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HOMOLOGICAL METHODS IN EQUATIONS OF MATHEMATICAL PHYSICS[1]

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Introduction

Mentioning (co)homology theory in the context of differential equations would sound a bit ridiculous some 30–40 years ago: what could be in common between the essentially analytical, dealing with functional spaces theory of partial differential equations (PDE) and rather abstract and algebraic cohomologies?

Nevertheless, the first meeting of the theories took place in the papers by D. Spencer and his school ([46, 17]), where cohomologies were applied to analysis of overdetermined systems of linear PDE generalizing classical works by Cartan [12]. Homology operators and groups introduced by Spencer (and called the *Spencer operators* and *Spencer homology* nowadays) play a basic role in all computations related to modern homological applications to PDE (see below).

Further achievements became possible in the framework of the geometrical approach to PDE. Originating in classical works by Lie, Bäcklund, Darboux, this approach was developed by A. Vinogradov and his co-workers (see [32, 61]). Treating a differential equation as a submanifold in a suitable jet bundle and using a nontrivial geometrical structure of the latter allows one to apply powerful tools of modern differential geometry to analysis of nonlinear PDE of a general nature. And not only this: speaking the geometrical language makes it possible to clarify underlying algebraic structures, the latter giving better and deeper understanding of the whole picture, [32, Ch. 1] and [58, 26].

It was also A. Vinogradov to whom the next homological application to PDE belongs. In fact, it was even more than an application: in a series of papers [59, 60, 63], he has demonstrated that the adequate language for Lagrangian formalism is a special spectral sequence (the so-called *Vinogradov $\mathcal{C}$-spectral sequence*) and obtained first spectacular results using this language. As it happened, the area of the $\mathcal{C}$-spectral sequence applications is much wider and extends to scalar differential invariants of geometric structures [57], modern field theory [5, 6, 3, 9, 18], etc. A lot of work was also done to specify and generalize Vinogradov's initial results, and here one could mention those by I. M. Anderson [1, 2], R. L. Bryant and P. A. Griffiths [11], D. M. Gessler [16, 15], M. Marvan [39, 40], T. Tsujishita [47, 48, 49], W. M. Tulczyjew [50, 51, 52].

Later, one of the authors found out that another cohomology theory ($\mathcal{C}$-cohomologies) is naturally related to any PDE [24]. The construction uses the fact that the infinite prolongation of any equation is naturally endowed with a flat connection (the *Cartan connection*). To such a connection, one puts into correspondence a differential complex based on the *Frölicher–Nijenhuis bracket* [42, 13]. The group H^0 for this complex coincides with the *symmetry algebra* of the equation at hand, the group H^1 consists of equivalence classes of *deformations* of the equation structure. Deformations of a special type are identified with *recursion operators* [43] for symmetries. On the other hand, this theory seems to be dual to the term E_1 of the Vinogradov $\mathcal{C}$-spectral sequence, while special cochain maps relating the former to the latter are Poisson structures on the equation [25].

Not long ago, the second author noticed ([56]) that both theories may be understood as *horizontal cohomologies* with suitable coefficients. Using this observation combined with the fact that the horizontal de Rham cohomology is equal to the cohomology of the *compatibility complex* for the universal linearization operator, he found a simple proof of the vanishing theorem for the term E_1 (the "k-line theorem") and gave a complete description of $\mathcal{C}$-cohomology in the "2-line situation".

Our short review will not be complete, if we do not mention applications of cohomologies to the singularity theory of solutions of nonlinear PDE ([35]), though this topics is far beyond the scope of these lecture notes.

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The idea to expose the above mentioned material in a lecture course at the Summer School in Levoča belongs to Prof. D. Krupka to whom we are extremely grateful.

We tried to give here a complete and self-contained picture which was not easy under natural time and volume limitations. To make reading easier, we included the Appendix containing basic facts and definitions from homological algebra. In fact, the material needs not 5 days, but 3–4 semester course at the university level, and we really do hope that these lecture notes will help to those who became interested during the lectures. For further details (in the geometry of PDE especially) we refer the reader to the books [32] and [34] (an English translation of the latter is to be published by the American Mathematical Society in 1999). For advanced reading we also strongly recommend the collection [19], where one will find a lot of cohomological applications to modern physics.

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1. Differential calculus over commutative algebras

Throughout this section we shall deal with a commutative algebra A over a field $\Bbbk$ of zero characteristic. For further details we refer the reader to [32, Ch. I] and [26].

1.1. **Linear differential operators.** Consider two A-modules P and Q and the group $\mathrm{Hom}_{\Bbbk}(P,Q)$. Two A-module structures can be introduced into this group:

$$(a\Delta)(p) = a\Delta(p), \quad (a^+\Delta)(p) = \Delta(ap), \tag{1.1}$$

where $a \in A$, $p \in P$, $\Delta \in \mathrm{Hom}_{\Bbbk}(P,Q)$. We also set

$$\delta_a(\Delta) = a^+\Delta - a\Delta, \quad \delta_{a_0,\dots,a_k} = \delta_{a_0} \circ \dots \circ \delta_{a_k},$$

$a_0, \dots, a_k \in A$. Obviously, $\delta_{a,b} = \delta_{b,a}$ and $\delta_{ab} = a^+\delta_b + b\delta_a$ for any $a, b \in A$.

Definition 1.1. A $\Bbbk$-homomorphism $\Delta\colon P \to Q$ is called a *linear differential operator* of order $\leq k$ over the algebra A, if $\delta_{a_0,\dots,a_k}(\Delta) = 0$ for all $a_0, \dots, a_k \in A$.

Proposition 1.1. *If M is a smooth manifold, ξ, ζ are smooth locally trivial vector bundles over M, $A = C^\infty(M)$ and $P = \Gamma(\xi), Q = \Gamma(\zeta)$ are the modules of smooth sections, then any linear differential operator acting from ξ to ζ is an operator in the sense of Definition* 1.1 *and vice versa.*

Exercise 1.1. Prove this fact.

Obviously, the set of all differential operators of order $\leq k$ acting from P to Q is a subgroup in $\mathrm{Hom}_{\Bbbk}(P,Q)$ closed with respect to both multiplications (1.1). Thus we obtain two modules denoted by $\mathrm{Diff}_k(P,Q)$ and $\mathrm{Diff}_k^+(P,Q)$ respectively. Since $a(b^+\Delta) = b^+(a\Delta)$ for any $a, b \in A$ and $\Delta \in \mathrm{Hom}_{\Bbbk}(P,Q)$, this group also carries the structure of an A-bimodule denoted by $\mathrm{Diff}_k^{(+)}(P,Q)$. Evidently, $\mathrm{Diff}_0(P,Q) = \mathrm{Diff}_0^+(P,Q) = \mathrm{Hom}_A(P,Q)$.

It follows from Definition 1.1 that any differential operator of order $\leq k$ is an operator of order $\leq l$ for all $l \geq k$ and consequently we obtain the embeddings $\mathrm{Diff}_k^{(+)}(P,Q) \subset \mathrm{Diff}_l^{(+)}(P,Q)$, which allow us to define the filtered bimodule $\mathrm{Diff}^{(+)}(P,Q) = \bigcup_{k\geq 0} \mathrm{Diff}_k^{(+)}(P,Q)$.

We can also consider the $\mathbb{Z}$-graded module associated to the filtered module $\mathrm{Diff}^{(+)}(P,Q)$: $\mathrm{Smbl}(P,Q) = \bigoplus_{k\geq 0} \mathrm{Smbl}_k(P,Q)$, where $\mathrm{Smbl}_k(P,Q) = \mathrm{Diff}_k^{(+)}(P,Q)/\mathrm{Diff}_{k-1}^{(+)}(P,Q)$, which is called the *module of symbols*. The elements of $\mathrm{Smbl}(P,Q)$ are called *symbols* of operators acting from P to Q. It easily seen that two module structures defined by (1.1) become identical in $\mathrm{Smbl}(P,Q)$.

The following properties of linear differential operator are directly implied by the definition:

Proposition 1.2. *Let P, Q and R be A-modules. Then:*

(1) *If $\Delta_1 \in \mathrm{Diff}_k(P, Q)$ and $\Delta_2 \in \mathrm{Diff}_l(Q, R)$ are two differential operators, then their composition $\Delta_2 \circ \Delta_1$ lies in $\mathrm{Diff}_{k+l}(P, R)$.*

(2) *The maps*

$$\mathrm{i}^{\cdot,+} \colon \mathrm{Diff}_k(P, Q) \to \mathrm{Diff}_k^+(P, Q), \ \ \mathrm{i}^{+,\cdot} \colon \mathrm{Diff}_k^+(P, Q) \to \mathrm{Diff}_k(P, Q)$$

generated by the identical map of $\mathrm{Hom}_{\Bbbk}(P, Q)$ are differential operators of order $\le k$.

Corollary 1.3. *There exists an isomorphism*

$$\mathrm{Diff}^+(P, \mathrm{Diff}^+(Q, R)) = \mathrm{Diff}^+(P, \mathrm{Diff}(Q, R))$$

generated by the operators $\mathrm{i}^{\cdot,+}$ and $\mathrm{i}^{+,\cdot}$.

Introduce the notation $\mathrm{Diff}_k^{(+)}(Q) = \mathrm{Diff}_k^{(+)}(A, Q)$ and define the map $\mathcal{D}_k \colon \mathrm{Diff}_k^+(Q) \to Q$ by setting $\mathcal{D}_k(\Delta) = \Delta(1)$. Obviously, $\mathcal{D}_k$ is an operator of order $\le k$. Let also

$$\psi \colon \mathrm{Diff}_k^+(P, Q) \to \mathrm{Hom}_A(P, \mathrm{Diff}_k^+(Q)), \quad \Delta \mapsto \psi_\Delta, \tag{1.2}$$

be the map defined by $(\psi_\Delta(p))(a) = \Delta(ap)$, $p \in P$, $a \in A$.

Proposition 1.4. *The map* (1.2) *is an isomorphism of A-modules.*

Proof. Compatibility of ψ with A-module structures is obvious. To complete the proof it suffices to note that the correspondence

$$\mathrm{Hom}_A(P, \mathrm{Diff}_k^+(Q)) \ni \varphi \mapsto \mathcal{D}_k \circ \varphi \in \mathrm{Diff}_k^+(P, Q)$$

is inverse to ψ. □

The homomorphism ψ_Δ is called Diff-*associated* to Δ.

Remark 1.1. Consider the correspondence $P \Rightarrow \mathrm{Diff}_k^+(P, Q)$ and for any A-homomorphism $f \colon P \to R$ define the homomorphism

$$\mathrm{Diff}_k^+(f, Q) \colon \mathrm{Diff}_k^+(R, Q) \to \mathrm{Diff}_k^+(P, Q)$$

by setting $\mathrm{Diff}_k^+(f, Q)(\Delta) = \Delta \circ f$. Thus, $\mathrm{Diff}_k^+(\cdot, Q)$ is a contravariant functor from the category of all A-modules to itself. Proposition 1.4 means that this functor is representable and the module $\mathrm{Diff}_k^+(Q)$ is its representative object. Obviously, the same is valid for the functor $\mathrm{Diff}^+(\cdot, Q)$ and the module $\mathrm{Diff}^+(Q)$.

From Proposition 1.4 we also obtain the following

Corollary 1.5. *There exists a unique homomorphism*

$$c_{k,l} = c_{k,l}(P) \colon \mathrm{Diff}_k^+(\mathrm{Diff}_l^+(P)) \to \mathrm{Diff}_{k+l}(P)$$

such that the diagram

$$\begin{array}{ccc} \mathrm{Diff}_k^+(\mathrm{Diff}_l^+(P)) & \xrightarrow{\mathcal{D}_k} & \mathrm{Diff}_l^+(P) \\ {\scriptstyle c_{k,l}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathcal{D}_l} \\ \mathrm{Diff}_{k+l}^+(P) & \xrightarrow{\mathcal{D}_{k+l}} & P \end{array}$$

is commutative.

Proof. It suffices to use the fact that the composition

$$\mathcal{D}_l \circ \mathcal{D}_k : \mathrm{Diff}_k(\mathrm{Diff}_l(P)) \longrightarrow P$$

is an operator of order $\leq k+l$ and to set $c_{k,l} = \psi_{\mathcal{D}_l \circ \mathcal{D}_k}$. $\square$

The map $c_{k,l}$ is called the *gluing homomorphism* and from the definition it follows that $(c_{k,l}(\Delta))(a) = (\Delta(a))(1)$, $\Delta \in \mathrm{Diff}_k^+(\mathrm{Diff}_l^+(P))$, $a \in A$.

Remark 1.2. The correspondence $P \Rightarrow \mathrm{Diff}_k^+(P)$ also becomes a (covariant) functor, if for a homomorphism $f\colon P \to Q$ we define the homomorphism $\mathrm{Diff}_k^+(f)\colon \mathrm{Diff}_k^+(P) \to \mathrm{Diff}_k^+(Q)$ by $\mathrm{Diff}_k^+(f)(\Delta) = f \circ \Delta$. Then the correspondence $P \Rightarrow c_{k,l}(P)$ is a natural transformation of functors $\mathrm{Diff}_k^+(\mathrm{Diff}_l^+(\cdot))$ and $\mathrm{Diff}_{k+l}^+(\cdot)$ which means that for any A-homomorphism $f\colon P \to Q$ the diagram

$$\begin{array}{ccc} \mathrm{Diff}_k^+(\mathrm{Diff}_l^+(P)) & \xrightarrow{\mathrm{Diff}_k^+(\mathrm{Diff}_l^+(f))} & \mathrm{Diff}_k^+(\mathrm{Diff}_l^+(Q)) \\ {\scriptstyle c_{k,l}(P)}\Big\downarrow & & \Big\downarrow{\scriptstyle c_{k,l}(Q)} \\ \mathrm{Diff}_{k+l}^+(P) & \xrightarrow{\mathrm{Diff}_{k+l}^+(f)} & \mathrm{Diff}_{k+l}^+(Q) \end{array}$$

is commutative.

Note also that the maps $c_{k,l}$ are compatible with the natural embeddings $\mathrm{Diff}_k^+(P) \to \mathrm{Diff}_s^+(P)$, $k \leq s$, and thus we can define the gluing $c_{*,*}\colon \mathrm{Diff}^+(\mathrm{Diff}^+(\cdot)) \to \mathrm{Diff}^+(\cdot)$.

1.2. **Multiderivations and the Diff-Spencer complex.** Let $A^{\otimes k} = A \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} A$, k times.

Definition 1.2. A $\Bbbk$-linear map $\nabla\colon A^{\otimes k} \to P$ is called a *skew-symmetric multiderivation* of A with values in an A-module P, if the following conditions hold:

(1) $\nabla(a_1, \ldots, a_i, a_{i+1}, \ldots, a_k) + \nabla(a_1, \ldots, a_{i+1}, a_i, \ldots, a_k) = 0$,
(2) $\nabla(a_1, \ldots, a_{i-1}, ab, a_{i+1}, \ldots, a_k) =$
$a\nabla(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_k) + b\nabla(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_k)$

for all $a, b, a_1, \ldots, a_k \in A$ and any i, $1 \leq i \leq k$.

The set of all skew-symmetric k-derivations forms an A-module denoted by $\mathrm{D}_k(P)$. By definition, $\mathrm{D}_0(P) = P$. In particular, elements of $\mathrm{D}_1(P)$ are called *P-valued derivations* and form a submodule in $\mathrm{Diff}_1(P)$ (but not in the module $\mathrm{Diff}_1^+(P)$!).

There is another, functorial definition of the modules $\mathrm{D}_k(P)$: for any $\nabla \in \mathrm{D}_k(P)$ and $a \in A$ we set $(a\nabla)(a_1, \dots, a_k) = a\nabla(a_1, \dots, a_k)$. Note first that the composition $\gamma_1 \colon \mathrm{D}_1(P) \hookrightarrow \mathrm{Diff}_1(P) \xrightarrow{\mathrm{i}^{\cdot,+}} \mathrm{Diff}_1^+(P)$ is a monomorphic differential operator of order ≤ 1. Assume now that the first-order monomorphic operators $\gamma_i = \gamma_i(P) \colon \mathrm{D}_i(P) \to \mathrm{D}_{i-1}(\mathrm{Diff}_1^+(P))$ were defined for all $i \leq k$. Assume also that all the maps γ_i are natural[4] operators. Consider the composition

$$\mathrm{D}_k(\mathrm{Diff}_1^+(P)) \xrightarrow{\gamma_k} \mathrm{D}_{k-1}(\mathrm{Diff}_1^+(\mathrm{Diff}_1^+(P))) \xrightarrow{\mathrm{D}_{k-1}(c_{1,1})} \mathrm{D}_{k-1}(\mathrm{Diff}_2^+(P)). \tag{1.3}$$

Proposition 1.6. *The following facts are valid:*

(1) $\mathrm{D}_{k+1}(P)$ *coincides with the kernel of the composition* (1.3).
(2) *The embedding* $\gamma_{k+1} \colon \mathrm{D}_{k+1}(P) \hookrightarrow \mathrm{D}_k(\mathrm{Diff}_1^+(P))$ *is a first-order differential operator.*
(3) *The operator* γ_{k+1} *is natural.*

The proof reduces to checking the definitions.

Remark 1.3. We saw above that the A-module $\mathrm{D}_{k+1}(P)$ is the kernel of the map $\mathrm{D}_{k-1}(c_{1,1}) \circ \gamma_k$, the latter being not an A-module homomorphism but a differential operator. Such an effect arises in the following general situation.

Let F be a functor acting on a subcategory of the category of A-modules. We say that F is *$\Bbbk$-linear*, if the corresponding map $\mathrm{F}_{P,Q} \colon \mathrm{Hom}_{\Bbbk}(P, Q) \to \mathrm{Hom}_{\Bbbk}(P, Q)$ is linear over $\Bbbk$ for all P and Q from our subcategory. Then we can introduce a new A-module structure in the the $\Bbbk$-module $\mathrm{F}(P)$ by setting $a \dot{} q = (\mathrm{F}(a))(q)$, where $q \in \mathrm{F}(P)$ and $\mathrm{F}(a) \colon \mathrm{F}(P) \to \mathrm{F}(P)$ is the homomorphism corresponding to the multiplication by a: $p \mapsto ap$, $p \in P$. Denote the module arising in such a way by $\mathrm{F}^{\cdot}(P)$.

Consider two $\Bbbk$-linear functors F and G and a natural transformation Δ: $P \Rightarrow \Delta(P) \in \mathrm{Hom}_{\Bbbk}(\mathrm{F}(P), \mathrm{G}(P))$.

Exercise 1.2. Prove that the natural transformation Δ induces a natural homomorphism of A-modules $\Delta^{\cdot} \colon \mathrm{F}^{\cdot}(P) \to \mathrm{G}^{\cdot}(P)$ and thus its kernel is always an A-module.

From Definition 1.2 on the preceding page it also follows that elements of the modules $\mathrm{D}_k(P)$, $k \geq 2$, may be understood as derivations $\Delta \colon A \to$

[4] This means that for any A-homomorphism $f \colon P \to Q$ one has $\gamma_i(Q) \circ \mathrm{D}_i(f) = \mathrm{D}_{i-1}(\mathrm{Diff}_1^+(f)) \circ \gamma_i(P)$.

$\mathrm{D}_{k-1}(P)$ satisfying $(\Delta(a))(b) = -(\Delta(b))(a)$. We call $\Delta(a)$ the *evaluation* of the multiderivation Δ at the element $a \in A$. Using this interpretation, define by induction on $k+l$ the operation $\wedge\colon \mathrm{D}_k(A) \otimes_A \mathrm{D}_l(P) \to \mathrm{D}_{k+l}(P)$ by setting

$$a \wedge p = ap,\ a \in \mathrm{D}_0(A) = A, p \in \mathrm{D}_0(P) = P,$$

and

$$(\Delta \wedge \nabla)(a) = \Delta \wedge \nabla(a) + (-1)^l \Delta(a) \wedge \nabla. \tag{1.4}$$

Using elementary induction on $k+l$, one can easily prove the following

Proposition 1.7. *The operation $\wedge$ is well defined and satisfies the following properties:*

(1) $\Delta \wedge (\Delta' \wedge \nabla) = (\Delta \wedge \Delta') \wedge \nabla$,

(2) $(a\Delta + a'\Delta') \wedge \nabla = a\Delta \wedge \nabla + a'\Delta' \wedge \nabla$,

(3) $\Delta \wedge (a\nabla + a'\nabla') = a\Delta \wedge \nabla + a'\Delta \wedge \nabla'$,

(4) $\Delta \wedge \Delta' = (-1)^{kk'} \Delta' \wedge \Delta$

for any elements $a, a' \in A$ and multiderivations $\Delta \in \mathrm{D}_k(A)$, $\Delta' \in \mathrm{D}_{k'}(A)$, $\nabla \in \mathrm{D}_l(P)$, $\nabla' \in \mathrm{D}_{l'}(P)$.

Thus, $\mathrm{D}_*(A) = \bigoplus_{k \geq 0} \mathrm{D}_k(A)$ becomes a $\mathbb{Z}$-graded commutative algebra and $\mathrm{D}_*(P) = \bigoplus_{k \geq 0} \mathrm{D}_k(P)$ is a graded $\mathrm{D}_*(A)$-module. The correspondence $P \Rightarrow \mathrm{D}_*(P)$ is a functor from the category of A-modules to the category of graded $\mathrm{D}_*(A)$-modules.

