Mathematical Physics

Lecture Note for phys301: Fall 2021

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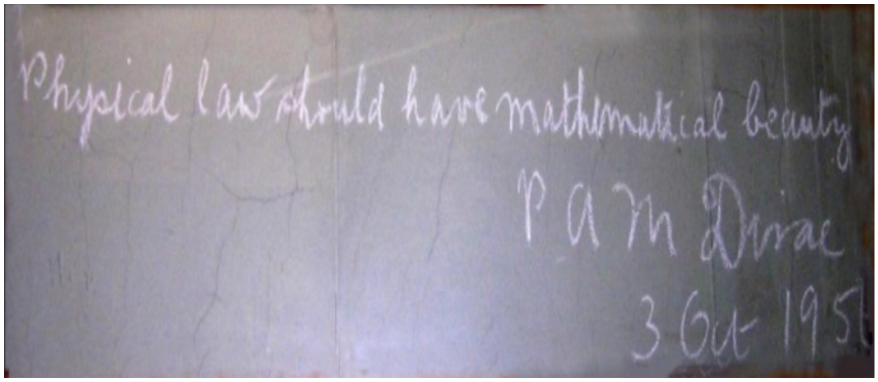


(Last revised Nov 18 , 2021)

Books and other Resources used in preparing these Lecture Notes:

- What I learned from my Ph.d mentor Richard Friedberg, Columbia University
- " Advanced Mathematics for Engineers and Scientists" by Murray R. Spiegel (McGraw-Hill Book Company) (I strongly recommend that you buy this book)
- " Atlas of Functions" by Jerome Spanier and Keith B. Oldham, Hemisphere publishing Corporation.
- "Mathematical Methods in Physical Sciences" by Mary Boas (Third Edition)
- "Mathematical Physics: Applications and Problems" by V. Balakrishnan (Springer)
- Various Wiki pages and other Google search resources
- David Griffith's books on Electro-Magnetic Theory and Quantum Mechanics and also Classical Dynamics by Marion.

<u>**Table of Content**</u>

Mathematical Physics

What is Mathematical Physics ??? What this course is about ???

The fact that physics requires mathematics at all levels makes the very definition of mathematical physics as a subject in the university physics curriculum rather fuzzy. Over the years, however, there has emerged a set of mathematical topics and techniques that are the most useful and widely applicable ones in various parts of physics. It is this repertoire or collection that constitutes “mathematical physics’”as the term is generally understood in its pedagogical sense. This course devoted to some of the topics of this core set.

Taking the phrase “mathematical physics” literally, this course is not an applied mathematics text in the conventional sense. It digresses into physics whenever the opportunity presents itself. Although numerous mathematical results are introduced and discussed, hardly any formal, rigorous proofs of theorems are presented. Instead, I have used specific examples and physical applications to illustrate and elaborate upon these results. The aim is to demonstrate how mathematics intertwines with physics in numerous instances. In my opinion, this is the fundamental justification for the very inclusion of mathematical physics as a subject .

Chapter 0

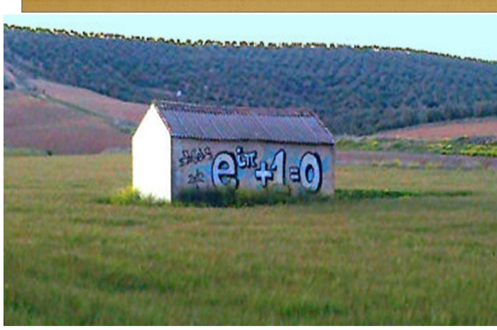
Laws of Nature & Mathematical Beauty

Some quotations

- “The so-called Pythagorean, who were the first to take up mathematics, not only advanced this subject, but saturated with it, they fancied that the principles of mathematics were the principles of all things. ”—Aristotle, Metaphysics 1–5, c. 350 BC
- “Philosophy is written in this grand book (the Universe) which stands continuously open to our gaze, but it cannot be read unless one first learns to understand the language in which it is written. It is written in the language of mathematics.” —Galileo Galilei, 1623
- “The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve” It is a testimony to the inherent simplicity and orderliness that pervades the fundamental science. ” – Eugene Wigner
- “Physical laws should have mathematical beauty. We must insist on it ”... P.A.M. Dirac
- “I wish you ladies and gentleman out there knew some of this mathematics. It is not just the logic and accuracy of it all you’re missing—it’s the poetry too.” —Richard Feynman (BBC interview, “A Novel Force in Nature”)
- “Mathematician’s patterns, like the painter’s or the poet’s must be beautiful; the ideas, like the colors or the words must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics...” H.G. Hardy

What is Mathematical Beauty ??

- In 1988, T̂he Mathematica Intelligence, a quarterly mathematics journal, carried out a poll about " the most beautiful theorem in mathematics in which twenty-four theorems were listed and readers were invited to award each a "score for beauty". The winner was " Euler equation".
- In 2004 the popular monthly magazine *Physics World* poll its reader to find *The greatest equations ever*, and even among physicists Euler's equation came a close second to the winning entry, Maxwell's equations for electromagnetism. Lagging far behind were the Pythagorean theorem, Einstein's equation, Newton's second law $F = ma$ and Botlzman law of entropy $S = klnW$.
- In 1933, at the age of 14, Richard Feynman described Euler equation as " The most remarkable formula in math", and in later years he referred to its close relative " Euler identity as *our jewel*.
- In words of Stanford mathematician Keith Devlin
  Like a shakespearian sonnet that captures the very essence of love, or a painting that brings out the beauty of human form that is far more than just skin deep, Euler's equationn reaches down into the very depths of existence " .



The Euler equation: $e^{i\pi} + 1 = 0$

Two irrational numbers e and π are related when you bring in complex number $i = \sqrt(-1)$......

Some Equations of Physics and Mathematical Beauty

$$\begin{aligned} F &= ma \\ E &= mc^2 \\ F &= G\frac{m_1 m_2}{r^2} \\ F &= k\frac{q_1 q_2}{r^2} \end{aligned}$$

Why $F \propto 1/r^2$

Why not $F \propto 1/r^\alpha$ where α can be any thing, integer, rational or irrational??

Why not some complicated function of r.

Do you know any equation that is "ugly"

Some Notations: Elegance & Simplicity

Some Examples:

- Coordinates: $(x, y, z) = (x_1, x_2, x_3) = (x_i, i = 1, 2, 3)$
- Unit Vectors: $(\hat{x}, \hat{y}, \hat{z}) = (\hat{x}_1, \hat{x}_2, \hat{x}_3) = (\hat{x}_i, i = 1, 2, 3)$
- $\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z = A_1 B_1 + A_2 B_2 + A_3 B_3 \equiv A_i B_i$
  (That is, repeated index represents a sum)
- Kronecker Delta function $\delta_{i,j}$

  $\delta_{i,j} = 0$ if $i \neq j$, $\delta_{ij} = 1$ if $i = j$.

  $\vec{A}.\vec{B} = A_i B_j \delta_{ij}$.
- Levi-Civita Symbol : ϵ_{ijk}
  See Wiki page link: `https://en.wikipedia.org/wiki/Levi-Civita_symbol`

  $\epsilon_{ijk} = 1, i \neq j \neq k$ and are in cyclic order: ($\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$)
  $\epsilon_{ijk} = -1, i \neq j \neq k$ and are not in cyclic order: ($\epsilon_{132} = \epsilon_{213} = \epsilon_{312} = -1$)

$\epsilon_{ijk} = 0$, if two of the indices are same: ($\epsilon_{122} = \epsilon_{111} = \epsilon_{133} ... = 0$)

$(\vec{A} \times \vec{B})_k = \epsilon_{ijk} A_i B_j$: This is the k^{th} component of $(\vec{A} \times \vec{B})$

Examples:

(1) Angular Momentum $\vec{L} = \vec{r} \times \vec{p}$

its i^{th} component $L_i = r_j p_k \epsilon_{ijk}$.

(2) Lorentz Force on an Electron: charge e, velocity $\vec{v}$, in a magnetic field $\vec{B}$: $\vec{F} = e\vec{v} \times \vec{B}$

its i^{th} component $F_i = e v_j B_k \epsilon_{ijk}$.

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Home Work Problem

(1) Use the formula $(\vec{A} \times \vec{B})_k = A_i B_j \epsilon_{ijk}$ to determine

(a) the three components of the Lorentz force $\vec{F} = e\vec{v} \times \vec{B}$ on an electron of charge e, given $\vec{v} = (1, 1, 1)$ and $\vec{B} = (2, -1, -2)$

(b) the three components of angular moment $\vec{L} = \vec{r} \times \vec{p}$ where $\vec{r} = (1, 2, 3)$ and $\vec{p} = (1, -1, -1)$.

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- If $a^2 + b^2 = 1$, set $a \equiv \cos\theta$ and $b \equiv \sin\theta$.
- If $a^2 - b^2 = 1$, $a \equiv \cosh x, b \equiv \sinh x$.

Example:

$$\begin{aligned}
f &= a\cos\theta + b\sin\theta \\
&= \sqrt{a^2+b^2}\Big(\frac{a}{\sqrt{a^2+b^2}}\cos\theta + \frac{b}{\sqrt{a^2+b^2}}\sin\theta\Big) \\
&\equiv \sqrt{a^2+b^2}(\sin\alpha\cos\theta + \cos\alpha\sin\theta), \quad \sin\alpha = \frac{a}{\sqrt{a^2+b^2}}, \ \ \cos\alpha\frac{b}{\sqrt{a^2+b^2}} \\
&= \sqrt{a^2+b^2}\sin(\theta+\alpha)
\end{aligned}$$

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Home Work Problem

(2) Given: $2\sin\theta + 5\cos\theta = A\sin(\theta+\phi)$, calculate A and ϕ. .

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An example of hidden Mathematical Beauty:

<u>Lorentz Transformation</u>

In physics, the Lorentz transformations are transformations from a coordinate frame in space-time to another frame that moves at a constant velocity v relative to the former. The transformations are named after the Dutch physicist Hendrik Lorentz.

The most common form of the transformation, parametrized by the real constant v representing a velocity confined to the x-direction, is expressed as:

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$x' = \gamma\,(x - vt)$$

$$y' = y$$

$$z' = z$$

where (t, x, y, z) and (t', x', y', z') are the coordinates of an event in two frames, where the primed frame is seen from the unprimed frame as moving with speed v along the x-axis, c is the speed of light, and $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ is called Lorentz factor.

When speed $v << c$, the Lorentz factor is negligibly different from 1, but as $v \to c$, γ grows without bound. The maximum value of v is equal to c, for the transformation to make sense.

Alternative way to write Lorentz Transformation

- (1)

  Expressing the speed as $\beta = \dfrac{v}{c}$, an equivalent form of the transformation is:

$ct' = \gamma\,(ct - \beta x)$

$x' = \gamma\,(x - \beta ct)$

$y' = y$

$z' = z.$

- (2)

  Let us write $ct = x_0, x = x_1, y = x_2, z = x_3$. Equations make a more symmetric form:

  $x_0' = \gamma\,(x_0 - \beta x)$

  $x_1' = \gamma\,(x_1 - \beta x_0)$

  $x_2' = x_2$

  $x_3' = x_3.$

- (3)

  We note: $\gamma^2 - (\beta\gamma)^2 = 1$

  This reminds us an identity in mathematics: $\cosh^2\alpha - \sinh^2\alpha = 1$,

  Therefore, let us make a following association and define an we angle α as,

$$\cosh\alpha = \gamma, \quad \sinh\alpha = \beta\gamma \tag{1}$$

Lorentz boost in the x-direction. It is given by,

$$\begin{aligned} x_0' &= x_0\cosh\alpha - x_1\sinh\alpha \\ x_1' &= -x_0\sinh\alpha + x_1\cosh\alpha \\ x_2' &= x_2 \\ x_3' &= x_3 \end{aligned} \tag{2}$$

And the corresponding matrix is,

$$\begin{bmatrix} x'_0 \\ x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \begin{bmatrix} \cosh\alpha & -\sinh\alpha & 0 & 0 \\ -\sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{3}$$

- (4) Lorentz transformation or the Lorentz "boosts" can be viewed as a rotation with angle of rotation α complex as shown below.

$$\tanh\alpha \;=\; \beta = \frac{e^{\alpha} - e^{-\alpha}}{e^{\alpha} + e^{-\alpha}}$$

Let us write $\alpha \equiv i\delta$, where δ is a real number. Then we have,

$$\tanh\alpha = \frac{e^{i\delta} - e^{-i\delta}}{e^{i\delta} + e^{-i\delta}} = \tan\delta = \beta \tag{4}$$

This gives,

$$\delta = \tan^{-1}(\beta) \tag{5}$$

Therefore. δ is a real number and $\alpha = i\delta$ is pure imaginary.

Recall:

Rotations in 3-dimensions

Consider simple rotations in three dimensions. Denoted by R, a rotation about z-axis by an angle θ of coordinates $(x, y, x) \to (x', y', z') = R(\theta)(x, y, z)$ is the equation,

$$\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \\ z' &= z \end{aligned} \tag{6}$$

Written in the matrix form, the equation is,

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{7}$$

———————————————————————-

Above equations show that Lorentz boost is “mathematically” equivalent to a rotation by an angle α which is pure imaginary number.

Lorentz boost as a form of “Rotation” in space-time.

This is *beautiful*!!!

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Home Work Problem

(3) An event in a laboratory frame occurs at $x_\mu = (x_0, x_1, x_2, x_3) = (0, 0, 0, 0)$. In a frame moving with velocity $\frac{1}{100}$ the speed of light along z-axis, calculate the space-time coordinates of event. Write the transformation matrix in terms of angle of rotation.

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I. CHAPTER I

Warming Up: Binomial & Taylor Expansion

We will review some fundamental concepts of mathematics that form the backbone of mathematical physics.

A. Binomial Expansion

History
Special cases of the binomial theorem were known since at least the 4th century BC when Greek mathematician Euclid mentioned the special case of the binomial theorem for exponent 2.There is evidence that the binomial theorem for cubes was known by the 6th century AD in India.

$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$., where

$\binom{n}{k} = \frac{n!}{(n-k)!k!}$

Examples

$(x+y)^0 = 1,$

$(x+y)^1 = x+y,$

$(x+y)^2 = x^2 + 2xy + y^2,$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3,$

$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4,$

Note: $(x+y)^n = x^n(1+(\frac{y}{x}))^n$. Therefore, Let us focus on $(1+x)^n$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots . \tag{8}$$

$(1+x)^0 = 1,$

$(1+x)^1 = 1+x,$

$(1+x)^2 = 1+2x+x^2,$

- Eq. (8) is a pretty formula. Note that when we put $n = 1$, all higher powers of x^2, x^3, ... etc have their coefficients go to zero.

- It also works when n is not an integer ....

  Around 1665, Isaac Newton generalized the binomial theorem to allow real exponents other than nonnegative integers. Also, when n is negative. For example:
  $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$ .....

- Note, that when n is not a positive integer, we do not get a Polynomial, but an infinite series.

- It is useful when x is small: $x << 1$,

  Examples:
  (1) $\frac{1}{1+x} \approx 1 - x$,($n = -1$) .

  (2) $\sqrt{1+x} \approx 1 + \frac{1}{2}x$, ($n = \frac{1}{2}$)

  (3) If $x << a$, $\frac{1}{(a-x)^3} = a^{-3}\frac{1}{(1-\frac{x}{a})^3} \approx a^{-3}(1 + 3\frac{x}{a})$.

NOTE: Truncations of binomial expansion are examples of Taylor series as we will be later.

B. Applications to Relativity

Below I list some important formulas for relativistic theory, that is, when particles move with speed comparable to speed of light.

Let m_0 be the mass of a particle in the frame in which it is at rest and m is its mass in the frame in which it moves with velocity v. Let E is the energy of the particle.

- $E = mc^2$ where $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \equiv \gamma m_0$. Note $E = m_0c^2$ when particle is at rest.
- Therefore, kinetic energy is $K = E - m_0c^2$.
- $E^2 = p^2c^2 + m_0^2c^4$

Problem (1): For a particle, moving with $\frac{1}{100}$ th speed of light, calculate the fractional change in the mass of the particle, compared to its rest mass:

Solution: Since the speed $v = \frac{c}{100}$, we can use the following approximation to calculate γ

$$\gamma = \frac{1}{\sqrt{1-(\frac{v}{c})^2}} \approx 1 + \tfrac{1}{2}(\tfrac{v}{c})^2, \quad (n = -\tfrac{1}{2}, x = -(\tfrac{v}{c})^2) \text{ using Eq. (8).}$$

$$\begin{aligned} m &= \gamma m_0 \approx m_0[1 + \frac{1}{2}(\frac{v}{c})^2] \\ \frac{m - m_0}{m_0} &= \frac{1}{2}(\frac{v}{c})^2 = .5.(.01)^2 = .5.10^{-4} \end{aligned}$$

Problem (2): The relativistic expression for energy of a particle of rest mass m_0 given by $E^2 = p^2c^2 + m_0^2c^4$. Show that for $v << c$, it reduces to $E = \frac{p^2}{2m} + m_0c^2 = \frac{1}{2}m_0v^2 + m_0c^2$.

Solution

$$\begin{aligned} E &= \sqrt{p^2c^2 + m_0^2c^4} \\ &= m_0c^2(1 + \frac{p^2}{m_0^2c^2})^{\frac{1}{2}} \\ &\approx m_0c^2(1 + \frac{1}{2}\frac{p^2}{m_0^2c^2}) \\ &= m_0c^2 + \frac{p^2}{2m_0} \end{aligned}$$

Note the following:

- Note, we have assumed that $x = \frac{p^2}{m_0^2 c^2} <<< 1$: check, if we put $p = m_0 v$, $x = (\frac{v}{c})^2$, which is the non-relativistic limit.

- Checking relativistic limit when energy is given. Suppose we say that an electron has kinetic energy of 10 Mev. Is it relativistic ?? How do you check this quickly ?? Note, you are given kinetic energy and not the velocity.

  Simply calculate the ratio of Kinetic energy and Rest mass energy.

  For example, rest mass energy of electron is $.5$ Mev. If Kinetic energy is much bigger than $.5$ Mev, electron is relativistic. This provides a very useful criterion for when to use relativistic expression for energy.

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Home Work Problem 1.1

(1) State which of the following particles have to be treated relativistically.

(a) An electron of kinetic energy 1 Mev, (b) Proton of Kinetic energy 1 Mev. (3) A neutron of kinetic energy 20 Mev.

(2) Given $f(x) = \frac{a}{(b-x)^{1.5}} = A + Bx$. Calculate A and B if $x <<< b$.

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C. Taylor Expansion

Refs: (1) Chapter I, page 8 of Murray and Spiegel;

(2) Wiki Link: `https://en.wikipedia.org/wiki/Taylor_series`

In 1715, Brook Taylor, a English mathematician provided a general method for constructing these series for all functions for which they exist. When a = 0, the series is also called a Maclaurin series - after Colin Maclaurin (Scottish mathematician) , who made extensive use of this special case of Taylor series in the 18th century.

Taylor series for a function $f(x)$ about $x = a$ is in general an infinite series:

$$\begin{aligned} f(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

where $f^{(n)}(a)$ denotes the nth derivative of f evaluated at the point a.

EXAMPLES:

- $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$,
- By integrating the above Maclaurin series, we find the Maclaurin series for $ln(1-x)$, where ln denotes the natural logarithm:

  $ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \cdots ., -1 \leq x < 1$
- Exponential function:

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots \end{aligned}$$

- Trigonometric Functions

$$\begin{aligned} \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \\ \cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \\ \tan^{-1} x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}, -1 \leq x \leq 1 \\ \cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \end{aligned}$$

  Note: $\sin x, \sinh x, \tan^{-1} x$ are odd functions and $\cos x, \cosh x$ are even functions. Graph them.

EXAMPLES:

(1) The first few Spherical Bessel functions are:

$$
\begin{aligned}
j_0(x) &= \frac{\sin x}{x} \\
j_1(x) &= \frac{\sin x}{x^2} - \frac{\cos x}{x} \\
j_2(x) &= (\frac{3}{x^3} - \frac{1}{x}) \sin x - \frac{3}{x^2} \cos x,
\end{aligned}
$$

Are they ill-defined at the origin ???

Check how these functions behave as $x \to 0$.

Using Taylor expansion of $\sin$ and $\cos x$, you can show that these functions do not diverge as $x \to \infty$. Let us work this for $j_1(x)$:

$$
\begin{aligned}
j_1(x) &= \frac{\sin x}{x^2} - \frac{\cos x}{x} \\
&= (x - x^3/3! + ...)/x^2 - (1 - x^2/2 + ...)/x \\
&= \frac{1}{x} - \frac{x^2}{3!} - \frac{1}{x} + \frac{x^2}{2} + ...O(x^n), \;\; n > 0 \\
&\to 0 \;\; as \;\; x \to 0
\end{aligned}
$$

NOTE: It is almost magical that although the various term in the these functions diverge, the divergences among various terms precisely cancel out and the functions are well behaved at $x = 0$.

We will be discussing these functions $j_n(x)$ later in the semester. For all n, these functions although appear singular at the origin are in fact well behaved at the origin.

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Home Work PROBLEMS (1.2)

(1) Sketch the following functions $f(x)$ for both positive and negative values of x, stating explicitly the behavior near $x = 0$ and at other special points such as near minima or maxima or points of discontinuities of the functions and $x \to \infty$ for non-periodic functions.

(1.1) $\sin x, \cos x, \tan x$

(1.2) $\ln x$, e^x, e^{-x^2}, $\cosh x$, $\sinh x$, $\tanh x$, $e^{-|x|}$, $\frac{1}{x}$, $\frac{1}{x^2}$.

(1.3) $\frac{\sin x}{x}$, $x^2 - x^4$.

(1.4) $f(x) = \frac{1}{x} - [\frac{1}{x}]$, where $[x]$ means integer part of x.

(2) Show that $f(x) = (\frac{3}{x^3} - \frac{1}{x}) \sin x - \frac{3}{x^2} \cos x$ is well defined at the origin.

(3) Use Taylor series method to obtain approximate value of the integrals

(3.1) $\int_0^1 \frac{1-e^{-x}}{x} dx$, (3.1) $\int_0^1 e^{-x^2} dx$

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II. CHAPTER II

Gaussian and Related Integrals and Functions

A. Basic Gaussian Integrals

The function e^{-x^2} , called a Gaussian, appears everywhere in mathematical sciences. In addition to playing a fundamental role in probability and statistics, it also appears very often in quantum mechanics. The fundamental Gaussian integral in its simplest form is

$$I = \int_0^\infty e^{-x^2} dx = \frac{1}{2} \int_{-\infty}^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \tag{9}$$

The integral cannot be evaluated by the usual method of integration by parts. Its value is determined as follows.

Consider the *square* of the integral and change to plane polar coordinates (ρ, ϕ) , where $\rho = \sqrt{x^2 + y^2}$ and $\tan\phi = \frac{y}{x}$. (This trick is attributed to Poisson.) The region of integration is the first quadrant in the xy-plane. The area element $dxdy$ in plane polar coordinates is $dxdy = \rho d\rho d\phi$. Thus

$$I^2 = \int_0^\infty dy \int_0^\infty e^{-(x^2+y^2)} dx = \int_0^\infty \rho \ d\rho \ e^{-\rho^2} \int_0^{\frac{\pi}{2}} d\phi = \frac{1}{4}\pi \tag{10}$$

Therefore, $I = \frac{\sqrt{\pi}}{2}$.

The simple result above has many interesting extensions that are useful in a remarkably large number of physical problems. Some are discussed below and also see home work problems.

