MODULE - 1
STATIC FORCE ANALYSIS

CONTENTS

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Objectives

- To analyze static force analysis of four bar chain mechanism
- To analyze static force analysis of slider crank mechanism

1.1. Introduction:

Relation between motion and forces causing is a fascinating subject. This study is a generally referred as dynamic. Modern Engineering aims at analysing and predicting dynamics behavior of physical systems

Theory of Mechanisms & Machines is used to understand the relationships between the geometry and motions of the parts of a machine or mechanism and forces which produce motion.

TOM (M&M theory) is divided into two parts:-

Kinematics of Machinery: Study of motion of the components and basic geometry of the mechanism and is not concerned with the forces which cause or affect motion. Study includes the determination of velocity and acceleration of the machine members

Dynamics of Machinery: Analyses the forces and couples on the members of the machine due to external forces (static force analysis) also analyses the forces andcouples due to accelerations of machine members (Dynamic force analysis)

Deflections of the machine members are neglected in general by treating machine members as rigidbodies (also called rigid body dynamics). In other words the link must be properly designed to withstand the forces without undue deformation to facilitate proper functioning of the system.

In order to design the parts of a machine or mechanism for strength, it is necessary to determine the forces and torques acting on individual links. Each component however small, should be carefully analysed for its role in transmitting force.

The forces associated with the principal function of the machine are usually known or assumed.

Ex:

a) Piston type of engine: gas force on the piston is known or assumed
b) QRM – Resistance of the cutting tool is assumed.a & b are called static forces.

Example of other static forces are:

i. Energy transmitted
ii. Forces due to assembly
iii. Forces due to applied loads
iv. Forces due to changes in temperature
v. Impact forces
vi. Spring forces
vii. Belt and pulley
viii. Weights of different parts

Apart from static forces, mechanism also experiences inertia forces when subjected to acceleration, called dynamic forces.

Static forces are predominant at lower speeds and dynamic forces are predominant athigher speeds.

Force analysis:

The analysis is aimed at determining the forces transmitted from one point to another, essentially from input point to out put point. This would be the starting point for strength design of a component/ system, basically to decide the dimensions of the components

Force analysis is essential to avoid either overestimation or under estimation of forces on machine member.

Under estimation: leads to design of insufficient strength and to early failure. Overestimation: machine component would have more strength than required. Over design leads to heavier machines, costlier and becomes not competitive

Graphical analysis of machine forces will be used here because of the simplification itoffers to a problem, especially in cases of complex machines. Moreover, the graphical analysis of forces is a direct application of the equations of equilibrium.

General Principle of force analysis:

A machine / mechanism is a three dimensional object, with forces acting in three dimensions. For a complete force analysis, all the forces are projected on to three mutually perpendicular planes. Then, for each reference plane, it is necessary that, the vector sum of theapplied forces in zero and that, the moment of the forces about any axis perpendicular to the reference plane or about any point in the plane is zero for equilibrium.

That is $\sum F = 0$ & $\sum M = 0$ or

$\sum F x=0$ & $\sum Fy=0$ and $\sum M=0$

A force is a vector quantity and three in properties define a force completely;Magnitude

Direction

Point of application

1.2. Static equilibrium.

Equilibrium

For a rigid body to be in Equilibrium

i) Sum of all the forces must be zero

ii) Sum of all the moments of all the forces about any axis must be zeroi.e, (i) $\sum F = 0$ (ii) $\sum M = 0$

$\boxed{\begin{array}{l}\sum Fx = 0 \\ \sum Fy = 0\end{array}}$ or $\sum TM = 0$, $\sum My = 0$

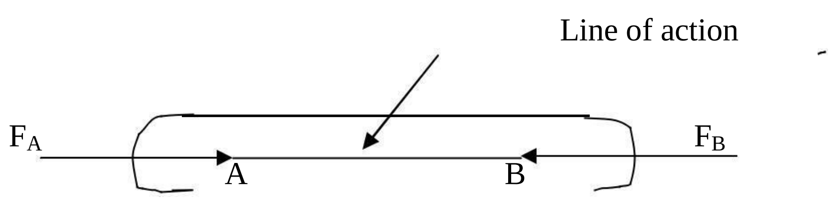
$\sum Fz = 0$ $\boxed{\sum Tz = 0}$ (For a planar system represented by 2D vectors)

Fx, Fy, Fz force Components along X, Y & Z axis

Similarly moments

1.3 Equilibrium of two and three force members.

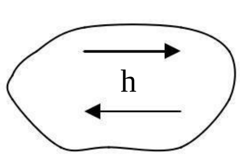
(i) Equilibrium of a body under the action of two forces only (no torque)



For body to the in Equilibrium under the action of 2 forces (only), the two forces must the equalopposite and collinear. The forces must be acting along the line joining A&B.

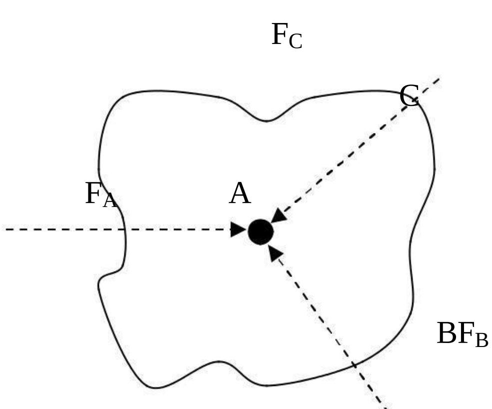
That is,

$F_A = - F_B$ (for equilibrium)



If this body is to be under equilibrium _h' should tend to zero

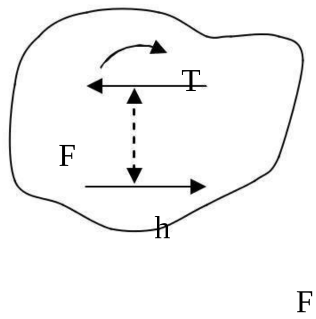
(ii) Equilibrium of a body under the action of three forces only (no torque / couple)



For equilibrium, the 3 forces must be concurrentand the force polygon will be a triangle.

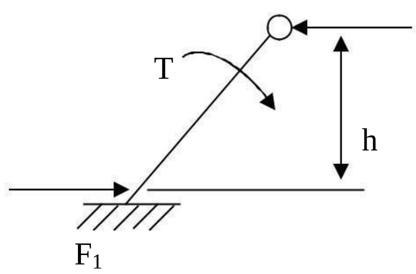
1.4 Members with two forces and torque.

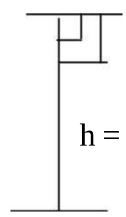
(iii) Equilibrium of a body acted upon by 2 forces and a torque.



For equilibrium, the two forces must form a counter couple. Therefore the forces must be equal, opposite and parallel and their senses must be so as to oppose the couple acting on the body

Example:





F2

h = Perpendicular distance between F1 & F_2

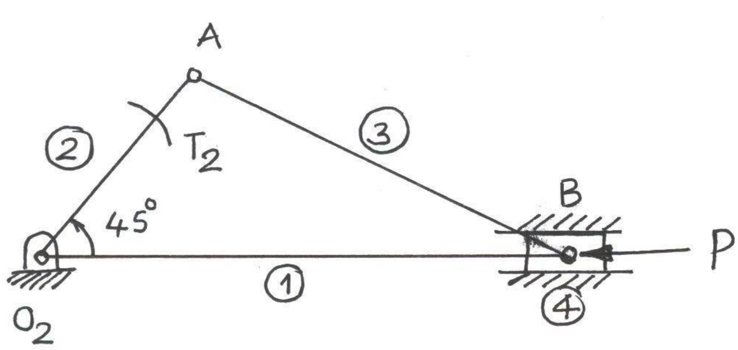
Free body diagram

The mass is separated from the system and all the forces acting on the mass are represented.

slider-crank mechanism with and without friction.

Problem No.1: Slider crank mechanism

Figure shows a slider crank mechanism in which the resultant gas pressure 8 x 10^4 Nm^{-2} acts on the piston of cross sectional area 0.1 m^2. The system is kept in equilibrium as a result of the couple applied to the crank 2, through the shaft at O_2. Determine forces acting on all the links (including the pins) and the couple on 2.

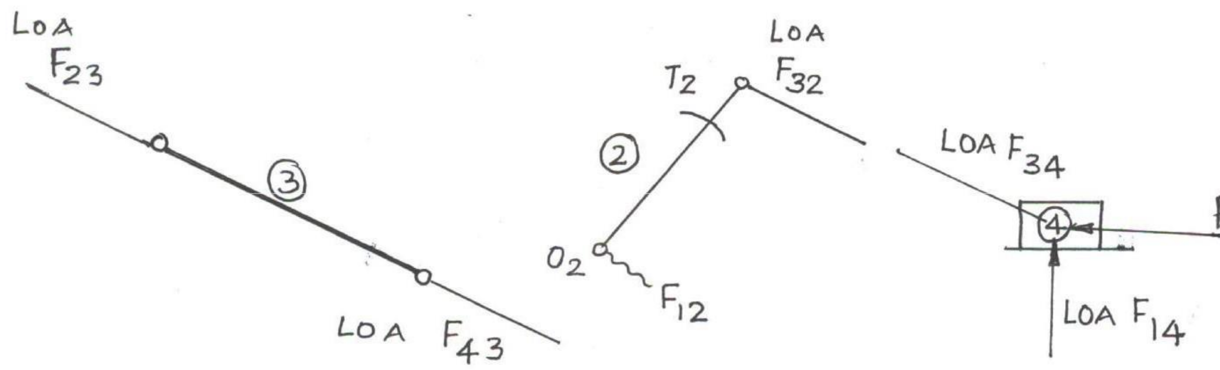


$$P = (8 \times 10^4) \times (0.1)$$

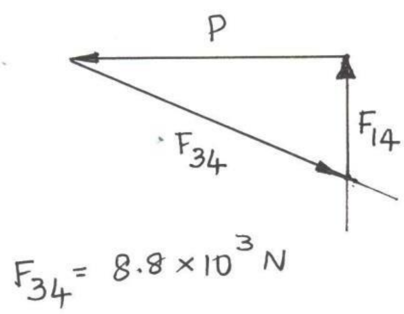
$$= 8 \times 10^3\ N$$

1.5 Free body diagrams.

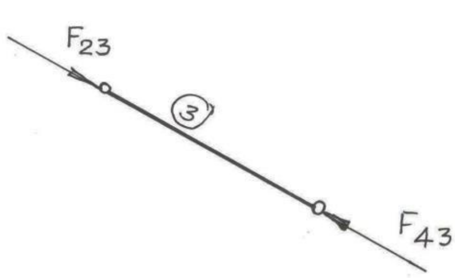
Free body diagram



Force triangle for the forces acting on 4 is drawn to some suitable scale.
Magnitude and direction of P known and lines of action of F_{34} & F_{14} known.



Measure the lengths of vectors and multiply by thescale factor to get the magnitudes of F_{14} & F_{34}.
Directions are also fixed.

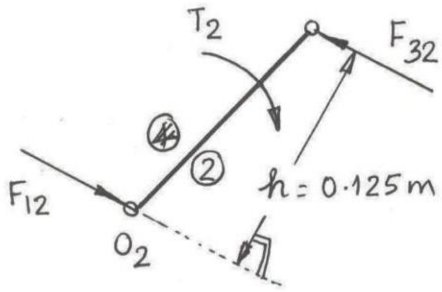


$i.e,\ F_{23} = -F_{32}$

Since link 3 is acted upon by only two forces, F_{43} and F_{23} are collinear, equal in magnitudeand opposite in direction

$$i.e.,\quad F_{43} = -F_{23} = 8.8 \times 10^3\ N$$

Also, $F_{23} = -F_{32}$ (equal in magnitude and opposite in direction



Link 2 is acted upon by 2 forces and a torque (stated in the problem), for equilibrium the two forces must be equal, parallel and opposite and their sense must oppose T_2.

There fore,

$$F_{32} = -F_{12} = 8.8 \times 10^3 \ N$$

F_{32} & F_{12} form a counter clock wise couple of magnitude,

$$F_{23} \times h = F_{12} \times h = 8.8 \times 10^3 \times 0.125 = 1100 Nm.$$

To keep 2 in equilibrium, T2 should act clockwise and magnitude is 1100Nm. Important to note;

i) h is measured perpendicular to F32 & F12;

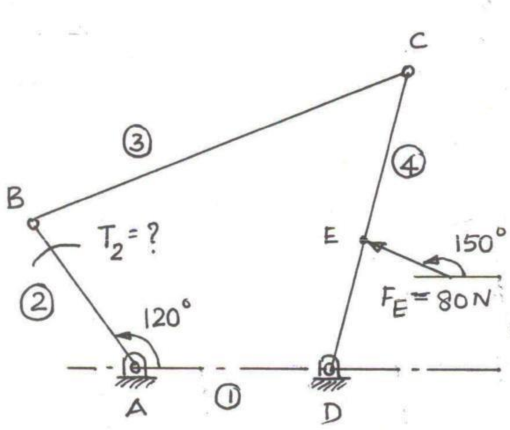
ii) always multiply back by scale factors.

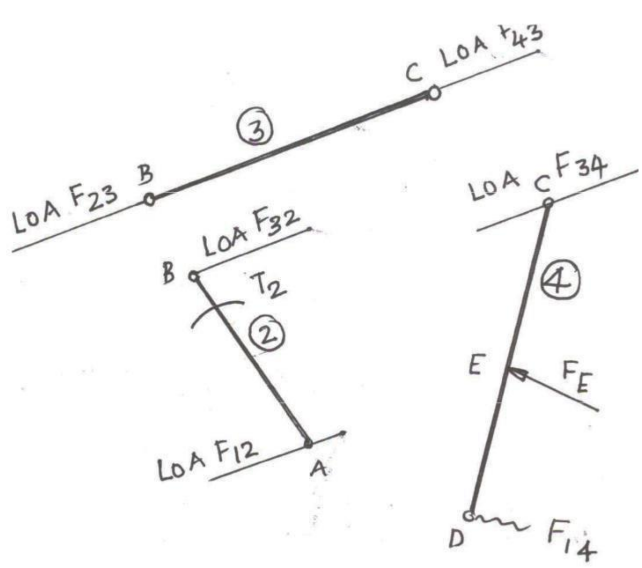
1.6 Static force analysis of four bar and single slider mechanism

Problem No 2. Four link mechanism.

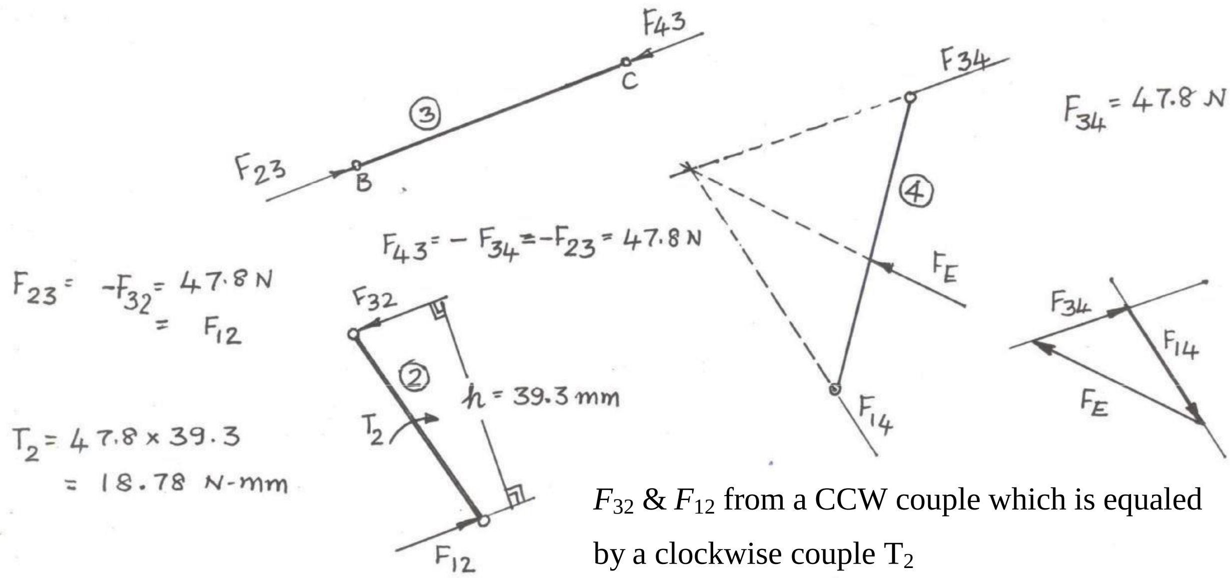
A four link mechanism is acted upon by forces as shown in the figure. Determine the torque $T_2$to be applied on link 2 to keep the mechanism in equilibrium.

AD=50mm, AB=40mm, BC=100mm, Dc=75mm, DE= 35mm,





Link 3 is acted upon by only two forces F_{23} & F_{43} and they must be collinear & along BC. Link 4 is acted upon by three forces F_{14}, F_{34} & F_4 and they must be concurrent. LOA F_{34} is known and F_E completely given.

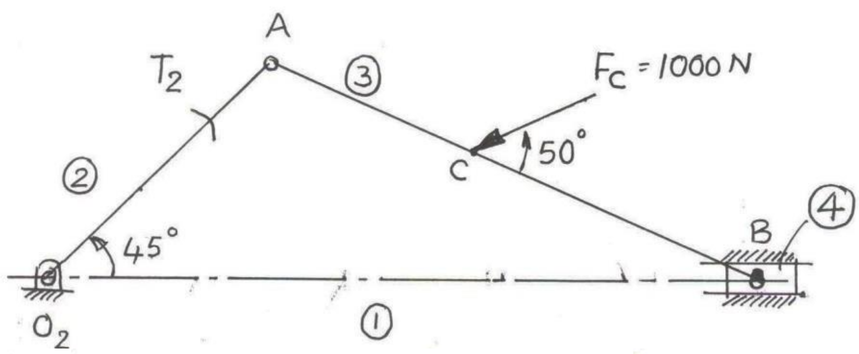


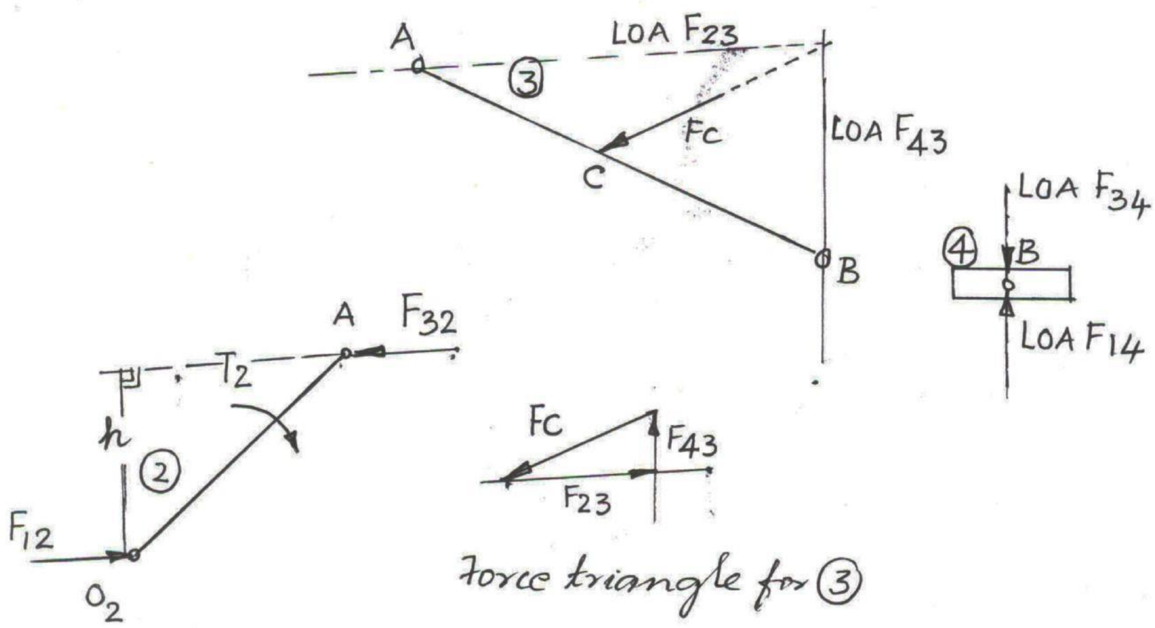
F_{32} & F_{12} from a CCW couple which is equaled by a clockwise couple T_2

Problem No 3.

Determine T_2 to keep the mechanism in equilibrium

AC=70mm,
AB=150mm,
O_2A= 40mm





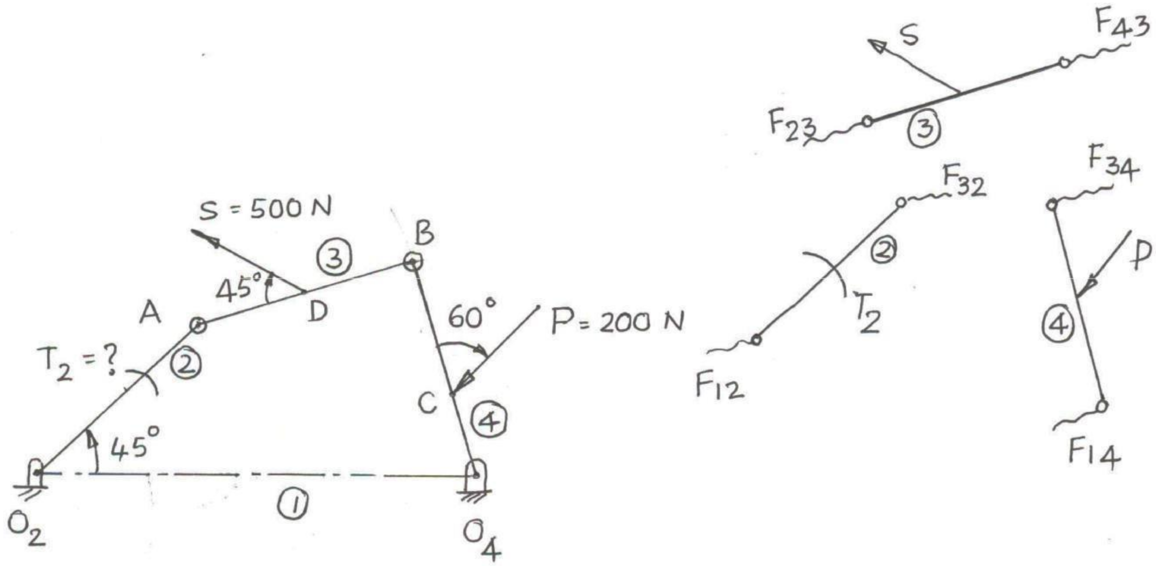
$$T_2 = F_{32} \times h = F_{12} \times h$$

F_{32} and F_{12} form a CCW couple and hence T_2 acts clock wise

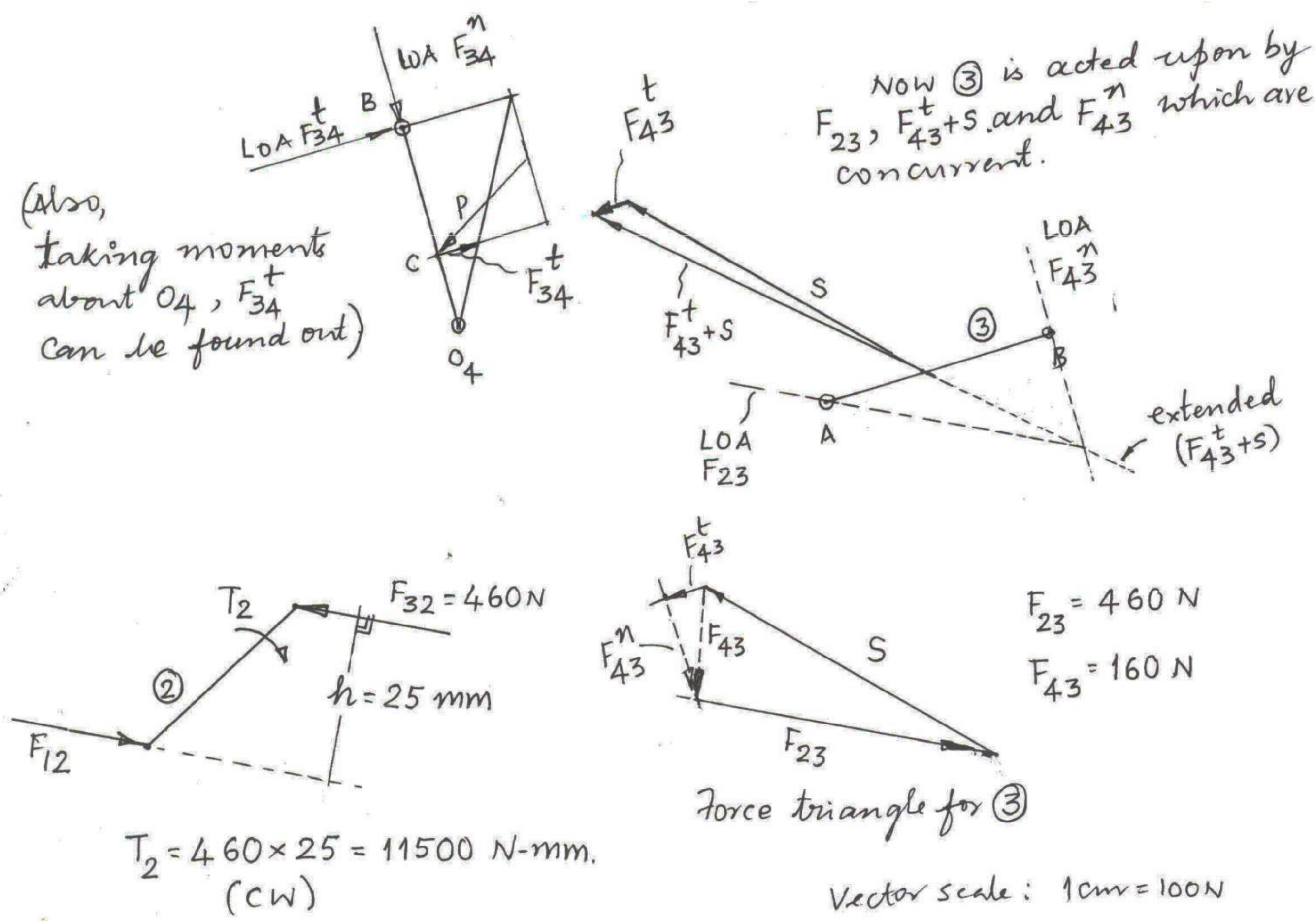
Problem No 4.

Determine the torque T_2 required to keep the given mechanism in equilibrium.

O_2A= 30mm, = AB =O_4B, O_2O_4 =60mm, $A\,O\,2\,O_4$ = 60º, BC = 19mm, AD=15mm.

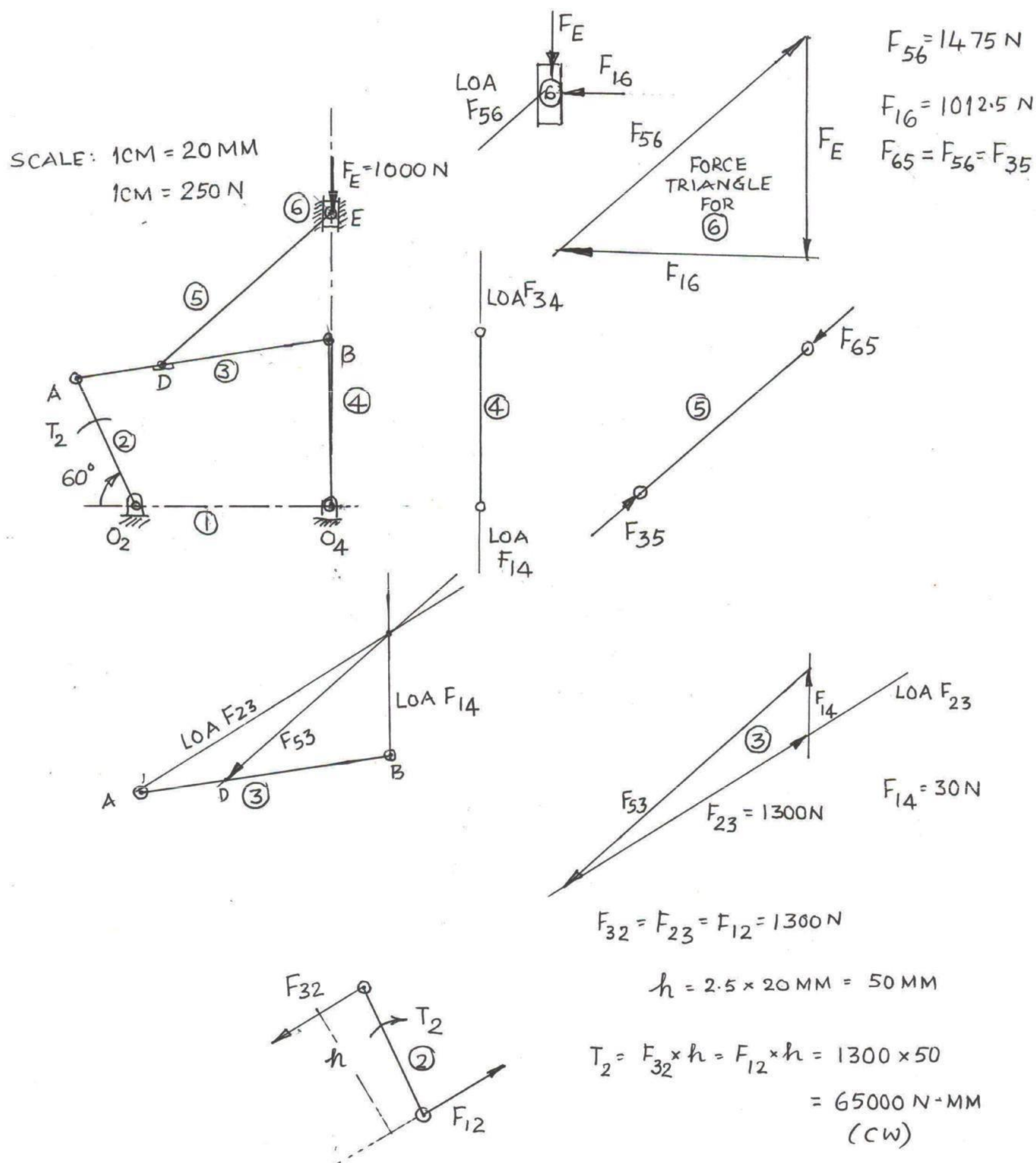


None of the links are acted upon by only 2 forces. Therefore links can‘t be analyzed individually.



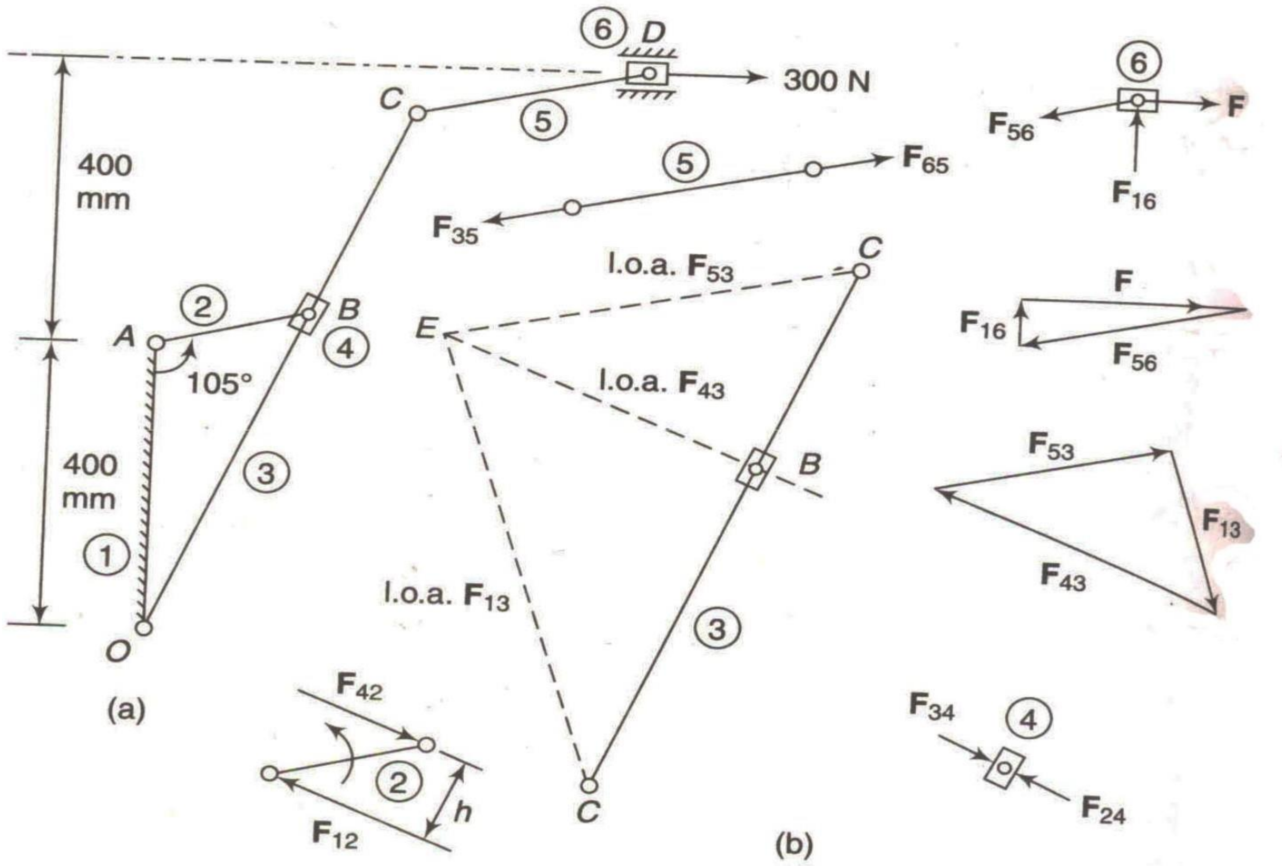
Problem No 5.

Determine the torque T_2 required to overcome the force F_E along the link 6. AD=30mm, AB=90mm, O_4 B=60mm, DE=80mm, O_2 A=50mm, O_2 O_4 =70mm



Problem No 6

For the static equilibrium of the quick return mechanism shown in fig. 12.11 (a), determine the input torque T_2 to be applied on link AB for a force of 300N on the slider D. The dimensions of the various links are OA=400mm, AB=200mm, OC=800mm, CD=300mm

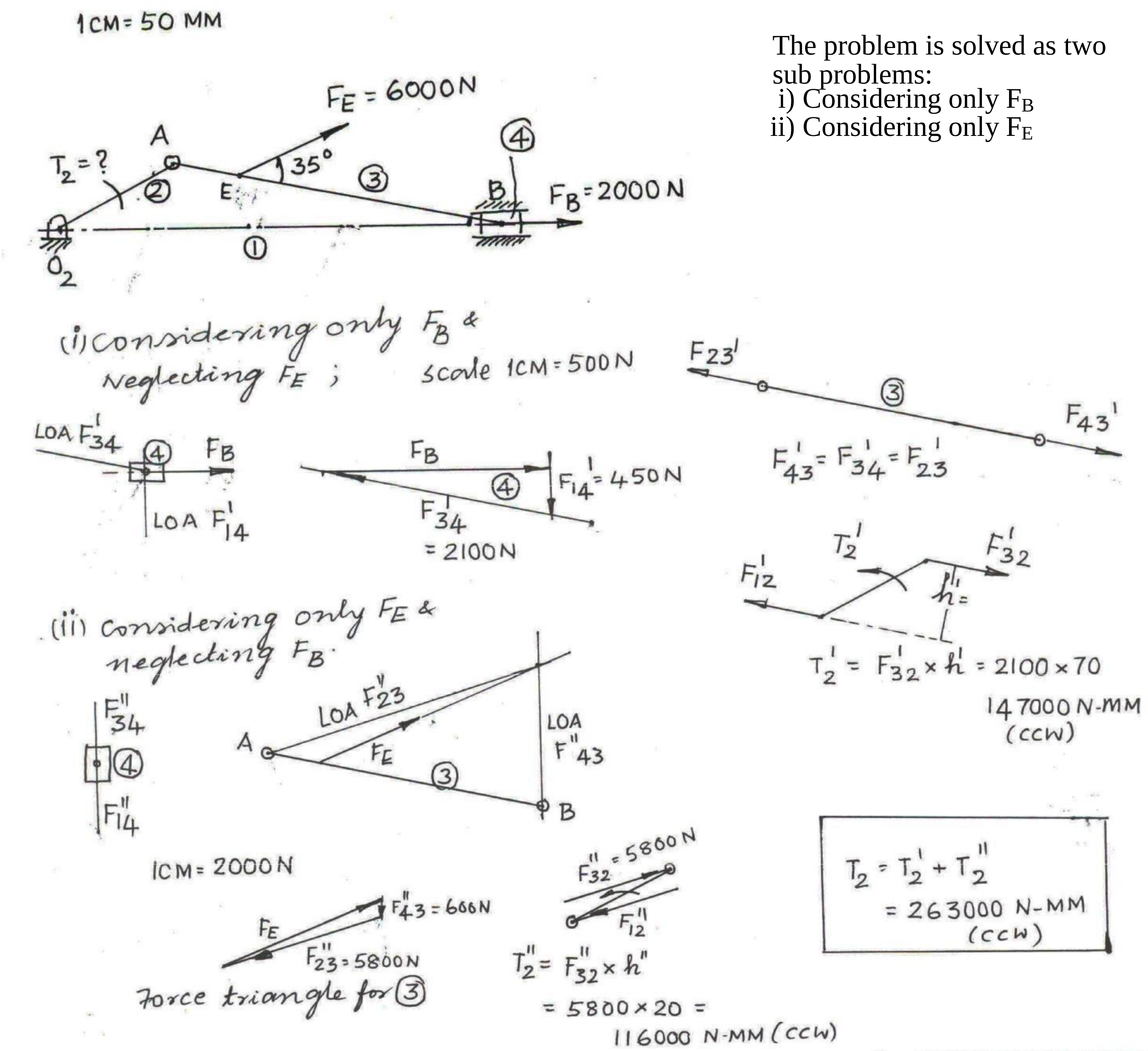


Than, torque on link 2,

$T_2 = F_{42}$x h = 403x120 = 48 360 N counter - clockwise

Problem No 7. Determine T_2 to keep the body in equilibrium. O_2A =100MM, AB=250MM, AE=50MM, $\angle AO_2B = 30^0$

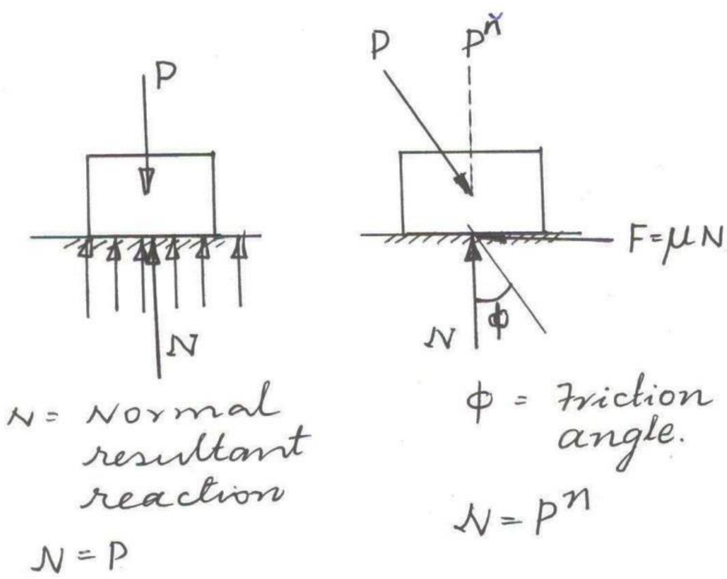
The problem is solved as two sub problems:
i) Considering only F_B
ii) Considering only F_E



1.7 Force Analysis considering friction.

If friction is considered in the analysis, the resultant force on a pin doesn't pass through the centre of the pin. Coefficient of friction is assumed to the known and is independent of loadand speed.

Friction in sliding member.

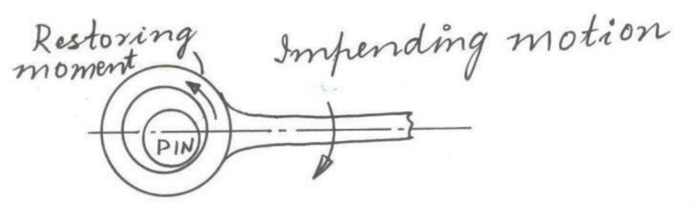


F = Frictional force coefficient of

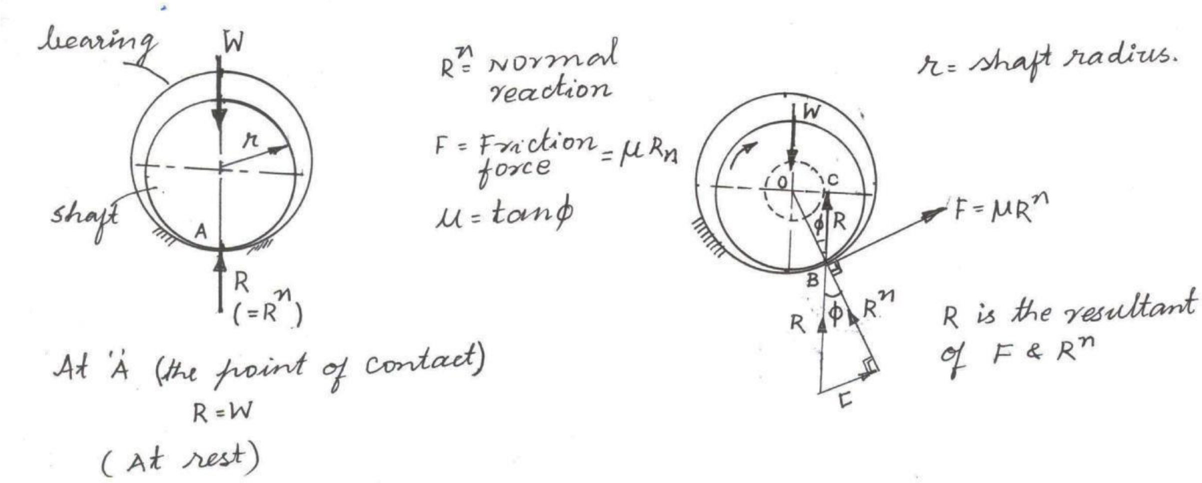
μ = friction

$$\tan\varphi = \mu = \frac{\mu N}{N}$$

Friction at pin points (bearings) & friction circle.



When a shaft revolves in a bearing, some power is lost due to friction between surfaces.



While rotating, the point of contact shifts to B; R^n passes through B.The resultant _R‘ is in a direction opposite to ω.

The circle drawn at O, with OC as radius is called _FRICTION CIRCLE‘For the shaft to be in equilibrium; W = R

Frictional moment M = R x OC

= W x OC

= W x r sin φ

= W x r tan φ

(sin φ ≈ tan φ, for smallφ) i.e, M = w x r x μ

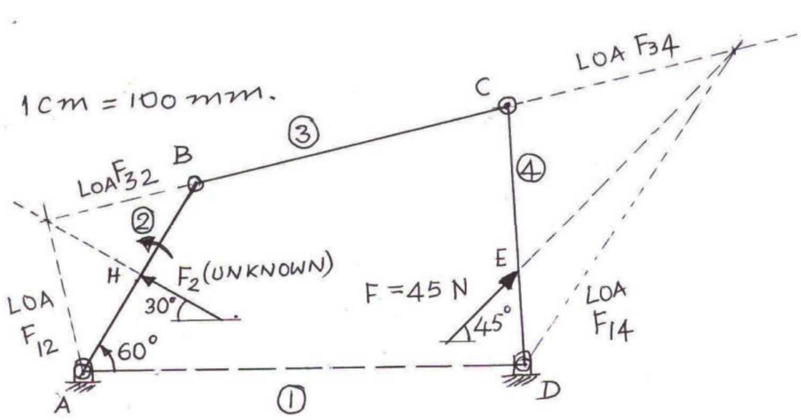
∴Radius of the friction circle (OC) = μr.

The friction circle is used to locate the line of action of the force between the shaft (pin) and thebearing or a pin joint. The direction of the force is always be tangent to it (friction axis) Frictionaxis: the new axis along which the thrust acts.