Let now $\nabla \in \mathrm{D}_k(\mathrm{Diff}_l^+(P))$ be a multiderivation. Define

$$(S(\nabla)(a_1, \dots, a_{k-1}))(a) = (\nabla(a_1, \dots, a_{k-1}, a)(1)), \tag{1.5}$$

$a, a_1, \dots, a_{k-1} \in A$. Thus we obtain the map

$$S\colon \mathrm{D}_k(\mathrm{Diff}_l^+(P)) \to \mathrm{D}_{k-1}(\mathrm{Diff}_{l+1}^+(P))$$

which can be represented as the composition

$$\mathrm{D}_k(\mathrm{Diff}_l^+(P)) \xrightarrow{\gamma_k} \mathrm{D}_{k-1}(\mathrm{Diff}_1^+(\mathrm{Diff}_l^+(P))) \xrightarrow{\mathrm{D}_{k-1}(c_{1,l})} \mathrm{D}_{k-1}(\mathrm{Diff}_{l+1}^+(P)). \tag{1.6}$$

Proposition 1.8. *The maps $S\colon \mathrm{D}_k(\mathrm{Diff}_l^+(P)) \to \mathrm{D}_{k-1}(\mathrm{Diff}_{l+1}^+(P))$ possess the following properties:*

(1) *S is a differential operator of order ≤ 1.*

(2) *$S \circ S = 0$.*

Proof. The first statement follows from (1.6), the second one is implied by (1.5). $\square$

Definition 1.3. The operator S is called the Diff-*Spencer operator*. The sequence of operators

$$0 \leftarrow P \xleftarrow{\mathcal{D}} \mathrm{Diff}^{+}(P) \xleftarrow{S} \mathrm{Diff}^{+}(P) \xleftarrow{S} \mathrm{D}_2(\mathrm{Diff}^{+}(P)) \leftarrow \cdots$$

is called the Diff-*Spencer complex*.

1.3. **Jets.** Now we shall deal with the functors $Q \Rightarrow \mathrm{Diff}_k(P, Q)$ and their representability.

Consider an A-module P and the tensor product $A \otimes_{\Bbbk} P$. Introduce an A-module structure in this tensor product by setting

$$a(b \otimes p) = (ab) \otimes p,\ \ a, b \in A,\ \ p \in P,$$

and consider the $\Bbbk$-linear map $\epsilon \colon P \to A \otimes_{\Bbbk} P$ defined by $\epsilon(p) = 1 \otimes p$. Denote by μ^k the submodule in $A \otimes_{\Bbbk} P$ generated by the elements of the form $(\delta_{a_0,\dots,a_k}(\epsilon))(p)$ for all $a_0, \dots, a_k \in A$ and $p \in P$.

Definition 1.4. The quotient module $(A \otimes_{\Bbbk} P)/\mu^k$ is called the *module of k-jets* for P and is denoted by $\mathcal{J}^k(P)$.

We also define the map $j_k \colon P \to \mathcal{J}^k(P)$ by setting $j_k(p) = \epsilon(p) \bmod \mu^k$. Directly from the definition of μ^k it follows that j_k is a differential operator of order $\leq k$.

Proposition 1.9. *There exists a canonical isomorphism*

$$\psi \colon \mathrm{Diff}_k(P, Q) \to \mathrm{Hom}_A(\mathcal{J}^k(P), Q), \quad \Delta \mapsto \psi^{\Delta}, \tag{1.7}$$

defined by the equality $\Delta = \psi^{\Delta} \circ j_k$ and called Jet-*associated to Δ.*

Proof. Note first that since the module $\mathcal{J}^k(P)$ is generated by the elements of the form $j_k(p)$, $p \in P$, the homomorphism ψ^{Δ}, if defined, is unique. To establish existence of ψ^{Δ}, consider the homomorphism

$$\eta \colon \mathrm{Hom}_A(A \otimes_{\Bbbk} P, Q) \to \mathrm{Hom}_{\Bbbk}(P, Q), \quad \eta(\varphi) = \varphi \circ \epsilon.$$

Since φ is an A-homomorphism, one has

$$\delta_a(\eta(\varphi)) = \delta_a(\varphi \circ \epsilon) = \varphi \circ \delta_a(\epsilon) = \eta(\delta_a(\varphi)), \quad a \in A.$$

Consequently, the element $\eta(\varphi)$ is an operator of order $\leq k$ if and only if $\varphi(\mu^k) = 0$, i.e., restricting η to $\mathrm{Diff}_k(P, Q) \subset \mathrm{Hom}_{\Bbbk}(P, Q)$ we obtain the desired isomorphism ψ. $\square$

The proposition proved means that the functor $Q \Rightarrow \mathrm{Diff}_k(P, Q)$ is representable and the module $\mathcal{J}^k(P)$ is its representative object.

Note that the correspondence $P \Rightarrow \mathcal{J}^k(P)$ is a functor itself: if $\varphi \colon P \to Q$ is an A-module homomorphism, we are able to define the homomorphism

$\mathcal{J}^k(\varphi)\colon \mathcal{J}^k(P) \to \mathcal{J}^k(Q)$ by the commutativity condition

$$\begin{array}{ccc} P & \xrightarrow{j_k} & \mathcal{J}^k(P) \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\mathcal{J}^k(\varphi)} \\ Q & \xrightarrow{j_k} & \mathcal{J}^k(Q) \end{array}$$

The universal property of the operator j_k allows us to introduce the natural transformation $c^{k,l}$ of the functors $\mathcal{J}^{k+l}(\cdot)$ and $\mathcal{J}^k(\mathcal{J}^l(\cdot))$ defined by the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{j_l} & \mathcal{J}^l(P) \\ {\scriptstyle j_{k+l}}\big\downarrow & & \big\downarrow{\scriptstyle j_k} \\ \mathcal{J}^{k+l}(P) & \xrightarrow{c^{k,l}} & \mathcal{J}^k(\mathcal{J}^l(P)) \end{array}$$

It is called the *co-gluing homomorphism* and is dual to the gluing one discussed in Remark 1.2 on page 8.

Another natural transformation related to functors $\mathcal{J}^k(\cdot)$ arises from the embeddings $\mu^l \hookrightarrow \mu^k$, $l \geq k$, which generate the projections $\nu_{l,k}\colon \mathcal{J}^l(P) \to \mathcal{J}^k(P)$ dual to the embeddings $\mathrm{Diff}_k(P,Q) \hookrightarrow \mathrm{Diff}_l(P,Q)$. One can easily see that if $f\colon P \to P'$ is an A-module homomorphism, then $\mathcal{J}^k(f) \circ \nu_{l,k} = \nu_{l,k} \circ \mathcal{J}^l(f)$. Thus we obtain the sequence of projections

$$\cdots \longrightarrow \mathcal{J}^k(P) \xrightarrow{\nu_{k,k-1}} \mathcal{J}^{k-1}(P) \longrightarrow \cdots \longrightarrow \mathcal{J}^1(P) \xrightarrow{\nu_{1,0}} \mathcal{J}^0(P) = P$$

and set $\mathcal{J}^\infty(P) = \operatorname{proj\,lim} \mathcal{J}^k(P)$. Since $\nu_{l,k} \circ j_l = j_k$, we can also set $j_\infty = \operatorname{proj\,lim} j_k\colon P \to \mathcal{J}^\infty(P)$.

Let $\Delta\colon P \to Q$ be an operator of order $\leq k$. Then for any $l \geq 0$ we have the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\Delta} & Q \\ {\scriptstyle j_{k+l}}\big\downarrow & & \big\downarrow{\scriptstyle j_l} \\ \mathcal{J}^{k+l}(P) & \xrightarrow{\psi_l^\Delta} & \mathcal{J}^l(Q) \end{array}$$

where $\psi_l^\Delta = \psi^{j_l \circ \Delta}$. Moreover, if $l' \geq l$, then $\nu_{l',l} \circ \psi_{l'}^\Delta = \psi_l^\Delta \circ \nu_{k+l',k+l}$ and we obtain the homomorphism $\psi_\infty^\Delta\colon \mathcal{J}^\infty(P) \to \mathcal{J}^\infty(Q)$.

Note that the co-gluing homomorphism is a particular case of the above construction: $c^{k,l} = \psi_k^{j_l}$. Thus, passing to the inverse limits, we obtain the co-gluing $c^{\infty,\infty}$:

$$\begin{array}{ccc} P & \xrightarrow{j_\infty} & \mathcal{J}^\infty(P) \\ {\scriptstyle j_\infty}\big\downarrow & & \big\downarrow{\scriptstyle j_\infty} \\ \mathcal{J}^\infty(P) & \xrightarrow{c^{\infty,\infty}} & \mathcal{J}^\infty(\mathcal{J}^\infty(P)) \end{array}$$

1.4. **Compatibility complex.** The following construction will play an important role below.

Consider a differential operator $\Delta\colon Q \to Q_1$ of order $\le k$. Without loss of generality we may assume that its Jet-associated homomorphism $\psi^\Delta\colon \mathcal{J}^k(Q) \to Q_1$ is epimorphic. Choose an integer $k_1 \ge 0$ and define Q_2 as the cokernel of the homomorphism $\psi^\Delta_{k_1}\colon \mathcal{J}^{k+k_1}(Q) \to \mathcal{J}^k(Q_1)$,

$$0 \to \mathcal{J}^{k+k_1}(Q) \xrightarrow{\psi^\Delta_{k_1}} \mathcal{J}^{k_1}(Q_1) \to Q_2 \to 0.$$

Denote the composition of the operator $j_{k_1}\colon Q_1 \to \mathcal{J}^{k_1}(Q_1)$ with the natural projection $\mathcal{J}^{k_1}(Q_1) \to Q_2$ by $\Delta_1\colon Q_1 \to Q_2$. By construction, we have

$$\Delta_1 \circ \Delta = \psi^{\Delta_1} \circ j_{k_1} \circ \Delta = \psi^{\Delta_1} \circ \psi^\Delta_{k_1} \circ j_{k+k_1}.$$

Exercise 1.3. Prove that Δ_1 is a compatibility operator for the operator Δ, i.e., for any operator ∇ such that $\nabla \circ \Delta = 0$ and $\operatorname{ord}\nabla \ge k_1$, there exists an operator $\Box$ such that $\nabla = \Box \circ \Delta_1$.

We can now apply the procedure to the operator Δ_1 and some integer k_2 obtaining $\Delta_2\colon Q_2 \to Q_3$, etc. Eventually, we obtain the complex

$$0 \longrightarrow Q \xrightarrow{\Delta} Q_1 \xrightarrow{\Delta_1} Q_2 \xrightarrow{\Delta_2} \cdots \longrightarrow Q_i \xrightarrow{\Delta_i} Q_{i+1} \longrightarrow \cdots$$

which is called the *compatibility complex* of the operator Δ.

1.5. **Differential forms and the de Rham complex.** Consider the embedding $\beta\colon A \to \mathcal{J}^1(A)$ defined by $\beta(a) = aj_1(1)$ and define the module $\Lambda^1 = \mathcal{J}^1(A)/\operatorname{im}\beta$. Let d be the composition of j_1 and the natural projection $\mathcal{J}^1(A) \to \Lambda^1$. Then $d\colon A \to \Lambda^1$ is a differential operator of order ≤ 1 (and, moreover, lies in $\mathrm{D}_1(\Lambda^1)$).

Let us now apply the construction of the previous subsection to the operator d setting all k_i equal to 1 and preserving the notation d for the operators d_i. Then we get the compatibility complex

$$0 \longrightarrow A \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \longrightarrow \cdots \longrightarrow \Lambda^k \xrightarrow{d} \Lambda^{k+1} \longrightarrow \cdots$$

which is called the *de Rham complex* of the algebra A. The elements of Λ^k are called *k-forms* over A.

Proposition 1.10. *For any $k \ge 0$, the module Λ^k is the representative object for the functor $\mathrm{D}_k(\cdot)$.*

Proof. It suffices to compare the definition of Λ^k with the description of $\mathrm{D}_k(P)$ given by Proposition 1.6 on page 9. $\square$

Remark 1.4. In the case $k = 1$, the isomorphism between $\mathrm{Hom}_A(\Lambda^1, \cdot)$ and $\mathrm{D}_1(\cdot)$ can be described more exactly. Namely, from the definition of the operator $d\colon A \to \Lambda^1$ and from Proposition 1.9 on page 11 it follows that any derivation $\nabla\colon A \to P$ is uniquely represented as the composition $\nabla = \varphi^\nabla \circ d$ for some homomorphism $\varphi^\nabla\colon \Lambda^1 \to P$.

As a consequence Proposition 1.10 on the page before, we obtain the following

Corollary 1.11. *The module Λ^k is the k-th exterior power of Λ^1.*

Exercise 1.4. Since $\mathrm{D}_k(P) = \mathrm{Hom}_A(\Lambda^k, P)$, one can introduce the pairing $\langle \cdot, \cdot \rangle\colon \mathrm{D}_k(P) \otimes \Lambda^k \to P$. Prove that the evaluation operation (see p. 10) and the wedge product are mutually dual with respect to this pairing, i.e.,

$$\langle X, da \wedge \omega \rangle = \langle X(a), \omega \rangle$$

for all $X \in \mathrm{D}_{k+1}(P)$, $\omega \in \Lambda^k$, and $a \in A$.

The following proposition establishes the relation of the de Rham differential to the wedge product.

Proposition 1.12 (the Leibniz rule)**.** *For any $\omega \in \Lambda^k$ and $\theta \in \Lambda^l$ one has*

$$d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^k \omega \wedge d\theta.$$

Proof. We first consider the case $l = 0$, i.e., $\theta = a \in A$. To do it, note that the wedge product $\wedge\colon \Lambda^k \otimes_A \Lambda^l \to \Lambda^{k+l}$, due to Proposition 1.10 on the preceding page, induces the natural embeddings of modules $\mathrm{D}_{k+l}(P) \to \mathrm{D}_k(\mathrm{D}_l(P))$. In particular, the embedding $\mathrm{D}_{k+1}(P) \to \mathrm{D}_k(\mathrm{D}_1(P))$ can be represented as the composition

$$\mathrm{D}_{k+l}(P) \xrightarrow{\gamma_{k+1}} \mathrm{D}_k(\mathrm{Diff}_1^+(P)) \xrightarrow{\lambda} \mathrm{D}_k(\mathrm{D}_1(P)),$$

where $(\lambda(\nabla))(a_1, \dots, a_k) = \nabla(a_1, \dots, a_k) - (\nabla(a_1, \dots, a_k))(1)$. In a dual way, the wedge product is represented as

$$\Lambda^k \otimes_A \Lambda^1 \xrightarrow{\lambda'} \mathcal{J}^1(\Lambda^k) \xrightarrow{\psi^d} \Lambda^{k+1},$$

where $\lambda'(\omega \otimes da) = (-1)^k (j_1(\omega a) - j_1(\omega) a)$. Then

$$\begin{aligned}(-1)^k \wedge \omega da = (-1)^k \psi^d(\lambda'(\omega \otimes da))& \\ = \psi^d(j_1(\omega a) - j_1(\omega) a) &= d(\omega a) - d(\omega) a.\end{aligned}$$

The general case is implied by the identity

$$d(\omega \wedge da) = (-1)^k d(d(\omega a) - d\omega \cdot a) = (-1)^{k+1} d(d\omega \cdot a).$$

$\square$

Let us return back to Proposition 1.10 on page 13 and consider the A-bilinear pairing

$$\langle\cdot,\cdot\rangle\colon \mathrm{D}_k(P)\otimes_A \Lambda^k \to P$$

again. Take a form $\omega \in \Lambda^k$ and a derivation $X \in \mathrm{D}_1(A)$. Using the definition of the wedge product in $\mathrm{D}_*(P)$ (see equality (1.4) on page 10), we can set

$$\langle \Delta, i_X\omega\rangle = (-1)^{k-1}\langle X\wedge\Delta,\omega\rangle \tag{1.8}$$

for an arbitrary $\Delta \in \mathrm{D}_{k-1}(P)$.

Definition 1.5. The operation $\mathrm{i}_X\colon \Lambda^k \to \Lambda^{k-1}$ defined by (1.8) is called the *internal product*, or *contraction*.

Proposition 1.13. *For any $X, Y \in \mathrm{D}_1(A)$ and $\omega\in\Lambda^k, \theta\in\Lambda^l$ one has*

(1) $\mathrm{i}_X(\omega\wedge\theta) = \mathrm{i}_X(\omega)\wedge\theta + (-1)^k\omega\wedge\mathrm{i}_X(\theta)$,

(2) $\mathrm{i}_X\circ\mathrm{i}_Y = -\mathrm{i}_Y\circ\mathrm{i}_X$

In other words, internal product is a derivation of the $\mathbb{Z}$-graded algebra $\Lambda^* = \bigoplus_{k\geq 0}\Lambda^k$ of degree -1 and $\mathrm{i}_X, \mathrm{i}_Y$ commute as graded maps.

Consider a derivation $X \in \mathrm{D}_1(A)$ and set

$$\mathrm{L}_X(\omega) = [\mathrm{i}_X, d](\omega) = \mathrm{i}_X(d(\omega)) + d(\mathrm{i}_X(\omega)),\ \ \omega\in\Lambda^*. \tag{1.9}$$

Definition 1.6. The operation $\mathrm{L}_X\colon \Lambda^*\to\Lambda^*$ defined by 1.9 is called the *Lie derivative.*

Directly from the definition one obtains the following properties of Lie derivatives:

Proposition 1.14. *Let $X, Y \in \mathrm{D}_1(A)$, $\omega,\theta\in\Lambda^*$, $a\in A$, $\alpha,\beta\in\Bbbk$. Then the following identities are valid:*

(1) $\mathrm{L}_{\alpha X+\beta Y} = \alpha\mathrm{L}_X + \beta\mathrm{L}_Y$,

(2) $\mathrm{L}_{aX} = a\mathrm{L}_X + da\wedge\mathrm{i}_X$,

(3) $\mathrm{L}_X(\omega\wedge\theta) = \mathrm{L}_X(\omega)\wedge\theta + \omega\wedge\mathrm{L}_X(\theta)$,

(4) $[d,\mathrm{L}_X] = d\circ\mathrm{L}_X - \mathrm{L}_X\circ d = 0$,

(5) $\mathrm{L}_{[X,Y]} = [\mathrm{L}_X,\mathrm{L}_Y]$, *where* $[X,Y] = X\circ Y - Y\circ X$,

(6) $\mathrm{i}_{[X,Y]} = [\mathrm{L}_X,\mathrm{i}_Y] = [\mathrm{i}_X,\mathrm{L}_Y]$.

To conclude this subsection, we present another description of the Diff-Spencer complex. Recall Remark 1.3 on page 9 and introduce the "dotted" structure into the modules $\mathrm{D}_k(\mathrm{Diff}^+_l(P))$ and note that $\mathrm{Diff}^+_l(P)^{\cdot} = \mathrm{Diff}_l(P)$. Define the isomorphism

$$\zeta\colon (\mathrm{D}_k(\mathrm{Diff}^+))^{\cdot}(P) = \mathrm{Hom}_A(\Lambda^k,\mathrm{Diff}^+)^{\cdot} = \mathrm{Diff}^+(\Lambda^k,P)^{\cdot} = \mathrm{Diff}(\Lambda^k,P).$$

Then we have

Proposition 1.15. *The above defined map ζ generates the isomorphism of complexes*

$$\begin{array}{ccccccc}
\cdots \longleftarrow & (\mathrm{D}_{k-1}(\mathrm{Diff}^+))^{\cdot}(P) & \xleftarrow{S^{\cdot}} & (\mathrm{D}_k(\mathrm{Diff}^+))^{\cdot}(P) & \longleftarrow \cdots \\
& \zeta\Big\downarrow & & \zeta\Big\downarrow & \\
\cdots \longleftarrow & \mathrm{Diff}(\Lambda^{k-1}, P) & \xleftarrow{v} & \mathrm{Diff}(\Lambda^k, P) & \longleftarrow \cdots
\end{array}$$

where $S^{\cdot}$ is the operator induced on "dotted" modules by the Diff-*Spencer operator, while $v(\nabla) = \nabla \circ d$.*

1.6. **Left and right differential modules.** From now on till the end of this section we shall assume the modules under consideration to be projective.

Definition 1.7. An A-module P is called a *left differential module*, if there exists an A-module homomorphism $\lambda\colon P \to \mathcal{J}^\infty(P)$ satisfying $\nu_{\infty,0}\circ\lambda = \mathrm{id}_P$ and such that the diagram

$$\begin{array}{ccc}
P & \xrightarrow{\lambda} & \mathcal{J}^\infty(P) \\
\lambda\Big\downarrow & & \Big\downarrow \mathcal{J}^\infty(\lambda) \\
\mathcal{J}^\infty(P) & \xrightarrow{c^{\infty,\infty}} & \mathcal{J}^\infty(\mathcal{J}^\infty(P))
\end{array}$$

is commutative.

Lemma 1.16. *Let P be a left differential module. Then for any differential operator $\Delta\colon Q_1 \to Q_2$ there exists an operator $\Delta_P\colon Q_1 \otimes_A P \to Q_2 \otimes_A P$ satisfying $(\mathrm{id}_Q)_P = \mathrm{id}_{Q\otimes_A P}$ for $Q = Q_1 = Q_2$ and*

$$(\Delta_2 \circ \Delta_1)_P = (\Delta_2)_P \circ (\Delta_1)_P$$

for any operators $\Delta_1\colon Q_1 \to Q_2$, $\Delta_2\colon Q_2 \to Q_3$.

Proof. Consider the map

$$\overline{\Delta}\colon Q_1 \otimes_A (A \otimes_{\Bbbk} P) \to Q_2 \otimes_A P, \quad q \otimes a \otimes p \mapsto \Delta(aq) \otimes p.$$

Since

$$\overline{\Delta}(q \otimes \delta_a(\epsilon)(p)) = \overline{\delta_a\Delta}(q \otimes 1 \otimes p), \quad p \in P, \quad q \in Q_1, \quad a \in A,$$

the map

$$\xi_P(\Delta)\colon Q_1 \otimes_A \mathcal{J}^\infty(P) \to Q_2 \otimes_A P$$

is well defined. Set now the operator Δ_P to be the composition

$$Q_1 \otimes_A P \xrightarrow{\mathrm{id}\otimes\lambda} Q_1 \otimes_A \mathcal{J}^\infty(P) \xrightarrow{\xi_P(\Delta)} Q_2 \otimes_A P,$$

which is a k-th order differential operator in an obvious way. Evidently, $(\mathrm{id}_Q)_P = \mathrm{id}_{Q\otimes_A P}$.

