ASTR 391 Lecture Notes
Phyical Astronomy, Spring 2020

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Contents

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1 Introduction to Astrophysics

1.1 *Observations and Observables*

Astronomy involves the observation of distant objects beyond Earth: from low-orbit spy satellites to our own Solar System to our Milky Way galaxy to other distant galaxies and out to the observable edge of the universe. A non-exhaustive list of some of the types of objects that are observed includes:

1. Planets and moons in our own Solar System
2. Stars (including our Sun)
3. Planets orbiting other stars
4. Remnants of 'dead' stars: white dwarfs, neutron stars, and black holes
5. Giant, cool clouds of gas and dust
6. Other galaxies beyond our Milky Way
7. Diffuse, hot gas: between stars, and between galaxies
8. The overall structure of the universe.

These observations are made using a variety of different techniques. Most frequent is the detection of electromagnetic radiation: everything from high-energy gamma- and X-rays, through ultraviolet, visible, and infrared light, and down to microwave and radio waves. When people speak of **multi-messenger astronomy**, they mean observations beyond merely electromagnetic detections. These other approaches involve the direct or indirect detection of high-energy, particles (i.e., not photons) such as the solar wind, cosmic rays, or neutrinos from . The newest set of observations includes the detection of gravitational waves from distant, massive, rapidly-rotating objects.

Because astronomy is an observationally-driven field, big advances and new discoveries often occur whenever technological capabilities improve substantially. Ancient astronomers, from well before Hipparchus down to Tycho Brahe, could only rely on what their own, unaided eyes could see. That changes with the invention of the telescope: astronomers still had to use their own eyes, but now they could see finer details (because of optical magnification) and study fainter objects (a telescope lens is larger than your eye's pupil, so it collects more light).

The next big revolution was astrophotography: a photographic setup can sit collecting light from a faint source for minutes or even hours, so much fainter and/or more distant objects could be studied than by just peering through a telescope. In the last century, the development of photoelectric detectors — first as 'single-pixel' devices and later as mega- or giga-pixel optical CCDs or infrared array detectors — has had at least as big an impact, by virtue of their dramatically enhanced sensitivity compared to photography. More recently still, other new technologies have also emerged such as interferometry to give the sharpest possible images at radio to infrared wavelengths, or adaptive optics which achieves something similar in the optical and infrared.

1.2 *Astronomy, Astrophysics, and Historical Baggage*

Astrophysics: effort to understand the nature of astronomical objects. Union of quite a few branches of physics — gravity, E&M, stat mech, quantum, fluid dynamics, relativity, nuclear, plasma — all matter, and have impact over a wide range of length and time scales.

Astronomy: providing the observational data upon which astrophysics is built. Thousands of years of history, with plenty of intriguing baggage.

Sexagesimal notation

Sexagesimal notation is one example. This means a Base-60 number system, and it qoriginated in Sumer in $\sim$3000 BC. Origin uncertain (how could it not be?), but we still use this today for time and angles: 60″ (arc-seconds) in 1′ (arc-minute), 60′ in 1^o , 360^o in one circle.

We use this angular notation to express where an object is on the Earth's surface, as well as where an astronomical object is on the sky. On Earth, this is through the latitude/longitude system – just a form of spherical coordinates, in which longitude corresponds to ϕ (the angle around the planet from the Greenwich meridian) and latitude corresponds to θ (the angle from the equator). **Celestrial coordinates** use a related approach: here, **Right Ascension** (RA or α) plays the role of ϕ or longitude (describing the angle from a point defined by the Earth's orbit) and **Declination** (Dec or δ) plays the role of θ or latitude (describing the angle north or south from a projection of the Earth's equator).

Further complicating things, all stars are in motion and so a star's coordinates slowly change over time. One subtle effect here is due to slow evolutions in the Earth's orbit. But for nearby stars, the largest effect is indeed due to their intrinsic motion; this motion of stars across the sky is called **proper motion**. The closest, fastest stars have proper motions of several arcsec per year, but proper motion is more usually measured in milliarcsec per year, or mas yr^{-1}. Sources far away (or especially, outside the Milky Way) have essentially zero proper motion.

The classic astronomer's website for learning the coordinates, proper motion, and many other useful details of a given star is **SIMBAD**[1]. For the more well-known stars, Wikipedia isn't a terrible source either.

The Magnitude System

Magnitudes are an even more notorious example of the weight of historical tradition. These aren't especially logical, but all astronomers use them through tradition (and force of habit) as a standard way of indicating the intrinsic and apparent brightness of stars and galaxies.

The magnitude system is originally based on the human eye by Hipparchus of Greece ($\sim$135 BC), who divided visible stars into six primary brightness bins. This arbitrary system continued for $\sim$2000 years, and it makes it fun to read old astronomy papers ("I observed a star of the first magnitude,"

[1] `http://simbad.u-strasbg.fr/simbad/sim-fid`

etc.). This was revised and made somewhat more quantitative by Pogson in 1856, who semi-arbitrarily decreed that a one-magnitude difference between two objects means that one is $\sim$2.512$\times$ brighter than the other. This is only an approximation to how the eye works! To further confuse things, thanks to Hipparchus smaller magnitudes mean brighter stars: so the system 'feels backwards.'

So given two stars with apparent brightnesses b_1 and b_2, their **apparent** (i.e., relative) **magnitudes** are related by:

$$(1) \quad m_1 - m_2 = -2.5 \log_{10} \frac{b_1}{b_2}.$$

So a 2.5 mag difference means one object is 10$\times$ brighter than the other. Not very intuitive! However, it turns out that a 2.5$\times$ brightness difference also roughly corresponds to a *sim*1 mag difference – a nice coincidence. For smaller variations milli-magnitudes (mmag) are sometimes uesd, and a 1 mmag = 10^{-3} mag difference corresponds roughly to one object being 1.001$\times$ brighter — so 1 mmag is about a part-in-a-thousand brightness difference.

An alternative way of presenting the apparent magnitude system is by defining some reference brightness and defining observed magnitudes relative to that brightness level. In the typical (Vega magnitude) system we will use in this class, the reference level is the brightness of the star Vega. If we call this reference brightness b_0, then the apparent magnitudes are defined by

$$(2) \quad m = -2.5 \log_{10} \frac{b/b_0}{.}$$

Absolute magnitudes refer not to how bright objects *look* to an observer, but to their intrinsic brightness (i.e., luminosities). We need a reference point, so we say that an object's absolute and apparent magnitudes are the same if the object is 10 pc away from us (again, this is arbitrary – and doesn't make so much sense for galaxies that are much larger than 10 pc!). Also, if a star is (e.g.) 5$\times$ further away its light will be spread over 5^2 more surface area throughout the universe. The absolute magnitude M of a a star or galaxy is

$$(3) \quad m\text{–}M = 2.5 \log_{10} \left(\frac{d}{10 \text{ pc}} \right)^2 = 5 \log 10 \frac{d}{10 \text{ pc}}.$$

We will see later than absolute magnitudes are directly related to the **luminosity** (L, the true intrinsic brightness) of a star in the same way that apparent magnitudes are related to apparent brightness:

$$(4) \quad M_1 - M_2 = -2.5 \log_{10} \frac{L_1}{L_2}.$$

Magnitudes can be either

- **bolometric** — relating the total electromagnetic power (i.e. luminosity) of the object (of course, we can never actually measure this unless we have detectors operating across an infinite wavelength/frequency range

– we need models!)

- **wavelength-dependent**, in which case the magnitude only relates the power in a specific wavelength range

Finally, to confuse things just a bit more there are two different kinds of magnitude systems. These make qualitatively different assumptions about how they set their zero-points, defining the magnitude of a given brightness. The older system (which you should assume in this class, unless told otherwise) is the Vega system, in which magnitudes at different wavelengths are always relative to a 10,000 K star (similar to the star Vega). The other, more modern scheme is that of **AB magnitudes**, in which a given magnitude at any wavelength always means the same flux density (a term we will define later).

1.3 *Fundamental Forces*

All of the fundamental physical forces are important for astronomy, but some are more important than others:

1. **Gravity**. By far the most important for astronomy! In this course we will consider the force of gravity more than any other force.
2. **Electromagnetism**. Also quite important, especially for energy transport via radiation but also for interactions involving charged particles and magnetic fields.
3. **Weak and Strong Nuclear Forces.** Not as critical for many situations considered in astronomy, but essential for understanding the nuclear fusion and fission that power stars and supernovae.

1.4 *Types of Particles*

- **Photons**. Electromagnetic particles that are also waves. Move at speed c, have **wavelength** λ and **frequency** ν such that $c = \lambda\nu$. Have energy $E = h\nu$ and momentum $p = E/c = h/\lambda$.
- **Protons**. Positively charged particles with mass approximately 1 amu. Found in the nuclei of all atoms, where it may be attracted to neutrons (or even other protons) via the strong nuclear force.
- **Neutrons**. Chargeless particles with mass approximately 1 amu. Found in the nuclei of all atoms except ^{1}H, where they are attracted to protons (as well as other neutrons) via the strong nuclear force. On their own, will decay relatively quickly via the weak nuclear force.
- **Electrons**. Negatively charged leptons, with $m_p/m_e \sim 1800$. Found in atoms and also found free at high temperatures when atoms become ionized.
- **Neutrinos**. Tiny, tiny, nearly-but-not-quite massless, chargeless particles. Come in three flavors (as do leptons) but in astrophysics we're mainly concerned with the **electron neutrino** ν_e and its antiparticle $\bar{\nu}_e$.

- **Antiparticles**. Every particle has a corresponding antiparticle with the same mass and opposite charge. When a particle and its antiparticle meet, they annihilate each other and release their full mass energy, $E = mc^2$. With enough energy available the reaction can be reversed and a particle and its antiparticle can be created!

- **Atoms and Ions.** The building blocks of the universe, and most of it hydrogen (with most of the rest helium).

- Molecules. Only stable at relatively cool temperatures ($\lesssim$ 4000 K), but found in planets, cool gas clouds, and the outer parts of the coolest stars.

1.5 *Concepts*

You will need to be familiar with the following (non-exhaustive) list of concepts in this course:

- **Pressure**. $P = F/A$, remember?

- **Density**. $\rho = M/V$. 'Nuff said.

- **Number density**. The number of particles per volume element, $n = N/V$. It is also related to mass density via $n = \rho/m = (M/m)1/V$.

- **Ideal gas**. We will assume most arrays of particles are ideal gases. You may be used to seeing the ideal gas law in the form of $PV = NRT$, where R is the ideal gas constant. In astronomy we often cast this instead in terms of the number density. Dividing by V, we then have $P = nk_BT$, where k_B is the Boltzmann constant.

- **Luminosity** just refers to the power output by some celestial body. Most commonly we'll refer to the luminosity of our own Sun, $L_\odot \approx 4 \times 10^{26}$ W.

- **Doppler Shift**: Waves of a given frequency and wavelength appear differently when emitted by a moving object. This applies to light waves too; at non-relativistic speeds, the observed wavelength

  $$\lambda_{obs} = \lambda_{emitted}\,(1 + v/c)\,, \tag{5}$$

  where v is the radial velocity. An object moving away from Earth has $v > 0$ and so $\lambda_{obs} > \lambda_{emitted}$: its light is redshifted. Light from objects moving toward the observer is said to be blueshifted.

- **Relativity** rears its head in several aspects. One is **mass-energy equivalence**: you've heard that $E = mc^2$. If you were to convert 10 kg of mass into energy in one second, you'd have a power (i.e., luminosity) of (10 kg)$(3 \times 10^8$ m s$^{-1})^2/1$ sec $\approx 10^{18}$ W. Only antimatter converts to energy so entirely, but even nuclear fusion and fission convert enough to be important for stars and supernovae.

- **Quantum mechanics**: We will address a few aspects of quantum mechanics; these will be introduced as we come across them. One key aspect already mentioned is the energy of a single photon, $E = h\nu$.

- **Thermodynamics**: One aspect is **thermal equilibrium**, whether things are at the same temperature. E.g., in a star there are both atoms (and ions, and free electrons) and also photons. If all of these particles have about the same temperature (i.e. thermal kinetic energy), then the mixture is in thermal equilibrium. Often this applies only in a small region (after all, a star is hotter inside than on its surface) and so in these cases we often speak of **local thermal equilibrium.** Another thermodynamic quantity we will sometimes use is that in a gas of temperature T, the average kinetic energy of a typical particle is just $\langle E \rangle = \frac{3}{2}kT$ — when doing an order-of-magnitude estimate we will sometimes approximate this further as $\langle E \rangle \approx kT$.

- **Order-of-magnitude estimation**: The ability to quickly and approximately calculate quantities of interest.

1.6 *OOMA: Order-of-Magnitude Astrophysics*

One of the key tools one should have in their toolkit is the ability to quickly and approximately estimate various quantities. Astronomy is a fun field because in many cases it's fine to calculate a quantity to within a factor of a few of the correct value. When we speak of an **order of magnitude**, we mean estimating an answer to within a factor of ten or so[2]. See your "OOMA" handout for a rundown of handy approximations to many important astronomical and physical quantities.

We'll spend a lot of time on stars, so it's important to understand some key scales to get ourselves correctly oriented. For example, consider the electron and neutron, which have $m_e \sim 10^{-30}$ kg and $m_n \sim 2 \times 10^{-27}$ kg. The ratio of these two is

$$\frac{m_n}{m_e} \approx 1800 \approx \frac{R_{\rm WD}}{R_{\rm NS}} \tag{6}$$

where $R_{\rm WD}$ and $R_{\rm NS}$ are the radii of a white dwarf and a neutron star: dead stellar remnants mainly supported by electrons and neutrons, respectively (we'll get to these more, later).

Meanwhile the mass of our Sun is $M_\odot \approx 2 \times 10^{30}$ kg. So while considering the masses involved might make it seem that objects in this course are astronomically far from the considerations of fundamental physics, this couldn't be further from the truth. In fact, many astrophysically large quantities can be almost purely derived from fundamental physical constants. E.g., the maximum possible mass of a white dwarf before it collapses under its own weight:

[2]Note that this is different from astronomical magnitudes, which we discussed previously.

$$M_{\mathrm{WD,max}} \approx \left(\frac{\hbar c}{G}\right)^{3/2} m_H^{-2} \tag{7}$$

(where m_H is the mass of a hydrogen atom), or the Schwarzschild radius (size of the event horizon) of a nonrotating black hole:

$$R_S = 2\frac{G}{c^2} M_{\mathrm{BH}}. \tag{8}$$

As another example, consider the gravitational acceleration at Earth's surface. If one can't remember that this is 9.8 m s^{-2}, we would hopefully still remember that $g \equiv GM/R^2$. If we can't recall $M_\oplus$ or $R_\oplus$, we might still remember that the Earth's circumference is $2\pi R_\oplus \approx 40,000$ km and the Earth has a mean density of about 5 g cm^{-3} = 5000 kg m^{-3} (rocks are ~5× denser than water). So

$$R_\oplus \approx \frac{40,000 \text{ km}}{2\pi} \sim \frac{40,000 \text{ km}}{6} \sim 6,500 \text{ km} \tag{9}$$

and

$$\begin{aligned} M_\oplus &= \rho_\oplus \left(\tfrac{4}{3}\pi R_\oplus^3\right) \\ &\sim \left(5 \times 10^3 \text{kg m}^{-3}\right)\left(4\left[6.5 \times 10^6 \text{ m}\right]^3\right) \\ &\sim 20 \times 10^{21} \times 250 \text{ kg} \\ &\sim 5 \times 10^{24} \text{ kg.} \end{aligned}$$

This is remarkably close (by astronomical standards) to the true value of 5.97×10^{24} kg. We could then finally estimate the surface gravity as

$$\begin{aligned} g_\oplus &= G\tfrac{M_\oplus}{R_\oplus^2} \\ &\approx \left(\tfrac{2}{3} \times 10^{-10} \text{ N m}^2 \text{ kg}^{-2}\right)\left(\tfrac{5\times10^{24} \text{ kg}}{[6.5\times10^6 \text{ m}]^2}\right) \end{aligned}$$

$$\sim \frac{2 \times 5}{3 \times 40} \times \frac{10^{-10} \times 10^{24}}{10^{12}} \; m\; s^{-2}$$

$$\sim 0.1 \times 10^2 \; m\; s^{-2} = 10 \; m\; s^{-2}.$$

Again, surprisingly close to the true value of 9.8 m s^{-2}! So some basic order-of-magnitude assumptions lead to a pretty good answer – this is often the case in astrophysics.

2 Size and Distance Scales

Space is big. Really big. So let's consider size scales in astrophysics.

2.1 *Size Scales*

- Bohr radius: the size of a hydrogen atom in Bohr's semiclassical model is $\hbar^2/m_e k_e e^2 \approx 0.5 \mathring{A} = 5 \times 10^{-11}$ m.

- Earth radius: Since the French Revolution defined $\pi R_\oplus = 20,000$ km, we have $R_\oplus \approx 6,300$ km$\approx 6.3 \times 10^6$ m.

- Solar radius: a rough rule of thumb is that $R_\odot \sim 100 R_\oplus$, and so $R_\odot \approx 7 \times 10^8$ m. (If you like gas giant planets such as Jupiter, $R_{\rm Jup} \sim 10 R_\oplus \sim 1/10 R_\odot$.)

- Astronomical unit: the distance from the Earth to the Sun. A rough rule of thumb is that 1 au$\approx 200 R_\odot$, and so 1 au$\approx 1.5 \times 10^{11}$ m. (This is also roughly 8 light-minutes.)

- Parsec: the fundamental unit of distance beyond the Solar System (no 'light years' here). Space is really empty: 1 pc$\approx 2 \times 10^5$ au! So 1 pc$\approx 3 \times 10^{16}$ cm.

As shown in Fig. 1 the parsec is observationally defined via **trigonometric parallax**, which is how astronomers measure the distance to the nearest stars. The Earth's orbit has a diameter of 2 au, so over half a year the Earth moves by that amount and an object at distance d will appear to shift position slightly by an angle 2θ. Then we have

$$\tan\theta = \frac{1 \text{ au}}{d}. \qquad (10)$$

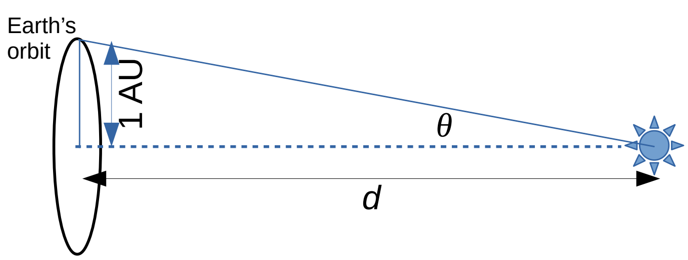

Figure 1: Trigonometric (parallax) distance measurements use the Earth's orbit to measure the changing angular position of distant objects. If the angular movement θ (**parallax**) is measured in arcsec, then the distance d will be measured in parsecs (pc).

Since θ is very small (try calculating a few examples!) when we meaure it in radians we can approximate $\tan\theta \approx \theta$, so we get

$$d = 1 \text{ AU}/\theta \quad (11)$$

or more conveniently

$$\frac{d}{\text{pc}} = \frac{1 \text{ arcsec}}{\theta}. \quad (12)$$

The name **parsec** comes from the distance to an object with a **PAR**allax of one arc**SEC**. It is also (by coincidence) roughly the average distance between stars in the Solar neighborhood. For example, α Centauri (the nearest star system) is 1.3 pc away. To the center of our Milky Way galaxy, though, is roughly 8,000 pc (8 kpc), and to the next-nearest galaxy of reasonable size is 620 kpc! Space is big.

2.2 *Cosmic Distance Ladder*

Distance is a key concept in astrophysics, and until recently often one of the least-precisely-known quantities we can measure. (Since ∼2015 this has begun to change somewhat thanks to ESA's **Gaia** mission, which is measuring trigonometric parallax for billions of objects with sub-milliarcsec precision – i.e., precision of $< 10^{-3}$ arcsec.

The **distance ladder** refers to the bootstrapping of distance measurements, from nearby stars to the furthest edges of the observable universe. It's a 'ladder' because each technique only has a limited range of applicability, as described below. Note that inside the Solar System we can just throw EM waves at something and wait for them to bounce back: the round-trip light travel time gives us the distance. This includes planetary radar, spacecraft communication, laser-ranging to the moon, and other examples; this isn't usually considered part of the distance ladder.

Parallax, already mentioned, is the first rung outside the Solar system. As noted this is being measured by Gaia for ∼1 billion stars across ∼half of the Galaxy. A revolution is underway!

Standard candles are objects with known intrinsic brightness (i.e., luminosity). If we know how bright something is, then we also know its absolute magnitude. By Eq. 3, it's easy to calculate its distance. There are relatively few truly standard candles, but a larger array of objects are standardizable: using other information we can correct for intrinsic variations between different objects. Some examples include:

- **Cepheid variables**: giant pulsating stars, whose pulsation period correlateds with luminosity.
- **Type Ia Supernovae**: one form of exploding stars (probably white dwarf)
- Distant Galaxies: it turns out that when we measure the rotation speed $v_{\rm rot}$ of various galaxies, we often find $L_{\rm gal} \propto v_{\rm rot}^2$: so measuring $v_{\rm rot}$ gives us the galaxy's intrinsic brightness.

Hubble's Law describes the expansion of the universe as measured by observations of very distant galaxies. Since the universe is expanding at a nearly-constant rate (more on that later on!), distant galaxies appear to be speeding away from us at a velocity $v \propto d$. So measuring v (a different v than v_{rot}, mind) gives us the distance. (Note that this doesn't work for the nearest galaxies, like Andromeda, which is moving toward us due to gravity dominating over cosmic expansion).

3 Stars: A Basic Overview

Stars are "balls of gas burning billions of miles away." Most of astrophysics that we will study in this class relates to star: observing them, making models to understand their interior processes, and studying their dead remnants, natal gas clouds, or stellar clusters with hundreds to trillions of members.

3.1 *Types of Observations*

The two main ways we observe stars (or almost anything else) are **photometry** and **spectroscopy**. To understand either of these, we first need to understand the spectra of stars: how much light they emit at different wavelengths. If you disperse light from the Sun or a hot incandescent bulk through a prism, you see a rainbow: light from shorter-wavelength blue/violet out to longer-wavelength orange/red. If you disperse light from a red LED or a laser, you would see that all of their light comes out at just a tiny range of wavelengths — essentially just a single color rather than a wide range (we call such single-colored light **monochromatic**).

For the dispersed Sunlight, we see the rainbow because our eyes are only sensitive to wavelengths from roughly 400–650 nm. But if we had panchromatic eyes we would see that the Sun (and all stars) emits lights over a broad range of wavelengths. Fig. 4 shows the amount of light coming from several different types of stars, indicating that considerable amounts of light are being emitted at invisible wavelengths.

Spectroscopy

Spectroscopy is the making of measurements such as those shown in Fig. 4, i.e. how much light do we see at each particular wavelength. Spectra are obtained by dispersing light through an optical element such as a prism, diffraction grating, or similar devices.

Each particular spectrum has its own characteristic **spectral resolution** R, which describes how well very narrow features in the spectrum can be resolved. Spectral resolution is typically defined as $R = \lambda/\Delta\lambda$, where $\Delta lambda$ is the width of the narrowest spectral feature that can be measured. If R is very low (say, < 100) then all but the broadest features are 'smoothed over' and washed out. Fig. 4 has R of a few thousand and many spectral features are visible in the spectrum. Typical spectrographic instruments will have $R = 10^3 - 10^4$, but some can go higher (or lower).

By measuring the location (i.e., the wavelength) of particular lines in an object's spectrum, we can learn something about what the object is made of, what temperature it is, and (through the Doppler shift, Eq. 5) how fast the object is moving toward us or away from us. E.g. the spectral line seen at 0.6563μm is the well-known Hα electronic transition line. Seeing it clearly indicates that hydrogen is in a star; unfortunately (for us), the lack of an elemental line (as in the M0V spectrum shown, which lacks Hα) may or may not indicate that that particular element is absent in the star.

Photometry

Photometry is the measurement of average brightness (usually very broad) spectral ranges. It often corresponds to (though is rarely described as) ultra-low-spectral resolution; a typical photometric bandpass would have $\Delta\lambda/\lambda \sim$ 20%. The common *V* (for visual) bandpass spans roughly 500–570 nm; many standard photometric bands (such as *V*) have single-letter names, and some of these are indicated on the OoMA handout and in the bottom of Fig. 4. If one measured the photometry of the stars in that figure, the measurements would essentially average the spectra down into just five points (one per indicated bandpass).

So obviously photometry is a cruder tool than spectroscopy: you typically wouldn't bet the farm that a star showed an Hα line (or not) juts based on photometry alone. On the other hand, astronomical targets are much fainter than the Sun and accurately measuring their faint, dispersed spectra can often be challenging; in contrast, it's comparatively easy to measure an object's

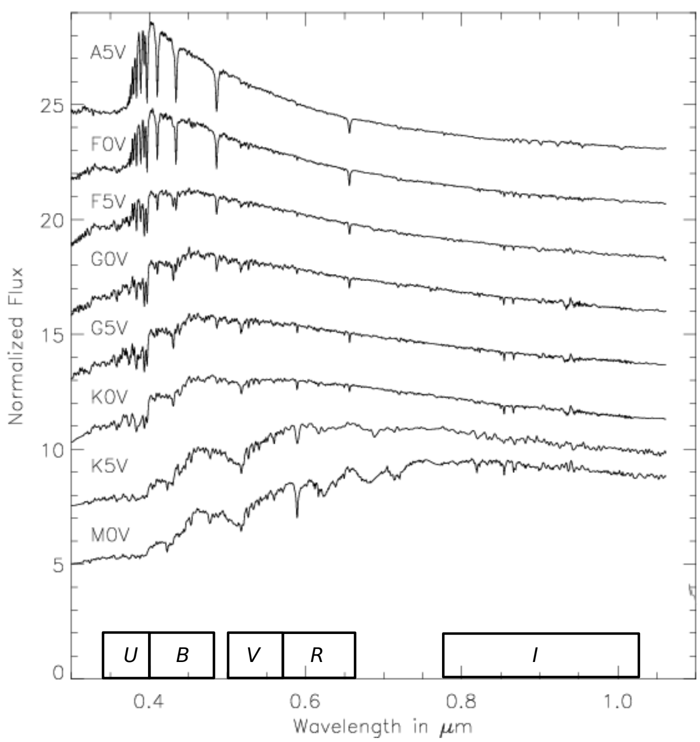

Figure 2: Optical-wavelength spectra of main-sequence stars across a range of spectral types. Wavelength is plotted in μm (10^{-6} m), so the range plotted corresponds to 300–1100 nm. The Sun is a G2V star, and so would be somewhere between the G0V and G5V spectra shown. Shown at bottom are the approximate spectral ranges of some common optical-wavelength photometric bands.

photometric properties. So much of early astronomy was built on photometry, with spectroscopy coming later.

The main way that photometry is used is via measurements of apparent magnitudes in the various photometric bandpasses. When one talks of a star's absolute or apparent magnitude without other reference, the **bolometric** measurement is typically meant — i.e., a measurement of the star's total luminosity integrated over all wavelengths. But one can also speak of a V-band or R-band magnitude; in this case it is assumed than an apparent magnitude is being referred to (unless otherwise specified). The hot A0V star Vega has an apparent magnitude of roughly zero in all optical-wavelength bandpasses; that is, $B \approx V \approx R \approx 0$ (and so forth). Again, the classic astronomer's website for learning the magnitudes, coordinates, and many other details of a given star is SIMBAD[3].

3.2 *Basic Properties of Stars*

As we will see later in the course, it typically takes just three parameters to describe the most salient aspects of a star. But just like constructing a coordinate system (Cartesian? spherical? cylindrical?) there are multiple ways to choose your fundamental stellar parameters – and any can be converted into the others (with greater or lesser degree of accuracy).

The intrinsic stellar parameters most commonly used are mass M, radius R, and either luminosity L or the **effective temperature** $T_{\rm eff}$. This last quantity is something like the average temperature in the outer layers of the star, and it is defined as

$$(13) \quad T_{\rm eff} = \left(\frac{L}{4\pi\sigma_{SB} R^2} \right)^{1/4}$$

where σ_{SB} is the Stefan-Boltzmann constant. As often as we can, we want to avoid having to remember complicated numbers like σ_{SB} – instead, astronomers like to put things in terms of other more familiar quantities. For stars, nothing is more familiar than the Sun. So instead we can rewrite Eq. 13 as

$$(14) \quad \frac{T_{{\rm eff},*}}{T_{{\rm eff},\odot}} = \left[\frac{L_*}{L_\odot} \left(\frac{R_\odot}{R_*} \right)^2 \right]^{1/4} .$$

Both of the above expressions are just rearrangements of a fundamental relation we will touch on soon, namely

$$(15) \quad L = 4\pi\sigma_{SB} T_{\rm eff}^4 R^2.$$

An alternative set of parameters often used in the study of **stellar evolution** (the birth, growth, and death of stars) would be the star's mass, age, and the amount of heavier-than-helium elements in the star. Since most of the universe (and stars) are made of H and He, astronomers call all elements heavier

[3] http://simbad.u-strasbg.fr/simbad/sim-fid

than He **metals**. Thus the relative enhancement level of these heavier elements is typically called **metallicity** (and is reported logarithmically).