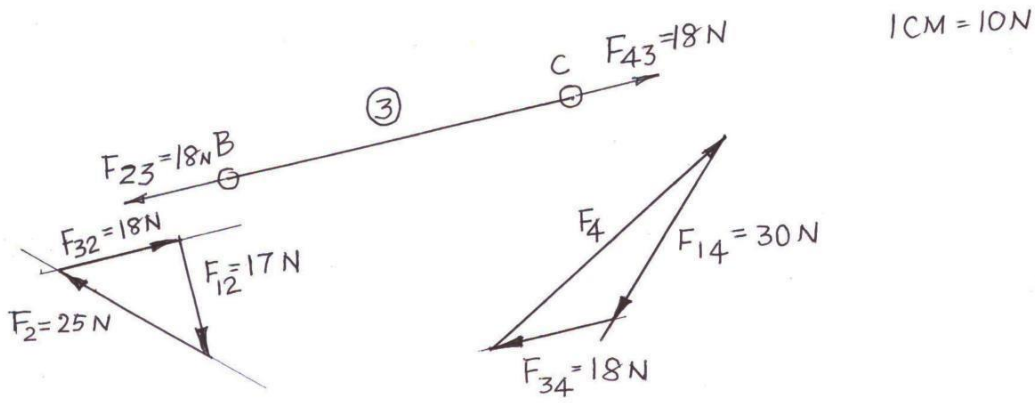
Problem No 8.

In a four bar mechanism ABCD, AB=350mm, BC=50mm, CD=400mm, AD=700mm, DE=150mm, $DAB = 60^0$, AD is fixed. Determine the force on link AB required at the mid point, in the direction shown, for static equilibrium. μ=0.4 for each revolving pair. Assume CCW impending motion of AB. Radius of each journal is 50mm.

Also find the torque on AB for its impending CW motion.Analysis for CCW motion



Solve the problem neglectingfriction to know the magnitudes and directions offorces



Radius of the friction circle = μ x journal radius = 0.4 x 50 = 20 mm

Analysis with Friction considered---AB rotates CCW, DC rotates CCW *ABC* decreasing, LBCD increasing

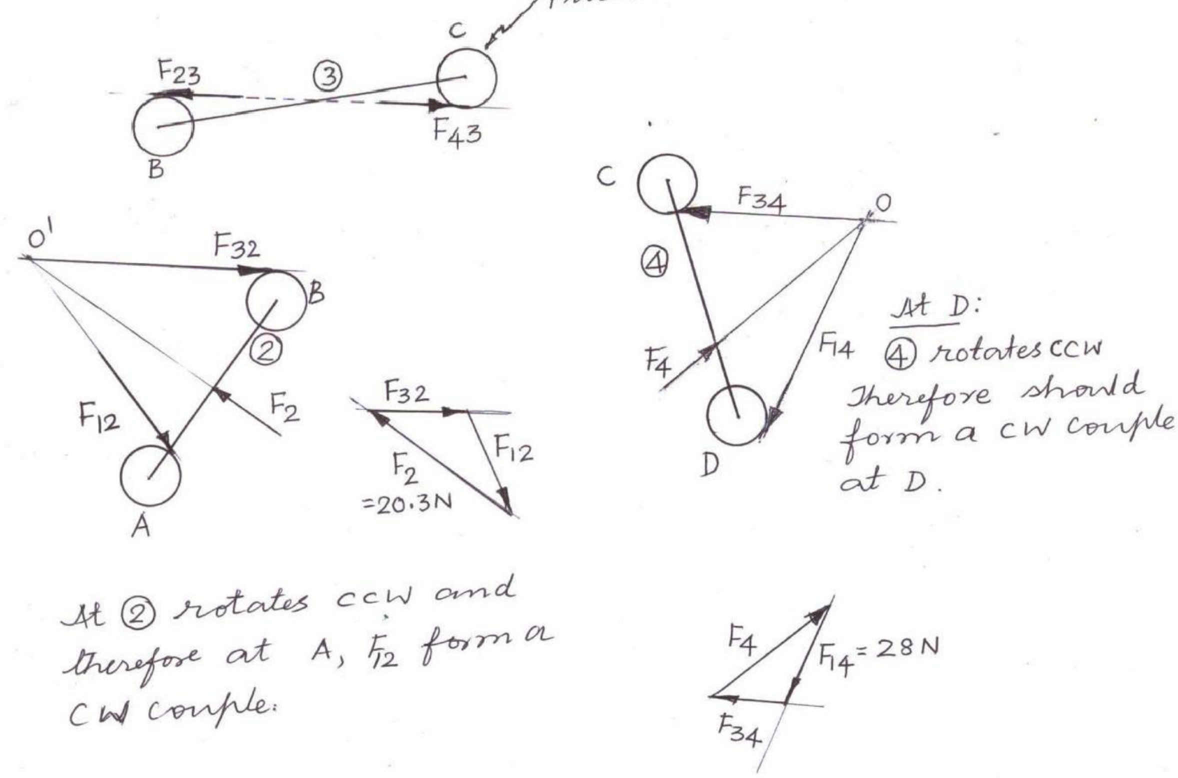
At C:

BCD increases & 3 rotates CW w.r.t 4
Therefore, F_{43} opposes the rotation of 4 by generating a CCW friction couple at C

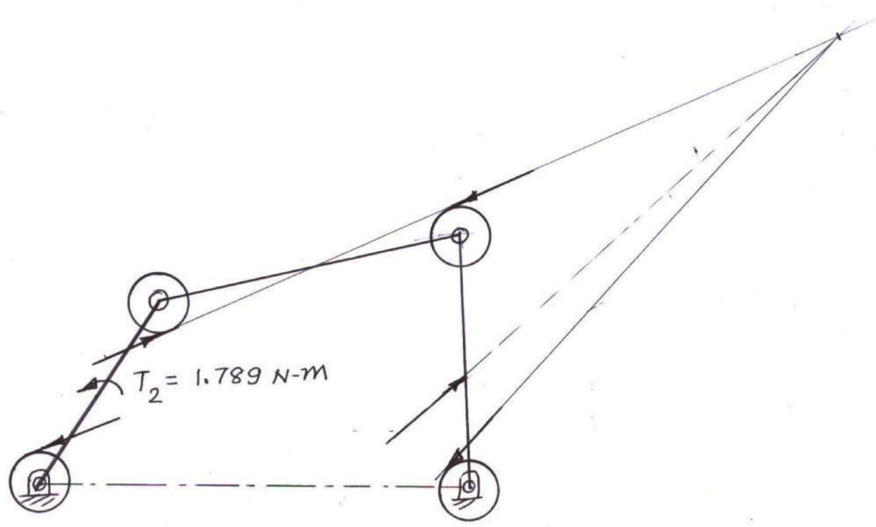
At B:

BCD decreases & 3 rotates CW w.r.t 2 Therefore, F_{23} forms a CCW friction couple at B

(not to scale)



For CW rotation of A



MODULE-2

Balancing of Rotating Masses

CONTENTS

2.1 Balancing of Rotating Masses

2.2 Static and dynamic balancing.

2.3 Balancing of single rotating mass by balancing masses in same plane and in different planes.

2.4 Balancing of several rotating masses by balancing masses in same plane and in different planes.

OBJECTIVES

- To study Importance of Balancing of rotating masses.
- To solve Various problems on Balancing of rotating masses.

2.1 Balancing of Rotating Masses

INTRODUCTION:

When man invented the wheel, he very quickly learnt that if it wasn‘t completely round and if it didn‘t rotate evenly about it‘s central axis, then he had a problem!

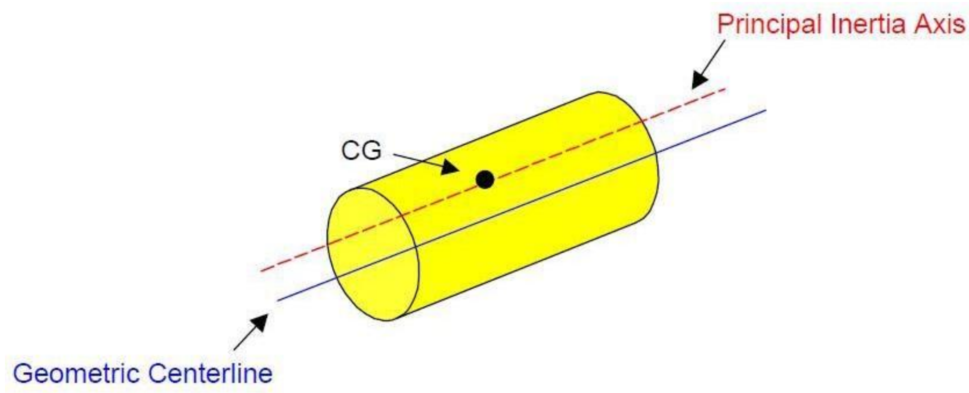
What the problem he had?

The wheel would vibrate causing damage to itself and it‘s support mechanism and in severe cases, is unusable.

A method had to be found to minimize the problem. The mass had to be evenly distributed about the rotating centerline so that the resultant vibration was at a minimum.

UNBALANCE:

The condition which exists in a rotor when vibratory force or motion is imparted to its bearings as a result of centrifugal forces is called unbalance or the uneven distribution of mass about a rotor‘s rotating centerline.



Rotating centerline:

The rotating centerline being defined as the axis about which the rotor would rotate if not constrained by its bearings. (Also called the Principle Inertia Axis or PIA).

Geometric centerline:

The geometric centerline being the physical centerline of the rotor.When the two centerlines are coincident, then the rotor will be in a state of balance. When they are apart, the rotor will be unbalanced.

Different types of unbalance can be defined by the relationship between the two centerlines. These include:

Static Unbalance – where the PIA is displaced parallel to the geometric centerline. (Shown above)

Couple Unbalance – where the PIA intersects the geometric centerline at the center of gravity. (CG)

Dynamic Unbalance – where the PIA and the geometric centerline do not coincide or touch.

The most common of these is dynamic unbalance.

Causes of Unbalance:

In the design of rotating parts of a machine every care is taken to eliminate any out of balance or couple, but there will be always some residual unbalance left in the finished part because of

i) slight variation in the density of the material or
ii) inaccuracies in the casting or
iii) inaccuracies in machining of the parts.

Why balancing is so important?

iii) A level of unbalance that is acceptable at a low speed is completely unacceptable at a higher speed.

iv) As machines get bigger and go faster, the effect of the unbalance is much more severe.

v) The force caused by unbalance increases by the square of the speed.

vi) If the speed is doubled, the force quadruples; if the speed is tripled the force increases by a factor of nine!

Identifying and correcting the mass distribution and thus minimizing the force and resultant vibration is very very important

2.2 Static and dynamic balancing.

BALANCING:

Balancing is the technique of correcting or eliminating unwanted inertia forces or moments in rotating or reciprocating masses and is achieved by changing the location of the mass centers.

The objectives of balancing an engine are to ensure:

That the centre of gravity of the system remains stationery during a complete revolution of the crank shaft and .That the couples involved in acceleration of the different moving parts balance each other.

Types of balancing:

Static Balancing:

Static balancing is a balance of forces due to action of gravity.

A body is said to be in static balance when its centre of gravity is in the axis of rotation.

Dynamic balancing:

Dynamic balance is a balance due to the action of inertia forces.

A body is said to be in dynamic balance when the resultant moments or couples, which involved in the acceleration of different moving parts is equal to zero.

The conditions of dynamic balance are met, the conditions of static balance are also met.

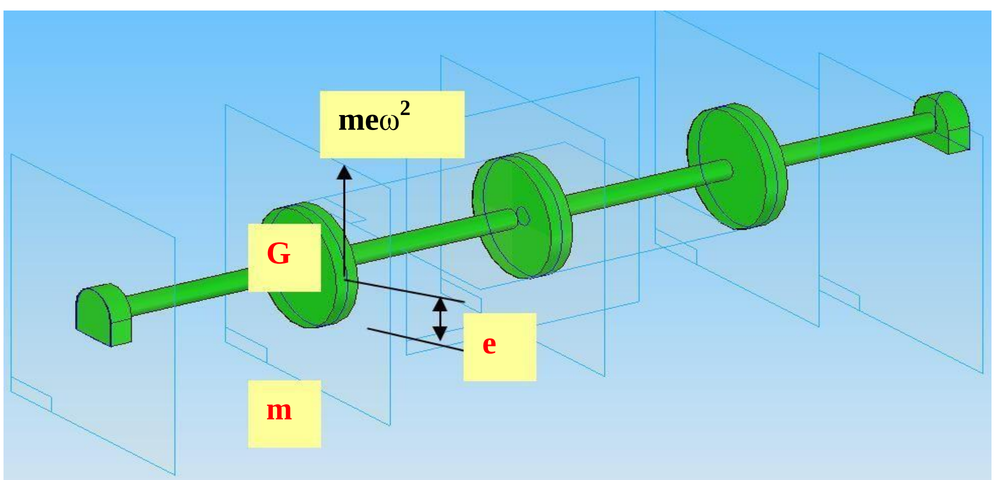
In rotor or reciprocating machines many a times unbalance of forces is produced due to inertia forces associated with the moving masses. If these parts are not properly balanced, the dynamic forces are set up and forces not only increase loads on bearings and stresses in the various components, but also unpleasant and dangerous vibrations.

Balancing is a process of designing or modifying machinery so that the unbalance is reduced to an acceptable level and if possible eliminated entirely.

BALANCING OF ROTATING MASSES

When a mass moves along a circular path, it experiences a centripetal acceleration and a force is required to produce it. An equal and opposite force called centrifugal force acts radially outwards and is a disturbing force on the axis of rotation. The magnitude of this remains constant but the direction changes with the rotation of the mass.

In a revolving rotor, the centrifugal force remains balanced as long as the centre of the mass of rotor lies on the axis of rotation of the shaft. When this does not happen, there is an eccentricity and an unbalance force is produced. This type of unbalance is common in steam turbine rotors, engine crankshafts, rotors of compressors, centrifugal pumps etc.



The unbalance forces exerted on machine members are time varying, impart vibratory motion and noise, there are human discomfort, performance of the machine deteriorate and detrimental effect on the structural integrity of the machine foundation.

Balancing involves redistributing the mass which may be carried out by addition or removal of mass from various machine members

Balancing of rotating masses can be of

1) Balancing of a single rotating mass by a single mass rotating in the same plane.
2) Balancing of a single rotating mass by two masses rotating in different planes.
3) Balancing of several masses rotating in the same plane
4) Balancing of several masses rotating in different planes

STATIC BALANCING

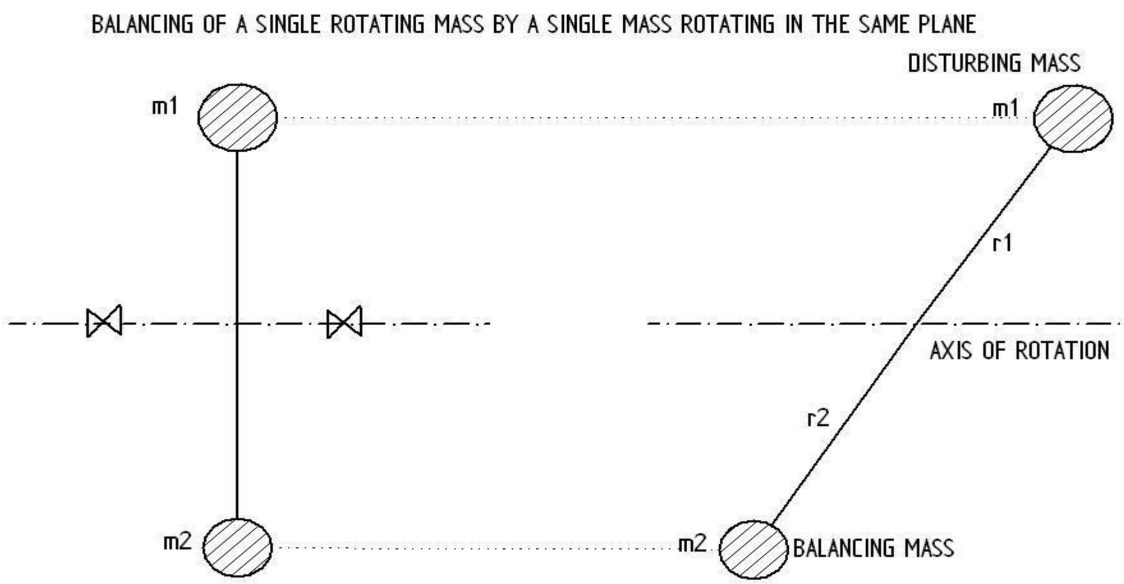
A system of rotating masses is said to be in static balance if the combined mass centre of the system lies on the axis of rotation

DYNAMIC BALANCING

When several masses rotate in different planes, the centrifugal forces, in addition to being out of balance, also form couples. A system of rotating masses is in dynamic balance when there does not exist any resultant centrifugal force as well as resultant couple.

2.3 Balancing of single rotating mass by balancing masses in same plane and in different planes.

CASE 1.
BALANCING OF A SINGLE ROTATING MASS BY A SINGLE MASS ROTATING IN THE SAME PLANE



Consider a disturbing mass m_1 which is attached to a shaft rotating at ω rad/s. Let

r_1 = radius of rotation of the mass m_1
=distance between the axis of rotation of the shaft and the centre of gravity of the mass m_1

The centrifugal force exerted by mass m_1 on the shaft is given by,

$$\mathbf{F_{c1} = m_1 \omega^2 r_1} \text{ ----------------------- } \mathbf{(1)}$$

This force acts radially outwards and produces bending moment on the shaft. In order to counteract the effect of this force F_{c1} , a balancing mass m_2 may be attached in the same plane of rotation of the disturbing mass m_1 such that the centrifugal forces due to the two masses are equal and opposite.

Let,

r_2 = radius of rotation of the mass m_2

= distance between the axis of rotation of the shaft and the centre of gravity of the mass m_2

Therefore the centrifugal force due to mass m_2 will be,

$F_{c2} = m_2 \omega^2 r_2$ −−−−−−−−−−−−−−−−−−−(2)

Equating equations (1) and (2), we get

$F_{c1} = F_{c2}$

$m \omega^2 r = m \omega^2 r$ or $m r = m r$ −−−−−−−−−−−−−−−−−(3)

The product $\mathbf{m_2 r_2}$ can be split up in any convenient way. As for as possible the radius of rotation of mass m_2 that is r_2 is generally made large in order to reduce the balancing mass m_2.

CASE 2:

BALANCING OF A SINGLE ROTATING MASS BY TWO MASSES ROTATING IN DIFFERENT PLANES.

There are two possibilities while attaching two balancing masses:

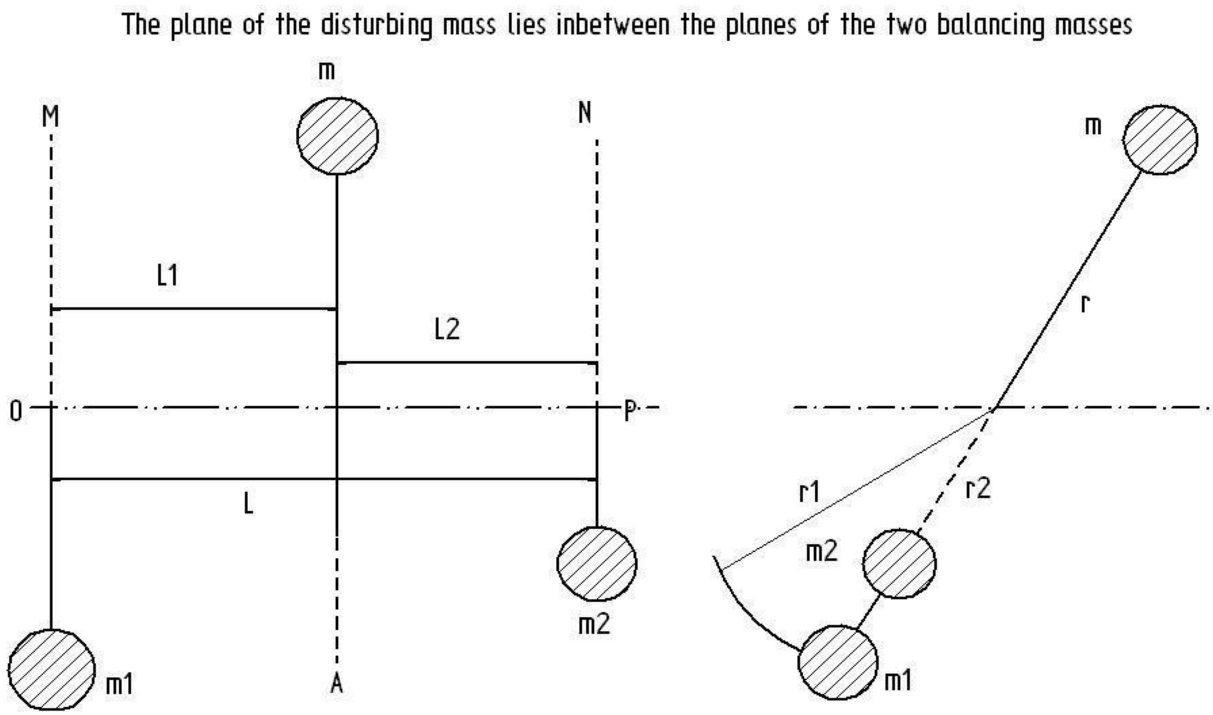
1. The plane of the disturbing mass may be in between the planes of the two balancing masses.

2. The plane of the disturbing mass may be on the left or right side of two planes containing the balancing masses.

In order to balance a single rotating mass by two masses rotating in different planes which are parallel to the plane of rotation of the disturbing mass i) the net dynamic force acting on the shaft must be equal to zero, i.e. the centre of the masses of the system must lie on the axis of rotation and this is the condition for static balancing ii) the net couple due to the dynamic forces acting on the shaft must be equal to zero, i.e. the algebraic sum of the moments about any point in the plane must be zero. The conditions i) and ii) together give dynamic balancing.

CASE 2(I):

THE PLANE OF THE DISTURBING MASS LIES IN BETWEEN THE PLANES OF THE TWO BALANCING MASSES.



Consider the disturbing mass m lying in a plane A which is to be balanced by two rotating masses m_1 and m_2 lying in two different planes M and N which are parallel to the plane A as shown.

Let r, r_1 and r_2 be the radii of rotation of the masses in planes A, M and N respectively.

Let L_1, L_2 and L be the distance between A and M, A and N, and M and N respectively.

Now,

The centrifugal force exerted by the mass m in plane A will be,

$$F_c = m\,\omega^2\,r \text{ ------------------- (1)}$$

Similarly,

The centrifugal force exerted by the mass m_1 in plane M will be,

$$F_{c1} = m_1\,\omega^2\,r_1 \text{ ------------------- (2)}$$

And the centrifugal force exerted by the mass m_2 in plane N will be,

$$F_{c2} = m_2\, \omega^2\, r_2 \text{ ------------------ } (3)$$

For the condition of static balancing,

$$F_c = F_{c1} + F_{c2}$$

$$\text{or } m\omega^2\, r = m\, \omega^2\, r + m\, \omega^2\, r$$

$$\text{i.e. } mr = m_1\, r_1 + m_2\, r_2 \text{ ---------------- } (4)$$

Now, to determine the magnitude of balancing force in the plane ‗M' or the dynamic force at the bearing ‗O' of a shaft, take moments about ‗ P ' which is the point of intersection of the plane N and the axis of rotation.

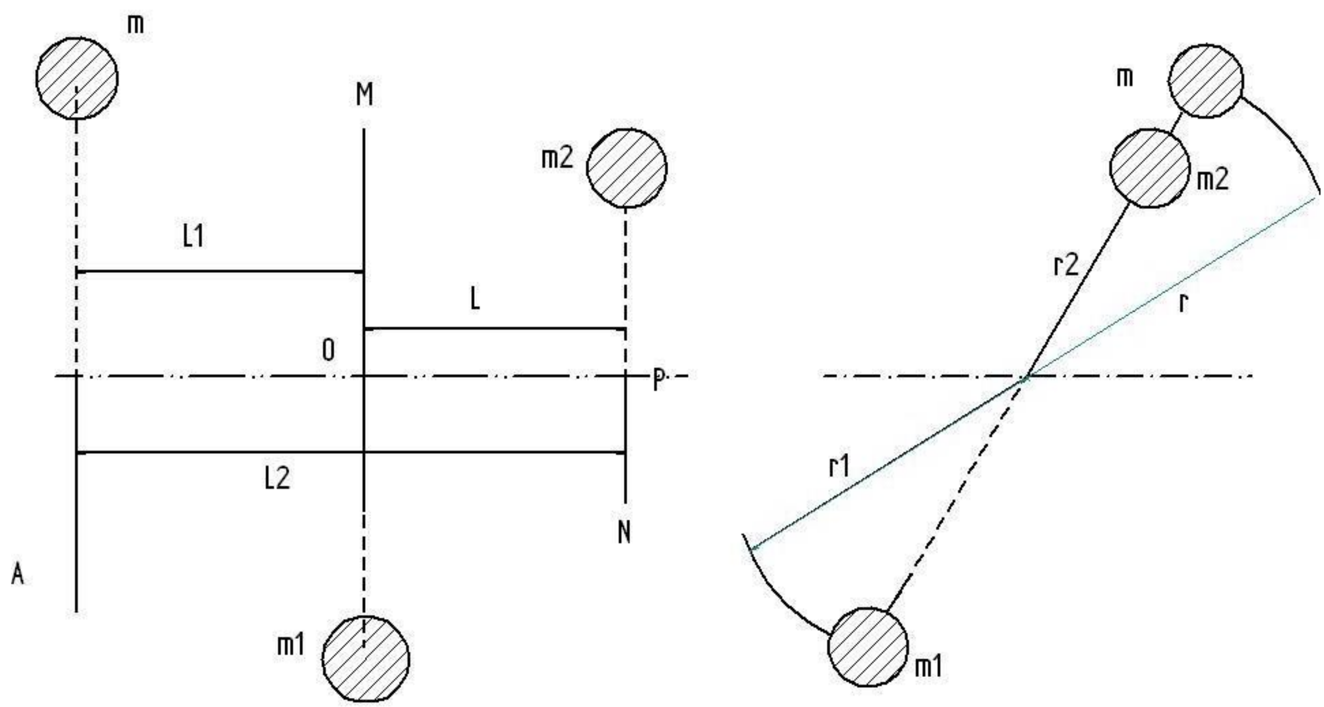
Similarly, in order to find the balancing force in plane ‗N' or the dynamic force at the bearing ‗P' of a shaft, take moments about ‗ O ' which is the point of intersection of the plane M and the axis of rotation

For dynamic balancing equations (5) or (6) must be satisfied along with equation (4).

CASE 2(II):

WHEN THE PLANE OF THE DISTURBING MASS LIES ON ONE END OF THE TWO PLANES CONTAINING THE BALANCING MASSES.

When the plane of the disturbing mass lies on one end of the planes of the balancing masses



For static balancing,

$$F_{c1} = F_c + F_{c2}$$

$$\text{or } m_1\,\omega^2\,r_1 = m\omega^2\,r + m_2\,\omega^2\,r_2$$

$$\text{i.e. } m_1\,r_1 = mr + m_2\,r_2 \text{ ------------------ } (1)$$

For dynamic balance the net dynamic force acting on the shaft and the net couple due to dynamic forces acting on the shaft is equal to zero.

To find the balancing force in the plane _M‘ or the dynamic force at the bearing _O‘ of a shaft, take moments about _P‘. i.e.

$F_{c1} \times L = F_c \times L_2$

or $m_1 \omega^2 r_1 \times L = m\omega^2 r \times L_2$

Therefore,

$$m_1 r_1 L = mrL_2 \qquad \text{or } m_1 r_1 = mr \frac{L_2}{L} \quad \text{---------(2)}$$

Similarly, to find the balancing force in the plane ‗N' , take moments about ‗O', i.e.,

$F_{c2} \times L = F_c \times L_1$

or $m_2 \omega^2 r_2 \times L = m\omega^2 r \times L_1$

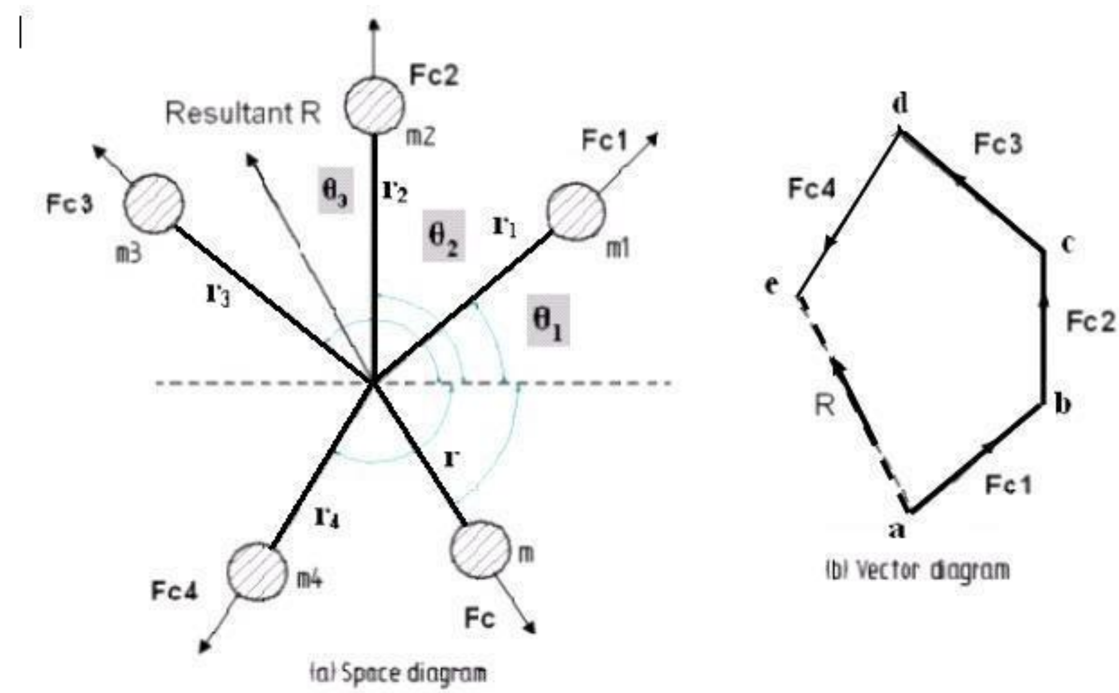
Therefore,

$$m_2 r_2 L = mrL_1 \qquad \text{or } m_2 r_2 = mr \frac{L_1}{L} \quad \text{---------(3)}$$

2.4 Balancing of several rotating masses by balancing masses in same plane and in different planes.

CASE 3:

BALANCING OF SEVERAL MASSES ROTATING IN THE SAME PLANE



BALANCING OF SEVERAL MASSES ROTATING IN THE SAME PLANE

Consider a rigid rotor revolving with a constant angular velocity ω rad/s. A number of masses say, four are depicted by point masses at different radii in the same transverse plane.

If m_1, m_2, m_3 and m_4 are the masses revolving at radii r_1, r_2, r_3 and r_4 respectively in the same plane.

The centrifugal forces exerted by each of the masses are F_{c1}, F_{c2}, F_{c3} and F_{c4} respectively.

Let F be the vector sum of these forces. i.e.

1. $=F_{c1} + F_{c2} + F_{c3} + F_{c4}$

$$= m_1 \omega^2 r_1 + m_2 \omega^2 r_2 + m_3 \omega^2 r_3 + m_4 \omega^2 r_4 \text{ ---------- } (1)$$

The rotor is said to be statically balanced if the vector sum F is zero. If the vector sum F is not zero, i.e. the rotor is unbalanced, then introduce a counterweight (balance weight) of mass _m' at radius _r' to balance the rotor so that,

$$m_1 \omega^2 r_1 + m_2 \omega^2 r_2 + m_3 \omega^2 r_3 + m_4 \omega^2 r_4 + m \omega^2 r = 0 \text{ ---------- } (2)$$

or

$$m_1 r_1 + m_2 r_2 + m_3 r_3 + m_4 r_4 + m r = 0 \text{ ----------------- } (3)$$

The magnitude of either _m' or _r' may be selected and the other can be calculated.
In general, if $\sum \mathbf{m_i r_i}$ is the vector sum of $\mathbf{m_1 r_1}$, $\mathbf{m_2 r_2}$, $\mathbf{m_3 r_3}$, $\mathbf{m_4 r_4}$ etc, then,

$$\sum m_i r_i + mr = 0 \text{ -------- } (4)$$

The above equation can be solved either analytically or graphically.

1. Analytical Method:

2. Resolve the centrifugal forces horizontally and vertically and find their sums, *i.e.* ΣH and ΣV . We know that

Sum of horizontal components of the centrifugal forces,

$$\Sigma H = m_1 \cdot r_1 \cos\theta_1 + m_2 \cdot r_2 \cos\theta_2 + \ldots\ldots$$

and sum of vertical components of the centrifugal forces,

$$\Sigma V = m_1 \cdot r_1 \sin\theta_1 + m_2 \cdot r_2 \sin\theta_2 + \ldots\ldots$$

3. Magnitude of the resultant centrifugal force,

$$F_C = \sqrt{(\Sigma H)^2 + (\Sigma V)^2}$$

4. If θ is the angle, which the resultant force makes with the horizontal, then

$$\tan\theta = \Sigma V / \Sigma H$$

5. The balancing force is then equal to the resultant force, but in ***opposite direction.***
6. Now find out the magnitude of the balancing mass, such that

$$F_C = m \cdot r$$

where m = Balancing mass, and

r = Its radius of rotation.

2. Graphical Method:

Step 1:

Draw the space diagram with the positions of the several masses, as shown.

Step 2:

Find out the centrifugal forces or product of the mass and radius of rotation exerted by each mass.

Step 3:

Now draw the vector diagram with the obtained centrifugal forces or product of the masses and radii of rotation. To draw vector diagram take a suitable scale.

Let ab, bc, cd, de represents the forces F_{c1}, F_{c2}, F_{c3} and F_{c4} on the vector diagram.

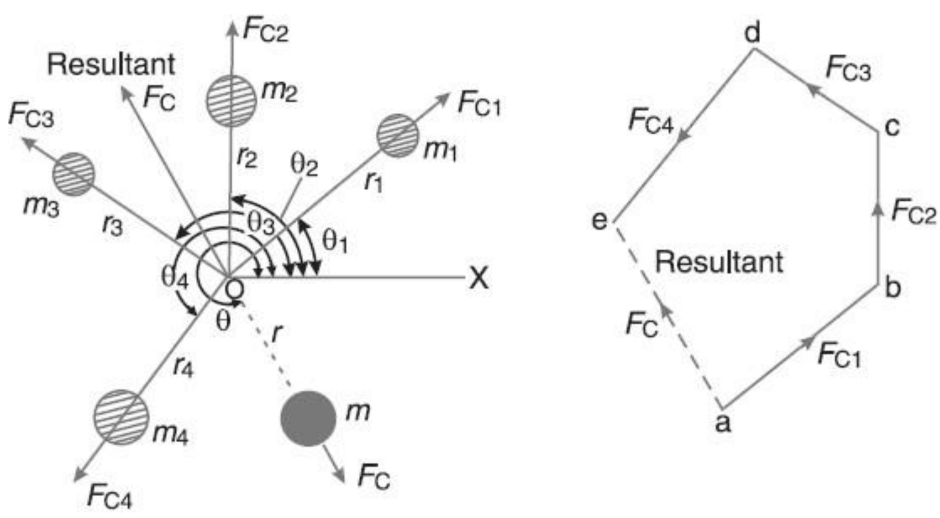
Draw _ab‘ parallel to force F_{c1} of the space diagram, at _b‘ draw a line parallel to force F_{c2}. Similarly draw lines cd, de parallel to F_{c3} and F_{c4} respectively.

Step 4:

As per polygon law of forces, the closing side _ae‘ represents the resultant force in magnitude and direction as shown in vector diagram.

Step 5:

The balancing force is then , equal and opposite to the resultant force.



(*a*) Space diagram. (*b*) Vector diagram.

Step 6:

Determine the magnitude of the balancing mass (m) at a given radius of rotation (r), such that,

$$F_c = m\omega^2 r$$

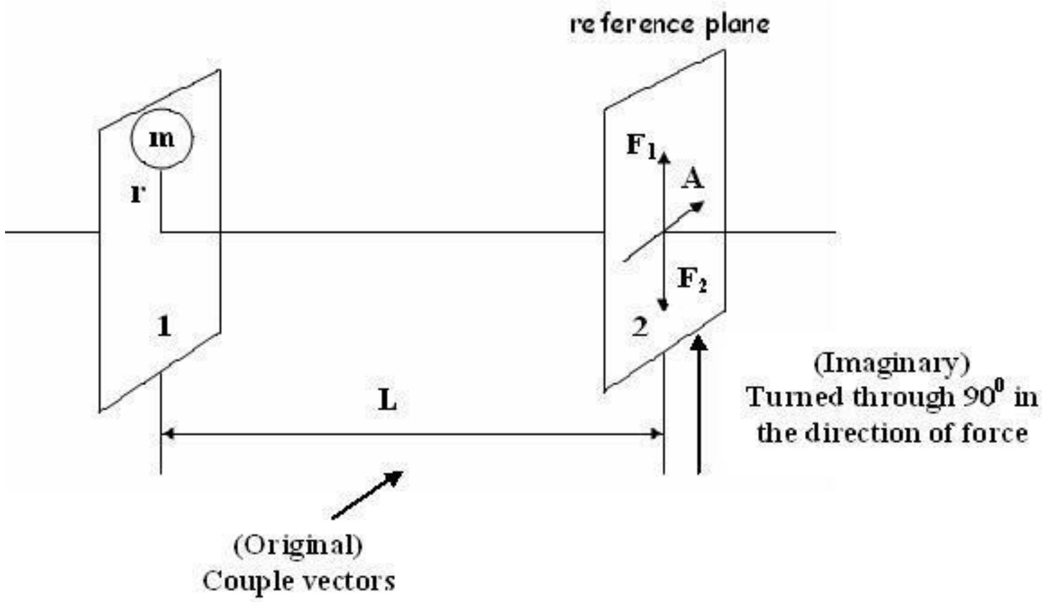
or

$$Mr = \text{resultant of } m_1 r_1, m_2 r_2, m_3 r_3 \text{ and } m_4 r_4$$

CASE 4:

BALANCING OF SEVERAL MASSES ROTATING IN DIFFERENT PLANES

When several masses revolve in different planes, they may be transferred to a reference plane and this reference plane is a plane passing through a point on the axis of rotation and perpendicular to it.



When a revolving mass in one plane is transferred to a reference plane, its effect is to cause a force of same magnitude to the centrifugal force of the revolving mass to act in the reference plane along with a couple of magnitude equal to the product of the force and the distance between the two planes.

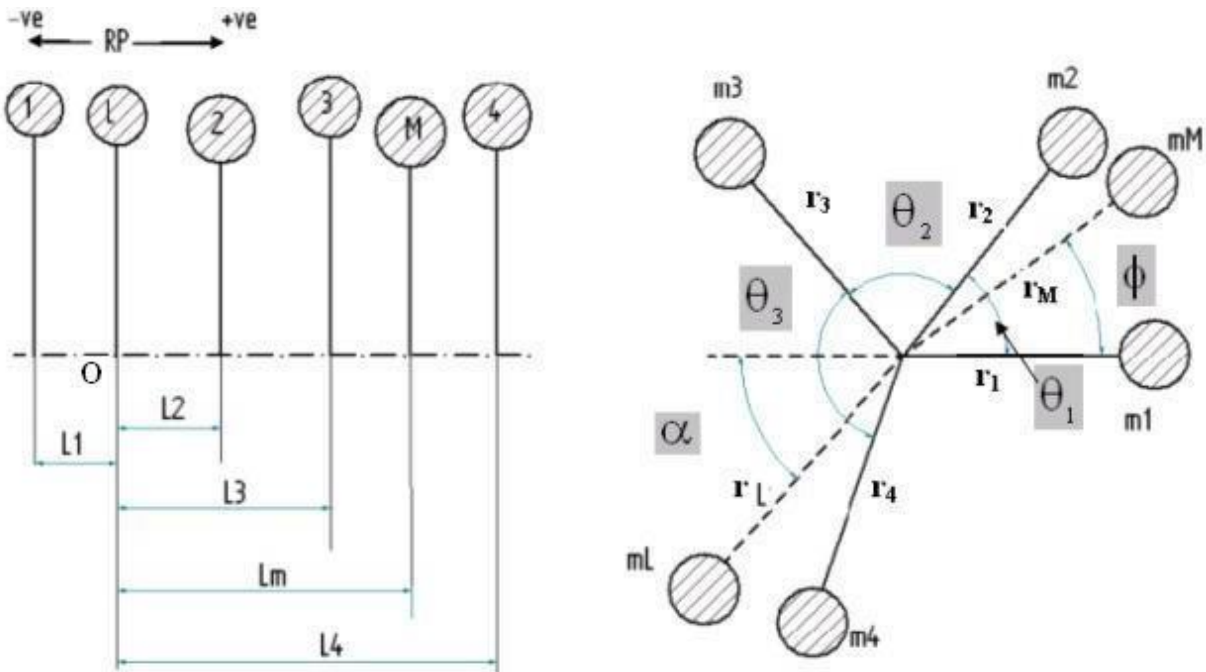
In order to have a complete balance of the several revolving masses in different planes,

1. the forces in the reference plane must balance, i.e., the resultant force must be zero and
2. the couples about the reference plane must balance i.e., the resultant couple must be zero.

A mass placed in the reference plane may satisfy the first condition but the couple balance is satisfied only by two forces of equal magnitude in different planes. Thus, in general, two planes are needed to balance a system of rotating masses.

Example:

Consider four masses m_1, m_2, m_3 and m_4 attached to the rotor at radii r_1, r_2, r_3 and r_4 respectively. The masses m_1, m_2, m_3 and m_4 rotate in planes 1, 2, 3 and 4 respectively.



(a) position of planes of masses

(b) Angular position of masses

a) Position of planes of masses

Choose a reference plane at _O‘ so that the distance of the planes 1, 2, 3 and 4 from _O‘ are L_1, L_2 , L_3 and L_4 respectively. The reference plane chosen is plane _L‘. Choose another plane _M‘ between plane 3 and 4 as shown.

Plane _M‘ is at a distance of L_m from the reference plane _L‘. The distances of all the other planes to the left of _L‘ may be taken as negative(-ve) and to the right may be taken as positive (+ve).

The magnitude of the balancing masses m_L and m_M in planes L and M may be obtained by following the steps given below.

Step 1:

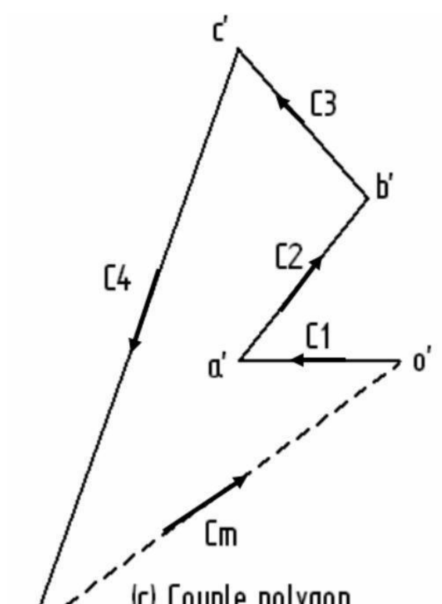
Tabulate the given data as shown after drawing the sketches of position of planes of masses and angular position of masses. The planes are tabulated in the same order in which they occur from left to right.