Now,

$$\begin{aligned}(\Delta_2 \circ \Delta_1)_P &= \xi_P(\Delta_2 \circ \Delta_1) \circ (\mathrm{id} \otimes \lambda)\\ &= \xi_P(\Delta_2) \circ \xi_{\mathcal{J}^\infty(P)}(\Delta_1) \circ (\mathrm{id} \otimes c^{\infty,\infty}) \circ (\mathrm{id} \otimes \lambda)\\ &= \xi_P(\Delta_2) \circ \xi_{\mathcal{J}^\infty(P)}(\Delta_1) \circ (\mathrm{id} \otimes \mathcal{J}^\infty(\lambda)) \circ (\mathrm{id} \circ \lambda)\\ &= \xi_P(\Delta_2) \circ (\mathrm{id} \otimes \lambda) \circ \xi_P(\Delta_1) \circ (\mathrm{id} \otimes \lambda) = (\Delta_2)_P \circ (\Delta_1)_P,\end{aligned}$$

which proves the second statement. □

Note that the lemma proved shows in particular that any left differential module is a left module over the algebra $\mathrm{Diff}(A)$ which justifies our terminology.

Due to the above result, any complex of differential operators $\cdots \to Q_i \to Q_{i+1} \to \cdots$ and a left differential module P generate the complex $\cdots \to Q_i \otimes_A P \to Q_{i+1} \otimes_A P \to \cdots$ "with coefficients" in P. In particular, since the co-gluing $c^{\infty,\infty}$ is in an obvious way co-associative, i.e., the diagram

$$\begin{array}{ccc}\mathcal{J}^\infty(P) & \xrightarrow{c^{\infty,\infty}(P)} & \mathcal{J}^\infty(\mathcal{J}^\infty(P))\\ {\scriptstyle c^{\infty,\infty}(P)}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{J}^\infty(c^{\infty,\infty}(P))}\\ \mathcal{J}^\infty(\mathcal{J}^\infty(P)) & \xrightarrow{c^{\infty,\infty}(\mathcal{J}^\infty(P))} & \mathcal{J}^\infty(\mathcal{J}^\infty(\mathcal{J}^\infty(P)))\end{array}$$

is commutative, $\mathcal{J}^\infty(P)$ is a left differential module with $\lambda = c^{\infty,\infty}$. Consequently, we can consider the de Rham complex with coefficients in $\mathcal{J}^\infty(P)$:

$$\begin{aligned}0 \to P \xrightarrow{j_\infty} \mathcal{J}^\infty(P) \to \Lambda^1 \otimes_A \mathcal{J}^\infty(P) \to \cdots&\\ \cdots \to \Lambda^i \otimes_A \mathcal{J}^\infty(P) \to \Lambda^{i+1} \otimes_A \mathcal{J}^\infty(P) \to \cdots&\end{aligned}$$

which is the inverse limit for the Jet-*Spencer complexes* of P

$$\begin{aligned}0 \to P \xrightarrow{j_k} \mathcal{J}^k(P) \xrightarrow{S} \Lambda^1 \otimes_A \mathcal{J}^{k-1}(P) \xrightarrow{S} \cdots&\\ \cdots \xrightarrow{S} \Lambda^i \otimes_A \mathcal{J}^{k-i}(P) \xrightarrow{S} \Lambda^{i+1} \otimes_A \mathcal{J}^{k-i-1}(P) \to \cdots,&\end{aligned}$$

where $S(\omega \otimes j_{k-i}(p)) = d\omega \otimes j_{k-i-1}(p)$.

Let $\Delta\colon P \to Q$ be a differential operator and $\psi^\Delta_\infty\colon \mathcal{J}^\infty(P) \to \mathcal{J}^\infty(Q)$ be the corresponding homomorphism. The kernel $E_\Delta = \ker \psi^\Delta_\infty$ inherits the left differential module structure of $\mathcal{J}^\infty(P)$ and we can consider the de Rham complex with coefficients in E_Δ:

$$0 \to E_\Delta \to \Lambda^1 \otimes_A E_\Delta \to \cdots \to \Lambda^i \otimes_A E_\Delta \to \Lambda^{i+1} \otimes_A E_\Delta \to \cdots \quad (1.10)$$

which is called the Jet-*Spencer complex* of the operator Δ.

Now we shall introduce the concept dual to that of left differential modules.

Definition 1.8. An A-module P is called a *right differential module*, if there exists an A-module homomorphism $\rho\colon \mathrm{Diff}^+(P) \to P$ that satisfies the equality $\rho\Big|_{\mathrm{Diff}_0^+(P)} = \mathrm{id}_P$ and makes the diagram

$$\begin{tikzcd} \mathrm{Diff}^+(\mathrm{Diff}^+(P)) \arrow[r, "c_{\infty,\infty}"] \arrow[d, "\mathrm{Diff}^+(\rho)"'] & \mathrm{Diff}^+(P) \arrow[d, "\rho"] \\ \mathrm{Diff}^+(P) \arrow[r, "\rho"] & P \end{tikzcd}$$

commutative.

Lemma 1.17. *Let P be a right differential module. Then for any differential operator $\Delta\colon Q_1 \to Q_2$ of order $\le k$ there exists an operator*

$$\Delta^P\colon \mathrm{Hom}_A(Q_2, P) \to \mathrm{Hom}_A(Q_1, P)$$

of order $\le k$ satisfying $\mathrm{id}_Q^P = \mathrm{id}_{\mathrm{Hom}_A(Q,P)}$ for $Q = Q_1 = Q_2$ and

$$(\Delta_2 \circ \Delta_1)^P = \Delta_1^P \circ \Delta_2^P$$

for any operators $\Delta_1\colon Q_1 \to Q_2$, $\Delta_2\colon Q_2 \to Q_3$.

Proof. Let us define the action of Δ^P by setting $\Delta^P(f) = \rho \circ \psi_{f\circ\Delta}$, where $f \in \mathrm{Hom}_A(Q_2, P)$. Obviously, this is a k-th order differential operator and $\mathrm{id}_Q^P = \mathrm{id}_{\mathrm{Hom}_A(Q,P)}$. Now,

$$\begin{aligned} (\Delta_2 \circ \Delta_1)^P &= \rho \circ \psi_{f\circ\Delta_2\circ\Delta_1} = \rho \circ c_{\infty,\infty} \circ \mathrm{Diff}^+(\psi_{f\circ\Delta_2}) \circ \psi_{\Delta_1} \\ &= \rho \circ \mathrm{Diff}^+(\rho \circ \psi_{f\circ\Delta_2}) \circ \psi_{\Delta_1} = \rho \circ \mathrm{Diff}^+(\Delta_2^P(f)) \circ \psi_{\Delta_1} \\ &= \Delta_1^P(\Delta_2^P(f)). \end{aligned}$$

Hence, $(\cdot)^P$ preserves composition. □

From the lemma proved it follows that any right differential module is a right module over the algebra $\mathrm{Diff}(A)$.

Let $\cdots \to Q_i \xrightarrow{\Delta_i} Q_{i+1} \to \cdots$ be a complex of differential operators and P be a right differential module. Then, by Lemma 1.17, we can construct the dual complex $\cdots \leftarrow \mathrm{Hom}_A(Q_i, P) \xleftarrow{\Delta_i^P} \mathrm{Hom}_A(Q_{i+1}, P) \leftarrow \cdots$ with coefficients in P. Note that the Diff-Spencer complex is a particular case of this construction. In fact, due to properties of the homomorphism $c_{\infty,\infty}$ the module $\mathrm{Diff}^+(P)$ is a right differential module with $\rho = c_{\infty,\infty}$. Applying Lemma 1.17 to the de Rham complex, we obtain the Diff-Spencer complex.

Note also that if $\Delta\colon P \to Q$ is a differential operator, then the cokernel C_Δ of the homomorphism $\psi_\Delta^\infty\colon \mathrm{Diff}^+(P) \to \mathrm{Diff}^+(Q)$ inherits the right differential module structure of $\mathrm{Diff}^+(Q)$. Thus we can consider the complex

$$0 \leftarrow \operatorname{coker}\Delta \xleftarrow{\mathcal{D}} C_\Delta \leftarrow \mathrm{D}_1(C_\Delta) \leftarrow \cdots \leftarrow \mathrm{D}_i(C_\Delta) \leftarrow \mathrm{D}_{i+1}(C_\Delta) \leftarrow \cdots$$

dual to the de Rham complex with coefficients in C_Δ. It is called the Diff-*Spencer complex* of the operator Δ.

1.7. **The Spencer cohomology.** Consider an important class of commutative algebras.

Definition 1.9. An algebra A is called *smooth*, if the module Λ^1 is projective and of finite type.

In this section we shall work over a smooth algebra A.

Take two Diff-Spencer complexes, of orders k and $k-1$, and consider their embedding

$$\begin{array}{ccccccccc}
0 & \longleftarrow & P & \longleftarrow & \mathrm{Diff}_k^+(P)) & \longleftarrow & \mathrm{D}_1(\mathrm{Diff}_{k-1}^+(P)) & \longleftarrow & \cdots \\
 & & \Big\| & & \Big\uparrow & & \Big\uparrow & & \\
0 & \longleftarrow & P & \longleftarrow & \mathrm{Diff}_{k-1}^+(P)) & \longleftarrow & \mathrm{D}_1(\mathrm{Diff}_{k-2}^+(P)) & \longleftarrow & \cdots
\end{array}$$

Then, if the algebra A is smooth, the direct sum of the corresponding quotient complexes is of the form

$$0 \leftarrow \mathrm{Smbl}(A, P) \xleftarrow{\delta} \mathrm{D}_1(\mathrm{Smbl}(A, P)) \xleftarrow{\delta} \mathrm{D}_2(\mathrm{Smbl}(A, P)) \leftarrow \cdots$$

By standard reasoning, exactness of this complex implies that of Diff-complexes.

Exercise 1.5. Prove that the operators δ are A-homomorphisms.

Let us describe the structure of the modules $\mathrm{Smbl}(A, P)$. For the time being, use the notation $D = \mathrm{D}_1(A)$. Consider the homomorphism $\alpha_k\colon P \otimes_A S^k(D) \to \mathrm{Smbl}_k(A, P)$ defined by

$$\alpha_k(p \otimes \nabla_1 \cdot \cdots \cdot \nabla_k) = \mathrm{smbl}_k(\Delta), \quad \Delta(a) = (\nabla_1 \circ \cdots \circ \nabla_k)(a)p,$$

where $a \in A$, $p \in P$, and $\mathrm{smbl}_k\colon \mathrm{Diff}_k(A, P) \to \mathrm{Smbl}_k(A, P)$ is the natural projection.

Lemma 1.18. *If A is a smooth algebra, the homomorphism α_k is an isomorphism.*

Proof. Consider a differential operator $\Delta\colon A \to P$ of order $\leq k$. Then the map $s_\Delta\colon A^{\otimes k} \to P$ defined by $s_\Delta(a_1, \ldots, a_k) = \delta_{a_1,\ldots,a_k}(\Delta)$ is a symmetric multiderivation and thus the correspondence $\Delta \mapsto s_\Delta$ generates a homomorphism

$$\mathrm{Smbl}_k(A, P) \to \mathrm{Hom}_A(S^k(\Lambda^1), P) = S^k(D) \otimes_A P, \tag{1.11}$$

which, as it can be checked by direct computation, is inverse to α_k. Note that the second equality in (1.11) is valid because A is a smooth algebra. $\square$

Exercise 1.6. Prove that the module $\mathrm{Smbl}_k(P,Q)$ is isomorphic to the module $S^k(D)\otimes_A \mathrm{Hom}_A(P,Q)$.

Exercise 1.7. Dualize Lemma 1.18 on the preceding page. Namely, prove that the kernel of the natural projection $\nu_{k,k-1}\colon \mathcal{J}^k(P)\to\mathcal{J}^{k-1}(P)$ is isomorphic to $S^k(\Lambda^1)\otimes_A P$, with the isomorphism $\alpha^k\colon S^k(\Lambda^1)\otimes_A P\to\ker\nu_{k,k-1}$ given by

$$\alpha^k(da_1\cdot\ldots\cdot da_k\otimes p)=\delta_{a_1,\ldots,a_k}(j_k)(p),\qquad p\in P.$$

Thus we obtain:

$$\mathrm{D}_i(\mathrm{Smbl}_k(P))=\mathrm{Hom}_A(\Lambda^i,P\otimes_A S^k(D))=P\otimes_A S^k(D)\otimes_A\Lambda^i(D).$$

But from the definition of the Spencer operator it easily follows that the action of the operator

$$\delta\colon P\otimes_A S^k(D)\otimes_A\Lambda^i(D)\to P\otimes_A S^{k+1}(D)\otimes_A\Lambda^{i-1}(D)$$

is expressed by

$$\delta(p\otimes\sigma\otimes\nabla_1\wedge\cdots\wedge\nabla_i)=\sum_{l=1}^{i}(-1)^{l+1}p\otimes\sigma\cdot\nabla_l\otimes\nabla_1\wedge\cdots\wedge\hat{\nabla}_l\wedge\cdots\wedge\nabla_i$$

where $p\in P$, $\sigma\in S^k(D)$, $\nabla_l\in D$ and the "hat" means that the corresponding term is omitted. Thus we see that the operator δ coincides with the Koszul differential (see the Appendix) which implies exactness of Diff-Spencer complexes.

The Jet-Spencer complexes are dual to them and consequently, in the situation under consideration, are exact as well. This can also be proved independently by considering two Jet-Spencer complexes of orders k and $k-1$ and their projection

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P & \longrightarrow & \mathcal{J}^k(P)) & \longrightarrow & \Lambda^1\otimes_A\mathcal{J}^{k-1}(P) & \longrightarrow & \cdots\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P & \longrightarrow & \mathcal{J}^{k-1}(P)) & \longrightarrow & \Lambda^1\otimes_A\mathcal{J}^{k-2}(P) & \longrightarrow & \cdots
\end{array}$$

Then the corresponding kernel complexes are of the form

$$0\to S^k(\Lambda^1)\otimes_A P\xrightarrow{\delta}\Lambda^1\otimes_A S^{k-1}(\Lambda^1)\otimes_A P\xrightarrow{\delta}\Lambda^2\otimes_A S^{k-2}(\Lambda^1)\otimes_A P\to\cdots$$

and are called the *δ-Spencer complexes* of P. These are complexes of A-homomorphisms. The operator

$$\delta\colon\Lambda^s\otimes_A S^{k-s}(\Lambda^1)\otimes_A P\to\Lambda^{s+1}\otimes_A S^{k-s-1}(\Lambda^1)\otimes_A P$$

is defined by $\delta(\omega \otimes u \otimes p) = (-1)^s \omega \wedge i(u) \otimes p$, where $i\colon S^{k-s}(\Lambda^1) \to \Lambda^1 \otimes S^{k-s-1}(\Lambda^1)$ is the natural inclusion. Dropping the multiplier P we get the de Rham complexes with polynomial coefficients. This proves that the δ-Spencer complexes and, therefore, the Jet-Spencer complexes are exact.

Thus we have the following

Theorem 1.19. *If A is a smooth algebra, then all* Diff*-Spencer complexes and* Jet*-Spencer complexes are exact.*

Now, let us consider an operator $\Delta\colon P \to P_1$ of order $\leq k$. Our aim is to compute the Jet-Spencer cohomology of Δ, i.e., the cohomology of the complex (1.10) on page 17.

Definition 1.10. A complex of $\mathcal{C}$-differential operators $\cdots \longrightarrow P_{i-1} \xrightarrow{\Delta_i} P_i \xrightarrow{\Delta_{i+1}} P_{i+1} \longrightarrow \cdots$ is called *formally exact*, if the complex

$$\cdots \longrightarrow \bar{\mathcal{J}}^{k_i+k_{i+1}+l}(P_{i-1}) \xrightarrow{\varphi_{\Delta_i}^{k_i+k_{i+1}+l}} \bar{\mathcal{J}}^{k_{i+1}+l}(P_i) \xrightarrow{\varphi_{\Delta_{i+1}}^{k_{i+1}+l}} \bar{\mathcal{J}}^{l}(P_{i+1}) \longrightarrow \cdots,$$

with $\operatorname{ord}\Delta_j \leq k_j$, is exact for any l.

Theorem 1.20. *Jet-Spencer cohomology of Δ coincides with the cohomology of any formally exact complex of the form*

$$0 \longrightarrow P \xrightarrow{\Delta} P_1 \longrightarrow P_2 \longrightarrow P_3 \longrightarrow \cdots$$

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Lambda^2 \otimes \mathcal{J}^\infty(P) & \longrightarrow & \Lambda^2 \otimes \mathcal{J}^\infty(P_1) & \longrightarrow & \Lambda^2 \otimes \mathcal{J}^\infty(P_2) & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle \bar{d}} & & \uparrow{\scriptstyle \bar{d}} & & \uparrow{\scriptstyle \bar{d}} & & \\
0 & \longrightarrow & \Lambda^1 \otimes \mathcal{J}^\infty(P) & \longrightarrow & \Lambda^1 \otimes \mathcal{J}^\infty(P_1) & \longrightarrow & \Lambda^1 \otimes \mathcal{J}^\infty(P_2) & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle \bar{d}} & & \uparrow{\scriptstyle \bar{d}} & & \uparrow{\scriptstyle \bar{d}} & & \\
0 & \longrightarrow & \mathcal{J}^\infty(P) & \longrightarrow & \mathcal{J}^\infty(P_1) & \longrightarrow & \mathcal{J}^\infty(P_2) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the i-th column is the de Rham complex with coefficients in the left differential module $\mathcal{J}^\infty(P_i)$. The horizontal maps are induced by the operators Δ_i. All the sequences are exact except for the terms in the left column and the bottom row. Now the standard spectral sequence arguments (see the Appendix) completes the proof. $\square$

Our aim now is to prove that in a sense all compatibility complexes are formally exact. To this end, let us discuss the notion of involutiveness of a differential operator.

The map $\psi_l^{\Delta}\colon \mathcal{J}^{k+l}(P) \to \mathcal{J}^l(P_1)$ gives rise to the map

$$\mathrm{smbl}_{k,l}(\Delta)\colon S^{k+l}(\Lambda^1) \otimes P \to S^l(\Lambda^1) \otimes P_1$$

called the *l-th prolongation of the symbol of* Δ.

Exercise 1.8. Check that 0-th prolongation map $\mathrm{smbl}_{k,0}\colon \mathrm{Diff}_k(P, P_1) \to \mathrm{Hom}(S^k(\Lambda^1) \otimes P, P_1)$ coincides with the natural projection of differential operators to their symbols, $\mathrm{smbl}_k\colon \mathrm{Diff}_k(P, P_1) \to \mathrm{Smbl}_k(P, P_1)$.

Consider the *symbolic module* $g^{k+l} = \ker \mathrm{smbl}_{k,l}(\Delta) \subset S^{k+l}(\Lambda^1) \otimes P$ of the operator Δ. It is easily shown that the subcomplex of the δ-Spencer complex

$$0 \longrightarrow g^{k+l} \xrightarrow{\delta} \Lambda^1 \otimes g^{k+l-1} \xrightarrow{\delta} \Lambda^2 \otimes g^{k+l-2} \xrightarrow{\delta} \cdots \tag{1.12}$$

is well defined. The cohomology of this complex in the term $\Lambda^i \otimes g^{k+l-i}$ is denoted by $H^{k+l,i}(\Delta)$ and is said to be *δ-Spencer cohomology of the operator* Δ.

Exercise 1.9. Prove that $H^{k+l,0}(\Delta) = H^{k+l,1}(\Delta) = 0$.

The operator Δ is called *involutive* (in the sense of Cartan), if $H^{k+l,i}(\Delta) = 0$ for all $i \geq 0$.

Definition 1.11. An operator Δ is called *formally integrable*, if for all l modules $E_{\Delta}^l = \ker \psi_{\Delta}^l \subset \mathcal{J}^{k+l}(P)$ and g^{k+l} are projective and the natural mappings $E_{\Delta}^l \to E_{\Delta}^{l-1}$ are surjections.

Till the end of this section we shall assume all the operators under consideration to be formally integrable.

Theorem 1.21. *If the operator Δ is involutive, then the compatibility complex of Δ is formally exact for all positive integers k_1, k_2, k_3, $\dots$.*

Proof. Suppose that the compatibility complex of Δ

$$P \xrightarrow{\Delta} P_1 \xrightarrow{\Delta_1} P_2 \xrightarrow{\Delta_2} \cdots$$

is formally exact in terms $P_1, P_2, \dots, P_{i-1}$. The commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & g^K & \longrightarrow & S^K \otimes P & \longrightarrow & S^{K-k} \otimes P_1 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_\Delta^{K-k} & \longrightarrow & \mathcal{J}^K(P) & \longrightarrow & \mathcal{J}^{K-k}(P_1) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_\Delta^{K-k-1} & \longrightarrow & \mathcal{J}^{K-1}(P) & \longrightarrow & \mathcal{J}^{K-k-1}(P_1) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & S^{k_i} \otimes P_i & \longrightarrow & P_{i+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathcal{J}^{k_i}(P_i) & \longrightarrow & P_{i+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathcal{J}^{k_i-1}(P_i) & \longrightarrow & 0 & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}$$

where $S^j = S^j(\Lambda^1)$, $K = k + k_1 + k_2 + \cdots + k_i$, shows that the complex

$$0 \to g^K \to S^K \otimes P \to S^{K-k} \otimes P_1 \to \cdots \to S^{k_i} \otimes P_i$$

is exact.

What we must to prove is that the sequences

$$S^{k_{i-1}+k_i+l} \otimes P_{i-1} \to S^{k_i+l} \otimes P_i \to S^l \otimes P_{i+1}$$

are exact for all $l \geq 1$. The proof is by induction on l, with the inductive step involving the standard spectral sequence arguments applied to the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S^l \otimes P_{i+1} & \xrightarrow{\delta} & \Lambda^1 \otimes S^{l-1} \otimes P_{i+1} & \xrightarrow{\delta} & \Lambda^2 \otimes S^{l-2} \otimes P_{i+1} & \xrightarrow{\delta} & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & S^{k_i+l} \otimes P_i & \xrightarrow{\delta} & \Lambda^1 \otimes S^{k_i+l-1} \otimes P_i & \xrightarrow{\delta} & \Lambda^2 \otimes S^{k_i+l-2} \otimes P_i & \xrightarrow{\delta} & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & S^{K+l} \otimes P_0 & \xrightarrow{\delta} & \Lambda^1 \otimes S^{K+l-1} \otimes P_0 & \xrightarrow{\delta} & \Lambda^2 \otimes S^{K+l-2} \otimes P_0 & \xrightarrow{\delta} & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & g^{K+l} & \xrightarrow{\delta} & \Lambda^1 \otimes g^{K+l-1} & \xrightarrow{\delta} & \Lambda^2 \otimes g^{K+l-2} & \xrightarrow{\delta} & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

□

Example 1.1. For the de Rham differential $d\colon A \to \Lambda^1$ the symbolic modules g^l are trivial. Hence, the de Rham differential is involutive and, therefore, the de Rham complex is formally exact.

