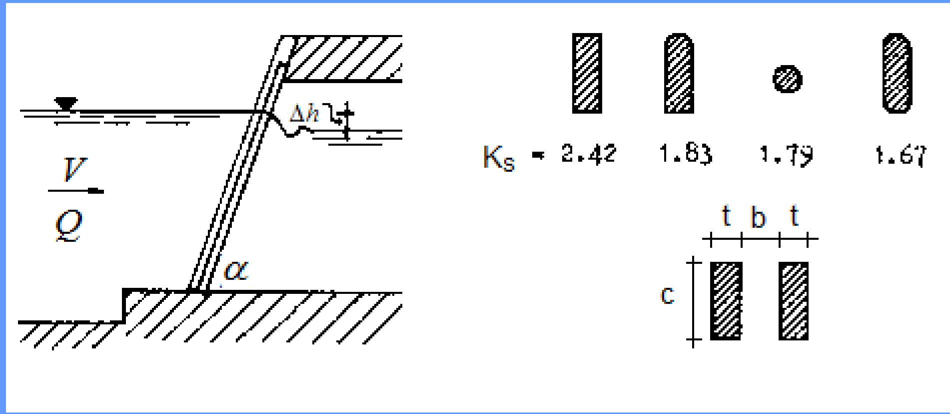
According to experience the velocity at the entrance of the rack should be between 0.25 m/s and 1.0 m/s. In order to obtain this velocity, the required trash rack area S is estimated by the formula derived by the continuity equation

$$K\phi sen(\alpha)S = \frac{Q}{V}; \qquad \phi = \frac{b}{b+t}$$

where S is the area in m^2, t the bar thickness, b the distance between bars, Q the discharge (m^3/s), V_0 the water velocity of approach and K_1 is a trash cleaning coefficient which, if the trash rack has an automatic cleaner, is equal to 0.80. α = angle of bar inclination, degree.

The head loss caused by the trash rack can be calculated by a formula developed by Kirschmer:

$$\Delta H = K_S\left(\frac{t}{b}\right)^{4/3}\left(\frac{V^2}{2g}\right)sen(\alpha)$$



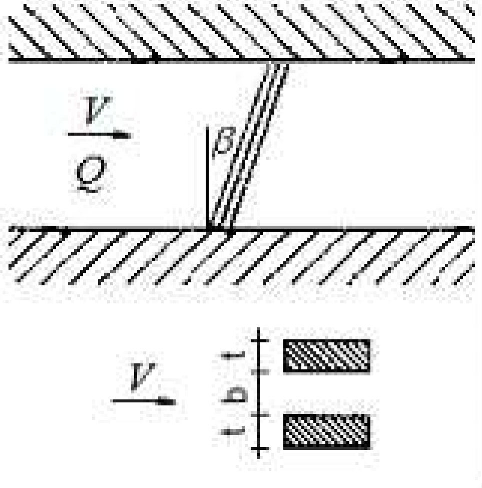
Where K_s = screen loss coefficient, given in sketch

For instance:

Let us consider a trash rack inclined 60° with the horizontal, made of stainless steel flat bars 12 mm thick and the width between bars is 70 mm. Q is 3 m^3/s

First we estimate the area S; if we ask for V = 1 m/s, we get S = 5.07 m^2; rounding to 6 m^2 we get V = 0.84 m/s. Accordingly if we use K_s = 2.4 we get Δh := 0.007 m.

If the grill is not perpendicular but makes an angle β with the water flow (β will have a maximum value of 90° for a grill located in the sidewall of a canal), there will be an additional head loss. The result of previous head loss should be multiplied by a correction factor provided in the following table (according to Mosonyi)



β \ t/b	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2
0°	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
10°	1.06	1.07	1.08	1.09	1.10	1.11	1.12	1.14	1.50
20°	1.14	1.16	1.18	1.21	1.24	1.26	1.31	1.43	2.25
30°	1.25	1.28	1.31	1.35	1.44	1.50	1.64	1.90	3.60
40°	1.43	1.48	1.55	1.64	1.75	1.88	2.10	2.56	5.70
50°	1.75	1.85	1.96	2.10	2.30	2.60	3.00	3.80	...
60°	2.25	2.41	2.62	2.90	3.26	3.74	4.40	6.05	...

Sudden Contraction in the transition between two pipes where d downstream < D upstream

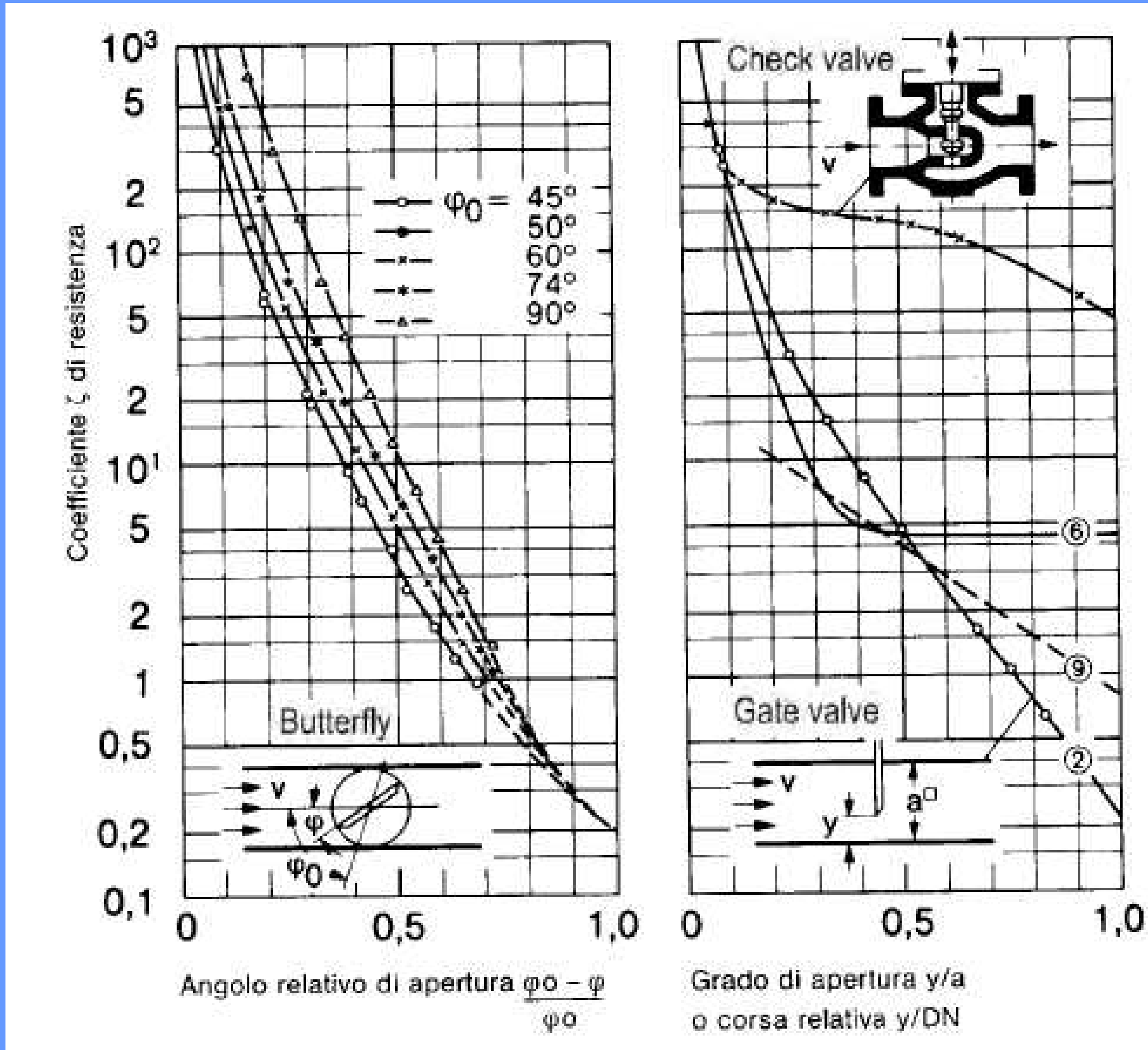
$$\Delta H = 0.42\left(1-\frac{d^2}{D^2}\right)\frac{V_2^2}{2g} \qquad \frac{d}{D} < 0.76$$

Sudden Expansion in the transition between two pipes (Borda's loss; where D downstream > d upstream)

$$\Delta H = \frac{1}{2g}(V_1 - V_2)^2 = \frac{V_2^2}{2g}\left(\frac{D_2^2}{d_1^2}-1\right)^2$$

$$\Delta H = k\frac{V^2}{2g}$$

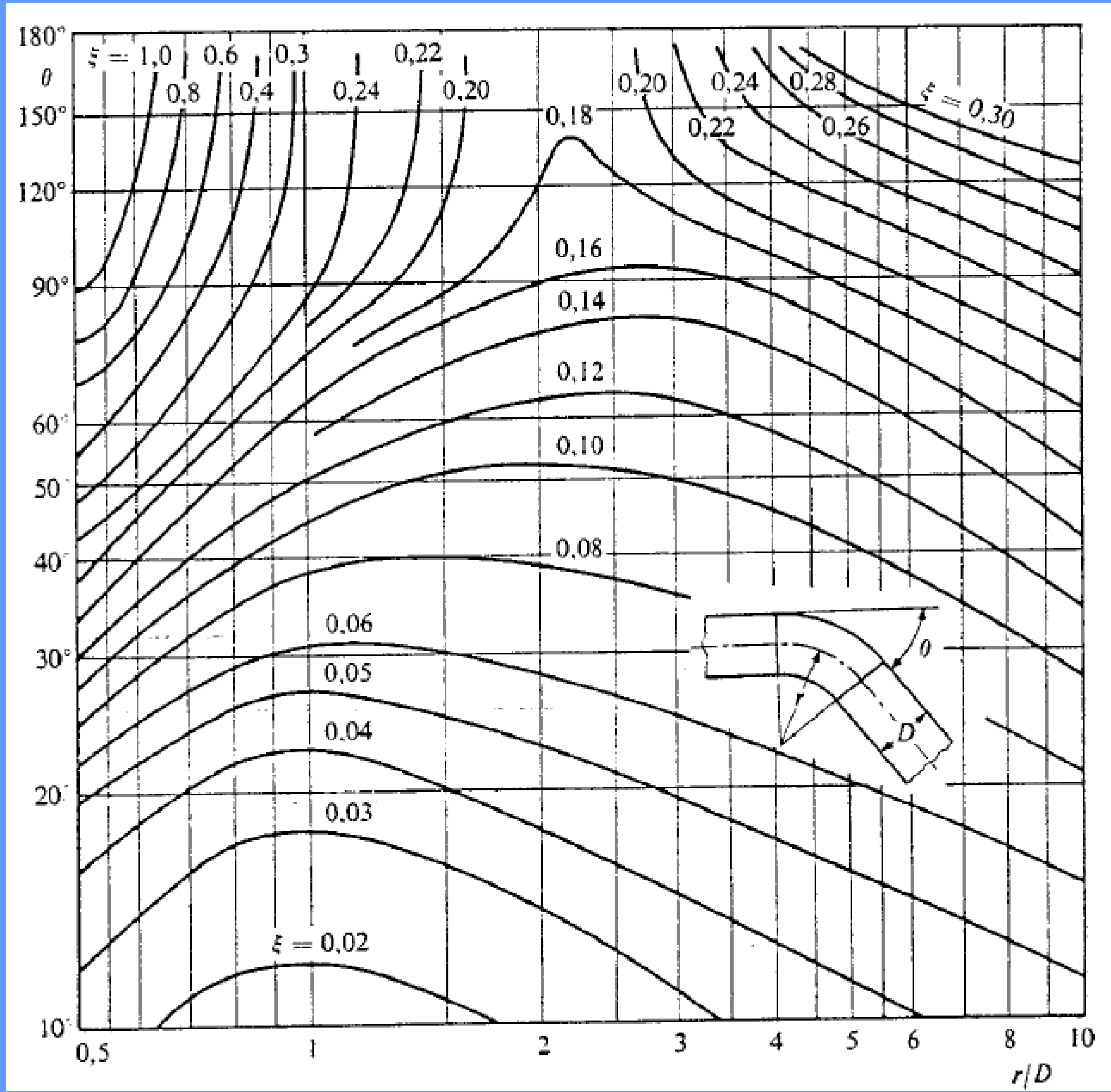
Valves or gates are used in small hydro schemes to isolate a component from the rest, so they are either entirely closed or entirely open. Flow regulation is assigned to the distributor vanes or to the needle valves of the turbine. The loss of head produced by water flowing through an open valve depends on the type and manufacture of the valve. Figure on the right shows the value of Kv for different kind of valves and different opening retios.



$$\Delta H = \xi\left(\frac{r}{D}, \theta, \frac{\varepsilon}{D}\right)\frac{V^2}{2g}$$

In a bend, pipe flow experiences increase of pressure along the outer wall and a decrease of pressure along the inner wall. This pressure unbalance causes a secondary current. Both movements together - the longitudinal flow and the secondary current - produces a spiral flow that, at a length of around 100 diameters, is dissipated by viscous friction. The head loss produced in these circumstances depends on the radius of the bend and on the diameter of the pipe. Furthermore, in view of the secondary circulation, there is a secondary friction loss, dependent of the relative roughness e/D.

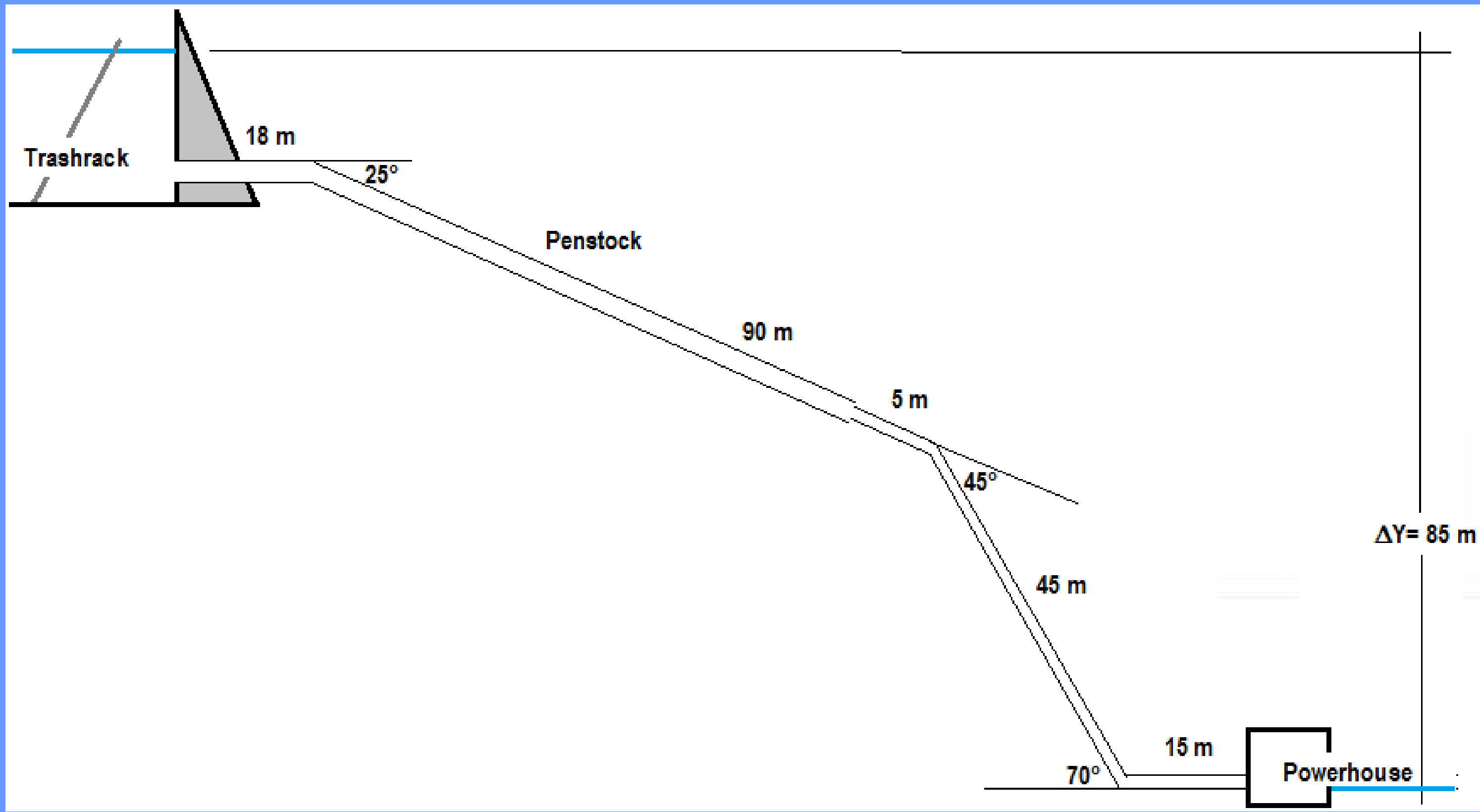
The problem is extremely complex when successive bends are placed one after another, close enough to prevent the flow from becoming stabilized



(Re = 10^6)

Pressure Flow: exercise 2

Considering a small hydropower scheme, the nominal discharge is 3 m^3/s and the gross head 85 m. The penstock is 1.5 m diameter in the first length and 1.2 m in the second one, with ε=0. The radius of curvature of the bend is four times the diameter of the pipe. At the entrance of the intake there is a trash rack inclined 600 with the horizontal. The rack is made of stainless steel flat bars, 12 mm thick and the width between bars is 70 mm. Estimate the total head loss



Pressure Flow: exercise 2

Z3	20		1.0985									
pipe												
D [m]	ε	ε/D	***A***	***L***	***L/D***	***Q***	***U***	***Re***	λ	$U^2/(2g)$	***J***	***JL***
1.5	0.0005	0.000333	1.767146	108	72	3	1.698	2521266	0.01561	0.146945	0.0015	0.1652
1.2	0.0005	0.000417	1.130973	65	54.17	3	2.653	3151583	0.01627	0.358752	0.0049	0.3162
											Total	**0.4813**
Head loss in trashrack			0.007	m								
pipe entrance			0.073	m								
I bend	(ξ=0.06)		0.009	m								
D contraction			0.054	m								
II bend	(ξ=0.1)		0.036	m								
III bend	(ξ=0.13)		0.047	m								
local head losses		**0.226**	m									
Total losses		**0.707**	m									
Gross head		85	m									

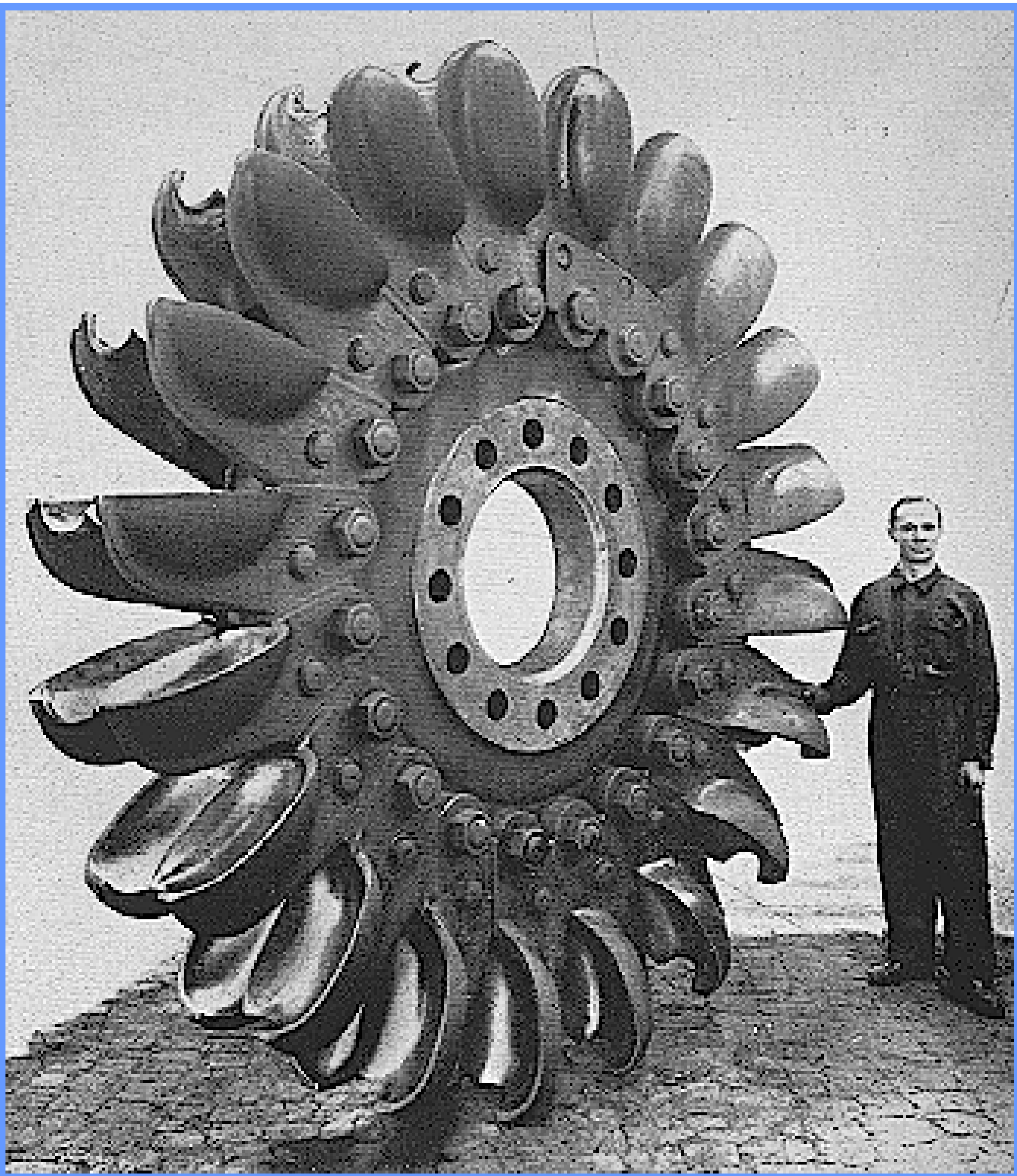
Conclusions:

- *The percentage head loss is reasonable and rather small*
- *In relative terms, if the pipe length is small, local head losses may be significative*
- *For instance, supposing 7200 hours/year of functioning, the head loss at the trashrack implies an yearly energy loss of 1483 kWh (9806*3*0.007/1000*7200)*

Hydraulic turbines

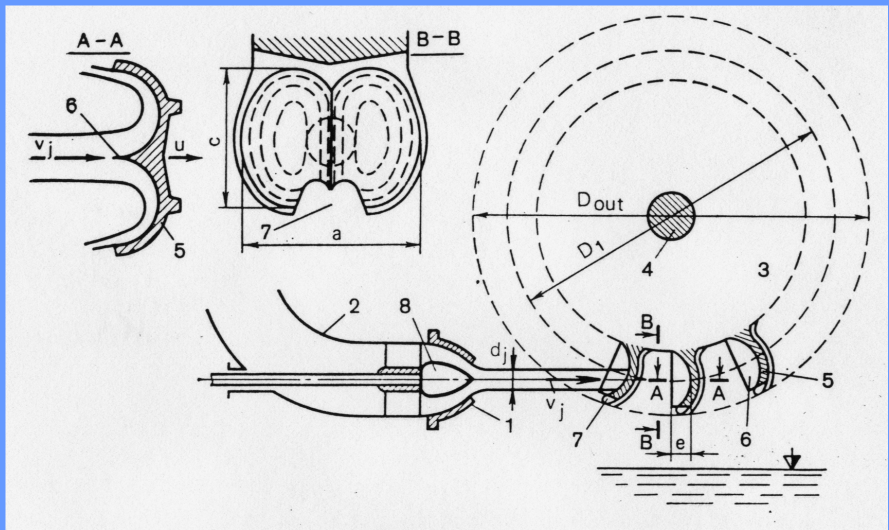
The purpose of a hydraulic turbine is to transform the water potential energy to mechanical rotational energy.