Plane 1	Mass (m) 2	Radius (r) 3	Centrifugal force/ω^2 (m r) 4	Distance from Ref. plane ‗L‘ (L) 5	Couple/ ω^2 (m r L) 6
1	m1	r1	m1 r1	- L_1	- $m_1 r_1 L_1$
L	mL	rL	mL rL	0	0
2	m_2	r_2	$m_2 r_2$	L_2	m2 r2 L2
3	m_3	r_3	$m_3 r_3$	L_3	m3 r3 L3
M	mM	rM	mM rM	LM	$m_M r_M L_M$
4	m_4	r_4	$m_4 r_4$	L_4	$m_4 r_4 L_4$

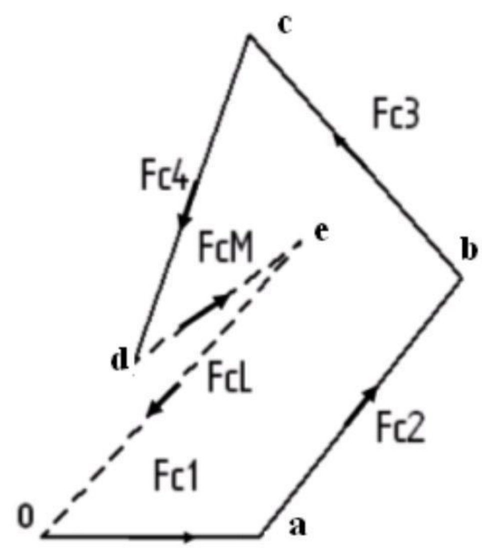
Step 2:

Construct the couple polygon first. (The couple polygon can be drawn by taking a convenient scale)

Add the known vectors and considering each vector parallel to the radial line of the mass draw the couple diagram. Then the closing vector will be ‗$m_M r_M L_M$‘.



(c) Couple polygon



(d) Force polygon

The vector d ‘o‘ on the couple polygon represents the balanced couple. Since the balanced couple C_M is proportional to $m_M r_M L_M$, therefore,

$$C_M = m_M r_M L_M = \text{vector } d' o'$$

$$\text{or} \quad m_M = \frac{\text{vector } d'o'}{r_M \, L_M}$$

From this the value of m_M in the plane M can be determined and the angle of inclination φ of this mass may be measured from figure (b).

Step 3:

Now draw the force polygon (The force polygon can be drawn by taking a convenient scale) by adding the known vectors along with ‗m_M r_M‘. The closing vector will be ‗m_L r_L‘. This represents the balanced force. Since the balanced force is proportional to ‗m_L r_L‘ ,

$$m_L \; r_L = \text{vector eo}$$

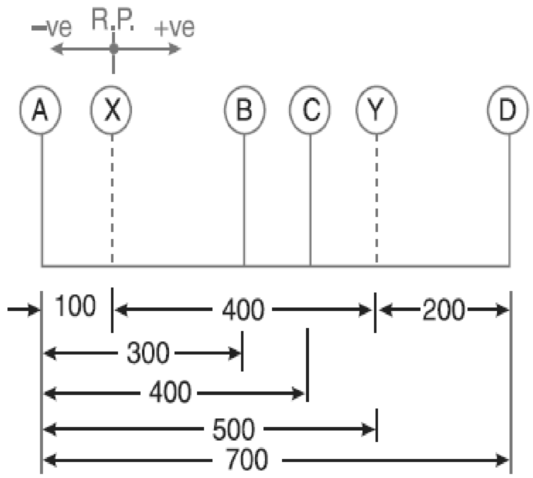
$$\text{or} \quad m_L = \frac{\text{vector eo}}{r_L}$$

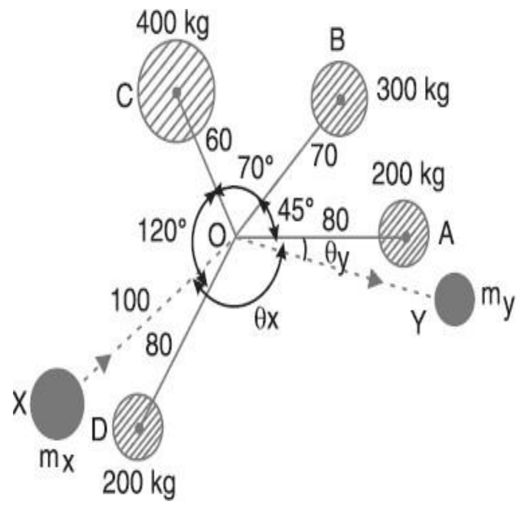
From this the balancing mass m_L can be obtained in plane ‗L‘ and the angle of inclination of this mass with the horizontal may be measured from figure (b).

Problems and solutions

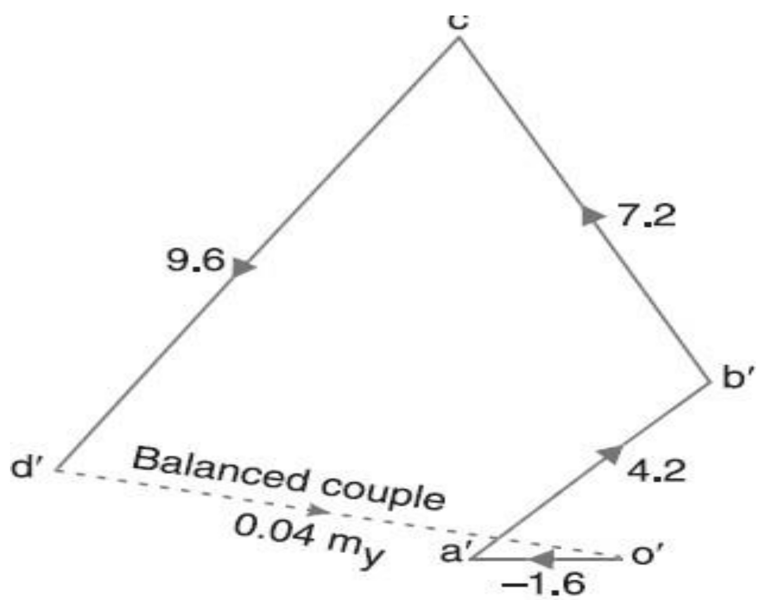
1. A shaft carries four masses A, B, C and D of magnitude 200 kg, 300 kg, 400 kg and 200 kg respectively and revolving at radii 80 mm, 70 mm, 60 mm and 80 mm in planes measured from A at 300 mm, 400 mm and 700 mm. The angles between the cranks measured anticlockwise are A to B 45°, B to C 70° and C to D 120°. The balancing masses are to be placed in planes X and Y. The distance between the planes A and X is 100 mm, between X and Y is 400 mm and between Y and D is 200 mm. If the balancing masses revolve at a radius of 100 mm, find their magnitudes and angular positions.

Given : m_A = 200 kg ; m_B = 300 kg ; m_C = 400 kg ;m_D = 200 kg ,r_A = 80 mmv= 0.08m ; r_B = 70 mm = 0.07 m ; r_C = 60 mm = 0.06 m ; r_D = 80 mm = 0.08 m ; r_X = r_Y = 100 mm = 0.1 m





Plane (1)	Mass (m) kg (2)	Radius (r) m (3)	Cent.force ÷ ω^2 (m.r) kg-m (4)	Distance from Plane x(l) m (5)	Couple ÷ ω^2 (m.r.l) kg-m^2 (6)
A	200	0.08	16	– 0.1	– 1.6
X(R.P.)	m_X	0.1	0.1 m_X	0	0
B	300	0.07	21	0.2	4.2
C	400	0.06	24	0.3	7.2
Y	m_Y	0.1	0.1 m_Y	0.4	0.04 m_Y
D	200	0.08	16	0.6	9.6

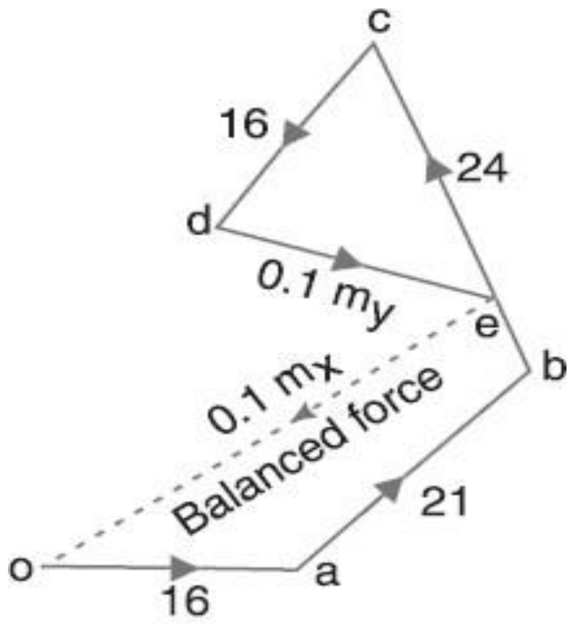


(c) Couple polygon.

By measurement, the angular position of m_Y is $\theta_Y = 12°$ in the clockwise direction from mass m_A (*i.e.* 200 kg).

$$0.04\, m_Y = \text{vector } d'o' = 7.3 \text{ kg-m}^2$$

$$m_Y = 182.5 \text{ kg}$$



(*d*) Force polygon.

$$0.1\, m_X = \text{vector } eo = 35.5 \text{ kg-m}$$

$m_X = 355$ kg

By measurement, the angular position of m_X is $\theta_X = 145°$ in the clockwise direction from mass m_A (*i.e.* 200 kg).

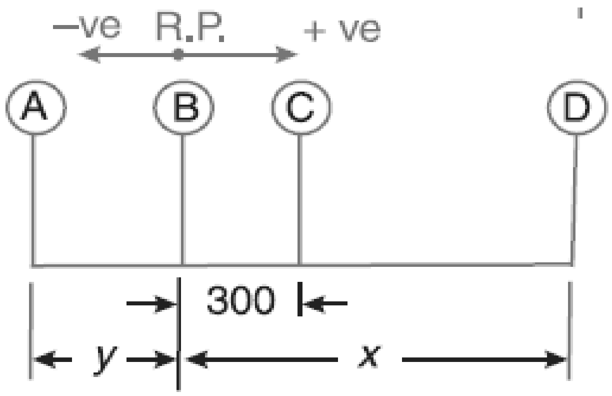
2. Four masses A, B, C and D as shown below are to be completely balanced. The planes containing masses B and C are 300 mm apart. The angle between planes containing B and C is 90°. B and C make angles of 210° and 120° respectively with D in the same sense. Find :

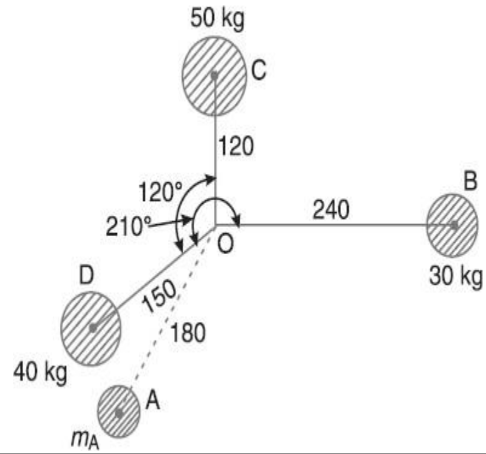
1. The magnitude and the angular position of mass A ; and

2. The position of planes A and D.

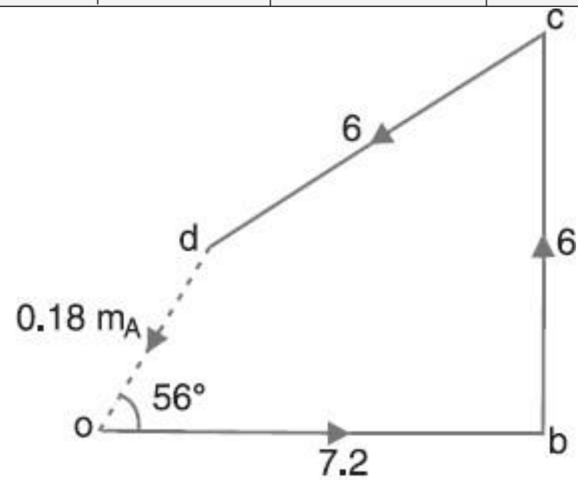
Given r_A = 180 mm = 0.18 m ; m_B = 30 kg ; r_B = 240 mm = 0.24 m ; m_C = 50 kg ; r_C = 120 mm = 0.12 m ; m_D = 40 kg ; r_D = 150 mm = 0.15 m ; ∠BOC = 90° ; ∠BOD = 210° ; ∠COD = 120°

	A	*B*	*C*	*D*
Mass (kg)	—	*30*	*50*	*40*
Radius (mm)	*180*	*240*	*120*	*150*





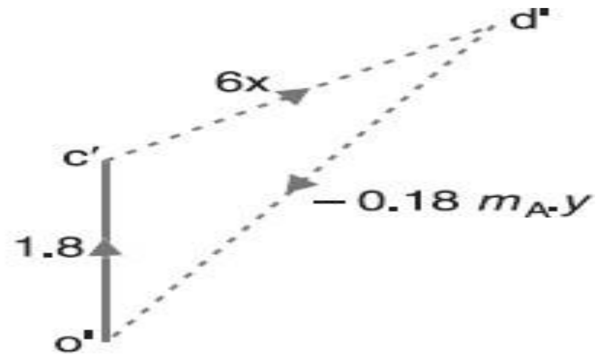
Plane (1)	Mass (m) kg (2)	Radius (r) m (3)	Cent.force $\div \omega^2$ (m.r) kg-m (4)	Distance from plane B (l) m (5)	Couple $\div \omega^2$ (m.r.l) kg-m^2 (6)
A	m_A	0.18	0.08 m_A	$-y$	$-0.18\, m_A y$
B (*R.P*)	30	0.24	7.2	0	0
C	50	0.12	6	0.3	1.8
D	40	0.15	6	x	$6x$



(*c*) Force polygon.

$0.18\, m_A$ = Vector *do* = 3.6 kg-m $m_A = 20$ kg

the angular position of mass *A* from mass *B* in the anticlockwise direction is $\angle AOB = 236°$



(d) Couple polygon.

$$6\,x = \text{vector } c'\,d' = 2.3 \text{ kg-m}^2$$

$$x = 0.383 \text{ m}$$

$$-\,0.18\, m_A . y = \text{vector } o'\,d' = 3.6 \text{ kg-m}^2$$

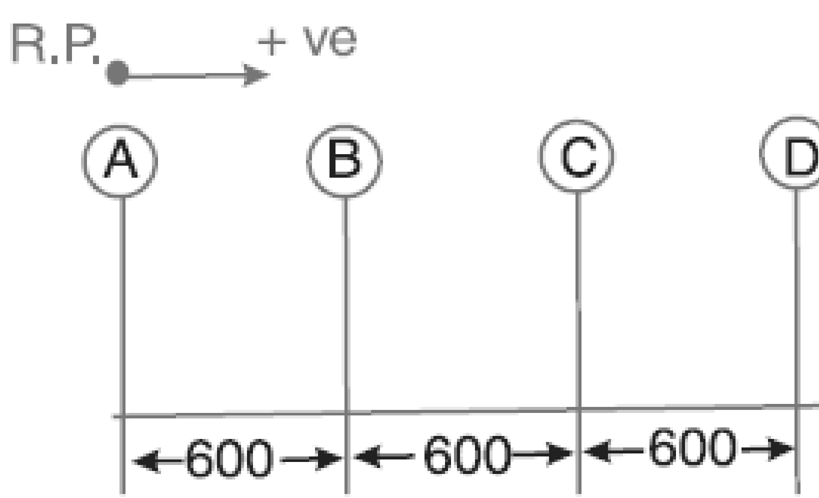
$$-\,0.18 \times 20\, y = 3.6$$

$$y = -\,1 \text{ m}$$

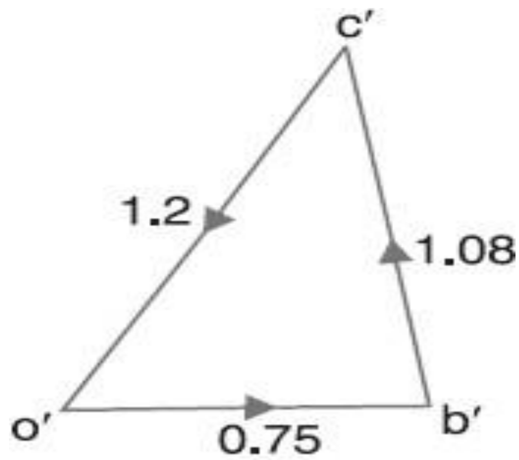
The negative sign indicates that the plane *A* is not towards left of *B* as assumed but it is 1000 mm towards right of plane *B*.

3. A, B, C and D are four masses carried by a rotating shaft at radii 100, 125, 200 and 150 mm respectively. The planes in which the masses revolve are spaced 600 mm apart and the mass of B, C and D are 10 kg, 5 kg, and 4 kg respectively. Find the required mass A and the relative angular settings of the four masses so that the shaft shall be in complete balance.

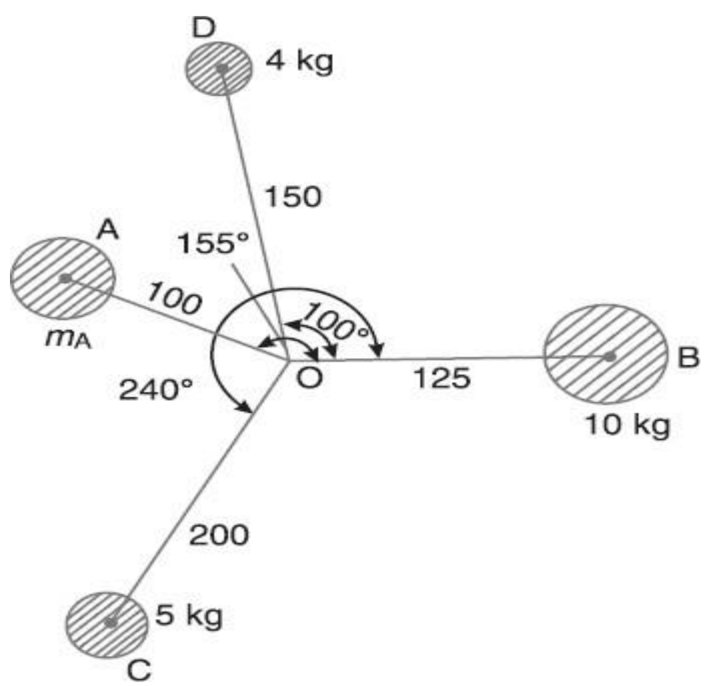
Given : $r_A = 100$ mm = 0.1 m ; $r_B = 125$ mm = 0.125 m ; $r_C = 200$ mm = 0.2 m ;
$r_D = 150$ mm = 0.15 m ; $m_B = 10$ kg ; $m_C = 5$ kg ; $m_D = 4$ kg



Plane (1)	Mass (m) kg (2)	Radius (r) m (3)	Cent. Force ÷ ω^2 (m.r)kg-m (4)	Distance from plane A (l)m (5)	Couple ÷ ω^2 (m.r.l) kg-m^2 (6)
A(R.P.)	m_A	0.1	0.1 m_A	0	0
B	10	0.125	1.25	0.6	0.75
C	5	0.2	1	1.2	1.2
D	4	0.15	0.6	1.8	1.08

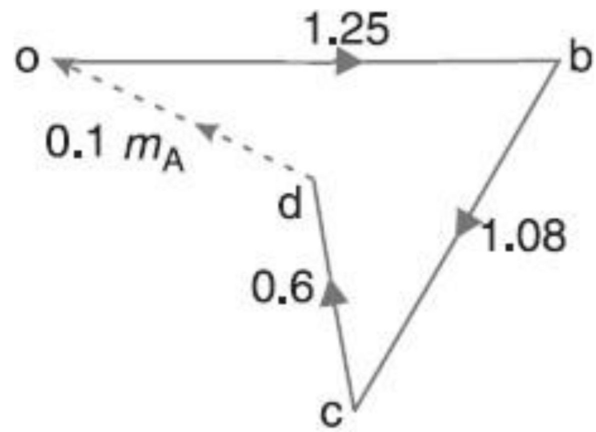


(c) Couple polygon.



$$\angle BOC = 240°$$

$$\angle BOD = 100°$$



(*d*) Force polygon.

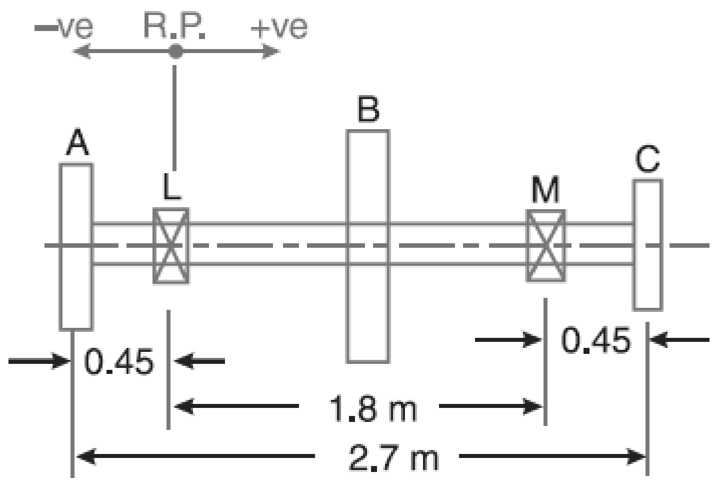
$$0.1\, m_A = 0.7 \text{ kg-m}^2$$

$$m_A = 7 \text{ kg}$$

$$\angle BOA = 155°$$

4. A shaft is supported in bearings 1.8 m apart and projects 0.45 m beyond bearings at each end. The shaft carries three pulleys one at each end and one at the middle of its length. The mass of end pulleys is 48 kg and 20 kg and their centre of gravity are 15 mm and 12.5 mm respectively from the shaft axis. The centre pulley has a mass of 56 kg and its centre of gravity is 15 mm from the shaft axis. If the pulleys are arranged so as to give static balance, determine : 1. relative angular positions of the pulleys, and 2. dynamic forces produced on the bearings when the shaft rotates at 300 r.p.m.

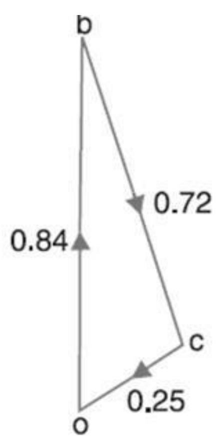
Given : m_A = 48 kg ; m_C = 20 kg ; r_A = 15 mm = 0.015 m ; r_C = 12.5 mm = 0.0125 m ; m_B = 56 kg ; r_B = 15 mm = 0.015 m ; N = 300 r.p.m. or $\omega = 2\pi \times 300/60 = 31.42$ rad/s



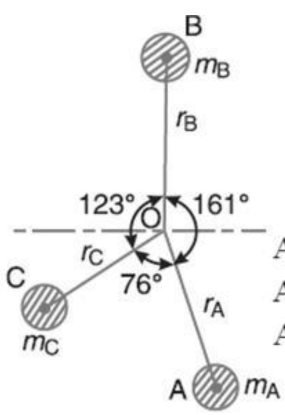
(*a*) Position of shaft and pulleys.

1. *Relative angular position of the pulleys*

Plane (1)	*Mass (m) kg* (2)	*Radius (r) m* (3)	*Cent. force* $\div \omega^2$ *(m.r) kg-m* (4)	*Distance from plane L(l)m* (5)	*Couple* $\div \omega^2$ *(m.r.l) kg-m*^{2} (6)
A	48	0.015	0.72	– 0.45	– 0.324
L(R.P)	m_L	r_L	$m_L.r_L$	0	0
B	56	0.015	0.84	0.9	0.756
M	m_M	r_M	$m_M.r_M$	1.8	1.8 $m_M.r_M$
C	20	0.0125	0.25	2.25	0.5625

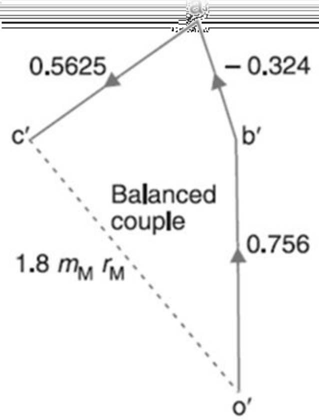


(*c*) Force polygon



Angle between pulleys *B* and *A* = 161°
Angle between pulleys *A* and *C* = 76° .
Angle between pulleys *C* and *B* = 123°

(*b*) Angular position of pulleys.



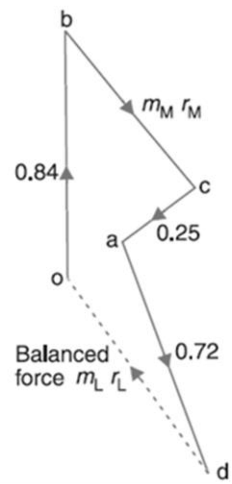
$$1.8\, m_M \cdot r_M = \text{vector } c'o' = 0.97 \text{ kg-m}^2$$

$$m_M \cdot r_M = 0.54 \text{ kg-m}^2$$

Dynamic force at the bearing M

$$= m_M \cdot r_M \cdot \omega^2 = 0.54\ (31.42)^2 = 533$$

(*d*) Couple polygon.



$$m_L \cdot r_L = 0.54 \text{ kg-m}$$

Dynamic force at the bearing L

$$= m_L \cdot r_L \cdot \omega^2 = 0.54\ (31.42)^2 = 533 \text{ N}$$

OUT COMES

1. Students will be able Check static and Dynamic balancing for Rotating systems.
2. Students able to solve problems on balancing of rotating masses

Exercise

1. What is meant by balancing of rotating masses?
2. Why rotating masses are to be dynamically balanced?
3. Define static balancing.
4. Define dynamic balancing.

FURTHER READING

1. Theory of Machines by S.S.Rattan, Third Edition, Tata McGraw Hill Education Private Limited.
2. Kinematics and Dynamics of Machinery by R. L. Norton, First Edition in SI units, Tata McGraw Hill Education Private Limited.
3. Primer on Dynamic Balancing –Causes, Corrections and Consequences‖ By Jim Lyons International Sales Manager IRD Balancing Div. EntekIRD International

MODULE 3

GYROSCOPE

CONTENTS

3.1 INTRODUCTION
3.2 GYROSCOPIC COUPLE
3.3 GYROSCOPIC EFFECT ON SHIP
3.4 GYROSCOPIC EFFECT ON AEROPLANE
3.5 STABILITY OF AUTOMOTIVE VEHICLE

3.1 INTRODUCTION

'Gyre' is a Greek word, meaning _circular motion' and Gyration means the whirling motion. A gyroscope is a spatial mechanism which is generally employed for the study of precessional motion of a rotary body. Gyroscope finds applications in gyrocompass, used in aircraft, naval ship, control system of missiles and space shuttle. The gyroscopic effect is also felt on the automotive vehicles while negotiating a turn.

A gyroscope consists of a rotor mounted in the inner gimbal. The inner gimbal is mounted in the outer gimbal which itself is mounted on a fixed frame as shown in Fig.1. When the rotor spins about X-axis with angular velocity ω rad/s and the inner gimbal precesses (rotates) about Y-axis, the spatial mechanism is forced to turn about Z-axis other than its own axis of rotation, and the gyroscopic effect is thus setup. The resistance to this motion is called gyroscopic effect.

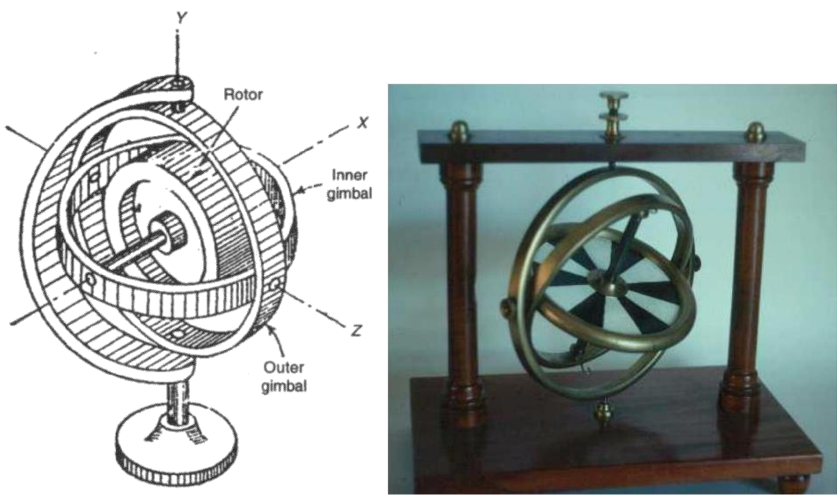


Fig. 1 Gyroscope mechanism

ANGULAR MOTION

A rigid body, (Fig.2) spinning at a constant angular velocity ω rad/s about a spin axis through the mass centre. The angular momentum ‗H‘ of the spinning body is represented by a **vector** whose magnitude is ‗Iω‘. I represents the mass amount of inertia of the rotor about the axis of spin.

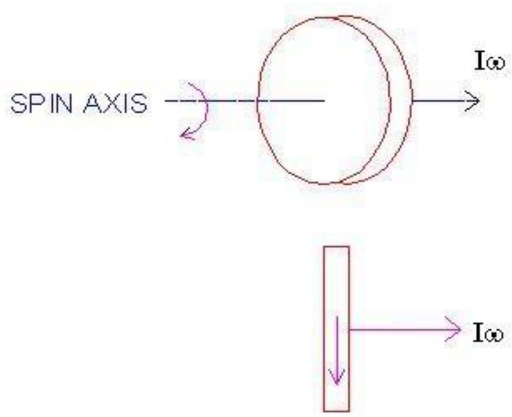


Fig.2 Spinning body

‗.‘ H= Iw

The direction of the angular momentum can be found from the right hand screw rule or the right hand thumb rule. Accordingly, if the fingers of the right hand are bent in the direction of rotation of rotor, then the thumb indicates the direction of momentum.

3.2 GYROSCOPIC COUPLE

Consider a rotary body of mass *m* having radius of gyration *k* mounted on the shaft supported at two bearings. Let the rotor spins (rotates) about X-axis with constant angular velocity rad/s. The X-axis is, therefore, called spin axis, Y-axis, precession axis and Z-axis, the couple or torque axis (Fig.3).

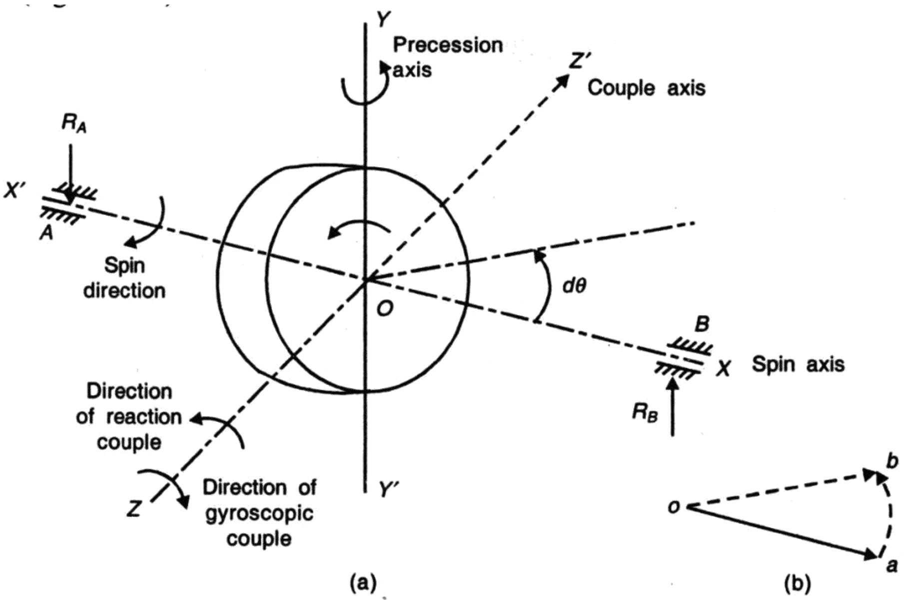


Fig. 3

Now, suppose the shaft axis (X-axis) precesses through a small angle $\delta\theta$ about Y-axis in the plane *XOZ*, then the angular momentum varies from *H* to $H + \delta H$, where δH is the change in the angular momentum, represented by vector ab [Figure 15.2(b)]. For the small value of angle of rotation 5^0, we can write

$$ab = oa \times \delta\theta$$

$$\delta H = H \times \delta\theta$$

$$= I\omega\delta\theta$$

However, the rate of change of angular momentum is:

$$C = \frac{dH}{dt} = \lim_{\delta t \to 0}\left(\frac{I\omega\,\delta\theta}{\delta t}\right)$$

$$= I\omega\frac{d\theta}{dt}$$

or **C = IWW$_p$**

where C = gyroscopic couple (N-m)
W = angular velocity of rotary body (rad/s) W_p = angular velocity of precession (rad/s)

Direction of Spin vector, Precession vector and Couple/Torque vector with forced precession

To determine the direction of spin, precession and torque/couple vector, right hand screw rule or right hand rule is used. The fingers represent the rotation of the disc and the thumb shows the direction of the spin, precession and torque vector (Fig.4).

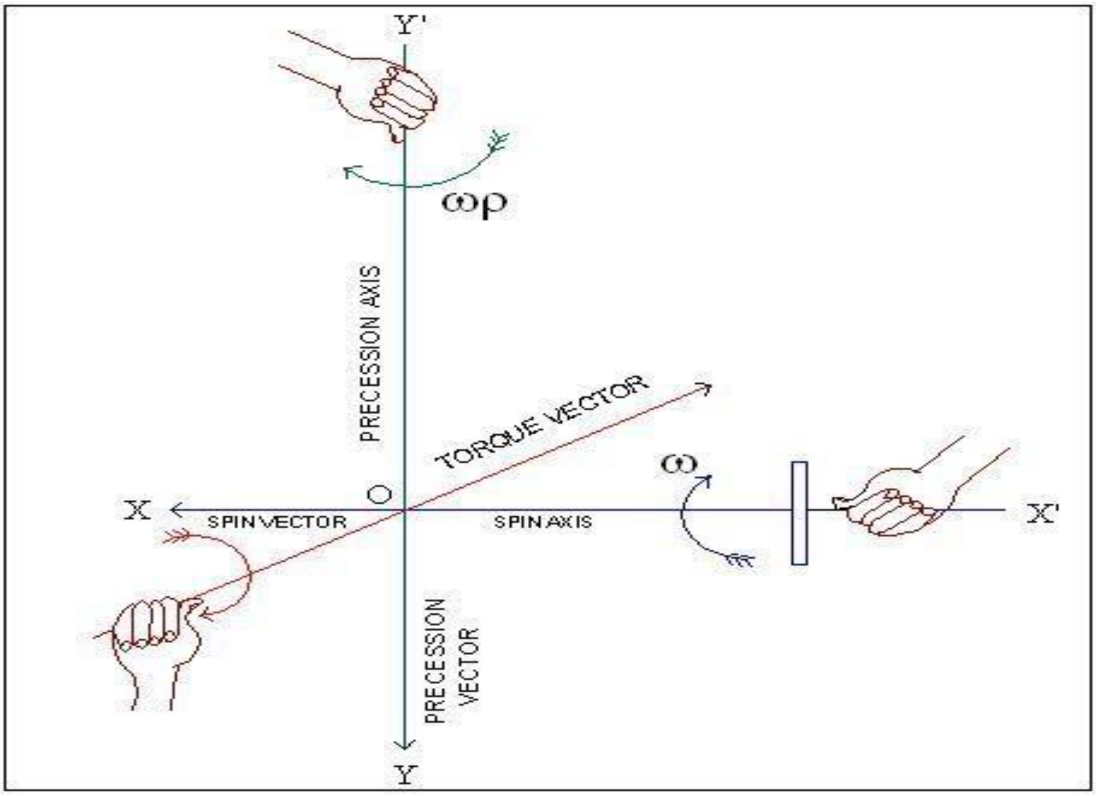


Fig.4. Direction of Spin vector, Precession vector and Couple/Torque vector

The method of determining the direction of couple/torque vector is as follows.

Case (i):

Consider a rotor rotating in anticlockwise direction when seen from the right (Fig.5 and Fig. 6), and to precess the spin axis about precession axis in clockwise and anticlockwise direction when seen from top. Then, to determine the active/reactive gyroscopic couple vector, the following procedure is used.

1. Turn the spin vector through 90^0 in the direction of precession on the XOZ plane
2. The turned spin vector will then correspond to the direction of active gyroscopic couple/torque vector
3. The reactive gyroscopic couple/torque vector is taken opposite to active gyro vector direction

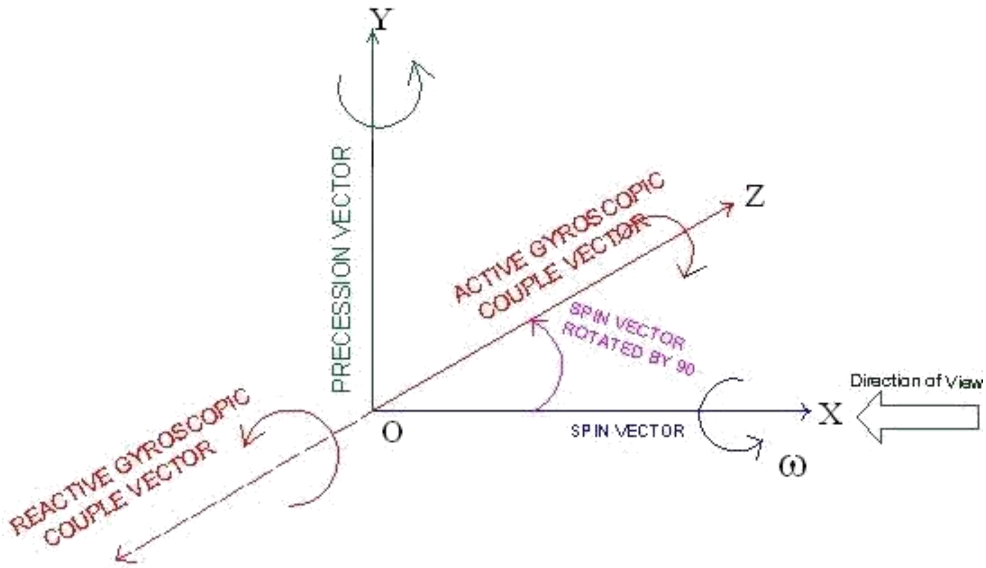


Fig. 5 Direction of active and reactive gyroscopic couple/torque vector

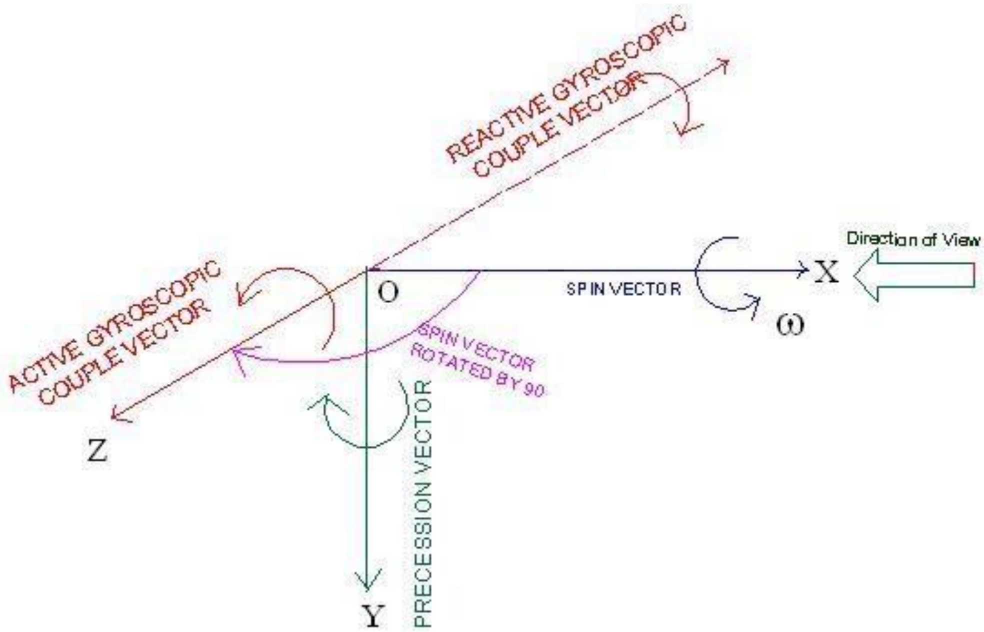


Fig. 6 Direction of active and reactive gyroscopic couple/torque vector

Case (ii):

Consider a rotor rotating in clockwise direction when seen from the right (Fig.7 and Fig. 8), and to precess the spin axis about precession axis in clockwise and anticlockwise direction when seen from top. Then, to determine the active/reactive gyroscopic couple vector,

1. Turn the spin vector through 90^0 in the direction of precession on the XOZ plane
2. The turned spin vector will then correspond to the direction of active gyroscopic couple/torque vector

The reactive gyroscopic couple/torque vector is taken opposite to active gyro vector direction

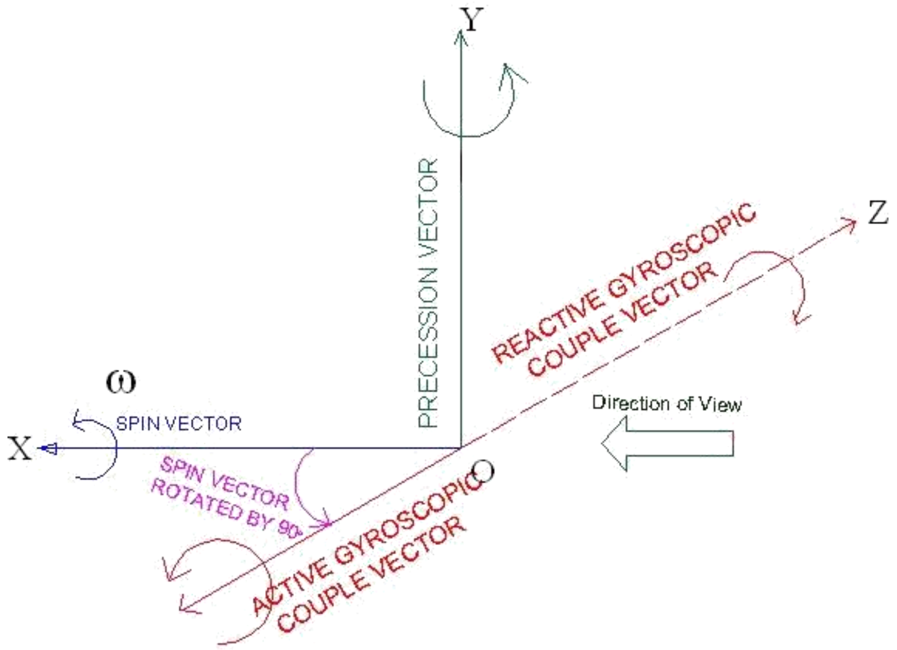


Fig. 7 Direction of active and reactive gyroscopic couple/torque vector

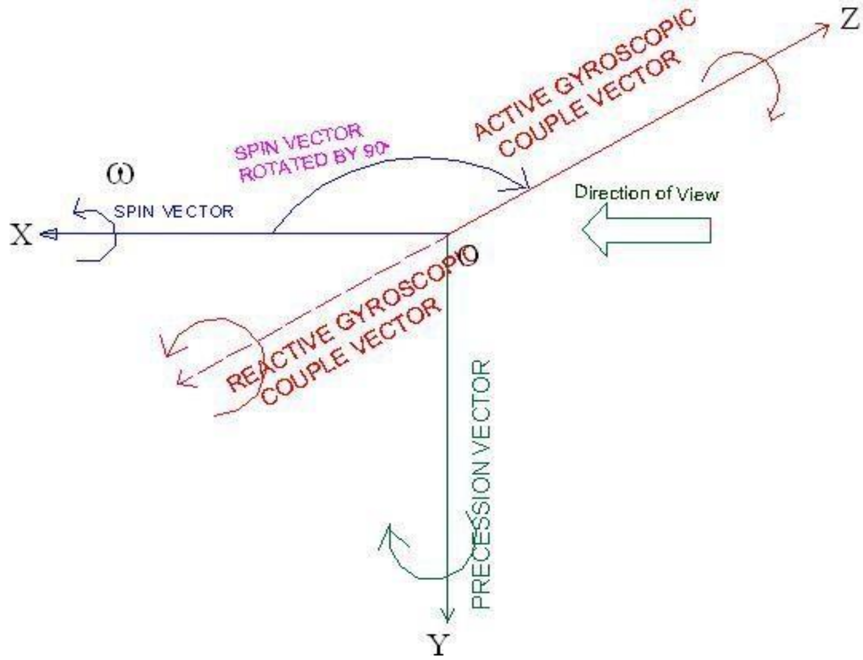


Fig. 8 Direction of active and reactive gyroscopic couple/torque vector

The resisting couple/ reactive couple will act in the direction opposite to that of the gyroscopic couple. This means that, whenever the axis of spin changes its direction, a **gyroscopic couple** is applied to it through the bearing which supports the spinning axis.