If one had to pick just two key parameters, they would likely be M and either $T_{\rm eff}$ or age.

The typical range of properties for stars is:

- **Mass**: From as low as 0.08 $M_\odot$ to as high as around $\sim 100 M_\odot$.
- $T_{\rm eff}$: From as low as about 2400 K to $> 30,000$ K. Many cooler objects exist; these include planets as well as **brown dwarfs** (objects with masses and temperatures between stars and planets).
- **Radius**: From as small as 0.08 $R_\odot$ (smaller than Jupiter!) to as large as $\sim 1000 R_\odot$ (much larger than the Earth's orbit around the Sun).

3.3 *Classification*

Classification is a key step toward understanding any new class of objects. When modern astronomy began, classification of the stars was a key goal — also an elusive one, until the physical processes became better understood. We're now going to begin to peel back the onion that is a Star. And the first step in peeling an onion is to look at it from the outside.

One of the first successful frameworks used photometry measurements of stellar flux density at different colors. Assuming again that stellar spectra are approximately blackbodies, the Planck function shows that we should see the hotter stars have bluer colors and be intrinsically brighter. This led to the **Hertzsprung-Russell diagram** (HR diagram), which plots absolute magnitude against color – we'll see the HR diagram again when we discuss stellar evolution.

It's fair to say that spectroscopy is one of our key tools for learning about astronomical objects, including stars. Fig. 4 shows a sequence of stars arranged from hot to cool: one can easily see the Wien peak shift with temperature, although none of the stars are perfect blackbodies. Other features come and go, determined (as we will see) mainly by stellar temperature but also surface gravity (or equivalently, surface pressure).

Table 1: Stellar spectral types.

SpT	$T_{\rm eff}$	Spectral features
O	$> 3 \times 10^4$	Ionized He or Si; no H (or only very weak)
B	$10^4 - 3 \times 10^4$	H Balmer lines, neutral He lines
A	$7500 - 10^4$	Strong H lines
F	$6000 - 7500$	H Balmer, first metal lines appear (Ca)
G	$5200 - 6000$	Fading H lines, increasing metal lines
K	$3700 - 5200$	Strong Ca and other metals, hydride molecules appear
M	$2400 - 3700$	Molecular bands rapidly strengthen: hydrides, TiO, H_2O
L	$1400 - 2400$	A melange of atomic and molecular bands; dust appears
T	$\sim 400 - 1400$	CH_4 strengthens, dust clears
Y	$\lesssim 400$	NH_3 strengthens

Through decades of refinement, spectra are now classified using Morgan-Keenan **spectral types**. These include a letter to indicate the approximate temperature, an Arabic numeral to refine the temperature, and a roman numeral to indicate the star's luminosity. The order of letters seems disjointed because stars were classified before the underlying physical causes were well-understood. The temperature sequence is OBAFGKMLTY, where the last three typically apply to brown dwarfs (intermediate in mass between planets and stars) and the rest apply to stars. Table 3.3 briefly describes each of the alphabetic spectral types. Additional resolution is added to the system through the use of numbers 0–9, so that F9–G0–G1 is a sequence of steadily decreasing T_{eff}. Finally, the Roman numerals described in Table 2 indicate the luminosity class, which typically correlates with the stellar radius (and inversely with the surface gravity).

Table 2: Stellar luminosity classes.

Lum	name	examples
VI	subdwarf	Kapteyn's Star (M1VI)
V	dwarf	Sun (G2V), Vega (A0V)
IV	subgiant	Procyon (F5IV)
III	giant	Arcturus (K1III)
II	bright giant	
I	supergiant	Rigel (B8Ia), Betelgeuse (M1Ia)
0	hypergiant	η Carinae, Pistol Star

4 Temperature, Luminosity, and Energy

Blackbody Emission

The stellar spectra plotted in Fig. are distinct but qualitatively similar in some respects. For example, if one squints at them to blur out the details of the various spectral absorption features, all the stellar spectra start out fairly faint at short wavelengths, rise to a maximum brightness at some intermediate wavelength, and then fade again toward longer wavelengths. This behavior is characteristic of a **blackbody** emission spectrum. Stars are not perfect blackbodies (they have spectral features, after all) but they are often reasonably close.

The particular shape of a blackbody spectrum is given by the **Planck blackbody function**

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/k_B T} - 1} \quad (16)$$

is of critical importance in astrophysics. The Planck function tells how bright an object with temperature T is as a function of frequency ν. Note that the Planck function can also be written in terms of wavelength λ, but you can't just replace the ν's in Eq. 16 with λ's: instead one must write the identity $\lambda B_\lambda = \nu B_\nu$ and calculate $B_\lambda(T)$ from there.

It's worth plotting $B(T)$ for a range of temperatures to see how the curve behaves, as shown in Fig. 3. One interesting result is that the wavelength of maximal intensity turns out to scale linearly with T. This so-called **Wien Peak** is approximately

$$\lambda_{\max} T \approx 3000 \mu m\ K \quad (17)$$

So radiation from a human body peaks at roughly 10μm in the mid-infrared, while that from a 6000 K, roughly Sun-like star peaks at 0.5μm = 500 nm — right in the response range of the human eye.

Another important correlation is the link between a blackbody's luminosity L and its temperature T. For any specific intensity I_ν, the bolometric flux F is given by Eqs. 92 and 93. When $I_\nu = B_\nu(T)$, the **Stefan-Boltzmann Law** directly follows:

$$F = \sigma_{SB} T^4 \quad (18)$$

where σ_{SB}, the Stefan-Boltzmann constant, is

$$\sigma_{SB} = \frac{2\pi^5 k_B^4}{15 c^2 h^3} \quad (19)$$

(or $\sim 5.67 \times 10^{-8}$ W m^{-2} K^{-4}).

Assuming **isotropic** emission (i.e., that the star shines equally brightly in all directions), the luminosity of a sphere with radius R and temperature T is

$$L = 4\pi R^2 F = 4\pi \sigma_{SB} R^2 T^4 \quad (20)$$

.

If we assume that the Sun is a blackbody with $R_\odot \approx 7 \times 10^8$ m and $T \approx$ 6000 K, then we would calculate

$$L_\odot \approx 4 \times 3 \times (6 \times 10^{-8}) \times (7 \times 10^8)^2 \times (6 \times 10^3)^4 \tag{21}$$

$$= 72 \times 10^{-8} \times (50 \times 10^{16}) \times (1000 \times 10^{12}) \tag{22}$$

$$= 3600 \times 10^{23} \text{ W} \tag{23}$$

which is surprisingly close to the IAU definition of $L_\odot = 3.828 \times 10^{26}$ W.

Soon we will discuss the detailed structure of stars. Again, their spectra (Fig. 4) show that they are not perfect blackbodies, but they are often pretty close. This leads to the common definition of an **effective temperature** linked to a star's size and luminosity by the Stefan-Boltzmann law, as shown by rearranging Eq. 20 to find Eq. 13. In other words, the effective temperature is the temperature of a blackbody with the same size and luminosity of the star.

4.1 *Units of Luminosity, Flux, and Blackbody Emission*

An important note on the units of these various quantities. Luminosity is the quantity that should be most familiar to you: this is just a power (energy per time) and measured in W=J s^{-1}. Specifically, the luminosity is the total power emitted by an object integrated over all wavelengths (or frequencies), from X-rays to radio waves.

We will also often talk about **flux**, which is the amount of power passing

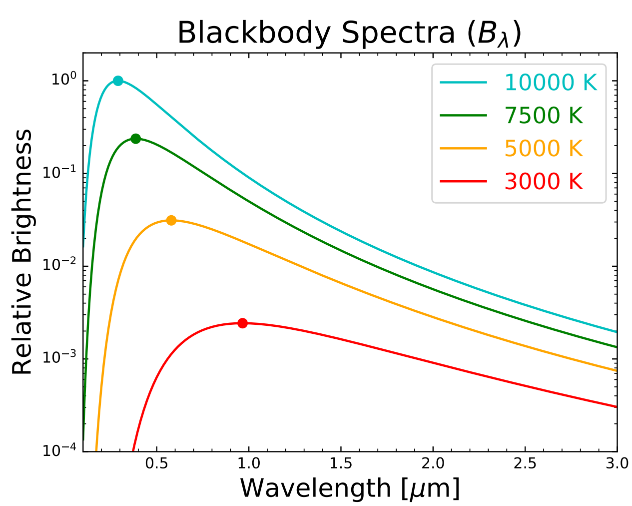

Figure 3: Blackbody spectra $B_\lambda(T)$ for a range of temperatures T. The temperatures used here correspond roughly to the range of $T_{\rm eff}$ spanned by the stars shown in Fig. 4. The circular points indicate the Wien peak of each blackbody.

through some area. If you know the luminosity of an object, the flux we measure from it is just the power spread out over some large surface area. For a star or similar object that radiates equally in all directions, the radiation goes out spherically and so the flux at some distance r is just

$$F = \frac{L}{4\pi r^2}. \tag{24}$$

The SI units of flux are W m^{-2}. One common example is the so-called **solar constant** (the flux incident on the Earth from the Sun). This is approximately

$$F_\oplus = \frac{L_\odot}{4\pi(1\ \mathrm{au})^2} \approx \frac{4 \times 10^{26}}{12 \times (1.5 \times 10^{11})^2} \approx \frac{10^4}{6}\ \mathrm{W\ m^{-2}}. \tag{25}$$

A more precise value is about 1400 W m^{-2}; this is a critical value for modeling weather, environmental behavior, and solar power generation.

As for the Planck function, it is measured in neither luminosity nor flux; instead its units are something called specific intensity that we will come back to later.

5 Energy Sources

Most objects we will consider in this class are powered by one of a few key mechanisms: nuclear fusion or fission, and the release of gravitational potential energy.

5.1 *Fusion and Fission*

Nuclear **fusion** and **fission** transmute an atom from one element to another. The names are similar but the concepts are opposite: fusion combines two atoms or their nuclei (e.g., H or He) to form one atom of a heavier element; fission splits one large nucleus (e.g. U or Pu) into two or more smaller particles. Both can release energy under the right circumstances

Both of these mechanisms are different avenues for liberating the mass energy inherent in all matter. Not to belabor the point, but

$$E = mc^2. \tag{26}$$

For individual particles we often measure these energies in units of **electron-volts** (or eV), with 1 J $\approx 6 \times 10^{18}$ eV. The two most-used mass energies are those of the electron and proton, where $m_e = 511$ keV and $m_p = 938$ MeV. These energies are both much greater than that carried by a single photon (whose energy is $E = h\nu$); a handy rule of thumb for photon energies is that

$$E_\gamma \approx \frac{1.2 \text{ eV}}{\lambda}, \tag{27}$$

so visible photons have an energy of about 2 eV each.

Fusion

Fusion is the most important energy source in stars, and in these cases fusion involves fairly low-mass atoms coming together. An example of a fusion reaction might be

$${}^2_1\mathrm{H} + {}^3_1\mathrm{H} \rightarrow {}^4_2\mathrm{He} + n + \Delta E \tag{28}$$

that releases an amount of energy ΔE. This extra energy is present because the masses of the input hydrogen isotopes *slightly* outweigh that of the helium produced:

$$\begin{aligned} \Delta m = & \; m_{\rm in} - m_{\rm out} & (29) \\ = & \; (2.014 \text{ amu} + 3.016 \text{ amu}) - (4.003 \text{ amu} + 1.009 \text{ amu}) & (30) \\ = & \; 0.018 \text{ amu.} & (31) \end{aligned}$$

One **atomic mass unit** (amu) has a mass-energy of 931 MeV, so our fusion reaction above releases (1.8%)(931 MeV) $\approx$ 16 MeV. Compared to the input mass-energy of $\sim 5 \times 931 \sim 4600$ MeV, that means our reaction has a mass-energy conversion efficiency of about 0.3%.

Most stars are mostly H, and the primary fusion source turns out to be the fusion of simple hydrogen nuclei ($^1_1$H) into helium nuclei ($^4_2$He, i.e., α **particles**). A very useful rule of thumb is that the conversion of H into He releases 0.7%mc^2.

Fission

In contrast to fusion, fission involves much heavier atoms. Fission is also less efficient than fusion, but still a potent source of energy. An classic fission raction is

$$(32)\quad {}^{235}_{92}\mathrm{U} \rightarrow {}^{231}_{90}\mathrm{Th} + {}^{4}_{2}\mathrm{He} + \Delta E.$$

In this case the change in mass is just $\Delta m = 0.005$ amu, corresponding to an energy release of about 4.5 MeV. Not too much less than was released in our fusion reaction above, but the input mass was much greater (∼235 mu) and so the efficiency is only 0.005/235∼ 2×10^{-5} — much less efficient than the conversion of mass energy in fusion!

5.2 *Gravitational Energy*

Another common source of energy we will often focus on is gravitational potential energy. Recall that for two point masses separated by r, the potential energy of the system is

$$(33)\quad U_G = -G\frac{m_1 m_2}{r}.$$

This is the energy gained as the two objects are brought close to each other (the gravitational force pulls along the direction of motion, so negative work is done). So U_G of the $Earth - Moon$ system is

$$(34)\quad U_{G,E-M} \sim (-\tfrac{2}{3} \times 10^{-10})\tfrac{(6\times10^{24})(6\times10^{22})}{3\times10^{8}}\ \mathrm{J}$$

$$(35)\quad \sim 8 \times 10^{28}\ \mathrm{J}.$$

Massive, astronomical objects have their own (quite considerable) self-gravity, so even a single object also has its own gravitational potential energy. For a sphere of uniform density, this can be shown to be

$$(36)\quad U_{G,\mathrm{sphere}} = -\frac{3}{5}\frac{GM^2}{R}.$$

Although few objects truly have uniform densities, Eq. 36 is often a reasonable first approximation. It also means that if an object changes size (expands or contracts), its gravitational potential energy will change. Since

$$(37)\quad \frac{dU}{dR} \propto +1/R^2,$$

as R decreases energy is released. This gravitational energy source will be important in a number of astrophysical contexts.

6 The Two-Body Problem and Kepler's Laws

Much of what we will discuss in this class involves orbiting objects: a moon around a planet, or several planets around their star, thousands of stars in a cluster, or billions in a galaxy. The first consideration of all these issues involves the so-called **two-body problem**, when two objects orbit each other due to their gravitational attraction. Observations of such systems are especially powerful because (as we will see) certain quantities — e.g. masses, radii, orbital periods — can be measured extremely precisely. For example, binary pulsars (two dead **neutron stars** orbiting each other) may have their masses measured with a precision of $10^{-3} M_{\odot}$ ($\sim 0.1\%$).

Our goal in the discussion below is to work through the gravitational two-body problem with an eye on features that are observationally testable, and on features specific to the $1/r^2$ nature of gravity. In the real world many "details" push us away from exact $1/r^2$ — e.g. physical sizes, non-spherical shapes, and general relativity.

6.1 *Kepler's Laws*

To set the stage, recall from introductory physics Kepler's three laws of orbital motion (not to be confused with Asimov's Three Laws of Robotics):

1. **Kepler's First Law**: objects orbit along elliptical trajectories, as shown in Fig. 4. The most relevant quantity here is the **semimajor axis** a, which will come up again and again. The **eccentricity** e is also important: if $e = 0$, the orbit is circular and $r(\phi) = a$ always. However, most orbits are at least slightly eccentric and so in general

$$r(\phi) = \frac{a\left(1 - e^2\right)}{1 + e\cos\phi}. \tag{38}$$

2. **Kepler's Second Law**: In a given time interval dt an object's orbit always sweeps out the same area dA across its orbital ellipse. That is, dA/dt is

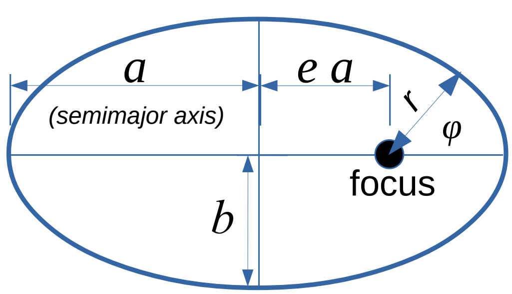

Figure 4: Elliptical trajectory of a two-body orbiting system. Important quantities are labeled, including the semimajor axis a, the orbital eccentricity e, and the polar coordinates r and ϕ as measured from the ellipse's focus.

constant; this will turn out to be

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\phi}{dt}. \tag{39}$$

3. **Kepler's Third Law**: The most useful of all Kepler's Laws in many situations we will encounter. This relates a, the orbital period P, and the total mass M_{tot} of the orbiting bodies, as a^3/P^2 equal to a constant. You may have seen this before as some variation on:

$$P^2 = \left(\frac{4\pi^2}{GM}\right) a^3 \tag{40}$$

6.2 *Deriving Kepler's Laws*

Our goal in what follows is to rigorously derive Kepler's Laws and so understand how we can describe the positions and velocities of objects orbiting in a two-body arrangement. Initially this may seem daunting: since in 3D space each object has three position coordinates and three velocity components, we have twelve degrees of freedom that might have to be explained. We need to reduce this number to make things tractable!

Kepler's 2nd Law

First, we can reduce the dimensionality by half by recognizing that both objects will orbit around their common center of mass (see Fig. 5). Whatever arbitrary origin we choose for our coordinate system (such that our objects have 3D positions $\vec{r_1}$ and $\vec{r_2}$), the position $\vec{R}$ of the center of mass in this reference frame will be

$$\vec{R} = \frac{m_1 \vec{r_1} + m_2 \vec{r_2}}{m_1 + m_2}. \tag{41}$$

position vector from one to the other will always be

$$\vec{r} = \vec{r_2} - \vec{r_1}. \tag{42}$$

Remember from introductory physics that if there are no external forces on the system, then the center of mass experiences no accelerations: $\ddot{\vec{R}} = 0$. And since physics is the same in all inertial reference frames, we have freedom to choose the 'easy' inertial reference frame in which $\dot{\vec{R}} = 0$ too – so the position of the center of mass never changes. Since physics is also the same in all locations, we can again pick the 'easy' reference frame in which $\vec{R} = 0$ too.

So the center of mass is at the origin now, and since it isn't moving and isn't accelerating it will always be at the origin. From Eq. 41, this means that

$$m_1 \vec{r_1} + m_2 \vec{r_2} = 0. \tag{43}$$

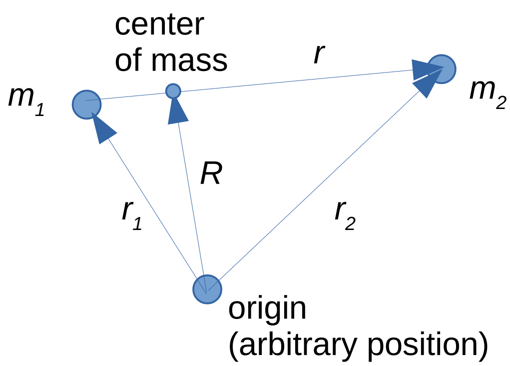

Figure 5: Coordinate system for two masses m_1 and m_2 separated by a distance r. Both objects will orbit around their common center-of-mass; in this case, $m_1 > m_2$ and so the center of mass is closer to m_1.

From Fig. 5 the position of one object relative to the other is just

$$\vec{r} = \vec{r_2} - \vec{r_1}. \tag{44}$$

Combining Eqs. 43 and 44 shows that both positions $\vec{r_1}$ and $\vec{r_2}$ are proportional to each other and also to the difference vector $\vec{r}$:

$$\vec{r_2} = \frac{\vec{r}}{1 + m_2/m_1} \tag{45}$$

and

$$\vec{r_1} = -\frac{m_2}{m_1}\vec{r_2}. \tag{46}$$

This means that in terms of the number of unknown quantities, we've reduced the two-body problem to an effective one-body problem with 'only' one set of unknown position and velocity coordinates.

We can go further: since gravity is a radially-acting force, $\vec{F_G} \parallel \vec{r}$. This means that torque will be zero (since $\vec{\tau} = \vec{r} \times \vec{F}$), and thus the **angular momentum** $\vec{L} = \vec{r} \times \vec{p}$ of the system will be constant in magnitude and direction: thus the orbit must be constrained to a (2D) plane, and so we're justified in using just r and ϕ to describe the orbit.

The area dA swept out by the orbit in a time interval dt is given by

$$\begin{aligned} dA = & \quad \frac{1}{2}\left|\vec{r} \times \vec{dr}\right| \\ = & \quad \frac{1}{2}\left|\vec{r} \times \vec{v}dt\right| \\ = & \quad \frac{dt}{2m}\left|\vec{r} \times m\vec{v}\right| \\ = & \quad \frac{1}{2}\frac{L}{m}dt = \text{constant.} \end{aligned}$$

So an equal area is swept out in any equal time interval – that's Kepler's Second Law.

One open question in the derivation immediately above is what mass we should use for m — since in fact we have two masses orbiting each other, m_1 and m_2. It turns out that this should be written in terms of the so-called **reduced mass** μ of the two-body system, where

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2} \tag{47}$$

and so

$$\frac{dA}{dt} = \frac{1}{2}\frac{L}{\mu} = \text{constant.} \tag{48}$$

Kepler's 3rd Law

Deriving this law involves rather more than we want to deal with in this class, but we can get close with some fairly basic approximations. Consider an object of mass m_2 in a simple, circular orbit with around a larger mass m_1 (we're neglecting that actually both objects orbit around their common center of mass). The orbit has semimajor axis a, which is also the (constant) separation r between the objects.

By Newton's first law, we must have$F_{\text{external}} = m a_r$ where (for uniform circular motion) a_r is the centripetal acceleration (not to be confused with a, the semimajor axis). So for a gravitational orbit, we have

$$\frac{G m_1}{m_2} a^2 = \frac{m_2 v_2^2}{a^2}. \tag{49}$$

The orbital velocity is related to the orbital period by $v = 2\pi a / P$. Plugging that in for v and rearranging, we find the almost-correct form:

$$P^2 \frac{G m_1}{4\pi^2} = a^3.$$

This is not quite right because we neglected the movement of the larger m_1 around the common center of mass. This isn't a big effect if m_2 is relatively small; a full derivation would show us that we should be using the total mass of the system, $M_{\text{tot}} = m_1 + m_2$. This gives us the "physicists's version" of Kepler's Third Law:

$$P^2 \frac{G M_{\text{tot}}}{4\pi^2} = a^3. \tag{50}$$

This isn't too bad, but we can make it even easier to use in many situations. We know that the Earth takes one year to orbit the Sun at a distance of 1 au, and that the total mass of the Earth-Sun system is $M_\odot + M_\oplus \approx M_\odot$. This means we can turn Eq. 50 into a set of ratios for the "astronomer's version" of

Kepler's Third Law:

$$\left(\frac{P}{1\ \mathrm{yr}}\right)^2 \left(\frac{M_{\mathrm{tot}}}{M_\odot}\right) = \left(\frac{a}{1\ \mathrm{au}}\right)^3 \tag{51}$$

Either Eq. 50 or Eq. 51 let astronomers calculate an object's mass merely by observing its orbital motion (i.e., a and P) — and either expression will give the correct answer when applied correctly.

6.3 *Introducing Energy Diagrams*

Energy Diagrams, or potential energy plots, are useful tools that we will use in a variety of situations in this class. Their main benefit is to tell us when various parameters are permitted (or prohibited) from taking on certain values due to the energy considerations of the system. These are most easily employed when we just have a single variable (e.g., energy is only a function of x), though they can be used in other situations too.

The steps are fairly straightforward:

1. Write down an expression for the potential energy U in your system.
2. Plot U vs. your dependent spatial variable (e.g., $U(x)$ vs. x).
3. Assume some amount of total mechanical energy (i.e., the sum of U and kinetic energy K).
4. Overplot a horizontal line indicating the total E_{mech} of the system.
5. Consider your plot and gain insight.

The insights gained should be the following:

- Forbidden regions: for any areas on the plot where $U > E_{\mathrm{mech}}$, this would imply $K < 0$ (which is impossible). Thus your system will never be located at these regions!
- Permitted regions: the opposite of forbidden regions, wherever $U \leq E_{\mathrm{mech}}$. The system could potentially be located anywhere in these regions. There could be one or multiple permitted regions, and they need not all be contiguous.
- Kinetic Energy: Since $K = E_{\mathrm{mech}} - U$, the difference between the horizontal E_{mech} and more complicated U curves gives the amount of kinetic energy at that position – and thus also how fast the system is moving.
- Turning points: Wherever $U = E_{\mathrm{mech}}$ exactly, the object has zero velocity. These are the points where the object would turn around if it had been in the adjacent Permitted Region.

Fig. 6 gives an example for a simple harmonic oscillator with $U(x) = 1/2kx^2$, with some nonzero amount of mechanical energy E_{mech}. At the indicated position x_1, there is some nonzero kinetic energy: so if the oscillator

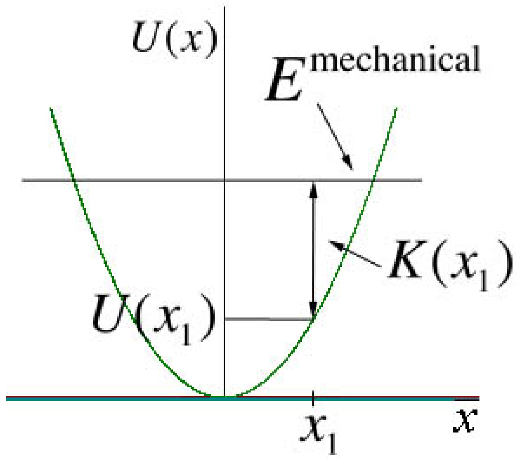

Figure 6: Energy diagram for the case of a simple harmonic oscillator, $U(x) = 1/2kx^2$, with some nonzero amount of mechanical energy E_{mech}.

were located at x_1, it would be moving – but more slowly than if it were at $x = 0$, since $E_{\text{mech}} - U(x)$ is less at x_1 than at $x = 0$. Regions with very large x are forbidden; this should make intuitive sense, since if you start a spring bouncing you don't expect it to stretch out to infinity all of a sudden.

6.4 *Energy of the Two-Body System*

In any system, the total mechanical energy is just $E_{\text{mecH}} = U + K$. In the two-body system in particular, our dependent variable is r and $U(r)$ takes its familiar form

$$U_g(r) = \frac{-Gm_1m_2}{r}. \tag{52}$$

Our kinetic energy is slightly more complicated. On an elliptical orbit the velocity $\vec{v}$ is typically neither totally radial nor azimuthal; in general

$$\begin{aligned} \vec{v} &= \vec{v}_r + \vec{v}_\phi \\ &= \dot{\vec{r}} + \dot{\phi}\vec{r}. \end{aligned}$$

So the total kinetic energy actually depends on r, inasmuch as

$$K = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2. \tag{53}$$

We can write the second part of this expression for K in terms of the an-

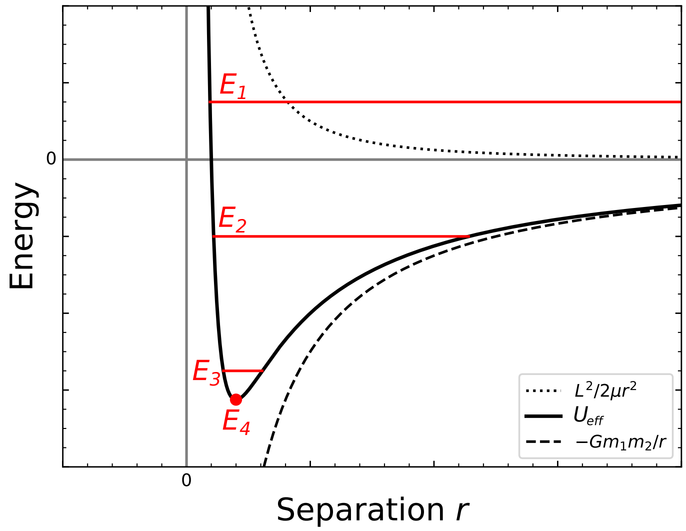

Figure 7: Energy diagram for the the two-body problem, showing the contributions of both the centripetal and gravitational terms to the final effective potential U_{eff}. Various permitted energy levels are indicated by E_1, E_2, etc.

gular momentum, which we determined earlier to be constant. We have

$$L = |\vec{r} \times \mu\vec{v}|$$
$$= \mu r v$$
$$= \mu r^2 \dot{\phi}.$$

So in our expression for $K(r)$, $\dot{\phi}^2 = L^2/\mu^2 r^4$ and thus Eq. 53 becomes

$$K = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2}. \tag{54}$$

Our expression for the total energy of the two-body system is then

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} - \frac{Gm_1}{m_2}r, \tag{55}$$

and we typically combine the last two terms together into an **effective potential** U_{eff}.