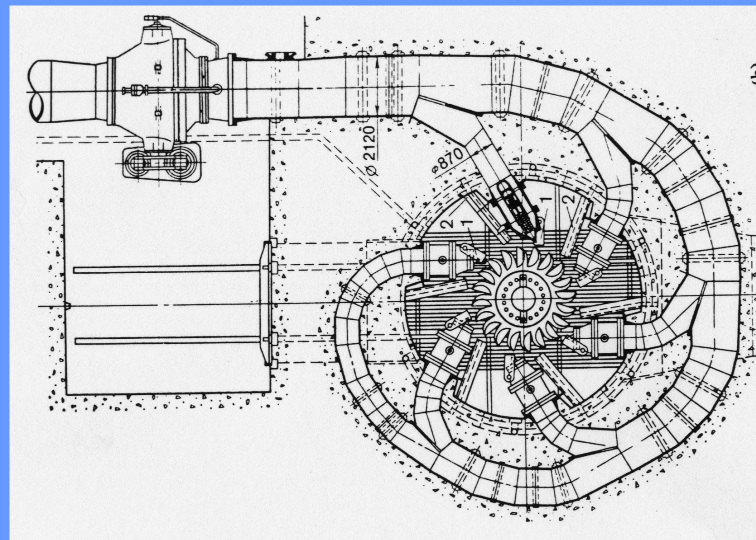
In the Pelton turbine the potential energy in water is converted into kinetic energy before entering the runner. The kinetic energy is in the form of a high-speed jet that strikes the buckets, mounted on the periphery of the runner. Turbines that operate in this way are called ***impulse turbines (turbine ad azione)****.*



SMALL HYDROPOWER PLANTS: Pelton turbines (after L. Pelton, 1880)

Pelton turbines are impulse turbines used for high heads from 60 m to more than 1 000 m. In this turbine one or more jets impinge on a wheel carrying on its periphery a large number of buckets (cucchiai). Each jet issues water through a nozzle (iniettore) with a needle valve to control the flow.
The axes of the nozzles are in the plan of the runner (girante). In case of an emergency stop of the turbine (e.g. in case of load rejection), the jet may be diverted by a deflector (tegolo deviatore) so that it does not impinge on the buckets and the runner cannot reach runaway speed. In this way the needle valve can be closed very slowly, so that overpressure surge in the pipeline is kept to an acceptable level (max 1.15 static pressure).
As any kinetic energy leaving the runner is lost, the buckets are designed to keep exit velocities to a minimum. One or two jet Pelton turbines can have horizontal or vertical axis. Three or more nozzles turbines have vertical axis. The maximum number of nozzles is 6 (not usual in small hydro). The efficiency of a Pelton is good from 30% to 100% of the maximum discharge for a one-jet turbine and from 10% to 100% for a multi-jet one.





CIMEGO Plant (110 MW, H 721 m, D=3.5m, n= 300gpm, d 0.31 m)

Balance of Rotational momentum for a control volume

$$\frac{D}{Dt}\left(\int_W (\vec{x}-\vec{x}_0)\wedge\rho\vec{V}dW\right)=\int_W (\vec{x}-\vec{x}_0)\wedge\rho\vec{g}dW+\int_S (\vec{x}-\vec{x}_0)\wedge\vec{\sigma}_n dS$$

$$\frac{\partial}{\partial t}\left(\int_W (\vec{x}-\vec{x}_0)\wedge\rho\vec{V}dW\right)-\int_S (\vec{x}-\vec{x}_0)\wedge(\rho\vec{V})\vec{V}\cdot\vec{n}dS=\int_W (\vec{x}-\vec{x}_0)\wedge\rho\vec{g}dW+\int_S (\vec{x}-\vec{x}_0)\wedge\vec{\sigma}_n dS$$

$$-\int_S (\vec{x}-\vec{x}_0)\wedge(\rho\vec{V})\vec{V}\cdot\vec{n}dS=\int_S (\vec{x}-\vec{x}_0)\wedge\vec{\sigma}_n dS$$

Moment of Momentum of entering jets

Resistive Torque exerted by the generator shaft (at 4)

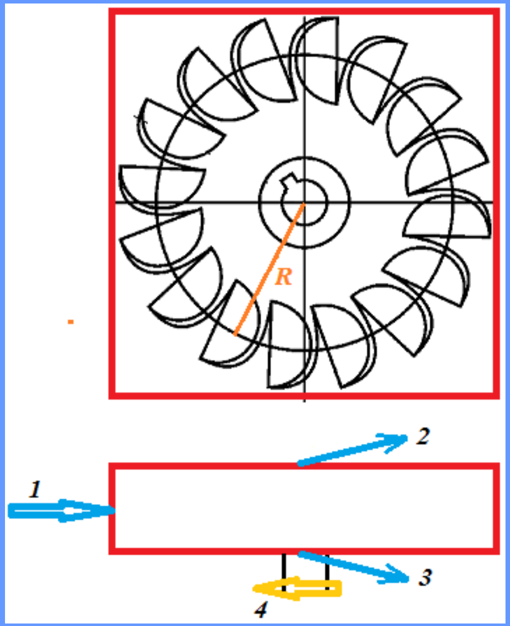
$$\int_S (\vec{x}-\vec{x}_0)\wedge(\rho\vec{V})\vec{V}\cdot\vec{n}dS$$

Angular Momentum of entering jets (at 1, 2 and 3)

$R\rho QV$ at position 1

$$R\rho\frac{Q}{2}[\omega R-(V-\omega R)\cos\vartheta]$$

at position 2 and 3

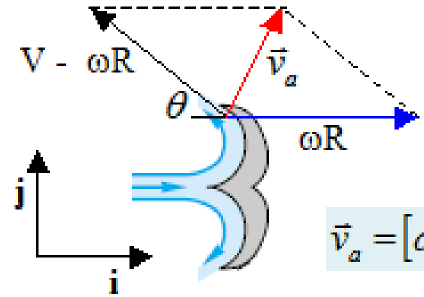


$$M = R\rho QV - R\rho\frac{Q}{2}[\omega R - (V-\omega R)\cos\vartheta]$$

$$M = R\rho Q(V-\omega R)(1+\cos\vartheta)$$

$$P = M\omega = R\rho Q(V-\omega R)(1+\cos\vartheta)\omega$$

$$\omega_1 = 0; \quad \omega_2 = \frac{V}{R}; \quad \omega_{max} = \frac{V}{2R}$$



$$\vec{v}_a = [\omega R - (V-\omega R)\cos\theta]\vec{i} + (V-\omega R)sen\theta\,\vec{j}$$

Accordingly, if θ=0

$$P_{max} = \gamma QH = \rho Q\frac{V^2}{2}$$

Which actually corresponds to the power by unit weight of the entering jet

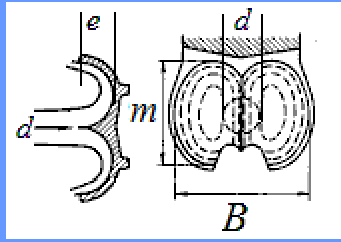
$$P = \gamma QH = \gamma Q\frac{V^2}{2g} = \rho Q\frac{V^2}{2}$$

And to the exit velocity

$$\vec{v}_a = [\omega_{max}R - (V-\omega_{max}R)] = \left[\frac{V}{2} - (V-\frac{V}{2})\right] = 0$$

ROADMAP for Pelton design:

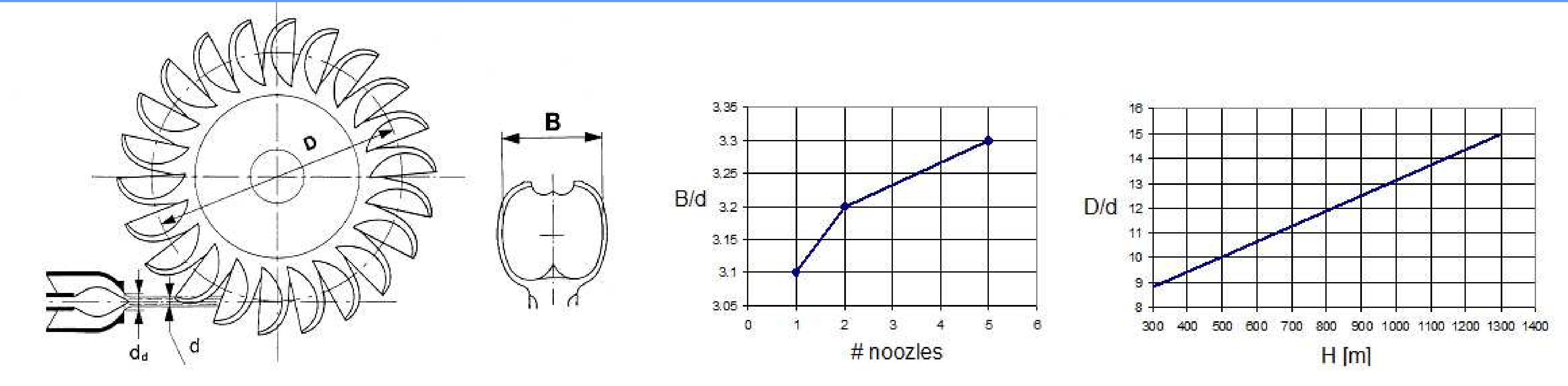
1) head and discharge are given, along with the type of turbine and the number of noozles (z). Increasing z decreases the dynamic forces on the runner and aereodynamic resistence; it increases flexibility on Q variations



$$Q, H, z \rightarrow U = \sqrt{2gH} \rightarrow q = Q/z \rightarrow q = \frac{\pi d^2}{4} U \rightarrow d = \sqrt{\frac{4q}{\pi U}}$$

2) The dimension B of the bucket and D of the runner are obtained by empirical rules of thumb as in the diagrams below

$$B/d = f(z); \quad D/d = f(H) \rightarrow B, D$$



Other rules of thumb are $3d \leq B \leq 4d; \quad e \cong d; \quad m \cong 0.7 \div 0.8\,B; \quad 8d \leq D \leq 20d;$

3) accordingly, one can obtain the optimal angular velocity and, from it, the number of rounds/minute, n

$$\omega = \frac{U}{2R} = \frac{U}{D} \rightarrow \omega = n\frac{2\pi}{60} \rightarrow n$$

4) And eventually one chooses the number of poles Z_p on the generator

$$50 = \frac{nZ_p}{60}$$

That must be an integer value. Accordingly one usually has to recalculate both the angular speed and the diameter D starting from the integer Z_p closer to the one found.
Let us apply this procedure to the ternary group of San Fiorano plant and using the design data let us estimate the force exerted on a single bucket by the impinging jet

Q	11.25	m^3/s			
H	1418	m			
z	4	for ternary group			
U	166.77	m/s	Force on a single bucket		
q	2.8125	m^3/s		234516.1	N
d	0.15	m		23.90582	tonn
B/d	3.26	B	0.48	m	
D/d	15	D	2.20	m	
temptative value			final value if Zp = 5		
omega	75.87	rad/sec	n	600	round/min
n	724.51	round/min	omega	62.83	rad/sec
Zp	4.141	poli	D	2.65	m

SMALL HYDROPOWER PLANTS: *Layout of a typical high head plant*

1 and 2 jets pelton wheels

HYDROPOWER PLANTS: low head run of river plant (Villanuova sul Clisi)

1

2

3

4

1) weir with flashboards
2) intake with flat gates as spillways
3) release of ecological flow
4) channel for cobbles and gravel settling, looking upstream

HYDROPOWER PLANTS: low head run of river plant (Villanuova sul Clisi)

5

6

7

8

5) channel for cobbles and gravel settling, looking downstream ***6) flat gates at junction between 6) and 8)***

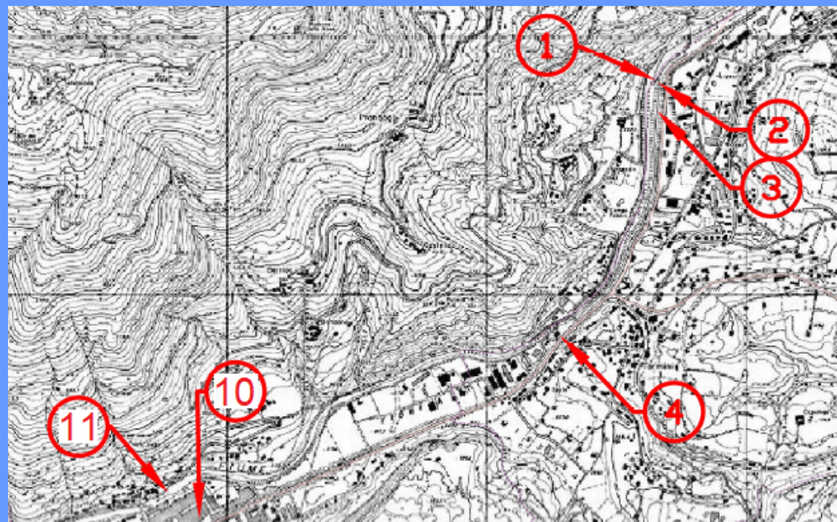
7) junction between 5) and 8). On the right the 2 sediment flushing gates ***8) power channel to the powerhouse***

HYDROPOWER PLANTS: *low head run of river plant (Villanuova sul Clisi)*

9

10

11



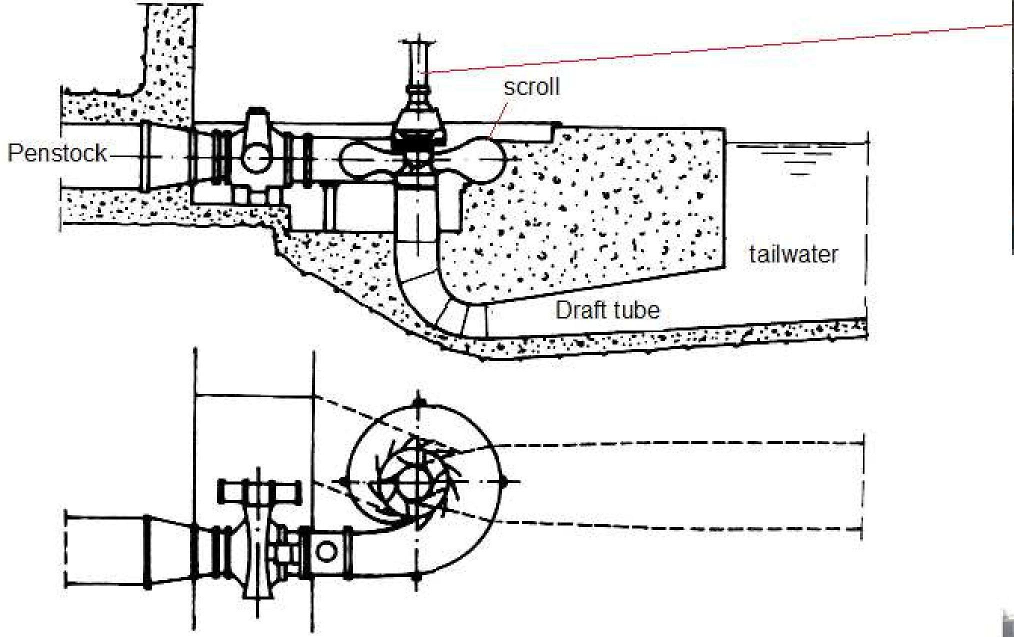
9) Trashrack before the powerhouse ***10) low head turbine***
11) Tail channel

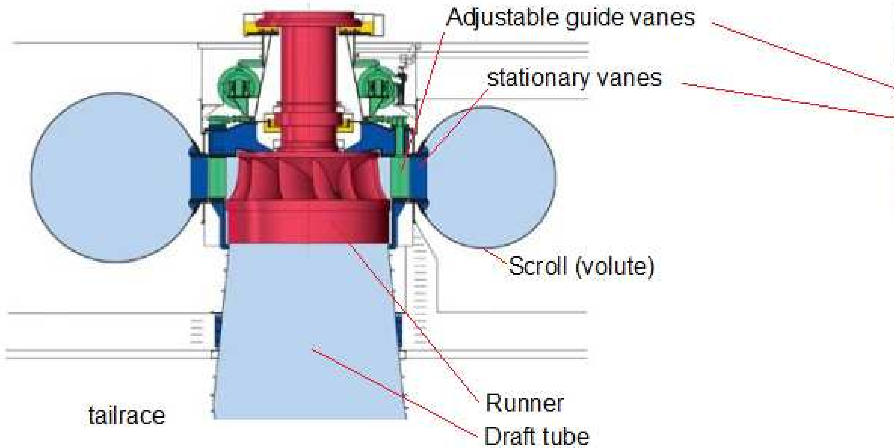
Francis Turbine

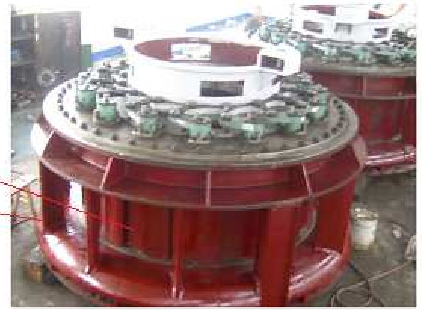
In other turbines, the water pressure can apply a force on the face of the runner blades, which decreases as it proceeds through the turbine. Turbines that operate in this way are called ***reaction turbines (turbine a reazione)****. The turbine casing, with the runner fully immersed in water, must be strong enough to withstand the operating pressure. Francis and Kaplan turbines belong to this category.*

In the following we shall present some fundamentals of the Francis functioning

SMALL HYDROPOWER PLANTS: Francis turbines







Balance of Rotational momentum for a control volume compring the runner (red circles below)

$$\frac{\partial}{\partial t}\left(\int_W (\vec{x}-\vec{x}_0)\wedge \rho\vec{V}dW\right) - \int_S (\vec{x}-\vec{x}_0)\wedge(\rho\vec{V})\vec{V}\cdot\vec{n}dS = \int_W (\vec{x}-\vec{x}_0)\wedge\rho\vec{g}dW + \int_S (\vec{x}-\vec{x}_0)\wedge\vec{\sigma}_n dS$$

$$-\int_S (\vec{x}-\vec{x}_0)\wedge(\rho\vec{V})\vec{V}\cdot\vec{n}dS = \int_S (\vec{x}-\vec{x}_0)\wedge\vec{\sigma}_n dS$$

Moment of Momentum of entering jets

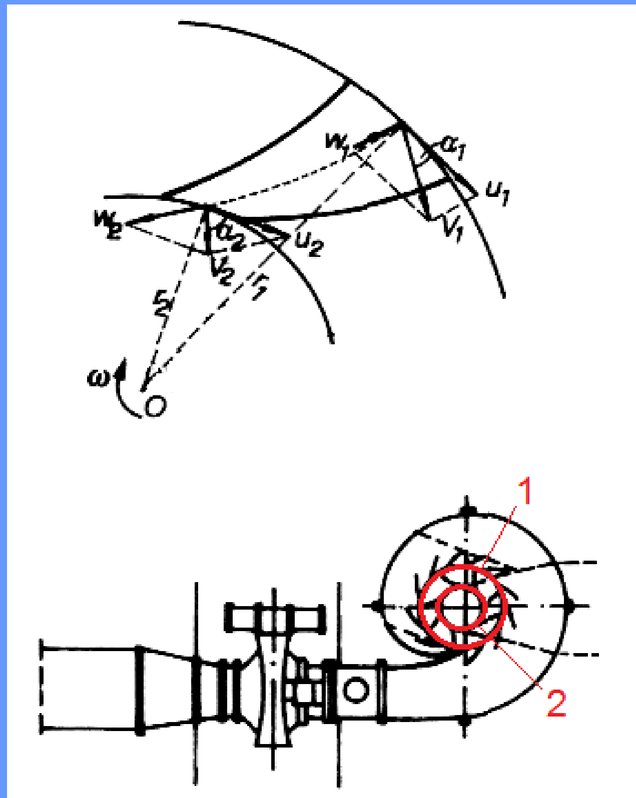
Resistive Torque exerted by the generator shaft on the runner

$$\int_S (\vec{x}-\vec{x}_0)\wedge(\rho\vec{V})\vec{V}\cdot\vec{n}dS$$

Moment of Momentum of entering flow (at 1 and flowing out at 2)

$$\begin{aligned} M &= \rho Q R_1 \wedge V_1 - \rho Q R_2 \wedge V_2 \\ &= \rho Q R_1 V_1 sen\beta_1 - \rho Q R_2 V_2 sen\beta_2 \\ &= \rho Q R_1 V_1 \cos\alpha_1 - \rho Q R_2 V_2 \cos\alpha_2 \end{aligned}$$

$$\begin{aligned} P = M\omega &= \rho Q R_1 \omega V_1 \cos\alpha_1 - \rho Q R_2 \omega V_2 \cos\alpha_2 \\ &= \rho Q\left(u_1 V_1 \cos\alpha_1 - u_2 V_2 \cos\alpha_2\right) \end{aligned}$$