Please note that, for analyzing the gyroscopic effect of the body, always reactive gyroscopic couple is considered.

.

Problem 1

A disc of 5 kg mass with radius of gyration 70 mm is mounted at span on a horizontal shaft spins at 720 rpm in clockwise direction when viewed from the right hand bearing. If the shaft precesses about the vertical axis at 30 rpm in clockwise direction when viewed from the top, determine the reactions at each bearing due to mass of the disc and gyroscopic effect.

Solution Angular velocity:

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 720}{60}$$

$$= 75.4 \text{ rad/s}$$

Angular velocity of precession: $$\omega_p = \frac{2\pi N_p}{60}$$

$$= \frac{2\pi \times 30}{60} = 3.14 \text{ rad/s}$$

Moment of inertia: $$I = mk^2$$

$$= 5 \times 0.07^2 = 0.0245 \text{ kg m}^2$$

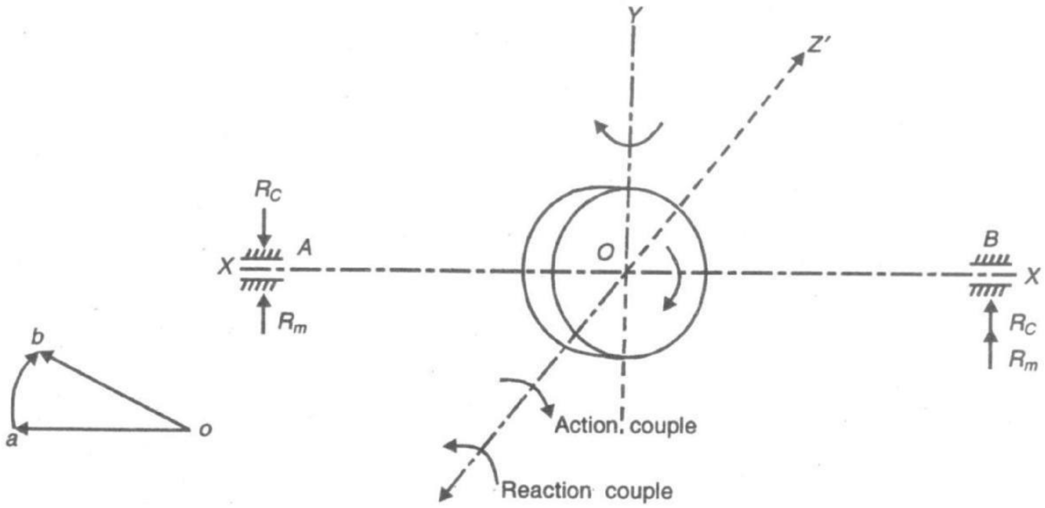


FIG.9a

Gyroscopic couple: $$C = I\,\omega\omega_p$$

$$= 0.0245 \times 75.4 \times 3.14$$

$$= 5.8 \text{ Nm}$$

This couple induces reaction R_c at the bearing support.

$$R_c \times \frac{120}{1000} = 5.8$$

or $$R_c = 48.3 \text{ N}$$

Reaction on the bearings due to weight of the disc, R_m = mg/2 = 5x9.81 /2 = 24.53 N

The angular momentum vector and induced reactive gyroscopic couple acting in anticlockwise direction is shown in Fig.9b.

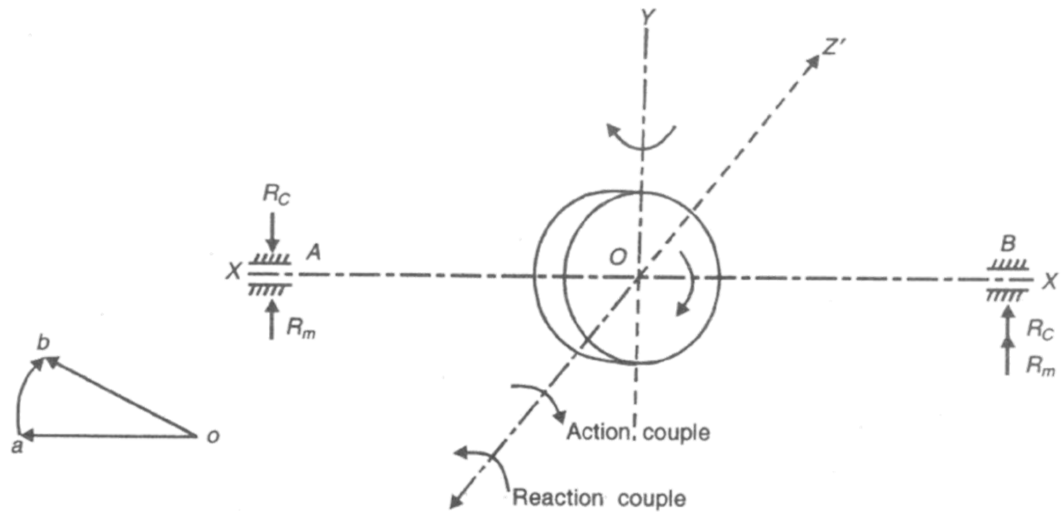


FIG.9b

Gyroscopic couple: $C = I\,\omega\omega_p$

$$= 0.0245 \times 75.4 \times 3.14$$

$$= 5.8 \text{ Nm}$$

This couple induces reaction R_c at the bearing support.

$$R_c \times \frac{120}{1000} = 5.8$$

or $R_c = 48.3$ N

The reaction R_c acts in upward direction at right hand bearing and in downward c at left hand bearing.

The reaction due to weight of the disc acts in upward direction. Therefore,

Reaction at bearing A: $R_A = R_c - R_m$

$$= 48.43 - 24.53$$

$$= \mathbf{23.9\ N(\downarrow)}$$

Reaction at bearing B: $R_B = R_c + R_m$

$$= 48.43 + 24.53$$

$$= \mathbf{72.96\ N(\uparrow)}$$

3.3 GYROSCOPIC EFFECT ON SHIP

Gyroscope is used for stabilization and directional control of a ship sailing in the rough sea. A ship, while navigating in the rough sea, may experience the following three different types of motion:

r Steering—The turning of ship in a curve while moving forward
r Pitching—The movement of the ship up and down from horizontal position in a vertical plane about transverse axis
(iii)Rolling—Sideway motion of the ship about longitudinal axis.

For stabilization of a ship against any of the above motion, the major requirement is that the gyroscope shall be made to precess in such a way that reaction couple exerted by the rotor opposes the disturbing couple which may act on the frame.

r Ship Terminology

Bow – It is the fore end of ship
Stern – It is the rear end of ship
Starboard – It is the right hand side of the ship looking in the direction of motion
Port – It is the left hand side of the ship looking in the direction of motion

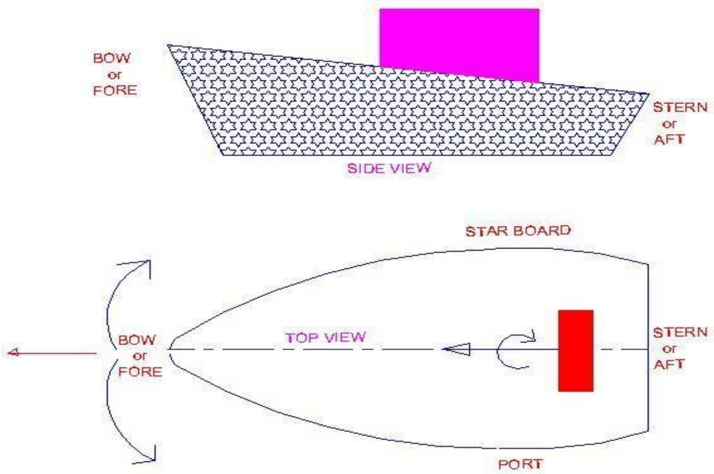


Fig. 10

Consider a gyro-rotor mounted on the ship along longitudinal axis (X-axis) as shown in Fig.10 and rotate in clockwise direction when viewed from rear end of the ship. The angular speed of the rotor is rad/s. The direction of angular momentum vector *oa*, based on direction of rotation of rotor, is decided using right hand thumb rule as discussed earlier.The gyroscopic effect during the three types of motion of ship is discussed.

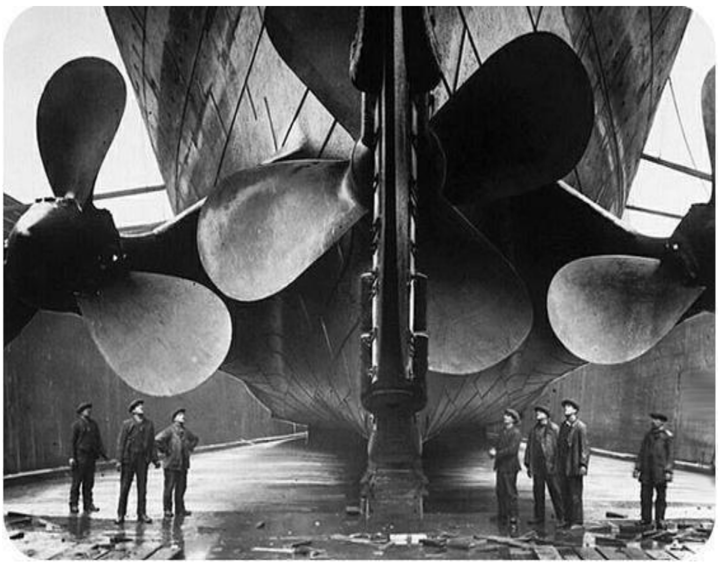


Fig.11

Gyroscopic effect on Steering of ship

+ ***Left turn with clockwise rotor***

When ship takes a left turn and the **rotor rotates in clockwise direction** viewed from stern, the gyroscopic couple act on the ship is analyzed in the following way.

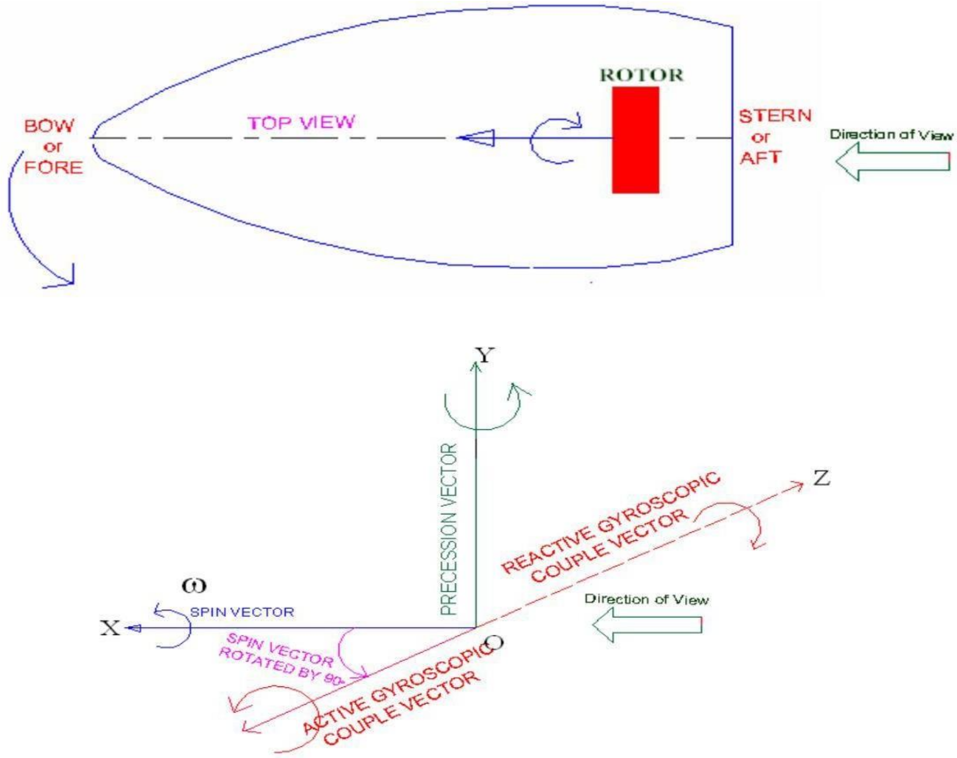


Fig. 12

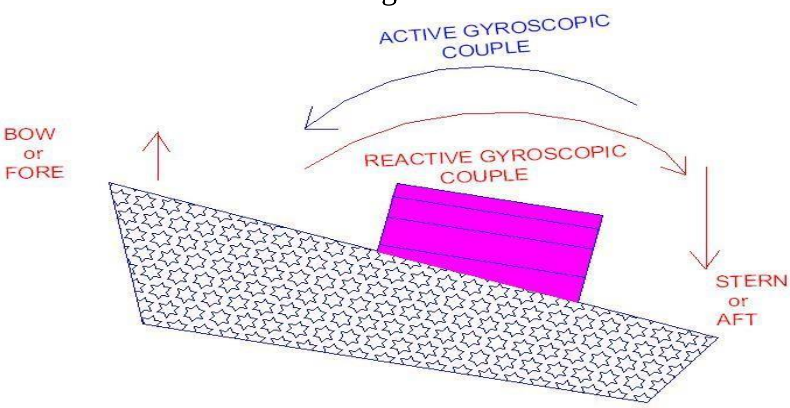


Fig. 13

Note that, always reactive gyroscopic couple is considered for analysis. From the above analysis (Fig.12), the couple acts over the ship between stern and bow. This reaction couple tends to raise the front end (bow) and lower the rear end (stern) of the ship.

(ii) Right turn with clockwise rotor

When ship takes a right turn and the **rotor rotates in clockwise direction** viewed from stern, the gyroscopic couple acts on the ship is analyzed (Fig 14). Again, the couple acts in vertical plane, means between stern and bow. Now the reaction couple tends to lower the bow of the ship and raise the stern.

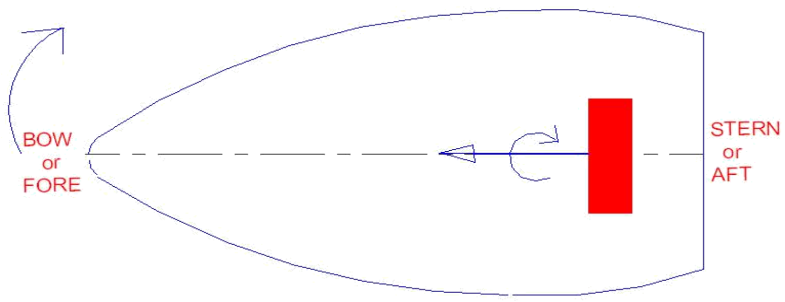


Fig. 14

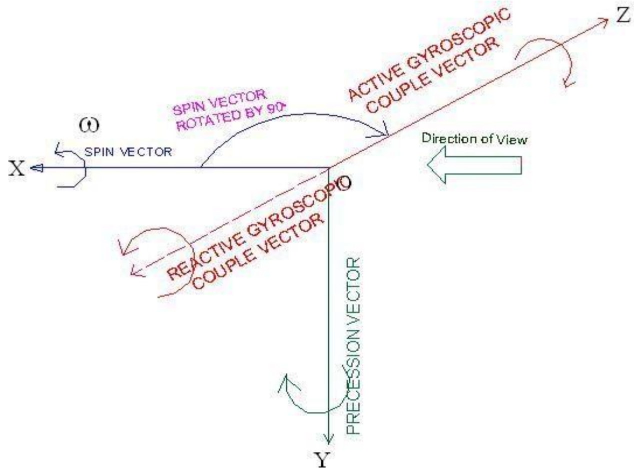


Fig.15

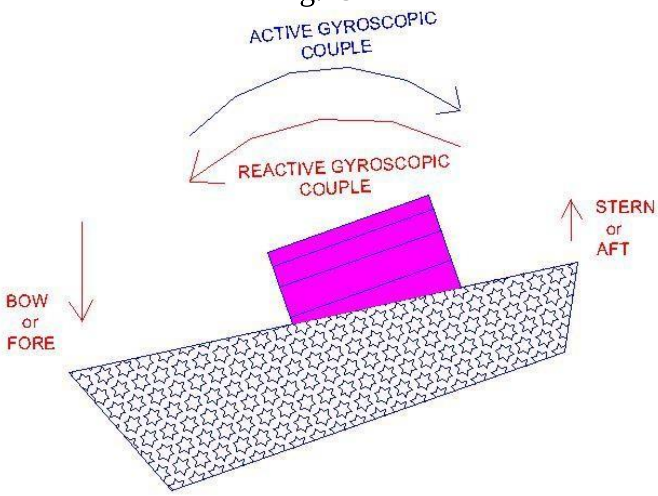


Fig. 16

Left turn with anticlockwise rotor

When ship takes a left turn and the **rotor rotates in anticlockwise direction** viewed from stern, the gyroscopic couple act on the ship is analyzed in the following way (Fig.18).

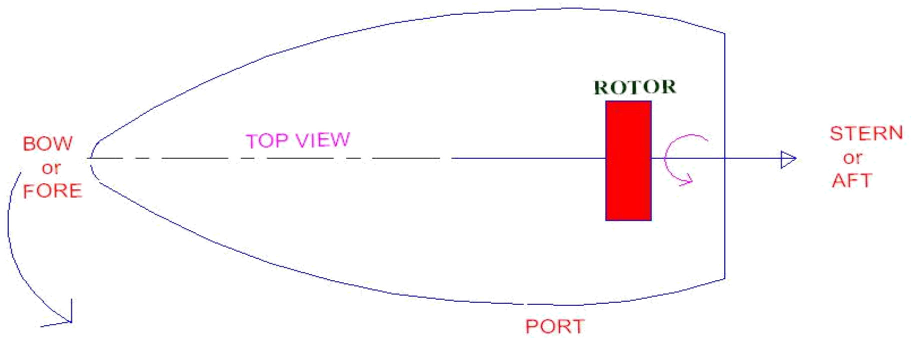


Fig. 17

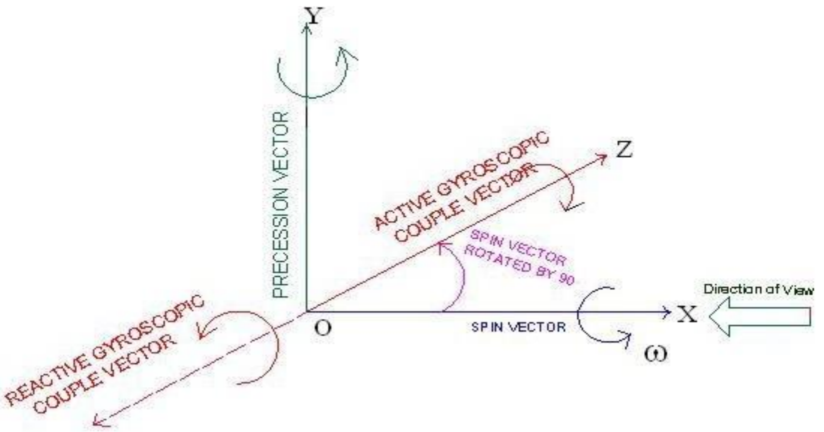


Fig.18

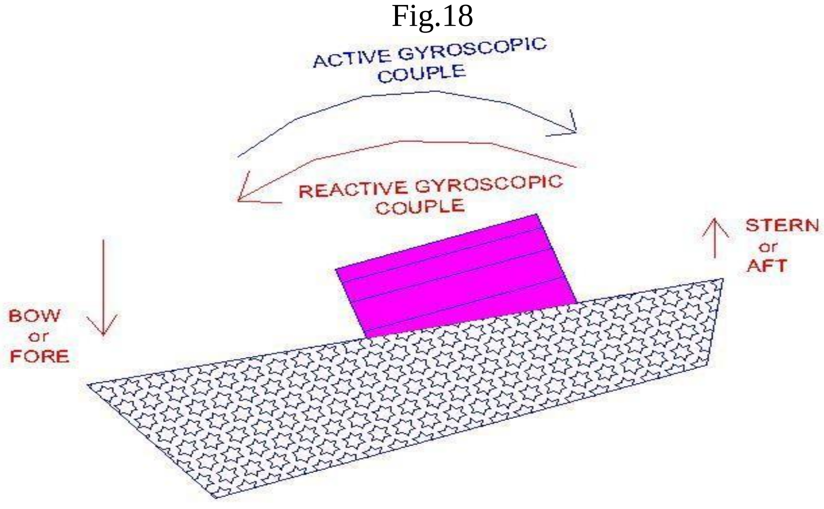


Fig. 19

The couple acts over the ship is between stern and bow. This reaction couple tends to press or dip the front end (bow) and raise the rear end (stern) of the ship.

(iv) Right turn with anticlockwise rotor

When ship takes a right turn and the **rotor rotates in anticlockwise direction** viewed from stern, the gyroscopic couple act on the ship is according to Fig 20. Now, the reaction couple tends to raise the bow of the ship and dip the stern.

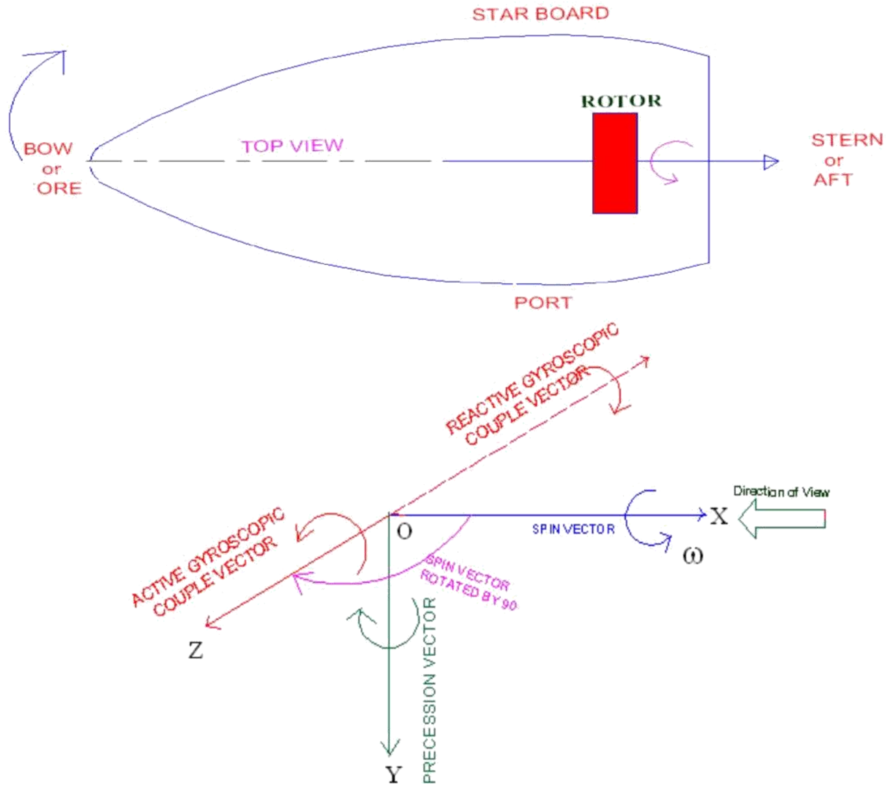


Fig.20

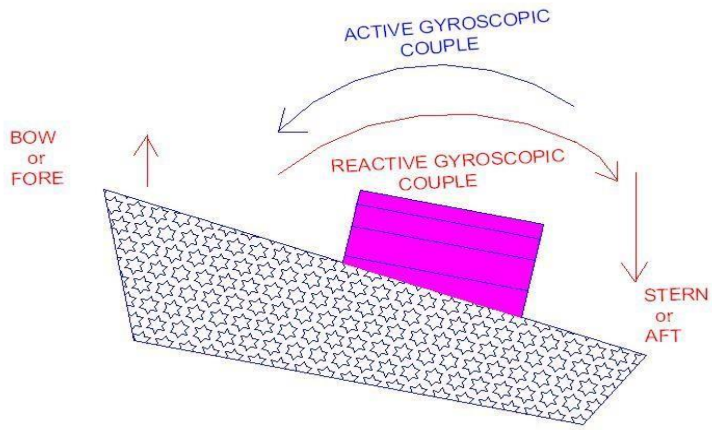


Fig. 21

Gyroscopic effect on Pitching of ship

The pitching motion of a ship generally occurs due to waves which can be approximated as sine wave. During pitching, the ship moves up and down from the horizontal position in vertical plane (Fig.22. & Fig. 23)

Fig.22 Pitching action of ship

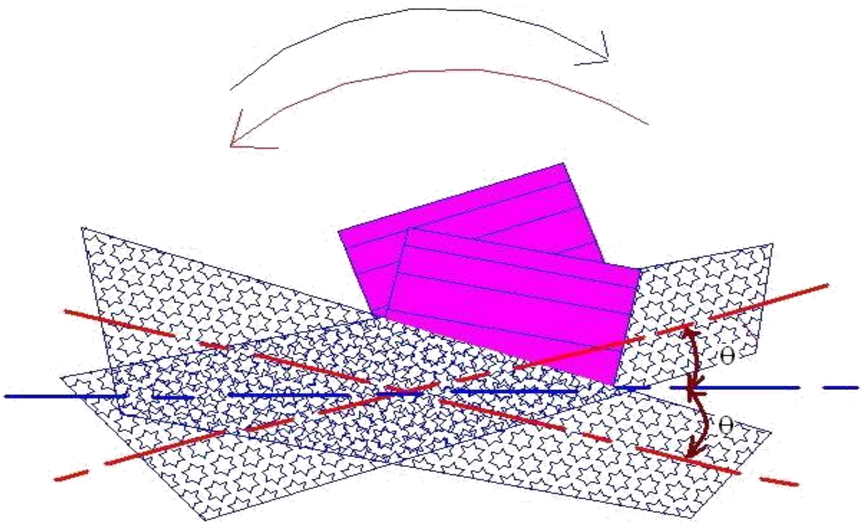


Fig.23 Pitching action of ship

Let θ = angular displacement of spin axis from its mean equilibrium position
A = amplitude of swing
(= angle in degree $\times \frac{2\pi}{360°}$)

and ω_0 = angular velocity of simple hormonic motion $\left(= \frac{2\pi}{\text{time period}}\right)$

The angular motion of the rotor is given as

$$\theta = A \sin \omega_0 t$$

Angular velocity of precess: $$\omega_p = \frac{d\theta}{dt}$$

$$= \frac{d}{dt}(A \sin \omega_0 t)$$

or $$\omega_p = A\omega_0 \cos \omega_0 t$$

The angular velocity of precess will be maximum when $\cos \omega_0 t = 1$

or $$\omega_{p\max} = A\omega_0$$

$$= A \times \frac{2\pi}{t}$$

Thus the gyroscopic couple: $$C = I\,\omega\,\omega_p$$

Consider a rotor mounted along the longitudinal axis and rotates in clockwise direction when seen from the rear end of the ship. The direction of momentum for this condition is shown by vector *ox* (Fig.24). When the ship moves up the horizontal position in vertical plane by an angle from the axis of spin, the rotor axis (X-axis) processes about Z-axis in XY-plane and for this case Z-axis becomes precession axis. The gyroscopic couple acts in anticlockwise direction about Y-axis and the reaction couple acts in opposite direction, i.e. in clockwise direction, which tends to move towards **right side** (Fig.25). However, when the ship pitches down the axis of spin, the direction of reaction couple is reversed and the ship turns towards **left side** (Fig. 26).

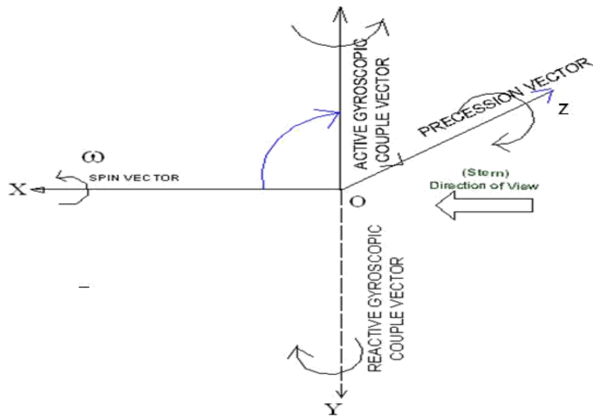


Fig. 24

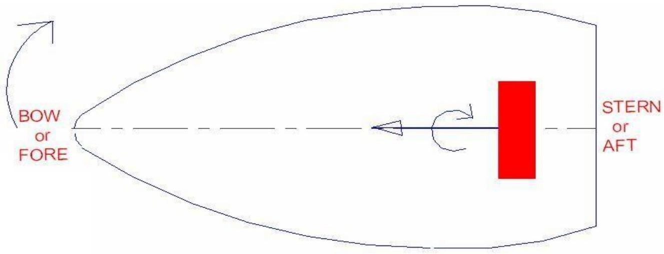


Fig. 25

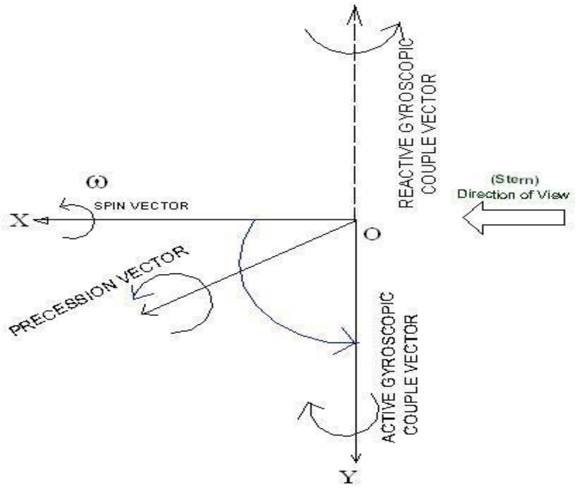


Fig.18

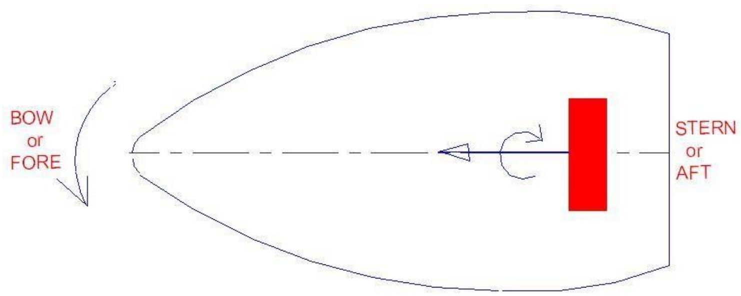


Fig.26

Similarly, for the anticlockwise direction of the rotor viewed from the rear end (Stern) of the ship, the analysis may be done.

Gyroscopic effect on Rolling of ship.

The axis of the rotor of a ship is mounted along the longitudinal axis of ship and therefore, there is **no** precession of this axis. Thus, no effect of gyroscopic couple on the ship frame is formed when the ship rolls.

Fig.27

Problem 2

A turbine rotor of a ship has a mass of 3500 kg and rotates at a speed of 2000 rpm. The rotor has a radius of gyration of 0.5 m and rotates in clockwise direction when viewed from the stern (rear) end. Determine the magnitude of gyroscopic couple and its direction for the following conditions

A When the ship runs at a speed of 12 knots and steers to the left in a curve of 70 m radius
A When the ship pitches 6° above and 6° below the horizontal position and the bow (Front) end is lowered. The pitching motion is simple harmonic with periodic time 30 sec.
(iii)When the ship rolls and at a certain instant, it has an angular velocity of 0.05 rad/s clockwise when viewed from the stern

Also find the maximum angular acceleration during pitching.

Solution Given, 1 knot = 1.86 kmph, the linear velocity of the ship:

$$V = 1.86 \times 12 = 22.32 \text{ kmph}$$
$$= \frac{22.32 \times 1000}{3600} = 6.2 \text{ m/s}$$

Angular velocity of the rotor: $\omega = \frac{2\pi N}{60} = \frac{2\pi \times 2000}{60}$

$$= 209.44 \text{ rad/s}$$

Precession velocity: $\omega_p = \frac{V}{R} = \frac{6.2}{70} = 0.08857$ rad/s

Moment of inertia: $I = mk^2 = 3500 \times 0.5^2 = 875$ kg m^2

Gyroscopic couple: $C = I\omega\omega_p$

$= 875 \times 209.44 \times 0.08857$

$= \mathbf{16231.34}$ **Nm**

When ship steers to the left, the reaction gyroscopic couple action is in anticlockwise direction and the bow of the ship is raised and stern is lowered, as shown in Fig.28.

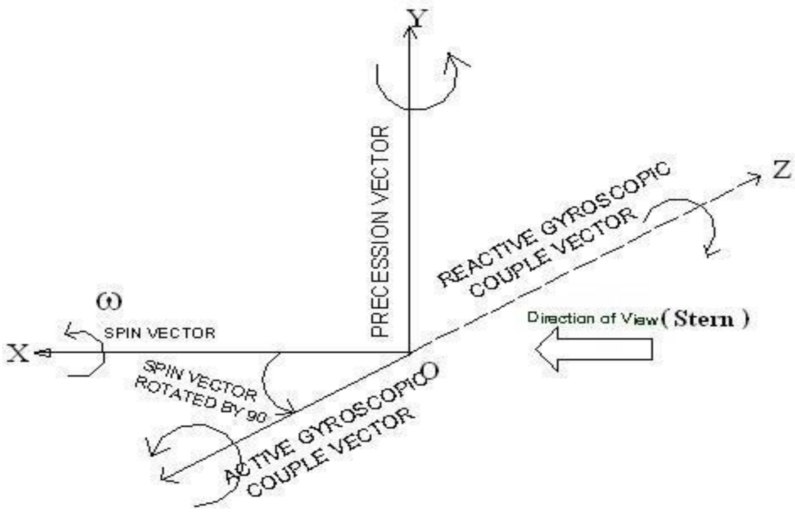


Fig.28

(ii) Amplitude of swing: $A = \frac{6° \times 2\pi}{360°} = 0.1047$ rad

Angular displacement: $\theta = A \sin \omega_0 t$

Angular velocity of precession: $\omega_p = \frac{d\theta}{dt} = A\omega_0 \cos\omega_0 t$

Maximum angular velocity of precession:

$$\omega_{pmax} = \omega_0 A$$

where $\omega_0 = \frac{2\pi}{\text{time period of oscillation}} = \frac{2\pi}{30}$

$= 0.2094$ rad/s

$W_{pmax} = 0.2094 \times 0.1047 = 0.022$ rad/s

Maximum couple for pitching:

Cmax = IWWpmax

4.875 x 209.44 x 0.022

5.4031.72 Nm

The effect of gyroscopic couple due to pitching is shown in Fig.29. The reactive gyroscopic couple will act in anticlockwise direction seen from top and it will turn ship towards the left side.

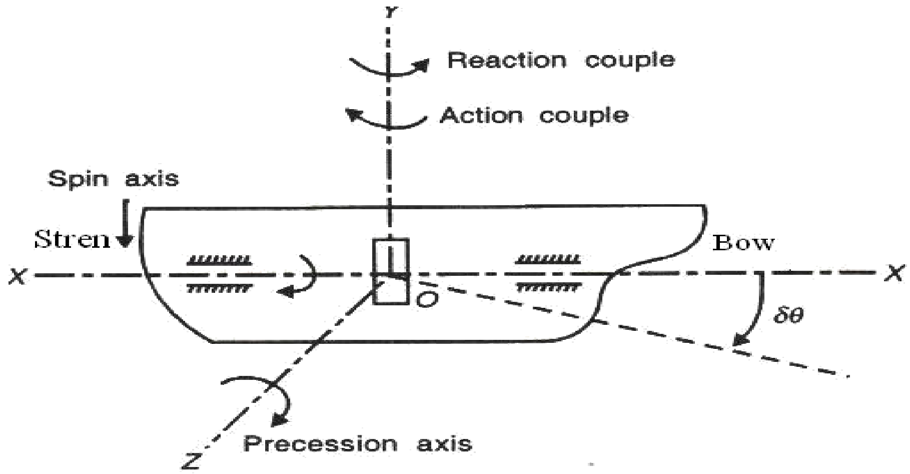


Fig.29

iii) Angular velocity of precession while the ship rolls is:

Wp = 0.05 rad/s

and gyroscopic couple : C = 1WWp
= 875 x 209.44 x 0.05
= 9163 Nm

Since the ship rolls in the same plane as the plane of spin, there is **no gyroscopic effect**.

Angular velocity of precess during pitching is:

$$\omega_p = \frac{d\theta}{dt} = A\omega_0 \cos \omega_0 t$$

Therefore, angular acceleration:

$$\alpha = \frac{d^2\theta}{dt^2} = -A\omega_0^2 \sin \omega_0 t$$

Maximum angular acceleration:

W_{max} = $-AW_0^2$
= 0.1047×0.2094^2
= 0.00459 rad/s^2

Problem 3

A ship is propelled by a rotor of mass of 2000 kg rotates at a speed of 2400 rpm. The radius of gyration of rotor is 0.4 m and spins clockwise direction when viewed from bow (front) end. Find the gyroscopic couple and its effect when;

= the ship takes left turn at a radius of 350 m with a speed of 35 kmph

= the ship pitches with the bow rising at an angular velocity of 1 rad/s (iii)the ship rolls at an angular velocity of 0.15 rad/s

Solution

Angular velocity:

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 2400}{60} = 251.33 \text{ rad/s}$$

Linear velocity: $V = 35 \text{ kmph} = \frac{35 \times 1000}{3600} = 9.72 \text{ m/s}$

Moment of inertia: $I = mk^2 = 2000 \times 0.4^2 = 320 \text{ kg m}^2$

Steering towards left

Angular velocity of precession: $\omega_p = \frac{V}{R} = \frac{9.72}{350} = 0.0278 \text{ rad/s}$

Gyroscopic couple:

$$C = I\omega\omega_p$$
$$= 320 \times 251.33 \times 0.0278$$
$$= 2235.8 \text{ Nm}$$

The reaction gyroscopic couple will act in anticlockwise and will tend to lower the bow as shown in Figure 30.

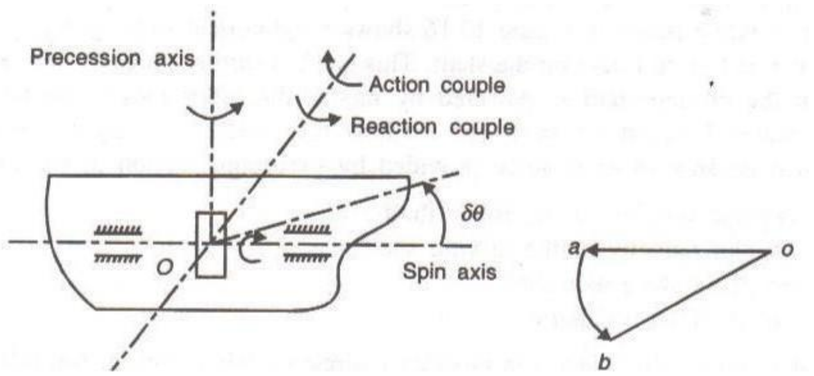


Fig.30

Pitching. Angular velocity of precession during pitching $\omega_p = 1.0$ rad/s

Gyroscopic couple: C = 320 x 251.33 x 1.0

= 80425.6 **Nm Ans.**

The reaction gyroscopic couple acting in anticlockwise direction will tend to turn the bow towards the Right side as shown in Figure 31.

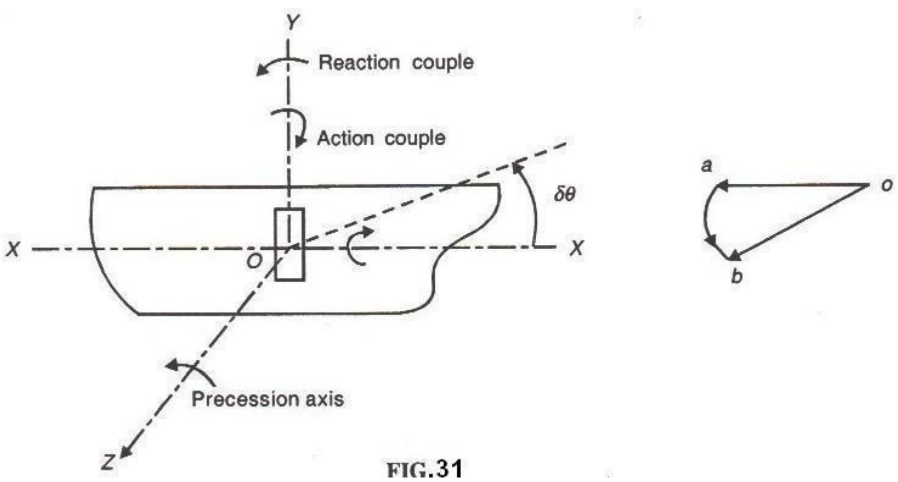


FIG.31

Rolling, Gyroscopic couple: C $= l6XQ_p$

$= 320 \times 251.33 \times 0.15 = 12063.84$ Nm

During rolling, the ship rolls in the same plane as the plane of spin and there will be no gyroscopic effect.

3.4 Gyroscopic Effect on Aeroplane

Aeroplanes are subjected to gyroscopic effect when it taking off, landing and negotiating left or right turn in the air.

Let

ω = Angular velocity of the engine rotating parts in rad/s,

m = Mass of the engine and propeller in kg,

r_W = Radius of gyration in m,

I = Mass moment of inertia of engine and propeller in kg m^2,

V = Linear velocity of the aeroplane in m/s,

R = Radius of curvature in m,

ω_p =Angular velocity of precession = $\frac{V}{R}$ rad/s

∴ Gyroscopic couple acting on the aero plane **= C = I WW$_p$**

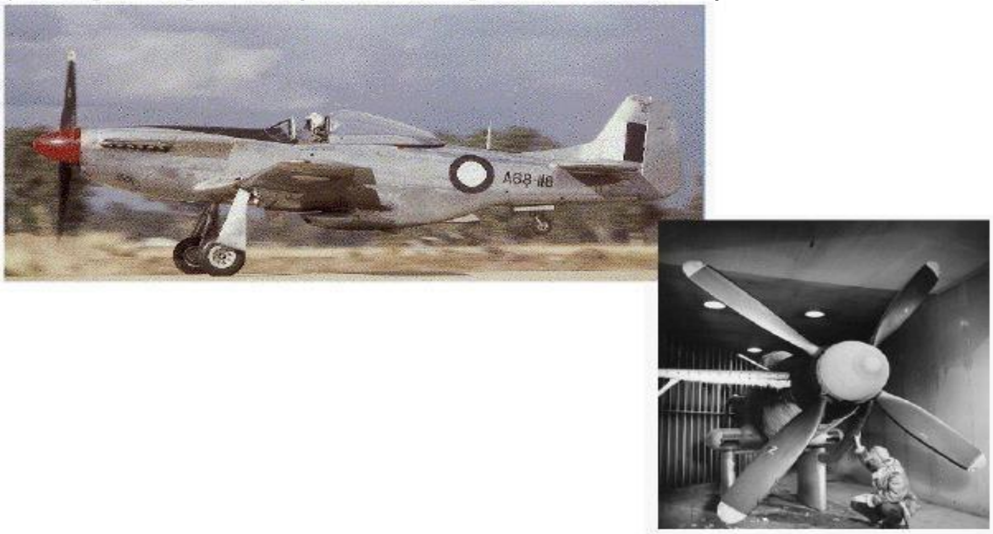
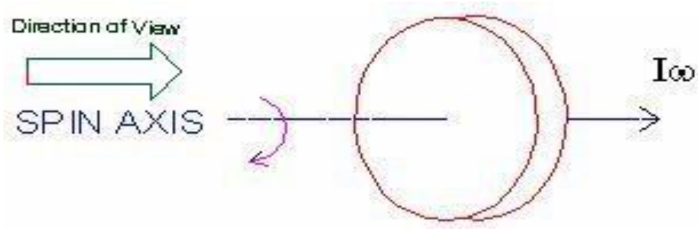
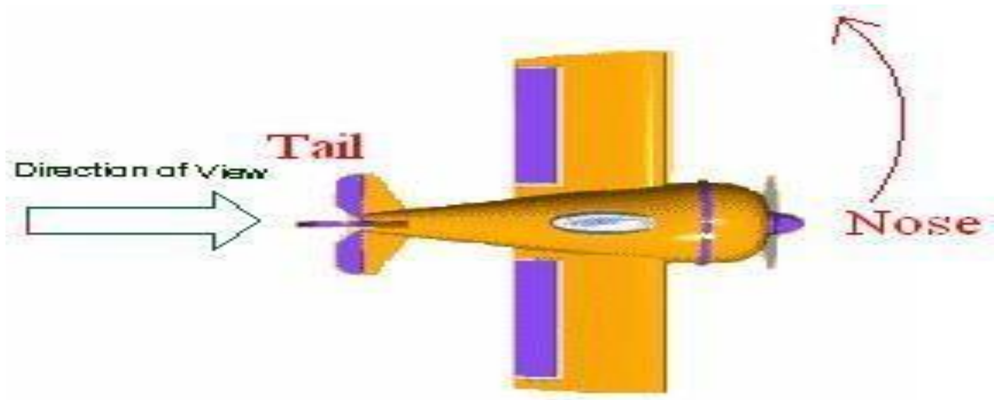


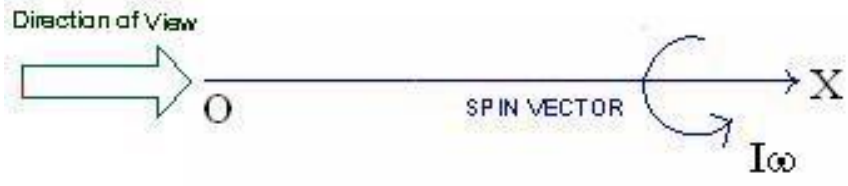
Fig.32

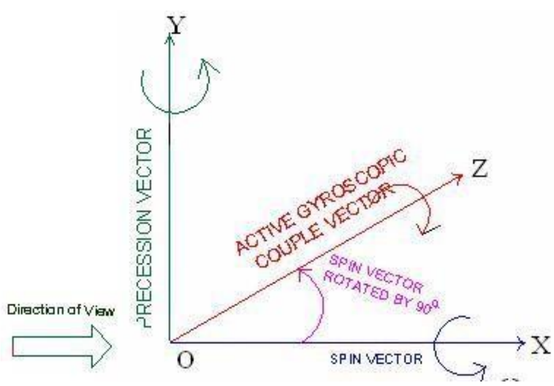
Let us analyze the effect of gyroscopic couple acting on the body of the aero plane for various conditions.