Fig. 7 shows the final energy diagram for two-body problem (compare to Fig. 6). In the extreme case that we have zero angular momentum, this means that the full potential would be the dashed gravitational term: $L = 0$ means that the objects are heading for a collision at $r = 0$. On the other hand, this means that so long as $L > 0$ there is always a forbidden zone at small r. Just because the Earth is gravitationally attracted to the Sun doesn't mean that we can just fall into it!

For nonzero L, the red lines show various permitted energy levels. As we know, a gravitationally bound system has negative total energy (Eq. 52). So if a two-body system has $U_{\rm eff} > 0$ (such as E_1 in the figure) the system is **unbound**: the objects can escape from each other away toward infinity. At the other extreme, there is some minimum possible energy (labeled here as E_4). At this energy, only a single separation r is permitted: thus the distance between the objects is always the same; they're on a circular orbit. As energy is added to a circular orbit, wider and wider ranges of r are accessible to the system (e.g. E_3 and E_2: adding energy to a circular orbit increases its eccentricity. When $U_{\rm eff} = 0$ the elliptical orbit becomes an unclosed parabola, and for any larger energy (e.g. E_1) the orbit becomes hyperbolic.

7 Binary Systems

Having dealt with the two-body problem, we'll leave the three-body problem to science fiction authors[4] and begin an in-depth study of stars. Our foray into Kepler's laws and two-body systems was appropriate, because about 50% of all stars are in binary (or higher-multiplicity) systems. With our fundamental dynamical model, plus data, we get a lot of stellar information from **binary stars** — that is, stars in two-star systems.

Note that there are increasing levels of stellar multiplicity: systems of three, four, or even more stars all orbiting each other in a complicated dance. These higher-order multiples are less common than binaries, and they are often arranged heirarchically: that is, a triple system will typically involve two stars closely-orbiting each other with the third at a much wider separation. The nearby α Centauri triple system (our closest stellar neighbors) is such a **heirarchical multiple**: the more massive stars α Cen A and α Cen B orbit each other with $a = 23$ au, while the smaller, cooler, lower-mass Proxima Centauri (the closest star to the Sun) orbits A & B at roughly $a \approx 8000$ au.

Stars in binaries are best characterized by mass M, radius R, and luminosity L. Note that an effective temperature T_{eff} is often used in place of L (see Eq. 13). An alternative set of parameters from the perspective of stellar evolution would be M; heavy-element enhancement metallicity [Fe/H], reported logarithmically; and age.

7.1 *Empirical Facts about binaries*

The distribution of stellar systems between singles, binaries, and higher-order multiples is roughly 55%, 35%, and 10% (for details see Raghavan et al. 2010[5]) – so the average number of stars per system is something like 1.6.

Orbital periods range from $<$ 1day to $\sim 10^{10}$ days ($\sim 3 \times 10^6$ yr). Any longer, and Galactic tides will disrupt the stable orbit (the Sun takes ~ 200 Myr to orbit the Milky Way). The periods of binary stars have a **log-normal distribution** – that is, the distribution is roughly Gaussian in $\log(P)$. For Sun-like stars, binaries are most common with $\log_{10}(P/d) = 4.8$ with a width of 2.3 dex (Duquennoy & Mayor 1992[6]) – that is, the most common orbital periods are roughly $10^{4.8 \pm 2.3}$ d.

There's also a wide range of eccentricities, from nearly circular to highly elliptical. For short periods, we see $e \approx 0$. This is due to tidal circularization: the stretching and squeezing of a 3D (non-pointlike) body by a companion's gravity 'steals' energy from the orbit, causing eccentric orbits to eventually circularize. Stars and planets aren't point-masses and aren't perfect spheres; tides represent the differential gradient of gravity across a physical object, and they bleed off orbital energy while conserving angular momentum. It turns out that this means e decreases as a consequence, as explained by the energy diagram analysis presented in Sec. 6.4.

[4]See: `https://en.wikipedia.org/wiki/The_Three-Body_Problem_(novel)` .
[5]`https://ui.adsabs.harvard.edu/abs/2010ApJS..190....1R/`
[6]`https://ui.adsabs.harvard.edu/abs/1991A%26A...248..485D/`

7.2 *Parameterization of Binary Orbits*

As discussed before in Sec. 6.2, two bodies orbiting in 3D requires 12 parameters, three for each body's position and velocity. Three of these map to the 3D position of the center of mass – we get these if we measure the binary's position on the sky and the distance to it. Three more map to the 3D velocity of the center of mass – we get these if we can track the motion of the binary through the Galaxy.

So we can translate any binary's motion into its center-of-mass rest frame, and we're left with **six numbers describing orbits** (see Fig. 9):

- P – the orbital period
- a – semimajor axis
- e – orbital eccentricity
- I – orbital inclination relative to the plane of the sky
- Ω – the longitude of the ascending node
- ω – the argument of pericenter

The period gives the relevant timescale; the next two parameters give us the shape of the ellipse; the last three describe the ellipse's orientation (three angles for 3D space, as you may have seen in classical mechanics).

7.3 *Binary Observations*

The best way to measure L comes from basic telescopic observations of the apparent bolometric flux F (i.e., integrated over all wavelengths). Then we have

$$F = \frac{L}{4\pi d^2} \tag{56}$$

where ideally d is known from parallax.

But the most precise way to measure M and R almost always involve stellar binaries (though asteroseismology can do very well, too). But if we can observe enough parameters to reveal the Keplerian orbit, we can get masses (and separation); if the stars also undergo eclipses, we also get sizes.

In general, how does this work? We have two stars with masses $m_1 > m_2$ orbiting their common center of mass on elliptical orbits. Kepler's third law says that

$$\frac{GM}{a^3} = \left(\frac{2\pi}{P}\right)^2 \tag{57}$$

so if we can measure P and a we can get M. For any type of binary, we usually want $P \lesssim 10^4$ days if we're going to track the orbit in one astronomer's career.

If the binary is nearby and we can directly see the elliptical motion of at least one component, then we have an **astrometric binary**. If we know the

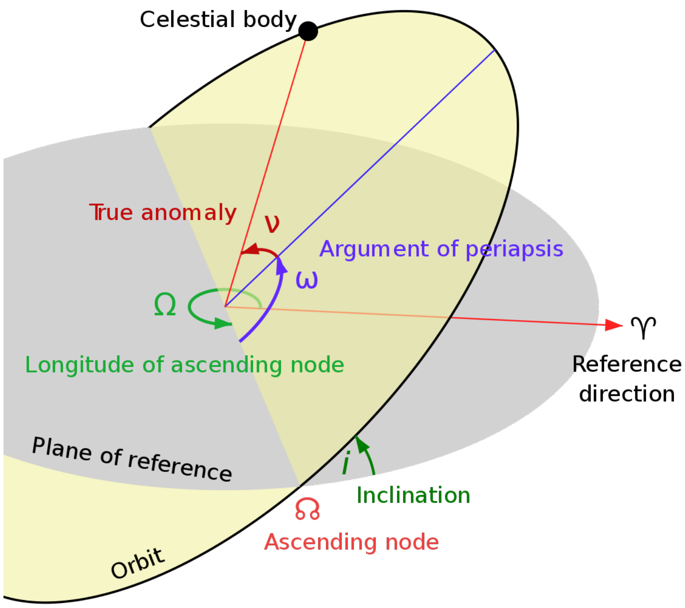

Figure 8: Geometry of an orbit. The observer is looking down along the z axis, so x and y point in the plane of the sky.

distance d, we can then directly determine a as well (or both a_1 and a_2 if we see both components). The first known astrometric binary was the bright, northern star Sirius – from its motion on the sky, astronomers first identified its tiny, faint, but massive white dwarf companion, Sirius b.

More often, two objects in a binary are so close that we can't separate the light well enough to see their astrometric motion. In these cases, we obtain spectra of the stars that let us measure the stars' Doppler shifts, and so measure the velocity of one or both stars. If we can only measure the periodic velocity shifts of one star (e.g. the other is too faint), then the **spectroscopic binary** is an "SB1". If we can measure the Doppler shifts of both stars, then we have an "SB2": we get the individual semimajor axes a_1 and a_2 of both components, and we can get the individual masses from $m_1 a_1 = m_2 a_2$.

If we have an SB1, we measure the radial velocity of the visible star. Assuming a circular orbit,

$$v_{r1} = \frac{2\pi a_1 \sin I}{P} \cos\left(\frac{2\pi t}{P}\right) \tag{58}$$

where P and v_{r1} are the observed quantities. What good is $a_1 \sin I$? We know that $a_1 = (m_2/M)a$, so from Kepler's Third Law we see that

$$\left(\frac{2\pi}{P}\right)^2 = \frac{Gm_2^3}{a_1^3 M^2} \tag{59}$$

Combining Eqs. 58 and 59, and throwing in an extra factor of $\sin^3 I$ to each side, we find

$$\frac{1}{G}\left(\frac{2\pi}{P}\right)^2 a_a^3 \sin^3 I = \frac{1}{G}\frac{v_{r1}^3}{(2\pi/P)} \tag{60}$$

$$= \frac{m_2^3 \sin^3 I}{M^2} \tag{61}$$

where this last term is the spectroscopic "mass function" – a single number built from observables that constrains the masses involved.

$$f_m = \frac{m_2^3 \sin^3 I}{(m_1+m_2)^2} \tag{62}$$

In the limit that $m_1 << m_2$ (e.g. a low-mass star or planet orbiting a more massive star), then we have

$$f_m \approx m_2 \sin^3 I \leq m_2 \tag{63}$$

Another way of writing this out in terms of the observed radial velocity semi-amplitude K (see Lovis & Fischer 2010) is:

$$K = \frac{28.4 \text{ m s}^{-1}}{(1-e^2)^{1/2}} \frac{m_2 \sin I}{M_{Jup}} \left(\frac{m_1+m_2}{M_\odot}\right)^{-2/3} \left(\frac{P}{1 \text{ yr}}\right)^{-1/3} \tag{64}$$

Fig. 9 shows the situation if the stars are eclipsing. In this example one star

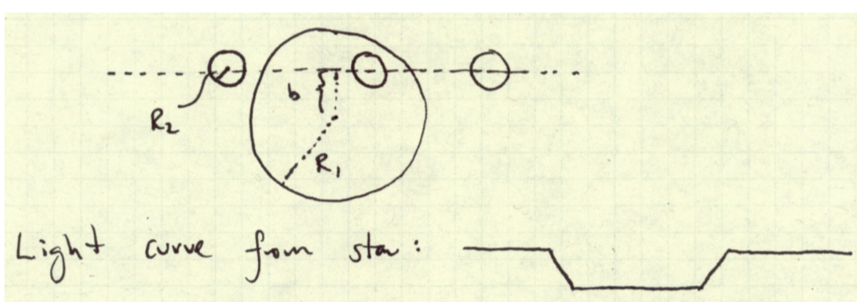

Figure 9: Geometry of an eclipse (top), and the observed light curve (bottom).

is substantially larger than the other; as the sizes become roughly equal (or as the impact parameter b reaches the edge of the eclipsed star), the transit looks less flat-bottomed and more and more V-shaped.

If the orbits are roughly circular then the duration of the eclipse (T_{14}) relates directly to the system geometry:

$$T_{14} \approx \frac{2R_1\sqrt{1-(b/R_1)^2}}{v_2} \quad (65)$$

while the fractional change in flux when one star blocks the other just scales as the fractional area, $(R_2/R_1)^2$.

There are a lot of details to be modeled here: the proper shape of the light curve, a way to fit for the orbit's eccentricity and orientation, also including the flux contribution during eclipse from the secondary star. Many of these details are simplified when considering extrasolar planets that transit their host stars: most of these have roughly circular orbits, and the planets contribute negligible flux relative to the host star.

Eclipses and spectroscopy together are very powerful: visible eclipses typically mean $I \approx 90^o$, so the $\sin I$ degeneracy in the mass function drops out and gives us an absolute mass. Less common is astrometry and spectroscopy – the former also determines I; this is likely to become much more common in the final Gaia data release (DR4, est. 2022).

8 Exoplanets, Planets, and the Solar System

The preceding discussion about binary systems is a good jumping-off point for an aside about planets within and beyond our Solar system.

The International Astronomical Union (IAU), the international organization of professional astronomers, defined a **planet** in 2006 to be anything that:

- is in orbit around the Sun,
- has sufficient mass to assume hydrostatic equilibrium (a nearly round shape), and
- has "cleared the neighborhood" around its orbit.

Note that by the first criterion, planets in the 'legal' sense of the term can only be found within our own Solar system. This would come as quite a shock to the many astronomers who study **extrasolar planets** (also known as **exoplanets**) that orbit other stars. Luckily these researchers don't treat this technical restriction too carefully, and so one frequently (and I think, accurately) hears of "planets" in other solar systems.

8.1 *The Solar System*

8.2 *Exoplanets*

9 Gravitational Waves

A subset of binary objects can be studied in an entirely different way than astrometry, spectroscopy, and eclipses: this is through **gravitational waves,** undulations in the fabric of spacetime itself caused by rapidly-orbiting, massive objects. For our description of that, I follow Choudhuri's textbook, parts of chapters 12 and 13. Note that in much of what follows, we skip details about a number of different factors (e.g. "projection tensors") that introduce angular dependencies, and enforce certain rules that radiation must obey. For a detailed treatment of all this, consult a modern gravitational wave textbook (even Choudhuri doesn't cover everything that follows, below).

Recall that in relativity we describe spacetime through the four-vector

$$x^i = (x^0, x^1, x^2, x^3) \tag{66}$$
$$= (ct, x, y, z) \tag{67}$$

(note that those are indices, not exponents!). The *special* relativistic metric that describes the geometry of spacetime is

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \tag{68}$$
$$= \eta_{ik} dx^i dx^k \tag{69}$$

But this is only appropriate for special (not general) relativity – and we definitely need GR to treat accelerating, inspiraling compact objects. For 8.901, we'll assume weak gravity and an only slightly modified form of gravity; "first-order general relativity." Then our new metric is

$$g_{ik} = \eta_{ik} + h_{ik} \tag{70}$$

where it's still true that $ds^2 = g_{ik} dx^i dx^k$, and h_{ik} is the GR perturbation. For ease of computation (see the textbook) we introduce a modified definition,

$$\bar{h}_{ik} = h_{ik} - \frac{1}{2}\eta_{ik} h \tag{71}$$

where here h is the trace (the sum of the elements on the main diagonal) of h_{ik}.

Now recall that Newtonian gravity gives rise to the gravitational Poisson equation

$$\nabla^2 \Phi = 4\pi G \rho \tag{72}$$

– this is the gravitational equivalent of Gauss' Law in electromagnetism. The GR equations above then lead to an equivalent expression in GR – the **inhomogeneous wave equation**,

$$\Box^2 \bar{h}_{ik} = -\frac{16\pi G}{c^4} T_{ik} \tag{73}$$

where $\Box^2 = -1\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2$ is the 4D differential operator and T_{ik} is the energy-momentum tensor, describing the distribution of energy and momentum in spacetime. This tensor is a key part of the **Einstein Equation** that describes how mass-energy leads to the curvature of spacetime, which unfortunately we don't have time to fully cover in 8.901.

One can solve Eq. 73 using the Green's function treatment found in almost all textbooks on electromagnetism. The solution is that

$$(74)\quad \bar{h_{ik}}(t, \vec{r}) = \frac{4G}{c^4}\int_S \frac{T_{ik}(t - |\vec{r} - \vec{r'}|/c, \vec{r}')}{|\vec{r} - \vec{r}'|} d^3r'$$

(where $t_r = t - |\vec{r} - \vec{r}'|/c$ is the 'Retarded Time'; see Fig. 10 for the relevant geometry). This result implies that the effects of gravitation propagate outwards at speed c, just as do the effects of electromagnetism.

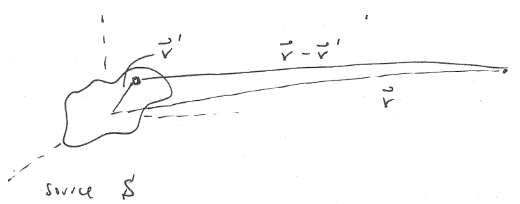

Figure 10: General geometry for Eq. 74.

We can simplify Eq. 74 in several ways. First, assuming than an observer is very far from the source implies $|\vec{r}| >> |\vec{r}'|$ for all points in the source S. Therefore,

$$(75)\quad \frac{1}{|\vec{r} - \vec{r}'|} \approx \frac{1}{r}$$

If the mass distribution (the source S) is also relatively small, then

$$(76)\quad t - |\vec{r} - \vec{r}'|/c \approx t - r/c$$

. Finally, general relativity tells us that the timelike components of h_{ik} do not radiate (see GR texts) – so we can neglect them in the analysis that follows.

Putting all this together, we have a simplified solution of Eq. 74, namely

$$(77)\quad \bar{h_{ik}} = \frac{4G}{c^4 r}\int_S T_{ik}(\vec{r}', t_R) d^3r'$$

We can combine this with one more trick. The properties of the stress-energy tensor (see text, again) turn out to prove that

$$(78)\quad \int_S T_{ik}(\vec{r}')d^3r' = \frac{1}{2}\frac{d^2}{dt^2}\int_S T_{00}(\vec{r}') \cdot x_i' x_k' d^3r'$$

This is possibly the greatest help of all, since in the limit of a weak-gravity source $T_{00} = \rho c^2$, where ρ is the combined density of mass and energy.

If we then define the **quadrupole moment tensor** as

$$(79) \quad I_{ik} = \int_S \rho(\vec{r}')x'_i x'_j d^3r'$$

then we have as a result

$$(80) \quad h_{ik} = \frac{2G}{c^4 r}\frac{d^2}{dt^2}I_{ik}$$

which is the **quadrupole formula** for the gravitational wave amplitude.

What is this quadrupole moment tensor, I_{ik}? We can use it when we treat a binary's motion as approximately Newtonian, and then use I_{ik} to infer how gravitational wave emission causes the orbit to change. If we have a circular binary orbiting in the xy plane, with separation r and m_1 at $(x > 0, y = 0)$. In the reduced description, we have a separation r, total mass M, reduced mass $\mu = m_1 m_2/M$, and orbital frequency $\Omega = \sqrt{GM/r^3}$. This means that the binary's position in space is

$$(81) \quad x_i = r(\cos\Omega t, \sin\Omega t, 0)$$

Treating the masses as point particles, we have $\rho = \rho_1 + \rho_2$ where

$$(82) \quad \rho_n = \delta(x - x_n)\delta(y - y_n)\delta(z)$$

so the moment tensor becomes simply $I_{ik} = \mu x_i x_k$, or

$$(83) \quad I_{ik} = \mu r^2 \begin{bmatrix} \cos^2\Omega t & \sin\Omega t\cos\Omega t & 0 \\ \sin\Omega t\cos\Omega t & \sin^2\Omega t & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

As we will see below, this result implies that gravitational waves are emitted at twice the orbital frequency.

9.1 *Gravitational Radiation*

Why gravitational *waves*? Eq. 73 above implies that in empty space, we must have simply

$$(84) \quad \Box^2 \bar{h}_{lm} = \Box^2 h_{lm} = 0$$

This implies the existence of the aforementioned propagating gravitational waves, in an analogous fashion to the implication of Maxwell's Equations for traveling electromagnetic waves. In particular, if we define the wave to be traveling in the x^3 direction then a plane gravitational wave has the form

$$(85) \quad h_{lm} = A_{lm}e^{ik(ct-x^3)}$$

(where i and k now have their usual wave meanings, rather than referring to indices). It turns out that the A_{lm} tensor can be written as simply

$$(86) \quad A_{lm} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & -a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The implication of just two variables in A_{lm} is that gravitational waves have just two polarizations, "+" and "×". This is why each LIGO and VIRGO detector needs just two arms – one per polarization mode.

Just like EM waves, GW also carry energy. The **Isaacson Tensor** forms part of the expression describing how much energy is being carried, namely:

$$(87) \quad \frac{dE}{dAdt} = \frac{1}{32\pi} c^3/G < \dot{h}_{ij}\dot{h}_{ij} >$$

This is meaningful only on distance scales of at least one wavelength, and when integrated over a large sphere (and accounting for better-unmentioned terms like the projection tensors), we have

$$(88) \quad \frac{dE}{dt} = \frac{1}{5}\frac{G}{c^5} < \dddot{I}_{ij}\dddot{I}_{ij} >$$

which is the **quadrupole formula** for the energy carried by gravitational waves.

9.2 *Practical Effects*

In practice, this means that the energy flux carried by a gravitational wave of frequency f and amplitude h is

$$(89) \quad F_{gw} = 3 \text{ mW m}^{-2} \left(\frac{h}{10^{-22}}\right)^2 \left(\frac{f}{1 \text{ kHz}}\right)^2$$

In contrast, the Solar Constant is about 1.4×10^6 mW m^{-2}. But the full moon is $\sim 10^6\times$ fainter than the sun, and gravitational waves carry energy comparable to that!

For a single gravitational wave event of duration τ, the observed "strain" (amplitude) h scales approximately as:

$$(90) \quad h = 10^{-21} \left(\frac{E_{GW}}{0.01 M_\odot c^2}\right)^{1/2} \left(\frac{r}{20 \text{ Mpc}}\right)^{-1} \left(\frac{f}{1 \text{ kHz}}\right)^{-1} \left(\frac{\tau}{1 \text{ ms}}\right)^{-1/2}$$

With today's LIGO strain sensitivity of $< 10^{-22}$, this means they should be sensitive to events out to at least the Virgo cluster (or further for stronger signals).

And as a final aside, note that the first detection of the presence of gravitational waves came not from LIGO but from observations of binary neutron stars. As the two massive objects rapidly orbit each other, gravitational waves steadily sap energy from the system, causing the orbits to steadily decay. When at least one of the neutron stars is a pulsar, this orbital decay can be measured to high precision.

10 Radiation

The number of objects directly detected via gravitational waves can be counted on two hands and a toe (11 as of early 2019). In contrast, billions and billions of astronomical objects have been detected via *electromagnetic* radiation. Throughout history and up to today, astronomy is almost completely dependent on EM radiation, as photons and/or waves, to carry the information we need to observatories on or near Earth.

To motivate us, let's compare two spectra of similarly hot sources, shown in Fig. 11.

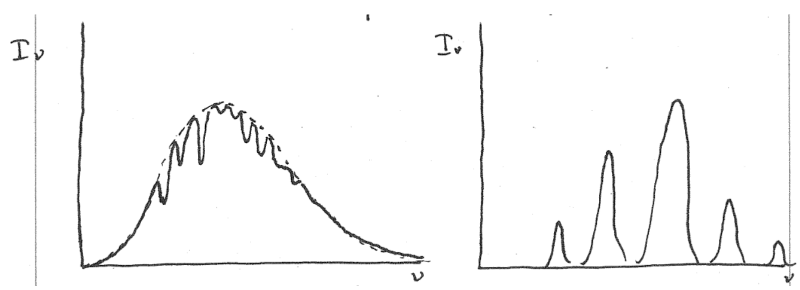

Figure 11: Toy spectra of two hot sources, $\sim 10^4$ K. Left: a nearly-blackbody A0 star with a few absorption lines. Right: central regions of Orion Nebula, showing only emission lines and no continuum.

In a sense we're moving backward: we'll deal later with how these photons are actually created. For now, our focus is on the **radiative transfer** from source to observer. We want to develop the language to explain and describe the difference between these spectra of two hot gas masses.

10.1 *Radiation from Space*

The light emitted from or passing through objects in space is almost the only way that we have to probe the vast majority of the universe we live in. The most distant object to which we have traveled and brought back samples, besides the moon, is a single asteroid. Collecting solar wind gives us some insight into the most tenuous outer layers of our nearby star, and meteorites on earth provide insight into planets as far away as Mars, but these are the only things from space that we can study in laboratories on earth. Beyond this, we have sent unmanned missions to land on Venus, Mars, and asteroids and comets. To study anything else in space we have to interpret the radiation we get from that source. As a result, understanding the properties of radiation, including the variables and quantities it depends on and how it behaves as it moves through space, is then key to interpreting almost all of the fundamental observations we make as astronomers.

Energy

To begin to define the properties of radiation from astronomical objects, we will start with the energy that we receive from an emitting source somewhere in space. Consider a source of radiation in the vacuum of space (for familiarity, you can think of the sun). At some point in space away from our source of radiation we want to understand the amount of energy dE that is received from this source. What is this energy proportional to?

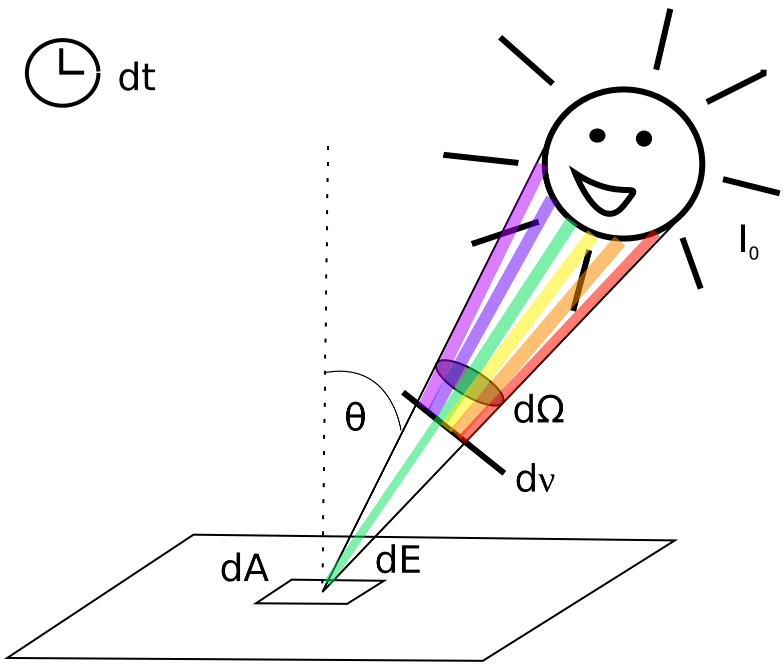

Figure 12: Description of the energy detected at a location in space for a period of time dt over an area dA arriving at an angle θ from an object with intensity I_0, an angular size $d\Omega$, through a frequency range $d\nu$ (in this case, only the green light).

As shown in Figure 12, our source of radiation has an intensity I_0 (we will get come back to this in a moment) over an apparent angular size (solid angle) of $d\Omega$. Though it may give off radiation over a wide range of frequencies, as is often the case in astronomy we only concern ourselves with the energy emitted in a specific frequency range $\nu+d\nu$ (think of using a filter to restrict the colors of light you see, or even just looking at something with your eyeball, which only detects radiation in the visible range). At the location of detection, the radiation passes through some area dA in space (an area perhaps like a spot on the surface of Earth) at an angle θ away from the normal to that surface. The last property of the radiation that we might want to consider is that we are detecting it over a given window of time (and many astronomical sources are time-variable). You might be wondering why the distance between our detector and the source is not being mentioned yet: we will get to this.

Considering these variables, the amount of energy that we detect will be proportional to the apparent angular size of our object, the range of frequencies over which we are sensitive, the time over which we collect the radiation, and the area over which we do this collection. The constant of proportionality is the **specific intensity** of our source: I_0. Technically, as this is the intensity just over a limited frequency range, we will write this as $I_{0,\nu}$.

In equation form, we can write all of this as:

$$dE_\nu = I_{0,\nu}\ cos\theta\ dA\ d\Omega\ d\nu\ dt \tag{91}$$

Here, the cos θ dA term accounts for the fact that the area that matters is actually the area "seen" from the emitting source. If the radiation is coming straight down toward our unit of area dA, it "sees" an area equal to that of the full dA (cos $\theta = 1$). However, if the radiation comes in at a different angle θ, then it "sees" our area dA as being tilted: as a result, the apparent area is smaller (cos $\theta < 1$). You can test this for yourself by thinking of the area dA as a sheet of paper, and observing how its apparent size changes as you tilt it toward or away from you.