B. Stirling Formula or Approximation: Application of Gaussian Integrals

$\ln n! = n \ln n - n + O(\ln n)$

Stirling's approximation is vital to a manageable formulation of statistical physics and thermodynamics. It vastly simplifies calculations involving logarithms of factorials where the factorial is huge. In statistical physics, we are typically discussing systems of 10^{22} particles.

With numbers of such orders of magnitude, this approximation is certainly valid, and also proves incredibly useful.

PROOF:

Let us start with the formula:

$$n! = \int_0^\infty x^n e^{-x}\,\mathrm{d}x.$$

You can check this for $n = 0, 1, 2, 3...$ by integrating by parts.

Using the identity $x = e^{\ln x}$, we have

$n! = \int_0^\infty e^{n\ln x - x}$

Change variables : $x = ny$, one obtains

$$\begin{aligned} n! &= \int_0^\infty e^{n\ln x - x}\,\mathrm{d}x \\ &= ne^{n\ln n}\int_0^\infty e^{n(\ln y - y)}\,\mathrm{d}y \equiv ne^{n\ln n}\int_0^\infty e^{nf(y)}\,\mathrm{d}y, \quad f(y) = \ln y - y \end{aligned}$$

Let us study the function $f(y) = \ln y - y$

$f(y)$ has a maxima at $y = y_0$ as,
$f'(y) = 1/y - 1 = 0$. That is $y_0 = 1$. $f''(y_0) = -\frac{1}{y_0^2} = -1 < 0$
Let us Taylor expand $f(y)$ about $y = y_0 = 1$

$$\begin{aligned} f(y) &= f(y_0) + f'(y_0)(y - y_0) + \frac{1}{2}f''(y_0)(y - y_0)^2 + ... \\ &= \ln y_0 - y_0 - \frac{1}{2y_0^2}(y - y_0)^2 + ... \\ &= -1 - \frac{1}{2}(y - 1)^2 + ... \end{aligned}$$

We can replace $f(y)$ by its Taylor expansion where we retain only quadratic terms. The resulting integral is a Gaussian integral.This is also known as Laplace's method.

———————————————————————————-

What is Laplace Method ??

Expand a function $f(y)$ about its maximum value, say y_0 (assuming such a maxima exists)

$$f(y) = f(y_0) + f'(y_0)(y - y_0) + \frac{1}{2}f''(y_0)(y - y_0)^2 + ...$$

If f has a global maximum at y_0, and so the derivative of f vanishes at y_0. Therefore, the function $f(y)$ may be approximated to quadratic order

$$f(y) \approx f(y_0) - \frac{1}{2}\left|f''(y_0)\right|(y - y_0)^2$$

$$\int_a^b e^{Mf(y)}\,dx \approx e^{Mf(y_0)}\int_a^b e^{-\frac{1}{2}M|f''(y_0)|(y-y_0)^2}\,dy$$

This latter integral is a Gaussian integral if the limits of integration go from $-\infty$ to $+\infty$ (which can be assumed because the exponential decays very fast away from y_0), and thus it can be calculated. We find

$$\int_a^b e^{Mf(y)}\,dy \approx \sqrt{\frac{2\pi}{M|f''(y_0)|}}e^{Mf(y_0)} \text{ as } M \to \infty.$$

We have used the formula, $\int_0^\infty e^{-ax^2}dx = \sqrt{\pi/a}, a > 0,$

NOTE: Laplace's method is itself a reduced form of a more general technique called the method of steepest descent or the saddle-point method, involving integration in the complex plane.

————————————————————————————–-

Applying Laplace method to Eq. (11) with $f(y) = \ln y - y$ and use the formula,

$\int_0^\infty e^{-ax^2}dx = \sqrt{\pi/a}, a > 0,$

Therefore,

$$n! \approx ne^{n\ln n}\int_0^\infty dx\;\, e^{n\ln x - x} = \sqrt{2\pi n}e^{n\ln n - n} \tag{11}$$

Taking log of this,

$\ln n! = n\ln n - n + O(\ln n)$

NOTE: The formula (11) is quite remarkable, because it is valid to an astonishing degree of accuracy even for relatively small values of n, including $n = 1$.When n is 10, the accuracy is already about 99.2%, and for $n = 100$, this becomes 99.92%., and so on.

The Stirling series for $n!$ is,

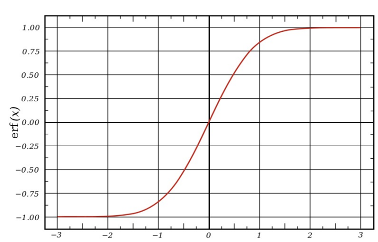
$$n! = e^{n\ln n - n}(2\pi n)^{\frac{1}{2}}[1 + \frac{1}{12n} + \frac{1}{288n^2} + O(n^{-3})....] \tag{12}$$

C. Error Function

In mathematics, the error function (also called the Gauss error function), often denoted by "erf", is a function of x variable defined as:

$\operatorname{erf} x = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-t^2}\,dt.$

In statistics, for non-negative values of x, the error function has the following interpretation: for a random variable Y that is normally distributed with mean 0 and standard deviation $\frac{1}{\sqrt{2}}$, erf x is the probability that Y falls in the range $[-x, x]$.



D. Γ functions

In mathematics, the Gamma function –represented by Γ, the capital letter gamma from the Greek alphabet) is one commonly used extension of the factorial function to complex numbers.

The gamma function is defined for all complex numbers except the non-positive integers. For any positive integer n

,

$\Gamma(n) = (n-1)!$.

For a complex number z,

$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx, \qquad \Re(z) > 0$.

Note that by setting $x^2 = u$ in Gaussian integral, Eq. (9),

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x^2} dx \tag{13}$$

$$= \sqrt{\pi} \tag{14}$$

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Home Work (2.1)

(1) $\int_\infty^\infty e^{-ax^2+bx} dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}$, $a > 0$, where b is any complex number.
If $a > 0$ and k is a real constant, show that
$\int_0^\infty dx e^{-ax^2} \cos kx = \sqrt{\frac{\pi}{a}} e^{\frac{-k^2}{4a}}$
Show that $\int_0^\infty dx e^{-ax^2} \sin kx$ cannot be evaluated by this procedure.

(2) Show that $\Gamma(1/2) = \int_{-\infty}^\infty e^{-x^2} dx$

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III. CHAPTER III

REAL NUMBERS

- Natural Numbers: Positive integers
- Integers: $0, \pm 1, \pm 2....$
- Rational Numbers
- Irrational Numbers & Its rational approximant

Five Important Numbers: $0, 1, i, \pi, e$

A. Integers

Pythagorean believed that all things were made of integers.

Integers are darlings of physicists – as very often , integers appearing in physical systems represent *quantum numbers*. Examples: quantization of angular momentum in central force problem, quantization of energy in bound state problems (such as Bohr model, particle in a box) and quantization of conductivity in quantum Hall effect..

There are some equations in mathematics whose solutions are integers. They are called Diophantine equation. The following Wiki link gives a good summary of such equations. Ref: <https://en.wikipedia.org/wiki/Diophantine_equation>

Example:Consider the famous equation:

$x^n + y^n = z^n$, (x, y, n) are integers.

For $n = 2$, there are infinitely many solutions (x, y, z): the Pythagorean triples as shown in Fig. (1). For further details, see <https://en.wikipedia.org/wiki/Tree_of_primitive_Pythagorean_triples>

For larger integer values of n, Fermat's Last Theorem (initially claimed in 1637 by Fermat and proved by Andrew Wiles in 1995) states there are no positive integer solutions (x, y, z).

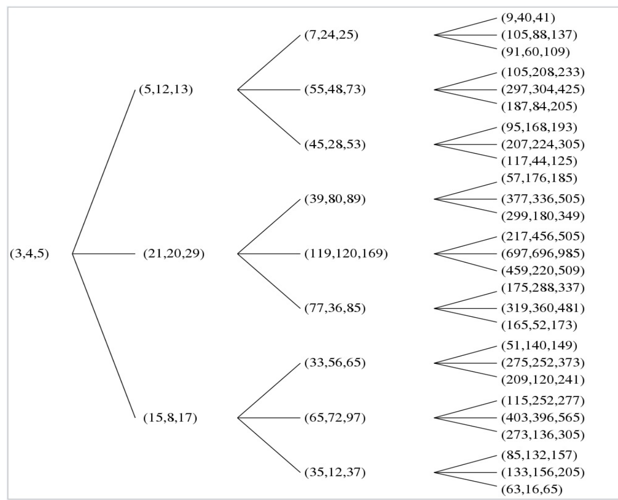
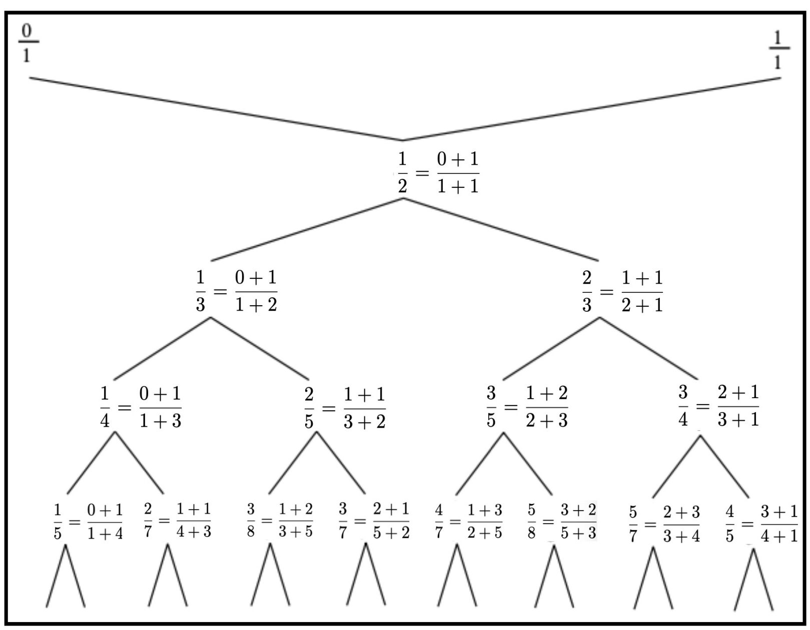


FIG. 1: All Pythagorean triplets, arranged in a tree

B. Rational Numbers: Farey Tree: Tree that generates all rationals.



- Discovered by Adolf Hurwitz in 1894, Farey tree generates all primitive rationals between 0 and 1. As shown in Fig. this hierarchical tree-like structure builds the entire set of rationals by starting with 0 and 1. Each successive row of the tree inherits all the Farey fractions from the level above it, and is enriched with some new fractions (all of which lie between 0 and 1) made by combining neighbors in the preceding row. To combine two fractions primitive $\frac{p_L}{q_L}$ and $\frac{p_R}{q_R}$, one simply adds their numerators, and also their denominators, so the the so called Farey sum $\frac{p_L+p_R}{q_L+q_R}$ gives a new primitive fraction. The n^{th} level of the Farey tree contains all fractions $\frac{p}{q}$, where $0 \leq p \leq q \leq n$, arranged horizontally in an increasing order.

- Given any two fractions $\frac{p_L}{q_L}$ and $\frac{p_R}{q_R}$ that satisfy

$$p_L q_R - p_R q_L = \pm 1, \tag{15}$$

then p_L and q_L are coprimes and so is p_R and q_R. This is because any common factor of p_L and q_L must divide the products $p_L q_R$ and $p_R q_L$ and hence the difference $p_L q_R - p_R q_L = \pm 1$. Any two fractions satisfying Eq. (15) are two neighboring fractions in the Farey tree and are known as the *friendly fractions*.

- Farey tree is constructed by applying the " Farey sum rule" to $\frac{p_L}{q_L}$ and $\frac{p_R}{q_R}$- the Farey parents that gives a new fraction $\frac{p_c}{q_c}$ - the Farey child:

$$\frac{p_c}{q_c} = \frac{p_L + q_R}{q_L + q_R}. \tag{16}$$

Analogous to the friendly pair $\frac{p_L}{q_L}$ and $\frac{p_R}{q_R}$, $\frac{p_c}{q_c}$ also forms friendly pair with each of its parents $\frac{p_L}{q_L}$ and $\frac{p_R}{q_R}$, satisfying the following two equations,

$$p_L q_c - p_c q_L = \pm 1 \tag{17}$$

$$p_c q_R - p_R q_c = \pm 1 \tag{18}$$

This implies that p_c and q_c are also coprime. In other words, the entire Farey tree consists of all fractions $\frac{p}{q}$ where p and q are coprime. These equations define a Farey triplet denoted as $[\frac{p_L}{q_L}, \frac{p_c}{q_c}, \frac{p_R}{q_R}]$ which will be referred as the "*friendly Farey triplet*".

C. Irrational Numbers: Continued Fraction Expansion

Every irrational number can be represented in precisely one way as an infinite continued fraction.

$$x = n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{n_4 + ...}}}}$$

$$\equiv [n_0; n_1, n_2, n_3, \ldots]$$

Calculating Continued Fractions: Gauss Map: $x_{i+1} = \frac{1}{x_i} - [\frac{1}{x_i}]$, where $[x]$ represents integer part of x.

Let x be an irrational number. To find its continued fraction expansion:

- Subtract the integer part and let $x_0 = x - [x] = x - n_0$
- $n_1 = [\frac{1}{x_0}]$
- $x_1 = \frac{1}{x_0} - n_0$
- $n_2 = [\frac{1}{x_1}]$
- $x_{i+1} = \frac{1}{x_i} - n_i$

Examples:

$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, ...]$
Golden-Mean: $(\sqrt{5} - 1)/2 = [1, 1, 1, 1, 1, 1, ...] = [\overline{1}]$
Silver-Mean: $\sqrt{2} - 1 = [2, 2, 2, ...] = [\overline{2}]$
(Diamond Mean): $2 - \sqrt{3} = [1, 2, 1, 2, 1, 2......] = [\overline{1, 2}]$

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Home Work: (3.1)

(1) Show that $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, ...]$

(2) Show that golden, silver and diamond means are solutions of quadratic equations.

(3) Show that rational approximants of golden-mean are ratios of Fibonacci numbers. Note $1, 2, 3, 5, 8, 13, 21, 34...F_n..$ are called Fibonacci numbers where $F_{n+1} = F_n + F_{n+1}$.

(4) Given integers p and q, let (x_0, y_0) is a solution of the linear Diophantine equation $px + qy = 1$. Show that there exists infinity of solutions of this linear equation.

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D. Continued Fraction Expansion: Best Rational Approximant of an Irrational Number

Hurwitz's Theorem

$x \approx \frac{p}{q}$- the rational approximants of an irrational obtained using continued fraction expansion is closer to x than any approximation with a smaller or equal denominator. That is, continued fraction expansion provides best rational approximants of irrational numbers as shown below for the most famous irrational number π.

There are many proofs of Hurwitz theorem. Ford circle representation of rational numbers, as described in the next section, provides a very nice proof. We will not discuss the proof. Those interested in the proof and more details, can read the Ford Circle paper on the web site (optional).

$$\pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, 1, 1, 15, 3, 13, 1,\\ 4, 2, 6, 6, 99, 1, 2, 2, 6, 3, 5, 1, 1, 6, 8, 1, 7, 1, 2, 3, 7, 1, 2, 1, 1, 12, 1, 1, 1,\\ 3, 1, 1, 8, 1, 1, 2, 1, 6, 1, 1, 5, 2, 2, 3, 1, 2, 4, 4, 16, 1, 161, 45, 1, 22, 1, 2, 2, 1, \ldots]$$

Truncated CF	**Rational value**	**Decimal value**	**Error**
[3]	3	3.00000000000000	$-1.41 \cdot 10^{-1}$
[3; 7]	$\frac{22}{7}$	3.14285714285714	$1.23 \cdot 10^{-3}$
[3; 7, 15]	$\frac{333}{106}$	3.14150943396226	$-8.32 \cdot 10^{-5}$
[3; 7, 15, 1]	$\frac{355}{113}$	3.14159292035398	$2.66 \cdot 10^{-7}$
[3; 7, 15, 1, 292]	$\frac{103993}{33102}$	3.14159265301190	$-5.77 \cdot 10^{-10}$
[3; 7, 15, 1, 292, 1]	$\frac{104348}{33215}$	3.14159265392142	$3.31 \cdot 10^{-10}$
[3; 7, 15, 1, 292, 1, 1]	$\frac{208341}{66317}$	3.14159265346744	$-1.22 \cdot 10^{-10}$
[3; 7, 15, 1, 292, 1, 1, 1]	$\frac{312689}{99532}$	3.14159265361894	$2.91 \cdot 10^{-11}$
[3; 7, 15, 1, 292, 1, 1, 1, 2]	$\frac{833719}{265381}$	3.14159265358108	$-8.71 \cdot 10^{-12}$
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1]	$\frac{1146408}{364913}$	3.14159265359140	$1.61 \cdot 10^{-12}$
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3]	$\frac{4272943}{1360120}$	3.14159265358939	$-4.04 \cdot 10^{-13}$
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1]	$\frac{5419351}{1725033}$	3.14159265358982	$2.22 \cdot 10^{-14}$
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14]	$\frac{80143857}{25510582}$	3.14159265358979	$-4.44 \cdot 10^{-16}$

q	$\frac{p}{q}$	$\frac{p}{q} - \pi$
1	$\boxed{\frac{3}{1}}$	$\boxed{-0.141593}$
2	$\frac{6}{2}$	-0.141593
3	$\frac{9}{3}$	-0.141593
4	$\frac{13}{4}$	$\boxed{0.108407}$
5	$\frac{16}{5}$	$\boxed{0.0584073}$
6	$\frac{19}{6}$	$\boxed{0.025074}$
7	$\boxed{\frac{22}{7}}$	$\boxed{0.00126449}$
8	$\frac{25}{8}$	-0.0165927
9	$\frac{28}{9}$	-0.0304815
10	$\frac{31}{10}$	-0.0415927
11	$\frac{35}{11}$	0.0402255
12	$\frac{38}{12}$	0.025074
13	$\frac{41}{13}$	0.0122535
14	$\frac{44}{14}$	0.00126449
15	$\frac{47}{15}$	-0.00825932

q	$\frac{p}{q}$	$\frac{p}{q} - \pi$
16	$\frac{50}{16}$	-0.0165927
17	$\frac{53}{17}$	-0.0239456
18	$\frac{57}{18}$	0.025074
19	$\frac{60}{19}$	0.0163021
20	$\frac{63}{20}$	0.00840735
21	$\frac{66}{21}$	0.00126449
22	$\frac{69}{22}$	-0.00522902
23	$\frac{72}{23}$	-0.0111579
24	$\frac{100}{24}$	-0.0165927
25	$\frac{79}{25}$	0.0184073
26	$\frac{82}{26}$	0.0122535
27	$\frac{85}{27}$	0.00655549
28	$\frac{88}{28}$	0.00126449
29	$\frac{91}{29}$	-0.00366162
30	$\frac{94}{30}$	-0.00825932

q	$\frac{p}{q}$	$\frac{p}{q} - \pi$
100	$\frac{314}{100}$	-0.00159265
101	$\frac{317}{101}$	-0.00297879
102	$\frac{320}{102}$	-0.00433775
103	$\frac{324}{103}$	0.00403841
104	$\frac{327}{104}$	0.00263812
105	$\frac{330}{105}$	0.00126449
106	$\boxed{\frac{333}{106}}$	$\boxed{-0.0000832196}$
107	$\frac{336}{107}$	-0.00140574
108	$\frac{339}{108}$	-0.00270376
109	$\frac{342}{109}$	-0.00397797
110	$\frac{346}{110}$	0.00386189
111	$\frac{349}{111}$	0.00255149
112	$\frac{352}{112}$	0.00126449
113	$\boxed{\frac{355}{113}}$	$\boxed{0.0000002667}$
114	$\frac{179}{57}$	-0.00124178
115	$\frac{361}{115}$	-0.00246222

Record approximations to π: Each rational in this list is a new record in the sense that it is closer to π than all rationals with smaller denominator. The list includes *all* such records up to denominator 16604. Rationals arising from the CF expansion of π are boxed.

$$\boxed{\frac{3}{1}}, \frac{13}{4}, \frac{16}{5}, \frac{19}{6}, \boxed{\frac{22}{7}}, \frac{179}{57}, \frac{201}{64}, \frac{223}{71}, \frac{245}{78}, \frac{267}{85}, \frac{289}{92}, \frac{311}{99}, \boxed{\frac{333}{106}}, \boxed{\frac{355}{113}}, \frac{52163}{16604}$$

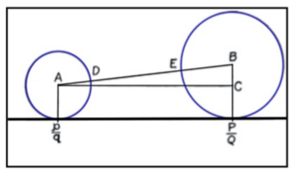
FIG. 2: Who was Hurwitz? Adolf Hurwitz - German mathematician and his daughter, 1912 with Einstein playing violin.

E. Ford Circles : Pictorial Representation of Rational Numbers

In 1938, an American mathematician Lester Ford showed that every primitive fraction $\frac{p}{q}$ (where p and q are relatively prime) can be represented by a circle in the $x - y$ plane, tangent to x-axis, with center at $(\frac{p}{q}, \frac{1}{2q^2})$ and radius $\frac{1}{2q^2}$.

The key characteristic of the Ford circles is the fact that two Ford circles representing two distinct fractions never intersect and are tangent only if the two fractions are Farey neighbors.

Proof of Hurwitz's theorem using Ford circles: See the paper by Ford, "Ford circles.pdf".



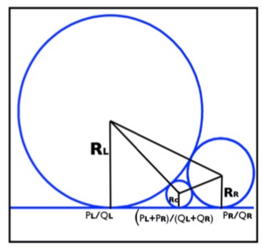


FIG. 3: Ford circle representation of fractions

IV. CHAPTER IV

COMPLEX NUMBERS

In sixteenth century, two Italian mathematicians Rafael Bombelli and Gerolamo Cardano gave the formal and not real solution of the simple quadratic equation,

$$z^2 + 1 = 0, \quad z = \sqrt{-1} \tag{19}$$

In the eighteenth century, Leonhard Euler denoted this imaginary number by i, i.e.

$$i = \sqrt{-1} \tag{20}$$

The number $z = x + iy$ is called a *complex number*. Numbers x and y are real and are called the real part of z and the imaginary part of z. In honor of his accomplishments, a moon crater was named Bombelli.

Complex numbers were coined in the 17th century by René Descartes as a derogatory term and regarded as fictitious or useless. However, soon after Descartes, their importance began to surface in the minds of some of the greatest mathematicians.

A. Polar Representation of Complex Number : Euler Formula

With $x = r\cos\theta$ and $y = r\sin\theta$,

$$z = x + iy = r\cos\theta + ir\sin\theta \tag{21}$$

Using Taylor series expansion,

$$\begin{aligned}
\cos\theta &= \sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n}}{2n!} \\
\sin\theta &= \sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n+1}}{(2n+1)!} \\
e^{i\theta} &= \sum_{n=0}^{\infty}\frac{i^n\theta^n}{n!}
\end{aligned}$$

(Note: In Taylor expansion of $\sin$ and $\cos$, the angle θ is in radians.)

We get the Euler formula,

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{22}$$

Therefore, every complex number can be written as,

$$a + ib = \sqrt{a^2+b^2}\ e^{i\phi}, \quad \phi = \tan^{-1}\frac{b}{a}$$

This is often called a *polar representation* of complex number. From Euler formula, we get

$$e^{in\theta} = \cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$$

NOTE: "n" in the above formula need not be an integer.

The identity $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin .n\theta$ is known as " De Moivre's theorem.