Case (i): PROPELLER rotates in CLOCKWISE direction when seen from rear end and Aeroplane turns towards LEFT


Direction of View
SPIN AXIS
Iω


Tail
Direction of View
Nose


Direction of View
O
SPIN VECTOR
X
Iω


Y
PRECESSION VECTOR
ACTIVE GYROSCOPIC COUPLE VECTOR
Z
SPIN VECTOR ROTATED BY 90°
Direction of View
O
SPIN VECTOR
X

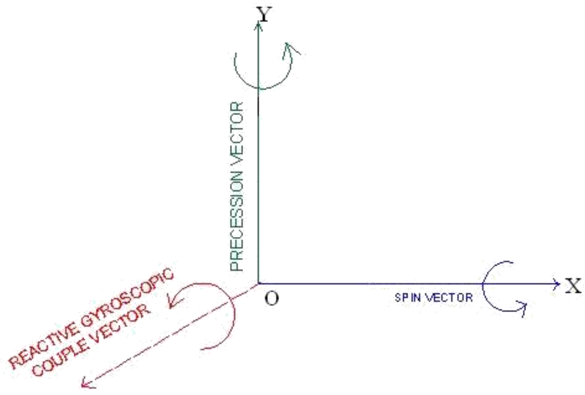


Fig.38

According to the analysis, the reactive gyroscopic couple tends to dip the tail and raise the nose of aeroplane.

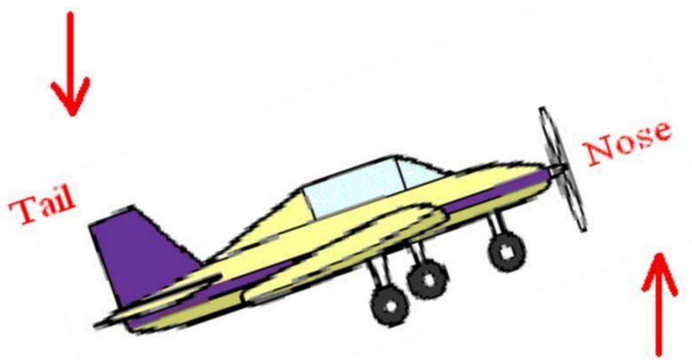


Fig.39

Case (ii): PROPELLER rotates in CLOCKWISE direction when seen from rear end and Aeroplane turns towards RIGHT

Fig.40

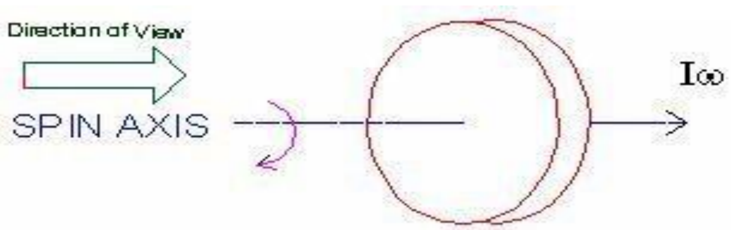


Fig.41

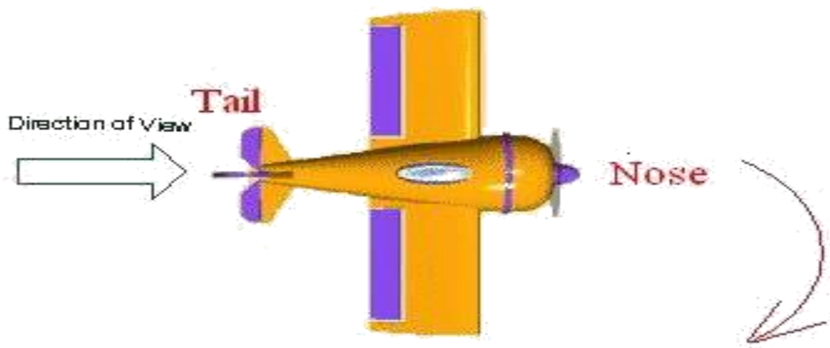


Fig.42

Fig.43

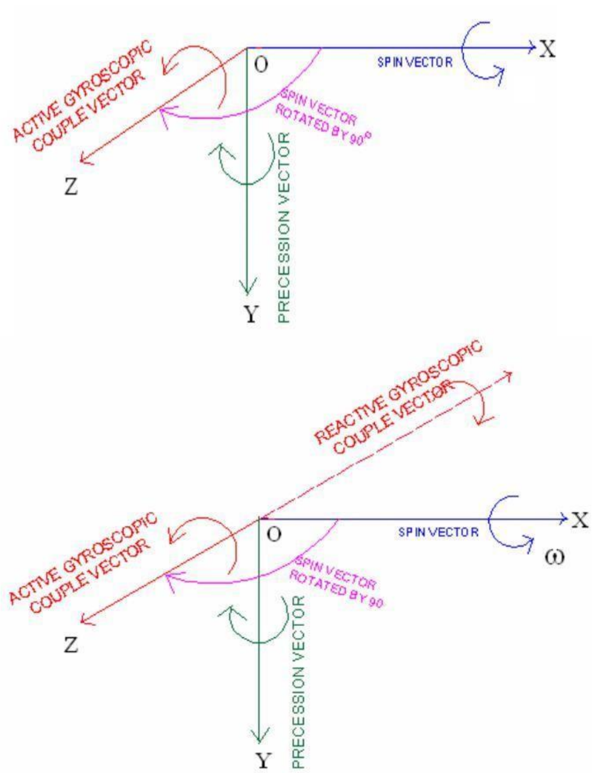


Fig. 44

According to the analysis, the reactive gyroscopic couple tends to raise the tail and dip the nose of aeroplane.

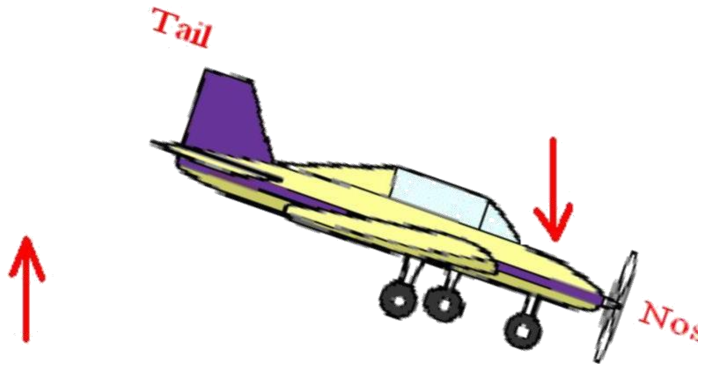


Fig.45

Case (iii): PROPELLER rotates in ANTICLOCKWISE direction when seen from rear end and Aeroplane turns towards LEFT

Fig.46

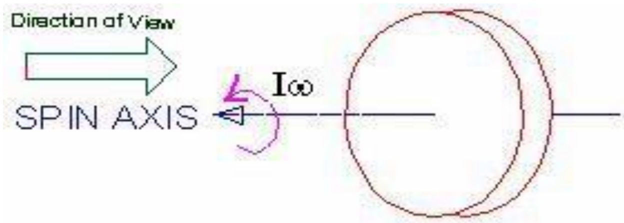


Fig.47

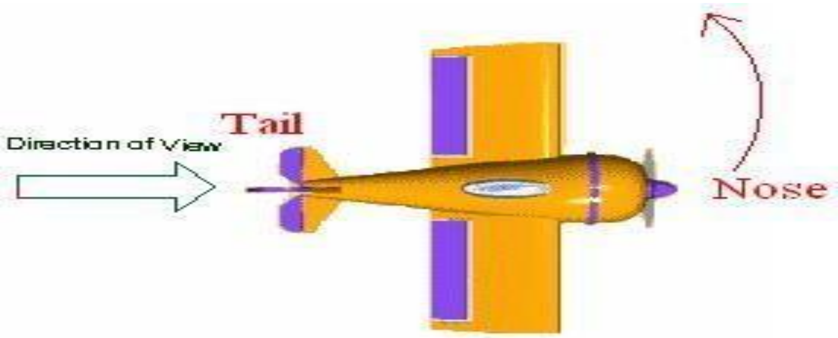


Fig.48

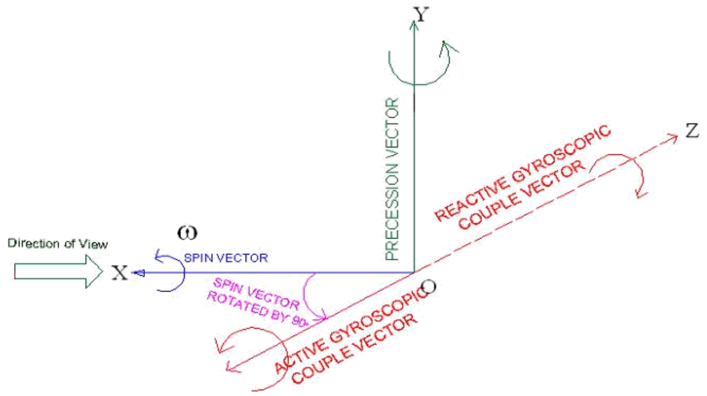


Fig.49

The analysis indicates, the reactive gyroscopic couple tends to raise the tail and dip the nose of aeroplane.

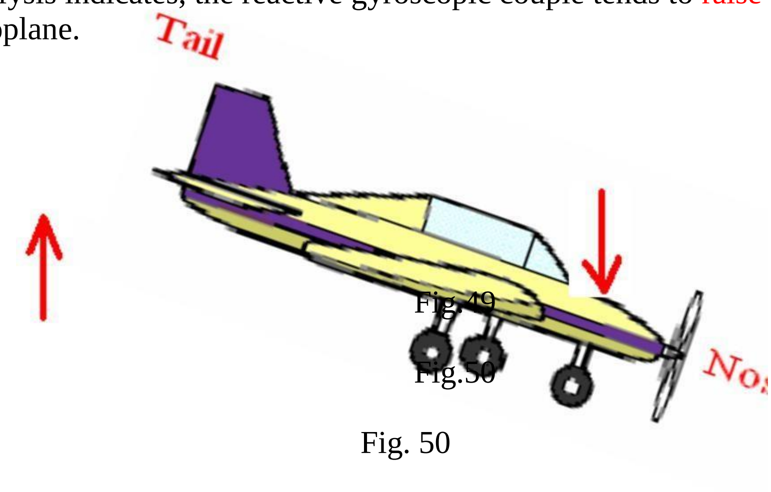


Fig. 50

Case (iv): **PROPELLER rotates in ANTICLOCKWISE direction when seen from rear end and Aeroplane turns towards RIGHT**

Fig.51

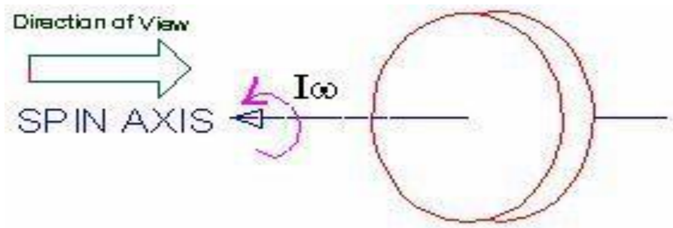


Fig.52

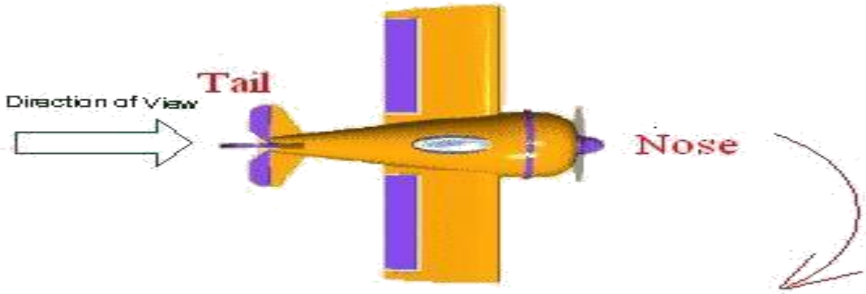


Fig.53

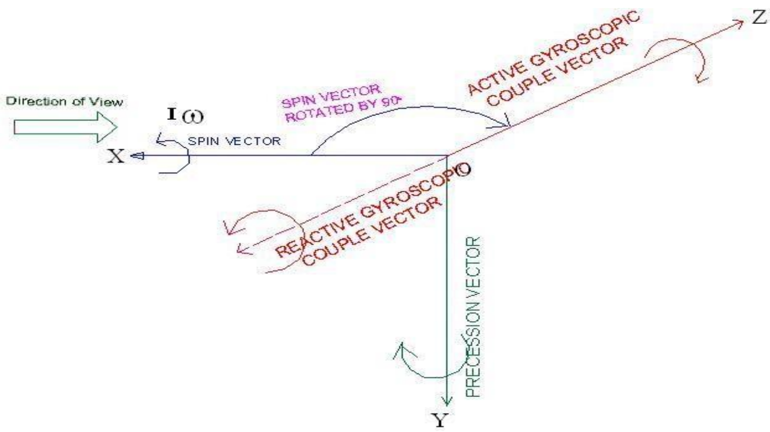


Fig.54

The analysis shows, the reactive gyroscopic couple tends to raise the tail and dip the nose of aeroplane.

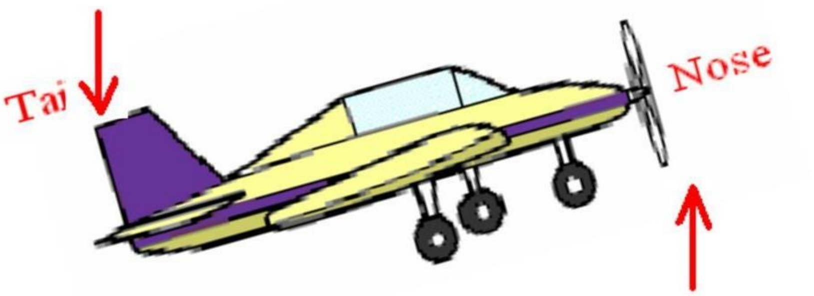


Fig.55

Case (v): PROPELLER rotates in CLOCKWISE direction when seen from rear end and Aeroplane takes off or nose move upwards

Fig.56

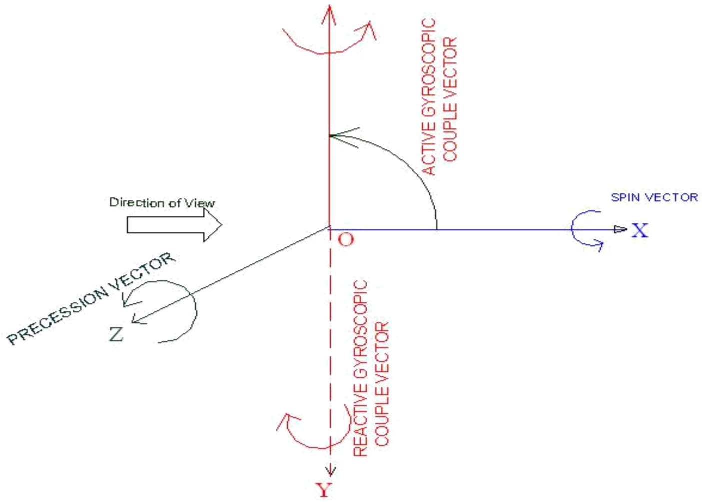


Fig.57

The analysis show, the reactive gyroscopic couple tends to turn the nose of aeroplane toward right

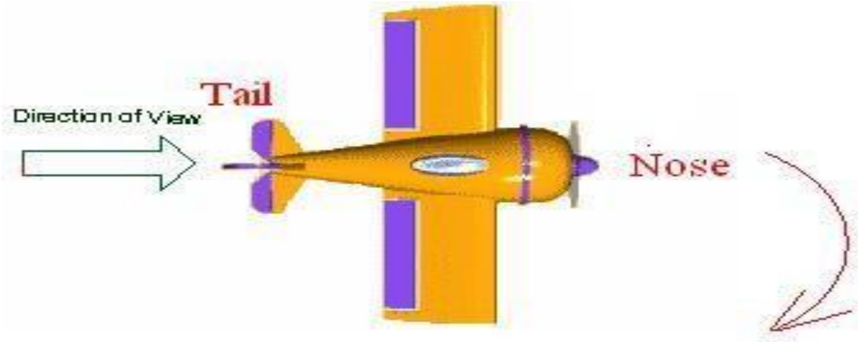
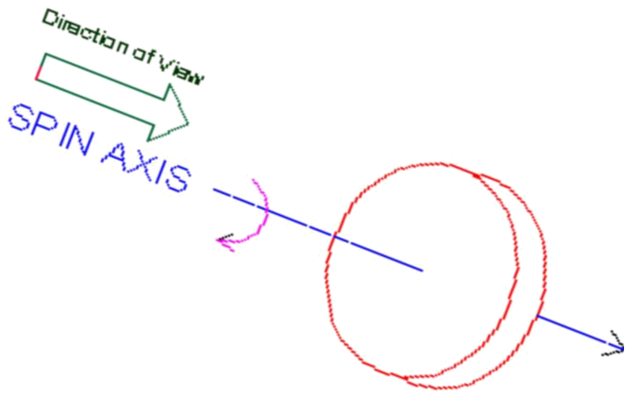


Fig.58

Case (vi): PROPELLER rotates in CLOCKWISE direction when seen from rear end and Aeroplane is landing or nose move downwards

Fig.59



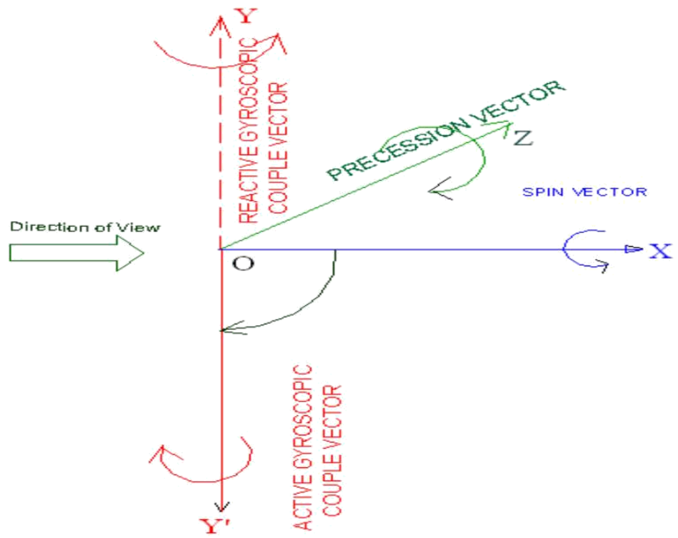


Fig. 61

The reactive gyroscopic couple tends to turn the nose of aeroplane toward left

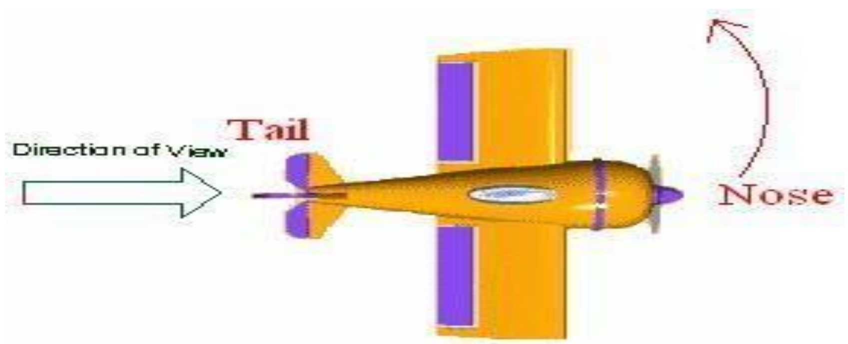


Fig.62

Case (vii): PROPELLER rotates in ANTICLOCKWISE direction when seen from rear end and Aeroplane takes off or nose move upwards

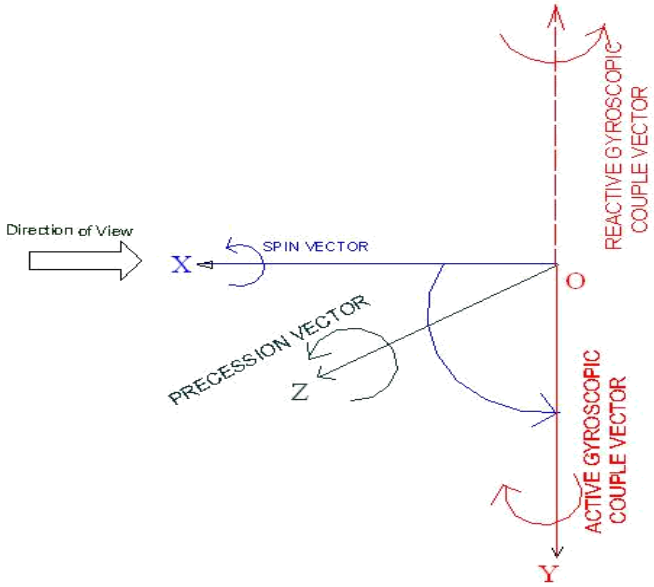


Fig.63

The reactive gyroscopic couple tends to turn the nose of aeroplane toward left

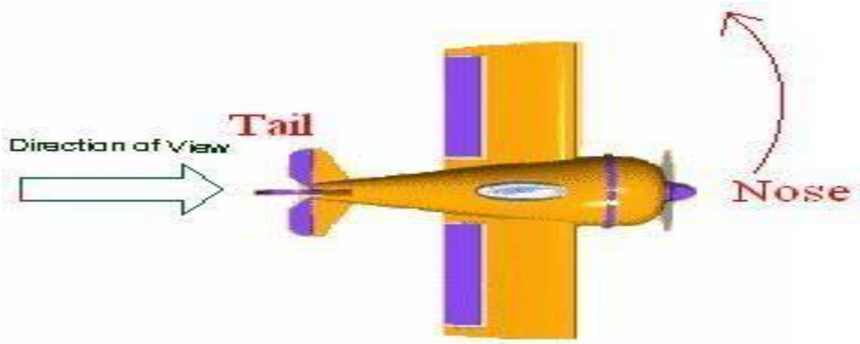


Fig.64

Case (viii): PROPELLER rotates in ANTICLOCKWISE direction when seen from rear end and Aeroplane is landing or nose move downwards

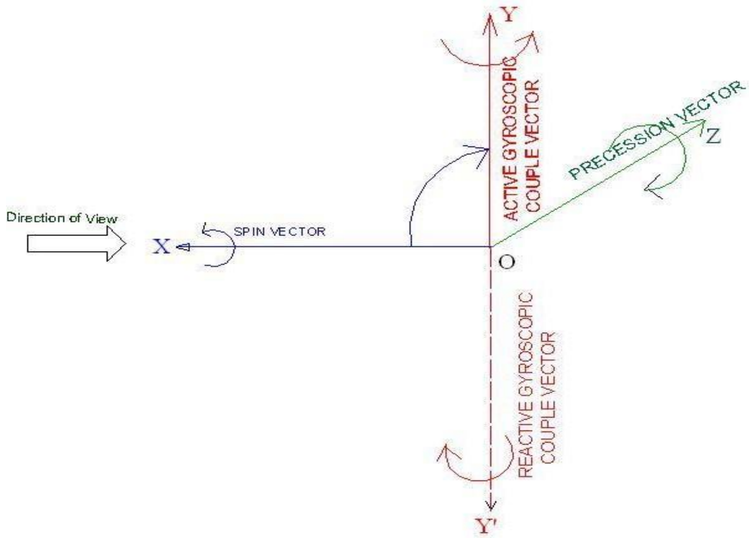


Fig.65

The analysis show, the reactive gyroscopic couple tends to turn the nose of aeroplane toward right

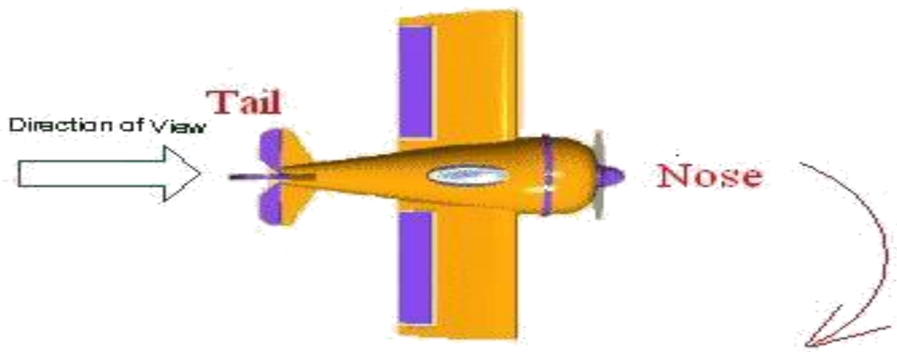


Fig.66

Problem 4

An aeroplane flying at a speed of 300 kmph takes right turn with a radius of 50 m. The mass of engine and propeller is 500 kg and radius of gyration is 400 mm. If the engine runs at 1800 rpm in clockwise direction when viewed from tail end, determine the gyroscopic couple and state its effect on the aeroplane. What will be the effect if the aeroplane turns to its left instead of right?

Solution Angular velocity of aeroplane engine:

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 1800}{60} = 188.49 \text{ rad/s}$$

Angular velocity of precession: $\omega_p = \frac{V}{R}$

or $\omega_p = \frac{300 \times 1000}{3600} \times \frac{1}{50}$

$= 1.67$ rad/s

Moment of inertia: $I = mk^2 = 500 \times 0.4^2$

$= 80 \text{ kg m}^2$

Gyroscopic couple: $c = I\omega\omega_p$

$= 80 \times 188.49 \times 1.67$

$= \mathbf{25182.26 \text{ Nm}}$ **Ans.**

Case (i): PROPELLER rotates in CLOCKWISE direction when seen from rear end and Aeroplane turns towards RIGHT

Fig.67

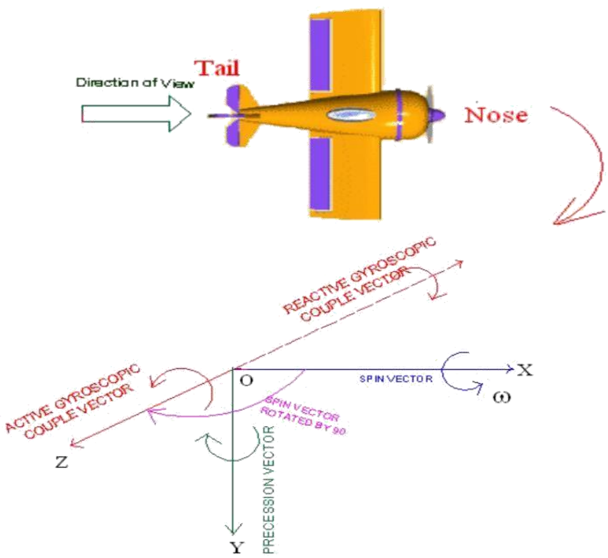


Fig.68

According to the analysis, the reactive gyroscopic couple tends to dip the nose and raise the tail of the aeroplane.

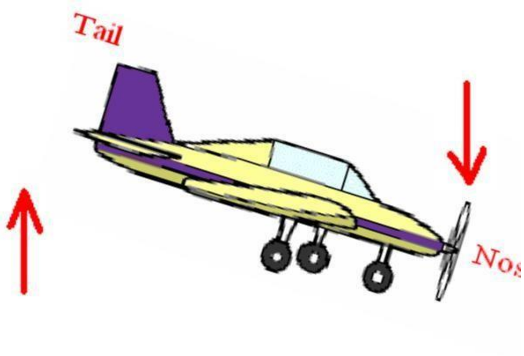


Fig.69

When aeroplane turns to its left, the magnitude of gyrocouple remains the same. However, the direction of reaction couple is reversed and it will raise the nose and dip the tail of the aeroplane.

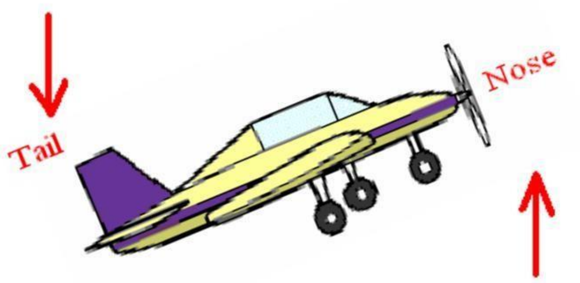


Fig.70

3.5 Stability of Automotive Vehicle

A vehicle running on the road is said to be stable when no wheel is supposed to leave the road surface. In other words, the resultant reactions by the road surface on wheels should act in upward direction. For a moving vehicle, one of the reaction is due to gyroscopic couple produced by the rotating wheels and rotating parts of the engine. Let us discuss stability of two and four wheeled vehicles when negotiating a curve/turn.

Stability of Two Wheeler negotiating a turn

Fig.71

Fig. 71 shows a two wheeler vehicle taking **left turn** over a curved path. The vehicle is inclined to the vertical for equilibrium by an angle known as angle of heel.

Let

m = Mass of the vehicle and its rider in kg,
W = Weight of the vehicle and its rider in newtons = *m.g*,
h = Height of the centre of gravity of the vehicle and rider, r_W = *Radius of the wheels*,
R = Radius of track or curvature,
I_W = *Mass moment of inertia of each wheel*,
I_E = *Mass moment of inertia of the rotating parts of the engine*,
ω_W = Angular velocity of the wheels,
ω_E = Angular velocity of the engine rotating parts,
G = Gear ratio = ω_E / ω_W,
v = Linear velocity of the vehicle = $\omega_W \times r_W$,
θ = Angle of heel. It is inclination of the vehicle to the vertical for equilibrium.

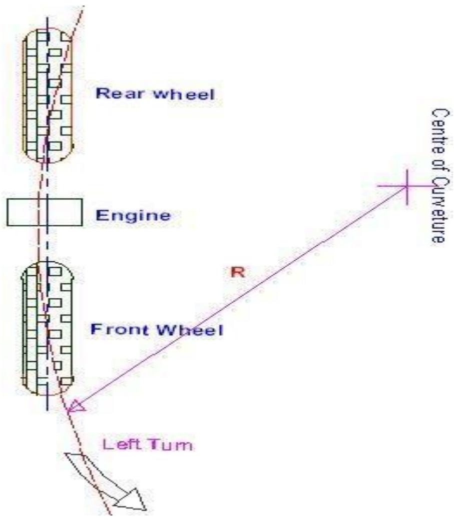


Fig.7
2

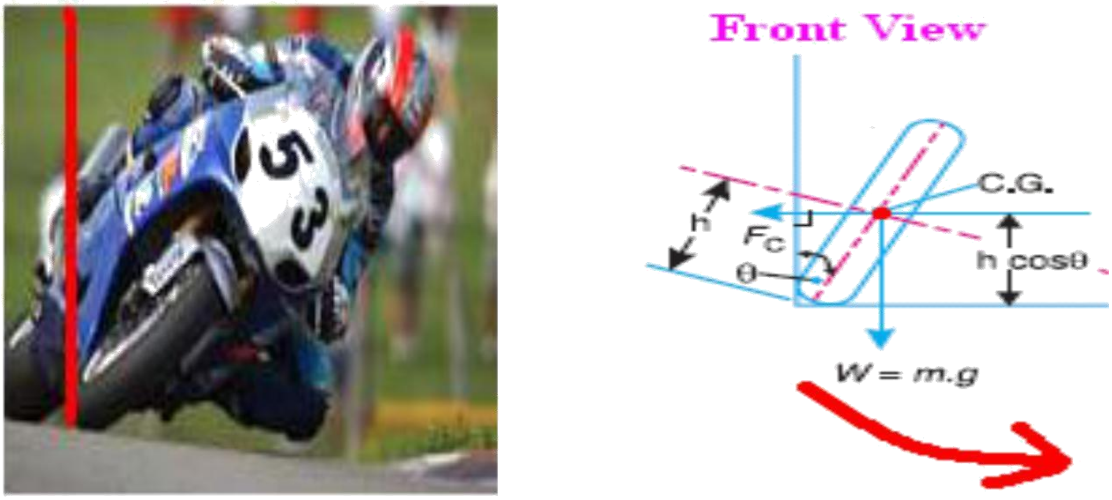


Fig.73

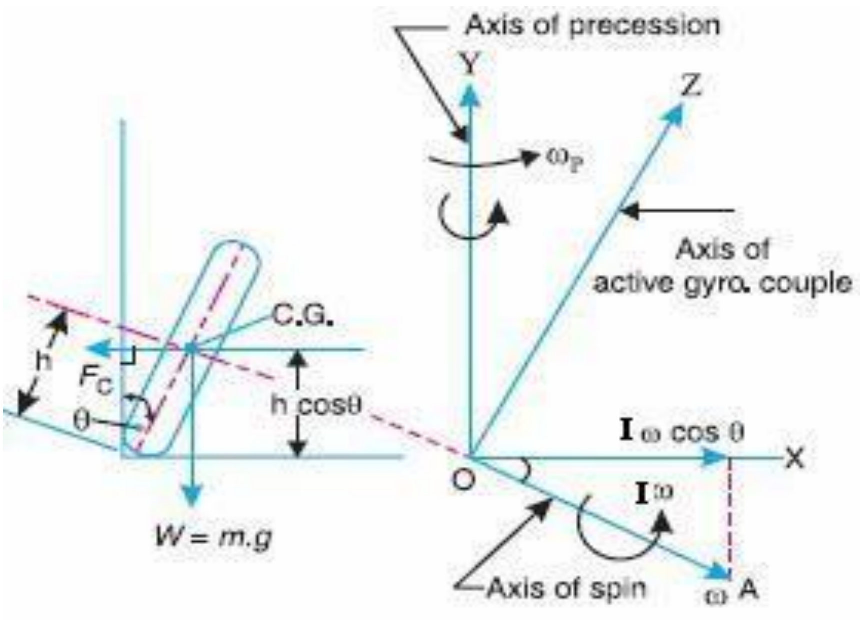


Fig.74

Let us consider the effect of the gyroscopic couple and centrifugal couple on the wheels.

1. Effect of Gyroscopic Couple

We know that, $V = \omega_W \times r_W$

$\omega_E = G \cdot \omega_W$ *or* $\omega_E = G \cdot v / r_W$

Angular momentum due to wheels = $2\ I_w \omega_W$

Angular momentum due to engine and transmission = $I_E\ \omega_E$

Total angular momentum ($I\ x\omega$) $= 2\ I_w\ \omega_W \pm I_E\ \omega_E$

$$= 2I_w \frac{v}{r_w} \pm I_E G \frac{v}{r_w}$$

$$= \frac{v}{r_w}(2I_w \pm GI_E)$$

Also, Velocity of precession = $\omega_p = \frac{V}{R}$

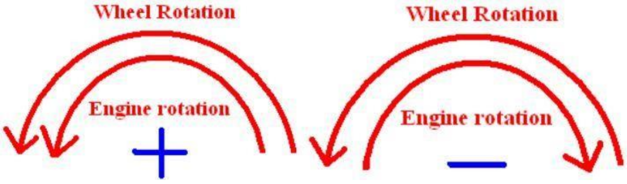
It is observed that, when the wheels move over the curved path, the vehicle is always inclined at an angle θ with the vertical plane as shown in Fig… This angle is known as ‗angle of heel'. In other words, the axis of spin is inclined to the horizontal at an angle θ , as shown in Fig.73 Thus, the angular momentum vector I ω due to spin is represented by OA inclined to OX at an angle θ. But, the precession axis is in vertical. Therefore, the spin vector is resolved along OX.

Gyroscopic Couple,

$$C_g = (I\omega)\cos\theta \times \omega_p$$

$$C_g = \frac{v^2}{Rr_w}(2I_w \pm GI_E)\cos\theta$$

Note: When the engine is rotating in the same direction as that of wheels, then the positive sign is used in the above equation. However, if the engine rotates in opposite direction to wheels, then negative sign is used.



The gyroscopic couple will act over the vehicle outwards i.e., in the anticlockwise direction when seen from the front of the two wheeler. This couple tends to overturn/topple the vehicle in the outward direction as shown in Fig…

Analysis:

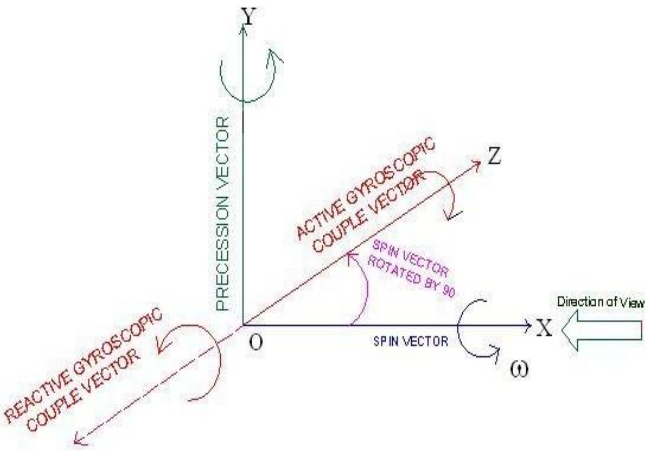


Fig.75

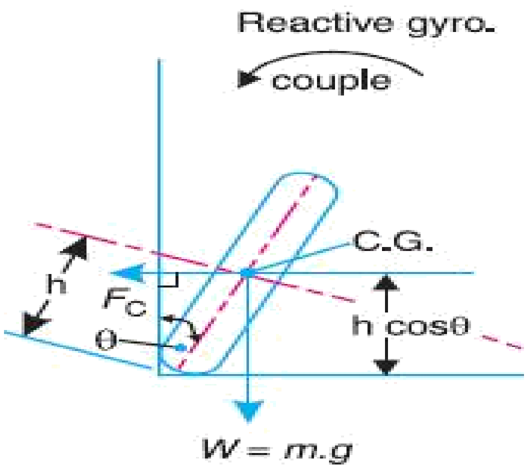


Fig.7
6

2. Effect of Centrifugal Couple

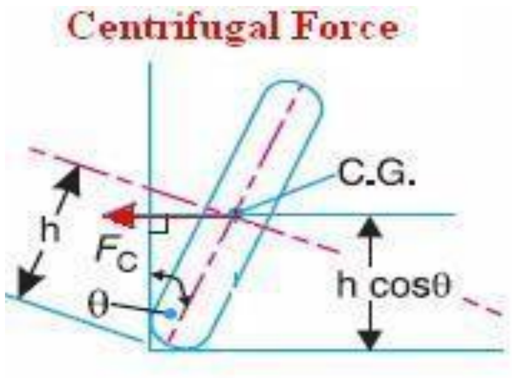


Fig. 77

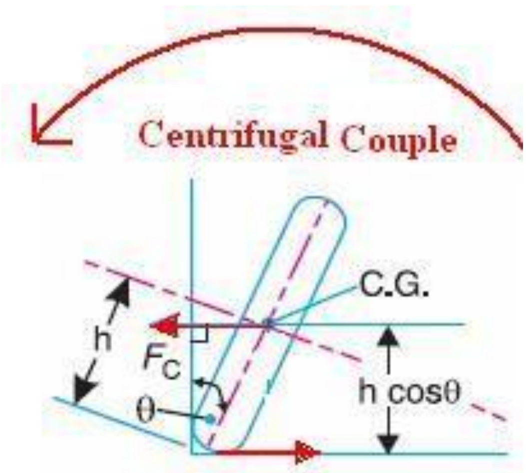
We have,

Centrifugal force, $F_c = \frac{mv^2}{R}$

or

Centrifugal Couple, $C_c = F_c \times h\cos\theta$

$$= \frac{mv^2}{R} h\cos\theta$$



The Centrifugal couple will act over the two wheeler outwards i.e., in the anticlockwise direction when seen from the front of the two wheeler. This couple tends to overturn/topple the vehicle in the outward direction as shown in Fig.78

Therefore, the total Over turning couple: $C = C_g + C_c$

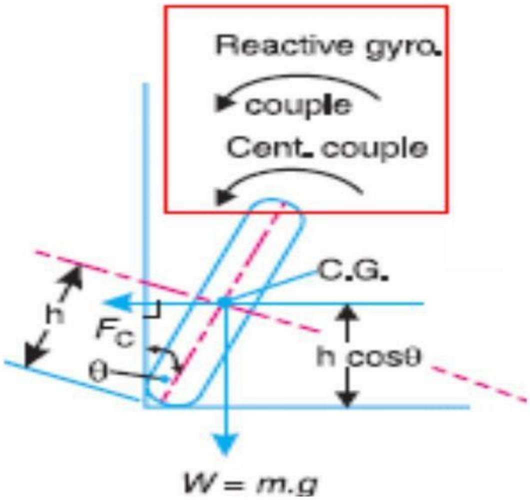


Fig.79

$$C = \frac{v^2}{Rr}(2I_w + GI_e)\cos\theta + \frac{mv^2}{R}h\cos\theta$$

For the vehicle to be in equilibrium, overturning couple should be equal to balancing couple acting in clockwise direction due to the weight of the vehicle and rider.

□ $C = mgh \sin\theta$

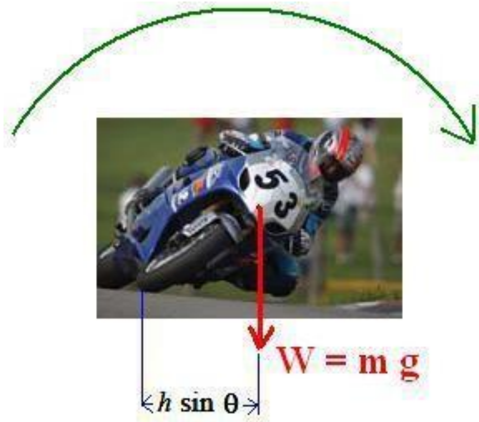


Fig.80

For the stability, overturning couple must be equal to balancing couple,

$$\boxed{\frac{v^2}{Rr_w}(2I_w + GI_e)\cos\theta + \frac{mv^2}{R}h\cos\theta = mgh\sin\theta}$$

Therefore, from the above equation, the value of angle of heel (θ) may be determined, so that the vehicle does not skid. Also, for the given value of θ, the maximum vehicle speed in the turn with out skid may be determined.

Problem 5

A motorcycle and its rider together weighs 2000 N and their combined centre of gravity is 550 mm above the road when motorcycle is upright. Each wheel is of 580 mm diameter and has a moment of inertia of 1.0 kgm^2. The moment of inertia of rotating parts of engine is 0.15 kg m^2. The engine rotates at 5 times the speed of the vehicle and the same sense. Determine the angle of heel necessary when motorcycle is taking a turn over a track of 35 m radius at a speed of 60 kmph.

