

Lecture Notes on Kinetic Theory and Magnetohydrodynamics of Plasmas

(Oxford MMathPhys/MSc in Mathematical and Theoretical Physics)

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These are the notes for my lectures on Kinetic Theory of Plasmas and on Magnetohydrodynamics, taught since 2014 as part of the MMathPhys programme at Oxford. Part I contains the lectures on plasma kinetics that formed part of the course on Kinetic Theory, taught jointly with Paul Dellar and James Binney. Part II is an introduction to magnetohydrodynamics, which was part of the course on Advanced Fluid Dynamics, taught jointly with Paul Dellar. These notes have evolved from two earlier courses: “Advanced Plasma Theory,” taught as a graduate course at Imperial College in 2008, and “Magnetohydrodynamics and Turbulence,” taught as a Mathematics Part III course at Cambridge in 2005-06. I will be grateful for any feedback from students, tutors or sympathisers.

PART I

Kinetic Theory of Plasmas

1. Kinetic Description of a Plasma

We would like to consider a gas consisting of charged particles—ions and electrons. In general, there may be many different species of ions, with different masses and charges, and, of course, only one type of electrons.

We shall index particle species by α ($\alpha = e$ for electrons, $\alpha = i$ for ions). Each is characterised by its mass m_α and charge $q_\alpha = Z_\alpha e$, where e is the magnitude of the electron charge and Z_α is a positive or negative integer (e.g., $Z_e = -1$).

1.1. Quasineutrality

We shall always assume that plasma is neutral overall:

$$\sum_{\alpha} q_{\alpha} N_{\alpha} = eV \sum_{\alpha} Z_{\alpha} \bar{n}_{\alpha} = 0, \quad (1.1)$$

where N_{α} is the number of the particles of species α , $\bar{n}_{\alpha} = N_{\alpha}/V$ is their mean number density and V the volume of the plasma. This condition is known as *quasineutrality*.

1.2. Weak Interactions

Interaction between charged particles is governed by the Coulomb potential:

$$\Phi(|\mathbf{r}_i^{(\alpha)} - \mathbf{r}_j^{(\alpha')}|) = -\frac{q_{\alpha} q_{\alpha'}}{|\mathbf{r}_i^{(\alpha)} - \mathbf{r}_j^{(\alpha')}|}, \quad (1.2)$$

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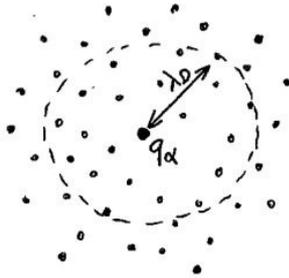


FIGURE 1. A particle amongst particles and its Debye sphere.

where by $\mathbf{r}_i^{(\alpha)}$ we mean the position of the i th particle of species α . We can safely anticipate that we will only be able to have a nice closed kinetic description if the gas is approximately ideal, i.e., if particles interact weakly, viz.,

$$k_B T \gg \Phi \sim \frac{e^2}{\Delta r} \sim e^2 n^{1/3}, \quad (1.3)$$

where k_B is the Boltzmann constant, which will henceforth be absorbed into the temperature T , and $\Delta r \sim n^{-1/3}$ is the typical interparticle distance. Let us see what this condition means and implies physically.

1.3. Debye Shielding

Let us consider a plasma in thermodynamic equilibrium (as one does in statistical mechanics, we will refuse to discuss, for the time being, how exactly it got there). Take one particular particle, of species α . It creates an electric field around itself, $\mathbf{E} = -\nabla\varphi$; all other particles are sitting in this field (Fig. 1)—and, indeed, also affecting it, as we will see below. In equilibrium, the densities of these particles ought to satisfy Boltzmann’s formula:

$$n_{\alpha'}(\mathbf{r}) = \bar{n}_{\alpha'} e^{-q_{\alpha'}\varphi(\mathbf{r})/T} \approx \bar{n}_{\alpha'} - \frac{\bar{n}_{\alpha'} q_{\alpha'} \varphi}{T}, \quad (1.4)$$

where $\bar{n}_{\alpha'}$ is the mean density of particles of species α' and $\varphi(\mathbf{r})$ is the electrostatic potential, which depends on the distance \mathbf{r} from our “central” particle. As $r \rightarrow \infty$, $\varphi \rightarrow 0$ and $n_{\alpha'} \rightarrow \bar{n}_{\alpha'}$. The exponential can be Taylor-expanded provided the weak-interaction condition (1.3) is satisfied ($e\varphi \ll T$).

By the Gauss–Poisson law, we have

$$\begin{aligned} \nabla \cdot \mathbf{E} = -\nabla^2 \varphi &= 4\pi q_\alpha \delta(\mathbf{r}) + 4\pi \sum_{\alpha'} q_{\alpha'} n_{\alpha'} \\ &\approx 4\pi q_\alpha \delta(\mathbf{r}) + \underbrace{4\pi \sum_{\alpha'} q_{\alpha'} \bar{n}_{\alpha'}}_{= 0 \text{ by quasineutrality}} - \underbrace{\left(\sum_{\alpha'} \frac{4\pi \bar{n}_{\alpha'} q_{\alpha'}^2}{T} \right)}_{\equiv 1/\lambda_D^2} \varphi. \end{aligned} \quad (1.5)$$

In the first line of this equation, the first term on the right-hand side is the charge density associated with the “central” particle and the second term the charge density of the rest of the particles. In the second line, we used the Taylor-expanded Boltzmann expression (1.4) for the particle densities and then the quasineutrality (1.1) to establish the vanishing of the second term. The combination that has arisen in the last term as a prefactor of φ has dimensions of inverse square length, so we define the *Debye length*

to be

$$\lambda_D \equiv \left(\sum_{\alpha} \frac{4\pi\bar{n}_{\alpha}q_{\alpha}^2}{T} \right)^{-1/2}. \quad (1.6)$$

Using also the obvious fact that the solution of Eq. (1.5) must be spherically symmetric, we recast this equation as follows

$$\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial \varphi}{\partial r} - \frac{1}{\lambda_D^2} \varphi = -4\pi q_{\alpha} \delta(\mathbf{r}). \quad (1.7)$$

The solution to this that asymptotes to the Coulomb potential $\varphi \rightarrow q_{\alpha}/r$ as $r \rightarrow 0$ and to zero as $r \rightarrow \infty$ is

$$\varphi = \frac{q_{\alpha}}{r} e^{-r/\lambda_D}. \quad (1.8)$$

Thus, in a quasineutral plasma, charges are shielded on typical distances $\sim \lambda_D$.

Obviously, this calculation only makes sense if the ‘‘Debye sphere’’ has many particles in it, viz., if

$$n\lambda_D^3 \gg 1. \quad (1.9)$$

Let us check that this is the case: indeed,

$$n\lambda_D^3 \sim n \left(\frac{T}{ne^2} \right)^{3/2} = \left(\frac{T}{n^{1/3}e^2} \right)^{3/2} \gg 1, \quad (1.10)$$

provided the weak-interaction condition (1.3) is satisfied. The quantity $n\lambda_D^3$ is called the *plasma parameter*.

1.4. Micro- and Macroscopic Fields

This calculation tells us something very important about electromagnetic fields in a plasma. Let $\mathbf{E}^{(\text{micro})}(\mathbf{r}, t)$ and $\mathbf{B}^{(\text{micro})}(\mathbf{r}, t)$ be the exact *microscopic fields* at a given location \mathbf{r} and time t . These fields are responsible for interactions between particles. On distances $l \ll \lambda_D$, these will be essentially just two-particle interactions—binary collisions between particles in a vacuum, just like in a neutral gas (except the interparticle potential is a Coulomb potential). In contrast, on distances $l \gtrsim \lambda_D$, individual particles’ fields are shielded and what remains are fields due to collective influence of large numbers of particles—*macroscopic fields*:

$$\mathbf{E}^{(\text{micro})} = \underbrace{\langle \mathbf{E}^{(\text{micro})} \rangle}_{\equiv \mathbf{E}} + \delta\mathbf{E}, \quad \mathbf{B}^{(\text{micro})} = \underbrace{\langle \mathbf{B}^{(\text{micro})} \rangle}_{\equiv \mathbf{B}} + \delta\mathbf{B}, \quad (1.11)$$

where the macroscopic fields \mathbf{E} and \mathbf{B} are averages over some intermediate scale l such that

$$\Delta r \sim n^{-1/3} \ll l \ll \lambda_D. \quad (1.12)$$

Such averaging is made possible by the condition (1.9).

Thus, plasma has a new feature compared to neutral gas: because the Coulomb potential is long-range ($\propto 1/r$), the fields decay on a length scale that is long compared to the interparticle distances ($\lambda_D \gg \Delta r \sim n^{-1/3}$ according to Eq. (1.9)) and so, besides interactions between individual particles, there are also collective effects: interaction of particles with mean macroscopic fields due to all other particles.

Before we use this approach to construct a description of the plasma as a continuum (on scales $\gtrsim \lambda_D$), let us check that particles travel sufficiently long distances between collisions in order to

feel the macroscopic fields, viz., that their mean free paths $\lambda_{\text{mfp}} \gg \lambda_{\text{D}}$. The mean free path can be estimated in terms of the collision cross-section σ :

$$\lambda_{\text{mfp}} \sim \frac{1}{n\sigma} \sim \frac{T^2}{ne^4} \quad (1.13)$$

because $\sigma \sim d^2$ and the effective distance d by which the particles have to approach each other in order to have significant Coulomb interaction is inferred by balancing the Coulomb potential energy with the particle temperature, $e^2/d \sim T$. Using Eqs. (1.13) and (1.6), we find

$$\frac{\lambda_{\text{mfp}}}{\lambda_{\text{D}}} \sim \frac{T^2}{ne^4} \left(\frac{ne^2}{T} \right)^{1/2} \sim n\lambda_{\text{D}}^3 \gg 1, \quad \text{q.e.d.} \quad (1.14)$$

Thus, it makes sense to talk about a particle travelling long distances experiencing the macroscopic fields exerted by the rest of the plasma collectively before being deflected by a much larger, but also much shorter-range, microscopic field of another individual particle.

1.5. Maxwell's Equations

The exact microscopic fields satisfy *Maxwell's equations* and, as Maxwell's equations are linear, so do the macroscopic fields: by direct averaging,

$$\nabla \cdot \langle \mathbf{E}^{(\text{micro})} \rangle = 4\pi \langle \sigma^{(\text{micro})} \rangle, \quad (1.15)$$

$$\nabla \cdot \langle \mathbf{B}^{(\text{micro})} \rangle = 0, \quad (1.16)$$

$$\nabla \times \langle \mathbf{E}^{(\text{micro})} \rangle + \frac{1}{c} \frac{\partial \langle \mathbf{B}^{(\text{micro})} \rangle}{\partial t} = 0, \quad (1.17)$$

$$\nabla \times \langle \mathbf{B}^{(\text{micro})} \rangle - \frac{1}{c} \frac{\partial \langle \mathbf{E}^{(\text{micro})} \rangle}{\partial t} = \frac{4\pi}{c} \langle \mathbf{j}^{(\text{micro})} \rangle. \quad (1.18)$$

The new quantities here are the averages of the microscopic charge density $\sigma^{(\text{micro})}$ and the microscopic current density $\mathbf{j}^{(\text{micro})}$. How do we calculate them?

Clearly, they depend on where all the particles are at any given time and how fast these particles move. We can assemble all this information in one function:

$$F_\alpha(\mathbf{r}, \mathbf{v}, t) = \sum_{i=1}^{N_\alpha} \delta^3(\mathbf{r} - \mathbf{r}_i^{(\alpha)}(t)) \delta^3(\mathbf{v} - \mathbf{v}_i^{(\alpha)}(t)), \quad (1.19)$$

where $\mathbf{r}_i^{(\alpha)}(t)$ and $\mathbf{v}_i^{(\alpha)}(t)$ are the exact phase-space coordinates of particle i of species α at time t , i.e., these are solutions of the exact equations of motion for all these particles moving in microscopic fields $\mathbf{E}^{(\text{micro})}(t, \mathbf{r})$ and $\mathbf{B}^{(\text{micro})}(t, \mathbf{r})$. The function F_α is called the *Klimontovich distribution function* and it is a random object (i.e., it fluctuates on scales $\ll \lambda_{\text{D}}$) because it depends on the exact particle trajectories, which depend on the exact microscopic fields. In terms of this distribution function,

$$\sigma^{(\text{micro})}(\mathbf{r}, t) = \sum_{\alpha} q_{\alpha} \int d^3\mathbf{v} F_{\alpha}(\mathbf{r}, \mathbf{v}, t), \quad (1.20)$$

$$\mathbf{j}^{(\text{micro})}(\mathbf{r}, t) = \sum_{\alpha} q_{\alpha} \int d^3\mathbf{v} \mathbf{v} F_{\alpha}(\mathbf{r}, \mathbf{v}, t). \quad (1.21)$$

We now need to average these quantities for use in Eqs. (1.15) and (1.18). We shall assume that the average over microscales (1.12) and the ensemble average (i.e., average over many different initial conditions) are the same. The ensemble average of F_α is an object familiar from the kinetic theory of gases, the so-called *one-particle distribution*

function:

$$\langle F_\alpha \rangle = f_{1\alpha}(\mathbf{r}, \mathbf{v}, t) \quad (1.22)$$

(we shall henceforth omit the subscript 1). If we learn how to compute f_α , then we can average Eqs. (1.20) and (1.21), substitute into Eqs. (1.15) and (1.18), and have the following set of macroscopic Maxwell's equations:

$$\nabla \cdot \mathbf{E} = 4\pi \sum_\alpha q_\alpha \int d^3\mathbf{v} f_\alpha(\mathbf{r}, \mathbf{v}, t), \quad (1.23)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (1.24)$$

$$\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0, \quad (1.25)$$

$$\nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c} \sum_\alpha q_\alpha \int d^3\mathbf{v} \mathbf{v} f_\alpha(\mathbf{r}, \mathbf{v}, t). \quad (1.26)$$

1.6. Vlasov–Landau Equation

We now need an evolution equation for $f_\alpha(\mathbf{r}, \mathbf{v}, t)$, hopefully in terms of the macroscopic fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$, so we can couple it to Eqs. (1.23–1.26) and thus have a closed system of equations describing our plasma.

The process of deriving it starts with Liouville's theorem and is a direct generalisation of the BBGKY procedure familiar from gas kinetics (e.g., Dellar 2016)[†] to the somewhat more cumbersome case of a plasma:

—many species α ;

—Coulomb potential for interparticle collisions (with some attendant complications to do with its long-range nature: in brief, use Rutherford's cross section and cut off long-range interactions at λ_D ; this is described in many textbooks and plasma-physics courses, e.g., Parra 2017a; Helander & Sigmar 2005);

—presence of forces due to the macroscopic fields \mathbf{E} and \mathbf{B} .

The result of this derivation is

$$\frac{\partial f_\alpha}{\partial t} + \{f_\alpha, H_{1\alpha}\} = \left(\frac{\partial f_\alpha}{\partial t} \right)_c. \quad (1.27)$$

The Poisson bracket contains $H_{1\alpha}$, the Hamiltonian for a single particle of species α moving in the *macroscopic* electromagnetic field—all the microscopic fields $\delta\mathbf{E}$ [‡] are gone into into the collision operator on the right-hand side, of which more will be said shortly (§1.7).

Technically speaking, we ought to be working with canonical variables, but dealing with canonical momenta is an unnecessary complication and so we shall stick to the (\mathbf{r}, \mathbf{v}) representation of the phase space. Eq. (1.27) then takes the form of Liouville's equation, but with microscopic fields hidden inside the collision operator:

$$\frac{\partial f_\alpha}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \cdot (\dot{\mathbf{r}} f_\alpha) + \frac{\partial}{\partial \mathbf{v}} \cdot (\dot{\mathbf{v}} f_\alpha) = \left(\frac{\partial f_\alpha}{\partial t} \right)_c, \quad (1.28)$$

where

$$\dot{\mathbf{r}} = \mathbf{v}, \quad \dot{\mathbf{v}} = \frac{q_\alpha}{m_\alpha} \left(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right). \quad (1.29)$$

[†] In §1.8, I will sketch Klimontovich's version of this procedure (Klimontovich 1967).

[‡] $\delta\mathbf{B}$ turns out to be irrelevant as long as the particle motion is non-relativistic, $v/c \ll 1$.

This gives us the *Vlasov–Landau equation*:

$$\boxed{\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \nabla f_\alpha + \frac{q_\alpha}{m_\alpha} \left(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right) \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}} = \left(\frac{\partial f_\alpha}{\partial t} \right)_c} \quad (1.30)$$

Any other macroscopic force that the plasma might be subject to (e.g., gravity) can be added to the Lorentz force in the third term on the left-hand side, as long as its divergence in velocity space is $(\partial/\partial \mathbf{v}) \cdot \text{force} = 0$. Eq. (1.30) is closed by Maxwell’s equations (1.23–1.26).

1.7. Collision Operator

Finally, a few words about the plasma collision operator, originally due to Landau (1936). It describes two-particle collisions both within the species α and with other species α' and so depends both on f_α and on all other $f_{\alpha'}$. Its derivation is left to you as an exercise in BBGKY’ing, calculating cross sections and velocity integrals (or in googling; shortcut: see Parra 2017a). In these Lectures, we shall largely focus on collisionless aspects of plasma kinetics. Whenever we find ourselves in need of invoking the collision operator, the important things about it for us will be its properties:

- *conservation of particles*,

$$\int d^3 \mathbf{v} \left(\frac{\partial f_\alpha}{\partial t} \right)_c = 0 \quad (1.31)$$

(within each species α);

- *conservation of momentum*,

$$\sum_\alpha \int d^3 \mathbf{v} m_\alpha \mathbf{v} \left(\frac{\partial f_\alpha}{\partial t} \right)_c = 0 \quad (1.32)$$

(same-species collisions conserve momentum, whereas different-species collisions conserve it only after summation over species—there is friction of one species against another; for example, the friction of electrons against the ions is the Ohmic resistivity of the plasma);

- *conservation of energy*,

$$\sum_\alpha \int d^3 \mathbf{v} \frac{m_\alpha v^2}{2} \left(\frac{\partial f_\alpha}{\partial t} \right)_c = 0; \quad (1.33)$$

- Boltzmann’s *H-theorem*: the kinetic entropy

$$S = - \sum_\alpha \int d^3 \mathbf{r} \int d^3 \mathbf{v} f_\alpha \ln f_\alpha \quad (1.34)$$

cannot decrease, and, as S is conserved by all the collisionless terms in Eq. (1.30), the collision operator must have the property that

$$\frac{dS}{dt} = - \sum_\alpha \int d^3 \mathbf{r} \int d^3 \mathbf{v} \left(\frac{\partial f_\alpha}{\partial t} \right)_c \ln f_\alpha \geq 0, \quad (1.35)$$

with equality obtained if and only if f_α is a local Maxwellian;

- unlike the Boltzmann operator for neutral gases, the Landau operator expresses the cumulative effect of many glancing (rather than “head-on”) collisions (due to the

long-range nature of the Coulomb interaction) and so it is a Fokker–Planck operator:†

$$\left(\frac{\partial f_\alpha}{\partial t}\right)_c = \sum_{\alpha'} \frac{\partial}{\partial v_i} \left(A_i^{(\alpha\alpha')} [f_{\alpha'}] + \frac{\partial}{\partial v_j} D_{ij}^{(\alpha\alpha')} [f_{\alpha'}] \right) f_\alpha, \quad (1.36)$$

where the drag $A_i^{(\alpha\alpha')} [f_{\alpha'}]$ and diffusion $D_{ij}^{(\alpha\alpha')} [f_{\alpha'}]$ coefficients are integral (in \mathbf{v} space) functionals of $f_{\alpha'}$. The Fokker–Planck form (1.36) of the Landau operator means that it describes diffusion in velocity space and so will erase sharp gradients in f_α with respect to \mathbf{v} —a property that we will find very important in §4.

1.8. Klimontovich's Version of BBGKY

By way of a technical digression, let us outline the (beginning of the) derivation of Eq. (1.30) due to Klimontovich (1967). Consider the Klimontovich distribution function (1.19) and calculate its time derivative: by the chain rule,

$$\begin{aligned} \frac{\partial F_\alpha}{\partial t} = & - \sum_i \frac{d\mathbf{r}_i^{(\alpha)}(t)}{dt} \cdot \left[\frac{\partial}{\partial \mathbf{r}} \delta^3(\mathbf{r} - \mathbf{r}_i^{(\alpha)}(t)) \delta^3(\mathbf{v} - \mathbf{v}_i^{(\alpha)}(t)) \right] \\ & - \sum_i \frac{d\mathbf{v}_i^{(\alpha)}(t)}{dt} \cdot \left[\frac{\partial}{\partial \mathbf{v}} \delta^3(\mathbf{r} - \mathbf{r}_i^{(\alpha)}(t)) \delta^3(\mathbf{v} - \mathbf{v}_i^{(\alpha)}(t)) \right]. \end{aligned} \quad (1.37)$$

First, because $\mathbf{r}_i^{(\alpha)}(t)$ and $\mathbf{v}_i^{(\alpha)}(t)$ obviously do not depend on the phase-space variables \mathbf{r} and \mathbf{v} , the derivatives $\partial/\partial \mathbf{r}$ and $\partial/\partial \mathbf{v}$ can be pulled outside, so the right-hand side of Eq. (1.37) can be written as a divergence in phase space. Secondly, the particle equations of motion give us

$$\frac{d\mathbf{r}_i^{(\alpha)}(t)}{dt} = \mathbf{v}_i^{(\alpha)}(t), \quad (1.38)$$

$$\frac{d\mathbf{v}_i^{(\alpha)}(t)}{dt} = \frac{q_\alpha}{m_\alpha} \left[\mathbf{E}^{(\text{micro})}(\mathbf{r}_i^{(\alpha)}(t), t) + \frac{\mathbf{v}_i^{(\alpha)}(t) \times \mathbf{B}^{(\text{micro})}(\mathbf{r}_i^{(\alpha)}(t), t)}{c} \right], \quad (1.39)$$

which we substitute into the right-hand side of Eq. (1.37)—*after* it is written in the divergence form. As the time derivatives of $\mathbf{r}_i^{(\alpha)}(t)$ and $\mathbf{v}_i^{(\alpha)}(t)$ inside the divergence multiply delta functions identifying $\mathbf{r}_i^{(\alpha)}(t)$ with \mathbf{r} and $\mathbf{v}_i^{(\alpha)}(t)$ with \mathbf{v} , we may replace $\mathbf{r}_i^{(\alpha)}(t)$ with \mathbf{r} and $\mathbf{v}_i^{(\alpha)}(t)$ with \mathbf{v} in the right-hand sides of Eqs. (1.38) and (1.39) when they go into Eq. (1.37). This gives (wrapping up all the sums of delta functions back into F_α)

$$\frac{\partial F_\alpha}{\partial t} = -\nabla \cdot (\mathbf{v} F_\alpha) - \frac{\partial}{\partial \mathbf{v}} \cdot \left[\frac{q_\alpha}{m_\alpha} \left(\mathbf{E}^{(\text{micro})}(\mathbf{r}, t) + \frac{\mathbf{v} \times \mathbf{B}^{(\text{micro})}(\mathbf{r}, t)}{c} \right) F_\alpha \right]. \quad (1.40)$$

Finally, because \mathbf{r} and \mathbf{v} are independent variables and the Lorentz force has zero divergence in \mathbf{v} space, we find that F_α satisfies *exactly*

$$\frac{\partial F_\alpha}{\partial t} + \mathbf{v} \cdot \nabla F_\alpha + \frac{q_\alpha}{m_\alpha} \left(\mathbf{E}^{(\text{micro})} + \frac{\mathbf{v} \times \mathbf{B}^{(\text{micro})}}{c} \right) \cdot \frac{\partial F_\alpha}{\partial \mathbf{v}} = 0. \quad (1.41)$$

† The simplest example that I can think of in which the collision operator is a velocity-space diffusion operator of this kind is the gas of Brownian particles [each with velocity described by Langevin's equation (10.20)]. This is treated in detail in §6.9 of Schekochihin (2016).

This is the *Klimontovich equation*. There is no collision integral here because microscopic fields are explicitly present. The equation is closed by the microscopic Maxwell's equations:

$$\nabla \cdot \mathbf{E}^{(\text{micro})} = 4\pi \sum_{\alpha} q_{\alpha} \int d^3\mathbf{v} F_{\alpha}(\mathbf{r}, \mathbf{v}, t), \quad (1.42)$$

$$\nabla \cdot \mathbf{B}^{(\text{micro})} = 0, \quad (1.43)$$

$$\nabla \times \mathbf{E}^{(\text{micro})} + \frac{1}{c} \frac{\partial \mathbf{B}^{(\text{micro})}}{\partial t} = 0, \quad (1.44)$$

$$\nabla \times \mathbf{B}^{(\text{micro})} - \frac{1}{c} \frac{\partial \mathbf{E}^{(\text{micro})}}{\partial t} = \frac{4\pi}{c} \sum_{\alpha} q_{\alpha} \int d^3\mathbf{v} \mathbf{v} F_{\alpha}(\mathbf{r}, \mathbf{v}, t). \quad (1.45)$$

Now we separate the microscopic fields into mean (macroscopic) and fluctuating parts according to Eq. (1.11) and also

$$F_{\alpha} = \underbrace{\langle F_{\alpha} \rangle}_{\equiv f_{\alpha}} + \delta F_{\alpha}. \quad (1.46)$$

Maxwell's equations are linear, so averaging them gives the same equations for \mathbf{E} and \mathbf{B} in terms of f_{α} [Eqs. (1.23–1.26)] and for $\delta\mathbf{E}$ and $\delta\mathbf{B}$ in terms of δF_{α} . Averaging the Klimontovich equation (1.45) gives the Vlasov–Landau equation:

$$\begin{aligned} \frac{\partial f_{\alpha}}{\partial t} + \mathbf{v} \cdot \nabla f_{\alpha} + \frac{q_{\alpha}}{m_{\alpha}} \left(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right) \cdot \frac{\partial f_{\alpha}}{\partial \mathbf{v}} \\ = -\frac{q_{\alpha}}{m_{\alpha}} \left\langle \left(\delta\mathbf{E} + \frac{\mathbf{v} \times \delta\mathbf{B}}{c} \right) \cdot \frac{\partial \delta F_{\alpha}}{\partial \mathbf{v}} \right\rangle \equiv \left(\frac{\partial f_{\alpha}}{\partial t} \right)_c. \end{aligned} \quad (1.47)$$

The macroscopic fields in the left-hand side satisfy the macroscopic Maxwell's equations (1.23–1.26). The microscopic fluctuating fields $\delta\mathbf{E}$ and $\delta\mathbf{B}$ inside the average in the right-hand side satisfy microscopic Maxwell's equations with fluctuating charge and current densities expressed in terms of δF_{α} . Thus, the right-hand side is quadratic in δF_{α} . In order to close this equation, we need an expression for the correlation function $\langle \delta F_{\alpha} \delta F_{\alpha'} \rangle$ in terms of f_{α} and $f_{\alpha'}$. This is basically what the BBGKY procedure plus truncation of velocity integrals based on an expansion in $1/n\lambda_D^3$ achieve. The result is the Landau collision operator (or the more precise Lenard–Balescu one; see Balescu 1963).

Further details are complicated, but my aim here was just to show how the fields are split into macroscopic and microscopic ones, with the former appearing explicitly in the kinetic equation and the latter wrapped up inside the collision operator. The presence of the macroscopic fields and the consequent necessity for coupling the kinetic equation with Maxwell's equations for these fields is the main mathematical difference between the kinetics of neutral gases and the kinetics of plasmas.

1.9. So What's New and What Now?

Let me summarise the new features that have appeared in the kinetic description of a plasma compared to that of a neutral gas.

- First, particles are charged, so they interact via Coulomb potential. The collision operator is, therefore, different: the cross-section is the Rutherford cross-section, most collisions are glancing (with interaction on distances up to the Debye length), leading to diffusion of the particle distribution function in velocity space. Mathematically, this is manifested in the collision operator in (1.30) having the Fokker–Planck structure (1.36).

One can spin out of the Vlasov–Landau equation (1.30) a theory that is analogous to what is done with Boltzmann's equation in gas kinetics (Dellar 2016): derive *fluid equations*, calculate viscosity, thermal conductivity, Ohmic resistivity, etc., of a collisionally dominated plasma, i.e., of a plasma in which the collision frequency of the particles is much greater than all other relevant time scales. This is done in the same way as in gas

kinetics, but now applying the Chapman–Enskog procedure to the Landau collision operator. This is quite a lot of work—and constitutes core textbook plasma-physics material (see Parra 2017a). In magnetised plasmas especially, the resulting fluid dynamics of the plasma are quite interesting and quite different from neutral fluids—we shall see some of this in Part II of these Lectures, while the classic treatment of the transport theory can be found in Braginskii (1965); a great textbook on this is Helander & Sigmar (2005).

- Secondly, Coulomb potential is long-range, so the electric and magnetic fields have a macroscopic (mean) part on scales longer than the Debye length—a particle experiencing these fields is not undergoing a collision in the sense of bouncing off another particle, but is rather interacting, via the fields, with the collective of all other particles. Mathematically, this manifests itself as a Lorentz-force term appearing in the right-hand side of the Vlasov–Landau kinetic equation (1.30). The macroscopic \mathbf{E} and \mathbf{B} fields that figure in it are determined by the particles via their mean charge and current densities that enter the macroscopic Maxwell’s equations (1.23–1.26).

In the case of neutral gas, all the interesting kinetic physics is in the collision operator, hence the focus on transport theory in gas-kinetic literature (see, e.g., the classic monograph by Chapman & Cowling 1991 if you want an overdose of this). In the collisionless limit, the kinetic equation for a neutral gas,

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f = 0, \quad (1.48)$$

simply describes particles with some initial distribution ballistically flying in straight lines in their initial directions of travel. In contrast, for a plasma, even the *collisionless kinetics* (and, indeed, especially the collisionless—or *weakly collisional*—kinetics) are interesting and nontrivial because, as the initial distribution starts to evolve, it gives rise to charge densities and currents, which modify \mathbf{E} and \mathbf{B} , which modify f_α , etc. This opens up a whole new conceptual world and it is on these effects involving interactions between particles and fields that we shall focus here, in pursuit of maximum novelty.†

We shall also be in pursuit of maximum simplicity (well, “as simple as possible, but not simpler”!) and so will mostly restrict our consideration to the “*electrostatic approximation*”:

$$\mathbf{B} = 0, \quad \mathbf{E} = -\nabla\varphi. \quad (1.49)$$

This, of course, eliminates a huge number of interesting and important phenomena without which plasma physics would not be the voluminous subject that it is, but we cannot do them justice in just a few lectures (most of Parra 2017b is devoted to collisionless magnetised plasmas).

Thus, we shall henceforth focus on a simplified kinetic system, called the *Vlasov–Poisson system*:

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \nabla f_\alpha - \frac{q_\alpha}{m_\alpha} (\nabla\varphi) \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}} = 0, \quad (1.50)$$

$$-\nabla^2\varphi = 4\pi \sum_\alpha q_\alpha \int d^3\mathbf{v} f_\alpha. \quad (1.51)$$

† Similarly interesting things happen when the field tying the particles together is gravity—an even more complicated situation because, while the potential is long-range, rather like the Coulomb potential, gravity is not shielded and so all particles feel each other at all distances. This gives rise to remarkably interesting theory (Binney 2016).

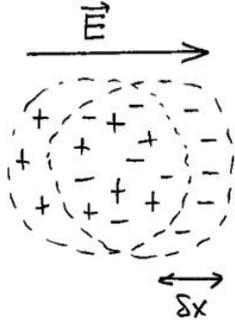


FIGURE 2. A displaced population of electrons will set up a quasineutrality-restoring electric field, leading to plasma oscillations.

Formally, considering a collisionless plasma[‡] would appear to be legitimate as long as the collision frequency is small compared to the characteristic frequencies of any other evolution that might be going on. What are the characteristic time scales (and length scales) in a plasma and what phenomena occur on these scales? This brings us to our next theme.

2. Equilibrium and Fluctuations

2.1. Plasma Frequency

Consider a plasma in equilibrium, in a happy quasineutral state. Suppose a population of electrons strays from this equilibrium and upsets quasineutrality a bit (Fig. 2). If they have shifted by distance δx , the restoring force on each electron will be

$$m_e \delta \ddot{x} = -eE = -4\pi e^2 n_e \delta x \quad \Rightarrow \quad \delta \ddot{x} = - \underbrace{\frac{4\pi e^2 n_e}{m_e}}_{\equiv \omega_{pe}^2} \delta x, \quad (2.1)$$

so there will be oscillations at what is known as the (electron) *plasma frequency*:

$$\omega_{pe} = \sqrt{\frac{4\pi e^2 n_e}{m_e}}. \quad (2.2)$$

Thus, we expect *fluctuations* of electric field in a plasma with characteristic frequencies $\omega \sim \omega_{pe}$ (these are Langmuir waves, we will derive their dispersion relation rigorously in §3.4). These fluctuations are due to *collective* motions of the particles—so they are still *macroscopic* fields in the nomenclature of §1.4.

The time scale associated with ω_{pe} is the scale of restoration of quasineutrality. The distance an electron can travel over this time scale before the restoring force kicks in, i.e., the distance over which quasineutrality can be violated, is (using the *thermal speed* $v_{the} \sim \sqrt{T/m_e}$ to estimate the electron's velocity)

$$\frac{v_{the}}{\omega_{pe}} \sim \sqrt{\frac{T}{m_e}} \sqrt{\frac{m_e}{e^2 n_e}} = \sqrt{\frac{T}{e^2 n_e}} \sim \lambda_D, \quad (2.3)$$

[‡] Or, we stress again, a weakly collisional plasma. The collision operator is dropped in Eq. (1.50), but let us not forget about it entirely even if the collision frequency is small; it will make a come back in §4.

the Debye length (1.6)—not surprising, as this is, indeed, the scale on which microscopic fields are shielded and plasma is quasineutral (§1.3).

Finally, let us check that the plasma oscillations happen on collisionless time scales. The collision frequency of the electrons is

$$\nu_e \sim \frac{v_{\text{the}}}{\lambda_{\text{mfp}}} = \frac{v_{\text{the}}}{\omega_{\text{pe}}} \frac{\omega_{\text{pe}}}{\lambda_{\text{mfp}}} \sim \frac{\lambda_{\text{D}}}{\lambda_{\text{mfp}}} \omega_{\text{pe}} \ll \omega_{\text{pe}}, \quad \text{q.e.d.}, \quad (2.4)$$

using Eqs. (2.3) and (1.14).

2.2. Slow vs. Fast

The plasma frequency ω_{pe} is only one of the characteristic frequencies (the largest) of the fluctuations that can occur in plasmas. We will think of the scales of all these fluctuations as short and of the associated variation in time and space as *fast*. They occur against the background of some *equilibrium state*,[†] which is either constant or varies *slowly* in time and space. The slow evolution and spatial variation of the equilibrium state can be due to some long time and length scales on which external conditions that gave rise to this state change or, as we will discover soon, it can be due to the average effect of a sea of small fluctuations.

Formally, what we are embarking on is an attempt to set up a mean-field theory, separating slow (large-scale) and fast (small-scale) parts of the distribution function:

$$f(\mathbf{r}, \mathbf{v}, t) = f_0(\epsilon^a \mathbf{r}, \mathbf{v}, \epsilon t) + \delta f(\mathbf{r}, \mathbf{v}, t), \quad (2.5)$$

where ϵ is some small parameter characterising the scale separation between fast and slow variation (note that this separation need not be the same for spatial and time scales, hence ϵ^a). To avoid clutter, we are dropping the species index where this does not lead to ambiguity.

For simplicity, we will drop the spatial dependence of the equilibrium distribution altogether and consider *homogeneous* systems:

$$f_0 = f_0(\mathbf{v}, \epsilon t), \quad (2.6)$$

which also means $\mathbf{E}_0 = 0$ (there is no equilibrium electric field). Equivalently, we restrict all our considerations to scales much smaller than the characteristic system size. Formally, this equilibrium distribution can be defined as the average of the exact distribution over the volume of space that we are considering and over time scales intermediate between the fast and the slow ones:[‡]

$$f_0(\mathbf{v}, t) = \langle f(\mathbf{r}, \mathbf{v}, t) \rangle \equiv \frac{1}{\Delta t} \int_{t-\Delta t/2}^{t+\Delta t/2} dt' \int \frac{d^3 \mathbf{r}}{V} f(\mathbf{r}, \mathbf{v}, t'), \quad (2.7)$$

where $\omega^{-1} \ll \Delta t \ll t_{\text{eq}}$, where t_{eq} is the equilibrium time scale.

2.3. Multiscale Dynamics

We will find it convenient to work in Fourier space:

$$\varphi(\mathbf{r}, t) = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \varphi_{\mathbf{k}}(t), \quad f(\mathbf{r}, \mathbf{v}, t) = f_0(\mathbf{v}, t) + \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \delta f_{\mathbf{k}}(\mathbf{v}, t). \quad (2.8)$$

[†] Or even just an initial state that is slow to change.

[‡] We use angle brackets to denote this average, but it should be clear that this is not the same thing as the average (1.11) that allowed us to separate macroscopic fields from microscopic ones. The latter was over sub-Debye scales, whereas our new average is over scales that are larger than fluctuation scales but smaller than the system scales; both fluctuations and equilibrium are macroscopic.

Then the Poisson equation (1.51) becomes

$$\varphi_{\mathbf{k}} = \frac{4\pi}{k^2} \sum_{\alpha} q_{\alpha} \int d^3\mathbf{v} \delta f_{\mathbf{k}\alpha} \quad (2.9)$$

and the Vlasov equation (1.50) written for $\mathbf{k} = 0$ (i.e., the spatial average of the equation) is

$$\frac{\partial f_0}{\partial t} + \frac{\partial \delta f_{\mathbf{k}=0}}{\partial t} = -\frac{q}{m} \sum_{\mathbf{k}} \varphi_{-\mathbf{k}} i\mathbf{k} \cdot \frac{\partial \delta f_{\mathbf{k}}}{\partial \mathbf{v}}. \quad (2.10)$$

Averaging also over time according to Eq. (2.7) eliminates fast variation and gives us

$$\boxed{\frac{\partial f_0}{\partial t} = -\frac{q}{m} \sum_{\mathbf{k}} \left\langle \varphi_{\mathbf{k}}^* i\mathbf{k} \cdot \frac{\partial \delta f_{\mathbf{k}}}{\partial \mathbf{v}} \right\rangle} \quad (2.11)$$

(note that $\varphi_{-\mathbf{k}} = \varphi_{\mathbf{k}}^*$ because $\varphi(\mathbf{r}, t)$ must be real).

The right-hand side of Eq. (2.11) gives us the slow evolution of the equilibrium (mean) distribution due to the effect of fluctuations. In practice, the main question is often how the equilibrium evolves and so we need a closed equation for the evolution of f_0 . This should be obtainable at least in principle because the fluctuating fields appearing in the right-hand side of Eq. (2.11) themselves depend on f_0 : indeed, writing the Vlasov equation (1.50) for the $\mathbf{k} \neq 0$ modes, we find the following evolution equation for the fluctuations:

$$\boxed{\frac{\partial \delta f_{\mathbf{k}}}{\partial t} + \underbrace{i\mathbf{k} \cdot \mathbf{v} \delta f_{\mathbf{k}}}_{\substack{\text{particle} \\ \text{streaming} \\ \text{(phase} \\ \text{mixing)}}} = \underbrace{\frac{q}{m} \varphi_{\mathbf{k}} i\mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}}_{\substack{\text{wave-particle} \\ \text{interaction} \\ \text{(linear)}}} + \underbrace{\frac{q}{m} \sum_{\mathbf{k}'} \varphi_{\mathbf{k}'} i\mathbf{k}' \cdot \frac{\partial \delta f_{\mathbf{k}-\mathbf{k}'}}{\partial \mathbf{v}}}_{\substack{\text{nonlinear} \\ \text{interactions}}}. \quad (2.12)$$

The three terms that control the evolution of the perturbed distribution function in Eq. (2.12) represent the three physical effects that we shall focus on in these Lectures. The second term on the left-hand side represents free ballistic motion of particles (“streaming”). It will give rise to the phenomenon of phase mixing (§4) and, in its interplay with plasma waves, to Landau damping and kinetic instabilities (§3). The first term on the right-hand side contains the interaction of the electric-field perturbations (waves) with the equilibrium particle distribution. The second term on the right-hand side of Eq. (2.12) has nonlinear interactions between fluctuating fields and the perturbed distribution—it is negligible when fluctuation amplitudes are small enough (which, sadly, they rarely are) and responsible for *plasma turbulence* when they are not.

The programme for determining the slow evolution of the equilibrium is “simple”: solve Eq. (2.12) together with Eq. (2.9), calculate the *correlation function* of the fluctuations, $\langle \varphi_{\mathbf{k}}^* \delta f_{\mathbf{k}} \rangle$, as a functional of f_0 , and use it to close Eq. (2.11); then proceed to solve the latter. Obviously, this is impossible to do in most cases. But it is possible to construct a hierarchy of approximations to the answer and learn much interesting physics in the process.

2.4. Hierarchy of Approximations

2.4.1. Linear Theory

Consider first infinitesimal perturbations of the equilibrium. All nonlinear terms can then be ignored, Eq. (2.11) turns into $f_0 = \text{const}$ and Eq. (2.12) becomes

$$\frac{\partial \delta f_{\mathbf{k}}}{\partial t} + i\mathbf{k} \cdot \mathbf{v} \delta f_{\mathbf{k}} = \frac{q}{m} \varphi_{\mathbf{k}} i\mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}, \quad (2.13)$$

the *linearised kinetic equation*. Solving this together with Eq. (2.9) allows one to find oscillating and/or growing/decaying† modes of the plasma perturbing a particular equilibrium. The theory for doing this is very well developed and contains some of the core ideas that give plasma physics its intellectual shape (§3).

Physically, the linear solutions will describe what happens over short term, viz., on times t such that

$$\omega^{-1} \ll t \ll t_{\text{eq}} \text{ or } t_{\text{nl}}, \quad (2.14)$$

where ω is the characteristic frequency of the perturbations, t_{eq} is the time over which the equilibrium starts getting modified by the perturbations (which depends on the amplitude to which they can grow: see the right-hand side of Eq. (2.11); if perturbations grow, they can modify the equilibrium by this mechanism so as to render it stable), and t_{nl} is the time at which perturbation amplitudes become large enough for nonlinear interactions between individual modes to matter (second term on the right-hand side of Eq. (2.12); if perturbations grow, they can saturate by this mechanism).

2.4.2. Quasilinear Theory (QLT)

Suppose

$$t_{\text{eq}} \ll t_{\text{nl}}, \quad (2.15)$$

i.e., *growing perturbations start modifying the equilibrium before they saturate nonlinearly*. Then the strategy is to solve Eq. (2.13) [together with Eq. (2.9)] for the perturbations, use the result to calculate their correlation function needed in the right-hand side of Eq. (2.11), then work out how the equilibrium therefore evolves and hence how large the perturbations must grow in order for this evolution to turn the equilibrium from one causing the instability to a stable one. This is a classic piece of theory, important conceptually—we will describe it in detail and do one example in §7 and another in Q8. In reality, however, it happens relatively rarely that unstable perturbations saturate at amplitudes small enough for the nonlinear interactions not to matter (i.e., for $t_{\text{nl}} \gg t_{\text{eq}}$).

2.4.3. Weak-Turbulence Theory

Sometimes, one is not lucky enough to get away with QLT (so $t_{\text{nl}} \lesssim t_{\text{eq}}$), but is lucky enough to have perturbations saturating nonlinearly at *small amplitude* such that‡

$$t_{\text{nl}} \gg \omega^{-1}, \quad (2.16)$$

i.e., *perturbations oscillate faster than they interact* (this can happen for example because propagating wave packets do not stay together long enough to break up completely in one encounter). In this case, one can do perturbation theory treating the nonlinear term in Eq. (2.12) as small and expanding in the small parameter $(\omega t_{\text{nl}})^{-1}$.

† We shall see (§4) that growing/decaying linear solutions imply the equilibrium distribution giving/receiving energy to/from the fluctuations.

‡ Note that the nonlinear time scale is typically inversely proportional to the amplitude; see Eq. (2.12).

Note that because the nonlinear term couples perturbations at different \mathbf{k} 's (scales), this theory will lead to broad (power-law) fluctuation spectra.

We will not have sufficient time for this: it is a pity as the *weak (or “wave”) turbulence theory* is quite an analytical *tour de force*—but it is a lot of work to do it properly! Classic texts on this are Kadomtsev (1965) (early but lucid) and Zakharov *et al.* (1992) (mathematically definitive); a recent textbook is Nazarenko (2011). Specifically on weak turbulence of Langmuir waves, there is a long, mushy review by Musher *et al.* (1995). I will provide an introduction to WT in §8.2 and subsequent sections.

2.4.4. Strong-Turbulence Theory

If perturbations manage to grow to a level at which

$$t_{\text{nl}} \sim \omega^{-1}, \quad (2.17)$$

we are facing *strong turbulence*. This is actually what mostly happens. Theory of such regimes tends to be of phenomenological/scaling kind, often in the spirit of the classic Kolmogorov (1941) theory of hydrodynamic turbulence.† Here are two examples, not necessarily the best or most relevant, just mine: Schekochihin *et al.* (2009, 2016). No one really knows how to do much beyond this sort of approach—and not for lack of trying (a recent but historically aware review is Krommes 2015).

3. Linear Theory: Plasma Waves, Landau Damping and Kinetic Instabilities

Enough idle chatter, let us calculate! In this section, we are concerned with the linearised Vlasov–Poisson system, Eqs. (2.13) and (2.9):

$$\frac{\partial \delta f_{\mathbf{k}\alpha}}{\partial t} + i\mathbf{k} \cdot \mathbf{v} \delta f_{\mathbf{k}\alpha} = \frac{q_\alpha}{m_\alpha} \varphi_{\mathbf{k}} i\mathbf{k} \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{v}}, \quad (3.1)$$

$$\varphi_{\mathbf{k}} = \frac{4\pi}{k^2} \sum_{\alpha} q_\alpha \int d^3\mathbf{v} \delta f_{\mathbf{k}\alpha}. \quad (3.2)$$

For compactness of notation, we will drop both the species index α and the wave number \mathbf{k} in the subscripts, unless they are necessary for understanding.

We will discover that that electrostatic perturbations in a plasma described by Eqs. (3.1) and (3.2) oscillate, can pass their energy to particles (damp) or even grow, sucking energy from the particles. We will also discover that it is useful to know some complex analysis.

3.1. Initial-Value Problem

We follow Landau’s original paper (Landau 1946) in considering an initial-value problem—because, as we will see, perturbations can be damped or grow, so it is not appropriate to think of them over $t \in [-\infty, +\infty]$ (and—NB!!!—the damped perturbations are *not* pure eigenmodes; see §4.3). So we look for $\delta f(\mathbf{v}, t)$ satisfying Eq. (3.1) with the initial condition

$$\delta f(\mathbf{v}, t = 0) = g(\mathbf{v}). \quad (3.3)$$

† A kind of exception is a very special case of strong Langmuir turbulence, which was very popular in the 1970s and 80s. The founding documents on this are, arguably, Zakharov (1972) and Kingsep *et al.* (1973), but there is a huge and sophisticated literature that followed and, alas, no particularly good review, but see Rudakov & Tsytovich (1978), Thornhill & ter Haar (1978), Goldman (1984) and Robinson (1997). I will give an introduction to this topic in §9.

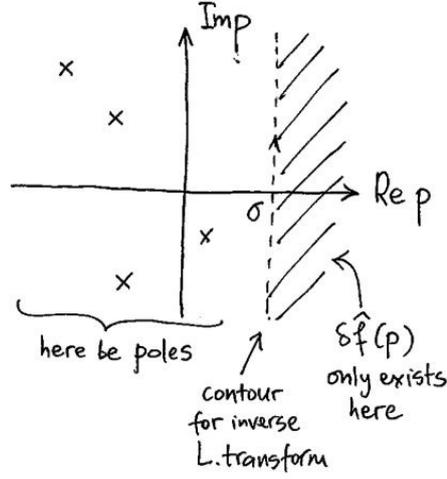


FIGURE 3. Layout of the complex- p plane: $\hat{\delta f}(p)$ is analytic for $\text{Re } p \geq \sigma$. At $\text{Re } p < \sigma$, $\hat{\delta f}(p)$ may have singularities (poles).

It is, therefore, appropriate to use *Laplace transform* to solve Eq. (3.1):

$$\boxed{\hat{\delta f}(p) = \int_0^{\infty} dt e^{-pt} \delta f(t)} . \quad (3.4)$$

It is a mathematical certainty that if there exists a real number $\sigma > 0$ such that

$$|\delta f(t)| < e^{\sigma t} \text{ as } t \rightarrow \infty, \quad (3.5)$$

then the integral (3.4) exists (i.e., is finite) for all values of p such that $\text{Re } p \geq \sigma$. The inverse Laplace transform, giving us back our distribution function as a function of time, is then

$$\delta f(t) = \frac{1}{2\pi i} \int_{-i\infty+\sigma}^{i\infty+\sigma} dp e^{pt} \hat{\delta f}(p), \quad (3.6)$$

where the integral is along a straight line in the complex plane parallel to the imaginary axis and intersecting the real axis at $\text{Re } p = \sigma$ (Fig. 3).

Since we expect to be able to recover our desired time-dependent function $\delta f(\mathbf{v}, t)$ from its Laplace transform, it is worth knowing the latter. To find it, we Laplace-transform Eq. (3.1):

$$\begin{aligned} \text{l.h.s.} &= \int_0^{\infty} dt e^{-pt} \frac{\partial \delta f}{\partial t} = [e^{-pt} \delta f]_0^{\infty} + p \int_0^{\infty} dt e^{-pt} \delta f = -g + p \hat{\delta f}, \\ \text{r.h.s.} &= -i\mathbf{k} \cdot \mathbf{v} \hat{\delta f} + \frac{q}{m} \hat{\varphi} i\mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}. \end{aligned} \quad (3.7)$$

Equating these two expressions, we find the solution:

$$\boxed{\hat{\delta f}(p) = \frac{1}{p + i\mathbf{k} \cdot \mathbf{v}} \left[i \frac{q}{m} \hat{\varphi}(p) \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} + g \right]} . \quad (3.8)$$

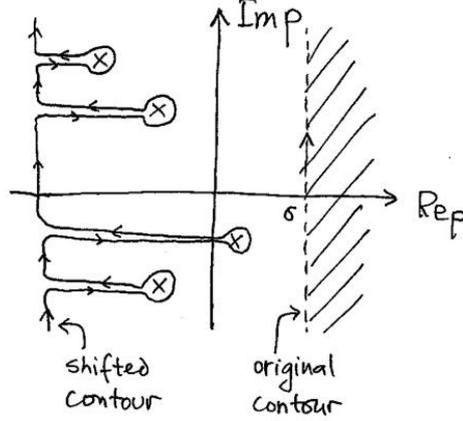


FIGURE 4. New contour for the inverse Laplace transform.

The Laplace transform of the potential, $\hat{\varphi}(p)$, itself depends on $\delta\hat{f}$ via Eq. (3.2):

$$\begin{aligned}\hat{\varphi}(p) &= \int_0^\infty dt e^{-pt} \varphi(t) = \frac{4\pi}{k^2} \sum_\alpha q_\alpha \int d^3\mathbf{v} \delta\hat{f}_\alpha(p) \\ &= \frac{4\pi}{k^2} \sum_\alpha q_\alpha \int d^3\mathbf{v} \frac{1}{p + i\mathbf{k} \cdot \mathbf{v}} \left[i \frac{q_\alpha}{m_\alpha} \hat{\varphi}(p) \mathbf{k} \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{v}} + g_\alpha \right].\end{aligned}\quad (3.9)$$

This is an algebraic equation for $\hat{\varphi}(p)$. Collecting terms, we get

$$\underbrace{\left[1 - \sum_\alpha \frac{4\pi q_\alpha^2}{k^2 m_\alpha} i \int d^3\mathbf{v} \frac{1}{p + i\mathbf{k} \cdot \mathbf{v}} \mathbf{k} \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{v}} \right]}_{\equiv \epsilon(p, \mathbf{k})} \hat{\varphi}(p) = \frac{4\pi}{k^2} \sum_\alpha q_\alpha \int d^3\mathbf{v} \frac{g_\alpha}{p + i\mathbf{k} \cdot \mathbf{v}}. \quad (3.10)$$

The prefactor in the left-hand side, which we denote $\epsilon(p, \mathbf{k})$, is called the *dielectric function*, because it encodes all the self-consistent charge-density perturbations that plasma sets up in response to an electric field. This is going to be an important function, so let us write it out beautifully:

$$\epsilon(p, \mathbf{k}) = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{k^2} \frac{i}{n_\alpha} \int d^3\mathbf{v} \frac{1}{p + i\mathbf{k} \cdot \mathbf{v}} \mathbf{k} \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{v}}, \quad (3.11)$$

where the plasma frequency of species α is defined by [cf. Eq. (2.2)]

$$\omega_{p\alpha}^2 = \frac{4\pi q_\alpha^2 n_\alpha}{m_\alpha}. \quad (3.12)$$

The solution of Eq. (3.10) for the potential is

$$\hat{\varphi}(p) = \frac{4\pi}{k^2 \epsilon(p, \mathbf{k})} \sum_\alpha q_\alpha \int d^3\mathbf{v} \frac{g_\alpha}{p + i\mathbf{k} \cdot \mathbf{v}}. \quad (3.13)$$

To calculate $\varphi(t)$, we need to inverse-Laplace-transform $\hat{\varphi}$: similarly to Eq. (3.6),

$$\varphi(t) = \frac{1}{2\pi i} \int_{-i\infty+\sigma}^{i\infty+\sigma} dp e^{pt} \hat{\varphi}(p). \quad (3.14)$$

How do we do this integral? Recall that $\delta\hat{f}$ and, therefore, $\hat{\varphi}$ only exists (i.e., is finite)

for $\text{Re } p \geq \sigma$, whereas at $\text{Re } p < \sigma$, it can have singularities, i.e., poles—let us call them p_i , indexed by i . If we analytically continue $\hat{\varphi}(p)$ everywhere to $\text{Re } p < \sigma$ except those poles, the result must have the form

$$\hat{\varphi}(p) = \sum_i \frac{c_i}{p - p_i} + A(p), \quad (3.15)$$

where c_i are some coefficients (residues) and $A(p)$ is the analytic part of the solution. The integration contour in Eq. (3.14) can be shifted to $\text{Re } p \rightarrow -\infty$ but with the proviso that it cannot cross the poles, as shown in Fig. 4).[†] Then the contributions to the integral from the vertical segments of the contour are exponentially small, the contributions from the segments leading towards and away from the poles cancel, and the contributions from the circles around the poles can, by Cauchy's formula, be expressed in terms of the poles and residues:

$$\varphi(t) = \sum_i c_i e^{p_i t}. \quad (3.16)$$

Thus, *initial perturbations of the potential will evolve* $\propto e^{p_i t}$, where p_i are poles of $\hat{\varphi}(p)$. In general, $p_i = -i\omega_i + \gamma_i$, where ω_i is a real frequency (giving wave-like behaviour of perturbations), $\gamma_i < 0$ represents damping and $\gamma_i > 0$ growth of the perturbations (instability).

Going back to Eq. (3.13), we realise that the poles of $\hat{\varphi}(p)$ are zeros of the dielectric function:

$$\epsilon(p_i, \mathbf{k}) = 0 \quad \Rightarrow \quad p_i = p_i(\mathbf{k}) = -i\omega_i(\mathbf{k}) + \gamma_i(\mathbf{k}). \quad (3.17)$$

To find the wave frequencies ω_i and the damping/growth rates γ_i , we must solve this equation, which is called the *plasma dispersion relation*.

We are not particularly interested in what c_i 's are, but they can be computed from the initial conditions, also via Eq. (3.13). The reason we are not particularly interested is that if we set up an initial perturbation with a given \mathbf{k} and then wait long enough, only the fastest-growing or, failing growth, the slowest-damped mode will survive, with all others having exponentially small amplitudes. Thus, a typical outcome of the linear theory is a mode oscillating at some frequency and growing or decaying at some rate. Since this is a linear solution, the prefactor in front of the exponential can be scaled arbitrarily and so does not matter.

3.2. Calculating the Dielectric Function: the “Landau Prescription”

In order to be able to solve $\epsilon(p, \mathbf{k}) = 0$, we must learn how to calculate $\epsilon(p, \mathbf{k})$ for any given p and \mathbf{k} . Before I wrote Eq. (3.15), I said that $\hat{\varphi}$, given by Eq. (3.13), had to be analytically continued to the entire complex plane from the area where its analyticity was guaranteed ($\text{Re } p \geq \sigma$), but I did not explain how this was to be done. In order to do it, we must learn how to calculate the velocity integral in Eq. (3.11)—if we want $\epsilon(p, \mathbf{k})$ and, therefore, its zeros p_i —and also how to calculate the similar integral in Eq. (3.13) containing g_α if we also want the coefficients c_i in Eq. (3.16).

First of all, let us turn these integrals into a 1D form. Given \mathbf{k} , we can always choose

[†] This is proven by making a closed loop out of the old and the new contours, joining them at $\pm i\infty$, and noting that this loop contains no poles.

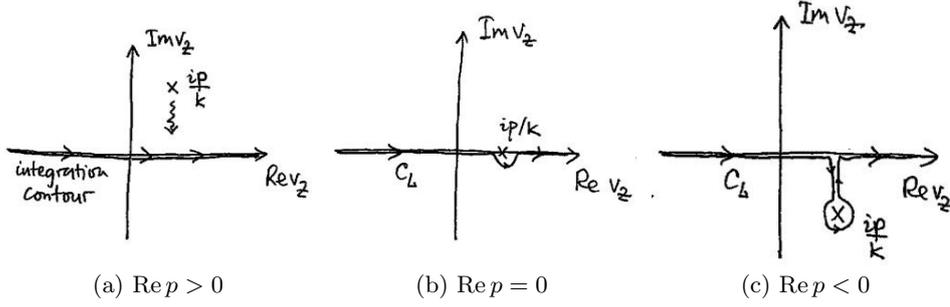


FIGURE 5. The Landau prescription for the contour of integration in Eq. (3.18).

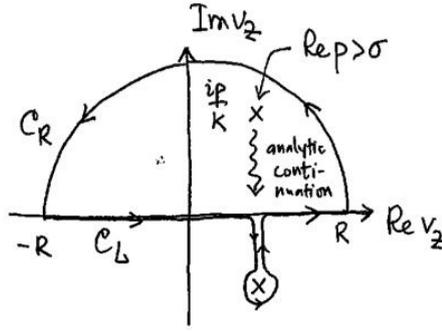


FIGURE 6. Proof of Landau's prescription [Eq. (3.19)].

the z axis to be along \mathbf{k} .[‡] Then

$$\begin{aligned} \int d^3\mathbf{v} \frac{1}{p + i\mathbf{k} \cdot \mathbf{v}} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} &= \int dv_z \frac{1}{p + ikv_z} k \frac{\partial}{\partial v_z} \underbrace{\int dv_x \int dv_y f_0(v_x, v_y, v_z)}_{\equiv F(v_z)} \\ &= -i \int_{-\infty}^{+\infty} dv_z \frac{F'(v_z)}{v_z - ip/k}. \end{aligned} \quad (3.18)$$

Assuming, reasonably, that $F'(v_z)$ is a nice (analytic) function everywhere, the integrand in Eq. (3.18) has one pole, $v_z = ip/k$. When $\text{Re } p \geq \sigma > 0$, this pole is harmless because, in the complex plane associated with the v_z variable, it lies above the integration contour, which is along the real axis, $v_z \in (-\infty, +\infty)$. We can think of analytic continuation of the above integral to $\text{Re } p < \sigma$ as moving the pole $v_z = ip/k$ down, towards and below the real axis. As long as $\text{Re } p > 0$, this can be done with impunity, in the sense that the pole stays above the integration contour, and so the natural analytic continuation is simply the same integral (3.18), still along the real axis. However, if the pole moves so far down that $\text{Re } p = 0$ or $\text{Re } p < 0$, we must deform the contour of integration in such a way as to keep the pole always above it, as shown in Fig. 5. This is called the *Landau prescription* and the contour thus deformed is called the *Landau contour*, C_L .

Let us prove that this is indeed an analytic continuation, i.e., that the integral (3.18), adjusted to be along C_L , is analytic for all values of p . Let us cut the Landau contour at $v_z = \pm R$ and close it in the upper half-plane with a semicircle C_R of radius R such that $R > \sigma/k$ (Fig. 6). Then, with integration running along the truncated C_L and counterclockwise along C_R , we get,

[‡] NB: This means that in what follows, $k \geq 0$ by definition.

by Cauchy's formula,

$$\int_{C_L} dv_z \frac{F'(v_z)}{v_z - ip/k} + \int_{C_R} dv_z \frac{F'(v_z)}{v_z - ip/k} = 2\pi i F'\left(\frac{ip}{k}\right). \quad (3.19)$$

Since analyticity is guaranteed for $\text{Re } p \geq \sigma$, the integral along C_R is analytic. The right-hand side is also analytic, by assumption. Therefore, the integral along C_L is analytic—this is the integral along the Landau contour if we take $R \rightarrow \infty$.

With the Landau prescription, our integral is calculated as follows:

$$\int_{C_L} dv_z \frac{F'(v_z)}{v_z - ip/k} = \begin{cases} \int_{-\infty}^{+\infty} dv_z \frac{F'(v_z)}{v_z - ip/k} & \text{if } \text{Re } p > 0, \\ \mathcal{P} \int_{-\infty}^{+\infty} dv_z \frac{F'(v_z)}{v_z - ip/k} + i\pi F'\left(\frac{ip}{k}\right) & \text{if } \text{Re } p = 0, \\ \int_{-\infty}^{+\infty} dv_z \frac{F'(v_z)}{v_z - ip/k} + i2\pi F'\left(\frac{ip}{k}\right) & \text{if } \text{Re } p < 0, \end{cases} \quad (3.20)$$

where the integrals are again over the real axis and the imaginary bits come from the contour making a half (when $\text{Re } p = 0$) or a full (when $\text{Re } p < 0$) circle around the pole. In the case of $\text{Re } p = 0$, or $ip = \omega$, the integral along the real axis is formally divergent and so we take its *principal value*, defined as

$$\mathcal{P} \int_{-\infty}^{+\infty} dv_z \frac{F'(v_z)}{v_z - \omega/k} = \lim_{\varepsilon \rightarrow 0} \left[\int_{-\infty}^{\omega/k - \varepsilon} + \int_{\omega/k + \varepsilon}^{+\infty} \right] dv_z \frac{F'(v_z)}{v_z - \omega/k}. \quad (3.21)$$

The difference between Eq. (3.21) and the usual Lebesgue definition of an integral is that the latter would be

$$\int_{-\infty}^{+\infty} dv_z \frac{F'(v_z)}{v_z - \omega/k} = \left[\lim_{\varepsilon_1 \rightarrow 0} \int_{-\infty}^{\omega/k - \varepsilon_1} + \lim_{\varepsilon_2 \rightarrow 0} \int_{\omega/k + \varepsilon_2}^{+\infty} \right] dv_z \frac{F'(v_z)}{v_z - \omega/k}, \quad (3.22)$$

and this, with, in general, $\varepsilon_1 \neq \varepsilon_2$, diverges logarithmically, whereas in Eq. (3.21), the divergences neatly cancel.