Intensity

Looking at Equation 91, we can figure out the units that the specific intensity must have: energy per time per frequency per area per solid angle. In SI units, this would be W Hz^{-1} m^{-2} sr^{-1}. Specific intensity is also sometimes referred to as **surface brightness**, as this quantity refers to the brightness over a fixed angular size on the source (in O/IR astronomy, surface brightness is measured in magnitudes per square arcsec). Technically, the specific intensity is a 7-dimensional quantity: it depends on position (3 space coordinates), direction (two more coordinates), frequency (or wavelength), and time. As we'll see below, we can equivalently parameterize the radiation with three coordinates of position, three of momentum (for direction, and energy/frequency), and time.

Flux

The **flux density** from a source is defined as the total energy of radiation received from all directions at a point in space, per unit area, per unit time, per frequency. Given this definition, we can modify equation 91 to give the flux density at a frequency ν:

$$F_\nu = \int_\Omega \frac{dE_\nu}{dA\ dt\ d\nu} = \int_\Omega I_\nu cos\theta d\Omega \tag{92}$$

The total flux at all frequencies (the **bolometric flux**) is then:

$$F = \int_\nu F_\nu\, d\nu \tag{93}$$

As expected the SI units of flux are W m^{-2}; e.g., the aforementioned Solar Constant (the flux incident on the Earth from the Sun) is roughly 1400 W m^{-2}.

The last, related property that one should consider (particularly for spatially well-defined objects like stars) is the **Luminosity**. The luminosity of a source is the total energy emitted per unit time. The SI unit of luminosity is just Watts. Luminosity can be determined from the flux of an object by integrating over its entire surface:

$$L = \int F \, dA \tag{94}$$

As with flux, there is also an equivalent luminosity density, L_ν, defined analogously to Eq. 93.

Having defined these quantities, we now ask how the flux you detect from a source varies as you increase the distance to the source. Looking at Figure 13, we take the example of our happy sun and imagine two spherical shells or bubbles around the sun: one at a distance R_1, and one at a distance R_2. The amount of energy passing through each of these shells per unit time is the same: in each case, it is equal to the luminosity of the sun, $L_\odot$. However, as $R_2 > R_1$, the surface area of the second shell is greater than the first shell. Thus, the energy is spread thinner over this larger area, and the flux (which by definition is the energy per unit area) must be smaller for the second shell. Comparing the equations for surface area, we see that flux decreases proportionally to $1/d^2$.

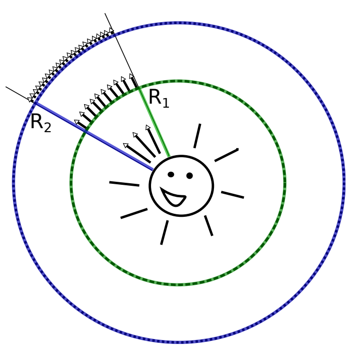

Figure 13: A depiction of the flux detected from our sun as a function of distance from the sun. Imagining shells that fully enclose the sun, we know that the energy passing through each shell per unit time must be the same (equal to the total luminosity of the sun). As a result, the flux must be less in the larger outer shell: reduced proportionally to $1/d^2$

10.2 *Conservation of Specific Intensity*

We have shown that the flux obeys an inverse square law with distance from a source. How does the specific intensity change with distance? The specific intensity can be described as the flux divided by the angular size of the source, or $I_\nu \propto F_\nu/\Delta\Omega$. We have just shown that the flux decreases with distance, proportional to $1/\mathrm{d}^2$. What about the angular source size? It happens that the source size also decreases with distance, proportional to $1/\mathrm{d}^2$. As a result, the specific intensity (just another name for surface brightness) is independent of distance.

Let's now consider in a bit more detail this idea that I_ν is conserved in empty space – this is a key property of radiative transfer. This means that in the absence of any material (the least interesting case!) we have $dI_\nu/ds = 0$, where s measures the path length along the traveling ray. And we also know from electrodynamics that a monochromatic plane wave in free space has a single, constant frequency ν. Ultimately our goal will be to connect I_ν to the flow of energy dE – this will eventually come by linking the energy flow to the number flow dN and the energy per photon,

$$dE = dN(h\nu) \tag{95}$$

We mentioned above that I_ν can be parameterized with three coordinates of position, three of momentum (for direction, and energy/frequency), and time. So $I_\nu = I_\nu(\vec{r}, \vec{p}, t)$. For now we'll neglect the dependence on t, assuming a constant radiation field – so our radiation field fills a particular six-dimensional phase space of $\vec{r}$ and $\vec{p}$.

This means that the particle distribution N is proportional to the phase space density f:

$$dN = f(\vec{r}, \vec{p})d^3r d^3p \tag{96}$$

By **Liouville's Theorem**, given a system of particles interacting with conservative forces, the phase space density $f(\vec{r}, \vec{p})$ is conserved along the flow of particles; Fig. 14 shows a toy example in 2D (since 6D monitors and printers aren't yet mainstream).

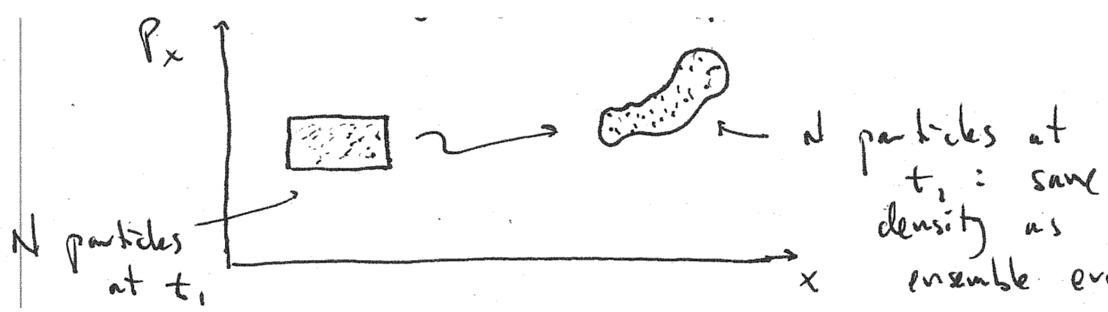

Figure 14: Toy example of Liouville's Theorem as applied to a 2D phase space of (x, p_x). As the system evolves from t_1 at left to t_2 at right, the density in phase space remains constant.

In our case, the particles relevant to Liouville are the photons in our ra-

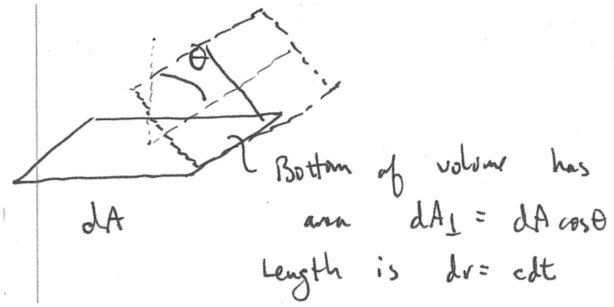

Figure 15: Geometry of the incident radiation field on a small patch of area dA.

diation field. Fig. 15 shows the relevant geometry. This converts Eq. 96 into

$$dN = f(\vec{r}, \vec{p}) c dt dA \cos\theta d^3 p \tag{97}$$

As noted previously, $\vec{p}$ encodes the radiation field's direction and energy (equivalent to frequency, and to linear momentum p) of the radiation field. So we can expand $d^3 p$ around the propagation axis, such that

$$d^3 p = p^2 dp d\Omega \tag{98}$$

This means we then have

$$dN = f(\vec{r}, \vec{p}) c dt dA \cos\theta p^2 dp d\Omega \tag{99}$$

Finally recalling that $p = h\nu / c$, and throwing everything into the mix along with Eq. 95, we have

$$dE = (h\nu) f(\vec{r}, \vec{p}) c dt dA \cos\theta \left(\frac{h\nu}{c}\right)^2 \left(\frac{h d\nu}{c}\right) d\Omega \tag{100}$$

We can combine this with Eq. 91 above, to show that specific intensity is directly proportional to the phase space density:

$$I_\nu = \frac{h^4 \nu^3}{c^2} f(\vec{r}, \vec{p}) \tag{101}$$

Therefore whenever phase space density is conserved, I_ν / ν^3 is conserved. And since ν is constant in free space, I_ν is conserved as well.

10.3 *Blackbody Radiation*

For radiation in thermal equilibrium, the usual statistical mechanics references show that the Bose-Einstein distribution function, applicable for photons, is:

$$n = \frac{1}{e^{h\nu/k_B T} - 1} \tag{102}$$

The phase space density is then

$$f(\vec{r}, \vec{p}) = \frac{2}{h^3} n \tag{103}$$

where the factor of two comes from two photon polarizations and h^3 is the elementary phase space volume. Combining Eqs. 101, 102, and 103 we find that in empty space

$$I_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/k_B T} - 1} \equiv B_\nu(T) \tag{104}$$

Where we have now defined $B_\nu(T)$, the **Planck blackbody function**. The Planck function says that the specific intensity (i.e., the surface brightness) of an object with perfect emissivity depends only on its temperature, T.

Finally, let's define a few related quantities for good measure:

(105)
$$J_\nu = \text{specific mean intensity}$$
(106)
$$= \frac{1}{4\pi} \int I_\nu d\Omega$$
(107)
$$= B_\nu(T)$$

(108)
$$u_\nu = \text{specific energy intensity}$$
(109)
$$= \int \frac{I_\nu}{c} d\Omega$$
(110)
$$= \frac{4\pi}{c} B_\nu(T)$$

(111)

$$P_\nu = \text{specific radiation pressure}$$

(112)

$$= \int \frac{I_\nu}{c} \cos^2 \theta d\Omega$$

(113)

$$= \frac{4\pi}{3c} B_\nu(T)$$

The last quantity in each of the above is of course only valid in empty space, when $I_\nu = B_\nu$. Note also that the correlation $P_\nu = u_\nu/3$ is valid whenever I_ν is isotropic, regardless of whether we have a blackbody radiation.

10.4 *Radiation, Luminosity, and Temperature*

(See Chp. 4).

11 Radiative Transfer

Radiation through empty space is what makes astronomy possible, but it isn't so interesting to study on its own. **Radiative transfer**, the effect on radiation of its passage through matter, is where things really get going.

11.1 *The Equation of Radiative Transfer*

We can use the fact that the specific intensity does not change with distance to begin deriving the radiative transfer equation. For light traveling in a vacuum along a path length s, we say that the intensity is a constant. As a result,

$$\frac{dI_\nu}{ds} = 0 \; (for\, radiation\, traveling\, through\, a\, vacuum) \tag{114}$$

This case is illustrated in the first panel of Figure 16. However, space (particularly objects in space, like the atmospheres of stars) is not a vacuum everywhere. What about the case when there is some junk between our detector and the source of radiation? This possibility is shown in the second panel of Figure 16. One quickly sees that the intensity you detect will be less than it was at the source. You can define an **extinction coefficient** α_ν for the space junk, with units of extinction (or fractional depletion of intensity) per distance (path length) traveled, or m^{-1} in SI units. For our purposes right now, we will assume that this extinction is uniform and frequency-independent (but in real life of course, it never is).

We also define

$$\alpha_\nu = n\sigma_\nu \tag{115}$$

$$= \rho\kappa_\nu \tag{116}$$

Where n is the number density of absorbing particles and σ_ν is their frequency-dependent cross-section, while ρ is the standard mass density and κ_ν is the

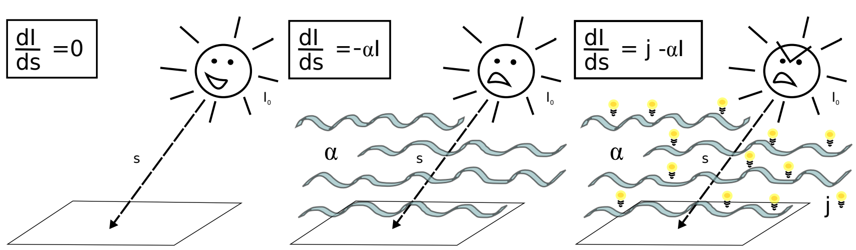

Figure 16: The radiative transfer equation, for the progressively more complicated situations of: (left) radiation traveling through a vacuum; (center) radiation traveling through a purely absorbing medium; (right) radiation traveling through an absorbing and emitting medium.

frequency-dependent **opacity**. Now, our equation of radiative transfer has been modified to be:

(117) $$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu \;(when\ there\ is\ absorbing\ material\ between\ us\ and\ our\ source)$$

As is often the case when simplifying differential equations, we then find it convenient to try to get rid of some of these pesky units by defining a new unitless constant: τ_ν, or **optical depth**. If α_ν is the fractional depletion of intensity per path length, τ_ν is just the fractional depletion. We then can define

(118) $$d\tau_\nu = \alpha_\nu ds$$

and re-write our equation of radiative transfer as:

(119) $$\frac{dI_\nu}{d\tau} = -I_\nu$$

Remembering our basic calculus, we see that this has a solution of the type

(120) $$I_\nu(s) = I_\nu(0)\exp\left(-\int_0^s d\tau_\nu\right)$$

(121) $$= I_\nu(0)e^{-\tau_\nu}\;(for\ an\ optically\ thin\ source)$$

So, at an optical depth of unity (the point at which something begins to be considered optically thick), your initial source intensity I_0 has decreased by a factor of *e*.

However, radiation traveling through a medium does not always result in a net decrease. It is also possible for the radiation from our original source to pass through a medium or substance that is not just absorbing the incident radiation but is also emitting radiation of its own, adding to the initial radiation field. To account for this, we define another coefficient: j_ν. This **emissivity coefficient** has units of energy per time per volume per frequency per solid angle. Note that these units (in SI: W m^{-3} Hz^{-1} sr^{-1}) are slightly different than the units of specific intensity. Including this coefficient in our radiative transfer equation we have:

(122) $$\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu$$

or, putting it in terms of the dimensionless optical depth τ, we have:

(123) $$\frac{dI_\nu}{d\tau_\nu} = \frac{j_\nu}{\alpha_\nu} - I_\nu$$

After defining the so-called **source function**

$$(124)\quad S_\nu = \frac{j_\nu}{\alpha_\nu}$$

we arrive at the final form of the **radiative transfer equation**:

$$(125)\quad \frac{dI_\nu}{d\tau_\nu} = S_\nu\, -\, I_\nu$$

11.2 *Solutions to the Radiative Transfer Equation*

What is the solution of this equation? For now, we will again take the simplest case and assume that the medium through which the radiation is passing is uniform (i.e., S_ν is constant). Given an initial specific intensity of $I_\nu(s=0) = I_{\nu,0}$, we obtain

$$(126)\quad I_\nu = I_{\nu,0}e^{-\tau_\nu} + S_\nu\left(1 - e^{-\tau_\nu}\right)\ \ (for\, constant\, source\, function)$$

What happens to this equation when τ is small? In this case, we haven't traveled very far through the medium and so should expect that absorption or emission hasn't had a strong effect. And indeed, in the limit that $\tau_\nu = 0$ we see that $I_\nu = I_{\nu,0}$.

What happens to this equation when τ becomes large? In this case, we've traveled through a medium so optically thick that the radiation has "lost all memory" of its initial conditions. Thus $e^{-\tau_\nu}$ becomes negligible, and we arrive at the result

$$(127)\quad I_\nu = S_\nu\ \ (for\, an\, optically\, thick\, source)$$

So the only radiation that makes it out is from the emission of the medium itself. What is this source function anyway? For a source in thermodynamic equilibrium, any opaque (i.e., optically thick) medium is a "black body" and so it turns out that $S_\nu = B_\nu(T)$, the Planck blackbody function. For an optically-thick source (say, a star like our sun) we can use Eq. 127 to then say that $I_\nu = B_\nu$.

The equivalence that $I_\nu = S_\nu = B_\nu$ gives us the ability to define key properties of stars – like their flux and luminosity – as a function of their temperature. As described in the preceding chapter, using Eq. 92 and 93 we can integrate the blackbody function to determine the flux of a star (or other blackbody) as a function of temperature, the Stefan-Boltzmann law:

$$(128)\quad F = \sigma T^4$$

Another classic result, the peak frequency (or wavelength) at which a star (or other blackbody) radiates, based on its temperature, can be found by differentiating the blackbody equation with respect to frequency (or wavelength). The result must be found numerically, and the peak wavelength can be expressed in **Wien's Law** as

(129) $$\lambda_{peak} = \frac{2.898 \times 10^{-3} \text{ m K}}{T}$$

We can improve on Eq. 126 and build a formal, general solution to the radiative transfer equation as follows. Starting with Eq. 125, we have

(130)
$$\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu$$

(131)
$$\frac{dI_\nu}{d\tau_\nu} e^{\tau_\nu} = S_\nu e^{\tau_\nu} - I_\nu e^{\tau_\nu}$$

(132)
$$\frac{d}{d\tau_\nu}\left(I_\nu e^{\tau_\nu}\right) = S_\nu e^{\tau_\nu}$$

We can integrate this last line to obtain the formal solution:

(133) $$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu(\tau'_\nu) e^{(\tau'_\nu - \tau_\nu)} d\tau'_\nu$$

As in our simpler approximations above, we see that the initial intensity $I_\nu(0)$ decays as the pathlength increases; at the same time we pick up an increasing contribution from the source function S_ν, integrated along the path. In practice S_ν can be fairly messy (i.e., when it isn't the Planck function), and it can even depend on I_ν. Nonetheless Eq. 133 lends itself well to a numerical solution.

11.3 *Kirchhoff's Laws*

We need to discuss one additional detail before getting started on stars and nebulae: **Kirchhoff's Law for Thermal Emission.** This states that a thermally emitting object in equilibrium with its surrounding radiation field has $S_\nu = B_\nu(T)$.

Note that the above statement does *not* require that our object's thermal radiation is necessarily blackbody radiation. Whether or not that is true depends on the interactions between photons and matter – which means it depends on the optical depth τ_ν.

Consider two lumps of matter, both at T. Object one is optically thick, i.e. $\tau_\nu >> 1$. In this case, Eq. 133 does indeed require that the emitted radiation has the form $I_\nu(\tau_\nu) = S_\nu = B_\nu(T)$ — i.e., blackbody radiation emerges from an optically thick object. This is mostly the case for a stellar spectrum, but not quite (as we'll see below).

First, let's consider the other scenario in which our second object is optically thin, i.e. $0 < \tau_\nu << 1$. If our initial specific intensity $I_\nu(0) = 0$, then we have

$$I_\nu(\tau_\nu) = 0 + S_\nu\,(1 - (1 - \tau_\nu)) \tag{134}$$

$$= \tau_\nu B_\nu(T) \tag{135}$$

Thus for an optically thin object, the emergent radiation will be blackbody radiation, scaled down by our low (but nonzero) τ_ν.

It's important to remember that τ_ν is frequency-dependent (hence the ν subscript!) due to its dependence on the extinction coefficient α_ν. So most astronomical objects represent a combination of the two cases discussed immediately above. At frequencies where atoms, molecules, etc. absorb light most strongly, α_ν will be higher than at other frequencies.

So in a simplistic model, assume we have a hot hydrogen gas cloud where α_ν is zero everywhere except at the locations of H lines. The location of these lines is given by the Rydberg formula,

$$\frac{1}{\lambda_{\mathrm{vac},1,2}} = R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right) \tag{136}$$

(where $R = 1/(91.2\text{ nm})$ is the Rydberg constant and $n_1 = 1, 2, 3, 4, 5$, etc. for the Lyman, Balmer, Paschen, and Brackett series, respectively).

In a thin gas cloud of temperature T, thickness s, and which is "backlit" by a background of empty space (so $I_{\nu,0} \approx 0$), from Eq. 135 all we will see is $\tau_\nu B_\nu(T) = \alpha_\nu s B_\nu(T)$ — so an **emission-line spectrum** which is zero away from the lines and has strong emission at the locations of each line.

What about in a stellar atmosphere? A single stellar T (an **isothermal atmosphere**) will yield just a blackbody spectrum, regardless of the form of α_ν. The simplest atmosphere yielding an interesting spectrum is sketched in Fig. 17: an optically thick interior at temperature T_H and a cooler, optically thin outer layer at $T_C < T_H$.

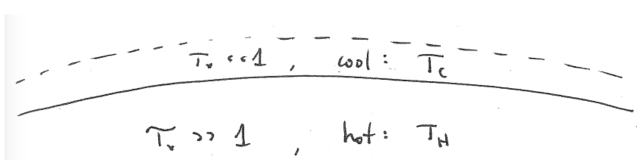

Figure 17: The simplest two-layer stellar atmosphere: an optically thick interior at temperature T_H and a cooler, optically thin outer layer at $T_C < T_H$.

The hot region is optically thick, so we have $I_\nu = S_\nu = B_\nu(T_H)$ emitted from the lower layer – again, regardless of the form of α_ν. The effect of the upper, cooler layer which has small but nonzero τ_ν is to slightly diminish the contribution of the lower layer while adding a contribution from the cooler layer:

$$I_\nu = I_\nu(0)e^{-\tau_\nu} + S_\nu\left(1 - e^{-\tau_\nu}\right) \tag{137}$$

$$= B_\nu(T_H)e^{-\tau_\nu} + B_\nu(T_C)\left(1 - e^{-\tau_\nu}\right) \tag{138}$$

$$\approx B_\nu(T_H)(1 - \tau_\nu) + B_\nu(T_C)\tau_\nu \tag{139}$$

$$\approx B_\nu(T_H) - \tau_\nu\left(B_\nu(T_H) - B_\nu(T_C)\right) \tag{140}$$

$$\approx B_\nu(T_H) - \alpha_\nu s\left(B_\nu(T_H) - B_\nu(T_C)\right) \tag{141}$$

So a stellar spectrum consists of two parts, roughly speaking. The first is $B_\nu(T_H)$, the contribution from the blackbody at the base of the atmosphere (the **spectral continuum**). Subtracted from this is a contribution wherever α_ν is strong – i.e., at the locations of strongly-absorbing lines. As we will see later, we can typically observe in a stellar atmosphere only down to $\tau_\nu \sim 1$. So at the line locations where (absorption is nonzero), we observe approximately $B_\nu(T_C)$. Thus in this toy model, the lines probe higher in the atmosphere (we can't observe as deeply into the star, because absorption is stronger at these frequencies – so we effectively observe the cooler, fainter upper layers). Meanwhile there is effectively no absorption in the atmosphere, so we see down to the hotter layer where emission is brighter. Fig. 18 shows a typical example.

Figure 18: Toy stellar spectrum (solid line) for the toy stellar model graphed in Fig. 17.

Note that our assumption has been that temperature in the star decreases with increasing altitude. More commonly, stellar models will parameterize an atmosphere in terms of its **pressure-temperature profile**, with pressure P decreasing monotonically with increasing altitude. An interesting phenomenon occurs when T increases with decreasing P (increasing altitude): in this case we have a **thermal inversion**, all the arguments above are turned on their heads, and the lines previously seen in absorption now appear in emission over the same continuum. Thermal inversions are usually a second-order correction to atmospheric models, but they are ubiquitous in the atmospheres of the Sun, Solar System planets, and exoplanets.

12 Stellar Classification, Spectra, and Some Thermodynamics

Questions you should be able to answer after this lecture:

- What is the difference between Thermal equilibrium and Thermodynamic equilibrium?
- What are the different temperatures that must be equal in Thermodynamic equilibrium?
- When is Local Thermodynamic Equilibrium valid for a region?

12.1 *Thermodynamic Equilibrium*

Our goal is to quantitatively explain the trends observed in Fig. 4. To do that, we need the tools provided to us by thermodynamics and statistical mechanics. We claimed earlier that $S_\nu = B_\nu(T)$, the source function is equal to the blackbody function, for a source in thermodynamic equilibrium. So, what are the conditions of thermodynamic equilibrium, and in what typical astronomical sources are these conditions satisfied?

There are two main conditions for thermodynamic equilibrium.

1. **Thermal equilibrium**: There is no heat transfer in a source: classically, it is at a constant, uniform temperature. However, as we will describe further in Section 15.2, for a star we generally take this just to mean that the temperature can vary spatially (but not in time), and that local energy losses (say, due to energy transport) are exactly balanced by gains (say, due to nuclear fusion).
2. Every temperature in the source is the same: the source is also in a radiation, ionization, and excitation equilibrium.

So, how do we define all of these different temperatures a source or system can have?

First, there is the **kinetic temperature** T_{kin}. This temperature describes the random motion of particles in a system. For a system in thermodynamic equilibrium, the distribution of speeds of particles (atoms or molecules) in this system is given by the **Maxwell-Boltzmann distribution**:

$$dN_v = 4\pi\, n \left(\frac{m}{2\pi\, k_B T_{kin}}\right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2\, k_B T_{kin}}\right) dv \quad (142)$$

Here, dN_v is the number of particles with mass m and number density n between speeds v and $v + dv$.

Second, there is the **excitation temperature** T_{ex}. This temperature describes the distribution of internal energies in the particles in a system. This internal energy can be the energy of different electronic states of an atom, or the energy of rotation or vibration in a molecule. For a system in thermodynamic equilibrium, the fraction of atoms (or molecules) occupying a particular energy state is given by the **Boltzmann distribution** (not to be confused with Eq. 142!):

$$\frac{N_1}{N_2} = \frac{g_1}{g_2} \exp\left(-\frac{E_1 - E_2}{k_B\, T_{ex}}\right) \tag{143}$$

Here, N_1 is the number of atoms or molecules in a state with an energy E_1 above the ground state, and N_2 is the number of atoms or molecules in a state with an energy E_2 above the ground state. The statistical weight of each state is given by g, which accounts for multiple configurations that might all have the same energy (i.e., the statistical degeneracy).

Next is T_{rad}, the **radiation temperature** in the system. This temperature is defined by an equation we have seen before: the Planck distribution, or the Blackbody law of Equation 16.

Finally, there is the **ionization temperature** T_i. This temperature describes the degree to which electrons are bound to the particles in a system. The fraction of the atoms in a gas which are ionized is given by the **Saha equation**, derived below and given as Eq. 171.

12.2 *Local Thermodynamic Equilibrium*

How typical is it for astronomical sources (like stars or planets or gas clouds) to be in thermodynamic equilibrium? In general, it is rare! Most sources are going to have significant temperature variations (for example, from the interior to the exterior of a star or planet). However, the situation is not hopeless, as in most sources, these changes are slow and smooth enough that over a small region, the two conditions we described are sufficiently satisfied. Such a situation is referred to as Local Thermodynamic Equilibrium or LTE.

When does LTE hold? First, for particles to have a Maxwell-Boltzmann distribution of velocities, and so to have a single kinetic temperature, the particles must have a sufficient opportunity to 'talk' to each other through collisions. Frequent collisions are also required for particles to have a uniform distribution of their internal energy states. The frequency of collisions is inversely proportional to the mean free path of the gas: the typical distance a particle travels before undergoing a collision. In general, for a region to be in LTE, the mean free path should be small compared to the distance over which the temperature varies appreciably. As LTE further requires that the radiation temperature is equal to the kinetic and excitation temperature, the matter and radiation must also be in equilibrium. For this to happen, not only must the mean free path for particles to undergo collisions with each other be small, but the mean free path for photons to undergo collisions with matter must be small as well. We have actually already introduced the mean free path for photons: it is equal to α^{-1}, where α was given in Equation 117 as the extinction coefficient, with units of fractional depletion of intensity per distance traveled. As intensity is depleted by being absorbed by matter, the inverse of the extinction coefficient describes the typical distance a photon will travel before interacting with matter.

Very qualitatively then, our two conditions for LTE are that the mean free path for particle-particle and particle-photon interactions must be less than the distance over which there is a significant temperature variation.

12.3 *Stellar Lines and Atomic Populations*

When we study stellar spectra, we examine how the strengths of various features change. Fig. 4 suggests that this is a continuous process as a function of T_{eff}. For example, we never see lines of both He I (i.e., neutral He) and Ca II (i.e., singly-ionized Ca, i.e. Ca^{+}) at the same time – these lines appear at completely different temperatures. What we want is a quantitative understanding of spectra.

When do we expect substantial excitation of these various atoms? Let's consider the electronic lines of atomic hydrogen. The H atom's energy levels are given by:

$$(144)\quad E_n = -\frac{13.6 \text{ eV}}{n^2}$$

which gives rise to the Rydberg formula (Eq. 136) for the locations of individual lines.