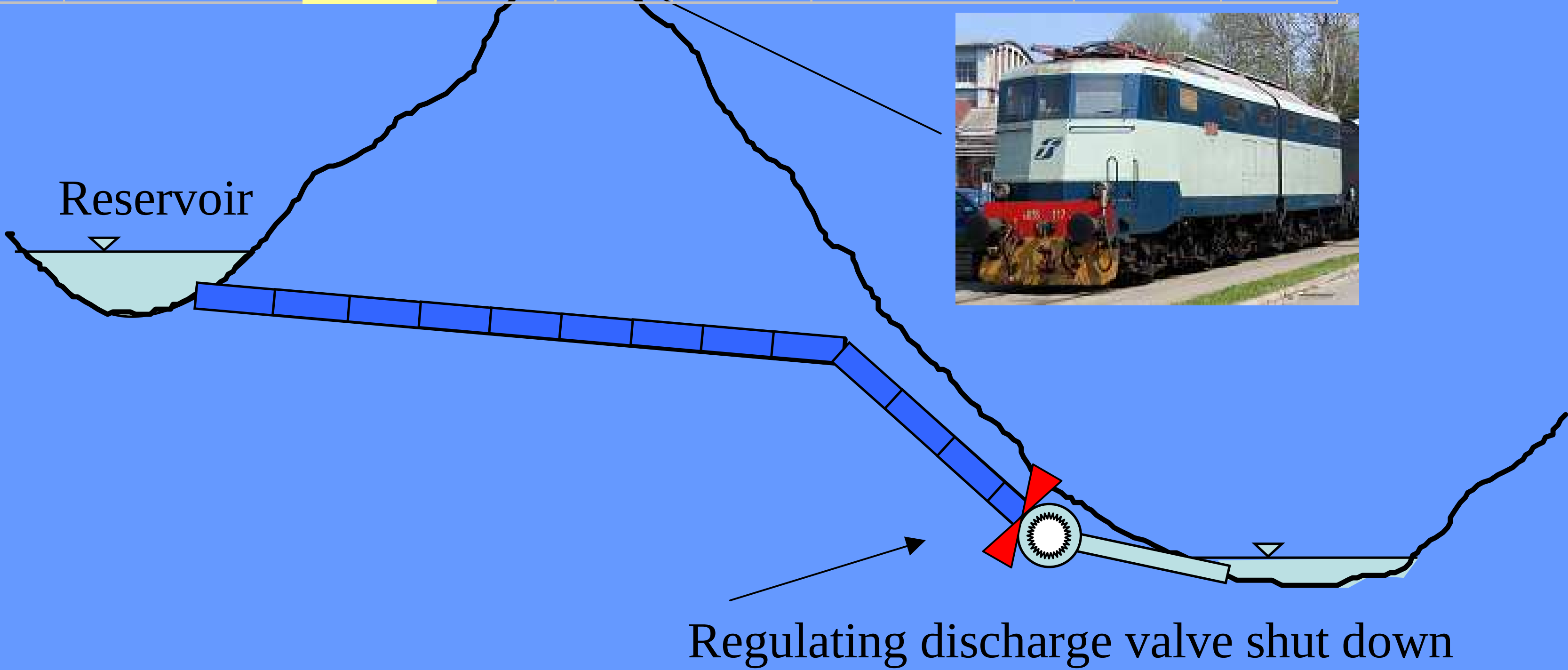


Ideal situation: α_1=0, α_2=90° which can be obtained only for the nominal discharge.

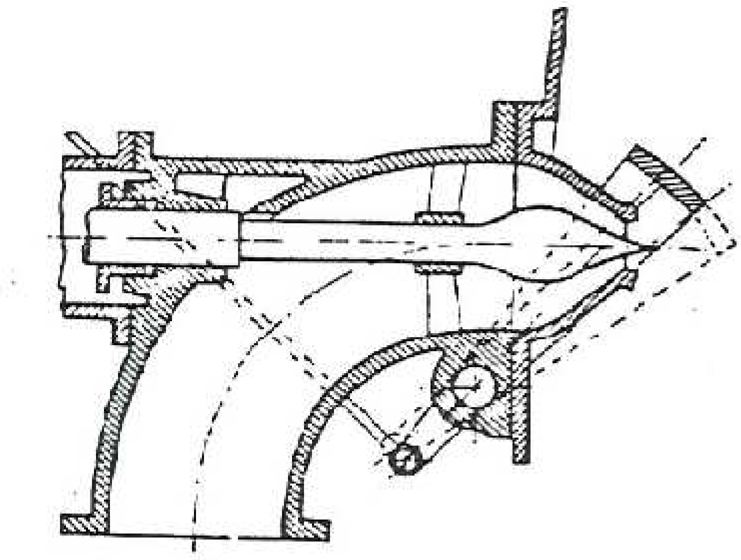
HYDROPOWER PLANTS: Unsteady flow

Momentum of water in a single penstock of San Fiorano at maximum Q

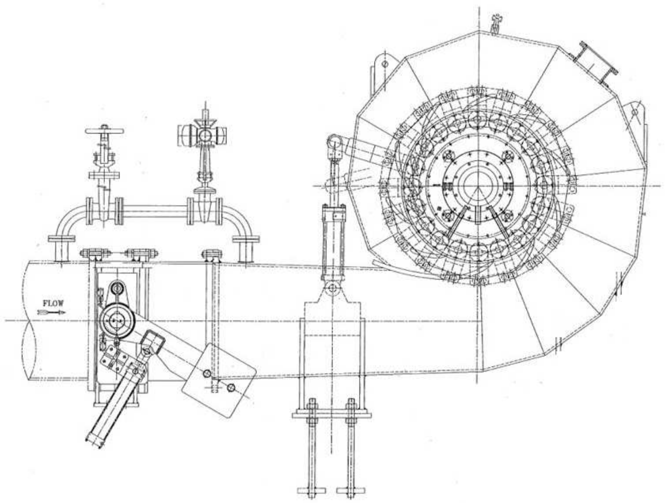
	Penstock					**Tunnel**		
	D	2.13		m		D	3.6	m
	Area	3.5632729		m^2		Area	10.1787602	m^2
	Q	22.5		m^3/s		L	4078	m
	L	2198		m		Q	45	m^3/s
	U	6.3144195		m/s		U	4.42097064	m/s
	Massa in movimento	7832073.9		kg		Massa in movimento	41508984.1	kg
(1)	Quantità di moto	49455000		kgm/s	**(1)**	Quantità di moto	183510000	kgm/s
	Massa locomotore	101000		kg	**(2)**	Quantità di moto loc.	2805555.56	kgm/s
	U	100 km/h	27.77778	m/s	**(1)/(2)**	Rapporto QdM	65.409505	
(2)	Quantità di moto loc.	2805555.6		kgm/s				
(1)/(2)	Rapporto QdM	17.627525						



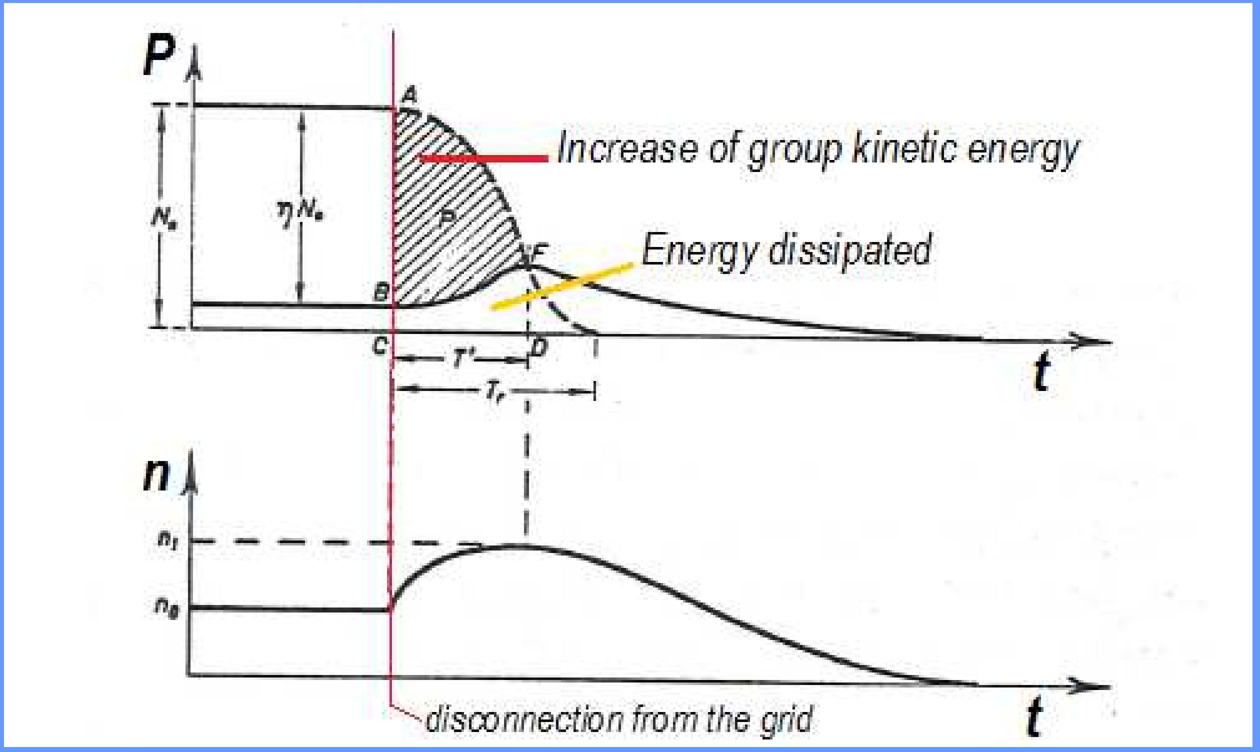
HYDROPOWER PLANTS: *regulation of discharge*



In a Pelton turbine the discharge is controlled through a needle valve (spina di regolazione o ago doble) and sudden shut down are operated through a deflector (tegolo deviatore)



In a Francis turbin the discharge is controlled by the distributor, that is formed by a certain number of mobile blades, whose group constitutes a concentrical ring around the impeller. Its function is to distribute, regulate or cut totally, the water flow that flows toward the impeller.

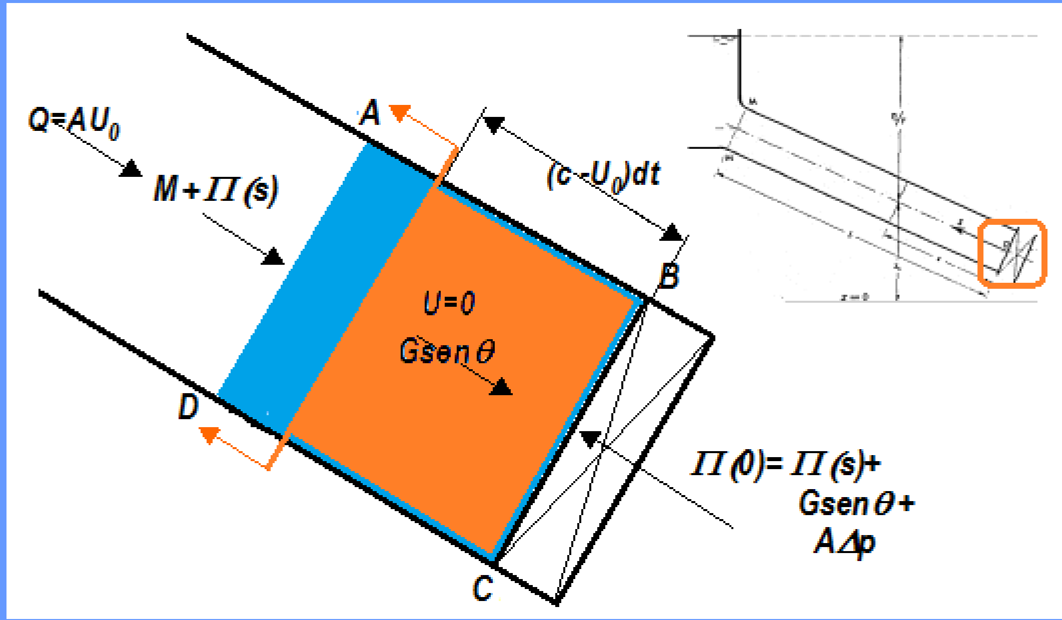


Time evolution of Power delivered (P) on the turbine axis and rotational speed (n) as a function of a sudden disconnection from the grid

Sudden shut down of a regulating valve in a penstock

The sudden U and Q reduction is comunicated to the incoming fluid through a pressure wave which travels with celerity c >>U_0.

Let us determine this pressure increase by applying the global momentum equation to the control volume ABCD, which comprises inside it the shock front.



$$-\vec{I}+\vec{M}+\vec{G}+\vec{\Pi}=0$$

$$-\frac{\partial}{\partial t}\left(\int_W \rho\vec{V}dW\right)+\int_S \rho\vec{V}(\vec{V}\cdot\vec{n})dS+\int_W \rho\vec{g}dW+\int_S \vec{\sigma}_n dS=0$$

$$\vec{I}=\lim_{dt\to 0}\frac{\mathbf{Q}_m(t+dt)-\mathbf{Q}_m(t)}{dt}=\lim_{dt\to 0}\frac{-\rho A\vec{U}_0(c-U_0)dt}{dt}=-\rho A\vec{U}_0(c-U_0)$$

$$\rho AU_0(c-U_0)+\rho QU_0-A\Delta p=0$$

$$\boxed{\Delta p=\rho cU_0}$$

Also known as Joukowsky's formula

Water hammer for San Fiorano			
	c	1086.000	m/s
	Q	22.500	m^3/s
	U	6.314	m/s
1	Δp_Jouk.	68.575	bar
	Δz	1418.000	m
2	Δp_hyd.	139.049	bar
1/2	Δp_Jouk/Δp_hyd.	0.493	

OIGAWA failure (1950): see paper:

Bonin, C. C. (1960). "Water-hammer damage to Oigawa Power Station." J. Engrg. for Power, 82(2), 111–119.

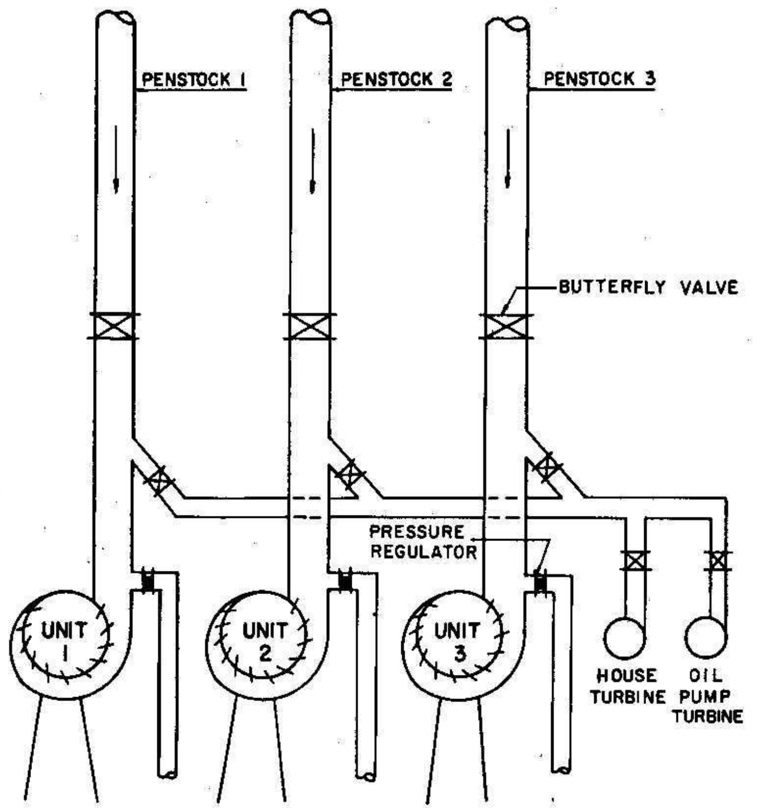


Fig. 1 Turbine water supply schematic diagram

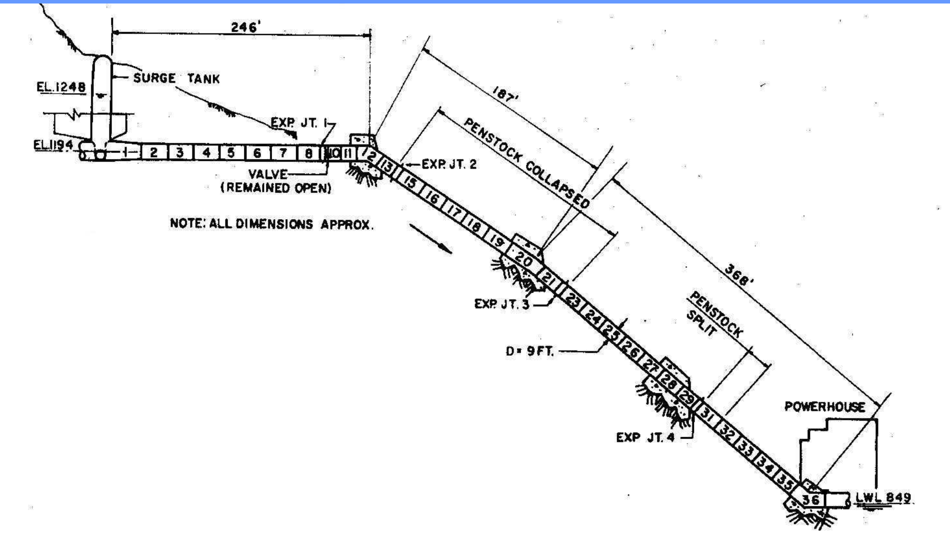


Fig. 2 Oigawa penstock profile

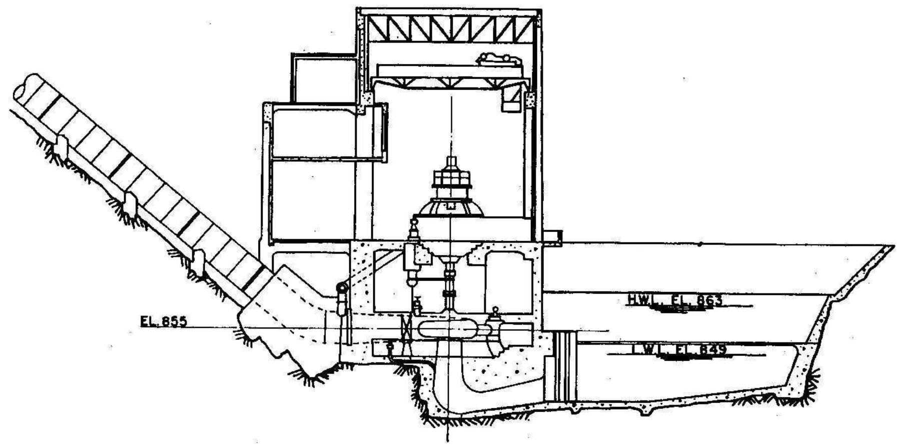
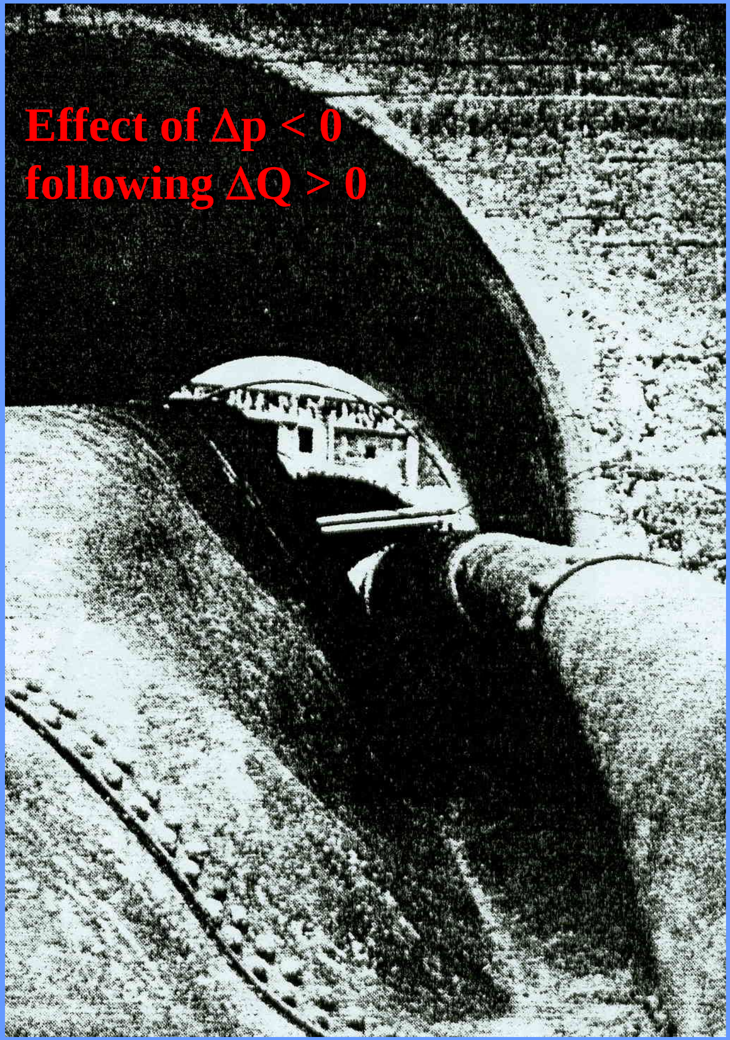
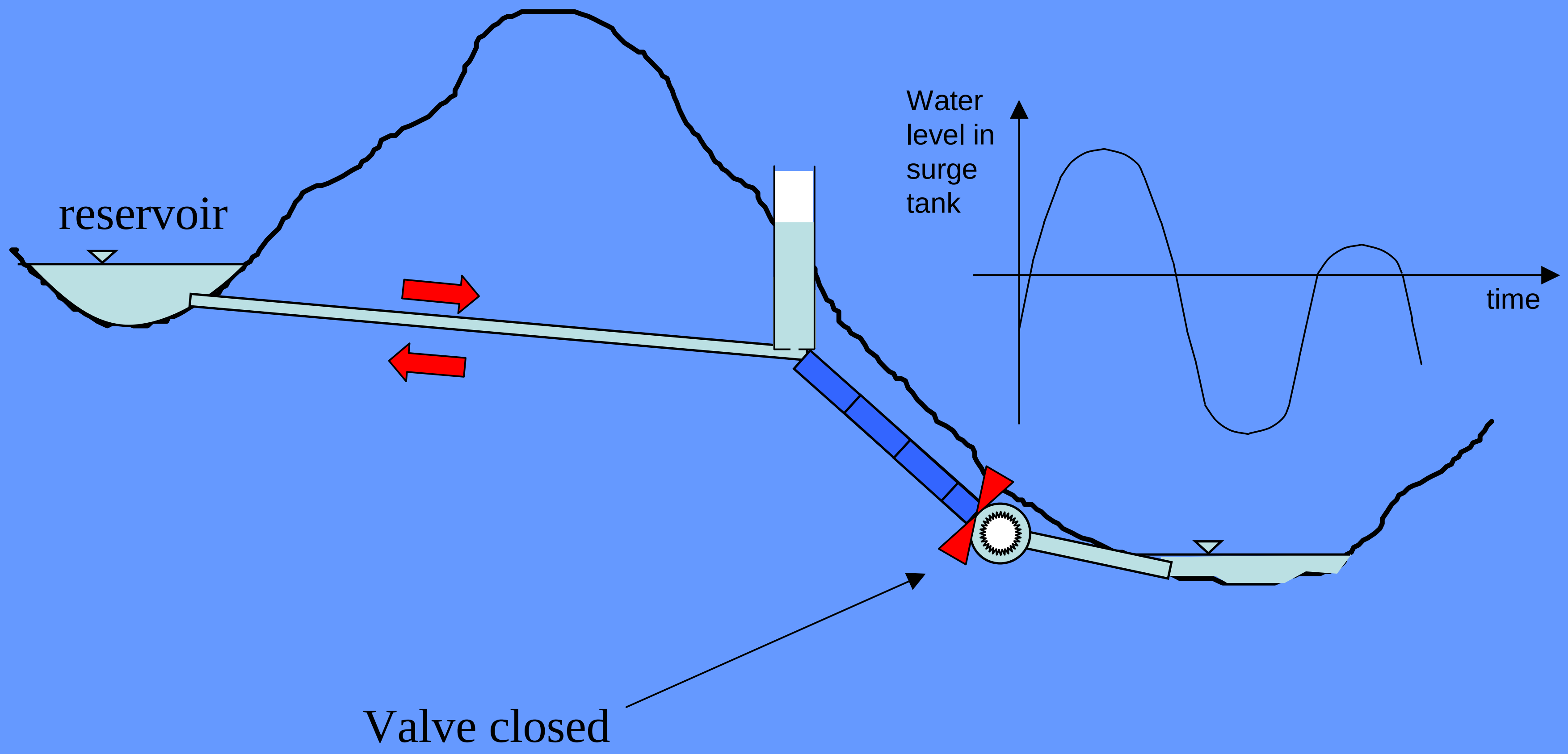