The $\text{Re } p = 0$ case in Eq. (3.20),

$$\int_{C_L} dv_z \frac{F'(v_z)}{v_z - \omega/k} = \mathcal{P} \int_{-\infty}^{+\infty} dv_z \frac{F'(v_z)}{v_z - \omega/k} + i\pi F'\left(\frac{\omega}{k}\right), \quad (3.23)$$

which tends to be of most use in analytical theory, is a particular instance of *Plemelj's formula*: for a real ζ and a well-behaved function f (no poles on or near the real axis),

$$\lim_{\varepsilon \rightarrow +0} \int_{-\infty}^{+\infty} dx \frac{f(x)}{x - \zeta \mp i\varepsilon} = \mathcal{P} \int_{-\infty}^{+\infty} dx \frac{f(x)}{x - \zeta} \pm i\pi f(\zeta), \quad (3.24)$$

also sometimes written as

$$\lim_{\varepsilon \rightarrow +0} \frac{1}{x - \zeta \mp i\varepsilon} = \mathcal{P} \frac{1}{x - \zeta} \pm i\pi \delta(x - \zeta), \quad (3.25)$$

Finally, armed with Landau's prescription, we are ready to calculate. Eq. (3.11) becomes

$$\epsilon(p, \mathbf{k}) = 1 - \sum_{\alpha} \frac{\omega_{p\alpha}^2}{k^2} \frac{1}{n_{\alpha}} \int_{C_L} dv_z \frac{F'_{\alpha}(v_z)}{v_z - ip/k} \quad (3.26)$$

and, analogously, Eq. (3.13) becomes

$$\hat{\varphi}(p) = -\frac{4\pi i}{k^3 \epsilon(p, \mathbf{k})} \sum_{\alpha} q_{\alpha} \int_{C_L} dv_z \frac{G_{\alpha}(v_z)}{v_z - ip/k}, \quad (3.27)$$

where $G_{\alpha}(v_z) = \int dv_x \int dv_y g_{\alpha}(v_x, v_y, v_z)$.

3.3. Solving the Dispersion Relation: Slow-Damping/Growth Limit

A particularly analytically solvable and physically interesting case is one in which, for $p = -i\omega + \gamma$, $\gamma \ll \omega$ (or, if $\omega = 0$, $\gamma \ll kv_{\text{th}\alpha}$), i.e., the case of slow damping or growth. In this limit, the dispersion relation is

$$\epsilon(p, \mathbf{k}) \approx \epsilon(-i\omega, \mathbf{k}) + i\gamma \frac{\partial}{\partial \omega} \epsilon(-i\omega, \mathbf{k}) = 0. \quad (3.28)$$

Setting the imaginary part of Eq. (3.28) to zero gives us the growth/damping rate in terms of the real frequency:

$$\boxed{\gamma = -\text{Im} \epsilon(-i\omega, \mathbf{k}) \left[\frac{\partial}{\partial \omega} \text{Re} \epsilon(-i\omega, \mathbf{k}) \right]^{-1}}. \quad (3.29)$$

Setting the real part of Eq. (3.28) to zero gives the equation for the real frequency:

$$\boxed{\text{Re} \epsilon(-i\omega, \mathbf{k}) = 0}. \quad (3.30)$$

Thus, we now only need $\epsilon(p, \mathbf{k})$ with $p = -i\omega$. Using Eq. (3.23), we get

$$\text{Re} \epsilon = 1 - \sum_{\alpha} \frac{\omega_{p\alpha}^2}{k^2} \frac{1}{n_{\alpha}} \mathcal{P} \int dv_z \frac{F'_{\alpha}(v_z)}{v_z - \omega/k}, \quad (3.31)$$

$$\text{Im} \epsilon = - \sum_{\alpha} \frac{\omega_{p\alpha}^2}{k^2} \frac{\pi}{n_{\alpha}} F'_{\alpha}\left(\frac{\omega}{k}\right). \quad (3.32)$$

Let us now consider a two-species plasma, consisting of electrons and a single species of ions. There will be two interesting limits:

- “*High-frequency*” electron waves: $\omega \gg kv_{\text{the}}$, where $v_{\text{the}} = \sqrt{2T_e/m_e}$ is the “thermal speed” of the electrons[†]; this limit will give us Langmuir waves (§3.4), slowly damped or growing (§3.5).

- “*Low-frequency*” ion waves: a particularly tractable limit will be that of “hot” electrons and “cold” ions, viz., $kv_{\text{the}} \gg \omega \gg kv_{\text{th}i}$, where $v_{\text{th}i} = \sqrt{2T_i/m_i}$ is the “thermal speed” of the ions; this limit will give us the sound (“ion-acoustic waves”; §3.8), which also can be damped or growing (§3.9).

3.4. Langmuir Waves

Consider the limit

$$\frac{\omega}{k} \gg v_{\text{the}}, \quad (3.33)$$

i.e., the phase velocity of the waves is much greater than the typical velocity of a particle from the “thermal bulk” of the distribution. This means that in Eq. (3.31), we can expand

[†] This is a standard well-defined quantity for a Maxwellian equilibrium distribution $F_e(v_z) = (n_e/\sqrt{\pi} v_{\text{the}}) \exp(-v_z^2/v_{\text{the}}^2)$, but if we wish to consider a non-Maxwellian F_e , let v_{the} be a typical speed characterising the width of the equilibrium distribution, defined by, e.g., Eq. (3.36).

in $v_z \sim v_{\text{the}}$ small compared to ω/k (higher values of v_z are cut off by the equilibrium distribution function). Note that $\omega \gg kv_{\text{the}}$ also implies $\omega \gg kv_{\text{thi}}$ because

$$\frac{v_{\text{thi}}}{v_{\text{the}}} = \sqrt{\frac{T_i m_e}{T_e m_i}} \ll 1 \quad (3.34)$$

as long as T_i/T_e is not huge.† Thus, Eq. (3.31) becomes

$$\begin{aligned} \text{Re } \epsilon &= 1 + \sum_{\alpha} \frac{\omega_{\text{p}\alpha}^2}{k^2} \frac{1}{n_{\alpha}} \frac{k}{\omega} \mathcal{P} \int dv_z F'_{\alpha}(v_z) \left[1 + \frac{kv_z}{\omega} + \left(\frac{kv_z}{\omega}\right)^2 + \left(\frac{kv_z}{\omega}\right)^3 + \dots \right] \\ &= 1 + \sum_{\alpha} \frac{\omega_{\text{p}\alpha}^2}{k\omega} \left[\underbrace{\frac{1}{n_{\alpha}} \int dv_z F'_{\alpha}(v_z)}_{=0} - \frac{k}{\omega} \underbrace{\frac{1}{n_{\alpha}} \int dv_z F_{\alpha}(v_z)}_{=1} \right. \\ &\quad \left. - 2 \frac{k^2}{\omega^2} \underbrace{\frac{1}{n_{\alpha}} \int dv_z v_z F_{\alpha}(v_z)}_{=0} - 3 \frac{k^3}{\omega^3} \underbrace{\frac{1}{n_{\alpha}} \int dv_z v_z^2 F_{\alpha}(v_z)}_{=v_{\text{th}\alpha}^2/2} + \dots \right] \\ &= 1 - \sum_{\alpha} \frac{\omega_{\text{p}\alpha}^2}{\omega^2} \left[1 + \frac{3}{2} \frac{k^2 v_{\text{th}\alpha}^2}{\omega^2} + \dots \right], \end{aligned} \quad (3.35)$$

where we have integrated by parts everywhere, assumed that there are no mean flows, $\langle v_z \rangle = 0$, and, in the last term, used

$$\langle v_z^2 \rangle = \frac{v_{\text{th}\alpha}^2}{2}, \quad (3.36)$$

which is indeed the case for a Maxwellian F_{α} or, if F_{α} is not a Maxwellian, can be viewed as the definition of $v_{\text{th}\alpha}$.

The ion contribution to Eq. (3.35) is small because

$$\frac{\omega_{\text{pi}}^2}{\omega_{\text{pe}}^2} = \frac{Zm_e}{m_i} \ll 1, \quad (3.37)$$

so ions do not participate in this dynamics at all. Therefore, to lowest order, the dispersion relation $\text{Re } \epsilon = 0$ becomes

$$1 - \frac{\omega_{\text{pe}}^2}{\omega^2} = 0 \quad \Rightarrow \quad \boxed{\omega^2 = \omega_{\text{pe}}^2 = \frac{4\pi e^2 n_e}{m_e}}, \quad (3.38)$$

the Tonks & Langmuir (1929) dispersion relation for what is known as *Langmuir, or plasma, oscillations*. This is the formal derivation of the result that we already had, on physical grounds, in §2.1.

We can do a little better if we retain the (small) k -dependent term in Eq. (3.35):

$$\text{Re } \epsilon \approx 1 - \frac{\omega_{\text{pe}}^2}{\omega^2} \left(1 + \underbrace{\frac{3}{2} \frac{k^2 v_{\text{the}}^2}{\omega^2}}_{\substack{\text{use} \\ \omega^2 \approx \omega_{\text{pe}}^2}} \right) = 0 \quad \Rightarrow \quad \boxed{\omega^2 \approx \omega_{\text{pe}}^2 (1 + 3k^2 \lambda_{\text{De}}^2)}, \quad (3.39)$$

where $\lambda_{\text{De}} = v_{\text{the}}/\sqrt{2}\omega_{\text{pe}} = \sqrt{T_e/4\pi e^2 n_e}$ is the “electron Debye length” [cf. Eq. (1.6)]. Eq. (3.39) is the Bohm & Gross (1949a) dispersion relation, describing an upgrade of

† For hydrogen plasma, $\sqrt{m_i/m_e} \approx 42$, more or less, the answer to the Ultimate Question of Life, Universe and Everything (Adams 1979).

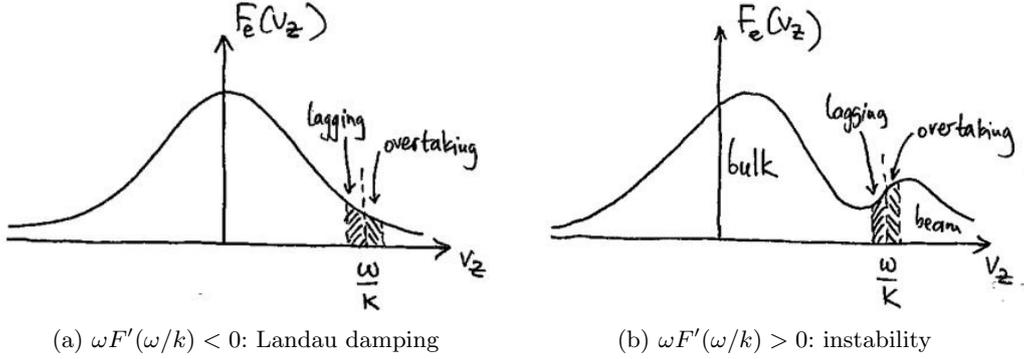


FIGURE 7. The Landau resonance (particle velocities equalling phase speed of the wave $v_z = \omega/k$) leads to damping of the wave if more particles lag just behind than overtake the wave and to instability in the opposite case.

the Langmuir oscillations to dispersive *Langmuir waves*, which have a non-zero group velocity (this effect is due to electron pressure: see Exercise 3.1).

Note that all this is only valid for $\omega \gg kv_{\text{the}}$, which we now see is equivalent to

$$k\lambda_{\text{De}} \ll 1. \quad (3.40)$$

Exercise 3.1. Langmuir hydrodynamics. Starting from the linearised kinetic equation for electrons and ignoring perturbations of the ion distribution functions completely, work out the fluid equations for electrons (i.e., evolution equations for the electron density n_e and velocity u_e) and show that you can recover the Langmuir waves (3.39) if you assume that electrons behave as a 1D adiabatic fluid (i.e., have the equation of state $p_e n_e^{-\gamma} = \text{const}$ with $\gamma = 3$). You can prove that they indeed do this by calculating their density and pressure directly from the Landau solution for the perturbed distribution function (see §§4.3 and 4.6), ignoring resonant particles. The “hydrodynamic” description of Langmuir waves will reappear in §9.

3.5. Landau Damping and Kinetic Instabilities

Now let us calculate the damping rate of Langmuir waves using Eqs. (3.29), (3.32) and (3.39):

$$\frac{\partial \text{Re } \epsilon}{\partial \omega} \approx \frac{2\omega_{\text{pe}}^2}{\omega^3}, \quad \text{Im } \epsilon \approx -\frac{\omega_{\text{pe}}^2}{k^2} \frac{\pi}{n_e} F'_e\left(\frac{\omega}{k}\right) \quad \Rightarrow \quad \boxed{\gamma \approx \frac{\pi \omega^3}{2 k^2 n_e} \frac{1}{F'_e\left(\frac{\omega}{k}\right)}}, \quad (3.41)$$

where ω is given by Eq. (3.39). Provided $\omega F'(\omega/k) < 0$ (as would be the case, e.g., for any distribution monotonically decreasing with $|v_z|$; see Fig. 7a), $\gamma < 0$ and so this is indeed a damping rate, the celebrated *Landau damping* (Landau 1946; it was confirmed experimentally two decades later, by Malmberg & Wharton 1964).

The same theory also describes a class of *kinetic instabilities*: if $\omega F'(\omega/k) > 0$, $\gamma > 0$, so perturbations grow exponentially with time. An iconic example is the *bump-on-tail instability* (Fig. 7b), which arises when a high-energy ($v_z \gg v_{\text{the}}$) electron beam is injected into a plasma and which we will study in great detail in §7.

We see that the damping or growth of plasma waves occurs via their interaction with the particles whose velocities coincide with the phase velocity of the wave (“*Landau resonance*”). Because such particles are moving in phase with the wave, its electric field is stationary in their reference frame and so can do work on these particles, giving its energy to them (damping) or receiving energy from them (instability). In contrast, other,

out-of-phase, particles experience no mean energy change over time because the field that they “see” is oscillating. It turns out (§3.6) that the process works in the spirit of socialist redistribution: the particles that are slightly lagging behind the wave will, on average, receive energy from it, damping the wave, whereas those overtaking the wave will give up their energy, amplifying the wave. The condition $\omega F'(\omega/k) < 0$ corresponds to the stragglers being more numerous than the strivers, leading to net damping; $\omega F'(\omega/k) > 0$ implies the opposite, leading to an instability (which then leads to a flattening of the distribution; see §7).

Let us note again that these results are quantitatively valid only in the limit (3.33), or, equivalently, (3.40). It makes sense that damping should be slow ($\gamma \ll \omega$) in the limit where the waves propagate much faster than the majority of the electrons ($\omega/k \gg v_{\text{the}}$) and so can interact only with a small number of particularly fast particles (for a Maxwellian equilibrium distribution, it is an exponentially small number $\sim e^{-\omega^2/k^2 v_{\text{the}}^2}$). If, on the other hand, $\omega/k \sim v_{\text{the}}$, the waves interact with the majority population and the damping should be strong: *a priori*, we might expect $\gamma \sim kv_{\text{the}}$.

A few years ago, Landau damping became a *cause célèbre* in the mathematics community and in the wider science world with the award of the Fields Medal in 2010 to Cédric Villani, who proved (with C. Mouhot) that, basically, Landau’s solution of the linearised Vlasov equation survived as a solution of the full nonlinear Vlasov equation for small enough and regular enough initial perturbations: see a “popular” account of this by Villani (2014). The regularity restriction is apparently important and the result can break down in interesting ways: see Bedrossian (2016). The culprit is plasma echo, of which more will be said in §§6.2 and 10 (without claim to mathematical rigour; see also Schekochihin *et al.* 2016 and Adkins & Schekochihin 2017).

Exercise 3.2. Stability of isotropic distributions. Prove that if $f_{0e}(v_x, v_y, v_z) = f_{0e}(v)$, i.e., is 3D-isotropic distribution, monotonic or otherwise, the Langmuir waves at $k\lambda_{De} \ll 1$ are always damped (this is solved in Lifshitz & Pitaevskii 1981; the statement of stability of isotropic distributions is in fact valid much more generally than just for long-wavelength Langmuir waves, as we will see in §5).

3.6. Physical Picture of Landau Damping

The following simple argument (Lifshitz & Pitaevskii 1981) illustrates the physical mechanism of Landau damping.

Consider an electron moving along the z axis, subject to a wave-like electric field:

$$\frac{dz}{dt} = v_z, \quad (3.42)$$

$$\frac{dv_z}{dt} = -\frac{e}{m_e} E(z, t) = -\frac{e}{m_e} E_0 \cos(\omega t - kz) e^{\varepsilon t}. \quad (3.43)$$

We have given the electric field a slow time dependence, $E \propto e^{\varepsilon t}$, but we will later take $\varepsilon \rightarrow +0$ —this describes the field switching on infinitely slowly from $t = -\infty$. We assume that the amplitude E_0 of the electric field is so small that it changes the electron’s trajectory only a little over several wave periods. Then we can solve the equations of motion perturbatively.

The lowest-order ($E_0 = 0$) solution is

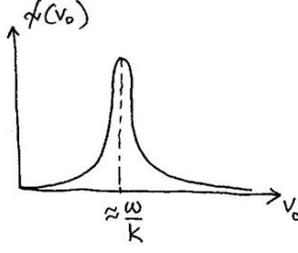
$$v_z(t) = v_0 = \text{const}, \quad z(t) = v_0 t. \quad (3.44)$$

In the next order, we let

$$v_z(t) = v_0 + \delta v_z(t), \quad z(t) = v_0 t + \delta z(t). \quad (3.45)$$

Eq. (3.43) becomes

$$\frac{d\delta v_z}{dt} = -\frac{e}{m_e} E(z(t), t) \approx -\frac{e}{m_e} E(v_0 t, t) = -\frac{eE_0}{m_e} \text{Re} e^{[i(\omega - kv_0) + \varepsilon]t}. \quad (3.46)$$

FIGURE 8. The function $\chi(v_0)$ defined in Eq. (3.49).

Integrating this gives

$$\begin{aligned}
 \delta v_z(t) &= -\frac{eE_0}{m_e} \int_0^t dt' \operatorname{Re} e^{[i(\omega - kv_0) + \varepsilon]t'} \\
 &= -\frac{eE_0}{m_e} \operatorname{Re} \frac{e^{[i(\omega - kv_0) + \varepsilon]t} - 1}{i(\omega - kv_0) + \varepsilon} \\
 &= -\frac{eE_0}{m_e} \frac{\varepsilon e^{\varepsilon t} \cos[(\omega - kv_0)t] - \varepsilon + (\omega - kv_0)e^{\varepsilon t} \sin[(\omega - kv_0)t]}{(\omega - kv_0)^2 + \varepsilon^2}. \quad (3.47)
 \end{aligned}$$

Integrating again,

$$\begin{aligned}
 \delta z(t) &= \int_0^t dt' \delta v_z(t') \\
 &= -\frac{eE_0}{m_e} \int_0^t dt' \operatorname{Re} \frac{e^{[i(\omega - kv_0) + \varepsilon]t'} - 1}{i(\omega - kv_0) + \varepsilon} \\
 &= -\frac{eE_0}{m_e} \left\{ \operatorname{Re} \frac{e^{[i(\omega - kv_0) + \varepsilon]t} - 1}{[i(\omega - kv_0) + \varepsilon]^2} - \frac{\varepsilon t}{(\omega - kv_0)^2 + \varepsilon^2} \right\} \\
 &= -\frac{eE_0}{m_e} \left\{ \frac{[\varepsilon^2 - (\omega - kv_0)^2] \{e^{\varepsilon t} \cos[(\omega - kv_0)t] - 1\} + 2\varepsilon(\omega - kv_0)e^{\varepsilon t} \sin[(\omega - kv_0)t]}{[(\omega - kv_0)^2 + \varepsilon^2]^2} \right. \\
 &\quad \left. - \frac{\varepsilon t}{(\omega - kv_0)^2 + \varepsilon^2} \right\}. \quad (3.48)
 \end{aligned}$$

The work done by the field on the electron per unit time, averaged over time, is the power gained by the electron:

$$\begin{aligned}
 \delta P(v_0) &= -e \langle E(z(t), t) v_z(t) \rangle \\
 &\approx -e \left\langle \left[E(v_0 t, t) + \delta z(t) \frac{\partial E}{\partial z}(v_0 t, t) \right] [v_0 + \delta v_z(t)] \right\rangle \\
 &= -eE_0 e^{\varepsilon t} \left\langle \underbrace{v_0 \cos[(\omega - kv_0)t]}_{\substack{\text{vanishes} \\ \text{under} \\ \text{averaging}}} + \underbrace{\delta v_z(t) \cos[(\omega - kv_0)t]}_{\substack{\text{only cos term} \\ \text{from} \\ \text{Eq. (3.47)} \\ \text{survives} \\ \text{averaging}}} + \underbrace{v_0 \delta z(t) k \sin[(\omega - kv_0)t]}_{\substack{\text{only sin term} \\ \text{from} \\ \text{Eq. (3.48)} \\ \text{survives} \\ \text{averaging}}} \right\rangle \\
 &= \frac{e^2 E_0^2}{2m_e} e^{2\varepsilon t} \left\{ \frac{\varepsilon}{(\omega - kv_0)^2 + \varepsilon^2} + \frac{2kv_0 \varepsilon (\omega - kv_0)}{[(\omega - kv_0)^2 + \varepsilon^2]^2} \right\} \\
 &= \frac{e^2 E_0^2}{2m_e} e^{2\varepsilon t} \frac{d}{dv_0} \underbrace{\frac{\varepsilon v_0}{(\omega - kv_0)^2 + \varepsilon^2}}_{\equiv \chi(v_0)}. \quad (3.49)
 \end{aligned}$$

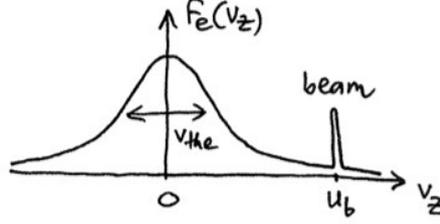


FIGURE 9. Electron distribution with a cold beam, Eq. (3.54).

We see that (Fig. 8)

—if the electron is lagging behind the wave, $v_0 \lesssim \omega/k$, then $\chi'(v_0) > 0 \Rightarrow \delta P(v_0) > 0$, so energy goes from the field to the electron (the wave is damped);

—if the electron is overtaking the wave, $v_0 \gtrsim \omega/k$, then $\chi'(v_0) < 0 \Rightarrow \delta P(v_0) < 0$, so energy goes from the electron to the field (the wave is amplified).

Now remember that we have a whole distribution of these electrons, $F(v_z)$. So the total power per unit volume going into (or out of) them is

$$\begin{aligned} P &= \int dv_z F(v_z) \delta P(v_z) = \frac{e^2 E_0^2 e^{2\varepsilon t}}{2m_e} \int dv_z F(v_z) \chi'(v_z) \\ &= -\frac{e^2 E_0^2 e^{2\varepsilon t}}{2m_e} \int dv_z F'(v_z) \chi(v_z). \end{aligned} \quad (3.50)$$

Noticing that, by Plemelj's formula (3.25), in the limit $\varepsilon \rightarrow +0$,

$$\chi(v_z) = \frac{\varepsilon v_z}{(\omega - kv_z)^2 + \varepsilon^2} = -\frac{iv_z}{2} \left(\frac{1}{kv_z - \omega - i\varepsilon} - \frac{1}{kv_z - \omega + i\varepsilon} \right) \rightarrow \pi \frac{\omega}{k^2} \delta\left(v_z - \frac{\omega}{k}\right), \quad (3.51)$$

we conclude

$$P = -\frac{e^2 E_0^2}{2m_e k^2} \pi \omega F'\left(\frac{\omega}{k}\right). \quad (3.52)$$

As in §3.5, we find damping if $\omega F'(\omega/k) < 0$ and instability if $\omega F'(\omega/k) > 0$.

Thus, around the wave-particle resonance $v_z = \omega/k$, the particles just lagging behind the wave receive energy from the wave and those just overtaking it give up energy to it. Therefore, qualitatively, damping occurs if the former particles are more numerous than the latter. We see that Landau's mathematics in §§3.1–3.5 led us to a result that makes physical sense.

3.7. Hot and Cold Beams

Let us return to the unstable situation, when a high-energy beam produces a bump on the tail of the distribution function and thus electrostatic perturbations can suck energy out of the beam and grow in the region of wave numbers where $v_0 < \omega/k < u_b$. Here v_0 is the point of minimum of the distribution in Fig. 7(b) and u_b is the velocity of the beam, which is the point of maximum of the bump when $u_b \gg v_{the}$. In view of Eq. (3.41), the instability will have a greater growth rate if the bump's slope is steeper, i.e., if the beam is colder.

Imagine modelling the beam with a little Maxwellian distribution with mean velocity u_b , added to the bulk distribution:†

$$F_e(v_z) = \frac{n_e - n_b}{\sqrt{\pi} v_{the}} \exp\left(-\frac{v_z^2}{v_{the}^2}\right) + \frac{n_b}{\sqrt{\pi} v_b} \exp\left[-\frac{(v_z - u_b)^2}{v_b^2}\right], \quad (3.53)$$

where n_b is the density of the beam, v_b is its width, and so $T_b = m_e v_b^2/2$ is its “temperature”, just like $T_e = m_e v_{the}^2/2$ is the temperature of the thermal bulk. A colder beam will have less

† The fact that we are working in 1D means that we are restricting our consideration to perturbations whose wave numbers \mathbf{k} are parallel to the beam's velocity. In general, allowing transverse wave numbers brings into play the transverse (electromagnetic) part of the dielectric tensor (see Q2). However, for non-relativistic beams, the fastest-growing modes will still be the longitudinal, electrostatic ones (see, e.g., Alexandrov *et al.* 1984, §32).

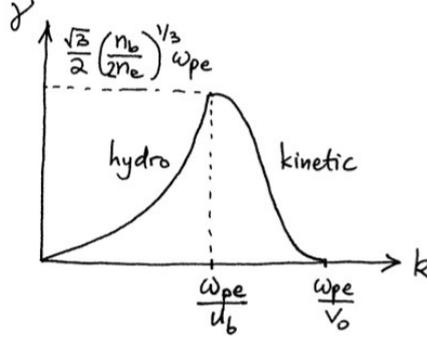


FIGURE 10. Sketch of the growth rate of the hydrodynamic and kinetic beam instabilities: Eq. (3.57) for $k < \omega_{pe}/u_b$, Eq. (3.58) for $k = \omega_{pe}/u_b$, and Eq. (3.41) for $\omega_{pe}/u_b < k < \omega_{pe}/v_0$, where v_0 is the point of the minimum of the distribution in Fig. 7(b) and u_b is the point of the maximum of the bump.

of a thermal spread around u_b . It turns out that if the width of the beam is sufficiently small, another instability appears, whose origin is hydrodynamic rather than kinetic. In the interest of having a full picture, let us work it out.

Consider a very simple limiting case of the distribution (3.53): let $v_b \rightarrow 0$ and $n_b \ll n_e$. Then (Fig. 9)

$$F_e(v_z) = F_M(v_z) + n_b \delta(v_z - u_b), \quad (3.54)$$

where F_M is the bulk Maxwellian from (3.53) (with density $\approx n_e$, neglecting n_b in comparison). Let us substitute the distribution (3.54) into the dielectric function (3.26), seek solutions with $p/k \gg v_{the}$, expand the part containing F_M in the same way as we did in §3.4†, neglect the ion contribution for the same reason as we did there, and deal with the term in the dielectric function containing $\delta'(v_z - u_b)$ by integrating by parts. The resulting dispersion relation is

$$\epsilon \approx 1 + \frac{\omega_{pe}^2}{p^2} - \frac{n_b}{n_e} \frac{\omega_{pe}^2}{(ku_b - ip)^2} = 0. \quad (3.55)$$

Since $n_b \ll n_e$, the last term can only matter for those perturbations that are close to resonance with the beam (this is called *Cherenkov's resonance*):

$$p = -iku_b + \gamma, \quad \gamma \ll ku_b. \quad (3.56)$$

This turns Eq. (3.55) into

$$1 - \frac{\omega_{pe}^2}{k^2 u_b^2} + \frac{n_b}{n_e} \frac{\omega_{pe}^2}{\gamma^2} = 0 \quad \Rightarrow \quad \gamma = \pm \sqrt{\frac{n_b}{n_e} \left(\frac{1}{k^2 u_b^2} - \frac{1}{\omega_{pe}^2} \right)^{-1/2}}. \quad (3.57)$$

The expression under the square root is positive and so there is indeed a growing mode only if $k < \omega_{pe}/u_b$. This is contrast to the case of a hot (or warm) beam in §3.5: there, having a kinetic instability required $\omega F_e'(\omega/k) > 0$, which was only possible at $k > \omega_{pe}/u_b$ (the phase speed of the perturbations had to be to the left of the bump's maximum). The new instability that we have found—the *hydrodynamic beam instability*—has the largest growth rate at $ku_b = \omega_{pe}$, i.e., when the beam and the plasma oscillations are in resonance, in which case, to resolve the singularity, we need to retain γ in the second term in Eq. (3.55). Doing so and expanding in γ , we get

$$\epsilon \approx 1 - \frac{\omega_{pe}^2}{(\omega_{pe}^2 + i\gamma)^2} + \frac{n_b}{n_e} \frac{\omega_{pe}^2}{\gamma^2} \approx \frac{2i\gamma}{\omega_{pe}} + \frac{n_b}{n_e} \frac{\omega_{pe}^2}{\gamma^2} = 0 \quad \Rightarrow \quad \boxed{\gamma = \left(\frac{\pm\sqrt{3} + i}{2}, -i \right) \left(\frac{n_b}{2n_e} \right)^{1/3} \omega_{pe}}. \quad (3.58)$$

† We can treat the Landau contour as simply running along the real axis because we are expecting to find a solution with $\text{Re } p > 0$ [see Eq. (3.20)], for reasons independent of the Landau resonance.

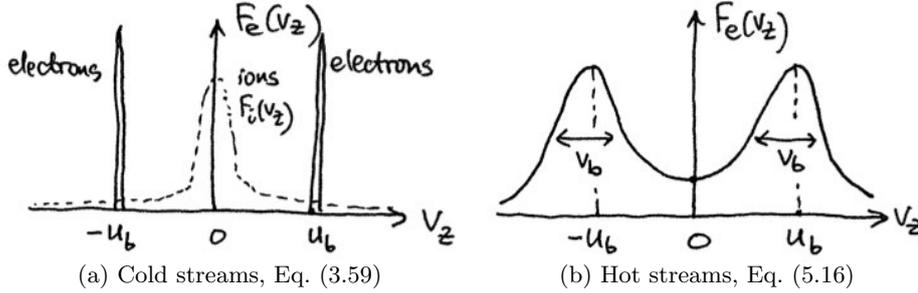


FIGURE 11. Two streams.

The unstable root ($\text{Re } \gamma > 0$) is the interesting one. The growth rate of the combined beam instability, hydrodynamic and kinetic, is sketched in Fig. 10.

Exercise 3.3. This instability is called “hydrodynamic” because it can be derived from fluid equations (cf. Exercise 3.1) describing cold electrons ($v_{\text{the}} = 0$) and a cold beam ($v_b = 0$). Convince yourself that this is the case.

Exercise 3.4. Using the model distribution (3.53), work out the conditions on v_b and n_b that must be satisfied in order for our derivation of the hydrodynamic beam instability to be valid, i.e., for Eq. (3.55) to be a good approximation to the true dispersion relation. What is the effect of finite v_b on the hydrodynamic instability? Sketch the growth rate of unstable perturbations as a function of k , taking into account both the hydrodynamic instability and the kinetic one, as well as the Landau damping.

Exercise 3.5. Two-stream instability. This is a popular instability[†] that arises, e.g., in a situation where the plasma consists of two cold counterstreams of electrons propagating against a quasineutrality-enforcing background of effectively immobile ions (Fig. 11a). Model the corresponding electron distribution by

$$F_e(v_z) = \frac{n_e}{2} [\delta(v_z - u_b) + \delta(v_z + u_b)] \quad (3.59)$$

and solve the resulting dispersion relation (where the ion terms can be neglected for the same reason as in §3.4). Find the wave number at which perturbations grow fastest and the corresponding growth rate. What happens, qualitatively, if the two streams are warm (have a finite thermal spread v_b ; Fig. 11b)? A nice fully tractable quantitative model of the latter situation is the double-Lorentzian distribution (5.16). Solve the dispersion relation for it to verify that your qualitative expectations are correct.

3.8. Ion-Acoustic Waves

Let us now see what happens at lower frequencies,

$$v_{\text{the}} \gg \frac{\omega}{k} \gg v_{\text{thi}}, \quad (3.60)$$

i.e., when the waves propagate slower than the bulk of the electron distribution but faster than the bulk of the ion one (Fig. 12). This is another regime in which we might expect to find weakly damped waves: they are out of phase with the majority of the ions, so $F'_i(\omega/k)$ might be small because $F_i(\omega/k)$ is small, while as far as the electrons are concerned, the phase speed of the waves is deep in the core of the distribution, perhaps

[†] It was discovered by engineers (Haefl 1949; Pierce & Hebenstreit 1949) and quickly adopted by physicists (Bohm & Gross 1949b). Buneman (1958) realised that a case with an electron and an ion stream (so with plasma carrying a current) is unstable in an analogous way. The kinetic version of the latter situation is the ion-acoustic instability derived in §3.9. In §5.1.4, we will discuss the stability of distributions featuring streams in a more general way.

close to its maximum at $v_z = 0$ (if that is where its maximum is) and so $F'_e(\omega/k)$ might turn out to be small because $F_e(v_z)$ changes slowly in that region.

To make this more specific, let us consider Maxwellian electrons:

$$F_e(v_z) = \frac{n_e}{\sqrt{\pi} v_{\text{the}}} \exp \left[-\frac{(v_z - u_e)^2}{v_{\text{the}}^2} \right], \quad (3.61)$$

where we are, in general, allowing the electrons to have a mean flow (current). We will assume that $u_e \ll v_{\text{the}}$ but allow $u_e \sim \omega/k$. We can anticipate that this will give us an interesting new effect. Indeed,

$$F'_e(v_z) = -\frac{2(v_z - u_e)}{v_{\text{the}}^2} F_e(v_z). \quad (3.62)$$

For resonant particles, $v_z = \omega/k$, the prefactor will indeed be small, so we can hope for $\gamma \ll \omega$, as anticipated above, but note that its sign will depend on the relative size of u_e and ω/k and so we might (we will!) get an instability (§3.9).

But let us not get ahead of ourselves: we must first calculate the real frequency $\omega(k)$ of these waves, from Eqs. (3.30) and (3.31):

$$\text{Re } \epsilon = 1 - \frac{\omega_{\text{pe}}^2}{k^2} \frac{1}{n_e} \mathcal{P} \int dv_z \frac{F'_e(v_z)}{v_z - \omega/k} - \underbrace{\frac{\omega_{\text{pi}}^2}{k^2} \frac{1}{n_i} \mathcal{P} \int dv_z \frac{F'_i(v_z)}{v_z - \omega/k}}_{\approx \frac{\omega_{\text{pi}}^2}{\omega^2} (1 + 3k^2 \lambda_{\text{Di}}^2)} = 0. \quad (3.63)$$

The last (ion) term in this equation can be expanded in $kv_z/\omega \ll 1$ in exactly the same way as it was done in Eq. (3.35). The expansion is valid provided

$$k\lambda_{\text{Di}} \ll 1, \quad (3.64)$$

and we will retain only the lowest-order term, dropping the $k^2 \lambda_{\text{Di}}^2$ correction. The second (electron) term in Eq. (3.63) is subject to the opposite limit, $v_z \gg \omega/k$, so, using Eq. (3.62),

$$\frac{\omega_{\text{pe}}^2}{k^2} \frac{1}{n_e} \mathcal{P} \int dv_z \frac{F'_e(v_z)}{v_z - \omega/k} \approx -\frac{\omega_{\text{pe}}^2}{k^2} \frac{1}{n_e} \mathcal{P} \int dv_z \frac{2(v_z - u_e)}{v_{\text{the}}^2 v_z} F_e(v_z) \approx -\frac{2\omega_{\text{pe}}^2}{k^2 v_{\text{the}}^2} = -\frac{1}{k^2 \lambda_{\text{De}}^2}, \quad (3.65)$$

where we have neglected $u_e \ll v_z$ because this integral is over the thermal bulk of the electron distribution.

With all these approximations, Eq. (3.63) becomes

$$\text{Re } \epsilon = 1 + \frac{1}{k^2 \lambda_{\text{De}}^2} - \frac{\omega_{\text{pi}}^2}{\omega^2} = 0. \quad (3.66)$$

The dispersion relation is then

$$\omega^2 = \frac{\omega_{\text{pi}}^2}{1 + 1/k^2 \lambda_{\text{De}}^2} = \frac{k^2 c_s^2}{1 + k^2 \lambda_{\text{De}}^2}, \quad (3.67)$$

where

$$c_s = \omega_{\text{pi}} \lambda_{\text{De}} = \sqrt{\frac{Z T_e}{m_i}} \quad (3.68)$$

is the *sound speed*, called that because, if we take $k\lambda_{\text{De}} \ll 1$, Eq. (3.67) describes a wave

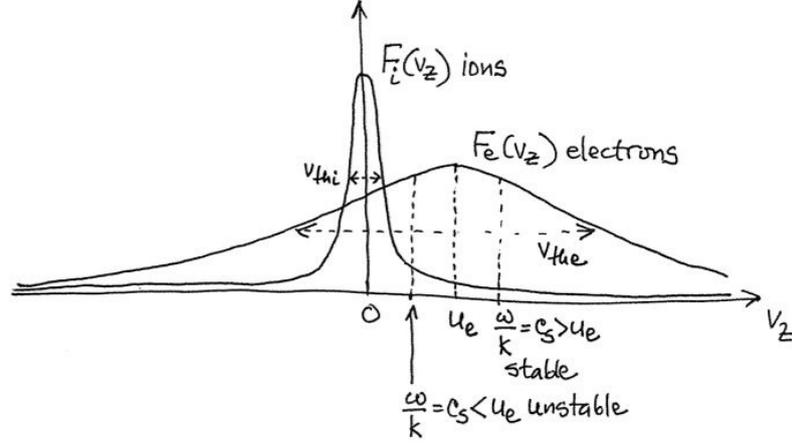


FIGURE 12. Ion-acoustic resonance: damping ($c_s > u_e$) or instability ($c_s < u_e$). Ion Landau damping is weak because $c_s \gg v_{thi}$, so in the tail of $F_i(v_z)$; electron damping/instability are also weak because $u_e, c_s \ll v_{the}$, so close to the peak $F_e(v_z)$.

that is very obviously a sound, or *ion-acoustic*, wave:

$$\boxed{\omega = \pm k c_s}. \quad (3.69)$$

The phase speed of this wave is the sound speed, $\omega/k = c_s$. That the expression (3.68) for c_s combines electron temperature and ion mass is a hint as to the underlying physics of sound propagation in plasma: ions provide the inertia, electrons the pressure (see Exercise 3.6).

We can now check under what circumstances the condition (3.60) is indeed satisfied:

$$\frac{c_s}{v_{the}} = \sqrt{\frac{Z m_e}{m_i}} \ll 1, \quad \frac{c_s}{v_{thi}} = \sqrt{\frac{Z T_e}{T_i}} \gg 1, \quad (3.70)$$

with the latter condition requiring that the ions should be colder than the electrons.

Exercise 3.6. Hydrodynamics of sound waves. Starting from the linearised kinetic equations for ions and electrons, work out the fluid equations for the plasma (i.e., the evolution equations for its mass density and mass flow velocity). Assuming $m_i \gg m_e$ and $T_i \ll T_e$, show that the sound waves (3.69) with c_s given by Eq. (3.68) are recovered if electrons have the equation of state of an isothermal fluid. Why should this be the case physically? Why is the equation of state for electrons different in a sound wave and in a Langmuir wave (see Exercise 3.1)? We will revisit ion hydrodynamics in §9.

3.9. Damping of Ion-Acoustic Waves and Ion-Acoustic Instability

Are ion acoustic waves damped? Can they grow? We have a standard protocol for answering this question: calculate $\text{Re } \epsilon$ and $\text{Im } \epsilon$ and substitute into Eq. (3.29). Using Eqs. (3.66) and (3.32), we find

$$\frac{\partial \text{Re } \epsilon}{\partial \omega} = \frac{2\omega_{pi}^2}{\omega^3}, \quad \text{Im } \epsilon = -\frac{\omega_{pe}^2}{k^2} \frac{\pi}{n_e} F'_e\left(\frac{\omega}{k}\right) - \frac{\omega_{pi}^2}{k^2} \frac{\pi}{n_i} F'_i\left(\frac{\omega}{k}\right). \quad (3.71)$$

The two terms in $\text{Im } \epsilon$ represent the interaction between the waves and, respectively, electrons and ions. The ion term is small both on account of $\omega_{pi} \ll \omega_{pe}$ and, assuming Maxwellian ions, of the exponential smallness of $F_i(\omega/k) \propto \exp[-(\omega/kv_{thi})^2]$. We are

then left with

$$\gamma = -\frac{\text{Im } \epsilon}{\partial(\text{Re } \epsilon)/\partial\omega} = -\sqrt{\pi} \frac{\omega^3}{k^2 v_{\text{the}}^3} \frac{m_i}{Z m_e} \left(\frac{\omega}{k} - u_e \right), \quad (3.72)$$

where we have used Eq. (3.62). In the long-wavelength limit, $k\lambda_{De} \ll 1$, we have $\omega = \pm kc_s$, and so, for the “+” mode,

$$\gamma = -\sqrt{\frac{\pi Z m_e}{8 m_i}} k (c_s - u_e). \quad (3.73)$$

If the electron flow is subsonic, $u_e < c_s$, this describes the *Landau damping of ion acoustic waves on hot electrons*. If, on the other hand, the electron flow is supersonic, the sign of γ reverses† and we discover the *ion-acoustic instability*: excitation of ion acoustic waves by fast electron current. The instability belongs to the same general class as, e.g., the bump-on-tail instability (§3.5) in that it involves waves sucking energy from particles, but the new conceptual feature here is that such energy conversion can result from a collaboration between different particle species (electrons supplying the energy, ions carrying the wave).

There is a host of related instabilities involving various combinations of electron and ion beams, currents, streams and counterstreams—an excellent treatment of them can be found in the textbook by Krall & Trivelpiece (1973) and in the review by Davidson (1983). We shall return to this topic in §5.1.4.

Exercise 3.7. Find the ion contribution to the damping of ion-acoustic waves. Under what conditions does it become comparable to, or larger than, the electron contribution?

3.10. Ion Langmuir Waves

Note that since

$$\frac{\lambda_{De}}{\lambda_{Di}} = \frac{v_{\text{the}} \omega_{pi}}{v_{\text{thi}} \omega_{pe}} = \sqrt{\frac{Z T_e}{T_i}}, \quad (3.74)$$

the condition (3.64) need not entail $k\lambda_{De} \ll 1$ in the limit of cold ions, Eq. (3.70)—in this case the size of the Debye sphere (1.6) is set by the ions, rather than the electrons, and so we can have perfectly macroscopic (in the language of §1.4) perturbations on scales both larger and smaller than λ_{De} . At larger scales, we have found sound waves (3.69). At smaller scales, $k\lambda_{De} \gg 1$, the dispersion relation (3.67) gives us *ion Langmuir oscillations*:

$$\omega^2 = \omega_{pi}^2 = \frac{4\pi Z^2 e^2 n_i}{m_i}, \quad (3.75)$$

which are analogous to the electron Langmuir oscillations (3.38) and, like them, turn into dispersive *ion Langmuir waves* if the small $k^2 \lambda_{Di}^2$ correction in Eq. (3.63) is retained, leading to the Bohm–Gross dispersion relation (3.39), but with ion quantities this time.

Exercise 3.8. Derive the dispersion relation for ion Langmuir waves. Investigate their damping/instability.

3.11. Summary of Electrostatic (Longitudinal) Plasma Waves

We have achieved what turns out to be a basically complete characterisation of electrostatic (also known as “longitudinal”, in the sense that $\mathbf{k} \parallel \mathbf{E}$) waves in an unmagnetised

† Recall that $k > 0$ by the choice of the z axis.

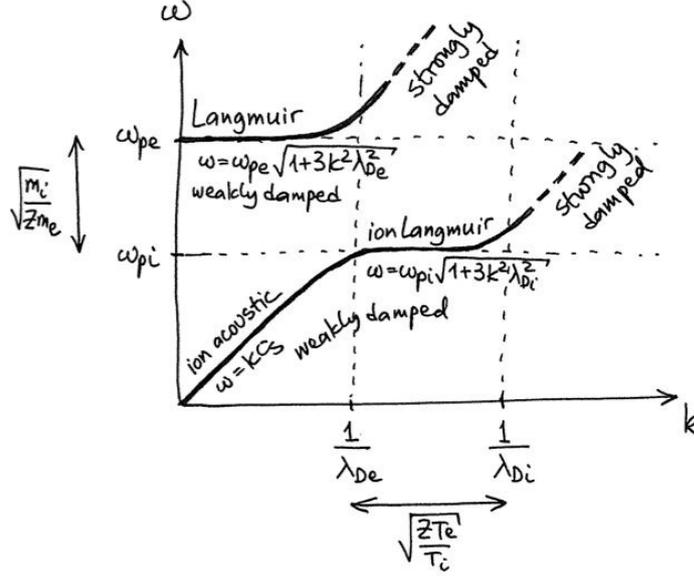


FIGURE 13. Electrostatic (longitudinal) plasma waves.

plasma. These are summarised in Fig. 13. In the limit of short wavelengths, $k\lambda_{De} \gg 1$ and $k\lambda_{Di} \gg 1$, the electron and ion branches, respectively, becomes dispersive, their damping rates increase and eventually stop being small. This corresponds to waves having phase speeds that are comparable to the speeds of the particles in the thermal bulk of their distributions, so a great number of particles are available to have Landau resonance with the waves and absorb their energy—the damping becomes strong.

Note that if the cold-ion condition $T_i \ll T_e$ is not satisfied, the sound speed is comparable to the ion thermal speed $c_s \sim v_{thi}$, and so the ion-acoustic waves are strongly damped at all wave numbers—it is well-nigh impossible to propagate sound through a collisionless hot plasma (no one will hear you scream)!

3.12. Plasma Dispersion Function: Putting Linear Theory on Industrial Basis

By the end of this section, we have entered the realm of practical calculation—it is now easy to imagine an industry of solving the plasma dispersion relation

$$\epsilon(p, \mathbf{k}) = 1 - \sum_{\alpha} \frac{\omega_{p\alpha}^2}{k^2} \frac{1}{n_{\alpha}} \int_{C_L} dv_z \frac{F'_{\alpha}(v_z)}{v_z - ip/k} = 0 \quad (3.76)$$

and similar dispersion relations arising from, e.g., considering electromagnetic perturbations (see Q2), magnetised plasmas (see Parra 2017b), different equilibria F_{α} (see Q3), etc.

A Maxwellian equilibrium is obviously an extremely important special case because that is, after all, the distribution towards which plasma is pushed by collisions on long time scales:

$$f_{0\alpha}(\mathbf{v}) = \frac{n_{\alpha}}{(\pi v_{th\alpha}^2)^{3/2}} e^{-v^2/v_{th\alpha}^2} \Rightarrow F_{\alpha}(v_z) = \frac{n_{\alpha}}{\sqrt{\pi} v_{th\alpha}} e^{-v_z^2/v_{th\alpha}^2}. \quad (3.77)$$

For this case, we would like to introduce a new “special function” that would incorporate the Landau prescription for calculating the velocity integral in Eq. (3.76) and that we could in some sense “tabulate” once and for all.†

† In the olden days, one would literally tabulate it (Fried & Conte 1961). In the 21st century, we could just teach a computer to compute it [see Eq. (3.86)] and have an app.

Taking F_α to be a Maxwellian and letting $u = v_z/v_{\text{th}\alpha}$ and $\zeta_\alpha = ip/kv_{\text{th}\alpha}$, we can rewrite the velocity integral in Eq. (3.76) as follows

$$\frac{1}{n_\alpha} \int_{C_L} dv_z \frac{F'_\alpha(v_z)}{v_z - ip/k} = -\frac{2}{\sqrt{\pi} v_{\text{th}\alpha}^2} \int_{C_L} du \frac{u e^{-u^2}}{u - \zeta_\alpha} = -\frac{2}{v_{\text{th}\alpha}^2} [1 + \zeta_\alpha Z(\zeta_\alpha)], \quad (3.78)$$

where the *plasma dispersion function* is defined to be

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int du \frac{e^{-u^2}}{u - \zeta}. \quad (3.79)$$

In these terms, the plasma dispersion relation (3.76) becomes

$$\epsilon = 1 + \sum_\alpha \frac{1 + \zeta_\alpha Z(\zeta_\alpha)}{k^2 \lambda_{D\alpha}^2} = 0. \quad (3.80)$$

Note that if the Maxwellian distribution (3.77) has a mean flow, as it did, e.g., in Eq. (3.61), this amounts to a shift by some mean velocity u_α and all one needs to do to adjust the above results is to shift the argument of Z accordingly:

$$\zeta_\alpha \rightarrow \zeta_\alpha - \frac{u_\alpha}{v_{\text{th}\alpha}}. \quad (3.81)$$

3.12.1. Some Properties of $Z(\zeta)$

It is not hard to see that

$$Z'(\zeta) = -\frac{1}{\sqrt{\pi}} \int du e^{-u^2} \frac{\partial}{\partial u} \frac{1}{u - \zeta} = -\frac{2}{\sqrt{\pi}} \int du \frac{u e^{-u^2}}{u - \zeta} = -2[1 + \zeta Z(\zeta)]. \quad (3.82)$$

Let us treat this identity as a differential equation: the integrating factor is e^{ζ^2} and so

$$e^{\zeta^2} Z(\zeta) = -2 \int_0^\zeta dt e^{t^2} + Z(0). \quad (3.83)$$

We know the boundary condition at $\zeta = 0$ from Eq. (3.23): for real ζ ,

$$\frac{1}{\sqrt{\pi}} \int du \frac{e^{-u^2}}{u - \zeta} = \frac{1}{\sqrt{\pi}} \underbrace{\mathcal{P} \int_{-\infty}^{+\infty} du \frac{e^{-u^2}}{u - \zeta}}_{= 0 \text{ for } \zeta = 0 \text{ because integrand is odd}} + i\sqrt{\pi} e^{-\zeta^2} \Rightarrow Z(0) = i\sqrt{\pi}. \quad (3.84)$$

Using this in Eq. (3.83) and changing the integration variable $t = -ix$, we find

$$Z(\zeta) = e^{-\zeta^2} \left(i\sqrt{\pi} + i \int_0^{i\zeta} dx e^{-x^2} \right) = 2i e^{-\zeta^2} \int_{-\infty}^{i\zeta} dx e^{-x^2}. \quad (3.85)$$

This turns out to be a uniformly valid expression for $Z(\zeta)$: our function is simply a complex erf! Here is a MATHEMATICA script for calculating it:

$$Z[\text{zeta}] := \text{I Sqrt}[\text{Pi}] \text{Exp}[-\text{zeta}^2] (1 + \text{I Erfi}[\text{zeta}]). \quad (3.86)$$

You can use this to code up Eq. (3.80) and explore, e.g., the strongly damped solutions ($\zeta \sim 1$, $\gamma \sim \omega$).

3.12.2. Asymptotics of $Z(\zeta)$

If you believe in preserving the ancient art of asymptotic theory, you will find most useful (as, effectively, we did in §§3.4–3.9) various limiting forms of $Z(\zeta)$. At small argument $|\zeta| \ll 1$,

the Taylor series is

$$Z(\zeta) = i\sqrt{\pi} e^{-\zeta^2} - 2\zeta \left(1 - \frac{2\zeta^2}{3} + \frac{4\zeta^4}{15} - \frac{8\zeta^6}{105} + \dots \right). \quad (3.87)$$

At large argument, $|\zeta| \gg 1$, $|\operatorname{Re} \zeta| \gg |\operatorname{Im} \zeta|$, the asymptotic series is

$$Z(\zeta) = i\sqrt{\pi} e^{-\zeta^2} - \frac{1}{\zeta} \left(1 + \frac{1}{2\zeta^2} + \frac{3}{4\zeta^4} + \frac{15}{8\zeta^6} + \dots \right). \quad (3.88)$$

All the results (for a Maxwellian equilibrium) that I derived in §§3.4–3.10 can be readily obtained from Eq. (3.80) by using the above limiting cases (see Q1). It is, indeed, a general practical strategy for studying this and similar plasma dispersion relations to look for solutions in the limits $\zeta_\alpha \rightarrow 0$ or $\zeta_\alpha \rightarrow \infty$, then check under what physical conditions the solutions thus obtained are valid (i.e., they indeed satisfy $|\zeta_\alpha| \ll 1$ or $|\zeta_\alpha| \gg 1$, $|\operatorname{Re} \zeta| \gg |\operatorname{Im} \zeta|$), and then fill in the non-asymptotic blanks in the same way that an experienced hunter espying antlers sticking out above the shrubbery can reconstruct, in contour outline, the rest of the hiding deer.

Exercise 3.9. Work out the Taylor series (3.87). A useful step might be to prove this interesting formula (which also turns out to be handy in other calculations; see, e.g., Q7):

$$\frac{d^m Z}{d\zeta^m} = \frac{(-1)^m}{\sqrt{\pi}} \int_{C_L} du \frac{H_m(u) e^{-u^2}}{u - \zeta}, \quad (3.89)$$

where $H_m(u)$ are Hermite polynomials.

Exercise 3.10. Work out the asymptotic series (3.88) using the Landau prescription (3.20) and expanding the principal-value integral similarly to the way it was done in Eq. (3.35). Work out also (or look up; e.g., Fried & Conte 1961) other asymptotic forms of $Z(\zeta)$, relaxing the condition $|\operatorname{Re} \zeta| \gg |\operatorname{Im} \zeta|$.

4. Energy, Entropy, Free Energy, Heating, Irreversibility and Phase Mixing

While we are done with the “computational” part of linear theory (calculating the rates at which field perturbations oscillate, damp or grow), we are not yet done with the “conceptual” part: what exactly is going on, mathematically and physically? The plan of addressing this question in this section is as follows.

- I will show that Landau damping of perturbations of a plasma in thermal equilibrium leads to the heating of this equilibrium—basically, that energy is conserved. This is not a surprise, but it is useful to see explicitly how this works (§4.1).

- I will then ask how it is possible to have heating (an irreversible process) in a plasma that was assumed collisionless and must conserve entropy. In other words, why, physically, is Landau damping a damping? This will lead us to consider entropy evolution in our system and to introduce an important concept of free energy (§4.2).

- In the above context, we will examine (§§4.3, 4.6) the Laplace-transform solution (3.8) for the perturbed distribution function and establish the phenomenon of *phase mixing*—emergence of fine structure in velocity (phase) space. This will allow collisions and, therefore, irreversibility back in (§4.5). We will also see that the Landau-damped solutions are *not* eigenmodes (while growing solutions can be), and so conclude that it made sense to insist on using an initial-value-problem formalism.

4.1. Energy Conservation and Heating

Let us go back to the full, nonlinear Vlasov–Poisson system, where we now restore the collision term:

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \nabla f_\alpha - \frac{q_\alpha}{m_\alpha} (\nabla \varphi) \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}} = \left(\frac{\partial f_\alpha}{\partial t} \right)_c, \quad (4.1)$$

$$-\nabla^2 \varphi = 4\pi \sum_\alpha q_\alpha \int d^3 \mathbf{v} f_\alpha. \quad (4.2)$$

Let us calculate the rate of change of the electric energy:

$$\begin{aligned} \frac{d}{dt} \int d^3 \mathbf{r} \frac{E^2}{8\pi} &= \underbrace{\int d^3 \mathbf{r} \frac{\nabla \varphi}{4\pi} \cdot \frac{\partial(\nabla \varphi)}{\partial t}}_{\text{by parts}} = - \underbrace{\int d^3 \mathbf{r} \frac{\varphi}{4\pi} \frac{\partial}{\partial t} \nabla^2 \varphi}_{\substack{\text{use} \\ \text{Eq. (4.2)}}} = \sum_\alpha q_\alpha \underbrace{\iint d^3 \mathbf{r} d^3 \mathbf{v} \varphi \frac{\partial f_\alpha}{\partial t}}_{\substack{\text{use} \\ \text{Eq. (4.1)}}} \\ &= \sum_\alpha q_\alpha \iint d^3 \mathbf{r} d^3 \mathbf{v} \varphi \left[\underbrace{-\mathbf{v} \cdot \nabla f_\alpha}_{\text{by parts}} + \underbrace{\frac{q_\alpha}{m_\alpha} (\nabla \varphi) \cdot \frac{\partial f}{\partial \mathbf{v}}}_{\substack{\text{vanishes} \\ \text{because} \\ f(\pm\infty) = 0}} + \underbrace{\left(\frac{\partial f_\alpha}{\partial t} \right)_c}_{\substack{\text{vanishes} \\ \text{because} \\ \text{number of} \\ \text{particles is} \\ \text{conserved}}} \right] \\ &= \sum_\alpha q_\alpha \iint d^3 \mathbf{r} d^3 \mathbf{v} f_\alpha \mathbf{v} \cdot \nabla \varphi = - \int d^3 \mathbf{r} \mathbf{E} \cdot \mathbf{j}, \end{aligned} \quad (4.3)$$

where \mathbf{j} is the current density. So the rate of change of the electric field is minus the rate at which electric field does work on the charges, a.k.a. Joule heating—not a surprising result. The energy goes into accelerating particles, of course: the rate of change of their kinetic energy is

$$\begin{aligned} \frac{dK}{dt} &= \sum_\alpha \iint d^3 \mathbf{r} d^3 \mathbf{v} \underbrace{\frac{m_\alpha v^2}{2} \frac{\partial f_\alpha}{\partial t}}_{\substack{\text{use} \\ \text{Eq. (4.1)}}} \\ &= \sum_\alpha \iint d^3 \mathbf{r} d^3 \mathbf{v} \frac{m_\alpha^2}{2} \left[\underbrace{-\mathbf{v} \cdot \nabla f_\alpha}_{\substack{\text{vanishes} \\ \text{because} \\ \text{full} \\ \text{divergence}}} + \underbrace{\frac{q_\alpha}{m_\alpha} (\nabla \varphi) \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}}}_{\text{by parts in } \mathbf{v}} + \underbrace{\left(\frac{\partial f_\alpha}{\partial t} \right)_c}_{\substack{\text{vanishes} \\ \text{because} \\ \text{energy is} \\ \text{conserved}}} \right] \\ &= - \sum_\alpha q_\alpha \iint d^3 \mathbf{r} d^3 \mathbf{v} f_\alpha \mathbf{v} \cdot \nabla \varphi = \int d^3 \mathbf{r} \mathbf{E} \cdot \mathbf{j}. \end{aligned} \quad (4.4)$$

Combining Eqs. (4.3) and (4.4) gives us the law of energy conservation:

$$\boxed{\frac{d}{dt} \left(K + \int d^3 \mathbf{r} \frac{E^2}{8\pi} \right) = 0}. \quad (4.5)$$

Exercise 4.1. Demonstrate energy conservation for the more general case in which magnetic-field perturbations are also allowed.

Thus, if the perturbations are damped, the energy of the particles must increase—this is usually called *heating*. Strictly speaking, heating refers to a slow increase in the mean temperature of the thermal equilibrium. Let us make this statement quantitative. Consider a Maxwellian plasma, homogeneous in space but possibly with some slow dependence on time (cf. §2):

$$f_{0\alpha} = \frac{n_\alpha}{(\pi v_{\text{th}\alpha}^2)^{3/2}} e^{-v^2/v_{\text{th}\alpha}^2} = n_\alpha \left(\frac{m_\alpha}{2\pi T_\alpha} \right)^{3/2} e^{-m_\alpha v^2/2T_\alpha}. \quad (4.6)$$

In a homogeneous system with a fixed volume, the density n_α is constant in time because the number of particles is constant: $d(Vn_\alpha)/dt = 0$. We allow, however, that the temperature is $T_\alpha = T_\alpha(t)$. The total kinetic energy of the particles is

$$K = V \sum_\alpha \underbrace{\int d^3\mathbf{v} \frac{m_\alpha v^2}{2} f_{0\alpha}}_{= \frac{3}{2} n_\alpha T_\alpha} + \sum_\alpha \iint d^3\mathbf{r} d^3\mathbf{v} \frac{m_\alpha v^2}{2} \delta f_\alpha. \quad (4.7)$$

Let us average this over time, as per Eq. (2.7): the perturbed part vanishes and we have

$$\langle K \rangle = V \sum_\alpha \frac{3}{2} n_\alpha T_\alpha. \quad (4.8)$$

Averaging also Eq. (4.5) and using Eq. (4.8), we get

$$\sum_\alpha \frac{3}{2} n_\alpha \frac{dT_\alpha}{dt} = - \frac{d}{dt} \frac{1}{V} \int d^3\mathbf{r} \frac{\langle E^2 \rangle}{8\pi}, \quad (4.9)$$

the heating rate of the equilibrium equals the rate of decrease of the mean energy of the perturbations.

We saw that the perturbations evolve according to Eq. (3.16). If we wait for a while, only the slowest-damped mode will matter, with all others exponentially small. Let us call its frequency and its damping rate $\omega_{\mathbf{k}}$ and $\gamma_{\mathbf{k}} < 0$, respectively, so $\mathbf{E}_{\mathbf{k}} \propto e^{-i\omega_{\mathbf{k}}t + \gamma_{\mathbf{k}}t}$. If we assume that $|\gamma_{\mathbf{k}}| \ll \omega_{\mathbf{k}}$, we may define the time average (2.7) in such a way that $\omega_{\mathbf{k}}^{-1} \ll \Delta t \ll |\gamma_{\mathbf{k}}|^{-1}$. Then Eq. (4.9) becomes

$$\sum_\alpha \frac{3}{2} n_\alpha \frac{dT_\alpha}{dt} = - \sum_{\mathbf{k}} 2\gamma_{\mathbf{k}} \frac{|\mathbf{E}_{\mathbf{k}}|^2}{8\pi} > 0 \quad \text{if } \gamma_{\mathbf{k}} < 0. \quad (4.10)$$

The Landau damping rate of the electric-field perturbation is the heating rate of the equilibrium.†

This result, while at first glance utterly obvious, might, on reflection, appear to be paradoxical: surely, the heating of the equilibrium implies increasing entropy—but the damping that is leading to the heating is collisionless and, in a collisionless system, in view of the H theorem, how can the entropy change?

4.2. Entropy and Free Energy

The kinetic entropy for each species of particles is defined to be

$$S_\alpha = - \iint d^3\mathbf{r} d^3\mathbf{v} f_\alpha \ln f_\alpha. \quad (4.11)$$

† Obviously, the damping of waves on particles of species α increases only the temperature of that species.

This quantity (or, indeed, the full-phase-space integral of any quantity that is a function only of f_α) can only be changed by collisions and, furthermore, the plasma-physics version of Boltzmann's H theorem establishes that

$$\frac{d}{dt} \sum_\alpha S_\alpha = - \sum_\alpha \iint d^3\mathbf{r} d^3\mathbf{v} \left(\frac{\partial f_\alpha}{\partial t} \right)_c \ln f_\alpha \geq 0, \quad (4.12)$$

where equality is achieved iff all f_α are Maxwellian with the same temperature $T_\alpha = T$.

Thus, *if collisions are ignored, the total entropy stays constant and everything that happens is, in principle, reversible. So how can there be net damping of waves and, worse still, net heating of the equilibrium particle distribution?!* Presumably, any damping solution can be turned into a growing solution by reversing all particle trajectories—so should the overall perturbation level stay constant?

As we already noted in §4.1, strictly speaking, heating is the change in the equilibrium temperature—and, therefore, equilibrium entropy. Indeed, for each species, the equilibrium entropy is

$$\begin{aligned} S_0 &= - \iint d^3\mathbf{r} d^3\mathbf{v} f_0 \ln f_0 = - \iint d^3\mathbf{r} d^3\mathbf{v} f_0 \left\{ \ln \left[n \left(\frac{m}{2\pi} \right)^{3/2} \right] - \frac{3}{2} \ln T - \frac{mv^2}{2T} \right\} \\ &= V \left[-n \ln n \left(\frac{m}{2\pi} \right)^{3/2} + \frac{3}{2} n \ln T + \frac{3}{2} n \right], \end{aligned} \quad (4.13)$$

where we have used $\int d^3\mathbf{v} (mv^2/2) f_0 = (3/2)nT$. Since $n = \text{const}$,

$$T \frac{dS_0}{dt} = V \frac{3}{2} n \frac{dT}{dt}, \quad (4.14)$$

so heating is indeed associated with the increase of S_0 .