To see conditions we need to excite these H atoms, we might make use of the relative probability of 2 atomic states with different energies (given by the Boltzmann distribution, Eq. 143). Statistical mechanics tells us that the statistical weight of each level in a hydrogen atom is

$$(145)\quad g_n = 2n^2$$

So for transitions between the ground state (-13.6 eV, $n = 1$) and the first excited state (-3.4 eV, $n = 2$) the relative fraction is given by

$$(146)\quad \frac{n_1}{n_2} = \frac{g_1}{g_2} \exp\left[-\left(E_1 - E_2\right)/k_B T\right]$$

When the levels are approximately equal, we then have

$$(147)\quad 1 = \frac{2}{8} \exp\left[10.2 \text{ eV}/k_B T\right]$$

The calculation above would thus imply that to get appreciable levels of excited hydrogen, we would need $T \approx 90,000$ K — much hotter than the observed temperatures of stars. In fact, H is totally ionized (not just mildly excited) even at much lower temperatures. Meanwhile, even A and F stars (with $T_{\text{eff}} \leq 10,000$ K) show prominent $n = 2$ hydrogen lines. We got the energetics right, but missed some other important thermodynamic quantities.

12.4 *The Saha Equation*

Let's investigate our hydrogen atom in further detail. From statistical mechanics, the distribution function of particles leads to the phase space density (see

Eqns. 102 and 103):

(148) $$f(\vec{r}, \vec{p}) = \frac{g}{h^3} \frac{1}{e^{[E-\mu]/k_B T \pm 1}}$$

where μ is the chemical potential and g is still the degeneracy factor:

(149) $$g = 2s + 1 (\text{for fermions})$$

(150) $$g = 2 (\text{for photons})$$

and where the $\pm$ operator is positive for Fermi-Dirac statistics and negative for Bose-Einstein statistics.

Again, we'll transform this six-dimensional density into a number density by integrating over momentum (see Eq. 98):

(151) $$n = 4\pi \int_0^\infty f(p) p^2 dp$$

But integrating Eq. 148 is going to be a bear of a job, so we'll make two additional approximations. First, we'll assume for now that all particles are non-relativistic – so their energy is given classically by

(152) $$E = \frac{p^2}{2m} + mc^2$$

And we'll also assume that we're dealing with large energies, such that $E - \mu >> k_B T$. In practice, this second point means we can neglect the ± 1 in the denominator of Eq. 148. Both these assumptions are reasonable for the gas in most stars. We'll come back later to some especially interesting astrophysical cases, when these assumptions no longer hold.

We can now make the attempt to calculate n from Eq. 151.

(153)
$$n = 4\pi \int_0^\infty f(p) p^2 dp$$

(154)
$$= \frac{4\pi g}{h^3} \int_0^\infty p^2 dp \exp\left(\frac{\mu}{k_B T}\right) \exp\left(-\frac{mc^2}{k_B T}\right) \exp\left(-\frac{p^2}{2mk_B T}\right)$$

(155)
$$= \frac{4\pi g}{h^3} \exp\left[\left(\mu - mc^2\right)/k_B T\right] \int_0^\infty p^2 \exp\left(-\frac{p^2}{2mk_B T}\right)$$

(156)
$$= \frac{g}{h^3} \left(2\pi m k_B T\right)^{3/2} \exp\left[\left(\mu - mc^2\right)/k_B T\right]$$

So now we have a relation between the number density and other relevant quantities. We can rearrange this expression to get

(157) $$\exp\left(\frac{\mu - mc^2}{k_B T}\right) = \frac{1}{g}\frac{n}{n_Q}$$

where n_Q is the "quantum density"

(158) $$n_Q \equiv \left(2\pi m k_B T/h^2\right)^{3/2}$$

When $n = n_Q$, then the spacing between particles $n^{-1/3}$ is roughly equal to the thermal de Broglie wavelength — the particles' wave functions start to overlap, quantum effects ramp up, and degeneracy effects become increasingly important.

Ideally we want to get rid of the pesky μ and set things in terms of other quantities. Recall from thermodynamics that the **chemical potential** μ is just the energy absorbed or released during reactions. At constant volume V and entropy S, μ is determined by the change in internal energy U:

(159) $$\mu \equiv \left(\frac{\partial U}{\partial n}\right)\bigg|_{V,S}$$

The implication is that in equilibrium, all chemical potentials in a reaction sum to zero. So given a notional reaction

A + B $\longleftrightarrow$ C + D

we must have both

(160) $$A + B = C + D$$

and also

(161) $\mu_A + \mu_B = \mu_C + \mu_D$

Just as energies flow to equalize temperature and reach thermal equilibrium, numbers of different particle species flow to reach chemical equilibrium. For the H system under consideration, the reaction to ionize our hydrogen is

e^- + p $\longleftrightarrow$ H + γ

In chemical equilibrium, we will then have

(162) $\mu_e + \mu_p = \mu_H$

since the chemical potential of a photon is zero. Why is this useful? Because we can rearrange Eq. 157 to find an expression for μ, and then use this in what follows (we're getting close). We find

(163) $\mu_i = m_i c^2 + k_B T \ln\left(\frac{n_i}{g_i n_{Q_i}}\right)$

and of course mass-energy must also be conserved in the reaction:

(164) $m_H c^2 = m_p c^2 + m_e c^2 + \epsilon_n$

where $\epsilon_1 = -13.6$ eV for full ionization. The statistical weights are a tad trickier, but for our fermions we have $g_p = g_e = 2$ while for ionizing atomic H we have $g_H = n^2 g_p g_e = 4$.

Requiring that the chemical potentials must balance, following Eq. 162 we then have:

(165)
$$m_p c^2 + m_e c^2 + k_B T\left[\ln\left(\frac{n_p}{2n_{Q_p}}\right) + \ln\left(\frac{n_e}{2n_{Q_e}}\right)\right] = m_H c^2 + k_B T \ln\left(\frac{n_H}{4n_{Q_H}}\right)$$

Bringing in the results of Eq. 164, we then have

(166) $\ln\left(\frac{n_p n_e}{4 n_{Q_p} n_{Q_e}}\right) = \frac{-13.6\text{ eV}}{k_B T} + \ln\left(\frac{n_H}{4n_{Q_H}}\right)$

Rearranging terms, we then have

(167) $\frac{n_p n_e}{n_H}\frac{n_{Q_H}}{n_{Q_p} n_{Q_e}} = e^{-(13.6\text{ eV}/k_B T)}$

We can simplify this one more step by recalling from Eq. 158 that $n_{Q_p} \approx n_{Q_H}$. This means that we have finally reached our goal:

(168) $\frac{n_p n_e}{n_H} = n_{Q_e} e^{-(13.6\text{ eV}/k_B T)}$

which is famous as the **Saha equation** for hydrogen ionization. This tells us how the relative number densities of p, e^-, and H atoms will depend on the temperature of the system of particles.

It's traditional to refactor Eq. 168 by defining yet two more terms, the baryon number

$$(169)\quad n_B = n_H + n_p$$

(which is conserved) and the ionization fraction

$$(170)\quad y = \frac{n_e}{n_B}$$

which goes from zero (all neutral H) to unity (full ionization). When we divide both sides of Eq. 168 by n_B, we find the classical form of the **Saha equation**,

$$(171)\quad \frac{y^2}{1-y} = \frac{n_{Q_e}}{n_B} e^{-(13.6 \text{ eV}/k_B T)}$$

In a stellar photosphere, decent estimates are that $n_B \sim 10^{16}$ cm^{-3} and $n_{Q_e} \approx 10^{21}$ cm^{-3} $\left(T/10^4 \text{ K}\right)^{3/2}$. Eq. 171 is easily solved or plotted with numerical tools — the result, shown in Fig. 19, is a steep function of temperature that indicates ionization setting in at much lower temperatures than inferred in Eq. 147 alone. Instead, we see essentially no ionization in the 5800 K Solar photosphere, but we expect an ionization fraction of 5% at 9,000 K, rising to 50% at 12,000 K and 95% at 16,000 K. Although the Saha equation is a toy model with only two level populations, it still does an excellent job in predicting that H lines should be absent (as they are) from the hottest O and B stars.

In general, we also want to be able to properly treat the fact that there

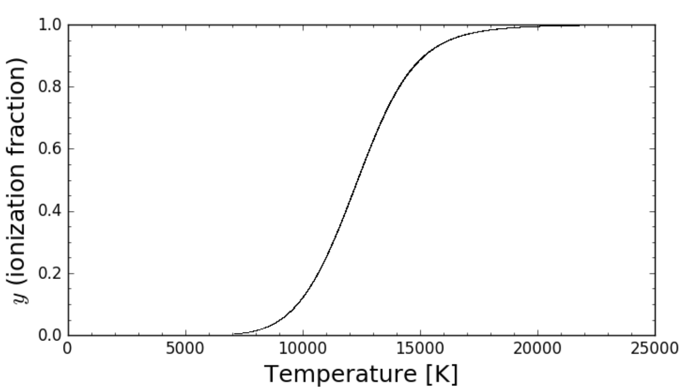

Figure 19: Ionization fraction y as a function of temperature T as inferred from the two-level Saha equation (Eq. 171).

are an infinite number of energy levels (not just two) between the ground state and full ionization. This means that we need to account for the **partition function** $Z(\bar{T})$,

(172) $Z(\bar{T}) \equiv \Sigma g_s e^{-E_s/k_B T}$

In principle one can calculate one's own partition functions, but in practice one often leaves that to the experts and borrows appropriately from the literature. So then the number density becomes

(173) $n = \left(\frac{2\pi m_e k_B T}{h^2}\right)^{3/2} e^{\mu/k_B T} Z(\bar{T})$

Using this new form to repeat the analysis above, equality of chemical potentials for the generic species A, B, and C will then yield

(174) $\frac{n_B n_C}{n_A} = \left(\frac{2\pi k_B T}{h^2}\right)^{3/2} \left(\frac{m_B m_C}{m_A}^{3/2}\right)^{3/2} \left(\frac{\bar{Z}_B \bar{Z}_C}{\bar{Z}_A}\right)$

If the partition function is dominated by a single state (as in our simple two-level example), we recover the earlier form:

(175) $\frac{n_B n_C}{n_A} = \left(\frac{2\pi k_B T}{h^2}\right)^{3/2} \left(\frac{m_B m_C}{m_A}^{3/2}\right)^{3/2} \frac{g_B g_C}{g_A} e^{(E_A - E_B - E_C)/k_B T}$

Note that for simple level-change reactions within atomic H, most factors cancel and we recover the usual Boltzmann distribution:

(176) $\frac{n_B}{n_A} = \frac{g_B}{g_A} e^{(E_A - E_B)/k_B T}$

As a few final remarks, note that the above analysis only applies for excitation caused by the thermal distribution of particles in our system. So this won't properly treat **photoionization** (i.e. ejection of an electron due to an incoming, highly-energetic photon). Also, everything here also requires mostly-classical conditions, i.e. $n << n_Q$ for all species involved. In very dense plasmas, pressure begins to affect electron orbital shapes and subsequently affects both intermediate energy levels as well as ionization.

13 Stellar Atmospheres

Having developed the machinery to understand the spectral lines we see in stellar spectra, we're now going to continue peeling our onion by examining its thin, outermost layer – the stellar atmosphere. Our goal is to understand how specific intensity I_ν varies as a function of increasing depth in a stellar atmosphere, and also how it changes depending on the angle relative to the radial direction.

13.1 *The Plane-parallel Approximation*

Fig. 20 gives a general overview of the geometry in what follows. The star is spherical (or close enough as makes no odds), but when we zoom in on a small enough patch the geometry becomes essentially plane-parallel. In that geometry, S_ν and I_ν depend on both altitude z as well as the angle θ from the normal direction. We assume that the radiation has no intrinsic dependence on either t or ϕ – i.e., the radiation is in steady state and is isotropic.

We need to develop a few new conventions before we can proceed. This is because in our definition of optical depth, $d\tau_\nu = \alpha_\nu ds$, the path length ds travels along the path. This Lagrangian description can be a bit annoying, so it's common to formulate our radiative transfer in a path-independent, Eulerian, prescription.

Let's call our previously-defined optical depth (Eq. 118) τ'_ν. We'll then create a slightly altered definition of optical depth – a vertical, ingoing optical depth (this is the convention). The new definition is almost identical to the old one:

(177) $d\tau_\nu = -\alpha_\nu dz$

But now our optical depth, is vertical and oriented to measure inward, toward the star's interior. In particular since $dz = ds \cos\theta$, relative to our old optical depth we now have

(178) $d\tau_\nu = -d\tau'_\nu \cos\theta$

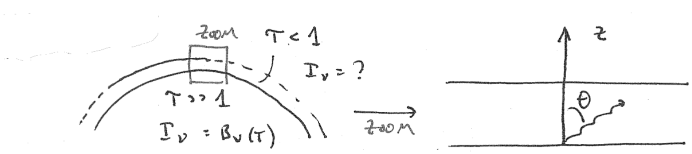

Figure 20: Schematic view of a stellar atmosphere, and at right a zoomed-in view showing the nearly plane-parallel nature on small scales.

Our radiative transfer equation, Eq. 125, now becomes

$$(179)\quad -\cos\theta \frac{dI_\nu}{d\tau_\nu} = S_\nu(\tau_\nu, \theta) - I_\nu(\tau_\nu, \theta)$$

It's conventional to also define $\mu = \cos\theta$, so our **radiative transfer equation** for stellar atmospheres now becomes

$$(180)\quad \mu \frac{dI_\nu}{d\tau_\nu} = I_\nu(\tau_\nu, \mu) - S_\nu(\tau_\nu, \mu).$$

We can solve this in an analogous manner to how we treated Eq. 132, multiplying all terms by $e^{-\tau_\nu/\mu}$, then rearranging to see that

$$(181)\quad \frac{d}{d\tau_\nu}\left(I_\nu e^{-\tau_\nu/\mu}\right) = -\frac{S_\nu}{\mu} e^{-\tau_\nu/\mu}.$$

Our solution looks remarkably similar to Eq. 133, except that we now explicitly account for the viewing angle μ:

$$(182)\quad I_\nu(\tau_\nu, \mu) = \int_{\tau_\nu}^{\infty} \frac{S_\nu(\tau'_\nu, \mu)}{\mu} e^{(\tau'_\nu - \tau_\nu)/\mu} d\tau'_\nu$$

So we now have at least a formal solution that could explain how I_ν varies as a function of the vertical optical depth τ_ν as well as the normal angle θ. It's already apparent that I_ν at a given depth is determined by the contributions from S_ν at all deeper levels, but these S_ν themselves depend on I_ν there. So we'd like to develop a more intuitive understanding than Eq. 182 provides.

Our goal will be to make a self-consistent model for S_ν and I_ν (or, as we'll see, I_ν and T). We'll again assume local thermodynamic equilibrium (LTE), so that

$$(183)\quad S_\nu = B_\nu(T) = B_\nu\left[T(\tau_\nu)\right]$$

(since T increases with depth into the star).

First, let's assume a simple form for S_ν so we can solve Eq. 182. We already tried a zeroth-order model for S_ν (i.e. a constant; see Eq. 126), so let's add a first-order perturbation, assuming that

$$(184)\quad S_\nu = a_\nu + b_\nu \tau_\nu$$

Where a_ν and b_ν are independent of τ_ν – for example, two blackbodies of different temperatures. When we plug this form into the formal solution of Eq. 182 and turn the crank, we find that the **emergent intensity** from the top of the star's atmosphere ($\tau_\nu = 0$) is

$$(185)\quad I_\nu(\tau_\nu = 0, \mu) = a_\nu + b_\nu \mu$$

Fig. 21 explains graphically what this solution means: namely, that the *an-*

gular dependence of a star's emergent radiation encodes the *depth* dependence of its atmosphere's source function. If the depth dependence is small, so will the angular dependence be – and the reverse will also hold. So if $b_\nu \approx 0$, I_ν will be nearly isotropic with θ.

This describes the phenomenon of **limb darkening**, wherein the center of a stellar disk appears brighter than the edge. This is commonly seen in photographs of the Sun – it often looks to the eye like merely shadow effects of a 3D sphere, but in fact this represents temperature stratification.

Another interesting consequence involves the fact that an observer can only typically observe down to $\tau_\nu \approx 1$. Because of the depth and angular dependencies we have just identified, this means that the surface where $\tau_\nu = 1$ (or any other constant value) occurs higher in the stellar atmosphere at the limb than at the disk center. Fig. 22 shows this effect. Since (as previously mentioned) temperature drops with decreasing pressure for most of a star's observable atmosphere, this means that we observe a cooler blackbody at the limb than at the center – and so the center appears brighter. (This is just a different way of thinking about the same limb-darkening effect mentioned above.) For the same reason, spectral lines look dark because at these lines α_ν is largest and so τ_ν occurs higher in the atmosphere, where temperatures are lower.

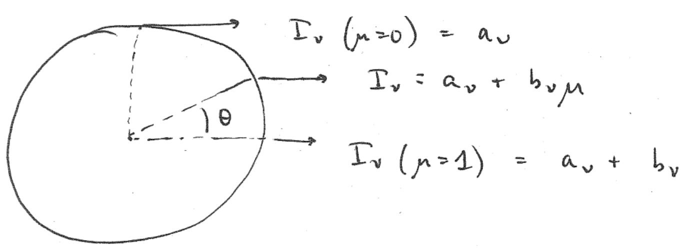

Figure 21: Emergent intensity as a function of θ assuming the linear model for S_ν given by Eq. 184.

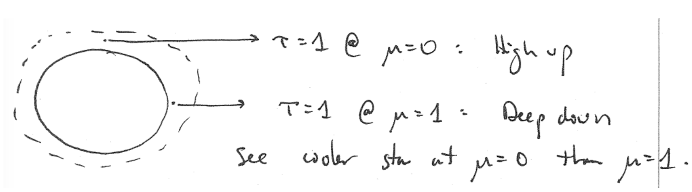

Figure 22: Depth dependence on the depth to which an observer can see into a stellar atmosphere: we see deeper at the center than at the limb.

13.2 *Gray Atmosphere*

Now let's try to build a more self-consistent atmospheric model. To keep things tractable, we'll compensate for adding extra complications by simplifying another aspect: we'll assume a **gray atmosphere** in which the absorption coefficient (and derived quantities are independent of frequency). So we will use α instead of α_ν, and τ in place of τ_ν.

In this case, the equation of radiative transfer still has the same form as in Eq. 180 above:

$$\mu \frac{dI}{d\tau} = I(\tau, \mu) - S(\tau, \mu) \tag{186}$$

With the difference that by ignoring frequency effects, we are now equivalently solving for the **bolometric quantities**

$$I = \int_\nu I_\nu \, d\nu \tag{187}$$

and

$$S = \int_\nu S_\nu \, d\nu \tag{188}$$

Up until now, we've always assumed LTE with a Planck blackbody source function whose temperature varies with depth. But we've only used ad hoc models for this source — now, let's introduce some physically meaningful constraints. Specifically, let's require that flux is conserved as it propagates through the atmosphere. This is equivalent to saying there is no energy generation in the atmosphere: we just input a bunch of energy at the base and let it transport through and escape from the top.

This requirement of **flux conservation** means that $\frac{dF}{dt} = 0$, where

$$F = \int I \cos\theta d\Omega = \int \mu I d\Omega \tag{189}$$

(by definition; see Eq. 92).

To apply this reasonable physical constraint, let's integrate Eq. 186 over all solid angles:

$$\int \mu \frac{\partial I}{\partial \tau} d\Omega = \int (I - S) d\Omega \tag{190}$$

which implies that

$$\frac{dF}{d\tau} = 4\pi \langle I \rangle - 4\pi S \tag{191}$$

which equals zero due to flux conservation. Note that S is isotropic, while in general I may not be (i.e. more radiation comes out of a star than goes into it from space). The perhaps-surprising implication is that in our gray atmosphere,

(192) $S = \langle I \rangle$

at all altitudes.

We can then substitute $\langle I \rangle$ for S in Eq. 186, to find

(193)
$$S = I - \mu \frac{dI}{d\tau}$$
(194)
$$\langle I \rangle = I - \mu \frac{dI}{d\tau}$$
(195)
$$\frac{1}{2} \int_{-1}^{1} I d\mu = I - \mu \frac{dI}{d\tau}$$

This is an integro-differential equation for gray atmospheres in radiative equilibrium. Though it looks odd, it is useful because an exact solution exists. After finding $\langle I \rangle = S$, we can use our formal solution to the radiative transfer equation to show that

(196) $S = \frac{3F_0}{4\pi} \left[\tau + q(\tau) \right]$

where F_0 is the input flux at the base of the atmosphere and $q(\tau)$ is the Hopf function, shown in Fig. 23. Let's examine an approximation to this function that provides a lot of insight into what's going on.

We'll start by examining the moments of the radiative transfer equation,

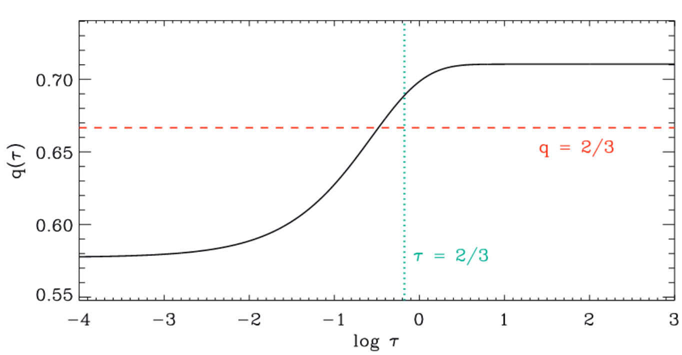

Figure 23: Hopf function $q(\tau)$, which has as its limits $q(0) = 1/\sqrt{3}$ and $q(\infty) = 0.7101....$

where moment n is defined as

$$\mu^{n+1} \int \frac{\partial I}{\partial \tau} d\Omega = \int \mu^n (I - S) d\Omega \tag{197}$$

We already did $n = 0$ back in Eq. 190, so let's consider $n = 1$. We'll need each of the following terms, given in Eqs. 198–200:

$$\int \mu S d\Omega = 2\pi S \int_{-1}^{1} \mu d\mu \tag{198}$$

which equals zero, since S is isotropic.

$$\int \mu I d\Omega \equiv F \tag{199}$$

(the definition of flux), and finally

$$\int \mu^2 I d\Omega = \int I \cos^2 \theta d\Omega \equiv c P_{\text{rad}} \tag{200}$$

from Eq. 113.

The first moment then becomes

$$c \frac{dP}{d\tau} = F \tag{201}$$

which we have required to be constant. Thus, we find that

$$P_{\text{rad}} = \frac{F}{c}(\tau + Q) \tag{202}$$

where here Q is a constant of integration; when certain assumptions are lifted, this becomes the Hopf function $q(\tau)$ of Fig. 23.

13.3 *The Eddington Approximation*

Eq. 202 is a potentially powerful result, because it tells us that in our gray, flux-conserving atmosphere the radiation pressure is just a linear function of the bolometric flux. This will become even more useful, since we are about to connect this back to S, and thence to I (a more useful observational diagnostic than P).

From the expression for the radiation pressure of a blackbody field (Eq. 113) we have that

$$P = \frac{4\pi}{3c} \int B_\nu d\nu \tag{203}$$

$$= \frac{4\pi}{3c} S \tag{204}$$

since we assume LTE, and thus our source function is the Planck function. *However*, remember that our atmosphere exhibits a temperature gradient, so our radiation field isn't actually a pure blackbody. We therefore make the key assumption — the **Eddington approximation** — that the temperature gradient is weak enough that the above expression for P is valid (correcting this assumption turns Q into $q(\tau)$).

Under these assumptions, we then combine Eq. 204 and 202 to find

(205)
$$S = \frac{3F}{4\pi}(\tau + Q)$$
(206)
$$= \frac{3F}{4\pi}\left(\tau + \frac{2}{3}\right)$$

Thus the stellar atmosphere's source function is just a linear function of optical depth – just as we had blithely assumed in Eq. 184 when introducing limb darkening. The value of 2/3 comes from a straightforward but tedious derivation described in Sec. 2.4.2 of the Choudhuri textbook.

We can also use Eq. 206 to clarify our previous discussion of limb darkening. Since we now know the particular linear dependence of S on τ, we can dispense with the arbitrary constants in Eq. 185 to show that the emergent intensity is

(207) $$I(\tau = 0, \mu) = \frac{3F}{4\pi}\left(\mu + \frac{2}{3}\right)$$

which shows decent agreement with observational data. This type of expression is called a **linear limb-darkening "law"**. Because of our assumptions this doesn't perfectly fit observed stellar limb-darkening profiles, so there is a whole family of various relations that people use (some physically justified, some empirical).

Finally, given the exact functional form of S in Eq. 206, we can now compute the stellar atmosphere's **thermal structure** – how its temperature changes with optical depth, pressure, or altitude. This relation is derived by relating S to the Stefan-Boltzmann flux F from Eq. 18:

(208)
$$S = \int S_\nu d\nu$$
(209)
$$= \int B_\nu d\nu$$
(210)
$$= \frac{\sigma_{SB} T^4}{\pi}$$

(as for that factor of π, see Sec. 1.3 of the Rybicki & Lightman). We now have

$S(T)$ as well as $S(\tau)$, so combining Eqs. 206, Eq. 210, and the Stefan-Boltzmann flux (Eq. 18) we obtain a relation that

$$T^4(\tau) = \frac{3}{4} T_{\text{eff}}^4 \left(\tau + \frac{2}{3} \right) \tag{211}$$

This gives us the thermal profile through the star's atmosphere. As we move deeper into the star the vertical optical depth τ increases (Eq. 177) and the temperature rises as well (Eq. 211). Note too that the atmospheric temperature $T = T_{\text{eff}}$ when $\tau = 2/3$. Earlier we have claimed that we see down to a depth of $\tau \approx 1$, so we have no refined that statement to say that we see into a stellar atmosphere down to the $\tau = 2/3$ surface.

13.4 *Frequency-Dependent Quantities*

We've achieved quite a bit, working only with frequency-integrated quantities: in particular, the temperature structure in Eq. 211 and the formal solution Eq. 182. However, although our earlier treatment of excitation and ionization of atomic lines (Sec. 12.4) qualitatively explains some of the trends in absorption lines seen in stellar spectra, we have so far only discussed line formation in the most qualitative terms.

We expect intensity to vary only slowly with frequency when temperatures are low. This because we expect the ratio $R(\nu)$ between intensities at two temperatures to scale as:

$$R(\nu) = \frac{I_\nu(T_B)}{I_\nu(T_A)} \tag{212}$$

$$\approx \frac{B_\nu(T_B)}{B_\nu(T_A)} \tag{213}$$

$$= \frac{e^{h\nu/kT_A} - 1}{e^{h\nu/kT_B} - 1} \tag{214}$$

This is consistent with the observed frequency dependence of limb darkening, which is seen to be much weaker at longer (infrared) wavelengths and stronger at shorter (e.g., blue-optical) wavelengths.

Let's now consider a more empirical way to make progress, based on the fact that we can observe the intensity emerging from the top of the atmosphere, $I_\nu(\tau_\nu = 0, \mu)$, across a wide range of frequencies. Expanding on our earlier, linear model of S_ν (Eq. 184), a fully valid expression for the source function is always

$$S_\nu = \sum_{n=0}^{\infty} a_{\nu,n} \tau_\nu^n \tag{215}$$

Putting this into our formal solution, Eq. 182, and invoking the definition of the gamma function gives

$$I_\nu(0,\mu) = \sum_{n=0}^{\infty} a_{\nu,n}(n!)\mu^n \tag{216}$$

So long as we are in LTE, then we also have $S_\nu(\tau_\nu) = B_\nu[T(\tau_\nu)]$. This lets us map out $T(\tau_\nu)$, which we can do for multiple frequencies – as shown in Fig. 24. Each T corresponds to a particular physical depth in the stellar atmosphere, so we have successfully identified a mapping between optical depth, frequency, and temperature.

Since for any ν we typically observe only down to a constant $\tau_\nu \approx 2/3$, we can rank the absorption coefficients α_{ν_i} for each of the ν_i sketched in Fig. 24. For any given τ_ν, $T(\tau_\nu)$ is greatest for ν_1 and least for ν_3. Thus we are seeing deepest into the star at ν_1 and α_{ν_1} must be relatively small, while on the other hand we see only to a shallow depth (where T is lower) at ν_3 and so α_{ν_3} must be relatively large.

13.5 *Opacities*

What affects a photon as it propagates out of a stellar atmosphere? So far we haven't talked much about the explicit frequency dependence of α_ν, but this is essential in order to interpret observations.