Polar form of complex number and the De Moivre's theorem are extremely useful in finding roots of complex numbers and also $\cos n\theta$, $\sin n\theta$ as illustrated below.

To multiply two complex numbers, multiply their lengths and add their angles.

$$\begin{aligned}(x_1+iy_1)(x_2+iy_2) &= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2) && (23)\\ &= r_1r_2e^{i(\theta_1+\theta_2)} && (24)\end{aligned}$$

Polar form also simplifies the division of two complex numbers:

$$\frac{a+ib}{c+id} = \sqrt{\frac{a^2+b^2}{c^2+d^2}}e^{i\phi_1 - i\phi_2}, \quad \tan\phi_1 = \frac{b}{a}, \quad \tan\phi_2 = \frac{d}{c}$$

Problem (1) : Find square root of $-(15+8i)$

Solution: Express the complex number in polar form:

$$\begin{aligned}-15-8i &= 17[\cos(\theta+2\pi n) + i\sin(\theta+2n\pi)], \quad \cos\theta = -\frac{15}{17}, \quad \sin\theta = -\frac{8}{17}\\ \sqrt{-15-8i} &= \sqrt{17}[\cos(\frac{\theta}{2}+\pi n) + i\sin(\frac{\theta}{2}+n\pi)]\end{aligned}$$

For $n = 1$, $\sqrt{-15-8i} = \sqrt{17}[\cos(\frac{\theta}{2}+\pi) + i\sin(\frac{\theta}{2}+\pi)] = -\sqrt{17}[\cos(\frac{\theta}{2}) + i\sin(\frac{\theta}{2})]$

For $n = 2$, $\sqrt{-15-8i} = -\sqrt{17}[\cos(\frac{\theta}{2}) + i\sin(\frac{\theta}{2})]$

Using the identity $2\cos^2\frac{\theta}{2} = 1+\cos\theta$ and $2\sin^2\frac{\theta}{2} = 1-\cos\theta$,

$$\begin{aligned} \cos\frac{\theta}{2} &= \pm\sqrt{\frac{1+\cos\theta}{2}} = \pm\sqrt{(1-\frac{15}{17})/2} = \pm\frac{1}{\sqrt{17}} \\ \sin\frac{\theta}{2} &= \sqrt{\frac{1-\cos\theta}{2}} = \pm\frac{4}{\sqrt{17}} \end{aligned}$$

Problem (2) : Show that $\sin 5\theta = \sin\theta(16\cos^4\theta - 2\cos^2\theta + 1)$

Solution: Use De Moivre's theorem , the binomial expansion and separate real and imaginary parts:

$$\begin{aligned} \cos 5\theta + i\sin 5\theta &= \cos^5\theta(1+i\tan\theta)^5 \\ &= \cos^5\theta(1+5i\tan\theta - 10\tan^2\theta - 10i\tan^3\theta + 5\tan^4\theta + i\tan^5\theta) \\ &= \cos^5\theta(1-10\tan^2\theta+5\tan^4\theta) + i\cos^5\theta(5\tan\theta - 10\tan^3\theta + \tan^5\theta) \end{aligned}$$

Equating the imaginary part,

$$\begin{aligned} \sin 5\theta &= 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta \\ &= \sin\theta[5\cos^4\theta - 10\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2] \\ &= \sin\theta[16\cos^4\theta - 12\cos^2\theta + 1] \end{aligned}$$

B. We can use complex numbers to represent Vectors in two-dimension

Firstly, the numbers $z = x + iy$ can be used to represent two-dimensional vectors with the x-axis representing the real numbers and the y-axis representing the pure-imaginary numbers. We will use symbol $\doteq$ to show this correspondence between a complex number and a vector:

$$\vec{v} \doteq x + iy \tag{25}$$

The geometrical interpretation of complex numbers as vectors is supported by the following two observations:

- **(1)** the addition of two complex numbers represents the addition of two vectors,

  That is, given two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, and two vectors $\vec{v}_1$ and $\vec{v}_2$,

$$\begin{aligned} z_1 + z_2 &= x_1 + iy_1 + x_2 + iy_2 = (x_1 + x_2) + i(y_1 + y_2) &(26)\\ z_1 + z_2 &\doteq \vec{v}_1 + \vec{v}_2 &(27)\end{aligned}$$

- **(2)** The dot and cross products of two vectors represented by z_1 and z_2 are the real and the imaginary parts of $z_1^* z_2$.

$$z_1^* z_2 = (x_1 - iy_1)(x_2 + iy_2) \;=\; (x_1 x_2 + y_1 y_2) + i(x_1 y_2 - y_1 x_2) \tag{28}$$

- **(3)** Complex Numbers can be used to represent rotations

  In 2D, if you want to rotate a vector $\vec{v}$ by an angle α, simply multiply it by $e^{i\alpha}$.

$$e^{i\alpha}(x + iy) = re^{i(\alpha+\theta)} \tag{29}$$

  Note: the product of two unit-length complex numbers represents a sequence of two rotations.

  Clearly, Leonhard Euler showed us a more elegant way to multiple complex numbers and rotate vectors.

  In a nutshell, we can use complex arithmetic to do a geometric operation of rotation.

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Home Work:**(4.1)**

(1.1) Show that $\cosh^{-1} x = \pm \ln(x + \sqrt{x^2 - 1})$.

(1.2) Show that $(-1 + \sqrt{3}i)^{10} = -512 + 512\sqrt{3}i$

(1.3) Evaluate $(-1 + i)^{1/3}$.

ANS:

$$2^{1/6} \cos(45^\circ + i \sin 45^\circ), \quad 2^{1/6} \cos(165^\circ + i \sin 165^\circ), \quad 2^{1/6} \cos(285^\circ + i \sin 285^\circ)$$

(1.4) Show that $(\frac{1-i}{1+i})^{10} = -1$

(1.5) Find the equation of a circle of radius 2 with center at $(-3, 4)$ in terms of $z = x + iy$.

(1.6) Using series expansion for e^x, prove the Euler formula $e^{i\phi} = \cos\phi + i \sin\phi$.

(1.7) Express in polar form: (a) $-5i$, (b) $-2 - 2i$.

(1.8) Calculate all roots of $i^{1/4}$.

(1.9) Find (x, y) if $(x + iy)^2 = 2i$.

(1.10) Show that $\cos^4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

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C. Solving some simple differential Equations using Complex functions as solutions

Consider some *linear* differential equations:

- (1) First Order: $\dot{x} + ax = 0$
- (2) Second order: Simple Harmonic Motion: $\ddot{x} + \omega^2 x = 0$
- (3) Second order: Damped Harmonic Motion: $\ddot{x} + 2bx + \omega^2 x = 0$
- (4) Second order: Cyclotron motion of an electron

In all these cases, let us try a solution $x = e^{\alpha t}$, where α is unknown.

Why exponential function?? : Because the first and also the second derivative of exponential function is also the same exponential function (multiplied by some constant). So clearly, the solutions can be expressed as exponential functions.

(1) Substituting $x = e^{\alpha t}$ in $\dot{x} + ax = 0$ gives: $(\alpha + a)e^{\alpha t} = 0$, that is $\alpha = -a$.

(2) Substituting $x = e^{\alpha x}$ in $\ddot{x} + \omega^2 x = 0$ gives $(\alpha^2 + \omega^2 = 0$, that is $\alpha = \pm i\omega$. Therefore, the general solution of this second order differential equation is

$x = Ae^{i\omega t} + Be^{-i\omega t}$ where A and B are constants.

NOTE:

$$Ae^{i\omega t} + Be^{-i\omega t} = \frac{A}{2}(\cos\omega t + \sin\omega t) + \frac{B}{2i}(\cos\omega t - \sin\omega t) \equiv C\cos\omega t + D\sin\omega t$$

where $C = \frac{A+B}{2}$ and $D = \frac{A-B}{2i}$. Note that the constants A and B may be complex numbers, but the constants C and C have to be real numbers if x represents a physically measurable quantity.

(3) Substituting $x = e^{\alpha t}$ in $\ddot{x} + 2bx + \omega^2 x = 0$ gives,
$\alpha^2 + 2b\alpha + \omega^2 = 0$. This is a quadratic equation in α with roots,
$\alpha = -b \pm \sqrt{b^2 - \omega^2} = -b + \pm i\sqrt{\omega^2 - b^2}$.

For weak damping b is small. The general solution is,
$e^{-\beta t}(Ae^{i\omega_1 t} + Be^{-i\omega_1 t}) \equiv Ce^{-\beta t}\cos(\omega_1 t + \phi)$, where $\omega_1 = \sqrt{\omega^2 - b^2}$.

NOTE: Above discussion of harmonic motion applies to a pendulum under the conditions of small amplitude.
$\dfrac{d^2\theta}{dt^2} + \dfrac{g}{l}\sin\theta = 0.$

If the maximal displacement of the pendulum is small, we can use the approximation

$\sin\theta \approx \theta$ and instead consider the equation

$\ddot{\theta} + \frac{g}{l}\theta \equiv \ddot{\theta} + \omega^2\theta.$

4 Consider an electron: mass m, charge q, subjected to magnetic field $\vec{B} = B_0\hat{z}$.

$$\begin{aligned} m\frac{d^2\vec{r}}{dt^2} &= q\vec{v} \times \vec{B} = qB_0\vec{v} \times \hat{z} \\ m(\ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z}) &= qB_0(\dot{y}\hat{x} - \dot{x}\hat{y}) \end{aligned}$$

$$m\ddot{x} = qB_0\dot{y}, \quad m\ddot{y} = -qB_0\dot{x}, \quad m\ddot{z} = 0$$

First we note that $m\ddot{z} = 0$ implies that the z-component of the velocity is constant:

$$\dot{z} = a, \quad z = at + b \tag{30}$$

where a and b are constants.

Writing $\dot{x} = v_x$ and $\dot{y} = v_y$

$$m\dot{v}_x = qB_0v_y, \quad m\dot{v}_y = -qB_0v_x \tag{31}$$

These are coupled equations in v_x and v_y. To uncouple them, define:

$$v_\pm = v_x \pm iv_y \equiv \dot{z} \tag{32}$$

The equations (31) can be written as :

$$\dot{v}_\pm = -i\frac{qB_0}{m}v_\pm \equiv -i\omega^2 v_\pm \tag{33}$$

Where $\omega^2 = \frac{qB_0}{m}$.

The solution of Eq. (33) is $v_\pm = Ae^{-i\omega t}$.

Let us define $Z = x + iy$ and $\dot{Z} = v_+$. Please distinguish between z- the z-component and the complex number $Z = x + iy$.

This gives,

$$
\begin{aligned}
\dot{Z} &= Ae^{-i\omega t} \\
Z &= i\frac{A}{\omega}e^{-i\omega t} + B \\
Z - B &= i\frac{A}{\omega}e^{-i\omega t} \\
|Z - B|^2 &= \frac{AA^*}{\omega^2}
\end{aligned}
$$

This is the equation of a circle with center at B and radius R where $R^2 = \frac{AA^*}{\omega^2}$.

Note that A and B are constants that are complex numbers to be determined by initial condition.

And from Eq. (30), we have $z = at + b$.

That is, electron moves in a spiral.

For a special case of initial condition where $v_z(t = 0) = 0$, we get $a = 0$. The resulting motion of the electron will be a circle.

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Home Work (4.2)

(1) Consider a simple pendulum executing a simple harmonic motion, $\theta = Ae^{i\omega t} + Be^{-i\omega t}$. Given $\theta(t = 0) = 0$ and $\theta(t = \frac{\pi}{2\omega}) = .01\pi$, find the constants A and B. Write the solution in the form $\theta = \theta_0 \sin(\omega t + \phi)$ and calculate θ_0 and ϕ.

(2) Consider an electron moving in an XY-plane with $v_x = v_y = v_0$. If we apply magnetic field along the z direction, the electron will move in a circle. If T is the period of the circular motion and if (x_0, y_0) is the initial location of the electron, calculate the equation of the circular orbit of the electron, in terms of T, v_0, and (x_0, y_0).

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V. CHAPTER V

Scalars, Vectors, Tensors and Spinors

A. What Are Scalars and Vectors

In our first introduction to vectors, we are told that a vector is a quantity with a magnitude and a direction— in contrast to a scalar, with which no direction is associated. We then proceed to physical examples of vectors such as velocity and force, which help us understand "intuitively" how to handle vectors. But here is the question that should be asked immediately when one is told that a vector is a quantity "with both a magnitude and a direction." Direction with respect to what? With respect to a given, fixed set of coordinate axes prescribed once and for all? If so, why is it that no such set is ever prescribed at the start of any text on mechanics, for instance?

The short answer is that relationships between vectors are valid in every frame of reference; and there is no need to specify any specific set of axes, precisely because the way vectors change from one set of axes to another is encoded in the very definition of a vector.

The fact is that the "definition" quoted above is seriously flawed. It does not give the true defining property of scalars and vectors.

The key point is to understand fundamental need for introducing such quantities. In a nutshell:

- It turns out that the laws of physical science are unchanged in form under various choices of coordinate systems, frames of reference, etc. That is, they are form- invariant (called covariant) under various groups of transformations.

- In order to make this property manifest, these laws must be relationships between quantities whose transformation properties are well-defined. That is, they must be expressed in terms of covariant quantities.

- Scalars, vectors, tensors and spinors., are precisely objects of this kind. In other words, there are NO physical quantities that cannot be classified as scalars, vectors, tensors and spinors.

Let us, therefore, restrict our attention Euclidean space of 3 dimensions.

Here are the proper definitions of a scalar and a vector in this case:

At the most elementary level, what we call scalars, vectors, and tensors are (sets of) quantities with specific transformation properties under rotations of the spatial coordinate axes. Clearly, in four space-time dimension, the invariance is with respect to Lorentz transformation

(I)A scalar is a quantity that is unchanged under a rotation of the coordinate axes about the origin of coordinates. They are also called tensors of rank zero.

(II) A vector that transforms in exactly the same way as the coordinates themselves transform, under a rotation of the coordinate axes. They are also called tensors of rank 1.

(III) A tensor of rank ≥ 2 is a set of quantities whose transformation properties under a rotation of the coordinate axes generalize in a straightforward manner that of a vector, as we shall see shortly. I reiterate that the precise definition of vectors, tensors, and other such objects is much more general. The space concerned need not be Euclidean, and many other groups of transformations may be considered. Of these, the most commonly occurring one is the Lorentz group of transformations in four-dimensional space–time arising from Special Relativity.

Physical quantities are either scalars, or vectors or tensors or spinors. There is NO other choice. For example, a physical quantity cannot be a sum of scalar and vector.

Since the designation, scalar, vector, tensor is tied to rotations, let us review rotations. The simplest example is a rotation about z-axis.

B. Review: Rotations in 3-dimensions

In our study of complex numbers, we learned that a complex number $z = x + iy$ represents a vector in XY-pane. That is,

$$\vec{v} \doteq z = x + iy = re^{i\phi} \tag{34}$$

Here (x, y) are the x and the y components of the vector and $r = \sqrt{x^2 + y^2}$ and $\tan\phi = \frac{y}{x}$.

Therefore, r is the magnitude of the vector and ϕ is the angle it makes with the x-axis.

We also learned that to rotate a vector by an angle θ counterclockwise, we simply multiply the complex number by $e^{i\theta}$.

That is, let a vector $\vec{v} = z = x + iy \rightarrow \vec{v'} = z' = x' + iy'$

$$z' = x' + iy' = ze^{i\theta} = (x + iy)(\cos\theta + i\sin\theta) \tag{35}$$

Equating the real and imaginary parts, we get:

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

We can write this as a matrix equation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \equiv R(\theta) \begin{bmatrix} x \\ y \end{bmatrix} \tag{36}$$

This is a rotation in three dimensions about z-axis by an angle θ of coordinates $(x, y, x) \rightarrow (x', y', z') = R(\theta)(x, y, z)$ is the equation,

$$\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \\ z' &= z \end{aligned} \tag{37}$$

Written in the matrix form, the equation is,

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \equiv R(\theta) \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{38}$$

$$x'_i = R_{ij}x_j (\equiv R_{i1}x_1 + R_{i2}x_2 + R_{i3}x_3) \tag{39}$$

Now, let us define scalar, vector and tensor as follows:

- A scalar or Tensor of rank 0 is a single-component object that remains unchanged under a rotation of the coordinate axes.
- A triplet $\vec{v} = (v_1, v_2, v_3)$ is a vector , if under a rotation R of the coordinate axes, the new components are given by

$$v_i' = R_{ij}v_j(\equiv R_{i1}v_1 + R_{i2}v_2 + R_{i3}v_3) \tag{40}$$

- Tensors of rank 2:

$$T_{ij}' = R_{ik}R_{jl}T_{kl} \tag{41}$$

- Tensors of rank 3:

$$T_{ijk}' = R_{il}R_{jm}R_{kn}T_{lmn} \tag{42}$$

C. Proper and Improper Rotations & Two types of Scalars and Two types of Vectors

Rotations for which determinant $R = 1$ are called continuous or proper rotations. They are obtainable "continuously from the identity transformation"—that is, they can be built up by a succession of infinitesimal rotations, starting from the identity transformation (or no rotation at all).

In contrast, transformations with determinant of $R = -1$ are called discontinuous or improper rotations. They cannot be built up continuously from the identity transformation: in general, they involve proper rotations together with reflections. For example, a reflection about the yz- plane corresponds to a transformation under which $x \to -x$, $y \to y$ and $z \to z$.

Scalars and Pseudoscalars; Polar and Axial Vectors

We can now make a finer distinction among scalars, depending on their transformation properties under proper and improper rotations, respectively. A true scalar is a quantity that remains unchanged under both proper and improper rotations; a pseudoscalar, on the other hand, remains unchanged under a proper rotation, but changes sign under an improper rotation. Similarly, the components of a vector transform like the coordinate themselves under both proper and improper rotations; a pseudovector behaves just like a vector under proper rotations, but has an extra change of sign under improper rotations. The same remark applies to tensors and pseudotensors of higher rank.

In the usual three-dimensional Euclidean space, one often uses the terms *polar vectors* and *axial vectors* for vectors and pseudovectors, respectively.

EXAMPLES:

- Examples of polar vectors include the position vector $\vec{r}$ of a point (naturally, since we have used this to define a polar vector), the velocity $\vec{v}$, the electric field $\vec{R}$, etc. Common examples of axial vectors are the orbital angular momentum $\vec{L} = \vec{r} \times \vec{p}$ and the magnetic field $\vec{b}$.

- The dot product of two polar vectors is a scalar. So is the dot product of two axial vectors. The dot product of a polar vector with an axial vector is a pseudoscalar.

All equations of physics are invariant under rotations.

Newton's equation: $m\frac{d\vec{v}}{dt} = \vec{F}$

It should now be clear that the invariance of Newton's equation of motion under coordinate rotations is made manifest by writing this equation as a relationship between vectors! If I find that $m\frac{d\vec{v}}{dt} = \vec{F}$ in my coordinate frame, and you find that the force and acceleration are $\vec{F\prime}$ and $\frac{d\vec{v\prime}}{dt}$, respectively, in your coordinate frame (which is tilted with respect to my set of axes), then it is *guaranteed* that $m\frac{d\vec{v\prime}}{dt} = \vec{F\prime}$.

In fact, if we know precisely how your frame is oriented with respect to mine, we can calculate both $\vec{F\prime}$ and $\frac{d\vec{v\prime}}{dt}$ from a knowledge of $\vec{F}$ and $\frac{d\vec{v}}{dt}$, because these quantities are vectors.

Note, in passing, that the time variable "t" is the same in both frames of reference. A spatial rotation does not affect the time, of course. Further, "t" would continue to remain the same in both frames even if the frames were moving with a uniform velocity with respect to each other, in Newtonian mechanics—but not so when special relativity is brought in. This is because Newtonian mechanics corresponds to the limit in which the fundamental speed $c \to \infty$. Newton's equations have to be changed in relativistic mechanics.

Returning to the deduction that $\vec{F}$ is a polar vector, we can use this fact to draw further conclusions. It is a manifest (but profound) fact that Newton's equation of motion remains valid for all kinds of forces—mechanical, electromagnetic, and so on. In particular, suppose the particle has a charge q (once again, assumed to be a scalar constant), and moves in an applied electric field E and magnetic field B. The Lorentz force on it is given by the familiar expression

$$\vec{F} = q[\vec{E} + \vec{v} \times \vec{B}]$$

D. Scalar and Vector products of vectors

Re: Murray Spiegel Chapter 5

Before discussing tensors further, let us review scalar (or dot) products and vector products.

- $\vec{A} \cdot \vec{B} = A_i B_j \delta_{ij}$ – Scalar
- $(\vec{A} \times \vec{B})$ – Vector. Its components are : $(\vec{A} \times \vec{B})_i = \epsilon_{ijk} A_j B_k$
- Consider triple vector product: $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$
  Its components can be written as:

$$\begin{aligned} \left(\vec{A} \times (\vec{B} \times \vec{C})\right)_i &= B_i A_j C_j - C_i A_k B_k \\ &= \epsilon_{ijk} A_j (\vec{B} \times \vec{C})_k \\ &= \epsilon_{ijk} \epsilon_{klm} A_j B_l C_m \end{aligned}$$

NOTE:
$\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$
$\vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C}$ - a scalar, referred as box product, denoted as $[\vec{A}\vec{B}\vec{C}]$

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Home Work (5.1)

(1) Show that vectors $\vec{v_1} = 3\hat{i} - 2\hat{j} + \hat{k}$, $\vec{v_2} = \hat{i} - 3\hat{j} + 5\hat{k}$ and $\vec{v_3} = 2\hat{i} + \hat{j} - 4\hat{k}$ form a triangle. What kind of triangle is this ??

Calculate $\vec{v_1} \times (\vec{v_2} \times \vec{v_3})$ and $(\vec{v_1} \times \vec{v_2}) \times \vec{v_3}$.

(2) Let $\vec{a}, \vec{b}, \vec{c}$ represents the sides of a triangle. Prove the law of sines for the triangle: that is $\frac{sinA}{a} = \frac{sinB}{b} = \frac{sinC}{c}$ where A, B, C are the angles of the vortices opposite the sides $\vec{a}, \vec{b}, \vec{c}$.

(3) Which of the following matrices represent rotation about z axis. Why ?. For matrices representing rotation, what is the angle of rotation.?

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & -\sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{4}} & \frac{1}{2} \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

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E. Tensors: Examples

(1) <u>Isotropic Tensors</u>

An isotropic tensor is one whose components remain unchanged in numerical value under rotations of the coordinate axes. There are only two independent isotropic Cartesian tensors in three-dimensional Euclidean space, δ_{ij} and ϵ_{ijk}.

The Kronecker delta δ_{ij} : Second Rank Tensor

$\delta_{ij} = 1$ if $i = j$

$\delta_{ij} = 0$, if $i \neq j$

The Levi-Civita symbol ϵ_{ijk} : Third Rank Tensor

(2) Physical quantities as Tensors

Numerous physical quantities are second-rank tensors. Examples include
– mechanical stress and strain;
– the moment of inertia of a mass distribution;
–the quadrupole moment of a charge distribution;
– the Maxwell stress tensor of an electromagnetic field;
– the "order parameter" in various types of liquid crystals;

Moment of Inertia

Physics students first encounter moment of inertial "I" in study of rotations of a rigid body. In their first study of rotation, they learn:

- Kinetic energy of a rotating body $K_R = \frac{1}{2}I\omega^2$. Translational kinetic energy $K_T = \frac{1}{2}mv^2$.
- $K = K_R + K_T$ when rotation axis passes through the center of mass. That is kinetic energy K and angular momentum $\vec{L}$ can be separated into motion of the center of mass and the motion around the center of mass.