Solution:

Velocity of vehicle : $v = \frac{60 \times 1000}{3600} = 16.67$ m/s

Angular velocity of wheel: $\omega = \frac{2v}{d} = \frac{2 \times 16.67}{0.58} = 57.48$ rad/s

Angular velocity of precession: $\omega_p = \frac{v}{R} = \frac{16.67}{35} = 0.476$ rad/s

(i) Gyroscopic couple due to two wheels:

$C_w = 2I_w WW_p \cos\theta$
2 x 1 .0 x 57.48 x 0.476 x cos θ
54.72 cosθ Nm

= Gyroscopic couple due to rotating parts of engine:

$C_E = I_E GWW_p \cos\theta$
0.15 x 5 x 57.48 x 0.476 x cosθ
20.52cosθ Nm

ii) Centrifugal force due to angular velocity of die wheel:

$$F_c = \frac{mv^2}{R} = \frac{2000 \times 16.67^2}{9.81 \times 35} = 1618.7 \text{ N}$$

Centrifugal couple: C_c = 1618.7 x 0.55 cosθ
= 890.28 cos θ Nm

Total overturning couple: $C = C_w + C_e + C_c$
T (54.72 + 20.52 + 890.28) cosθ
T 965.52 cosθ Nm

Balancing couple $= mgh \sin\theta$

$$= \frac{2000}{9.81} \times 9.81 \times 0.55 \sin\theta$$

$= 1100 \sin\theta$ Nm

For the stability of motorcycle, overturning couple should be equal to resisting couple.

1100 sinθ = 965.52 cosθ

or

$$\tan\theta = \frac{965.52}{1100} = 0.877$$

heel angle: $\theta = \mathbf{41.27°}$

Problem 6

A motor cycle with its rider has a mass of 300 kg. The centre of gravity of the machine and rider combined being 0.6 m above the ground with machine in vertical position. Moment of inertia of each wheel is 0.525 kg m^2 and the rolling diameter of 0.6 m. The engine rotates 6 times the speed of the road wheels and in the same sense. The engine rotating parts have a mass moment of inertia of 0.1686 kg m^2. Find (i) the angle of heel necessary if the vehicle is running at 60 km/hr round a curve of 30 m (ii) If the road and tyre friction allow for the angle of heel not to exceed 50^o, what is the maximum road velocity of the motor cycle.

Solution:

m = 300 kg, h = 0.6 m, I_w = 0.525 kg m^2 , dw=0.6 m; r_w = 0.3 m, G = 6, I_E = 0.1686 m , V= 60km/hr = 16.66 m/s, R = 30 m (i) θ = ? (ii) θ = 50^o V=?

1. Angle of heel,

We have,

$$\boxed{\frac{v^2}{Rr_w}(2I_w + GI_e)\cos\theta + \frac{mv^2}{R} h\cos\theta = mgh\sin\theta}$$

$$\therefore \frac{16.66^2}{30}\left[\frac{2x0.525 + 6x0.1685}{0.3} + 300\,x\,0.6\right]\cos\theta = 300x9.81x0.6x\sin\theta$$

$$\theta = 45^o$$

A) Given, $\theta = 50^o$, V=?,

$$\boxed{\frac{v^2}{Rr_w}(2I_w + GI_e)\cos\theta + \frac{mv^2}{R} h\cos\theta = mgh\sin\theta}$$

$$\therefore \frac{V^2}{30}\left[\frac{2\,x\,0.525 + 6\,x\,0.1685}{0.3} + 300\,x\,0.6\right]\cos 50 = 300\,x\,9.81\,x\,0.6\,x\sin 50$$

$$\therefore V = 66\ Kmph$$

Stability of Four Wheeled Vehicle negotiating a turn.

Stable condition

Unstable Condition

Fig.81

Consider a four wheels automotive vehicle as shown in Figure 82. The engine is mounted at the rear with its crank shaft parallel to the rear axle. The centre of gravity of the vehicle lies vertically above the ground where total weight of the vehicle is assumed to be acted upon.

Let

m = Mass of the vehicle (kg)
W = Weight of the vehicle (N) = *m.g,*
h = Height of the centre of gravity of the vehicle (m)
r_W = *Radius of the wheels (m)*
R = Radius of track or curvature (m)
I_W = *Mass moment of inertia of each wheel* ($kg\text{-}m^2$)
I_E = *Mass moment of inertia of the rotating parts of the engine* ($kg\text{-}m^2$)
ω_W = Angular velocity of the wheels
(rad/s) ω_E = Angular velocity of the engine
(rad/s)
G = Gear ratio = ω_E / ω_W,
v = Linear velocity of the vehicle (m/s)= $\omega_W \times$
r_W, x = Wheel track (m)
b = Wheel base (m)

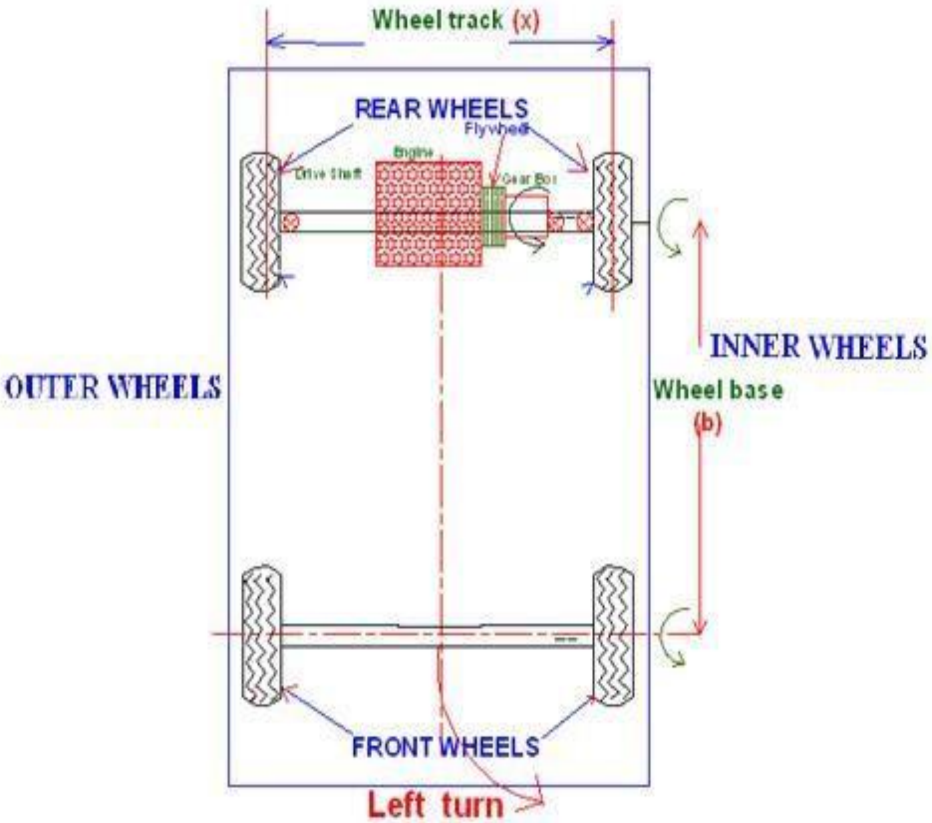


Fig.82

1. Reaction due to weight of Vehicle

Weight of the vehicle. Assuming that weight of the vehicle *(mg)* is equally distributed over four wheels. Therefore, the force on each wheel acting downward is m*g/4* and the reaction by the road surface on the wheel acts in upward direction.

$$R_w = \frac{mg}{4}$$

x **Effect of Gyroscopic couple due to Wheel**

Gyroscopic couple due to four wheels is,

$$C_w = 4\, I_w W W_p$$

(iii) **Effect of Gyroscopic Couple due to Engine**

Gyroscopic couple due to rotating parts of the engine

$$C_E = I_E\, W W_p = I_E\, G\, W W_p$$

Therefore, Total gyroscopic couple:

$$C_g = C_w + C_E = W W_p\,(4 I_W \pm I_E G)$$

When the wheels and rotating parts of the engine rotate in the same direction, then positive sign is used in the above equation. Otherwise negative sign should be considered.

Assuming that the vehicle takes a left turn, the reaction gyroscopic couple on the vehicle acts between outer and inner wheels.

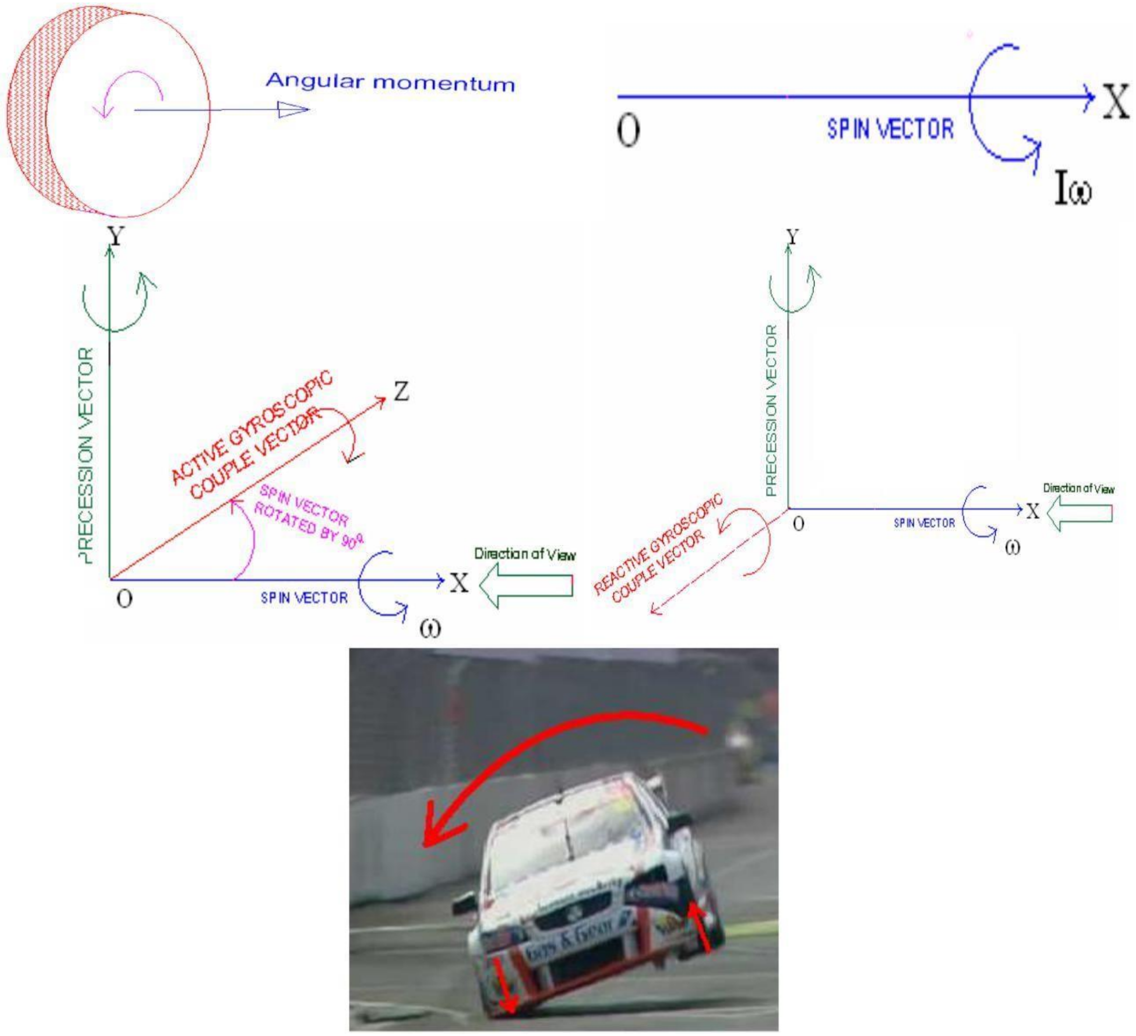


Fig.83

This gyroscopic couple tends to press the outer wheels and lift the inner wheels.

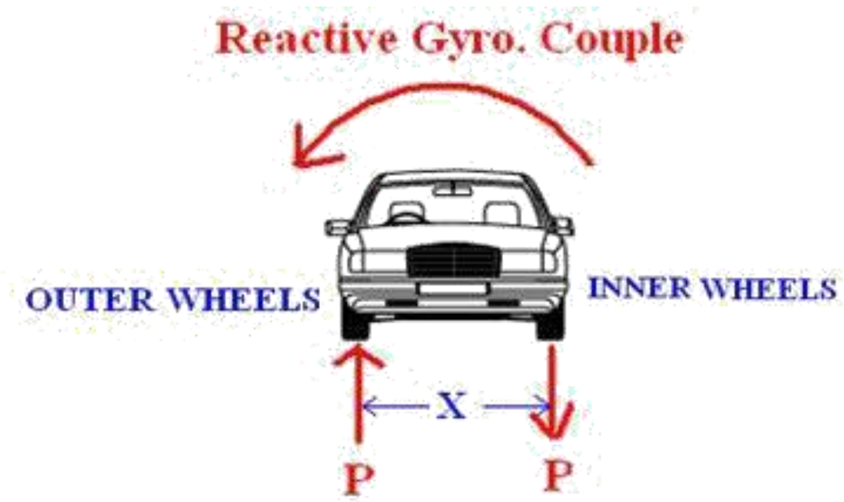


Fig.84

Due to the reactive gyroscopic couple, vertical reactions on the road surface will be produced. The reaction will be vertically upwords on the outer wheels and vertically downwords on the inner wheels. Let the magnitude of this reaction at the two outer and inner wheels be P Newtons, then,

$$\mathbf{P} \ x \ X = C_g$$

$$\mathbf{P} = \frac{C_g}{x}$$

Road reaction on each outer/Inner wheel,

$$\frac{P}{2} = \frac{C_g}{2X}$$

Effect of Centrifugal Couple

When a vehicle moves on a curved path, a centrifugal force acts on the vehicle in outward direction through the centre of gravity of the vehicle(Fig…)

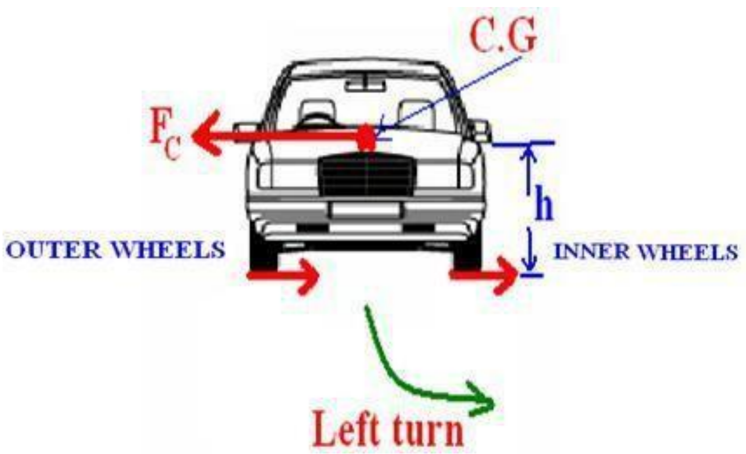


Fig.85

Centrifugal force,

$$F_c = m\omega_p^2 R = \frac{mv^2}{R}$$

This force forms a Centrifugal couple.

$$C_c = \frac{mv^2 h}{R}$$

This centrifugal couple tends to press the outer and lift the inner

Fig.86

Due to the centrifugal couple, vertical reactions on the road surface will be produced. The reaction will be vertically upwords on the outer wheels and vertically downwords on the inner wheels. Let the magnitude of this reaction at the two outer and inner wheels be F Newtons, then,

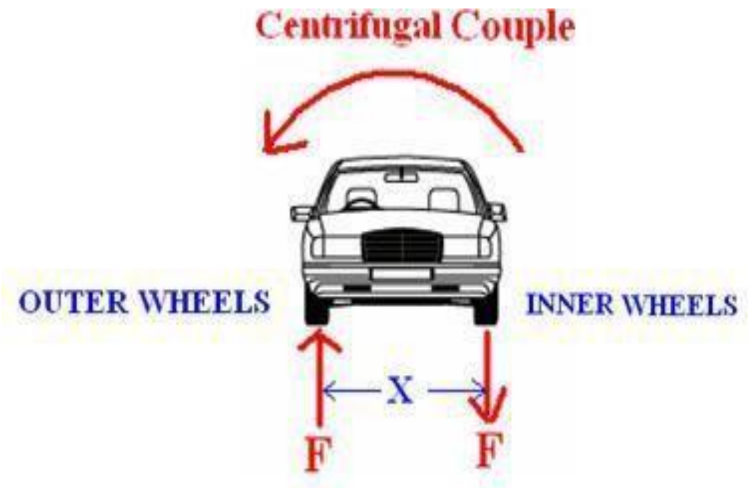


Fig.87

Road reaction on each outer/Inner wheel,

$$\frac{F}{2} = \frac{C_c}{2X}$$

The reactions on the outer/inner wheels are as follows,

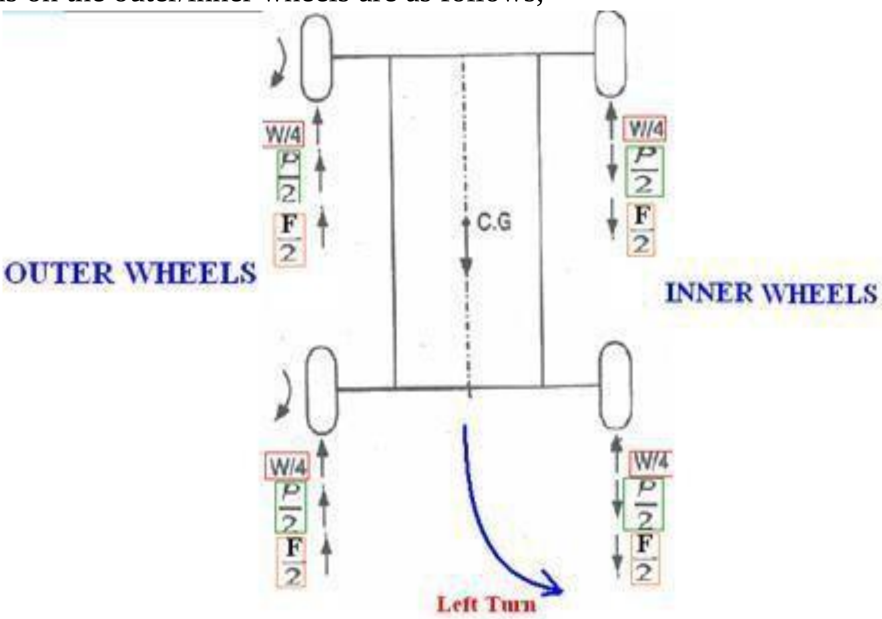


Fig.88

Total vertical reaction at each outer wheels

$$P_o = \frac{W}{4} + \frac{P}{2} + \frac{Q}{2}$$

Total vertical reaction at each inner wheels

$$P_i = \frac{W}{4} - \frac{P}{2} - \frac{Q}{2}$$

Problem 7

An automobile car is travelling along a track of 100 m mean radius. The moment of inertia of 500 mm diameter wheel is 1.8 kg m^2. The engine axis is parallel to the rear axle and crank shaft rotates in the same sense as the wheel. The moment of inertia of rotating parts of the engine is 1 kg m^2. The gear ratio is 4 and **the mass of** the vehicle is 1500 kg. If the centre of gravity of the vehicle is 450 mm above the road level and width of the track of the vehicle is 1.4 m, determine the limiting speed of the vehicle for condition that all four wheels maintain contact with the road surface.

Solution Let □ = limiting velocity of the vehicle.

Angular velocity: $\omega = \frac{v}{r} = \frac{v}{0.25}$ rad/s

Precession velocity: $\omega_p = \frac{v}{R} = \frac{v}{100}$ rad/s

(i) Reaction due to gyroscopic couple:

(a) Gyroscopic couple due to four wheels:

$$C_w = 4I_w\omega\omega_p$$
$$= 4 \times 2 \times \frac{v}{0.25} \times \frac{v}{100} = 0.32\ v^2\ \text{Nm}$$

(b) Gyroscopic couple due to engine parts:

$$C_e = I_e G\omega\omega_p$$
$$= 1 \times 4 \times \frac{v}{0.25} \times \frac{v}{100} = 0.16\ v^2\ \text{Nm}$$

Total gyroscopic couple:

$$C_g = C_w + C_e$$
$$= 0.32v^2 + 0.16v^2 = 0.48v^2\ \text{Nm}$$

Reaction due to total gyroscopic couple on each outer wheel:

$$R_g = \frac{C_g}{2b} = \frac{0.48v^2}{2 \times 1.5} = 0.16v^2\text{N}\ (\uparrow)$$

Reaction due to total gyroscopic couple on each inner wheel:

$$C_g = 0.16\ v^2\text{N}\ (\downarrow)$$

(ii) Reaction due to centrifugal couple:

Centrifugal force: $$F_c = \frac{mv^2}{R} = \frac{1500 \times v^2}{100} = 15v^2\ \text{N}$$

Overturning couple due to centrifugal force:

$$C_c = F_c \text{ x } h$$

$$= 15\ v^2 \text{x } 0.45 = 6.75\ v^2\ \text{Nm}$$

Vertical downward reaction on each inner wheel is:

$$R_c = \frac{C_c}{2b} = \frac{6.75v^2}{2 \times 1.5} = 2.25v^2\ \text{N}\ (\downarrow)$$

(iii) Reaction due to weight of the vehicle:

$$R_w = \frac{mg}{4} = \frac{1500 \times 9.81}{4} = 3678.75\ \text{N}\ (\uparrow)$$

The limiting condition to avoid lifting of inner wheels from the road surface is:

$$R_i = R_w - R_c - R_g > 0$$

or

$$R_w > R_c + R_g$$

$$3678.75 \geq 2.25v^2 + 0.16\ v^2$$

$$v = 39.07\ \text{m/s}\quad \text{or}\quad \mathbf{140.65\ kmph}$$

or

Problem 8

A four wheeled motor vehicle of mass 2000 kg has a wheel base of 2.5 m, track width 1.5m and height of c.g is 500 mm above the ground level and lies 1 m from the front axle. Each wheel has an effective diameter of 0.8m and a moment of inertia of 0.8 kgm2. The drive shaft, engine flywheel rotating at 4 times the speed of road wheel in clockwise direction when viewed from the front and is equivalent to a mass of 75 kg having a radius of gyration of 100mm.If the vehicle is taking a right turn of 60 m radius at 60kmph, determine the load on each wheel.

Solution,

Since the C.G of the vehicle is 1 m from the front,

The percentage of weight on the front wheels = (2.5-1)/2.5 x 100
= 60%

The percentage of weight on the rear wheels = 40 %

Total weight on the front wheels = 11772 N

Total weight on the rear wheels = 7848 N

Weight on each of front wheel = 5886 N = $W_F/2$

Weight on each of rear wheel = 3924 N = $W_R/2$

The road reaction due to weight of the vehicle is always upwards

Effect of Gyroscopic couple due to Wheel,

$$\mathbf{C_W = 4I_W.\ W_W.\ W_P}$$
$$\mathbf{= 37.1\ Nm}$$

Gyroscopic couple due to wheels acts between outer and inner wheels.

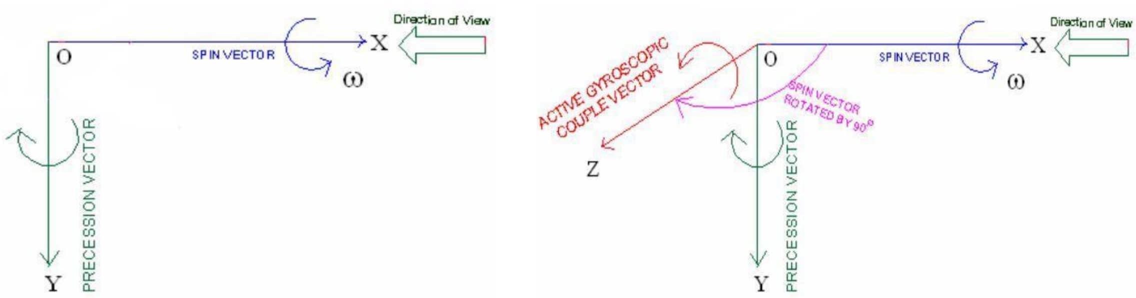


Fig.89

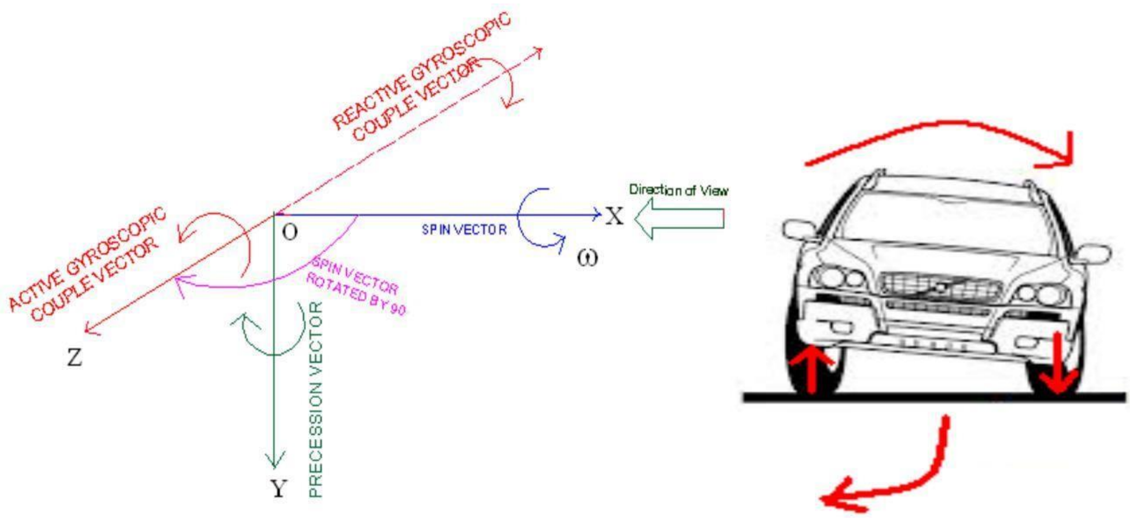


Fig.90

The gyroscopic couple tends to press the outer and lift the inner wheels

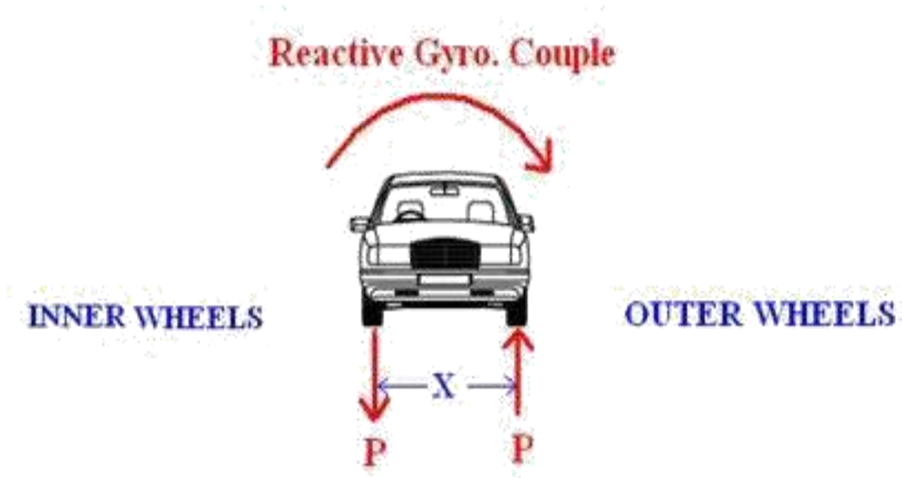


Fig. 91

The road reaction is vertically upward for outer wheels and downward for inner wheels

Road reaction on each outer/Inner wheel,

$$\frac{P}{2} = \frac{C_W}{2X} = \mathbf{12.37\ N}$$

Effect of Gyroscopic Couple due to Engine

Gyroscopic couple due to engine

$\mathbf{C_E = I_E.\ W_E.\ W_P}$
$\mathbf{C_E = I_E.G.\ W_W.\ W_P}$
$\mathbf{= 34.7\ N\ m}$

Gyroscopic couple due to engine acts between Front and Rear wheels.

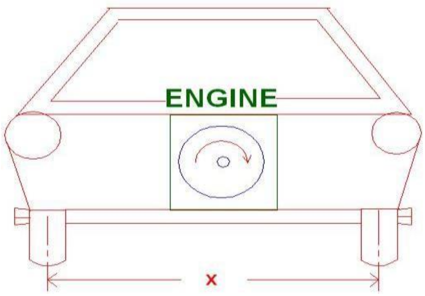


Fig. 92

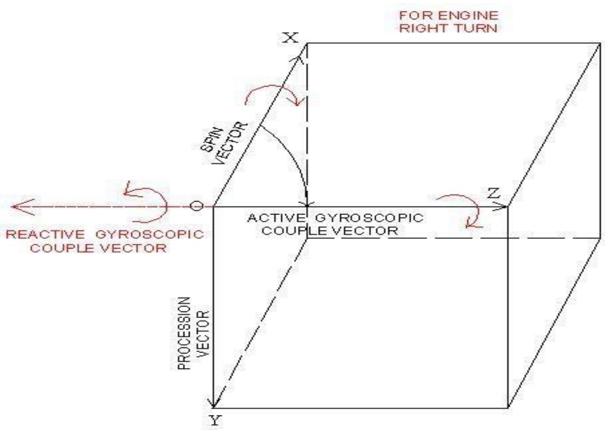


Fig. 93

The couple tends to press Rear wheels and Lift front wheels

Fig. 94

The road reaction is vertically upward for REAR and downward for FRONT wheels.

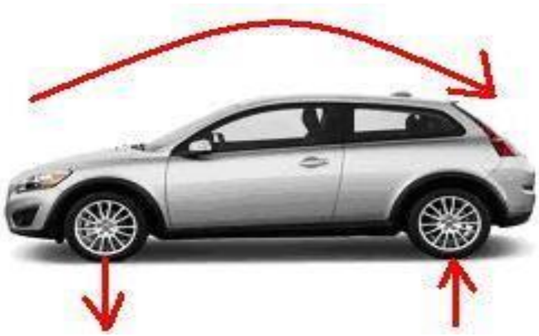


Fig.95

Road reaction on each Front/Rear wheels

$$\frac{Q}{2} = \frac{C_E}{2b} = 6.94\ \mathrm{N}$$

Effect of Centrifugal Couple

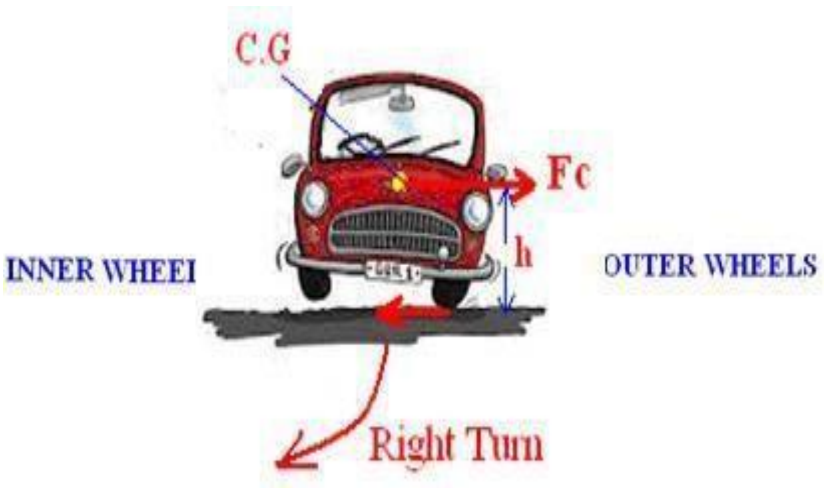


Fig.96

Centrifugal force, $F_c = \frac{MV^2}{R} = \mathbf{9263}$ N

Centrifugal Couple $C_C = \frac{mV^2}{R} \times h = \mathbf{4631.5}$ N

The gyroscopic couple tends to press the outer and lift the inner wheels.

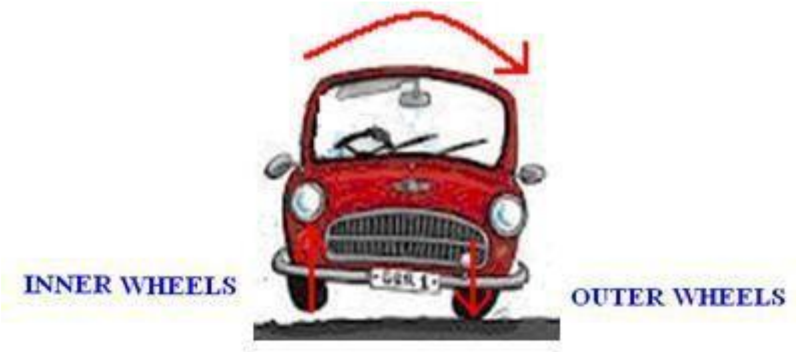


Fig.97

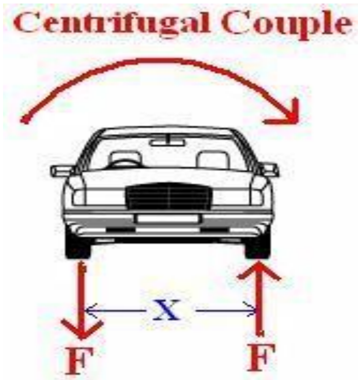


Fig. 98

The road reaction is vertically upward for outer wheels and downward for inner

wheels. Road reaction on each outer/Inner wheel

$$\frac{F}{2} = \frac{C_C}{2X} = \mathbf{1543.8}\text{N}$$

Engine crank shaft rotates clockwise direction seen from front, and Vehicle takes RIGHT turn

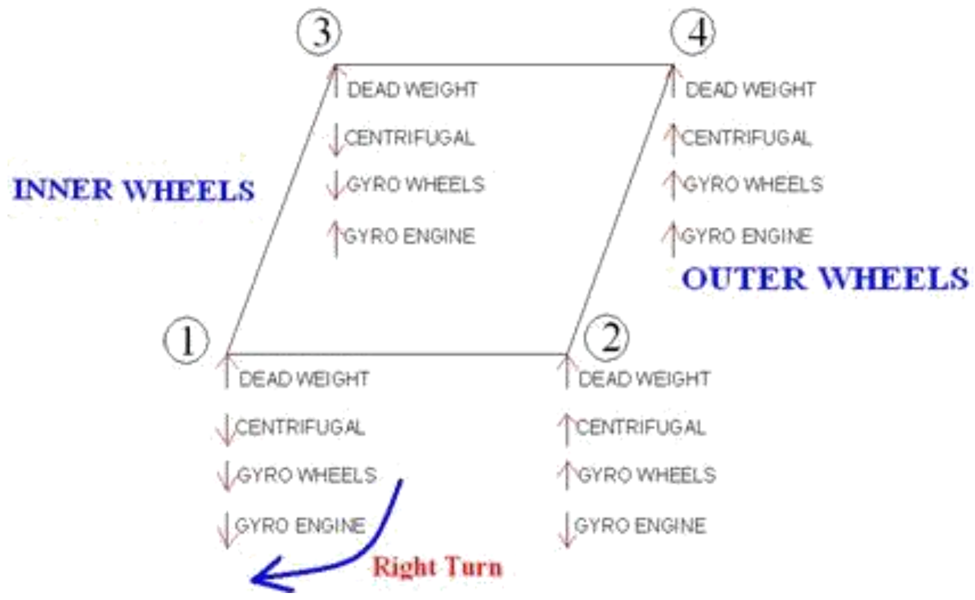


Fig.99

Load on front wheel 1 = 4322.86 N
Load on front wheel 2 = 7435.26 N
Load on rear wheel 3 = 2374.74 N
Load on rear wheel 4 = 5487.14 N

Problem 9

A section of an electric rail track of gauge 1.5 m has a left hand curve of radius 300 m, the superelevation of the outer rail being 260 mm. The approach to the curve is along a straight length of track, over the last 50 m there is a uniform increase in elevation of the outer rail from level track to the super elevation of 260 mm. Each motor used for traction has a rotor of mass 550 kg and radius of gyration 300 mm. The motor shaft is parallel to the axes of the running wheels. It is supported in bearings 780 mm apart and runs at four times the wheel speed but in opposite direction. The diameter of running wheel is 1.2 m. Determine the forces on the bearings due to gyroscopic action when the train is travelling at 90 kmph (a) on the last 50 m of approach track (b) on the curve track.

Solution Angular velocity:

$$\omega = \frac{\text{Gear ratio} \times v}{r}$$

$$= \frac{4 \times 90 \times 1000}{3600 \times 0.6} = 166.67 \text{ rad/s}$$

Let □p = angular velocity of precession.

Moment of inertia: I= mk^2 = 550 x 0.3^2 = 49.5 kg m^2

Gyroscopic couple:

$$C = I\omega\omega_p$$
$$= 49.5 \times 166.67 \times \omega_p$$
$$= 8250.16\ \omega_p \text{ Nm}$$

$$P = \frac{8250.16\,\omega_p}{0.78}$$
$$= 10577.1\ \omega_p \text{ N}$$

Forces on bearings:

(a) Angle turned by engine shaft in the last 50 m track

$$= \frac{0.26}{1.5} = 0.1734 \text{ rad}$$

Time taken to cover this distance $\dfrac{50}{90/3.6} = 2$ sec

Velocity of precession: $\omega_p = \dfrac{0.1734}{2} = 0.0867$

Forces on bearings: P = 10577.1 x 0.0867 = 917.03 N

The change in momentum is represented by vector *oa* and *ob* as shown in Figure 15.18.

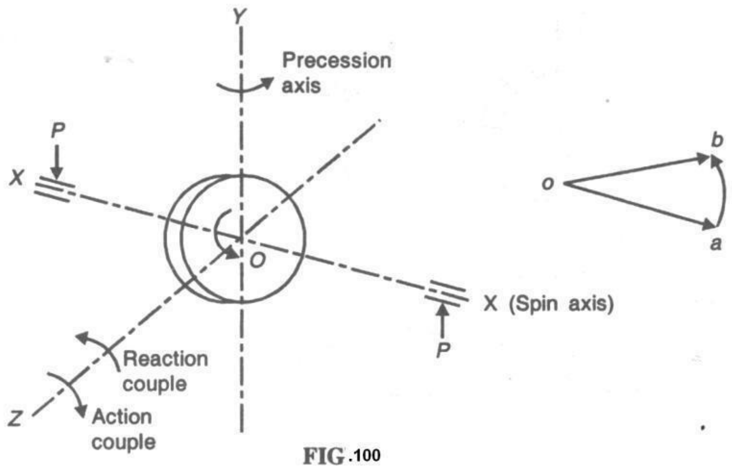


FIG.100

The couple required for precession is, therefore, acting in clockwise looking upward direction. The reaction couple acts in anticlockwise direction looking downward as the forces on the bearings are in the directions shown in Figure 100.

i) When electric rail moves on curved path, the effective angular velocity of precession about the axis perpendicular to the axis of rotation is:

$$\omega_p = \frac{v}{R}\cos\theta$$

where is angle due to superelevation of outer rail. Referring to Figure 15.19.

$$\cos\theta = \frac{AB}{AC} = \frac{1.4773}{1.5} = 0.9848$$

or $$\omega_p = \frac{90 \times 1000}{3600 \times 300} \times 0.9848 = 0.08206 \text{ rad/s}$$

Effective angular velocity of spin =

Therefore,

Forces on bearings: P = 10577.1 Wp
= 10577.1 x 0.08206
= 867.95 N **Ans.**

The change in angular momentum vector and reaction couple shown in Figure 15.19 shows direction of forces on the bearings.

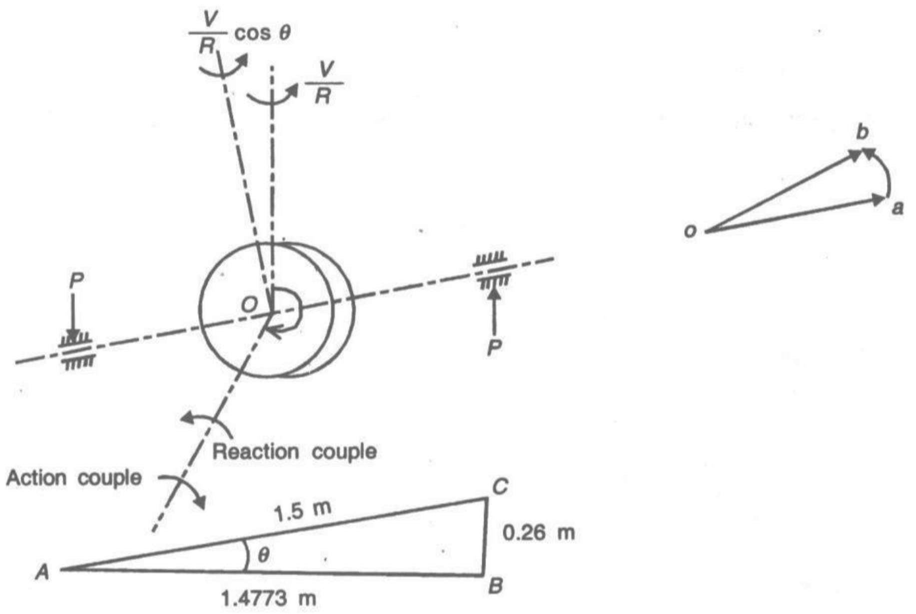


Fig.101

Problem 10.

A four wheeled trolley of total weight 20 kN running on rails of 1 m gauge rounds a curve of 30 m at 40 kmph on a track of embankment slope of 10^0. The wheels have external diameter of 0.6 m and each pair of axle weighs 2000 N and has a radius of gyration of 0.25 m. The height of the C.G of trolley above the wheel is 1 m. Calculate the reaction on the each rail due to gyroscopic and centrifugal couple.