From the definition of α_ν in Eq. 116, one sees that

$$n\sigma_\nu = \rho\kappa_\nu \tag{217}$$

A lot of work in radiative transfer is about calculating the opacity κ_ν given ρ, T, and composition. In practice most desired opacities are tabulated and one uses a simple look-up table for ease of calculation. Nonetheless we can still consider some of the basic cases. These include:

1. Thomson (electron) scattering
2. Bound-bound reactions
3. Bound-free: photoionization & recombination

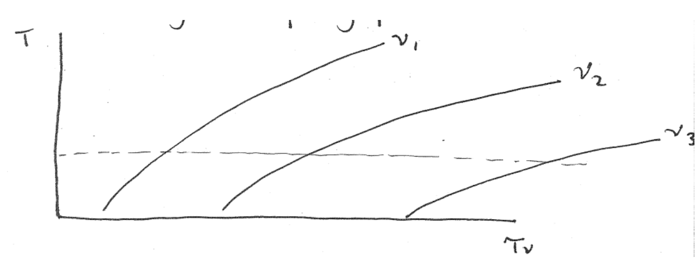

Figure 24: Notional atmospheric structure, $T(\tau_\nu)$, with different frequencies ν_i.

4. Free-free: Bremsstrahlung

Thomson scattering

The simplest effect is **Thomson scattering**, also known as electron scattering. In this interaction a photon hits a charged particle, shakes it up a bit (thus taking energy out of the radiation field), and is then re-radiated away. The basic (frequency-independent) cross-section is derived in many textbooks (e.g. Rybicki & Lightman, Sec. 3.4), which shows that in cgs units,

$$\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2 \tag{218}$$

or approximately $2/3 \times 10^{-24}$ cm^2 (or 2/3 of a "barn").

Notice that $\sigma_T \propto m^{-2}$, so the lightest charge-carriers are the most important – this means electrons. Eq. 218 suggests that in a notional medium composed solely of electrons, we would have

$$\kappa_\nu = \frac{n_e \sigma_T}{\rho_e} = \frac{\sigma_T}{m_e} \tag{219}$$

In any real astrophysical situation our medium will contain a wide range of particles, not just electrons. So in actuality we have

$$\kappa_\nu = \frac{n_e \sigma_T}{\rho_{\rm tot}} \equiv \frac{1}{\mu_e}\frac{\sigma_T}{m_p} \tag{220}$$

where we have now defined the **mean molecular weight of the electron** to be

$$\mu_e = \frac{\rho_{tot}}{n_e m_p} \tag{221}$$

The quantity μ_e represents the mean mass of the plasma per electron, in units of m_p (note that this is a bit different from the mean molecular weight for ions, which is important in stellar interior calculations). But in a fully ionized H-only environment, $n_e = N$ cm^{-3} while $\rho = N m_p$ cm^{-3} — so $\mu_e = 1$. Meanwhile in a fully ionized, 100% He plasma, $n_e = 2N$ cm^{-3} while $\rho = 4N$ cm^{-3} — so in this case, $\mu_e = 2$.

Bound-bound transitions

As the name implies, these involve changes between energy levels that still leave all particles bound. Most stellar opacity sources are of this type, which give rise to lines such as those depicted schematically in Fig. 25. In all cases the *intrinsic* line width $\Delta\nu \sim \hbar/\tau$ is given by the Heisenberg uncertainty principle and by τ, the typical lifetime of the state. Depending on the system being studied. But τ can change considerably depending on the system under analysis.

The general expression for the line's cross-section will be that

$$\sigma_\nu = \frac{\pi e^2}{m_e c} f \Phi(\nu - \nu_0) \quad (222)$$

where f is the transition's dimensionless oscillator strength (determined by the atomic physics) and $\Phi(\nu - \nu_0)$ is the line profile shape as sketched in Fig. 25. Note that Eq. 222 will also sometimes be written not in terms of f but rather as

$$\sigma_\nu = \frac{B_{LU} h\nu}{4\pi} \Phi(\nu - \nu_0) \quad (223)$$

where B_{LU} is the "Einstein B" coefficient for the transition from the lower to the upper state.

Regardless, the lifetime of the state may be intrinsic (and long-lived) if the particles involved are isolated and non-interacting, and undergo only **spontaneous emission**. This gives rise to the narrowest lines, which are said to be **naturally broadened.**

When conditions are denser and the particle interaction timescale $\lesssim \tau$, then collisions perturb the energy levels and so slightly higher- or lower-energy photons can couple to the particles involved. This leads to **pressure broadening** (or collisional broadening), which leads (as the name implies) to broader lines in higher-pressure environments.

Finally, particle velocities will impart a range of Doppler shifts to the observed line profile, causing various types of extrinsic broadening. In general these can all be lumped under the heading of **Doppler broadening**, in which the line width is set by the material's velocity,

$$\frac{\Delta\nu}{\nu_0} = \frac{v_r}{c} \quad (224)$$

This is an important effect for the accretion (or other) disks around black holes and around young stars, and also for the nearly-solid-body rotation of individual stars.

There's a lot more to say about bound-bound transitions than we have time for here. But whatever the specific situation, our approach will always be

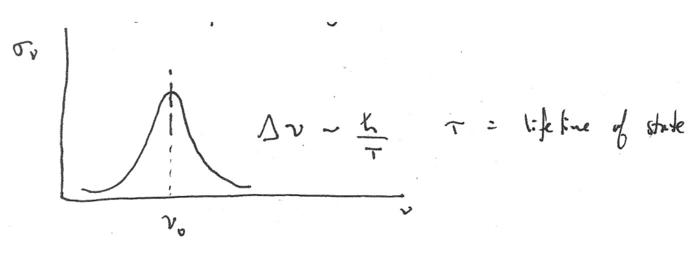

Figure 25: Schematic of σ_ν for bound-bound reactions, showing a line centered at ν_0 and with intrinsic width $\Delta\nu \sim \hbar/\tau$.

Figure 26: Bremsstrahlung (braking radiation) — an electron decelerates near an ion and emits a photon.

the following: use the Saha and Boltzmann equations to establish the populations in the available energy levels; then use atomic physics to determine the oscillator strength f and line profile $\Phi(\nu - \nu_0)$.

Bremsstrahlung

If you speak German, you might recognize that this translates as "braking radiation" — and Bremsstrahlung (or "free-free") is radiation caused by the deceleration of charged particles (typically electrons), as shown schematically in Fig. 26. Under time reversal, this phenomenon also represents absorption of a photon and acceleration of the electron. Typically this is modeled as occurring as the e^- is near (but not bound to) a charged but much more massive ion, which is assumed to be stationary during the interaction. Rybicki & Lightman devote a whole chapter to Bremsstrahlung, but we'll just settle for two useful rules of thumb:

$$\alpha_\nu^{ff} \approx 0.018 T^{-3/2} Z^2 n_e n_i \nu^{-2} \bar{g_{ff}} \tag{225}$$

and

$$\epsilon_\nu^{ff} \approx (6.8 \times 10^{-38}) Z^2 n_e n_i T^{-1/2} e^{-h\nu/kT} \bar{g_{ff}}$$

where Z is the ionic charge and $\bar{g_{ff}}$ is the **Gaunt factor**, typically of order unity.

Free-free absorption is dependent on both temperature and density, and is often commonly described by Kramer's opacity law:

$$\kappa_{ff} = \frac{1}{2}\kappa_{ff,0}(1+X)\langle\frac{\mathcal{Z}^2}{\mathcal{A}}\rangle \rho T^{-7/2} \tag{226}$$

Bound-free

In this case, electrons transition between a bound (possibly excited) state and the free (i.e., ionized) state. If the initial state is bound, then an incoming

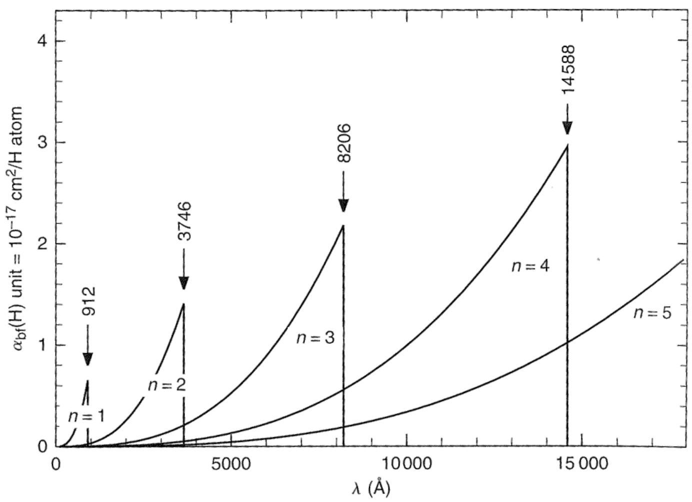

Figure 27: Extinction coefficient α_ν for bound-free transitions of the H atom (from Gray's *Stellar Photospheres*, Fig. 8.2). The characteristic scaling with $\nu^{-3} \propto \lambda^3$ is clearly apparent.

photon comes in and (possibly) ejects an electron. Thus the e^- begins within a series of discretized, quantum, atomic energy levels and ends unbound, with a continuum of energy levels available to it. A full derivation shows that for a given bound transition we find $\sigma_\nu \propto \nu^{-3}$. But as ν decreases toward the ionization threshold ν_i (e.g., 13.6 ev$/h$ for H in the ground state), then σ_ν will sharply drop when the photon is no longer able to ionize. But assuming there is some excited hydrogen (e.g., ν_2=13.6$/4h$ for H in the $n = 2$ state), then there will be a second one-sided peak located at ν_2, and so on as shown in Fig. 27.

H-minus opacity

For years after astronomers first turned their spectrographs toward the Sun, it was unclear which processes explained the observed Solar opacity. It was apparent that the opacity was fairly large, despite the fact that H and He are almost entirely neutral in the Solar photosphere and bound-bound transitions also weren't able to explain the data.

The solution turned out to be the **negative hydrogen ion**, H^-, which is stable because the normal H atom is highly polarized and can hold another e^-. The electron is bound only weakly, with a dissociation energy of just 0.75 eV (no stable, excited states exist). Thus all photons with $\lambda \lesssim 1.7\mu$m can potentially break this ion and, being absorbed, contribute to an overall continuum opacity that is strongest from 0.4–1.4μm. The magnitude of the total H^- opacity depends sensitively on the ion's abundance: it drops off steeply in stars much hotter than the Sun (when most H^- is ionized) and in the very coolest stars (when no free e^- are available to form the ion). In addition to being a key opacity source in many stars, H^- has only recently been recognized as a key opacity source in the atmospheres of the hottest extrasolar planets (see e.g. Lothringer et al., 2018).

14 Timescales in Stellar Interiors

Having dealt with the stellar photosphere and the radiation transport so relevant to our observations of this region, we're now ready to journey deeper into the inner layers of our stellar onion. Fundamentally, the aim we will develop in the coming chapters is to develop a connection between M, R, L, and T in stars (see Table 14 for some relevant scales).

More specifically, our goal will be to develop equilibrium models that describe stellar structure: $P(r)$, $\rho(r)$, and $T(r)$. We will have to model gravity, pressure balance, energy transport, and energy generation to get everything right. We will follow a fairly simple path, assuming spherical symmetric throughout and ignoring effects due to rotation, magnetic fields, etc.

Before laying out the equations, let's first think about some key timescales. By quantifying these timescales and assuming stars are in at least short-term equilibrium, we will be better-equipped to understand the relevant processes and to identify just what stellar equilibrium means.

14.1 *Photon collisions with matter*

This sets the timescale for radiation and matter to reach equilibrium. It depends on the **mean free path** of photons through the gas,

$$\ell = \frac{1}{n\sigma} \tag{227}$$

So by dimensional analysis,

$$\tau_\gamma \approx \frac{\ell}{c} \tag{228}$$

If we use numbers roughly appropriate for the average Sun (assuming full

Table 3: Relevant stellar quantities.

Quantity	Value in Sun	Range in other stars
M	2×10^{33} g	$0.08 \lesssim (M/M_\odot) \lesssim 100$
R	7×10^{10} cm	$0.08 \lesssim (R/R_\odot) \lesssim 1000$
L	4×10^{33} erg s^{-1}	$10^{-3} \lesssim (L/L_\odot) \lesssim 10^6$
T_eff	5777 K	3000 K $\lesssim (T_\mathrm{eff}/\mathrm{K}) \lesssim$ 50,000 K
ρ_c	150 g cm^{-3}	$10 \lesssim (\rho_c/\mathrm{g\ cm^{-3}}) \lesssim 1000$
T_c	1.5×10^7 K	$10^6 \lesssim (T_c/\mathrm{K}) \lesssim 10^8$

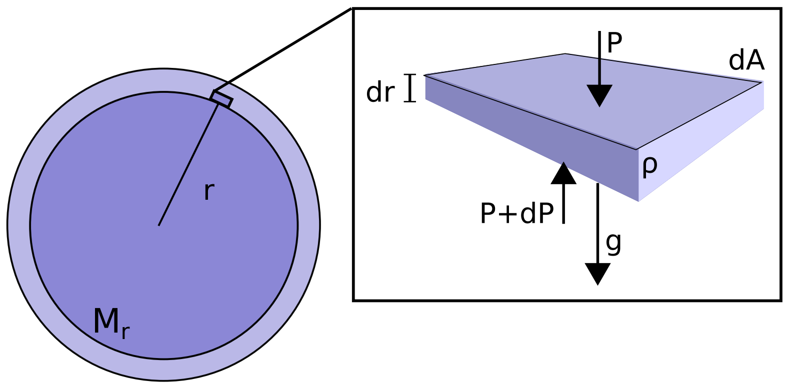

Figure 28: The state of hydrostatic equilibrium in an object like a star occurs when the inward force of gravity is balanced by an outward pressure gradient. This figure illustrates that balance for a packet of gas inside of a star

ionization, and thus Thomson scattering), we have

$$\ell = \frac{1}{n\sigma} \tag{229}$$

$$= \frac{m_p}{\rho\sigma_T} \tag{230}$$

$$= \frac{1.7 \times 10^{-24}\text{ g}}{(1.4\text{ g cm}^{-3})(2/3 \times 10^{-24}\text{ cm}^{-2})} \tag{231}$$

$$\approx 2\text{ cm} \tag{232}$$

So the matter-radiation equilibration timescale is roughly $\tau_\gamma \approx 10^{-10}$ s. Pretty fast!

14.2 *Gravity and the free-fall timescale*

For stars like the sun not to be either collapsing inward due to gravity or expanding outward due to their gas pressure, these two forces must be in balance. This condition is known as hydrostatic equilibrium. This balance is illustrated in Figure 28

As we will see, gravity sets the timescale for fluid to come into mechanical equilibrium. When we consider the balance between pressure and gravity on a small bit of the stellar atmosphere with volume $V = Adr$ (sketched in Fig. 28), we see that in equilibrium the vertical forces must cancel.

The small volume element has mass dm and so will feel a gravitational force equal to

$$F_g = \frac{GM_r\, dm}{r^2} \tag{233}$$

where M_r is the mass of the star enclosed within a radius r,

$$M(r) \equiv 4\pi \int_{r'=0}^{r'=r} \rho(r') r'^2 dr' \tag{234}$$

Assuming the volume element has a thickness dr and area dA, and the star has a uniform density ρ, then we can replace dm with $\rho dr dA$. This volume element will also feel a mean pressure which we can define as dP, where the pressure on the outward facing surface of this element is P and the pressure on the inward facing surface of this element is $P + dP$. The net pressure force is then $dPdA$, so

$$F_P(r) = F_g(r) \tag{235}$$

$$A\left(P(r) - P(r+dr)\right) = -\rho V g \tag{236}$$

$$= \rho A dr g \tag{237}$$

(238)

which yields the classic expression for **hydrostatic equilibrium**,

$$\frac{dP}{dr} = \rho(r) g(r) \tag{239}$$

where

$$g \equiv -\frac{GM(r)}{r^2} \tag{240}$$

and $M(r)$ is defined as above.

When applying Eq. 239 to stellar interiors, it's common to recast it as

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2} \tag{241}$$

In Eqs. 239 and 241 the left hand side is the pressure gradient across our volume element, and the right hand side is the gravitational force averaged over that same volume element. So it's not that pressure balances gravity in a star, but rather gravity is balanced by the gradient of increasing pressure from the center to the surface.

The gradient dP/dr describes the pressure profile of the stellar interior in equilibrium. What if the pressure changes suddenly – how long does it take

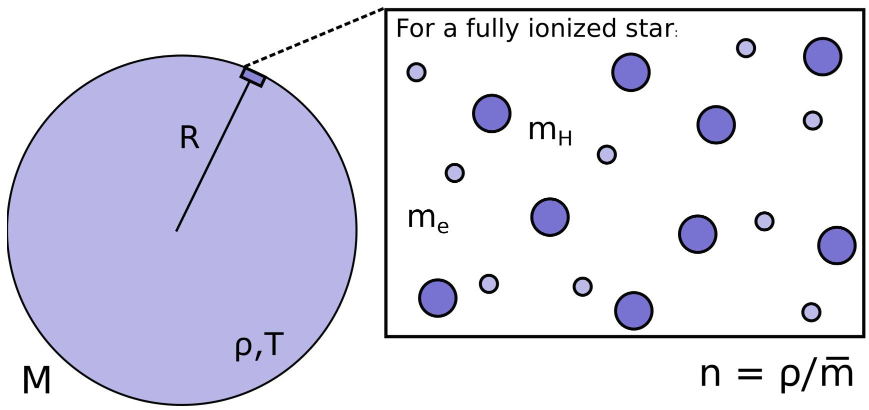

Figure 29: A simple model of a star having a radius R, mass M, constant density ρ, a constant temperature T, and a fully ionized interior. This simple model can be used to derive a typical free-fall time and a typical sound-crossing time for the sun.

us to re-establish equilibrium? Or equivalently: if nothing were holding up a star, how long would it take to collapse under its own gravity? Looking at Figure 29, we can model this as the time it would take for a parcel of gas on the surface of a star, at radius R, to travel to its center, due to the gravitational acceleration from a mass M.

Looking at Figure 29, we can model this as the time it would take for a parcel of gas on the surface of a star, at radius R, to travel to its center, due to the gravitational acceleration from a mass M. To order of magnitude, we can combine the following two equations

$$a = -\frac{GM}{r^2} \tag{242}$$

and

$$d = -\frac{1}{2}at^2. \tag{243}$$

Setting both r and d equal to the radius of our object R, and assuming a constant density $\rho = \frac{3M}{4\pi R^3}$, we find

$$\tau_{ff} \sim \frac{1}{\sqrt{G\rho}} \tag{244}$$

which is within a factor of two of the exact solution,

$$\tau_{ff} = \sqrt{\frac{3\pi}{32G\rho}} \tag{245}$$

Note that the free-fall timescale does not directly depend on the mass of an object or its radius (or in fact, the distance from the center of that object). It only depends on the density. Since $G \approx 2/3 \times 10^{-7}$ (cgs units), with $\langle \rho_\odot \rangle \approx 1$ g cm^{-3}, the average value is $\tau_{dyn} \sim 30$ min.

In real life, main-sequence stars like the sun are stable and long-lived structures that are not collapsing. Even if you have a cloud of gas that is collapsing under its own gravity to form a star, it does not collapse all the way to $R = 0$ thanks to its internal hydrostatic pressure gradient.

14.3 *The sound-crossing time*

We have an expression for the time scale upon which gravity will attempt to force changes on a system (such changes can either be collapse, if a system is far out of hydrostatic equilibrium and gravity is not significantly opposed by pressure, or contraction, if a system is more evenly balanced). What is the corresponding time scale upon which pressure will attempt to cause a system to expand?

The pressure time scale in a system can be characterized using the sound speed (as sound is equivalent to pressure waves in a medium). This isothermal sound speed is given by the relation

$$(246)\quad c_s = \sqrt{\frac{P}{\rho}}$$

Although gas clouds in the interstellar medium may be reasonably approximated as isothermal, the same is not true for stars. We will ignore that fact for now, but will return to this point later.

Referring back to Figure 29, we can define the sound-crossing time for an object as the time it takes for a sound wave to cross the object. Using a simple equation of motion $d = vt$ and approximating $2R$ just as R we can then define a sound-crossing time as

$$(247)\quad \tau_s \sim R\sqrt{\frac{\rho}{P}}$$

Using the ideal gas equation, we can substitute $\frac{\rho}{\bar{m}}kT$ for P and get an expression for the sound crossing time in terms of more fundamental parameters for an object:

$$(248)\quad \tau_s \sim R\sqrt{\frac{\bar{m}}{kT}}$$

Unlike the free-fall time we derived earlier, the sound-crossing time depends directly upon the size of the object, and its temperature. At the center of the Sun, $T_c \approx 1.5 \times 10^7$ K and $\bar{m} \sim m_p$ and so the sound-crossing timescale is roughly 30 min.

Note that by Eq. 245 we see that τ_s is also approximately equal to the free-fall timescale τ_{ff}. For an object not just to be in hydrostatic equilibrium but to remain this way, the pressure must be able to respond to changes in gravity, and vice versa. This response requires that a change in one force is met with a change in another force on a timescale that is sufficiently fast to restore the force balance. In practice, this means that for objects in hydrostatic equilibrium, the free-fall time is more or less equivalent to the sound-crossing time. In that way, a perturbation in pressure or density can be met with a corresponding response before the object moves significantly out of equilibrium.

14.4 *Radiation transport*

If photons streamed freely through a star, they'd zip without interruption from the core to the stellar surface in $R_\odot/c \approx 2$ s. But as we saw above in Eq. 232, the photons actually scatter every ~1 cm. With each collision they "forget" their history, so the motion is a random walk with N steps. So for a single photon[7] to reach the surface from the core requires

$$\ell\sqrt{N} \sim R_\odot \tag{249}$$

which implies that the **photon diffusion timescale** is

$$\tau_{\gamma,\mathrm{diff}} \sim \frac{N\ell}{c} \sim \frac{R_\odot^2}{\ell}\frac{1}{c} \tag{250}$$

or roughly 10^4 yr.

14.5 *Thermal (Kelvin-Helmholtz) timescale*

The thermal timescale answers the question, How long will it take to radiate away an object's gravitational binding energy? This timescale also governs the contraction of stars and brown dwarfs (and gas giant planets) by specifying the time it takes for the object to radiate away a significant amount of its gravitational potential energy. This is determined by the **Kelvin-Helmholtz timescale**. This thermal time scale can generally be given as:

$$\tau_{KH} = \frac{E}{L}, \tag{251}$$

where E is the gravitational potential energy released in the contraction to its final radius and L is the luminosity of the source. Approximating the Sun as a uniform sphere, we have

$$\tau_{KH} \sim \frac{GM_\odot^2}{R_\odot}\frac{1}{L_\odot} \tag{252}$$

[7] This is rather poetic – of course a given photon doesn't survive to reach the surface, but is absorbed and re-radiated as a new photon $\sim (R_\odot/\ell)^2$ times. Because of this, it may be better to think of the timescale of Eq. 250 as the **radiative energy transport timescale**.

which is roughly 3×10^7 yr.

Before nuclear processes were known, the Kelvin-Helmholtz timescale was invoked to argue that the Sun could be only a few 10^7 yr old – and therefore much of geology and evolutionary biology (read: Darwin) must be wrong. There turned out to be missing physics, but τ_{KH} turns out to still be important when describing the contraction of large gas clouds as they form new, young stars.

The time that a protostar spends contracting depends upon its mass, as its radius slowly contracts. A 0.1 $M_\odot$ star can take 100 million years on the Hayashi track to finish contracting and reach the main sequence. On the other hand, a 1 $M_\odot$ star can take only a few million years contracting on the Hayashi track before it develops a radiative core, and then spends up to a few tens of millions of years on the Henyey track before reaching the main sequence and nuclear burning equilibrium. The most massive stars, 10 $M_\odot$ and above, take less than 100,000 years to evolve to the main sequence.

14.6 *Nuclear timescale*

The time that a star spends on the main sequence – essentially the duration of the star's nuclear fuel under a constant burn rate – is termed the **the nuclear timescale**. It is a function of stellar mass and luminosity, essentially analogous to the thermal time scale of Equation 251. Here, the mass available (technically, the mass difference between the reactants and product of the nuclear reaction) serves as the energy available, according to $E = mc^2$.

If we fuse 4 protons to form one He^4 nucleus (an **alpha particle**), then the fractional energy change is

$$\frac{\Delta E}{E} = \frac{4m_p c^2 - m_{He} c^2}{4 m_p c^2} \approx 0.007 \tag{253}$$

This is a handy rule of thumb: fusing H to He liberates roughly 0.7% of the available mass energy. As we will see, in more massive stars heavier elements can also fuse; further rules of thumb are that fusing He to C and then C to Fe (through multiple intermediate steps) each liberates another 0.1% of mass energy. But for a solar-mass star, the main-sequence nuclear timescale is

$$\tau_{nuc} = \frac{\Delta E}{E_{\rm tot}} \approx \frac{0.007 M_\odot c^2}{L_\odot} \approx 10^{11} \text{ yr} \tag{254}$$

which implies a main-sequence lifetime of roughly 100 billion years. The actual main-sequence lifetime for a $1 M_\odot$ star is closer to 10 billion years; it turns out that significant stellar evolution typically occurs by the time ~10% of a star's mass has been processed by fusion.

14.7 *A Hierarchy of Timescales*

So if we arrange our timescales, we find a strong separation of scales:

$$\begin{array}{ccccccccc} \tau_{nuc} & \gg & \tau_{KH} & \gg & \tau_{\gamma,diff} & \gg & \tau_{dyn} & \gg & \tau_\gamma \\ 10^{11} \text{ yr} & \gg & 3 \times 10^7 \text{ yr} & \gg & 10^4 \text{ yr} & \gg & 30 \text{ min} & \gg & 10^{-10} \text{ s} \end{array}$$

This separation is pleasant because it means whenever we consider one timescale, we can assume that the faster processes are in equilibrium while the slower processes are static.

Much excitement ensues when this hierarchy breaks down. For example, we see convection occur on τ_{dyn} which then fundamentally changes the thermal transport. Or in the cores of stars near the end of their life, τ_{nuc} becomes much shorter. If it gets shorter than τ_{dyn}, then the star has no time to settle into equilibrium – it may collapse.

14.8 *The Virial Theorem*

In considering complex systems as a whole, it becomes easier to describe important properties of a system in equilibrium in terms of its energy balance rather than its force balance. For systems in equilibrium– not just a star now, or even particles in a gas, but systems as complicated as planets in orbit, or clusters of stars and galaxies– there is a fundamental relationship between the internal, kinetic energy of the system and its gravitational binding energy.

This relationship can be derived in a fairly complicated way by taking several time derivatives of the moment of inertia of a system, and applying the equations of motion and Newton's laws. We will skip this derivation, the result of which can be expressed as:

$$\frac{d^2I}{dt^2} = 2\langle K\rangle + \langle U\rangle, \tag{255}$$

where $\langle K\rangle$ is the time-averaged kinetic energy, and $\langle U\rangle$ is the time-averaged gravitational potential energy. For a system in equilibrium, $\frac{d^2I}{dt^2}$ is zero, yielding the form more traditionally used in astronomy:

$$\langle K\rangle = -\frac{1}{2}\langle U\rangle \tag{256}$$

The relationship Eq. 256 is known as the Virial Theorem. It is a consequence of the more general fact that whenever $U \propto r^n$, we will have

$$\langle K\rangle = \frac{1}{n}\langle U\rangle \tag{257}$$

And so for gravity with $U \propto r^{-1}$, we have the Virial Theorem, Eq. 256.

When can the Virial Theorem be applied to a system? In general, the system must be in equilibrium (as stated before, this is satisfied by the second time derivative of the moment of inertia being equal to zero). Note that this is not necessarily equivalent to the system being stationary, as we are considering the time-averaged quantities $\langle K\rangle$ and $\langle U\rangle$. This allows us to apply the Virial Theorem to a broad diversity of systems in motion, from atoms swirling within a star to stars orbiting in a globular cluster, for example. The system also generally must be isolated. In the simplified form we are using, we don't consider so-called 'surface terms' due to an additional external pressure from a medium in which our system is embedded. We also assume that there are not any other sources of internal support against gravity in the system apart from the its internal, kinetic energy (there is no magnetic field in the source, or rotation). Below, we introduce some of the many ways we can apply this tool.

Virial Theorem applied to a Star

For stars, the Virial Theorem relates the internal (i.e. thermal) energy to the gravitational potential energy. We can begin with the equation of hydrostatic equilibrium, Eq. 239. We multiply both sides by $4\pi r^3$ and integrate as follows

$$\int_0^R \frac{dP}{dr} 4\pi r^3 dr = -\int_0^R \left(\frac{GM(r)}{r}\right)\left(4\pi r^2 \rho(r)\right) dr \tag{258}$$

The left-hand side can be integrated by parts,

$$\int_0^R \frac{dP}{dr} 4\pi r^3 dr = 4\pi r^3 P|_0^R - 3\int_0^R P 4\pi r^2 dr \tag{259}$$

and since $r(0) =$ and $P(R) = 0$, the first term equals zero. We can deal with the second term by assuming that the star is an ideal gas, replacing $P = nkT$, and using the thermal energy density

$$u = \frac{3}{2} nkT = \frac{3}{2} P \tag{260}$$

This means that the left-hand side of Eq. 258 becomes

$$-2\int_0^R u(4\pi r^2 dr) = -2E_{th} \tag{261}$$

Where E_{th} is the total thermal energy of the star.