  What if the axis of of rotation does not pass through the center of mass

- Moment of inertia depend upon the axis of rotation. So clearly, moment of inertia is not a scalar quantity, like the mass which does not depend upon any axis.

  Naively, you may think that in three dimension, since are three independent axes of rotations, there are three independent moments of inertial, say I_1, I_2, I_3. This is just like a vector that has three independent components in three dimension.

However, the story of moment of inertia is more complicated !!

In your undergraduate classical mechanics class, you learn:

- $K_R = \frac{1}{2} I_{ij}\omega_i\omega_j$
- $L_i = I_{ij}\omega_j$

NOTE: Repeated index is summed over.

These equations imply that moment of inertia has NINE components: I_{ij} : a second rank tensor.

You get cross terms such as I_{12} etc when axis of rotations does not pass through the center of mass !!! We will see this in an example shortly.

$$\mathbf{I} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}$$

Formulas for I_{ij} for continuous and discrete distribution of masses

(a) If the mass of a rigid body is distributed over the volume V, given by the density function $\rho(r)$, then I_{ij} (about the origin of coordinates) is given by

$$I_{ij} = \int dV(r^2\delta_{ij} - x_i x_j)\rho(r), \tag{43}$$

where the x_i are the position coordinates of the volume element dV , located at a distance $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ from the origin.
Note that we are using the notation $x_1 = x, x_2 = y, x_3 = z$.

The I_{ij} is a symmetric tensor, that is $I_{ij} = I_{ji}$. That means that there are only 6-independent components of I_{ij}.

- Let $i = j = 1$,

$$\begin{aligned} I_{11} &= \int dV(r^2 - x_1^2)\rho(r) \\ &= \int dV(x_1^2 + x_2^2 + x_3^2 - x_1^2)\rho(r) \\ &= \int dV(x_2^2 + x_3^2)\rho(r) \\ &\equiv \int dV(y^2 + z^2)\rho(r) \end{aligned}$$

- Let $i = 1$ and $j = 2$, since $\delta_{12} = 0$, we have

$$I_{12} = -\int dV x_1 x_2 \rho(r) \equiv -\int dV yz\rho(r) \tag{44}$$

(b) If a body consists of N point masses m_α, the moment of inertia tensor is given by

$$I_{ij} = \sum_{\alpha=1}^{N} m_\alpha[(r^\alpha)^2\delta_{ij} - x_i^\alpha x_j^\alpha], \tag{45}$$

where $(r^\alpha)^2 = (x_1^\alpha)^2 + (x_2^\alpha)^2 + (x_3^\alpha)^2$.

Note that the superscript α labels the N-particles and the subscript i and j label the (x, y, x) components of a vector $\vec{r}$.

F. Example: Moment of Inertia of a Cube

Ref: Classical Dynamics by Marion

Consider a homogeneous cube of density ρ, mass M and side of length b. Two cases:

Case I : Let us choose corner of the cube to be the origin and let three adjacent edges lie along the coordinate axes.

Using the formula (43), the diagonal elements are:

$$
\begin{aligned}
I_{11} &= \rho \int_0^b dx_3 \int_0^b \; dx_2 \; (x_2^2 + x_3^2) \int_0^b dx_1 \\
&= \frac{2}{3}\rho b^5 = \frac{2}{3} M b^2
\end{aligned}
$$

It is obvious that $I_{11} = I_{22} = I_{33} = \frac{2}{3}Mb^2$

The off diagonal elements are:

$$
\begin{aligned}
I_{12} &= -\rho \int_0^b x_1 dx_1 \int_0^b x_2 dx_2 \int_0^b dx_3 \\
&= -\frac{1}{4}\rho b^5 = -\frac{1}{4} M b^2
\end{aligned}
$$

It is again obvious that all of diagonal elements are equal to $-\frac{1}{4}Mb^2$.

Calculate the kinetic energy in terms of $\omega_1, \omega_2, \omega_3$.

$$
\begin{aligned}
K_R &= I_{ij}\omega_i\omega_j \\
&= \frac{2}{3}Mb^2(\omega_1^2 + \omega_2^2 + \omega_3^2) - \frac{2}{4}Mb^2(\omega_1\omega_2 + \omega_2\omega_3 + \omega_3\omega_1)
\end{aligned}
$$

Case II : Let us choose center of the cube to be the origin and axes of symmetry to be the coordinate axes: choose x_1 axis to be the diagonal of the cube and the other two axes lie in a plane normal to the diagonal. Their orientation is arbitrary.

Show that $I_{11} = \frac{1}{6}Mb^2$ and $I_{22} = I_{33} = \frac{11}{12}Mb^2$ and off diagonal elements are zero.

G. Monopole, Dipole and Quadrupole Moments of Charge Distribution

The multipole expansion of the potential due to a general charge distribution in electrostatics provides another example of the use of tensors.

Let $\phi(r)$ be the electrostatic potential at the point r due to a static charge distribution specified by a charge density $\rho(r)$ in space. Now, it turns out that the basic partial differential equation satisfied by the potential is Poisson's equation. The solution of the equation is:

$$\phi(r) = \frac{1}{4\pi\epsilon_0}\int dV' \frac{\rho(r')}{|r-r'|} \tag{46}$$

The multipole expansion of the potential can be written as

$$\phi(r) = \frac{1}{4\pi\epsilon_0}[\frac{Q}{r} + \frac{P_i x_i}{r^3} + \frac{Q_{ij}x_i x_j}{r^5} + ...] \tag{47}$$

$$\begin{aligned} Q &= \int dV \rho(r) \\ P_i &= \int dV x_i \rho(r) \\ Q_{ij} &= \frac{1}{2}\int dV (3x_i x_j - r^2\delta_{ij})\rho(r) \end{aligned}$$

Very far away from the charge distribution, potential looks like that of a charge Q located at the origin, to leading order.

- The dipole, quadrupole and all higher moments of a spherically symmetric charge distribution vanish identically.
- Dipole, quadrupole and (higher order tensors) represent the effects of the departure from spherical symmetry of the charge distribution.

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Home Work: (5.1)

(1) Show that moment of inertia obtained in Case II corresponds to diagonalization of the matrix of case I

(2) Calculate the inertia tensor of a rod of length l and mass M about axes passing through its center of mass, using formula (43). You can assume that rod has negligible radius.

If the rod is a solid cylinder of radius R, what will be the Inertia tensor ??

(3) Calculate the inertia tensor of a sphere of radius R and mass M about axes passing through its center of mass, using formula (43).

(4) Consider a pendulum composed of a rigid rod of length l and mass M with a mass m_1 at its end. Another mass m_2 is placed halfway down the rod. The pendulum with the one end of the rod fixed swings in the plane. Calculate the moment of inertia of the pendulum about an axis perpendicular to its plane of swing.

(5) Consider an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Let the mass of the ellipsoid M is uniformly distributed. Show that the the moment of inertial tensor about its axes of symmetries is a diagonal matrix where $I_{11} = \frac{1}{5}(b^2 + c^2), I_{22} = \frac{1}{5}(c^2 + a^2), I_{33} = \frac{1}{5}(a^2 + b^2)$,

HHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHH

H. Spinor

Unlike vectors and tensors, a spinor transforms to its negative when the space is continuously rotated through a complete turn by 360 degrees. In other words, a spinor is like an arbitrary vector lying on a M{obius strip; it has to go around the strip twice to get back where it started. Christened as spinors by Ehrenfest during quantum revolution, such quantities have been known to mathematicians long before that period. It appears that in 1897, Felix Klein originally designed the spinor to simplify the treatment of the classical spinning top.

A more thorough understanding of spinors as mathematical objects is credited to Elie Cartan - a french mathematician who studied spinors in the context of rotation group.

Spinors are essential in quantum physics as they represent quantum spins - an essential property of all fundamental particles.

If you rotate spin of say electrons by θ, its wave function rotates only by $\frac{\theta}{2}$. That is, a 2π rotation of spin leaves spin unchanged but its wave function acquires a minus sign. The wave function of electron is a spinor.

When the concept of quantum spin was discovered, Pauli put forward the idea that the wave function ψ of an electron could be represented by two *complex components* that changes sign under 2π rotation,

Physically, the sign change after 2π rotation does not pose any difficulty as in quantum mechanics, only quantities of the form $|\psi|^2$ have physical meaning rather than ψ itself. Therefore, even if $\psi \to -\psi$ after a rotation of 2π, $|\psi|^2$ goes back to its original value.

The two-component object ψ that describes the state of a particle with spin-$1/2$ is clearly not a vector or a tensor. Paul Ehrenfest whose student Goudsmit along with Uhlenbeck proposed spin degree of freedom, coined the term *spinor* to describe such complex quantities.

Fermions like protons, electrons and quarks comprise all the ordinary matter in the universe. But despite this nearly universal dominance in Nature, they, that is, their wave functions, do not obey the behavior typical of scalar, vector and tensor quantities, but that of spinors. Their transformation properties are described by Lorentz transformation.

cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc

VI. CHAPTER VI

Fourier Transformation

Fig.1 Baron Jean Baptiste
Joseph Fourier
(1768–1830)

Fourier Series

"*Fourier's theorem is not only one of the most beautiful results of modern analysis, but it may be said to furnish an indispensable instrument in the treatment of nearly every recondite question in modern physics.*" Lord Kelvin

HISTORY:

Fourier series occupy a unique place in the history of mathematics. It is considered the beginning of the subject we now call Mathematical physics which got started with the publication of Jean-Baptiste Joseph (1768–1830) Fourier's treatise, *Théorie Analytique de la Chaleur (Analytic Theory of Heat)* in 1822.

Fourier (1768–1830) introduced the Fourier series for the purpose of solving the heat equation in a metal plate. Through Fourier's research the fact was established that an arbitrary function can be represented by a trigonometric series. The first announcement of this great discovery was made by Fourier in 1807, before the French Academy. The heat equation is a partial differential equation. Prior to Fourier's work, no solution to the heat equation was known in the general case, although particular solutions were known if the heat source behaved in a simple way, in particular, if the heat source was a sine or cosine wave. These simple solutions are now sometimes called eigensolutions. Fourier's idea was to model a complicated heat source as a superposition (or linear combination) of simple sine and cosine waves, and to write the solution as a superposition of the corresponding eigensolutions. This superposition or linear combination is called the Fourier series.

A. Introduction

Ref: Chapter 7 of Murray and Spiegel

Fourier series expresses ANY periodic function in terms of simplest known periodic functions, namely sines and cosines.

For simplicity, let us consider a periodic function $f(x)$ of period 2π. Its Fourier series is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{48}$$

NOTE: Each term in the above series repeats itself after 2π as the terms $\sin nx$ and $\cos nx$ have period $\frac{2\pi}{n}$. That is period of $\sin 2x$ is π and period of $\sin 3x$ is $\frac{2\pi}{3}$.

Fourier series can also be written in a more compact form as:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{49}$$

The coefficients (a_n, b_n, c_n) are Fourier coefficients.

Note that summation index n runs only over positive values in Eq. (50) and over positive, negative as well as zero in Eq. (49).

How to determine these coefficients??? Let us calculate b_1

Multiply Eq. (50) by $\sin x$ (b_1 is the coefficient of $\sin x$) and integrate over x:

$$
\begin{aligned}
\int_0^{2\pi} f(x)\sin x \ dx &= a_0 \int_0^{2\pi} \sin x dx + \sum_{n=1}^{\infty} a_n \int_0^{2\pi} \sin x \cos nx \ dx. + \sum_{n=1}^{\infty} b_n \int_0^{2\pi} \sin x \sin nx \ dx. \\
&= 0 + 0 + \sum_{n=1}^{\infty} b_n \int_0^{2\pi} \sin x \sin nx \ dx \\
&= \sum_{n=1}^{\infty} b_n \delta_{1n} \int_0^{2\pi} \sin^2 x \ dx \\
&= b_1 \int_0^{2\pi} \sin^2 x \ dx \\
&= \pi b_1
\end{aligned}
$$

$$
b_1 \ = \tfrac{1}{\pi} \ \int_0^{2\pi} f(x) \sin x \ dx
$$

- $\frac{a_0}{2} = \frac{1}{2\pi} \int_0^{2\pi} f(x)$, which is average of $f(x)$ over a period.
- $a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \ dx$
- $b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \ dx$

NOTE: Fourier coefficients are obtained using following integrals:

$$
\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} (\sin mx) dx &\equiv \ < \sin mx >= 0 \\
\frac{1}{2\pi} \int_0^{2\pi} (\sin mx) dx &\equiv \ < \cos mx >= 0 \\
\frac{1}{2\pi} \int_0^{2\pi} (\sin^2 x) dx &\equiv \ < \sin^2 x >= \frac{1}{2} \\
\frac{1}{2\pi} \int_0^{2\pi} (\cos^2 x) dx &\equiv \ < \cos^2 x >= \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} (\sin mx)(\sin nx) dx &= \frac{1}{2} \delta_{mn} \\
\frac{1}{2\pi} \int_0^{2\pi} (\cos mx)(\cos nx) dx &= \frac{1}{2} \delta_{mn} \\
\int_0^{2\pi} (\cos mx)(\sin nx) dx &= 0
\end{aligned}
$$

- If $f(x)$ is an even function, that is $f(x) = f(-x)$, then $b_n = 0$.

- If $f(x)$ is an odd function, that is $f(x) = -f(-x)$, then $a_n = 0$.

- If $f(x)$ is real, that is $f(x) = f^*(x)$, $f_n = f_n^*$.

- Note that the index n in the sine-cosine basis runs only over the nonnegative integers.

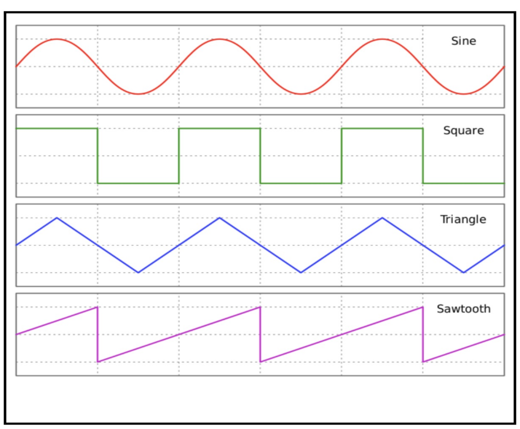
B. Why do we want from the Fourier Transform?- The Spectrum

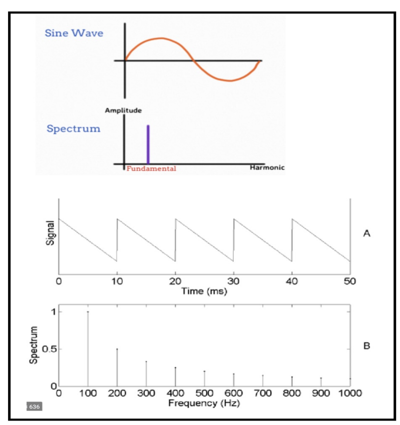
- We desire a measure of the frequencies present in a wave. This is the definition of the term, the "spectrum."

- The spectrum contains equally spaced components, and their spacing is equal to one divided by the period of the waveform.

- The lowest of the components above zero frequency is called the fundamental, and the others are called harmonics.

- The word "anharmonic" is used to characterizes waves which contain many frequencies. Anharmonic waves are sums of sinusoidal.





C. Examples

(I) $f(x) = \sin^2 x$. This is an even function. Its Fourier series consists of only cosine terms.

Since we know that $\sin^2 x = \frac{1-\cos 2x}{2}$, we get $a_0 = 1, a_1 = 0, a_2 = -\frac{1}{2}$

(II) $\cos^3 x$: It is an even function.

Using the identity, $\cos^3 x = \frac{3}{4}\cos x + \frac{3}{4}\cos 3x$, we get:

$a_0 = 1, a_1 = \frac{3}{4}, a_2 = 0, a_3 = \frac{3}{4}$.

(III) **Building Square Wave using Sine Functions**:

Consider a square wave:

$f(x) = 1,\ \ 0 < x < \pi$ and

$f(x) = -1,\ \ \pi < x < 2\pi$

Solution: This is an odd function. Its Fourier series consists of only sine terms.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \tag{50}$$

Calculating Fourier Coefficients b_n

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} dx\ \ f(x)\sin nx \\
&= \frac{1}{\pi}\int_{0}^{2\pi} dx\ \ f(x)\sin nx \\
&= \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\ \ dx + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\sin nx\ \ dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} \sin nx\ \ dx - \frac{1}{\pi}\int_{\pi}^{2\pi} \sin nx\ \ dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} \sin nx\ \ dx \\
&= \frac{2}{n\pi}(\cos n\pi - \cos 0)
\end{aligned}$$

$b_{2n} = 0, \quad b_{2n-1} = -\frac{4}{(2n-1)\pi}$

That is, $b_2, b_4, b_6 .... = 0, \quad b_1 = -\frac{4}{\pi}, \quad b_3 = -\frac{4}{3\pi}, \quad b_5 = -\frac{4}{5\pi}$

$f(x) = -\frac{4}{\pi}\sin x - \frac{4}{3\pi}\sin 3x - \frac{4}{5\pi}\sin 5x ....$

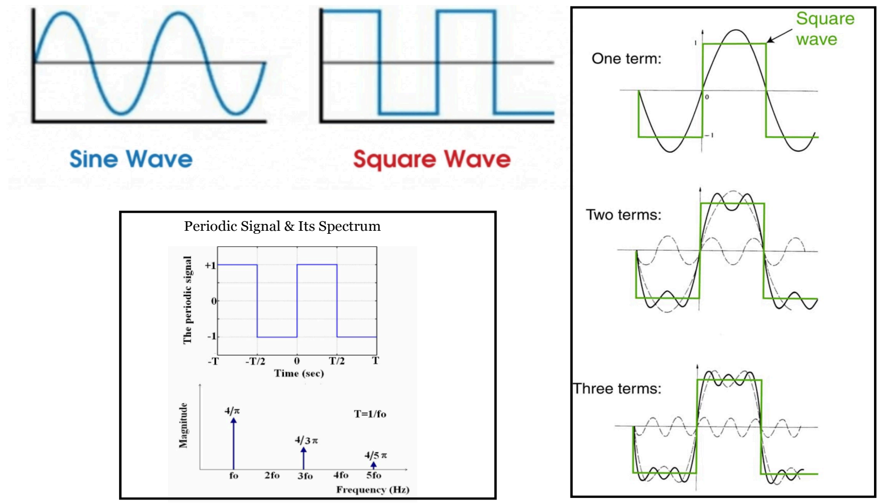


FIG. 4: Shows the building of square wave using Fourier coefficients b_1, b_3, b_5 of the Fourier series. At the point of discontinuity, Fourier series gives $f(x) = 0$, the the average value of the function. Graph of Fourier coefficients is called the *spectrum of the function* as frequency $f = \frac{1}{T}$.

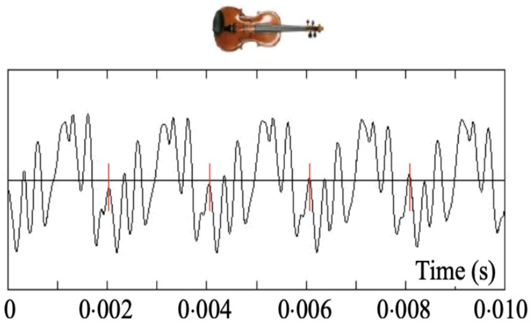
HOME WORK : Chapter 6

(1) Consider a saw-tooth wave: $f(x) = \frac{x}{\pi}$, where $-\pi < x < \pi$. Show that its Fourier coefficients are $b_n = -\frac{2}{n\pi}(-1)^n$. Graph the function using first three harmonics, similar to Fig. (4) for square wave.

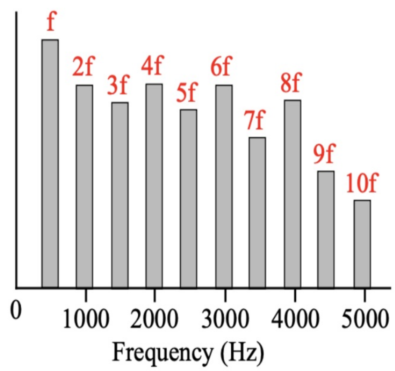
D. Applications in Musics

"*Music is the sound of mathematics* "

NOTE: When two musical instruments play the same note, the notes have the same frequency for both instruments, but the two instruments sound different; they have different timbres - Fourier spectrum.

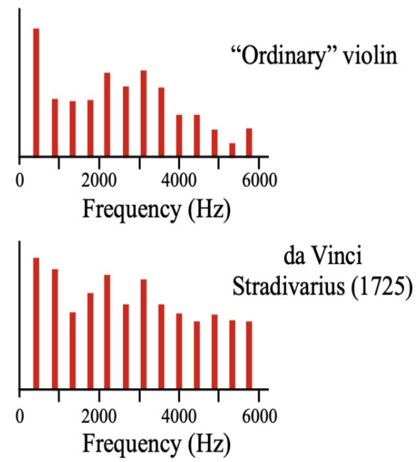


Waveform of a violin playing B_4 (494 Hz).



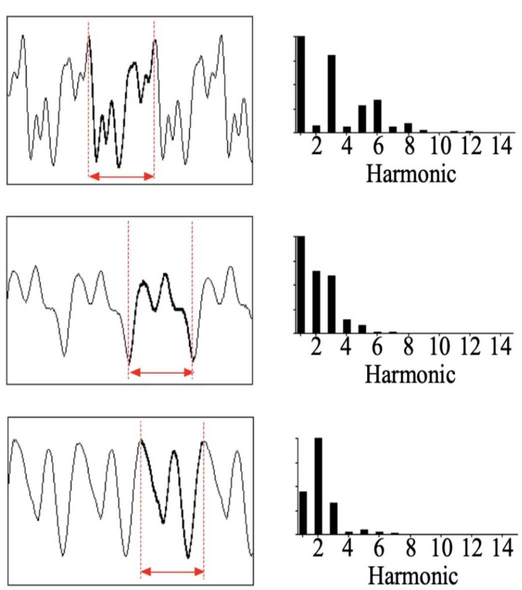
The corresponding frequency spectrum.

It is these small differences in frequency spectra that differentiate an "ordinary" instrument from a "quality" instrument.

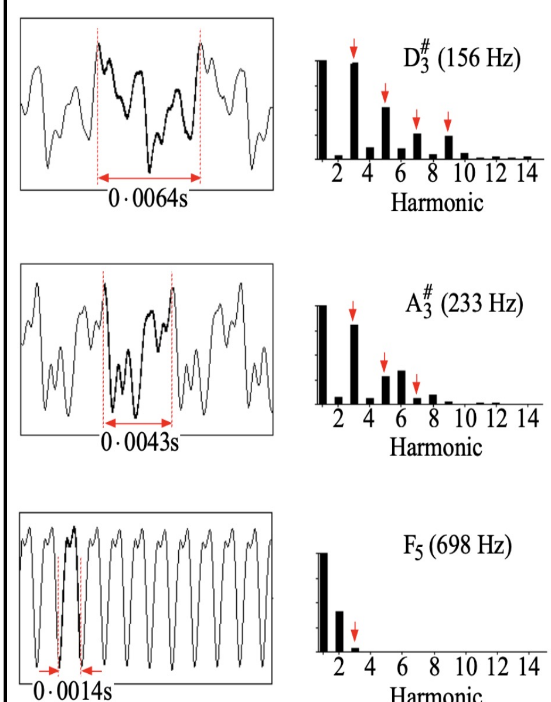


Comparison of the harmonics of an "ordinary" violin with the da Vinci Stradivarius (1725) both playing A_4 (440 Hz).