Fig. 3 Oigawa powerhouse

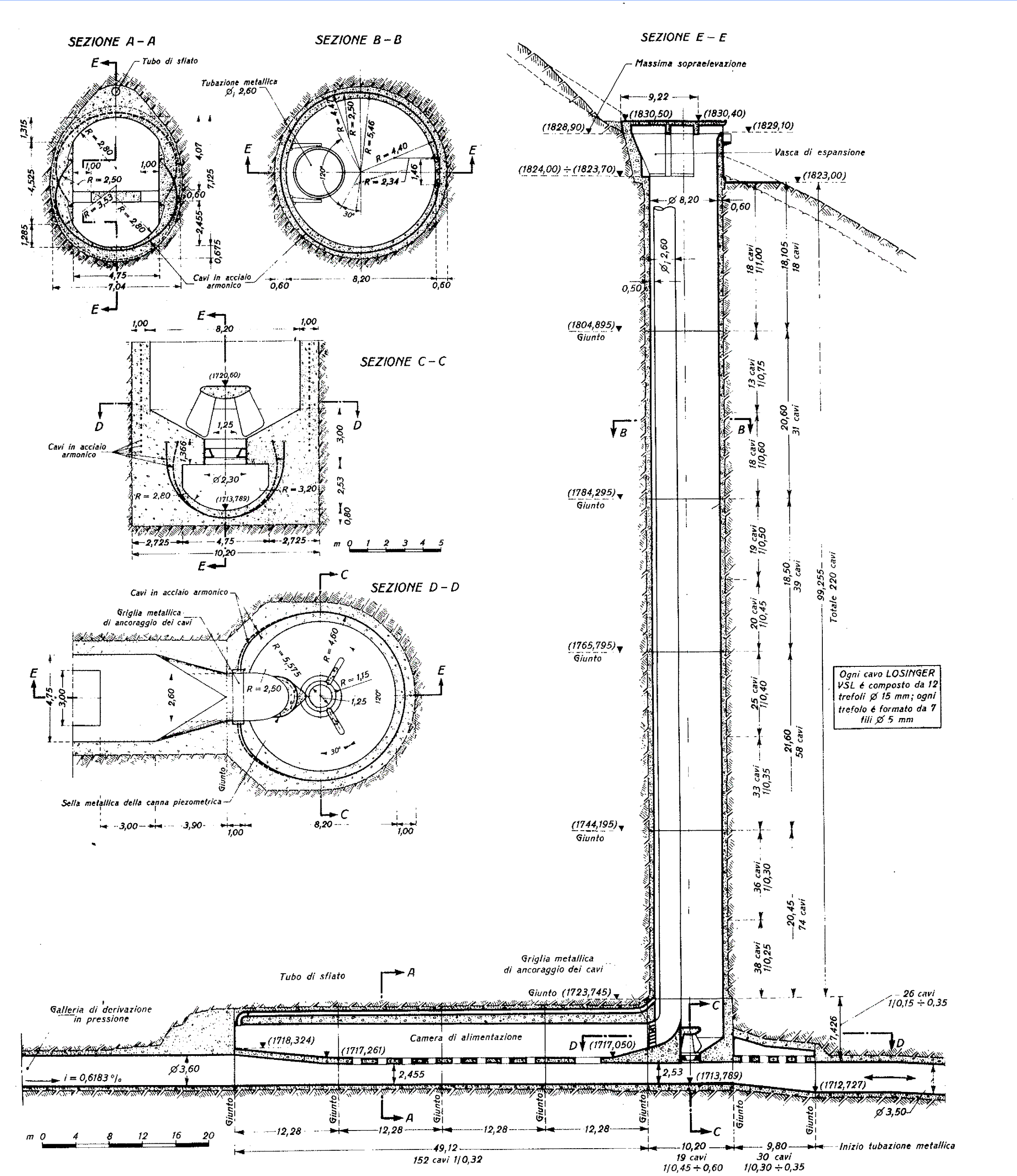
OIGAWA failure (1950)



Transient flow in a large, high head, hydropower plant: role of Surge tank (pozzo piezometrico)



LARGE HYDROPOWER PLANTS: tunnel and surge chamber

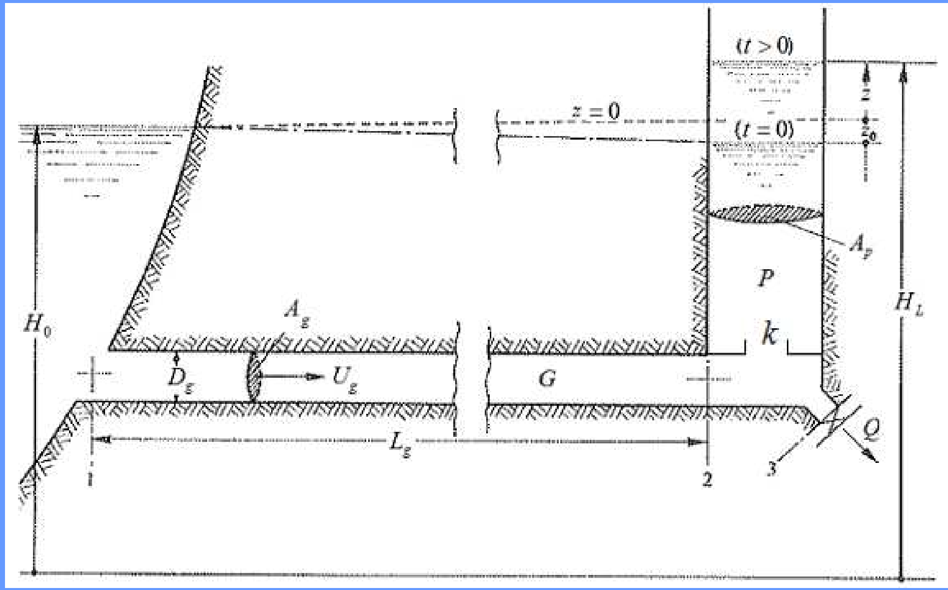


San Fiorano surge tank

Transient flow in a large, high head, hydropower plant: role of Surge chamber (pozzo piezometrico)

• *Mass oscillation between the low pressure tunnel and the surge tank*
•*Qualitative analysis: sudden startup and shutdown with the maximum operative discharge*
•*Hypothesis: fluid is incompressible (ε=0); the boundary geometry does not vary in time; operating valve at the end of the tunnel; velocity is negligible within the surge chamber*

$$\begin{cases} \dfrac{\partial H}{\partial s} = -\dfrac{1}{g}\dfrac{\partial U}{\partial t} - J \\ \dfrac{\partial \rho A}{\partial t} + \dfrac{\partial \rho A U}{\partial s} = 0 \\ J = \dfrac{\lambda U^2}{8g\overline{R}} \\ \lambda = f(\mathrm{Re}, \dfrac{\varepsilon}{\overline{R}}) \end{cases}$$



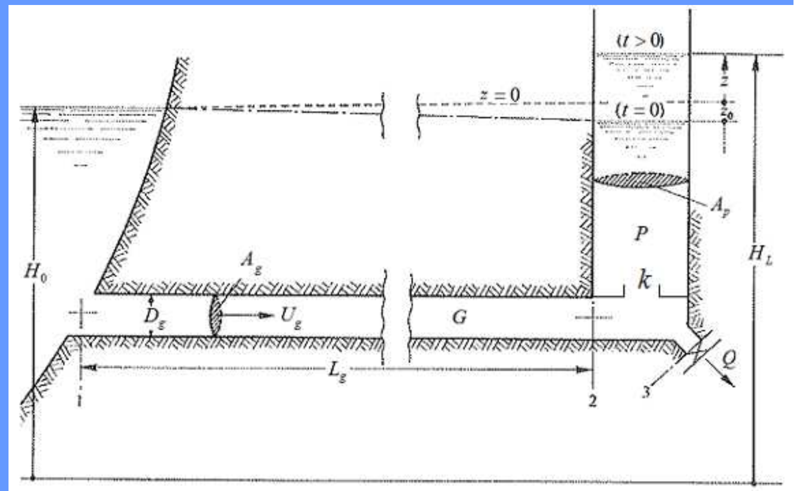
• *Under these hypothesis, from the continuity equation we get* $\dfrac{\partial AU}{\partial s} = 0$

And from the energy equation $H(L) - H(0) = -\dfrac{1}{g}\dfrac{dU}{dt}L \mp JL \mp kQ^2$ *where - holds when the flow is from the reservoir to the surge chamber (U > 0) and + when the flow is reversed from the chamber to the reservoir (U < 0)*

$$\begin{cases} \dfrac{\partial H}{\partial s} = -\dfrac{1}{g}\dfrac{\partial U}{\partial t} - J \\ \dfrac{\partial \rho A}{\partial t} + \dfrac{\partial \rho A U}{\partial s} = 0 \\ J = \dfrac{\lambda U^2}{8g\bar{R}} \\ \lambda = f(\mathrm{Re}, \dfrac{\varepsilon}{\bar{R}}) \end{cases}$$

$$H(L) - H(0) = -\frac{1}{g}\frac{dU}{dt}L \mp JL \mp kQ^2 \qquad \frac{\partial AU}{\partial s} = 0$$

$$z(L) + \frac{U^2}{2g} = -\frac{1}{g}\frac{dU}{dt}L - \frac{\lambda U|U|}{2gD}L - kQ|Q| \qquad Q = Q(t)$$



$$\frac{\partial A_g U}{\partial s} = 0 \;\rightarrow\; A_g U_g = A_p \frac{dz}{dt} \;\rightarrow\; U_g = \frac{A_p}{A_g}\frac{dz}{dt}$$

$$z(L) + \frac{L}{g}\frac{A_p}{A_g}\frac{d^2 z}{dt^2} + \underbrace{\frac{1}{2g}\left(\frac{A_p}{A_g}\right)^2\left(\frac{dz}{dt}\right)^2}_{1} + \underbrace{\frac{\lambda L}{2gD}\left(\frac{A_p}{A_g}\right)^2 \frac{dz}{dt}\left|\frac{dz}{dt}\right|}_{2} + \underbrace{kA_p^2 \frac{dz}{dt}\left|\frac{dz}{dt}\right|}_{3} = 0$$

The terms encircled above are head losses:

1. *kinetic term at the surge tank,*
2. *distributed head losses in the tunnel,*
3. *localized head loss at the surge chamber entrance.*

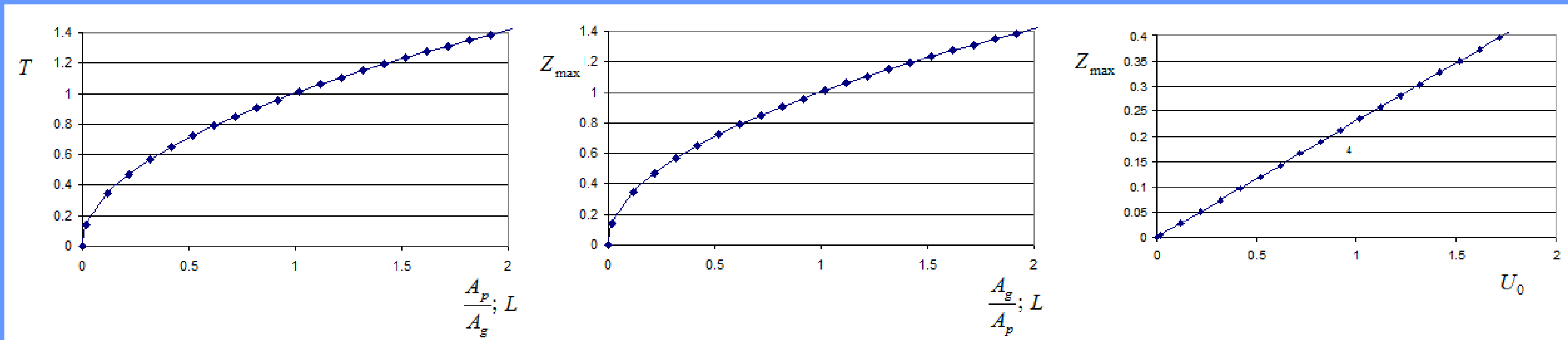
Let us initially disregard the terms representing energy losses and kinetic energy at the surge tank

$$z+\frac{L}{g}\frac{A_p}{A_g}\frac{d^2z}{dt^2}=0$$

$$\frac{d^2z}{dt^2}+\left(\sqrt{\frac{gA_g}{LA_p}}\right)z=0; \qquad \frac{d^2z}{dt^2}+\omega^2 z=0; \qquad \omega=\sqrt{\frac{gA_g}{LA_p}}; \qquad \boxed{T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{LA_p}{gA_g}}}$$

$$z(t)=C_1 sen\,\omega t+C_2\cos\omega t \qquad + i.c. \quad z(0)=0; \quad \frac{dz}{dt}(0)=\frac{A_g}{A_p}U_0$$

$$z(t)=U_0\sqrt{\frac{LA_g}{gA_p}}sen\,\omega t \qquad \boxed{Z_{\max}=\pm U_0\sqrt{\frac{LA_g}{gA_p}}}$$

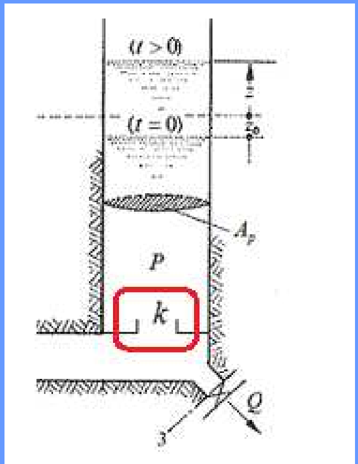


When the water level reaches the maximum elevation in the surge tank, the potential energy increment equals the initial kinetic energy content in the low pressure tunnel

$$Z_{\max} = \pm U_0 \sqrt{\frac{LA_g}{gA_p}}; \qquad \Delta E_p = \frac{1}{2}\gamma A_p Z_{\max}^2 = \frac{1}{2}\rho A_g L U_0^2$$

This is a consequence of the hypothesis on the absence of energy dissipation within the system; The maximum oscillations are important because they are considered in the plant design, in order to find out maximum pressures in the tunnel (Zmax > 0) and to avoid air entrance (Zmax < 0) within the tunnel.

The contraction at the basis of the surge tank is important because it decreases both T and the maximum elevation. However its value must be chosen carefully. An optimal contraction is such that the piezometric head is kept constant throughout the oscillation at the surge tank entrance so that z+head loss = Z_{max}



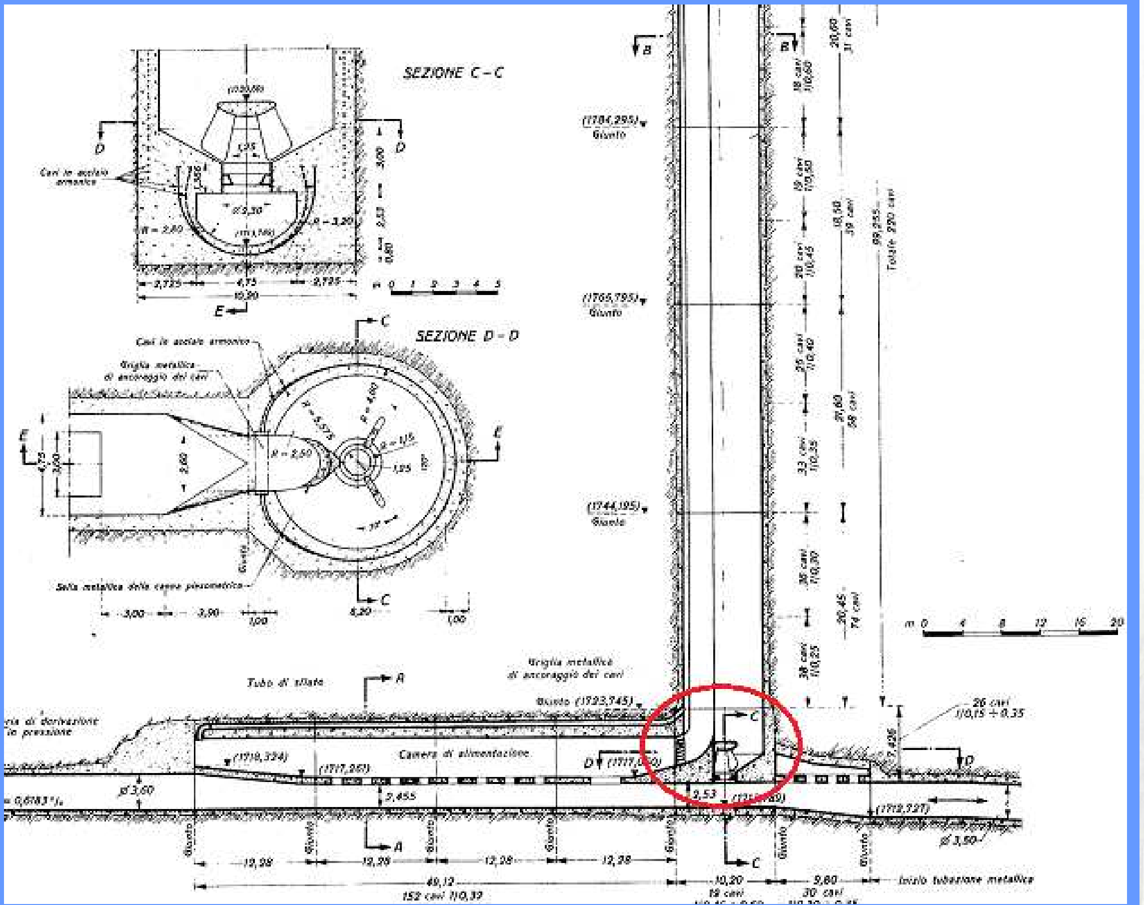
$$\boxed{z(L) + kA_p^2\left(\frac{dz}{dt}\right)^2} + \frac{L}{g}\frac{A_p}{A_g}\frac{d^2z}{dt^2} = 0 \qquad Z_{\max} + \frac{L}{g}\frac{A_p}{A_g}\frac{d^2z}{dt^2} = 0 \qquad z = \frac{A_g}{A_p}t\left(U_0 - \frac{Z_{\max}gt}{2L}\right)$$

$$z = Z_{\max} \quad se\ t = \frac{1}{2}\frac{2LU_0}{Z_{\max}g}; \qquad \boxed{Z_{\max} = \frac{1}{\sqrt{2}}U_0\sqrt{\frac{LA_g}{gA_p}}}$$

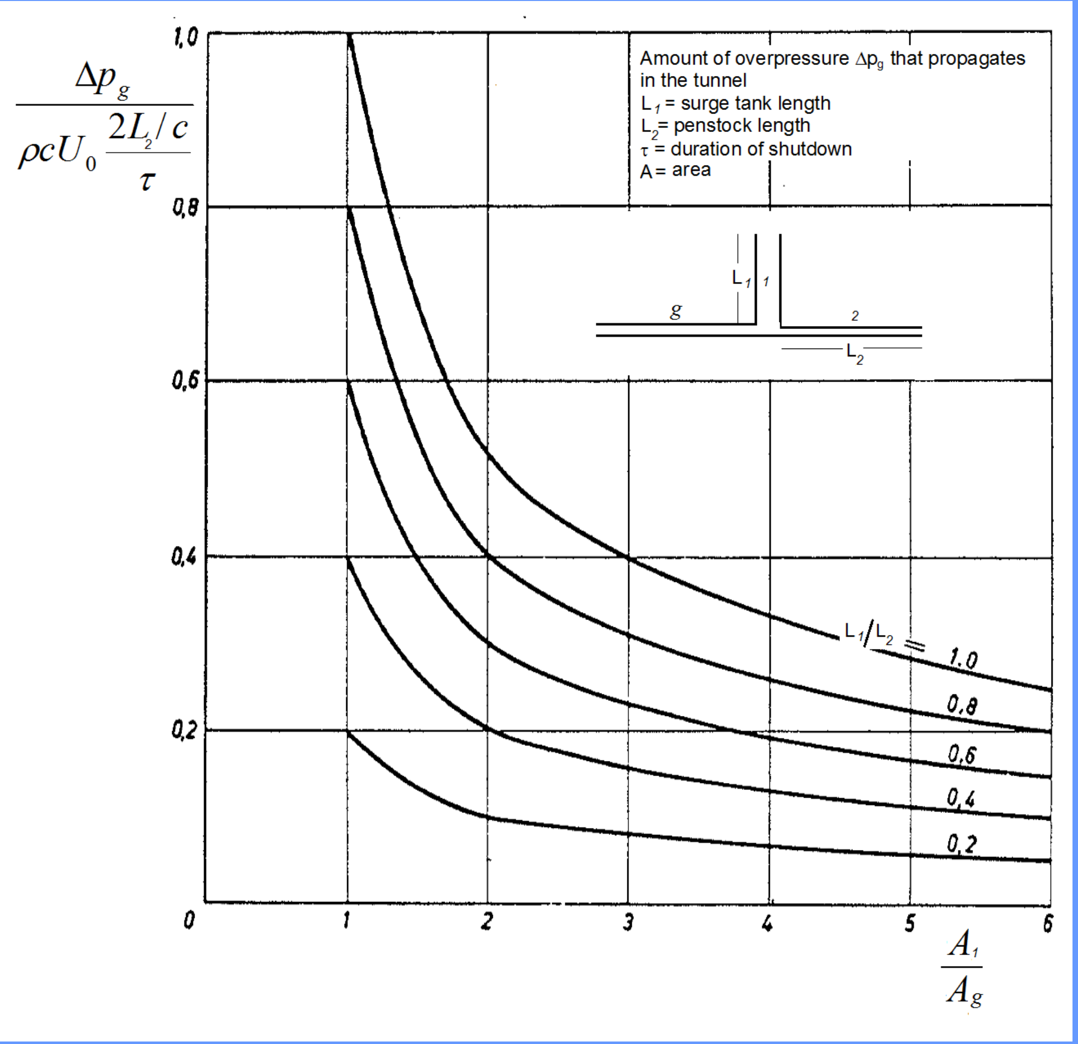
Its shape is found case by case experimentally.