As for the right-hand side of Eq. 258, we can simplify it considerably by recalling that

$$\Phi_g = -\frac{GM(r)}{r} \tag{262}$$

and

$$dM = 4\pi r^2 \rho(r) dr. \tag{263}$$

Thus the right-hand side of Eq. 258 becomes simply

$$\int_0^R \Phi_g(M') dM' = E_{\mathrm{grav}} \tag{264}$$

And so merely from the assumptions of hydrostatic equilibrium and an ideal gas, it turns out that

(265) $E_{\text{grav}} = -2E_{th}$

or alternatively,

(266) $E_{tot} = -E_{th} = E_{\text{grav}}/2$

The consequence is that the total energy of the bound system is negative, and that it has negative heat capacity – a star heats up as it loses energy! Eq. 266 shows that if the star radiates a bit of energy so that E_{tot} decreases, E_{th} increases while E_{grav} decreases by even more. So energy was lost from the star, causing its thermal energy to increase while it also becomes more strongly gravitationally bound. This behavior shows up in all gravitational systems with a thermal description — from stars to globular clusters to Hawking radiation near a black hole to the gravitational collapse of a gas cloud into a star.

Virial Theorem applied to Gravitational Collapse

We can begin by restating the Virial Theorem in terms of the average total energy of a system $\langle E \rangle$:

(267) $\langle E \rangle = \langle K \rangle + \langle U \rangle = \frac{1}{2}\langle U \rangle$

A classic application of this relationship is then to ask, if the sun were powered only by energy from its gravitational contraction, how long could it live? To answer this, we need to build an expression for the gravitational potential energy of a uniform sphere: our model for the gravitational potential felt at each point inside of the sun. We can begin to put this into equation form by considering what the gravitational potential is for an infinitesimally thin shell of mass at the surface of a uniformly-dense sphere.

Using dM as defined previously, the differential change in gravitational potential energy that this shell adds to the sun is

(268) $dU = -\frac{GM(r)dM}{r}.$

The simplest form for $M(r)$ is to assume a constant density. In this case, we can define

(269) $M(r) = \frac{4}{3}\pi r^3 \rho$

To determine the total gravitational potential from shells at all radii, we must integrate Equation 268 over the entire size of the sphere from 0 to R, substituting our expressions for dM and $M(r)$ from Equations 263 and 269:

$$U = -\frac{G(4\pi\rho)^2}{3}\int_0^R r^4 dr. \tag{270}$$

Note that if this were not a uniform sphere, we would have to also consider ρ as a function of radius: $\rho(r)$ and include it in our integral as well. That would be a more realistic situation for a star like our sun, but we will keep it simple for now.

Performing this integral, and replacing the average density ρ with the quantity $\frac{3M}{4\pi R^3}$, we then find

$$U = -\frac{G(3M)^2}{R^6}\frac{R^5}{5} = -\frac{3}{5}\frac{GM^2}{R} \tag{271}$$

which is the gravitational potential (or binding energy) of a uniform sphere. All together, this is equivalent to the energy it would take to disassemble this sphere, piece by piece, and move each piece out to a distance of infinity (at which point it would have zero potential energy and zero kinetic energy).

To understand how this relates to the energy available for an object like the sun to radiate as a function of its gravitational collapse, we have to perform one more trick, and that is to realize that Equation 267 doesn't just tell us about the average energy of a system, but how that energy has evolved. That is to say,

$$\Delta E = \frac{1}{2}\Delta U \tag{272}$$

So, the change in energy of our sun as it collapsed from an initial cloud to its current size is half of the binding energy that we just calculated. How does our star just lose half of its energy as it collapses, and where does it go? The Virial Theorem says that as a cloud collapses it turns half of its potential energy into kinetic energy (Equation 256). The other half then goes into terms that are not accounted for in the Virial Theorem: radiation, internal excitation of atoms and molecules and ionization (see the Saha Equation, Equation 171).

Making the simplistic assumption that all of the energy released by the collapse goes into radiation, then we can calculate the energy available purely from gravitational collapse and contraction to power the luminosity of the sun. Assuming that the initial radius of the cloud from which our sun formed is not infinity, but is still large enough that the initial gravitational potential energy is effectively zero, the energy which is radiated from the collapse is half the current gravitational potential energy of the sun, or

$$E_{radiated} = -\frac{3}{10}\frac{GM_\odot^2}{R_\odot} \tag{273}$$

Eq. 273 therefore links the Virial Theorem back to the Kelvin-Helmholtz timescale of Sec. 14.5. For the sun, this is a total radiated energy of $\sim 10^{41}$ J. If we assume that the sun radiates this energy at a rate equal to its current luminosity ($\sim 10^{26}$ W) then we can calculate that the sun could be powered at its current luminosity just by this collapse energy for 10^{15} s, or 3×10^{7} years. While this is a long time, it does not compare to our current best estimates for the age of the earth and sun: $\sim$4.5 billion years. As an interesting historical footnote, it was Lord Kelvin who first did this calculation to estimate the age of the sun (back before we knew that the sun must be powered by nuclear fusion). He used this calculation to argue that the Earth must only be a few million years old, he attacked Charles Darwin's estimate of hundreds of millions of years for the age of the earth, and he argued that the theory of evolution and natural selection must be bunk. In the end of course, history has shown who was actually correct on this point.

15 Stellar Structure

Questions you should be able to answer after these lectures:

- What equations, variables, and physics describe the structure of a star?
- What are the two main types of pressure in a star, and when is each expected to dominate?
- What is an equation of state, and what is the equation of state that is valid for the sun?

15.1 *Formalism*

One of our goals in this class is to be able to describe not just the observable, exterior properties of a star, but to understand all the layers of these cosmic onions — from the observable properties of their outermost layers to the physics that occurs in their cores. This next part will then be a switch from some of what we have done before, where we have focused on the "surface" properties of a star (like size, total mass, and luminosity), and considered many of these to be fixed and unchanging. Our objective is to be able to describe the entire internal structure of a star in terms of its fundamental physical properties, and to model how this structure will change over time as it evolves.

Before we define the equations that do this, there are two points that may be useful to understand all of the notation being used here, and the way in which these equations are expressed.

First, when describing the evolution of a star with a set of equations, we will use mass as the fundamental variable rather than radius (as we have mostly been doing up until this point.) It is possible to change variables in this way because mass, like radius, increases monotonically as you go outward in a star from its center. We thus will set up our equations so that they follow individual, moving shells of mass in the star. There are several benefits to this. For one, it makes the problem of following the evolution of our star a more well-bounded problem. Over a star's lifetime, its radius can change by orders of magnitude from its starting value, and so a radial coordinate must always be defined with respect to the hugely time-varying outer extent of the star. In contrast, as our star ages, assuming its mass loss is insignificant, its mass coordinate will always lie between zero and its starting value M — a value which can generally be assumed to stay constant for most stars over most of stellar evolution. Further, by following shells of mass that do not cross over each other, we implicitly assume conservation of mass at a given time, and the mass enclosed by any of these moving shells will stay constant as the star evolves, even as the radius changes. This property also makes it easier to follow compositional changes in our star.

In general, the choice to follow individual fluid parcels rather than reference a fixed positional grid is known as adopting Lagrangian coordinates instead of Eulerian coordinates. For a **Lagrangian** formulation of a problem:

- This is a particle-based description, following individual particles in a fluid over time
- Conservation of mass and Newton's laws apply directly to each particle being followed
- However, following each individual particle can be computationally expensive
- This expense can be somewhat avoided for spherically-symmetric (and thus essentially '1D') problems

In contrast, for a **Eulerian** formulation of a problem:

- This is a field-based description, recording changes in properties at each point on a fixed positional grid in space over time
- The grid of coordinates is not distorted by the fluid motion
- Problems approached in this way are generally less computationally expensive, and are generally easier for 2D and 3D problems

There are thus trade-offs for choosing each formulation. For stellar structure, Lagrangian coordinates are generally preferred, and we will rely heavily on equations expressed in terms of a stellar mass variable going forward.

Second, it might be useful to just recall the difference between the two types of derivatives that you may encounter in these equations. The first is a partial derivative, written as ∂f. The second is a total derivative, written as df. To illustrate the difference, let's assume that f is a function of a number of variables: $f(x,t)$. The partial derivative of f with respect to x is just $\frac{\partial f}{\partial x}$. Here, we have assumed in taking this derivative that x is held fixed with time and does not vary. However, most of the quantities that we will deal with in the equations of stellar structure *do* vary with time. The use of a partial derivative with respect to radius or mass indicates that we are considering the change in this space(like) coordinate for an instantaneous, fixed time value. In contrast, the total derivative does not hold any variables to be fixed, and considers how all of the dependent variables changes as a function of the variable considered. Note that when you see a quantity like $\dot{r}$ in an equation, this is actually the partial rather than total derivative with respect to time.

15.2 *Equations of Stellar Structure*

In this class, we will define four fundamental equations of stellar structure, and several additional relationships that, taken all together, will define the structure of a star and how it evolves with time. Depending on the textbook that you consult, you will find different versions of these equations using slightly different variables, or in a slightly different format.

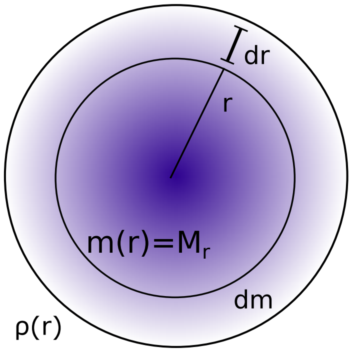

Figure 30: An illustration of a shell with mass dm and thickness dr. The mass enclosed inside of the shell is $m(r)$ (or M_r, depending on how you choose to write it). Assume that this object has a density structure $\rho(r)$

Mass continuity

The first two equations of stellar structure we have already seen before, as the conversion between the mass and radius coordinates

$$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho} \tag{274}$$

and as the equation of hydrostatic equilibrium (Eq. 239), now recast in terms of mass:

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4} \tag{275}$$

Eq. 274 and its variant forms are known variously as the **Mass Continuity Equation** or the **Equation of Conservation of Mass**. Either way, this is the first of our four fundamental equations of stellar structure, and relates our mass coordinate m to the radius coordinate r, as shown in Fig. 30.

Note that up until now we have been generally either been assuming a uniform constant density in all of the objects we have considered, or have been making approximations based on the average density $\langle\rho\rangle$. However, to better and more realistically describe stars we will want to use density distributions that are more realistic (e.g., reaching their highest value in the center of the star, and decreasing outward to zero at the edge of the star). This means we should start trying to think about ρ as a function rather than a constant (even when it is not explicitly written as $\rho(r)$ or $\rho(m)$ in the following equations).

Hydrostatic equilibrium

The second equation of stellar structure (Eq. 275, the equation of hydrostatic equilibrium) concerns the motion of a star, and we derived it in Sec. 14.2. As we noted earlier, stars can change their radii by orders of magnitude over the course of their evolution. As a result, we must consider how the interiors of stars move due the forces of pressure and gravity. We have already seen a specific case for this equation: the case in which gravity and pressure are balanced such that there is no net acceleration, and the star is in hydrostatic equilibrium (Equation 239).

We want to first consider a more general form of Eq. 275 that allows for the forces to be out of balance and thus there to be a net acceleration, and second to change variables from a dependence on radius to a dependence on mass. We can begin by rewriting our condition of force balance in Equation 239 as

$$0 = -\frac{Gm(r)}{r^2} - \frac{1}{\rho}\frac{\partial P}{\partial r}. \tag{276}$$

Each term in this equation has units of acceleration. Thus, this equation can be more generally written as

$$\ddot{r} = -\frac{Gm(r)}{r^2} - \frac{1}{\rho}\frac{\partial P}{\partial r}. \tag{277}$$

Using Equation 263 we can recast this expression in terms of a derivative with respect to m rather than r. This gives us the final form that we will use:

$$\ddot{r} = -\frac{Gm(r)}{r^2} - 4\pi r^2 \frac{\partial P}{\partial m}. \tag{278}$$

This is the most general form of our second equation of stellar structure. When $\ddot{r}$ is zero we are in equilibrium and so we obtain Eq. 275, the equation of **hydrostatic equilibrium**. This more general form, Eq. 278, is sometimes referred to as the **Equation of Motion** or the **Equation of Momentum Conservation**.

The Thermal Transport Equation

We also need to know how the temperature profile of a star changes with depth. If we do that, we can directly connect the inferred profile of temperature vs. optical depth (Eq. 211) to a physical coordinate within the star.

Assume there is a luminosity profile (determined by the energy equation, to be discussed next), such that the flux at radius r is

$$F(r) = \frac{L(r)}{4\pi r^2} \tag{279}$$

In a plane-parallel atmosphere, we learned (Eq. 202) that the flux is related to the gradient of the radiation pressure. The assumptions we made then don't restrict the applicability of that relation only to the outer atmosphere, so we can apply it anywhere throughout the interior of our star. The only (minor) adjustment is that we replace dz with dr since we are now explicitly considering a spherical geometry, so we now have

$$F = -\frac{c}{\alpha}\frac{dP_{rad}}{dr} \tag{280}$$

Since we know that $P_{rad} = 4/3c\ \sigma_{SB} T^4$ (Eq. 305), we see that

$$\frac{dP_{rad}}{dr} = \frac{16\sigma_{SB}}{3c} T^3 \frac{dT}{dr}. \tag{281}$$

When combined with Eq. 279, we find the thermal profile equation,

$$\frac{dT}{dr} = -\frac{3\rho\kappa L(r)}{64\pi\sigma_{SB} T^3 r^2} \tag{282}$$

The Energy Equation

Eq. 282 shows that we need to know the luminosity profile in order to determine the thermal profile. In the outer photosphere we earlier required that flux is conserved (Sec. 13.2), but go far enough in and all stars (until the ends of their lives) are liberating extra energy via fusion.

Thus the next equation of stellar structure concerns the generation of energy within a star. As with the equation of motion, we will first begin with a simple case of equilibrium. In this case, we are concerned with the thermodynamics of the star: this is the equation for Thermal Equilibrium, or a constant flow of heat with time for a static star (a situation in which there is no work being done on any of our mass shells).

Consider the shell dm shown in Fig. 30. Inside of this shell we define a quantity ϵ_m that represents the net local gain of energy per time per unit mass (SI units of J s^{-1} kg^{-1}) due to local nuclear processes. Note that sometimes the volumetric power ϵ_r will also sometimes be used, but the power per unit mass ϵ_m is generally the more useful form. Regardless, we expect either ϵ to be very large deep in the stellar core and quickly go to zero in the outer layers where fusion is negligible – in those other regions, $\epsilon = 0$, L is constant, and we are back in the flux-conserving atmosphere of Sec. 13.2.

We then consider that the energy per time entering the shell is L_r (note that like M_r, this is now a local and internal rather than global or external property: it can be thought of as the luminosity of the star as measured at a radius r inside the star) and the energy per time that exits the shell is now $L_r + dL_r$ due to this local gain from nuclear burning in the shell. To conserve energy, we must then have (note that these are total rather than partial derivatives as there is no variation with time):

$$\frac{dL_r}{dm} = \epsilon_m. \tag{283}$$

This is the equation for **Thermal Equilibrium** in a star. While Thermal Equilibrium and Hydrostatic Equilibrium are separate conditions, it is generally unlikely that a star will be in Thermal Equilibrium without already being in Hydrostatic equilibrium, thus guaranteeing that there is no change in the energy flow in the star with time or with work being done. In general, Thermal Equilibrium and Eq. 283 require that any local energy losses in the shell (typically from energy propagating outward in the star) are exactly balanced by the rate of energy production in that shell due to nuclear burning. On a macroscopic scale, it means that the rate at which energy is produced in the center of the star is exactly equal to the star's luminosity: the rate at which that energy exits the surface.

How likely is it that a star satisfies this requirement? While a star may spend most of its life near Thermal Equilibrium while it is on the main sequence, most of the evolutionary stages it goes through do not satisfy Eq. 283: for example, pre-main sequence evolution (protostars) and post-main sequence evolution (red giants). How can we describe conservation of energy for an object that is not in Thermal equilibrium?

Following standard texts (e.g., Prialnik), we can make use of u, the internal energy density in a shell in our star. We can change u either by doing work on the shell, or by having it absorb or emit heat. We have already described how the heat in the shell can change with L_r and ϵ_m. Similarly, the incremental work done on the shell can be defined as a function of pressure and the incremental change in volume:

(284)
$$dW = -PdV$$
(285)
$$= -P\left(\frac{dV}{dm}dm\right)$$
(286)
$$= -P\ d\left(\frac{1}{\rho}\right)dm$$

The change in internal energy per unit mass (du) is equal to the work done per unit mass ($\frac{dW}{dm}$), so finally we can rewrite Eq. 286 as:

(287)
$$du = -Pd\left(\frac{1}{\rho}\right)$$

Taking the time derivative of each side,

(288)
$$\frac{du}{dt} = -P\frac{d}{dt}\left(\frac{1}{\rho}\right)$$

Compression of the shell will decrease dV, and thus require energy to be added to the shell, while expansion increases dV and is a way to release energy in the shell.

Changes in the internal energy of the shell u with time can then be described in terms of the both the work done on the shell and the changes in heat:

(289)
$$\frac{du}{dt} = \epsilon_m - \frac{\partial L_r}{\partial m} - P\frac{d}{dt}\left(\frac{1}{\rho}\right)$$

The general form of Eq. 289 is the next equation of stellar structure, known either as the **Energy Equation** or the **Equation of Conservation of Energy**.

You may also sometimes see this equation written in various other forms, such as in terms of the temperature T and entropy S of the star. In this form, you then have

$$\frac{\partial L_r}{\partial m} = \epsilon_m - T\frac{dS}{dt} \tag{290}$$

Chemical Composition

An additional relationship that is useful for determining stellar evolution is the change in a star's composition.This relation will be less of an 'equation' for the purposes of this class, and more a rough depiction of how the composition of a star can vary with time.

We can define the composition of a star using a quantity called the mass fraction of a species:

$$X_i = \frac{\rho_i}{\rho}. \tag{291}$$

Here, ρ_i is the partial density of the i^{th} species.

Particles in a star are defined by two properties: their baryon number $\mathcal{A}$ (or the number of total protons and neutrons they contain) and their charge $\mathcal{Z}$. Using the new notation of baryon number, we can rewrite

$$n = \frac{\rho}{\bar{m}}, \tag{292}$$

as the corresponding partial number density of the i^{th} species:

$$n_i = \frac{\rho_i}{\mathcal{A}_i\, m_H}. \tag{293}$$

We can then slightly rewrite our expression for the composition as

$$X_i = n_i \frac{\mathcal{A}_i}{\rho} m_H. \tag{294}$$

Changes in composition must obey (at least) two conservation laws. Conservation of charge:

$$\mathcal{Z}_i + \mathcal{Z}_j = \mathcal{Z}_k + \mathcal{Z}_l. \tag{295}$$

and conservation of baryon number:

$$\mathcal{A}_i + \mathcal{A}_j = \mathcal{A}_k + \mathcal{A}_l. \tag{296}$$

If you also consider electrons, there must also be a conservation of lepton number.

Without attempting to go into a detailed formulation of an equation for the rate of change of X we can see that it must depend on the starting composition and the density, and (though it does not explicitly appear in these equations) the temperature, as this will also govern the rate of the nuclear reactions responsible for the composition changes (analogous to the collision timescale $t_{col} = \frac{v}{nA}$ as shown in Figure 37, in which the velocity of particles is set by the gas temperature). This leads us to our last 'equation' of stellar structure, which for us will just be a placeholder function $\mathbf{f}$ representing that the change in composition is a function of these variables:

(297) $\dot{\mathbf{X}} = \mathbf{f}(\rho, T, \mathbf{X}).$

Technically, this $\mathbf{X}$ is a vector representing a series of equations for the change of each X_i.

The final fundamental relation we need in order to derive the structure of a star is an expression for the temperature gradient, which will be derived a bit later on.

15.3 *Pressure*

We have already seen a relationship for the gas pressure for an ideal gas, $P = nkT$. However, now that we have begun talking more about the microscopic composition of the gas we can actually be more specific in our description of the pressure. Assuming the interior of a star to be largely ionized, the gas will be composed of ions (e.g., H^+) and electrons. Their main interactions ('collisions') that are responsible for pressure in the star will be just between like particles, which repel each other due to their electromagnetic interaction. As a result, we can actually separate the gas pressure into the contribution from the ion pressure and the electron pressure:

(298) $P_{gas} = P_e + P_{ion}$

For a pure hydrogen star, these pressures will be equivalent, however as the metallicity of a star increases, the electron pressure will be greater than the ion pressure, as the number of free electrons per nucleon will go up (for example, for helium, the number of ions is half the number of electrons).

Assuming that both the ions and electrons constitute an ideal gas, we can rewrite the ideal gas equation for each species:

(299) $P_e = n_e kT$

and

(300) $P_{ion} = n_{ion} kT$

However, this is not the full story: there is still another source of pressure in addition to the gas pressure that we have not been considering: the pressure from radiation.

Considering this pressure then at last gives us the total pressure in a star:

$$P = P_{ion} + P_e + P_{rad} \tag{301}$$

We can determine the radiation pressure using an expression for pressure that involves the momentum of particles:

$$P = \frac{1}{3}\int_0^\infty v\ p\ n(p)\ dp \tag{302}$$

Here v is the velocity of the particles responsible for the pressure, p is their typical momentum, and $n(p)$ is the number density of particles in the momentum range $(p, p + dp)$. We first substitute in values appropriate for photons ($v = c$, $p = \frac{h\nu}{c}$). What is $n(p)$? Well, we know that the Blackbody (Planck) function (Equation 16) has units of energy per volume per interval of frequency per steradian. So, we can turn this into number of particles per volume per interval of momentum by (1) dividing by the typical energy of a particle (for a photon, this is $h\nu$), then (3) multiplying by the solid angle 4π, and finally (4) using $p = \frac{E}{c}$ to convert from energy density to momentum density.

$$P_{rad} = \frac{1}{3}\, 4\pi \int_0^\infty c \left(\frac{h\nu}{c}\right)\left(\frac{1}{h\nu}\right)\left(\frac{1}{c}\right)\frac{2h\nu^3}{c^2}\left[e^{\frac{h\nu}{kT}} - 1\right]^{-1} d\nu \tag{303}$$

Putting this all together,

$$P_{rad} = \frac{1}{3}\ \left(\frac{4}{c}\right)\left[\pi \int_0^\infty \left(\frac{1}{c}\right)\frac{2h\nu^3}{c^2}\left[e^{\frac{h\nu}{kT}} - 1\right]^{-1} d\nu\right] \tag{304}$$

Here, the quantity in brackets is the same integral that is performed in order to yield the Stefan-Boltzmann law (Equation 126). The result is then

$$P_{rad} = \frac{1}{3}\ \left(\frac{4}{c}\right)\sigma T^4 \tag{305}$$

The quantity $\frac{4\sigma}{c}$ is generally defined as a new constant, a.

We can also define the specific energy (the energy per unit mass) for radiation, using the relation

$$u_{rad} = 3\frac{P_{rad}}{\rho} \tag{306}$$

When solving problems using the Virial theorem, we have encountered a similar expression for the internal energy of an ideal gas:

$$KE_{gas} = \frac{3}{2}NkT \tag{307}$$

From the ideal gas law for the gas pressure ($P = nkT$), we can see that the specific internal energy $\frac{KE}{\bar{m}}$ then can be rewritten in a similar form:

$$u_{gas} = \frac{P_{gas}}{\rho} \tag{308}$$

15.4 *The Equation of State*

In a star, an equation of state relates the pressure, density, and temperature of the gas. These quantities are generally dependent on the composition of the gas as well. An **equation of state** then has the general dependence $P = P(\rho, T, X)$. The simplest example of this is the ideal gas equation. Inside some stars radiation pressure will actually dominate over the gas pressure, so perhaps our simplest plausible (yet still general) equation of state would be

$$P = P_{gas} + P_{rad} \tag{309}$$

$$= nkT + \frac{4F}{3c} \tag{310}$$

$$= \frac{\rho kT}{\mu m_p} + \frac{4\sigma_{SB}}{3c}T^4 \tag{311}$$

where μ is now the **mean molecular weight per particle** – e.g., $\mu = 1/2$ for fully ionized H.

But a more general and generally applicable equation of state is often that of an adiabatic equation of state. As you might have encountered before in a physics class, an adiabatic process is one that occurs in a system without any exchange of heat with its environment. In such a thermally-isolated system, the change in internal energy is due only to the work done on or by a system. Unlike an isothermal process, an adiabatic process will by definition change the temperature of the system. As an aside, we have encountered both adiabatic and isothermal processes before, in our description of the early stages of star formation. The initial collapse of a star (on a free-fall time scale) is a roughly isothermal process: the optically thin cloud is able to essentially radiate all of the collapse energy into space unchecked, and the temperature does not substantially increase. However, once the initial collapse is halted when the star becomes optically thick, the star can only now radiate a small fraction of its collapse energy into space at a time. It then proceeds to contract nearly adiabatically.

Adiabatic processes follow an equation of state that is derived from the first law of thermodynamics: for a closed system, the internal energy is equal to the amount of heat supplied, minus the amount of work done.

As no heat is supplied, the change in the specific internal energy (energy per unit mass) u comes from the work done by the system. We basically already derived this in Equation 287:

$$du = -Pd\left(\frac{1}{\rho}\right) \tag{312}$$

As we have seen both for an ideal gas and from our expression for the radiation pressure, the specific internal energy is proportional to $\frac{P}{\rho}$:

$$u = \phi\frac{P}{\rho} \tag{313}$$

Where ϕ is an arbitrary constant of proportionality. If we take a function of that form and put it into Equation 312 we recover an expression for P in terms of ρ for an adiabatic process:

$$P \propto \rho^{\frac{\phi+1}{\phi}} \tag{314}$$

We can rewrite this in terms of an adiabatic constant K_a and an adiabatic exponent γ_a:

$$P = K_a\rho^{\gamma_a} \tag{315}$$

For an ideal gas, $\gamma_a = \frac{5}{3}$.

This adiabatic relation can also be written in terms of volume:

$$PV^{\gamma_a} = K_a \tag{316}$$

This can be compared to the corresponding relationship for an ideal gas, in which PV = constant.

15.5 *Summary*

In summary, we have a set of coupled stellar structure equations (Eq. 274, Eq. 278, Eq. 282, Eq. 289, and Eq. 315):

(317) $$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}$$

(318) $$\ddot{r} = -\frac{Gm(r)}{r^2} - 4\pi r^2 \, \frac{\partial P}{\partial m}.$$

(319) $$\frac{dT}{dr} = -\frac{3\rho\kappa L(r)}{64\pi\sigma_{SB} T^3 r^2}$$

(320) $$\frac{du}{dt} = \epsilon_m - \frac{\partial L_r}{\partial m} - P\frac{d}{dt}\left(\frac{1}{\rho}\right)$$

(321) $$P = K_a \rho^{\gamma_a}$$

If we can solve these together in a self-consistent way, we have good hope of revealing the unplumbed depths of many stars. To do this we will also need appropriate boundary conditions. Most of these are relatively self-explanatory:

(322) $$M(0) = 0$$

(323) $$M(R) = M_{tot}$$

(324) $$L(0) = 0$$

(325) $$L(R) = 4\pi R^2 \sigma_{SB} T_{\rm eff}^4$$

(326) $$\rho(R) = 0$$

(327) $$P(R) \approx 0$$

(328) $$T(R) \approx T_{\rm eff}$$

(329)

To explicitly solve the equations of stellar structure even with all these constraints in hand is still a beast of a task. In practice one integrates numerically, given some basic models (or tabulations) of opacity and energy generation.

16 Stability, Instability, and Convection

Now that we have the fundamental equations of stellar structure, we would like to examine some interesting situations in which they apply. One such interesting regime is the transition from stable stars to instability, either in a part of the star or throughout its interior. We will examine this by answering the following question: if we perturb the system (or a part of it), does it settle back into equilibrium?

16.1 *Thermal stability*

Suppose we briefly exceed thermal equilibrium; what happens? In equilibrium, the input luminosity from nuclear burning balances the energy radiated away:

$$\frac{dE_{\rm tot}}{dt} = L_{\rm nuc} - L_{\rm rad} = 0 \tag{330}$$

If the star briefly overproduces energy, then (at least briefly) $L_{\rm nuc} > L_{\rm rad}$ and we overproduce a clump ΔE of energy. Over a star's main-sequence lifetime its core temperature steadily rises, so this slight imbalance is happening all the time. Whenever it does, the star must be responding on the photon diffusion timescale, $\tau_{\gamma,{\rm diff}} \approx 10^4$ yr... but how?