Example 1.2. Consider the geometric situation and suppose that the manifold M is a (pseudo-)Riemannian manifold. For an integer p consider the operator $\Delta = d{*}d\colon \Lambda^p \to \Lambda^{n-p}$, where $*$ is the Hodge star operator on the modules of differential forms. Let us show that the complex

$$\bar{\Lambda}^p \xrightarrow{\Delta} \bar{\Lambda}^{n-p} \xrightarrow{d} \bar{\Lambda}^{n-p+1} \xrightarrow{d} \Lambda^{n-p+2} \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n \longrightarrow 0$$

is formally exact and, thus, is the compatibility complex for the operator Δ. In view of the previous example we must prove that the image of the map $\mathrm{smbl}(\Delta)\colon S^{l+2} \otimes \Lambda^p \to S^l \otimes \Lambda^{n-p}$ coincides with the image of the map $\mathrm{smbl}(d)\colon S^{l+1} \otimes \Lambda^{n-p-1} \to S^l \otimes \Lambda^{n-p}$ for all $l \geq 0$. Since $\Delta * = d{*}d{*} = d({*}d{*} + d)$, it is sufficient to show that the map $\mathrm{smbl}({*}d{*} + d)\colon S^{l+1} \otimes (\Lambda^{n-p+1} \oplus \Lambda^{n-p-1}) \to S^l \otimes \Lambda^{n-p}$ is an epimorphism. Consider $\mathrm{smbl}(L)\colon S^l \otimes \Lambda^{n-p} \to S^l \otimes \Lambda^{n-p}$, where $L = ({*}d{*} + d)({*}d{*} \pm d)$ is the Laplace operator. From coordinate considerations it easily follows that the symbol of the Laplace operator is epimorphic, and so the symbol of the operator ${*}d{*} + d$ is also epimorphic.

The condition of involutiveness is not necessary for the formal exactness of the compatibility complex due to the following

Theorem 1.22 (δ-Poincaré lemma)**.** *If the algebra A is Noetherian, then for any operator Δ there exists an integer $l_0 = l_0(m, n, k)$, where $m = \operatorname{rank} P$, such that $H^{k+l,i}(\Delta) = 0$ for $l \geq l_0$ and $i \geq 0$.*

Proof can be found, e.g., in [32, 10]. Thus, from the proof of Theorem 1.21 on page 22 we see that for sufficiently large integer k_1 the compatibility complex is formally exact for any operator Δ.

We shall always assume that compatibility complexes are formally exact.

1.8. **Geometrical modules.** There are several directions to generalize or specialize the above described theory. Probably, the most important one, giving rise to various interesting specializations, is associated with the following concept.

Definition 1.12. An abelian subcategory $\mathcal{M}(A)$ of the category of all A-modules is said to be *differentially closed*, if

(1) it is closed under tensor product over A,
(2) it is closed under the action of the functors $\mathrm{Diff}_k^{(+)}(\cdot,\cdot)$ and $\mathrm{D}_i(\cdot)$,
(3) the functors $\mathrm{Diff}_k^{(+)}(P,\cdot), \mathrm{Diff}_k^{(+)}(\cdot, Q)$ and $\mathrm{D}_i(\cdot)$ are representable in $\mathcal{M}(A)$, whenever P, Q are objects of $\mathcal{M}(A)$.

As an example consider the following situation. Let M be a smooth (i.e., C^∞-class) finite-dimensional manifold and set $A = C^\infty(M)$. Let $\pi : E \to M$, $\xi : F \to M$ be two smooth locally trivial finite-dimensional vector bundles over M and $P = \Gamma(\pi), Q = \Gamma(\xi)$ be the corresponding A-modules of smooth sections.

One can prove that the module $\mathrm{Diff}_k^{(+)}(P, Q)$ coincides with the module of k-th order differential operators acting from the bundle π to ξ (see Proposition 1.1 on page 6). Further, the module $\mathrm{D}(A)$ coincides with the module of vector fields on the manifold M.

However if one constructs representative objects for the functors such as $\mathrm{Diff}_k(P,\cdot)$ and $\mathrm{D}_i(\cdot)$ in the category of all A-modules, the modules $\mathcal{J}^k(P)$ and Λ^i will not coincide with "geometrical" jets and differential forms.

Exercise 1.10. Show that in the case $M = \mathbb{R}$ the form $d(\sin x) - \cos x\, dx$ is nonzero.

Definition 1.13. A module P over $C^\infty(M)$ is called *geometrical*, if

$$\bigcap_{x \in M} \mu_x P = 0,$$

where μ_x is the ideal in $C^\infty(M)$ consisting of functions vanishing at point $x \in M$.

Denote by $\mathcal{G}(M)$ the full subcategory of the category of all modules whose objects are geometrical $C^\infty(M)$-modules. Let P be an A-module and set

$$\mathcal{G}(P) = P \Big/ \bigcap_{x \in M} \mu_x P.$$

Evidently, $\mathcal{G}(P)$ is a geometrical module while the correspondence $P \Rightarrow \mathcal{G}(P)$ is a functor from the category of all $C^\infty(M)$-modules to the category $\mathcal{G}(M)$ of geometrical modules.

Proposition 1.23. *Let M be a smooth finite-dimensional manifold and $A = C^\infty(M)$. Then*

1. *The category $\mathcal{G}(A)$ of geometrical A-modules is differentially closed.*
2. *The representative objects for the functors $\mathrm{Diff}_k(P, \cdot)$ and $\mathrm{D}_i(\cdot)$ in $\mathcal{G}(A)$ coincide with $\mathcal{G}(\mathcal{J}^k(P))$ and $\mathcal{G}(\Lambda^i)$ respectively.*
3. *The module $\mathcal{G}(\Lambda^i)$ coincides with the module of differential i-forms on M.*
4. *If $P = \Gamma(\pi)$ for a smooth locally trivial finite-dimensional vector bundle $\pi : E \to M$, then the module $\mathcal{G}(\mathcal{J}^k(P))$ coincides with the module $\Gamma(\pi_k)$, where $\pi_k : J^k(\pi) \to M$ is the bundle of k-jets for the bundle π (see Section 3.1).*

Exercise 1.11. Prove (1), (2), and (3) above.

The situation described in this Proposition will be referred to as the *geometrical* one.

Another example of a differentially closed category is the category of filtered geometrical modules over a filtered algebra. This category is essential to construct differential calculus over manifolds of infinite jets and infinitely prolonged differential equations (see Sections 3.3 and 3.8 respectively).

Remark 1.5. The logical structure of the above described theory is obviously generalized to the supercommutative case. For a noncommutative generalization see [54, 55].

2. Algebraic model for Lagrangian formalism

Using the above introduced algebraic concepts, we shall construct now an algebraic model for Lagrangian formalism; see also [53]. For geometric motivations, we refer the reader to Section 7 and to Subsection 7.5 especially.

2.1. **Adjoint operators.** Consider an A-module P and the complex of A-homomorphisms

$$0 \to \mathrm{Diff}^+(P, A) \xrightarrow{w} \mathrm{Diff}^+(P, \Lambda^1) \xrightarrow{w} \mathrm{Diff}^+(P, \Lambda^2) \xrightarrow{w} \cdots, \tag{2.1}$$

where, by definition, $w(\nabla) = d \circ \nabla \in \mathrm{Diff}^+(P, \Lambda^{i+1})$ for the operator $\nabla \in \mathrm{Diff}^+(P, \Lambda^i)$. Let $\hat{P}_n$, $n \geq 0$, be the cohomology module of this complex at the term $\mathrm{Diff}^+(P, \Lambda^n)$.

Any operator $\Delta\colon P \to Q$ determines the natural cochain map

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \mathrm{Diff}^+(Q, \Lambda^{i-1}) & \xrightarrow{w} & \mathrm{Diff}^+(Q, \Lambda^i) & \longrightarrow & \cdots \\
 & & \Big\downarrow{\scriptstyle \tilde{\Delta}} & & \Big\downarrow{\scriptstyle \tilde{\Delta}} & & \\
\cdots & \longrightarrow & \mathrm{Diff}^+(P, \Lambda^{i-1}) & \xrightarrow{w} & \mathrm{Diff}^+(P, \Lambda^i) & \longrightarrow & \cdots
\end{array}$$

where $\tilde{\Delta}(\nabla) = \nabla \circ \Delta \in \mathrm{Diff}^+(P, \Lambda^i)$ for $\nabla \in \mathrm{Diff}^+(Q, \Lambda^i)$.

Definition 2.1. The cohomology map $\Delta_n^*\colon \hat{Q}_n \to \hat{P}_n$ induced by $\tilde{\Delta}$ is called the (n-th) *adjoint operator* for Δ.

Below we assume n to be fixed and omit the corresponding subscript. The main properties of the adjoint operator are described by

Proposition 2.1. *Let P, Q and R be A-modules. Then*

(1) *If $\Delta \in \mathrm{Diff}_k(P, Q)$, then $\Delta^* \in \mathrm{Diff}_k(\hat{Q}, \hat{P})$.*
(2) *If $\Delta_1 \in \mathrm{Diff}(P, Q)$ and $\Delta_2 \in \mathrm{Diff}(Q, R)$, then $(\Delta_2 \circ \Delta_1)^* = \Delta_1^* \circ \Delta_2^*$.*

Proof. Let $[\nabla]$ denote the cohomology class of $\nabla \in \mathrm{Diff}^+(P, \Lambda^n)$, where $w(\nabla) = 0$.

(1) Let $a \in A$. Then

$$\begin{aligned}
\delta_a(\Delta^*)([\nabla]) &= \Delta^*([\nabla]) - \Delta^*(a[\nabla]) = [\nabla \circ a \circ \Delta] - [\nabla \circ \Delta \circ a] \\
&= (a \circ \Delta)^*([\nabla]) - (\Delta \circ a)^*([\nabla]) = -\delta_a(\Delta^*)([\nabla]).
\end{aligned}$$

Consequently, $\delta_{a_0,\dots,a_k}(\Delta^*) = (-1)^{k+1}(\delta_{a_0,\dots,a_k}(\Delta))^*$ for any $a_0, \dots, a_k \in A$.

(2) The second statement is implied by the following identities:

$$(\Delta_2 \circ \Delta_1)^*([\nabla]) = [\nabla \circ \Delta_2 \circ \Delta_1] = \Delta_1^*([\nabla \circ \Delta_2]) = \Delta_1^*(\Delta_2^*([\nabla])),$$

which concludes the proof. ∎

Example 2.1. Let $a \in A$ and $a = a_P\colon P \to P$ be the operator of multiplication by a: $p \mapsto ap$. Then obviously $a_P^* = a_{\hat{P}}$.

Example 2.2. Let $p \in P$ and $p\colon A \to P$ be the operator acting by $a \mapsto ap$. Then, by Proposition 2.1 (1) on the preceding page, $p^* \in \operatorname{Hom}_A(\hat{P}, \hat{A})$. Thus there exists a natural paring $\langle\cdot,\cdot\rangle\colon P \otimes_A \hat{P} \to \hat{A}$ defined by $\langle p, \hat{p}\rangle = p^*(\hat{p})$, $\hat{p} \in \hat{P}$.

2.2. **Berezinian and integration.** Consider a complex of differential operators $\cdots \to P_k \xrightarrow{\Delta_k} P_{k+1} \to \cdots$. Then, by Proposition 2.1 on the page before, $\cdots \leftarrow \hat{P}_k \xleftarrow{\Delta_k^*} \hat{P}_{k+1} \leftarrow \cdots$ is a complex of differential operators as well. This complex called *adjoint* to the initial one.

Definition 2.2. The complex adjoint to the de Rham complex of the algebra A is called the *complex of integral forms* and is denoted by

$$0 \leftarrow \Sigma_0 \xleftarrow{\delta} \Sigma_1 \xleftarrow{\delta} \cdots,$$

where $\Sigma_i = \hat{\Lambda}^i, \delta = d^*$. The module $\Sigma_0 = \hat{A}$ is called the *Berezinian* (or the *module of the volume forms*) and is denoted by $\mathcal{B}$.

Assume that the modules under consideration are projective and of finite type. Then we have $\hat{P} = \operatorname{Hom}_A(P, \mathcal{B})$. In particular, $\Sigma_i = \hat{\Lambda}^i = \mathrm{D}_i(\mathcal{B})$.

Let us calculate the Berezinian in the geometrical situation (see Subsection 1.8), when $A = C^\infty(M)$.

Theorem 2.2. *If $A = C^\infty(M)$, M being a smooth finite-dimensional manifold, then*

1. *$\hat{A}_s = 0$ for $s \neq n = \dim M$.*
2. *$\hat{A}_n = \mathcal{B} = \Lambda^n$, i.e., the Berezinian coincides with the module of forms of maximal degree. This isomorphism takes each form $\omega \in \Lambda^n$ to the cohomology class of the zero-order operator $\omega\colon A \to \Lambda^n$, $f \mapsto f\omega$.*

The proof is similar to that of Theorem 1.19 on page 21 and is left to the reader.

In the geometrical situation there exists a natural isomorphism $\Lambda^i \to \mathrm{D}_{n-i}(\Lambda^n) = \Sigma_i$ which takes $\omega \in \Lambda^i$ to the homomorphism $\omega\colon \Lambda^{n-i} \to \Lambda^n$ defined by $\omega(\eta) = \eta \wedge \omega$, $\eta \in \Lambda^{n-i}$.

Exercise 2.1. Show that $\langle \omega_1, \omega_2\rangle = \omega_1 \wedge \omega_2$, $\omega_1 \in \Lambda^i$, $\omega_2 \in \Lambda^{n-i}$.

Exercise 2.2. Prove that $d_i^* = (-1)^{i+1} d_{n-i-1}$, where $d_i\colon \Lambda^i \to \Lambda^{i+1}$ is the de Rham differential.

Thus, in the geometrical situation the complex of integral forms coincides (up to a sign) with the de Rham complex.

Exercise 2.3. Prove the coordinate formula for the adjoint operator:

(1) if $\Delta = \sum_\sigma a_\sigma \dfrac{\partial^{|\sigma|}}{\partial x_\sigma}$ is a scalar operator, then $\Delta^* = \sum_\sigma (-1)^{|\sigma|} \dfrac{\partial^{|\sigma|}}{\partial x_\sigma} \circ a_\sigma$;

(2) if $\Delta = \|\Delta_{ij}\|$ is a matrix operator, then $\Delta^* = \|\Delta^*_{ji}\|$.

The operator $\mathcal{D}\colon \mathrm{Diff}^+(\Lambda^k) \to \Lambda^k$ defined on page 8 generates the map $\int\colon \mathcal{B} \to H^*(\Lambda^\bullet)$ from the Berezinian to the de Rham cohomology group of A. Namely, for any operator $\nabla \in \mathrm{Diff}(A, \Lambda^n)$ satisfying $d \circ \nabla = 0$ we set $\int[\nabla] = [\nabla(1)]$, where $[\cdot]$ denotes the cohomology class.

Proposition 2.3. *The map $\int\colon \mathcal{B} \to H^*(\Lambda^\bullet)$ possesses the following properties:*

(1) *If $\omega \in \Sigma_1$, then $\int \delta\omega = 0$.*
(2) *For any differential operator $\Delta\colon P \to Q$ and elements $p \in P$, $\hat{q} \in \hat{Q}$ the identity*

$$\int \langle \Delta(p), \hat{q} \rangle = \int \langle p, \Delta^*(\hat{q}) \rangle$$

holds.

Proof. (1) Let $\omega = [\nabla] \in \Sigma_1$. Then $\delta\omega = [\nabla \circ d]$ and consequently $\int \omega = [\nabla d(1)] = 0$.

(2) Let $\hat{q} = [\nabla]$ for some operator $\nabla\colon Q \to \Lambda^n$. Then

$$\begin{aligned}\int \langle \Delta(p), \hat{q} \rangle = \int [\nabla\Delta(p)] = \int \nabla \circ \Delta \circ p \\ = \int \langle p, [\nabla \circ \Delta] \rangle = \int \langle p, \Delta^*(\hat{q}) \rangle,\end{aligned}$$

which completes the proof. □

Remark 2.1. Note that the Berezinian $\mathcal{B}$ is a differential right module (see Subsection 1.6) and the complex of integral forms may be understood as the complex dual to the de Rham complex with coefficients in $\mathcal{B}$.

Exercise 2.4. Show that in the geometrical situation the right action of vector fields can also be defined via $X(\omega) = -\mathrm{L}_X(\omega)$, where L_X is the Lie derivative.

Now we establish a relationship between the de Rham cohomology and the homology of the complex of integral forms.

Proposition 2.4 (algebraic Poincaré duality)**.** *There exists a spectral sequence $(E^r_{p,q}, d^r_{p,q})$ with*

$$E^2_{p,q} = H_p((\Sigma_\bullet)_{-q}),$$

the homology of complexes of integral forms, and converging to the de Rham cohomology $H(\Lambda^\bullet)$.

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Diff}^+(A,A) & \xrightarrow{w} & \mathrm{Diff}^+(A,\Lambda^1) & \xrightarrow{w} & \mathrm{Diff}^+(A,\Lambda^2) & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle \tilde d} & & \uparrow{\scriptstyle \tilde d} & & \uparrow{\scriptstyle \tilde d} & & \\
0 & \longrightarrow & \mathrm{Diff}^+(\Lambda^1,A) & \xrightarrow{w} & \mathrm{Diff}^+(\Lambda^1,\Lambda^1) & \xrightarrow{w} & \mathrm{Diff}^+(\Lambda^1,\Lambda^2) & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle \tilde d} & & \uparrow{\scriptstyle \tilde d} & & \uparrow{\scriptstyle \tilde d} & & \\
0 & \longrightarrow & \mathrm{Diff}^+(\Lambda^2,A) & \xrightarrow{w} & \mathrm{Diff}^+(\Lambda^2,\Lambda^1) & \xrightarrow{w} & \mathrm{Diff}^+(\Lambda^2,\Lambda^2) & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle \tilde d} & & \uparrow{\scriptstyle \tilde d} & & \uparrow{\scriptstyle \tilde d} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

where the differential $\tilde{d}\colon \mathrm{Diff}^+(\Lambda^{k+1}, P) \to \mathrm{Diff}^+(\Lambda^k, P)$ is defined by $\tilde{d}(\Delta) = \Delta \circ d$. The statement follows easily from the standard spectral sequence arguments. $\square$

2.3. **Green's formula.** Let Q be an A-module. Then a natural homomorphism $\xi_Q\colon Q \to \hat{\hat{Q}}$ defined by $\xi_Q(q)(\hat{q}) = \langle q, \hat{q}\rangle$ exists. Consequently, to any operator $\Delta\colon P \to \hat{Q}$ there corresponds the operator $\Delta^\circ\colon Q \to \hat{P}$, where $\Delta^\circ = \Delta^* \circ \xi_Q$. This operator will also be called *adjoint* to Δ.

Remark 2.2. In the geometrical situation the two notions of adjointness coincide.

Example 2.3. Let $\hat{q} \in \hat{Q}$ and $\hat{q}\colon A \to \hat{Q}$ be the zero-order operator defined by $a \mapsto a\hat{q}$. The adjoint operator is $\hat{q}$ itself understood as an element of $\mathrm{Hom}_A(Q, \mathcal{B})$.

Proposition 2.5. *The correspondence $\Delta \mapsto \Delta^\circ$ possesses the following properties:*

(1) *Let $\Delta \in \mathrm{Diff}(P, \hat{Q})$ and $\Delta(p) = [\nabla_p]$, where $\nabla_p \in \mathrm{Diff}(Q, \Lambda^i)$. Then $\Delta^\circ(q) = [\square_q]$, where $\square_q \in \mathrm{Diff}(P, \Lambda^i)$ and $\square_q(p) = \nabla_p(q)$.*
(2) *For any $\Delta \in \mathrm{Diff}(P, \hat{Q})$, one has $(\Delta^\circ)^\circ = \Delta$.*
(3) *For any $a \in A$, one has $(a\Delta)^\circ = \Delta^\circ \circ a$.*
(4) *If $\Delta \in \mathrm{Diff}_k(P, \mathcal{B})$, then $\Delta^\circ = j_k^* \circ (a\Delta)$.*
(5) *If $X \in \mathrm{D}_1(\mathcal{B})$, then $X + X^\circ = \delta X \in \mathrm{Diff}_0(A, \mathcal{B}) = \mathcal{B}$.*

Proof. Statements (1), (3), and (4) are the direct consequences of the definition. Statement (2) is implied by (1). Let us prove (5).

Evidently, $\delta_a(j_1) = j_1(a) - aj_1(1) \in \mathcal{J}^1(A)$. Hence for an operator $\Delta \in \mathrm{Diff}_1(A, P)$ one has $(\delta_a(j_1))^*(\Delta) = \Delta(a) - a\Delta(1) = (\delta_a\Delta)(1)$. Consequently,

$$\delta_a(X + X^\circ)(1) = (\delta_a X)(1) + (\delta_a(j_1^*))(X) = (\delta_a X)(1) - \delta_a(j_1)^*(X) = 0$$

and finally $\delta X = j_1^*(X) = X^\circ(1) = X + X^\circ$. $\square$

Note that Statements (1) and (4) of Proposition 2.5 on the facing page can be taken for the definition of Δ°.