Note that a smaller contraction would cause a pressure increase and overpressure propagation along the tunnel

<u>*optimal contraction*</u>



Usually L is given on the basis of the plant layout, A_g on the basis of the actual head loss in the tunnel and A_p is the parameter to choose, also in order to prevent overpressure propagation in the tunnel.

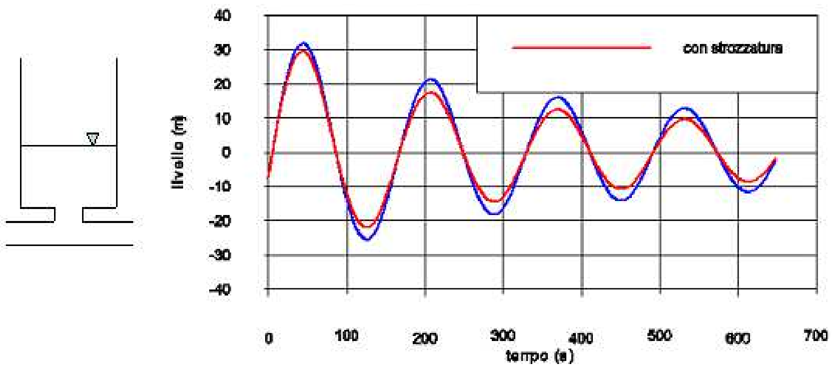


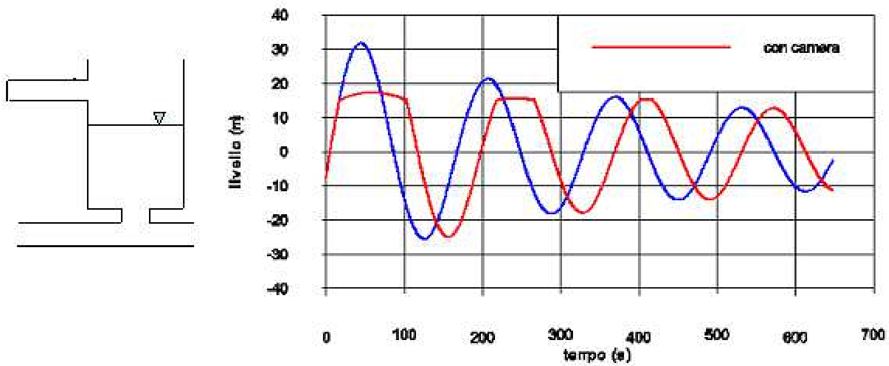
This theoretical graph shows the strong overpressure reduction in the tunnel even when the surge tank cross section area is slightly larger than the cross section area of the penstock

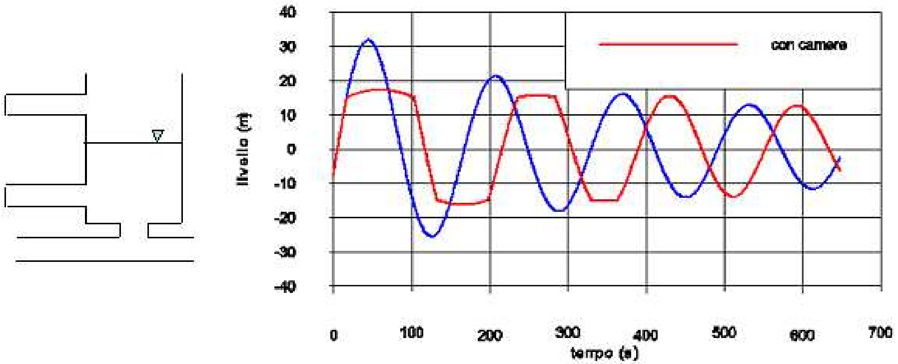
Real problems are studied by using either physical or numerical models. The numerical solution of the complete equation is rather straightforward. For instance, by a simple Finite Difference Method:

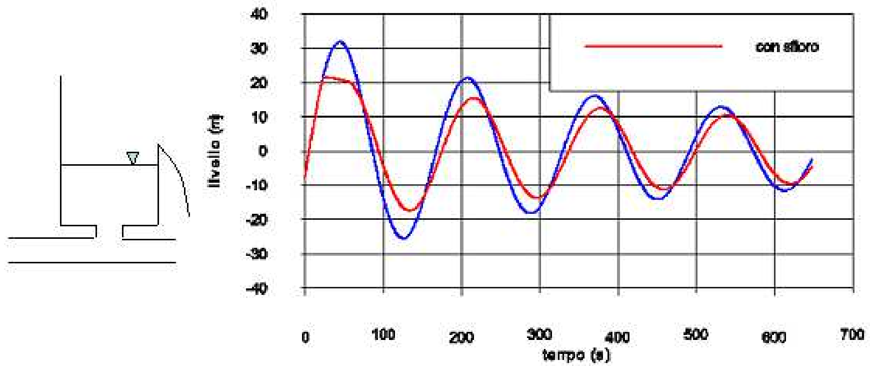
$$z(L)+\frac{1}{2g}\left(\frac{A_p}{A_g}\right)^2\left(\frac{z_{t+1}-z_t}{\Delta t}\right)^2+\frac{L}{g}\frac{A_p}{A_g}\left(\frac{z_{t+1}-2z_t+z_{t-1}}{\Delta t^2}\right)+\frac{z_{t+1}-z_t}{\Delta t}\left|\frac{z_{t+1}-z_t}{\Delta t}\right|\left[kA_p^2+\frac{\lambda L}{2gD}\left(\frac{A_p}{A_g}\right)^2\right]=0$$

Different types of surge chambers (cilindric, upper expansion chamber, upper and lower, with overflowing weir)

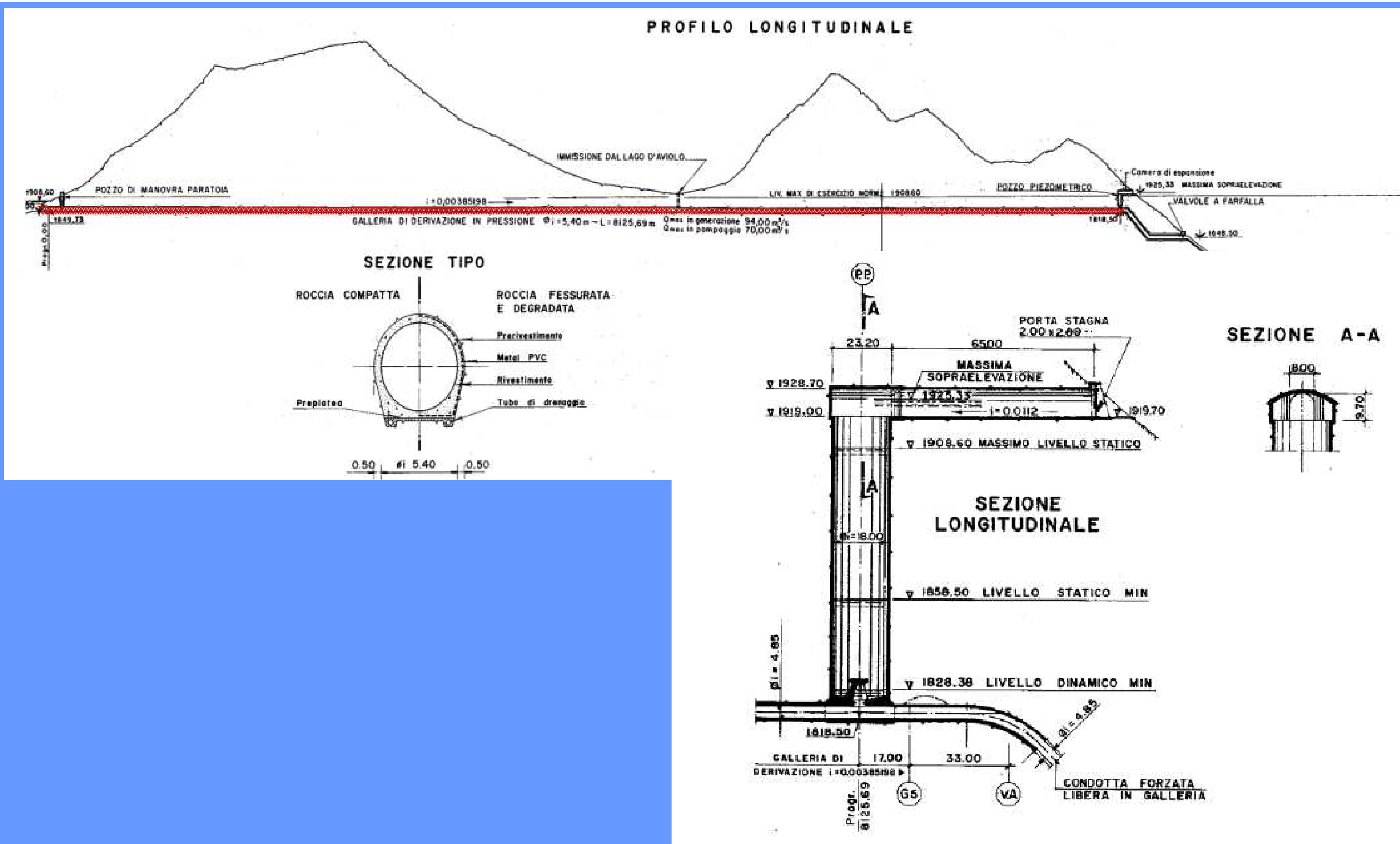








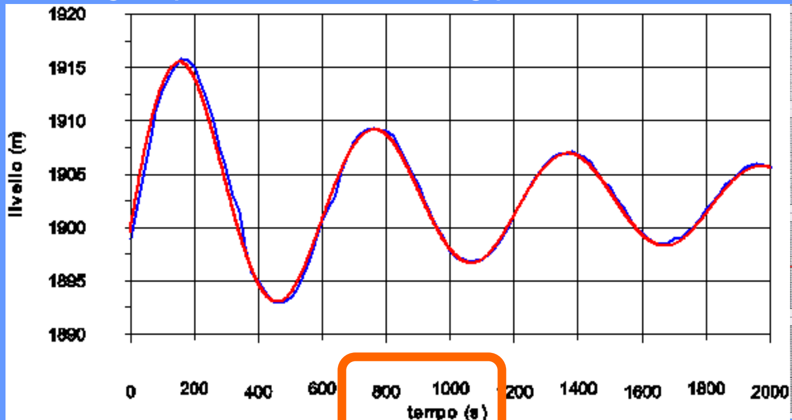
HYDROPOWER PLANTS: numerical simulation; Edolo surge chamber

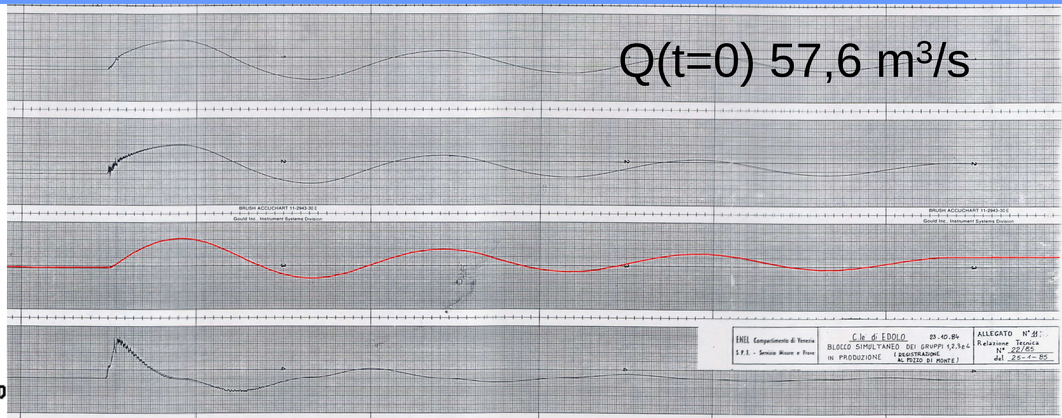


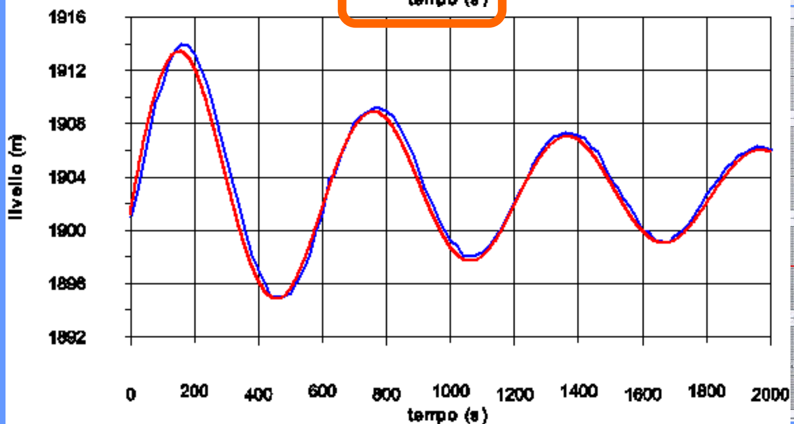
Pozzo piezometrico di monte dell'impianto di Edolo

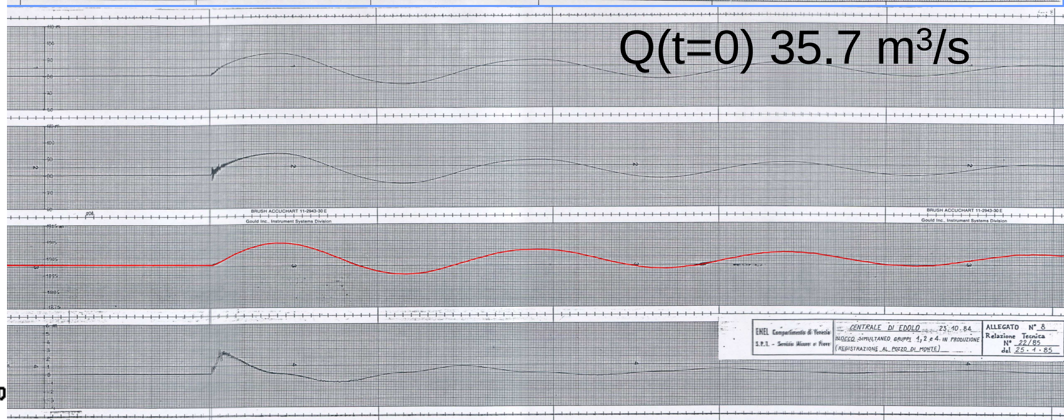
HYDROPOWER PLANTS: numerical simulation; Edolo surge chamber

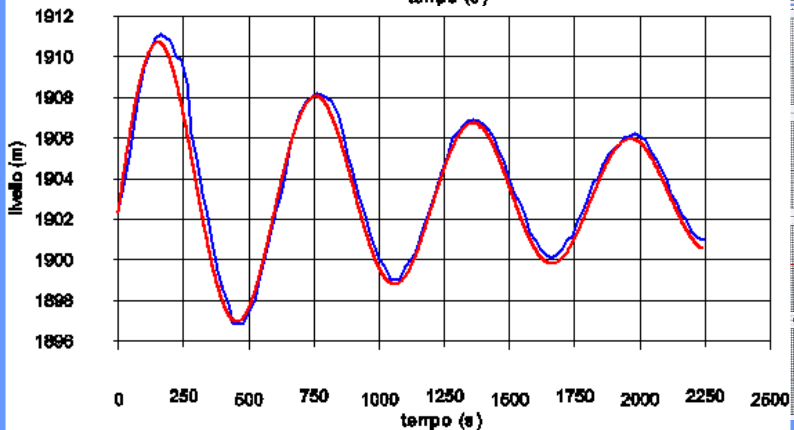
4/3/2 groups shut down during production: Observed ——— *Simulated* ———

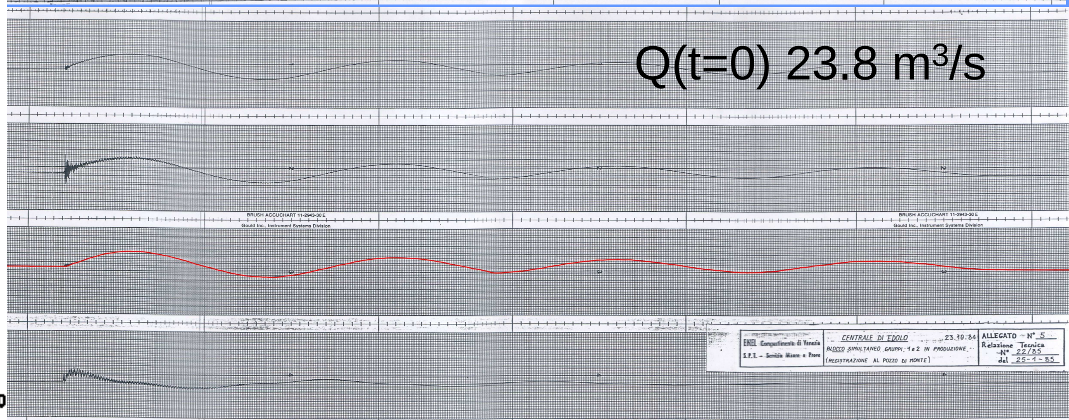








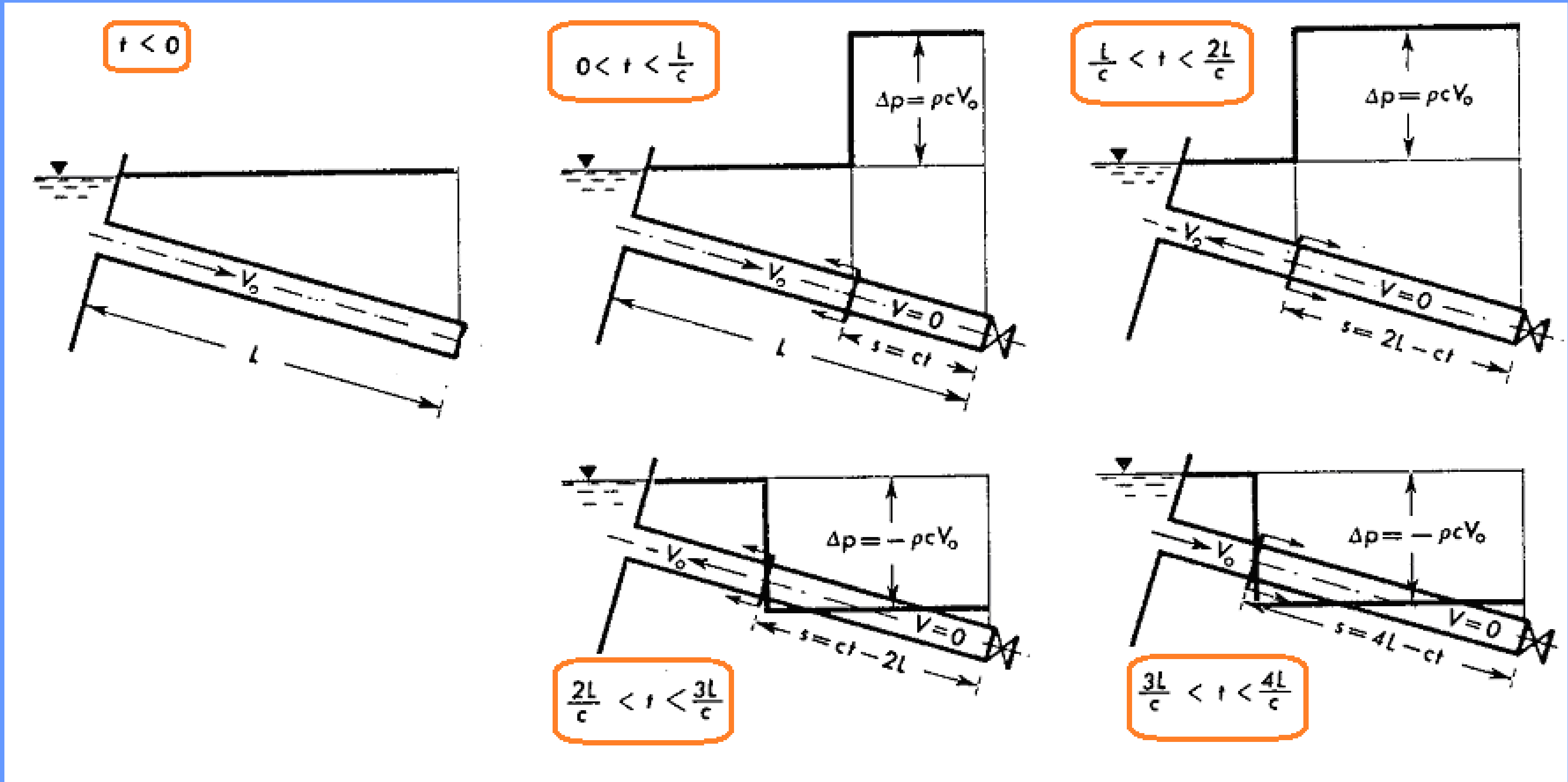




• Hypothesis: fluid is compressible ($\varepsilon \neq 0$, with ε bulk modulus of elasticity of the fluid, weakly dependent on pressure p).