Since, according to Eq. (4.10), this can be successfully accomplished by collisionless damping and since entropy overall can only increase due to collisions, we must search for the “missing entropy” (or, rather, for the missing *decrease* in entropy) in the perturbed part of the distribution. The mean entropy associated with the perturbed distribution is

$$\begin{aligned} \langle \delta S \rangle &= \langle S - S_0 \rangle = - \iint d^3\mathbf{r} d^3\mathbf{v} \langle (f_0 + \delta f) \ln(f_0 + \delta f) - f_0 \ln f_0 \rangle \\ &= - \iint d^3\mathbf{r} d^3\mathbf{v} \left\langle (f_0 + \delta f) \left[\ln f_0 + \frac{\delta f}{f_0} - \frac{\delta f^2}{2f_0} + \dots \right] - f_0 \ln f_0 \right\rangle \\ &\approx - \iint d^3\mathbf{r} d^3\mathbf{v} \frac{\langle \delta f^2 \rangle}{2f_0}, \end{aligned} \quad (4.15)$$

after expanding to second order in small $\delta f/f_0$ and using $\langle \delta f \rangle = 0$. The total entropy of each species, $S = S_0 + \delta S$, can only be changed by collisions, so, if collisions are ignored, any heating of a given species, i.e., an increase in its S_0 [see Eq. (4.14)] must be compensated by a *decrease* in its δS . The latter can only be achieved by *increasing* the mean square amplitude of the perturbations of the distribution function: indeed, using Eqs. (4.14) and (4.15), we find†

$$\begin{aligned} T \left(\frac{dS_0}{dt} + \frac{d\langle \delta S \rangle}{dt} \right) &= V \frac{3}{2} n \frac{dT}{dt} - \frac{d}{dt} \iint d^3\mathbf{r} d^3\mathbf{v} \frac{T \langle \delta f^2 \rangle}{2f_0} \\ &= - \iint d^3\mathbf{r} d^3\mathbf{v} T \left\langle \left(\frac{\partial f}{\partial t} \right)_c \ln f \right\rangle. \end{aligned} \quad (4.16)$$

† T can be brought inside the time derivative in the second term because $\langle \delta f^2 \rangle / f_0 \ll f_0$.

If the right-hand side is ignored, T can only increase if $\langle \delta f^2 \rangle$ increases too.

It is useful to work out the collision term in Eq. (4.16) in terms of f_0 and δf : using the fact that $\langle \delta f \rangle = 0$ by definition and the conservation of the particle number by the collision operator, we get

$$\begin{aligned} \iint d^3\mathbf{r} d^3\mathbf{v} T \left\langle \left(\frac{\partial f}{\partial t} \right)_c \ln f \right\rangle &\approx \iint d^3\mathbf{r} d^3\mathbf{v} \left[T \left(\frac{\partial f_0}{\partial t} \right)_c \ln f_0 + \left\langle \frac{T\delta f}{f_0} \left(\frac{\partial \delta f}{\partial t} \right)_c \right\rangle \right] \\ &= V \int d^3\mathbf{v} \frac{mv^2}{2} \left(\frac{\partial f_0}{\partial t} \right)_c + \iint d^3\mathbf{r} d^3\mathbf{v} \left\langle \frac{T\delta f}{f_0} \left(\frac{\partial \delta f}{\partial t} \right)_c \right\rangle. \end{aligned} \quad (4.17)$$

The second term is the collisional damping of δf , of which more will be said soon. The first term is the collisional energy exchange between the equilibrium distributions of different species (intra-species collisions conserve energy, but there can be friction between species). If the species under consideration is α , this energy exchange can be represented as $\sum_{\alpha'} \nu_{\alpha\alpha'} (T_\alpha - T_{\alpha'})$ (e.g., Helander & Sigmar 2005) and will act to equilibrate temperatures between species as the system strives toward thermal equilibrium. If the collision frequencies $\nu_{\alpha\alpha'}$ are small, this will be a slow effect. Due to overall energy conservation, the energy-exchange terms vanish exactly if Eq. (4.17) is summed over species.

Finally, let us sum Eq. (4.16) over species and use Eq. (4.9) to relate the total heating to the rate of change of the electric-perturbation energy:

$$\boxed{\frac{d}{dt} \left[\underbrace{\sum_{\alpha} \iint d^3\mathbf{r} d^3\mathbf{v} \frac{T_{\alpha} \langle \delta f_{\alpha}^2 \rangle}{2f_{0\alpha}} + \int d^3\mathbf{r} \frac{\langle E^2 \rangle}{8\pi}}_{\equiv W} \right] = \sum_{\alpha} \iint d^3\mathbf{r} d^3\mathbf{v} \left\langle \frac{T_{\alpha} \delta f_{\alpha}}{f_{0\alpha}} \left(\frac{\partial \delta f_{\alpha}}{\partial t} \right)_c \right\rangle \leq 0}, \quad (4.18)$$

where we used Eq. (4.17) in the right-hand side (with the total equilibrium collisional energy-exchange terms vanishing upon summation over species). The right-hand side must be non-positive-definite because collisions cannot decrease entropy [Eq. (4.12)].

Eq. (4.18) is a way to express the idea that, except for the effect of collisions, the change in the electric-perturbation energy (= -heating) must be compensated by the change in $\langle \delta f^2 \rangle$, in terms of a *conservation law* of a quadratic positive-definite quantity, W , that measures the total amount of perturbation in the system (a quadratic norm of the perturbed solution).[†] Examining the quantity under the time derivative in Eq. (4.18), we realise that it is the *free energy* of the perturbed state, comprising the entropy of the perturbed distribution and the energy of the electric field:

$$W = \mathcal{E} - \sum_{\alpha} T_{\alpha} \langle \delta S_{\alpha} \rangle, \quad \mathcal{E} = \int d^3\mathbf{r} \frac{\langle E^2 \rangle}{8\pi}. \quad (4.19)$$

It is quite a typical situation in non-equilibrium systems that there is an energy-like (quadratic in the relevant fields and positive definite) quantity, which is conserved except for dissipation. For example, in hydrodynamics, the motions of a fluid are governed by the Navier–Stokes equation:

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \Delta \mathbf{u}, \quad (4.20)$$

where \mathbf{u} is velocity, ρ mass density ($\rho = \text{const}$ for an incompressible fluid), p pressure and μ the

[†] Note that the existence of such a quantity implies that the Maxwellian equilibrium is stable: if a quadratic norm of the perturbed solution cannot grow, clearly there cannot be any exponentially growing solutions. This is known as Newcomb's theorem, first communicated to the world in the paper by Bernstein (1958, Appendix I). A generalisation of this principle to isotropic distributions is the subject of Q5(c) and of §5.2.2.

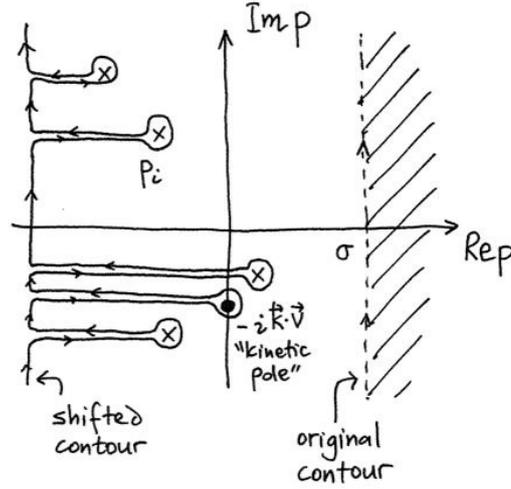


FIGURE 14. Shifting the integration contour in Eq. (4.23). This is analogous to Fig. 4 but note the additional “kinetic” pole.

dynamical viscosity of the fluid. The conservation law is

$$\frac{d}{dt} \int d^3 \mathbf{r} \frac{\rho u^2}{2} = -\mu \int d^3 \mathbf{r} |\nabla \mathbf{u}|^2 \leq 0. \quad (4.21)$$

The conserved quadratic quantity is kinetic energy and the negative-definite dissipation (leading to net entropy production) is due to viscosity.

Thus, as the electric perturbations decay via Landau damping, the perturbed distribution function must grow. This calls for going back to our solution for it (§3.1) and analysing carefully the behaviour of δf .

4.3. Structure of Perturbed Distribution Function

Start with our solution (3.8) for $\delta f(p)$ and substitute into it the solution (3.15) for $\varphi(p)$:

$$\delta \hat{f}(p) = \underbrace{\frac{1}{p + i \mathbf{k} \cdot \mathbf{v}}}_{\substack{\text{“kinetic”} \\ \text{(ballistic)} \\ \text{pole}}} \left\{ i \frac{q}{m} \left[\underbrace{\sum_i \frac{c_i}{p - p_i}}_{\substack{\text{poles} \\ \text{representing} \\ \text{linear} \\ \text{modes,} \\ \text{Eq. (3.17)}}} + A(p) \right] \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} + g \right\}. \quad (4.22)$$

To compute the inverse Laplace transform, Eq. (3.6), we adopt the same method as in §3.1 (Fig. 4), viz., shift the contour to large negative $\text{Re } p$ as shown in Fig. 14 and use

Cauchy's formula:

$$\begin{aligned} \delta f(t) &= \frac{1}{2\pi i} \int_{-i\infty+\sigma}^{i\infty+\sigma} dp e^{pt} \delta \hat{f}(p) \\ &= i \frac{q}{m} \underbrace{\sum_i \frac{c_i e^{p_i t}}{p_i + i\mathbf{k} \cdot \mathbf{v}} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}}_{\substack{\text{eigenmode-like} \\ \text{solution, comes from} \\ \text{the poles of } \varphi(p)}} + e^{-i\mathbf{k} \cdot \mathbf{v} t} \underbrace{\left[g - i \frac{q}{m} \sum_i \frac{c_i}{p_i + i\mathbf{k} \cdot \mathbf{v}} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} \right]}_{\substack{\text{ballistic response,} \\ \text{comes from} \\ p = -i\mathbf{k} \cdot \mathbf{v}}}. \end{aligned} \quad (4.23)$$

Note that the analytic part of $\hat{\varphi}(p)$ must be $A(-i\mathbf{k} \cdot \mathbf{v}) = 0$ in order for the above solution to satisfy $\delta f(t=0) = g$.

The solution (4.23) teaches us two important things.

1) First, *the Landau-damped solution is not an eigenmode*. Even though the evolution of the potential, Eq. (3.16), does look like a sum of damped eigenmodes of the form $\varphi \propto e^{p_i t}$, $\text{Re } p_i < 0$, the full solution of the Vlasov–Poisson system does *not* decay: there is a part of $\delta f(t)$, the “ballistic response” $\propto e^{-i\mathbf{k} \cdot \mathbf{v} t}$, that oscillates without decaying—in fact, we shall see in §4.6 that δf even has a growing part! It is this part that is responsible for keeping free energy conserved, as per Eq. (4.18) without collisions. Thus, you may think of Landau damping as a process of transferring (free) energy from the electric-field perturbations to the perturbations of the distribution function.

In contrast to the case of damping, a growing solution ($\text{Re } p_i > 0$) can be viewed as an eigenmode because, after a few growth times, the first term in Eq. (4.23) will be exponentially larger than the ballistic term. This will allow us to ignore the latter in our treatment of QLT (§7.1)—a handy, although not necessary (see Q8) simplification. Note that reversibility is not an issue for the growing solutions: so, there may be (and often are) damped solutions as well, so what? We only care about the growing modes because they will be all that is left if we wait long enough.

2) These δf perturbations—and this is the second lesson of Eq. (4.23)—have *fine structure in velocity (phase) space*. This structure gets finer with time: roughly speaking, if $\delta f \propto e^{-i\mathbf{k} \cdot \mathbf{v} t}$, then

$$\frac{1}{\delta f} \frac{\partial \delta f}{\partial v} \sim ikt \rightarrow \infty \quad \text{as } t \rightarrow \infty. \quad (4.24)$$

This phenomenon is called *phase mixing*. You can think of the basic mechanism responsible for it as a shearing in phase space: the homogeneous part of the linearised kinetic equation,

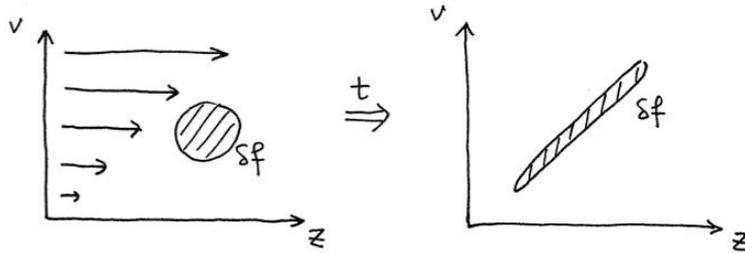
$$\frac{\partial \delta f}{\partial t} + v \frac{\partial \delta f}{\partial z} = \dots, \quad (4.25)$$

describes advection of δf by a linear shear flow in the (z, v) plane. This turns any δf structure in this plane into long thin filaments, with large gradients in v (Fig. 15).

4.4. Landau Damping is Phase Mixing

Phase mixing helps us make sense of the notion that, even though φ is the velocity integral of δf , the former can be decaying while the latter is not:

$$\varphi = \frac{4\pi}{k^2} \sum_{\alpha} q_{\alpha} \underbrace{\int d^3 \mathbf{v} \delta f_{\alpha}}_{\substack{\text{fine} \\ \text{structure} \\ \text{cancels}}} \propto e^{-\gamma t} \rightarrow 0. \quad (4.26)$$

FIGURE 15. Shearing of δf in phase space.

The velocity integral over the fine structure increasingly cancels as time goes on—a perturbation initially “visible” as φ phase-mixes away, disappearing into the negative entropy associated with the fine velocity dependence of δf [Eq. (4.15)].

More generally speaking, one can similarly argue that the refinement of velocity dependence of δf causes lower velocity moments of δf (density, flow velocity, temperature, heat flux, and so on) to decrease with time, transferring free energy to higher moments (ever higher as time goes on). One way to formalise this statement neatly is in terms of Hermite moments: since Hermite polynomials are orthogonal, the free energy of the perturbed distribution can be written as a sum of “energies” of the Hermite moments [see Eq. (10.32)]. It is then possible to represent the Landau-damped perturbations as having a broad spectrum in Hermite space, with the majority of the free energy residing in high-order moments—infinite in the formal limit of zero collisionality and infinite time (see Q7 and Kanekar *et al.* 2015).

Since the m th-order Hermite moment can, for $m \gg 1$, be asymptotically represented as a cos function in v space oscillating with the “frequency” $\sqrt{2m}/v_{\text{th}}$ [Eq. (10.33)], Eq. (4.24) implies that the typical moment in which the free energy resides grows with time as $m \sim (kv_{\text{th}}t)^2$.

Since Landau damping is a long-time effect of this phase-mixing process, it cannot be captured by any “fluid” approximation to the kinetic system involving a truncation of the hierarchy of moment equations at some finite-order moment—it is an essentially kinetic effect “beyond all orders”.

4.5. Role of Collisions

As ever larger velocity-space gradients emerge, it becomes inevitable that at some point they will become so large that *collisions can no longer be ignored*. Indeed, the Landau collision operator is a Fokker–Planck (diffusion) operator in velocity space [Eq. (1.36)] and so it will eventually wipe out the fine structure in \mathbf{v} , however small is the collision frequency ν . Let us estimate how long this takes.

The size of the velocity-space gradients of δf due to ballistic response is given by Eq. (4.24). Then the collision term is

$$\left(\frac{\partial \delta f}{\partial t}\right)_c \sim \nu v_{\text{th}}^2 \frac{\partial^2 \delta f}{\partial v^2} \sim -\nu v_{\text{th}}^2 k^2 t^2 \delta f. \quad (4.27)$$

Solving for the time evolution of the perturbed distribution function due to collisions, we get

$$\frac{\partial \delta f}{\partial t} \sim -\nu (kv_{\text{th}}t)^2 \delta f \quad \Rightarrow \quad \delta f \sim \exp\left(-\frac{1}{3} \nu k^2 v_{\text{th}}^2 t^3\right) \equiv e^{-(t/t_c)^3}. \quad (4.28)$$

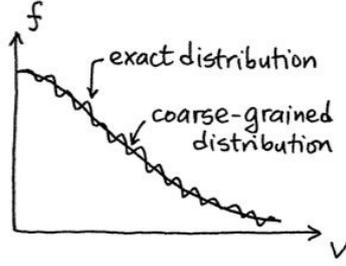


FIGURE 16. Coarse graining of the distribution function.

Therefore, the characteristic collisional decay time is

$$t_c \sim \frac{1}{\nu^{1/3}(kv_{\text{th}})^{2/3}}. \quad (4.29)$$

Note that $t_c \ll \nu^{-1}$ provided $\nu \ll kv_{\text{th}}$, i.e., t_c is within the range of times over which our “collisionless” theory is valid. After time t_c , “collisionless” damping becomes irreversible because the part of δf that is fast-varying in velocity space is lost (entropy has grown) and so it is no longer possible, even in principle, to invert all particle trajectories, have the system retrace back its steps, “phase-unmix” and thus “undamp” the damped perturbation.

In a sufficiently collisionless system, phase unmixing is, in fact, possible if nonlinearity is allowed—giving rise to the beautiful phenomenon of *plasma echo*, in which perturbations can first appear to be damped away but then come back from phase space (§6.2). This effect is a source of much preoccupation to pure mathematicians (Villani 2014; Bedrossian 2016): indeed the validity of the linearised Vlasov equation (3.1) as a sensible approximation to the full nonlinear one, Eq. (2.12), is in question if the velocity derivative $\partial\delta f/\partial\mathbf{v}$ in the last term of the latter starts growing uncontrollably. Phase unmixing has also recently turned out to have interesting consequences for the role of Landau damping in plasma turbulence (Schekochihin *et al.* 2016; Adkins & Schekochihin 2017).

Some rather purist theoreticians sometimes choose to replace collisional estimates of the type discussed above by a stipulation that $\delta f(\mathbf{v})$ must be “*coarse-grained*” beyond some suitably chosen scale in \mathbf{v} (Fig. 16)—this is equivalent to saying that the formation of the fine-structured phase-space part of δf constitutes a loss of information and so leads to growth of entropy (i.e., the loss of negative entropy associated with $\langle\delta f^2\rangle$). Somewhat non-rigorously, this means that we can just consider the ballistic term in Eq. (4.23) to have been wiped out and use the coarse-grained (i.e., velocity-space-averaged) version of δf :[†]

$$\bar{\delta f} = i \frac{q}{m} \sum_i \frac{c_i e^{p_i t}}{p_i + i\mathbf{k} \cdot \mathbf{v}} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}. \quad (4.30)$$

We can check that the correct solution for the potential, Eq. (3.16), can be recovered from this:

$$\begin{aligned} \varphi &= \frac{4\pi}{k^2} \sum_{\alpha} q_{\alpha} \int d^3\mathbf{v} \bar{\delta f}_{\alpha} \\ &= \sum_i c_i e^{p_i t} \left[\underbrace{\sum_{\alpha} i \frac{4\pi q_{\alpha}^2}{m_{\alpha} k^2} \int d^3\mathbf{v} \frac{1}{p_i + i\mathbf{k} \cdot \mathbf{v}} \mathbf{k} \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{v}} - 1 + 1}_{= -\epsilon(p_i, \mathbf{k}) = 0 \text{ by definition of } p_i} \right] = \sum_i c_i e^{p_i t}. \end{aligned} \quad (4.31)$$

[†] With an understanding that any integral involving the resonant denominator must be taken along the Landau contour (see Q8).

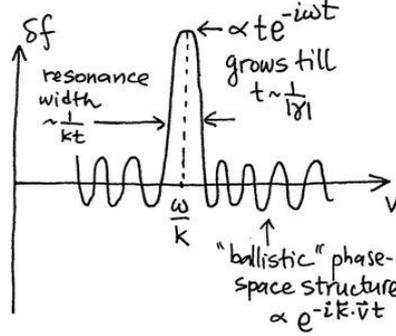


FIGURE 17. Emergence of the Case–van Kampen mode.

If you are wondering how this works without the coarse-graining kludge, read on.

4.6. Further Analysis of δf : the Case–van Kampen Mode

Having given a rather qualitative analysis of the structure and consequences of the solution (4.23), I anticipate a degree of dissatisfaction from a perceptive reader. Yes, there is a non-decaying piece of δf . But conservation of free energy in a collisionless system in the face of Landau damping in fact requires $\langle \delta f^2 \rangle$ to grow, not just to fail to decay [see Eq. (4.18)]. How do we see that this does indeed happen? The analysis that follows addresses this question. These considerations are not really necessary for most practical plasma physics calculations (see, however, Q8), but it may be necessary for your piece of mind and comfort with this whole conceptual framework.

Let us rearrange the solution (4.23) as follows:

$$\delta f(t) = i \frac{q}{m} \sum_i c_i \frac{e^{p_i t} - e^{-i\mathbf{k}\cdot\mathbf{v}t}}{p_i + i\mathbf{k}\cdot\mathbf{v}} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} + g e^{-i\mathbf{k}\cdot\mathbf{v}t}. \quad (4.32)$$

The second term is the ballistic evolution of the initial perturbation (particles flying apart in straight lines at their initial velocities)—the homogeneous solution of the kinetic equation (3.1). This develops a lot of fine-scale velocity-space structure, but obviously does not grow. The first term, the particular solution arising from the (linear) wave-particle interaction, is more interesting, especially around the resonances $\text{Re } p_i + \mathbf{k}\cdot\mathbf{v} = 0$.

Consider one of the modes, $p_i = -i\omega + \gamma$, and assume $\gamma \ll \mathbf{k}\cdot\mathbf{v} \sim \omega$. This allows us to introduce “intermediate” times:

$$\frac{1}{\mathbf{k}\cdot\mathbf{v}} \ll t \ll \frac{1}{\gamma}. \quad (4.33)$$

This means that the wave has had time to oscillate, phase mixing has got underway, but the perturbation has not yet been damped away completely. We have then, for the relevant piece of the perturbed distribution (4.32),

$$\delta f \propto \frac{e^{p_i t} - e^{-i\mathbf{k}\cdot\mathbf{v}t}}{p_i + i\mathbf{k}\cdot\mathbf{v}} = -ie^{-i\omega t} \frac{e^{\gamma t} - e^{-i(\mathbf{k}\cdot\mathbf{v}-\omega)t}}{\mathbf{k}\cdot\mathbf{v} - \omega - i\gamma} \approx -ie^{-i\omega t} \frac{1 - e^{-i(\mathbf{k}\cdot\mathbf{v}-\omega)t}}{\mathbf{k}\cdot\mathbf{v} - \omega}, \quad (4.34)$$

with the last, approximate, expression valid at the intermediate times (4.33), assuming also that, even though we might be close to the resonance, we shall not come closer than γ , viz., $|\mathbf{k}\cdot\mathbf{v} - \omega| \gg \gamma$. Respecting this ordering, but taking $|\mathbf{k}\cdot\mathbf{v} - \omega| \ll 1/t$, we find

$$\delta f \propto t e^{-i\omega t}. \quad (4.35)$$

Thus, δf has a peak that *grows with time*, emerging from the sea of fine-scale but constant-amplitude structures (Fig. 17). The width of this peak is obviously $|\mathbf{k} \cdot \mathbf{v} - \omega| \sim 1/t$ and so δf around the resonance develops a sharp structure, which, in the formal limit $t \rightarrow \infty$ (but respecting $\gamma t \ll 1$, i.e., with infinitesimal damping), tends to a delta function:

$$\boxed{\delta f \propto -ie^{-i\omega t} \frac{1 - e^{-i(\mathbf{k} \cdot \mathbf{v} - \omega)t}}{\mathbf{k} \cdot \mathbf{v} - \omega} \rightarrow e^{-i\omega t} \pi \delta(\mathbf{k} \cdot \mathbf{v} - \omega)} \quad \text{as } t \rightarrow \infty. \quad (4.36)$$

Here is a “formal” proof:

$$\frac{1 - e^{-ixt}}{x} = \underbrace{\frac{1 - \cos xt}{x}}_{\substack{\text{finite as} \\ t \rightarrow \infty, \\ \text{even at} \\ x = 0}} + i \underbrace{\frac{\sin xt}{x}}_{\substack{= it \text{ at} \\ x = 0, \\ \text{so dom-} \\ \text{inant}}} \approx \frac{e^{ixt} - e^{-ixt}}{2x} = \frac{i}{2} \int_{-t}^t dt' e^{ixt'} \rightarrow i\pi \delta(x) \quad \text{as } t \rightarrow \infty. \quad (4.37)$$

The delta-function solution (4.36) is an instance of a *Case–van Kampen mode* (van Kampen 1955; Case 1959).[†] Note that writing the solution in the vicinity of the resonance in this form is tantamount to stipulating that any integral taken with respect to \mathbf{v} (or \mathbf{k}) and involving δf must always be done along the Landau contour, circumventing the pole from below [cf. Eq. (3.23)]. We will find the representation (4.36) of δf useful in working out the quasilinear theory of Landau damping (in Q8).

If we restore finite damping, all this goes on until $t \sim 1/\gamma$, with the delta function reaching the height $\propto 1/\gamma$ and width $\propto \gamma$. In the limit $t \gg 1/\gamma$, the damped part of the solution decays, $e^{\gamma t} \rightarrow 0$, and we are left with just the ballistic part, the second term in Eq. (4.23).

4.7. Free-Energy Conservation for a Landau-Damped Solution

Finally, let us convince ourselves that, if we ignore collisions, we can recover Eq. (4.18) with a zero right-hand side from the full collisionless Landau-damped solution given by Eqs. (3.16) and (4.32). For simplicity, let us consider the case of electron Langmuir waves and prove that

$$\boxed{\frac{d}{dt} \int d^3\mathbf{v} \frac{T|\delta f_{\mathbf{k}}|^2}{2f_0} = -\frac{d}{dt} \frac{|\mathbf{E}_{\mathbf{k}}|^2}{8\pi} = -2\gamma_{\mathbf{k}} \frac{|\mathbf{E}_{\mathbf{k}}|^2}{8\pi}}. \quad (4.38)$$

Ignoring the term in Eq. (4.32) involving g as it obviously cannot give us a growing amplitude, letting the relevant root of the dispersion relation be $p_i = -i\omega_{pe} + \gamma_{\mathbf{k}}$, where $\gamma_{\mathbf{k}}$ is given by Eq. (3.41), and assuming a Maxwellian f_0 , we may write the solution (4.32) for electrons ($q = -e$) as

$$\delta f_{\mathbf{k}} \approx \frac{e}{m_e} \underbrace{c_i e^{p_i t}}_{= \varphi_{\mathbf{k}}} \frac{1 - e^{-i(\mathbf{k} \cdot \mathbf{v} - ip_i)t}}{\mathbf{k} \cdot \mathbf{v} - ip_i} \frac{2\mathbf{k} \cdot \mathbf{v}}{v_{\text{the}}^2} f_0 \approx \frac{e\varphi_{\mathbf{k}}}{T} \mathbf{k} \cdot \mathbf{v} \underbrace{\frac{1 - e^{-i(\mathbf{k} \cdot \mathbf{v} - \omega_{pe})t}}{\mathbf{k} \cdot \mathbf{v} - \omega_{pe}}}_{\approx i\pi \delta(\mathbf{k} \cdot \mathbf{v} - \omega_{pe})} f_0. \quad (4.39)$$

[†] Here we come into passing contact with an alternative (to Landau’s) formalism, due to van Kampen, for treating collisionless plasmas. The objective is more mathematical rigour—but even if this is of limited appeal to you, the book by van Kampen & Felderhof (1967) is still a good read and a good chance to question and reexamine your understanding of how it all works.

We are going to have to compute $|\delta f_{\mathbf{k}}|^2$ and squaring delta functions is a dangerous game belonging to the class of games that one must play *veeery carefully*. Here is how:

$$\left| \frac{1 - e^{-ixt}}{x} \right|^2 = \frac{1 - e^{ixt}}{x} \frac{1 - e^{-ixt}}{x} \approx \underbrace{\frac{1 - e^{ixt}}{x}}_{= -it \text{ at } x=0} i\pi\delta(x) = t\pi\delta(x). \quad (4.40)$$

Using this prescription,

$$\begin{aligned} \int d^3\mathbf{v} \frac{T|\delta f_{\mathbf{k}}|^2}{2f_0} &= \int d^3\mathbf{v} \frac{e^2|\varphi_{\mathbf{k}}|^2}{m_e v_{\text{the}}^2} (\mathbf{k} \cdot \mathbf{v})^2 t\pi\delta(\mathbf{k} \cdot \mathbf{v} - \omega_{pe}) f_0 \\ &= 2t \frac{k^2|\varphi_{\mathbf{k}}|^2}{8\pi} \underbrace{\frac{\omega_{pe}^4}{k^3} \frac{\pi}{n_e v_{\text{the}}^2} F\left(\frac{\omega_{pe}}{k}\right)}_{= -\gamma_{\mathbf{k}}, \text{ Eq. (3.41)}} = -2\gamma_{\mathbf{k}} t \frac{|\mathbf{E}_{\mathbf{k}}|^2}{8\pi}. \end{aligned} \quad (4.41)$$

Thus, the entropic part of the free energy grows secularly with time and its time derivative satisfies Eq. (4.38), q.e.d.

5. General Kinetic Stability Theory

In §3, we learned how to perturb some given equilibrium distribution $f_{0\alpha}$ infinitesimally and find the evolution of the perturbation: damping, growth, oscillation (as well as the interesting phase-space happenings that came to light in §4). Let us now pose the question in a more general way. In a collisionless plasma, there can be infinitely many possible equilibria, including quite complicated ones. If we set one up, will it persist, i.e., is it stable? If it is not stable, what modification do we expect it to undergo in order to become stable? Other than solving the dispersion relation (3.17) to answer the first question and developing various types of nonlinear theories to answer the second (along the lines advertised in §2.4 and developed in §7 and subsequent sections), both of which can be quite complicated and often intractable technical challenges, do we have at our disposal any general principles that allow us to pronounce on stability, linear or otherwise? Is there a general insight that we can cultivate as to what sort of distributions are likely to be stable or unstable and to what sorts of perturbations?

We have had glimpses of such general principles already. For example, in §3.5, we ascertained, by explicit calculation, that we could encounter a situation with a (small) growth rate if the equilibrium distribution had a positive derivative somewhere along the direction (z) of the wave number of the perturbation, viz., $v_z F'_e(v_z) > 0$. We developed this further in §3.7, finding not only hot but also cold beams and streams to trigger instabilities. In Exercise 3.2, we dropped a hint that general statements could perhaps be made about certain general classes of distributions: 3D-isotropic equilibria are stable (we shall prove this shortly). In Q5, we found that isotropic, monotonically decreasing equilibria are stable not just against infinitesimal (linear), electrostatic perturbations, but also against small but finite electromagnetic ones, giving us a taste of a powerful nonlinear constraint. How general are these statements? Are they sufficient or also necessary criteria? Is there a universal stability litmus test? Let us attack the problem of kinetic stability with an aspiration to to generality (although still, for now, for electrostatic perturbations).

5.1. Linear Stability: Nyquist's Method

We start with the relatively modest ambition to determine linear stability of generic equilibria, i.e., their stability against infinitesimal perturbations. This comes down to

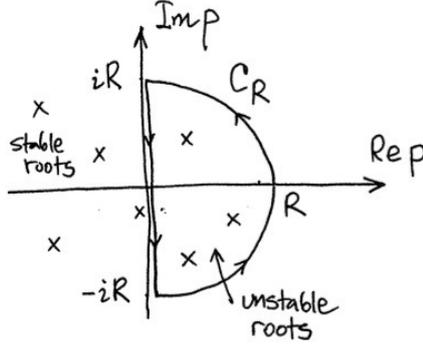


FIGURE 18. Integration contour for Eq. (5.4).

the question of whether the dispersion relation (3.17) has any unstable solutions: roots with growth rates $\gamma_i(\mathbf{k}) > 0$.

It is going to be useful to write the dielectric function (3.26) as follows

$$\epsilon(p, \mathbf{k}) = 1 - \frac{\omega_{pe}^2}{k^2} \int_{C_L} dv_z \frac{\bar{F}'(v_z)}{v_z - ip/k}, \quad (5.1)$$

$$\bar{F} = \frac{1}{n_e} \sum_{\alpha} Z_{\alpha}^2 \frac{m_e}{m_{\alpha}} F_{\alpha} = \frac{F_e}{n_e} + \frac{Z m_e}{m_i} \frac{F_i}{n_i}, \quad (5.2)$$

where the last expression in (5.2) is for the case of a two-species plasma. Thus, the distribution functions of different species come into the linear problem additively, weighted by their species' charges and (inverse) masses.

Let us develop a method (due to Nyquist 1932) for counting zeros of $\epsilon(p)$ (we will henceforth suppress \mathbf{k} in the argument) in the half-plane $\text{Re } p > 0$ (the unstable roots of the dispersion relation). We observe that $\epsilon(p)$ is analytical (be virtue of our efforts in §3.2 to make it so) and that if $p = p_i$ is its zero of order N_i , then in its vicinity,

$$\epsilon(p) = \text{const} (p - p_i)^{N_i} + \dots \Rightarrow \frac{\partial \ln \epsilon(p)}{\partial p} = \frac{N_i}{p - p_i} + \dots, \quad (5.3)$$

so zeros of ϵ are poles of $\partial \ln \epsilon(p) / \partial p$; the latter function has no other poles because ϵ is analytical. If we now integrate this function over a closed contour C_R running along the imaginary axis (and just to the right of it: $p = -i\omega + 0$) in the complex p plane from iR to $-iR$ and then along a semicircle of radius R back to iR (Fig. 18), we will, in the limit $R \rightarrow \infty$, capture all these poles:

$$\lim_{R \rightarrow \infty} \int_{C_R} dp \frac{\partial \ln \epsilon(p)}{\partial p} = 2\pi i \sum_i N_i = 2\pi i N, \quad (5.4)$$

where N is the total number of zeros of $\epsilon(p)$ in the half-plane $\text{Re } p > 0$. It turns out (as we shall prove in a moment) that the contribution to the integral over C_R from the semicircle vanishes at $R \rightarrow \infty$ and so we need only integrate along the imaginary axis:

$$\boxed{2\pi i N = \int_{+i\infty+0}^{-i\infty+0} dp \frac{\partial \ln \epsilon(p)}{\partial p} = \ln \frac{\epsilon(-i\infty)}{\epsilon(+i\infty)}}. \quad (5.5)$$

Proof. All we need to show is that

$$|p| \frac{\partial \ln \epsilon(p)}{\partial p} \rightarrow 0 \quad \text{as } |p| \rightarrow \infty, \operatorname{Re} p > 0. \quad (5.6)$$

Indeed, using Eq. (5.1) and the Landau integration rule (3.20), we have in this limit:

$$\epsilon(p) = 1 - \frac{\omega_{pe}^2}{k^2} \int_{-\infty}^{+\infty} dv_z \bar{F}'(v_z) \frac{ik}{p} \left(1 - \frac{ikv_z}{p} + \dots \right) \approx 1 + \frac{1}{p^2} \sum_{\alpha} \omega_{p\alpha}^2, \quad (5.7)$$

where we have integrated by parts and used $\int dv_z F_{\alpha} = n_{\alpha}$. Manifestly, the condition (5.6) is satisfied.

Note that, along the imaginary axis $p = -i\omega$, by the same expansion and using also the Plemelj formula (3.23), we have

$$\epsilon(-i\omega) \approx 1 - \frac{1}{\omega^2} \sum_{\alpha} \omega_{p\alpha}^2 - i\pi \frac{\omega_{pe}^2}{k^2} \bar{F}'\left(\frac{\omega}{k}\right) \rightarrow 1 \mp i0 \quad \text{as } \omega \rightarrow \mp\infty. \quad (5.8)$$

This is going to be useful shortly.

In view of Eq. (5.8) and of our newly proven formula (5.5), as the function $\epsilon(-i\omega)$ runs along the real line in ω , it changes from

$$\epsilon(i\infty) = 1 - i0 \quad \text{at } \omega = -\infty, \quad (5.9)$$

where we have arbitrarily fixed the phase, to

$$\epsilon(-i\infty) = e^{2\pi i N} + i0 \quad \text{at } \omega = +\infty, \quad (5.10)$$

where N is the number of times the function

$$\epsilon(-i\omega) = 1 - \frac{\omega_{pe}^2}{k^2} \left[\mathcal{P} \int_{-\infty}^{+\infty} dv_z \frac{\bar{F}'(v_z)}{v_z - \omega/k} + i\pi \bar{F}'\left(\frac{\omega}{k}\right) \right] \quad (5.11)$$

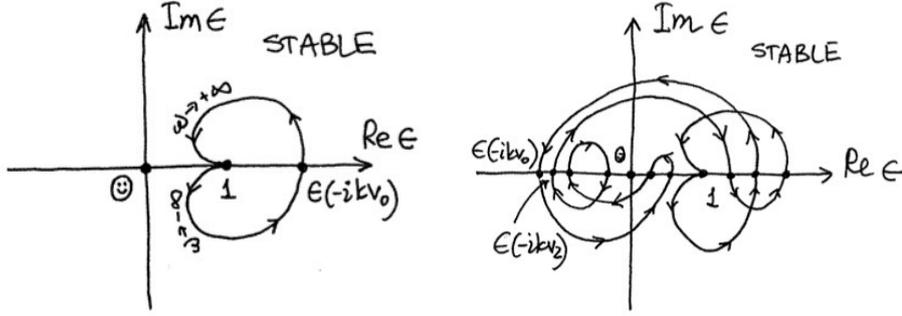
circles around the origin in the complex ϵ plane. Since N is also the number of unstable roots of the dispersion relation, this gives us a way to count these roots by sketching $\epsilon(-i\omega)$ —this sketch is called the *Nyquist diagram*. Two examples of Nyquist diagrams implying stability are given in Fig. 19: the curve $\epsilon(-i\omega)$ departs from $1 - i0$ and comes back to $1 + i0$ via a path that, however complicated, never makes a full circle around zero. Two examples of unstable situations appear in Fig. 21(b,d): in these cases, zero is circumnavigated, implying that the equilibrium distribution \bar{F} is unstable (at a given value of k).

In order to work out whether the Nyquist curve circles zero (and how many times), all one needs to do is find $\operatorname{Re} \epsilon(-i\omega)$ at all points ω where $\operatorname{Im} \epsilon(-i\omega) = 0$, i.e., where the curve intersects the real line, and hence sketch the Nyquist diagram. We shall see in a moment, with the aid of some important examples, how this is done, but let us do a little bit of preparatory work first.

It follows immediately from Eq. (5.11) that these crossings happen whenever $\omega/k = v_*$ is a velocity at which $\bar{F}(v_z)$ has an extremum, $\bar{F}'(v_*) = 0$. At these points, the dielectric function (5.11) is real and can be expressed so:

$$\boxed{\epsilon(-ikv_*) = 1 + \frac{\omega_{pe}^2}{k^2} P(v_*)}. \quad (5.12)$$

Here $P(v_*)$ is (minus) the principal-value integral in Eq. (5.11), which can be manipulated



(a) Single-maximum, stable equilibrium

(b) Strange but stable equilibrium

FIGURE 19. Two examples of Nyquist diagrams showing stability (because failing to circle zero): (a) the case of a monotonically decreasing distribution (§5.1.1, Fig. 20a); (b) another stable case, even though very complicated (it also illustrates the argument in §5.1.3).

as follows:

$$\begin{aligned}
 P(v_*) &= -\mathcal{P} \int_{-\infty}^{+\infty} dv_z \frac{\bar{F}'(v_z)}{v_z - v_*} = -\mathcal{P} \int_{-\infty}^{+\infty} dv_z \frac{1}{v_z - v_*} \frac{\partial}{\partial v_z} [\bar{F}(v_z) - \bar{F}(v_*)] \\
 &= \int_{-\infty}^{+\infty} dv_z \frac{\bar{F}(v_*) - \bar{F}(v_z)}{(v_z - v_*)^2}, \tag{5.13}
 \end{aligned}$$

where we have integrated by parts; the additional term $\bar{F}(v_*)$ was inserted under the derivative in order to eliminate the boundary terms arising in this integration by parts around the pole $v_z = v_*$.[†]

Now we are ready to analyse particular (and, as we shall see, also generic) equilibrium distributions $\bar{F}(v_z)$.

5.1.1. Stability of Monotonically Decreasing Distributions

Consider first a distribution function that has a single maximum at $v_z = v_0$ and monotonically decays in both directions away from it (Fig. 20a): $\bar{F}'(v_0) = 0$, $\bar{F}''(v_0) < 0$. This means that, besides at $\omega = \mp\infty$, $\text{Im } \epsilon(-i\omega) \propto \bar{F}'(\omega/k)$ also vanishes at $\omega = kv_0$. It is then clear that

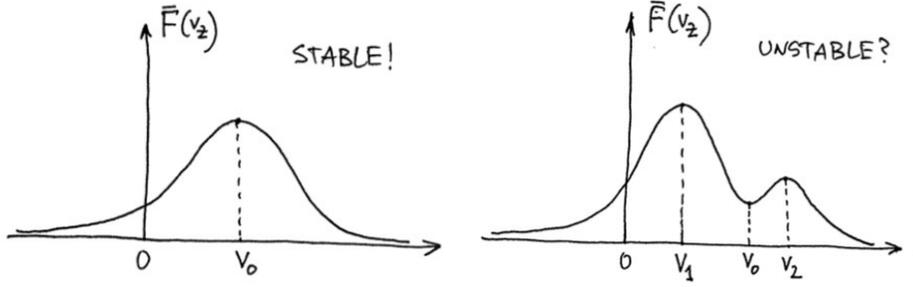
$$\epsilon(-ikv_0) = 1 + \frac{\omega_{pe}^2}{k^2} P(v_0) > 1 \tag{5.14}$$

because $\bar{F}(v_0) > \bar{F}(v)$ for all v_z and so $P(v_0) > 0$. Thus, the Nyquist curve departs from $1 - i0$ at $\omega = -\infty$, intersects the real line once at $\omega = kv_0$ and then comes back to $1 + i0$ without circling zero; the corresponding Nyquist diagram is sketched in Fig. 19(a). Conclusion:

Monotonically decreasing distributions are stable against electrostatic perturbations.

We do not in fact need all this mathematical machinery just to prove the stability of monotonically decreasing distributions (in §5.2.1, we will see that this is a very robust result)—but it will come handy when dealing with less simple cases. Parenthetically, let me give two alternative proofs of stability, the first of which (Exercise 5.1) is useful technically and the second (§5.1.2) conceptually.

[†] Note that in the final expression in Eq. (5.13), there is no longer a need for principal-value integration because, v_* being a point of extremum of \bar{F} , the numerator of the integrand is quadratic in $v_z - v_*$ in the vicinity of v_* .



(a) Single-maximum distribution (§5.1.1) (b) Single-minimum distribution (§5.1.3)

FIGURE 20. Two examples of equilibrium distributions.

Exercise 5.1. Direct proof of linear stability of monotonically decreasing distributions. (a) Consider the dielectric function (5.1) with $p = -i\omega + \gamma$ and assume $\gamma > 0$ (so the Landau contour is just the real axis). Work out the real and imaginary parts of the dispersion relation $\epsilon(p) = 0$ and show that it can never be satisfied if $v_z \bar{F}'(v_z) \leq 0$, i.e., that any equilibrium distribution that has a maximum at zero and decreases monotonically on both sides of it is stable against electrostatic perturbations.‡

(b) What if the maximum is at $v_z = v_0 \neq 0$?

(c) Recall Exercise 3.2 and show that all homogeneous, 3D-isotropic (in velocity) equilibria are stable against electrostatic perturbations (with no need to assume long wave lengths).

(d) Prove, in the same way, that isotropic equilibria are also stable against *electromagnetic* perturbations. You will need to derive the transverse dielectric function in the same way as in Q2 or Q3, but for a general equilibrium distribution $f_{0\alpha}(v_x, v_y, v_z)$; failing that, you can look it up in a book, e.g., Krall & Trivelpiece (1973) or Davidson (1983).

5.1.2. Energy Dissipation

Coming soon. See Lifshitz & Pitaevskii (1981, §30)

5.1.3. Penrose's Instability Criterion

We would like to learn how to test for stability generic distributions that have multiple minima and maxima: the simplest of them is shown in Fig. 20b, evoking the bump-on-tail situation discussed in §3.5 and thus posing a risk (but, as we are about to see, not a certainty!) of being unstable.

The Nyquist curve $\epsilon(-i\omega)$ departs from $1 - i0$ at $\omega = -\infty$, then crosses the real line for the first time at $\omega = kv_1$, corresponding to the leftmost maximum of \bar{F} .† This crossing is upwards, from the lower to the upper half-plane, and it is not hard to see that a maximum will always correspond to such an upward crossing and a minimum to a downward one, from the upper to the lower half-plane: this follows directly from the change of sign of $\text{Im } \epsilon$ in Eq. (5.11) because $\bar{F}'(\omega/k)$ goes from positive to negative at any point of maximum and vice versa at any minimum. After a few crossings back and forth, corresponding to local minima and maxima (if any), the Nyquist curve will come to the downward crossing corresponding to the *global* minimum (other than at $v_z = \pm\infty$) of the distribution function at, say, $\omega = kv_0$. If at this point $P(v_0) > 0$, then $\epsilon(-ikv_0) > 1$ and the same is true at all other crossing points v_* because v_0 is the global minimum

‡ This kind of argument can also be useful in stability considerations applying to more complicated situations, e.g., magnetised plasmas (Bernstein 1958).

† For the distribution sketched in Fig. 20(b), this maximum is global, so $P(v_1) > 0$ and, therefore, $\epsilon(-ikv_1) > 1$. This is the rightmost such crossing when v_1 is the global maximum.

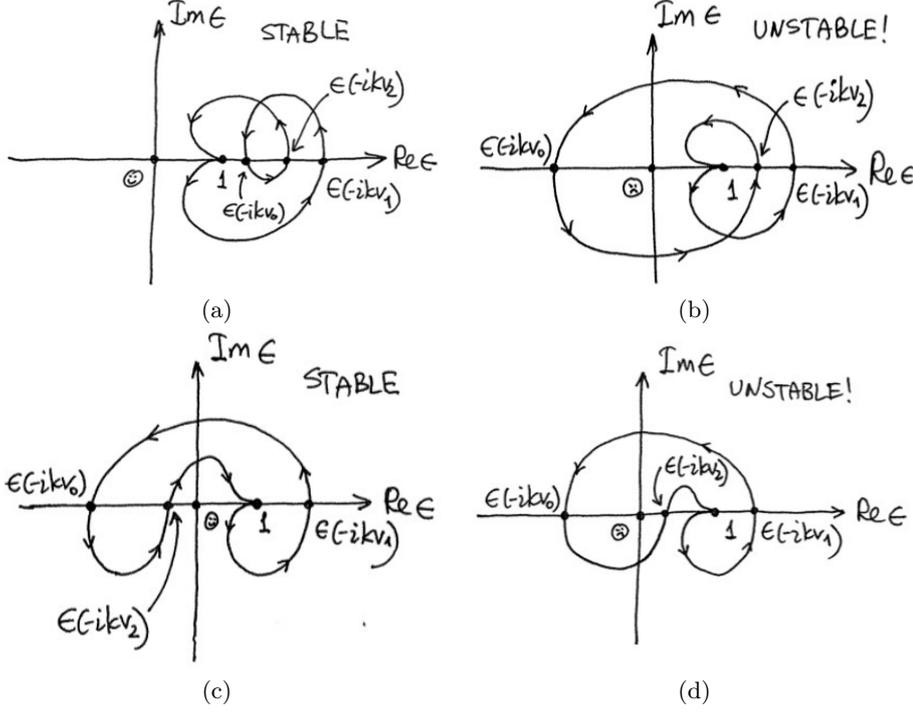


FIGURE 21. Various possible forms of the Nyquist diagram for a single-minimum distribution sketched in Fig. 20b: (a) $\epsilon(-ikv_0) > 1$, stable; (b) $\epsilon(-ikv_0) < 0$, $\epsilon(-ikv_2) > 1$, unstable; (c) $\epsilon(-ikv_0) < \epsilon(-ikv_2) < 0$, stable; (d) $\epsilon(-ikv_0) < 0 < \epsilon(-ikv_2) < 1$, unstable.

of \bar{F} and so $P(v_*) > P(v_0) > 0$ for all other extrema. In this situation, illustrated in Fig. 21(a), the Nyquist curve never circumnavigates zero and, therefore, $P(v_0) > 0$ is a sufficient condition of stability. It is also the necessary one, which is proved in the following way.

Suppose $P(v_0) < 0$. Then, in Eq. (5.12), we can always find a range of k that are small enough that $\epsilon(-ikv_0) < 0$, so the downward crossing at v_0 happens on the negative side of zero in the ϵ plane. After this downward crossing, the Nyquist curve will make more crossings, until it finally comes to rest at $1 + i0$ as $\omega = +\infty$. Let us denote by $v_2 > v_0$ the point of extremum for which the corresponding crossing occurs at a point on the $\text{Re } \epsilon$ axis that is closest to (but always will be to the right of) $\epsilon(-ikv_0) < 0$. If $\epsilon(-ikv_2) > 0$, then there is no way back, zero has been fully circumnavigated and so there must be at least one unstable root (see Fig. 21b,d). If $\epsilon(-ikv_2) < 0$, there is in principle some wiggle room for the Nyquist curve to avoid circling zero (see Fig. 21c for a single-minimum distribution of Fig. 20b—or Fig. 19b for some serious wiggles). However, since $P(v_2) > P(v_0)$ for any v_2 (because v_0 is the global minimum of \bar{F}), we can always increase k in Eq. (5.12) just enough so $\epsilon(-ikv_2) > 0$ even though $\epsilon(-ikv_0) < 0$ still (this corresponds to turning Fig. 21c into Fig. 21d). Thus, if $P(v_0) < 0$, there will always be some range of k inside which there is an instability.

We have obtained a sufficient and necessary condition of instability of an equilibrium

$\bar{F}(v_z)$ against electrostatic perturbations: if v_0 is the point of global minimum of \bar{F} ,[†]

$$P(v_0) = \int_{-\infty}^{+\infty} dv_z \frac{\bar{F}(v_0) - \bar{F}(v_z)}{(v_z - v_0)^2} < 0 \quad \Leftrightarrow \quad \bar{F} \text{ is unstable} \quad . \quad (5.15)$$

This is the famous *Penrose's instability criterion* (famous criterion, not the famous Penrose; it was proved by *Oliver* Penrose 1960, in a stylistically somewhat different way than I did it here). Note that considerations of the kind presented above can be used to work out the wave-number intervals, corresponding to various troughs in \bar{F} , in which instabilities exist.

Intuitively, the criterion (5.15) says that, in order for a distribution to be unstable, it needs to have a trough and this trough must be deep enough. Thus, if $F(v_0) = 0$, i.e., if the distribution has a “hole”, it is always unstable (an extreme example of this is the two-stream instability; see Exercise 3.5). Another corollary is that you cannot stabilise a distribution by just adding some particles in a narrow interval around v_0 , as this would create two minima nearby, which, the filled interval being narrow, are still going to render the system unstable. To change that, you must fill the trough substantially with particles—hence the tendency to flatten bumps into plateaux, which we will discover in §7 (this answers, albeit in very broad strokes, the question posed at the beginning of §5 about the types of stable distributions towards which the unstable ones will be pushed as the instabilities saturate).

Exercise 5.2. Consider a single-minimum distribution like one in Fig. 20(b), but with the global maximum on the right and the lesser maximum on the left of the minimum. Draw various possible Nyquist diagrams and convince yourself that Penrose's criterion works. If you enjoy this, think of a distribution that would give rise to the Nyquist diagram in Fig. 19(b).

Exercise 5.3. What happens if the distribution function \bar{F} has an inflection point, i.e., $\bar{F}(v_0) = 0$, $\bar{F}'(v_0) = 0$, $\bar{F}''(v_0) = 0$?

Exercise 5.4. What happens if the distribution function has a trough with a flat bottom (i.e., a flat minimum over some interval of velocities)?

5.1.4. Bumps, Beams, Streams and Flows

An elementary example of the use of Penrose's criterion is the two-stream instability, first introduced in Exercise 3.5. The case of two cold streams, Eq. (3.59) and Fig. 11(a), is obviously unstable because there is a gaping hole in this distribution. What if we now give these streams some thermal width? This can be modeled by the double-Lorentzian distribution (Fig. 11b)

$$F_e(v_z) = \frac{n_e v_b}{2\pi} \left[\frac{1}{(v_z - u_b)^2 + v_b^2} + \frac{1}{(v_z + u_b)^2 + v_b^2} \right], \quad (5.16)$$

which is particularly easy to handle analytically. For the moment, we will consider the ions to be infinitely heavy, so $\bar{F} = F_e$.

Since the distribution (5.16) is symmetric, it can only have its minimum at $v_0 = 0$. Asking that it should indeed be a minimum, rather than a maximum, i.e., $\bar{F}'(0) > 0$, we

[†] Another way of putting this is: *a distribution \bar{F} is unstable iff it has a minimum at some v_0 for which $P(v_0) < 0$* . Obviously, if $P(v_0) < 0$ at some minimum, it is also so at the global minimum.

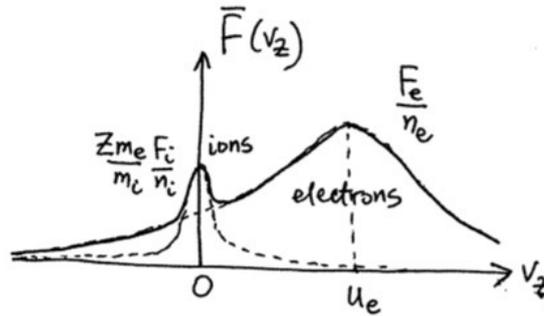


FIGURE 22. Combined distribution (5.2) for cold ions and hot electrons (cf. Fig. 12).

find that the condition for this is

$$u_b > \frac{v_b}{\sqrt{3}}. \quad (5.17)$$

Otherwise, the two streams are too wide (in velocity space) and the distribution is monotonically decreasing, so, according to §5.1.1, it is stable.

If the condition (5.17) is satisfied, the distribution has two bumps, but is this enough to make it unstable? Substituting this distribution into Penrose's criterion (5.15) and doing the integral exactly[†], we get the necessary and sufficient instability condition:

$$P(0) = -\frac{u_b^2 - v_b^2}{(u_b^2 + v_b^2)^2} < 0 \quad \Leftrightarrow \quad \boxed{u_b > v_b}. \quad (5.18)$$

Thus, if the streams are sufficiently fast and/or their thermal spread is sufficiently narrow, an instability will occur, but it is not quite enough just to have a little trough. Note that Penrose's criterion does not differentiate between hydrodynamic (cold) and kinetic (hot) instability mechanisms (§3.7).

Exercise 5.5. Use Nyquist's method to work out the range of wave numbers at which perturbations will grow for the two-stream instability (you will find the answer in Jackson 1960—yes, *that* Jackson). Convince yourself that this is all in accord with the explicit solution of the dispersion relation that you might have already obtained in Exercise 3.5.

It is obvious how these considerations can be generalised to more complicated situations, e.g., to cases where the streams have different velocities, where one of them is, in fact, the thermal bulk of the distribution and the other is a little bump on its tail (§3.7), where there are more than two streams, etc. The streams also need not be composed of the particles of the same species: indeed, as we saw in Eq. (5.1), in the linear theory, the distributions of all species are additively combined into \bar{F} with weights that are inversely proportional to their masses [see Eq. (5.2)]. Thus, the ion-acoustic instability (§3.9) is also just a kind of two-stream—or, if you like, bump-on-tail—instability, with the entire hot and mighty electron distribution making up the magnificent bump on the tail of the cold, m_e/m_i -weighted ion one (Fig. 22).[‡] When the streams/beams have thermal spreads, they are more commonly thought of as mean flows—or currents, if the electron flows are not compensated by ion ones.

[†] The easiest way is to turn the integration path along the real axis into a loop by completing it with a semicircle at positive or negative complex infinity, where the integrand vanishes, and use Cauchy's formula.

[‡] In fact, when the two species' temperatures are the same, there is still an instability, whose criterion can again be obtained by the Nyquist-Penrose method: see Jackson (1960).

Exercise 5.6. Construct an equilibrium distribution to model your favorite plasma system with flows and/or beams and investigate its stability: find the growth rate as a function of wave number, instability conditions, etc.

5.1.5. Anisotropies

So we have found that various holes, bumps, streams, beams, flows, currents and other such nonmonotonic features in the (combined, multispecies) equilibrium distribution present an instability risk, unless they are sufficiently small, shallow, wide and/or close enough to the thermal bulk. All of these are, of course, *anisotropic* features—indeed, as we saw in Exercise 5.1(c,d), 3D-isotropic distributions are harmless, instability-wise. It turns out that anisotropies of the distribution function in velocity space are dangerous even when the distribution decays monotonically in all directions.† However, the instabilities that occur in such situations are *electromagnetic*, rather than electrostatic, and so require an investigation into the properties of the transverse dielectric function of the kind derived in Q2 or Q3, but for a general equilibrium. The corresponding instability criterion is derived in Q4, by a somewhat adjusted version of Nyquist’s method. A nice treatment of anisotropy-driven instabilities can be found in Krall & Trivelpiece (1973) and an even more thorough one in Davidson (1983). In §5.2.4, we will show in quite a simple way that, at least in principle, there is energy to be extracted from anisotropic distributions.

5.2. Nonlinear Stability: Thermodynamic Method

Let us now change tack completely and ask the stability question while forbidding ourselves any recourse to linear theory. The general idea is to find, for a given initial equilibrium distribution f_0 , an upper bound on the amount of energy that might be transferred into electromagnetic perturbations (not necessarily small). If that bound is zero, the system is stable; if it is not zero but is sharp enough to be nontrivial, it gives us a constraint on the amplitude of the perturbations in the saturated state.

Here is how it is done.‡ Let us introduce some function

$$H = \underbrace{\int d^3\mathbf{r} \frac{E^2 + B^2}{8\pi}}_{\equiv \mathcal{E}} + \underbrace{\iint d^3\mathbf{r} d^3\mathbf{v} [\mathcal{A}(\mathbf{r}, \mathbf{v}, f) - \mathcal{A}(\mathbf{r}, \mathbf{v}, f_*)]}_{\equiv A[f, f_*]} = \mathcal{E} + A[f, f_*], \quad (5.19)$$

where f_* is some trial distribution, which will represent our best guess about the properties of the stable distribution towards which the system will want to evolve and/or in the general vicinity of which we are interested in investigating stability. The function $\mathcal{A}(\mathbf{r}, \mathbf{v}, f)$ is chosen in such way that for any f ,

$$A[f, f_*] \geq 0. \quad (5.20)$$

If it is also chosen so that H is conserved by the (collisionless) Vlasov–Maxwell equations, then $H(t) = H(0)$ and the inequality (5.20) gives us the following bound on the field

† In Q3, you have an opportunity to derive the most famous of the instabilities triggered by anisotropy.

‡ These ideas appear to have crystallised in the papers by T. K. Fowler in early 1960s (see his review, Fowler 1968; his reminiscences and speculations on the subject 50 years later can be found in Fowler 2016), although a number of founding fathers of plasma physics were thinking along these lines around the same time (references are given in opportune places below).

energy at time t :

$$\boxed{\mathcal{E}(t) - \mathcal{E}(0) = A[f_0, f_*] - A[f(t), f_*] \leq A[f_0, f_*]}, \quad (5.21)$$

where f_0 is the initial ($t = 0$) equilibrium that is under investigation.

The bound (5.21) implies stability if $A[f_0, f_*] = 0$, i.e., certainly for $f_0 = f_*$. This proves the stability of any f_* for which a functional $A[f, f_*]$ satisfying (5.20) and giving a conserved H can be produced.

Physically, the above construction is nontrivial if our bound on the energy is smaller than the total initial kinetic energy of the particles:

$$A[f_0, f_*] < \sum_{\alpha} \iint d^3\mathbf{r} d^3\mathbf{v} \frac{m_{\alpha} v^2}{2} f_{0\alpha} \equiv K(0). \quad (5.22)$$

It is obvious that we cannot extract from a distribution more energy than $K(0)$, but the above tells us that, in fact, one can extract less. Thus, $A[f_0, f_*]$ can be thought of as an upper bound on the *available energy* of the distribution f_0 . The sharper it can be made, the closer we are to learning something useful. Thus, the idea is to identify some suitable functional $A[f, f_*]$ for which H is conserved and some class of trial distributions f_* for which (5.20) holds, then minimise $A[f_0, f_*]$ within that class, subject to whatever physical constraints one can reasonably expect to hold: e.g., conservation of particles, momentum, any other (possibly approximate) invariants that the system might possess (e.g., its adiabatic invariants; see Helander 2017).[†]

To make some steps towards practical implementation of this programme, let us investigate how to choose \mathcal{A} in such way as to ensure conservation of H :

$$\frac{dH}{dt} = \frac{d\mathcal{E}}{dt} + \sum_{\alpha} \iint d^3\mathbf{r} d^3\mathbf{v} \frac{\partial \mathcal{A}}{\partial f_{\alpha}} \frac{\partial f_{\alpha}}{\partial t} = \sum_{\alpha} \iint d^3\mathbf{r} d^3\mathbf{v} \left(\frac{\partial \mathcal{A}}{\partial f_{\alpha}} - \frac{m_{\alpha} v^2}{2} \right) \frac{\partial f_{\alpha}}{\partial t} = 0. \quad (5.23)$$

The second equality was obtained by using the conservation of total energy,

$$\frac{d}{dt}(\mathcal{E} + K) = 0, \quad K = \sum_{\alpha} \iint d^3\mathbf{r} d^3\mathbf{v} \frac{m_{\alpha} v^2}{2} f_{\alpha}, \quad (5.24)$$

where K is the kinetic energy of the particles. Eq. (5.23) tells us how to choose \mathcal{A} :

$$\boxed{\mathcal{A}(\mathbf{r}, \mathbf{v}, f) = \sum_{\alpha} \left[\frac{m_{\alpha} v^2}{2} f_{\alpha} + G_{\alpha}(f_{\alpha}) \right]}, \quad (5.25)$$

where $G_{\alpha}(f_{\alpha})$ are arbitrary functions of f_{α} . These can be added here because Vlasov's equation has an infinite number of invariants: for any (sufficiently smooth) $G_{\alpha}(f_{\alpha})$,

$$\frac{d}{dt} \iint d^3\mathbf{r} d^3\mathbf{v} G_{\alpha}(f_{\alpha}) = 0. \quad (5.26)$$

This follows from the fact that, in the absence of collisions, the kinetic equation (1.30) expresses the conservation of phase volume in (\mathbf{r}, \mathbf{v}) space (the flow in this phase space is divergence-free).

[†] Krall & Trivelpiece (1973) comment with a slight air of resignation that, with the rules of the game much vaguer than in linear theory, the thermodynamical approach to stability is “more art than science”. In the Russian translation of their textbook, this statement provokes a stern footnote from the scientific editor (A. M. Dykhne), observing that the *right* way to put it would be “more art than craft”.

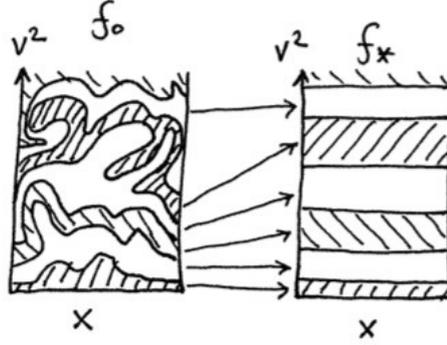


FIGURE 23. Gardner's rearrangement of the distribution, conserving phase-space volume.

Exercise 5.7. Prove the conservation law (5.26), assuming that the system is isolated.