Note now that from Proposition 1.15 on page 16 it follows that the modules $\mathrm{D}_i(P)$, $i \geq 2$, can be described as

$$\mathrm{D}_i(P) = \{\, \nabla \in \mathrm{Diff}_1(\Lambda^{i-1}, P) \mid \nabla \circ d = 0 \,\}.$$

Taking $\mathcal{B}$ for P, one can easily show that $\delta\nabla = \nabla^\circ(1)$ and the last equality holds for $i = 1$ as well. Proposition 2.5 on the facing page shows that the correspondence $\Delta \mapsto \Delta^\circ$ establishes an isomorphism between the modules $\mathrm{Diff}(P, \hat{Q})$ and $\mathrm{Diff}^+(Q, \hat{P})$ which, taking into account Proposition 1.15 on page 16, means that the Diff-Spencer complex of the module $\hat{P}$ is isomorphic to the complex

$$0 \longleftarrow \hat{P} \xleftarrow{\mu} \mathrm{Diff}(P, \mathcal{B}) \xleftarrow{\omega} \mathrm{Diff}(P, \Sigma_1) \xleftarrow{\omega} \mathrm{Diff}(P, \Sigma_2) \longleftarrow \cdots, \tag{2.2}$$

where $\omega(\nabla) = \delta \circ \nabla$, $\mu(\nabla) = \nabla^\circ(1)$. From Theorem 1.19 on page 21 one immediately obtains

Theorem 2.6. *Complex* (2.2) *is exact.*

Remark 2.3. Let $\Delta\colon P \to Q$ be a differential operator. Then obviously the following commutative diagram takes place:

$$\begin{tikzcd}
0 & \hat{Q} \arrow[l] \arrow[d, "\Delta^*"'] & \mathrm{Diff}(Q, \mathcal{B}) \arrow[l, "\mu"'] \arrow[d] & \mathrm{Diff}(Q, \Sigma_1) \arrow[l, "\omega"'] \arrow[d] & \cdots \arrow[l, "\omega"'] \\
0 & \hat{P} \arrow[l] & \mathrm{Diff}(P, \mathcal{B}) \arrow[l, "\mu"'] & \mathrm{Diff}(P, \Sigma_1) \arrow[l, "\omega"'] & \cdots \arrow[l, "\omega"']
\end{tikzcd}$$

As a corollary of Theorem 2.6 we obtain

Theorem 2.7 (Green's formula)**.** *If* $\Delta \in \mathrm{Diff}(P, \hat{Q}), p \in P, q \in Q,$ *then*

$$\langle q, \Delta(p)\rangle - \langle \Delta^\circ(q), p\rangle = \delta G$$

for some integral 1-form $G \in \Sigma_1$.

Proof. Consider an operator $\nabla \in \mathrm{Diff}(A, \mathcal{B})$. Then $\nabla - \nabla^\circ(1)$ lies in $\ker \mu$ and consequently there exists an operator $\square \in \mathrm{Diff}(A, \Sigma_1)$ satisfying $\nabla - \nabla^\circ(1) = \omega(\square) = \delta \circ \square$. Hence, $\nabla(1) - \nabla^\circ(1) = \delta G$, where $G = \square(1)$. Setting $\nabla(a) = \langle q, \Delta(ap)\rangle$ we obtain the result. $\square$

Remark 2.4. The integral 1-form G is dependent on p and q. Let us show that we can choose G in such a way that the map $p \times q \mapsto G(p,q)$ is a bidifferential operator. Note first that the map $\omega\colon \mathrm{Diff}^+(A,\Sigma_1) \to \mathrm{Diff}^+(A,\mathcal{B})$ is an A-homomorphism. Since the module $\mathrm{Diff}^+(A,\mathcal{B})$ is projective, there exists an A-homomorphism $\varkappa\colon \operatorname{im}\omega \to \mathrm{Diff}^+(A,\Sigma_1)$ such that $\omega \circ \varkappa = \mathrm{id}$. We can put $\square = \varkappa(\nabla - \nabla(1))$. Thus $G = \varkappa(\nabla - \nabla(1))(1)$. This proves the required statement.

Remark 2.5. From algebraic point of view, we see that in the geometrical situations there is the multitude of misleading isomorphisms, e.q., $\mathcal{B} = \Lambda^n$, $\Delta^\circ = \Delta^*$, etc. In generalized settings, for example, in supercommutative situation (see Subsection 7.9 on page 132), these isomorphisms disappear.

2.4. **The Euler operator.** Let P and Q be A-modules. Introduce the notation

$$\mathrm{Diff}_{(k)}(P,Q) = \underbrace{\mathrm{Diff}(P,\dots,\mathrm{Diff}(P},Q)\dots)_{k\ \text{times}}$$

and set $\mathrm{Diff}_{(*)}(P,Q) = \bigoplus_{k=0}^{\infty} \mathrm{Diff}_{(k)}(P,Q)$. A differential operator $\nabla \in \mathrm{Diff}_{(k)}(P,Q)$ satisfying the condition

$$\nabla(p_1,\dots,p_i,p_{i+1},\dots,p_k) = \sigma\nabla(p_1,\dots,p_{i+1},p_i,\dots,p_k)$$

is called *symmetric*, if $\sigma = 1$, and *skew-symmetric*, if $\sigma = -1$ for all i. The modules of symmetric and skew-symmetric operators will be denoted by $\mathrm{Diff}^{\mathrm{sym}}_{(k)}(P,Q)$ and $\mathrm{Diff}^{\mathrm{alt}}_{(k)}(P,Q)$, respectively. From Theorem 2.6 on the preceding page and Corollary 1.3 on page 7 it follows that for any k the complex

$$0 \leftarrow \mathrm{Diff}_{(k)}(P,\mathcal{B}) \xleftarrow{\omega} \mathrm{Diff}_{(k)}(P,\Sigma_1) \xleftarrow{\omega} \mathrm{Diff}_{(k)}(P,\Sigma_2) \xleftarrow{\omega} \cdots, \qquad (2.3)$$

where $\omega(\nabla) = \delta \circ \nabla$, is exact in all positive degrees, while its 0-homology is of the form $H_0(\mathrm{Diff}_{(k)}(P,\Sigma_\bullet)) = \mathrm{Diff}_{(k-1)}(P,\hat{P})$. This result can be refined in the following way.

Theorem 2.8. *The symmetric*

$$0 \leftarrow \mathrm{Diff}^{\mathrm{sym}}_{(k)}(P,\mathcal{B}) \xleftarrow{\omega} \mathrm{Diff}^{\mathrm{sym}}_{(k)}(P,\Sigma_1) \xleftarrow{\omega} \mathrm{Diff}^{\mathrm{sym}}_{(k)}(P,\Sigma_2) \xleftarrow{\omega} \cdots \qquad (2.4)$$

and skew-symmetric

$$0 \leftarrow \mathrm{Diff}^{\mathrm{alt}}_{(k)}(P,\mathcal{B}) \xleftarrow{\omega} \mathrm{Diff}^{\mathrm{alt}}_{(k)}(P,\Sigma_1) \xleftarrow{\omega} \mathrm{Diff}^{\mathrm{alt}}_{(k)}(P,\Sigma_2) \xleftarrow{\omega} \cdots \qquad (2.5)$$

are acyclic complexes in all positive degrees, while the 0*-homologies denoted by* $L^{\mathrm{sym}}_k(P)$ *and* $L^{\mathrm{alt}}_k(P)$ *respectively are of the form*

$$L^{\mathrm{sym}}_k = \{\, \nabla \in \mathrm{Diff}^{\mathrm{sym}}_{(k-1)}(P,\hat{P}) \mid (\nabla(p_1,\dots,p_{k-2}))^\circ = \nabla(p_1,\dots,p_{k-2}) \,\},$$

$$L^{\mathrm{alt}}_k = \{\, \nabla \in \mathrm{Diff}^{\mathrm{alt}}_{(k-1)}(P,\hat{P}) \mid (\nabla(p_1,\dots,p_{k-2}))^\circ = -\nabla(p_1,\dots,p_{k-2}) \,\}$$

for $k > 1$ and

$$L_1^{\mathrm{sym}}(P) = L_1^{\mathrm{alt}}(P) = \hat{P}.$$

Proof. We shall consider the case of symmetric operators only, since the case of skew-symmetric ones is proved in the same way exactly.

Obviously, the complex (2.4) is a direct summand in (2.3) on the facing page and due to this fact the only thing we need to prove is that the diagram

$$\begin{tikzcd}
\mathrm{Diff}_{(k-1)}(P, \hat{P}) \arrow[d, "\rho'"'] & \mathrm{Diff}_{(k)}(P, \mathcal{B}) \arrow[l, "\mu_{(k-1)}"'] \arrow[d, "\rho"] \\
\mathrm{Diff}_{(k-1)}(P, \hat{P}) & \mathrm{Diff}_{(k)}(P, \mathcal{B}) \arrow[l, "\mu_{(k-1)}"']
\end{tikzcd}$$

is commutative. Here

$$\begin{aligned}
\mu_{(k-1)}(\nabla)(p_1, \dots, p_{k-1}) &= (\nabla(p_1, \dots, p_{k-1}))^\circ(1),\\
\rho(\nabla)(p_1, \dots, p_{k-1}, p_k) &= \nabla(p_1, \dots, p_k, p_{k-1}),\\
\rho'(\nabla)(p_1, \dots, p_{k-2}) &= (\nabla(p_1, \dots, p_{k-2}))^\circ.
\end{aligned}$$

Note that $\mu_{(k-1)} = \mathrm{Diff}_{(k-1)}(\mu)$, where μ is defined in (2.2) on page 31. To prove commutativity, it suffices to consider the case $k = 2$. Let $\nabla \in \mathrm{Diff}_{(2)}(P, \mathcal{B})$ and $\nabla(p_1, p_2) = [\Delta_{p_1,p_2}]$. Then $\mu_{(1)}(\nabla)(p_1) = [\Delta'_{p_1}]$, where $\Delta'_{p_1}(p_2) = \Delta_{p_1,p_2}(1)$. Further, $\rho'(\mu_{(1)}(\nabla)) = [\Delta''_{p_1}]$, where

$$\Delta''_{p_1}(p_2) = \Delta'_{p_2}(p_1) = \Delta_{p_2,p_1}(1).$$

On the other hand, one has $\rho(\nabla)(p_1, p_2) = \nabla(p_2, p_1)$ and $\mu_{(1)}(\rho(\nabla))(p_1) = [\square_{p_1}]$, where $\square_{p_1}(p_2) = \Delta_{p_2,p_1}(1)$. $\square$

Definition 2.3. The elements of the space $\mathcal{L}ag(P) = \bigoplus_{k=1}^{\infty} L_k^{\mathrm{sym}}(P)$ are called *Lagrangians* of the module P. An operator $L \in \mathrm{Diff}_{(*)}^{\mathrm{sym}}(P, \mathcal{B})$ is called a *density* of a Lagrangian $\mathcal{L}$, if $\mathcal{L} = L \bmod \operatorname{im} \omega$. The natural correspondence $\mathbf{E}\colon \mathrm{Diff}_{(*)}^{\mathrm{sym}}(P, \mathcal{B}) \to \mathrm{Diff}_{(*)}^{\mathrm{sym}}(P, \hat{P})$, $L \mapsto \mathcal{L}$ is called the *Euler operator*, while operators of the form $\Delta = \mathbf{E}(L)$ are said to be *Euler–Lagrange operators.*

Theorem 2.8 on the facing page implies the following

Corollary 2.9. *For any projective A-module P one has:*

1. *An operator $\Delta \in \mathrm{Diff}_{(*)}^{\mathrm{sym}}(P, \hat{P})$ is an Euler–Lagrange operator if and only if Δ is self-adjoint, i.e., if $\Delta \in L_*^{\mathrm{sym}}(P)$.*
2. *A density $L \in \mathrm{Diff}_{(*)}^{\mathrm{sym}}(P, \mathcal{B})$ corresponds to a trivial Lagrangian, i.e., $\mathbf{E}(L) = 0$, if and only if L is a total divergence, i.e., $L \in \operatorname{im} \omega$.*

2.5. **Conservation laws.** Denote by $\mathcal{F}$ the commutative algebra of nonlinear operators[5] $\mathrm{Diff}^{\mathrm{sym}}_{(*)}(P, A)$. Then for any A-module Q one has

$$\mathrm{Diff}^{\mathrm{sym}}_{(*)}(P, Q) = \mathcal{F} \otimes_A Q.$$

Let $\Delta \in \mathcal{F} \otimes_A Q$ be a differential operator and let us set $\mathcal{F}_\Delta = \mathcal{F}/\mathfrak{a}$, where $\mathfrak{a}$ denotes the ideal in $\mathcal{F}$ generated by the operators of the form $\square \circ \Delta$, $\square \in \mathrm{Diff}(Q, A)$.

Thus, fixing P, we obtain the functor $Q \Rightarrow \mathcal{F} \otimes_A Q$ and fixing an operator $\Delta \in \mathrm{Diff}_{(*)}(P, Q)$ we get the functor $Q \Rightarrow \mathcal{F}_\Delta \otimes_A Q$ acting from the category $\mathcal{M}_A$ to $\mathcal{M}_\mathcal{F}$ and to $\mathcal{M}_{\mathcal{F}_\Delta}$ respectively, where $\mathcal{M}$ denotes the category of all modules over the corresponding algebra. These functors in an obvious way generate natural transformations of the functors $\mathrm{Diff}^{(+)}_k(\cdot)$, $\mathrm{D}_k(\cdot)$, etc., and of their representative objects $\mathcal{J}^k(P)$, Λ^k, etc. For example, to any operator $\nabla\colon Q_1 \to Q_2$ there correspond operators $\mathcal{F} \otimes \nabla\colon \mathcal{F} \otimes_A Q_1 \to \mathcal{F} \otimes_A Q_2$ and $\mathcal{F}_\Delta \otimes \nabla\colon \mathcal{F}_\Delta \otimes_A Q_1 \to \mathcal{F}_\Delta \otimes_A Q_2$.

These natural transformations allow us to lift the theory of linear differential operators from A to $\mathcal{F}$ and to restrict the lifted theory to $\mathcal{F}_\Delta$. They are in parallel to the theory of $\mathcal{C}$-differential operators (see the next section).

The natural embeddings

$$\mathrm{Diff}^{\mathrm{sym}}_{(k)}(P, R) \hookrightarrow \mathrm{Diff}^{\mathrm{sym}}_{(k-1)}(P, \mathrm{Diff}(P, R))$$

generate the map $\ell\colon \mathcal{F} \otimes_A R \to \mathcal{F} \otimes_A \mathrm{Diff}(P, R)$, $\varphi \mapsto \ell_\varphi$, which is called the *universal linearization*. Using this map, we can rewrite Corollary 2.9 (1) on page 33 in the form $\ell_\Delta = \ell^\circ_\Delta$ while the Euler operator is written as $\mathbf{E}(L) = \ell^\circ_L(1)$. Note also that $\ell_{\varphi\psi} = \varphi\ell_\psi + \psi\ell_\varphi$ for any $\varphi, \psi \in \mathcal{F} \otimes_A R$.

Definition 2.4. The group of *conservation laws* for the algebra $\mathcal{F}_\Delta$ (or for the operator Δ) is the first homology group of the complex of integral forms

$$0 \leftarrow \mathcal{F}_\Delta \otimes_A \mathcal{B} \leftarrow \mathcal{F}_\Delta \otimes_A \Sigma_1 \leftarrow \mathcal{F}_\Delta \otimes_A \Sigma_2 \leftarrow \cdots \tag{2.6}$$

with coefficients in $\mathcal{F}_\Delta$.

[5] In geometrical situation, this algebra is identified with the algebra of polynomial functions on infinite jets (see the next section).

3. Jets and nonlinear differential equations. Symmetries

We expose here main facts concerning geometrical approach to jets (finite and infinite) and to nonlinear differential operators. We shall confine ourselves with the case of vector bundles, though all constructions below can be carried out—with natural modifications—for an arbitrary locally trivial bundle π (and even in more general settings). For further reading, the books [32, 34] together with the paper [62] are recommended.

3.1. **Finite jets.** Let M be an n-dimensional smooth, i.e., of the class C^∞, manifold and $\pi\colon E \to M$ be a smooth m-dimensional vector bundle over M. Denote by $\Gamma(\pi)$ the $C^\infty(M)$-module of sections of the bundle π. For any point $x \in M$ we shall also consider the module $\Gamma_{\mathrm{loc}}(\pi; x)$ of all *local* sections at x.

For a section $\varphi \in \Gamma_{\mathrm{loc}}(\pi; x)$ satisfying $\varphi(x) = \theta \in E$, consider its graph $\Gamma_\varphi \subset E$ and all sections $\varphi' \in \Gamma_{\mathrm{loc}}(\pi; x)$ such that

(a) $\varphi(x) = \varphi'(x)$;

(b) the graph $\Gamma_{\varphi'}$ is tangent to Γ_φ with order k at θ.

Conditions (a) and (b) determine equivalence relation $\sim_x^k$ on $\Gamma_{\mathrm{loc}}(\pi; x)$ and we denote the equivalence class of φ by $[\varphi]_x^k$. The quotient set $\Gamma_{\mathrm{loc}}(\pi; x)/\sim_x^k$ becomes an $\mathbb{R}$-vector space, if we put

$$[\varphi]_x^k + [\psi]_x^k = [\varphi + \psi]_x^k,\ a[\varphi]_x^k = [a\varphi]_x^k, \quad \varphi, \psi \in \Gamma_{\mathrm{loc}}(\pi; x), \quad a \in \mathbb{R},$$

while the natural projection $\Gamma_{\mathrm{loc}}(\pi; x) \to \Gamma_{\mathrm{loc}}(\pi; x)/\sim_x^k$ becomes a linear map. We denote this quotient space by $J_x^k(\pi)$. Obviously, $J_x^0(\pi)$ coincides with $E_x = \pi^{-1}(x)$.

The tangency class $[\varphi]_x^k$ is completely determined by the point x and partial derivatives at x of the section φ up to order k. From here it follows that $J_x^k(\pi)$ is finite-dimensional with

$$\dim J_x^k(\pi) = m \sum_{i=0}^{k} \binom{n+i-1}{n-1} = m\binom{n+k}{k}. \tag{3.1}$$

Definition 3.1. The element $[\varphi]_x^k \in J_x^k(\pi)$ is called the *k-jet* of the section $\varphi \in \Gamma_{\mathrm{loc}}(\pi; x)$ at the point x.

The k-jet of φ at x can be identified with the k-th order Taylor expansion of the section φ. From the definition it follows that it is independent of coordinate choice.

Consider now the set

$$J^k(\pi) = \bigcup_{x \in M} J_x^k(\pi) \tag{3.2}$$

and introduce a smooth manifold structure on $J^k(\pi)$ in the following way. Let $\{\mathcal{U}_\alpha\}_\alpha$ be an atlas in M such that the bundle π becomes trivial over each $\mathcal{U}_\alpha$, i.e., $\pi^{-1}(\mathcal{U}_\alpha) \simeq \mathcal{U}_\alpha \times F$, where F is the "typical fiber". Choose a basis $e_1^\alpha, \dots, e_m^\alpha$ of local sections of π over $\mathcal{U}_\alpha$. Then any section of $\pi\,|_{\mathcal{U}_\alpha}$ is representable in the form $\varphi = u^1 e_1^\alpha + \dots + u^m e_m^\alpha$ and the functions $x_1, \dots, x_n, u^1, \dots, u^m$, where $x_1, \dots, x_n$ are local coordinates in $\mathcal{U}_\alpha$, constitute a local coordinate system in $\pi^{-1}(\mathcal{U}_\alpha)$. Let us define the functions $u_\sigma^j \colon \bigcup_{x \in \mathcal{U}_\alpha} J_x^k(\pi) \to \mathbb{R}$, where $\sigma = i_1 \dots i_r$, $|\sigma| = r \le k$, by

$$u_\sigma^j([\varphi]_x^k) = \left.\frac{\partial^{|\sigma|} u^j}{\partial x_{i_1} \cdots \partial x_{i_r}}\right|_x . \tag{3.3}$$

Then these functions, together with local coordinates $x_1, \dots, x_n$, determine the map $f_\alpha \colon \bigcup_{x \in \mathcal{U}_\alpha} J_x^k(\pi) \to \mathcal{U}_\alpha \times \mathbb{R}^N$, where N is the number defined by (3.1) on the page before. Due to computation rules for partial derivatives under coordinate transformations, the map

$$(f_\alpha \circ f_\beta^{-1})\,|_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta} \colon (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times \mathbb{R}^N \to (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times \mathbb{R}^N$$

is a diffeomorphism preserving the natural projection $(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times \mathbb{R}^N \to (\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$. Thus we have proved the following result:

Proposition 3.1. *The set $J^k(\pi)$ defined by* (3.2) *is a smooth manifold while the projection $\pi_k \colon J^k(\pi) \to M$, $\pi_k \colon [\varphi]_x^k \mapsto x$, is a smooth vector bundle.*

Definition 3.2. Let $\pi \colon E \to M$ be a smooth vector bundle, $\dim M = n$, $\dim E = n + m$.

(1) The manifold $J^k(\pi)$ is called the *manifold of k-jets* for π;
(2) The bundle $\pi_k \colon J^k(\pi) \to M$ is called the *bundle of k-jets* for π;
(3) The above constructed local coordinates $\{x_i, u_\sigma^j\}$, $i = 1, \dots, n$, $j = 1, \dots, m$, $|\sigma| \le k$, are called the *special coordinate system* on $J^k(\pi)$ associated to the trivialization $\{\mathcal{U}_\alpha\}_\alpha$ of the bundle π.

Obviously, the bundle π_0 coincides with π.

Since tangency of two manifolds with order k implies tangency with less order, there exists a map

$$\pi_{k,l} \colon J^k(\pi) \to J^l(\pi), \quad [\varphi]_x^k \mapsto [\varphi]_x^l, \quad k \ge l,$$

which is a smooth fiber bundle. If $k \ge l \ge s$, then obviously

$$\pi_{l,s} \circ \pi_{k,l} = \pi_{k,s}, \quad \pi_l \circ \pi_{k,l} = \pi_k. \tag{3.4}$$

On the other hand, for any section $\varphi \in \Gamma(\pi)$ (or $\in \Gamma_{\mathrm{loc}}(\pi; x)$) we can define the map $j_k(\varphi) \colon M \to J^k(\pi)$ by setting $j_k(\varphi)(x) = [\varphi]_x^k$. Obviously, $j_k(\varphi) \in \Gamma(\pi_k)$ (respectively, $j_k(\varphi) \in \Gamma_{\mathrm{loc}}(\pi_k; x)$).