•slope friction is negligible; pipe is prismatic and has an elastic behaviour.

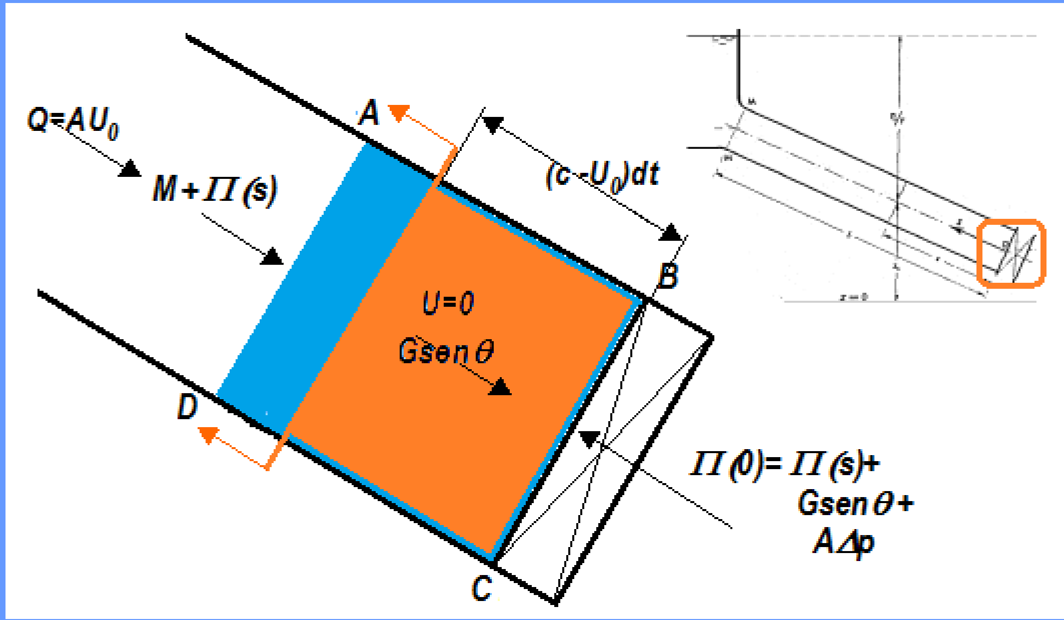
•Sudden closure of the discharge regulating valve



Sudden shut down of a regulating valve in a penstock

The sudden U and Q reduction is comunicated to the incoming fluid through a pressure wave which travels with celerity c >>U_0.

Let us determine this pressure increase by applying the global momentum equation to the control volume ABCD, which comprises inside it the shock front.



$$-\vec{I}+\vec{M}+\vec{G}+\vec{\Pi}=0$$

$$-\frac{\partial}{\partial t}\left(\int_W \rho\vec{V}dW\right)+\int_S \rho\vec{V}(\vec{V}\cdot\vec{n})dS+\int_W \rho\vec{g}dW+\int_S \vec{\sigma}_n dS=0$$

$$\vec{I}=\lim_{dt\to 0}\frac{\mathbf{Q}_m(t+dt)-\mathbf{Q}_m(t)}{dt}=\lim_{dt\to 0}\frac{-\rho A\vec{U}_0(c-U_0)dt}{dt}=-\rho A\vec{U}_0(c-U_0)$$

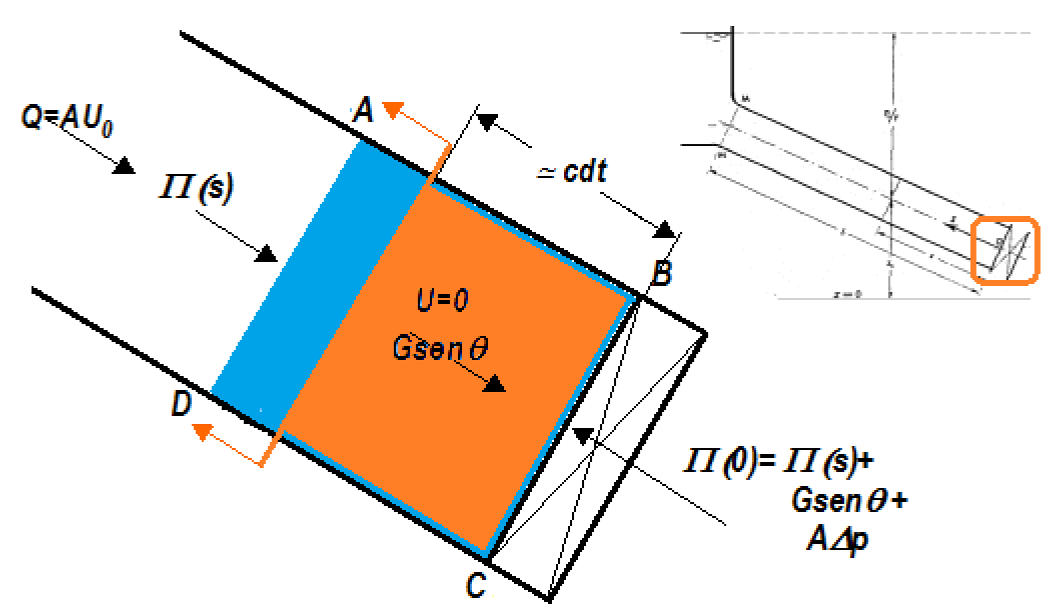
$$\rho AU_0(c-U_0)+\rho QU_0-A\Delta p=0$$

$$\boxed{\Delta p=\rho cU_0}$$

Also known as Joukowsky's formula

Note that if we accept that $c \gg U_0$ then $(c - U_0) \cong c$. This is consistent with the approximation that we shall do in the following, when we shall make the hypothesis that $c/U_0 \gg 1$.

*In such a case, however, **M,** which is proportional to U_0^2 is negligeable with respect to a term containing cU_0. Accordingly, if the momentum equation must be applied in an approximate fashion according to these hypothesis, the final result does not change.*



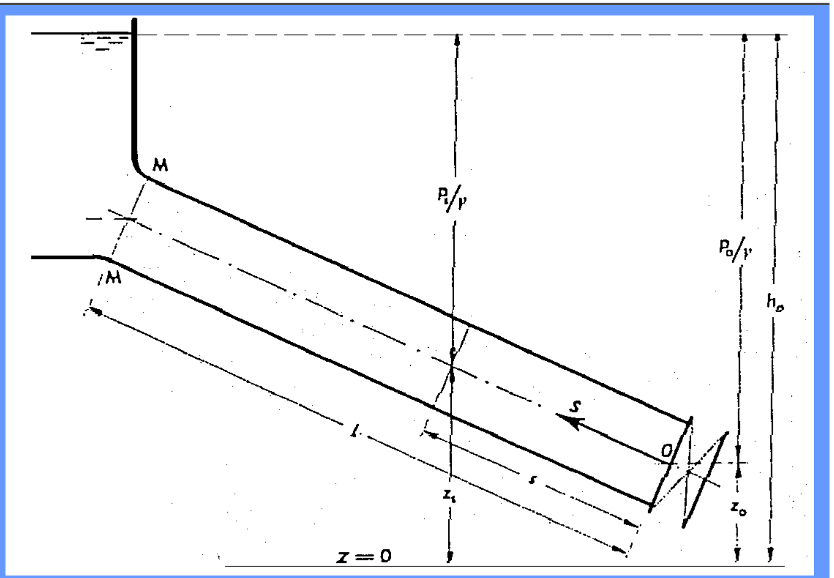
$$-\vec{I} + \vec{G} + \vec{\Pi} = 0$$

$$-\frac{\partial}{\partial t}\left(\int_W \rho \vec{V} dW\right) + \int_W \rho \vec{g} dW + \int_S \vec{\sigma}_n dS = 0 \qquad \cong$$

$$\vec{I} = \lim_{dt \to 0} \frac{\mathbf{Q}_m(t+dt) - \mathbf{Q}_m(t)}{dt} = \lim_{dt \to 0} \frac{-\rho A \vec{U}_0 c dt}{dt} = -\rho A \vec{U}_0 c$$

$$\rho A U_0 c - A \Delta p = 0$$

$$\boxed{\Delta p = \rho c U_0}$$ as before



- Hypothesis: fluid is compressible ($\varepsilon \neq 0$, with ε bulk modulus of elasticity of the liquid, weakly dependent on pressure p)

$$\frac{1}{\varepsilon} = -\frac{1}{V}\left.\frac{\partial V}{\partial p}\right|_{S=\cos t} = \frac{1}{\rho}\left.\frac{\partial \rho}{\partial p}\right|_{S=\cos t} ; \qquad \varepsilon \propto 10^9\ Pa$$

- slope friction is negligible;

$$J = \frac{\lambda U|U|}{8g\bar{R}} \cong 0$$

- dependent variables involved have a propagatory behaviour; c, elastic wave celerity, is much larger than U, that is the average mass transfer velocity

$$f(s+ds, t+dt) = f(s,t); \quad \frac{\partial f}{\partial t} + \frac{ds}{dt}\frac{\partial f}{\partial s} = 0; \quad \frac{ds}{dt} \equiv c; \quad c >> U; \quad \left|\frac{c}{U}\right| \equiv \frac{1}{Ma} = \left|\frac{\frac{\partial f}{\partial t}}{U\frac{\partial f}{\partial s}}\right| >> 1; \quad \left|\frac{\partial f}{\partial t}\right| >> \left|U\frac{\partial f}{\partial s}\right|$$

- pipe is prismatic and with elastic behaviour (Young's modulus of elasticity E);

$$d\sigma = E\frac{dD}{D}$$

- Two boundary conditions: h constant upstream (surge tank level) and Q(t) at the penstock outlet controlled by the regulating valve

- *Accordingly, the following system of equations can be derived*

$$\begin{cases} \dfrac{\partial H}{\partial s} = -\dfrac{1}{g}\dfrac{\partial U}{\partial t}; & \dfrac{\partial h}{\partial s} + \dfrac{U}{g}\dfrac{\partial U}{\partial s} = -\dfrac{1}{g}\dfrac{\partial U}{\partial t}; \quad \dfrac{\partial h}{\partial s} = -\dfrac{1}{g}\dfrac{\partial U}{\partial t}; \\ \dfrac{\partial \rho A}{\partial t} + \dfrac{\partial \rho A U}{\partial s} = 0; & \dfrac{\partial \rho A}{\partial t} + U\dfrac{\partial \rho A}{\partial s} + \rho A \dfrac{\partial U}{\partial s} = 0; \quad \rho \dfrac{\partial A}{\partial p}\dfrac{\partial p}{\partial t} + A\dfrac{\partial \rho}{\partial p}\dfrac{\partial p}{\partial t} + \rho A \dfrac{\partial U}{\partial s} = 0; \end{cases}$$

$$\begin{cases} \dfrac{\partial h}{\partial s} = -\dfrac{1}{g}\dfrac{\partial U}{\partial t}; \\ \left(\dfrac{1}{A}\dfrac{\partial A}{\partial p} + \dfrac{1}{\rho}\dfrac{\partial \rho}{\partial p}\right)\dfrac{\partial p}{\partial t} + \dfrac{\partial U}{\partial s} = 0; \end{cases}$$

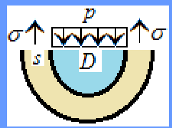
- *and, by observing that, being z constant*

$$h = z + \frac{p}{\gamma}; \qquad \frac{\partial h}{\partial t} = \frac{1}{\gamma}\left(1 - \frac{p}{\varepsilon}\right)\frac{\partial p}{\partial t} \cong \frac{1}{\gamma}\frac{\partial p}{\partial t}$$

$$\begin{cases} \dfrac{\partial h}{\partial s} = -\dfrac{1}{g}\dfrac{\partial U}{\partial t}; \\ \left(\dfrac{1}{A}\dfrac{\partial A}{\partial p} + \dfrac{1}{\rho}\dfrac{\partial \rho}{\partial p}\right)\gamma\dfrac{\partial h}{\partial t} + \dfrac{\partial U}{\partial s} = 0; \end{cases}$$

- *now we have to consider the stress-strain relationship of the pipe. We have made the assumption of a linear relationship according to Young's modulus E and of validity of Mariotte's equation*

$$d\sigma = E\frac{dD}{D}; \quad 2\sigma s = pD; \quad \frac{\partial A}{\partial p} = \frac{dA}{dD}\frac{dD}{d\sigma}\frac{\partial \sigma}{\partial p} = \frac{\pi D}{2}\frac{D}{E}\frac{D}{2s} = \frac{\pi D^3}{4Es}; \quad \frac{1}{A}\frac{\partial A}{\partial p} = \frac{D}{Es};$$



- *So that we obtain*

$$\begin{cases} \dfrac{\partial h}{\partial s} = -\dfrac{1}{g}\dfrac{\partial U}{\partial t}; \\ \dfrac{\rho}{\varepsilon}\left(\dfrac{\varepsilon D}{Es}+1\right) g \dfrac{\partial h}{\partial t} + \dfrac{\partial U}{\partial s} = 0; \end{cases} \qquad \begin{cases} \dfrac{\partial h}{\partial s} = -\dfrac{1}{g}\dfrac{\partial U}{\partial t}; \\ \dfrac{g}{c^2}\dfrac{\partial h}{\partial t} = -\dfrac{\partial U}{\partial s}; \end{cases}$$

where we have introduced the elastic wave celerity c

$$\frac{\rho}{\varepsilon}\left(\frac{\varepsilon D}{Es}+1\right) = \left(\frac{\sqrt{\frac{\varepsilon D}{Es}+1}}{\sqrt{\frac{\varepsilon}{\rho}}}\right)^2 \equiv \frac{1}{c^2}$$

- *And, if we reverse the positive s direction we eventually obtain the <u>Allievi's simplified system for water hammer analysis</u>*

$$\boxed{\begin{cases} \dfrac{\partial h}{\partial s} = \dfrac{1}{g}\dfrac{\partial U}{\partial t}; \\ \dfrac{g}{c^2}\dfrac{\partial h}{\partial t} = \dfrac{\partial U}{\partial s}; \end{cases}}$$

Actually, this system is equivalent to <u>d'Alambert second order PDE</u>, the prototype of hyperbolic equations

$$\boxed{\frac{\partial^2 h}{\partial t^2} = c^2\frac{\partial^2 h}{\partial s^2}; \qquad \frac{\partial^2 U}{\partial t^2} = c^2\frac{\partial^2 U}{\partial s^2}}$$

- *Analytical general solution of d'Alambert equation*

$$\frac{\partial^2 h}{\partial t^2} = c^2 \frac{\partial^2 h}{\partial s^2}$$

$$\boxed{h(s,t) = h_0(s,0) + F(s-ct) + f(s+ct)}$$

- *In order to obtain the solution for U we consider the first equation of Allievi's system*

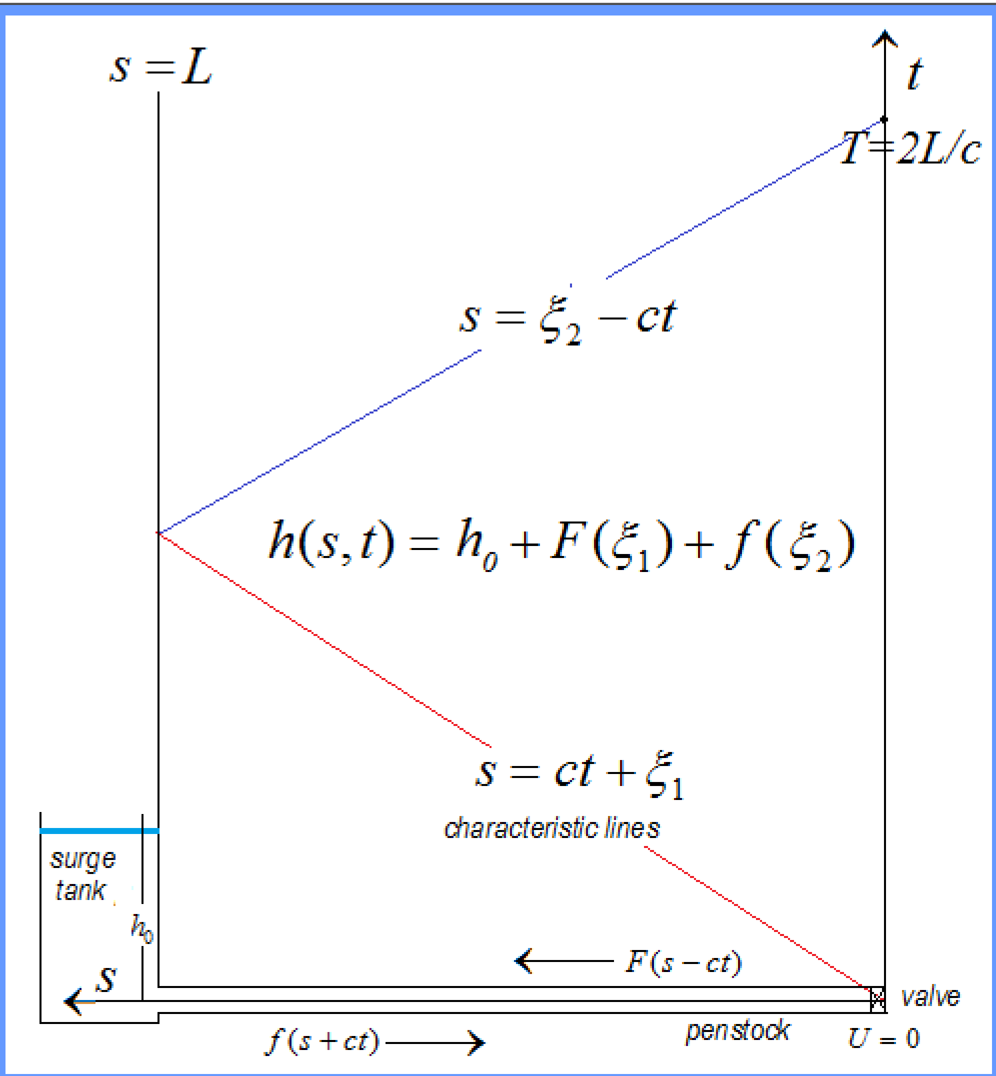
$$\frac{\partial h}{\partial s} = \frac{1}{g}\frac{\partial U}{\partial t}$$

$$\frac{\partial h}{\partial s} = \frac{\partial F}{\partial s} + \frac{\partial f}{\partial s} = -\frac{1}{c}\frac{\partial F}{\partial t} + \frac{1}{c}\frac{\partial f}{\partial t}$$

$$-\frac{1}{c}\frac{\partial F}{\partial t} + \frac{1}{c}\frac{\partial f}{\partial t} = \frac{1}{g}\frac{\partial U}{\partial t}$$

- *and by integration with respect to t, we get the general solution*

$$\boxed{U(s,t) = U_0(s,0) + \frac{g}{c}\left[f(s+ct) - F(s-ct)\right]}$$



- *F is a p wave that moves upstream, being unchanged when s-ct =ξ_1. In other words, when s=ct + ξ_1.*
f is a p wave that moves downstream, being unchanged when s+ct =ξ_2. In other words, when s= ξ_2.- ct.