The existence of an infinite number of conservation laws suggests that the evolution of a collisionless system in phase space is much more constrained than that of a collisional one. In the latter case, the evolution is constrained only by conservation of particles, momentum and energy and the requirement that entropy must not decrease. We shall return shortly to the question of how available energy might be related to entropy.

A quick sanity check is to choose $G_\alpha(f_\alpha) = 0$. The inequality (5.20) is then certainly satisfied for $f_* \propto \delta(\mathbf{v})$ and the bound (5.21) becomes

$$\mathcal{E}(t) - \mathcal{E}(0) \leq K(0), \quad (5.27)$$

i.e., one cannot extract any more energy than the total energy contained in the distribution—indeed, one cannot. Let us now move on to more nontrivial results.

5.2.1. Gardner's Theorem

Gardner (1963), in a classic two-page paper, proved that if the equilibrium distributions of all species are isotropic and decrease monotonically as functions of the particle energy $\varepsilon_\alpha = m_\alpha v^2/2$, the system is stable:†

$$\boxed{\frac{\partial f_{0\alpha}}{\partial \varepsilon_\alpha} < 0 \Rightarrow \text{stability}}. \quad (5.28)$$

Proof. For every species (suppressing species indices), let us again take $G(f) = 0$ in Eq. (5.25), but construct a nontrivial f_* that satisfies (5.20) with $f(t)$ at any time t since the beginning of its evolution from the initial distribution f_0 .

Let us define f_* to be a monotonically decreasing function of v^2 (i.e., energy) such that for any $\Lambda > 0$, the volume of the region in the phase space (\mathbf{r}, \mathbf{v}) where $f_* > \Lambda$ is the same as the volume of the phase-space region where $f_0 > \Lambda$. Then f_* is the distribution with the smallest kinetic energy [Eq. (5.24)], denoted here by K_* , that can be reached

† The stability of *Maxwellian* equilibria against small perturbations was first proved by W. Newcomb, whose argument was published as Appendix I of Bernstein (1958) (and followed by Fowler 1963, who proved stability against large perturbations). Gardner (1963) attributes the first appearance of the stability condition (5.28) to an obscure 1960 report by M. N. Rosenbluth, although the same condition was derived also by Kruskal & Oberman (1958), more or less in the manner described in §5.2.2. Many great minds were clearly thinking alike in those glory days of plasma physics.

from f_0 while preserving phase-space volume:

$$K(t) \geq K_*. \quad (5.29)$$

Indeed, while the phase-space volume occupied by any given value of the probability density is the same for f_0 and for f_* , the corresponding energy is always lower for f_* than for f_0 or for any other f that can evolve from it, because in f_* , the values of the probability density are rearranged in such a way as to put the largest of them at the lowest values of v^2 , thus minimising the velocity integral in Eq. (5.24).

A vivid analogy is to think of the evolution of f under the collisionless kinetic equation (1.28) as the evolution of a mixture of “fluids” of different densities (values of f) advected in a 6D phase-space (\mathbf{r}, \mathbf{v}) by a divergence-free flow $(\dot{\mathbf{r}}, \dot{\mathbf{v}})$. The lowest-energy state is the one in which these fluids are arranged in layers of density decreasing with increasing v^2 , the heaviest at the bottom, the lightest at the top (Fig. 23).

In view of (5.29) and since \mathcal{A} is given by Eq. (5.25) with $G(f) = 0$,

$$A[f, f_*] = K(t) - K_* \geq 0, \quad (5.30)$$

so (5.20) holds and the bound (5.21) follows. When $f_0 = f_*$, i.e., the equilibrium distribution satisfies (5.28), the system is stable, q.e.d.

Note that the condition (5.28) is sufficient, but not necessary, as we already know from, e.g., Exercise 5.1(c,d).

In a fresh-off-the-press paper, Helander (2017) has developed a beautiful scheme for calculating “ground states” (the states of minimum energy) of Vlasov’s equation, i.e., for determining f_* and then calculating K_* to work out specific values of the available energy implied by the upper bound (5.21), $A[f_0, f_*] = K(0) - K_*$.

5.2.2. Small Perturbations

There is a neat development (due, it seems, to Kruskal & Oberman 1958 and Fowler 1963) of the formalism presented at the beginning of this section that leads again to Gardner’s result (5.28), but puts us in contact with some familiar themes from §4.

Let us investigate the stability of isotropic distributions with respect to small (but not necessarily infinitesimal) perturbations, i.e., we take $f(t) = f_0 + \delta f$, $\delta f \ll f_0$, and also $f_* = f_0$, so the bound (5.21) will imply stability if we can find $G(f)$ such that (5.20) holds.

In Eq. (5.25), we expand

$$G(f) = G(f_0) + G'(f_0)\delta f + G''(f_0)\frac{\delta f^2}{2} + \dots \quad (5.31)$$

and use this to obtain, keeping terms up to second order,

$$A[f(t), f_0] = \sum_{\alpha} \iint d^3\mathbf{r} d^3\mathbf{v} \left\{ \left[\frac{m_{\alpha}v^2}{2} + G'_{\alpha}(f_{0\alpha}) \right] \delta f_{\alpha} + G''_{\alpha}(f_{0\alpha}) \frac{\delta f_{\alpha}^2}{2} \right\}. \quad (5.32)$$

Suppose we contrive to pick $G_{\alpha}(f_{0\alpha})$ in such a way that

$$G'_{\alpha}(f_{0\alpha}) = -\frac{m_{\alpha}v^2}{2} \equiv -\varepsilon_{\alpha}, \quad (5.33)$$

obliterating the first-order term in Eq. (5.32). Then, since $f_{0\alpha} = f_{0\alpha}(\varepsilon_{\alpha})$ by assumption (it is isotropic), differentiating the above condition with respect to $f_{0\alpha}$ gives

$$G''_{\alpha}(f_{0\alpha}) = -\frac{1}{\partial f_{0\alpha}/\partial \varepsilon_{\alpha}} \Rightarrow A[f(t), f_0] = \sum_{\alpha} \iint d^3\mathbf{r} d^3\mathbf{v} \frac{\delta f_{\alpha}^2}{2(-\partial f_{0\alpha}/\partial \varepsilon_{\alpha})}. \quad (5.34)$$

We see that $A[f(t), f_0] \geq 0$ and, therefore, Eq. (5.21) with $f_* = f_0$ implies stability if, again, $f_{0\alpha}(\varepsilon_\alpha)$ is monotonically decreasing for all species.

Besides stability, we have also found an interesting quadratic conserved quantity for our system:

$$H = \mathcal{E} + A[f, f_0] = \int d^3\mathbf{r} \frac{E^2 + B^2}{8\pi} + \sum_\alpha \iint d^3\mathbf{r} d^3\mathbf{v} \frac{\delta f_\alpha^2}{2(-\partial f_{0\alpha}/\partial \varepsilon_\alpha)}. \quad (5.35)$$

The condition (5.28) makes H positive definite and so no wonder the system is stable: perturbations around f_0 have a conserved norm! For a Maxwellian equilibrium, $-\partial f_{0\alpha}/\partial \varepsilon_\alpha = f_{0\alpha}/T_\alpha$, so this H is none other than W , (the electromagnetic version of) our free energy (4.19), and so it is tempting to think of Eq. (5.35) as providing a natural generalisation of free energy to non-Maxwellian plasmas.

In Q5, the results of this section are obtained in a more straightforward way, directly from the Vlasov–Maxwell equations.

This style of thinking has been having a revival lately: see, e.g., the discussion of firehose and mirror stability of a magnetised plasma in Kunz *et al.* (2015). Generalised energy invariants like H are important not just for stability calculations, but also for theories of turbulence in near-collisionless environments that are not particularly close to thermal (Maxwellian) equilibrium, e.g., the solar wind (see, e.g., Schekochihin *et al.* 2009).

5.2.3. Finite Perturbations

One might wonder at this point whether the condition (5.33) is fulfillable and also whether anything can be done without assuming small perturbations. An answer to both questions is provided by the following argument.

The realisation in §5.2.2 that our conserved quantity H is a generalisation of free energy nudges us in the direction of a particular choice of functions $G_\alpha(f_\alpha)$ and trial equilibria $f_{*\alpha}$, fully inspired by conventional thermodynamics. Namely, in Eq. (5.25), let

$$G_\alpha(f_\alpha) = T_\alpha f_\alpha \left(\ln \frac{f_\alpha}{C_\alpha} - 1 \right), \quad f_{*\alpha} = C_\alpha \exp \left(-\frac{m_\alpha v^2}{2T_\alpha} \right), \quad (5.36)$$

where C_α and T_α are constants independent of space. It is then certainly true that $G'_\alpha(f_{*\alpha}) = -\varepsilon_\alpha$. It is also straightforward to show that the inequality (5.20) is always satisfied: essentially, this follows from the fact that the Maxwellian distribution $f_{*\alpha}$ maximises entropy, $-\iint d^3\mathbf{r} d^3\mathbf{v} f_\alpha \ln f_\alpha$, subject to fixed energy, $1/T_\alpha$ being the corresponding Lagrange multiplier.

Exercise 5.8. Prove that if G_α and $f_{*\alpha}$ are given by Eq. (5.36), then

$$A[f, f_*] = \sum_\alpha \iint d^3\mathbf{r} d^3\mathbf{v} \left[\frac{m_\alpha v^2}{2} (f_\alpha - f_{*\alpha}) + G_\alpha(f_\alpha) - G_\alpha(f_{*\alpha}) \right] \geq 0 \quad (5.37)$$

for any values of C_α and T_α .

Thus, Eq. (5.21) provides an upper bound on the energy of any electromagnetic fields that can be extracted from the initial distribution $f_{0\alpha}$. In order to make this bound as sharp as possible, one picks the constants C_α and T_α so as to minimise $A[f_0, f_*]$ subject to constraints that cannot change: e.g., the number of particles of each species:

$$C_\alpha = \left(\frac{m_\alpha}{2\pi T_\alpha} \right)^{3/2} \frac{1}{V} \iint d^3\mathbf{r} d^3\mathbf{v} f_{0\alpha}. \quad (5.38)$$

5.2.4. Anisotropic Equilibria

Let me give an example of the use of this scheme for deriving an upper bound on the energy of unstable perturbations for a case of an anisotropic initial distribution—the case that, at the end of §5.1, I had to relegate to Q4 as it needed substantial extra work if it were to be handled by the method developed there.

Consider a *bi-Maxwellian distribution*, a useful and certainly the simplest model for anisotropic equilibria:

$$f_{0\alpha} = n_\alpha \left(\frac{m_\alpha}{2\pi} \right)^{3/2} \frac{1}{T_{\perp\alpha} T_{\parallel\alpha}^{1/2}} \exp \left(-\frac{m_\alpha v_\perp^2}{2T_{\perp\alpha}} - \frac{m_\alpha v_\parallel^2}{2T_{\parallel\alpha}} \right), \quad (5.39)$$

where $T_{\perp\alpha}$ and $T_{\parallel\alpha}$ are the “temperatures” of particle motion perpendicular and parallel to some special direction. Is this distribution unstable? (Yes: see Q3.) To obtain an upper bound on the energy available for extraction from it, we substitute the distribution function (5.39) into Eq. (5.36), use also Eq. (5.38), and find

$$A[f_0, f_*] = V \sum_\alpha n_\alpha \left(\ln \frac{T_\alpha^{3/2}}{T_{\perp\alpha} T_{\parallel\alpha}^{1/2}} + T_{\perp\alpha} + \frac{T_{\parallel\alpha}}{2} - \frac{3}{2} T_\alpha \right). \quad (5.40)$$

This is minimised by $T_\alpha = T_{\perp\alpha}^{2/3} T_{\parallel\alpha}^{1/3}$, resulting in the following estimate of the available energy:†

$$\mathcal{E}(t) - \mathcal{E}(0) \leq \min_{T_\alpha} A[f_0, f_*] = \frac{3}{2} V \sum_\alpha n_\alpha \left(\frac{2}{3} T_{\perp\alpha} + \frac{1}{3} T_{\parallel\alpha} - T_{\perp\alpha}^{2/3} T_{\parallel\alpha}^{1/3} \right). \quad (5.41)$$

The bound is zero when $T_{\perp\alpha} = T_{\parallel\alpha}$ and is always positive otherwise because it is the difference between an arithmetic and a geometric mean of the two temperatures. We do not, of course, have any way of knowing how good an approximation this is to the true saturated level of whatever instability might exist here in any particular physical regime (if any),‡ but this does hint rather suggestively that temperature anisotropy is a source of free energy.

Further examples of such calculations can be found in Krall & Trivelpiece (1973, §9.14) and Fowler (1968). A certain further development of the methodology discussed above allows one to derive upper bounds not just on the energy of perturbations but also on their growth rates (Fowler 1964, 1968).

6. Nonlinear Theory: Two Pretty Nuggets

Nonlinear theory of anything is, of course, hard—indeed, in most cases, intractable. These days, an impatient researcher’s answer to being faced with a hard question is to outsource it to a computer. This sometimes leads to spectacular successes, but also, somewhat more frequently, to spectacular confusion about how to interpret the output. In dealing with a steady stream of data

† Helander (2017) shows that Gardner’s minimum-energy distribution $f_{*\alpha}$ for this case is a Maxwellian (which, in general, it need not be) and so this estimate of the available energy coincides with Gardner’s $K(0) - K_*$ bound. This is an interesting, if perhaps somewhat anomalous, example of a system “wanting” to go to a Maxwellian equilibrium even in the absence of collisions.

‡ Nominally, this calculation applies with equal validity to many different instabilities that can be triggered by temperature anisotropy in both unmagnetised and magnetised plasmas—and indeed also to anisotropic distributions that can, in fact, be proved stable (which is common in magnetised plasmas where the externally imposed magnetic field is sufficiently large).

produced by the ever more powerful machines, one is sometimes helped by the residual memory of analytical results obtained in the prehistoric era when computation was harder than theory and plasma physicists had to find ingenious ways to solve nonlinear problems “by hand”—which usually required finding ingenious ways of posing problems that were solvable. These could be separated into two broad categories: interesting particular cases of nonlinear behaviour involving usually just a few interacting waves and systems of very many waves amenable to some approximate statistical treatment.¶ Here I will give two very pretty examples of the former, before moving on to an extended presentation of the latter in §7 onwards.

6.1. Nonlinear Landau Damping

Coming soon. See O’Neil (1965); Mazitov (1965).

6.2. Plasma Echo

Coming soon. See Gould *et al.* (1967); Malmberg *et al.* (1968).

7. Quasilinear Theory

7.1. General Scheme of QLT

In §§3 and 4, we discussed at length the structure of the linear solution corresponding to a Landau-damped initial perturbation. This could be adequately done for a Maxwellian plasma and we have found that, after some interesting transient time-dependent phase-space dynamics, perturbations damp away and their energy turns into heat, increasing somewhat the temperature of the equilibrium (see, however, Q8).

We now consider a different problem: an *unstable* (and so decidedly non-Maxwellian) equilibrium distribution giving rise to exponentially growing perturbations. The specific example on which we shall focus is the *bump-on-tail instability*, which involves generation of unstable Langmuir waves with phase velocities corresponding to instances of positive derivative of the equilibrium distribution function (Fig. 24). The energy of the waves grows exponentially:

$$\frac{\partial |\mathbf{E}_{\mathbf{k}}|^2}{\partial t} = 2\gamma_{\mathbf{k}} |\mathbf{E}_{\mathbf{k}}|^2, \quad \gamma_{\mathbf{k}} = \frac{\pi \omega_{pe}^3}{2 k^2} \frac{1}{n_e} F' \left(\frac{\omega_{pe}}{k} \right), \quad (7.1)$$

where $F(v_z) = \int dv_x \int dv_y f_0(\mathbf{v})$ [see Eq. (3.41)]. In the absence of collisions, the only way for the system to achieve a nontrivial steady state (i.e., such that $|\mathbf{E}_{\mathbf{k}}|^2$ is not just zero everywhere) is by adjusting the equilibrium distribution so that

$$\gamma_{\mathbf{k}} = 0 \quad \Leftrightarrow \quad F' \left(\frac{\omega_{pe}}{k} \right) = 0 \quad (7.2)$$

at all k where $|\mathbf{E}_{\mathbf{k}}|^2 \neq 0$, say, $k \in [k_2, k_1]$. If we translate this range into velocities, $v = \omega_{pe}/k$, we see that the equilibrium must develop a flat spot:

$$F'(v) = 0 \quad \text{for} \quad v \in [v_1, v_2] = \left[\frac{\omega_{pe}}{k_1}, \frac{\omega_{pe}}{k_2} \right]. \quad (7.3)$$

This is called a *quasilinear plateau* (§7.4). Obviously, the rest of the equilibrium distribution may (and will) also be modified in some, to be determined, way (§§7.6, 7.7).

These modifications of the original (initial) equilibrium distributions can be accomplished by the growing fluctuations via the feedback mechanism already discussed in §2.3,

¶ The third kind is asking for general criteria of certain kinds of behaviour, such as stability or otherwise—we dabbled in this type of nonlinear theory in §5.2.

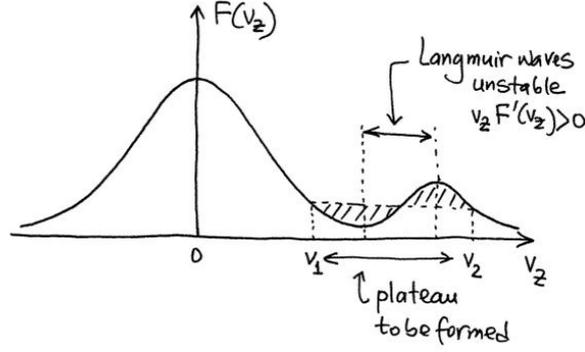


FIGURE 24. An unstable distribution with a bump on its tail.

namely, the equilibrium distribution will evolve slowly according to Eq. (2.11):

$$\frac{\partial f_0}{\partial t} = -\frac{q}{m} \sum_{\mathbf{k}} \left\langle \varphi_{\mathbf{k}}^* i\mathbf{k} \cdot \frac{\partial \delta f_{\mathbf{k}}}{\partial \mathbf{v}} \right\rangle. \quad (7.4)$$

The time averaging here [see Eq. (2.7)] is over $\omega_{pe}^{-1} \ll \Delta t \ll \gamma_{\mathbf{k}}^{-1}$.

The general scheme of QLT is:

- start with an unstable equilibrium f_0 ,
- use the linearised Eqs. (3.1) and (3.2) to work out the linear solution for the growing perturbations $\varphi_{\mathbf{k}}$ and $\delta f_{\mathbf{k}}$ in terms of f_0 ,
- use this solution in Eq. (7.4) to evolve f_0 , leading, if everything works as it is supposed to, to an ever less unstable equilibrium.

We keep only the fastest growing mode (all others are exponentially small after a while), and so the solution (3.16) for the electric perturbations is

$$\varphi_{\mathbf{k}} = c_{\mathbf{k}} e^{(-i\omega_{\mathbf{k}} + \gamma_{\mathbf{k}})t}. \quad (7.5)$$

In the solution (4.23) for the perturbed distribution function, we may ignore the ballistic term because the exponentially growing piece (the first term) will eventually leave all this velocity-space structure behind,† so

$$\delta f_{\mathbf{k}} = i \frac{q}{m} \frac{c_{\mathbf{k}} e^{(-i\omega_{\mathbf{k}} + \gamma_{\mathbf{k}})t}}{-i\omega_{\mathbf{k}} + \gamma_{\mathbf{k}} + i\mathbf{k} \cdot \mathbf{v}} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} = \frac{q}{m} \frac{\varphi_{\mathbf{k}}}{\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}} - i\gamma_{\mathbf{k}}} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}. \quad (7.6)$$

Substituting the solution (7.6) into Eq. (7.4), we get

$$\frac{\partial f_0}{\partial t} = -\frac{q^2}{m^2} \sum_{\mathbf{k}} |\varphi_{\mathbf{k}}|^2 i\mathbf{k} \cdot \frac{\partial}{\partial \mathbf{v}} \frac{1}{\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}} - i\gamma_{\mathbf{k}}} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{D}(\mathbf{v}) \cdot \frac{\partial f_0}{\partial \mathbf{v}}. \quad (7.7)$$

† See, however, Q8, where we work out how to avoid having to wait for this to happen: in fact, the results below are valid for $\gamma_{\mathbf{k}} t \lesssim 1$ as well.

This is a diffusion equation in velocity space, with a velocity-dependent diffusion matrix

$$\begin{aligned}
D(\mathbf{v}) &= -\frac{q^2}{m^2} \sum_{\mathbf{k}} i\mathbf{k}\mathbf{k} |\varphi_{\mathbf{k}}|^2 \frac{1}{\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}} - i\gamma_{\mathbf{k}}} \\
&= -\frac{q^2}{m^2} \sum_{\mathbf{k}} i\mathbf{k}\mathbf{k} |\varphi_{\mathbf{k}}|^2 \frac{1}{2} \left(\frac{1}{\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}} - i\gamma_{\mathbf{k}}} + \underbrace{\frac{1}{-\mathbf{k} \cdot \mathbf{v} - \omega_{-\mathbf{k}} - i\gamma_{-\mathbf{k}}}}_{\text{here we changed variables } \mathbf{k} \rightarrow -\mathbf{k}} \right) \\
&= -\frac{q^2}{m^2} \sum_{\mathbf{k}} \frac{\mathbf{k}\mathbf{k}}{k^2} |\mathbf{E}_{\mathbf{k}}|^2 \frac{i}{2} \left(\frac{1}{\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}} - i\gamma_{\mathbf{k}}} - \frac{1}{\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}} + i\gamma_{\mathbf{k}}} \right) \\
&= \frac{q^2}{m^2} \sum_{\mathbf{k}} \frac{\mathbf{k}\mathbf{k}}{k^2} |\mathbf{E}_{\mathbf{k}}|^2 \operatorname{Im} \frac{1}{\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}} - i\gamma_{\mathbf{k}}} \\
&= \frac{q^2}{m^2} \sum_{\mathbf{k}} \frac{\mathbf{k}\mathbf{k}}{k^2} |\mathbf{E}_{\mathbf{k}}|^2 \frac{\gamma_{\mathbf{k}}}{(\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}})^2 + \gamma_{\mathbf{k}}^2}. \tag{7.8}
\end{aligned}$$

To obtain these expressions, we used the fact that the wave-number sum could just as well be over $-\mathbf{k}$ instead of \mathbf{k} and that $\omega_{-\mathbf{k}} = -\omega_{\mathbf{k}}$, $\gamma_{-\mathbf{k}} = \gamma_{\mathbf{k}}$ (because $\varphi_{-\mathbf{k}} = \varphi_{\mathbf{k}}^*$, where $\varphi_{\mathbf{k}}$ is given by Eq. (7.5)). The matrix D is manifestly positive definite—this adds credence to our *a priori* expectation that a plateau will form: diffusion will smooth the bump in the equilibrium distribution function.

The question of validity of the QL approximation is quite nontrivial and rife with subtle issues, all of which I have swept under the carpet. They mostly have to do with whether coupling between waves [the last term in Eq. (2.12)] will truly remain unimportant throughout the quasilinear evolution, especially as the plateau regime is approached and the growth rate of the waves becomes infinitesimally small. If you wish to investigate further—and in the process gain a finer appreciation of nonlinear plasma theory,—the article by Besse *et al.* (2011) (as far as I know, the most recent substantial contribution to the topic) is a good starting point, from which you can follow the paper trail backwards in time and decide for yourself whether you trust the QLT.

7.2. Conservation Laws

When we get to the stage of solving a specific problem (§7.3), we shall see that paying attention to energy and momentum budgets leads one to important discoveries about the QL evolution of the particle distribution. With this prospect in mind, as well as by way of a consistency check, let us show that Eqs. (7.7–7.8) conserve energy and momentum.

7.2.1. Energy Conservation

The rate of change of the particle energy associated with the equilibrium distribution is

$$\begin{aligned}
\frac{dK}{dt} &\equiv \frac{d}{dt} \sum_{\alpha} \iint d^3\mathbf{r} d^3\mathbf{v} \frac{m_{\alpha}v^2}{2} f_{0\alpha} = \sum_{\alpha} \iint d^3\mathbf{r} d^3\mathbf{v} \frac{m_{\alpha}v^2}{2} \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{D}_{\alpha}(\mathbf{v}) \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{v}} \\
&= - \sum_{\alpha} \iint d^3\mathbf{r} d^3\mathbf{v} m_{\alpha} \mathbf{v} \cdot \mathbf{D}_{\alpha}(\mathbf{v}) \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{v}} \\
&= -V \sum_{\alpha} \frac{q_{\alpha}^2}{m_{\alpha}} \sum_{\mathbf{k}} \frac{|\mathbf{E}_{\mathbf{k}}|^2}{k^2} \int d^3\mathbf{v} \underbrace{\operatorname{Im} \frac{\mathbf{k} \cdot \mathbf{v}}{\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}} - i\gamma_{\mathbf{k}}}}_{\substack{\text{add and subtract} \\ \omega_{\mathbf{k}} + i\gamma_{\mathbf{k}} \text{ in the} \\ \text{numerator}}} \mathbf{k} \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{v}} \\
&= -V \sum_{\mathbf{k}} \frac{|\mathbf{E}_{\mathbf{k}}|^2}{4\pi} \operatorname{Im} \left[(\omega_{\mathbf{k}} + i\gamma_{\mathbf{k}}) \underbrace{\sum_{\alpha} \frac{\omega_{p\alpha}^2}{k^2} \frac{1}{n_{\alpha}} \int d^3\mathbf{v} \frac{1}{\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}} - i\gamma_{\mathbf{k}}}}_{= 1 - \epsilon(-i\omega_{\mathbf{k}} + \gamma_{\mathbf{k}}, \mathbf{k}) = 1} \mathbf{k} \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{v}} \right] \\
&\quad \text{because } -i\omega_{\mathbf{k}} + i\gamma_{\mathbf{k}} \text{ is a solution of} \\
&\quad \text{dispersion relation } \epsilon = 0 \\
&= -V \sum_{\mathbf{k}} 2\gamma_{\mathbf{k}} \frac{|\mathbf{E}_{\mathbf{k}}|^2}{8\pi} = -\frac{d}{dt} \int d^3\mathbf{r} \frac{E^2}{8\pi}, \quad \text{q.e.d.}, \tag{7.9}
\end{aligned}$$

viz., the total energy $K + \int d^3\mathbf{r} E^2/8\pi = \text{const.}$ This will motivate §7.6.

7.2.2. Momentum Conservation

Since unstable distributions like the one with a bump on its tail can carry net momentum, it is useful to calculate its rate of change:

$$\begin{aligned}
\frac{d\mathcal{P}}{dt} &\equiv \frac{d}{dt} \sum_{\alpha} \iint d^3\mathbf{r} d^3\mathbf{v} m_{\alpha} \mathbf{v} f_{0\alpha} = \sum_{\alpha} \iint d^3\mathbf{r} d^3\mathbf{v} m_{\alpha} \mathbf{v} \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{D}_{\alpha}(\mathbf{v}) \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{v}} \\
&= - \sum_{\alpha} \iint d^3\mathbf{r} d^3\mathbf{v} m_{\alpha} \mathbf{D}_{\alpha}(\mathbf{v}) \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{v}} \\
&= -V \sum_{\alpha} \frac{q_{\alpha}^2}{m_{\alpha}} \sum_{\mathbf{k}} \frac{\mathbf{k}|\mathbf{E}_{\mathbf{k}}|^2}{k^2} \int d^3\mathbf{v} \operatorname{Im} \frac{1}{\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}} - i\gamma_{\mathbf{k}}} \mathbf{k} \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{v}} \\
&= -V \sum_{\mathbf{k}} \frac{\mathbf{k}|\mathbf{E}_{\mathbf{k}}|^2}{4\pi} \operatorname{Im} \underbrace{\sum_{\alpha} \frac{\omega_{p\alpha}^2}{k^2} \frac{1}{n_{\alpha}} \int d^3\mathbf{v} \frac{1}{\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}} - i\gamma_{\mathbf{k}}}}_{= 1 - \epsilon(-i\omega_{\mathbf{k}} + \gamma_{\mathbf{k}}, \mathbf{k}) = 1} \mathbf{k} \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{v}} = 0, \quad \text{q.e.d.}, \tag{7.10}
\end{aligned}$$

so momentum can only be redistributed between particles. This will motivate §7.7.

7.3. Quasilinear Equations for the Bump-on-Tail Instability in 1D

What follows is the iconic QL calculation due to Vedenov *et al.* (1962) and Drummond & Pines (1962).

These two papers, published in the same year, are a spectacular example of the “great minds think alike” principle. They both appeared in the Proceedings of the 1961 IAEA conference in Salzburg, one of those early international gatherings in which the Soviets (grudgingly allowed out) and the Westerners were telling each other about their achievements in the recently declassified controlled-nuclear-fusion research. The entire Proceedings are now online (www.naweb.iaea.org/napc/physics/FEC/1961.pdf)—they are a remarkable historical document and a great read, containing, besides the papers (in three languages), a record of discussions that were held. The Vedenov *et al.* (1962) paper is in Russian, but you will find a very similar exposition in English in the review by Vedenov (1963) published shortly thereafter. Two other lucid

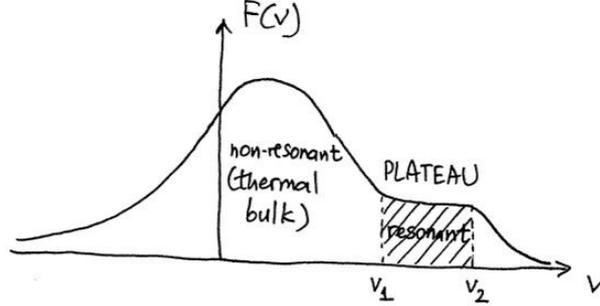


FIGURE 25. Quasilinear plateau.

accounts of quasilinear theory belonging to the same period are in the books by Kadomtsev (1965) and by Sagdeev & Galeev (1969).

As promised in §7.1, we shall consider electron Langmuir oscillations in 1D, triggered by the bump-on-tail instability, so $\mathbf{k} = k\hat{z}$, $\omega_k = \omega_{pe}$, γ_k is given by Eq. (7.1), and the QL diffusion equation (7.7) becomes

$$\boxed{\frac{\partial F}{\partial t} = \frac{\partial}{\partial v} D(v) \frac{\partial F}{\partial v}}, \quad (7.11)$$

where $F(v)$ is the 1D version of the distribution function, $v = v_z$ and the diffusion coefficient, now a scalar, is given by

$$\boxed{D(v) = \frac{e^2}{m_e^2} \sum_k |E_k|^2 \text{Im} \frac{1}{kv - \omega_{pe} - i\gamma_k}}. \quad (7.12)$$

As we explained when discussing Eq. (7.1), if the fluctuation field has reached a steady state, it must be the case that

$$\frac{\partial |E_k|^2}{\partial t} = 2\gamma_k |E_k|^2 = 0 \quad \Leftrightarrow \quad |E_k|^2 = 0 \quad \text{or} \quad \gamma_k = 0, \quad (7.13)$$

i.e., either there are no fluctuations or there is no growth (or damping) rate. The result is a non-zero spectrum of fluctuations in the interval $k \in [k_2, k_1]$ and a plateau in the distribution function, Eq. (7.3), in the corresponding velocity interval $v \in [v_1, v_2] = [\omega_{pe}/k_1, \omega_{pe}/k_2]$ (Fig. 25). The particles in this interval are *resonant* with Langmuir waves; those in the (“thermal”) bulk of the distribution outside this interval are *non-resonant*. We will have solved the problem completely if we find

- F^{plateau} , the value of the distribution function in the interval $[v_1, v_2]$,
- the extent of the plateau $[v_1, v_2]$,
- the functional form of the spectrum $|E_k|^2$ in the interval $[k_2, k_1]$,
- any modifications of the distribution function $F(v)$ of the nonresonant particles.

7.4. Resonant Region: QL Plateau and Spectrum

Consider first the velocities $v \in [v_1, v_2]$ for which $|E_{k=\omega_{pe}/v}|^2 \neq 0$. If L is the linear size of the system, the wave-number sum in Eq. (7.12) can be replaced by an integral

according to

$$\sum_k = \sum_k \frac{\Delta k}{2\pi/L} = \frac{L}{2\pi} \int dk. \quad (7.14)$$

Defining the continuous energy spectrum of the Langmuir waves[†]

$$W(k) = \frac{L}{2\pi} \frac{|E_k|^2}{4\pi}, \quad (7.15)$$

we rewrite the QL diffusion coefficient (7.12) in the following form

$$D(v) = \frac{e^2}{m_e^2 v} \frac{1}{v} \operatorname{Im} \int dk \frac{4\pi W(k)}{k - \omega_{pe}/v - i\gamma_k/v} = \frac{e^2}{m_e^2} \frac{4\pi^2}{v} W\left(\frac{\omega_{pe}}{v}\right). \quad (7.16)$$

The last expression is obtained by applying Plemelj's formula (3.25) to the wave-number integral taken in the limit $\gamma_k/v \rightarrow +0$.[‡] Substituting now this expression into Eq. (7.11) and using also Eq. (7.1) to express

$$\gamma_k = \frac{\pi \omega_{pe}^3}{2 k^2 n_e} F'\left(\frac{\omega_{pe}}{k}\right) \Rightarrow \frac{\partial F}{\partial v} = \left[\frac{2 k^2}{\pi \omega_{pe}^3} n_e \gamma_k \right]_{k=\omega_{pe}/v}, \quad (7.17)$$

we get

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial v} \frac{e^2}{m_e^2} \frac{4\pi^2}{v} W\left(\frac{\omega_{pe}}{v}\right) \left[\frac{2 k^2}{\pi \omega_{pe}^3} n_e \gamma_k \right]_{k=\omega_{pe}/v} = \frac{\partial}{\partial v} \frac{\omega_{pe}}{m_e v^3} \underbrace{2\gamma_{\omega_{pe}/v} W\left(\frac{\omega_{pe}}{v}\right)}_{= \partial W / \partial t}. \quad (7.18)$$

Rearranging, we arrive at

$$\frac{\partial}{\partial t} \left[F - \frac{\partial}{\partial v} \frac{\omega_{pe}}{m_e v^3} W\left(\frac{\omega_{pe}}{v}\right) \right] = 0. \quad (7.19)$$

Thus, during QL evolution, the expression in the square brackets stays constant in time. Since at $t = 0$, there are no waves, $W = 0$, we find

$$F(t, v) = F(0, v) + \frac{\partial}{\partial v} \frac{\omega_{pe}}{m_e v^3} W\left(t, \frac{\omega_{pe}}{v}\right) \rightarrow F^{\text{plateau}} \quad \text{as } t \rightarrow \infty. \quad (7.20)$$

In the saturated state ($t \rightarrow \infty$), $W(\omega_{pe}/v) = 0$ outside the interval $v \in [v_1, v_2]$. Therefore, Eq. (7.20) gives us two implicit equations for v_1 and v_2 :

$$\boxed{F(0, v_1) = F(0, v_2) = F^{\text{plateau}}} \quad (7.21)$$

and, after integration over velocities, also an equation for F^{plateau} .[¶]

$$\int_{v_1}^{v_2} dv [F^{\text{plateau}} - F(0, v)] = 0 \Rightarrow \boxed{F^{\text{plateau}} = \frac{1}{v_2 - v_1} \int_{v_1}^{v_2} dv F(0, v)}. \quad (7.22)$$

Finally, integrating Eq. (7.20) with respect to v and using the boundary condition

[†] Why the prefactor is $1/4\pi$, rather than $1/8\pi$, will become clear in §7.6.

[‡] In fact, the wave-number integral must be taken along the Landau contour (i.e., keeping the contour below the pole) regardless of the sign of γ_k : see Q8, where we work out the QL theory for Landau-damped, rather than growing, perturbations.

[¶] This is somewhat reminiscent of the ‘‘Maxwell construction’’ in thermodynamics of real gases: the plateau sits at such a level that the integral under it, i.e., the number of particles involved, stays the same as it was for the same velocities in the initial state; see Fig. 24.

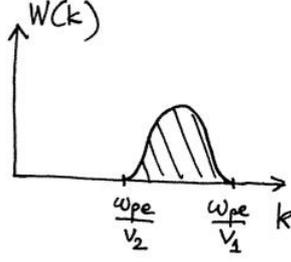


FIGURE 26. Quasilinear spectrum.

$W(\omega_{pe}/v_1) = 0$, we get, at $t \rightarrow \infty$,

$$W\left(\frac{\omega_{pe}}{v}\right) = \frac{m_e v^3}{\omega_{pe}} \int_{v_1}^v dv' [F^{\text{plateau}} - F(0, v')]. \quad (7.23)$$

Hence the spectrum is

$$\boxed{W(k) = \frac{m_e \omega_{pe}^2}{k^3} \int_{v_1}^{\omega_{pe}/k} dv [F^{\text{plateau}} - F(0, v)]} \quad \text{for } k \in \left[\frac{\omega_{pe}}{v_2}, \frac{\omega_{pe}}{v_1} \right] \quad (7.24)$$

and $W(k) = 0$ everywhere else (Fig. 26).

Thus, we have completed the first three items of the programme formulated at the end of §7.3. What about the particle distribution outside the resonant region? How is it modified by the quasilinear evolution? Is it modified at all? The following calculation shows that it must be.

7.5. Energy of Resonant Particles

Since feeding the instability requires extracting energy from the resonant particles, their energy must change. We calculate this change by taking the $m_e v^2/2$ moment of Eq. (7.20):

$$\begin{aligned} K_{\text{res}}(\infty) - K_{\text{res}}(0) &= \int_{v_1}^{v_2} dv \frac{m_e v^2}{2} [F^{\text{plateau}} - F(0, v)] \\ &= \int_{v_1}^{v_2} dv \frac{m_e v^2}{2} \frac{\partial}{\partial v} \frac{\omega_{pe}}{m_e v^3} W\left(\frac{\omega_{pe}}{v}\right) \\ &= -\omega_{pe} \int_{v_1}^{v_2} dv \frac{1}{v^2} W\left(\frac{\omega_{pe}}{v}\right) \\ &= - \int_{\omega_{pe}/v_2}^{\omega_{pe}/v_1} dk W(k) = -2 \sum_{\mathbf{k}} \frac{|E_{\mathbf{k}}|^2}{8\pi} \equiv -2 \mathcal{E}(\infty). \end{aligned} \quad (7.25)$$

Thus, only half of the energy lost by the resonant particles goes into the electric-field energy of the waves,

$$\boxed{\mathcal{E}(\infty) = \frac{K_{\text{res}}(0) - K_{\text{res}}(\infty)}{2}}. \quad (7.26)$$

Since the energy must be conserved overall [Eq. (7.9)], we must account for the missing half: this is easy to do physically, as, obviously, the electric energy of the waves is their potential energy, which is half of their total energy—the other half being the kinetic energy of the oscillatory plasma motions associated with the wave (Exercise 3.1 will help you make this explicit). These oscillations are enabled by the non-resonant, “thermal-bulk” particles, and so we must be able to show that, as a result of QL evolution, these

particles pick up the total of $\mathcal{E}(\infty)$ of energy—in formal terms, one might say that the plasma is *heated*.[†]

7.6. Heating of Non-Resonant Particles

Consider the thermal bulk of the distribution, $v \ll v_1$ (assuming that the bump is indeed far out in the tail of the distribution). The QL diffusion coefficient (7.12) becomes, assuming now $\gamma_k, kv \ll \omega_{pe}$ and using the last expression in Eq. (7.8),

$$\begin{aligned} D(v) &= \frac{e^2}{m_e^2} \sum_k |E_k|^2 \frac{\gamma_k}{(kv - \omega_{pe})^2 + \gamma_k^2} \approx \frac{e^2}{m_e^2} \sum_k |E_k|^2 \frac{\gamma_k}{\omega_{pe}^2} \\ &= \frac{e^2}{m_e^2 \omega_{pe}^2} \sum_k \frac{1}{2} \frac{\partial |E_k|^2}{\partial t} = \frac{4\pi e^2}{m_e^2 \omega_{pe}^2} \frac{d}{dt} \sum_k \frac{|E_k|^2}{8\pi} = \frac{1}{m_e n_e} \frac{d\mathcal{E}}{dt}, \end{aligned} \quad (7.27)$$

independent of v . The QL evolution equation (7.11) for the bulk distribution is then[‡]

$$\frac{\partial F}{\partial t} = \frac{1}{m_e n_e} \frac{d\mathcal{E}}{dt} \frac{\partial^2 F}{\partial v^2}. \quad (7.28)$$

Eq. (7.28) describes slow diffusion of the bulk distribution, i.e., as the wave field grows, the bulk distribution gets a little broader (which is what heating is). Namely, the “thermal” energy satisfies

$$\frac{dK_{\text{th}}}{dt} = \frac{d}{dt} \int dv \frac{m_e v^2}{2} F = \frac{1}{m_e n_e} \frac{d\mathcal{E}}{dt} \underbrace{\int dv \frac{m_e v^2}{2} \frac{\partial^2 F}{\partial v^2}}_{= m_e n_e \text{ (by parts twice)}} = \frac{d\mathcal{E}}{dt}. \quad (7.29)$$

Integrating this with respect to time, we find that the missing half of the energy lost by the resonant particles indeed goes into the heating of the thermal bulk:

$$\boxed{K_{\text{th}}(\infty) - K_{\text{th}}(0) = \mathcal{E}(\infty) = \frac{K_{\text{res}}(0) - K_{\text{res}}(\infty)}{2}}. \quad (7.30)$$

Overall, the energy is, of course, conserved:

$$K_{\text{th}}(\infty) + K_{\text{res}}(\infty) + \mathcal{E}(\infty) = K_{\text{th}}(0) + K_{\text{res}}(0), \quad (7.31)$$

as it should be, in accordance with Eq. (7.9).

Eq. (7.28) can be explicitly solved: changing the time variable to $\tau = \mathcal{E}(t)/m_e n_e$ turns it into a simple diffusion equation

$$\frac{\partial F}{\partial \tau} = \frac{\partial^2 F}{\partial v^2}. \quad (7.32)$$

[†] This is slightly loose language. Technically speaking, since there are no collisions, this is not really heating, i.e., the exact total entropy does not increase. The “thermal” energy that increases is the energy of plasma oscillations, which are mean “fluid” motions of the plasma, whereas “true” heating would involve an increase in the energy of particle motions around the mean.

[‡] Note that this implies $d \int dv F(v)/dt = 0$, so the number of these particles is conserved, there is no exchange between the non-resonant and resonant populations.

Letting the initial distribution be a Maxwellian and ignoring the bump on its tail, the solution is

$$\begin{aligned} F(\tau, v) &= \int dv' F(0, v') \frac{e^{-(v-v')^2/4\tau}}{\sqrt{4\pi\tau}} = \int dv' \frac{n_e}{\sqrt{\pi v_{\text{the}}^2 4\pi\tau}} \exp\left[-\frac{v'^2}{v_{\text{the}}^2} - \frac{(v-v')^2}{4\tau}\right] \\ &= \frac{n_e}{\sqrt{\pi(v_{\text{the}}^2 + 4\tau)}} \exp\left[-\frac{v^2}{v_{\text{the}}^2 + 4\tau}\right]. \end{aligned} \quad (7.33)$$

Since

$$v_{\text{the}}^2 + 4\tau = \frac{2T_e}{m_e} + \frac{4\mathcal{E}(t)}{m_e n_e} = \frac{2}{m_e} \left[T_e + \frac{2\mathcal{E}(t)}{n_e} \right], \quad (7.34)$$

one concludes that an initially Maxwellian bulk stays Maxwellian but its temperature grows as the wave energy grows, reaching in saturation

$$\boxed{T_e(\infty) = T_e(0) + \frac{2\mathcal{E}(\infty)}{n_e}}. \quad (7.35)$$

7.7. Momentum Conservation

The bump-on-tail configuration is in general asymmetric in v and so the particles in the bump carry a net mean momentum. Let us find out whether this momentum changes. Taking the $m_e v$ moment of Eq. (7.20), we calculate the total momentum lost by the resonant particles:

$$\begin{aligned} \mathcal{P}_{\text{res}}(\infty) - \mathcal{P}_{\text{res}}(0) &= \int_{v_1}^{v_2} dv m_e v [F^{\text{plateau}} - F(0, v)] \\ &= \int_{v_1}^{v_2} dv m_e v \frac{\partial}{\partial v} \frac{\omega_{\text{pe}}}{m_e v^3} W\left(\frac{\omega_{\text{pe}}}{v}\right) \\ &= -\omega_{\text{pe}} \int_{v_1}^{v_2} dv \frac{1}{v^3} W\left(\frac{\omega_{\text{pe}}}{v}\right) \\ &= - \int_{\omega_{\text{pe}}/v_2}^{\omega_{\text{pe}}/v_1} dk \frac{kW(k)}{\omega_{\text{pe}}} < 0. \end{aligned} \quad (7.36)$$

This is negative, so momentum is indeed lost. Since it cannot go into electric field [Eq. (7.10)], it must all get transferred to the thermal particles. Let us confirm this.

Going back to the QL diffusion equation for the non-resonant particles, Eq. (7.28), at first glance, we have a problem: the diffusion coefficient is independent of v and so momentum is conserved. However, one should never take zero for an answer when dealing with asymptotic expansions—indeed, it turns out here that we ought to work to higher order in our calculation of $D(v)$. Keeping next-order terms in Eq. (7.27), we get

$$\begin{aligned} D(v) &= \frac{e^2}{m_e^2} \sum_k |E_k|^2 \frac{\gamma_k}{(kv - \omega_{\text{pe}})^2 + \gamma_k^2} = \frac{e^2}{m_e^2} \sum_k |E_k|^2 \frac{\gamma_k}{\omega_{\text{pe}}^2} \left(1 + \frac{2kv}{\omega_{\text{pe}}} + \dots\right) \\ &\approx \frac{4\pi e^2}{m_e^2 \omega_{\text{pe}}^2} \frac{d}{dt} \left[\sum_k \frac{|E_k|^2}{8\pi} + v \sum_k \frac{k|E_k|^2}{4\pi\omega_{\text{pe}}} \right] = \frac{1}{m_e n_e} \frac{d}{dt} \left[\mathcal{E} + v \int dk \frac{kW(k)}{\omega_{\text{pe}}} \right]. \end{aligned} \quad (7.37)$$

Thus, there is a wave-induced drag term in the QL diffusion equation (7.11), which indeed turns out to impart to the thermal particles the small additional momentum that, according to Eq. (7.36), the resonant particles lose when rearranging themselves to

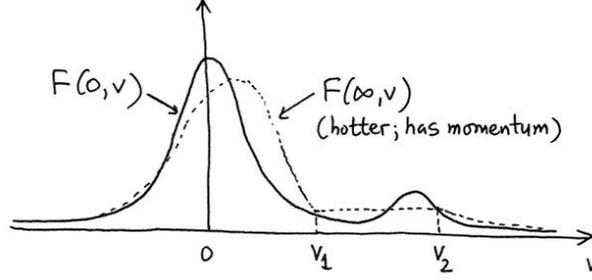


FIGURE 27. The initial distribution and the final outcome of the QL evolution: its bulk hotter and shifted towards the plateau in the tail.

produce the QL plateau:

$$\begin{aligned} \frac{d\mathcal{P}_{\text{th}}}{dt} &= \frac{d}{dt} \int dv m_e v F = \int dv m_e v \frac{\partial}{\partial v} D(v) \frac{\partial F}{\partial v} = -m_e \int dv D(v) \frac{\partial F}{\partial v} \\ &= - \left[\frac{d}{dt} \int dk \frac{kW(k)}{\omega_{pe}} \right] \frac{1}{n_e} \int dv v \frac{\partial F}{\partial v} = \frac{d}{dt} \int dk \frac{kW(k)}{\omega_{pe}}, \end{aligned} \quad (7.38)$$

whence, integrating and comparing with Eq. (7.36),

$$\boxed{\mathcal{P}_{\text{th}}(\infty) - \mathcal{P}_{\text{th}}(0) = \int dk \frac{kW(k)}{\omega_{pe}} = \mathcal{P}_{\text{res}}(0) - \mathcal{P}_{\text{res}}(\infty)}. \quad (7.39)$$

This means that *the thermal bulk of the final distribution is not only slightly broader (hotter) than that of the initial one (§7.6), but it is also slightly shifted towards the plateau (Fig. 27).*

In a collisionless plasma, this is the steady state. However, as this steady state is approached, $\gamma_k \rightarrow 0$, so the QL evolution becomes ever slower and even a very small collision frequency can become important. Eventually, collisions will erode the plateau and return the plasma to a global Maxwellian equilibrium—which is the fate of all things.

8. Kinetics of Quasiparticles

Let us reimagine our system of particles and waves as a mixture of two interacting gases: “true” particles (electrons) and *quasiparticles*, or *plasmons*, which will be the “quantised” version of Langmuir waves. If each of these plasmons has momentum $\hbar\mathbf{k}$ and energy $\hbar\omega_{\mathbf{k}}$, we can declare

$$N_{\mathbf{k}} = \frac{V|\mathbf{E}_{\mathbf{k}}|^2/4\pi}{\hbar\omega_{\mathbf{k}}} \quad (8.1)$$

to be the mean occupation number of plasmons with wave number \mathbf{k} (in a box of volume V). The total energy of the plasmons is then

$$\sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} N_{\mathbf{k}} = V \sum_{\mathbf{k}} \frac{|\mathbf{E}_{\mathbf{k}}|^2}{4\pi}, \quad (8.2)$$

twice the total electric energy in the system (twice because it includes the energy of the mean oscillatory motion of electrons within a wave; see discussion at the end of §7.5).



(a) absorption of a plasmon by an electron (b) emission of a plasmon by an electron

FIGURE 28. Diagrams for Eq. (8.6).

Similarly, the total momentum of the plasmons is

$$\sum_{\mathbf{k}} \hbar \mathbf{k} N_{\mathbf{k}} = V \sum_{\mathbf{k}} \frac{\mathbf{k} |\mathbf{E}_{\mathbf{k}}|^2}{4\pi\omega_{\mathbf{k}}}. \quad (8.3)$$

This is indeed in line with our previous calculations [Eq. (7.39)]. Note that the role of \hbar here is simply to define a splitting of wave energy into individual plasmons—this can be done in an arbitrary way, provided \hbar is small enough to ensure $N_{\mathbf{k}} \gg 1$. Since there is nothing quantum-mechanical about our system, all our results will in the end have to be independent of \hbar , so we will use \hbar as an arbitrarily small parameter, in which it will be convenient to expand, expecting it eventually to cancel out in all physically meaningful relationships.

8.1. QLT in the Language of Quasiparticles

We may now think of the QL evolution (or indeed generally of the nonlinear evolution) of our plasma in terms of interactions between plasmons and electrons. These are *resonant* electrons; the thermal bulk only participates via its supporting role of enabling oscillatory plasma motions associated with plasmons. The electrons are described by their distribution function $f_0(\mathbf{v})$, which we can, to make our formalism nicely uniform, recast in terms of occupation numbers: if the wave number corresponding to velocity \mathbf{v} is $\mathbf{p} = m_e \mathbf{v} / \hbar$, then its occupation number is

$$n_{\mathbf{p}} = \left(\frac{2\pi\hbar}{m_e} \right)^3 f_0(\mathbf{v}) \quad \Rightarrow \quad \sum_{\mathbf{p}} n_{\mathbf{p}} = \frac{V}{(2\pi)^3} \int d^3\mathbf{p} n_{\mathbf{p}} = V \int d^3\mathbf{v} f_0(\mathbf{v}) = V n_e. \quad (8.4)$$

It is understood that $n_{\mathbf{p}} \ll 1$ (our electron gas is non-degenerate).

The QL evolution of the plasmon and electron distributions is controlled by two processes: absorption or emission of a plasmon by an electron (known as Cherenkov absorption/emission). Diagrammatically, these can be depicted as shown in Fig. 28. As we know from §7.2, they are subject to momentum conservation, $\mathbf{p} = \mathbf{k} + (\mathbf{p} - \mathbf{k})$, and energy conservation:

$$\begin{aligned} 0 = \varepsilon_{\mathbf{p}}^e - \varepsilon_{\mathbf{k}}^l - \varepsilon_{\mathbf{p}-\mathbf{k}}^e &= \frac{\hbar^2 p^2}{2m_e} - \hbar\omega_{\mathbf{k}} - \frac{\hbar^2 |\mathbf{p} - \mathbf{k}|^2}{2m_e} = \hbar \left(-\omega_{\mathbf{k}} + \frac{\hbar \mathbf{p} \cdot \mathbf{k}}{m_e} - \frac{\hbar k^2}{2m_e} \right) \\ &= \hbar(\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}}) + O(\hbar^2). \end{aligned} \quad (8.5)$$

This is the familiar resonance condition $\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}} = 0$. The superscripts e and l stand for electrons and (Langmuir) plasmons.

We may now write an equation for the evolution of the plasmon occupation number:

$$\begin{aligned} \frac{\partial N_{\mathbf{k}}}{\partial t} = & - \underbrace{\sum_{\mathbf{p}} \mathfrak{w}(\mathbf{p} - \mathbf{k}, \mathbf{k} \rightarrow \mathbf{p}) \delta(\varepsilon_{\mathbf{p}-\mathbf{k}}^e + \varepsilon_{\mathbf{k}}^l - \varepsilon_{\mathbf{p}}^e) n_{\mathbf{p}-\mathbf{k}} N_{\mathbf{k}}}_{\text{Fig. 28(a)}} \\ & + \underbrace{\sum_{\mathbf{p}} \mathfrak{w}(\mathbf{p} \rightarrow \mathbf{k}, \mathbf{p} - \mathbf{k}) \delta(\varepsilon_{\mathbf{p}}^e - \varepsilon_{\mathbf{k}}^l - \varepsilon_{\mathbf{p}-\mathbf{k}}^e) n_{\mathbf{p}} (N_{\mathbf{k}} + 1)}_{\text{Fig. 28(b)}}, \end{aligned} \quad (8.6)$$

where \mathfrak{w} are the probabilities of absorption and emission and must be equal:

$$\mathfrak{w}(\mathbf{p} - \mathbf{k}, \mathbf{k} \rightarrow \mathbf{p}) = \mathfrak{w}(\mathbf{p} \rightarrow \mathbf{k}, \mathbf{p} - \mathbf{k}) \equiv w(\mathbf{p}, \mathbf{k}). \quad (8.7)$$

The first term in the right-hand side of Eq. (8.6) describes the absorption of one of (indistinguishable) $N_{\mathbf{k}}$ plasmons by one of $n_{\mathbf{p}-\mathbf{k}}$ electrons, the second term describes the emission by one of $n_{\mathbf{p}}$ electrons of one of $N_{\mathbf{k}} + 1$ plasmons. The $+1$ is, of course, a small correction to $N_{\mathbf{k}} \gg 1$ and can be neglected, although sometimes, in analogous but more complicated calculations, it has to be kept because lowest-order terms cancel. Using Eqs. (8.7), (8.5) and (8.4), we find

$$\begin{aligned} \frac{\partial N_{\mathbf{k}}}{\partial t} &= \sum_{\mathbf{p}} w(\mathbf{p}, \mathbf{k}) \delta(\varepsilon_{\mathbf{p}}^e - \varepsilon_{\mathbf{k}}^l - \varepsilon_{\mathbf{p}-\mathbf{k}}^e) (n_{\mathbf{p}} - n_{\mathbf{p}-\mathbf{k}}) N_{\mathbf{k}} \\ &= V \int d^3 \mathbf{v} w\left(\frac{m_e \mathbf{v}}{\hbar}, \mathbf{k}\right) \delta(\hbar(\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}})) \left[f_0(\mathbf{v}) - f_0\left(\mathbf{v} - \frac{\hbar \mathbf{k}}{m_e}\right) \right] N_{\mathbf{k}} \\ &= V \int d^3 \mathbf{v} w\left(\frac{m_e \mathbf{v}}{\hbar}, \mathbf{k}\right) \frac{1}{\hbar} \delta(\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}}) \frac{\hbar}{m_e} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} N_{\mathbf{k}} \\ &= \underbrace{\frac{V}{m_e} w\left(\frac{m_e \omega_{pe}}{\hbar k}, k\right) F'\left(\frac{\omega_{pe}}{k}\right)}_{= 2\gamma_{\mathbf{k}}} N_{\mathbf{k}}. \end{aligned} \quad (8.8)$$

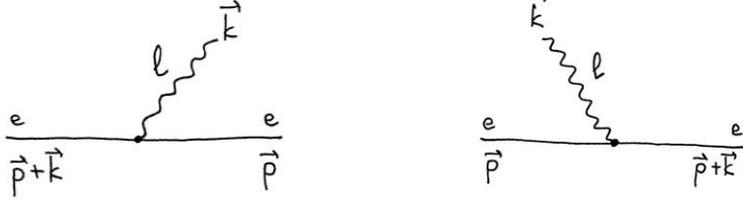
Note that \hbar has disappeared from our equations, after being used as an expansion parameter.

Since $N_{\mathbf{k}} \propto |\mathbf{E}_{\mathbf{k}}|^2$ [Eq. (8.1)], the prefactor in Eq. (8.8) is clearly just the (twice) growth or damping rate of the waves. Comparing with Eq. (7.1), we read off the expression for the absorption/emission probability:

$$w\left(\frac{m_e \omega_{pe}}{\hbar k}, k\right) = \frac{\pi m_e \omega_{pe}^3}{V n_e k^2}. \quad (8.9)$$

Thus, our calculation of Landau damping could be thought of as a calculation of this probability. Whether there is damping or an instability is decided by whether it is absorption or emission of plasmons that occurs more frequently—and that depends on whether, for any given \mathbf{k} , there are more electrons that are slightly slower or slightly faster than the plasmons with wave number \mathbf{k} . Note that getting the correct sign of the damping rate is automatic in this approach, since the probability w must obviously be positive.

The evolution equation for the occupation number of electrons can be derived in a similar fashion, if we itemise the processes that lead to an electron ending up in a state with a given wave number $\mathbf{p} = m_e \mathbf{v} / \hbar$ or moving from this state to one with a different wave number. The four relevant diagrams are the two in Fig. 28 and the additional two shown in Fig. 29. The absorption and emission probabilities are the same as before and



(a) emission of a plasmon by an electron (b) absorption of a plasmon by an electron

FIGURE 29. Additional diagrams for Eq. (8.10).

so are the energy conservation conditions. We have

$$\begin{aligned}
\frac{\partial n_{\mathbf{p}}}{\partial t} &= \underbrace{\sum_{\mathbf{k}} w(\mathbf{p} - \mathbf{k}, \mathbf{k} \rightarrow \mathbf{p}) \delta(\varepsilon_{\mathbf{p}-\mathbf{k}}^e + \varepsilon_{\mathbf{k}}^l - \varepsilon_{\mathbf{p}}^e) n_{\mathbf{p}-\mathbf{k}} N_{\mathbf{k}}}_{\text{Fig. 28(a)}} \\
&+ \underbrace{\sum_{\mathbf{k}} w(\mathbf{p} + \mathbf{k} \rightarrow \mathbf{k}, \mathbf{p}) \delta(\varepsilon_{\mathbf{p}+\mathbf{k}}^e - \varepsilon_{\mathbf{k}}^l - \varepsilon_{\mathbf{p}}^e) n_{\mathbf{p}+\mathbf{k}} (N_{\mathbf{k}} + 1)}_{\text{Fig. 29(a)}} \\
&- \underbrace{\sum_{\mathbf{k}} w(\mathbf{p}, \mathbf{k} \rightarrow \mathbf{p} + \mathbf{k}) \delta(\varepsilon_{\mathbf{p}}^e + \varepsilon_{\mathbf{k}}^l - \varepsilon_{\mathbf{p}+\mathbf{k}}^e) n_{\mathbf{p}} N_{\mathbf{k}}}_{\text{Fig. 29(b)}} \\
&- \underbrace{\sum_{\mathbf{k}} w(\mathbf{p} \rightarrow \mathbf{k}, \mathbf{p} - \mathbf{k}) \delta(\varepsilon_{\mathbf{p}}^e - \varepsilon_{\mathbf{k}}^l - \varepsilon_{\mathbf{p}-\mathbf{k}}^e) n_{\mathbf{p}} (N_{\mathbf{k}} + 1)}_{\text{Fig. 28(b)}} \\
&\approx \sum_{\mathbf{k}} w(\mathbf{p} + \mathbf{k}, \mathbf{k}) \delta(\varepsilon_{\mathbf{p}+\mathbf{k}}^e - \varepsilon_{\mathbf{k}}^l - \varepsilon_{\mathbf{p}}^e) (n_{\mathbf{p}+\mathbf{k}} - n_{\mathbf{p}}) N_{\mathbf{k}} \\
&- \sum_{\mathbf{k}} w(\mathbf{p}, \mathbf{k}) \delta(\varepsilon_{\mathbf{p}}^e - \varepsilon_{\mathbf{k}}^l - \varepsilon_{\mathbf{p}-\mathbf{k}}^e) (n_{\mathbf{p}} - n_{\mathbf{p}-\mathbf{k}}) N_{\mathbf{k}} \\
&\approx \sum_{\mathbf{k}} \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{p}} w(\mathbf{p}, \mathbf{k}) \delta(\varepsilon_{\mathbf{p}}^e - \varepsilon_{\mathbf{k}}^l - \varepsilon_{\mathbf{p}-\mathbf{k}}^e) \mathbf{k} \cdot \frac{\partial n_{\mathbf{p}}}{\partial \mathbf{p}} N_{\mathbf{k}}, \tag{8.10}
\end{aligned}$$

where we have expanded twice in small \mathbf{k} (i.e., in \hbar). This is a diffusion equation in \mathbf{p} (or, equivalently, $\mathbf{v} = \hbar \mathbf{p} / m_e$) space. In view of Eq. (8.4), Eq. (8.10) has the same form as Eq. (7.7), viz.,

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{D}(\mathbf{v}) \cdot \frac{\partial f_0}{\partial \mathbf{v}}, \tag{8.11}$$

where the diffusion matrix is

$$\mathbf{D}(\mathbf{v}) = \sum_{\mathbf{k}} \mathbf{k} \mathbf{k} \frac{\hbar N_{\mathbf{k}}}{m_e^2} w\left(\frac{m_e \mathbf{v}}{\hbar}, \mathbf{k}\right) \delta(\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}}) = \frac{e^2}{m_e^2} \sum_{\mathbf{k}} \frac{\mathbf{k} \mathbf{k}}{k^2} |\mathbf{E}_{\mathbf{k}}|^2 \pi \delta(\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}}). \tag{8.12}$$

The last expression is identical to the resonant form of the QL diffusion matrix (7.8) [cf. Eqs. (7.16) and (10.45)]. To derive it, we used the definition (8.1) of $N_{\mathbf{k}}$ and the absorption/emission probability (8.7), already known from linear theory.

Thus, we are able to recover the (resonant part of the) QL theory from our new electron-plasmon interaction approach. There is more to this approach than a pretty “field-theoretic” reformulation of already-derived earlier results. The diagram technique

and the interpretation of the nonlinear state of the plasma as arising from interactions between particles and quasiparticles can be readily generalised to situations in which the nonlinear interactions in Eq. (2.12) cannot be neglected and/or more than one type of waves is present. In this new language, the nonlinear interactions would be manifested as interactions between plasmons (rather than only between plasmons and electrons) contributing to the rate of change of $N_{\mathbf{k}}$. There are many possibilities: four-plasmon interactions, interactions between plasmons and phonons (sound waves), as well as between the latter and electrons and/or ions, etc. Some of these will be further explored in §8.2 and onwards. A comprehensive monograph on this subject is Tsytovich (1995) (see also Kingsep 2004).

This is a good place to stop these lectures, although it is not, of course, the end of plasma kinetics: weak and strong turbulence theory, magnetised plasma waves, “drift” and “gyro-” kinetics—there are vast expanses of interesting physics and interesting theory to explore beyond this basic introduction. Some of these topics are covered by Parra (2017*b*) and others call for further reading (e.g., on turbulence: Kadomtsev 1965; Sagdeev & Galeev 1969; Tsytovich 1995; Krommes 2015; on gyrokinetics: Howes *et al.* 2006; Abel *et al.* 2013).