Definition 3.3. The section $j_k(\varphi)$ is called the *k-jet of the section φ*. The correspondence $j_k\colon \Gamma(\pi) \to \Gamma(\pi_k)$ is called the *k-jet operator*.

From the definition it follows that

$$\pi_{k,l} \circ j_k(\varphi) = j_l(\varphi), \quad j_0(\varphi) = \varphi, \quad k \geq l, \tag{3.5}$$

for any $\varphi \in \Gamma(\pi)$.

Let $\varphi, \psi \in \Gamma(\pi)$ be two sections, $x \in M$ and $\varphi(x) = \psi(x) = \theta \in E$. It is a tautology to say that the manifolds Γ_φ and Γ_ψ are tangent to each other with order $k+l$ at θ or that the manifolds $\Gamma_{j_k(\varphi)}, \Gamma_{j_k(\psi)} \subset J^k(\pi)$ are tangent with order l at the point $\theta_k = j_k(\varphi)(x) = j_k(\psi)(x)$.

Definition 3.4. Let $\theta_k \in J^k(\pi)$. An *R-plane* at θ_k is an n-dimensional plane tangent to a manifold of the form $\Gamma_{j_k(\varphi)}$ such that $[\varphi]_x^k = \theta_k$.

Immediately from definitions we obtain the following result.

Proposition 3.2. *Consider a point $\theta_k \in J^k(\pi)$. Then the fiber of the bundle $\pi_{k+1,k}\colon J^{k+1}(\pi) \to J^k(\pi)$ over θ_k coincides with the set of all R-planes at θ_k.*

For $\theta_{k+1} \in J^{k+1}(\pi)$, we shall denote the corresponding R-plane at $\theta_k = \pi_{k+1,k}(\theta_{k+1})$ by $L_{\theta_{k+1}} \subset T_{\theta_k}(J^k(\pi))$.

3.2. **Nonlinear differential operators.** Let us consider now the algebra of smooth functions on $J^k(\pi)$ and denote it by $\mathcal{F}_k = \mathcal{F}_k(\pi)$. Take another vector bundle $\pi'\colon E' \to M$ and consider the pull-back $\pi_k^*(\pi')$. Then the set of sections of $\pi_k^*(\pi')$ is a module over $\mathcal{F}_k(\pi)$ and we denote this module by $\mathcal{F}_k(\pi, \pi')$. In particular, $\mathcal{F}_k(\pi) = \mathcal{F}_k(\pi, \mathbf{1}_M)$, where $\mathbf{1}_M$ is the trivial one-dimensional bundle over M.

The surjections $\pi_{k,l}$ and π_k for all $k \geq l \geq 0$ generate the natural embeddings $\nu_{k,l} = \pi_{k,l}^*\colon \mathcal{F}_l(\pi, \pi') \to \mathcal{F}_k(\pi, \pi')$ and $\nu_k = \pi_k^*\colon \Gamma(\pi') \to \mathcal{F}_k(\pi, \pi')$. Due to (3.4) on the facing page, we have the equalities

$$\nu_{k,l} \circ \nu_{l,s} = \nu_{k,s}, \quad \nu_{k,l} \circ \nu_l = \nu_k, \quad k \geq l \geq s. \tag{3.6}$$

Identifying $\mathcal{F}_l(\pi, \pi')$ with its image in $\mathcal{F}_k(\pi, \pi')$ under $\nu_{k,l}$, we can consider $\mathcal{F}_k(\pi, \pi')$ as a filtered module,

$$\Gamma(\pi') \hookrightarrow \mathcal{F}_0(\pi, \pi') \hookrightarrow \cdots \hookrightarrow \mathcal{F}_{k-1}(\pi, \pi') \hookrightarrow \mathcal{F}_k(\pi, \pi'), \tag{3.7}$$

over the filtered algebra $C^\infty(M) \hookrightarrow \mathcal{F}_0 \hookrightarrow \cdots \hookrightarrow \mathcal{F}_{k-1} \hookrightarrow \mathcal{F}_k$ with the embeddings $\mathcal{F}_k \cdot \mathcal{F}_l(\pi, \pi') \subset \mathcal{F}_{\max(k,l)}(\pi, \pi')$. Let $F \in \mathcal{F}_k(\pi, \pi')$. Then we have the correspondence

$$\Delta = \Delta_F\colon \Gamma(\pi) \to \Gamma(\pi'), \quad \Delta(\varphi) = j_k(\varphi)^*(F), \quad \varphi \in \Gamma(\pi). \tag{3.8}$$

Definition 3.5. A correspondence Δ of the form (3.8) on the page before is called a (*nonlinear*) *differential operator* of order $\leq k$ acting from the bundle π to the bundle π'. In particular, when $\Delta(a\varphi + b\psi) = a\Delta(\varphi) + b\Delta(\psi)$, $a, b \in \mathbb{R}$, the operator Δ is said to be *linear*.

Example 3.1. Let us show that the k-jet operator $j_k \colon \Gamma(\pi) \to \Gamma(\pi_k)$ (Definition 3.3 on the preceding page) is differential. To do this, recall that the total space of the pull-back $\pi_k^*(\pi_k)$ consists of points $(\theta_k, \theta_k') \in J^k(\pi) \times J^k(\pi)$ such that $\pi_k(\theta_k) = \pi_k(\theta_k')$. Consequently, we may define the *diagonal section* $\rho_k \in \mathcal{F}_k(\pi, \pi_k)$ of the bundle $\pi_k^*(\pi_k)$ by setting $\rho_k(\theta_k) = \theta_k$. Obviously, $j_k = \Delta_{\rho_k}$, i.e.,

$$j_k(\varphi)^*(\rho_k) = j_k(\varphi), \quad \varphi \in \Gamma(\pi).$$

The operator j_k is linear.

Example 3.2. Let $\tau^* \colon T^*M \to M$ be the cotangent bundle of M and $\tau_p^* \colon \bigwedge^p T^*M \to M$ be its p-th exterior power. Then the *de Rham differential* d is a first order linear differential operator acting from τ_p^* to τ_{p+1}^*, $p \geq 0$.

Let us prove now that composition of nonlinear differential operators is a differential operator again. Let $\Delta \colon \Gamma(\pi) \to \Gamma(\pi')$ be a differential operator of order $\leq k$. For any $\theta_k = [\varphi]_x^k \in J^k(\pi)$, set

$$\Phi_\Delta(\theta_k) = [\Delta(\varphi)]_x^0 = (\Delta(\varphi))(x). \tag{3.9}$$

Evidently, the map Φ_Δ is a morphism of fiber bundles (but not of vector bundles!), i.e., $\pi' \circ \Phi_\Delta = \pi_k$.

Definition 3.6. The map Φ_Δ is called the *representative morphism* of the operator Δ.

For example, for $\Delta = j_k$ we have $\Phi_{j_k} = \mathrm{id}_{J^k(\pi)}$. Note that there exists a one-to-one correspondence between nonlinear differential operators and their representative morphisms: one can easily see it just by inverting equality (3.9). In fact, if $\Phi \colon J^k(\pi) \to E'$ is a morphism of π to π', a section $\varphi \in \mathcal{F}(\pi, \pi')$ can be defined by setting $\varphi(\theta_k) = (\theta_k, \Phi(\theta_k)) \in J^k(\pi) \times E'$. Then, obviously, Φ is the representative morphism for $\Delta = \Delta_\varphi$.

Definition 3.7. Let $\Delta \colon \Gamma(\pi) \to \Gamma(\pi')$ be a k-th order differential operator. Its *l-th prolongation* is the composition $\Delta^{(l)} = j_l \circ \Delta \colon \Gamma(\pi) \to \Gamma(\pi_l)$.

Lemma 3.3. *For any k-th order differential operator Δ, its l-th prolongation is a $(k+l)$-th order operator.*

Proof. In fact, for any $\theta_{k+l} = [\varphi]_x^{k+l} \in J^{k+l}(\pi)$ set $\Phi_\Delta^{(l)}(\theta_{k+l}) = [\Delta(\varphi)]_x^l \in J^l(\pi)$. Then the operator, for which the morphism $\Phi_\Delta^{(l)}$ is representative, coincides with $\Delta^{(l)}$. $\square$

Corollary 3.4. *The composition $\Delta' \circ \Delta$ of two nonlinear differential operators $\Delta\colon \Gamma(\pi) \to \Gamma(\pi')$ and $\Delta'\colon \Gamma(\pi') \to \Gamma(\pi'')$ of orders $\le k$ and $\le k'$ respectively is a $(k+k')$-th order differential operator.*

Proof. Let $\Phi_\Delta^{(k')}\colon J^{k+k'}(\pi) \to J^{k'}(\pi')$ be the representative morphism for $\Delta^{(k')}$. Then the operator $\square$, for which the composition $\Phi_{\Delta'} \circ \Phi_\Delta^{(k')}$ is the representative morphism, coincides with $\Delta' \circ \Delta$. ∎

The following obvious proposition describes main properties of prolongations and representative morphisms.

Proposition 3.5. *Let $\Delta\colon \Gamma(\pi) \to \Gamma(\pi')$ and $\Delta'\colon \Gamma(\pi') \to \Gamma(\pi'')$ be two differential operators of orders k and k' respectively. Then:*

(1) $\Phi_{\Delta'\circ\Delta} = \Phi_{\Delta'} \circ \Phi_\Delta^{(k')}$,
(2) $\Phi_\Delta^{(l)} \circ j_{k+l}(\varphi) = \Delta^{(l)}(\varphi)$ *for any* $\varphi \in \Gamma(\pi)$ *and* $l \ge 0$,
(3) $\pi_{l,l'} \circ \Phi_\Delta^{(l)} = \Phi_\Delta^{(l')} \circ \pi_{k+l,k+l'}$, *i.e., the diagram*

$$\begin{array}{ccc} J^{k+l}(\pi) & \xrightarrow{\Phi_\Delta^{(l)}} & J^{l}(\pi') \\ {\scriptstyle \pi_{k+l,k+l'}}\big\downarrow & & \big\downarrow{\scriptstyle \pi'_{l,l'}} \\ J^{k+l'}(\pi) & \xrightarrow{\Phi_\Delta^{(l')}} & J^{l'}(\pi') \end{array} \tag{3.10}$$

is commutative for all $l \ge l' \ge 0$.

3.3. **Infinite jets.** We now pass to infinite limit in all previous constructions.

Definition 3.8. The *space of infinite jets* $J^\infty(\pi)$ of the fiber bundle $\pi\colon E \to M$ is the inverse limit of the sequence

$$\cdots \longrightarrow J^{k+1}(\pi) \xrightarrow{\pi_{k+1,k}} J^k(\pi) \longrightarrow \cdots \longrightarrow J^1(\pi) \xrightarrow{\pi_{1,0}} E \xrightarrow{\pi} M,$$

i.e., $J^\infty(\pi) = \operatorname{proj\,lim}_{\{\pi_{k,l}\}} J^k(\pi)$.

Thus a *point* θ of $J^\infty(\pi)$ is a sequence of points $\{\theta_k\}_{k\ge0}$, $\theta_k \in J^k(\pi)$, such that $\pi_{k,l}(\theta_k) = \theta_l$, $k \ge l$. Points of $J^\infty(\pi)$ can be understood as m-dimensional formal series and can be represented in the form $\theta = [\varphi]_x^\infty$, $\varphi \in \Gamma_{\mathrm{loc}}(\pi)$.

A *special coordinate system* associated to a trivialization $\{\mathcal{U}_\alpha\}_\alpha$ is given by the functions $x_1, \dots, x_n, \dots, u_\sigma^j, \dots$.

A *tangent vector* to $J^\infty(\pi)$ at a point θ is defined as a system of vectors $\{w, v_k\}_{k\ge0}$ tangent to M and to $J^k(\pi)$ respectively such that $(\pi_k)_* v_k = w$, $(\pi_{k,l})_* v_k = v_l$ for all $k \ge l \ge 0$.

A *smooth bundle* ξ over $J^\infty(\pi)$ is a system of bundles $\eta\colon Q \to M$, $\xi_k\colon P_k \to J^k(\pi)$ together with smooth maps $\Psi_k\colon P_k \to Q$, $\Psi_{k,l}\colon P_k \to P_l$, $k \geq l \geq 0$, such that

$$\Psi_l \circ \Psi_{k,l} = \Psi_k, \quad \Psi_{k,l} \circ \Psi_{l,s} = \Psi_{k,s}, \quad k \geq l \geq s \geq 0$$

For example, if $\eta\colon Q \to M$ is a bundle, then the pull-backs $\pi_k^*(\eta)\colon \pi_k^*(Q) \to J^k(\pi)$ together with natural projections $\pi_k^*(Q) \to \pi_l^*(Q)$ and $\pi_k^*(Q) \to Q$ form a bundle over $J^\infty(\pi)$. We say that ξ is a *vector bundle* over $J^\infty(\pi)$, if η and all ξ_k are vector bundles while the maps Ψ_k and $\Psi_{k,l}$ are fiberwise linear.

A *smooth map* of $J^\infty(\pi)$ to $J^\infty(\pi')$, where $\pi\colon E \to M$, $\pi'\colon E' \to M'$, is defined as a system F of maps $F_{-\infty}\colon M \to M'$, $F_k\colon J^k(\pi) \to J^{k-s}(\pi')$, $k \geq s$, where $s \in \mathbb{Z}$ is a fixed integer called the *degree of* F, such that

$$\pi_{k-r,k-s-1} \circ F_k = F_{k-1} \circ \pi_{k,k-1}, \quad k \geq s+1,$$

and

$$\pi_{k-s} \circ F_k = F_{-\infty} \circ \pi_k, \quad k \geq s.$$

For example, if $\Delta\colon \Gamma(\pi) \to \Gamma(\pi')$ is a differential operator of order s, then the system of maps $F_{-\infty} = \mathrm{id}_M$, $F_k = \Phi_\Delta^{(k-s)}$, $k \geq s$ (see the previous subsection), is a smooth map of $J^\infty(\pi)$ to $J^\infty(\pi')$.

A *smooth function* on $J^\infty(\pi)$ is an element of the direct limit $\mathcal{F} = \mathcal{F}(\pi) = \operatorname{inj\,lim}_{\{\pi_{k,l}^*\}} \mathcal{F}_k(\pi)$, where $\mathcal{F}_k(\pi)$ is the algebra of smooth functions on $J^k(\pi)$. Thus, a smooth function on $J^\infty(\pi)$ is a function on $J^k(\pi)$ for some finite but an arbitrary k. The set $\mathcal{F} = \mathcal{F}(\pi)$ of such functions is identified with $\bigcup_{k=0}^\infty \mathcal{F}_k(\pi)$ and forms a commutative filtered algebra. Using duality between smooth manifolds and algebras of smooth functions on these manifolds, we deal in what follows with the algebra $\mathcal{F}(\pi)$ rather than with the manifold $J^\infty(\pi)$ itself.

From this point of view, a *vector field* on $J^\infty(\pi)$ is a filtered derivation of $\mathcal{F}(\pi)$, i.e., an $\mathbb{R}$-linear map $X\colon \mathcal{F}(\pi) \to \mathcal{F}(\pi)$ such that

$$X(fg) = fX(g) + gX(f), \quad f, g \in \mathcal{F}(\pi), \quad X(\mathcal{F}_k(\pi)) \subset \mathcal{F}_{k+l}(\pi)$$

for all k and some $l = l(X)$. The latter is called the *filtration degree* of the field X. The set of all vector fields is a filtered Lie algebra over $\mathbb{R}$ with respect to commutator $[X, Y]$ and is denoted by $\mathrm{D}(\pi) = \bigcup_{l \geq 0} \mathrm{D}^{(l)}(\pi)$.

Differential forms of degree i on $J^\infty(\pi)$ are defined as elements of the filtered $\mathcal{F}(\pi)$-module $\Lambda^i = \Lambda^i(\pi) = \bigcup_{k\geq 0} \Lambda^i(\pi_k)$, where $\Lambda^i(\pi_k) = \Lambda^i(J^k(\pi))$ and the module $\Lambda^i(\pi_k)$ is considered to be embedded into $\Lambda^i(\pi_{k+1})$ by the map $\pi_{k+1,k}^*$. Defined in such a way, these forms possess all basic properties of differential forms on finite-dimensional manifolds. Let us mention the most important ones:

(1) The module $\Lambda^i(\pi)$ is the i-th exterior power of $\Lambda^1(\pi)$, $\Lambda^i(\pi) = \bigwedge^i \Lambda^1(\pi)$. Respectively, the operation of *wedge product* $\wedge\colon \Lambda^p(\pi) \otimes \Lambda^q(\pi) \to \Lambda^{p+q}(\pi)$ is defined and $\Lambda^*(\pi) = \bigoplus_{i\geq 0} \Lambda^i(\pi)$ becomes a supercommutative $\mathbb{Z}$-graded algebra.

(2) The module $\mathrm{D}(\pi)$ is dual to $\Lambda^1(\pi)$, i.e.,

$$\mathrm{D}(\pi) = \mathrm{Hom}^{\phi}_{\mathcal{F}(\pi)}(\Lambda^1(\pi), \mathcal{F}(\pi)), \tag{3.11}$$

where $\mathrm{Hom}^{\phi}_{\mathcal{F}(\pi)}(\cdot,\cdot)$ denotes the module of filtered homomorphisms over $\mathcal{F}(\pi)$. Moreover, equality (3.11) is established in the following way: there is a derivation $d\colon \mathcal{F}(\pi) \to \Lambda^1(\pi)$ (the *de Rham differential* on $J^\infty(\pi)$) such that for any vector field X there exists a uniquely defined filtered homomorphism f_X satisfying $f_X \circ d = X$.

(3) The operator d is extended up to maps $d\colon \Lambda^i(\pi) \to \Lambda^{i+1}(\pi)$ in such a way that the sequence

$$0 \longrightarrow \mathcal{F}(\pi) \xrightarrow{d} \Lambda^1(\pi) \longrightarrow \cdots \longrightarrow \Lambda^i(\pi) \xrightarrow{d} \Lambda^{i+1}(\pi) \longrightarrow \cdots$$

becomes a complex, i.e., $d\circ d = 0$. This complex is called the *de Rham complex* on $J^\infty(\pi)$. The latter is a derivation of the superalgebra $\Lambda^*(\pi)$.

Using algebraic techniques (see Section 1), we can introduce the notions of *inner product* and *Lie derivative* and to prove their basic properties (cf. Proposition 1.14 on page 15). We can also define *linear differential operators* over $J^\infty(\pi)$ as follows. Let P and Q be two filtered $\mathcal{F}(\pi)$-modules and $\Delta \in \mathrm{Hom}^{\phi}_{\mathcal{F}(\pi)}(P, Q)$. Then Δ is called a linear differential operator of order $\leq k$ acting from P to Q, if

$$(\delta_{f_0} \circ \delta_{f_1} \circ \cdots \circ \delta_{f_k})\Delta = 0$$

for all $f_0, \ldots, f_k \in \mathcal{F}(\pi)$, where, as in Section 1, $(\delta_f \Delta)p = \Delta(fp) - f\Delta(p)$. We write $k = \mathrm{ord}(\Delta)$.

Due to existence of filtrations in the algebra $\mathcal{F}(\pi)$, as well as in modules P and Q, one can define differential operators of *infinite order* acting from P to Q. Namely, let $P = \{P_l\}$, $Q = \{Q_l\}$, $P_l \subset P_{l+1}$, $Q_l \subset Q_{l+1}$, P_l and Q_l being $\mathcal{F}_l(\pi)$-modules. Let $\Delta \in \mathrm{Hom}^{\phi}_{\mathcal{F}(\pi)}(P, Q)$ and s be filtration of Δ, i.e., $\Delta(P_l) \subset Q_{l+s}$. We can always assume that $s \geq 0$. Suppose now that $\Delta_l = \Delta\,|_{P_l} : P_l \to Q_l$ is a linear differential operator of order o_l over $\mathcal{F}_l(\pi)$ for any l. Then we say that Δ is a linear differential operator of order growth o_l. In particular, if $o_l = \alpha l + \beta$, $\alpha, \beta \in \mathbb{R}$, we say that Δ is of constant growth α.

Distributions. Let $\theta \in J^\infty(\pi)$. The tangent plane to $J^\infty(\pi)$ at the point θ is the set of all tangent vectors to $J^\infty(\pi)$ at this point (see above). Denote such a plane by $T_\theta = T_\theta(J^\infty(\pi))$. Let $\theta = \{x, \theta_k\}$, $x \in M$, $\theta_k \in J^k(\pi)$ and

$v = \{w, v_k\}$, $v' = \{w', v'_k\} \in T_\theta$. Then the linear combination $\lambda v + \mu v' = \{\lambda w + \mu w', \lambda v_k + \mu v'_k\}$ is again an element of T_θ and thus T_θ is a vector space. A correspondence $\mathcal{T} : \theta \mapsto \mathcal{T}_\theta \subset T_\theta$, where $\mathcal{T}_\theta$ is a linear subspace, is called a *distribution* on $J^\infty(\pi)$. Denote by $\mathcal{T}\mathrm{D}(\pi) \subset \mathrm{D}(\pi)$ the submodule of vector fields lying in $\mathcal{T}$, i.e., a vector field X belongs to $\mathcal{T}\mathrm{D}(\pi)$ if and only if $X_\theta \in \mathcal{T}_\theta$ for all $\theta \in J^\infty(\pi)$. We say that the distribution $\mathcal{T}$ is *integrable*, if it satisfies the formal Frobenius condition: for any vector fields $X, Y \in \mathcal{T}\mathrm{D}(\pi)$ their commutator lies in $\mathcal{T}\mathrm{D}(\pi)$ as well, or $[\mathcal{T}\mathrm{D}(\pi), \mathcal{T}\mathrm{D}(\pi)] \subset \mathcal{T}\mathrm{D}(\pi)$.