Solution,

Weight of trolley = N = 20000 N
Wheel track = 2x
= 1 m
Radius of curve = R = 30 m
Trolley velocity = 40 kmph= 11.1 m/s
Track of embankment slope of = θ = 10^0
Diameter of wheel = d = 0.6 m
Weight of each pair of wheels = W_1 = 2000 N= mg
Radius of gyration k_g = 0.25 m
Height of C.G from wheel base = 1 m

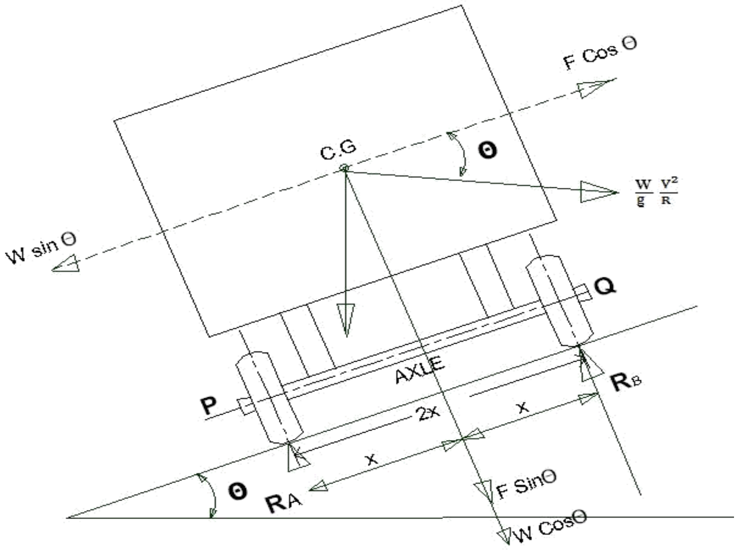


Fig.102

Referring to above Fig. 102,

Consider, the total effect of weight of trolley and that of centrifugal force F,

∴ The reaction RA and RB at the wheels X and Y,

Resolving forces perpendicular to the track,

R_A+R_B = mg Cos θ + F Sin θ

= mg cos θ + m $\frac{V^2}{R}$ sin θ

= mg $\left(\cos\theta + \frac{V^2}{gR}\sin\theta\right)$

= 20000$\left[0.9848 + \frac{11.1^2}{9.81*30} * 0.1736\right]$

R_A+R_B = 21.158 N

Taking moments about Q,

$R_A * 2x = (F \sin\theta + mg \cos\theta)\, x - (F \cos\theta + mg \sin\theta)\, h$

$$R_A = \frac{\left(\frac{mv^2}{R}\sin\theta + mg\cos\theta\right)}{2} - \frac{h}{2x}\left(\frac{mv^2}{R}\cos\theta - mg\ sin\theta\right)$$

$$= \frac{mg\left(\frac{v^2}{gR}\sin\theta + \cos\theta\right)}{2} - \frac{hmg}{2x}\left(\frac{v^2}{gR}\cos\theta - sin\theta\right)$$

$$= \frac{20000}{2}\left[\frac{11.1^2}{9.81*30}\, x\ 0.1736 + 0.9848\right] - \frac{1*20000}{1}\left[\frac{11.1^2}{9.81*40} * 0.9848 - 0.1736\right]$$

R_A = 5751 N

R_B = 15407 N

Let the force at each pair of wheels or each rail due to gyroscopic couple = F_g

∴ Gyroscopic couple applied = $I\omega \cos\theta\, \omega_p$

∴ $F_g * 2x = I\omega \cos\theta \, \omega_p$

$= \frac{I\omega \cos\theta \, \omega p}{2x}$

But, $I = mk^2_g = \frac{2000}{9.81} * 0.25^2 = 12.74 \text{ kg m}^2$

$\omega p = \frac{V}{R} = \frac{11.1}{30} = 0.37$ rad/s

$\omega = \frac{V}{\frac{d}{2}} = \frac{11.1}{\frac{30}{2}} = 37$ rad/s

$F_g = \frac{12.74 * 37 * 0.9848 * 0.37}{1}$

$= 172$ N

∴ Reaction on inner rail = $R_A - F_g$

ii) 5751 – 172

5479 N

∴ Reaction on outer rail = $R_A + F_g$

1) 15407 + 172

15579 N

MODULE-4
INTRODUCTION AND FREE VIBRATION

4.1 Introduction
Objectives
4.2 Definitions
4.3 Analysis of a Single Degree of Freedom System for Free Vibrations
4.3.1 Elements of Lumped Parameter Vibratory System
4.3.2 Undamped Free Vibration
4.3.3 Damped Free Vibration
4.3.4 Free Transverse Vibration due to a Point Load on a Simply Supported Shaft
4.3.5 Free Torsional Vibration of a Single Rotor System
4.4 Causes of Vibration in Machines
4.5 The Harmful Effects of Vibrations
4.6 Vibration Control
4.7 Summary
4.8 Key Words
4.9 Answers to SAQs

4.1 INTRODUCTION

In earlier units, you have studied various mechanisms and machines. The IC engine is one of them which converts thermal energy of fossil fuels to power. It produces highlyfluctuating torque. Even the machines having rotating parts are never completely balanced. From static and dynamic analysis of such machines, it is known that these machines transmit forces to the ground through structure. These forces are periodic in nature.

You know that in a simple pendulum, bob starts to and fro motion or we can say oscillations when bob is disturbed from its equilibrium position. It executes oscillationsat natural frequency. It keeps on oscillating until its motion dies out. If such a system is subjected to the periodic forces it responds to the impressed frequency which makes system to execute forced vibration at forcing frequency. If impressed frequency is equalto the natural frequency, resonance occurs which results in large oscillations and due to this it results in excessive dynamic stresses.

This unit deals with oscillatory behaviour of the dynamic systems. All the bodies havingmass and elasticity are capable of vibration. In studying mechanical vibrations, the bodies are treated as elastic bodies instead of rigid bodies. The bodies have mass also.
Because of mass it they can possess kinetic energy by virtue of their velocity. They can possess elastic strain energy which is comparable to the potential energy. The change ofpotential energy into kinetic energy and vice-versa keeps the body vibrating without external excitation (force or disturbance). If the cause of vibration is known, the remedy to control it can be made.

Vibration of a system is undesirable because of unwanted noise, high stresses, undesirable wear, etc. It is of great importance also in diagnostic maintenance.

Objectives

After studying this unit, you should be able to

- analyse a system for mechanical vibration, determine degree of freedom of a system,

- determine natural frequency of a system,
- analyse and study dynamical behaviour of a system, and
- control vibration in a system.

4.2 DEFINITIONS

Periodic Motion

The motion which repeats after a regular interval of time is called periodic motion.

Frequency

The number of cycles completed in a unit time is called frequency. Its unit iscycles per second (cps) or Hertz (Hz).

Time Period

Time taken to complete one cycle is called periodic time. It is represented inseconds/cycle.

Amplitude

The maximum displacement of a vibrating system or body from the meanequilibrium position is called amplitude.

Free Vibrations

When a system is disturbed, it starts vibrating and keeps on vibrating thereafterwithout the action of external force. Such vibrations are called free vibrations.

Natural Frequency

When a system executes free vibrations which are undamped, the frequency ofsuch a system is called natural frequency.

Forced Vibrations

The vibrations of the system under the influence of an external force are calledforced vibrations. The frequency of forced vibrations is equal to the forcing frequency.

Resonance

When frequency of the exciting force is equal to the natural frequency of the system it is called resonance. Under such conditions the amplitude of vibrationbuilds up dangerously.

Degree of Freedom

The degree of freedom of a vibrating body or system implies the number of independent coordinates which are required to define the motion of the body orsystem at given instant.

Simple Harmonic Motion

It is a to and fro periodic motion of a particle in which :

(a) acceleration is proportional to the displacement from the mean position.

(b) Acceleration is always directed towards a fixed point which is the mean equilibrium position.

It can be represented by an expression having a periodic function like sine orcosine.

$x = X \sin \omega t$

where X is the amplitude.

Diagramatically it can be represented as shown in Figure 7.1.

when $\omega t = 0, \pi$ or $2\pi \Rightarrow x = 0$

when $\omega t = \frac{\pi}{2}, \Rightarrow x = X$

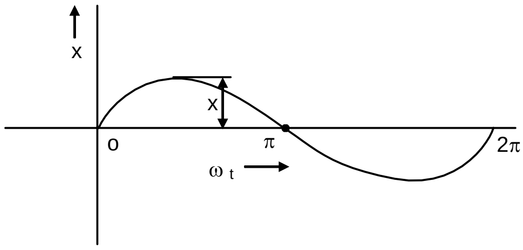


Figure : Simple Harmonic Motion

SAQ 1

At which phase angle, amplitude occurs for a sinusoidal function?

4.3 ANALYSIS OF A SINGLE DEGREE OF FREEDOM SYSTEMS FOR FREE VIBRATIONS

A practical system is very complicated. Therefore, before proceeding to analyse the system it is desirable to simplify it by modeling the system. The modeling of the system is carried over in such a manner that the result is acceptable within the desirable accuracy. Instead of considering distributed mass, a lumped mass is easier to analyse, whose dynamic behaviour can be determined by one independent principal coordinate, in a single degree freedom system. It is important to study the single degree freedom system for a clear understanding of basic features of a vibration problem.

4.3.1 Elements of Lumped Parameter Vibratory System

The elements constituting a lumped parameter vibratory system are :

The Mass

The mass is assumed to be rigid and concentrated at the centre of gravity.

The Spring

It is assumed that the elasticity is represented by a helical spring. When deformed it stores energy. The energy stored in the spring is given by

$$PE = \frac{1}{2} k x^2$$

where k is stiffness of the spring. The force at the spring is given by

$$F = k x$$

The springs work as energy restoring element. They are treated massless.

The Damper

In a vibratory system the damper is an element which is responsible for loss ofenergy in the system. It converts energy into heat due to friction which may beeither sliding friction or viscous friction. A vibratory system stops vibration because of energy conversion by damper. There are two types of dampers.

Viscous Damper

A viscous damper consists of viscous friction which converts energy intoheat due to this. For this damper, force is proportional to the relative velocity.

$F_d \propto$ relative velocity (v)

$$F_d = cv$$

where c is constant of proportionality and it is called coefficient ofdamping.

The coefficient of viscous damping is defined as the force in 'N' whenvelocity is 1 m/s.

Coulumb's Damper

The dry sliding friction acts as a damper. It is almost a constant force but direction is always opposite to the sliding velocity. Therefore, direction offriction changes due to change in direction of velocity.

The Excitation Force

It is a source of continuous supply of energy to the vibratory system. It is anexternal periodic force which acts on the vibratory system.

It is important to study the single degree freedom system for a clear understandingof basic features of a vibration problem.

4.3.2 Undamped Free Vibration

There are several methods to analyse an undapmed system.

Methodology

Method Based on Newton's II Law

According to the Newton's II law, the rate of change of linear momentum isproportional to the force impressed upon it

$$\frac{d}{dt}(mv) \propto \text{Net force in direction of the velocity}$$

Using $v = \frac{dx}{dt} = \dot{x}$

$$\therefore \quad \frac{d\dot{x}}{dt} = (m\ddot{x}) = c \sum F$$

where c is constant of proportionality.

or $m\ddot{x} = c \sum F$

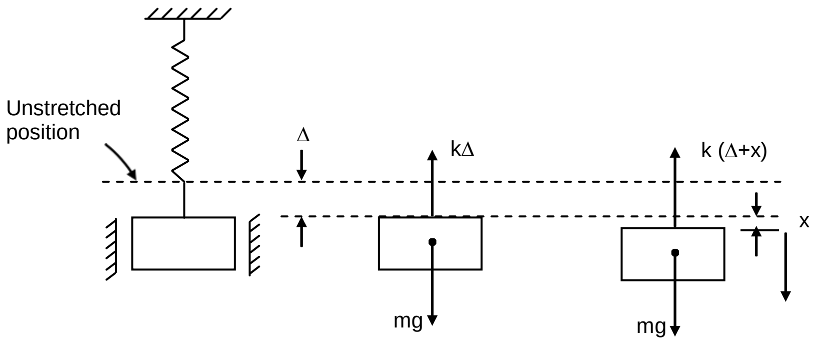
For proper units in a system $c = 1$

$$m\ddot{x} = \sum F$$

The direction of forces $m\ddot{x}$ and $\sum F$ are same. A model which represents undamped single degree of freedom system shall have two elements, i.e.helical spring and mass. The mass is constrained to move only in one direction as shown in Figure 7.2. The mass is in static condition in Figure 7.2(a). The free body diagram of the mass is shown in

Figure 7.2(b). The body is in equilibrium under the action of the two forces. Here 'Δ' is the extension of the spring after suspension of the mass on the spring.

Therefore, $k\,\Delta = mg$ . . . (7.1)



(a) Spring Mass (b) Static Condition (c) Dynamic Condition

Figure : Undamped Free Vibration

figure represents the dynamic condition of the body. In this case, the body is moving down with acceleration ' $\ddot{x}$ ' also in downward direction, therefore,

$$m\ddot{x} = \sum F \text{ in direction of } \ddot{x}$$

or $$m\dot{x} = mg - k\,(x + \Delta)$$

Incorporating Eq. (7.1) in Eq. (7.2)

$$m\ddot{x} = -\,k\,x$$

or $$m\ddot{x} + k\,x = 0$$

Method Based on D'Alembert's Principle

The free body diagram of the mass in dynamic condition can be drawn as follows :

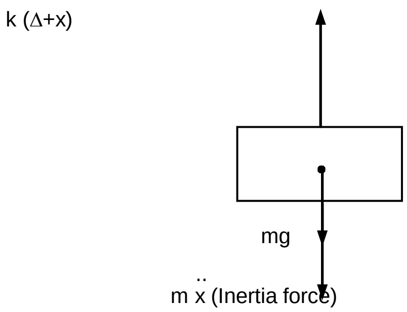


Figure : Free Body Diagram

The free body diagram of mass is shown in Figure 7.3. The force equation can be written as follows :

$$m\ddot{x} + mg = k\,(x + \Delta)$$

Incorporating Eq. (7.1) in Eq. (7.4), the following relation is obtained.

$$m\ddot{x} + kx = 0$$

This equation is same as we got earlier.

Energy Method

This method is applicable to only the conservative systems. In conservative systems there is no loss of energy and therefore total energy remains constant. When a mechanical system is in motion, the total energy of the system is partly kinetic and partly potential (elastic strain energy). The kinetic energy is due to the mass (m) and velocity ($\dot{x}$). The potential energyis due to spring stiffness and relative movement between the two ends of the spring.

Energy (E) = T + U = constant (C)

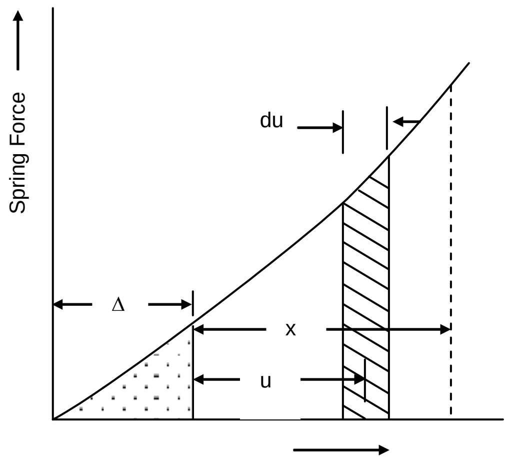
where T = Kinetic energy of the system, and'

U = Elastic strain energy.

Since total energy remains constant

$$\therefore \quad \frac{dE}{dt} = 0 \quad \text{or} \quad \frac{d}{dt}(T + U) = 0$$

$$T = \frac{1}{2} m (\dot{x})^2$$



Deflection

Figure : Spring Force

Deflection Diagram

The potential energy of the system consists of two points :

(a) loss/gain in PE of mass, and

(b) strain energy of spring.

Consider an infinitesimal element du at $x = u$.

From Figure 7.4

Spring force $(F_u) = k\,(u + \Delta)$

Work done $dW = k\,(u + \Delta) \times du$

$$\therefore \quad U = \int_0^x dW - \text{loss of PE of mass}$$

$$= \int_0^x k\,(u + \Delta)\,du - mg\,x$$

$$\therefore \quad U = \int_0^x (ku + mg)\,du - mg\,x \qquad [\because k\,\Delta = mg]$$

$$\text{or} \qquad U = \frac{1}{2}(kx^2) + mg\,x - mg\,x$$

or $$U = \frac{1}{2} kx^2$$

$$\frac{d}{dt}\left(\frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2\right) = 0$$

$$\therefore \qquad \frac{1}{2} m \times 2\dot{x} \times \ddot{x} + \frac{1}{2} k \times 2x \times \dot{x} = 0$$

$$\text{or} \qquad m\ddot{x} + kx = 0$$

This is the same equation as we got earlier.

Rayleigh's Method

It is a modified energy method. It may be noted that in a conservative system potential energy is maximum when kinetic energy is minimum and vice-versa. Therefore, equating maximum kinetic energy with maximum potential energy.

$$\therefore \qquad \frac{1}{2} m (\dot{x}_{max})^2 = \frac{1}{2} k (x_{max})^2$$

$$\text{and} \qquad x_{max} = X$$

$$\therefore \qquad \frac{1}{2} m (X\omega)^2 = \frac{1}{2} k X^2$$

$$\text{or} \qquad \omega = \sqrt{\frac{k}{m}}$$

Solution of Differential Equation

The differential equation of single degree freedom undamped system is given by

$$m\ddot{x} + kx = 0$$

$$\text{or} \qquad \ddot{x} + \left(\frac{k}{m}\right) x = 0$$

when coefficient of acceleration term is unity, the underroot of coefficient of x is equal to the natural circular frequency, i.e. 'ω_n'

$$\therefore \qquad \omega_n = \sqrt{\frac{k}{m}}$$

Therefore, Eq. (7.7) becomes

$$\ddot{x} + \omega_n^2 \, x = 0$$

The equation is satisfied by functions sin $\omega_n t$ and cos $\omega_n t$. Therefore, solution of Eq. (7.9) can be written as

$$x = A \sin \omega_n t + B \cos \omega_n t$$

where A and B are constants. These constants can be determined from initial onditions. The system shown in Figure can be disturbed in two ways :

(a) by pulling mass by distance 'X', and

(b) by hitting mass by means of a fast moving object with a velocity \ say 'V'.

Considering case (a)

$t = 0, \qquad x = X$ and $\dot{x} = 0$

$\therefore \quad X = B \quad \text{and} \quad A = 0$

$\therefore \quad x = X \cos \omega_n t$ 177

Theory of Machines

Considering case (b)

$t = 0,\quad x = 0$ and $\dot{x} = V$

$B = 0$ and $A = \dfrac{V}{\omega_n}$

$$\therefore \quad x = \frac{V}{\omega_n} \sin \omega_n t$$

Behaviour of Undamped System

onsider the system shown in Figure . The system has been disturbed by pulling the mass by distance '*X*'. The solution of the system in this case is given by Eq. (7.11) which is

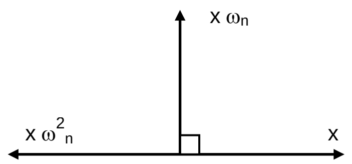
$$x = X \cos \omega_n t$$

$$\therefore \quad \dot{x} = - X \omega_n \sin \omega_n t = X \omega_n \cos\left(\omega_n t + \frac{\pi}{2}\right)$$

and

$$\ddot{x} = - X \omega_n^2 \cos \omega_n t = X \omega_n^2 \cos (\omega_n t + \pi)$$

These expressions indicate that velocity vector leads displacement by $\dfrac{\pi}{2}$ and acceleration leads displacement by 'π'. The maximum velocity is $(X\,\omega_n)$ and maximum acceleration is $(X\,\omega_n^2)$.



(a)

(b)

(c)

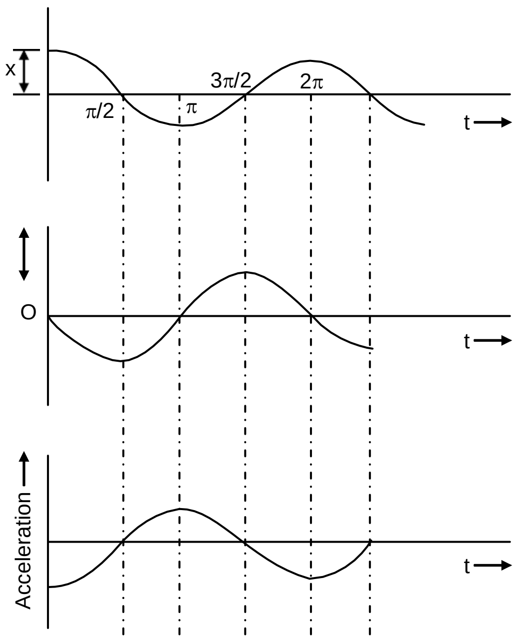


Figure : Plots of Displacement, Velocity and Acceleration

Figure 7.6 shows the plots of displacement, velocity and acceleration, with respect to time. The following observations can be made from these diagrams :

(a) A body, if disturbed, will never stop vibrating.

(b) When displacement is maximum, velocity is zero and acceleration is maximum in direction opposite to displacement.

(c) When displacement is zero, velocity is maximum and acceleration is zero.

4.3.3 Damped Free Vibration

In undamped free vibrations, two elements (spring and mass) were used but in damped third element which is damper in addition to these are used. The three element model is shown in Figure 7.7. In static equilibrium

$$k\,\Delta = mg$$

$$m\ddot{x} = mg - k\,(x + \Delta) - c\dot{x}$$

$\therefore$ $$m\ddot{x} = -\,kx - c\dot{x}$$

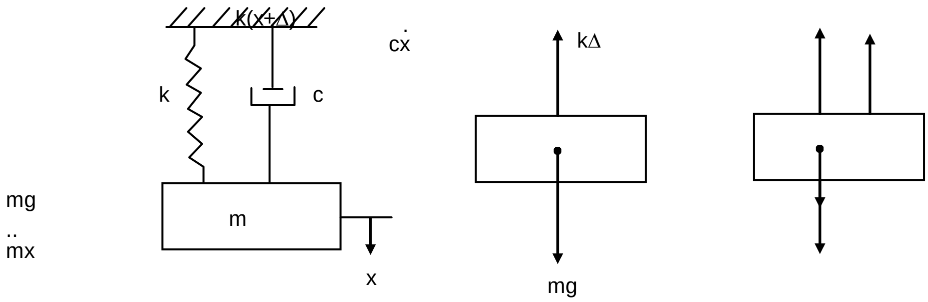
or $$m\ddot{x} + c\dot{x} + kx = 0$$

Let $$x = X\,e^{st}$$

$$ms^2 + cs + k = 0$$

or $$s^2 + \frac{c}{m}\,s + \frac{k}{m} = 0$$

$\therefore$ $$s_{1,2} = -\left(\frac{c}{2m}\right) \pm \frac{1}{2}\sqrt{\left(\frac{c}{m}\right)^2 - 4\left(\frac{k}{m}\right)}$$



(a) **(b)** **(c)**

Figure : Damped Free Vibration

$$x = X_1\, e^{\left[-\left(\frac{c}{2m}\right) + \frac{1}{2}\sqrt{\left(\frac{c}{m}\right)^2 - 4\left(\frac{k}{m}\right)}\right]t} + X_2\, e^{\left[-\left(\frac{c}{2m}\right) - \frac{1}{2}\sqrt{\left(\frac{c}{m}\right)^2 - 4\left(\frac{k}{m}\right)}\right]t}$$

$$= e^{-\left(\frac{c}{2m}\right)t}\left[X_1\, e^{\frac{1}{2}\left\{\sqrt{\left(\frac{c}{m}\right)^2 - 4\left(\frac{k}{m}\right)}\right\}t} + X_2\, e^{-\frac{1}{2}\left\{\sqrt{\left(\frac{c}{m}\right)^2 - 4\left(\frac{k}{m}\right)}\right\}t}\right]$$

The nature of this solution depends on the term in the square root. There are three possible cases :

(a) $\left(\frac{c}{m}\right)^2 > 4\left(\frac{k}{m}\right) -$ Overdamped case

(b) $\left(\frac{c}{m}\right)^2 = 4\left(\frac{k}{m}\right)$ – Critically damped case

(c) $\left(\frac{c}{m}\right)^2 < 4\left(\frac{k}{m}\right)$ – Underdamped case

Let the critical damping coefficient be C_c, therefore,

$$\left(\frac{C_c}{m}\right)^2 = 4\left(\frac{k}{m}\right)$$

or $$C_c = 2\sqrt{km} = 2\sqrt{\frac{k}{m}m^2} = 2m\sqrt{m^2}\sqrt{\frac{k}{m}} = 2m\,\omega_n$$

or $$C_c = 2\sqrt{km} = 2m\,\omega_n$$

Almost all the systems are underdamped in practice.

Therefore, $$\sqrt{\left(\frac{c}{m}\right)^2 - 4\left(\frac{k}{m}\right)} = i\sqrt{4\left(\frac{k}{m}\right) - \left(\frac{c}{m}\right)^2}$$

The ratio of damping coefficient (c) to the critical damping coefficient is called damping factor 'ζ'.

$$\zeta = \frac{C}{C_c}$$

$$= 4 \qquad \sqrt{4\omega_n^2 - \left(\frac{c}{C_c}\times\frac{C_c}{m}\right)^2} \qquad \sqrt{4\omega_n^2 - \zeta^2\times\left(\frac{2m\,\omega_n}{m}\right)^2}$$

$$= 2\omega_n\sqrt{1-\zeta^2}$$

$$\therefore \qquad x = e^{-\frac{c}{2m}t}\left[X_1\,e^{(i\,\omega_n\sqrt{1-\zeta^2})\,t} + X_2\,e^{(-i\,\omega_n\sqrt{1-\zeta^2})\,t}\right]$$

Let $$\omega_n\sqrt{1-\zeta^2} = \omega_d \quad \text{(say)}$$

where ω_d is natural frequency of the damped free vibrations.

Therefore, for under-damped case

$$x = e^{-\frac{c}{2m}t}\left[X_1\,e^{i\,\omega_d\,t} + X_2\,e^{-i\,\omega_d\,t}\right]$$

For critically damped system

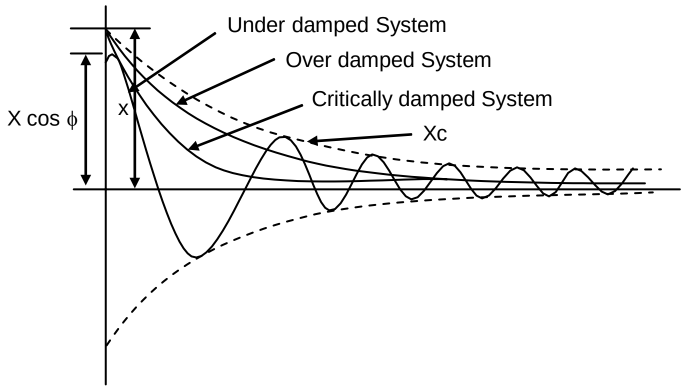
$$x = (X_1 + X_2\,t)\,e^{-\frac{c}{2m}t}$$

For overdamped system

$$x \quad e^{-\frac{c}{2m}t}\left[\quad\sqrt{\quad}\qquad\sqrt{\quad}\quad\right]$$

$$\frac{C}{2m} = \frac{C}{C_c}\times\frac{C_c}{2m} = \zeta\times\frac{2m\,\omega_n}{2m} = \zeta\,\omega_n$$

$$\therefore \quad x \quad e^{-\zeta \omega_n t} \Big\lfloor \quad \sqrt{\quad} \quad \sqrt{\quad} \Big\rfloor$$



Figure

The Eq. (7.19) can also be written as

$$x = X\, e^{-\zeta\, \omega_n\, t} \cos(\omega_d\, t + \phi)$$

where X and ϕ are constants. X represents amplitude and ϕ phase angle.

Let at $t = t,\ x = x_0.$

$$\therefore \quad x_0 = X\, e^{-\zeta\, \omega_n\, t} \cos(\omega_d\, t + \phi)$$

After one time period

$$t = t + t_p \quad \text{and} \quad x = x_1$$

$$\therefore \quad x_1 = X\, e^{-\zeta\, \omega_n\, (t + t_p)} \cos\{\omega_d\, (t + t_p) + \phi\}$$

Dividing Eq. (7.24) by Eq. (7.25)

$$\frac{x_0}{x_1} = \frac{X\, e^{-\zeta\, \omega_n\, (t + t_p)} \cos \omega_d\; t + \phi}{X\, e^{-\zeta\, \omega_n\, (t + t_p)} \cos\{\omega_d\, (t + t_p)\, \phi\}}$$

Since $t_p = \dfrac{1}{f_p} = \dfrac{2\pi}{\omega_d}$

or $\omega_d\, t_p = 2\pi$

$$\therefore \quad \frac{x_0}{x_1} = e^{\zeta\, \omega_n\, t_p}\, \frac{\cos(\omega_d\, t + \phi)}{\cos\{\omega_d\, t + 2\pi + \phi\}}$$

Since $\cos\theta = \cos(2\pi + \theta)$

$$\therefore \quad \cos(\omega_d\, t + \phi) = \cos\{\omega_d\, t + 2\pi + \phi\}$$

$$\therefore \quad \frac{x_0}{x_1} = e^{\zeta\, \omega_n\, t_p}$$

or $$L_n\left(\frac{x_0}{x_1}\right) = \zeta\, \omega_n\, t_p = \zeta\, \omega_n\, \frac{2\pi}{\omega_d} = \frac{2\pi\, \omega_n\, \zeta}{\omega_n\sqrt{1 - \zeta^2}}$$

or $$L_n\left(\frac{x_0}{x_1}\right) = \frac{2\pi\, \zeta}{\sqrt{1 - \zeta^2}} \qquad \ldots (7.26)$$

$\dfrac{2\pi\, \zeta}{\sqrt{1 - \zeta^2}}$ is called logarithmic decrement.

It can be proved that

$$\ln \frac{x_0}{x_n} = \frac{2n\pi\zeta}{\sqrt{1-\zeta^2}}$$

If $\zeta < 0.3 \quad \ln \frac{x_0}{x_1} \approx 2\pi\zeta$

Figure 7.8 represents displacement time diagram for the above mentioned three cases. For over-damped and critically damped system mass returns to its original position slowly and there is no vibration. Vibration is possible only in the under-damped system because the roots of Eq. (7.14) are complex and solution consists of periodic functions (Eq. (7.22)).

4.3.4 Free Transverse Vibration due to a Point Load on a Simply **Supported Shaft**

In this type of vibration, all the particles vibrate along paths perpendicular to the shaft axis. The shaft may be having single to several supports. It may be carrying its own load, a single point load or several point loads come in this category. Now these cases are to be dealt with separately.

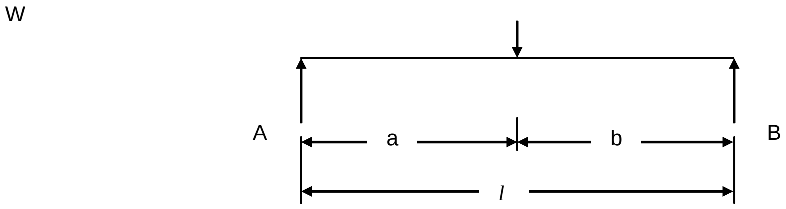


Figure : Free Transverse Vibration

Consider a very light shaft *AB* of length ‘*l*’ carrying a point load ‘*W*’ at a distance ‘*a*’ from the support *A* and at a distance ‘*b*’ from the support *B*.

$$a + b = l$$

and the deflection

$$\delta = \frac{W\,a^2\,b^2}{3E\,I\,l}$$

The natural circular frequency for the system is given by

$$\omega_n = \sqrt{\frac{k}{\left(\frac{W}{g}\right)}}$$

or $$\omega_n = \sqrt{\frac{kg}{W}} = \sqrt{\frac{g}{\left(\frac{W}{k}\right)}}$$

or $$\omega_n = \sqrt{\frac{g}{\delta}}$$

where $$\delta = \frac{W}{k}$$

$\therefore$ $$f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$$

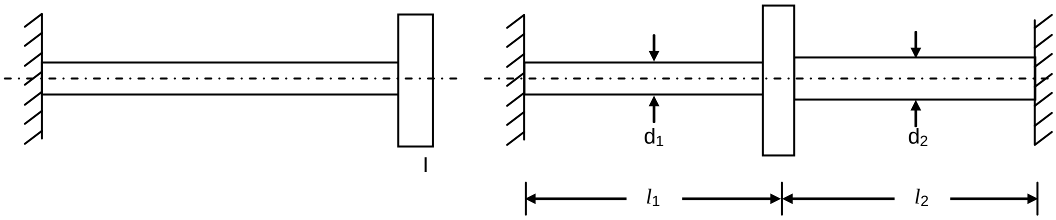
$= 4.985$ Hz

$\sqrt{\delta}$

The mass of the beam was neglected for determination of the above mentioned natural frequency.

4.3.5 Free Torsional Vibration of a Single Rotor System

In torsional vibration, all the particles of the system vibrate along circular arcs having their centers along the axis of rotation. Figure 7.10 represents a single rotor systems. In both the cases (a) and (b), there is only one inertia '*I*'.



(a) **(b)**

Figure : Free Torsional Vibration

In part (a) it is supported by one shaft segment and in part (b) it is supported by the two shaft segments.

The differential equation for the rotor shown in Figure 7.10(a) can be obtained by considering two couples, i.e. inertia couple and torsional elastic couple. If shaft is twisted slightly say by angle 'θ', the couple is given by

$$(k_t\,\theta)$$

where k_t is torsional stiffness which is given by

$$k_t = \frac{T}{\theta} = \frac{G\,J}{l}$$

where G is modulus of rigidity,

J is polar moment of inertia, and

l is length of shaft.

The differential equation for the rotor given in Figure 7.10(a) is

$$I\,\ddot{\theta} + k_t\,\theta = 0$$

or $$\theta + \frac{GJ}{l}\,\theta = 0$$

∴ $$\omega_n = \sqrt{\frac{GJ}{l}}$$

For the shaft shown in Figure 7.10(b), the two segments are acting like parallel springs. Therefore, the differential equation for this will be

$$I\,\ddot{\theta} + \left(\frac{G\,J_1}{l_1} + \frac{G\,J_2}{l_2}\right)\theta = 0$$

or $$\ddot{\theta} + \frac{G}{I}\left(\frac{J_1}{l_1} + \frac{J_2}{l_2}\right)\theta = 0$$

or $$\omega_n = \sqrt{\frac{G}{I}\left(\frac{J_1}{l_1} + \frac{J_2}{l_2}\right)}$$

SAQ 2

(a) What is the difference between energy method and Rayleigh's method?

(b) By how much angle acceleration and velocity lead displacement?

(c) Along which curve amplitude decays in under-damped system?

4.4 CAUSES OF VIBRATION IN MACHINES

There are various sources of vibration in an industrial environment :

(a) Impact processes such as pile driving and blasting.

(b) Rotating or reciprocating machinery such as engines, compressors and motors.

(c) Transportation vehicles such as trucks, trains and aircraft.

(d) Flow of fluids through pipes and without pipes.

(e) Natural calamities such as earthquakes.

4.5 THE HARMFUL EFFECTS OF VIBRATIONS

There are various harmful effects of vibration :

(a) Excessive wear of bearings.

(b) Formation of cracks in machines, buildings and structure, etc.

(c) Loosening of fasteners in mechanical systems.

(d) Structural and mechanical failures in machines and buildings.

(e) Frequent and costly maintenance of machines.

(f) Electronic malfunctions through failure of solder joints.

(g) Abrasion of insulation around electric conductors, causing soots.

(h) The occupational exposure of humans to vibration leads to pain, discomfort and reduction in working efficiency.

4.6 VIBRATION CONTROL

The vibration can sometimes be eliminated on the basis of theoretical analysis. However, in eliminating the vibration may be too high. Therefore, a designer must compromise the manufacturing costs involved between an acceptable amount of vibration and a reasonable manufacturing cost. The following steps may be taken to control vibrations :

(a) The first group of methods attempts to reduce the excitation level at the source. The balancing of inertial forces, smoothening of fluid flows and proper lubrication at joints are effective methods and should be applied whenever possible.

(b) A suitable modification of parameters may also reduce the excitation level. The system parameters namely inertia, stiffness and damping are suitably chosen or modified to reduce the response to a given excitation.

(c) In this method, transmission of path of vibration is modified. It is popularly known as vibration isolation.

As mentioned above, the first attempt is made to reduce vibration at the source. In some cases, this can be easily achieved by either balancing or an increase in the precision of machine element. The use of close tolerances and better surface finish for machine parts make the machine less susceptible to vibration. This method may not be feasible in some cases like earthquake excitation, atmospheric turbulence, road roughness, engine combustion instability.

After reduction of excitation at the source, we need to look for a method to further control the vibration. Such a selection is guided by the factors predominantly governing the vibration level.

Example 4.1

Determine the natural frequency of spring mass pulley system shown in

Solution

By Energy Method

$$\text{Total energy } (E) = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} kx^2$$

$$x = r\theta$$

or $$\dot{x} = r\dot{\theta}$$

$$\ddot{x} = r\ddot{\theta}$$

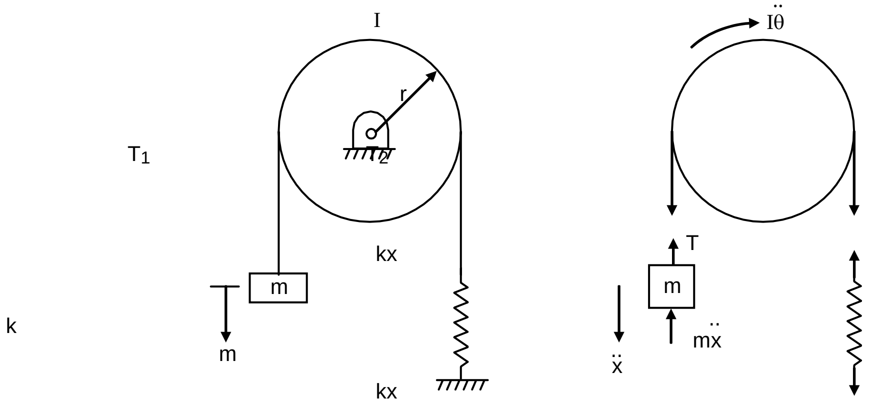
$\therefore$ $$E = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} \frac{I}{r^2} \dot{x}^2 + \frac{1}{2} kx^2$$

$$\frac{dE}{dt} = \frac{1}{2} m\, 2\dot{x}\,\ddot{x} + \frac{1}{2} \frac{I}{r^2} 2\dot{x}\,\ddot{x} + \frac{1}{2} k\, 2x\,\dot{x} = 0$$

or $$\left(m + \frac{1}{r^2}\right)\ddot{x} + k\,x = 0$$

or $$\ddot{x} + \frac{k}{\left(m + \frac{I}{r^2}\right)} x = 0$$

$\therefore$ $$\omega_n = \sqrt{\frac{k}{\left(m + \frac{1}{r^2}\right)}}$$



By D'Alembert's Principle

$$(T_1 - T_2)\, r = I\,\ddot{\theta} \quad \text{and} \quad r\,\ddot{\theta} = \ddot{x}$$

$$T_1 = -m\ddot{x} \quad \text{and} \quad T_2 = k\,x$$

$$-m\ddot{x} - k\,x = \frac{I}{r^2}\ddot{x}$$

$$\text{or} \qquad m\ddot{x} + \frac{I}{r^2}\ddot{x} + kx = 0$$

or $$\left(m + \frac{I}{r^2}\right)\ddot{x} + k\,x = 0$$

or $$\ddot{x} + \frac{k}{m + \frac{I}{r^2}}\,x = 0$$

$$\therefore \qquad \omega_n = \sqrt{\frac{k}{m + \frac{I}{r^2}}}$$

Example

Determine the effect of mass of the spring on the natural frequency of spring mass system.

Solution

Let m_s be the mass in kg per unit length.

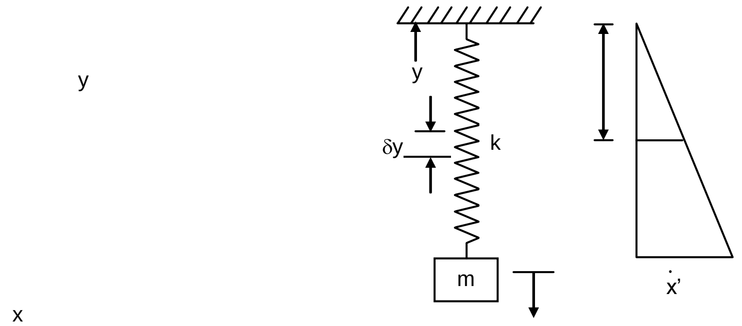
Figure 7.12 shows a spring mass system. Let the velocity distribution be linear therefore, the total energy 'E' is given by

$$E = \frac{1}{2}\,m\,\dot{x}^2 + \frac{1}{2}\int_0^l (m_s\,\delta_y)\,\dot{y}^2 + \frac{1}{2}\,k\,x^2$$

$$= \frac{1}{2}\,m\,\dot{x}^2 + \frac{1}{2}\int_0^l m_s\left(\frac{\dot{x}\,y}{l}\right)^2 dy + \frac{1}{2}\,k\,x^2$$

$$= \frac{1}{2}\,m\,\dot{x}^2 + \frac{1}{2}\,m_s\,\dot{x}^2\int_0^l \frac{y^2}{l^2}\,dy + \frac{1}{2}\,k\,x^2$$

$$= \frac{1}{2}\left(m + \frac{m_s\,l}{3}\right)\dot{x}^2 + \frac{1}{2}\,k\,x^2$$



$$\frac{dE}{dt} = \frac{1}{2}\left(m + \frac{m_s\,l}{3}\right)\ddot{x}\times 2\dot{x} + \frac{1}{2}\,k\,2x\,\dot{x} = 0$$

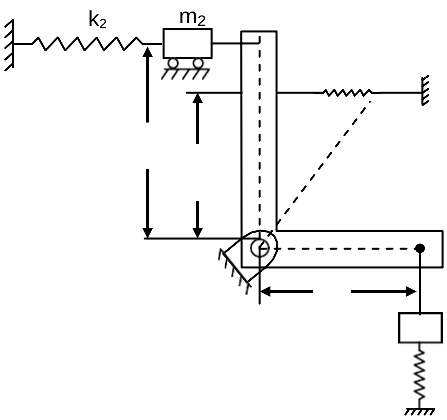
or $$\left(m + \frac{m_s\,l}{3}\right)\ddot{x} + k\,x = 0$$

or $$\ddot{x} + \frac{k}{\left(m + \frac{m_s\,l}{3}\right)}\,x = 0$$

$$\omega_n = \sqrt{\frac{k}{\left(m + \frac{m_s\,l}{3}\right)}}$$

Example

Figure 7.13 shows an indicator mechanisms. The bell crank arm is pivoted at *O* and has mass moment of inertia *I*. Find natural frequency of the system.



Solution

Let θ be the angular displacement of bell crank arm.

$$KE = \frac{1}{2} I \theta^2 + \frac{1}{2} m_2 (a\theta)^2 + \frac{1}{2} m_1 (c\theta)^2$$

$$PE = \frac{1}{2} k_1 (c\theta)^2 + \frac{1}{2} k_2 (a\theta)^2 + \frac{1}{2} k_3 (b\theta)^2$$

Total energy (E) = $KE + PE$

and $$\frac{dE}{dt} = 0$$

$\therefore$ $$(I + m_2 a^2 + m_1 c^2)\ddot{\theta} + (k_1 c^2 + k_2 a^2 + k_3 b^2)\theta = 0$$

or $$\ddot{\theta} + \left(\frac{k_1 c^2 + k_2 a^2 + k_3 b^2}{I + m_2 a^2 + m_1 c^2}\right)\theta = 0$$

$\therefore$ $$\omega_n = \sqrt{\frac{k_1 c^2 + k_2 a^2 + k_3 b^2}{I + m_2 a^2 + m_1 c^2}} \text{ rad/sec.}$$

Example

A damped system has following elements :

Mass = 4 kg; k = 1 kN/m; C = 40 N-sec/m

Determine :

(a) damping factor,

(b) natural frequency of damped oscillation,

(c) logarithmic decrement, and

(d) number of cycles after which the original amplitude is reduced to 20%.

Solution

Given data :

m = 4 kg; k = 1 kN/m; C = 40 N-sec/m

$$C_c = 2\sqrt{k\,m} = 2\sqrt{1000 \times 4} = 126.49 \text{ Ns/m}$$

(a) Damping factor

$$\zeta = \frac{40}{126.49} = 0.316$$

(b) $$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{4}} = 15.8 \text{ r/s}$$

$$\therefore \quad \omega_d = \omega_n \sqrt{1-\zeta^2} = 15.8\sqrt{1-(0.316)^2} = 14.99 \text{ r/s}$$

$$\therefore \quad f_d = \frac{\omega_d}{2\pi} = 2.386 \text{ cps or Hz}$$

(c) Logarithmic decrement $(\delta) = \dfrac{2\pi\,\zeta}{\sqrt{1-\zeta^2}}$

$$= \frac{2\pi \times 0.316}{\sqrt{1-0.316^2}} = 2.0928$$

(d) $$\delta = \ln \frac{x_1}{x_2} = \frac{1}{n} \ln \frac{x_1}{x_n}$$

or $$2.0928 = \frac{1}{n} \ln 5$$

or $$n = \frac{\ln 5}{2.0928} = 0.769$$

4.7 SUMMARY

A system which has mass and elasticity can start vibrating if it is disturbed. The natural frequencies of a system depend on the degrees of freedom of a system. For a multi-degree of freedom system, there will be several natural frequencies. For a two-degree of freedom system, there will be two natural frequencies.

The vibration can be linear, transverse or rotational depending on the type of the system. The methods of analysis constitutes applications of Newton's law, D'Alembert's principle, energy method and Rayleigh's method. All the methods can in general be used to analyse the system but it can be easily analysed by using a particular method. Therefore, selection of a particular method is always desirable for a given system. The energy method and Rayleigh's method can be used for a conservative system where there is no energy loss but a practical system cannot be conservative in ideal sense. The cause of vibration, their harmful effects and remedies have also been mentioned for practical utility to control vibrations.

4.8 KEY WORDS

Periodic Motion : It is the motion which repeats after a regular interval of time.

Frequency : It is the number of cycles completed in a unit time.

Time Period : It is the time taken to complete one cycle.

Amplitude : It is maximum displacement of a vibrating system

		from the position of mean equilibrium position.
Free Vibration	**:**	It is the vibration of the system which takes place without any external force after the disturbance.