8.2. *Weak Turbulence in the Language of Quasiparticles*

Work in progress. See books by Zakharov *et al.* (1992); Tsytovich (1995); Kingsep (2004); Nazarenko (2011).

8.3. *General Scheme for Calculating Probabilities in WT*

9. Langmuir Turbulence

Coming soon. See Zakharov (1972); Kingsep (2004).

9.1. *Zakharov’s Equations*

9.2. *Secondary Instability of a Langmuir Wave*

9.3. *Weak Langmuir Turbulence*

9.4. *Solitons*

9.5. *Langmuir Collapse*

9.6. *Kingsep–Rudakov–Sudan Turbulence*

See Kingsep *et al.* (1973).

9.7. *Pelletier’s Equilibrium Ensemble*

See Pelletier (1980).

9.8. *Theories Galore*

10. Stochastic Echo and Phase-Space Turbulence

Plasma Kinetics Problem Set

1. Industrialised linear theory with the Z function. Consider a two-species plasma close to Maxwellian equilibrium. Rederive all the results obtained in §§3.4, 3.5, 3.8, 3.9, 3.10 (including the exercises) starting from Eq. (3.80) and using the asymptotic expansions (3.87) and (3.88) of the plasma dispersion function.

Namely, consider the limits $\zeta_e \gg 1$ or $\zeta_e \ll 1$ and $\zeta_i \gg 1$, find solutions in these limits and establish the conditions on the wave number of the perturbations and on the equilibrium parameters under which these solutions are valid.

In particular, for the case of $\zeta_e \ll 1$ and $\zeta_i \gg 1$, obtain general expressions for the wave frequency and damping without assuming $k\lambda_{De}$ to be either small or large. Recover from your solution the cases considered in §§3.8–3.9 and §3.10.

Find also the ion contribution to the damping of the ion acoustic and Langmuir waves and comment on the circumstances in which it might be important to know what it is.

Convince yourself that you believe the sketch of longitudinal plasma waves in Fig. 13. If you feel computationally inclined, solve the plasma dispersion relation (3.80) numerically [using, e.g., Eq. (3.86)] and see if you can reproduce Fig. 13.

You may wish to check your results against some textbook: e.g., Krall & Trivelpiece (1973) and Alexandrov *et al.* (1984) give very thorough treatments of the linear theory (although in rather different styles than I did).

2. Transverse plasma waves. Go back to the Vlasov–Maxwell, rather than Vlasov–Poisson, system and consider *electromagnetic* perturbations in a Maxwellian unmagnetised plasma (unmagnetised in the sense that in equilibrium, $\mathbf{B}_0 = 0$):

$$\frac{\partial \delta f_\alpha}{\partial t} + i\mathbf{k} \cdot \mathbf{v} \delta f_\alpha + \frac{q_\alpha}{m_\alpha} \left(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right) \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{v}} = 0, \quad (10.1)$$

where \mathbf{E} and \mathbf{B} satisfy Maxwell’s equations (1.23–1.26) with charge and current densities determined by the perturbed distribution function δf_α .

(a) Consider an initial-value problem for such perturbations and show that the equation for the Laplace transform of \mathbf{E} can be written in the form†

$$\hat{\epsilon}(p, \mathbf{k}) \cdot \hat{\mathbf{E}}(p) = \left(\begin{array}{l} \text{terms associated with initial} \\ \text{perturbations of } \delta f_\alpha, \mathbf{E} \text{ and } \mathbf{B} \end{array} \right), \quad (10.2)$$

where the *dielectric tensor* $\hat{\epsilon}(p, \mathbf{k})$ is, in tensor notation,

$$\epsilon_{ij}(p, \mathbf{k}) = \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) \epsilon_{\text{TT}}(p, k) + \frac{k_i k_j}{k^2} \epsilon_{\text{LL}}(p, k) \quad (10.3)$$

and the *longitudinal dielectric function* $\epsilon_{\text{LL}}(p, k)$ is the familiar electrostatic one, given by Eq. (3.80), while the *transverse dielectric function* is

$$\boxed{\epsilon_{\text{TT}}(p, k) = 1 + \frac{1}{p^2} \left[k^2 c^2 - \sum_\alpha \omega_{p\alpha}^2 \zeta_\alpha Z(\zeta_\alpha) \right]}. \quad (10.4)$$

† In Q3, dealing with the Weibel instability, you will have to do essentially the same calculation, but with a non-Maxwellian equilibrium. To avoid doing the work twice, you could do that question first and then specialise to a Maxwellian $f_{0\alpha}$. However, the algebra is a bit hairier for the non-Maxwellian case, so it may be useful to do the simpler case first, to train your hand—and also to have a way to cross-check the later calculation.

(b) Hence solve the transverse dispersion relation, $\epsilon_{\text{TT}}(p, k) = 0$, and show that, in the high-frequency limit ($|\zeta_e| \gg 1$), the resulting waves are simply the light waves, which, at long wave lengths, turn into plasma oscillations. What is the wave length above which light can “feel” that it is propagating through plasma?—this is called the *plasma (electron) skin depth*, d_e . Are these waves damped?

(c) In the low-frequency limit ($|\zeta_e| \ll 1$), show that perturbations are aperiodic (have zero frequency) and damped. Find their damping rate and show that this result is valid for perturbations with wave lengths longer than the plasma skin depth ($kd_e \ll 1$). Explain physically why these perturbations fail to propagate.

3. Weibel instability. Weibel (1958) realised that transverse plasma perturbations can go unstable if the equilibrium distribution is anisotropic with respect to some special direction $\hat{\mathbf{n}}$, namely if $f_{0\alpha} = f_{0\alpha}(v_{\perp}, v_{\parallel})$, where $v_{\parallel} = \mathbf{v} \cdot \hat{\mathbf{n}}$, $v_{\perp} = |\mathbf{v}_{\perp}|$ and $\mathbf{v}_{\perp} = \mathbf{v} - v_{\parallel}\hat{\mathbf{n}}$. The anisotropy can be due to some beam or stream of particles injected into the plasma, it also arises in collisionless shocks or, generically, when plasma is sheared or non-isotropically compressed by some external force. The simplest model for an anisotropic distribution of the required type is a *bi-Maxwellian*:†

$$f_{0\alpha} = \frac{n_{\alpha}}{\pi^{3/2} v_{\text{th}\perp\alpha}^2 v_{\text{th}\parallel\alpha}} \exp\left(-\frac{v_{\perp}^2}{v_{\text{th}\perp\alpha}^2} - \frac{v_{\parallel}^2}{v_{\text{th}\parallel\alpha}^2}\right), \quad (10.5)$$

where, formally, $v_{\text{th}\perp\alpha} = \sqrt{2T_{\perp\alpha}/m_{\alpha}}$ and $v_{\text{th}\parallel\alpha} = \sqrt{2T_{\parallel\alpha}/m_{\alpha}}$ are the two “thermal speeds” in a plasma characterised by two effective temperatures $T_{\perp\alpha}$ and $T_{\parallel\alpha}$ (for each species).

(a) Using exactly the same method as in Q2, consider electromagnetic perturbations in a bi-Maxwellian plasma, assuming their wave vectors to be parallel to the direction of anisotropy, $\mathbf{k} \parallel \hat{\mathbf{n}}$. Show that the dielectric tensor again has the form (10.3) and the longitudinal dielectric function is again given by Eq. (3.80), while the transverse dielectric function is

$$\epsilon_{\text{TT}}(p, k) = 1 + \frac{1}{p^2} \left[k^2 c^2 + \sum_{\alpha} \omega_{\text{p}\alpha}^2 \left(1 - \frac{T_{\perp\alpha}}{T_{\parallel\alpha}} [1 + \zeta_{\alpha} Z(\zeta_{\alpha})] \right) \right]. \quad (10.6)$$

(b) Show that in one of the tractable asymptotic limits, this dispersion relation has a zero-frequency, purely growing solution with the growth rate

$$\gamma = \frac{k v_{\text{th}\parallel e} T_{\parallel e}}{\sqrt{\pi} T_{\perp e}} (\Delta_e - k^2 d_e^2), \quad (10.7)$$

where $\Delta_e = T_{\perp e}/T_{\parallel e} - 1$ is the fractional temperature anisotropy, which must be positive in order for the instability to occur. Find the maximum growth rate and the corresponding wave number. Under what condition(s) is the asymptotic limit in which you worked indeed a valid approximation for this solution?

(c) Are there any other unstable solutions? (cf. Weibel 1958)

(d) What happens if the electrons are isotropic but ions are not?

† In Q4, you will need the dielectric tensor in terms of a general equilibrium distribution $f_{0\alpha}(v_x, v_y, v_z)$. If you are planning to do that question, it may save time (at the price of a very slight increase in algebra) to do the derivation with a general $f_{0\alpha}$ and then specialise to the bi-Maxwellian (10.5). You can check your algebra by looking up the result in Krall & Trivelpiece (1973) or in Davidson (1983).

(e**) If you want a challenge and a test of stamina, work out the case of perturbations whose wave number is *not* necessarily in the direction of the anisotropy ($\mathbf{k} \times \hat{\mathbf{n}} \neq 0$). Are the $\mathbf{k} \parallel \hat{\mathbf{n}}$ or some oblique perturbations the fastest growing? This is a lot of algebra, so only do it if you enjoy this sort of thing. The dispersion relation for this general case appears to be in the Appendix of Ruyer *et al.* (2015), but they only solve it numerically; no one seems to have looked at asymptotic limits. This could be the start of a dissertation.

4* Criterion of instability of anisotropic distributions.

Coming soon. See Krall & Trivelpiece (1973, §9.10).

5. Free energy and stability. (a) Starting from the linearised Vlasov–Poisson system and assuming a Maxwellian equilibrium, show by direct calculation from the equations, rather than via expansion of the entropy function and the use of energy conservation (as was done in §4.2), that free energy is conserved:

$$\frac{d}{dt} \int d^3\mathbf{r} \left[\sum_{\alpha} \int d^3\mathbf{v} \frac{T_{\alpha} \delta f_{\alpha}^2}{2f_{0\alpha}} + \frac{|\nabla\varphi|^2}{8\pi} \right] = 0. \quad (10.8)$$

This is an exercise in integrating by parts.

(b) Now consider the full Vlasov–Maxwell equations and prove, again for a Maxwellian plasma plus small perturbations,

$$\boxed{\frac{d}{dt} \int d^3\mathbf{r} \left[\sum_{\alpha} \int d^3\mathbf{v} \frac{T_{\alpha} \delta f_{\alpha}^2}{2f_{0\alpha}} + \frac{|\mathbf{E}|^2 + |\mathbf{B}|^2}{8\pi} \right]} = 0. \quad (10.9)$$

(c) Consider the same problem, this time with an equilibrium that is not Maxwellian, but merely isotropic, i.e., $f_{0\alpha} = f_{0\alpha}(v)$, or, in what will prove to be a more convenient form,

$$f_{0\alpha} = f_{0\alpha}(\varepsilon_{\alpha}), \quad (10.10)$$

where $\varepsilon_{\alpha} = m_{\alpha}v^2/2$ is the particle energy. Find an integral quantity quadratic in perturbed fields and distributions that is conserved by the Vlasov–Maxwell system under these circumstances and that turns into the free energy (10.9) in the case of a Maxwellian equilibrium (if in difficulty, you will find the answer in, e.g., Davidson 1983 or in Kruskal & Oberman 1958, which appears to be the original source). Argue that

$$\frac{\partial f_{0\alpha}}{\partial \varepsilon_{\alpha}} < 0 \quad (10.11)$$

is a sufficient condition for stability of small ($\delta f_{\alpha} \ll f_{0\alpha}$, but not necessarily infinitesimal) perturbations in such a plasma.

6* Fluctuation-dissipation relation for a collisionless plasma. Let us consider a linear kinetic system in which perturbations are stirred up by an external force, which we can think of as an imposed (time-dependent) electric field $\mathbf{E}_{\text{ext}} = -\nabla\chi$. The perturbed distribution function then satisfies

$$\frac{\partial \delta f_{\alpha}}{\partial t} + \mathbf{v} \cdot \nabla \delta f_{\alpha} - \frac{q_{\alpha}}{m_{\alpha}} (\nabla \varphi_{\text{tot}}) \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{v}} = 0, \quad (10.12)$$

where $\varphi_{\text{tot}} = \varphi + \chi$ is the total potential, whose self-consistent part, φ , obeys the usual Poisson equation

$$-\nabla^2 \varphi = 4\pi \sum_{\alpha} q_{\alpha} \int d^3\mathbf{v} \delta f_{\alpha} \quad (10.13)$$

and the equilibrium $f_{0\alpha}$ is assumed to be Maxwellian.

(a) By considering an initial-value problem for Eqs. (10.12) and (10.13) with zero initial perturbation, show that the Laplace transforms of φ_{tot} and χ are related by

$$\hat{\varphi}_{\text{tot}}(p) = \frac{\hat{\chi}(p)}{\epsilon(p)}, \quad (10.14)$$

where $\epsilon(p)$ is the dielectric function given by (3.80).

(b) Consider a time-periodic external force,

$$\chi(t) = \chi_0 e^{-i\omega_0 t}. \quad (10.15)$$

Working out the relevant Laplace transforms and their inverses [see Eq. (3.14)], show that, after transients have decayed, the total electric field in the system will oscillate at the same frequency as the external force and be given by

$$\varphi_{\text{tot}}(t) = \frac{\chi_0 e^{-i\omega_0 t}}{\epsilon(-i\omega_0)}. \quad (10.16)$$

(c) Now consider the *plasma-kinetic Langevin problem*: assume the external force to be a white noise, i.e., a random process with the time-correlation function

$$\langle \chi(t)\chi^*(t') \rangle = 2D \delta(t - t'). \quad (10.17)$$

Show that the resulting steady-state mean-square fluctuation level in the plasma will be

$$\langle |\varphi_{\text{tot}}(t)|^2 \rangle = \frac{D}{\pi} \int_{-\infty}^{+\infty} \frac{d\omega}{|\epsilon(-i\omega)|^2}. \quad (10.18)$$

This is a kinetic *fluctuation-dissipation relation*: given a certain level of external stirring, parametrised by D , this formula predicts the fluctuation energy in terms of D and of the internal dissipative properties of the plasma, encoded by its dielectric function.

(d) For a system in which the Landau damping is weak, $|\gamma| \ll kv_{\text{th}\alpha}$, calculate the integral (10.18) using Plemelj's formula (3.25) to show that

$$\langle |\varphi_{\text{tot}}(t)|^2 \rangle = D \sum_i \frac{1}{|\gamma_i|} \left[\frac{\partial \text{Re} \epsilon(-i\omega)}{\partial \omega} \right]_{\omega=\omega_i}^{-2}, \quad (10.19)$$

where $p_i = -i\omega_i + \gamma_i$ are the weak-damping roots of the dispersion relation.

Here is a reminder of how the *standard Langevin problem* can be solved using Laplace transforms. The Langevin equation is

$$\left[\frac{\partial \varphi}{\partial t} + \gamma \varphi = \chi(t) \right], \quad (10.20)$$

where φ describes some quantity, e.g., the velocity of a Brownian particle, subject to a damping rate γ and an external force χ . In the case of a Brownian particle, χ is assumed to be a white noise, as per Eq. (10.17). Assuming $\varphi(t=0) = 0$, the Laplace-transform solution of Eq. (10.20) is

$$\hat{\varphi}(p) = \frac{\hat{\chi}(p)}{p + \gamma}. \quad (10.21)$$

Considering first a non-random oscillatory force (10.15), we have

$$\hat{\chi}(p) = \int_0^\infty dt e^{-pt} \chi(t) = \frac{\chi_0}{p + i\omega_0} \Rightarrow \hat{\varphi}(p) = \frac{\chi_0}{(p + \gamma)(p + i\omega_0)}. \quad (10.22)$$

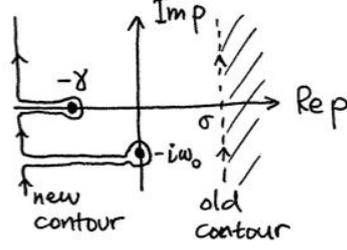


FIGURE 30. Shifting the integration contour in Eq. (10.23).

The inverse Laplace transform of $\hat{\varphi}$ is calculated by shifting the integration contour to large negative $\text{Re } p$ while not allowing it to cross the two poles, $p = -\gamma$ and $p = -i\omega_0$, in a manner analogous to that explained in §3.1 (Fig. 4) and shown in Fig. 30. The integral is then dominated by the contributions from the poles:

$$\varphi(t) = \frac{1}{2\pi i} \int_{-i\infty+\sigma}^{i\infty+\sigma} dp e^{pt} \hat{\varphi}(p) = \chi_0 \left(\frac{e^{-i\omega_0 t}}{-i\omega_0 + \gamma} + \frac{e^{-\gamma t}}{-\gamma + i\omega_0} \right) \rightarrow \frac{\chi_0 e^{-i\omega_0 t}}{-i\omega_0 + \gamma} \quad \text{as } t \rightarrow \infty, \quad (10.23)$$

which is quite obviously the right solution of Eq. (10.20) with a periodic force (the second term in the brackets is the decaying transient needed to enforce the zero initial condition).

In the more complicated case of a white-noise force [Eq. (10.17)],

$$\begin{aligned} \langle |\varphi(t)|^2 \rangle &= \frac{1}{(2\pi)^2} \left\langle \left| \int_{-i\infty+\sigma}^{i\infty+\sigma} dp e^{pt} \frac{\hat{\chi}(p)}{p + \gamma} \right|^2 \right\rangle \\ &= \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} d\omega' e^{[-i(\omega - \omega') + 2\sigma]t} \frac{\langle \hat{\chi}(-i\omega + \sigma) \hat{\chi}^*(-i\omega' + \sigma) \rangle}{(-i\omega + \sigma + \gamma)(i\omega' + \sigma + \gamma)}, \end{aligned} \quad (10.24)$$

where we have changed variables $p = -i\omega + \sigma$ and similarly for the second integral. The correlation function of the Laplace-transformed force is, using Eq. (10.17),

$$\langle \hat{\chi}(p) \hat{\chi}^*(p') \rangle = \int_0^\infty dt \int_0^\infty dt' e^{-pt - p'^* t'} \langle \chi(t) \chi^*(t') \rangle = 2D \int_0^\infty dt e^{-(p+p'^*)t} = \frac{2D}{p + p'^*}, \quad (10.25)$$

provided $\text{Re } p > 0$ and $\text{Re } p' > 0$. Then Eq. (10.24) becomes

$$\begin{aligned} \langle |\varphi(t)|^2 \rangle &= \frac{D}{2\pi^2} \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} d\omega' \frac{e^{[-i(\omega - \omega') + 2\sigma]t}}{[-i(\omega - \omega') + 2\sigma](-i\omega + \sigma + \gamma)(i\omega' + \sigma + \gamma)} \\ &= \frac{D}{\pi} \int_{-\infty}^{+\infty} d\omega' \frac{e^{(i\omega' + \sigma)t}}{(i\omega' + \sigma + \gamma)} \frac{1}{2\pi i} \int_{-i\infty+\sigma}^{i\infty+\sigma} dp \frac{e^{pt}}{(p + i\omega' + \sigma)(p + \gamma)} \\ &= \frac{D}{\pi} \int_{-\infty}^{+\infty} d\omega' \frac{e^{(i\omega' + \sigma)t}}{(i\omega' + \sigma + \gamma)} \left[\frac{e^{-(i\omega' + \sigma)t}}{-i\omega' - \sigma + \gamma} + \frac{e^{-\gamma t}}{\gamma + i\omega' + \sigma} \right], \end{aligned} \quad (10.26)$$

where we have reverted to the p variable in one of the integrals and then performed the integration by the same manipulation of the contour as in Eq. (10.23). We now note that, since there are no exponentially growing solutions in this system, $\sigma > 0$ can be chosen arbitrarily small. Taking $\sigma \rightarrow +0$ and neglecting the decaying transient in Eq. (10.26), we get, in the limit $t \rightarrow \infty$,

$$\langle |\varphi(t)|^2 \rangle = \frac{D}{\pi} \int_{-\infty}^{+\infty} \frac{d\omega'}{|-i\omega' + \gamma|^2} = \frac{D}{\pi} \int_{-\infty}^{+\infty} \frac{d\omega'}{\omega'^2 + \gamma^2} = \frac{D}{\gamma}. \quad (10.27)$$

Note that, while the integral in Eq. (10.27) is doable exactly, it can, for the case of weak damping, also be computed via Plemelj's formula.

Eq. (10.27) is the standard Langevin fluctuation-dissipation relation. It can also be obtained without Laplace transforms, either by directly integrating Eq. (10.20) and correlating $\varphi(t)$ with

itself or by noticing that

$$\frac{\partial}{\partial t} \frac{\langle \varphi^2 \rangle}{2} + \gamma \langle \varphi^2 \rangle = \langle \chi(t) \varphi(t) \rangle = \left\langle \chi(t) \int_0^t dt' [-\gamma \varphi(t') + \chi(t')] \right\rangle = D, \quad (10.28)$$

where we have used Eq. (10.17) and the fact that $\langle \chi(t) \varphi(t') \rangle = 0$ for $t' \leq t$, by causality. Eq. (10.27) is the steady-state solution to the above, but Eq. (10.28) also teaches us that, if we interpret $\langle \varphi^2 \rangle / 2$ as energy, D is the power that is injected into the system by the external force. Thus, fluctuation-dissipation relations such as Eq. (10.27) tells us what fluctuation *energy* will persist in a dissipative system if a certain amount of *power* is pumped in.

7* Phase-mixing spectrum. Here we study the velocity-space structure of the perturbed distribution function δf derived in Q6.

In order to do this, we need to review the *Hermite transform*:

$$\delta f_m = \frac{1}{n} \int dv_z \frac{H_m(u) \delta f(v_z)}{\sqrt{2^m m!}}, \quad u = \frac{v_z}{v_{\text{th}}}, \quad H_m(u) = (-1)^m e^{u^2} \frac{d^m}{du^m} e^{-u^2}, \quad (10.29)$$

where H_m is the Hermite polynomial of (integer) order m . We are only concerned with the v_z dependence of δf (where z , as always, is along the wave number of the perturbations—in this case set by the wave number of the force); all v_x and v_y dependence is Maxwellian and can be integrated out. The inverse transform is given by

$$\delta f(v_z) = \sum_{m=0}^{\infty} \frac{H_m(u) F(v_z)}{\sqrt{2^m m!}} \delta f_m, \quad F(v_z) = \frac{n}{\sqrt{\pi} v_{\text{th}}} e^{-u^2}. \quad (10.30)$$

Because $H_m(u)$ are orthogonal polynomials, viz.,

$$\frac{1}{n} \int dv_z H_m(u) H_{m'}(u) F(v_z) = 2^m m! \delta_{mm'}, \quad (10.31)$$

they have a Parseval theorem and so the contribution of the perturbed distribution function to the free energy [Eq. (4.18)] can be written as

$$\int d^3 \mathbf{v} \frac{T |\delta f|^2}{2 f_0} = \frac{nT}{2} \sum_m |\delta f_m|^2. \quad (10.32)$$

In a plasma where perturbations are constantly stirred up by a force, Landau damping must be operating all the time, removing energy from φ to provide “dissipation” of the injected power. The process of phase mixing that accompanies Landau damping must then lead to a certain fluctuation level $\langle |\delta f_m|^2 \rangle$ in the Hermite moments of δf . Lower m 's correspond to “fluid” quantities: density ($m = 0$), flow velocity ($m = 1$), temperature ($m = 2$). Higher m 's correspond to finer structure in velocity space: indeed, for $m \gg 1$, the Hermite polynomials can be approximated by trigonometric functions,

$$H_m(u) \approx \sqrt{2} \left(\frac{2m}{e} \right)^{m/2} \cos \left(\sqrt{2m} u - \frac{\pi m}{2} \right) e^{u^2/2}, \quad (10.33)$$

and so the Hermite transform is somewhat analogous to a Fourier transform in velocity space with “frequency” $\sqrt{2m}/v_{\text{th}}$.

(a) Show that in the kinetic Langevin problem described in Q6(c), the mean square fluctuation level of the m -th Hermite moment of the perturbed distribution function is given by

$$\langle |\delta f_m(t)|^2 \rangle = \frac{q^2 D}{T^2 \pi 2^m m!} \int_{-\infty}^{+\infty} d\omega \left| \frac{\zeta Z^{(m)}(\zeta)}{\epsilon(-i\omega)} \right|^2, \quad \zeta = \frac{\omega}{k v_{\text{th}}}, \quad (10.34)$$

where $Z^{(m)}(\zeta)$ is the m -th derivative of the plasma dispersion function [note Eq. (3.89)].

(b**) Show that, assuming $m \gg 1$ and $\zeta \ll \sqrt{2m}$,

$$Z^{(m)}(\zeta) \approx \sqrt{2\pi} i^{m+1} \left(\frac{2m}{e}\right)^{m/2} e^{i\zeta\sqrt{2m}-\zeta^2/2} \quad (10.35)$$

and, therefore, that

$$\boxed{\langle |\delta f_m(t)|^2 \rangle \approx \frac{\text{const}}{\sqrt{m}}}. \quad (10.36)$$

Thus, the *Hermite spectrum* of the free energy is shallow and, in particular, the total free energy diverges—it has to be regularised by collisions. This is a manifestation of a copious amount of fine-scale structure in velocity space (note also how this shows that Landau-damped perturbations involve all Hermite moments, not just the “fluid” ones).

Deriving Eq. (10.35) is a (reasonably hard) mathematical exercise: it involves using Eqs. (3.89) and (10.33) and manipulating contours in the complex plane. This is a treat for those who like this sort of thing. Getting to Eq. (10.36) will also require the use of Stirling’s formula.

The Hermite order at which the spectrum (10.36) must be cut off due to collisions can be quickly deduced as follows. We saw in §4.5 that the typical velocity derivative of δf can be estimated according to Eq. (4.24) and the time it takes for this perturbation to be wiped out by collisions is given by Eq. (4.29). But, in view of Eq. (10.33), the velocity gradients probed by the Hermite moment m are of order $\sqrt{2m}/v_{\text{th}}$. The collisional cut off m_c in Hermite space can then be estimated so:

$$m_c \sim v_{\text{th}}^2 \frac{\partial^2}{\partial v^2} \sim (kv_{\text{th}}t_c)^2 \sim \left(\frac{kv_{\text{th}}}{\nu}\right)^{2/3}. \quad (10.37)$$

Therefore, the total free energy stored in phase space diverges: using Eqs. (10.32) and (10.36),

$$\frac{1}{n} \int d^3\mathbf{v} \frac{\delta f^2}{2f_0} = \frac{1}{2} \sum_m \langle |\delta f_m|^2 \rangle \sim \int^{m_c} dm \frac{\text{const}}{\sqrt{m}} \propto \nu^{-1/3} \rightarrow \infty \quad \text{as } \nu \rightarrow +0. \quad (10.38)$$

In contrast, the total free-energy dissipation rate is finite, however small is the collision frequency: estimating the right-hand side of Eq. (4.18), we get

$$\frac{1}{n} \int d^3\mathbf{v} \frac{\delta f}{f_0} \left(\frac{\partial \delta f}{\partial t}\right)_c \sim -\nu \sum_m \langle |\delta f_m|^2 \rangle \propto \nu \int^{m_c} dm \sqrt{m} \sim kv_{\text{th}}. \quad (10.39)$$

Thus, the kinetic system can collisionally produce entropy at a rate that is entirely independent of the collision frequency.

If you find phase-space turbulence and generally life in Hermite space as fascinating as I do, you can learn more from Kanekar *et al.* (2015) (on fluctuation-dissipation relations and Hermite spectra) and from Schekochihin *et al.* (2016) and Adkins & Schekochihin (2017) (on what happens when a nonlinearity strikes).

8. Quasilinear theory of Landau damping. In §7, we discussed the QL theory of an unstable system, in which, whatever the size of the initial electric perturbations, they eventually grow large enough to affect the equilibrium distribution and modify it so as to suppress further growth. In a *stable* equilibrium, any initial perturbations will be Landau-damped, but, if they are sufficiently large to start with, they can also affect f_0 quasilinearly in a way that will slow down this damping.

Consider, in 1D, an initial spectrum $W(0, k)$ of plasma oscillations (waves) excited in the wave-number range $[k_2, k_1] = [\omega_{pe}/v_2, \omega_{pe}/v_1] \ll \lambda_{De}^{-1}$, with total electric energy $\mathcal{E}(0)$. Modify the QL theory of §7 to show the following.

(a) A steady state can be achieved in which the distribution develops a plateau in the velocity interval $[v_1, v_2]$ (Fig. 31). Find F^{plateau} in terms of v_1 , v_2 and the initial

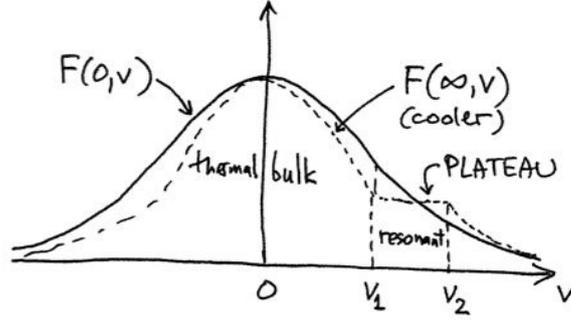


FIGURE 31. Initial stable equilibrium distribution and the final outcome of the QL evolution of a system with Landau-damped electric perturbations.

distribution $F(0, v)$. What is the energy of the waves in this steady state? What is the lower bound on initial electric energy $\mathcal{E}(0)$ below which the perturbations would just decay without forming a fully-fledged plateau?

(b) Derive the evolution equation for the thermal (nonresonant) bulk of the distribution and show that it cools during the QL evolution, with the total thermal energy declining by the same amount as the electric energy of the waves:

$$K_{\text{th}}(t) - K_{\text{th}}(0) = -[\mathcal{E}(0) - \mathcal{E}(t)]. \quad (10.40)$$

Identify where all the energy lost by the thermal particles and the waves goes and thus confirm that the total energy in the system is conserved. Why, physically, do thermal particles lose energy?

(c) Show that we must have

$$\frac{\mathcal{E}(0)}{n_e T_e} \gg \frac{\gamma_k}{\omega_{pe}} \frac{\delta v}{v} \quad (10.41)$$

in order for the wave energy to change only by a small fraction before saturating and

$$\frac{\mathcal{E}(0)}{n_e T_e} \gg \frac{\gamma_k}{\omega_{pe}} \left(\frac{\delta v}{v} \right)^3 \frac{1}{(k \lambda_{De})^2} \quad (10.42)$$

in order for the QL evolution to be faster than the damping. Here $\delta v = v_2 - v_1$ and $v \sim v_1 \sim v_2$.

This question requires some nuance in handling the calculation of the QL diffusion coefficient. In §7.1, we used the expression for $\delta f_{\mathbf{k}}$, Eq. (7.6), in which only the eigenmode-like part was retained, while the phase-mixing terms were dropped on the grounds that we could always just wait long enough for them to be eclipsed by the term containing an exponentially growing factor $e^{\gamma_{\mathbf{k}} t}$. When we are dealing with *damped* perturbations, there is no point in waiting because the exponential term is getting smaller, while the phase-mixing terms do not decay (except by collisions, see §§4.3 and 4.5, but we are not prepared to wait for that).

Let us, therefore, bite the bullet and use the full expression for the perturbed distribution function, Eq. (4.23), where we single out the slowest-damped mode and assume that all others, if any, will be damped fast enough never to produce significant QL effects:

$$\delta f_{\mathbf{k}} = i \frac{q}{m} \varphi_{\mathbf{k}} \frac{1 - e^{-i(\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}})t - \gamma_{\mathbf{k}} t}}{\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}} - i\gamma_{\mathbf{k}}} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} + e^{-i\mathbf{k} \cdot \mathbf{v} t} (g_{\mathbf{k}} + \dots), \quad (10.43)$$

where “...” stand for any possible undamped, phase-mixing remnants of other modes. When the solution (10.43) is substituted into Eq. (7.4), where it is multiplied by $\varphi_{\mathbf{k}}^*$ and time averaged

[according to Eq. (2.7)], the second term vanishes because, for resonant particles ($\mathbf{k} \cdot \mathbf{v} \approx \omega_{\mathbf{k}}$), it contains no resonant denominators and so is smaller than the first term, whereas for the nonresonant particles, it is removed by time averaging (check that this works at least for $|\gamma_{\mathbf{k}}|t \lesssim 1$ and indeed beyond that). Keeping only the first term in the expression (10.43), substituting it into Eq. (7.4) and going through a calculation analogous to that given in Eq. (7.8), we find that the diffusion matrix is (check this)

$$\mathbf{D}(\mathbf{v}) = \frac{q^2}{m^2} \sum_{\mathbf{k}} \frac{\mathbf{k}\mathbf{k}}{k^2} |\mathbf{E}_{\mathbf{k}}|^2 \text{Im} \left\langle \frac{1 - e^{-i(\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}})t - \gamma_{\mathbf{k}}t}}{\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}} - i\gamma_{\mathbf{k}}} \right\rangle, \quad (10.44)$$

which is a generalisation of the penultimate line of Eq. (7.8). For nonresonant particles, the phase-mixing term is eliminated by time averaging and we end up with the old result: the last line of Eq. (7.8). For resonant particles, assuming $|\gamma_{\mathbf{k}}| \ll |\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}}| \ll \omega_{\mathbf{k}} \sim \mathbf{k} \cdot \mathbf{k}$ and $|\gamma_{\mathbf{k}}|t \ll 1$, we may adopt the approximation (4.37), which we have previously used to analyse the structure of δf . This gives us

$$\mathbf{D}(\mathbf{v}) = \frac{q^2}{m^2} \sum_{\mathbf{k}} \frac{\mathbf{k}\mathbf{k}}{k^2} |\mathbf{E}_{\mathbf{k}}|^2 \pi \delta(\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}}), \quad (10.45)$$

which is the same result as Eq. (7.16)—including, importantly, the sign, which we would have gotten wrong had we just mechanically applied Plemelj's formula to Eq. (7.12) with $\gamma_{\mathbf{k}} < 0$. This is equivalent to saying that the k integral in Eq. (7.16) should be taken along the Landau contour, rather than simply along the real line.

Note that the above construction was done assuming $|\gamma_{\mathbf{k}}|t \ll 1$, i.e., all the QL action has to occur before the initial perturbations decay away (which is reasonable). Note also that there is nothing above that would not apply to the case of unstable perturbations ($\gamma_{\mathbf{k}} > 0$) and so we conclude that results of §7, derived formally for $\gamma_{\mathbf{k}}t \gg 1$, in fact also hold on shorter time scales ($\gamma_{\mathbf{k}}t \ll 1$, but, obviously, still $\omega_{\mathbf{k}}t \gg 1$).

9. Quasilinear theory of Weibel instability. (a) Starting from the Vlasov equations including magnetic perturbations, show that the slow evolution of the equilibrium distribution function is described by the following diffusion equation:

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{D}(\mathbf{v}) \cdot \frac{\partial f_0}{\partial \mathbf{v}}, \quad (10.46)$$

where the QL diffusion tensor is

$$\mathbf{D}(\mathbf{v}) = \frac{q^2}{m^2} \sum_{\mathbf{k}} \frac{1}{i(\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}}) + \gamma_{\mathbf{k}}} \left(\mathbf{E}_{\mathbf{k}}^* + \frac{\mathbf{v} \times \mathbf{B}_{\mathbf{k}}^*}{c} \right) \left(\mathbf{E}_{\mathbf{k}} + \frac{\mathbf{v} \times \mathbf{B}_{\mathbf{k}}}{c} \right) \quad (10.47)$$

and $\omega_{\mathbf{k}}$ and $\gamma_{\mathbf{k}}$ are the frequency and the growth rate, respectively, of the fastest-growing mode.

(b) Consider the example of the low-frequency electron Weibel instability with wave numbers \mathbf{k} parallel to the anisotropy direction [Eq. (10.7)]. Take $\mathbf{k} = k\hat{z}$ and $\mathbf{B}_{\mathbf{k}} = B_{\mathbf{k}}\hat{y}$ and, denoting $\Omega_{\mathbf{k}} = eB_{\mathbf{k}}/m_e c$ (the Larmor frequency associated with the perturbed magnetic field), show that Eq. (10.46) becomes

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v_x} \left(D_{xx} \frac{\partial f_0}{\partial v_x} + D_{xz} \frac{\partial f_0}{\partial v_z} \right) + \frac{\partial}{\partial v_z} D_{zz} \frac{\partial f_0}{\partial v_z}, \quad (10.48)$$

where the coefficients of the QL diffusion tensor are

$$D_{xx} = \sum_{\mathbf{k}} \frac{\gamma_{\mathbf{k}}}{k^2} |\Omega_{\mathbf{k}}|^2, \quad D_{xz} = - \sum_{\mathbf{k}} \frac{2\gamma_{\mathbf{k}}v_x v_z}{k^2 v_z^2 + \gamma_{\mathbf{k}}^2} |\Omega_{\mathbf{k}}|^2, \quad D_{zz} = \sum_{\mathbf{k}} \frac{\gamma_{\mathbf{k}} v_x^2}{k^2 v_z^2 + \gamma_{\mathbf{k}}^2} |\Omega_{\mathbf{k}}|^2. \quad (10.49)$$

(c) Suppose the electron distribution function f_0 is initially bi-Maxwellian, Eq. (10.5),

with $0 < T_{\perp}/T_{\parallel} - 1 \ll 1$ (as should be the case for this instability to work). As QL evolution starts, we may define the temperatures of the evolving distribution according to

$$T_{\perp} = \frac{1}{n} \int d^3\mathbf{v} \frac{m(v_x^2 + v_y^2)}{2} f_0, \quad T_{\parallel} = \frac{1}{n} \int d^3\mathbf{v} m v_z^2 f_0. \quad (10.50)$$

Show that initially, viz., before f_0 has time to change shape significantly so as no longer to be representable as a bi-Maxwellian, the two temperatures will evolve approximately (using $\gamma_k \ll kv_{\text{th}}$) according to

$$\boxed{\frac{\partial T_{\perp}}{\partial t} = -\lambda T_{\perp}, \quad \frac{\partial T_{\parallel}}{\partial t} = 2\lambda T_{\perp}}, \quad \text{where} \quad \lambda(T_{\perp}, T_{\parallel}) = \sum_k \frac{2\gamma_k |\Omega_k|^2}{k^2 v_{\text{th}\parallel}^2}. \quad (10.51)$$

Thus, QL evolution will lead, at least initially, to the reduction of the temperature anisotropy, thus weakening the instability (these equations should not be used to trace $T_{\perp}/T_{\parallel} - 1$ all the way to zero because there is no reason why the QL evolution should preserve the bi-Maxwellian shape of f_0).

Note that, even modulo the caveat about the bi-Maxwellian not being a long-term solution, this does not give us a way to estimate (or even guess) what the saturated fluctuation level will be. The standard Weibel lore is that saturation occurs when the approximations that were used to derive the linear theory (Q3) break down, namely, when magnetic field becomes strong enough to magnetise the plasma, rendering the Larmor scale $\rho_e = v_{\text{the}}/\Omega_k$ associated with the fluctuations small enough to be comparable to the latter's wavelengths $\sim k^{-1}$. Using the typical values of k from Eq. (10.7), we can write this condition as follows

$$\Omega_k \sim kv_{\text{the}} \sim \sqrt{\Delta_e} \frac{v_{\text{the}}}{d_e} \quad \Leftrightarrow \quad \boxed{\frac{1}{\beta_e} \equiv \frac{B^2/8\pi}{n_e T_e} \sim \Delta_e}. \quad (10.52)$$

Thus, Weibel instability will produce fluctuations the ratio of whose magnetic-energy density to the electron-thermal-energy density (customarily referred to as the inverse of ‘‘plasma beta,’’ $1/\beta_e$) is comparable to the electron pressure anisotropy Δ_e . Because at that point the fluctuations will be relaxing this pressure anisotropy at the same rate as they can grow in the first place [in Eq. (10.51), $\lambda \sim \gamma_k$], the QL approach is not valid anymore.

These considerations are, however, usually assumed to be qualitatively sound and lead people to believe that, even in collisionless plasmas, the anisotropy of the electron distribution must be largely self-regulating, with unstable Weibel fluctuations engendered by the anisotropy quickly acting to isotropise the plasma (or at least the electrons).

This is all currently very topical in the part of the plasma-astronomy world preoccupied with collisionless shocks, origin of the cosmic magnetism, hot weakly collisional environments such as the intergalactic medium (in galaxy clusters) or accretion flows around black holes and many other interesting subjects.

(d) Eqs. (10.51) say that the total mean kinetic energy,

$$\int d^3\mathbf{v} \frac{mv^2}{2} f_0 = n \left(T_{\perp} + \frac{T_{\parallel}}{2} \right), \quad (10.53)$$

does not change. But fluctuations are generated and grow at the rate γ_k ! Without much further algebra, can you tell whether you should therefore doubt the result (10.51)?

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PART II

Magnetohydrodynamics

11. MHD Equations

Like hydrodynamics from gas kinetics, MHD can be derived systematically from the Vlasov–Maxwell–Landau equations for a plasma in the limit of large collisionality + a number of additional assumptions (see, e.g., Goedbloed & Poedts 2004; Parra 2017*a*). Here I will adopt a purely fluid approach—partly to make these lectures self-consistent and partly because there is a certain beauty in it: we need to know relatively little about the properties of the constituent substance in order to spin out a very sophisticated and complete theory about the way in which it flows. This approach is also more generally applicable because the substance that we will be dealing with need not be gaseous, like plasma—you may also think of liquid metals, various conducting solutions, etc.

So, we declare an interest in the flow of a conducting fluid and attempt to be guided in our description of it by the very basic things: conservation laws of mass, momentum and energy plus Maxwell’s equations for the electric and magnetic fields. This will prove sufficient for most of our purposes. So we consider a fluid characterised by the following quantities:

ρ —mass density,
 \mathbf{u} —flow velocity,
 p —pressure,
 σ —charge density,
 \mathbf{j} —current density,
 \mathbf{E} —electric field,
 \mathbf{B} —magnetic field.

Our immediate objective is to find a set of closed equations that would allow us to determine all of these quantities as functions of time and space within the fluid.

11.1. Conservation of Mass

This is the most standard of all arguments in fluid dynamics (Fig. 32):

$$\frac{d}{dt} \underbrace{\int_V d^3\mathbf{r} \rho}_{\text{mass inside a volume of fluid}} = - \underbrace{\int_{\partial V} (\rho\mathbf{u}) \cdot d\mathbf{S}}_{\text{mass flux in/out}} = - \int_V d^3\mathbf{r} \nabla \cdot (\rho\mathbf{u}). \quad (11.1)$$

As this equation holds for any V , however small, it can be converted into a differential relation

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{u}) = 0}. \quad (11.2)$$

This is the *continuity equation*.

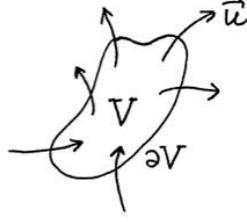


FIGURE 32. Stuff flowing in and out of a volume of fluid.

11.2. Conservation of Momentum

A similar approach:

$$\begin{aligned}
 \frac{d}{dt} \underbrace{\int_V d^3\mathbf{r} \rho \mathbf{u}}_{\text{momentum inside a volume of fluid}} &= - \underbrace{\int_{\partial V} (\rho \mathbf{u} \mathbf{u}) \cdot d\mathbf{S}}_{\substack{\text{Reynolds stress} \\ \text{momentum flux through boundary} \\ \text{(fluid flow carrying its own momentum)}}} - \underbrace{\int_{\partial V} p d\mathbf{S}}_{\substack{\text{pressure on boundary} \\ \text{(momentum flux due to internal motion)}}} - \underbrace{\int_{\partial V} \boldsymbol{\Pi} \cdot d\mathbf{S}}_{\text{viscous stress}} + \underbrace{\int_V d^3\mathbf{r} \mathbf{F}}_{\substack{\text{all other forces (E\&M} \\ \text{will come in here)}}} \\
 &= \int_V d^3\mathbf{r} [-\nabla \cdot (\rho \mathbf{u} \mathbf{u}) - \nabla p - \nabla \cdot \boldsymbol{\Pi} + \mathbf{F}]. \tag{11.3}
 \end{aligned}$$

In differential form, this becomes

$$\begin{aligned}
 \underbrace{\frac{\partial}{\partial t} \rho \mathbf{u}}_{\substack{= \rho \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \rho \\ = \rho \frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \cdot \nabla (\rho \mathbf{u})}} &= \underbrace{-\nabla \cdot (\rho \mathbf{u} \mathbf{u})}_{\substack{-\rho \mathbf{u} \cdot \nabla \mathbf{u} \\ - \mathbf{u} \cdot \nabla (\rho \mathbf{u})}} - \nabla p - \nabla \cdot \boldsymbol{\Pi} + \mathbf{F}, \tag{11.4}
 \end{aligned}$$

and so, finally,

$$\boxed{\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p - \nabla \cdot \boldsymbol{\Pi} + \mathbf{F}}. \tag{11.5}$$

$\equiv \frac{d\mathbf{u}}{dt}$
 convective derivative

This is the *momentum equation*.

One part of this equation does have to be calculated from some knowledge of the microscopic properties of the constituent fluid or gas—the viscous stress. For a gas, it is done in kinetic theory (e.g., Lifshitz & Pitaevskii 1981; Dellar 2016; Schekochihin 2016):

$$\boldsymbol{\Pi} = -\rho\nu \left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T - \frac{2}{3} \nabla \cdot \mathbf{u} \mathbf{1} \right], \tag{11.6}$$

where ν is the kinematic (Newtonian) viscosity. In what follows, we will never require the explicit form of $\boldsymbol{\Pi}$ (except perhaps in §11.10).

In a magnetised plasma (i.e., such that its collision frequency \ll Larmor frequency of the gyrating charges), the viscous stress is much more complicated and anisotropic with respect to the direction of the magnetic field: because of their Larmor motion, charged particles diffuse

differently across and along the field. This gives rise to the so-called *Braginskii (1965) stress* (see, e.g., Helander & Sigmar 2005; Parra 2017a).

11.3. Electromagnetic Fields and Forces

The fact that the fluid is conducting means that it can have distributed charges (σ) and currents (\mathbf{j}) and so the electric (\mathbf{E}) and magnetic (\mathbf{B}) fields will exert body forces on the fluid. Indeed, for one particle of charge q , the Lorentz force is

$$\mathbf{f}_L = q \left(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right), \quad (11.7)$$

and if we sum this over all particles (or, to be precise, average over their distribution and sum over species), we will get

$$\mathbf{F} = \sigma \mathbf{E} + \frac{\mathbf{j} \times \mathbf{B}}{c}. \quad (11.8)$$

This body force (force density) goes into Eq. (11.5) and so we must know \mathbf{E} , \mathbf{B} , σ and \mathbf{j} in order to compute the fluid flow \mathbf{u} .

Clearly it is a good idea to bring in Maxwell's equations:

$$\nabla \cdot \mathbf{E} = 4\pi\sigma \quad (\text{Gauss}), \quad (11.9)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (11.10)$$

$$\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E} \quad (\text{Faraday}), \quad (11.11)$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \quad (\text{Ampère-Maxwell}). \quad (11.12)$$

To these, we must append Ohm's law in its simplest form: The electric field in the frame of a fluid element moving with velocity \mathbf{u} is

$$\mathbf{E}' = \mathbf{E} + \frac{\mathbf{u} \times \mathbf{B}}{c} = \eta \mathbf{j}, \quad (11.13)$$

where \mathbf{E} is the electric field in the laboratory frame and η is the *Ohmic resistivity*.

Normally, the resistivity, like viscosity, has to be computed from kinetic theory (see, e.g., Helander & Sigmar 2005; Parra 2017a) or tabulated by assiduous experimentalists. In a magnetised plasma, the simple form (11.13) of Ohm's law is only valid at spatial scales longer than the Larmor radii and time scales longer than the Larmor periods of the particles (see, e.g., Goedbloed & Poedts 2004; Parra 2017b).

Eqs. (11.9–11.13) can be reduced somewhat if we assume (quite reasonably for most applications) that our fluid flow is *non-relativistic*. Let us stipulate that all fields evolve on time scales $\sim \tau$, have spatial scales $\sim l$ and that the flow velocity is

$$u \sim \frac{l}{\tau} \ll c. \quad (11.14)$$

Then, from Ohm's law (11.13),

$$E \sim \frac{u}{c} B \ll B, \quad (11.15)$$

so *electric fields are small compared to magnetic fields*.

In Ampère–Maxwell’s law (11.12),

$$\frac{\left| \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \right|}{|\nabla \times \mathbf{B}|} \sim \frac{\frac{1}{c} \frac{1}{\tau} \frac{u}{c} B}{\frac{1}{l} B} \sim \frac{u^2}{c^2} \ll 1, \quad (11.16)$$

so *the displacement current is negligible* (note that at this point we have ordered out light waves; see Q2 in Kinetic Theory). This allows us to revert to the pre-Maxwell form of Ampère’s law:

$$\boxed{\mathbf{j} = \frac{c}{4\pi} \nabla \times \mathbf{B}}. \quad (11.17)$$

Thus, the current is no longer an independent field, there is a one-to-one correspondence $\mathbf{j} \leftrightarrow \mathbf{B}$.

Finally, comparing the electric and magnetic parts of the Lorentz force (11.8), and using Gauss’s law (11.9) to estimate $\sigma \sim E/l$, we get

$$\frac{|\sigma \mathbf{E}|}{\left| \frac{1}{c} \mathbf{j} \times \mathbf{B} \right|} \sim \frac{\frac{1}{l} E^2}{\frac{1}{c} \frac{1}{l} B^2} \sim \frac{E^2}{B^2} \sim \frac{u^2}{c^2} \ll 1. \quad (11.18)$$

Thus, the MHD body force is

$$\boxed{\mathbf{F} = \frac{\mathbf{j} \times \mathbf{B}}{c} = \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi}}. \quad (11.19)$$

This goes into Eq. (11.5) and we note with relief that σ , \mathbf{j} and \mathbf{E} have all fallen out of the momentum equation—we only need to know \mathbf{B} .

11.4. Maxwell Stress and Magnetic Forces

Let us take a break from formal derivations to consider what Eq. (11.19) teaches us about the sort of new dynamics that our fluid will experience as a result of being conducting. To see this, it is useful to play with the expression (11.19) in a few different ways.

By simple vector algebra,

$$\mathbf{F} = \underbrace{\frac{\mathbf{B} \cdot \nabla \mathbf{B}}{4\pi}}_{\text{“magnetic tension”}} - \underbrace{\frac{\nabla B^2}{8\pi}}_{\text{“magnetic pressure”}} = -\nabla \cdot \underbrace{\left(\frac{B^2}{8\pi} \mathbf{I} - \frac{\mathbf{B}\mathbf{B}}{4\pi} \right)}_{\text{“Maxwell stress”}}, \quad (11.20)$$

where the last expression was obtained with the aid of $\nabla \cdot \mathbf{B} = 0$. Thus, the action of the Lorentz force in a conducting fluid amounts to a new form of stress. Mathematically, this “*Maxwell stress*” is somewhat similar to the kind of stress that would arise from a suspension in the fluid of elongated molecules—e.g., polymer chains, or other kinds of “balls on springs” (see, e.g., Dellar 2017; the analogy can be made rigorous: see Ogilvie & Proctor 2003). Thus, we expect that the magnetic field threading the fluid will impart to it a degree of “elasticity.”

Exactly what this means dynamically becomes obvious if we rewrite the magnetic tension and pressure forces in Eq. (11.20) in the following way. Let $\mathbf{b} = \mathbf{B}/B$ be the unit vector in the direction of \mathbf{B} (the unit tangent to the field line). Then

$$\mathbf{B} \cdot \nabla \mathbf{B} = B \mathbf{b} \cdot \nabla (B \mathbf{b}) = B^2 \mathbf{b} \cdot \nabla \mathbf{b} + \mathbf{b} \mathbf{b} \cdot \nabla \frac{B^2}{2} \quad (11.21)$$

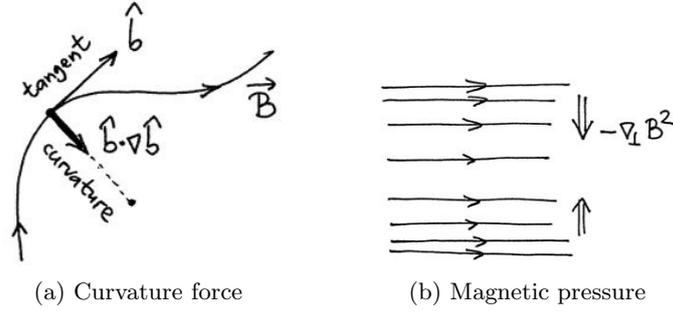


FIGURE 33. Magnetic forces.

and, putting this back into Eq. (11.20), we get

$$\mathbf{F} = \underbrace{\frac{B^2}{4\pi} \mathbf{b} \cdot \nabla \mathbf{b}}_{\text{"curvature force"}} - \underbrace{(\mathbf{1} - \mathbf{b}\mathbf{b}) \cdot \nabla \frac{B^2}{8\pi}}_{\substack{\equiv \nabla_{\perp} \\ \text{magnetic} \\ \text{pressure force}}}. \quad (11.22)$$

Thus, we learn that the Lorentz force consists of two distinct parts (Fig. 33):

- *curvature force*, so called because $\mathbf{b} \cdot \nabla \mathbf{b}$ is the vector curvature of the magnetic field line—the implication being that field lines, if bent, will want to straighten up;
- *magnetic pressure*, whose presence implies that field lines will resist compression or rarefaction (the field wants to be uniform in strength).

Note that both forces act perpendicularly to \mathbf{B} , as they must, since magnetic field never exerts a force along itself on a charged particle [Eq. (11.7)].

So this is the effect of the field on the fluid. What is the effect of the fluid on the field?

11.5. Evolution of Magnetic Field

Returning to deriving MHD equations, we use Ohm's law (11.13) to express \mathbf{E} in terms of \mathbf{u} , \mathbf{B} and \mathbf{j} in the right-hand side of Faraday's law (11.11). We then use Ampere's law (11.17) to express \mathbf{j} in terms of \mathbf{B} . The result is

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left(\mathbf{u} \times \mathbf{B} - \frac{c^2 \eta}{4\pi} \nabla \times \mathbf{B} \right). \quad (11.23)$$

After using also $\nabla \cdot \mathbf{B} = 0$ to get $\nabla \times (\nabla \times \mathbf{B}) = -\nabla^2 \mathbf{B}$ and renaming $c^2 \eta / 4\pi \rightarrow \eta$, the *magnetic diffusivity*, we arrive at the *magnetic induction equation* (due to Hertz):

$$\boxed{\frac{\partial \mathbf{B}}{\partial t} = \underbrace{\nabla \times (\mathbf{u} \times \mathbf{B})}_{\text{advection}} + \underbrace{\eta \nabla^2 \mathbf{B}}_{\text{diffusion}}}. \quad (11.24)$$

Note that if $\nabla \cdot \mathbf{B} = 0$ is satisfied initially, any solution of Eq. (11.24) will remain divergence-free at all times.

11.6. Magnetic Reynolds Number

The relative importance of the diffusion term (obvious what this does) and the advection term (to be discussed in the next few sections) in Eq. (11.24) is measured by a

dimensionless number:

$$\frac{|\nabla \times (\mathbf{u} \times \mathbf{B})|}{|\eta \nabla^2 \mathbf{B}|} \sim \frac{\frac{u}{l} B}{\frac{\eta}{l^2} B} = \frac{ul}{\eta} \equiv \text{Rm}, \quad (11.25)$$

called the *magnetic Reynolds number*. In nature, it can take a very broad range of values:

$$\begin{aligned} \text{liquid metals in industrial contexts (metallurgy): } & \text{Rm} \sim 10^{-3} \dots 10^{-1}, \\ \text{planet interiors: } & \text{Rm} \sim 100 \dots 300, \\ \text{solar convective zone: } & \text{Rm} \sim 10^6 \dots 10^9, \\ \text{interstellar medium ("warm" phase): } & \text{Rm} \sim 10^{18}, \\ \text{intergalactic medium (cores of galaxy clusters): } & \text{Rm} \sim 10^{29}, \\ \text{laboratory "dynamo" experiments (see §11.10): } & \text{Rm} \sim 1 \dots 100 \text{ (and growing)}. \end{aligned}$$

Generally speaking, when flow velocities are large/distances are large/resistivities are low, $\text{Rm} \gg 1$ and it makes sense to discuss "*ideal MHD*," i.e., the limit $\eta \rightarrow 0$. In fact, η often needs to be brought back in to deal with instances of large $\nabla \mathbf{B}$, which arise naturally from solutions of ideal MHD equations (see §11.10, §15.2 and Parra 2017*a*), but let us consider the ideal case for now to understand what the advective part of the induction equation does to \mathbf{B} .

11.7. Lundquist Theorem

The ideal ($\eta = 0$) version of the induction equation (11.24),

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}), \quad (11.26)$$

implies that *fluid elements that lie on a field line initially will remain on this field line*, i.e., "the magnetic field moves with the flow."

Proof. Unpacking the double vector product in Eq. (11.26),

$$\frac{\partial \mathbf{B}}{\partial t} = -\mathbf{u} \cdot \nabla \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{u} - \mathbf{B} \nabla \cdot \mathbf{u} + \mathbf{u} \nabla \cdot \mathbf{B}, \quad (11.27)$$

or, using the notation for the "convective derivative" [Eq. (11.5)],

$$\frac{d\mathbf{B}}{dt} \equiv \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} - \mathbf{B} \nabla \cdot \mathbf{u}. \quad (11.28)$$

The continuity equation (11.2) can be rewritten in a somewhat reminiscent form

$$\frac{d\rho}{dt} \equiv \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \rho = -\rho \nabla \cdot \mathbf{u} \quad \Rightarrow \quad \nabla \cdot \mathbf{u} = -\frac{1}{\rho} \frac{d\rho}{dt}. \quad (11.29)$$

The last expression is now used for $\nabla \cdot \mathbf{u}$ in Eq. (11.28):

$$\frac{d\mathbf{B}}{dt} = \mathbf{B} \cdot \nabla \mathbf{u} + \frac{\mathbf{B}}{\rho} \frac{d\rho}{dt}. \quad (11.30)$$

Multiplying this equation by $1/\rho$ and noting that

$$\frac{1}{\rho} \frac{d\mathbf{B}}{dt} - \frac{\mathbf{B}}{\rho^2} \frac{d\rho}{dt} = \frac{d}{dt} \frac{\mathbf{B}}{\rho}, \quad (11.31)$$

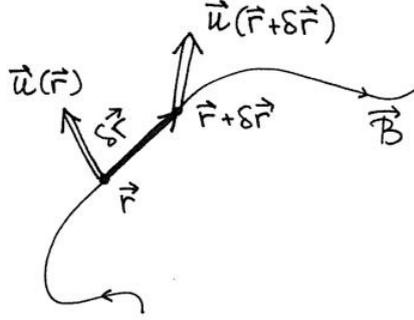


FIGURE 34. Pressure-balanced perturbations.

we arrive at

$$\boxed{\frac{d}{dt} \frac{\mathbf{B}}{\rho} = \frac{\mathbf{B}}{\rho} \cdot \nabla \mathbf{u}}. \quad (11.32)$$

Let us compare the evolution of the vector \mathbf{B}/ρ with the evolution of an infinitesimal Lagrangian separation vector in a moving fluid: the convective derivative is the Lagrangian time derivative, so

$$\frac{d}{dt} \delta \mathbf{r}(t) = \mathbf{u}(\mathbf{r} + \delta \mathbf{r}) - \mathbf{u}(\mathbf{r}) \approx \delta \mathbf{r} \cdot \nabla \mathbf{u}. \quad (11.33)$$

Thus, $\delta \mathbf{r}$ and \mathbf{B}/ρ satisfy the same equation. This means that if two fluid elements are initially on the same field line,

$$\delta \mathbf{r} = \text{const} \frac{\mathbf{B}}{\rho}, \quad (11.34)$$

then they will stay on the same field line, q.e.d.

This means that in MHD, the fluid flow will be entraining the magnetic-field lines with it—and, as saw in §11.4, the field lines will react back on the fluid:
 —when the fluid tries to bend the field, the field will want to spring back,
 —when the fluid tries to compress or rarefy the field, the field will resist as if it possessed (perpendicular) pressure.

This is the sense in which MHD fluid is “elastic”: it is threaded by magnetic-field lines, which move with it and act as elastic bands.

11.8. Flux Freezing

There is an essentially equivalent formulation of the result of §11.7 that highlights the fact that the ideal induction equation (11.26) is a *conservation law*—conservation of *magnetic flux*.

The magnetic flux through a surface S (Fig. 35a) is, by definition,

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} \quad (11.35)$$

($d\mathbf{S} \equiv \hat{\mathbf{n}} dS$, where $\hat{\mathbf{n}}$ is a unit normal pointing out of the surface). *The flux Φ depends on the loop ∂S , but not on the choice of the surface spanning it.* Indeed, if we consider two surfaces, S_1 and S_2 , spanning the same loop ∂S (Fig. 35b) and define $\Phi_{1,2} = \int_{S_{1,2}} \mathbf{B} \cdot d\mathbf{S}$,

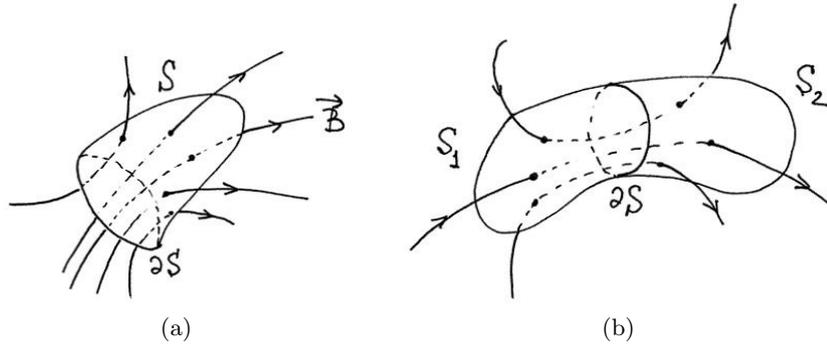


FIGURE 35. Magnetic flux through a loop ∂S (a) is independent of the surface spanning the loop (b).