This condition can expressed in a dual way as follows. Let us set

$$\mathcal{T}^1\Lambda(\pi) = \{\, \omega \in \Lambda^1(\pi) \mid \mathrm{i}_X\omega = 0,\ X \in \mathcal{T}\mathrm{D}(\pi) \,\}$$

and consider the ideal $\mathcal{T}\Lambda^*(\pi)$ generated in $\Lambda^*(\pi)$ by $\mathcal{T}^1\Lambda(\pi)$. Then the distribution $\mathcal{T}$ is integrable if and only if the ideal $\mathcal{T}\Lambda^*(\pi)$ is differentially closed: $d(\mathcal{T}\Lambda^*(\pi)) \subset \mathcal{T}\Lambda^*(\pi)$.

Finally, we say that a submanifold $N \subset J^\infty(\pi)$ is an *integral manifold* of $\mathcal{T}$, if $T_\theta N \subset \mathcal{T}_\theta$ for any point $\theta \in N$. An integral manifold N is called *locally maximal* at a point $\theta \in N$, if there no neighborhood $\mathcal{U} \subset N$ of θ is embedded to other integral manifold N' such that $\dim N \leq \dim N'$.

3.4. **Nonlinear equations and their solutions.** Let $\pi\colon E \to M$ be a vector bundle.

Definition 3.9. A submanifold $\mathcal{E} \subset J^k(\pi)$ is called a (*nonlinear*) *differential equation* of order k in the bundle π. We say that $\mathcal{E}$ is a *linear equation*, if $\mathcal{E} \cap \pi_x^{-1}(x)$ is a linear subspace in $\pi_x^{-1}(x)$ for all $x \in M$. In other words, $\mathcal{E}$ is a linear subbundle in the bundle π_k.

We shall always assume that $\mathcal{E}$ is projected surjectively to E under $\pi_{k,0}$.

Definition 3.10. A (local) section f of the bundle π is called a (local) *solution* of the equation $\mathcal{E}$, if its graph lies in $\mathcal{E}$: $j_k(f)(M) \subset \mathcal{E}$.

We say that the equation $\mathcal{E}$ is *determined*, if $\operatorname{codim} \mathcal{E} = \dim \pi$, that it is *overdetermined*, if $\operatorname{codim} \mathcal{E} > \dim \pi$, and that it is *underdetermined*, if $\operatorname{codim} \mathcal{E} < \dim \pi$.

Obviously, in a special coordinate system these definitions coincide with "usual" ones.

One of the ways to represent differential equations is as follows. Let $\pi'\colon \mathbb{R}^r \times \bar{\mathcal{U}} \to \bar{\mathcal{U}}$ be the trivial r-dimensional bundle. Then the set of functions $(F^1, \dots, F^r)$ can be understood as a section φ of the pull-back $(\pi_k|_{\mathcal{U}})^*(\pi')$, or as a nonlinear operator $\Delta = \Delta_\varphi$ defined in $\mathcal{U}$, while the equation $\mathcal{E}$ is characterized by the condition

$$\mathcal{E} \cap \mathcal{U} = \{\, \theta_k \in \mathcal{U} \mid \varphi(\theta_k) = 0 \,\}. \tag{3.12}$$

More generally, any equation $\mathcal{E} \subset J^k(\pi)$ can be represented in the form similar to (3.12) on the facing page. Namely, for any equation $\mathcal{E}$ there exists a fiber bundle $\pi' \colon E' \to M$ and a section $\varphi \in \mathcal{F}_k(\pi, \pi)$ such that $\mathcal{E}$ coincides with the set of zeroes for φ: $\mathcal{E} = \{\varphi = 0\}$. In this case we say that $\mathcal{E}$ is *associated to the operator* $\Delta = \Delta_\varphi \colon \Gamma(\pi) \to \Gamma(\pi')$ and use the notation $\mathcal{E} = \mathcal{E}_\Delta$.

Example 3.3. Let $\pi = \tau_p^* \colon \bigwedge^p T^*M \to M$, $\pi' = \tau_{p+1}^* \colon \bigwedge^{p+1} T^*M \to M$ and $d \colon \Gamma(\pi) = \Lambda^p(M) \to \Gamma(\pi') = \Lambda^{p+1}(M)$ be the de Rham differential (see Example 3.2 on page 38). Thus we obtain a first-order equation $\mathcal{E}_d$ in the bundle τ_p^*. Consider the case $p = 1, n \geq 2$ and choose local coordinates $x_1, \ldots, x_n$ in M. Then any form $\omega \in \Lambda^1(M)$ is represented as $\omega = u^1 dx_1 + \cdots + u^n dx_n$ and we have $\mathcal{E}_d = \{\, u_i^j = u_j^i \mid i < j \,\}$. This equation is underdetermined when $n = 2$, determined for $n = 3$ and overdetermined for $n > 3$.

Example 3.4. Consider an arbitrary vector bundle $\pi \colon E \to M$ and a differential form $\omega \in \Lambda^p(J^k(\pi))$, $p \leq \dim M$. Then the condition $j_k(\varphi)^*(\omega) = 0$, $\varphi \in \Gamma(\pi)$, determines a $(k+1)$-st order equation $\mathcal{E}_\omega$ in the bundle π. Consider the case $p = \dim M = 2$, $k = 1$ and choose a special coordinate system x, y, u, u_x, u_y in $J^1(\pi)$. Let $\varphi = \varphi(x, y)$ be a local section and

$$
\begin{aligned}
\omega = A du_x \wedge du_y + (B_1 du_x + B_2 du_y) \wedge du + du_x \wedge (B_{11} dx + B_{12} dy) \\
+ du_y \wedge (B_{21} dx + B_{22} dy) + du \wedge (C_1 dx + C_2 dy) + D dx \wedge dy,
\end{aligned}
$$

where A, B_i, B_{ij}, C_i, D are functions of x, y, u, u_x, u_y. Then we have

$$
\begin{aligned}
j_1(\varphi)^*\omega = \Big(A^\varphi(\varphi_{xx}\varphi_{yy} - \varphi_{xy}^2) + (\varphi_y B_1^\varphi + B_{12}^\varphi)\varphi_{xx} \\
- (\varphi_x B_2^\varphi + B_{12}^\varphi)\varphi_{yy} + (\varphi_y B_2^\varphi - \varphi_x B_1^\varphi + B_{22}^\varphi - B_{11}^\varphi)\varphi_{xy} \\
+ \varphi_x C_2^\varphi - \varphi_y C_1^\varphi + D^\varphi) \Big) dx \wedge dy,
\end{aligned}
$$

where $F^\varphi = j_1(\varphi)^* F$ for any $F \in \mathcal{F}_1(\pi)$. Hence, the equation $\mathcal{E}_\omega$ is of the form

$$
a(u_{xx}u_{yy} - u_{xy}^2) + b_{11}u_{xx} + b_{12}u_{xy} + b_{22}u_{yy} + c = 0, \tag{3.13}
$$

where $a = A, b_{11} = u_y B_1 + B_{12}$, $b_{12} = u_y B_2 - u_x B_1 + B_{22} - B_{11}$, $b_{22} = u_x B_2 + B_{12}$, $c = u_x C_2 - u_y C_1 + D$ are functions on $J^1(\pi)$. Equation (3.13) is the so-called two-dimensional *Monge–Ampere equation* and obviously any such an equation can be represented as $\mathcal{E}_\omega$ for some $\omega \in \Lambda^1(J^1(\pi))$ (see [36] for more details).

Example 3.5. Consider again a bundle $\pi \colon E \to M$ and a section $\nabla \colon E \to J^1(\pi)$ of the bundle $\pi_{1,0} \colon J^1(\pi) \to E$. Then the graph $\mathcal{E}_\nabla = \nabla(E) \subset J^1(\pi)$ is a first-order equation in the bundle π. Let $\theta_1 \in \mathcal{E}_\nabla$. Then, due to

Proposition 3.2 on page 37, the point θ_1 is identified with the pair (θ_0, L_{θ_1}), where $\theta_0 = \pi_{1,0}(\theta_1) \in E$, while L_{θ_1} is the R-plane at θ_0 corresponding to θ_1. Hence, the section ∇ (or the equation $\mathcal{E}_\nabla$) may be understood as a distribution of horizontal (i.e., nondegenerately projected to T_xM under $(\pi_k)_*$, where $x = \pi_k(\theta_k)$) n-dimensional planes on E: $\mathcal{T}_\nabla \colon E \ni \theta \mapsto \theta_1 = L_{\nabla(\theta)}$. In other words, ∇ is a *connection* in the bundle π. A solution of the equation $\mathcal{E}_\nabla$, by definition, is a section $\varphi \in \Gamma(\pi)$ such that $j_1(\varphi)(M) \subset \nabla(E)$. It means that at any point $\theta = \varphi(x) \in \varphi(M)$ the plane $\mathcal{T}_\nabla(\theta)$ is tangent to the graph of the section φ. Thus, solutions of $\mathcal{E}_\nabla$ coincide with integral manifolds of $\mathcal{T}_\nabla$.

In a local coordinate system $(x_1, \dots, x_n, u^1, \dots, u^m, \dots, u_i^j, \dots)$, $i = 1, \dots, n$, $j = 1, \dots, m$, the equation $\mathcal{E}_\nabla$ is represented as

$$u_i^j = \nabla_i^j(x_1, \dots, x_n, u^1, \dots, u^m),\ i = 1, \dots, n, j = 1, \dots, m, \tag{3.14}$$

∇_i^j being smooth functions.

Example 3.6. As we saw in the previous example, to solve the equation $\mathcal{E}_\nabla$ is the same as to find integral n-dimensional manifolds of the distribution $\mathcal{T}_\nabla$. Hence, the former to be solvable, the latter is to satisfy the Frobenius theorem. Thus, for solvable $\mathcal{E}_\nabla$ we obtain conditions on the section $\nabla \in \Gamma(\pi_{1,0})$. Let write down these conditions in local coordinates.

Using representation (3.14), note that $\mathcal{T}_\nabla$ is given by 1-forms

$$\omega^j = du^j - \sum_{i=1}^n \nabla_i^j dx_i,\ j = 1, \dots, m.$$

Hence, the integrability conditions may be expressed as

$$d\omega^j = \sum_{i=1}^m \rho_i^j \wedge \omega_i,\ j = 1, \dots, m,$$

for some 1-forms ρ_i^i. After elementary computations, we obtain that the functions ∇_i^j must satisfy the following relations:

$$\frac{\partial \nabla_\alpha^j}{\partial x_\beta} + \sum_{\gamma=1}^m \nabla_\alpha^\gamma \frac{\partial \nabla_\beta^j}{\partial u^\gamma} = \frac{\partial \nabla_\beta^j}{\partial x_\alpha} + \sum_{\gamma=1}^m \nabla_\beta^\gamma \frac{\partial \nabla_\alpha^j}{\partial u^\gamma} \tag{3.15}$$

for all $j = 1, \dots, m$, $1 \le \alpha < \beta \le m$. Thus we got a naturally constructed first-order equation $\mathcal{I}(\pi) \subset J^1(\pi_{1,0})$, whose solutions are horizontal n-dimensional distributions in $E = J^0(\pi)$.

3.5. **Cartan distribution on $J^k(\pi)$.** We shall now introduce a very important structure on $J^k(\pi)$ responsible for "individuality" of these manifolds.

Definition 3.11. Let $\pi \colon E \to M$ be a vector bundle. Consider a point $\theta_k \in J^k(\pi)$ and the span $\mathcal{C}_{\theta_k}^k \subset T_{\theta_k}(J^k(\pi))$ of all R-planes at the point θ_k.

(1) The correspondence $\mathcal{C}^k = \mathcal{C}^k(\pi)\colon \theta_k \mapsto \mathcal{C}^k_{\theta_k}$ is called the *Cartan distribution* on $J^k(\pi)$.
(2) Let $\mathcal{E} \subset J^k(\pi)$ be a differential equation. Then the correspondence $\mathcal{C}^k(\mathcal{E})\colon \mathcal{E} \ni \theta_k \mapsto \mathcal{C}^k_{\theta_k} \cap T_{\theta_k}\mathcal{E} \subset T_{\theta_k}\mathcal{E}$ is called the *Cartan distribution* on $\mathcal{E}$. We call elements of the Cartan distributions *Cartan planes*.
(3) A point $\theta_k \in \mathcal{E}$ is called *regular*, if the Cartan plane $\mathcal{C}^k_{\theta_k}(\mathcal{E})$ is of maximal dimension. We say that $\mathcal{E}$ is a *regular equation*, if all its points are regular.

In what follows, we deal with regular equations or with neighborhoods of regular points. As it can be easily seen, for any regular point there exists a neighborhood of this point all points of which are regular.

Let $\theta_k \in J^k(\pi)$ be represented in the form

$$\theta_k = [\varphi]^k_x, \quad \varphi \in \Gamma(\pi), \quad x = \pi_k(\theta_k). \tag{3.16}$$

Then, by definition, the Cartan plane $\mathcal{C}^k_{\theta_k}$ is spanned by the vectors

$$j_k(\varphi)_{*,x}(v), \quad v \in T_x M, \tag{3.17}$$

for all $\varphi \in \Gamma_{\mathrm{loc}}(\pi)$ satisfying (3.16).

Let $x_1, \dots, x_n, \dots, u^j_\sigma, \dots,\ j = 1, \dots, m,\ |\sigma| \le k$, be a special coordinate system in a neighborhood of θ_k. The vectors of the form (3.17) can be expressed as linear combinations of the vectors

$$\frac{\partial}{\partial x_i} + \sum_{|\sigma|\le k} \sum_{j=1}^{m} \frac{\partial^{|\sigma|+1}\varphi^j}{\partial x_\sigma \partial x_i} \frac{\partial}{\partial u^j_\sigma}, \tag{3.18}$$

where $i = 1, \dots, n$. Using this representation, we prove the following result:

Proposition 3.6. *For any point $\theta_k \in J^k(\pi)$, $k \ge 1$, the Cartan plane $\mathcal{C}^k_{\theta_k}$ is of the form $\mathcal{C}^k_{\theta_k} = (\pi_{k,k-1})^{-1}_*(L_{\theta_k})$, where L_{θ_k} is the R-plane at the point $\pi_{k,k-1}(\theta_k) \in J^{k-1}(\pi)$ determined by the point θ_k.*

Proof. Denote the vector (3.18) by $v^{k,\varphi}_i$. It is obvious that for any two sections φ and φ' satisfying (3.16) the difference $v^{k,\varphi}_i - v^{k,\varphi'}_i$ is a $\pi_{k,k-1}$-vertical vector and any such a vector can be obtained in this way. On the other hand, the vectors $v^{k-1,\varphi}_i$ do not depend on section φ satisfying (3.16) and form a basis in the space L_{θ_k}. □

Remark 3.1. From the result proved it follows that the Cartan distribution on $J^k(\pi)$ can be locally considered as generated by the vector fields

$$D^{(k-1)}_i = \frac{\partial}{\partial x_i} + \sum_{|\sigma|\le k-1} \sum_{j=1}^{m} u^j_{\sigma i} \frac{\partial}{\partial u^j_\sigma}, \quad V^s_\tau = \frac{\partial}{\partial u^s_\tau}, \quad |\tau| = k, s = 1, \dots, m. \tag{3.19}$$

From here, by direct computations, it follows that $[V^s_\tau, D^{(k-1)}_i] = V^s_{\tau-i}$, where

$$V^s_{\tau-i} = \begin{cases} V_{\tau'}, & \text{if } \tau = \tau' i, \\ 0, & \text{if } \tau \text{ does not contain } i. \end{cases}$$

But, as it follows from Proposition 3.6 on the preceding page, vector fields V^j_σ for $|\sigma| \le k$ do not lie in $\mathcal{C}^k$. Thus, the Cartan distribution on $J^k(\pi)$ is not integrable.

Introduce 1-forms in special coordinates on $J^{k+1}(\pi)$:

$$\omega^j_\sigma = du^j_\sigma - \sum_{i=1}^n u^j_{\sigma i} dx_i, \tag{3.20}$$

where $j = 1, \dots, m$, $|\sigma| < k$. From the representation (3.19) on the page before we immediately obtain the following important property of the forms introduced:

Proposition 3.7. *The system of forms* (3.20) *annihilates the Cartan distribution on* $J^k(\pi)$*, i.e., a vector field* X *lies in* $\mathcal{C}^k$ *if and only if* $\mathrm{i}_X \omega^j_\sigma = 0$ *for all* $j = 1, \dots, m, |\sigma| < k$.

Definition 3.12. The forms (3.20) are called the *Cartan forms* on $J^k(\pi)$ associated to the special coordinate system x_i, u^j_σ.

Note that the $\mathcal{F}_k(\pi)$-submodule generated in $\Lambda^1(J^k(\pi)$ by the forms (3.20) is independent of the choice of coordinates.

Definition 3.13. The $\mathcal{F}_k(\pi)$-submodule generated in $\Lambda^1(J^k(\pi))$ by the Cartan forms is called the *Cartan submodule*. We denote this submodule by $\mathcal{C}\Lambda^1(J^k(\pi))$.

We shall now describe maximal integral manifolds of the Cartan distribution on $J^k(\pi)$. Let $N \subset J^k(\pi)$ be an integral manifold of the Cartan distribution. Then from Proposition 3.7 it follows that the restriction of any Cartan form ω to N vanishes. Similarly, the differential $d\omega$ vanishes on N. Therefore, if vector fields X, Y are tangent to N, then $d\omega\,|_N\,(X, Y) = 0$.

Definition 3.14. Let $\mathcal{C}^k_{\theta_k}$ be the Cartan plane at $\theta \in J^(\pi)$.

(1) Two vectors $v, w \in \mathcal{C}^k_{\theta_k}$ are said to be *in involution*, if $d\omega_{\theta_k}(v, w) = 0$ for any $\omega \in \mathcal{C}\Lambda^1(J^k(\pi))$.
(2) A subspace $W \subset \mathcal{C}^k_{\theta_k}$ is said to be *involutive*, if any two vectors $v, w \in W$ are in involution.
(3) An involutive subspace is called *maximal*, if it cannot be embedded into any other involutive subspace.

Consider a point $\theta_k = [\varphi]_x^k \in J^k(\pi)$. Then from Proposition 3.7 on the facing page it follows that the direct sum decomposition $\mathcal{C}_{\theta_k}^k = T_{\theta_k}^v \oplus T_{\theta_k}^{\varphi}$ takes place, where $T_{\theta_k}^v$ denotes the tangent plane to the fiber of the projection $\pi_{k,k-1}$ passing through the point θ_k, while $T_{\theta_k}^{\varphi}$ is the tangent plane to the graph of $j_k(\varphi)$. Hence, the involutiveness is sufficient to be checked for the following pairs of vectors $v, w \in \mathcal{C}_{\theta_k}^k$:

(1) $v, w \in T_{\theta_k}^v$;
(2) $v, w \in T_{\theta_k}^{\varphi}$;
(3) $v \in T_{\theta_k}^v$, $w \in T_{\theta_k}^{\varphi}$.

Note now that the tangent space $T_{\theta_k}^v$ is identified with the tensor product $S^k(T_x^*) \otimes E_x$, $x = \pi_k(\theta_k) \in M$, where T_x^* is the fiber of the cotangent bundle to M at x, E_x is the fiber of the bundle π at the same point while S^k denotes the k-th symmetric power. Then any vector $w \in T_x M$ determines the map $\delta_w \colon S^k(T_x^*) \otimes E_x \to S^{k-1}(T_x^*) \otimes E_x$ by

$$\delta_w(\rho_1 \cdot \ldots \cdot \rho_k) \otimes e = \sum_{i=1}^{k} \rho_1 \cdot \ldots \cdot \langle \rho_i, w \rangle \cdot \ldots \cdot \rho_k \otimes e,$$

where the dot "$\cdot$" denotes multiplication in $S^k(T_x^*)$, $\rho_i \in T_x^*$, $e \in E_x$ while $\langle \cdot, \cdot \rangle$ is the natural pairing between T_x^* and T_x.

Proposition 3.8. *Let $v, w \in \mathcal{C}_{\theta_k}^k$. Then:*

(1) *All pairs $v, w \in T_{\theta_k}^v$ are in involution.*
(2) *All pairs $v, w \in T_{\theta_k}^{\varphi}$ are in involution too. If $v \in T_{\theta_k}^v$ and $w \in T_{\theta_k}^{\varphi}$, then they are in involution if and only if $\delta_{(\pi_k)_*(w)} v = 0$.*

Proof. Note first that the involutiveness conditions are sufficient to check for the Cartan forms (3.20) on the preceding page only. The all three results follow from the representation (3.19) on page 45 by straightforward computations. ☐

Let $\theta_k \in J^k(\pi)$ and F_{θ_k} be the fiber of the bundle $\pi_{k,k-1}$ passing through the point θ_k while $H \subset T_x M$ be a linear subspace. Using the linear structure, we identify the fiber F_{θ_k} of the bundle $\pi_{k,k-1}$ with its tangent space and define the space

$$\mathrm{Ann}(H) = \{\, v \in F_{\theta_k} \mid \delta_w v = 0,\ \forall w \in H \,\}.$$

Then, as it follows from Proposition 3.8, the following description of maximal involutive subspaces takes place:

Corollary 3.9. *Let $\theta_k = [\varphi]_x^k$, $\varphi \in \Gamma_{\mathrm{loc}}(\pi)$. Then any maximal involutive subspace $V \subset \mathcal{C}_{\theta_k}^k(\pi)$ is of the form $V = j_k(\varphi)_*(H) \oplus \mathrm{Ann}(H)$ for some $H \subset T_x M$.*

If V is a maximal involutive subspace, then the corresponding space H is obviously $\pi_{k,*}(V)$. We call the dimension of H the *type* of the maximal involutive subspace V and denote it by $\mathrm{tp}(V)$.

Proposition 3.10. *Let V be a maximal involutive subspace. Then*

$$\dim V = m\binom{n-r+k-1}{k} + r,$$

where $n = \dim M$, $m = \dim \pi$, $r = \mathrm{tp}(V)$.