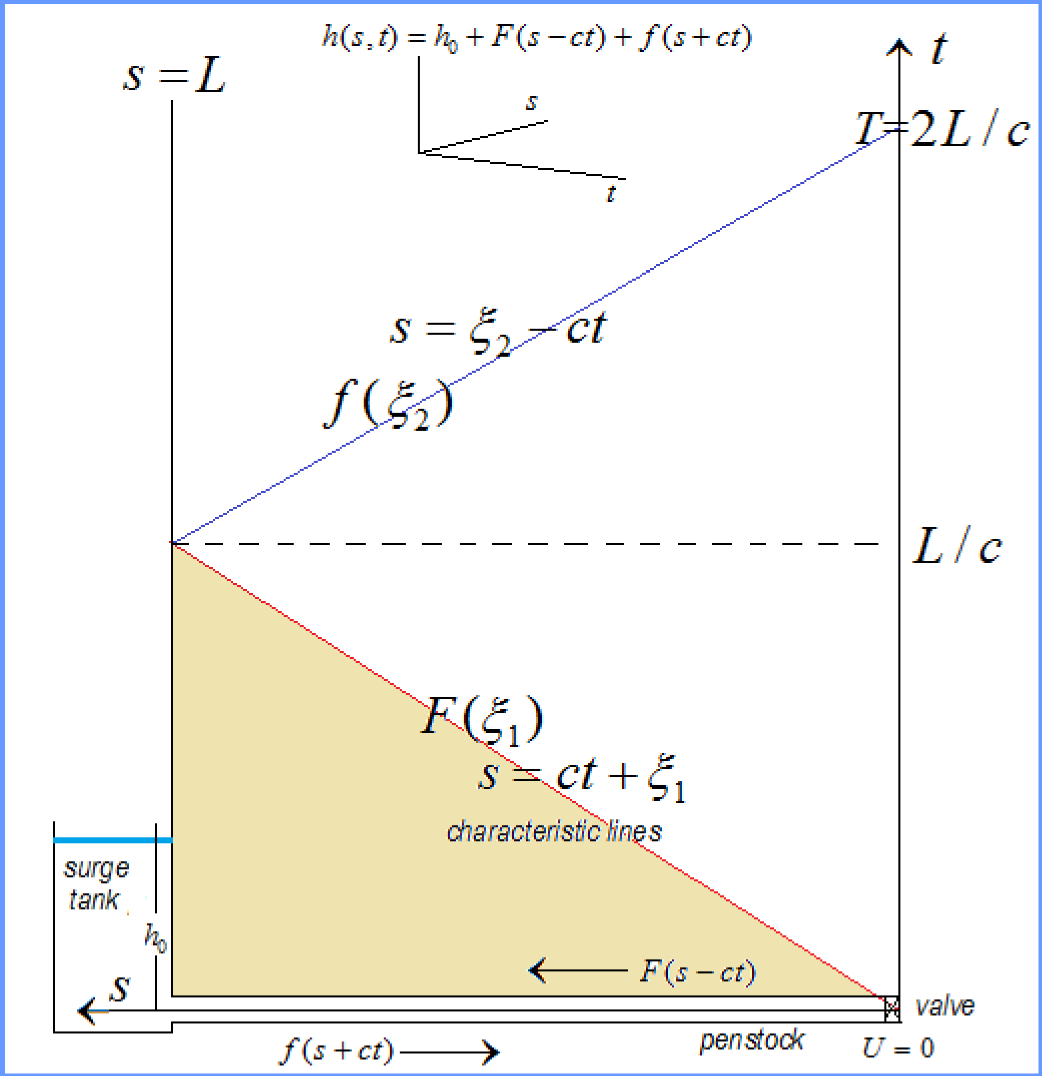
$$\boxed{h(s,t) = h_0 + F(s-ct) + f(s+ct)}$$

• *F is a generic pressure wave that moves upstream, being unchanged when s-ct =ξ_1. In other words, when s=ct + ξ_1. This implies that it propagates along the penstock without being modified. The value associated to F depends on the boundary conditions at the regulating valve at s=0. If t <s/c the flow is not affected by the downstream variation and is still steady*

•*f is a pressure wave that moves downstream, being unchanged when s+ct =ξ_2. In other words, when s= ξ_2- ct. It is generated by reflection of F at the upstream boundary (surge tank). Actually if s = L, then h=h_0. Accordingly*

$$\boxed{h(L,t) = h_0; \qquad f(L+ct) = -F(L-ct)}$$

Accordingly, one can conclude that 1) the descending f wave has the same value of the ascending F wave but with opposite sign. 2) for each s, f=0 while t <(2L-s)/c.



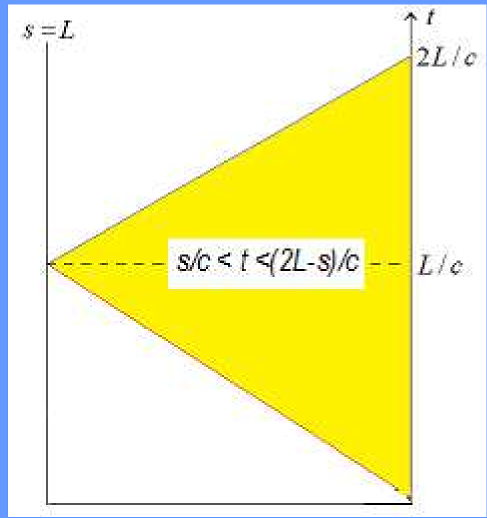
Let us consider a variation of the flowing discharge. The discharge is controlled at the penstock outlet (s=0) through a relationship

$$Q(t) = \Sigma(t)\sqrt{2gh(0,t)}$$

Where Σ is proportional to the valve opening, that is the quantity whose variation in time can be controlled.The variation of Q implies a corresponding variation of the average velocity U immediately upstream in the penstock; At each s along the penstock, while s/c < t <(2L-s)/c, f=0; accordingly:

$$\begin{cases} h(s,t) = h_0 + F(s-ct) \\ \frac{c}{g}\left[U_0 - U(s,t)\right] = F(s-ct) \end{cases}$$

$$\boxed{\Delta h(s,t) = +\frac{c}{g}\left[U_0 - U(s,t)\right]}$$



this variation directly provides the pressure variation $\Delta h(0,t)$ at the penstock outlet (s=0) as well as for each s along the penstock

$$\Delta Q \rightarrow \Delta U \rightarrow \Delta h(0,t)$$

IMPORTANT CONCLUSIONS:

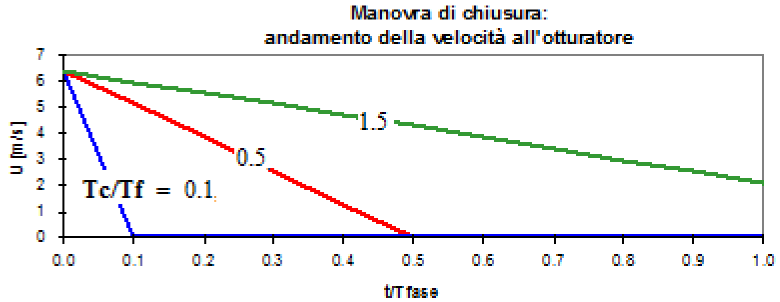
Δh is a function of ΔU, irrespective of the dependence of U(t) on t.

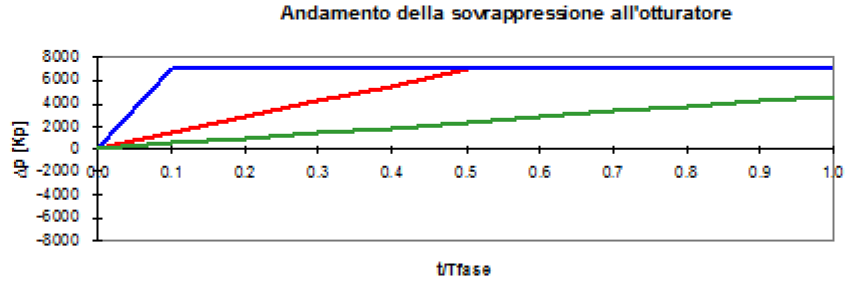
If Q(t) = 0 (total and sudden valve closure), $\Delta p = \rho c U_0$

If Q(0)=0 and Q(t)=Q_0 (total and sudden valve opening), $\Delta p = -\rho c U_0$

s = 0, valve closure at t=τ, 0< τ< T$\equiv$2L/c (Pipe period, tempo di fase)

fluido		condotta		legge lineare di chiusura dell'otturatore	
ρ	998	spessore	0.030 m	Tc/Tf	0.1, 0.5, 1.5
ε	2E+ 9 Pa	D	2.11 m		
c	1085.7 m/s	L	2189 m		
		E [Pa]	2E+11 Pa		
h(0)	1400 m				
U(0)	6.42 m/s				





•T$\equiv$2L/c = 4.03 s for S. Fiorano power plant

•If the valve closure time is less than 2L/c, $\Delta p=\rho c U_0$ irrespective of the closure duration (rapid valve closure -manovra brusca-, i.e. where the time taken to close the valve is less than or equal to the pipe period)

•If the valve closure time is larger than 2L/c (slow valve closure - chiusura lenta), we must take into account also the descending f waves, as we shall soon see after dealing with Allievi's interlocked equation.

•When the pipe geometrical properties vary in space, an effective celerity c must be computed, so that the pipe period is equivalent to the real one, whilst the equivalent D could be computed requiring that the kinetic energy is the same of the one present within the real pipe.

$$\frac{2L}{c} = \sum_{i=1,n} \frac{2L_i}{c_i} \qquad \rho \frac{LQ^2}{2D^2} = \sum_{i=1,n} \rho \frac{L_i Q_i^2}{2D_i^2}$$

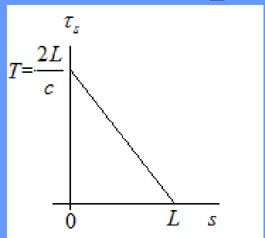
- Considering these properties, an important, general relationship can be derived between f and F

$$f(s+ct_1) = f\left[L + c\left(t_1 - \frac{L-s}{c}\right)\right]$$

$$f\left[L + c\left(t_1 - \frac{L-s}{c}\right)\right] = -F\left[L - c\left(t_1 - \frac{L-s}{c}\right)\right]$$

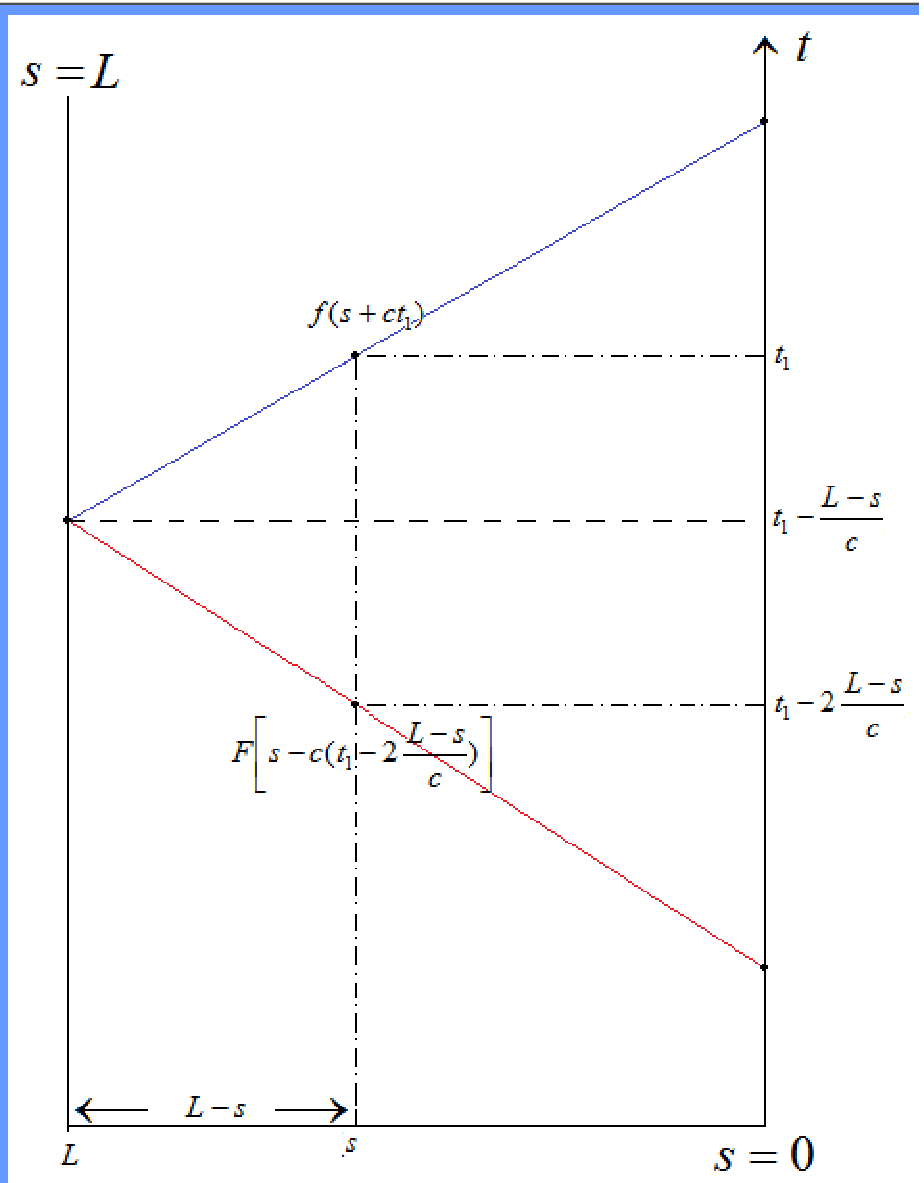
$$F\left[L - c\left(t_1 - \frac{L-s}{c}\right)\right] = F\left[s - c\left(t_1 - 2\frac{L-s}{c}\right)\right]$$

$$\boxed{\begin{aligned}&\tau_s = 2\frac{L-s}{c}\\ &f(s+ct) = -F\left[s - c(t - \tau_s)\right]\end{aligned}}$$



- Accordingly, the solution can be obtained as a function of F only

$$\boxed{\begin{aligned}&h(s,t) = h_0 + F(s-ct) - F\left[s - c(t - \tau_s)\right]\\ &\frac{c}{g}\left(U_0 - U(s,t)\right) = F(s-ct) + F\left[s - c(t - \tau_s)\right]\end{aligned}}$$



Let us now consider the situation that arises when $t > (2L-s)/c$. In such a case, at each station s along the penstock $f \neq 0$; For simplicity's sake, <u>let us limit our analysis to the penstock outlet</u>, immediately upstream of the regulating valve, where <u>s= 0</u>.

$$h(t) = h_0 + F(-ct) - F\left[-c(t - \frac{2L}{c})\right] = h_0 + F(t) - F(t - \frac{2L}{c})$$

$$\frac{c}{g}\left(U_0 - U(t)\right) = F(-ct) + F\left[-c(t - \frac{2L}{c})\right] = F(t) + F(t - \frac{2L}{c})$$

Let us write these equations for a time series built as $0 < t_1 < T\,;\; t_2 = t_1 + T;\;\; t_3 = t_2 + T;\ldots\;\; t_n = t_{n-1} + T;$

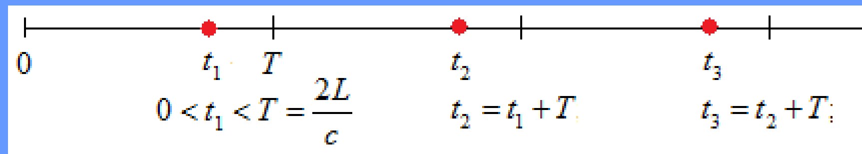
For simplicity's sake, in the following we shall write h_n or F_n for $h(t_n)$ and $F(t_n)$. The equations above can then be rewritten for the different t_i, considering that $F_0 = 0$, because $t_0 < 0$

$$h_1 - h_0 = F_1 \qquad \frac{c}{g}\left(U_0 - U_1\right) = F_1$$

$$h_2 - h_0 = F_2 - F_1 \qquad \frac{c}{g}\left(U_0 - U_2\right) = F_2 + F_1$$

....

$$h_n - h_0 = F_n - F_{n-1} \qquad \frac{c}{g}\left(U_0 - U_n\right) = F_n + F_{n-1}$$



And, by substituting F_i one can easily get a set of equations that are linked together that can be solved if the solution of the first is inserted in the second, and the solution of the second is inserted into the third and so on

$$h_1 - h_0 = F_1 \qquad \frac{c}{g}(U_0 - U_1) = F_1 \quad \Rightarrow \quad h_1 - h_0 = \frac{c}{g}(U_0 - U_1)$$

$$h_2 - h_0 = F_2 - F_1 \qquad \frac{c}{g}(U_0 - U_2) = F_2 + F_1 \quad \Rightarrow \quad h_2 + h_1 - 2h_0 = \frac{c}{g}(U_1 - U_2)$$

....

$$h_n - h_0 = F_n - F_{n-1} \qquad \frac{c}{g}(U_0 - U_n) = F_n + F_{n-1} \quad \Rightarrow \quad h_n + h_{n-1} - 2h_0 = \frac{c}{g}(U_{n-1} - U_n)$$

$$\frac{h_n}{h_0} + \frac{h_{n-1}}{h_0} - 2 = \frac{cU_0}{gh_0}\left(\frac{U_{n-1}}{U_0} - \frac{U_n}{U_0}\right)$$

The final equation can be further worked out making it dimensionless and considering the boundary condition also in dimensionless form

$$Q(t,0) = U_t A = \Sigma(t)\sqrt{2gh_t}\,; \quad \rightarrow \quad \frac{U_t}{U_0} = \frac{\Sigma_t}{\Sigma_0}\sqrt{\frac{h_t}{h_0}}; \quad \rightarrow \quad \frac{h_n}{h_0} + \frac{h_{n-1}}{h_0} - 2 = \frac{cU_0}{gh_0}\left(\frac{\Sigma_{n-1}}{\Sigma_0}\sqrt{\frac{h_{n-1}}{h_0}} - \frac{\Sigma_n}{\Sigma_0}\sqrt{\frac{h_n}{h_0}}\right)$$

Finally, by introducing the dummy variable z_n one eventually gets the Allievi interlocked equations (- Equazioni concatenate di Allievi -)

$$z_n = \sqrt{\frac{h_n}{h_0}};$$

$$\boxed{z_n^2 + z_{n-1}^2 - 2 = \frac{cU_0}{gh_0}\left(\frac{\Sigma_{n-1}}{\Sigma_0} z_{n-1} - \frac{\Sigma_n}{\Sigma_0} z_n\right)}$$

Use of Allievi's equation

$$z_n = \sqrt{\frac{h_n}{h_0}}; \qquad \boxed{z_n^2 + z_{n-1}^2 - 2 = \frac{cU_0}{gh_0}\left(\frac{\Sigma_{n-1}}{\Sigma_0} z_{n-1} - \frac{\Sigma_n}{\Sigma_0} z_n\right)}$$

By using this recursive equation one can get the value of the piezometric head at the penstock outlet at time $t_{i,}$ i=1,2..n, spaced of T one from the other. However, given that $t_{i,}$ can be chosen at will under the condition 0< $t_{i,}$ <T, this equation practically provides the function h(0,t).

In order to use it, one must know the valve closing law Σ_t/Σ_0, that starts from 1 if the valve is initially open. If the valve is initially closed, the equation are made dimensionless with respect to the final steady condition in order to avoid the situation Σ_0=0.

After choosing t_1, being z_0 =1, the equation above provides z_1 from which one computes h_1. Then one iterates obtaining h_2 and so forth…

Note that:

1) after that the valve is closed, Σ_t/Σ_0, =0 so that the equation above simplifies as. This equation clearly implies that the piezometric head oscillates without damping around h_0 with a periodicity 2T

$$\frac{h_n + h_{n-1}}{2} = h_0$$

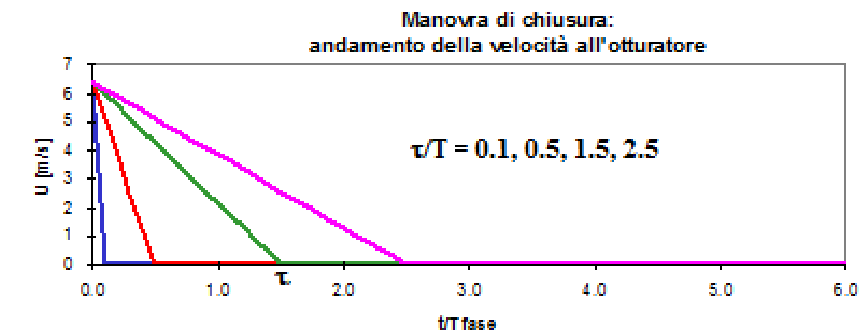
2) If a complete shut down at time τ occurs with τ < T, choosing $\tau < t_1$ <T, being Σ_1/Σ_0=0, one gets

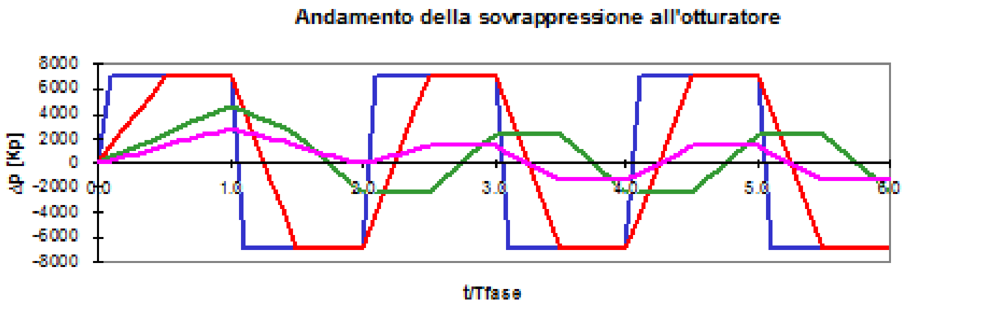
$$z_1^2 + 1 - 2 = \frac{cU_0}{gh_0}(1-0)$$

that immediately provides Joukowsky equation.

Let us now apply Allievi's interlocked equations at s= 0,in correspondence of different valve closure time τ

fluido			condotta			legge lineare di chiusura dell'otturatore
ρ	998		spessore	0.030	m	τ/T = 0.1, 0.5, 1.5, 2.5
ε	2000000000	Pa	D	2.11	m	
c	1085.7	m/s	L	2189	m	
h(0)	1400	m	E [Pa]	2E+11	Pa	
U(0)	6.42	m/s				





When τ > T, the maximum pressure at the penstock outlet occurs at t=T and is given by the Michaud's relation

$$\Delta p = \pm \rho c U_0 \frac{T}{\tau} = \pm \frac{2\rho U_0 L}{\tau}$$

It clearly shows the three quantity (L,U_0, τ) on which to operate in order to keep Δp as low as possible.
The same approach that has been used by Allievi for evaluating h(t) at the outlet can be used also for intermediate station s along the penstock. However, in the following we shall introduce a different type of numerical procedure, that is more general and that will be used also for unsteady open channel flow

Many international standards advocate the use of quantitative simulation for design and stability studies on hydroelectric plant in order to predict the dynamic performance of a planned installation and to allows design constraints to be revised before construction begins. In this direction it must be observed that analytical solutions are of little use but in the simplest cases so that it is often necessary to use numerical methods instead. In this direction, the Method of Characteristics appears as the best method available for this type of problems. It basically converts the partial differential equation into a ordinary differential equations whose validity is constrained to a set of characteristic curves, related to the velocity of the travelling wave.Typically, the method of finite differences is then applied to discretise the equations (temporally and spatially) so that computer numerical integration techniques can be applied.As a thorough reference on the application of this method to power plants, download from the website: Evangelisti, G., Teoria Generale del Colpo d'ariete col metodo delle caratteristiche, Energia Elettrica, 2, 65-90 1965, and the companion paper published on the same Journal, Energia Elettrica, 3, 145-162, 1965.