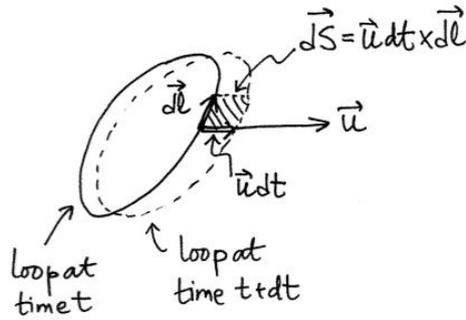


FIGURE 36. A loop moving with the fluid.

then the flux out of the volume V enclosed by $S_1 \cup S_2 = \partial V$ is

$$\Phi_2 - \Phi_1 = \int_{\partial V} \mathbf{B} \cdot d\mathbf{S} = \int_V d^3r \nabla \cdot \mathbf{B} = 0, \quad \text{q.e.d.} \quad (11.36)$$

Alfvén's Theorem. *Flux through any loop moving with the fluid is conserved.*

Proof. Let $S(t)$ be a surface spanning the loop at time t . If the loop moves with the fluid (Fig. 36), at the slightly later time $t + dt$ it is spanned (for example) by the surface

$$S(t + dt) = S(t) \cup \text{ribbon traced by the loop as it moves over time } dt. \quad (11.37)$$

Then the flux at time t is

$$\Phi(t) = \int_{S(t)} \mathbf{B}(t) \cdot d\mathbf{S} \quad (11.38)$$

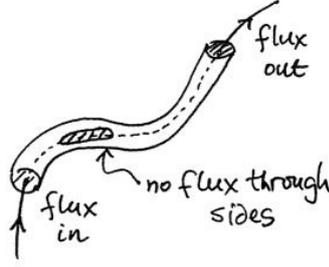


FIGURE 37. Flux tube.

and at the later time,

$$\begin{aligned}
 \Phi(t + dt) &= \int_{S(t+dt)} \mathbf{B}(t + dt) \cdot d\mathbf{S} \\
 &= \underbrace{\int_{S(t)} \mathbf{B}(t + dt) \cdot d\mathbf{S}}_{= \Phi(t)} + \underbrace{\int_{\text{ribbon}} \mathbf{B}(t + dt) \cdot \underbrace{d\mathbf{S}}_{= \mathbf{u} dt \times d\mathbf{l}}}_{= dt \int_{\partial S(t)} \mathbf{B}(t) \cdot (\mathbf{u} \times d\mathbf{l})} \\
 &= \underbrace{\int_{S(t)} \mathbf{B}(t) \cdot d\mathbf{S}}_{= \Phi(t)} + dt \int_{S(t)} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \underbrace{dt \int_{\partial S(t)} \mathbf{B}(t) \cdot (\mathbf{u} \times d\mathbf{l})}_{= -dt \int_{\partial S(t)} (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}} \\
 &= \Phi(t) + dt \int_{S(t)} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} - dt \int_{\partial S(t)} (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} \\
 &= \Phi(t) + dt \int_{S(t)} [\nabla \times (\mathbf{u} \times \mathbf{B})] \cdot d\mathbf{S}.
 \end{aligned} \tag{11.39}$$

Therefore,

$$\frac{d\Phi}{dt} = \frac{\Phi(t + dt) - \Phi(t)}{dt} = \int_{S(t)} \underbrace{\left[\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) \right]}_{\substack{\text{ideal induction} \\ \text{equation (11.26)}}} \cdot d\mathbf{S} = 0, \quad \text{q.e.d.} \tag{11.40}$$

This result means that *field lines are frozen into the flow*. Indeed, consider a flux tube enclosing a field line (Fig. 37). As the tube deforms, the field line stays inside it because fluxes through the ends and sides of the tube cannot change.

Note that *Ohmic diffusion breaks flux freezing*.

11.9. Amplification of Magnetic Field by Fluid Flow

An interesting physical consequence of these results is that *flows of conducting fluid can amplify magnetic fields*. For example, consider a flow that stretches an initial cylindrical tube of length l_1 and cross section S_1 into a long thin spaghetti of length l_2 and cross section S_2 (Fig. 38). By conservation of flux,

$$B_1 S_1 = B_2 S_2. \tag{11.41}$$

By conservation of mass,

$$\rho_1 l_1 S_1 = \rho_2 l_2 S_2. \tag{11.42}$$

Therefore,

$$\frac{B_2}{\rho_2 l_2} = \frac{B_1}{\rho_1 l_1} \Rightarrow \frac{B_2}{B_1} = \frac{\rho_2 l_2}{\rho_1 l_1}. \tag{11.43}$$

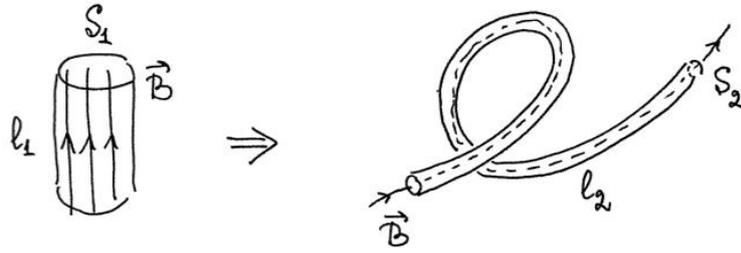


FIGURE 38. Amplification of magnetic field by stretching.

In an incompressible fluid, $\rho_2 = \rho_1$, and the field is amplified by a factor l_2/l_1 . In a compressible fluid, the field can also be amplified by compression.

Going back to the induction equation in the form (11.27),

$$\frac{\partial \mathbf{B}}{\partial t} + \underbrace{\mathbf{u} \cdot \nabla \mathbf{B}}_{\text{advection}} = \underbrace{\mathbf{B} \cdot \nabla \mathbf{u}}_{\text{stretching}} - \underbrace{\mathbf{B} \nabla \cdot \mathbf{u}}_{\text{compression}}, \quad (11.44)$$

the three terms in it are responsible for, in order, advection of the field by the flow (i.e., the flow carrying the field around with it), “stretching” (amplification) of the field by velocity gradients that make fluid elements longer and, finally, compression or rarefaction of the field by convergent or divergent flows (unless $\nabla \cdot \mathbf{u} = 0$, as it is in an incompressible fluid).

Hence arises the famous problem of *MHD dynamo*: are there fluid flows that lead to sustained amplification of the magnetic fields? The answer is yes—but the flow must be 3D (the absence of dynamo action in 2D is a theorem, the simplest version of which is due to Zeldovich 1956; see Q4). Magnetic fields of planets, stars, galaxies, etc. are all believed to owe their origin and sustainment to this effect. This topic requires (and merits) a more detailed treatment (§11.10), but for now let us flag two important aspects:

- resistivity, however small, turns out to be impossible to neglect because large gradients of \mathbf{B} appear as the field is advected by the flow;
- the amplification of the field is checked by the Lorentz force once the field is strong enough that it can act back on the flow, viz., when their energy densities become comparable:

$$\frac{B^2}{8\pi} \sim \frac{\rho u^2}{2}. \quad (11.45)$$

11.10. MHD Dynamo

I will fill this in at some point, but for now, read Schekochihin & Cowley (2007, §3) and see §15.3 for further references. You will also find a (somewhat outdated) preview in my handwritten lecture notes available here:

<http://www-thphys.physics.ox.ac.uk/people/AlexanderSchekochihin/notes/leshouches07.pdf>.

11.11. Conservation of Energy

Let us summarise the equations that we have derived so far, namely Eqs. (11.2), (11.5) and (11.24), expressing conservation of

$$\text{mass} \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (11.46)$$

$$\begin{aligned} \text{momentum} \quad \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) &= -\nabla p - \nabla \cdot \mathbf{\Pi} + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi} \\ &= -\nabla \cdot \underbrace{\left[\left(p + \frac{B^2}{8\pi} \right) \mathbf{I} - \frac{\mathbf{B}\mathbf{B}}{4\pi} + \mathbf{\Pi} \right]}_{\text{total stress}}, \end{aligned} \quad (11.47)$$

$$\text{and flux} \quad \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}. \quad (11.48)$$

To complete the system, we need an equation for p , which has to come from the one conservation law that we have not yet utilised: *conservation of energy*.

The total *energy density* is

$$\varepsilon = \underbrace{\frac{\rho u^2}{2}}_{\text{kinetic}} + \underbrace{\frac{p}{\gamma - 1}}_{\text{internal}} + \underbrace{\frac{E^2}{8\pi}}_{\text{electric}} + \underbrace{\frac{B^2}{8\pi}}_{\text{magnetic}}, \quad (11.49)$$

where the electric energy can (and, for consistency with §11.3, must) be neglected because $E^2/B^2 \sim u^2/c^2 \ll 1$. We follow the same logic as we did in §§11.1 and 11.2:

$$\begin{aligned} \frac{d}{dt} \underbrace{\int_V d^3\mathbf{r} \varepsilon}_{\text{energy inside a volume of fluid}} &= - \underbrace{\int_{\partial V} \left(\frac{\rho u^2}{2} + \frac{p}{\gamma - 1} \right) \mathbf{u} \cdot d\mathbf{S}}_{\text{fluid flow carrying its kinetic and internal energy through boundary}} - \underbrace{\int_{\partial V} [(p\mathbf{I} + \mathbf{\Pi}) \cdot \mathbf{u}] \cdot d\mathbf{S}}_{\text{work done on boundary by pressure and viscous stress}} \\ &\quad - \underbrace{\int_{\partial V} \mathbf{q} \cdot d\mathbf{S}}_{\text{heat flux}} - \underbrace{\int_{\partial V} \frac{c}{4\pi} (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{S}}_{\text{Poynting flux}}. \end{aligned} \quad (11.50)$$

Like the viscous stress $\mathbf{\Pi}$, the heat flux \mathbf{q} must be calculated kinetically (in a plasma) or tabulated (in an arbitrary complicated substance). In a gas, $\mathbf{q} = -\kappa \nabla T$, but it is more complicated in a magnetised plasma (see, e.g., Braginskii 1965; Helander & Sigmar 2005; Parra 2017a).

Note that the magnetic energy and the work done by the Lorentz force are not included in the first two terms on the right-hand side of Eq. (11.50) because all of that must already be correctly accounted for by the Poynting flux. Indeed, since $c\mathbf{E} = -\mathbf{u} \times \mathbf{B} + \eta \nabla \times \mathbf{B}$

[Eq. (11.13), with η renamed as in Eq. (11.24)], we have

$$\begin{aligned} \int_{\partial V} \frac{c}{4\pi} (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{S} &= \underbrace{\int_{\partial V} \frac{B^2}{8\pi} \mathbf{u} \cdot d\mathbf{S}}_{\substack{\text{magnetic energy} \\ \text{flow}}} + \underbrace{\int_{\partial V} \left[\left(\frac{B^2}{8\pi} \mathbf{I} - \frac{\mathbf{B}\mathbf{B}}{4\pi} \right) \cdot \mathbf{u} \right] \cdot d\mathbf{S}}_{\text{work done by Maxwell stress}} \\ &+ \underbrace{\int_{\partial V} \eta \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi} \cdot d\mathbf{S}}_{\substack{\text{resistive slippage accounting} \\ \text{for field not being precisely} \\ \text{frozen into flow}}}. \end{aligned} \quad (11.51)$$

After application of Gauss's theorem and shrinking of the volume V to infinitesimality, we get the differential form of Eq. (11.50):

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{\rho u^2}{2} + \frac{p}{\gamma - 1} + \frac{B^2}{8\pi} \right) &= -\nabla \cdot \left[\frac{\rho u^2}{2} \mathbf{u} + \frac{\gamma}{\gamma - 1} p \mathbf{u} + \mathbf{\Pi} \cdot \mathbf{u} + \mathbf{q} \right. \\ &\quad \left. + \frac{B^2 \mathbf{I} - \mathbf{B}\mathbf{B}}{4\pi} \cdot \mathbf{u} + \eta \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi} \right]. \end{aligned} \quad (11.52)$$

It remains to separate the evolution equation for p by using the fact that we know the equations for ρ , \mathbf{u} and \mathbf{B} and so can deduce the rates of change of the kinetic and magnetic energies.

11.11.1. Kinetic Energy

Using Eqs. (11.46) and (11.47),

$$\begin{aligned} \frac{\partial}{\partial t} \frac{\rho u^2}{2} &= \frac{u^2}{2} \frac{\partial \rho}{\partial t} + \rho \mathbf{u} \cdot \frac{\partial \mathbf{u}}{\partial t} \\ &= -\frac{u^2}{2} \nabla \cdot (\rho \mathbf{u}) - \rho \mathbf{u} \cdot \nabla \frac{u^2}{2} - \mathbf{u} \cdot \left\{ \nabla \cdot \left[\left(p + \frac{B^2}{8\pi} \right) \mathbf{I} - \frac{\mathbf{B}\mathbf{B}}{4\pi} + \mathbf{\Pi} \right] \right\} \\ &= -\nabla \cdot \left[\cancel{\frac{\rho u^2}{2} \mathbf{u}} + \cancel{p \mathbf{u}} + \cancel{\left(\frac{B^2}{8\pi} \mathbf{I} - \frac{\mathbf{B}\mathbf{B}}{4\pi} \right) \cdot \mathbf{u}} + \cancel{\mathbf{\Pi} \cdot \mathbf{u}} \right] \\ &\quad + \underbrace{p \nabla \cdot \mathbf{u}}_{\substack{\text{compressional} \\ \text{work}}} + \underbrace{\left(\frac{B^2}{8\pi} \mathbf{I} - \frac{\mathbf{B}\mathbf{B}}{4\pi} \right) : \nabla \mathbf{u}}_{\substack{\text{energy exchange} \\ \text{with magnetic field}}} + \underbrace{\mathbf{\Pi} : \nabla \mathbf{u}}_{\substack{\text{viscous} \\ \text{dissipation}}}. \end{aligned} \quad (11.53)$$

The flux terms (energy flows and work by stresses on boundaries) that have been crossed out cancel with corresponding terms in Eq. (11.52) once Eq. (11.53) is subtracted from it.

11.11.2. *Magnetic Energy*

Using the induction equation (11.48),

$$\begin{aligned} \frac{\partial B^2}{\partial t} \frac{1}{8\pi} &= \frac{\mathbf{B}}{4\pi} \cdot [-\mathbf{u} \cdot \nabla \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{u} - \mathbf{B} \nabla \cdot \mathbf{u} + \eta \nabla^2 \mathbf{B}] \\ &= -\nabla \cdot \left[\frac{B^2}{8\pi} \mathbf{u} + \eta \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi} \right] - \underbrace{\left(\frac{B^2}{8\pi} \mathbf{1} - \frac{\mathbf{B}\mathbf{B}}{4\pi} \right) : \nabla \mathbf{u}}_{\text{energy exchange with velocity field}} - \underbrace{\eta \frac{|\nabla \times \mathbf{B}|^2}{4\pi}}_{\text{Ohmic dissipation}}. \end{aligned} \quad (11.54)$$

Again, the crossed out flux terms will cancel with corresponding terms in Eq. (11.52). The metamorphosis of the resistive term into a flux term and an Ohmic dissipation term is a piece of vector algebra best checked by expanding the divergence of the flux term. Finally, the \mathbf{u} -to- \mathbf{B} energy exchange term (penultimate on the right-hand side) corresponds precisely to the \mathbf{B} -to- \mathbf{u} exchange term in Eq. (11.53) and cancels with it if we add Eqs. (11.53) and (11.54).

11.11.3. *Thermal Energy*

Subtracting Eqs. (11.53) and (11.54) from Eq. (11.52), consummating the promised cancellations, and mopping up the remaining $\nabla \cdot (p\mathbf{u})$ and $p\nabla \cdot \mathbf{u}$ terms, we end up with the desired evolution equation for the thermal (internal) energy:

$$\underbrace{\frac{d}{dt} \frac{p}{\gamma - 1}}_{\text{advection of internal energy}} = \underbrace{-\nabla \cdot \mathbf{q}}_{\text{heat flux}} - \underbrace{\frac{\gamma}{\gamma - 1} p \nabla \cdot \mathbf{u}}_{\text{compressional heating}} - \underbrace{\boldsymbol{\Pi} : \nabla \mathbf{u}}_{\text{viscous heating}} + \underbrace{\eta \frac{|\nabla \times \mathbf{B}|^2}{4\pi}}_{\text{Ohmic heating}}. \quad (11.55)$$

A further rearrangement and the use of the continuity equation (11.46) to express $\nabla \cdot \mathbf{u} = -d \ln \rho / dt$ turn Eq. (11.55) into

$$\boxed{\frac{d}{dt} \ln \frac{p}{\rho^\gamma} = \frac{\gamma - 1}{p} \left(-\nabla \cdot \mathbf{q} - \boldsymbol{\Pi} : \nabla \mathbf{u} + \eta \frac{|\nabla \times \mathbf{B}|^2}{4\pi} \right)}. \quad (11.56)$$

This form of the thermal-energy equation has very clear physical content: the left-hand side represents advection of the entropy of the MHD fluid by the flow—each fluid element behaves adiabatically, except for the sundry non-adiabatic effects on the right-hand side. The latter are the heat flux in/out of the fluid element and the dissipative (viscous and resistive) heating, leading to entropy production. Note that the form of the viscous stress $\boldsymbol{\Pi}$ ensures that the viscous heating is always positive [see, e.g., Eq. (11.6)]. In these Lectures, we will, for the most part, focus on ideal MHD and so use the adiabatic version of Eq. (11.56), with the right-hand side set to zero.

Let us reiterate the equations of *ideal MHD*, now complete:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (11.57)$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi}, \quad (11.58)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}), \quad (11.59)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \frac{p}{\rho^\gamma} = 0. \quad (11.60)$$

In what follows, we shall study various solutions and asymptotic regimes of these rather nice equations.

12. MHD in a Straight Magnetic Field

Eqs. (11.57–11.60) have a very simple static, uniform *equilibrium solution*:

$$\rho_0 = \text{const}, \quad p_0 = \text{const}, \quad \mathbf{u}_0 = 0, \quad \mathbf{B}_0 = B_0 \hat{\mathbf{z}} = \text{const}. \quad (12.1)$$

We will turn to more nontrivial equilibria in due course, but first we shall study this one carefully—because it is very generic in the sense that many other, more complicated, equilibria *locally* look just like this.

12.1. MHD Waves

If you have an equilibrium solution of any set of equations, your first reflex ought to be to perturb it and see what happens: the system might support waves, instabilities, possibly interesting nonlinear behaviour of small perturbations (e.g., §§7–10).

So we now seek solutions to Eqs. (11.57–11.60) in the form

$$\rho = \rho_0 + \delta\rho, \quad p = p_0 + \delta p, \quad \mathbf{u} = \frac{\partial \boldsymbol{\xi}}{\partial t}, \quad \mathbf{B} = B_0 \hat{\mathbf{z}} + \delta \mathbf{B}, \quad (12.2)$$

where we have introduced the fluid *displacement* field $\boldsymbol{\xi}^\dagger$. To start with, we consider all

† Thinking in terms of displacements makes sense in MHD but not in (homogeneous) hydrodynamics because in the latter case, just displacing a fluid element produces no back reaction, whereas in MHD, because magnetic fields are frozen into the fluid and are elastic, displacing fluid elements causes magnetic restoring forces to switch on.

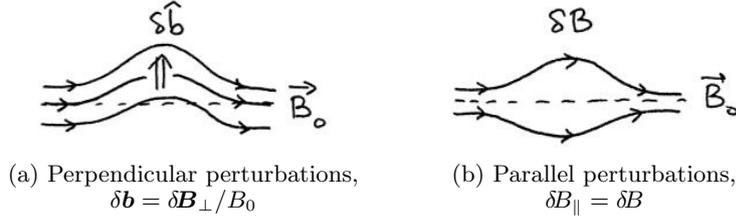


FIGURE 39. Magnetic perturbations.

perturbations to be infinitesimal and so linearise Eqs. (11.57–11.60) as follows.

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \rho = -\rho \nabla \cdot \mathbf{u} &\Rightarrow \frac{\partial \delta \rho}{\partial t} = -\rho_0 \nabla \cdot \frac{\partial \boldsymbol{\xi}}{\partial t} \\ &\Rightarrow \boxed{\frac{\delta \rho}{\rho_0} = -\nabla \cdot \boldsymbol{\xi}}, \end{aligned} \quad (12.3)$$

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \frac{p}{\rho^\gamma} = 0 &\Rightarrow \frac{\partial \delta p}{\partial t} \frac{\delta p}{p_0} = \gamma \frac{\partial \delta \rho}{\partial t} \frac{\delta \rho}{\rho_0} \\ &\Rightarrow \boxed{\frac{\delta p}{p_0} = -\gamma \nabla \cdot \boldsymbol{\xi}}, \end{aligned} \quad (12.4)$$

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} - \mathbf{B} \nabla \cdot \mathbf{u} &\Rightarrow \frac{\partial \delta \mathbf{B}}{\partial t} = B_0 \nabla_\parallel \frac{\partial \boldsymbol{\xi}}{\partial t} - \hat{\mathbf{z}} B_0 \nabla \cdot \frac{\partial \boldsymbol{\xi}}{\partial t} \\ &\Rightarrow \frac{\delta \mathbf{B}}{B_0} = \nabla_\parallel \boldsymbol{\xi} - \hat{\mathbf{z}} \nabla \cdot \boldsymbol{\xi} = \nabla_\parallel \boldsymbol{\xi}_\perp - \hat{\mathbf{z}} \nabla_\perp \cdot \boldsymbol{\xi}_\perp \\ &\Rightarrow \boxed{\frac{\delta \mathbf{B}_\perp}{B_0} = \nabla_\parallel \boldsymbol{\xi}_\perp}, \quad \boxed{\frac{\delta B_\parallel}{B_0} = -\nabla_\perp \cdot \boldsymbol{\xi}_\perp}, \end{aligned} \quad (12.5)$$

where \parallel and \perp denote projections onto the direction (z) of \mathbf{B}_0 and onto the plane (x, y) perpendicular to it, respectively. Eqs. (12.5) tell us that parallel displacements produce no perturbation of the magnetic field—obviously not, because the magnetic field is carried with the fluid flow and nothing will happen if you displace a straight uniform field parallel to itself.

The physics of magnetic-field perturbations become clearer if we observe that

$$\frac{\delta \mathbf{B}}{B_0} = \frac{\delta(B\mathbf{b})}{B_0} = \delta \mathbf{b} + \hat{\mathbf{z}} \frac{\delta B}{B_0}. \quad (12.6)$$

The perturbed field-direction vector $\delta \mathbf{b}$ must be perpendicular to $\hat{\mathbf{z}}$ (otherwise the field direction is unperturbed; formally this is shown by perturbing the equation $\mathbf{b}^2 = 1$). Therefore, the perpendicular and parallel perturbations of the magnetic field are the perturbations of its direction and strength, respectively (Fig. 39):

$$\frac{\delta \mathbf{B}_\perp}{B_0} = \delta \mathbf{b}, \quad \frac{\delta B_\parallel}{B_0} = \frac{\delta B}{B_0}. \quad (12.7)$$

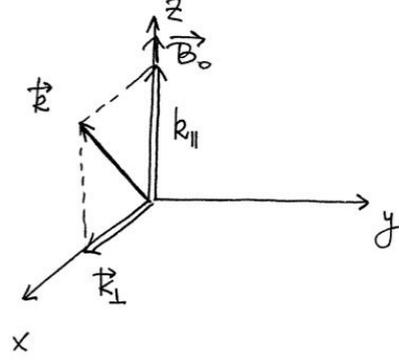


FIGURE 40. Coordinate system for the treatment of MHD waves.

Finally, linearising Eq. (11.58) gives us

$$\begin{aligned}
 \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) &= -\nabla p - \frac{\nabla B^2}{8\pi} + \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{4\pi} \quad (12.8) \\
 = \rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} &= \underbrace{-\nabla \delta p}_{\substack{= \gamma p_0 \nabla \nabla \cdot \boldsymbol{\xi}, \\ \text{Eq. (12.4)}}} - \frac{B_0^2}{4\pi} \nabla \frac{\delta B}{B_0} + \frac{B_0^2}{4\pi} \nabla_{\parallel} \left(\delta \mathbf{b} + \hat{\mathbf{z}} \frac{\delta B}{B_0} \right) \\
 &= \frac{B_0^2}{4\pi} \left(-\nabla_{\perp} \frac{\delta B}{B_0} + \nabla_{\parallel} \delta \mathbf{b} \right) \\
 &= \frac{B_0^2}{4\pi} (\nabla_{\perp} \nabla_{\perp} \cdot \boldsymbol{\xi}_{\perp} + \nabla_{\parallel}^2 \boldsymbol{\xi}_{\perp}), \\
 &\quad \text{Eqs. (12.7) and (12.5)}
 \end{aligned}$$

Assembling all this, we get

$$\boxed{\frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = c_s^2 \nabla \nabla \cdot \boldsymbol{\xi} + v_A^2 (\nabla_{\perp} \nabla_{\perp} \cdot \boldsymbol{\xi}_{\perp} + \nabla_{\parallel}^2 \boldsymbol{\xi}_{\perp})}, \quad (12.9)$$

where two special velocities have emerged:

$$c_s = \sqrt{\frac{\gamma p_0}{\rho_0}}, \quad v_A = \frac{B_0}{\sqrt{4\pi \rho_0}}, \quad (12.10)$$

the *sound speed* and the *Alfvén speed*, respectively. The former is familiar from fluid dynamics, while the latter is another speed, arising in MHD, at which perturbations can travel. We shall see momentarily how this happens.

Let us seek wave-like solutions of Eq. (12.9), $\boldsymbol{\xi} \propto \exp(-i\omega t + i\mathbf{k} \cdot \mathbf{r})$. For such perturbations,

$$\omega^2 \boldsymbol{\xi} = c_s^2 \mathbf{k} \mathbf{k} \cdot \boldsymbol{\xi} + v_A^2 (\mathbf{k}_{\perp} \mathbf{k}_{\perp} \cdot \boldsymbol{\xi}_{\perp} + k_{\parallel}^2 \boldsymbol{\xi}_{\perp}). \quad (12.11)$$

Without loss of generality, let $\mathbf{k} = (k_{\perp}, 0, k_{\parallel})$ (i.e., by definition, x is the direction of \mathbf{k}_{\perp} ; see Fig. 40). Then Eq. (12.11) becomes

$$\omega^2 \xi_x = c_s^2 k_{\perp} (k_{\perp} \xi_x + k_{\parallel} \xi_{\parallel}) + v_A^2 k_{\perp}^2 \xi_x, \quad (12.12)$$

$$\omega^2 \xi_y = v_A^2 k_{\parallel}^2 \xi_y, \quad (12.13)$$

$$\omega^2 \xi_{\parallel} = c_s^2 k_{\parallel} (k_{\perp} \xi_x + k_{\parallel} \xi_{\parallel}). \quad (12.14)$$

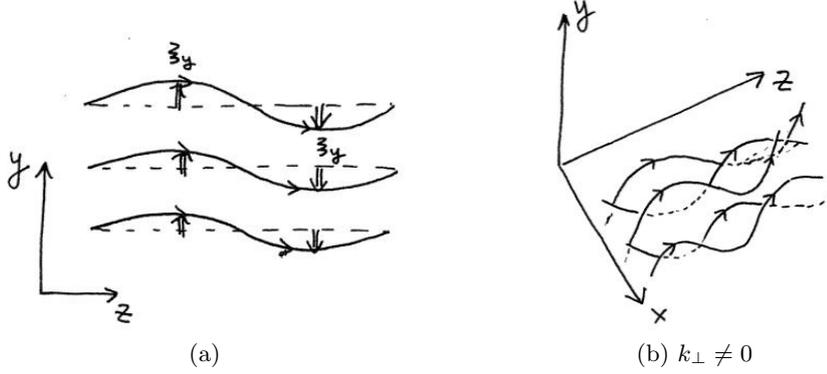


FIGURE 41. Alfvén waves.

The perturbations of the rest of the fields are

$$\frac{\delta\rho}{\rho_0} = -i\mathbf{k} \cdot \boldsymbol{\xi} = -i(k_{\perp}\xi_x + k_{\parallel}\xi_{\parallel}), \quad (12.15)$$

$$\frac{\delta p}{p_0} = \gamma \frac{\delta\rho}{\rho_0}, \quad (12.16)$$

$$\delta\mathbf{b} = ik_{\parallel}\boldsymbol{\xi}_{\perp} = ik_{\parallel} \begin{pmatrix} \xi_x \\ \xi_y \\ 0 \end{pmatrix}, \quad (12.17)$$

$$\frac{\delta B}{B_0} = -ik_{\perp}\xi_x. \quad (12.18)$$

12.1.1. Alfvén Waves

We start by spotting, instantly, that Eq. (12.13) decouples from the rest. Therefore, $\boldsymbol{\xi} = (0, \xi_y, 0)$ is an eigenvector, with two associated eigenvalues

$$\boxed{\omega = \pm k_{\parallel}v_A}, \quad (12.19)$$

representing *Alfvén waves* propagating parallel and antiparallel to \mathbf{B}_0 . An Alfvénic perturbation is (Fig. 41a)

$$\boldsymbol{\xi} = \xi_y\hat{\mathbf{y}}, \quad \delta\rho = 0, \quad \delta p = 0, \quad \delta B = 0, \quad \delta\mathbf{b} = ik_{\parallel}\xi_y\hat{\mathbf{y}}, \quad (12.20)$$

i.e., it is incompressible and only involves magnetic field lines behaving as elastic strings, springing back against perturbing motions, due to the restoring curvature force. Note that these waves can have $k_{\perp} \neq 0$ even though their dispersion relation (12.19) does not depend on k_{\perp} (Fig. 41b).

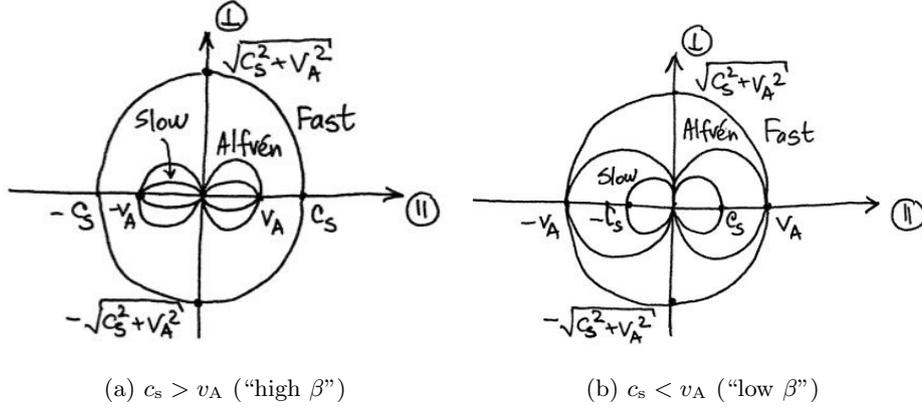
12.1.2. Magnetosonic Waves

Eqs. (12.12) and (12.14) form a closed 2D system:

$$\omega^2 \begin{pmatrix} \xi_x \\ \xi_{\parallel} \end{pmatrix} = \begin{pmatrix} c_s^2 k_{\perp}^2 + v_A^2 k^2 & c_s^2 k_{\parallel} k_{\perp} \\ c_s^2 k_{\parallel} k_{\perp} & c_s^2 k_{\parallel}^2 \end{pmatrix} \cdot \begin{pmatrix} \xi_x \\ \xi_{\parallel} \end{pmatrix}. \quad (12.21)$$

The resulting dispersion relation is

$$\omega^4 - k^2(c_s^2 + v_A^2)\omega^2 + c_s^2 v_A^2 k^2 k_{\parallel}^2 = 0. \quad (12.22)$$


 FIGURE 42. Friedrichs diagram: radius ω/k , angle $\cos \theta = k_{\parallel}/k$.

This has four solutions:

$$\omega^2 = \frac{1}{2} k^2 \left[c_s^2 + v_A^2 \pm \sqrt{(c_s^2 + v_A^2)^2 - 4c_s^2 v_A^2 \cos^2 \theta} \right], \quad \cos^2 \theta = \frac{k_{\parallel}^2}{k^2}. \quad (12.23)$$

The two "+" solutions are the "fast magnetosonic waves" and the two "-" ones are the "slow magnetosonic waves".

Since both sound and Alfvén speeds are involved, it is obvious that the key parameter demarcating different physical regimes will be their ratio, or, conventionally, the ratio of the thermal to magnetic energies in the MHD medium, known as the *plasma beta*:

$$\beta = \frac{p_0}{B_0^2/8\pi} = \frac{2}{\gamma} \frac{c_s^2}{v_A^2}. \quad (12.24)$$

The magnetosonic waves can be conveniently summarised by the so-called *Friedrichs diagram*, a graph of Eq. (12.23) in polar coordinates where the phase speed ω/k is radius and the angle is θ , the direction of propagation with respect to \mathbf{B}_0 (Fig. 42).

Clearly, magnetosonic waves contain perturbations of both the magnetic field and of the "hydrodynamic" quantities ρ , p , \mathbf{u} , but working them all out for the case of general oblique propagation ($\theta \sim 1$) is a bit messy. The physics of what is going on is best understood via a few particular cases.

12.1.3. Parallel Propagation

Consider $k_{\perp} = 0$ ($\theta = 0$). Then $(\xi_x, 0, 0)$ and $(0, 0, \xi_{\parallel})$ are eigenvectors of the matrix in Eq. (12.21) and the two corresponding waves are

- another *Alfvén wave*, this time with perturbation in the x direction (which, however, is not physically different from the y direction when $k_{\perp} = 0$):

$$\omega^2 \xi_x = k_{\parallel}^2 v_A^2 \xi_x \quad \Rightarrow \quad \omega = \pm k_{\parallel} v_A, \quad (12.25)$$

$$\boldsymbol{\xi} = \xi_x \hat{\mathbf{x}}, \quad \delta \rho = 0, \quad \delta p = 0, \quad \delta B = 0, \quad \delta \mathbf{b} = i k_{\parallel} \xi_x \hat{\mathbf{x}} \quad (12.26)$$

(at high β , this is the slow wave, at low β , this is the fast wave);

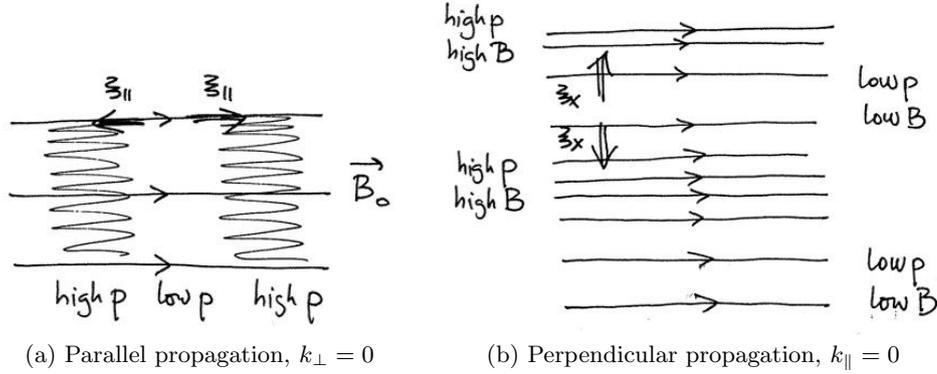


FIGURE 43. Sound waves.

- the parallel-propagating *sound wave* (Fig. 43a):

$$\omega^2 \xi_{\parallel} = k_{\parallel}^2 c_s^2 \xi_{\parallel} \Rightarrow \boxed{\omega = \pm k_{\parallel} c_s}, \quad (12.27)$$

$$\boldsymbol{\xi} = \xi_{\parallel} \hat{\mathbf{z}}, \quad \frac{\delta \rho}{\rho_0} = -ik_{\parallel} \xi_{\parallel}, \quad \frac{\delta p}{p_0} = \gamma \frac{\delta \rho}{\rho_0}, \quad \delta B = 0, \quad \delta \mathbf{b} = 0 \quad (12.28)$$

(at high β , this is the fast wave, at low β , this is the slow wave); the magnetic field does not participate here at all.

12.1.4. Perpendicular Propagation

Now consider $\boxed{k_{\parallel} = 0}$ ($\theta = 90^\circ$). Then $(\xi_x, 0, 0)$ is again an eigenvector of the matrix in Eq. (12.21)[†]. The resulting fast magnetosonic wave is again a *sound wave*, but because it is perpendicular-propagating, both thermal and magnetic pressures get involved, the perturbations are compressions/rarefactions in both the fluid and the field, and the speed at which they travel is a combination of the sound and Alfvén speeds (with the latter now representing the magnetic pressure response):

$$\omega^2 \xi_x = k_{\perp}^2 (c_s^2 + v_A^2) \xi_x \Rightarrow \boxed{\omega = \pm k_{\perp} \sqrt{c_s^2 + v_A^2}}, \quad (12.29)$$

$$\boldsymbol{\xi} = \xi_x \hat{\mathbf{x}}, \quad \frac{\delta \rho}{\rho_0} = -ik_{\perp} \xi_x, \quad \frac{\delta p}{p_0} = \gamma \frac{\delta \rho}{\rho_0}, \quad \frac{\delta B}{B_0} = -ik_{\perp} \xi_x, \quad \delta \mathbf{b} = 0. \quad (12.30)$$

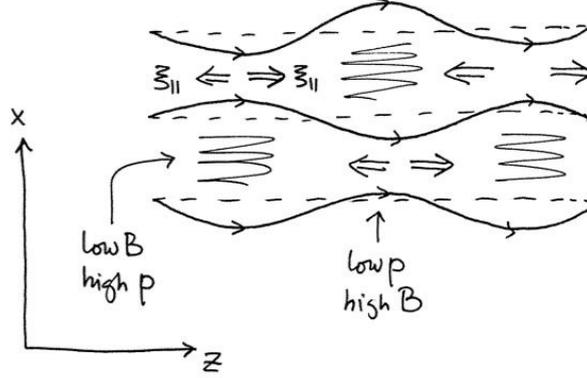
Note that the thermal and magnetic compressions are in phase and there is no bending of the magnetic field (Fig. 43b).

12.1.5. Anisotropic Perturbations: $k_{\parallel} \ll k_{\perp}$

Taking $k_{\parallel} = 0$ in §12.1.4 was perhaps a little radical as we lost all waves apart from the fast one. As we are about to see, a lot of babies were thrown out with this particular bathwater.

So let us consider MHD waves in the limit $\boxed{k_{\parallel} \ll k_{\perp}}$. This turns out to be an extremely relevant regime, because, in a strong magnetic field, realistically excitable perturbations, both linear and nonlinear, tend to be highly elongated in the direction of the field. Going

[†] As is $(0, 0, \xi_{\parallel})$, but with $\omega = 0$; we will deal with this mode in §12.3.4.

FIGURE 44. Slow wave in the anisotropic limit $k_{\parallel} \ll k_{\perp}$: pressure balanced, $\xi_x \ll \xi_{\parallel}$.

back to Eq. (12.23) and enforcing this limit, we get

$$\begin{aligned} \omega^2 &= \frac{1}{2} k^2 (c_s^2 + v_A^2) \left[1 \pm \sqrt{1 - \frac{4c_s^2 v_A^2}{(c_s^2 + v_A^2)^2} \frac{k_{\parallel}^2}{k^2}} \right] \\ &\approx \frac{1}{2} k^2 (c_s^2 + v_A^2) \left[1 \pm 1 \mp \frac{2c_s^2 v_A^2}{(c_s^2 + v_A^2)^2} \frac{k_{\parallel}^2}{k^2} \right]. \end{aligned} \quad (12.31)$$

The upper sign gives the familiar *fast wave*

$$\omega = \pm k \sqrt{c_s^2 + v_A^2}. \quad (12.32)$$

This is just the magnetically enhanced sound wave that was considered in §12.1.4. The small corrections to it due to k_{\parallel}/k are not particularly interesting.

The lower sign in Eq. (12.31) gives the *slow wave*

$$\omega = \pm k_{\parallel} \frac{c_s v_A}{\sqrt{c_s^2 + v_A^2}}, \quad (12.33)$$

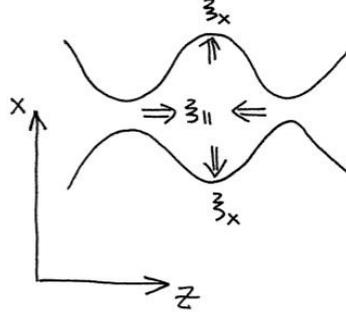
which is more interesting. Let us find the corresponding eigenvector: from Eq. (12.14),

$$\begin{aligned} (\omega^2 - k_{\parallel}^2 c_s^2) \xi_{\parallel} &= k_{\parallel} k_{\perp} c_s^2 \xi_x. \\ &= -k_{\parallel}^2 \frac{c_s^4}{c_s^2 + v_A^2}, \\ &\text{Eq. (12.33)} \end{aligned} \quad (12.34)$$

Therefore, the displacements are mostly parallel:

$$\frac{\xi_x}{\xi_{\parallel}} = -\frac{k_{\parallel}}{k_{\perp}} \frac{c_s^2}{c_s^2 + v_A^2} \ll 1. \quad (12.35)$$

Using this equation together with Eqs. (12.15–12.18), we find that the perturbations in

FIGURE 45. Slow wave in the high- β limit: pressure balanced, $\xi_x \sim \xi_{\parallel}$.

the remaining fields are

$$\frac{\delta\rho}{\rho_0} = -i(k_{\perp}\xi_x + k_{\parallel}\xi_{\parallel}) = -i\frac{v_A^2}{c_s^2 + v_A^2}k_{\parallel}\xi_{\parallel}, \quad (12.36)$$

$$\frac{\delta p}{p_0} = \gamma\frac{\delta\rho}{\rho_0}, \quad (12.37)$$

$$\delta\mathbf{b} = ik_{\parallel}\xi_x\hat{\mathbf{x}} = -i\frac{k_{\parallel}}{k_{\perp}}\frac{c_s^2}{c_s^2 + v_A^2}k_{\parallel}\xi_{\parallel}\hat{\mathbf{x}} \rightarrow 0, \quad (12.38)$$

$$\frac{\delta B}{B_0} = -ik_{\perp}\xi_x = i\frac{c_s^2}{c_s^2 + v_A^2}k_{\parallel}\xi_{\parallel}. \quad (12.39)$$

Thus, to lowest order in k_{\parallel}/k_{\perp} , this wave involves no bending of the magnetic field, but has a pressure/density perturbation and a magnetic-field-strength perturbation—the latter in counter-phase to the former (Fig. 44). To be precise, *the slow-wave perturbations are pressure balanced*:

$$\delta\left(p + \frac{B^2}{8\pi}\right) = p_0\frac{\delta p}{p_0} + \frac{B_0^2}{4\pi}\frac{\delta B}{B_0} = \rho_0\left(c_s^2\frac{\delta\rho}{\rho_0} + v_A^2\frac{\delta B}{B_0}\right) = 0. \quad (12.40)$$

The same is, of course, already obvious from the momentum equation (12.8), where, in the limit $k_{\parallel} \ll k_{\perp}$ and $\omega \ll kc_s$ (“incompressible” perturbations; see §12.2), the dominant balance is

$$\nabla_{\perp}\left(p + \frac{B^2}{8\pi}\right) = 0. \quad (12.41)$$

Finally, the Alfvén waves in the limit of anisotropic propagation are just the same as ever (§12.1.1)—they are unaffected by k_{\perp} , while being perfectly capable of having perpendicular variation (Fig. 41b).

12.1.6. High- β Limit: $c_s \gg v_A$

Another limit in which high-frequency acoustic response (fast waves) and low-frequency, pressure-balanced Alfvénic response (slow and Alfvén waves) are separated is $\beta \gg 1 \Leftrightarrow c_s \gg v_A$ †. In this limit, the approximate expression (12.31) for the magnetosonic frequencies is still valid, but because v_A/c_s , rather than k_{\parallel}/k , is small. The rest of the

† This limit is astrophysically very interesting because magnetic fields locally produced by plasma motions in various astrophysical environments (e.g., interstellar and intergalactic media) can only be as strong energetically as the motions that make them [see Eq. (11.45) and §11.10] and so, the latter being subsonic, $v_A \sim u \ll c_s$.

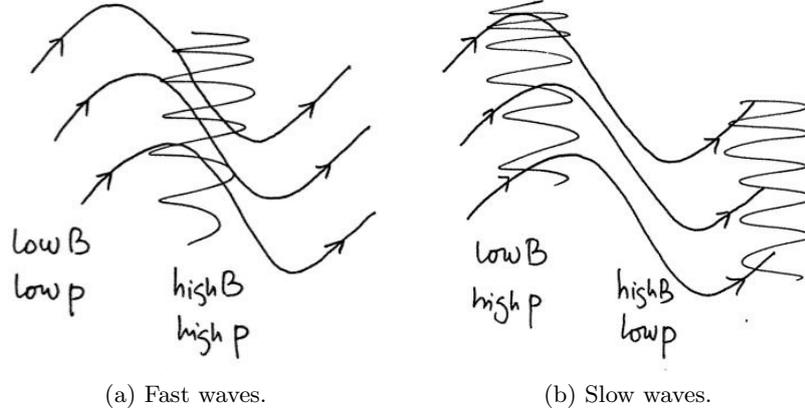


FIGURE 46. General oblique magnetosonic waves.

calculations in §12.1.5 are also valid, with the following simplifications arising from v_A being negligible compared to c_s .

The upper sign in Eq. (12.31) again gives us the *fast wave*, which, this time, is a pure *sound wave*:

$$\omega = \pm k c_s. \quad (12.42)$$

This is natural because, at high β , the magnetic pressure is negligible compared to thermal pressure and sound can propagate oblivious of the magnetic field.

The lower sign in Eq. (12.31) yields the *slow wave*: Eq. (12.33) is still valid and becomes, for $v_A \ll c_s$,

$$\omega = \pm k_{\parallel} v_A. \quad (12.43)$$

Because the slow wave's dispersion relation in this limit looks exactly like that of an Alfvén wave [Eq. (12.19)], it is called the *pseudo-Alfvén wave*. The similarity is deceptive as the nature of the perturbation (the eigenvector) is completely different. Substituting $\omega^2 = k_{\parallel}^2 v_A^2$ into Eq. (12.14), we find

$$k_{\perp} \xi_x + k_{\parallel} \xi_{\parallel} = \frac{v_A^2}{c_s^2} k_{\parallel} \xi_{\parallel} \ll k_{\parallel} \xi_{\parallel}. \quad (12.44)$$

This just says that, to lowest order in $1/\beta$, $\nabla \cdot \xi = 0$, i.e., the perturbations are incompressible. In contrast to the anisotropic case [Eq. (12.35)], the perpendicular and parallel displacements are now comparable (at least in general, assuming $k_{\parallel} \sim k_{\perp}$):

$$\frac{\xi_x}{\xi_{\parallel}} = -\frac{k_{\parallel}}{k_{\perp}}. \quad (12.45)$$

Also in contrast to the anisotropic case, the density and pressure perturbations are now

vanishingly small, but the field can be bent as well as compressed:

$$\frac{\delta\rho}{\rho_0} = -i(k_\perp\xi_x + k_\parallel\xi_\parallel) = -i\frac{v_A^2}{c_s^2}k_\parallel\xi_\parallel \rightarrow 0, \quad (12.46)$$

$$\frac{\delta p}{p_0} = \gamma \frac{\delta\rho}{\rho_0} \rightarrow 0, \quad (12.47)$$

$$\delta\mathbf{b} = ik_\parallel\xi_x\hat{\mathbf{x}} = -i\frac{k_\parallel}{k_\perp}k_\parallel\xi_\parallel, \quad (12.48)$$

$$\frac{\delta B}{B_0} = -ik_\perp\xi_x = ik_\parallel\xi_\parallel. \quad (12.49)$$

The δB and $\delta\mathbf{b}$ perturbations are in counter-phase, as are ξ_\parallel and ξ_x (Fig. 45). It is easy to check that pressure balance (12.40) is again maintained by these perturbations.

In the more general case of oblique propagation ($k_\parallel \sim k_\perp$) and finite beta ($\beta \sim 1$), the fast and slow magnetosonic waves generally have comparable frequencies and contain perturbations of all relevant fields, with the fast waves tending to have the perturbations of the thermal and magnetic pressure in phase and slow waves in counter-phase (Fig. 46).

12.2. Incompressible MHD

Enough linear theory! We shall now occupy ourselves with the behaviour of *finite* (although still small) perturbations of a straight-field equilibrium. While we abandon linearisation (i.e., the neglect of nonlinear terms), much of what the linear theory has taught us about the basic responses of an MHD fluid remains true and useful. In particular, the linear relations between the perturbation amplitudes of various fields provide us with a guidance as to the *relative size* of finite perturbations of these fields. This makes sense if, while allowing the nonlinearities back in, we do not assume the linear physics to be completely negligible, i.e., if we allow the linear and nonlinear time scales to compete (§12.2.4). We shall see that solutions for which this is the case satisfy self-consistent equations, so can be expected to be realisable (and, as we know from experimental, observational and numerical evidence, *are* realised).

12.2.1. Subsonic Ordering

We start by constructing nonlinear equations that describe the incompressible limit, i.e., fields and motions that are subsonic: both their phase speeds and flow velocities will be assumed small compared to the speed of sound [Eq. (12.32)]:

$$\frac{\omega/k}{\sqrt{c_s^2 + v_A^2}} \ll 1, \quad \text{Ma} \equiv \frac{u}{\sqrt{c_s^2 + v_A^2}} \ll 1. \quad (12.50)$$

In this limit, we expect all fast-wave-like perturbations to disappear (in a similar way to the sound waves disappearing in the incompressible Navier–Stokes hydrodynamics) and for the MHD dynamics to contain only Alfvénic and slow-wave-like perturbations. We saw in §§12.1.5 and 12.1.6 that, linearly, fast and slow waves are well separated either in the limit of $k_\parallel/k_\perp \ll 1$ or in the limit of $\beta \gg 1$. Indeed, comparing the Alfvén [Eq. (12.19)] and slow-wave [Eq. (12.33)] frequencies to the sound (fast-wave) frequency (12.32), we get

$$\frac{\omega_{\text{Alfvén}}}{\omega_{\text{fast}}} \sim \frac{k_\parallel v_A}{k\sqrt{c_s^2 + v_A^2}} \sim \frac{k_\parallel}{k} \frac{1}{\sqrt{1 + \beta}}, \quad \frac{\omega_{\text{slow}}}{\omega_{\text{fast}}} \sim \frac{k_\parallel c_s v_A}{k(c_s^2 + v_A^2)} \sim \frac{k_\parallel}{k} \frac{\sqrt{\beta}}{1 + \beta}, \quad (12.51)$$

both of which are small in either of the two limits, satisfying the first of the conditions (12.50).

The second condition (12.50) involves the “*magnetic Mach number*” Ma (generalised to compare the flow velocity to the speed of sound in a magnetised fluid), which measures the size of the perturbations themselves—in the linear theory, this was arbitrarily small, but now we will need to relate it to our other small parameter(s), k_{\parallel}/k or $1/\beta$. This means that we would like to construct an asymptotic *ordering* in which there will be some prescription as to how small, or otherwise, various (relative) perturbations and small parameters are—not by themselves, i.e., compared to 1, but compared to each other (compared to 1, the small parameters can all formally be taken to be as small as we desire).

The general strategy for ordering perturbations with respect to each other will be to use the linear relations obtained in the two incompressible limits ($k_{\parallel}/k \ll 1$ or $\beta \gg 1$). If we do not specifically expect one perturbation to be larger or smaller than another on some physical grounds (like the properties of the linear response), we must order them the same; this does not stop us later constructing subsidiary expansions in which they might be different. For example, MHD equations themselves were an expansion a number of small parameters, in particular u/c [see Eq. (11.14)]. However, at the time of deriving them, we did not want to rule out sonic or supersonic motions and so, effectively, we ordered $\text{Ma} \sim 1$, $k_{\parallel}/k \sim 1$ and $\beta \sim 1$, as far as the u/c expansion was concerned, i.e., Ma , k_{\parallel}/k , $1/\beta \gg u/c$. Now we are constructing a subsidiary expansion in these other parameters, keeping in mind that they are allowed to be small but not as small as the small parameter already used in the derivation of the MHD equations, which are our starting point.

12.2.2. Ordering of Alfvénic Perturbations

Since the Alfvénic perturbations decouple completely from the rest (§12.1.1), linear theory does not give us a way to relate u_y to u_{\parallel} , so we will exercise the no-prejudice principle stated above and assume

$$u_y \sim u_{\parallel}, \quad (12.52)$$

i.e., the Mach numbers for the Alfvénic and slow-wave-like motions are comparable. We can, however, relate u_y to $\delta \mathbf{b}$, via the curvature-force response [Eq. (12.20)]:

$$|\delta \mathbf{b}| \sim k_{\parallel} \xi_y \sim \frac{k_{\parallel} u_y}{\omega} \sim \frac{u_y}{v_A} \sim \text{Ma} \sqrt{1 + \beta}. \quad (12.53)$$

12.2.3. Ordering of Slow-Wave-Like Perturbations

For slow-wave-like perturbations, in either the anisotropic or the high- β limit, from Eqs. (12.14) and (12.33),

$$\nabla \cdot \mathbf{u} \sim \omega(k_{\perp} \xi_x + k_{\parallel} \xi_{\parallel}) \sim \frac{\omega^2}{k_{\parallel} c_s^2} \omega \xi_{\parallel} \sim \frac{v_A^2}{c_s^2 + v_A^2} k_{\parallel} u_{\parallel} \sim \frac{k_{\parallel} u_{\parallel}}{1 + \beta}. \quad (12.54)$$

Thus, the divergence of the flow velocity is small (the dynamics are incompressible) in all three of our (potentially) small parameters:

$$\frac{\nabla \cdot \mathbf{u}}{k \sqrt{c_s^2 + v_A^2}} \sim \frac{k_{\parallel}}{k} \frac{1}{1 + \beta} \text{Ma}. \quad (12.55)$$

From this, we can immediately obtain an ordering for the density and pressure perturbations: using Eqs. (12.3), (12.4), (12.33) and Eq. (12.54) [cf. Eqs. (12.36) and (12.46)],

$$\frac{\delta\rho}{\rho_0} \sim \frac{\delta p}{p_0} \sim \nabla \cdot \boldsymbol{\xi} \sim \frac{\nabla \cdot \mathbf{u}}{\omega} \sim \frac{\text{Ma}}{\sqrt{\beta}}. \quad (12.56)$$

The magnetic-field-strength (magnetic-pressure) perturbation is, using Eqs. (12.39) and (12.33) [cf. Eq. (12.49)],

$$\frac{\delta B}{B_0} \sim k_{\perp} \xi_x \sim \frac{c_s^2}{c_s^2 + v_A^2} \frac{k_{\parallel} u_{\parallel}}{\omega} \sim \sqrt{\beta} \text{Ma}, \quad (12.57)$$

or, perhaps more straightforwardly, from pressure balance (12.40) and using Eq. (12.56),

$$\frac{\delta B}{B_0} = -\frac{\beta}{2} \frac{\delta p}{p_0} \sim \sqrt{\beta} \text{Ma}. \quad (12.58)$$

Finally, in a similar fashion, using Eqs. (12.17) and (12.57) [cf. Eqs. (12.38) and (12.48)], we find

$$|\delta \mathbf{b}| \sim k_{\parallel} \xi_x \sim \frac{k_{\parallel}}{k_{\perp}} \sqrt{\beta} \text{Ma} \quad (12.59)$$

for slow-wave-like perturbations. Note that in all interesting limits this is overridden by the Alfvénic perturbations (12.53).

12.2.4. Ordering of Time Scales

Let us recall that our motivation for using linear relations between perturbations to determine their relative sizes in a nonlinear regime was that linear response will lose its exclusive sway but remain non-negligible. In formal terms, this means that we must order the linear and nonlinear time scales to be comparable[†]. The nonlinearities in MHD equations are advective, i.e., of the form $\mathbf{u} \cdot \nabla(\text{stuff})$ and similar, so the rate of nonlinear interaction is $\sim ku$ (in the case of anisotropic perturbations, $\sim k_{\perp} u_{\perp}$). Ordering this to be comparable to the frequencies of the Alfvén and slow waves [see Eq. (12.51)] gives us

$$\omega_{\text{Alfvén}} \sim ku \quad \Rightarrow \quad \text{Ma} \sim \frac{k_{\parallel}}{k} \frac{1}{\sqrt{1+\beta}}, \quad (12.60)$$

$$\omega_{\text{slow}} \sim ku \quad \Rightarrow \quad \text{Ma} \sim \frac{k_{\parallel}}{k} \frac{\sqrt{\beta}}{1+\beta}. \quad (12.61)$$

Note that the first of these relations overrides the second in all interesting limits.

12.2.5. Summary of Subsonic Ordering

Thus, the ordering of the time scales determines the size of the perturbations via Eq. (12.60). Using this restriction on Ma, we may summarise our subsonic ordering as follows[‡]

$$\boxed{\text{Ma} \equiv \frac{u}{\sqrt{c_s^2 + v_A^2}} \sim \frac{|\delta \mathbf{b}|}{\sqrt{1+\beta}} \sim \frac{1}{\sqrt{\beta}} \frac{\delta B}{B} \sim \sqrt{\beta} \frac{\delta p}{\rho_0} \sim \sqrt{\beta} \frac{\delta p}{p_0} \sim \frac{k_{\parallel}}{k} \frac{1}{\sqrt{1+\beta}} \ll 1} \quad (12.62)$$

[†] In the context of MHD turbulence theory, this principle, applied at each scale, is known as the *critical balance* (see §12.4).

[‡] Note that it is not absolutely necessary to work out detailed linear theory of a set of equations in order to be able to construct such orderings: it is often enough to know roughly where you are going and simply balance terms representing the physics that you wish to keep (or expect to have to keep). An example of this approach is given in §12.2.9.

and $\omega \sim ku$. The ordering can be achieved either in the limit of $k_{\parallel}/k \ll 1$ or $1/\beta \ll 1$, or both. Note that if one of these parameters is small, the other can be order unity or even large (as long as it is not larger than the inverse of the small one).

The case of anisotropic perturbations and arbitrary β applies in a broad range of plasmas, from magnetically confined fusion ones (tokamaks, stellarators) to space (e.g., the solar corona or the solar wind). We shall consider the implications of this ordering in §12.3.

The case of high β applies, e.g., to high-energy galactic and extragalactic plasmas. It is the direct generalisation to MHD of incompressible Navier–Stokes hydrodynamics, i.e., in this case, all one needs to do is solve MHD equations assuming $\rho = \text{const}$ and $\nabla \cdot \mathbf{u} = 0$. We shall consider this case now.

12.2.6. Incompressible MHD Equations

Assuming $\beta \gg 1$, our ordering becomes

$$\boxed{\frac{u}{c_s} \sim \frac{\omega}{kc_s} \sim \frac{1}{\sqrt{\beta}} \sim \text{Ma}, \quad |\delta \mathbf{b}| \sim \frac{\delta B}{B_0} \sim \sqrt{\beta} \text{Ma} \sim 1, \quad \frac{\delta \rho}{\rho_0} \sim \frac{\delta p}{p_0} \sim \frac{\text{Ma}}{\sqrt{\beta}} \sim \text{Ma}^2}.}$$
(12.63)

Thus, the density and pressure perturbations are minuscule, while magnetic perturbations are order unity—magnetic fields are relatively easy to bend (i.e., subsonic motions can tangle the field substantially in this regime). Because of this, it will not make sense to split \mathbf{B} into \mathbf{B}_0 and $\delta \mathbf{B}$ explicitly, we will treat the magnetic field as a single field, with no need for a strong mean component.

Let us examine the MHD equations (11.57–11.60) under the ordering (12.63).

Since $\omega \sim ku$, the convective derivative $d/dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ survives intact in all equations, allowing the advective nonlinearity to enter.

The continuity equation (11.57) simply reiterates our earlier statement that the velocity field is divergenceless to lowest order:

$$\nabla \cdot \mathbf{u} = -\frac{1}{\rho} \frac{d\rho}{dt} \sim \omega \frac{\delta \rho}{\rho_0} \sim \text{Ma}^3 kc_s \rightarrow 0. \quad (12.64)$$

The momentum equation (11.58) becomes

$$\left(1 + \frac{\delta \rho}{\rho_0}\right) \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right) = -\nabla \cdot \underbrace{\left(\frac{c_s^2}{\gamma} \frac{\delta p}{p_0} + \frac{B^2}{8\pi\rho_0}\right)}_{\equiv \tilde{p}} + \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{4\pi\rho_0}. \quad (12.65)$$

The density perturbation in the left-hand side is $\sim \text{Ma}^2$ and so negligible compared to unity. The remaining terms in this equation are all the same order ($\sim \text{Ma}^2 kc_s^2$) and so they must all be kept. The total “pressure” \tilde{p} is determined by enforcing $\nabla \cdot \mathbf{u} = 0$ [Eq. (12.64)]. Namely, our equations are

$$\boxed{\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \tilde{p} + \mathbf{B} \cdot \nabla \mathbf{B}}, \quad (12.66)$$

where

$$\nabla^2 \tilde{p} = -\nabla \nabla : (\mathbf{u}\mathbf{u} - \mathbf{B}\mathbf{B}) \quad (12.67)$$

and we have rescaled the magnetic field to velocity units, $\mathbf{B}/\sqrt{4\pi\rho_0} \rightarrow \mathbf{B}$.

In the induction equation, best written in the form (11.27), all terms are the same order $\sim kuB \sim \text{Ma} kc_s B$ except the one containing $\nabla \cdot \mathbf{u}$, which is $\sim \text{Ma}^3 kc_s B$ and so

must be neglected. We are left with

$$\boxed{\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u}}. \quad (12.68)$$

Finally, the internal-energy equation (11.60), which, keeping only the lowest-order terms, becomes

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \left(\frac{\delta p}{p_0} - \gamma \frac{\delta \rho}{\rho_0} \right) = 0, \quad (12.69)$$

can be used to find $\delta \rho / \rho_0$, once $\delta p / p_0 = \gamma(\tilde{p} - B^2/2)/c_s^2$ is calculated from the solution of Eqs. (12.66–12.68). Note that $\delta \rho / \rho_0$ is merely a spectator quantity, not required to solve Eqs. (12.66–12.68), which form a closed set.

Eqs. (12.66–12.68) are the equations of *incompressible MHD* (let us call it *iMHD*). Note that while they have been obtained in the limit of $\beta \gg 1$, all β dependence has disappeared from them—basically, they describe subsonic dynamics on top of an infinite heat bath.

Exercise 12.1. Show that iMHD conserves the sum of kinetic and magnetic energies,

$$\frac{d}{dt} \int d^3 \mathbf{r} \left(\frac{u^2}{2} + \frac{B^2}{2} \right) = 0. \quad (12.70)$$

Exercise 12.2. Check that you can obtain the right waves, viz., Alfvén (§12.1.1) and pseudo-Alfvén (§12.1.6), directly from iMHD.

12.2.7. Elsasser MHD

The iMHD equations possess a remarkable symmetry. Let us introduce *Elsasser (1950) fields*

$$\mathbf{Z}^\pm = \mathbf{u} \pm \mathbf{B} \quad (12.71)$$

and rewrite Eqs. (12.66) and (12.68) as evolution equations for \mathbf{Z}^\pm : after trivial algebra,

$$\boxed{\frac{\partial \mathbf{Z}^\pm}{\partial t} + \mathbf{Z}^\mp \cdot \nabla \mathbf{Z}^\pm = -\nabla \tilde{p}} \quad (12.72)$$

and, since $\nabla \cdot \mathbf{Z}^\pm = 0$,

$$\nabla^2 \tilde{p} = -\nabla \nabla : \mathbf{Z}^+ \mathbf{Z}^-. \quad (12.73)$$

Thus, one can think of iMHD as representing the evolution of two incompressible “velocity fields” advecting each other.

Let us restore the separation of the magnetic field into its mean and perturbed parts, $\mathbf{B} = \mathbf{B}_0 + \delta \mathbf{B} = v_A \hat{\mathbf{z}} + \delta \mathbf{B}$ (recall that \mathbf{B} is in velocity units). Then

$$\mathbf{Z}^\pm = \pm v_A \hat{\mathbf{z}} + \delta \mathbf{Z}^\pm \quad (12.74)$$

and Eq. (12.72) becomes

$$\boxed{\frac{\partial \delta \mathbf{Z}^\pm}{\partial t} \mp v_A \nabla_{\parallel} \delta \mathbf{Z}^\pm + \delta \mathbf{Z}^\mp \cdot \nabla \delta \mathbf{Z}^\pm = -\nabla \tilde{p}}. \quad (12.75)$$

Thus, $\delta \mathbf{Z}^\pm$ are finite, counter-propagating (at the Alfvén speed v_A) perturbations—and they interact nonlinearly only with each other, not with themselves. If we let, say, $\delta \mathbf{Z}^- =$

$0 \Leftrightarrow \mathbf{u} = \delta\mathbf{B}$, then $\delta\mathbf{Z}^+$ satisfies

$$\frac{\partial\delta\mathbf{Z}^+}{\partial t} - v_A \nabla_{\parallel} \delta\mathbf{Z}^+ = 0, \quad (12.76)$$

and similarly for $\delta\mathbf{Z}^-$ (propagating at $-v_A$) if $\delta\mathbf{Z}^+ = 0$. Therefore,

$$\delta\mathbf{Z}^{\pm} = \mathbf{f}(\mathbf{r} \pm v_A t \hat{\mathbf{z}}), \quad \delta\mathbf{Z}^{\mp} = 0, \quad (12.77)$$

where \mathbf{f} is an arbitrary function, are *exact nonlinear solutions* of iMHD. They are called *Elsasser states*. Physically, they are isolated Alfvén-wave packets that propagate along the guide field and never interact (because they all travel at the same speed and so can never catch up with or overtake one another). In order to have any interesting *nonlinear* dynamics, the system must have *counter-propagating* Alfvén-wave packets (see §12.4).