Clarinet, flute and bassoon playing the same note B_3 (247 Hz). Although the waveforms are very different the frequency sequence is the same in all cases is *harmonic*, i.e., f, 2f, 3f, 4f, etc.



Clarinet playing three different notes. The odd harmonics are enhanced at the lower frequencies:



E. Applications in crystallography: Reciprocal Lattice & Definition of a crystal

In a crystal, atoms are periodically arranged.

Therefore, electron density $\rho(r)$ is a periodic function of r. The density $\rho(r)$ can be expanded as a Fourier series.

For simplicity, let us just consider 1D lattice of periodicity a

$$\rho(x) = n_0 + \sum_{n>0} C_n \cos \frac{2\pi n x}{a} + S_n \sin \frac{2\pi n x}{a} = \sum_n A_n e^{i\frac{2\pi n x}{a}} \tag{51}$$

Since $\rho(r)$ is real, $A_n = A_n^*$.

The point $\frac{2\pi n}{a}$ is a point in the Fourier space and defines a lattice known as the *reciprocal lattice* of the crystal.

For one-dimensional lattice of period a, reciprocal lattice has periodicity $\frac{2\pi}{a}$.

In 3D, instead of a single vector a (in x-direction) , a periodic lattice is defined by three vectors $(\vec{a_1}, \vec{a_2}, \vec{a_3})$

$$\rho(\vec{r}) = \sum_G A_G e^{-i\vec{G}\cdot\vec{r}} \tag{52}$$

For a cubic crystal, these three vectors lie along $\hat{x}, \hat{y}, \hat{z}$ directions as shown in Fig. (5).

Show that the reciprocal lattice vectors are $\vec{b_i} = 2\pi \frac{\vec{a_j} \times \vec{a_k}}{[\vec{a_1}\vec{a_2}\vec{a_3}]}$.

- The simple cubic lattice, with cubic primitive cell of side a, has for its reciprocal a simple cubic lattice with a cubic primitive cell of side $\frac{2\pi}{a}$
- The reciprocal to a simple hexagonal Bravais lattice with lattice constants c and a is another simple hexagonal lattice with lattice constants $\frac{2\pi}{c}$ and $\frac{4\pi}{a\sqrt{3}}$ rotated through 30 degrees about the c axis with respect to the direct lattice.
- The reciprocal lattice to a BCC lattice is the FCC lattice, with a cube side of $\frac{4\pi}{a}$.

Crystal structure is determined experimentally by studying diffraction of x-rays through the crystal known as Bragg diffraction. Crystal acts like diffraction grating and the constructive interference pattern makes a reciprocal lattice and thus encodes the structure of the crystal.

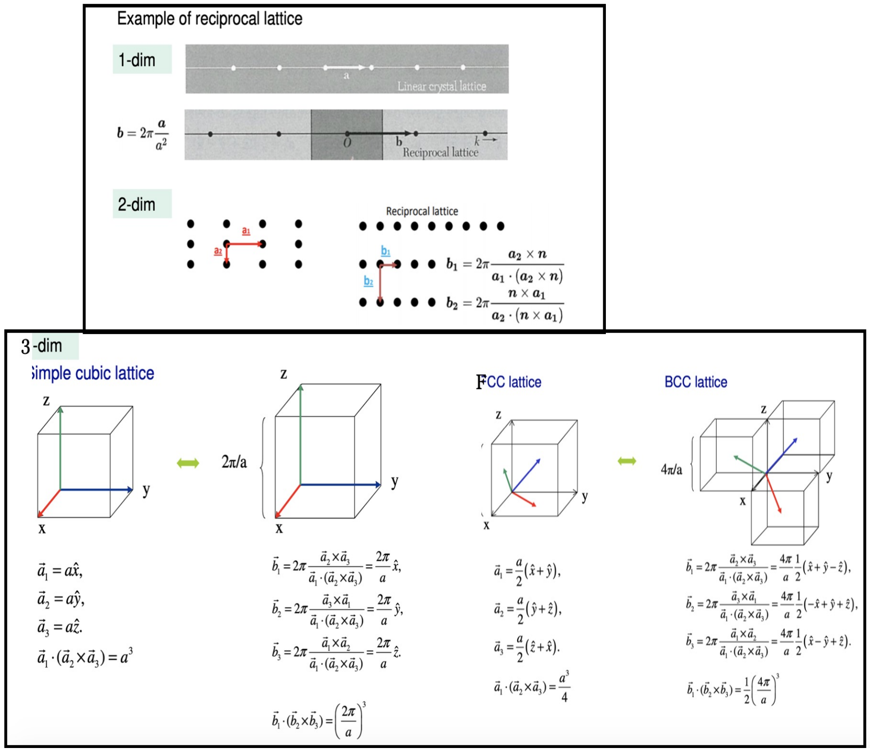


FIG. 5: Some reciprocal lattices in one, two and three dimension. The BCC and the FCC stands for body-centered and face-centered lattices. Examples of metals that have the BCC structure include Lithium (Li), Sodium (Na), Potassium (K), Chromium (Cr). Some examples of the metals having FCC structure are Aluminum (Al), Copper (Cu), Gold (Au), Lead (Pb) and Nickel (Ni)

The Bragg law $2d\sin\theta = n\lambda$ occurs only for wavelengths $\lambda \leq 2d$. Here d is the periodicity of the crystalline lattice. Since $d \approx 10^{-8}$, that is why we need x-rays. Bragg's diffraction pattern is a consequence of the periodicity of the lattice. However, as discussed below, diffraction pattern or reciprocal lattice can also result in *quasicrystals*.

F. Quasicrystals

Crystallography is the experimental science of determining the arrangement of atoms in crystalline solids. Crystallographic methods now depend on analysis of the diffraction patterns of a sample targeted by a beam of some type. X-rays are most commonly used; other beams used include electrons or neutrons. Crystallographers often explicitly state the type of beam used, as in the terms X-ray crystallography, neutron diffraction and electron diffraction.

A crystal is a solid where the atoms form a periodic arrangement. Diffraction patterns of crystals are periodic patterns in Fourier space, or the “reciprocal space”.

However, after the discovery of quasicrystal, the definition of a crystal is changed. In 1992, in order to include quasicrystals, the International Union of Crystallography changed the definition of a crystal, retaining only the criterion of an essentially sharp diffraction pattern. Schechtman was awarded the Nobel Prize in Chemistry in 2011.

A quasicrystal consists of arrays of atoms that are ordered but not strictly periodic. They have many attributes in common with ordinary crystals, such as displaying a discrete pattern in x-ray diffraction, and the ability to form shapes with smooth, flat faces.

Quasicrystals are most famous for their ability to show five-fold symmetry, which is impossible for an ordinary periodic crystal (see crystallographic restriction theorem).

The International Union of Crystallography has redefined the term ”crystal” to include both ordinary periodic crystals and quasicrystals (”any solid having an essentially discrete diffraction diagram, that is a reciprocal lattice.

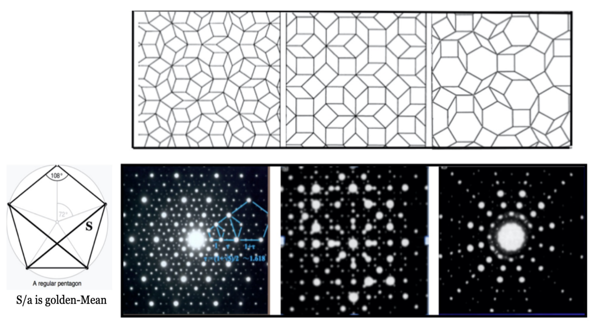


Figure 4.9. 5-fold, 8-fold, and 12-fold lattices (left to right) and the diffraction patterns corresponding to them. In the diffraction pattern coming from a 10-fold symmetry (bottom left), the golden mean has been marked explicitly on the pattern.

G. Fourier Integrals: Fourier Series for any Arbitrary Function that is not periodic

Ref: Chapter 8 of Murray and Spiegel

What happens when the function is not periodic ?

- $\pm L \to \pm\infty$
- $\frac{2\pi n}{L} \to k$
- $\frac{1}{L}\sum_{n=-\infty}^{\infty} \to \frac{1}{2\pi}\int_{\infty}^{\infty} dk$

- $c_n \to \tilde{f}(k)$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \tilde{f}(k) \ dk \tag{53}$$

$\tilde{f}(k)$ is called the Fourier transform of $f(x)$ and is given by,

$$\tilde{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x) \ dx \tag{54}$$

H. Gaussian: Fourier Transform of a Gaussian is a Gaussian

$$f(x) = \frac{1}{\sqrt{2\pi b^2}} e^{-\frac{x^2}{2b^2}}, \quad \tilde{f}(k) = e^{-\frac{1}{2}b^2k^2} \tag{55}$$

$$f(x) = \frac{1}{\sqrt{2\pi b^2}} e^{-\frac{(x-a)^2}{2b^2}}, \quad \tilde{f}(k) = e^{-iak-\frac{1}{2}b^2k^2} \tag{56}$$

I. Dirac Delta Function: Fourier Transform of a constant

Consider a special case of Fourier transform, namely the Fourier transform of a constant, say unity. That is, set $f(x) = 1$ in the equations (53) and (54).

$$1 = \int_{-\infty}^{\infty} e^{ikx} \tilde{f}(k) \ dx, \quad \tilde{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \ dk \tag{57}$$

This function was introduced by physicist Paul Dirac as a tool for modeling idealized point particles and instantaneous impulses. Known as the Dirac δ function, it is given by

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \ e^{ikx}, \tag{58}$$

which is also constrained to satisfy the identity

$$\int_{-\infty}^{\infty} \delta(x) \, dx = 1. \tag{59}$$

Here is a summary of statements that helps in understanding δ-function.

- Dirac δ-function can be loosely thought of as a function on the real line which is zero everywhere except at the origin, where it is infinite,

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}.$$

- It can also be viewed as as the weak limit of a sequence of bump functions, which are zero over most of the real line, with a tall spike at the origin, with the area enclosed by the bumps equal to unity. Or a sequence of Gaussians $F_m = \frac{1}{\sqrt{m^2\pi}} e^{-\frac{x^2}{m^2}}$ converge to a δ-function as $m \to \infty$ as $\frac{1}{\sqrt{m^2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{m^2}} dx = 1$.

- An explicit form for the delta function is always to be understood as something that makes sense only when it occurs in an integral such as:

$$\int_{-\infty}^{\infty} dx \ \delta(x-a) f(x) = f(a) \tag{60}$$

- Dirac δ-function is not a true mathematical function, some mathematicians objected to it as nonsense until Laurent Schwartz developed the theory of distributions.

- it follows that the Fourier transform of $\delta(x)$ is just unity.

- Three-dimensional δ-function:

$$\delta^3(\vec{r}) = \frac{1}{(2\pi)^3} \int d^3k \ e^{i\vec{k}\cdot\vec{r}} \tag{61}$$

- The Kronecker delta function δ_{ij} , which is usually defined on a discrete domain and takes values 0 and 1, is the discrete analog of the Dirac delta function.

J. The Occurrence of the δ-Function in Physical Problems

Why does the δ-function appears so naturally in physical problems? Here is a familiar instance. Consider the basic problem of electrostatics: given a static charge density $\rho(\vec{r})$ in free space, what is the corresponding electrostatic potential $\phi(\vec{r})$ at any arbitrary point $\vec{r} = (x, y, z)$ r = (x, y, z)? From Maxwell's equations, we know that ϕ satisfies Poisson's equation, namely,

$$\nabla^2\phi = -\rho(\vec{r})/\epsilon_0 \tag{62}$$

What does one do in the case of a point charge q located at some point $\vec{r_0} = (x_0, y_0, z_0)$.

A point charge is an idealization in which a finite amount of charge q is supposed to be packed into zero volume. The charge density must therefore be infinite at the point $\vec{r_0}$, and zero elsewhere. The delta function comes to our aid. We may write, in this case,

$$\rho(\vec{r}) = e\delta(\vec{r} - \vec{r_0}) = e\delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \tag{63}$$

K. Fourier Expansion as a preamble to Special Functions: Orthogonal Functions

Fourier expansion sets a stage for the chapters on " special functions", providing an introductory and expressionary statement that lays down the underlying philosophy of the topic of special functions.

- Two vectors $\vec{A}$ and $\vec{B}$ are orthogonal if $\vec{A}\cdot\vec{B} = A_iB_i = 0$. As a generalization, two functions $A(x)$ and $B(x)$ are orthogonal if $\int A(x)B(x)dx = 0$.

  A vector $\vec{V}$ is called a unit vector, or a normalized vector if $\vec{V}.\vec{V} = 1$, that is, its magnitude is unity. Extending this concept to functions, $V(x)$ is a normalized function if $\int |V(x)|^2 dx = 1$.

  If we have a set of functions, they form an *orthonormal* (orthogonal and normalized) set if,

$$\int A_n(x)A_m(x)dx = \delta_{m,n} \tag{64}$$

  Compare this with the Fourier expansion where the functions were $(\sin mx, \cos mx)$ or e^{imx} obey the following relations that can be interpreted as the "orthonormality" condition :

$$\frac{1}{\pi}\int_0^{2\pi}(\sin mx)(\sin nx)dx \; = \; \delta_{mn}, \qquad \frac{1}{\pi}\int_0^{2\pi}(\cos mx)(\cos nx)dx = \delta_{mn}$$

- Just as any vector in 3D can be expanded as $\vec{V} = V_x\hat{x} + V_y\hat{y} + V_z\hat{z} \equiv V_i\hat{x}_i$, Any function $f(x)$ can be expanded in a set of orthonormal functions $\phi_n(x)$,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n\phi_n(x)$$

Note: It is important that the set is “complete” . We will define the term “completeness” later.

L. Appendix: Fourier Series for a Periodic Function of period $2L$

Consider a periodic function $f(x) = f(x+2L) = f(x+4L).... = f(x+2mL)$, m is an integer.

In its simplest form, Fourier series of a periodic function is given by:

$$f(x) \; = \; \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}, \tag{65}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-\frac{in\pi x}{L}} dx \tag{66}$$

Note the following:

- Check that Eq. (65) satisfies the condition that $f(x) = f(x+2L)$:

$$f(x+2L) \; = \; \sum_{n=-\infty}^{\infty} f_n e^{\frac{i\pi n(x+2L)}{L}} = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{\pi n x}{L}} e^{i2\pi n} = f(x), \quad as \;\, e^{i2\pi n} = 1$$

- If we take the period to be 2π, the Eqs. (65) and (66) become:

$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x}, \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-in\pi x} dx$

- To prove formula (66): Multiply Eq. (65) by $e^{-\frac{im\pi x}{L}}$ and integrate over x

$$\begin{aligned}\int_{-L}^{L} f(x) e^{-\frac{im\pi x}{L}} dx \; &= \; \sum_{n=-\infty}^{n=\infty} c_n \int_{-L}^{L} e^{\frac{i(n-m)\pi x}{L}} dx \\ &= \; \sum_{n=-\infty}^{n=\infty} c_n \delta_{mn} \int_{-L}^{L} dx \\ &= \; 2L c_m\end{aligned}$$

$$c_m \; = \; \frac{1}{2L}\int_{-L}^{L} f(x) e^{-\frac{im\pi x}{L}} dx$$

Note that we have used the following identity in the above equation:

$$\int_{-L}^{L} e^{\frac{i(n-m)\pi x}{L}} dx \;=\; \int_{-L}^{L} \big[\cos\frac{(n-m)\pi x}{L} + i\sin\frac{(n-m)\pi x}{L}\big] dx$$

$$=\; 0, \quad m \neq n$$

Using Euler formula, Fourier series can be also written in terms of sine and cosine:

$$\begin{aligned} f(x) \;&=\; \sum_{n=-\infty}^{\infty} f_n\big(\cos\frac{n\pi x}{L} + i\sin\frac{\pi n x}{L}\big) \\ &\equiv\; \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\tfrac{\pi}{L}nx + b_n \sin\tfrac{\pi}{L}nx. \end{aligned}$$

Fourier coefficients (a_n, b_n) are given by

$$\begin{aligned} a_n &= \frac{1}{L}\int_0^{2L} f(x)\cdot\cos\left(\tfrac{\pi}{L}nx\right)\, dx \\ b_n &= \frac{1}{L}\int_0^{2L} f(x)\cdot\sin\left(\tfrac{\pi}{L}nx\right)\, dx. \end{aligned} \tag{67}$$

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Home Work (6.2)

(2) Prove equation (56)

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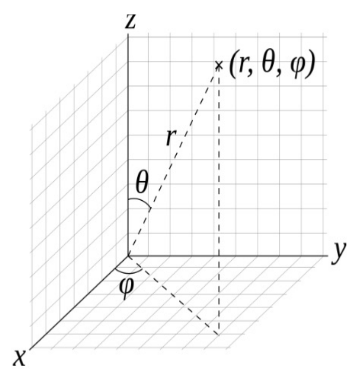
VII. CHAPTER VII

The What and Why of Curvilinear Coordinate Systems



The name "curvilinear coordinates" was coined by the French mathematician Lamé (1795-1870), derived from the fact that the coordinate surfaces of the curvilinear systems are curved. Well-known examples of curvilinear coordinate systems in three-dimensional Euclidean space are cylindrical (ρ, ϕ, z) and spherical coordinates (r, ϕ, θ).

The curvilinear coordinates may be derived from a set of Cartesian coordinates by using a transformation that is locally invertible (a one-to-one map) at each point. This means that one can convert a point given in a Cartesian coordinate system to its curvilinear coordinates and back.

Cylindrical Coordinates

$$\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \\ z &= z \end{aligned} \tag{68}$$

$\rho = \sqrt{x^2 + y^2}$

NOTE: Special case: Polar coordinates in 2D. Set $z = 0$

Spherical Coordinates

$$\begin{aligned} x &= r\cos\phi\,\sin\theta, \\ y &= r\sin\phi\,\sin\theta, \\ z &= r\cos\theta. \end{aligned} \tag{69}$$

$$\begin{aligned} r &= \sqrt{x^2+y^2+z^2}, \\ \theta &= \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}, \\ \phi &= \mathrm{Arctan}(y/x). \end{aligned} \tag{70}$$

Why do we need Spherical or Cylindrical coordinates ??

Although rectangular coordinate system may seem like the simplest system, some problems in physics are soluble only if other coordinate systems are used. Consider gravitational field of a sphere, or electric field due to a charged sphere...
Or, Consider gravitational field of a cylinder, or electric field due to a charged cylinder...
Planetary motion, H-atom, Bohr model of an electron , circling a proton.
Normal modes of a drum...

A. Orthogonal Unit Vectors in Cylindrical and Spherical coordinate system

Rectangular, cylindrical and spherical coordinates are "orthogonal coordinate systems" in which the unit vectors are normal to each other. The familiar case of the Cartesian coordinate system, in which $(\hat{x},\hat{y},\hat{z}) \equiv (\hat{i},\hat{j},\hat{k})$ comprise an orthogonal basis.

We will now construct unit orthogonal vectors for cylindrical and spherical coordinates. In these curvilinear coordinate systems, the unit vectors are position-dependent, i.e., their directions do not remain the same at all points in space, unlike the Cartesian basis. However, at each point, the basis vectors form a mutually orthogonal triad.

If a vector, $\vec{r}$ depends on a parameters u, then a unit vector that points in the "direction" of increasing u is the unit tangent vector given by

$$\hat{u} = \frac{\frac{\partial \vec{r}}{\partial u}}{|\frac{\partial \vec{r}}{\partial u}|} \tag{71}$$

Rectangular: $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$$\hat{x} = \frac{\frac{\partial \vec{r}}{\partial x}}{|\frac{\partial \vec{r}}{\partial x}|}, \quad \hat{y} = \frac{\frac{\partial \vec{r}}{\partial y}}{|\frac{\partial \vec{r}}{\partial y}|}, \quad \hat{z} = \frac{\frac{\partial \vec{r}}{\partial z}}{|\frac{\partial \vec{r}}{\partial z}|}$$

Cylindrical: $\vec{r} = \rho\cos\phi\hat{x} + \rho\sin\phi\hat{y} + \hat{z}$

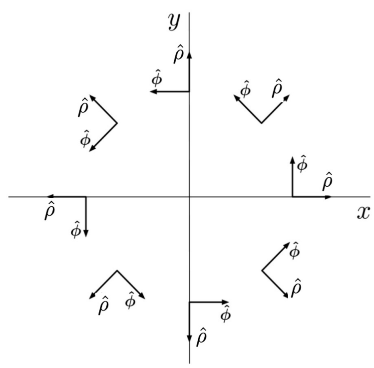


FIG. 6: Unit vectors $(\hat{\rho}, \hat{\phi})$ in two dimension. $\hat{\rho}$ is unit vector pointing in the direction of increasing ρ and $\hat{\phi}$ is a unit vector pointing in the direction of increasing ϕ

$$\hat{\rho} = \frac{\frac{\partial \vec{r}}{\partial \rho}}{|\frac{\partial \vec{r}}{\partial \rho}|} \quad \hat{\phi} = \frac{\frac{\partial \vec{r}}{\partial \phi}}{|\frac{\partial \vec{r}}{\partial \phi}|} \quad \hat{\theta} = \frac{\frac{\partial \vec{r}}{\partial \theta}}{|\frac{\partial \vec{r}}{\partial \theta}|}$$

$$\begin{aligned}
\hat{\rho} &= \cos\phi\hat{x} + \sin\phi\hat{y} \\
\hat{\phi} &= -\sin\phi\hat{x} + \cos\phi\hat{y} \\
\hat{z} &= \hat{z}
\end{aligned}$$

$$\begin{aligned}
\hat{x} &= \cos\phi\hat{\rho} - \sin\phi\hat{\phi} \\
\hat{y} &= \sin\phi\hat{\rho} + \cos\phi\hat{\phi}
\end{aligned}$$

Spherical: $\vec{r} = r\cos\phi\sin\theta\hat{x} + r\sin\phi\sin\theta\hat{y} + r\cos\theta\hat{z}$

$$\hat{r} = \frac{\frac{\partial\vec{r}}{\partial r}}{|\frac{\partial\vec{r}}{\partial r}|},\quad \hat{\phi} = \frac{\frac{\partial\vec{r}}{\partial\phi}}{|\frac{\partial\vec{r}}{\partial\phi}|},\quad \hat{\theta} = \frac{\frac{\partial\vec{r}}{\partial\theta}}{|\frac{\partial\vec{r}}{\partial\theta}|}$$

$$\hat{r} = \sin\theta\,(\cos\phi\hat{x} + \sin\phi\hat{y}) + \cos\theta\hat{z}$$

$$\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y}$$

$$\hat{\theta} = \cos\theta\,(\cos\phi\hat{x} + \sin\phi\hat{y}) - \sin\theta\hat{z}$$

B. Vector Calculus

Vector calculus studies various differential operators defined on scalar or vector fields, which are typically expressed in terms of the del operator (∇).

The four basic operators are:

(1) $\mathrm{grad}(f) = \nabla f$

$$\mathrm{grad}(f) = \nabla f = \left(\tfrac{\partial}{\partial x},\ \tfrac{\partial}{\partial y},\ \tfrac{\partial}{\partial z}\right)f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \equiv (\partial_1, \partial_2, \partial_3)f$$

(2) $\mathrm{div}(\mathbf{F}) = \nabla\cdot\mathbf{F}$

$$\mathrm{div}\,\mathbf{F} = \nabla\cdot\mathbf{F} = \left(\tfrac{\partial}{\partial x},\ \tfrac{\partial}{\partial y},\ \tfrac{\partial}{\partial z}\right)\cdot\left(F_x,\ F_y,\ F_z\right) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \equiv \partial_i F_i$$

That is $(x, y, z) \to (1, 2, 3)$ and repeated index is summed over, that is $\partial_i F_i = \partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3$.