Natural Frequency	:	It is the frequency of vibration of a system which is undamped and without external excitation when it is disturbed.
Forced Vibration	:	It is the vibration of a system which is due to external excitation.
Resonance	:	When forcing frequency is equal to the natural frequency, resonance takes place.
Degree of Freedom	:	It is equal to the number of independent coordinates which are required to define the motion of the system.
Mode of Vibration	:	It is the way, the system vibrates in the free vibrations.
Conservative System	:	It is the system for which total energy remains constant.
Damper	:	It is the element which is responsible for decay in energy.

MODULE-5

FORCED VIBRATION (SINGLE DEGREE OF FREEDOM SYSTEM)

In this chapter, the steady state response of harmonically excited single degree of freedom systems will be discussed. Simpler phasor diagram method will be used to obtain the steady state response. Response due to rotating unbalance, whirling of shafts, vibration isolations will also be discussed.

Steady state response due to Harmonic Oscillation:

Consider a spring-mass-damper system as shown in figure 1. The equation of motion of this system subjected to a harmonic forcing $F\sin\omega t$ can be given by

$$m\ddot{x} + kx + c\dot{x} = F\sin\omega t \qquad (1)$$

where, *m*, *k* and *c* is the mass, spring stiffness and damping coefficient of the system.

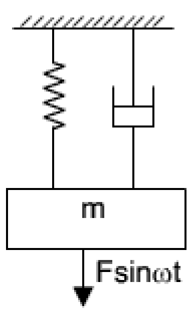


Figure 1 Harmonically excited system

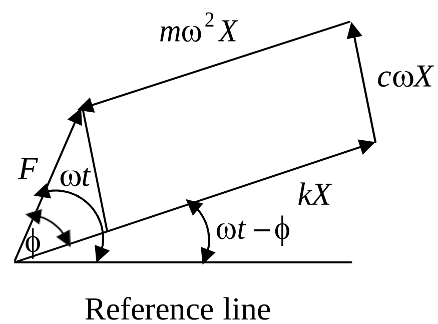


Figure 2: Force polygon

The steady state response of the system can be determined by solving equation (1) in may different ways. Here a simpler graphical method is used which will give physical understanding to this dynamic problem. From solution of differential equations it is known that the steady state solution (particular integral) will be in the form

$$x = X\sin(\omega t - \varphi) \qquad (2)$$

As each term of equation (1) represents a forcing term viz., first term represent the inertia force, second term the spring force, third term the damping force and term in the right hand side is the applied force, one may draw a close polygon as shown in figure 2 considering the equilibrium of the system under the action of these forces. Considering equation (2),

- spring force = $kX\sin(\omega t - \phi)$
- damping force = $c\omega X\cos(\omega t - \phi)$

- inertia force = $-m\omega^2 X \sin(\omega t-\phi)$

From Figure 2

$$X=\frac{F}{\sqrt{\left(k-m\omega^2\right)^2+\left(c\omega\right)^2}}$$

$$\phi=\tan^{-1}\frac{c\omega}{k-m\omega^2}$$

$$X=\frac{F/k}{\sqrt{\left(1-\frac{m}{k}\omega^2\right)^2+\left(\frac{c\omega}{k}\right)^2}} \qquad (3)$$

$$\Rightarrow \frac{Xk}{F}=\frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]+\left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2}} \qquad (4)$$

$$\omega_n=\sqrt{\frac{k}{m}}$$

$c_c=2m\omega_n$ = Critical damping

$\zeta=\frac{c}{c_c}$ = damping factor or damping ratio

$$\frac{c\omega}{k}=\frac{c}{c_c}*\frac{c_c}{k}=2\zeta\frac{\omega}{\omega_n}$$

$$\frac{c_c}{k}=\frac{2m\omega_n}{k}=\frac{2\omega_n}{\omega_n^2}=\frac{2}{\omega_n}$$

$$\tan\phi=\frac{\left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]}{1-\left(\frac{\omega}{\omega_n}\right)^2} \qquad (5)$$

As the ratio F/k is the static deflection (Xo)f the spring, $Xk/F=X/X_0$ is known as the magnification factor or amplitude ratio of the system. Figure 3 shows the magnification ~ frequency ratio and phase angle (ϕ)~ frequency ratio plot. It is clear that for undamped system the magnification factor tends to infinity when the frequency of external excitation equals natural frequency of the system. But for underdamped systems the maximum amplitude of excitation has a definite value and it occurs at a frequency $\frac{\omega}{\omega_n}<1.$ For frequency of external excitation very less than the natural frequency of the system, with increase in frequency ratio, the dynamic deflection (X) dominates the static deflection (X_0), the magnification factor increases till it reaches a maximum value at resonant frequency after which the magnification factor decreases and for very high value of frequency ratio (say $\frac{\omega}{\omega_n}>2$, the vibration is very much attenuated.

One may observe that with increase in damping ratio, the resonant response amplitude decreases.

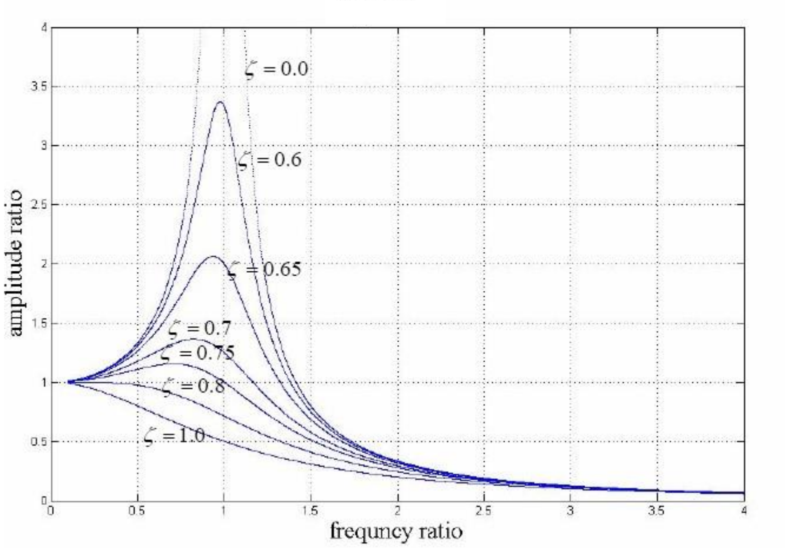


Figure 2

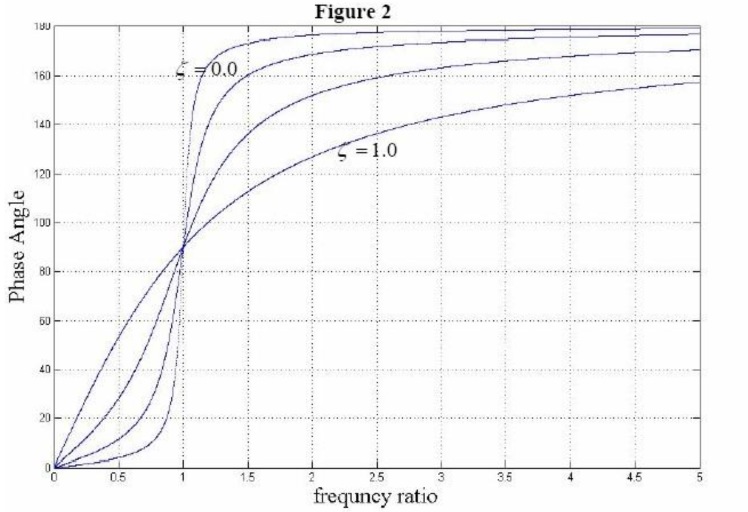


Figure 3: Magnification factor ~ frequency ratio and phase angle ~frequency ratio for different damping ratio.

So for a underdamped system the total response of the system which is the combination of transient response and steady state response can be given by

$$x(t) = x_1 e^{-\zeta\omega_n t} \sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \phi_1\right) + \frac{F_0}{k} \frac{\sin(\omega t - \phi)}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2}} \quad (6)$$

It may be noted that as $t \to \infty$, the first part of equation (6) tends to zero and second part remains. From phase angle ~frequency ratio plot it is clear that, for very low value of frequency ratio, phase angle tends to zero and at resonant frequency it is 90^0 and for very high value of frequency ratio it is 180^0.

Example 1: Find the resonant frequency ratio (value of frequency ratio for which the steady state response will be maximum) for a spring-mass-damper system.

Solution: The steady state solution for a single degree of freedom system can be given by

$$X = \frac{F}{\sqrt{\left(K - m\omega^2\right)^2 + \left(c\omega\right)^2}}$$

$$= \frac{F/K}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

$$= \frac{F/K}{\sqrt{\left(1 - r^2\right)^2 + \left(2\zeta r\right)^2}} \quad \text{where, } r = \frac{\omega}{\omega_n}.$$

X will be maximum if the denominator is minimum.

Hence $\frac{d}{dr}\left(\left(1-r^2\right)^2 + \left(2\zeta r\right)^2\right) = 0$

Or, $\frac{d}{dr}\left(1 + r^4 - 2r^2 + 4\zeta^2 r^2\right) = 4r^3 - 4r + 8\zeta^2 r = 0$

Hence, r = 0 or, $r^2 = 1 - 2\zeta^2$ or, $\underline{\underline{r = \sqrt{1-2\zeta^2}}}$ (7)

For $r = \sqrt{1-2\zeta^2}$, $X_{max} = \frac{F/K}{\sqrt{\left(1-1+2\zeta^2\right)^2 + 4\zeta^2\left(1-2\zeta^2\right)}}$

$$= \frac{F/K}{\sqrt{4\zeta^4 + 4\zeta^2 - 8\zeta^4}} = \frac{F/K}{\sqrt{4\zeta^2 - 4\zeta^4}}$$

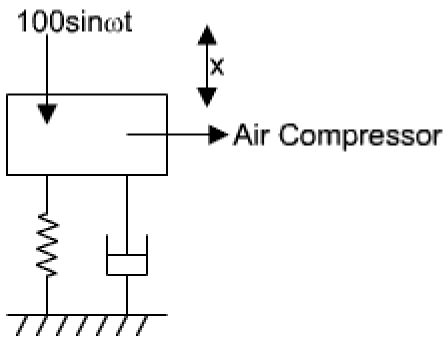
$$= \frac{F/K}{2\zeta\sqrt{1-\zeta^2}} \quad (8)$$

So the peak magnification factor $== \frac{1}{2\zeta\sqrt{1-\zeta^2}}$ which occur at a frequency ratio of

$r = \sqrt{1-2\zeta^2}$. Hence for underdamped system, it occurs when the external excitation frequency is slightly less than the natural frequency.

Example 2: An air compressor of mass 100 kg mounted on an elastic foundation. It has been observed that, when a harmonic force of amplitude 100N is applied to the compressor, the maximum steady state displacement of 5 mm occurred at a frequency of 300 rpm. Determine the equivalent stiffness and damping constants of the foundation.

Sol: The air compressor can be represented as a spring mass damper system as shown in figure below.



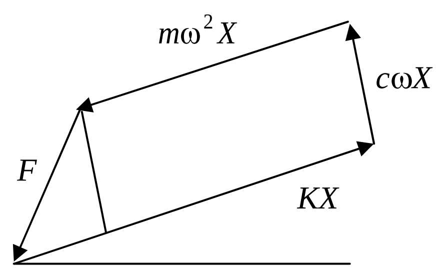
X = Steady state displacement = 5 mm
F = Forcing amplitude = 100 N
ω_{max} = frequency for max displacement = $\frac{2\pi \; x \; 300}{60} = 10\pi$ rad/s.
We have to determine K_{eq} and $C_{eq.}$

The system can be modeled as a single dof system as shown in the above figure and the steady state solution can be given by

$$X = \frac{F}{\sqrt{\left(K - m\omega^2\right)^2 + (c\omega)^2}}$$

$$= \frac{F/K}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

$$= \frac{F/K}{\sqrt{\left(1-r^2\right)^2 + (2\zeta r)^2}}$$



X will be maximum if the denominator is minimum.

Hence $\frac{d}{dr}\left(\left(1-r^2\right)^2 + (2\zeta r)^2\right) = 0$

Or, $\frac{d}{dr}\left(1 + r^4 - 2r^2 + 4\zeta^2 r^2\right) = 4r^3 - 4r + 8\zeta^2 r = 0$

Or, r = 0 or, $r^2 = 1-2\zeta^2$ or, $\underline{\underline{r = \sqrt{1-2\zeta^2}}}$

For $r = \sqrt{1-2\zeta^2}$, $X_{max} = \dfrac{F/K}{\sqrt{\left(1-1+2\zeta^2\right)^2 + 4\zeta^2\left(1-2\zeta^2\right)}}$

$$= \frac{F/K}{\sqrt{4\zeta^4 + 4\zeta^2 - 8\zeta^4}} = \frac{F/K}{\sqrt{4\zeta^2 - 4\zeta^4}}$$

$$= \frac{F/K}{2\zeta\sqrt{1-\zeta^2}}$$

So given

$X_{max} = 5\times10^{-3} = \dfrac{F/K}{2\zeta\sqrt{1-\zeta^2}}$ $\qquad \omega_n^2 = K/M = K/100$

Also, $r = \dfrac{\omega}{\omega_n} = \dfrac{10\pi}{\sqrt{K/100}} = \sqrt{1-2\zeta^2}$

$\Rightarrow 5\times10^{-3} = \dfrac{100}{K2\zeta\sqrt{1-\zeta^2}}$

and $\dfrac{10000\pi^2}{K} = \sqrt{1-2\zeta^2}$

or, $K = \dfrac{100}{5\times10^{-3}2\zeta\sqrt{1-\zeta^2}} = \dfrac{10000\pi^2}{\sqrt{1-2\zeta^2}}$

or, $\dfrac{1}{\zeta^2\left(1-\zeta^2\right)} = \dfrac{\pi^4}{\left(1-2\zeta^2\right)^2}$

or, $1+4\zeta^4-4\zeta^2-\pi^4\left(\zeta^2-\zeta^4\right)=0$

or, $\left(4+\pi^4\right)\zeta^4-\left(4+\pi^4\right)\zeta^2+1=0$

or, $101.4091\zeta^4 - 101.4091\zeta^2 + 1 = 0$

or, $\zeta^2 = \dfrac{101.4091 \pm \sqrt{(101.4091)^2 - 4(101.4091)(1)}}{2\times101.4091}$

$= 9.9603 \times 10^{-3}$ or $\zeta = 0.0998$

$K = \dfrac{100\times10^{-3}}{10\zeta\sqrt{1-\zeta^2}} = \dfrac{10^4\pi^2}{1-2\zeta^2} = \dfrac{10^4\pi^2}{1-2(0.0998)^2}$

$=100.7 \times 10^3$ N/m. = 100.7 KN/m.

$C = 2\times0.0998\times100\times\sqrt{\dfrac{100.7\times10^3}{100}}$ where $\dfrac{C}{m} = 2\zeta\omega_n$

= **633.396 N.S/m Ans.**

<u>Rotating Unbalance</u>

One may find many rotating systems in industrial applications. The unbalanced force in such a system can be represented by an eccentric mass *m* with eccentricity *e*, which is rotating with angular velocity ω as shown in Figure 4.

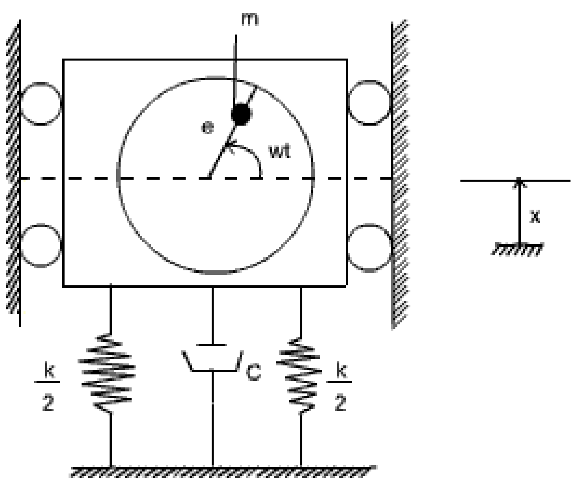


Figure 4. Vibrating system with rotating unbalance.

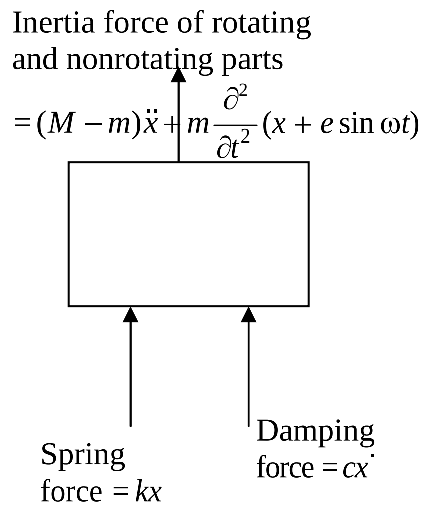


Figure 5. Freebody diagram of the system

Let x be the displacement of the nonrotating mass (*M-m*) from the static equilibrium position, then the displacement of the rotating mass *m* is *x + e sin*ω *t.*

From the freebody diagram of the system shown in figure 5, the equation of motion is

$$(M-m)\ddot{x} + m\frac{\partial^2}{\partial t^2}(x + e\sin\omega t) + kx + c\dot{x} = 0 = 0 \qquad (9)$$

or $M\ddot{x} + kx + c\dot{x} = me\omega^2 \sin\omega t$ (10)

This equation is same as equation (1) where *F* is replaced by $me\omega^2$. So from the force polygon as shown in figure 6

$$me\omega^2 = \sqrt{\{(-M\omega^2 + k)^2 + c\omega^2\}X^2} \qquad (11)$$

$$\text{or,}\ X = \frac{me\omega^2}{\sqrt{(k - M\omega^2)^2 + (c\omega)^2}} \qquad (12)$$

$$\text{or,}\ \frac{X}{e} = \frac{m\omega / M}{\sqrt{(\frac{k}{M} - \omega^2)^2 + (\frac{c}{M}\omega)^2}} \qquad (13)$$

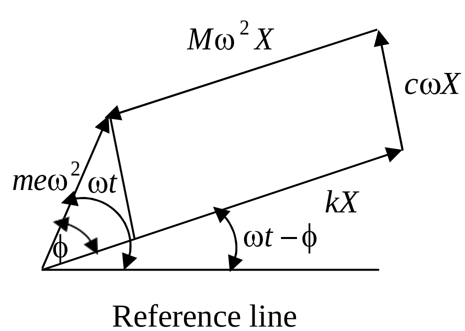


Figure 6: Force polygon

or, $\frac{X}{e}\frac{M}{m} = \frac{\omega/\omega_n}{\sqrt{(1-(\frac{\omega}{\omega_n})^2)^2 + (\frac{2\zeta\omega}{\omega_n})^2}}$ (14)

and $\tan\phi = \frac{2\zeta\frac{\omega}{\omega_n}}{1-(\frac{\omega}{\omega_n})^2}$ (15)

So the complete solution becomes

$$x(t) = x_1 e^{-\zeta\omega_n t}\sin(\sqrt{1-\zeta^2}\,\omega_n t + \phi_1) + \frac{me\omega^2}{\sqrt{(k-M\omega^2)^2 + (c\omega^{2)}}}\sin(\omega t - \phi)$$

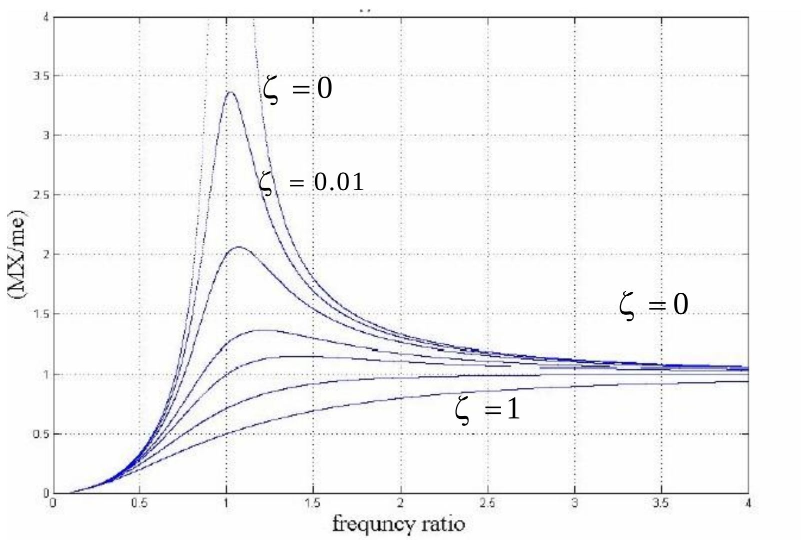


Figure 7: $\frac{MX}{me} \sim \frac{\omega}{\omega_n}$ plot for system with rotating unbalance

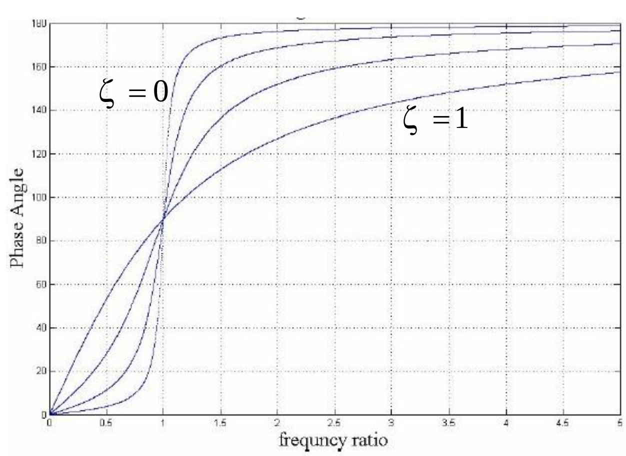


Figure 8 Phase angle ~ frequency ratio plot for system with rotating unbalance

It may be noted from figure 8 that, for a system with very very low damping, it is very unsafe to run the machine near the natural frequency ratio greater than 2, the system vibration reduces to $X = me/M$ and phase angle tends to 180^0.

Whirling of shaft:

Whirling is defined as the rotation of the plane made by the bent shaft and the line of the centre of the bearing. It occurs due to a number of factors, some of which may include (i) eccentricity, (ii) unbalanced mass, (iii) gyroscopic forces, (iv) fluid friction in bearing, viscous damping.

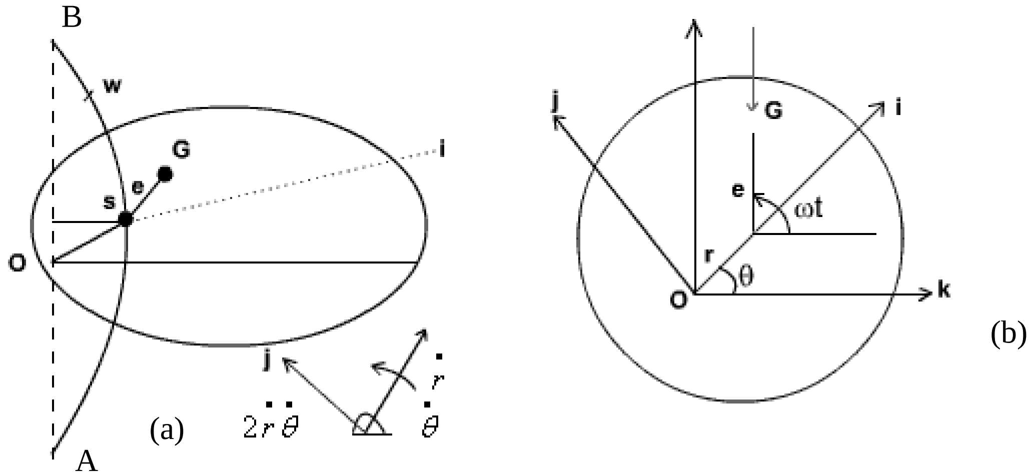


Figure 9: Whirling of shaft

Consider a shaft AB on which a disc is mounted at **s**. **G** is the mass center of the disc, which is at a distance ***e*** from **s.** As mass center of the disc is not on the shaft center, when the shaft rotates, it will be subjected to a centrifugal force. This force will try to bend the shaft. Now the shafts neutral axis, which is represented by line ASB, is different from the line joining the bearing centers AOB. The rotation of the plane containing the line joining bearing centers and the bend shaft (in this case it is AOBsA) is called the whirling of the shaft.

Considering unit vectors **i**, **j**, k as shown in the above figure 9(b), the acceleration of point G can be given by

$a_G = a_S + a_{G/S}$

$$=\left[\ddot{r}-r\dot{\theta}^2-e\omega^2\cos(\omega t-\theta)\right]\boldsymbol{i}+\left[r\ddot{\theta}-e\omega^2\sin(\omega t-\theta)+2\dot{r}\dot{\theta}\right]\boldsymbol{j} \tag{16}$$

which is acting along radial direction k, which will give rise to restoring torque, assuming a viscous damping for a to be acting at S. The EOM in radial direction

$$m\left[\ddot{r}-r\dot{\theta}^2-e\omega^2\cos(\omega t-\theta)\right]+kr+C\dot{r}=0 \tag{17}$$

$$m\left[r\ddot{\theta}+2\dot{r}\dot{\theta}-e\omega^2\sin(\omega t-\theta)\right]+cr\dot{\theta}=0 \tag{18}$$

$$\ddot{r}+\frac{c}{m}\dot{r}+\left(\frac{k}{m}-\dot{\theta}^2\right)=e\omega^2\cos(\omega t-\theta) \tag{19}$$

$$r\ddot{\theta}+\left(\frac{c}{m}r+2\dot{r}\right)\dot{\theta}=e\omega^2\sin(\omega t-\theta) \tag{20}$$

Considering the synchronous whirl case, i.e., $\dot{\theta}=\omega$

So, $\theta=(\omega t-\phi)$ (21)

ϕ is a phase angle between e and r.

Taking $\ddot{\theta}=\ddot{r}=\dot{r}=0$, from equation (19)

$$\left(\frac{k}{m}-\omega^2\right)=e\omega^2\cos\phi \tag{22}$$

$$\& \quad \frac{c}{m}r\omega=e\omega^2\sin\phi \tag{23}$$

$$\Rightarrow \tan\phi=\frac{\frac{c}{m}\omega}{\left(\frac{k}{m}-\omega^2\right)}=\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2} \tag{24}$$

$\omega_n=\sqrt{\frac{k}{m}}$ and $\zeta=\frac{c}{c_e}$

$$\cos\phi = \frac{\frac{c}{m}\omega}{\sqrt{\left(\frac{k}{m}-\omega^2\right)^2+\left(\frac{c}{m}\omega\right)^2}} \qquad (25)$$

Now equation (22) reduces to

$$\left(\frac{\mathrm{k}}{\mathrm{m}}-\omega^2\right)r = e\omega^2\frac{\left(\frac{k}{m}-\omega^2\right)}{\sqrt{\left(\frac{k}{m}-\omega^2\right)^2+\left(\frac{c}{m}\omega\right)^2}} = \frac{e\omega^2}{\sqrt{(\omega_n{}^2-\omega^2)+(2\zeta\frac{\omega}{\omega_n})^2}} \qquad (26)$$

$$r = \frac{me\omega^2}{\sqrt{(k-m\omega^2)^2+\left(c\omega^2\right)^2}} \qquad (27)$$

$$\text{or,}\ \frac{r}{e} = \frac{\omega^2/\omega_n{}^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2}} \qquad (28)$$

$$\tan\phi = \frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2} \qquad (29)$$

The eccentricity line e=SG leads the displacement line r = OS by phase angle ϕ which depends on the amount of damping and the rotation speed ratio ω/ω_n. When the rotational speed equals to the natural frequency or critical speed, the amplitude is restrained by damping only. From equation (29 at very high speed $\omega >> \omega_n, \phi \to 180^0$ and the center of mass G tends to approach the fixed point O and the shaft center S rotates about it in a circle of radius e.

Support Motion:

Many machine components or instruments are subjected to forces from the support. For example while moving in a vehicle, the ground undulation will cause vibration, which will be transmitted, to the passenger. Such a system can be modeled by a spring-mass damper system as shown in figure 10. Here the support motion is considered in the form of $y = Y\sin\omega t$, which is transmitted to mass *m*, by spring (stiffness k) and damper (damping coefficient c). Let x be the vibration of mass about its equilibrium position.

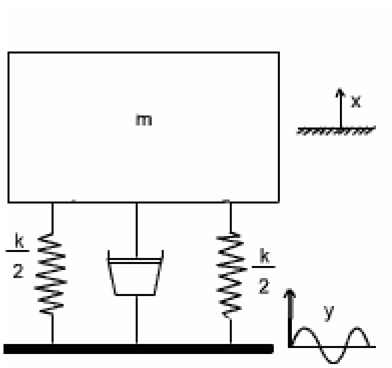


Figure 10: A system subjected to support motion

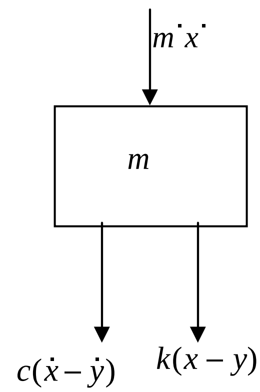


Figure 11: Freebody diagram

Now to derive the equation of motion, from the freebody diagram of the mass as shown figure 11

$$m\ddot{x} = -k(x-y) - c(\dot{x}-\dot{y}) \quad (30)$$

let z=x-y (31)

$$m\ddot{z} + kz + c\dot{z} = -m\ddot{y} = m\omega^2 y\sin \quad (32)$$

$$m\ddot{z} + kz + c\dot{z} = m\omega^2 y\sin\omega t \quad (33)$$

As equation (33) is similar to equation (1), solution of equation (33) can be written as

$$z = Z\sin(\omega t - \phi) \quad (34)$$

$$Z = \frac{m\omega^2 y}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}} \text{ and } \tan\phi = \frac{c\omega}{k-m\omega^2} \quad (35)$$

If the absolute motion x of the mass is required, we can solve for $x = z + y$. Using the exponential form of harmonic motion

$$y = Ye^{i\omega t} \tag{36}$$

$$z = Ze^{i(\omega t - \phi)} = (Ze^{-i\phi})e^{i\omega t} \tag{37}$$

$$x = Xe^{i(\omega t - \psi)} = (Xe^{-i\psi})e^{i\omega t} \tag{38}$$

Substituting equation (38) in (30) one obtains

$$\left\{ m(Ze^{-i\phi})\omega^2 + k(Ze^{-i\phi}) + ci\omega(Ze^{-i\phi}) \right\} e^{i\omega t} = m\omega^2 Ye^{i\omega t} \tag{39}$$

$$Ze^{-i\phi}(k - m\omega^2 + ic\omega) = m\omega^2 Y \tag{40}$$

$$Ze^{-i\phi} = \frac{m\omega^2 Y}{k - m\omega^2 + ic\omega} \tag{41}$$

$$x = (Ze^{-i\phi} + Y)e^{i\omega t} \tag{42}$$

$$x = \left(\frac{k - m\omega^2 + ic\omega + m\omega^2}{k - m\omega^2 + ic\omega}\right)Ye^{i\omega t}$$

$$= X(\cos\psi - i\sin\psi)e^{i\omega t} \tag{43}$$

The steady state amplitude and Phase from this equation are

$$\left|\frac{X}{Y}\right| = \sqrt{\frac{k^2 + (c\omega)^2}{(k - m\omega^2)^2 + (c\omega)^2}} \tag{44}$$

$$\tan\psi = \frac{mc\omega^3}{k(k - m\omega^2) + (c\omega)^2} \tag{45}$$

$$\left|\frac{X}{Y}\right| = \sqrt{\frac{1 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2}} \tag{46}$$

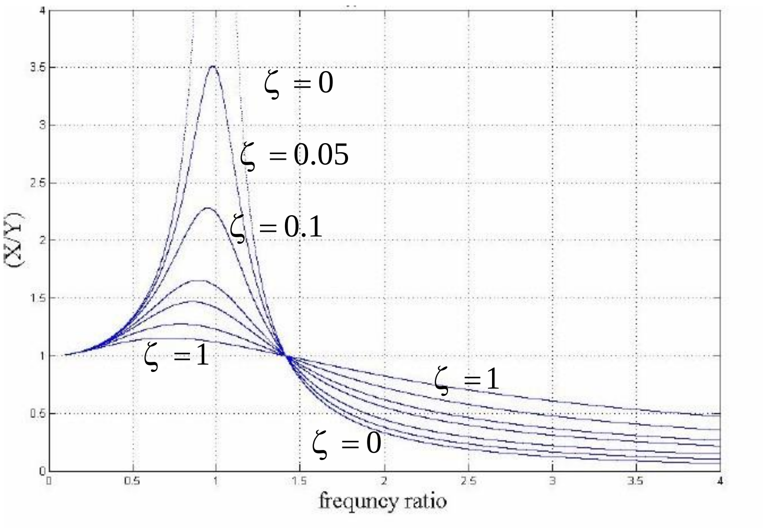


Figure 12: Amplitude ratio ~ frequency ratio plot for system with support motion

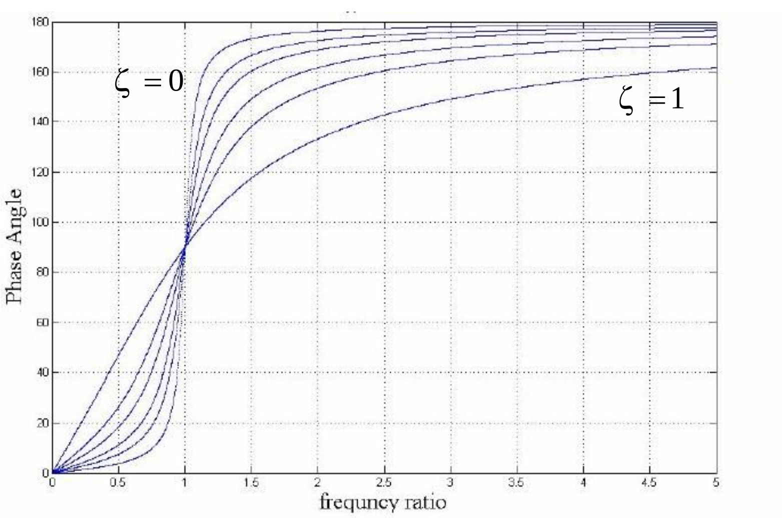


Figure 13: Phase angle ~ frequency ratio plot for system with support motion

From figure 12, it is clear that when the frequency of support motion nearly equal to the natural frequency of the system, resonance occurs in the system. This resonant amplitude decreases with increase in damping ratio for $\frac{\omega}{\omega_n} < \sqrt{2}$. At $\frac{\omega}{\omega_n} = \sqrt{2}$, irrespective of damping factor, the mass vibrate with an amplitude equal to that of the support and for $\frac{\omega}{\omega_n} > \sqrt{2}$, amplitude ratio becomes less than 1, indicating that the mass will vibrate with an amplitude less than the support motion.

But with increase in damping, in this case, the amplitude of vibration of the mass will increase. So in order to reduce the vibration of the mass, one should operate the system at a frequency very much greater than $\sqrt{2}$ times the natural frequency of the system. This is the principle of vibration isolation.

Vibration Isolation:

In many industrial applications, one may find the vibrating machine transmit forces to ground which in turn vibrate the neighbouring machines. So in that contest it is necessary to calculate how much force is transmitted to ground from the machine or from the ground to the machine.

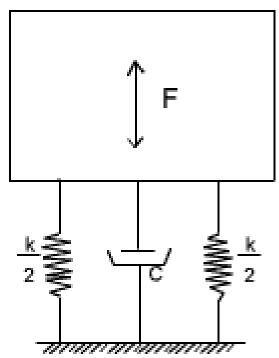


Figure 13 : A vibrating system

Figure 13 shows a system subjected to a force $F = F_0 \sin\omega t$ and vibrating with $x = X\sin(\omega t - \phi)$. This force will be transmitted to the ground only by the spring and damper.

Force transmitted to the ground

$$F_t = \sqrt{(KX)^2 + (c\omega X)^2} = KX\sqrt{1 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2} \tag{47}$$

It is known from equation (3) that for a disturbing force $F = F_0 \sin\omega t$, the amplitude of resulting oscillation

$$X = \frac{F_0 / K}{\sqrt{[1 - (\frac{\omega}{\omega_n})^2]^2 + (\frac{2\zeta\omega}{\omega_n})^2}} \tag{48}$$

Substituting equation (48) in (47) and defining the transmissibility TR as the ratio of the force transmitted Force to the disturbing force one obtains

$$\left|\frac{F_t}{F_0}\right| = \sqrt{\frac{1+\left(\frac{2\zeta\omega}{\omega_n}\right)^2}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left(\frac{2\zeta\omega}{\omega_n}\right)^2}} \qquad (49)$$

Comparing equation (49) with equation (46) for support motion, it can be noted that

$$\text{TR}=\left|\frac{F_t}{F_0}\right|=\left|\frac{X}{Y}\right| \qquad (50)$$

When damping is negligible

$$TR=\frac{1}{\left(\frac{\omega}{\omega_n}\right)^2-1}, \qquad (51)$$

$\frac{\omega}{\omega_n}$ to be used always greater than $\sqrt{2}$

Replacing $\omega_n^2 = g/\Delta$

$$TR=\frac{1}{(2\pi f)^2\frac{\Delta}{g}-1} \qquad (52)$$

To reduce the amplitude *X* of the isolated mass m without changing TR, *m* is often mounted on a large mass M. The stiffness *K* must then be increased to keep ratio *K/(m+M)* constant. The amplitude X is, however reduced, because *K* appears in the denominator of the expression

$$X=\frac{F_0/K}{\sqrt{[1-(\frac{\omega}{\omega_n})^2]^2+(\frac{2\zeta\omega}{\omega_n})^2}} \qquad (53)$$

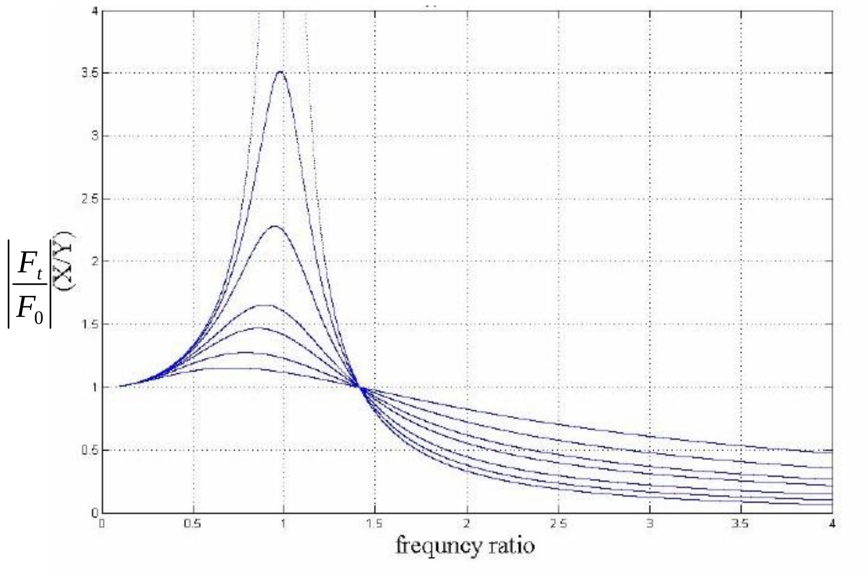


Figure 14: Transmissibility ~frequency ratio plot

Figure 14 shows the variation transmissibility with frequency ratio and it can be noted that vibration will be isolated when the system operates at a frequency ratio higher than $\sqrt{2}$.

Equivalent Viscous Damping:

In the previous sections, it is assumed that the energy dissipation takes place due to viscous type of damping where the damping force is proportional to velocity. But there are systems where the damping takes place in many other ways. For example, one may take surface to surface contact in vibrating systems and take Coulomb friction into account. Also in many cases energy is dissipated in joints also, which is a form of structural damping. In these cases one may still use the derived equations by considering an equivalent viscous damping. This can be achieved by equating the energy dissipated in the original and the equivalent system.

The primary influence of damping on the oscillatory systems is that of limiting the amplitude at resonance. Damping has little influence on the response in the frequency regions away from resonance. In case of viscous damping, the amplitude at resonance is

$$X = \frac{F_0}{c\omega_n} = \frac{F_0}{2\zeta k} \tag{54}$$

For other type of damping, no such simple expression exists. It is possible to however, to approximate the resonant amplitude by substituting an equivalent damping *Ceq* in the foregoing equation.

The equivalent damping *Ceq* is found by equating the energy dissipated by the viscous damping to that of the nonviscous damping with assumed harmonic motion.

$$\pi Ceq\omega X^2 = W_d \tag{55}$$

Where W_d must be evaluated from the particular type of damping.

Structural Damping:

When materials are cyclically stressed, energy is dissipated internally within the material itself. Experiments by several investigators indicate that for most structural metals such as steel and aluminum, the energy dissipated per cycle is independent of the frequency over a wide frequency range and proportional to the square of the amplitude of vibration. Internal damping fitting this classification is called solid damping or structural damping. With the energy dissipation per cycle proportional to the square of the vibration amplitude, the loss coefficient is a constant and the shape of the hysteresis curve remains unchanged with amplitude and independent of the strain rate. Energy dissipated by structural damping can be written as

$$W_d = \alpha X^2 \tag{56}$$

Where α is a constant with units of force displacement.

By the concept of equivalent viscous damping

$$W_d = \alpha X^2 = \pi c_{eq} \omega X^2 \qquad \text{or,} \quad c_{eq} = \frac{\alpha}{\pi\omega} \tag{57}$$

Coulomb Damping:

Coulomb damping is mechanical damping that absorbs energy by sliding friction, as opposed to viscous damping, which absorbs energy in fluid, or viscous, friction. Sliding friction is a constant value regardless of displacement or velocity. Damping of large complex structures with non-welded joints, such as airplane wings, exhibit coulomb damping.

Work done per cycle by the Coulomb force F_d

$$W_d = 4F_d X \tag{58}$$

For calculating equivalent viscous damping

$$\pi Ceq\omega X^2 = 4F_d X \tag{59}$$

From the above equation equivalent viscous damping is found

$$c_{eq} = \frac{4F_d}{\pi\omega X} \tag{60}$$

Summary

Some important features of steady state response for harmonically excited systems are as follows-

- The steady state response is always of the form $x(t) = X \sin(\omega t - \phi)$. Where it is having same frequency as of forcing. X is amplitude of the response, which is strongly dependent on the frequency of excitation, and on the properties of the spring—mass system.
- There is a phase lag ϕ between the forcing and the system response, which depends on the frequency of excitation and the properties of the spring-mass system.
- The steady state response of a forced, damped, spring mass system is independent of initial conditions

In this chapter response due to rotating unbalance, support motion, whirling of shaft and equivalent damping are also discussed.

Exercise Problems

1. An underdamped shock absorber is to be designed for a motor cycle of mass 200Kg. When the shock absorber is subjected to an initial vertical velocity due to a road bump, the resulting displacement-time curve is to be as indicated in fig(b). Find the necessary stiffness and damping constants of the shock absorber if the damped period of vibration is to be 2 s and amplitude x_1 is to be reduced to one-fourth in one half cycle (i.e $x_{1.5}=x_1/4$). Also find the minimum initial velocity that leads to a maximum displacement of 250 mm.
2. Develop equation of motion for a spring mass system with Coulomb damping.
3. An electronic instrument of mass m= 8kg is placed on four elastic support pad of special rubber. The force displacement curve of each pad is given by $F = (5x+1000x^2)*10^3$. Determine the spring constant between the instrument and ground in the vertical direction.
4. A machine of 100kg mass is supported on springs of total stiffness 700 KN/m and has an unbalanced rotating element, which results in a disturbing force of 350N at a speed of 3000 rev/min. Assuming a damping factor of $\xi = 0.20$, determine (a) its amplitude of motion due to the unbalance, (b) the transmissibility, and (c) the transmitted force.
5. Find the steady state response of the spring mass damper system to a force $F = 5\sin 4t + 10\cos 4t$.
6. If the steady state response of a linear system to a force of $F = 5\sin 2t$ is $4\sin(2t + 0.25)$, what will be the response if a force of $F = 10\sin 2t$ will act on it..

Computer Assignment

1. Develop a general-purpose program, to find the free vibration response of a viscously damped system. Use the program to find response of a system with m= 450 Kg, K= 26519.2, c= 1000.0, x_0 =0.539657, v_0(initial velocity)=1.0.
2. Find the free vibration response of a critically damped and over damped system with the above mentioned values of m and k.
3. Plot magnification factor vs. frequency ratio and |X/Y| or $|F_t/F_0|$ for different values of $\frac{\omega}{\omega_n}$