12.2.8. Cross-Helicity

Eqs. (12.72) manifestly support *two* conservation laws:

$$\frac{d}{dt} \int d^3\mathbf{r} \frac{|\mathbf{Z}^{\pm}|^2}{2} = 0, \quad (12.78)$$

i.e., the energy of each Elsasser field is individually conserved. This can be reformulated as conservation of the total energy,

$$\frac{d}{dt} \int d^3\mathbf{r} \frac{1}{2} \left(\frac{|\mathbf{Z}^+|^2}{2} + \frac{|\mathbf{Z}^-|^2}{2} \right) = \frac{d}{dt} \int d^3\mathbf{r} \left(\frac{u^2}{2} + \frac{B^2}{2} \right) = 0, \quad (12.79)$$

and of a new quantity, known as the *cross-helicity*:

$$\boxed{\frac{d}{dt} \int d^3\mathbf{r} \frac{1}{2} \left(\frac{|\mathbf{Z}^+|^2}{2} - \frac{|\mathbf{Z}^-|^2}{2} \right) = \frac{d}{dt} \int d^3\mathbf{r} \mathbf{u} \cdot \mathbf{B} = 0} . \quad (12.80)$$

In the Elsasser formulation, the cross-helicity is a measure of energy *imbalance* between the two Elsasser fields[†]—this is observed, for example, in the solar wind, where there is significantly more energy in the Alfvénic fluctuations propagating away from the Sun than towards it (see, e.g., Wicks *et al.* 2013).

Exercise 12.3. To see why we needed incompressibility to get this new conservation law, work out the time evolution equation for $\int d^3\mathbf{r} \mathbf{u} \cdot \mathbf{B}$ from the general (compressible) MHD equations and hence the condition under which the cross-helicity is conserved.

12.2.9. Stratified MHD

It is quite instructive to consider a very simple example of non-uniform MHD equilibrium: the case of a *stratified atmosphere*. Let us introduce gravity into MHD equations, viz., the momentum equation (11.58) becomes

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla \left(p + \frac{B^2}{8\pi} \right) + \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{4\pi} - \rho g \hat{\mathbf{z}} \quad (12.81)$$

(uniform gravitational acceleration pointing downward, against the z direction). We wish to consider a static equilibrium inhomogeneous in the z direction and threaded by a uniform magnetic field (which may be zero):

$$\rho_0 = \rho_0(z), \quad p_0 = p_0(z), \quad \mathbf{u}_0 = 0, \quad \mathbf{B}_0 = B_0 \mathbf{b}_0 = \text{const}, \quad (12.82)$$

[†] Cross-helicity can also be interpreted as a topological invariant, counting the linkages between flux tubes and vortex tubes analogously to what magnetic helicity does for the flux tubes alone (see §13.2).

where \mathbf{b}_0 is at some general angle to $\hat{\mathbf{z}}$ and $p_0(z)$ and $\rho_0(z)$ are constrained by the vertical force balance:

$$\boxed{\frac{dp_0}{dz} = -\rho_0 g} \quad \Rightarrow \quad g = -\frac{p_0}{\rho_0} \frac{d \ln p_0}{dz} = \frac{c_s^2}{\gamma} \frac{1}{H_p}, \quad (12.83)$$

where it has been opportune to define the pressure scale height H_p . We shall now seek time-dependent solutions of the MHD equations for which

$$\rho = \rho_0(z) + \delta\rho, \quad \frac{\delta\rho}{\rho_0} \ll 1, \quad p = p_0(z) + \delta p, \quad \frac{\delta p}{p_0} \ll 1, \quad (12.84)$$

and the spatial variation of all perturbations occurs on scales that are small compared to the pressure scale height H_p or the analogously defined density scale height $H_\rho = -(d \ln \rho_0 / dz)^{-1}$ (for ordering purposes, we denote them both H):

$$\boxed{kH \gg 1}. \quad (12.85)$$

After the equilibrium pressure balance is subtracted from Eq. (12.81), this equation becomes, under any ordering in which $\delta\rho \ll \rho_0$,

$$\boxed{\rho_0 \frac{d\mathbf{u}}{dt} = -\nabla \left(\delta p + \frac{B^2}{8\pi} \right) + \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{4\pi} - \delta\rho g \hat{\mathbf{z}}}. \quad (12.86)$$

The last term is the *buoyancy (Archimedes) force*. In order for this new feature to give rise to any nontrivial new physics, it must be ordered comparable to all the other terms in the equation: using Eq. (12.83) to express $g \sim p_0 / \rho_0 H$, we find

$$\delta\rho g \sim k\delta p \quad \Rightarrow \quad \frac{\delta\rho}{\rho_0} \sim kH \frac{\delta p}{p_0} \gg \frac{\delta p}{p_0}, \quad (12.87)$$

$$\delta\rho g \sim \frac{kB^2}{4\pi} \quad \Rightarrow \quad \frac{\delta\rho}{\rho_0} \sim \frac{kH}{\beta} \ll 1 \quad \Rightarrow \quad \beta \gg kH \gg 1. \quad (12.88)$$

So we learn that the density perturbations must now be much larger than the pressure perturbations, but, in order for the former to remain small and for the magnetic field to be in the game, β must be high (it is in anticipation of this that we did not split \mathbf{B} into \mathbf{B}_0 and $\delta\mathbf{B}$, expecting them to be of the same order).

Let us now expand the internal-energy equation (11.60) in small density and pressure perturbations. Denoting $s = p/\rho^\gamma = s_0(z) + \delta s$ (entropy density) and introducing the entropy scale height

$$\frac{1}{H_s} \equiv \frac{d \ln s_0}{dz} = -\frac{1}{H_p} + \frac{\gamma}{H_\rho} \quad (12.89)$$

(assumed positive), we find[†]

$$\frac{d \delta s}{dt} = -\frac{u_z}{H_s}, \quad \frac{\delta s}{s_0} = \frac{\delta p}{p_0} - \gamma \frac{\delta\rho}{\rho_0} \approx -\gamma \frac{\delta\rho}{\rho_0}. \quad (12.90)$$

The last, approximate, expression follows from the smallness of pressure perturbations [Eq. (12.87)]. This then gives us

$$\boxed{\frac{d \delta\rho}{dt} = \frac{u_z}{\gamma H_s}}. \quad (12.91)$$

But, on the other hand, the continuity equation (11.57) is

$$\frac{d \delta\rho}{dt} = -\nabla \cdot \mathbf{u} + \frac{u_z}{H_\rho} \quad \Rightarrow \quad \nabla \cdot \mathbf{u} = u_z \left(\frac{1}{H_\rho} - \frac{1}{\gamma H_s} \right) = \frac{u_z}{\gamma H_p} \quad \Rightarrow \quad \frac{\nabla \cdot \mathbf{u}}{ku} \sim \frac{1}{kH} \ll 1. \quad (12.92)$$

Thus, the dynamics are incompressible again and the role of the continuity equation is to tell us that we must find δp from the momentum equation (12.86) by enforcing $\nabla \cdot \mathbf{u} = 0$ to lowest

[†] We are able to take equilibrium quantities in and out of spatial derivatives because $kH \gg 1$ and the perturbations are small.

order. The difference with iMHD (§12.2.6) is that $\delta\rho/\rho$ now participates in the dynamics via the buoyancy force and must be found self-consistently from Eq. (12.91).

Finally, we rewrite our newly found simplified system of equations for a stratified, high- β atmosphere, in the following neat way:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \tilde{p} + \mathbf{B} \cdot \nabla \mathbf{B} + a \hat{\mathbf{z}}, \quad (12.93)$$

$$\nabla^2 \tilde{p} = -\nabla \nabla : (\mathbf{u}\mathbf{u} - \mathbf{B}\mathbf{B}) + \frac{\partial a}{\partial z}, \quad (12.94)$$

$$\frac{\partial a}{\partial t} + \mathbf{u} \cdot \nabla a = -N^2 u_z, \quad N = \frac{c_s}{\gamma \sqrt{H_s H_p}}, \quad (12.95)$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u}, \quad (12.96)$$

where we have rescaled $\mathbf{B}/\sqrt{4\pi\rho_0} \rightarrow \mathbf{B}$ and denoted the Archimedes acceleration

$$a = -\frac{\delta\rho}{\rho_0} g = -\frac{\delta\rho}{\rho_0} \frac{c_s^2}{\gamma H_p}, \quad (12.97)$$

a quantity also known as *the buoyancy* of the fluid. We shall call Eqs. (12.93–12.96) the equations of *stratified MHD (SMHD)*.

A new frequency N , known as the *Brunt–Väisälä frequency*, has appeared in our equations.† In order for all the linear and nonlinear time scales that are present in our equations to coexist legitimately within our ordering, we must demand that the Alfvén, Brunt–Väisälä and nonlinear time scales all be comparable:

$$kv_A \sim N \sim ku \quad \Rightarrow \quad \boxed{\frac{1}{\sqrt{\beta}} \sim \frac{1}{kH} \sim \text{Ma}}. \quad (12.98)$$

This gives us a relative ordering between all the small parameters that have appeared so far, including the new one, $1/kH$. Using Eq. (12.91) and recalling Eq. (12.87), let us summarise the ordering of the perturbations:

$$\frac{u}{c_s} \sim \frac{\delta\rho}{\rho_0} \sim \text{Ma}, \quad \frac{\delta p}{p_0} \sim \text{Ma}^2, \quad |\delta \mathbf{b}| \sim \frac{\delta B}{B_0} \sim 1. \quad (12.99)$$

The difference with the iMHD high- β ordering (12.63) is that the density perturbations have now been promoted to dynamical relevance, thankfully without jeopardising incompressibility (i.e., still ordering out the sonic perturbations). The ordering (12.99) can be thought of as a generalisation to MHD of the *Boussinesq approximation* in hydrodynamics.

Further investigations of the SMHD equations are undertaken in Q5.

12.3. Reduced MHD

We now turn to the anisotropic ordering, $k_{\parallel}/k \ll 1$ (while $\beta \sim 1$, in general), for which we studied the linear theory in §12.1.5. Specialising to this case from our general ordering (12.62), we have

$$\boxed{\text{Ma} \sim \frac{u_{\perp}}{c_s} \sim \frac{u_{\parallel}}{c_s} \sim |\delta \mathbf{b}| \sim \frac{\delta B}{B_0} \sim \frac{\delta\rho}{\rho_0} \sim \frac{\delta p}{p_0} \sim \frac{\omega}{k_{\perp} c_s} \sim \frac{k_{\parallel}}{k_{\perp}} \ll 1}. \quad (12.100)$$

Starting again with the continuity equation (11.57), dividing through by ρ_0 and order-

† N is real because we assumed $H_s > 0$ (a “stably stratified atmosphere”), otherwise the atmosphere becomes convectively unstable—this happens when the equilibrium entropy decreases upwards (cf. §14.3, Q8 and Q5c).

ing all terms, we get

$$\underbrace{\left(\underbrace{\frac{\partial}{\partial t}}_{\text{Ma}} + \underbrace{\mathbf{u}_\perp \cdot \nabla_\perp}_{\text{Ma}} + \underbrace{u_\parallel \cdot \nabla_\parallel}_{\text{Ma}^2} \right)}_{\text{Ma}^2} \underbrace{\frac{\delta \rho}{\rho_0}}_{\text{Ma}} = - \underbrace{\left(1 + \frac{\delta \rho}{\rho_0} \right)}_{\text{Ma}} \underbrace{\left(\underbrace{\nabla_\perp \cdot \mathbf{u}_\perp}_{\text{Ma}} + \underbrace{\nabla_\parallel u_\parallel}_{\text{Ma}^2} \right)}. \quad (12.101)$$

Thus, to lowest order, the perpendicular velocity field is 2D-incompressible:

$$\mathcal{O}(\text{Ma}) : \quad \boxed{\nabla_\perp \cdot \mathbf{u}_\perp = 0}. \quad (12.102)$$

In the next order (which we will need in §12.3.2),

$$\mathcal{O}(\text{Ma}^2) : \quad (\nabla \cdot \mathbf{u})_2 = - \left(\frac{\partial}{\partial t} + \mathbf{u}_\perp \cdot \nabla_\perp \right) \frac{\delta \rho}{\rho_0} = - \frac{d}{dt} \frac{\delta \rho}{\rho_0}, \quad (12.103)$$

where, to leading order, the convective derivative now involves only perpendicular advection.

Eq. (12.102) implies that \mathbf{u}_\perp can be written in terms of a *stream function*:

$$\boxed{\mathbf{u}_\perp = \hat{\mathbf{z}} \times \nabla_\perp \Phi}. \quad (12.104)$$

Similarly, for the magnetic field, we have

$$0 = \nabla \cdot \mathbf{B} = \underbrace{\nabla_\perp \cdot \delta \mathbf{B}_\perp}_{\text{Ma}} + \underbrace{\nabla_\parallel \delta B_\parallel}_{\text{Ma}^2} \approx \nabla_\perp \cdot \delta \mathbf{B}_\perp, \quad (12.105)$$

so $\delta \mathbf{B}_\perp$ is also 2D-solenoidal and can be written in terms of a *flux function*:

$$\boxed{\frac{\delta \mathbf{B}_\perp}{\sqrt{4\pi\rho_0}} = \hat{\mathbf{z}} \times \nabla_\perp \Psi}. \quad (12.106)$$

Note that $\Psi = -A_\parallel / \sqrt{4\pi\rho_0}$, the parallel component of the vector potential.

Thus, Alfvénically polarised perturbations, \mathbf{u}_\perp and $\delta \mathbf{B}_\perp$ (see §12.1.1), can be described by two scalar functions, Φ and Ψ . Let us work out the evolution equations for them.

12.3.1. Alfvénic Perturbations

We start with the induction equation, again most useful in the form (11.27). Dividing through by B_0 , we have

$$\frac{d}{dt} \frac{\delta \mathbf{B}}{B_0} = \mathbf{b} \cdot \nabla \mathbf{u} - \mathbf{b} \nabla \cdot \mathbf{u}. \quad (12.107)$$

Throwing out the obviously subdominant $\delta \mathbf{b}$ contribution in the last term on the right-hand side (i.e., approximating $\mathbf{b} \approx \hat{\mathbf{z}}$ in that term), then taking the perpendicular part of the remaining equation, we get

$$\frac{d}{dt} \frac{\delta \mathbf{B}_\perp}{B_0} = \mathbf{b} \cdot \nabla \mathbf{u}_\perp. \quad (12.108)$$

As we saw above, the convective derivative is with respect to the perpendicular velocity only and, in view of the stream-function representation of the latter [Eq. (12.104)], for any function f , we have, to leading order,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{u}_\perp \cdot \nabla_\perp f = \frac{\partial f}{\partial t} + \hat{\mathbf{z}} \cdot (\nabla_\perp \Phi \times \nabla_\perp f) = \frac{\partial f}{\partial t} + \{\Phi, f\}, \quad (12.109)$$

where the ‘‘Poisson bracket’’ is

$$\{\Phi, f\} = \frac{\partial \Phi}{\partial x} \frac{\partial f}{\partial y} - \frac{\partial \Phi}{\partial y} \frac{\partial f}{\partial x}. \quad (12.110)$$

Similarly, to leading order,

$$\mathbf{b} \cdot \nabla f = \frac{\partial f}{\partial z} + \delta \mathbf{b} \cdot \nabla_{\perp} f = \frac{\partial f}{\partial z} + \frac{1}{v_A} \hat{\mathbf{z}} \cdot (\nabla_{\perp} \Psi \times \nabla_{\perp} f) = \frac{\partial f}{\partial z} + \frac{1}{v_A} \{\Psi, f\}. \quad (12.111)$$

Finally, using Eqs. (12.109) and (12.111) in Eq. (12.108) and expressing $\delta \mathbf{B}_{\perp}$ in terms of Ψ [Eq. (12.106)] and \mathbf{u}_{\perp} in terms of Φ [Eq. (12.104)], it is a straightforward exercise to show (after ‘‘uncurling’’ Eq. (12.108)) that[†]

$$\boxed{\frac{\partial \Psi}{\partial t} + \{\Phi, \Psi\} = v_A \frac{\partial \Phi}{\partial z}}. \quad (12.112)$$

Turning now to the momentum equation (11.58), taking its perpendicular part and dividing through by $\rho \approx \rho_0$, we get

$$\underbrace{\frac{d\mathbf{u}_{\perp}}{dt}}_{\text{Ma}^2} = \frac{1}{\rho_0} \left[-\nabla_{\perp} \left(p + \frac{B^2}{8\pi} \right) + \frac{\mathbf{B} \cdot \nabla \delta \mathbf{B}_{\perp}}{4\pi} \right] = - \underbrace{\nabla_{\perp} \left(\frac{c_s^2}{\gamma} \frac{\delta p}{p_0} + v_A^2 \frac{\delta B}{B_0} \right)}_{\text{Ma}} + \underbrace{v_A^2 \mathbf{b} \cdot \nabla \frac{\delta \mathbf{B}_{\perp}}{B_0}}_{\text{Ma}^2}. \quad (12.113)$$

To lowest order,

$$\mathcal{O}(\text{Ma}) : \quad \nabla_{\perp} \left(\frac{c_s^2}{\gamma} \frac{\delta p}{p_0} + v_A^2 \frac{\delta B}{B_0} \right) = 0 \quad \Rightarrow \quad \boxed{\frac{\delta p}{p_0} = -\gamma \frac{v_A^2}{c_s^2} \frac{\delta B}{B_0}}. \quad (12.114)$$

This is a statement of pressure balance, which is physically what has been expected [see Eq. (12.41)] and which will be useful in §12.3.2. In the next order, Eq. (12.113) contains the perpendicular gradient of the second-order contribution to the total pressure. To avoid having to calculate it, we take the curl of Eq. (12.113) and thus obtain

$$\mathcal{O}(\text{Ma}^2) : \quad \nabla_{\perp} \times \frac{d\mathbf{u}_{\perp}}{dt} = v_A^2 \nabla_{\perp} \times \left(\mathbf{b} \cdot \nabla \frac{\delta \mathbf{B}_{\perp}}{B_0} \right). \quad (12.115)$$

Finally, using again Eqs. (12.104), (12.106), (12.109) and (12.111) in Eq. (12.115), some slightly tedious algebra leads us to

$$\boxed{\frac{\partial}{\partial t} \nabla_{\perp}^2 \Phi + \{\Phi, \nabla_{\perp}^2 \Phi\} = v_A \frac{\partial}{\partial z} \nabla_{\perp}^2 \Psi + \{\Psi, \nabla_{\perp}^2 \Psi\}}. \quad (12.116)$$

Note that $\nabla_{\perp}^2 \Phi$ is the vorticity of the flow \mathbf{u}_{\perp} and so the above equation is the MHD generalisation of the 2D Euler equation.

To summarise the equations (12.116) and (12.112) in their most compact form, we have

$$\frac{d}{dt} \nabla_{\perp}^2 \Phi = v_A \mathbf{b} \cdot \nabla \nabla_{\perp}^2 \Psi, \quad (12.117)$$

$$\frac{d\Psi}{dt} = v_A \frac{\partial \Phi}{\partial z}, \quad (12.118)$$

[†] Another easy route to this equation is to start from the induction equation in the form (11.59), let $\mathbf{B} = \nabla \times \mathbf{A}$, ‘‘uncurl’’ Eq. (11.59) and take the z component of the resulting evolution equation for \mathbf{A} .

where the convective time derivative d/dt and the parallel spatial derivative $\mathbf{b} \cdot \nabla$ are given by Eqs. (12.109) and (12.111), respectively. Beautifully, these *nonlinear* equations describing Alfvénic perturbations have decoupled completely from everything else: we do not need to know $\delta\rho$, δp , u_{\parallel} or δB in order to solve for \mathbf{u}_{\perp} and $\delta\mathbf{B}_{\perp}$. *Alfvénic dynamics are self-contained*.

Eqs. (12.117) and (12.118) are called the Equations of *Reduced MHD (RMHD)*. They were originally derived in the context of tokamak plasmas (Kadomtsev & Pogutse 1974; Strauss 1976) and are extremely popular as a simple paradigm for MHD is a strong guide field—not just in tokamaks, but also in space[†].

12.3.2. Compressive Perturbations

What about the rest of our fields—in the linear language, the slow-wave-like perturbations (§12.1.5)? While we do not need them to compute the Alfvénic perturbations, we might still wish to know them for their own sake.

Returning to the induction equation (12.107) and taking its z component, we get

$$\frac{d}{dt} \frac{\delta B_{\parallel}}{B_0} = \mathbf{b} \cdot \nabla u_{\parallel} - \nabla \cdot \mathbf{u} \quad \Rightarrow \quad \boxed{\frac{d}{dt} \left(\frac{\delta B}{B_0} - \frac{\delta\rho}{\rho_0} \right) = \mathbf{b} \cdot \nabla u_{\parallel}}, \quad (12.119)$$

where all terms are $\mathcal{O}(\text{Ma}^2)$, $\delta B_{\parallel} \approx \delta B$ to leading order and we used Eq. (12.103) to express $\nabla \cdot \mathbf{u}$. The derivatives d/dt and $\mathbf{b} \cdot \nabla$ contain the nonlinearities involving Φ and Ψ , which we already know from Eqs. (12.117) and (12.118).

To find an equation for u_{\parallel} , we take the z component of the momentum equation (11.58):

$$\underbrace{\frac{du_{\parallel}}{dt}}_{\text{Ma}^2} = \frac{1}{\rho_0} \left[\underbrace{\frac{\partial}{\partial z} \left(p + \frac{B^2}{8\pi} \right)}_{\text{Ma}^3} + \underbrace{\frac{\mathbf{B} \cdot \nabla \delta B_{\parallel}}{4\pi}}_{\text{Ma}^2} \right] \quad \Rightarrow \quad \boxed{\frac{du_{\parallel}}{dt} = v_A^2 \mathbf{b} \cdot \nabla \frac{\delta B}{B_0}}. \quad (12.120)$$

The parallel pressure gradient is $\mathcal{O}(\text{Ma}^3)$ because there is pressure balance [Eq. (12.114)] to lowest order.

Finally, let us bring in the energy equation (11.60), as yet unused. To leading order, it is

$$\frac{d}{dt} \frac{\delta s}{s_0} = \frac{d}{dt} \left(\frac{\delta p}{p_0} - \gamma \frac{\delta\rho}{\rho_0} \right) = 0 \quad \Rightarrow \quad \boxed{\frac{d}{dt} \left(\frac{\delta\rho}{\rho_0} + \frac{v_A^2}{c_s^2} \frac{\delta B}{B_0} \right) = 0}, \quad (12.121)$$

where, to obtain the final version of the equation, we substituted Eq. (12.114) for $\delta p/p_0$.

Eqs. (12.119–12.121) are a complete set of equations for δB , u_{\parallel} and $\delta\rho$, given Φ and Ψ . These equations are linear in the Lagrangian frame associated with the Alfvénic perturbations, provided the parallel distances are measured along perturbed field lines. Physically, they tell us that *slow waves propagate along perturbed field lines and are passively (i.e., without acting back) advected by the perpendicular Alfvénic flows*.

Exercise 12.4. Check that the linear relationships between various perturbations in a slow wave derived in §12.1.5 are manifest in Eqs. (12.119–12.121).

[†] In the latter context, they are used most prominently as a description of Alfvénic turbulence at small scales (see §12.4), for which the RMHD equations can be shown to be the correct description even if the plasma is collisionless and in general requires kinetic treatment (Schekochihin *et al.* 2009; Kunz *et al.* 2015).

In what follows, when we refer to RMHD, we will mean all five equations (12.117–12.118) and (12.119–12.121).

12.3.3. Elsasser Fields and the Energetics of RMHD

The Elsasser approach (§12.2.7) can be adapted to the RMHD system. Defining *Elsasser potentials*

$$\boxed{\zeta^\pm = \Phi \pm \Psi} \quad \Leftrightarrow \quad \delta \mathbf{Z}_\perp^\pm = \mathbf{u}_\perp \pm \frac{\delta \mathbf{B}_\perp}{\sqrt{4\pi\rho_0}} = \hat{\mathbf{z}} \times \nabla_\perp \zeta^\pm, \quad (12.122)$$

it is a straightforward exercise to show that

$$\frac{\partial}{\partial t} \nabla_\perp^2 \zeta^\pm \mp v_A \frac{\partial}{\partial z} \nabla_\perp^2 \zeta^\pm = -\frac{1}{2} [\{\zeta^+, \nabla_\perp^2 \zeta^-\} + \{\zeta^-, \nabla_\perp^2 \zeta^+\} \mp \nabla_\perp^2 \{\zeta^+, \zeta^-\}]. \quad (12.123)$$

The key corollary of this equation is the same as in §12.2.7, although here it applies to perpendicular perturbations only: *only counter-propagating Alfvénic perturbations can interact and any finite-amplitude perturbation composed of just one Elsasser field is a nonlinear solution.*

The energies of these Elsasser fields are individually conserved (cf. §12.2.8),

$$\frac{d}{dt} \int d^3\mathbf{r} |\nabla_\perp \zeta^\pm|^2 = \frac{d}{dt} \int d^3\mathbf{r} |\delta \mathbf{Z}_\perp^\pm|^2 = 0, \quad (12.124)$$

i.e., when the two fields do interact, they scatter each other nonlinearly, but do *not* exchange energy.

There is an Elsasser-like formulation for the slow waves as well†:

$$\boxed{\delta Z_\parallel^\pm = u_\parallel \pm \frac{\delta B}{\sqrt{4\pi\rho_0}} \sqrt{1 + \frac{v_A^2}{c_s^2}}}. \quad (12.125)$$

Then, from Eqs. (12.119–12.121), one gets, after more algebra,

$$\begin{aligned} \frac{\partial \delta Z_\parallel^\pm}{\partial t} \mp \frac{c_s v_A}{\sqrt{c_s^2 + v_A^2}} \frac{\partial \delta Z_\parallel^\pm}{\partial z} = \\ -\frac{1}{2} \left[\left(1 \mp \frac{1}{\sqrt{1 + v_A^2/c_s^2}} \right) \{\zeta^+, \delta Z_\parallel^\pm\} + \left(1 \pm \frac{1}{\sqrt{1 + v_A^2/c_s^2}} \right) \{\zeta^-, \delta Z_\parallel^\pm\} \right]. \end{aligned} \quad (12.126)$$

Note the (expected) appearance of the slow-wave phase speed [cf. Eq. (12.33)] in the left-hand side. Thus, slow waves interact only with Alfvénic perturbations—when $v_A \ll c_s$, only with the counterpropagating ones, but at finite β , because the slow waves are slower, a co-propagating Alfvénic perturbation can catch up with a slow one, have its way with it in passing and speed on (it’s a tough world).

There is no energy exchange in these interactions: the “+” and “−” slow-wave energies are individually conserved:

$$\frac{d}{dt} \int d^3\mathbf{r} |\delta Z_\parallel^\pm|^2 = 0. \quad (12.127)$$

† At high β , $v_A \ll c_s$, so we recover from Eqs. (12.125) and (12.122) the Elsasser fields as defined for iMHD in Eq. (12.71).

12.3.4. Entropy Mode

There are only two equations in Eq. (12.126), whereas we had three equations (12.119–12.121) for our three compressive fields δB , u_{\parallel} and $\delta\rho$. The third equation, Eq. (12.121), was in fact for the entropy perturbation:

$$\boxed{\frac{d\delta s}{dt} = 0}, \quad \frac{\delta s}{s_0} = -\gamma \left(\frac{\delta\rho}{\rho_0} + \frac{v_A^2}{c_s^2} \frac{\delta B}{B_0} \right). \quad (12.128)$$

We see that δs is a decoupled variable, independent from ζ^{\pm} or $\delta Z_{\parallel}^{\pm}$ (because it is the only one that involves $\delta\rho/\rho_0$). Eq. (12.128) says that δs is a *passive scalar field*, simply carried around by the Alfvénic velocity \mathbf{u}_{\perp} (via d/dt). At high β , this is just a density perturbation.

The associated linear mode is not a wave: its dispersion relation is

$$\boxed{\omega = 0}. \quad (12.129)$$

This is the (famously often forgotten) 7th MHD mode, known as the *entropy mode* (there are 7 equations in MHD, so there must be 7 linear modes: two fast waves, two Alfvén waves, two slow waves and one entropy mode).

Exercise 12.5. Go back to §12.1 and find where we overlooked this mode.

Since the entropy mode is decoupled, its “energy” (variance) is individually conserved:

$$\frac{d}{dt} \int d^3\mathbf{r} |\delta s|^2 = 0. \quad (12.130)$$

Thus, *in RMHD, the (nonlinear) evolution of all perturbations is constrained by 5 separate conservation laws: $\int d^3\mathbf{r} |\delta Z_{\perp}^{\pm}|^2$, $\int d^3\mathbf{r} |\delta Z_{\parallel}^{\pm}|^2$ and $\int d^3\mathbf{r} |\delta s|^2$ are all invariants.*

Such are the simplifications allowed by anisotropy.

12.4. MHD Turbulence

RMHD is a good starting point for developing the theory of *MHD turbulence*—a phenomenon observed with great precision in the solar wind and believed ubiquitous in the Universe.

I am writing a tutorial review of this topic, which should be available by the end of the summer 2017. For now, see §15.3 for further references.

13. MHD Relaxation

So far, we have only considered MHD in a straight field against the background of constant density and pressure (except in §12.2.9, where this was generalised slightly). As any more complicated (static) equilibrium will locally look like this, what we have done has considerable universal significance. Now we shall occupy ourselves with a somewhat less universal (i.e., dependent on the circumstances of a particular problem) and more “large-scale” (compared to the dynamics of wavy perturbations) question: *what kind of (static) equilibrium states are there and into which of those states will an MHD fluid normally relax?*

13.1. Static MHD Equilibria

Let us go back to the MHD equations (11.57–11.60) and seek static equilibria, i.e., set $\mathbf{u} = 0$ and $\partial/\partial t = 0$. The remaining equations are

$$\boxed{-\nabla p + \frac{\mathbf{j} \times \mathbf{B}}{c} = 0, \quad \mathbf{j} = \frac{c}{4\pi} \nabla \times \mathbf{B}, \quad \nabla \cdot \mathbf{B} = 0} \quad (13.1)$$

(the force balance, Ampère's law and the solenoidality-of- \mathbf{B} constraint). These are 7 equations for 7 unknowns (p , \mathbf{B} , \mathbf{j}), so a complete set. Density is irrelevant because nothing moves and so inertia does not matter.

The force-balance equation has two immediate general consequences:

$$\mathbf{B} \cdot \nabla p = 0, \quad (13.2)$$

so *magnetic surfaces are surfaces of constant pressure*, and

$$\mathbf{j} \cdot \nabla p = 0, \quad (13.3)$$

so *currents flow along those surfaces*.

Eq. (13.2) implies that if magnetic field lines are stochastic and fill the volume of the system, then $p = \text{const}$ across the system and so the force balance becomes

$$\mathbf{j} \times \mathbf{B} = 0. \quad (13.4)$$

Such equilibria are called *force-free* and turn out to be very interesting, as we shall discover soon (from §13.1.2 onwards).

13.1.1. MHD Equilibria in Cylindrical Geometry

As the simplest example of an inhomogeneous equilibrium, let us consider the case of cylindrical and axial symmetry:

$$\frac{\partial}{\partial \theta} = 0, \quad \frac{\partial}{\partial z} = 0. \quad (13.5)$$

Solenoidality of the magnetic field then rules out it having a radial component:

$$\nabla \cdot \mathbf{B} = \frac{1}{r} \frac{\partial}{\partial r} r B_r = 0 \quad \Rightarrow \quad r B_r = \text{const} \quad \Rightarrow \quad B_r = 0. \quad (13.6)$$

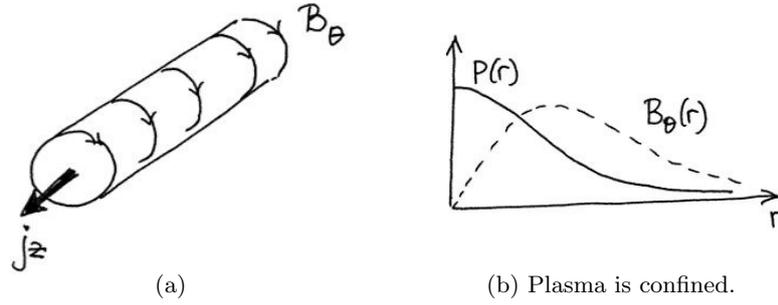
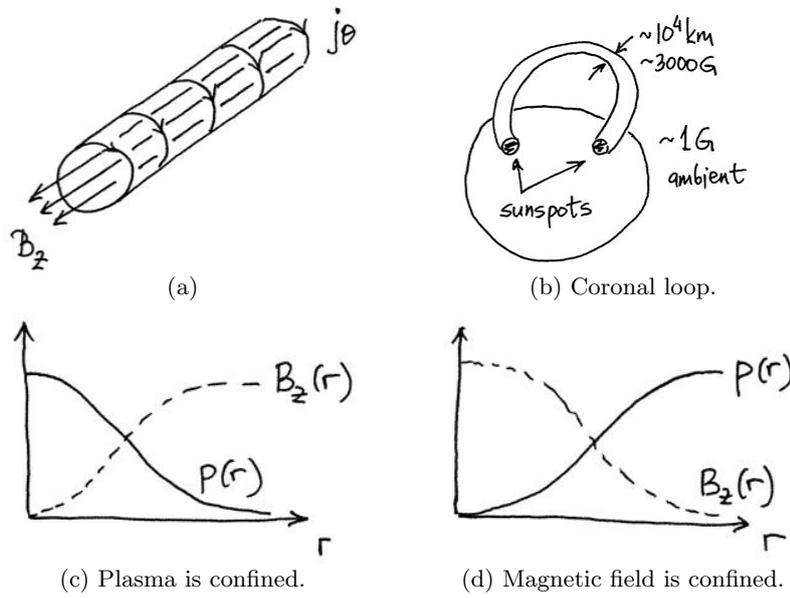
Ampère's law tells us that currents do not flow radially either:

$$\mathbf{j} = \frac{c}{4\pi} \nabla \times \mathbf{B} \quad \Rightarrow \quad \begin{cases} j_r = 0, \\ j_\theta = -\frac{c}{4\pi} \frac{\partial B_z}{\partial r}, \\ j_z = \frac{c}{4\pi} \frac{1}{r} \frac{\partial}{\partial r} r B_\theta. \end{cases} \quad (13.7)$$

Finally, the radial pressure balance gives us

$$\begin{aligned} \frac{\partial p}{\partial r} &= \frac{(\mathbf{j} \times \mathbf{B})_r}{c} = \frac{j_\theta B_z - j_z B_\theta}{c} = \frac{1}{4\pi} \left(-B_z \frac{\partial B_z}{\partial r} - \frac{B_\theta}{r} \frac{\partial}{\partial r} r B_\theta \right) \\ &= -\frac{\partial}{\partial r} \frac{B_z^2}{8\pi} - \frac{B_\theta^2}{4\pi r} - \frac{\partial}{\partial r} \frac{B_\theta^2}{8\pi} \quad \Rightarrow \quad \boxed{\frac{\partial}{\partial r} \left(p + \frac{B^2}{8\pi} \right) = -\frac{B_\theta^2}{4\pi r}}. \end{aligned} \quad (13.8)$$

This simply says that the total pressure gradient is balanced by the tension force. A general equilibrium for which this is satisfied is called a *screw pinch*.

FIGURE 47. z pinch.FIGURE 48. θ pinch.

One simple particular case of this is the z pinch (Fig. 47a). This is achieved by letting a current flow along the z axis, giving rise to an azimuthal field:

$$j_\theta = 0, \quad j_z = \frac{c}{4\pi r} \frac{\partial}{\partial r} r B_\theta \Rightarrow B_\theta = \frac{4\pi}{c} \frac{1}{r} \int_0^r dr' r' j_z(r'), \quad B_z = 0. \quad (13.9)$$

Eq. (13.8) becomes

$$\boxed{\frac{\partial p}{\partial r} = -\frac{1}{c} j_z B_\theta}. \quad (13.10)$$

The “pinch” comes from magnetic loops and is due to the curvature force: the loops want to contract inwards, the pressure gradient opposes this and so *plasma is confined* (Fig. 47b). This configuration will, however, prove to be very badly unstable (§14.4)—which does not stop it from being a popular laboratory set up for short-term confinement experiments (see, e.g., review by Haines 2011).

Another simple particular case is the θ pinch (Fig. 48a). This is achieved by imposing a straight but radially non-uniform magnetic field in the z direction and, therefore,

azimuthal currents:

$$B_\theta = 0, \quad j_z = 0, \quad j_\theta = -\frac{c}{4\pi} \frac{\partial B_z}{\partial r}. \quad (13.11)$$

Eq. (13.8) is then just a pressure balance, pure and simple:

$$\boxed{\frac{\partial}{\partial r} \left(p + \frac{B_z^2}{8\pi} \right) = 0}. \quad (13.12)$$

In this configuration, we can confine the plasma (Fig. 48c) or the magnetic flux (Fig. 48d). The latter is what happens, for example, in flux tubes that link sunspots (Fig. 48b). The θ pinch is a stable configuration (Q9).

The more general case of a screw pinch [Eq. (13.8)] is a superposition of z and θ pinches, with both magnetic fields and currents wrapping themselves around cylindrical flux surfaces.

The next step in complexity is to assume axial, but not cylindrical symmetry ($\partial/\partial\theta = 0$, $\partial/\partial z \neq 0$). This is explored in Q7.

13.1.2. Force-Free Equilibria

Another interesting and elegant class of equilibria arises if we consider situations in which ∇p is negligible and can be completely omitted from the force balance. This can happen in two possible sets of circumstances:

—pressure is the same across the system, e.g., because the field lines are stochastic [a previously mentioned consequence of Eq. (13.2)];

—low $\beta = p/(B^2/8\pi) \ll 1$, so thermal energy is negligible compared to magnetic energy and so p is irrelevant.

A good example of the latter situation is the *solar corona*, where $\beta \sim 1-10^{-6}$ (assuming $n \sim 10^9 \text{ cm}^{-3}$, $T \sim 10^2 \text{ eV}$ and $B \sim 1-10^3 \text{ G}$, the lower value applying in the photosphere, the upper one in the coronal loops; see Fig. 48b)

In such situations, the equilibrium is purely magnetic, i.e., the magnetic field is “*force-free*,” which implies that the current must be parallel to the magnetic field:

$$\boxed{\mathbf{j} \times \mathbf{B} = 0} \quad \Rightarrow \quad \mathbf{j} \parallel \mathbf{B} \quad \Rightarrow \quad \frac{4\pi}{c} \mathbf{j} = \nabla \times \mathbf{B} = \alpha(\mathbf{r})\mathbf{B}, \quad (13.13)$$

where $\alpha(\mathbf{r})$ is an arbitrary scalar function. Taking the divergence of the last equation tells us that

$$\mathbf{B} \cdot \nabla \alpha = 0, \quad (13.14)$$

so the function $\alpha(\mathbf{r})$ is constant on magnetic surfaces. If \mathbf{B} is chaotic and volume-filling, then $\alpha = \text{const}$ across the system.

The case of $\alpha = \text{const}$ is called the *linear force-free field*. In this case, taking the curl of Eq. (13.13) and then iterating it once gives us

$$-\nabla^2 \mathbf{B} = \alpha \nabla \times \mathbf{B} = \alpha^2 \mathbf{B} \quad \Rightarrow \quad \boxed{(\nabla^2 + \alpha^2) \mathbf{B} = 0}, \quad (13.15)$$

so the magnetic field satisfies a Helmholtz equation (to solve which, one must, of course, specify some boundary conditions).

Thus, there is, potentially, a large zoo of MHD equilibria. Some of them are stable, some are not, and, therefore, some are more interesting and/or more relevant than others.

How does one tell? A good question to ask is as follows. Suppose we set up some initial configuration of magnetic field (by, say, switching on some current-carrying coils, driving currents inside plasma, etc.)—*to what (stable) equilibrium will this system eventually relax?*

In general, any initially arranged magnetic configuration will exert forces on the plasma, these will drive flows, which in turn will move the magnetic fields around; eventually, everything will settle into some static equilibrium. We expect that, normally, some amount of the energy contained in the initial field will be lost in such a relaxation process because the flows will be dissipating, the fields diffusing and/or reconnecting, etc.—the losses occur due to the resistive and viscous terms in the non-ideal MHD equations derived in §11. Thus, one expects that *the final relaxed static state will be a minimum-energy state* and so we must be able to find it by minimising magnetic energy:

$$\int d^3\mathbf{r} \frac{B^2}{8\pi} \rightarrow \min. \quad (13.16)$$

Clearly, if the relaxation occurred without any constraints, the solution would just be $\mathbf{B} = 0$. In fact, there are constraints. These constraints are *topological*: if you think of magnetic field lines as a tangled mess, you will realise that, while you can change this tangle by moving field lines around, you cannot easily undo linkages, knots, etc.—anything that, to be undone, would require the field lines to have “ends”. This intuition can be turned into a quantitative theory once we discover that the induction equation (11.59) has an invariant that involves the magnetic field only.

13.2. Helicity

Magnetic helicity in a volume V is defined as

$$H = \int_V d^3\mathbf{r} \mathbf{A} \cdot \mathbf{B}, \quad (13.17)$$

where \mathbf{A} is the vector potential, $\nabla \times \mathbf{A} = \mathbf{B}$.

13.2.1. Helicity Is Well Defined

This is not obvious because \mathbf{A} is not unique: a gauge transformation

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla\chi, \quad (13.18)$$

with χ an arbitrary scalar function, leaves \mathbf{B} unchanged and so does not affect physics. Under this transformation, helicity stays invariant:

$$H \rightarrow H + \int_V d^3\mathbf{r} \mathbf{B} \cdot \nabla\chi = H + \int_{\partial V} d\mathbf{S} \cdot \mathbf{B} \chi = H, \quad (13.19)$$

provided \mathbf{B} at the boundary is parallel to the boundary, i.e., provided the volume V encloses the field (nothing sticks out).

13.2.2. Helicity Is Conserved

Let us come back to the induction equation, Eq. (11.23) (in which we retain resistivity to keep track of non-ideal effects, i.e., of the breaking of flux conservation):

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B} - \eta \nabla \times \mathbf{B}). \quad (13.20)$$

“Uncurling” this equation, we get

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{u} \times \mathbf{B} - \eta \nabla \times \mathbf{B} + \nabla\chi. \quad (13.21)$$

Using Eqs. (13.20) and (13.21), we have

$$\begin{aligned}
\frac{\partial}{\partial t} \mathbf{A} \cdot \mathbf{B} &= \mathbf{B} \cdot (\mathbf{u} \times \mathbf{B} - \eta \nabla \times \mathbf{B} + \nabla \chi) + \mathbf{A} \cdot [\nabla \times (\mathbf{u} \times \mathbf{B} - \eta \nabla \times \mathbf{B})] \\
&= -\eta \mathbf{B} \cdot (\nabla \times \mathbf{B}) + \nabla \cdot (\mathbf{B} \chi) \\
&\quad - \nabla \cdot [\mathbf{A} \times (\mathbf{u} \times \mathbf{B} - \eta \nabla \times \mathbf{B})] + (\mathbf{u} \times \mathbf{B} - \eta \nabla \times \mathbf{B}) \cdot \underbrace{(\nabla \times \mathbf{A})}_{=\mathbf{B}} \\
&= \nabla \cdot [\mathbf{B} \chi - \mathbf{u} \mathbf{A} \cdot \mathbf{B} + \mathbf{B} \mathbf{A} \cdot \mathbf{u} + \eta \mathbf{A} \times (\nabla \times \mathbf{B})] - 2\eta \mathbf{B} \cdot (\nabla \times \mathbf{B}). \quad (13.22)
\end{aligned}$$

Integrating this and using Gauss's theorem, we get

$$\begin{aligned}
\frac{\partial}{\partial t} \int_V d^3\mathbf{r} \mathbf{A} \cdot \mathbf{B} &= \int_{\partial V} d\mathbf{S} \cdot [\mathbf{B} \chi - \mathbf{u} \mathbf{A} \cdot \mathbf{B} + \mathbf{B} \mathbf{A} \cdot \mathbf{u} + \eta \mathbf{A} \times (\nabla \times \mathbf{B})] \\
&\quad - 2\eta \int_V d^3\mathbf{r} \mathbf{B} \cdot (\nabla \times \mathbf{B}). \quad (13.23)
\end{aligned}$$

The surface integral vanishes provided both \mathbf{u} and \mathbf{B} are parallel to the boundary (no fields stick out and no flows cross). The resistive term in the surface integral can also be ignored either by arranging V appropriately or simply by taking it large enough so $\mathbf{B} \rightarrow 0$ on ∂V , or, indeed, by taking $\eta \rightarrow +0$. Thus,

$$\boxed{\frac{dH}{dt} = -2\eta \int d^3\mathbf{r} \mathbf{B} \cdot (\nabla \times \mathbf{B})}, \quad (13.24)$$

magnetic helicity is conserved in ideal MHD†.

Furthermore, it turns out that even in resistive MHD, helicity is “better conserved” than energy, in the following sense. As we saw in §11.11.2, the magnetic energy evolves according to

$$\frac{d}{dt} \int d^3\mathbf{r} \frac{B^2}{8\pi} = \left(\begin{array}{c} \text{energy exchange terms} \\ \text{and fluxes} \end{array} \right) - 2\eta \int d^3\mathbf{r} |\nabla \times \mathbf{B}|^2. \quad (13.25)$$

The first term on the right-hand side contains various fluxes and energy exchanges with the velocity field [see Eq. (11.54)], all of which eventually decay as the system relaxes (flows decay by viscosity). The second term represents Ohmic heating. If η is small but the Ohmic heating is finite, it is finite because magnetic field develops fine-scale gradients: $\nabla \sim \eta^{-1/2}$, so

$$-2\eta \int d^3\mathbf{r} |\nabla \times \mathbf{B}|^2 \rightarrow \text{const} \quad \text{as} \quad \eta \rightarrow +0. \quad (13.26)$$

But then the right-hand side of Eq. (13.24)

$$-2\eta \int d^3\mathbf{r} \mathbf{B} \cdot (\nabla \times \mathbf{B}) = \mathcal{O}(\eta^{1/2}) \rightarrow 0 \quad \text{as} \quad \eta \rightarrow +0. \quad (13.27)$$

Thus, as an initial magnetic configuration relaxes, while its energy can change quickly (on dynamical times), its helicity changes only very slowly in the limit of small η . *The constancy of H (as $\eta \rightarrow +0$) provides us with the constraint subject to which the energy will need to be minimised.*

Before we use this idea, let us discuss what the conservation of helicity means physically, or, rather, topologically.

† The resistive term in the right-hand side of Eq. (13.24) is $\propto \int d^3\mathbf{r} \mathbf{B} \cdot \mathbf{j}$, a quantity known as the *current helicity*.

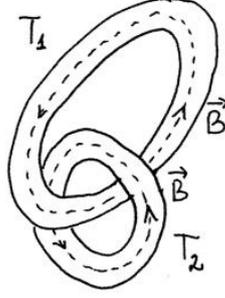


FIGURE 49. Linked flux tubes.

13.2.3. Helicity Is a Topological Invariant

Consider two linked flux tubes, T_1 and T_2 (Fig. 49). The helicity of T_1 is the product of the fluxes through T_1 and T_2 :

$$\begin{aligned}
 H_1 &= \int_{T_1} d^3\mathbf{r} \mathbf{A} \cdot \mathbf{B} = \int_{T_1} \underbrace{d\mathbf{l}}_{\mathbf{b}d\mathbf{l}} \cdot \underbrace{d\mathbf{S}}_{\mathbf{b}d\mathbf{S}} \underbrace{\mathbf{A} \cdot \mathbf{B}}_{\mathbf{A} \cdot \mathbf{b}B} \\
 &= \int_{T_1} \mathbf{A} \cdot \mathbf{b}d\mathbf{l} \mathbf{B} \cdot \mathbf{b}d\mathbf{S} = \int_{T_1} \mathbf{A} \cdot d\mathbf{l} \mathbf{B} \cdot d\mathbf{S} = \Phi_1 \underbrace{\int_{T_1} \mathbf{A} \cdot d\mathbf{l}}_{= \int_{S_1} \mathbf{B} \cdot d\mathbf{S}} = \Phi_1 \Phi_2. \quad (13.28) \\
 &= \Phi_{\text{through hole in } T_1}
 \end{aligned}$$

By the same token, in general, in a system of many linked tubes, the helicity of tube i is

$$H_i = \Phi_i \Phi_{\text{through hole in tube } i} = \Phi_i \sum_j \Phi_j N_{ij}, \quad (13.29)$$

where N_{ij} is the number of times tube j passes through the hole in tube i . The total helicity of this entire assemblage of flux tubes is then

$$H = \sum_{ij} \Phi_i \Phi_j N_{ij}. \quad (13.30)$$

Thus, H is the number of linkages of the flux tubes weighted by the field strength in them. It is in this sense that it is a topological invariant.

Note that the cross-helicity $\int d^3\mathbf{r} \mathbf{u} \cdot \mathbf{B}$ (§12.2.8) can similarly be interpreted as counting the linkages between flux tubes (\mathbf{B}) and vortex tubes ($\boldsymbol{\omega} = \nabla \times \mathbf{u}$). The current helicity $\int d^3\mathbf{r} \mathbf{B} \cdot \mathbf{j}$ [appearing in the right-hand side of Eq. (13.24)] counts the number of linkages between current loops. The latter is not an MHD invariant though.

13.3. J. B. Taylor Relaxation

Let us now work out the equilibrium to which an MHD system will relax by *minimising magnetic energy subject to constant helicity*:

$$\delta \int_V d^3\mathbf{r} (B^2 - \alpha \mathbf{A} \cdot \mathbf{B}) = 0, \quad (13.31)$$

where α is the Lagrange multiplier introduced to enforce the constant-helicity constraint. Let us work out the two terms:

$$\begin{aligned}\delta \int_V d^3\mathbf{r} B^2 &= 2 \int_V d^3\mathbf{r} \mathbf{B} \cdot \delta\mathbf{B} = 2 \int_V d^3\mathbf{r} \mathbf{B} \cdot (\nabla \times \delta\mathbf{A}) \\ &= 2 \int_V d^3\mathbf{r} [-\nabla \cdot (\mathbf{B} \times \delta\mathbf{A}) + (\nabla \times \mathbf{B}) \cdot \delta\mathbf{A}] \\ &= -2 \int_{\partial V} d\mathbf{S} \cdot (\mathbf{B} \times \delta\mathbf{A}) + 2 \int_V d^3\mathbf{r} (\nabla \times \mathbf{B}) \cdot \delta\mathbf{A},\end{aligned}\quad (13.32)$$

$$\begin{aligned}\delta H = \delta \int_V d^3\mathbf{r} \mathbf{A} \cdot \mathbf{B} &= \int_V d^3\mathbf{r} (\mathbf{B} \cdot \delta\mathbf{A} + \mathbf{A} \cdot \delta\mathbf{B}) = \int_V d^3\mathbf{r} [\mathbf{B} \cdot \delta\mathbf{A} + \mathbf{A} \cdot (\nabla \times \delta\mathbf{A})] \\ &= \int_V d^3\mathbf{r} [\mathbf{B} \cdot \delta\mathbf{A} - \nabla \cdot (\mathbf{A} \times \delta\mathbf{A}) + \underbrace{(\nabla \times \mathbf{A}) \cdot \delta\mathbf{A}}_{=\mathbf{B}}] \\ &= - \int_{\partial V} d\mathbf{S} \cdot (\mathbf{A} \times \delta\mathbf{A}) + 2 \int_V d^3\mathbf{r} \mathbf{B} \cdot \delta\mathbf{A}.\end{aligned}\quad (13.33)$$

Now, since

$$\frac{\partial \delta\mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) = \nabla \times \left(\frac{\partial \boldsymbol{\xi}}{\partial t} \times \mathbf{B} \right) \quad (13.34)$$

for small displacements, we have $\delta\mathbf{A} = \boldsymbol{\xi} \times \mathbf{B}$, whence

$$\mathbf{B} \times \delta\mathbf{A} = B^2 \boldsymbol{\xi} - \mathbf{B} \cdot \boldsymbol{\xi} \mathbf{B}, \quad (13.35)$$

$$\mathbf{A} \times \delta\mathbf{A} = \mathbf{A} \cdot \mathbf{B} \boldsymbol{\xi} - \mathbf{A} \cdot \boldsymbol{\xi} \mathbf{B}. \quad (13.36)$$

Therefore, the surface terms in Eqs. (13.32) and (13.33) vanish if \mathbf{B} and $\boldsymbol{\xi}$ are parallel to the boundary ∂V , i.e., if the volume V encloses both \mathbf{B} and the plasma—there are no displacements through the boundary.

This leaves us with

$$\delta \int_V d^3\mathbf{r} (B^2 - \alpha \mathbf{A} \cdot \mathbf{B}) = 2 \int_V d^3\mathbf{r} (\nabla \times \mathbf{B} - \alpha \mathbf{B}) \cdot \delta\mathbf{A} = 0, \quad (13.37)$$

which instantly implies that \mathbf{B} is a linear force-free field:

$$\nabla \times \mathbf{B} = \alpha \mathbf{B} \quad \Rightarrow \quad \boxed{\nabla^2 \mathbf{B} = -\alpha^2 \mathbf{B}}. \quad (13.38)$$

Thus, our system will relax to a linear force-free state determined by Eq. (13.38) and system-specific boundary conditions. Here $\alpha = \alpha(H)$ depends on the (fixed by initial conditions) value of H via the equation

$$H(\alpha) = \int d^3\mathbf{r} \mathbf{A} \cdot \mathbf{B} = \frac{1}{\alpha} \int d^3\mathbf{r} B^2, \quad (13.39)$$

where \mathbf{B} is the solution of Eq. (13.38) (since $\nabla \times \mathbf{B} = \alpha \mathbf{B} = \alpha \nabla \times \mathbf{A}$, we have $\mathbf{B} = \alpha \mathbf{A} + \nabla \chi$ and the χ term vanishes under volume integration).

Thus, the prescription for finding force-free equilibria is

- solve Eq. (13.38), get $\mathbf{B} = \mathbf{B}(\alpha)$, parametrically dependent on α ,
- calculate $H(\alpha)$ according to Eq. (13.39),
- set $H(\alpha) = H_0$, where H_0 is the initial value of helicity, hence calculate $\alpha = \alpha(H_0)$ and complete the solution by using this α in $\mathbf{B} = \mathbf{B}(\alpha)$.

Note that it is possible for this procedure to return multiple solutions. In that case, the

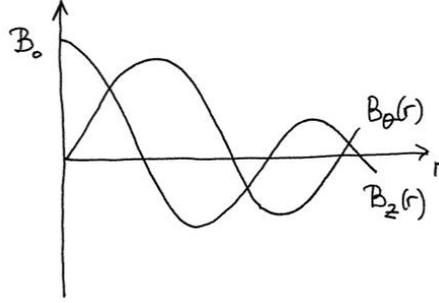


FIGURE 50. Relaxed cylindrical pinch.

solution with the smallest energy must be the right one (if a system relaxed to a local minimum, one can always imagine it being knocked out of it by some perturbation and falling to a lower energy).

13.4. Relaxed Force-Free State of a Cylindrical Pinch

Let us illustrate how the procedure derived in §13.3 works by considering again the case of cylindrical and axial symmetry [Eq. (13.5)]. The z component of Eq. (13.38) gives us the following equation for $B_z(r)$:

$$B_z'' + \frac{1}{r} B_z' + \alpha^2 B_z = 0. \quad (13.40)$$

This is a Bessel equation, whose solution, subject to $B_z(0) = B_0$ and $B_z(\infty) = 0$, is

$$\boxed{B_z(r) = B_0 J_0(\alpha r)}. \quad (13.41)$$

We can now calculate the azimuthal field as follows

$$\alpha B_\theta = (\nabla \times \mathbf{B})_\theta = -B_z' \quad \Rightarrow \quad \boxed{B_\theta(r) = B_0 J_1(\alpha r)}. \quad (13.42)$$

This gives us an interesting twisted field (Fig. 50), able to maintain itself in equilibrium without help from pressure gradients.

Finally, we calculate its helicity according to Eq. (13.39): assuming that the length of the cylinder is L , its radius R and so its volume $V = \pi R^2 L$, we have

$$\begin{aligned} H &= \frac{1}{\alpha} \int d^3 \mathbf{r} B^2 = \frac{2\pi L B_0^2}{\alpha} \int_0^R dr r [J_0^2(\alpha r) + J_1^2(\alpha r)] \\ &= \frac{B_0^2 V}{\alpha^2} \left[J_0^2(\alpha R) + 2J_1^2(\alpha R) + J_2^2(\alpha R) - \frac{2}{\alpha R} J_1(\alpha R) J_2(\alpha R) \right]. \end{aligned} \quad (13.43)$$

If we solve this for $\alpha = \alpha(H)$, our solution is complete.

Exercise 13.1. Work out what happens in the general case of $\partial/\partial\theta \neq 0$ and $\partial/\partial z \neq 0$ and whether the simple symmetric solution obtained above is the correct relaxed, minimum-energy state (not always, it turns out). This is not a trivial exercise. The solution is in Taylor & Newton (2015, §9), where you will also find much more on the subject of J. B. Taylor relaxation, relaxed states and much besides—all from the original source.

There are other useful variational principles—other in the sense that the constraints that are imposed are different from helicity conservation. The need for those arises when one considers magnetic equilibria in domains that do not completely enclose the field lines, i.e., at the boundary, $d\mathbf{S} \cdot \mathbf{B} \neq 0$. One example of such a variational principle, also yielding a force-free

field (although not necessarily a linear one), is given in Q1(e). A specific example of such a field arises in Q7(f).

13.5. Parker's Problem and Topological MHD

Coming soon... On topology in MHD, a very mathematically minded student might enjoy the book by Arnold & Khesin (1999).

14. MHD Stability and Instabilities

We now wish to take a more general view of the MHD stability problem: given some static† equilibrium (some ρ_0 , p_0 , \mathbf{B}_0 and $\mathbf{u}_0 = 0$), will this equilibrium be stable to small perturbations of it, i.e., will these perturbations grow or decay?

There are two ways to answer this question:

1) Carry out the normal-mode analysis, i.e., linearise the MHD equations around the given equilibrium, just as we did when we studied MHD waves in §12.1, and see if any of the frequencies (solutions of the dispersion relation) turn out to be complex, with positive imaginary parts (growth rates). This approach has the advantage of being direct and also of yielding specific information about rates of growth or decay, the character of the growing and decaying modes, etc. However, for spatially complicated equilibria, this is often quite difficult to do and one might be willing to settle for less: just being able to prove that some configuration is stable or that certain types of perturbations might grow. Hence the the second approach:

2) Check whether, for a given equilibrium, all possible perturbations will lead to the energy of the system *increasing*. If so, then the equilibrium is stable—this is called *the energy principle* and we shall prove it shortly. If, on the other hand, certain perturbations lead to the energy *decreasing*, that equilibrium is unstable. The advantage of this second approach is that we do not need to solve the (linearised) MHD equations in order to pronounce on stability, just to examine the properties of the perturbed energy functional.

It should be already quite clear how to do the normal-mode analysis, at least conceptually, so we shall focus on the second approach.

14.1. Energy Principle

Recall what the total energy is in MHD (§11.11):

$$\mathcal{E} = \int d^3\mathbf{r} \left[\frac{\rho u^2}{2} + \frac{B^2}{8\pi} + \frac{p}{\gamma - 1} \right] \equiv \int d^3\mathbf{r} \frac{\rho u^2}{2} + W. \quad (14.1)$$

As we saw in §12.1, all perturbations of an MHD system away from equilibrium can be expressed in terms of small displacements $\boldsymbol{\xi}$ —we will work this out shortly for a general equilibrium, but for now, let us accept that this will be true‡. As $\mathbf{u} = \partial\boldsymbol{\xi}/\partial t$ by definition of $\boldsymbol{\xi}$, we have

$$\mathcal{E} = \int d^3\mathbf{r} \frac{1}{2} \rho_0 \left| \frac{\partial\boldsymbol{\xi}}{\partial t} \right|^2 + W_0 + \delta W_1[\boldsymbol{\xi}] + \delta W_2[\boldsymbol{\xi}, \boldsymbol{\xi}] + \dots, \quad (14.2)$$

† A treatment of the more general case of a dynamic equilibrium, $\mathbf{u}_0 \neq 0$, can be found in the excellent textbook by Davidson (2016).

‡ In fact, also the fully nonlinear dynamics can be completely expressed in terms of displacements if the MHD equations are written in Lagrangian coordinates (Newcomb 1962), but we have omitted this topic for lack of time.

where we have kept terms up to second order in ξ and so W_0 is the equilibrium part of W (i.e., its value for $\xi = 0$), $\delta W_1[\xi]$ is linear in ξ , $\delta W_2[\xi, \xi]$ is bilinear (quadratic), etc. Energy must be conserved to all orders, so

$$\frac{d\mathcal{E}}{dt} = \int d^3\mathbf{r} \underbrace{\rho_0 \frac{\partial^2 \xi}{\partial t^2}}_{\equiv \mathbf{F}[\xi]} \cdot \frac{\partial \xi}{\partial t} + \delta W_1 \left[\frac{\partial \xi}{\partial t} \right] + \delta W_2 \left[\frac{\partial \xi}{\partial t}, \xi \right] + \delta W_2 \left[\xi, \frac{\partial \xi}{\partial t} \right] + \dots = 0. \quad (14.3)$$

This must be true at all times, including at $t = 0$, when ξ and $\partial \xi / \partial t$ can be chosen independently (MHD equations are second-order in time if written in terms of displacements). Therefore, for arbitrary functions ξ and η ,

$$\int d^3\mathbf{r} \eta \cdot \mathbf{F}[\xi] + \delta W_1[\eta] + \delta W_2[\eta, \xi] + \delta W_2[\xi, \eta] + \dots = 0. \quad (14.4)$$

In the first order, this tells us that

$$\delta W_1[\eta] = 0, \quad (14.5)$$

which is good to know because it means that δW_1 disappears from Eq. (14.2) (there are no first-order energy perturbations). In the second order, we get

$$\int d^3\mathbf{r} \eta \cdot \mathbf{F}[\xi] = -\delta W_2[\eta, \xi] - \delta W_2[\xi, \eta]. \quad (14.6)$$

Let $\eta = \xi$. Then Eq. (14.6) implies

$$\boxed{\delta W_2[\xi, \xi] = -\frac{1}{2} \int d^3\mathbf{r} \xi \cdot \mathbf{F}[\xi]}. \quad (14.7)$$

This is the part of the perturbed energy in Eq. (14.2) that can be both positive and negative. The *Energy Principle* is

$$\boxed{\delta W_2[\xi, \xi] > 0 \text{ for any } \xi \Leftrightarrow \text{equilibrium is stable}} \quad (14.8)$$

(Bernstein *et al.* 1958). Before we are in a position to prove this, we must do some preparatory work.