(3) $\mathrm{curl}(\mathbf{F}) = \nabla\times\mathbf{F}$

$$\begin{aligned}
\nabla \times \mathbf{F} &= \left(\tfrac{\partial}{\partial x},\ \tfrac{\partial}{\partial y},\ \tfrac{\partial}{\partial z}\right) \times \left(F_x,\ F_y,\ F_z\right) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \\
&= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k}
\end{aligned}$$

(4) $\nabla^2 f = \nabla \cdot \nabla f \equiv \partial_i \partial_i f$

$\nabla^2 f = (\nabla \cdot \nabla) f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

The Laplacian Operator

The operator ∇^2 formed by taking the scalar product of the del operator with itself is called the Laplacian. It is important to note that ∇^2 is a scalar differential operator, in contrast to ∇, which is a vector operator.

Why do gradients, curls, divergences and Laplace operators appear in physics ???

You would have noticed that physical laws involving scalar and vector fields almost always involve the divergence, curl, and Laplacian of these fields. Why should this be so?

The invariance of physical laws under rotations of the coordinate axes requires that they be expressed in terms of quantities such as scalars, vectors, and tensors. These quantities have definite transformation laws under such rotations.

Physical laws generally involve differential equations.

- When solving Newton's equation, you want to know $(x(t), y(t), z(t))$. Their derivatives give Newton's equation. Newton's equations depend on only one variable, namely t and are ordinary differential equation.

Newton's equation of motion remains valid for all kinds of forces—mechanical, electromagnetic, and so on. In particular, suppose the particle has a charge e (once again, assumed to be a scalar constant), and moves in an applied electric field E and magnetic field B. The Lorentz force on it is given by the familiar expression,

$\vec{F} = e\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}.$

- When you want to know electric and magnetic field $E_i(x, y, z, t)$ and $B_i(x, y, z, t)$, you solve Maxwell's equations:

In the region $\rho = 0, \vec{J} = 0$

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$
$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

That is, you see curl and divergence of the fields. Taking the curl of the curl equations, and using the curl of the curl identity we obtain

$$\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = 0$$
$$\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = 0$$

The quantity $\mu_0 \varepsilon_0 = \frac{1}{c^2}$

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = 0$$
$$\frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = 0$$

Where do the "gradients" and the "curl come:

$$\vec{E} = -\nabla \phi, \quad \nabla \times \vec{A} = \vec{B}.$$

Scale Factors

The differential operators can be expressed nicely in terms of " Scale Factors" (h_1, h_2, h_3) which are defined below.

Rectangular Coordinates:

$$\begin{aligned}
\vec{r} &= x\hat{x} + y\hat{y} + z\hat{z} \\
d\vec{r} &= \frac{\partial \vec{r}}{\partial x}dx + \frac{\partial \vec{r}}{\partial y}dy + \frac{\partial \vec{r}}{\partial z}dz \\
&= \hat{x}\ dx + \hat{y}\ dy + \hat{z}\ dz \\
&\equiv h_1 dx\ \hat{x} + h_2 dy\ \hat{y} + h_3 dz\ \hat{z}, \quad (h_1, h_2, h_3 = 1, 1, 1)
\end{aligned}$$

Spherical Coordinates: $(\hat{r}, \hat{\phi}, \hat{\theta})$

$$\left|\frac{\partial \mathbf{r}}{\partial r}\right| = 1, \quad \left|\frac{\partial \mathbf{r}}{\partial \theta}\right| = r, \quad \left|\frac{\partial \mathbf{r}}{\partial \phi}\right| = r\sin\theta.$$

$\hat{r} = \frac{\frac{\partial \vec{r}}{\partial r}}{|\frac{\partial \vec{r}}{\partial r}|}, \quad \hat{\phi} = \frac{\frac{\partial \vec{r}}{\partial \phi}}{|\frac{\partial \vec{r}}{\partial \phi}|}, \quad \hat{\theta} = \frac{\frac{\partial \vec{r}}{\partial \theta}}{|\frac{\partial \vec{r}}{\partial \theta}|}$

$$\begin{aligned}
\vec{r} &= r\cos\phi\sin\theta\hat{x} + r\sin\phi\sin\theta\hat{y} + r\cos\theta\hat{z} \\
d\vec{r} &= \frac{\partial \vec{r}}{\partial r}dr + \frac{\partial \vec{r}}{\partial \phi}d\phi + \frac{\partial \vec{r}}{\partial \theta}d\theta \\
&= dr\ \hat{r} + \ rd\phi\ \hat{\phi} + \ r\sin\theta d\theta\ \hat{\theta} \\
&\equiv h_1 dr\ \hat{r} + h_2 d\theta\ \hat{\theta} + \ h_3 d\phi\ \hat{\phi}, \quad (h_1, h_2, h_3 = 1, r, r\sin\theta)
\end{aligned}$$

Cylindrical Coordinates: $(\hat{\rho}, \hat{\phi}, \hat{z})$

$$\begin{aligned}
d\vec{r} &= \frac{\partial \vec{r}}{\partial \rho}d\rho + \frac{\partial \vec{r}}{\partial \phi}d\phi + \frac{\partial \vec{r}}{\partial z}dz \\
&= d\rho\ \hat{\rho} + rd\phi\ \hat{\phi} + dz\ \hat{z} \\
&\equiv h_1 d\rho\ \hat{\rho} + h_2 d\phi\ \hat{\phi} + h_3 dz\ \hat{\theta}, \quad (h_1, h_2, h_3 = 1, \rho, 1)
\end{aligned}$$

C. General Formulas for Gradient, Divergence, Curl and Laplacian

Let us denote the coordinates the unit vectors as $\hat{e}_i$. In rectangular coordinates , $\hat{e}_i = \hat{x}_i$.

- $\vec{\nabla} f = \hat{e}_i \frac{1}{h_i} \partial_i f$
- $\vec{\nabla} \cdot \vec{V} = \frac{1}{h_1 h_2 h_3} \partial_i [\frac{(h_1 h_2 h_3)}{h_i} V_i]$
- $\vec{\nabla} \times \vec{V} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix}$
- $\nabla^2 f = \frac{1}{h_1 h_2 h_3} \partial_i (\frac{h_1 h_2 h_3}{h_i^2} \partial_i f)$

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Chapter 7 Home Work

(7.1) Prove that the unit vectors in cylindrical and spherical coordinate systems are orthogonal.

(7.2) For cylindrical coordinate unit vectors, prove the followings:

$$\begin{aligned} \dot{\hat{\rho}} &= \dot{\phi}\hat{\phi} \\ \dot{\hat{\phi}} &= -\dot{\phi}\hat{\rho} \end{aligned}$$

(7.3) For spherical coordinate unit vectors, prove the followings:

$$\begin{aligned} \dot{\hat{r}} &= \dot{\theta}\hat{\theta} + \sin\theta\dot{\phi}\hat{\phi} \\ \dot{\hat{\phi}} &= -\sin\theta\dot{\phi}\hat{r} - \cos\theta\dot{\phi}\hat{\theta} \\ \dot{\hat{\theta}} &= -\dot{\theta}\hat{r} + \cos\theta\dot{\phi}\hat{\phi} \end{aligned}$$

(7.3) Represent the vector $\vec{V} = z\hat{x} - 2x\hat{y} + y\hat{z}$ in cylindrical coordinate systems .

(7.4) Express the velocity and acceleration of a particle in cylindrical coordinates.

(7.5) Evaluate the integral $\int_V (x^2 + y^2 + z^2) dV$ over the volume of a sphere centered at origin and of radius R.

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VIII. CHAPTER VIII

Differential Equations & Partial Differential Equations

In introductory physics class such as phys106, you encounter ordinary differential equations for $\vec{r}(t) = (x(t), y(t), z(t))$. Projectile motion in 3D reduces to three ordinary differential equations in x, y, z. There is only one independent variable, namely t.

But then you encounter fields: Electric fields- E , magnetic Fields- B , Gravitational field....

Or, you will encounter wave equations: sound waves, waves in a string, drums, represented by some function $u(x, y, z, t)$. That is, (x, y, z, t) are all independent variables.

Note:

- $\vec{E} = \vec{E}(x, y, z, t)$ and $\vec{B} = \vec{B}(x, y, z, t)$.
- $\vec{E} = (E_x, E_y, E_z) = (E_x(x, y, z), E_y(x, y, z), E_z(x, y, z))$ and same for the magnetic and gravitation field...

Differential equations you encounter are " partial differential equation".

- Laplace Equation; $u = u(x, y, z)$.
  $\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$
- The scalar wave equation is
  $\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \nabla^2 u$
  An even more compact notation sometimes used in physics reads simply

  $$\Box u = 0,$$

  where all operators are combined into the d'Alembert operator:

  $$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2.$$

- Helmholtz equation. It corresponds to the linear partial differential equation:

  $\nabla^2 f = -k^2 f$

  When the equation is applied to waves, k is known as the wave number. The Helmholtz equation has a variety of applications in physics, including the wave equation and the diffusion equation,

- Heat equation
  $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2},$
- Poisson's equation:

  In three-dimensional Cartesian coordinates, it takes the form
  $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\varphi(x,y,z) = f(x,y,z).$
- Schrödinger Equation

  $$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = -\frac{\hbar^2}{2m}\nabla^2\Psi(\mathbf{r},t) + V(\mathbf{r})\Psi(\mathbf{r},t)\,.$$

A. Maxwell Equations

$\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} - \nabla^2\mathbf{E} = 0$
$\frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} - \nabla^2\mathbf{B} = 0$
There each component of $\vec{E}$ and $\vec{B}$ satisfies wave equation.

IX. CHAPTER IX

How to solve partial differential equations???

For physicists, separation of variables is a useful and favorite method for solving partial differential equations.

Why???

Special functions of mathematical physics , such as Bessel functions, Legendre polynomials, spherical harmonics emerge in solutions of partial differential equations that describe physical physical systems (previous chapter), using separation of variables. Interestingly, mathematicians have studied these functions starting mostly from 18th century, without any reference to physical problems.

Each special function encodes the symmetry of the problem. For example, Bessel functions appear in problems involving cylindrical symmetry. Legendre polynomials and spherical harmonics appear in problems with spherical symmetry.

Separation of Variable

Consider linear differential equations satisfied by a function $F(x, y, z, t)$.

- The first step is to look for solutions in the form of products:

$$F(x, y, z, t) = X(x)Y(y)Z(z)T(t) \tag{72}$$

  It sounds absurd to look for solutions of this type, as we do not expect solutions in general to be of this type.

- Solution obtained is not the general solution. The general solution is a sum of all possible solutions.

  Since the equation is a linear equation, if we have many solutions, their sum is also a solution.

  That is, by combing or pasting all solutions, we hope to get general solution that satisfies our boundary condition.

- Depending on the symmetry of the problem, you can choose rectangular, cylindrical or spherical coordinates. For example, the Cartesian axes might be separated,

  $F(\mathbf{r}) \equiv F(x, y, z) = X(x)Y(y)Z(z)$

  or radial and angular coordinates might be separated:

  $F(\mathbf{r}) = R(r)\Theta(\theta)\Phi(\phi)$.

  We will begin with the simple example of wave equation in 1D where the wave function is $u = u(x, t)$ and we will look for solutions like

$$u(x, t) = X(x)T(t) \tag{73}$$

A. Wave Equation in One-Dimension

Let us solve this by separation of variable:

The wave equation in one space dimension can be written as follows:

$$\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad u = u(x, t) \tag{74}$$

B. Solution by Separation of Variables

$$u(x, t) = X(x)T(t) \tag{75}$$

Substitute this in Eq. (74):

$$\frac{1}{v^2}X\frac{d^2T}{dt^2} = T\frac{d^2X}{dx^2} \tag{76}$$

Divide both sides by XT,

$$\frac{1}{v^2}\frac{1}{T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2} \tag{77}$$

Note the first term depends only on t and the second term depends only on x. That is, we have an equation like

$$F(x) = G(t) \tag{78}$$

where x and t are independent of each other. Therefore, we must have $F(x) = G(t) =$ Constant. Let us call that constant to be $-k^2$.

$$\frac{d^2X}{dx^2} + k^2X = 0$$
$$\frac{d^2T}{dt^2} + k^2v^2T \equiv \frac{d^2T}{dt^2} + \omega^2T = 0$$

Where we have set $\omega = kv$. Solutions are:

$$X(x) = A\sin kx + B\cos kx \equiv X_k(x) \tag{79}$$
$$T(t) = C\sin\omega t + D\cos\omega t \equiv T_\omega(t), \tag{80}$$

Therefore, separation of variables give us a family of solutions, parametrized by k or ω.

$$u(x,t) = X(x)T(t) \equiv u_\omega = X_kT_\omega. \tag{81}$$

General solution is a linear combination of all possible solutions. If there is no restriction on ω, we integrate over all possible values of ω:

$$u(x,t) = \int_{-\infty}^{\infty} s(\omega)u_\omega(x,t)\mathrm{d}\omega$$

EXAMPLE: For a string fastened at $x = 0$ and $x = l$, we have to impose the condition,

$$u(x=0) = u(x=l) = 0 \tag{82}$$

Substituting this in Eq. (79), gives $B = 0, kl = n\pi$ and therefore, $u(x,t) = u_n(x,t)$

General solution is a sum over all n.

$$u(x,t) = \sum_{-\infty}^{\infty} s_nu_n(x,t)$$

This example shows how "integers" appear in the solution.

C. Special Feature of the One-Dimensional Wave Equation

The one-dimensional wave equation is unusual for a partial differential equation in that a relatively simple general solution may be found.

The 1D equation $(v^2\partial_x^2 - v^2\partial_t^2)u = 0$ can be factored as:

$$(v\partial_x - \partial_t)(v\partial_x + \partial_t)u = 0 \tag{83}$$

Defining new variables:

$$\xi = x - vt$$

$$\eta = x + vt$$

changes the wave equation into

$$\frac{\partial^2 u}{\partial\xi\partial\eta} \equiv \partial_\xi\partial_\eta u = 0,$$

which leads to the general solution of the form

$$u(\xi, \eta) = F(\xi) + G(\eta)$$

or equivalently:

$$u(x,t) = F(x - vt) + G(x + vt).$$

In other words, solutions of the 1D wave equation are sums of a right traveling function F and a left traveling function G. F and G satisfies first order differential equations.

As the wave equation may be ”factored” into two One-way wave equations:

$$\left[\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}\right]\left[\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right]u = 0.$$

As a result, if we define V thus,

$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = V$,. This gives

$\frac{\partial V}{\partial t} - v\frac{\partial V}{\partial x} = 0.$

From this, V must have the form $G(x + vt)$, and from this the correct form of the full solution u can be deduced.

Or, you can write the solution as:

The total wave function for this eigenmode is then the linear combination

$u_\omega(x,t) = e^{-i\omega t}\left(Ae^{-ikx} + Be^{ikx}\right) = Ae^{-i(kx+\omega t)} + Be^{i(kx-\omega t)},$

where complex numbers A, B depend in general on any initial and boundary conditions of the problem.

D. Laplace Equation in rectangular coordinates in two dimension

$\nabla^2 V = \partial_x^2 V + \partial_y^2 V = 0.$

Solution by separation of variables using rectangular coordinates:

$$V(x,y) = X(x)Y(y) \tag{84}$$

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} \tag{85}$$

Divide both sides by XY,

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0 \tag{86}$$

Note the first term depends only on x and the second term depends only on y. That is, we have an equation like

$$F(x) + G(y) = 0 \tag{87}$$

where x and y are independent of each other. Therefore, we must have $F(x) = -G(y) =$ Constant. Let us call that constant to be k^2.

$$\begin{aligned}\frac{d^2X}{dx^2} - k^2 X &= 0 \\ \frac{d^2Y}{dy^2} + k^2 Y &= 0\end{aligned}$$

$$X(x) = Ae^{kx} + Be^{-kx} \tag{88}$$

$$Y(y) = C\sin ky + D\cos ky \tag{89}$$

General solution is

$$V(x,y) = \int (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky)dk \tag{90}$$

where A, B, C, D are constants, independent of (x, y), but may depend upon k.

EXAMPLE: Solve the Laplace equation in two dimension, subject to following boundary conditions:

(1) $V = 0$ when $y = 0$

(2) $V = 0$ when $y = \pi$

(3) $V = f(y)$ when $x = 0$

(4) $V \to 0$ as $x \to \infty$.

(See Introduction to Electrodynamics by Griffith. page 127 , second edition).

Solution:

Boundary conditions (1) and (2) implies that

$D = 0, k\pi = n\pi$. That is $k \equiv k_n = n$

Boundary condition (4) implies that $A = 0$. Boundary condition (4) gives (from Eq. (90)):

$$V(0,y) = f(y) = \sum_{n=1}^{\infty} G_n \sin ny \tag{91}$$

How to determine G_n ???

Note that (91) represents Fourier series expansion of a function where G_n are Fourier coefficients, given by

$$G_n = \frac{2}{\pi}\int_0^\pi dy\ \ f(y)\sin ny \tag{92}$$

NOTE: The statement that general solution is a sum of all possible solutions obtained by separation of variables is due to the property of *completeness* of set of solutions obtained by separation of variables.

That is , family of solutions above, labeled by k form a "complete set". That means that any function can be written as a linear combination of them.

Remark: Separation of variables in 2D (or one space one time) is particularly simple as it involves only one separation constant, so the factored solutions form one parameter family of solutions. This makes fitting boundary conditions by series relatively simple.

In three dimension, we have three equations and in the separation of variable there are two separation constants, instead of one in two dimension. At least one of the equation may depend upon both constants, as we will see below.

E. Laplace Equation in three dimension

$$\nabla^2 V = \partial_x^2 V + \partial_y^2 V + \partial_z^2 V = 0 \tag{93}$$

Solve the equation using separation of variables:

$$V(x,y,x) = X(x)Y(y)Z(z) \tag{94}$$

$$YZ\frac{d^2X}{dx^2} + ZX\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2} = 0 \tag{95}$$

Divide by XYZ,

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dx^2} = 0 \tag{96}$$

We have a situation like,

$$F(x) + G(y) + H(z) = 0 \tag{97}$$

for all values of (x, y, z) where x , y and z are independent of each other. Therefore, we must have each term to be a constant. Let us call these constants to be k_1^2, k_2^2 and k_3^2 where $k_1^2 + k_2^2 + k_3^2 = 0$

$$\begin{aligned}
\frac{d^2X}{dx^2} + k_1^2 X &= 0 \\
\frac{d^2Y}{dy^2} + k_2^2 Y &= 0 \\
\frac{d^2Z}{dz^2} + k_3^2 Z &= 0,
\end{aligned}$$

If $k_i^2 > 0$, the solutions of these equations are linear combination of sine and cosine functions.

If $k_i^2 < 0$, the solutions are linear combination of $e^{k_i x}$ and $e^{-k_i x}$. See problem (9.2) below where boundary conditions determine the nature of solutions.

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Chapter 9 Home Work

(9.1) For a string fastened at $x = -\frac{l}{2}$ and $x = \frac{l}{2}$, write down the general solution of the wave propagating in the string with velocity v.

(9.2) An infinitely long (infinite long along x direction), square metal pipe of sides of lengths π is grounded, that is , maintained at zero potential except the end of the pipe at $x = 0$ is maintained at a potential $V_0(y, z)$. Show that the potential inside the pipe is given by

$$V(x, y, z) = \sum_{m,n} C_{m,n} e^{-\sqrt{n^2+m^2}x} \sin ny \sin mz, \tag{98}$$

where

$$C_{n,m} = \frac{2^2}{\pi^2} \int_0^{\pi} V_0(y, z) \sin ny \sin mz \;\; dy \;\; dz \tag{99}$$

Note the boundary conditions:

(1) $V = 0$ when $y = z = 0$

(2) $V(x, y, z) = 0$ when $y = z = \pi$

(3) $V(x, y, z) \to 0$ as $x \to \infty$

(4) $V(x, y, z) = V_0(x, y)$ when $x = 0$.

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X. CHAPTER X

Laplace's Equation in spherical polar coordinates

Spherical polar coordinates: (r, θ, φ)

Laplace equation, expressed in curvilinear coordinates is:

$$\begin{aligned}\nabla^2 V &= \frac{1}{h_1 h_2 h_3}\partial_i(\frac{h_1 h_2 h_3}{h_i^2}\partial_i V) \\ &= \frac{1}{h_1 h_2 h_3}[\partial_1(\frac{h_2 h_3}{h_1}\partial_1 V) + \partial_2(\frac{h_3 h_1}{h_2}\partial_2 V) + \partial_3(\frac{h_1 h_2}{h_3}\partial_3 V)] = 0,\end{aligned}$$

where, in spherical polar coordinates, $(h_1, h_2, h_3 = 1, r, r\sin\theta)$. Substituting these values of (h_1, h_2, h_3) in above equation,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0. \tag{100}$$

Eliminating the common factor of $\frac{1}{r^2}$, we get

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0. \tag{101}$$

A. Solving Laplace equation in spherical-polar coordinates using separation of variables

Consider the problem of finding solutions of the form

$V(r, \theta, \phi) = R(r)Y(\theta, \phi)$

Substituting this in Eq. (101), and dividing by RY separates the equation as sum of two functions like $F_1(r) + F_2(\theta, \phi) = 0$. This implies that $F_1 = -F_2 =$ a constant. Let us call this constant to be λ.

$$\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = \lambda \tag{102}$$

$$\frac{1}{Y}\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial Y}{\partial \theta}\right) + \frac{1}{Y}\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial \phi^2} = -\lambda. \tag{103}$$

For the $Y(\theta, \phi)$ equation, let us again seek a solution in the form of separation of variables:

$$Y(\theta, \phi) = \Theta(\theta)\Phi(\phi) \tag{104}$$

Substituting this in Eq. (103), and dividing by $\Theta\Phi$, the equation separates into sum of two functions of the form $f_1(\theta) + f_2(\phi) = 0$. This implies that $f_1 = -f_2 =$ constant. Let us call that constant to be m^2.

- Φ-equation

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0 \tag{105}$$

  Since Φ is a periodic function of period 2π, m must be an integer.

- Θ-equation

$$\lambda \sin^2\theta + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = m^2$$

  Dividing by $\sin^2\theta$ and multiplying by Θ, the above equation can be written as:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}\Theta + \lambda\Theta = 0 \tag{106}$$

  Eq. (106) is a well known equation in mathematics, for functions known as *associated Legendre function* if $\lambda = l(l+1)$ and the solutions are denoted as $P_l^m(\theta, \phi)$. That is, the constant λ must be a product of two successive integers. Also, $|m| \leq l$. That is, for some non-negative integer with $l \geq |m|$). Only under these conditions, solutions of the equation are finite at $\theta = 0, \pi$. Setting λ equal to $l(l+1)$, we have:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}\Theta + l(l+1)\Theta = 0. \tag{107}$$

- R-equation

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - l(l+1)R = 0 \tag{108}$$

The equation for R has solutions of the form,

$R(r) = Ar^l + Br^{-l-1}$. Requiring the solution to be regular throughout in 3D imposes the condition that the constant $B = 0$.

B. Spherical Harmonics

The function $P_\ell^m(\cos\theta)\, e^{im\phi}$ are known as *spherical harmonics* and are denoted as $Y_\ell^m(\theta, \phi)$. The normalized spherical harmonics are:

$$Y_\ell^m(\theta, \phi) = (-1)^m \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)\, e^{im\phi}$$

Orthononmality of spherical harmonics:

$$\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} Y_\ell^m\, Y_{\ell'}^{m'*} \sin\theta d\theta d\phi = \delta_{\ell\ell'}\, \delta_{mm'}.$$

Legendre Polynomials

When $m = 0$, $P_l^m(\theta) = P_l(cos\theta)$ where $P_l(\cos\theta)$ are the Legendre polynomials in $\cos\theta$.