From the virial theorem, we know that

$$\Delta E = -\Delta E_{\rm th} = \frac{1}{2}\Delta E_{\rm grav} \tag{331}$$

So if nuclear processes inject an extra ΔE into the star, we know we will *lose* an equivalent amount ΔE of thermal energy and simultaneously *gain* $2\Delta E$ of gravitational energy. Thus the star must have cooled, and – since its mass has not appreciably changed –its radius must have expanded.

Of the two, the temperature change is the more relevant for thermal stability because nuclear reaction rates depend very sensitively on temperature. For Sun-like stars, $\epsilon \propto T^{16}$ – so even a slight cooling will strongly diminish the nuclear energy production rate and will tend to bring the star back into thermal equilibrium. This makes sense, because stars are stable during their slow, steady evolution on the main sequence.

16.2 *Mechanical and Dynamical Stability*

Suppose that a fluid element of the star is briefly pushed away from hydrostatic equilibrium; what happens? We expect the star to respond on the dynamical timescale, $\tau_{\rm dyn} \approx 30$ min... but how?

Let's consider a toy model of this scenario, in which we squeeze the star slightly and see what happens. (A full analysis would require us to compute a full eigenspectrum of near-equilibrium Navier-Stokes equations, and is definitely beyond the scope of our discussion here.) If we start with Eq. 275, we can integrate to find $P(M)$:

(332)
$$P(M) = \int_0^P dP$$

(333)
$$= \int_{M_{\rm tot}}^{M} \frac{dP}{dM} dM$$

(334)
$$= -\int_{M_{\rm tot}}^{M} \frac{GM}{4\pi r^4} dM$$

Initially, in equilibrium the gas pressure must be equal to the pressure required to maintain hydrostatic support – i.e., we must have $P_{hydro} = P_{gas}$.

If we squeeze the star over a sufficiently short period of time (shorter than the thermal diffusion timescale), heat transfer won't occur during the squeezing and so the contraction is adiabatic. If the contraction is also homologous, then we will have

$r \longleftrightarrow r' = r(1-\epsilon)$

$\rho \longleftrightarrow \rho' = \rho(1+3\epsilon)$

How will the star's pressure respond? Since the contraction was sufficiently rapid, we have an adiabatic equation of state

(335) $P \propto \rho^{\gamma_{ad}}$,

where

(336) $\gamma_{ad} \equiv \frac{c_P}{c_V}$

where c_P and c_V are the heat capacities at constant pressure and volume, respectively. Statistical mechanics shows that we have $\gamma_{ad} = 5/3$ for an ideal monoatomic gas, and 4/3 for photon radiation (or a fully relativistic, degenerate gas).

Thus our perturbed star will have a new internal gas pressure profile,

(337)
$$P'_{gas} = P_{gas}(1+3\epsilon)^{\gamma_{ad}}$$

(338)
$$\approx P_{gas}(1+3\gamma_{ad}\epsilon)$$

Will this new gas pressure be enough to maintain hydrostatic support of the star? To avoid collapse, we need $P_{gas} > P_{hydro}$ always. We know from Eq. 334 that the new *hydrostatic* pressure required to maintain equilibrium will be

(339)
$$P'_{hydro} = -\int_{M_{\rm tot}}^{M} \frac{GM}{4\pi(r')^4} dM$$

(340)
$$= -(1+4\epsilon)\int_{M_{\rm tot}}^{M} \frac{GM}{4\pi r^4} dM$$

(341)
$$= (1+4\epsilon)P_{hydro}$$

Thus our small perturbation may push us out of equilibrium! The new pressures are in the ratio

(342) $$\frac{P'_{gas}}{P'_{hydro}} \approx \frac{1+3\gamma_{ad}\epsilon}{1+4\epsilon}$$

and so the star will only remain in equilibrium so long as

(343) $$\gamma_{ad} > \frac{4}{3}$$

This is pretty close; a more rigorous treatment of the same question yields

(344) $$\int_{0}^{M} \left(\gamma_{ad} - \frac{4}{3}\right) \frac{P}{\rho} dM > 0.$$

Regardless of the exact details, Eqs. 343 and 344 indicate that a star comes ever closer to collapse as it becomes more fully supported by relativistic particles (whether photon radiation, or a relativistic, degenerate gas).

The course reading from Sec. 3.6 of Prialnik shows another possible source of instability, namely via partial ionization of the star. γ_{ad} can also drop below 4/3, thus also leading to collapse, via the reaction $H \longleftrightarrow H^+ + e^-$.

In this reaction, both c_v and c_P change because added heat can go into ionization rather then into increasing the temperature. c_v changes more rapidly than c_P, so γ_{ad} gets smaller (as low as 1.2 or so). Qualitatively speaking, a stellar contraction reverses some of the ionization, reducing the number of particles and also reducing the pressure opposing the initial squeeze. As in the relativistic support case, when $\gamma_{ad} \leq 4/3$, the result is instability.

16.3 *Convection*

Not all structural instabilities lead to stellar collapse. One of the most common instabilities is almost ubiquitous in the vast majority of stars: convection. **Convection** is easily visualized by bringing a pot of water to a boil, and dropping in dark beans, rice grains, or other trace particles. We will now show how the same situation occurs inside of stars.

Convection is one of several dominant modes of energy transport inside of stars. Up until now, we have considered energy transport only by radiation, as described by Eq. 282,

$$\frac{dT}{dr} = -\frac{3\rho\kappa L(r)}{64\pi\sigma_{SB}T^3r^2}. \tag{345}$$

In a few cases energy can also be transported directly by conduction, which is important in the dense, degenerate white dwarfs and neutron stars. Whereas radiation transports heat via photon motions and conduction transports heat through microscopic particle motion, convection transports heat via bulk motions of large parcels of gas or fluid.

When a blob of stellar material is pushed upwards by some internal perturbation, how does it respond: will it sink back down, or continue to rise? Again, a consideration of different timescales is highly relevant here. An outward motion typically corresponds to a drop in both pressure and temperature. The pressure will equilibrate on $\tau_{dyn} \approx 30$ min, while heat will flow on the much slower $\tau_{\gamma,diff} \approx 10^4$ yr. So the motion is approximately adiabatic, and a rising blob will transport heat from the lower layers of the star into the outer layers.

The fluid parcel begins at r with some initial conditions $P(r)$, $\rho(r)$, and $T(r)$. After moving outward to $(r + dr)$ the parcel's temperature will remain unchanged even as the pressure rapidly equilibrates, so that the new pressure $P'(r + dr) = P(r + dr)$. Meanwhile (as in the previous sections) we will have an adiabatic equation of state (Eq. 335), which determines the parcel's new density ρ'.

The gas parcel will be stable to this radial perturbation so long as $\rho' > \rho(r + dr)$. Otherwise, if the parcel is less dense than its surroundings, it will be like a child's helium balloon and continue to rise: instability! A full analysis shows that this stability requirement can be restated in terms of P and ρ as

$$\left(\frac{dP/dr}{d\rho/dr}\right) < \frac{dP}{d\rho}|_{\text{adiabatic}} \tag{346}$$

$$< \gamma_{ad}\frac{P}{\rho}. \tag{347}$$

Since dP/dr and $d\rho/dr$ are both negative quantities, this can be rearranged as

$$\frac{\rho}{\gamma_{ad}P}\frac{dP}{dr} > \frac{d\rho}{dr}. \tag{348}$$

If we also assume that the stellar material is approximately an ideal gas, then

$P = \rho kT/\mu m_p$ and so

$$\left|\frac{dT}{dr}\right| < \frac{T}{P}\left|\frac{dP}{dr}\right|\left(1 - \frac{1}{\gamma_{ad}}\right) \quad (349)$$

Eq. 349 is the **Schwarzschild stability criterion** against convection. The absolute magnitudes are not strictly necessary, but can help to mentally parse the criterion: as long as the thermal profile is shallower than the modified pressure profile, the star will remain stable to radial perturbations of material.

Modeling convection

Fully self-consistent models of stellar convection are an active area of research and require considerable computational resources to accurately capture the three-dimensional fluid dynamics. The simplest model of convection is to assume that the process is highly efficient – so much so that it drives the system to saturate the Schwarzschild criterion, and so

$$\frac{dT}{dr} = \frac{T}{P}\frac{dP}{dr}\left(1 - \frac{1}{\gamma_{ad}}\right) \quad (350)$$

The somewhat *ad hoc,* but long-tested, framework of **mixing length theory** (MLT) allows us to refine our understanding of convection. In MLT one assumes that gas parcels rise some standard length ℓ, deliver their heat there, and sink again. Accurately estimating ℓ can be as much art as science; at least for nearly Solar stars, ℓ can be calibrated against a host of other observations.

Another way to understand convection comes from examining the relevant equations of stellar structure. Since the star is unstable to convection when the thermal profile becomes too steep, let's consider the thermal transport equation:

$$\frac{dT}{dr} = -\frac{3}{64\pi}\frac{\rho\kappa}{\sigma_{SB}T^3}\frac{L}{r^2} \quad (351)$$

Convection may occur either when $|dT/dr|$ is especially large, or when the Schwarzschild criterion's factor of $(1 - 1/\gamma_{ad})$ is especially small:

1. **Large** κ **and/or low** T: sometimes met in the outer layers (of Sun-like stars);
2. **Large** $F \equiv L/r^2$: potentially satisfied near cores
3. **Small** γ_{ad}: near ionization layers and molecular dissociation layers.

Overall, we usually see convection across a range of stellar types. Descending along the main sequence, energy transport in the hottest (and most massive) stars is dominated by radiation. Stars of somewhat lower mass (but still with $M_* > M_\odot$) will retain radiative outer atmospheres but acquire interior convective regions. By the time one considers stars of roughly Solar mass, we see a convective exterior that surrounds an internal radiative core. Many years of Solar observations shows the outer surface of the Sun bubbling away, just like a boiling pot. Large, more evolved stars (e.g., red giants) also have convective outer layers; in these cases, the size of the convective cells $\ell \gtrsim R_\odot$!

As one considers still lower masses along the main sequence, the convection region deepens; below spectral types of M2V-M3V, the stars become **fully convective** — i.e., $\ell = R_*$. The Gaia DR2 color-magnitude diagram shows a narrow break in the main sequence which is interpreted as a direct observational signature of the onset of full convection for these smallest, coolest stars. These stars therefore have a fully adiabatic equation of state throughout their interior.

16.4 *Another look at convection vs. radiative transport*

Again, we already know that radiative transport is the default mechanism for getting energy from a star's center to its surface. However, it turns out that within a star, a second mechanism can take over from radiative transport and become dominant. To understand when this happens, we need to bring back two concepts we have previously discussed.

First, we have the temperature gradient. We will use the version that defines the temperature change as a function of radius:

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\kappa\rho}{T^3}\frac{L_r}{4\pi r^2} \tag{352}$$

Second, we have the definition of an adiabatic process: a process in which no heat is exchanged between a system and its environment.

Again, we begin by considering a blob of gas somewhere within a star. It has a temperature T_{blob} and is surrounded by gas at an ambient, local temperature T_*. At this point, they are in thermal equilibrium so that $T_{blob} = T_*$. What happens if this blob is given a quick nudge upward so that now it is warmer than the gas around it: $T_{blob} > T_*$? Just as warm air does, we expect that it will rise. In order for the blob to stop rising, it must become cooler than its environment.

There are two ways for our blob to cool. One way is for it to radiate (that is, exchange heat with its environment). The other way is for it to do work on its environment (essentially, to expand in order to reach pressure equilibrium with its surroundings). Which one is going to be more effective in a star? To answer this, we can just look at time scales. Heat exchange will occur on a roughly thermal (or Kelvin-Helmholtz) time scale. For the sun, this time scale is on the order of ten million years. In contrast, work can be done on the blob's environment on a dynamical time scale (technically, the sound-crossing time scale, as this work is done by the expansion of the blob due to pressure). For the sun, this time scale is only about 30 minutes. The enormous difference in magnitude of these scales suggests that there will be almost no chance for the blob to exchange heat with its environment over the time scale in which it expands to reach pressure equilibrium with its environment: our blob will expand and cool nearly entirely adiabatically.

Once the blob has expanded enough to cool down to the ambient temperature, it will cease its upward motion and become stable again. The question is: how quickly will this happen? If an overly warm blob can quickly become cooler than the surrounding gas, then it will not travel far, and upward gas motions will be swiftly damped out. As the gas does not then move in bulk, energy in the star is transported just through the radiation field. However, if the blob cannot quickly become cooler than the ambient gas, it will rise and rise until it encounters a region where it finally satisfies this criterion. This sets up a convective zone in the star: an unstable situation that results in significant movement of gas in the star (think of a pot of water boiling). Warm gas travels upward through this region and eventually reaches the top of the convective zone (which could be the surface of the star, or a region inside the star where the physical conditions have changed significantly) where it is able to cool and return downward. Through the work that it does on its environment over this journey, it carries significant energy from the inner to the outer regions of the star. For this 'convective' zone of the star (which could be a small region or the entire star) convective transport is then the primary means by which energy is transported.

To determine whether convection will dominate, we compare the temperature gradient of the star (the ambient change in temperature as a function of radius) and the rate at which a parcel of gas will cool adiabatically (the so-called adiabatic temperature gradient). These are shown visually in Figure 31 for both convective stability and instability.

If the adiabatic temperature gradient is steeper than the temperature gradient in a star (as set by purely radiative energy transport) then the rate at which a blob of gas will rise and expand and cool will be more rapid than the rate at which the ambient gas in the star cools over the same distance. As a result, if a blob experiences a small displacement upward, it will very quickly become cooler than its surroundings, and sink back to its original position. No significant motion or convection will occur (this region is convectively stable) and the star will continue to transport energy radiatively.

However, if the adiabatic temperature gradient is shallower than the (radiative) temperature gradient in a star, then the rate at which a parcel of gas expands and cools as it rises will be slower than the rate at which the surrounding gas of the star cools over the same distance. Because of this, if a blob is displaced upward, it will remain hotter than its surroundings after it adiabatically expands to reach pressure equilibrium with its surroundings, and it will continue to rise. This sets up convection in the star: as long as the adiabatic temperature gradient is shallower than the radiative temperature gradient, the blob will rise. Only when the blob reaches an area of the star with different physics (such that the temperature gradient becomes shallower than the adiabatic gradient) will it stop rising. The region in which

$$\left(\frac{dT}{dr}\right)_{ad} < \left(\frac{dT}{dr}\right)_{*} \tag{353}$$

defines the convective zone and the region in which convection dominates the energy transport.

Convection can then be favored in several ways. One way is through mak-

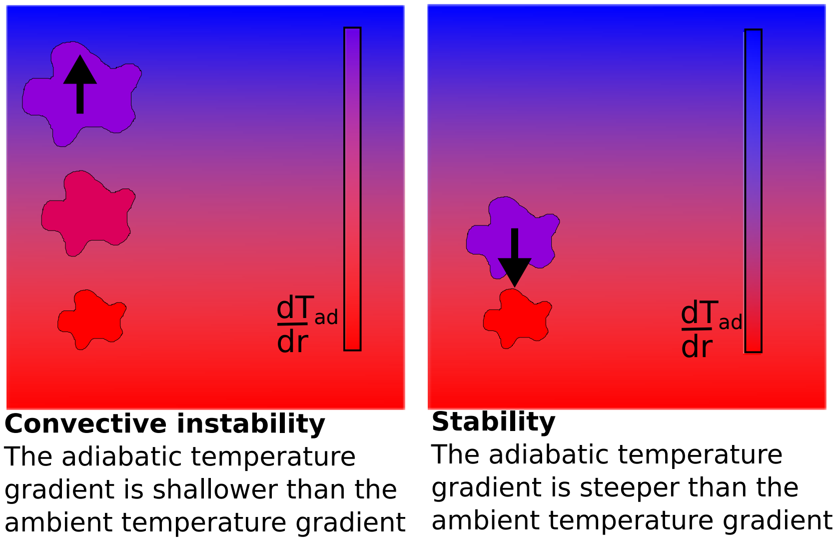

Figure 31: **Left**: An illustration of convective instability in a star. The ambient (background) temperature gradient, set by radiative transport of energy in the star, is steeper than the rate at which a parcel of gas can cool adiabatically. Because of this, a parcel of gas that becomes slightly warmer than the gas around it will rise uncontrollably, resulting in convection, which then is responsible for transporting energy in that region of the star. **Right**: An illustration of a stable situation. The ambient (background) temperature gradient, is shallower than the rate at which a parcel of gas can cool adiabatically. Because of this, a parcel of gas that becomes slightly warmer than the gas around it will very quickly become cooler than the gas around it, and will not rise significantly. In this situation, radiative energy transport dominates.

ing the adiabatic temperature gradient more shallow (this is set by the equation of state for the gas, and requires a deviation from the ideal gas law that lowers the adiabatic exponent). While this can and does occur, it is beyond the scope of this class, so we will not consider this in more detail. Alternatively then, we can ask what causes the temperature gradient of a star to steepen? Looking at Equation 352, we can see that the temperature gradient in a star is proportional to a number of variables, including the opacity κ and the energy flux L_r. Regions of high opacity are in fact a significant cause of convective zones in stars. As we saw in Section 11.1, many of the processes that cause opacity in stars favor conditions in which there are bound electrons. This will occur in cooler regions of a star, particularly in regions where the gas (Hydrogen or Helium) is only partially ionized. In fact, partially ionized gas also has a slightly lower adiabatic exponent than fully ionized gas, which further contributes to the development of convective instability. The sun's outer layers are convective for these reasons (its core is radiative, as this region is fully ionized). Cooler stars like red dwarfs are actually fully convective from their core to surface. As we will discuss further when we reach the topic of nuclear burning in stars, nuclear processes that release substantial amounts of energy can significantly increase L_r and thus also drive convection. This is the reason that stars more massive than the sun (which have a slightly different fusion reaction occurring) have convective cores.

Note that in defining convective instability we can also swap variables, and instead of temperature, consider the density and pressure of the blob (as both of these are connected to the temperature through our equation of state (the ideal gas equation, which is a good description of the conditions in the interior of stars). This approach is taken by Prialnik, and is the basis of the historical argument first made by Karl Schwarzschild in evaluating the convective stability of stars. Here, we assume that we have a blob that has pressure and density equal to the ambient values in the surrounding gas. When it is displaced upward, its pressure now exceeds the ambient pressure, and it expands adiabatically to reach pressure equilibrium with its surroundings. In expanding, not only has its pressure decreased, but its density as well. If the blob is now less dense than its surroundings, it will experiences a force that will displace it upward. However, if the blob remains more dense than its surroundings, it will instead experience a downward displacement force. This argument is in a sense more physical, as we are not appealing to 'warm air rising' but rather the underlying physical mechanism: the Archimedes buoyancy law. Using these variables, our condition for convective instability is now

$$\left(\frac{dP}{dr}\right)_{ad} < \left(\frac{dP}{dr}\right)_{*} . \tag{354}$$

All of this actually has an interesting application not just to stars, but to earth's atmosphere as well: the formation of thunderstorms! The formation of extremely tall (up to 12 miles!) clouds that lead to severe thunderstorms and tornadoes are driven by a convective instability in earth's lower atmosphere. The conditions that lead to this convective instability can be measured, and factor into forecasts of severe weather outbreaks. The criterion for convective instability is exactly the same as we just discussed for stars: the adiabatic temperature gradient, or the rate at which a parcel of gas displaced from ground level will cool as it rises, must be less than the temperature gradient (or profile) of the atmosphere:

$$\left(\frac{dT}{dr}\right)_{ad} < \left(\frac{dT}{dr}\right)_{atm} \tag{355}$$

The conditions that lead to thunderstorms have two things going on that make this more likely. First, weather systems that lead to thunderstorms are typically driven by the approach of a cold air mass (a cold front) that is pushing like a wedge into the upper atmosphere. This steepens the temperature gradient of the atmosphere: problem #1. The second thing that happens in advance of these weather systems is the buildup of a moist air mass in advance of the cold front, which drives high humidity. As you may have experienced firsthand (think of how quickly it gets cold at night in the desert, or alternatively, how warm a humid summer night can be, and how hard it is to stay cool on a humid day) air with a high moisture content is better at retaining heat, and thus cools more slowly. In essence, it has a shallower adiabatic temperature gradient: problem #2. Together, these two conditions are a recipe for strong convection. Humid air that is heated near the sunbaked ground will dramatically rise, unchecked, into the upper atmosphere, depositing energy and water vapor to make enormous, powerful cumulonimbus (thunderhead) clouds. The strength of the convection is measured by meteorologists with the CAPE (Convective Available Potential Energy) index. It measures this temperature differential, and uses it to determine how strong the upward buoyancy force will be. An extremely large CAPE for a given region could be a reason to issue a tornado watch.

16.5 *XXXX – extra material on convection in handwritten notes*

17 Polytropes

Much of the challenge in making self-consistent **stellar model**s comes from the connection between T and L. The set of so-called **polytrope** models derives from assuming that we can just ignore the thermal and luminosity equations of stellar structure. This assumption is usually wrong, but it is accurate in some cases, useful in others, and historically was essential for making early progress toward understanding stellar interiors. A polytrope model assumes that for some proportionality constant K and index γ (or equivalently, n),

(356)
$$P = K\rho^{\gamma}$$

(357)
$$= K\rho^{1+1/n}$$

We have already discussed at least two types of stars for which a polytrope is an accurate model. For **fully convective** stars, energy transport is dominated by bulk motions which are essentially **adiabatic** (since $\tau_{dyn} \langle \tau_{\gamma,\text{diff}}$); thus $\gamma = \gamma_{ad} = 5/3$. It turns out that the same index also holds for degenerate objects (white dwarfs and neutron stars); in the non-relativistic limit these also have $\gamma = 5/3$, even though heat transport is dominated by conduction not convection. When degenerate interiors become fully relativistic, γ approaches $4/3$ and (as we saw previously) the stars can come perilously close to global instability.

The key equation in polytrope models is that of hydrostatic equilibrium (Eq. 239),

$$\frac{dP}{dr} = -\frac{GM}{r^2}\rho$$

which when rearranged yields

(358) $$\frac{r^2}{\rho}\frac{dP}{dr} = -GM.$$

Taking the derivative of each side, we have

(359) $$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -GdM,$$

and substituting in the mass-radius equation (Eq. 274) for dM gives

(360) $$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho.$$

It is then customary to define the density in terms of a dimensionless density function $\phi(r)$, such that

(361) $$\rho(r) = \rho_c \phi(r)^n$$

and n is the polytrope index of Eq. 17. Note that $\phi(r = R) = 0$ defines the stellar surface. In the interior, $\phi(r = 0) = 1$, so ρ_c is the density at the center of the star. This can be determined by calculating M_* via

$$\int \rho(r) dV = M_* \tag{362}$$

and then solving for ρ_c.

Combining Eq. 361 with Eq. 17 above gives

$$P(r) = K\rho_c^{1+1/n}\phi(r)^{n+1} = P_C\phi^{n+1}. \tag{363}$$

Plugging this back into Eq. 360 and rearranging yields the formidable-looking

$$\lambda^2 \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = -\phi^n \tag{364}$$

where we have defined

$$\lambda = \left(\frac{K(n+1)\rho_c^{1/n-1}}{4\pi G}\right)^{1/2}. \tag{365}$$

When one also then defines

$$r = \lambda\xi, \tag{366}$$

then we finally obtain the famous **Lane-Emden Equation**

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\phi}{d\xi}\right) = -\phi^n. \tag{367}$$

The solutions to the Lane-Emden equation are the set of functions $\phi(\xi)$, each of which corresponds to a different index n and each of which completely specifies a star's density profile in the polytrope model via Eq. 361. The solution for a given n is conventionally denoted $\phi_n(\xi)$. Each solution also determines the temperature profile $T(r)$.

What are the relevant boundary conditions for $\phi(\xi)$, and what are the possible values of this dimensionless ξ anyway? Well, just as with $\phi(r)$ we must also have that $\phi(\xi = 0) = 1$, and analogously we will have $\phi(\xi = \xi_{\text{surf}}) = 0$. As for ξ_{surf} (the value of ξ at the stellar surface), its value will depend on the particular form of the solution, $\phi(\xi)$. A final, useful boundary condition is that we have no cusp in the central density profile – i.e., the density will be a smooth function from $r = +\epsilon$ to $-\epsilon$. So our boundary conditions are thus

(368)
$$\phi(\xi = 0) = 1$$

(369)
$$\phi(\xi = \xi_{\rm surf}) = 0$$

(370)
$$\frac{d\phi}{d\xi}|_{\xi=0} = 0$$

Just three analytic forms of $\phi(\xi)$ exist, corresponding to $n = 0,\ 1,\ 5$. Solutions give finite stellar mass only for $n \leq 5$. Textbooks on stellar interiors give examples of these various solutions. One example is $n = 1$, for which the solution is

(371) $$\phi_1(\xi) = a_0 \frac{\sin \xi}{\xi} + a_1 \frac{\cos \xi}{\xi}$$

where a_0 and a_1 are determined by the boundary conditions. A quick comparison to those conditions, above, shows that the solution is

(372) $$\phi_1(\xi) = \frac{\sin \xi}{\xi}$$

which is the well-known sinc function. For a reasonable stellar model in which ρ only decreases with increasing r, this also tells us that for $n = 1$, $\xi_{\rm surf} = \pi$.

The point of this dense thicket of ϕ's and ξ's is that once n is specified, you only have to solve the Lane-Emden equation once. (And this has already been done – Fig. 32 shows the solutions for $n = 0$ to 5.) Merely by scaling K and ρ_c one then obtains an entire family of stellar structure models for each ϕ_n – each model in the family has its own central density and total mass, even though the structure of all models in the family (i.e., for each n) are homologously related.

For example, for $n = 1$ and a star with $R_* = R_\odot$, the definition of ξ means that $\xi_{\rm surf} = \pi = \lambda R_\odot$. If we found the central density ρ_c using the known stellar mass M_* and Eq. 362, then we will have everything we need to find K (via Eq. 365) and so find a full expression for the pressure profile (via Eq. 363) and the associated density profile (via Eq.).

The thermal profile $T(r)$ can then also be determined by assuming that the stellar interior is an ideal gas with

(373) $$P = nkT = \frac{\rho k T}{\mu m_p}.$$

A superficial examination here seems to imply that despite our initial polytropic assumption that $P \propto \rho^{1+1/n}$, we instead have $P \propto \rho$ (independent of n). The 'solution' to this quandry is to recognize that of course T is not constant throughout the star, but is also varying with r. Given our expression for $\rho(r)$

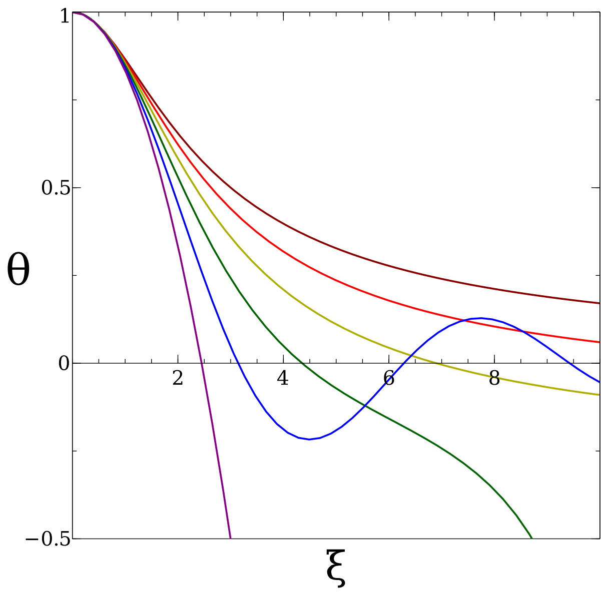

Figure 32: Solutions to the Lane-Emden Equation, here denoted by θ instead of ϕ, for $n = 0$ (most concentrated) to 5 (least concentrated). The applicability of each curve to stellar interiors ends at the curve's first zero-crossing. Figure from Wikipedia, used under a Creative Commons CCO 1.0 license.

in Eq. 361, we then have

(374) $$P = \rho_c \phi^n \frac{k_B T}{\mu m_p} = K \rho_c^{1+1/n} \phi^{n+1}$$

and so we then have

(375) $$T = \frac{K \mu m_p}{k_B} \rho_c^{1/n} \phi.$$

So $T \propto \phi$, $\rho \propto \phi^n$, and $P \propto \phi^{n+1}$. It would be an interesting exercise to use these relations to calculate the explicit **thermal profile** of a polytropic model...