14.1.1. Properties of the Force Operator $\mathbf{F}[\xi]$

Since the right-hand side of Eq. (14.6) is symmetric with respect to swapping $\xi \leftrightarrow \eta$, so must be the left-hand side:

$$\int d^3\mathbf{r} \eta \cdot \mathbf{F}[\xi] = \int d^3\mathbf{r} \xi \cdot \mathbf{F}[\eta]. \quad (14.9)$$

Therefore, operator $\mathbf{F}[\xi]$ is self-adjoint. Since, by definition,

$$\mathbf{F}[\xi] = \rho_0 \frac{\partial^2 \xi}{\partial t^2}, \quad (14.10)$$

the eigenmodes of this operator satisfy

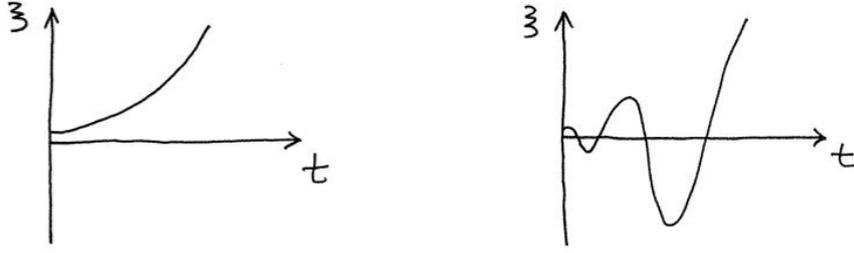
$$\xi(t, \mathbf{r}) = \xi_n(\mathbf{r}) e^{-i\omega_n t} \Rightarrow \mathbf{F}[\xi_n] = -\rho_0 \omega_n^2 \xi_n. \quad (14.11)$$

As always for self-adjoint operators, we can prove a number of useful statements.

1) The eigenvalues $\{\omega_n^2\}$ are real.

Proof. If Eq. (14.11) holds, so must

$$\mathbf{F}[\xi_n^*] = -\rho_0 (\omega_n^2)^* \xi_n^*, \quad (14.12)$$



(a) Instability.

(b) "Overstability" (does not happen in MHD).

FIGURE 51. MHD instabilities.

provided \mathbf{F} has no complex coefficients (we shall confirm this explicitly in §14.2.1). Taking the full scalar products (including integrating over space) of Eq. (14.11) with ξ_n^* and of Eq. (14.12) with ξ_n and subtracting one from the other, we get

$$\begin{aligned}
 -[\omega_n^2 - (\omega_n^2)^*] \underbrace{\int d^3\mathbf{r} \rho_0 |\xi_n|^2}_{>0} &= \int d^3\mathbf{r} \xi_n^* \cdot \mathbf{F}[\xi_n] - \int d^3\mathbf{r} \xi_n \cdot \mathbf{F}[\xi_n^*] = 0 \\
 \Rightarrow \boxed{\omega_n^2 = (\omega_n^2)^*}, & \quad \text{q.e.d.} \tag{14.13}
 \end{aligned}$$

This result implies that, if any MHD equilibrium is *unstable*, at least one of the eigenvalues must be $\omega_n^2 < 0$ and, since it is guaranteed to be real, any MHD instability will give rise to purely growing modes (Fig. 51a), rather than growing oscillations (also known as "overstabilities"; see Fig. 51b).

2) The eigenmodes $\{\xi_n\}$ are orthogonal.

Proof. Taking the full scalar products of Eq. (14.11) with ξ_m (assuming $m \neq n$ and non-degeneracy of $\omega_{m,n}^2$), and of the analogous equation

$$\mathbf{F}[\xi_m] = -\rho_0 \omega_m^2 \xi_m \tag{14.14}$$

with ξ_n and subtracting them, we get†

$$\begin{aligned}
 -\underbrace{(\omega_n^2 - \omega_m^2)}_{\neq 0} \int d^3\mathbf{r} \rho_0 \xi_n \cdot \xi_m &= \int d^3\mathbf{r} \xi_m \cdot \mathbf{F}[\xi_n] - \int d^3\mathbf{r} \xi_n \cdot \mathbf{F}[\xi_m] = 0 \\
 \Rightarrow \boxed{\int d^3\mathbf{r} \rho_0 \xi_n \cdot \xi_m = \delta_{nm} \int d^3\mathbf{r} \rho_0 |\xi_n|^2}, & \quad \text{q.e.d.} \tag{14.15}
 \end{aligned}$$

14.1.2. Proof of the Energy Principle (14.8)

Let us assume completeness of the set of eigenmodes $\{\xi_n\}$ (not, in fact, an indispensable assumption, but we shall not worry about this nuance here; see Kulsrud 2005, §7.2). Then any displacement at any given time t can be decomposed as

$$\xi(t, \mathbf{r}) = \sum_n a_n(t) \xi_n(\mathbf{r}). \tag{14.16}$$

† Note that in view of Eq. (14.13), we can take $\{\xi_n\}$ to be real.

The energy perturbation (14.7) is

$$\begin{aligned}\delta W_2[\boldsymbol{\xi}, \boldsymbol{\xi}] &= -\frac{1}{2} \int d^3\mathbf{r} \boldsymbol{\xi} \cdot \mathbf{F}[\boldsymbol{\xi}] = -\frac{1}{2} \sum_{nm} a_n a_m \int d^3\mathbf{r} \boldsymbol{\xi}_n \cdot \mathbf{F}[\boldsymbol{\xi}_m] \\ &= \frac{1}{2} \sum_{nm} a_n a_m \omega_m^2 \underbrace{\int d^3\mathbf{r} \rho_0 \boldsymbol{\xi}_n \cdot \boldsymbol{\xi}_m}_{\text{use Eq. (14.15)}} = \frac{1}{2} \sum_n a_n^2 \omega_n^2 \int d^3\mathbf{r} \rho_0 |\boldsymbol{\xi}_n|^2.\end{aligned}\quad (14.17)$$

By the same token,

$$K[\boldsymbol{\xi}, \boldsymbol{\xi}] \equiv \frac{1}{2} \int d^3\mathbf{r} \rho_0 |\boldsymbol{\xi}|^2 = \frac{1}{2} \sum_n a_n^2 \int d^3\mathbf{r} \rho_0 |\boldsymbol{\xi}_n|^2. \quad (14.18)$$

Then, if we arrange $\omega_1^2 \leq \omega_2^2 \leq \dots$, the smallest eigenvalue is

$$\omega_1^2 = \min_{\boldsymbol{\xi}} \frac{\delta W_2[\boldsymbol{\xi}, \boldsymbol{\xi}]}{K[\boldsymbol{\xi}, \boldsymbol{\xi}]}. \quad (14.19)$$

Therefore,

- condition (14.8) is *sufficient* for stability because, if $\delta W_2[\boldsymbol{\xi}, \boldsymbol{\xi}] > 0$ for all possible $\boldsymbol{\xi}$, then the smallest eigenvalue $\omega_1^2 > 0$, and so all eigenvalues are positive, $\omega_n^2 \geq \omega_1^2 > 0$;
- condition (14.8) is *necessary* for stability because, if the equilibrium is stable, then all eigenvalues are positive, $\omega_n^2 > 0$, whence $\delta W_2[\boldsymbol{\xi}, \boldsymbol{\xi}] > 0$ in view of Eq. (14.17), q.e.d.

14.2. Explicit Calculation of δW_2

Now that we know that we need the sign of δW_2 to ascertain stability (or otherwise), it is worth working out δW_2 as an explicit function of $\boldsymbol{\xi}$. It is a second-order quantity, but Eq. (14.7) tells us that all we need to calculate is $\mathbf{F}[\boldsymbol{\xi}]$ to first order in $\boldsymbol{\xi}$, i.e., we just need to linearise the MHD equations around an arbitrary static equilibrium. The procedure is the same as in §12.1, but without assuming ρ_0 , p_0 and \mathbf{B}_0 to be spatially homogeneous.

14.2.1. Linearised MHD Equations

Thus, generalising somewhat the procedure adopted in Eqs. (12.3–12.5), we have

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 &\Rightarrow \frac{\partial \delta \rho}{\partial t} = -\nabla \cdot \left(\rho_0 \frac{\partial \boldsymbol{\xi}}{\partial t} \right) \\ &\Rightarrow \boxed{\delta \rho = -\nabla \cdot (\rho_0 \boldsymbol{\xi})},\end{aligned}\quad (14.20)$$

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) p = -\gamma p \nabla \cdot \mathbf{u} &\Rightarrow \frac{\partial \delta p}{\partial t} = -\frac{\partial \boldsymbol{\xi}}{\partial t} \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \frac{\partial \boldsymbol{\xi}}{\partial t} \\ &\Rightarrow \boxed{\delta p = -\boldsymbol{\xi} \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \boldsymbol{\xi}},\end{aligned}\quad (14.21)$$

$$\begin{aligned}\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) &\Rightarrow \frac{\partial \delta \mathbf{B}}{\partial t} = \nabla \times \left(\frac{\partial \boldsymbol{\xi}}{\partial t} \times \mathbf{B}_0 \right) \\ &\Rightarrow \boxed{\delta \mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0)}.\end{aligned}\quad (14.22)$$

Note that again $\delta \rho$, δp and $\delta \mathbf{B}$ are all expressed as linear operators on $\boldsymbol{\xi}$ —and so $\delta W = \delta \int d^3\mathbf{r} [B^2/8\pi + p/(\gamma - 1)]$ must also be some operator involving $\boldsymbol{\xi}$ and its gradients but not $\partial \boldsymbol{\xi}/\partial t$ (as we assumed in §14.1).

Finally, we deal with the momentum equation (to which we add gravity as this will

give some interesting instabilities):

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi} + \rho \mathbf{g}. \quad (14.23)$$

This gives us

$$\begin{aligned} \mathbf{F}[\boldsymbol{\xi}] &= \rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = -\nabla \delta p + \frac{(\nabla \times \mathbf{B}_0) \times \delta \mathbf{B}}{4\pi} + \frac{(\nabla \times \delta \mathbf{B}) \times \mathbf{B}_0}{4\pi} + \delta \rho \mathbf{g} \\ &= \nabla (\boldsymbol{\xi} \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \boldsymbol{\xi}) - \mathbf{g} \nabla \cdot (\rho_0 \boldsymbol{\xi}) + \frac{\mathbf{j}_0 \times \delta \mathbf{B}}{c} + \frac{(\nabla \times \delta \mathbf{B}) \times \mathbf{B}_0}{4\pi}, \end{aligned} \quad (14.24)$$

where $\mathbf{j}_0 = c(\nabla \times \mathbf{B}_0)/4\pi$, we have used Eqs. (14.20) and (14.21) for $\delta \rho$ and δp , respectively, and $\delta \mathbf{B}$ is given by Eq. (14.22).

14.2.2. Energy Perturbation

Now we can use Eq. (14.24) in Eq. (14.7) to calculate explicitly

$$\begin{aligned} \delta W_2 &= \frac{1}{2} \int d^3 \mathbf{r} \left[\underbrace{-\boldsymbol{\xi} \cdot \nabla (\boldsymbol{\xi} \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \boldsymbol{\xi})}_{= (\boldsymbol{\xi} \cdot \nabla p_0) \nabla \cdot \boldsymbol{\xi} + \gamma p_0 (\nabla \cdot \boldsymbol{\xi})^2} + (\mathbf{g} \cdot \boldsymbol{\xi}) \nabla \cdot (\rho_0 \boldsymbol{\xi}) \right. \\ &\quad \left. - \underbrace{\frac{(\mathbf{j}_0 \times \delta \mathbf{B}) \cdot \boldsymbol{\xi}}{c}}_{= \frac{\mathbf{j}_0 \cdot (\boldsymbol{\xi} \times \delta \mathbf{B})}{c}} - \underbrace{\frac{(\nabla \times \delta \mathbf{B}) \times \mathbf{B}_0}{4\pi} \cdot \boldsymbol{\xi}}_{= \frac{(\nabla \times \delta \mathbf{B}) \cdot (\boldsymbol{\xi} \times \mathbf{B}_0)}{4\pi}} \right]. \end{aligned} \quad (14.25)$$

by parts

$$\begin{aligned} &= \frac{(\delta \mathbf{B} \times \nabla) \cdot (\boldsymbol{\xi} \times \mathbf{B}_0)}{4\pi} \\ &= \frac{\delta \mathbf{B} \cdot [\nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0)]}{4\pi} \\ &= \frac{|\delta \mathbf{B}|^2}{4\pi} \text{ by Eq. (14.22)} \end{aligned}$$

Thus, we have arrived at a standard textbook (e.g., Kulsrud 2005) expression for the energy perturbation (this expression is non-unique because one can do various integrations by parts):

$$\begin{aligned} \delta W_2 &= \frac{1}{2} \int d^3 \mathbf{r} \left[(\boldsymbol{\xi} \cdot \nabla p_0) \nabla \cdot \boldsymbol{\xi} + \gamma p_0 (\nabla \cdot \boldsymbol{\xi})^2 + (\mathbf{g} \cdot \boldsymbol{\xi}) \nabla \cdot (\rho_0 \boldsymbol{\xi}) \right. \\ &\quad \left. + \frac{\mathbf{j}_0 \cdot (\boldsymbol{\xi} \times \delta \mathbf{B})}{c} + \frac{|\delta \mathbf{B}|^2}{4\pi} \right], \end{aligned} \quad (14.26)$$

where $\delta \mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0)$. Note that two of the terms inside the integral (the second and the fifth) are positive-definite and so always stabilising. The terms that are not sign-definite and so potentially destabilising involve equilibrium gradients of pressure, density and magnetic field (currents). It is perhaps not a surprise to learn that Nature might dislike gradients—while it is of course not a rule that all such inhomogeneities render the system unstable, we will see that they often do, usually when gradients exceed certain critical thresholds.

All we need to do now is calculate δW_2 according to Eq. (14.26) for any equilibrium

that interests us and see if it can be negative for any class of perturbations (or show that it is positive for all perturbations).

14.3. Interchange Instabilities

As the first and simplest example of how one does stability calculations using the Energy Principle, we will (perhaps disappointingly) consider a purely hydrodynamic situation: the stability of a simple hydrostatic equilibrium describing a generic *stratified atmosphere*:

$$\rho_0 = \rho_0(z) \quad \text{and} \quad p_0 = p_0(z) \quad \text{satisfying} \quad \boxed{\frac{dp_0}{dz} = -\rho_0 g} \quad (14.27)$$

(gravity acts downward, against the z direction, $\mathbf{g} = -g\hat{\mathbf{z}}$).

14.3.1. Formal Derivation of the Schwarzschild Criterion

With $\mathbf{B}_0 = 0$ and the hydrostatic equilibrium (14.27), Eq. (14.26) becomes

$$\begin{aligned} \delta W_2 &= \frac{1}{2} \int d^3\mathbf{r} \left[\xi_z p'_0 \nabla \cdot \boldsymbol{\xi} + \gamma p_0 (\nabla \cdot \boldsymbol{\xi})^2 - g \xi_z (\rho'_0 \xi_z + \rho_0 \nabla \cdot \boldsymbol{\xi}) \right] \\ &= \frac{1}{2} \int d^3\mathbf{r} \left[2p'_0 \xi_z \nabla \cdot \boldsymbol{\xi} + \gamma p_0 (\nabla \cdot \boldsymbol{\xi})^2 - \rho'_0 g \xi_z^2 \right], \end{aligned} \quad (14.28)$$

where we have used $\rho_0 g = -p'_0$. We see that δW_2 depends on ξ_z and $\nabla \cdot \boldsymbol{\xi}$. Let us treat them as independent variables and minimise δW_2 with respect to them (i.e., *seek the most unstable possible situation*):

$$\frac{\partial}{\partial (\nabla \cdot \boldsymbol{\xi})} \left[\begin{array}{l} \text{integrand of} \\ \text{Eq. (14.28)} \end{array} \right] = 2p'_0 \xi_z + 2\gamma p_0 (\nabla \cdot \boldsymbol{\xi}) = 0 \quad \Rightarrow \quad \nabla \cdot \boldsymbol{\xi} = -\frac{p'_0}{\gamma p_0} \xi_z. \quad (14.29)$$

Substituting this back into Eq. (14.28), we get

$$\begin{aligned} \delta W_2 &= \frac{1}{2} \int d^3\mathbf{r} \left[-\frac{p_0'^2}{\gamma p_0} - \rho'_0 g \right] \xi_z^2 = \frac{1}{2} \int d^3\mathbf{r} \frac{\rho_0 g}{\gamma} \underbrace{\left[\frac{p'_0}{p_0} - \gamma \frac{\rho'_0}{\rho_0} \right]}_{= \frac{d}{dz} \ln \frac{p_0}{\rho_0^\gamma}} \xi_z^2. \end{aligned} \quad (14.30)$$

By the Energy Principle, the system is stable iff

$$\delta W_2 > 0 \quad \Leftrightarrow \quad \boxed{\frac{d \ln s_0}{dz} > 0}, \quad (14.31)$$

where $s_0 = p_0/\rho_0^\gamma$ is the entropy function. Eq. (14.31) is the *Schwarzschild criterion* for convective stability[†]. If this criterion is broken, there will be an instability, called the *interchange instability*.

This calculation illustrates both the power and the weakness of the method:

—on the one hand, we have obtained a stability criterion quite quickly and without having to solve the underlying equations,

—on the other hand, while we have established the condition for instability, we have as yet absolutely no idea what is going on physically.

[†] We studied perturbations of a stably stratified atmosphere in §12.2.9 and Q5, where we indeed saw that these perturbations did not grow provided the entropy scale length $1/H_s = d \ln s_0 / dz$ was positive.

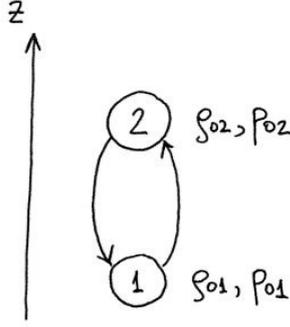


FIGURE 52. Interchange instability.

14.3.2. Physical Picture

We can remedy the latter problem by examining what type of displacements give rise to $\delta W_2 < 0$ when the Schwarzschild criterion is broken. Recalling Eqs. (14.20) and (14.21) and specialising to the displacements given by Eq. (14.29) (as they are the ones that minimise δW_2), we get

$$\frac{\delta p}{\rho_0} = -\frac{1}{\rho_0} \nabla \cdot (\rho_0 \xi) = -\frac{\rho'_0}{\rho_0} \xi_z - \nabla \cdot \xi = \frac{1}{\gamma} \left(-\gamma \frac{\rho'_0}{\rho_0} + \frac{p'_0}{p_0} \right) = \frac{1}{\gamma} \underbrace{\frac{d \ln s_0}{dz}}_{< 0 \text{ (unstable)}} \xi_z, \quad (14.32)$$

$$\frac{\delta p}{p_0} = -\frac{\xi \cdot \nabla p_0}{p_0} - \gamma \nabla \cdot \xi = -\frac{p'_0}{p_0} \xi_z - \gamma \nabla \cdot \xi = 0. \quad (14.33)$$

Thus, the offending perturbations maintain themselves in pressure balance (i.e., they are *not* sound waves) and locally increase or decrease density for blobs of fluid that fall ($\xi_z < 0$) or rise ($\xi_z > 0$), respectively.

This gives us some handle on the situation: if we imagine a blob of fluid slowly rising (slowly, so $\delta p = 0$) from the denser nether regions of the atmosphere to the less dense upper ones, then we can ask whether staying in pressure balance with its surroundings will require the blob to expand ($\delta \rho < 0$) or contract ($\delta \rho > 0$). If it is the latter, it will fall back down, pulled by gravity; if the former, then it will keep rising (buoyantly). The direction of the entropy gradient determines which of these two scenarios is realised.

14.3.3. Intuitive Rederivation of the Schwarzschild Criterion

We can use this physical intuition to derive the Schwarzschild criterion directly. Consider two blobs, at two different vertical locations, lower (1) and upper (2), where the equilibrium densities and pressures are ρ_{01}, p_{01} and ρ_{02}, p_{02} . Now interchange these two blobs (Fig. 52). Inside the blobs, the new densities and pressures are ρ_1, p_1 and ρ_2, p_2 .

Requiring the blobs to stay in pressure balance with their local surroundings gives

$$p_1 = p_{02}, \quad p_2 = p_{01}. \quad (14.34)$$

Requiring the blobs to rise or fall adiabatically, i.e., to satisfy $p/\rho^\gamma = \text{const}$, and then using pressure balance (14.34) gives

$$\frac{p_{01}}{\rho_{01}^\gamma} = \frac{p_1}{\rho_1^\gamma} = \frac{p_{02}}{\rho_1^\gamma} \Rightarrow \frac{\rho_1}{\rho_{01}} = \left(\frac{p_{02}}{p_{01}} \right)^{1/\gamma}. \quad (14.35)$$

Requiring that the buoyancy of the rising blob overcome gravity, i.e., that the weight

of the displaced fluid be larger than the weight of the blob,

$$\rho_{02}g > \rho_{1}g, \quad (14.36)$$

gives the condition for instability:

$$\rho_1 < \rho_{02} \Leftrightarrow \frac{\rho_1}{\rho_{02}} = \frac{\rho_{01}}{\rho_{02}} \left(\frac{p_{02}}{p_{01}} \right)^{1/\gamma} < 1 \Leftrightarrow \boxed{\frac{p_{02}}{\rho_{02}^\gamma} < \frac{p_{01}}{\rho_{01}^\gamma}}. \quad (14.37)$$

This is exactly the same as the Schwarzschild condition (14.31) for the interchange instability (and this is why the instability is called that).

Note that, while this is of course a much simpler and more intuitive argument than the application of the Energy Principle, it only gives us a particular example of the kind of perturbation that would be *unstable* under particular conditions, not any general criterion of what equilibria might be guaranteed to be stable.

In Q8, we will explore how the above considerations can be generalised to an equilibrium that also features a non-zero magnetic field.

14.4. Instabilities of a Pinch

As our second (also classic) example, we consider the stability of a z -pinch equilibrium (§13.1.1, Fig. 47):

$$\mathbf{B}_0 = B_0(r)\hat{\boldsymbol{\theta}}, \quad \mathbf{j}_0 = j_0(r)\hat{\mathbf{z}} = \frac{c}{4\pi r} (rB_0)'\hat{\mathbf{z}}, \quad p_0'(r) = -\frac{1}{c} j_0 B_0 = -\frac{B_0(rB_0)'}{4\pi r}. \quad (14.38)$$

Since we are going to have to work in cylindrical coordinates, we must first write all

the terms in Eq. (14.26) in these coordinates and with the equilibrium (14.38):

$$\begin{aligned} (\boldsymbol{\xi} \cdot \nabla p_0)(\nabla \cdot \boldsymbol{\xi}) &= \xi_r p_0' \left(\frac{1}{r} \frac{\partial}{\partial r} r \xi_r + \cancel{\frac{1}{r} \frac{\partial \xi_\theta}{\partial \theta}} + \frac{\partial \xi_z}{\partial z} \right) \\ &= p_0' \frac{\xi_r^2}{r} + p_0' \xi_r \left(\frac{\partial \xi_r}{\partial r} + \frac{\partial \xi_z}{\partial z} \right), \end{aligned} \quad (14.39)$$

$$\gamma p_0 (\nabla \cdot \boldsymbol{\xi})^2 = \gamma p_0 \left(\frac{1}{r} \frac{\partial}{\partial r} r \xi_r + \frac{1}{r} \frac{\partial \xi_\theta}{\partial \theta} + \frac{\partial \xi_z}{\partial z} \right)^2, \quad (14.40)$$

$$\delta \mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0) = \hat{\mathbf{r}} \left(\frac{1}{r} \frac{\partial}{\partial \theta} \xi_r B_0 \right) + \hat{\boldsymbol{\theta}} \left(-\frac{\partial}{\partial z} \xi_z B_0 - \frac{\partial}{\partial r} \xi_r B_0 \right) + \hat{\mathbf{z}} \left(\frac{1}{r} \frac{\partial}{\partial \theta} \xi_z B_0 \right), \quad (14.41)$$

$$\begin{aligned} \frac{\mathbf{j}_0 \cdot (\boldsymbol{\xi} \times \delta \mathbf{B})}{c} &= \underbrace{\frac{j_0}{c}}_{= -\frac{p_0'}{B_0}} (\xi_r \delta B_\theta - \xi_\theta \delta B_r) = p_0' \left[\xi_r \left(\frac{\partial \xi_z}{\partial z} + \frac{\partial \xi_r}{\partial r} + \xi_r \frac{B_0'}{B_0} \right) + \cancel{\xi_\theta \frac{1}{r} \frac{\partial \xi_r}{\partial \theta}} \right], \\ &= p_0' \xi_r \left(\frac{\partial \xi_z}{\partial z} + \frac{\partial \xi_r}{\partial r} \right) + \frac{p_0' B_0'}{B_0} \xi_r^2, \end{aligned} \quad (14.42)$$

$$\begin{aligned} \frac{|\delta \mathbf{B}|^2}{4\pi} &= \frac{B_0^2}{4\pi r^2} \left[\left(\frac{\partial \xi_r}{\partial \theta} \right)^2 + \left(\frac{\partial \xi_z}{\partial \theta} \right)^2 \right] + \underbrace{\frac{B_0^2}{4\pi} \left(\frac{\partial \xi_z}{\partial z} + \frac{\partial \xi_r}{\partial r} + \xi_r \frac{B_0'}{B_0} \right)^2}_{= \frac{B_0^2}{4\pi} \left(\frac{\partial \xi_z}{\partial z} + \frac{\partial \xi_r}{\partial r} \right)^2 + \frac{2B_0 B_0'}{4\pi} \xi_r \left(\frac{\partial \xi_z}{\partial z} + \frac{\partial \xi_r}{\partial r} \right) + \frac{B_0'^2}{4\pi} \xi_r^2}. \end{aligned} \quad (14.43)$$

The terms that are crossed out have been dropped because they combine into a full derivative with respect to θ and so, upon substitution into Eq. (14.26), vanish under integration. Assembling all this together, we have

$$\begin{aligned} \delta W_2 &= \frac{1}{2} \int d^3 \mathbf{r} \left\{ \underbrace{\left(p_0' + \frac{p_0' r B_0'}{B_0} + \frac{r B_0'^2}{4\pi} \right)}_{= 2p_0' + \frac{B_0^2}{4\pi r}} \frac{\xi_r^2}{r} + 2 \underbrace{\left(p_0' + \frac{B_0 B_0'}{4\pi} \right)}_{= -\frac{B_0^2}{4\pi r}} \xi_r \left(\frac{\partial \xi_z}{\partial z} + \frac{\partial \xi_r}{\partial r} \right) \right. \\ &\quad \left. + \gamma p_0 (\nabla \cdot \boldsymbol{\xi})^2 + \frac{B_0^2}{4\pi r^2} \left[\left(\frac{\partial \xi_r}{\partial \theta} \right)^2 + \left(\frac{\partial \xi_z}{\partial \theta} \right)^2 \right] + \frac{B_0^2}{4\pi} \left(\frac{\partial \xi_z}{\partial z} + \frac{\partial \xi_r}{\partial r} \right)^2 \right\} \\ &= \frac{1}{2} \int d^3 \mathbf{r} \left\{ 2p_0' \frac{\xi_r^2}{r} + \frac{B_0^2}{4\pi} \left(\frac{\partial \xi_z}{\partial z} + \frac{\partial \xi_r}{\partial r} - \frac{\xi_r}{r} \right)^2 \right. \\ &\quad \left. + \gamma p_0 (\nabla \cdot \boldsymbol{\xi})^2 + \frac{B_0^2}{4\pi r^2} \left[\left(\frac{\partial \xi_r}{\partial \theta} \right)^2 + \left(\frac{\partial \xi_z}{\partial \theta} \right)^2 \right] \right\}, \end{aligned} \quad (14.44)$$

where, in simplifying the first two terms in the integrand, we used the equilibrium equation (14.38):

$$p_0' = -\frac{B_0^2}{4\pi r} - \frac{B_0 B_0'}{4\pi} \quad \Rightarrow \quad \frac{r B_0'^2}{4\pi} = -\frac{p_0' r B_0'}{B_0} - \frac{B_0 B_0'}{4\pi} = -\frac{p_0' r B_0'}{B_0} + p_0' + \frac{B_0^2}{4\pi r}. \quad (14.45)$$

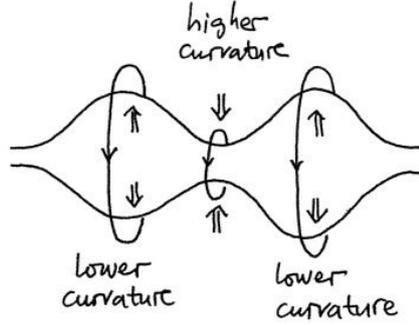


FIGURE 53. Sausage instability.

Finally, after a little further tidying up,

$$\delta W_2 = \frac{1}{2} \int d^3 \mathbf{r} \left\{ 2p'_0 \frac{\xi_r^2}{r} + \frac{B_0^2}{4\pi} \left(r \frac{\partial}{\partial r} \frac{\xi_r}{r} + \frac{\partial \xi_z}{\partial z} \right)^2 + \gamma p_0 \left(\frac{1}{r} \frac{\partial}{\partial r} r \xi_r + \frac{1}{r} \frac{\partial \xi_\theta}{\partial \theta} + \frac{\partial \xi_z}{\partial z} \right)^2 + \frac{B_0^2}{4\pi r^2} \left[\left(\frac{\partial \xi_r}{\partial \theta} \right)^2 + \left(\frac{\partial \xi_z}{\partial \theta} \right)^2 \right] \right\}. \quad (14.46)$$

14.4.1. Sausage Instability

Let us first consider axisymmetric perturbations: $\partial/\partial\theta = 0$. Then δW_2 depends on two variables only:

$$\xi_r \quad \text{and} \quad \eta \equiv \frac{\partial \xi_r}{\partial r} + \frac{\partial \xi_z}{\partial z}. \quad (14.47)$$

Indeed, unpacking all the r derivatives in Eq. (14.46), we get

$$\delta W_2 = \frac{1}{2} \int d^3 \mathbf{r} \left[2p'_0 \frac{\xi_r^2}{r} + \frac{B_0^2}{4\pi} \left(\eta - \frac{\xi_r}{r} \right)^2 + \gamma p_0 \left(\eta + \frac{\xi_r}{r} \right)^2 \right]. \quad (14.48)$$

We shall treat ξ_r and η as independent variables and minimise δW_2 with respect to η :

$$\frac{\partial}{\partial \eta} \left[\text{integrand of Eq. (14.48)} \right] = 2 \frac{B_0^2}{4\pi} \left(\eta - \frac{\xi_r}{r} \right) + 2\gamma p_0 \left(\eta + \frac{\xi_r}{r} \right) = 0 \quad \Rightarrow \quad \eta = \frac{1 - \gamma\beta/2}{1 + \gamma\beta/2} \frac{\xi_r}{r}, \quad (14.49)$$

where, as usual, $\beta = 8\pi p_0/B_0^2$. Putting this back into Eq. (14.48), we get

$$\begin{aligned} \delta W_2 &= \int d^3 \mathbf{r} p_0 \left[\frac{rp'_0}{p_0} + \frac{1}{\beta} \left(\frac{\gamma\beta}{1 + \gamma\beta/2} \right)^2 + \frac{\gamma}{2} \left(\frac{2}{1 + \gamma\beta/2} \right)^2 \right] \frac{\xi_r^2}{r^2} \\ &= \int d^3 \mathbf{r} p_0 \left(r \frac{d \ln p_0}{dr} + \frac{2\gamma}{1 + \gamma\beta/2} \right) \frac{\xi_r^2}{r^2}. \end{aligned} \quad (14.50)$$

There will be an instability ($\delta W_2 < 0$) if (but not only if, because we are considering the restricted set of axisymmetric displacements)

$$\boxed{-r \frac{d \ln p_0}{dr} > \frac{2\gamma}{1 + \gamma\beta/2}}, \quad (14.51)$$

i.e., when the pressure gradient is too steep, the equilibrium is unstable.

What sort of instability is this? Recall that the perturbations that we have identified

as making $\delta W_2 < 0$ are axisymmetric, have some radial and axial displacements and are *compressible*: from Eq. (14.49),

$$\nabla \cdot \boldsymbol{\xi} = \eta + \frac{\xi_r}{r} = \frac{2}{1 + \gamma\beta/2} \frac{\xi_r}{r}. \quad (14.52)$$

They are illustrated in Fig. 53. The mechanism of this aptly named *sausage instability* is clear: squeezing the flux surfaces inwards increases the curvature of the azimuthal field lines, this exerts stronger curvature force, leading to further squeezing; conversely, expanding outwards weakens curvature and the plasma can expand further.

Exercise 14.1. Convince yourself that the displacements that have been identified cause magnetic perturbations that are consistent with the cartoon in Fig. 53.

14.4.2. Kink Instability

Now consider non-axisymmetric perturbations ($\partial/\partial\theta \neq 0$) to see what other instabilities might be there. First of all, since we now have θ variation, δW_2 depends on ξ_θ . However, in Eq. (14.46), ξ_θ only appears in the third term, where it is part of $\nabla \cdot \boldsymbol{\xi}$, which enters quadratically and with a positive coefficient γp_0 . We can treat $\nabla \cdot \boldsymbol{\xi}$ as an independent variable, alongside ξ_r and ξ_z , and minimise δW_2 with respect to it. Obviously, the energy perturbation is minimal when

$$\nabla \cdot \boldsymbol{\xi} = 0, \quad (14.53)$$

i.e., the most dangerous non-axisymmetric perturbations are incompressible (unlike for the case of the axisymmetric sausage mode in §14.4.1: there we could not—and did not—have such incompressible perturbations because we did not have ξ_θ at our disposal, to be chosen in such a way as to enforce incompressibility).

To carry out further minimisation of δW_2 , it is convenient to Fourier transform our displacements in the θ and z directions—both are directions of symmetry (i.e., the equilibrium profiles do not vary in these directions), so this can be done with impunity:

$$\boldsymbol{\xi} = \sum_{m,k} \boldsymbol{\xi}_{mk}(r) e^{i(m\theta + kz)}. \quad (14.54)$$

Then Eq. (14.46) (with $\nabla \cdot \boldsymbol{\xi} = 0$) becomes, by Parseval's theorem (the operator $\mathbf{F}[\boldsymbol{\xi}]$ being self-adjoint; see §14.1.1),

$$\delta W_2 = \frac{1}{2} \sum_{m,k} 2\pi L_z \int_0^\infty dr r \left\{ 2p'_0 \frac{|\xi_r|^2}{r} + \frac{B_0^2}{4\pi} \left[\left| r \frac{\partial \xi_r}{\partial r} + ik\xi_z \right|^2 + \frac{m^2}{r^2} (|\xi_r|^2 + |\xi_z|^2) \right] \right\}. \quad (14.55)$$

As ξ_z and ξ_z^* only appear algebraically in Eq. (14.55) (no r derivatives), it is easy to minimise δW_2 with respect to them: setting the derivative of the integrand with respect to either ξ_z or ξ_z^* to zero, we get

$$-ik \left(r \frac{\partial \xi_r}{\partial r} + ik\xi_z \right) + \frac{m^2}{r^2} \xi_z = 0 \quad \Rightarrow \quad \xi_z = \frac{ikr^3}{m^2 + k^2r^2} \frac{\partial \xi_r}{\partial r}. \quad (14.56)$$

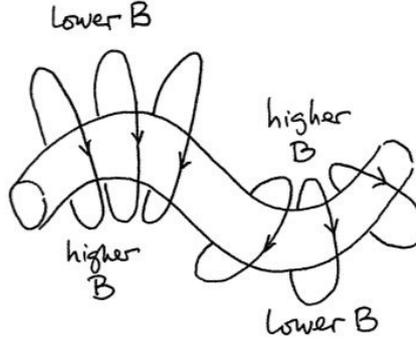


FIGURE 54. Kink instability.

Putting this back into Eq. (14.55) and assembling terms, we get

$$\begin{aligned} \delta W_2 = \sum_{m,k} \pi L_z \int_0^\infty dr r \left\{ 2p_0 \left(\frac{rp'_0}{p_0} + \frac{m^2}{\beta} \right) \frac{|\xi_r|^2}{r^2} \right. \\ \left. + \frac{B_0^2}{4\pi} \left[\underbrace{\left(1 - \frac{k^2 r^2}{m^2 + k^2 r^2} \right)^2 + \frac{m^2 k^2 r^2}{(m^2 + k^2 r^2)^2}}_{= \frac{m^2}{m^2 + k^2 r^2}} \right] \left| r \frac{\partial \xi_r}{\partial r} \right|^2 \right\}. \end{aligned} \quad (14.57)$$

The second term here is always stabilising. The most unstable modes will be ones with $k \rightarrow \infty$, for which the stabilising term is as small as possible. The remaining term will allow $\delta W_2 < 0$ and, therefore, an instability, if

$$\boxed{-r \frac{d \ln p_0}{dr} > \frac{m^2}{\beta}}. \quad (14.58)$$

Again, the equilibrium is unstable if the pressure gradient is too steep. The most unstable modes are ones with the smallest m , viz., $m = 1$.

Note that another way of writing the instability condition (14.58) is

$$-rp'_0 = \frac{B_0^2}{4\pi} + \frac{rB_0 B'_0}{4\pi} > m^2 \frac{B_0^2}{8\pi} \Rightarrow \boxed{r \frac{d \ln B_0}{dr} > \frac{m^2}{2} - 1}, \quad (14.59)$$

where we have used the equilibrium equation (14.38).

What does this instability look like? The unstable perturbations are incompressible:

$$\nabla \cdot \xi = 0 \Rightarrow \frac{1}{r} \frac{\partial}{\partial r} r \xi_r + \frac{im}{r} \xi_\theta + ik \xi_z = 0. \quad (14.60)$$

Setting $m = 1$ and using Eq. (14.56), we find

$$i \xi_\theta = -\frac{\partial}{\partial r} r \xi_r + \underbrace{\frac{k^2 r^4}{m^2 + k^2 r^2}}_{\approx r^2 \text{ as } k \rightarrow \infty} \frac{\partial \xi_r}{\partial r} \approx -2 \xi_r \quad \text{and} \quad \xi_z \ll \xi_r. \quad (14.61)$$

The basic cartoon (Fig. 54) is as follows: the flux surfaces are bent, with a twist (to

remain uncompressed). The bending pushes the magnetic loops closer together and thus increases magnetic pressure in concave parts and, conversely, decreases it in the convex ones. Plasma is pushed from the areas of higher B to those with lower B , thermal pressure in the latter (convex) areas becomes uncompensated, the field lines open up further, etc. This is called the *kink instability*.

Similar methodology can be used to show that, unlike the z pinch, the θ pinch (§13.1.1, Fig. 48) is always stable: see Q9.

15. Further Reading

What follows is *not* a literature survey, but rather just a few pointers for the keen and the curious.

15.1. MHD Instabilities

There are very many of these, easily a whole course's worth. They are an interesting topic. A founding text is the old, classic, super-meticulous monograph by Chandrasekhar (2003). In the context of **toroidal (fusion) plasmas**, you want to learn the so-called **ballooning theory**, a tour de force of theoretical plasma physics, which, like the relaxation theory, is associated with J. B. Taylor's name (so his lectures, Taylor & Newton 2015, are a good starting point; the original paper on the subject is Connor *et al.* 1979). In the unlikely event that you have an appetite for more **energy-principle calculations** in the style of §14.4, the book by Freidberg (2014) will teach you more than you ever wanted to know. In **astrophysics**, MHD instabilities have been a hot topic since the early 1990s, not least due the realisation by Balbus & Hawley (1991) that the **magnetorotational instability (MRI)** is responsible for triggering turbulence and, therefore, maintaining momentum transport in accretion flows—so the lecture notes by Balbus (2015) are an excellent place to start learning about this subject (this is also an opportunity to learn how to handle equilibria that are *not static*, e.g., most interestingly, featuring **rotating and shear flows**)[†].

As with everything in physics, the frontier in this subject is nonlinear phenomena. One very attractive theoretical topic has been the theory of **explosive instabilities and erupting flux tubes** by S. C. Cowley and his co-workers: the founding (quite pedagogically written) paper was Cowley & Artun (1997), the key recent one is Cowley *et al.* (2015); follow the paper trail from there for various refinements and applications (from space to tokamaks).

15.2. Resistive MHD

Most of our discussion revolved around properties of ideal MHD equations. It is, in fact, quite essential to study resistive effects, even when resistivity is very small, because many ideal solutions have a natural tendency to develop ever smaller spatial gradients, which can only be regularised by resistivity (we touched on this, e.g., in §13.2.2). The key linear result here is the **tearing mode**, a resistive instability associated with the propensity of magnetic-field lines to reconnect—change their topology in such a way as to release some of their energy. This is covered in the lectures by Parra (2017a); other good places to read about it are Taylor & Newton (2015) again, the original paper by Furth *et al.* (1963), or standard textbooks (e.g., Sturrock 1994, §17).

[†] Another excellent set of lecture notes on **astrophysical fluid dynamics** is Ogilvie (2016), this one originating from Cambridge Part III.

Here again the frontier is nonlinear: the theory of **magnetic reconnection**: tearing modes, in their nonlinear stage, tend to lead to formation of *current sheets* (which is, in fact, a general tendency of *X*-point solutions in MHD), and how reconnection happens after that has been a subject of active research since mid-20th century. Magnetic reconnection is believed to be a key player in a host of plasma phenomena, from solar flares to the so-called “sawtooth crash” in tokamaks, to MHD turbulence. Kulsrud (2005, §14) has a good introduction to the history and the basics of the subject from a live witness and key contributor. There has been much going on in it in the last decade, many of the advances occurring on the collisionless reconnection front requiring kinetic theory (some key names to search for in the extensive recent literature are W. Daughton, J. Drake, J. Egedal), but even within MHD, the discovery of the **plasmoid instability** (amounting to the realisation that current sheets are tearing unstable; see Loureiro *et al.* 2007) has led to a new theory of resistive MHD reconnection (Uzdensky *et al.* 2010), a development that I (obviously) find important.

15.3. *Dynamo Theory and MHD Turbulence*

These are topics on which I hope to fill in the gaps in these notes (§§11.10 and 12.4). A (somewhat outdated) review that will give you some idea of what will be there and also supply some key further references is Schekochihin & Cowley (2007) (although I plan to include also a detailed introduction to the analytical theory of the Kazantsev–Kraichnan dynamo). Looking at other available reviews, on both topics, **MHD turbulence and turbulent dynamo**, one can do worse than read Tobias *et al.* (2012), although I do not necessarily endorse everything they say, in substance or in style—this subject is very much alive and this is a view from one particular tribe (here is a recent review of the same subject by an opposing tribe, which should give you a sense of the intensity of the debates: Beresnyak & Lazarian 2015; and here, finally, is an attempt to reconcile them: Mallet *et al.* 2016, braving the danger of ending up in cross-fire).

A classic (and mostly timeless) text on the **mean-field dynamo** theory (amplification of large-scale magnetic fields by small-scale turbulence) is Moffatt (1978).

15.4. *Lagrangian MHD and MHD Action Principle*

Coming back to more academic topics, this is a mathematically attractive formulation of MHD, on which there is an excellent classic paper by Newcomb (1962). This formalism, besides shedding some conceptual light, turns out to give us some useful analytical tools, e.g., for the treatment of explosive MHD instabilities (Pfirsch & Sudan 1993; Cowley & Artun 1997).

15.5. *Hall MHD, Electron MHD, Braginskii MHD*

These and other “two-fluid” approximations of plasma dynamics have to do with with (i) what happens at scales where different species (ions and electrons) cannot be considered to move together (**Hall/Electron MHD**; see, e.g., Q6) and (ii) how momentum transport (viscosity) and energy transport (heat conduction) operate in a *magnetised* plasma, i.e., a plasma where the Larmor motion of particles dominates over their Coulomb collisions, even though the latter might be faster than the fluid motions (**Braginskii 1965 MHD**). In general, this is a kinetic subject, although certain limits can be treated by fluid approximations. An introduction to these topics is given in Parra (2017*b*) and Parra (2017*a*) (see also Goedbloed & Poedts 2004, §3 and the excellent monograph by Helander & Sigmar 2005).

15.6. Double-Adiabatic MHD and Onwards to Kinetics

A conceptually interesting and important paradigm is the so-called **double-adiabatic MHD** (or **CGL equations**, after the original authors Chew *et al.* 1956; see also Kulsrud 1983). This deals with a situation in a magnetised plasma (in the sense defined in §15.5) when pressure becomes anisotropic, with pressures perpendicular and parallel to the local direction of the magnetic field evolving each according to its own, separate equation, replacing the adiabatic law (11.60) and based on the conservation of the adiabatic invariants of the Larmor-gyrating particles. The **dynamics of pressure-anisotropic plasma**, based on CGL equations or, which is usually more correct physically, on the full kinetic description (and its reduced versions, e.g., **Kinetic MHD**; see Parra 2017*b*, also Kulsrud 1983), are another current frontier, with applications to weakly collisional astrophysical plasmas (from interplanetary to intergalactic). A key feature that makes this topic both interesting and difficult is that pressure anisotropies in high- β plasmas trigger small-scale instabilities (in particular, the Alfvén wave becomes unstable—the so-called **firehose instability**), which break the fluid approximation and leave us without a good mean-field theory for the description of macroscopic motions in such environments (for a short introduction to these issues, see Schekochihin *et al.* 2010, although this subject is developing so fast that anything written 7 years ago is already partially obsolete; you can read Squire *et al.* 2017 for a taste of how hairy things become in what concerns even such staples as Alfvén waves).

Magnetohydrodynamics Problem Set

1. Clebsch Coordinates. As $\nabla \cdot \mathbf{B} = 0$, it is always possible to find two scalar functions $\alpha(\mathbf{r})$ and $\beta(\mathbf{r})$ such that

$$\mathbf{B} = \nabla\alpha \times \nabla\beta. \quad (15.1)$$

(a) Argue that any magnetic field line can be described by the equations

$$\alpha = \text{const}, \quad \beta = \text{const}. \quad (15.2)$$

This means that (α, β, ℓ) , where ℓ is the distance (arc length) along the field line, are a good set of curvilinear coordinates, known as the *Clebsch coordinates*.

(b) Show that the magnetic flux through any area S in the (x, y) plane is

$$\Phi = \int_{\tilde{S}} d\alpha d\beta, \quad (15.3)$$

where \tilde{S} is the area S in new coordinates after transforming $(x, y) \rightarrow (\alpha(x, y, 0), \beta(x, y, 0))$.

(c) Show that if Eq. (15.1) holds at time $t = 0$ and α and β are evolved in time according to

$$\frac{d\alpha}{dt} = 0, \quad \frac{d\beta}{dt} = 0, \quad (15.4)$$

where d/dt is the convective derivative, then Eq. (15.1) correctly describes the magnetic field at all $t > 0$.

(d) Argue from the above that magnetic flux is frozen into the flow and magnetic field lines move with the flow.

(e) Show that the field that minimises the magnetic energy within some domain subject to the constraint that the values of α and β are fixed at the boundary of this domain (i.e., that the “footprints” of the field lines are fixed) is a force-free field.

A prototypical example of the kind of fields that arise from the variational principle in (e) is the “arcade” fields describing magnetic loops sticking out of the Sun’s surface, with footpoints anchored at the surface. One such field will be considered in Q7(f) and more can be found in Sturrock (1994, §13).

2. Uniform Collapse. A simple model of star formation envisions a sphere of galactic plasma with number density $n_{\text{gal}} = 1 \text{ cm}^{-3}$ undergoing a gravitational collapse to a spherical star with number density $n_{\text{star}} = 10^{26} \text{ cm}^{-3}$. The magnetic field in the galactic plasma is $B_{\text{gal}} \sim 3 \times 10^{-6} \text{ G}$. Assuming that flux is frozen, estimate the magnetic field in a star. Find out if this is a good estimate. If not, how, in your view, could we account for the discrepancy?

3. Flux Concentration. Consider a simple 2D model of incompressible convective motion (Fig. 55):

$$\mathbf{u} = U \left(-\sin \frac{\pi x}{L} \cos \frac{\pi z}{L}, 0, \cos \frac{\pi x}{L} \sin \frac{\pi z}{L} \right). \quad (15.5)$$

(a) In the neighbourhood of the stagnation point $(0, 0, 0)$, linearise the flow, assume vertical magnetic field, $\mathbf{B} = (0, 0, B(t, x))$ and derive an evolution equation for $B(t, x)$, including both advection by the flow and Ohmic diffusion. Suppose the field is initially

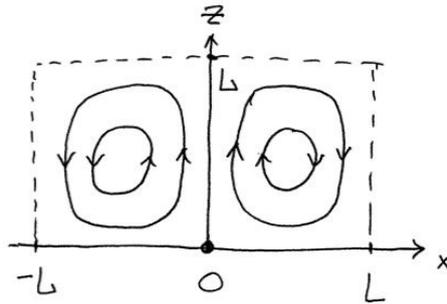


FIGURE 55. Convective cells from Q3.

uniform, $B(t = 0, x) = B_0 = \text{const}$. It should be clear to you from your equation that magnetic field is being swept towards $x = 0$. What is the time scale of this sweeping? Given the magnetic Reynolds number $\text{Rm} = UL/\eta \gg 1$, show that flux conservation holds on this time scale.

(b) Find the steady-state solution of your equation. Assume $B(x) = B(-x)$ and use flux conservation to determine the constants of integration (in terms of B_0 and Rm). What is the width of the region around $x = 0$ where the flux is concentrated? What is the magnitude of the field there?

(c*) Obtain the time-dependent solution of your equation for B and confirm that it indeed converges to your steady-state solution. Find the time scale on which this happens.

Hint. The following changes of variables may prove useful: $\xi = \sqrt{\pi \text{Rm}} x/L$, $\tau = \pi U t/L$, $X = \xi e^\tau$, $s = (e^{2\tau} - 1)/2$.

(d) Can you think of a quick heuristic argument based on the induction equation that would tell you that all these answers were to be expected?

4. Zeldovich's Antidynamo Theorem. Consider an arbitrary 2D velocity field: $\mathbf{u} = (u_x, u_y, 0)$. Assume incompressibility. Show that, in a finite system (i.e., in a system that can be enclosed within some volume outside which there are no fields or flows), this velocity field cannot be a dynamo, i.e., any initial magnetic field will always eventually decay.

Hint. Consider separately the evolution equations for B_z and for the magnetic field in the (x, y) -plane. Show that B_z decays by working out the time evolution of the volume integral of B_z^2 . Then write B_x, B_y in terms of one scalar function (which must be possible because $\partial B_x/\partial x + \partial B_y/\partial y = 0$) and show that it decays as well.

5. MHD Waves in a Stratified Atmosphere. The generalisation of iMHD to the case of a stratified atmosphere is explained in §12.2.9. Convince yourself that you understand how the SMHD equations and the SMHD ordering arise and then study them as follows.

(a) Work out all SMHD waves (both their frequencies and the corresponding eigenvectors). It is convenient to choose the coordinate system in such a way that $\mathbf{k} = (k_x, 0, k_z)$, where z is the vertical direction (the direction of gravity). The mean magnetic field $\mathbf{B}_0 = B_0 \mathbf{b}_0$ is assumed to be straight and uniform, at a general angle to z . We continue referring to the projection of the wave number onto the magnetic-field direction as $k_{\parallel} = \mathbf{k} \cdot \mathbf{b}_0 = k_x b_{0x} + k_z b_{0z}$. Note that in the case of $B_0 = 0$, you are dealing with

stratified hydrodynamics, not MHD—the waves that you obtain in this case are the well known *gravity waves*, or “*g-modes*”.

(b) Explain the physical nature of the perturbations (what makes the fluid oscillate) in the special cases (i) $k_z = 0$ and $\mathbf{b}_0 = \hat{\mathbf{z}}$, (ii) $k_z = 0$ and $\mathbf{b}_0 = \hat{\mathbf{x}}$, (iii) $k_x = 0$, (iv) $k_z \neq 0$, $k_x \neq 0$ and $\mathbf{b}_0 = \hat{\mathbf{z}}$.

(c) Under what conditions are the perturbations you have found unstable? What is the physical mechanism for the instability? What role does the magnetic field play (stabilising or destabilising) and why? Cross-check your answers with §14.3 and Q8.

(d) Find the conserved energy (a quadratic quantity whose integral over space stays constant) for the full nonlinear SMHD equations (12.93–12.96). Give a physical interpretation of the quantity that you have obtained—why should it be conserved?

6. Electron MHD. In certain physical regimes (roughly realised, for example, in the solar-wind and other kinds of astrophysical turbulence at scales smaller than the ion Larmor radius; see Schekochihin *et al.* 2009, Boldyrev *et al.* 2013), plasma turbulence can be described by an approximation in which the magnetic field is frozen into the electron flow \mathbf{u}_e , while ions are considered motionless, $\mathbf{u}_i = 0$. In this approximation, Ohm’s law becomes†

$$\mathbf{E} = -\frac{\mathbf{u}_e \times \mathbf{B}}{c}. \quad (15.6)$$

Here \mathbf{u}_e can be expressed directly in terms of \mathbf{B} because the current density in a plasma consisting of motionless hydrogen ions ($n_i = n_e$) and moving electrons is

$$\mathbf{j} = en_e(\mathbf{u}_i - \mathbf{u}_e) = -en_e\mathbf{u}_e, \quad (15.7)$$

but, on the other hand, \mathbf{j} is known via Ampère’s law. Here n_e is the electron number density and e the electron charge.

(a) Using this and Faraday’s law, show that the evolution equation for the magnetic field in this approximation is

$$\boxed{\frac{\partial \mathbf{B}}{\partial t} = -d_i \nabla \times [(\nabla \times \mathbf{B}) \times \mathbf{B}]}, \quad (15.8)$$

where the magnetic field has been rescaled to Alfvénic velocity units, $\mathbf{B}/\sqrt{4\pi m_i n_i} \rightarrow \mathbf{B}$, and $d_i = c/\omega_{pi}$ is the ion inertial scale (“ion skin depth”), $\omega_{pi} = \sqrt{4\pi e^2 n_i/m_i}$. Eq. (15.8) is the equation of *Electron MHD (EMHD)*, completely self-consistent for \mathbf{B} .

(b) Show that magnetic energy is conserved by Eq. (15.8). Is magnetic helicity conserved? Does J. B. Taylor relaxation work and what kind of field will be featured in the relaxed state? Is it obvious that this field is a good steady-state solution of Eq. (15.8)?

(c) Consider infinitesimal perturbations of a straight-field equilibrium, $\mathbf{B} = B_0 \hat{\mathbf{z}} + \delta \mathbf{B}$, and show that they are helical waves with the dispersion relation

$$\boxed{\omega = \pm k_{\parallel} v_A k d_i}. \quad (15.9)$$

These are called *Kinetic Alfvén Waves (KAW)*.

(d) Now consider finite perturbations and argue that the appropriate ordering in which

† Strictly speaking, the generalised Ohm’s law in this approximation also contains an electron-pressure gradient (see, e.g., Goedbloed & Poedts 2004), but that vanishes upon substitution of \mathbf{E} into Faraday’s law.

linear and nonlinear physics can coexist while perturbations remain small is

$$|\delta\mathbf{b}| \sim \frac{\delta B}{B} \sim \frac{k_{\parallel}}{k} \ll 1. \quad (15.10)$$

Under this ordering, show that the magnetic field can be represented as

$$\frac{\delta\mathbf{B}}{B_0} = \frac{1}{v_A} \hat{\mathbf{z}} \times \nabla_{\perp} \Psi + \hat{\mathbf{z}} \frac{\delta B}{B} \quad (15.11)$$

and the evolution equations for Ψ and $\delta B/B_0$ are

$$\boxed{\frac{\partial \Psi}{\partial t} = v_A^2 d_i \mathbf{b} \cdot \nabla \frac{\delta B}{B_0}, \quad \frac{\partial}{\partial t} \frac{\delta B}{B_0} = -d_i \mathbf{b} \cdot \nabla \nabla_{\perp}^2 \Psi}, \quad (15.12)$$

where $\mathbf{b} \cdot \nabla$ is given by Eq. (12.111). These are the equations of *Reduced Electron MHD*.

(e) Check that the conservation of magnetic energy and the KAW dispersion relation (15.9) are recovered from Eqs. (15.12). Is there any other conservation law?

7. Grad–Shafranov Equation. Consider static MHD equilibria [Eqs. (13.1)] in cylindrical coordinates (r, θ, z) and assume axisymmetry, $\partial/\partial\theta = 0$.

(a) Using the solenoidality of the magnetic field, show that any axisymmetric such field can be expressed in the form

$$\mathbf{B} = I\nabla\theta + \nabla\psi \times \nabla\theta, \quad (15.13)$$

where I and ψ are functions of r and z and $\nabla\theta = \hat{\boldsymbol{\theta}}/r$ ($\hat{\boldsymbol{\theta}}$ is the unit basis vector in the θ direction). Show that magnetic surfaces are surfaces of $\psi = \text{const}$.

(b) Using the force balance, show that $\nabla I \times \nabla\psi = 0$ and $\nabla p \times \nabla\psi = 0$ and hence argue that

$$I = I(\psi) \quad \text{and} \quad p = p(\psi) \quad (15.14)$$

are functions of ψ only (i.e., they are constant on magnetic surfaces).

(c) Again from the force balance, show that $\psi(r, z)$ satisfies the *Grad–Shafranov equation*

$$\boxed{-\left(\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2}\right) = 4\pi r^2 \frac{dp}{d\psi} + I \frac{dI}{d\psi}}. \quad (15.15)$$

This defines the shape of an axisymmetric equilibrium, given the profiles $p(\psi)$ and $I(\psi)$.

(d) Show that in cylindrical symmetry ($\partial/\partial\theta = 0$, $\partial/\partial z = 0$), Eq. (15.15) reduces to Eq. (13.8).

(e) Assume $I(\psi) = \text{const}$ (so the azimuthal field $B_{\theta} = I/r$ is similar to the magnetic field from a central current) and $p(\psi) = a\psi$, where a is some constant. Find a solution of Eq. (15.15) that gives rise to magnetic surfaces that resemble nested tori, but with “D-shaped” cross section (Fig. 56; this looks a bit like the modern tokamaks). If you stipulate that p must vanish at $r = 0$ and at $r = R$ along the $z = 0$ axis and also at $z = \pm L$ along the $r = 0$ axis and that the maximum pressure at $r < R$ is p_0 , show that the corresponding magnetic surfaces are described by

$$\psi = 2\sqrt{\frac{2\pi p_0}{1 + R^2/4L^2}} r^2 \left(1 - \frac{r^2}{R^2} - \frac{z^2}{L^2}\right). \quad (15.16)$$

Where is the (azimuthal) magnetic axis of these surfaces? What is the value of a ?

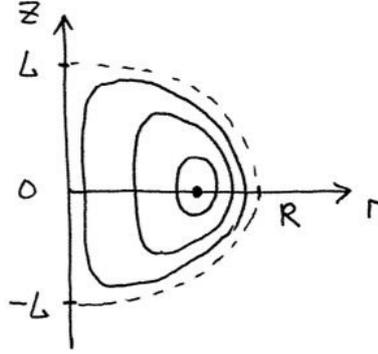


FIGURE 56. A simple equilibrium from Q7, superficially resembling the poloidal cross-section of a tokamak.

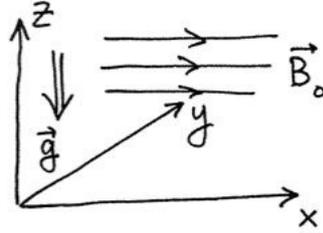


FIGURE 57. Magnetised atmosphere.

(f) Seek solutions to Eq. (15.15) that are linear force-free fields. Show that in this case, Eq. (15.15) reduces to the Bessel equation (a substitution $\psi = rf(r, z)$ will prove useful). Set $B_z(0, 0) = B_0$. Find solutions of two kinds: (i) ones in a semi-infinite domain $z \geq 0$, with the field vanishing exponentially at $z \rightarrow \infty$; (ii) ones periodic in z . If you also impose the boundary condition $B_r = 0$ at $r = R$, how can this be achieved? Can either of these solutions be the result of J. B. Taylor relaxation of an MHD system? If so, how would one decide whether it is more or less likely to be the correct relaxed state than the solution derived in §13.4?

You will find the solution of the type (i) in Sturrock (1994, §13) (who also shows how to construct many other force-free fields, useful in various physical and astrophysical contexts). Think of this solution in the context of Q1(e). The solution of type (ii) is a particular case of the general ($\partial/\partial\theta \neq 0$) equilibrium solution derived and discussed in Taylor & Newton (2015, §9). However, the axisymmetric solution is not very useful because, as they show, depending on the values of helicity and of R , the true relaxed state is either the cylindrically and axially symmetric solution derived in §13.4 or one which also has variation in the θ direction.

8. Magnetised Interchange Instability. Consider the same set up as in §14.3, but now the stratified atmosphere is threaded by straight horizontal magnetic field (Fig. 57):

$$\rho_0 = \rho_0(z), \quad p_0 = p_0(z), \quad \mathbf{B}_0 = B_0(z)\hat{\mathbf{x}}, \quad \frac{d}{dz} \left(p_0 + \frac{B_0^2}{8\pi} \right) = -\rho_0 g. \quad (15.17)$$

We shall be concerned with the stability of this equilibrium.

(a) For simplicity, assume $\partial\xi/\partial x = 0$. This rules out any perturbations of the magnetic-

field direction, $\delta \mathbf{b} = 0$, so there will be no field-line bending, no restoring curvature forces. For this restricted set of perturbations, work out δW_2 and observe that, like in the unmagnetised case considered in §14.3, it depends only on $\nabla \cdot \boldsymbol{\xi}$ and ξ_z . Minimise δW_2 with respect to $\nabla \cdot \boldsymbol{\xi}$ and show that

$$\frac{d}{dz} \ln \frac{p_0}{\rho_0^\gamma} + \frac{2}{\beta} \frac{d}{dz} \ln \frac{B_0}{\rho_0} < 0 \quad (15.18)$$

is a sufficient condition for instability (the *magnetised interchange instability*). Would you be justified in expecting stability if the condition (15.18) were not satisfied?

(b) Explain how this instability operates and rederive the condition for instability by considering interchanging blobs (or, rather, flux tubes), in the spirit of §14.3.3.

If field-line bending is allowed ($\partial \xi / \partial x \neq 0$), another instability emerges, the *Parker (1966) instability*. Do investigate.

9. Stability of the θ Pinch. Consider the following cylindrically and axially symmetric equilibrium:

$$\mathbf{B}_0 = B_0(r) \hat{\mathbf{z}}, \quad \mathbf{j}_0 = j_0(r) \hat{\boldsymbol{\theta}} = -\frac{c}{4\pi} B_0'(r) \hat{\boldsymbol{\theta}}, \quad \frac{d}{dr} \left(p_0 + \frac{B_0^2}{8\pi} \right) = 0 \quad (15.19)$$

(a θ pinch; see §13.1.1, Fig. 48). Consider general displacements of the form

$$\boldsymbol{\xi} = \boldsymbol{\xi}_{mk}(r) e^{im\theta + ikz}. \quad (15.20)$$

Show that the θ pinch is always stable. Specifically, you should be able to show that

$$\delta W_2 = 2\pi L_z \int_0^\infty dr r \left\{ \gamma p_0 |\nabla \cdot \boldsymbol{\xi}|^2 + \frac{B_0^2}{4\pi} \left[k^2 (|\xi_r|^2 + |\xi_\theta|^2) + \left| \frac{\xi_r}{r} + \frac{\partial \xi_r}{\partial r} + \frac{im\xi_\theta}{r} \right|^2 \right] \right\} > 0, \quad (15.21)$$

where L_z is the length of the cylinder.

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Appendix A. Kolmogorov Turbulence

When I fill up this section, it will be an updated version of my lectures that, in handwritten form, can be found here: <http://www-thphys.physics.ox.ac.uk/people/Alexander-Schekochihin/notes/SummerSchool07/>. In the meanwhile, §§33-34 of Landau & Lifshitz (1987) contain almost everything you need to know, but if you want to know more, books by Frisch (1995) and Davidson (2004) are modern classics that one cannot go wrong by reading.

In a somewhat lateral way, the scientific biography of Robert Kraichnan by Eyink & Frisch (2010) is an excellent read about turbulence, putting the subject (or at least its antecedents) in a broader context of theoretical physics and containing very many relevant references (especially early ones). These authors are not, however, quite as broad-minded or (in my view) up to date as I would have liked on the subject of intermittency, even in their contextual referencing. My favorite intermittency (in MHD) paper is, obviously, Mallet & Schekochihin (2017), whence you can follow the references back in time (or look instead at the last of the 5 lectures linked above).

A.1. Dimensional Theory of the Kolmogorov Cascade

A.2. Exact Laws

A.3. Intermittency

A.4. Turbulent Mixing

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