Spherical harmonics play a very important role in physics problems. Next chapter will be devoted to studying properties of these functions.

C. General solution of Laplace Equation in spherical polar coordinates

The general solution to Laplace's equation is a linear combination of the spherical harmonic functions multiplied by the appropriate scale factor r^l,

$$V(r, \theta, \varphi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} (C_\ell^m r^\ell + D_\ell^m r^{-\ell-1}) Y_l^m(\theta, \phi) \tag{109}$$

XI. CHAPTER XI

More on Spherical Harmonics & Legendre functions

$$Y_0^0(\theta,\varphi) = \sqrt{\frac{1}{4\pi}}$$

$$\begin{aligned}
Y_1^{-1}(\theta,\varphi) &= \frac{1}{2}\sqrt{\frac{3}{2\pi}}\ \sin\theta\, e^{-i\varphi} \\
Y_1^{0}(\theta,\varphi) &= \frac{1}{2}\sqrt{\frac{3}{\pi}}\ \cos\theta \\
Y_1^{1}(\theta,\varphi) &= \frac{-1}{2}\sqrt{\frac{3}{2\pi}}\ \sin\theta\, e^{i\varphi} \\
Y_2^{-2}(\theta,\varphi) &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\ \sin^2\theta\, e^{-2i\varphi} \\
Y_2^{-1}(\theta,\varphi) &= \frac{1}{2}\sqrt{\frac{15}{2\pi}}\ \sin\theta\ \cos\theta\, e^{-i\varphi} \\
Y_2^{0}(\theta,\varphi) &= \frac{1}{4}\sqrt{\frac{5}{\pi}}\,(3\cos^2\theta - 1) \\
Y_2^{1}(\theta,\varphi) &= \frac{-1}{2}\sqrt{\frac{15}{2\pi}}\ \sin\theta\ \cos\theta\, e^{i\varphi} \\
Y_2^{2}(\theta,\varphi) &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\ \sin^2\theta\, e^{2i\varphi}
\end{aligned}$$

A. Spherical Harmonics in Rectangular Coordinates

$Y_l^m(\theta,\phi)$ have very simple representation in rectangular coordinates:

- $r^l Y_l^m$ is a homogeneous polynomial function of (x, y, z) of degree l.
- ϕ dependence of Y_l^m is $e^{im\phi}$.
- Given l, $|m| \le l$

NOTE: $x = r\sin\theta\cos\phi,\ \ y = r\sin\theta\sin\phi,\ \ z = r\cos\theta$.

$$\begin{aligned}
x + iy &= r\sin\theta(\cos\phi + i\sin\phi) \\
&= r\sin\theta e^{i\phi}
\end{aligned}$$

$$(x+iy)^m = r^m \sin^m\theta e^{im\phi}$$

Examples of homogeneous polynomial functions $f(x, y, z)$

Degree zero: $f(x, y, z)$ is a constant, independent of (x, y, z)

Degree one: $f(x, y, z) = ax + by + cz$

Degree two: $f(x, y, z) = ax^2 + by^2 + cz^2 + Axy + Byz + Czx$. This includes r^2.

NOTE: r is not a homogeneous polynomial function of (x, y, z) as $r = \sqrt{x^2 + y^2 + z^2}$

If homogeneous polynomial functions have ϕ dependence as $e^{im\phi}$

Degree one :

if $l = 1$ and $m = 0$, $f(x, y, z) = z$

if $l = 1$ and $m = 1$, $f(x, y, z) = (x + iy)$

if $l = 1$ and $m = -1$, $f(x, y, z) = (x - iy)$

Degree two:

if $l = 2$ and $m = 0$, $f(x, y, z) = az^2 + br^2$

if $l = 2$ and $m = 1$, $f(x, y, z) = z(x + iy)$

if $l = 2$ and $m = -1$, $f(x, y, z) = z(x - iy)$

if $l = 2$ and $m = 2$, $f(x, y, z) = (x + iy)^2$

if $l = 2$ and $m = -2$, $f(x, y, z) = (x - iy)^2$

NOTE: We cannot have terms like xy or zx or zy as it will not give correct ϕ dependence.

Degree Three:

if $l = 3$ and $m = 0$, $f(x, y, z) = az^3 + bzr^2$

if $l = 3$ and $m = 1$, $f(x, y, z) = (az^2 + br^2)(x + iy)$

if $l = 3$ and $m = -1$, $f(x, y, z) = (az^2 + br^2)(x - iy)$

if $l = 3$ and $m = 2$, $f(x, y, z) = z(x + iy)^2$

if $l = 3$ and $m = -2$, $f(x, y, z) = z(x - iy)^2$

EXAMPLES: Constructing spherical harmonics, without worrying about normalization. Let us denote unnormalized spherical harmonics as Y_{lm}.

- $l = 0$, therefore, $m = 0$ since $-m \leq l \leq m$. Therefore, Y_{00} is a homogeneous polynomial function of (x, y, z) of degree zero. Therefore, Y_{00} is a constant.
- $l = 1$: Therefore, rY_{1m} is a homogeneous function of (x, y, z) of degree one
  $m = 1$, $Y_{1,1} = \frac{x+iy}{r}$
  $m = -1$, $Y_{1,-1} = \frac{x-iy}{r}$
  $m = 0$, $Y_{1,0} = \frac{z}{r}$
- $l = 2$: Therefore, rY_{2m} is a homogeneous function of (x, y, z) of degree two
  $m = 0$, $Y_{2,0} = \frac{az^2+br^2}{r^2}$
  $m = 1$, $Y_{2,1} = \frac{(x+iy)z}{r^2}$
  $m = 2$, $Y_{2,2} = \frac{(x+iy)^2}{r^2}$
- $Y_{l,\pm l} = \frac{(x \pm iy)^l}{r^l}$
- $Y_{l,l-1} = z\frac{(x \pm iy)^{l-1}}{r^l}$

Here a, b are constants that can be determined from normalization equation for spherical harmonics.

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(11 a) Write $\frac{x}{r}$ and $\frac{y}{r}$ as sum of two spherical harmonics.

(11 b) Write $\frac{xz}{r}$ and $\frac{yz}{r}$ as sum of two spherical harmonics.

(11 c) Write $Y_{10,10}$ and $Y_{100,99}$ in (x, y, z) coordinates.

(11 d) Write $Y_{4,m}$, $|m| \leq l$ in (x, y, z) coordinates.

Ignore normalization in the above problems.

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B. Importance of Spherical harmonics in Physics :Do the integers (l, m) have physical significance ??

- Since the spherical harmonics form a complete set of orthogonal functions and thus an orthonormal basis, each function defined on the surface of a sphere can be written as a sum of these spherical harmonics. This is similar to periodic functions defined on a circle that can be expressed as a sum of circular functions (sines and cosines) via Fourier series.
- In QM the integer m is the quantum number for the z-component of the angular momentum $L_z$and $l(l+1)$ represent possible values of L^2- square of angular momentum.
- In classical mechanics, and electro-magnetic theory, they appear in multipole expansions. You will see many examples in Griffith's book.

C. Legendre Polynomials

Legendre polynomials occur in the solution of Laplace's equation when separated in spherical polar coordinates. An example is the problem of the static potential, in a charge-free region of space, using the method of separation of variables, where the boundary conditions have axial symmetry (no dependence on an azimuthal angle). Here $\hat{z}$ is the axis of symmetry and θ is the angle between the position of the observer and the $\hat{z}$ axis , the solution for the potential will be

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-(\ell+1)} \right) P_\ell(\cos\theta) \,.$$

See Griffith, Introduction to Electrodynamics, section 3.3.2 and section 3.4.

<u>Other ways to define Legendre functions</u>

In addition to solutions to some differential equations, special functions can be defined in many ways, and the various definitions highlight different aspects as well as suggest generalizations and connections to different mathematical structures and physical and numerical applications. Below we look at three additional definitions of Legendre functions.

Below, we will replace the argument $\cos\theta$ in $P_l(\cos\theta)$ by x where $1 \leq x \leq 1$.
That is, $P_l(\cos\theta) \equiv P_l(x)$,

- (1) Definition via generating function: Generating functions expressed as a power series whose coefficients are special functions.

  The function $\frac{1}{\sqrt{1-2xt+t^2}}$ is called generating function of Legendre polynomials as

  $$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{l=0}^{\infty} P_l(x)t^l,$$

  That is,

  $$(1-2xt+t^2)^{-\frac{1}{2}} = P_0(x)t^0 + P_1(x)t^1 + P_2(x)t^2 + ...$$

  That is, the Legendre polynomials can also be defined as the coefficients in a formal expansion in powers of t, $(-1 < t < 1)$,

  Binomial expansion of $\frac{1}{\sqrt{1-2xt+t^2}}$ in t:

  Recall : if $z < 1$, $(1+z)^n = 1 + nz + \frac{n(n-1)}{2!}z^2 + \frac{n(n-1)(n-2)}{3!}z^3 + ...$, where n need not be integer.

  With $n = -\frac{1}{2}$, and $z = (-2xt + t^2)$,

  $(1-2xt+t^2)^{-\frac{1}{2}} = 1 - \frac{1}{2}(-2xt+t^2) - \frac{1}{2}\frac{(-1/2-1)}{2!}(-2xt+t^2)^2 + ...$

  Collecting coefficients of $t, t^2, t^3 ...$,

  $$\begin{aligned}(1-2xt+t^2)^{-\frac{1}{2}} &= 1 + (-\frac{1}{2})(-2xt) + (-\frac{1}{2})t^2 + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2!}(2xt)^2 + ...\\ &= 1 \quad + \quad xt \quad + \quad \frac{1}{2}(3x^2-1)t^2 + ...\\ &= P_0(x) + P_1(x)t + P_2(x)t^2 + ...\end{aligned}$$

The coefficient of t^l is a polynomial in x of degree l, denoted as $P_l(x)$.

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x. \\ P_2(x) &= \frac{1}{2}(3x^2 - 1) \end{aligned}$$

NOTE the followings:

- $P_l(x)$ has a definite parity, that is , it contains only even powers of x when l is even and contains only odd-powers of x when l is odd.

Legendre polynomials have been known in mathematics for a long time. They were first investigated in connection with Newton's law of universal gravitation in three dimensions by Laplace in 1792.

$V(\vec{r}) = \sum_i \frac{m_i}{|\vec{r}_i - \vec{r}|}$.

Each term in the above summation is an individual Newtonian potential for a point mass. Legendre had investigated the expansion of the potential in powers of $r = |\vec{r}|$ and $r_1 = |\vec{r}_1|$

Laplace discovered that if $r \leq r_1$, then

$$\frac{1}{|\vec{r}_1 - \vec{r}|} = P_0(\cos\gamma)\frac{1}{r_1} + P_1(\cos\gamma)\frac{r}{r_1^2} + P_2(\cos\gamma)\frac{r^2}{r_1^3} + \cdots = \tfrac{1}{r_1}\textstyle\sum_{n=0}^{\infty} P_n(\cos\gamma)(\frac{r}{r_1})^n$$

where γ is the angle between the vectors $\vec{r}$ and $\vec{r}_1$.

This formula can be derived from the formula for generating function using the following identity:

$$(\vec{r} - \vec{r}_1)^2 = r^2 + r_1^2 - 2\vec{r}\cdot\vec{r}_1 \tag{110}$$

- (2) Integral Representation of Legendre Polynomials

$$P_l(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(x+\sqrt{x^2-1}\cos\theta)^l d\theta \tag{111}$$

- (3) Rodrigues' formula:
  $P_l(x) = \dfrac{1}{2^l l!}\dfrac{d^l}{dx^l}(x^2-1)^l$.

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(11.1) Using the generating function formula, show that the Legendre Polynomials P_3, P_4 are given by

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$ and $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

(11.2) Show that $P_1(\cos 88) \approx A\pi$. Calculate A.

(11.3) Prove that for $r_1 > r$,
$\frac{1}{|\vec{r}_1-\vec{r}|} = \frac{1}{r_1}\sum_{n=0}^{\infty} P_n(\cos\gamma)(\frac{r}{r_1})^n$, where
$r = |\vec{r}|$ and $r_1 = |\vec{r}_1|$ and $\cos\gamma = \hat{r}\cdot\hat{r}_1$.

(11.4) Given $P_n(x)$ is a polynomial of degree n, use Eq. (**??**), to determine $P_0(x), P_1(x), P_2(x)$.

(11.5) Show that $P_2(\cos\theta) = \frac{1+3\cos 2\theta}{4}$

(11.6) Consider a hollow sphere of radius R. Let $\cos^2\frac{\theta}{2}$ is the potential on the surface of the sphere. Find the potential inside and outside sphere. (See Griffith, page 139-141, second edition)

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<u>Orthogonality and completeness</u>

The standardization $P_n(1) = 1$ fixes the normalization of the Legendre polynomials in the interval $-1 \le x \le 1$. Since they are also orthogonal with respect to the same norm, the two statements can be combined into the single equation,

$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \frac{2}{2n+1}\delta_{mn}$

That the polynomials are complete means the following. Given any piecewise continuous function $f(x)$ with finitely many discontinuities in the interval $[-1, 1]$, the sequence of sums

$f(x) = \displaystyle\sum_{\ell=0}^{\infty} a_\ell P_\ell(x)$, where

$a_\ell = \dfrac{2\ell+1}{2}\displaystyle\int_{-1}^{1} f(x)P_\ell(x)\,dx$

This completeness property is often stated in the form

$\displaystyle\sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} P_\ell(x)P_\ell(y) = \delta(x-y)$, with $-1 \le x \le 1$ and $-1 \le y \le 1$.

Example:

$$\frac{1}{\sqrt{1-x^2}} = \sum c_n P_n(x), \quad |x| < 1,$$

where,

$$c_n = (n+\frac{1}{2})\pi[\frac{(n-1)!!}{n!!}], \quad n = 0, 2, 4, ... \tag{112}$$

XII. CHAPTER XII

Bessel Functions

Just like spherical harmonics and Legendre functions appear in physics in problems involving spherical symmetry, Bessel functions appear in problems with cylindrical symmetry. Bessel functions are solutions of the radial part of Laplace equation when separated in cylindrical coordinates. They also appear in two dimensions such as in vibrations in circular membrane.

Below, we will discuss other ways to define Bessel function, also known as *cylindrical Bessel function*.

A. Various Definitions of Bessel Functions

- (1) Integral Representation: Define Bessel function (BF) $J_m(u)$ by its angular representation:

$$J_m(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin\phi}e^{-im\phi}d\phi \equiv < e^{ix\sin(\phi)}e^{-im\phi} >_\phi \tag{113}$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos[x\sin\phi - m\phi]d\phi \tag{114}$$

Second relation follows from the fact that

$$\frac{1}{\pi}\int_0^{2\pi} \sin[x\sin\phi - m\phi]d\phi = 0$$

- (2) Generating Functions: Functions expressed as a power series whose coefficients are $J_m(x)$:

There are various forms of generating functions as shown in (A), (B) and (C) below:

(A) Equation (113) implies the following,

$$e^{ix\sin\phi} = \sum_{-\infty}^{\infty} J_m(x)e^{im\phi} \tag{115}$$

(B) If we define $t = e^{i\phi}$, Eq (115) leads to another form of generating function, a function of a variable t when expanded as a power series in t, its m^{th} coefficient is $J_m(u)$.

$$e^{\frac{x}{2}(t-t^{-1})} = \sum_{m=-\infty}^{\infty} J_m(x)t^m \tag{116}$$

Using this formula, we can prove that $J_{-m}(x) = (-1)^m J_m(x)$.

(C) Eq. (115) leads to the another definition of Bessel functions.

For Bessel functions of even orders:

$$\cos(x\cos\phi) \;=\; J_0(x) + 2\sum_{m=1}^{\infty}(-1)^m J_{2m}(x)\cos 2m\phi \tag{117}$$

For Bessel functions of odd orders:

$$\sin(x\cos\phi) \;=\; 2\sum_{m=0}^{\infty}(-1)^m J_{2m+1}(x)\cos(2m+1)\phi \tag{118}$$

Therefore $J_m(x)$ are the Fourier coefficients of the periodic functions $\cos(x\cos\phi)$ and $\sin(x\cos\phi)$. By setting $\phi = 0$, we get,

$$\cos x \;=\; J_0(x) - 2J_2(x) - 2J_4(x) - 2J_6(x) + ... \tag{119}$$

$$\sin x \;=\; 2J_1(x) - 2J_3(x) - 2J_5(x) - 2J_7(x) + ... \tag{120}$$

Home Work: Prove equations (117) and (118).

- By expanding the l.h.s of Eq. (116) in a Taylor series about $t = 0$ and comparing the coefficient of t^n, we obtain

$$J_m(x) = \sum_{k=0}^{k=\infty}(-1)^k \frac{(x/2)^{n+2k}}{k!(n+k)!} \tag{121}$$

For $m = 0$,

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + ... \tag{122}$$

This series converges for all x.

(D) Series expansion for Bessel Functions, where m can also be a half-integer.

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{m+2k}}{k!\Gamma(m+k+1)}, \tag{123}$$

where for $m > 0$, $\Gamma(m+k+1) = (m+k)!$ is the *gamma function*. $\Gamma(1) = 1$. When m is an half-integer, show that

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

Note that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

- Note the following important results:

$$J_{-m}(x) = (-1)^m J_m(x), \quad J_m(-x) = (-)^m J_m(x) \tag{124}$$

$J_m(x) \to 0$ and $J_m(x) \to \infty$

- $J_m(x)$ as $x \to 0$

Expand $e^{ix\sin(\phi)}$ as a Taylor series expansion in x,

$$\begin{aligned}
e^{ix\sin(\phi)} &= \sum_l < \frac{1}{l!}(ix\sin\phi)^l \\
&= \sum_l \frac{x^l}{2^l l!} < (e^{i\phi} - e^{-i\phi})^l > \\
&= \sum_l < \frac{1}{l!}(ix\sin\phi)^l \\
&= \sum_l \frac{x^l}{2^l l!} e^{il\phi} < (1 - e^{-2i\phi})^l > \\
&= \sum_l \frac{x^l}{2^l l!} e^{il\phi} (1 - le^{-2i\phi} + \frac{l^2}{2!} e^{-4i\phi} + ...)
\end{aligned}$$

$$
\begin{aligned}
J_m(x) &= \frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin\phi}e^{-im\phi}d\phi \;\equiv <e^{ix\sin\phi}e^{-im\phi}>_\phi \\
&= \sum_l \frac{x^l}{2^l l!} < (e^{i\phi}-e^{-i\phi})^l e^{-im\phi} > \\
&= \sum_l \frac{x^l}{2^l l!} < (1-e^{-2i\phi})^l e^{i(l-m)\phi} > \\
&= \sum_l \frac{x^l}{2^l l!} < e^{il\phi}(1-le^{-2i\phi}+\frac{l^2}{2!}e^{-4i\phi}+.....)e^{-im\phi} > \\
&= \sum_l \frac{x^l}{2^l l!} < (1-le^{-2li\phi}+......)e^{i(l-m)\phi} > \\
&= \sum_l \frac{x^l}{2^l l!}(\delta_{lm}-l\delta_{l,m+2}+......) \\
&\approx \frac{x^m}{2^m m!}
\end{aligned}
$$

Show that if as $x \to 0$, $J_{-m}(x) \to (-1)^m \frac{x^m}{2^m m!}$

- $J_m(x)$: as $x \to \infty$

  We will use Laplace method:(See page 21-22 of the lecture notes)

  We start with Eq. (113). As $x \to \infty$, the main contribution to the above integral comes when the argument of the exponential $(x\sin\phi - m\phi)$ is extremum. Denoting this argument as $f(\phi)$, that is, $f(\phi) = i(x\sin(\phi) - m\phi)$, we find the value of ϕ where $f(\phi)$ is an extremum and then expand $f(\phi)$ about that point.

$$
\begin{aligned}
f(\phi) &= i(x\sin(\phi) - m\phi) \\
f'(\phi) &= i(x\cos(\phi) - m) \\
f''(\phi) &= -ix\sin(\phi)
\end{aligned}
$$

The solution for $f'(\phi) = i(x\cos\phi - m) = 0$.
That is, $\cos\phi = \frac{m}{x}$.

As $x \to \infty$, $\cos\phi \to 0$. This gives two solutions
$\phi = \phi_1 = \pi/2$ and $\phi = \phi_2 = -\pi/2$.

We now write $f(\phi) = f(\phi_j) + \frac{f''}{2!}(\phi - \phi_j)^2 + ...$, $(j = 1, 2)$, we get,

$$\begin{aligned} J_m(x) &= \frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin\phi} e^{-im\phi} d\phi \equiv \frac{1}{2\pi}\int_0^{2\pi} e^{f(\phi)} \\ J_m(x) &= \frac{1}{2\pi}[\sqrt{\frac{2\pi}{-f''(\phi_1)}} f(\phi_1) + \frac{1}{2\pi}[\sqrt{\frac{2\pi}{-f''(\phi_2)}} f(\phi_2) \\ &= \frac{1}{\sqrt{ix}} e^{i(x - m\pi/2)} + \frac{1}{\sqrt{ix}} e^{-i(x-m\pi/2)} \\ &= \sqrt{\frac{2}{\pi x}} \cos(x - m\pi/2 - \pi/4) \end{aligned} \tag{125}$$

NOTE: For $m = \frac{1}{2}$,

$$J_{\frac{1}{2}}(x) \to \sqrt{\frac{2}{\pi x}} \cos(x - \pi/4 - \pi/4) = \sqrt{\frac{2}{\pi x}} \sin x \tag{126}$$

Zeros or Roots of Bessel's Functions

The formula (125) shows that $J_n(x)$ has an infinite number of zeros and the difference between the successive roots approaches π as x becomes large.

The first four approximate values of zeros for $m = 0$ obtained from (125) are

$3/4\pi = 2.355, 7/4\pi = 5.49, 11/4\pi = 8.637, 15/4\pi = 1.77$

The corresponding numerical values are

$2.405, 5.520, 8.654, 11.792$..... that is,

$J_0(2.405) = 0$, $J_0(5.520) = 0$, etc.....

The zeros of Bessel functions are important in problems such as vibrational modes of a circular membrane or a drum as one seeks solutions that vanish at the boundaries.

Application: Vibrations of a circular membrane:

Consider a circular membrane of radius R. Let the vibrations of the membrane are described by $f(\rho, \phi, t) = AJ_0(k\rho)e^{i\phi}\cos\omega t$ where $\omega = kv$. Here v is the speed of propagation of the vibrations. Calculate the frequencies of vibrations of the membrane.

Solution: Note that at the boundary of the membrane, J_0 must vanish. That is, $J_0(kR) = 0$. therefore, kR must coincide with one of the roots of the $J_0(\rho k)$.

The two lowest frequencies ω_1 and ω_2 are:

$k_1R = 2.405$, $\omega_1 = 2.405v/R$

$k_2R = 5.520$, $\omega_2 = 5.520v/R$

XIII. CHAPTER XIII

Spherical Bessel Functions

Consider Helmholtz equation:

$$\nabla^2 f = -k^2 f$$

When the equation is applied to waves, k is known as the wave number. The Helmholtz equation has a variety of applications in physics, including the wave equation and the diffusion. It also describes particle confined to a spherical box.