Let us start from the following set of simplified equations, where, however, friction is present

$$\begin{cases} \dfrac{\partial h}{\partial s} = -\dfrac{1}{g}\dfrac{\partial U}{\partial t} - J; \\ \dfrac{g}{c^2}\dfrac{\partial h}{\partial t} = -\dfrac{\partial U}{\partial s}; \end{cases} \qquad \begin{cases} g\dfrac{\partial h}{\partial s} + \dfrac{\partial U}{\partial t} + Jg = 0; \\ \dfrac{\partial h}{\partial t} + \dfrac{c^2}{g}\dfrac{\partial U}{\partial s} = 0; \end{cases}$$

The following procedure changes the two PDEs into a single Ordinary Differential Equation. To this purpose let us multiply the first eq. times the Lagrange multiplier λ and let us add it to the second

$$\lambda g\frac{\partial h}{\partial s} + \lambda\frac{\partial U}{\partial t} + \lambda Jg + \frac{\partial h}{\partial t} + \frac{c^2}{g}\frac{\partial U}{\partial s} = 0$$

Our aim is now to obtain two terms like

$$\frac{DU}{Dt}=\frac{\partial U}{\partial t}+\frac{ds}{dt}\frac{\partial U}{\partial t}; \qquad \frac{Dh}{Dt}=\frac{\partial h}{\partial t}+\frac{ds}{dt}\frac{\partial h}{\partial t}$$

Whose physical meaning is lagrangian. And to this purpose we work out the previous equation

$$\lambda\frac{\partial U}{\partial t}+\frac{c^2}{g}\frac{\partial U}{\partial s}+\frac{\partial h}{\partial t}+\lambda g\frac{\partial h}{\partial s}=-\lambda Jg$$

We are looking λ values such that the following conditions hold

$$\lambda\frac{DU}{Dt}=\lambda\frac{\partial U}{\partial t}+\frac{c^2}{g}\frac{\partial U}{\partial s} \qquad \rightarrow \frac{ds}{dt}=\frac{c^2}{\lambda g}$$

$$\frac{Dh}{Dt}=\frac{\partial h}{\partial t}+\lambda g\frac{\partial h}{\partial s} \qquad \rightarrow \frac{ds}{dt}=\lambda g$$

This is true when

$$\frac{c^2}{\lambda g}=\lambda g \qquad \rightarrow \lambda^2=\frac{c^2}{g^2} \qquad \rightarrow \lambda=\pm\frac{c}{g}$$

In correspondence of these values we can rewrite equation (1) with validity constrained along the corresponding ds/dt line, (i.e., the characteristic line)

$$\lambda=\pm\frac{c}{g} \qquad \rightarrow \frac{ds}{dt}=\pm c$$

$$\begin{cases} \dfrac{DU}{Dt}+\dfrac{g}{c}\dfrac{Dh}{Dt}=-Jg \\ \dfrac{ds}{dt}=c \qquad\qquad C^+ \end{cases} ; \qquad \begin{cases} +\dfrac{DU}{Dt}-\dfrac{g}{c}\dfrac{Dh}{Dt}=-Jg \\ \dfrac{ds}{dt}=-c \qquad\qquad C^- \end{cases}$$

These equations are true only along the lines C^+ and C^- in space. This property allows to trasnform the original PDE problem into a much simpler ODE problem. Actually, If one consider the meaning of the substantial derivative operator, that corresponds to a lagrangian variation along the flow direction, one can easily accepts that along the 2 characteristic lines C^+ and C^- the 2 systems can be written in a lagrangian way as

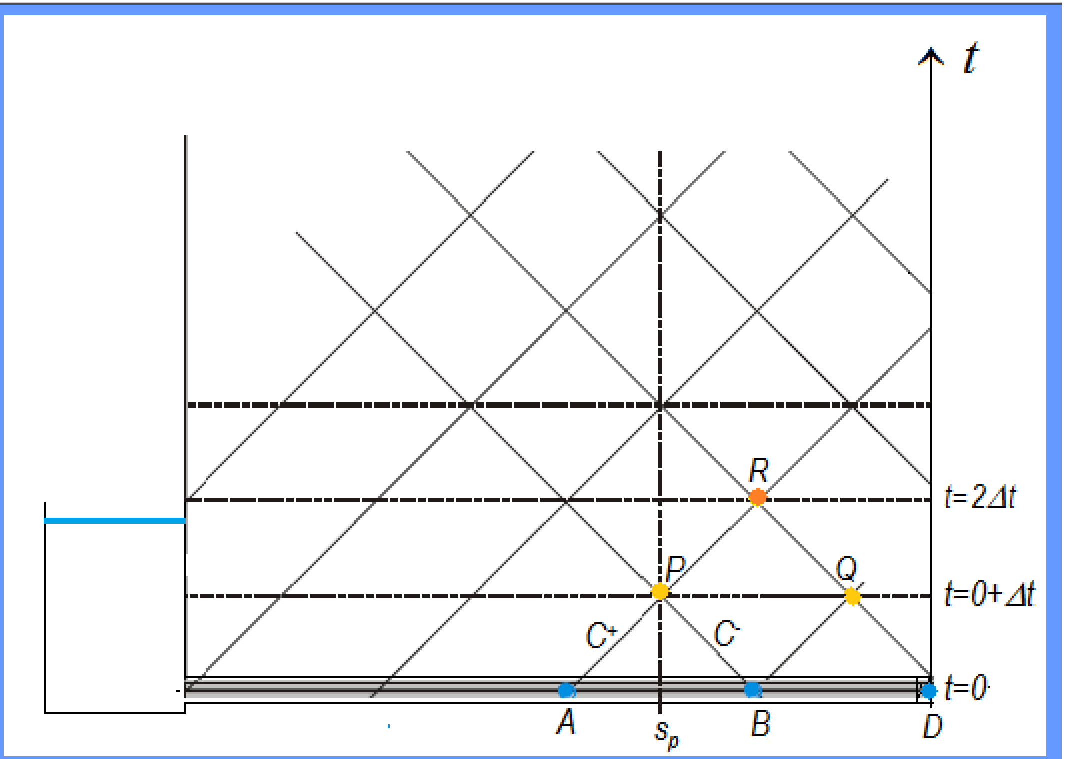
$$\begin{cases} dU+\dfrac{g}{c}dh=-Jgdt \\ \dfrac{ds}{dt}=c \qquad\qquad C^+ \end{cases} ; \qquad \begin{cases} +dU-\dfrac{g}{c}dh=-Jgdt \\ \dfrac{ds}{dt}=-c \qquad\qquad C^- \end{cases}$$

These equations are the basis for the finite-difference solution of the water hammer problem. Let us suppose, for simplicity's sake that the penstock is prismatic with uniform properties. In such a case c is constant and the 2 characteristic lines, C^+ and C^- , define two sets of straight lines in the s-t plane along which pressure and velocity variations propagate.

Let us now suppose that h and U are known along the penstock at t=0.

Let us take two points, A and B, at t=0, and let us draw C^+ through A and C^- through B. These two lines will cross at P, at time $t=0+\Delta t$, at the point,

$$\begin{cases} s_P - s_A = c\Delta t \\ s_P - s_B = -c\Delta t \end{cases}$$



Whose solution provides s_p and Δt. In order to compute U and h at P, U_P and h_P, one solves the system

$$\begin{cases} U_P - U_A + \dfrac{g}{c}\left(h_P - h_A\right) = -J_A g\Delta t \\ U_P - U_B - \dfrac{g}{c}\left(h_P - h_B\right) = -J_B g\Delta t \end{cases}$$

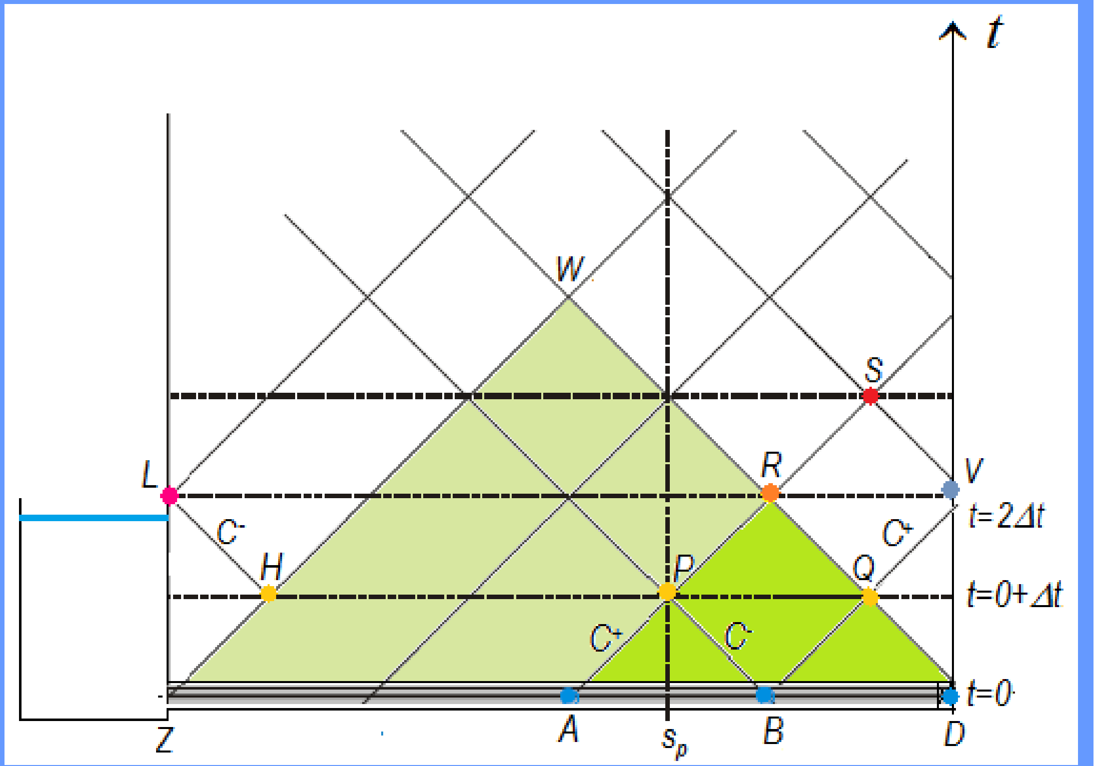
Where J is evaluated in A and B if an explicit approximation is deemed adequate, otherwise in P by solving a non linear system. Due to reversal of the flow direction, the slope friction must be evaluated as

$$\frac{\lambda U\left|U\right|}{2gD}$$

where λ is here the friction coefficient given, e.g., by the Colebrook-White equation

By applying the same procedure to B and D one gets the solution at Q. Iterating between P and Q finally one gets the solution at R.

It is interesting to note that, however, the solution in S cannot be computed on the basis of the initial condition along the penstock, as can be graphically appreciated observing that S is outside of the green triangle ARD. Actually, the knowledge of the initial condition allows to find the solution within the pale green domain ZWD only.



In order to march in time outside this domain one must use the boundary conditions. Let us consider, for instance the solution in S. In order to compute , U_S and h_S, first one has to compute the solution in V. This can be done using the downstream boundary condition in V with the equation along C^+ through Q

$$\begin{cases} s_V - s_Q = c\Delta t \\ s_V = L \end{cases}$$

that provides Δt

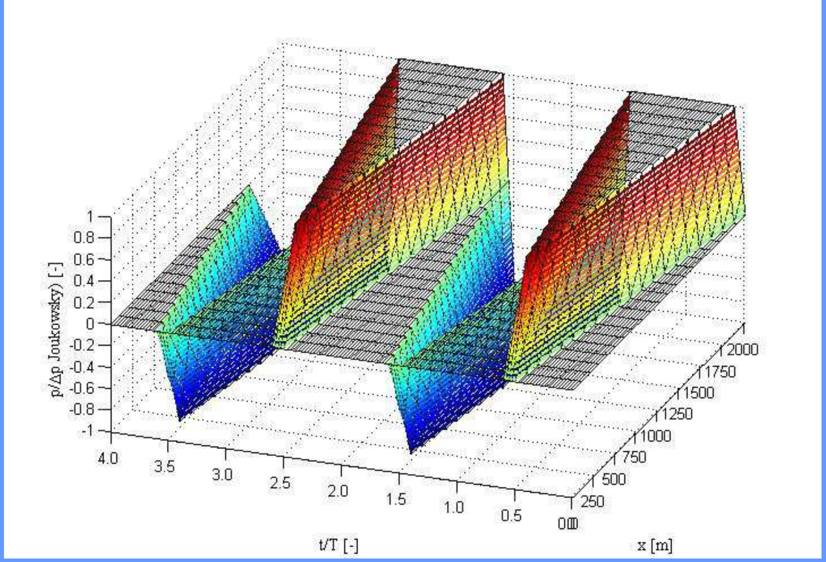
$$\begin{cases} U_V - U_Q + \dfrac{g}{c}\left(h_V - h_Q\right) = -J_Q g \Delta t \\ U_V = \dfrac{\Sigma(t)}{A}\sqrt{2gh_V} \end{cases}$$

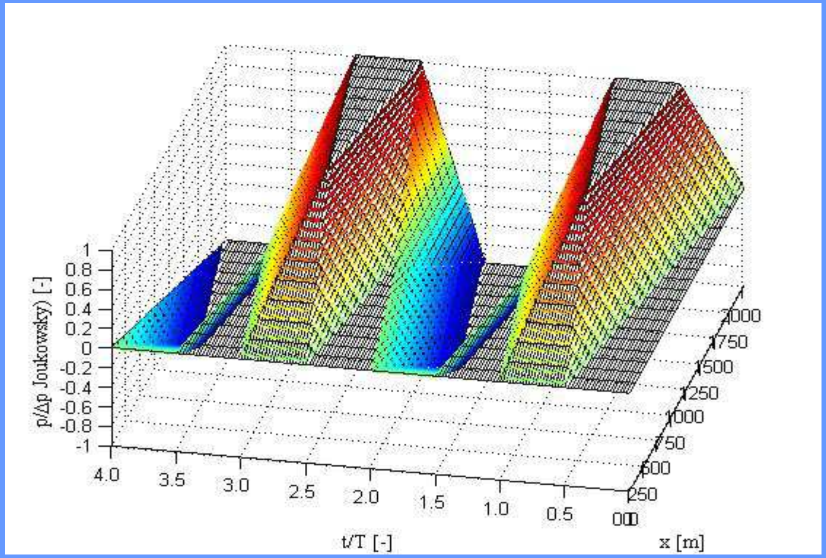
When the solution is known in V, the solution in S can be computed as done before, using the solution in R and in V.

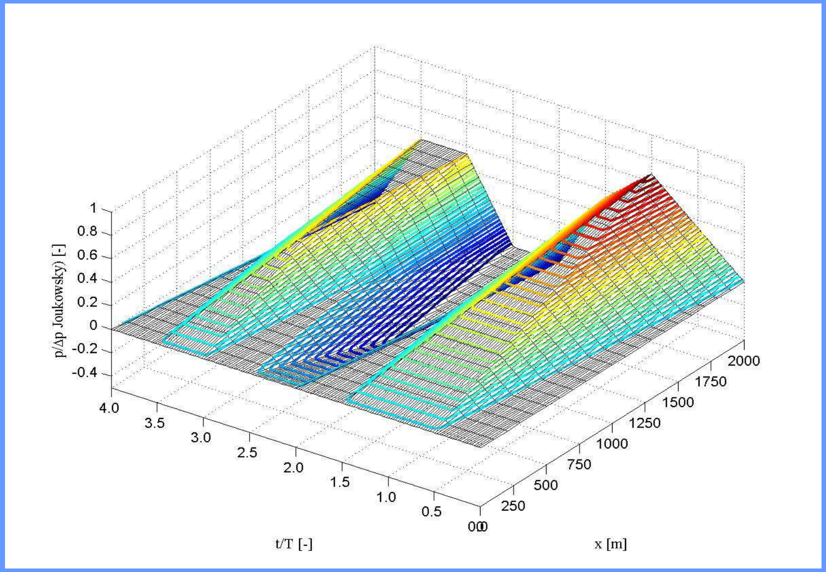
In a similar way one computes the solution in L, by using a system between the equation along C^- through H and the boundary conditon at L, where $h(L) = h_{surge\_tank}(t)$

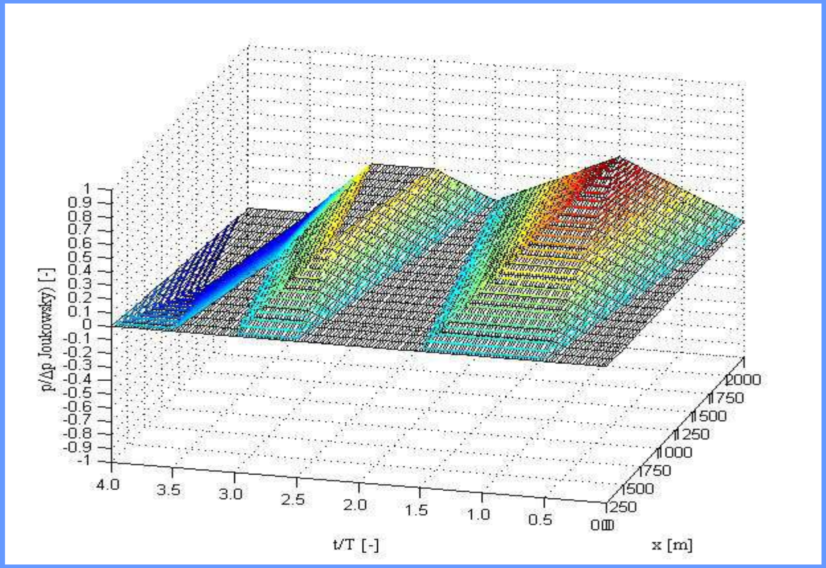
HYDROPOWER PLANTS: *Unsteady Flow - the Method of Characteristics*

Propagation of pressure waves for τ/T=0.1, 0.5,1.5 and 2.5





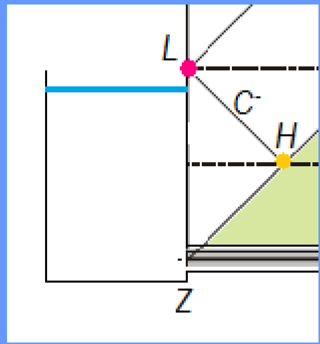




$$\begin{cases} s_L = 0 \\ s_L - s_H = -c\Delta t \end{cases}$$

$$\begin{cases} h_L = h_L(t) \\ U_L - U_H - \dfrac{g}{c}\left(h_L - h_H\right) = -J_H g \Delta t \end{cases}$$

B.C.: level in the surge tank at $t = 2\,\Delta t$



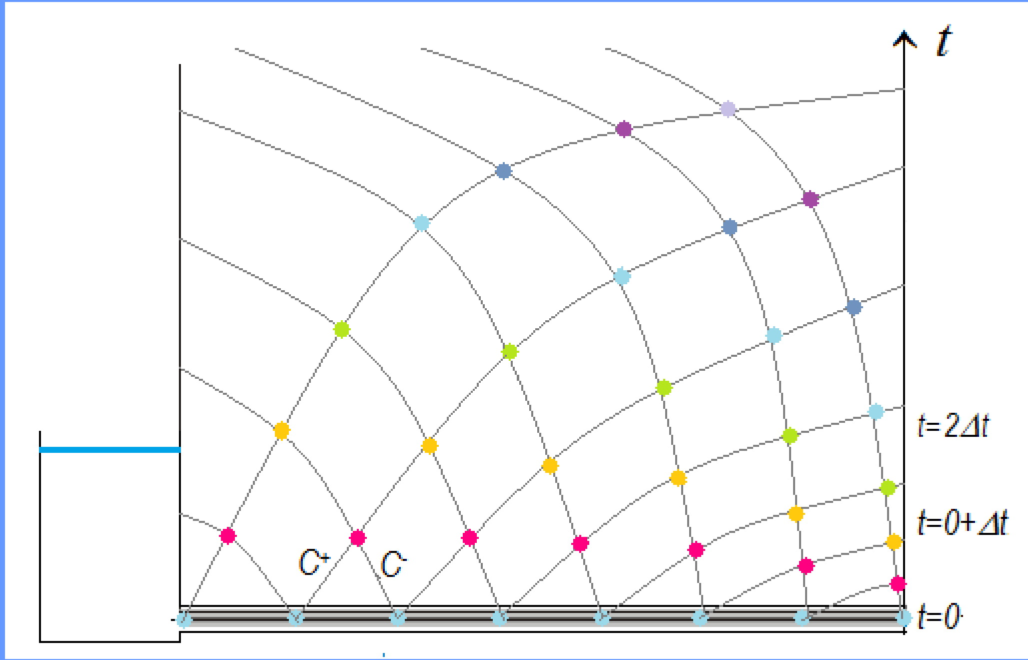
These two systems provide Δt and U_L

Advantages of the Method of Characteristics

General approach for hyperbolic equations, used also in flood propagation

A numerical approximation of the governing equation can easily deal with the irregularities of real applications.

Between the numerical methods, the major strength of the method of characteristics is that it tracks information about the solution along approximations to the characteristics. This makes the numerical method of characteristics generally better at dealing with a solution that has sharp fronts.



Its disadvantages are that if c varies in space it produces an approximation on an irregularly spaced set of points in the xt plane (see figure on the right, where c varies in space). In such a case it is somewhat more complicated to implement than a direct finite-difference method because it further requires an interpolation algorithm, that can decrease the quality of the final result.

If this not the case, the method of characteristics is straightforward to implement by a finite difference approximation of the governing system of equations as seen in the examples before.