When solving the Helmholtz equation in spherical coordinates by separation of variables,

$f(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$, the general solution has the form:

$$f(r, \theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(a_{\ell m} j_\ell(kr) + b_{\ell m} y_\ell(kr)\right) Y_\ell^m(\theta, \varphi).$$

The two linearly independent solutions to this equation are called the spherical Bessel functions j_n and y_n. These two solutions respectively correspond to solutions that regular (do not diverge) at the origin and at infinity.

Note that as we take the limit $k \to 0$
$\nabla^2 f = -k^2 f$ reduces to Laplace equation $\nabla^2 f = 0$ with solution given by Eq. (109), namely,

$$f(r, \theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (C_\ell^m r^\ell + D_\ell^m r^{-\ell-1}) Y_l^m(\theta, \phi)$$

This shows that two solutions, j_n, y_n are the regular and the irregular (singular at the origin).

In addition to the definition as solutions of the differential equation, there many other definitions of spherical Bessel functions.

- (I) Integral Representation

$$
\begin{aligned}
j_l(x) &= \frac{1}{2i^l}\int_0^{\pi} e^{ix\cos\theta} P_l(\cos\theta)\sin\theta d\theta \\
&= \frac{1}{2i^l}\int_{-1}^{1} e^{ixy} P_l(y) dy
\end{aligned}
$$

(Ref: See page 622 of Morse and Feshbach, Methods of Theoretical Physics)

Calculate $j_0(x)$ using integral representation

$$
\begin{aligned}
j_0(x) &= \frac{1}{2}\int_0^{\pi} e^{ix\cos\theta}\sin\theta d\theta \\
&= -\frac{1}{2}\int_1^{-1} e^{ixy} dy, \quad y = \cos\theta \\
&= \frac{e^{ix} - e^{-ix}}{2ix} \\
&= \frac{\sin x}{x}
\end{aligned}
$$

NOTE: There is another form of integral representation,

$$
j_l(x) = \frac{x^n}{2^{l+1} l!}\int_0^{\pi} \cos(x\ cos\theta)\sin^{2l+1}\theta\ \ d\theta \tag{127}
$$

- (II) Generating Function: A function when expanded in powers of t have their coefficients as the spherical Bessel function:

The spherical Bessel functions have the generating function (for $2t < x$):

$$
\frac{1}{x}\cos\left(\sqrt{x^2 - 2xt}\right) = \frac{\cos x}{x} + \sum_{n=1}^{\infty} \frac{t^n}{n!} j_{n-1}(x)
$$

To calculate j_0 using this formula, we need to calculate the coefficient of t by expanding l.h.s. From the binomial expansion about $t = 0$,

$\sqrt{x^2 - 2xt} = x(1 - 2\frac{t}{x})^{\frac{1}{2}} \approx (x - t)$, and $\sin t \approx t$ and $\cos t \approx 1$, we get

$$\begin{aligned}\cos(\sqrt{x^2-2xt}) &\approx \cos[x(1-\frac{t}{x})]\\ &= \cos(x-t)\\ &= \cos x\cos t+\sin x\sin t\\ &\approx \cos x + t\sin x\end{aligned}$$

From the generating function, we equate the coefficients of t from the l.h.s, which is $\frac{sinx}{x}$ and the coefficient of t from the r.h.s, which is $j_0(x)$, we get

$j_0(x) = \frac{\sin x}{x}$.

- (III) Spherical Bessel functions are related to Bessel functions as

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x) \tag{128}$$

NOTE: Above formula gives, $j_0(x) = \sqrt{\frac{\pi}{2x}} J_{\frac{1}{2}}(x)$. Using Eq. (126), we see that as $x \to \infty$, $j_0(x) \to \frac{\sin x}{x}$, which is the exact formula for $j_0(x)$.

- (IV) Rayleigh's formulas

$$j_n(x) = (-x)^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \frac{\sin x}{x} \tag{129}$$

<u>limit as $x \to 0$</u>

$j_n(x) \to \frac{x^n}{1.3.5......(2n+1)} \equiv \frac{x^n}{(2n+1)!!}$

<u>Examples</u>

$j_0(x) = \dfrac{\sin x}{x}$.
$j_1(x) = \dfrac{\sin x}{x^2} - \dfrac{\cos x}{x}$,
$j_2(x) = \left(\dfrac{3}{x^2}-1\right)\dfrac{\sin x}{x} - \dfrac{3\cos x}{x^2}$,
$j_3(x) = \left(\dfrac{15}{x^3}-\dfrac{6}{x}\right)\dfrac{\sin x}{x} - \left(\dfrac{15}{x^2}-1\right)\dfrac{\cos x}{x}$

NOTE: $j_n(x)$ are even functions for n-even and are odd-functions for n-odd.

Show that

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}} \sin x, \quad J_{-\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}} \cos x, \tag{130}$$

$\underline{x \to \infty}$

Using $j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x)$, show that

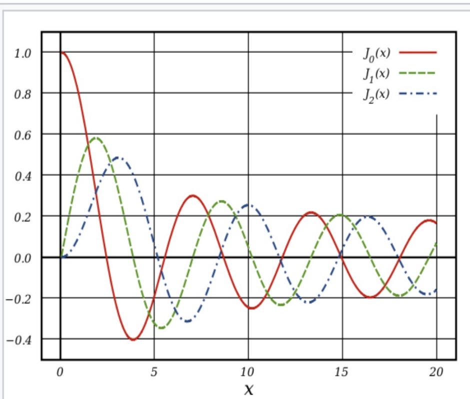
$$j_n(x) \to \frac{1}{x} \cos[x - \frac{\pi}{2}(n+1)] \tag{131}$$

Example: Waves inside a sphere

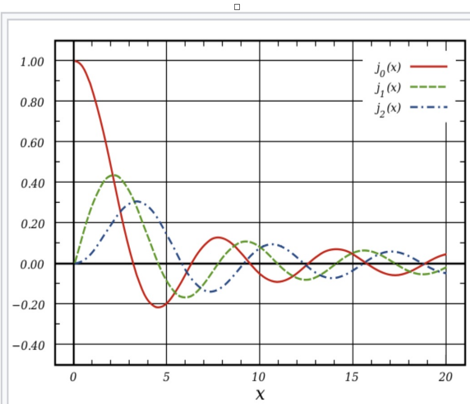
Consider a sphere of radius R. Given that the sound waves of speed v inside the sphere are described by $f(r, \theta, \phi) = A_{k,l,m} j_l(kr) Y_l^m(\theta, \phi)$, calculate the resonance frequencies of waves propagating inside the sphere.

For zeros of spherical Bessel function, use the asymptotic formula,
$j_l(x) \to \frac{1}{x} \cos[x - \frac{\pi}{2}(l+1)]$.

Solution: Frequencies $\omega =$******



Plot of Bessel function of the first kind, $J_\alpha(x)$, for integer orders $\alpha = 0, 1, 2$



Spherical Bessel functions of the first kind, $j_n(x)$, for $n = 0, 1, 2$

A. Common Characteristics of Special Functions; F_n

We have studied following special functions:

(I) Spherical harmonics: $Y_{lm}(\theta, \phi)$ and Legendre functions $P_l(\theta)$ (which are special case of Y_{lm}.

(II) Bessel Function: $J_m(r)$

(III) Spherical Bessel Functions: $j_m(r)$

Let us summarize them, by looking at them from common perspective. Their discussion involves:

Generating function,

Orthonormality,

Integral representation,

Series expansion,

Recursion relations.

- Generating Function: what is meant by a generating function ?

A function $g_u(t)$ of a variable t when expanded as a power series in t, its n^{th} coefficient is $J_n(u)$.

$$\frac{1}{\sqrt{1+t^2-2tx}} = \sum_{m=0}^{\infty} P_m(x) t^m, \quad t < 1$$

$$e^{\frac{x}{2}(t-t^{-1})} = \sum_{m=-\infty}^{\infty} J_m(x) t^m$$

$$\frac{1}{x}\cos(\sqrt{x^2-2xt}) = \frac{1}{x}\cos x + \sum_{m=1}^{\infty} j_{m-1}(x)\frac{t^m}{m!}$$

- Integral Representation: Define Special functions by their angular representation:

$$\begin{aligned} J_m(x) &= < e^{ix \sin\phi} e^{-im\phi} >_\phi \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{ix \sin\phi} e^{-im\phi} d\phi \end{aligned}$$

$$j_l(x) = \frac{1}{2i^l} \int_{-1}^{1} [e^{ix \cos\phi} P_l(\cos\phi)] d\cos\phi$$

$$P_l(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (x + \sqrt{x^2 - 1} \cos\theta)^l d\theta \tag{132}$$

HOME WORK:

Use integral representation to calculate Bessel, spherical Bessel and Legendre functions for $m = 0, 1, 2$.

- Normalization and Orthogonality :

  Functions $F_n(x)$ are orthogonal if

$$\int_{x_0}^{x_1} w(x) F_n(x) F_m(x) dx = N_n \delta_{n,m} \tag{133}$$

The orthogonality property allows the functions $F_n(x)$ to be used as a *basis set* for expanding a wide range of functions - a generalized Fourier expansion:

$$f(x) = \sum C_n F_n(x)$$

The constants C_n are the generalized coefficients of Fourier expansion and are given by,

$$c_n = \frac{1}{N_n} \int f(x) w(x) F_n(x) dx$$

$$\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} Y_\ell^m \, Y_{\ell'}^{m'*} \, d\Omega \;=\; \delta_{\ell\ell'}\,\delta_{mm'}$$

$$\int_0^1 xJ_n(x)J_m(x)dx \;=\; \delta_{n,m}\frac{|J'(z_{n,k})|^2}{2}, \quad J_n(z_{n,k})=0$$

$$\int_0^\infty x^2 j_n(ux)j_n(vx)\,dx \;=\; \frac{\pi}{2u^2}\delta(u-v)$$

- Recursion Relation. Given F_1, and F_2, we can determine $F_3, F_4, ...F_n....$

$$J_{n+1}(x) \;=\; \frac{2n}{x}J_n(x) - J_{n-1}(x)$$

$$\frac{2n+1}{x}j_n(x) \;=\; j_{n-1}(x) + j_{n+1}(x)$$

$$(n+1)P_{n+1}(x) \;=\; (2n+1)xP_n(x) - nP_{n-1}(x)$$

- $\displaystyle\sum_{m=-\ell}^{\ell} Y_{\ell m}^*(\mathbf{x})\,Y_{\ell m}(\mathbf{x}) = \frac{2\ell+1}{4\pi}$

  which generalizes the identity $\cos^2\theta + \sin^2\theta = 1$ for $l = 1$.

- Addition Theorem of Spherical harmonics

  A mathematical result of considerable interest and use is called the addition theorem for spherical harmonics. Given two vectors $\vec{r}$ and $\vec{r'}$, with spherical coordinates r, θ, φ and (r', θ', φ'), respectively, the angle γ between them is given by the relation

  $$\cos\gamma = \cos\theta'\cos\theta + \sin\theta\sin\theta'\cos(\varphi-\varphi')$$

  in which the role of the trigonometric functions appearing on the right-hand side is played by the spherical harmonics and that of the left-hand side is played by the Legendre polynomials.

$$P_\ell(\hat{\mathbf{x}} \cdot \hat{\mathbf{y}}) = \frac{4\pi}{2\ell + 1} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\hat{\mathbf{y}})\, Y_{\ell m}^*(\hat{\mathbf{x}})$$

XIV. CHAPTER XIV

Matrices & Normal Modes

A. Matrices

Matrices make frequent appearance in mathematical sciences such as in describing rotations (orthogonal matrices, unitary matrices) , in solving coupled equations and also in representation of an operator in quantum mechanics (Hermitian matrices).

- A matrix R is **orthogonal** if $RR^T = 1$ where the superscript T denotes the transport.
- A matrix H is **Hermitian** if $H = H^\dagger$ where † stands for transpose plus complex conjugation.
- A matrix U is **Unitary** if $UU^\dagger = 1$

Eigenvalues & Eigenvectors

See page 348 and 349 of Murray and Spiegel.

*** Incomplete *******

B. Finding normal modes: Two coupled masses

As an application of matrices, we will discuss normal modes of two coupled masses.
Ref: Normal Modes by David Mermin

In the past, we have talked about the oscillations of one mass. We saw that there were various possible motions, depending on what was influencing the mass (spring, damping, driving forces). In this chapter we'll look at oscillations (generally without damping or driving) involving more than one object.

2.1 Two masses

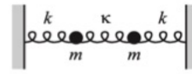


Figure 1

For a single mass on a spring, there is one natural frequency, namely $\sqrt{k/m}$. (We'll consider undamped and undriven motion for now.) Let's see what happens if we have two equal masses and three spring arranged as shown in Fig. 1. The two outside spring constants are the same, but we'll allow the middle one to be different. In general, all three spring constants could be different, but the math gets messy in that case.

Let x_1 and x_2 measure the displacements of the left and right masses from their respective equilibrium positions. We can assume that all of the springs are unstretched at equilibrium, but we don't actually have to, because the spring force is linear (see Problem [to be added]). The middle spring is stretched (or compressed) by $x_2 - x_1$, so the $F = ma$ equations on the two masses are

$$\begin{aligned} m\ddot{x}_1 &= -kx_1 - \kappa(x_1 - x_2), \\ m\ddot{x}_2 &= -kx_2 - \kappa(x_2 - x_1). \end{aligned} \tag{1}$$

Concerning the signs of the κ terms here, they are equal and opposite, as dictated by Newton's third law, so they are either both right or both wrong. They are indeed both right, as can be seen by taking the limit of, say, large x_2. The force on the left mass is then in the positive direction, which is correct.

These two $F = ma$ equations are "coupled," in the sense that both x_1 and x_2 appear in both equations. How do we go about solving for $x_1(t)$ and $x_2(t)$? There are (at least) two ways we can do this.

2.1.1 First method

This first method is quick, but it works only for simple systems with a sufficient amount of symmetry. The main goal in this method is to combine the $F = ma$ equations in well-chosen ways so that x_1 and x_2 appear only in certain unique combinations. It sometimes involves a bit of guesswork to determine what these well-chosen ways are. But in the present problem, the simplest thing to do is add the $F = ma$ equations in Eq. (1), and it turns out that this is in fact one of the two useful combinations to form. The sum yields

$$m(\ddot{x}_1 + \ddot{x}_2) = -k(x_1 + x_2) \implies \frac{d^2}{dt^2}(x_1 + x_2) = -\frac{k}{m}(x_1 + x_2). \tag{2}$$

The variables x_1 and x_2 appear here only in the unique combination, $x_1 + x_2$. And furthermore, this equation is simply a harmonic-motion equation for the quantity $x_1 + x_2$. The solution is therefore

$$x_1(t) + x_2(t) = 2A_\text{s}\cos(\omega_\text{s} t + \phi_\text{s}), \qquad \text{where } \boxed{\omega_\text{s} \equiv \sqrt{\frac{k}{m}}} \tag{3}$$

The "s" here stands for "slow," to be distinguished from the "fast" frequency we'll find below. And we've defined the coefficient to be $2A_\text{s}$ so that we won't have a bunch of factors of $1/2$ in our final answer in Eq. (6) below.

No matter what complicated motion the masses are doing, the quantity $x_1 + x_2$ always undergoes simple harmonic motion with frequency ω_s. This is by no means obvious if you look at two masses bouncing back and forth in an arbitrary manner.

The other useful combination of the $F = ma$ equations is their difference, which conveniently is probably the next thing you might try. This yields

$$m(\ddot{x}_1 - \ddot{x}_2) = -(k + 2\kappa)(x_1 - x_2) \implies \frac{d^2}{dt^2}(x_1 - x_2) = -\frac{k + 2\kappa}{m}(x_1 - x_2). \qquad (4)$$

The variables x_1 and x_2 now appear only in the unique combination, $x_1 - x_2$. And again, we have a harmonic-motion equation for the quantity $x_1 - x_2$. So the solution is (the "f" stands for "fast")

$$x_1(t) - x_2(t) = 2A_\text{f} \cos(\omega_\text{f} t + \phi_\text{f}), \qquad \text{where } \boxed{\omega_\text{f} \equiv \sqrt{\frac{k + 2\kappa}{m}}} \qquad (5)$$

As above, no matter what complicated motion the masses are doing, the quantity $x_1 - x_2$ always undergoes simple harmonic motion with frequency ω_f.

We can now solve for $x_1(t)$ and $x_2(t)$ by adding and subtracting Eqs. (3) and (5). The result is

$$\begin{aligned} x_1(t) &= A_\text{s} \cos(\omega_\text{s} t + \phi_\text{s}) + A_\text{f} \cos(\omega_\text{f} t + \phi_\text{f}), \\ x_2(t) &= A_\text{s} \cos(\omega_\text{s} t + \phi_\text{s}) - A_\text{f} \cos(\omega_\text{f} t + \phi_\text{f}). \qquad (6) \end{aligned}$$

The four constants, A_s, A_f, ϕ_s, ϕ_f are determined by the four initial conditions, $x_1(0)$, $x_2(0)$, $\dot{x}_1(0)$, $\dot{x}_1(0)$.

The above method will clearly work only if we're able to guess the proper combinations of the $F = ma$ equations that yield equations involving unique combinations of the variables. Adding and subtracting the equations worked fine here, but for more complicated systems with unequal masses or with all the spring constants different, the appropriate combination of the equations might be far from obvious. And there is no guarantee that guessing around will get you anywhere. So before discussing the features of the solution in Eq. (6), let's take a look at the other more systematic and fail-safe method of solving for x_1 and x_2.

2.1.2 Second method

This method is longer, but it works (in theory) for any setup. Our strategy will be to look for simple kinds of motions where both masses move with the same frequency. We will then build up the most general solution from these simple motions. For all we know, such motions might not even exist, but we have nothing to lose by trying to find them. We will find that they do in fact exist. You might want to try to guess now what they are for our two-mass system, but it isn't necessary to know what they look like before undertaking this method.

Let's guess solutions of the form $x_1(t) = A_1 e^{i\omega t}$ and $x_2(t) = A_2 e^{i\omega t}$. For bookkeeping purposes, it is convenient to write these solutions in vector form:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{i\omega t}. \tag{7}$$

We'll end up taking the real part in the end. We can alternatively guess the solution $e^{\alpha t}$ without the i, but then our α will come out to be imaginary. Either choice will get the job done. Plugging these guesses into the $F = ma$ equations in Eq. (1), and canceling the factor of $e^{i\omega t}$, yields

$$\begin{aligned} -m\omega^2 A_1 &= -kA_1 - \kappa(A_1 - A_2), \\ -m\omega^2 A_2 &= -kA_2 - \kappa(A_2 - A_1). \end{aligned} \tag{8}$$

In matrix form, this can be written as

$$\begin{pmatrix} -m\omega^2 + k + \kappa & -\kappa \\ -\kappa & -m\omega^2 + k + \kappa \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{9}$$

At this point, it seems like we can multiply both sides of this equation by the inverse of the matrix. This leads to $(A_1, A_2) = (0, 0)$. This is obviously a solution (the masses just sit there), but we're looking for a nontrivial solution that actually contains some motion. The only way to escape the preceding conclusion that A_1 and A_2 must both be zero is if the inverse of the matrix doesn't exist. Now, matrix inverses are somewhat messy things (involving cofactors and determinants), but for the present purposes, the only fact we need to know about them is that they involve dividing by the determinant. So if the determinant is zero, then the inverse doesn't exist. This is therefore what we want. Setting the determinant equal to zero gives the quartic equation,

$$\begin{aligned}
\begin{vmatrix} -m\omega^2+k+\kappa & -\kappa \\ -\kappa & -m\omega^2+k+\kappa \end{vmatrix} = 0 \quad &\Longrightarrow \quad (-m\omega^2+k+\kappa)^2-\kappa^2=0 \\
&\Longrightarrow \quad -m\omega^2+k+\kappa = \pm\kappa \\
&\Longrightarrow \quad \omega^2 = \frac{k}{m} \quad \text{or} \quad \frac{k+2\kappa}{m}. \qquad (10)
\end{aligned}$$

The four solutions to the quartic equation are therefore $\omega = \pm\sqrt{k/m}$ and $\omega = \pm\sqrt{(k+2\kappa)/m}$

For the case where $\omega^2 = k/m$, we can plug this value of ω^2 back into Eq. (9) to obtain

$$\kappa\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad (11)$$

Both rows of this equation yield the same result (this was the point of setting the determinant equal to zero), namely $A_1 = A_2$. So (A_1, A_2) is proportional to the vector $(1, 1)$.

For the case where $\omega^2 = (k+2\kappa)/m$, Eq. (9) gives

$$\kappa\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad (12)$$

Both rows now yield $A_1 = -A_2$. So (A_1, A_2) is proportional to the vector $(1, -1)$.

With $\omega_\text{s} \equiv \sqrt{k/m}$ and $\omega_\text{f} \equiv \sqrt{(k+2\kappa)/m}$, we can write the general solution as the sum of the four solutions we have found. In vector notation, $x_1(t)$ and $x_2(t)$ are given by

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{i\omega_\text{s}t} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-i\omega_\text{s}t} + C_3\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{i\omega_\text{f}t} + C_4\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-i\omega_\text{f}t}. \qquad (13)$$

We now perform the usual step of invoking the fact that the positions $x_1(t)$ and $x_2(t)$ must be real for all t. This yields that standard result that $C_1 = C_2^* \equiv (A_\text{s}/2)e^{i\phi_\text{s}}$ and $C_3 = C_4^* \equiv (A_\text{f}/2)e^{i\phi_\text{f}}$. We have included the factors of 1/2 in these definitions so that we won't have a bunch of factors of 1/2 in our final answer. The imaginary parts in Eq. (13) cancel, and we obtain

$$\boxed{\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = A_\text{s}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos(\omega_\text{s}t+\phi_\text{s}) + A_\text{f}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos(\omega_\text{f}t+\phi_\text{f})} \qquad (14)$$

Therefore,

$$
\begin{aligned}
x_1(t) &= A_\mathrm{s}\cos(\omega_\mathrm{s}t + \phi_\mathrm{s}) + A_\mathrm{f}\cos(\omega_\mathrm{f}t + \phi_\mathrm{f}), \\
x_2(t) &= A_\mathrm{s}\cos(\omega_\mathrm{s}t + \phi_\mathrm{s}) - A_\mathrm{f}\cos(\omega_\mathrm{f}t + \phi_\mathrm{f}).
\end{aligned} \tag{15}
$$

This agrees with the result in Eq. (6).

As we discussed in Section 1.1.5, we could have just taken the real part of the $C_1(1,1)e^{i\omega_\mathrm{s}t}$ and $C_3(1,-1)e^{i\omega_\mathrm{f}t}$ solutions, instead of going through the "positions must be real" reasoning. However, you should continue using the latter reasoning until you're comfortable with the short cut of taking the real part.

Remark: Note that Eq. (9) can be written in the form,

$$
\begin{pmatrix} k+\kappa & -\kappa \\ -\kappa & k+\kappa \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = m\omega^2\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}. \tag{16}
$$

So what we did above was solve for the *eigenvectors* and *eigenvalues* of this matrix. The eigenvectors of a matrix are the special vectors that get carried into a multiple of themselves what acted on by the matrix. And the multiple (which is $m\omega^2$ here) is called the eigenvalue. Such vectors are indeed special, because in general a vector gets both stretched (or shrunk) *and* rotated when acted on by a matrix. Eigenvectors don't get rotated at all. ♣

A third method of solving our coupled-oscillator problem is to solve for x_2 in the first equation in Eq. (1) and plug the result into the second. You will get a big mess of a fourth-order differential equation, but it's solvable by guessing $x_1 = Ae^{i\omega t}$.