Proof. Let us choose local coordinates in M in such a way that the vectors $\partial/\partial x_1, \dots, \partial/\partial x_r$ form a basis in H. Then, in the corresponding special system in $J^k(\pi)$, coordinates along $\mathrm{Ann}(H)$ will consist of those functions u^j_σ, $|\sigma| = k$, for which multi-index σ does not contain indices $1, \dots, r$. $\square$

Let $N \subset J^k(\pi)$ be a maximal integral manifold of the Cartan distribution and $\theta_k \in N$. Then the tangent plane to N at θ_k is a maximal involutive plane. Let its type be equal to $r(\theta_k)$.

Definition 3.15. The number $\mathrm{tp}(N) = \max\limits_{\theta_k \in N} r(\theta_k)$ is called the *type* of the maximal integral manifold N of the Cartan distribution.

Obviously, the set $g(N) = \{\, \theta_k \in N \mid r(\theta_k) = \mathrm{tp}(N) \,\}$ is everywhere dense in N. We call the points $\theta_k \in g(N)$ *generic*. Let θ_k be such a point and $\mathcal{U}$ be its neighborhood in N consisting of generic points. Then:

(1) $N' = \pi_{k,k-1}(N)$ is an integral manifold of the Cartan distribution on $J^{k-1}(\pi)$;
(2) $\dim(N') = \mathrm{tp}(N)$ and
(3) $\pi_{k-1}|_{N'} : N' \to M$ is an immersion.

Theorem 3.11. *Let $N \subset J^{k-1}(\pi)$ be an integral manifold of the Cartan distribution on $J^k(\pi)$ and $\mathcal{U} \subset N$ be an open domain consisting of generic points. Then*

$$\mathcal{U} = \{\, \theta_k \in J^k(\pi) \mid L_{\theta_k} \supset T_{\theta_{k-1}}\mathcal{U}' \,\},$$

where $\theta_{k-1} = \pi_{k,k-1}(\theta_k)$, $\mathcal{U}' = \pi_{k,k-1}(\mathcal{U})$.

Proof. Let $M' = \pi_{k-1}(\mathcal{U}') \subset M$. Denote its dimension (which equals to $\mathrm{tp}(N)$) by r and choose local coordinates in M in such a way that the submanifold $\mathcal{V}'$ is determined by the equations $x_{r+1} = \cdots = x_n = 0$ in these coordinates. Then, since $\mathcal{U}' \subset J^{k-1}(\pi)$ is an integral manifold and $\pi_{k-1}|_{\mathcal{U}'} : \mathcal{U}' \to \mathcal{V}'$ is a diffeomorphism, in corresponding special coordinates the manifold $\mathcal{U}'$ is given by the equations

$$u^j_\sigma = \begin{cases} \dfrac{\partial^{|\sigma|}\varphi^j}{\partial x_\sigma}, & \text{if } \sigma \text{ does not contain } r+1, \dots, n, \\ 0 & \text{otherwise,} \end{cases}$$

for all $j = 1, \dots, m$, $|\sigma| \le k-1$ and some smooth function $\varphi = \varphi(x_1, \dots, x_r)$. Hence, the tangent plane H to $\mathcal{U}'$ at θ_{k-1} is spanned by the vectors of the form (3.18) on page 45 with $i = 1, \dots, r$. Consequently, a point θ_k, such that $L_{\theta_k} \supset H$, is determined by the coordinates

$$u^j_\sigma = \begin{cases} \dfrac{\partial^{|\sigma|}\varphi^j}{\partial x_\sigma}, & \text{if } \sigma \text{ does not contain } r+1, \dots, n, \\ \text{arbitrary real numbers} & \text{otherwise,} \end{cases}$$

where $j = 1, \dots, m$, $|\sigma| \le k$. Hence, if θ_k, θ'_k are two such points, then the vector $\theta_k - \theta'_k$ lies in $\operatorname{Ann}(H)$, as it follows from the proof of Proposition 3.10 on the preceding page. As it can be easily seen, any integral manifold of the Cartan distribution projecting to $\mathcal{U}'$ is contained in $\mathcal{U}$, which concludes the proof. $\square$

Remark 3.2. Note that maximal integral manifolds N of type $\dim M$ are exactly graphs of jets $j_k(\varphi), \varphi \in \Gamma_{\mathrm{loc}}(\pi)$. On the other hand, if $\operatorname{tp}(N) = 0$, then N coincides with a fiber of the projection $\pi_{k,k-1}\colon J^k(\pi) \to J^{k-1}(\pi)$.

3.6. **Classical symmetries.** Having the basic structure on $J^k(\pi)$, we can now introduce transformations preserving this structure.

Definition 3.16. Let $\mathcal{U}, \mathcal{U}' \subset J^k(\pi)$ be open domains.

(1) A diffeomorphism $F\colon \mathcal{U} \to \mathcal{U}'$ is called a *Lie transformation*, if it preserves the Cartan distribution, i.e., $F_*(\mathcal{C}^k_{\theta_k}) = \mathcal{C}^k_{F(\theta_k)}$ for any point $\theta_k \in \mathcal{U}$.

Let $\mathcal{E}$, $\mathcal{E}' \subset J^k(\pi)$ be differential equations.

(2) A Lie transformation $F\colon \mathcal{U} \to \mathcal{U}$ is called a (*local*) *equivalence*, if $F(\mathcal{U} \cap \mathcal{E}) = \mathcal{U}' \cap \mathcal{E}'$.
(3) A (local) equivalence is called a (*local*) *symmetry*, if $\mathcal{E} = \mathcal{E}'$ and $\mathcal{U} = \mathcal{U}'$.

Below we shall not distinguish between local and global versions of the concepts introduced.

Example 3.7. Consider the case $J^0(\pi) = E$. Then, since any n-dimensional horizontal plane in $T_\theta E$ is tangent to some section of the bundle π, the Cartan plane $\mathcal{C}^0_\theta$ coincides with the whole space $T_\theta E$. Thus the Cartan distribution is trivial in this case and any diffeomorphism of E is a Lie transformation.

Example 3.8. Since the Cartan distribution on $J^k(\pi)$ is locally determined by the Cartan forms (see (3.20) on page 46), the condition of F to be a Lie transformation can be reformulated as

$$F^*\omega^j_\sigma = \sum_{\alpha=1}^{m} \sum_{|\tau|<k} \lambda^{j,\alpha}_{\sigma,\tau}\omega^\alpha_\tau, \quad j = 1, \ldots, m, \quad |\sigma| < k, \tag{3.21}$$

where $\lambda^{j,\alpha}_{\sigma,\tau}$ are smooth functions on $J^k(\pi)$. Equations (3.21) are the base for computations in local coordinates.

In particular, if $\dim \pi = 1$ and $k = 1$, equations (3.21) reduce to the only condition $F^*\omega = \lambda\omega$, where $\omega = du - \sum_{i=1}^n u_i\, dx_i$. Hence, Lie transformations in this case are just *contact transformations* of the natural contact structure in $J^1(\pi)$.

Let $F\colon J^k(\pi) \to J^k(\pi)$ be a Lie transformation. Then graphs of k-jets are taken by F to n-dimensional maximal manifolds. Let θ_{k+1} be a point of $J^{k+1}(\pi)$ and represent θ_{k+1} as a pair $(\theta_k, L_{\theta_{k+1}})$, or, which is the same, as a class of graphs of k-jets tangent to each other at θ_k. Then the image $F_*(L_{\theta_{k+1}})$ will almost always be an R-plane at $F(\theta_k)$. Denote the corresponding point in $J^{k+1}(\pi)$ by $F^{(1)}(\theta_{k+1})$.

Definition 3.17. Let $F\colon J^k(\pi) \to J^k(\pi)$ be a Lie transformation. The above defined map $F^{(1)}\colon J^{k+1}(\pi) \to J^{k+1}(\pi)$ is called the 1-*lifting* of F.

The map $F^{(1)}$ is a Lie transformation at the domain of its definition, since almost everywhere it takes graphs of $(k+1)$-jets to graphs of the same kind. Hence, for any $l \geq 1$ we can define $F^{(l)} = (F^{(l-1)})^{(1)}$ and call this map the l-*lifting* of F.

Theorem 3.12. *Let $\pi\colon E \to M$ be an m-dimensional vector bundle over an n-dimensional manifold M and $F\colon J^k(\pi) \to J^k(\pi)$ be a Lie transformation. Then:*

1. *If $m > 1$ and $k > 0$, then the map F is of the form $F = G^{(k)}$ for some diffeomorphism $G\colon J^0(\pi) \to J^0(\pi)$;*
2. *If $m = 1$ and $k > 1$, then the map F is of the form $F = G^{(k-1)}$ for some contact transformation $G\colon J^1(\pi) \to J^1(\pi)$.*

Proof. Recall that fibers of the projection $\pi_{k,k-1}\colon J^k(\pi) \to J^{k-1}(\pi)$ for $k \geq 1$ are maximal integral manifolds of the Cartan distribution of type 0 (see Remark 3.2 on the page before). Further, from Proposition 3.10 on page 48 it follows in the cases $m > 1$, $k > 0$ and $m = 1$, $k > 1$ that they are integral manifolds of maximal dimension, provided $n > 1$. Therefore, the map F is $\pi_{k,\epsilon}$-fiberwise, where $\epsilon = 0$ for $m > 1$ and $\epsilon = 1$ for $m = 1$.

Thus there exists a map $G\colon J^\epsilon(\pi) \to J^\epsilon(\pi)$ such that $\pi_{k,\epsilon}\circ F = G\circ\pi_{k,\epsilon}$ and G is a Lie transformation in an obvious way. Let us show that $F = G^{(k-\epsilon)}$. To do this, note first that in fact, by the same reasons, the transformation F generates a series of Lie transformations $G_l\colon J^l(\pi) \to J^l(\pi)$, $l = \epsilon, \ldots, k$, satisfying $\pi_{l,l-1} \circ G_l = G_{l-1} \circ \pi_{l,l-1}$ and $G_k = F, G_\epsilon = G$. Let us compare the maps F and $G_{k-1}^{(1)}$.

From Proposition 3.6 on page 45 and the definition of Lie transformations we obtain

$$F_*((\pi_{k,k-1})_*^{-1}(L_{\theta_k})) = F_*(\mathcal{C}_{\theta_k}^k) = \mathcal{C}_{F(\theta_k)} = (\pi_{k,k-1})_*^{-1}(L_{F(\theta_k)})$$

for any $\theta_k \in J^k(\pi)$. But

$$F_*((\pi_{k,k-1})_*^{-1}(L_{\theta_k})) = (\pi_{k,k-1})_*^{-1}(G_{k-1,*}(L_{\theta_k}))$$

and consequently $G_{k-1,*}(L_{\theta_k}) = L_{F(\theta_k)}$. Hence, by the definition of 1-lifting we have $F = G_{k-1}^{(1)}$. Using this fact as a base of elementary induction, we obtain the result of the theorem for $\dim M > 1$.

Consider the case $n = 1$, $m = 1$ now. Since all maximal integral manifolds are one-dimensional in this case, it should be treated in a special way. Denote by $\mathcal{V}$ the distribution consisting of vector fields tangent to the fibers of the projection $\pi_{k,k-1}$. We must show that

$$F_*\mathcal{V} = \mathcal{V} \tag{3.22}$$

for any Lie transformation F, which is equivalent to F being $\pi_{k,k-1}$-fiberwise.

Let us prove (3.22). To do it, consider an arbitrary distribution $\mathcal{P}$ on a manifold N and introduce the notation

$$\mathcal{P}D = \{ X \in \mathrm{D}(N) \mid X \text{ lies in } \mathcal{P} \} \tag{3.23}$$

and

$$D_{\mathcal{P}} = \{ X \in \mathrm{D}(N) \mid [X, Y] \in \mathcal{P},\ \forall Y \in \mathcal{P}D \}. \tag{3.24}$$

Then one can show (using coordinate representation, for example) that $D\mathcal{V} = D\mathcal{C}^k \cap D_{[D\mathcal{C}^k, D\mathcal{C}^k]}$ for $k \geq 2$. But Lie transformations preserve the distributions at the right-hand side of the last equality and consequently preserve $D\mathcal{V}$. $\square$

Definition 3.18. Let $\pi\colon E \to M$ be a vector bundle and $\mathcal{E} \subset J^k(\pi)$ be a k-th order differential equation.

(1) A vector field X on $J^k(\pi)$ is called a *Lie field*, if the corresponding one-parameter group consists of Lie transformations.
(2) A Lie field is called an *infinitesimal symmetry* of the equation $\mathcal{E}$, if it is tangent to $\mathcal{E}$.

Since in the sequel we shall deal with infinitesimal symmetries only, we shall call them just *symmetries*. By definition, one-parameter groups of transformations corresponding to symmetries preserve generalized solutions[6].

Let X be a Lie field on $J^k(\pi)$ and $F_t\colon J^k(\pi) \to J^k(\pi)$ be its one-parameter group. Then we can construct the l-lifting $F_t^{(l)}\colon J^{k+l}(\pi) \to J^{k+l}(\pi)$ and the corresponding Lie field $X^{(l)}$ on $J^{k+l}(\pi)$. This field is called the *l-lifting* of the field X. As we shall see a bit later, liftings of Lie fields are defined globally and can be described explicitly. An immediate consequence of the definition and of Theorem 3.12 on page 50 is

Theorem 3.13. *Let $\pi\colon E \to M$ be an m-dimensional vector bundle over an n-dimensional manifold M and X be a Lie field on $J^k(\pi)$. Then:*

(1) *If $m > 1$ and $k > 0$, the field X is of the form $X = Y^{(k)}$ for some vector field Y on $J^0(\pi)$;*
(2) *If $m = 1$ and $k > 1$, the field X is of the form $X = Y^{(k-1)}$ for some contact vector field Y on $J^1(\pi)$.*

To finish this subsection, we describe coordinate expressions for Lie fields. Let $(x_1, \dots, x_n, \dots, u_\sigma^j, \dots)$ be a special coordinate system in $J^k(\pi)$. Then from (3.21) on page 50 it follows that

$$X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{m} \sum_{|\sigma| \le k} X_\sigma^j \frac{\partial}{\partial u_\sigma^j}$$

is a Lie field if and only if

$$X_{\sigma i}^j = D_i(X_\sigma^j) - \sum_{\alpha=1}^{n} u_{\sigma\alpha}^j D_i(X_\alpha), \tag{3.25}$$

where

$$D_i = \frac{\partial}{\partial x_i} + \sum_{j=1}^{m} \sum_{|\sigma| \ge 0} u_{\sigma i}^j \frac{\partial}{\partial u_\sigma^j} \tag{3.26}$$

are the so-called *total derivatives.*

Exercise 3.1. It is easily seen that the operators (3.26) do not preserve the algebras $\mathcal{F}_k$: they are derivations acting from $\mathcal{F}_k$ to $\mathcal{F}_{k+1}$. Prove that nevertheless for any contact field on $J^1(\pi)$, $\dim \pi = 1$, or for an arbitrary vector field on $J^0(\pi)$ (regardless of the dimension of π) the formulas above determine a vector field on $J^k(\pi)$.

Recall now that a contact field X on $J^1(\pi)$ is completely determined by its *generating function* $f = \mathrm{i}_X \omega$, where $\omega = du - \sum_i u_i \, dx_i$ is the Cartan

[6]A *generalized solution* of an equation $\mathcal{E}$ is a maximal integral manifold $N \subset \mathcal{E}$ of the Cartan distribution on $\mathcal{E}$; see [35].

(contact) form on $J^1(\pi)$. The contact field corresponding to $f \in \mathcal{F}_1(\pi)$ is denoted by X_f and is given by the expression

$$X_f = -\sum_{i=1}^{n} \frac{\partial f}{\partial u_{1_i}} \frac{\partial}{\partial x_i} + \left(f - \sum_{i=1}^{n} u_i \frac{\partial f}{\partial u_i} \right) \frac{\partial}{\partial u} + \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} + u_i \frac{\partial f}{\partial u} \right) \frac{\partial}{\partial u_i}. \tag{3.27}$$

Thus, starting with a field (3.27) in the case $\dim \pi = 1$ or with an arbitrary field on $J^0(\pi)$ for $\dim \pi > 1$ and using (3.25) on the facing page, we can obtain efficient expressions for Lie fields.

Remark 3.3. Note that in the multi-dimensional case $\dim \pi > 1$ we can introduce the functions $f^j = \mathrm{i}_X \omega^j$, where $\omega^j = du^j - \sum_i u_i^j \, dx_i$ are the Cartan forms on $J^1(\pi)$. Such a function may be understood as an element of the module $\mathcal{F}_1(\pi, \pi)$. The local conditions of a section $f \in \mathcal{F}_1(\pi, \pi)$ to generate a Lie field is as follows:

$$\frac{\partial f^\alpha}{\partial u_i^\alpha} = \frac{\partial f^\beta}{\partial u_i^\beta}, \quad \frac{\partial f^\alpha}{\partial u_i^\beta} = 0, \quad \alpha \neq \beta.$$

We call f the *generating function* (though, strictly speaking, the term *generating section* should be used) of the Lie field X, if X is a lifting of the field X_f.

Let us write down the conditions of a Lie field to be a symmetry. Assume that an equation $\mathcal{E}$ is given by the relations $F^1 = 0, \dots, F^r = 0$, where $F^j \in \mathcal{F}_k(\pi)$. Then X is a symmetry of $\mathcal{E}$ if and only if

$$X(F^j) = \sum_{\alpha=1}^{r} \lambda_\alpha^j F^\alpha, \quad j = 1, \dots, r,$$

where λ_α^j are smooth functions, or $X(F^j)\,|_{\mathcal{E}} = 0$, $j = 1, \dots, r$. These conditions can be rewritten in terms of generating sections and we shall do it later in a more general situation.

3.7. **Prolongations of differential equations.** Prolongations are differential consequences of a given differential equation. Let us give a formal definition.

Definition 3.19. Let $\mathcal{E} \subset J^k(\pi)$ be a differential equation of order k. Define the set

$$\mathcal{E}^1 = \{ \, \theta_{k+1} \in J^{k+1}(\pi) \mid \pi_{k+1,k}(\theta_{k+1}) \in \mathcal{E}, \; L_{\theta_{k+1}} \subset T_{\pi_{k+1,k}(\theta_{k+1})} \mathcal{E} \, \}$$

and call it the *first prolongation* of the equation $\mathcal{E}$.

If the first prolongation $\mathcal{E}^1$ is a submanifold in $J^{k+1}(\pi)$, we define the second prolongation of $\mathcal{E}$ as $(\mathcal{E}^1)^1 \subset J^{k+2}(\pi)$, etc. Thus the *$l$-th prolongation* is a subset $\mathcal{E}^l \subset J^{k+l}(\pi)$.

Let us redefine the notion of l-th prolongation directly. Namely, take a point $\theta_k \in \mathcal{E}$ and consider a section $\varphi \in \Gamma_{\mathrm{loc}}(\pi)$ such that the graph of $j_k(\varphi)$ is tangent to $\mathcal{E}$ with order l. Let $\pi_k(\theta_k) = x \in M$. Then $[\varphi]_x^{k+l}$ is a point of $J^{k+l}(\pi)$ and the set of all points obtained in such a way obviously coincides with $\mathcal{E}^l$, provided all intermediate prolongations $\mathcal{E}^1, \dots, \mathcal{E}^{l-1}$ be well defined in the sense of Definition 3.19 on the page before.

Assume now that locally $\mathcal{E}$ is given by the equations $F^1 = 0, \dots, F^r = 0$, $F^j \in \mathcal{F}_k(\pi)$ and $\theta_k \in \mathcal{E}$ is the origin of the chosen special coordinate system. Let $u^1 = \varphi^1(x_1, \dots, x_n), \dots, u^m = \varphi^m(x_1, \dots, x_n)$ be a local section of the bundle π. Then the equations of the first prolongation are

$$\frac{\partial F^j}{\partial x_i} + \sum_{\alpha,\sigma} \frac{\partial F^j}{\partial u^\alpha_\sigma} u^\alpha_{\sigma i} = 0, \quad i = 1, \dots, n, \quad j = 1, \dots, r,$$

combined with the initial equations $F^r = 0$. From here and by comparison with the coordinate representation of prolongations for nonlinear differential operators (see Subsection 3.2), we obtain the following result:

Proposition 3.14. *Let $\mathcal{E} \subset J^k(\pi)$ be a differential equation. Then*

1. *If the equation $\mathcal{E}$ is determined by a differential operator $\Delta\colon \Gamma(\pi) \to \Gamma(\pi')$, then its l-th prolongation is given by the l-th prolongation $\Delta^{(l)}\colon \Gamma(\pi) \to \Gamma(\pi'_l)$ of the operator Δ.*
2. *If $\mathcal{E}$ is locally described by the system $F^1 = 0, \dots, F^r = 0$, $F^j \in \mathcal{F}_k(\pi)$, then the system*

$$D_\sigma F^j = 0, \quad |\sigma| \le l, j = 1, \dots, r, \tag{3.28}$$

where $D_\sigma = D_{i_1} \circ \dots \circ D_{i_{|\sigma|}}$, $\sigma = i_1 \dots i_{|\sigma|}$, corresponds to $\mathcal{E}^l$. Here D_i stands for the i-th total derivative (*see* (3.26) *on page* 52).

From the definition it follows that for any $l \ge l' \ge 0$ one has $\pi_{k+l,k+l'}(\mathcal{E}^l) \subset \mathcal{E}^{l'}$ and consequently one has the maps $\pi_{k+l,k+l'}\colon \mathcal{E}^l \to \mathcal{E}^{l'}$.

Definition 3.20. An equation $\mathcal{E} \subset J^k(\pi)$ is called *formally integrable*, if

1. all prolongations $\mathcal{E}^l$ are smooth manifolds and
2. all the maps $\pi_{k+l+1,k+l}\colon \mathcal{E}^{l+1} \to \mathcal{E}^l$ are smooth fiber bundles.

Definition 3.21. The inverse limit $\operatorname{proj\,lim}_{l\to\infty} \mathcal{E}^l$ with respect to projections $\pi_{l+1,l}$ is called the *infinite prolongation* of the equation $\mathcal{E}$ and is denoted by $\mathcal{E}^\infty \subset J^\infty(\pi)$.

3.8. **Basic structures on infinite prolongations.** Let $\pi\colon E \to M$ be a vector bundle and $\mathcal{E} \subset J^k(\pi)$ be a k-th order differential equation. Then we have embeddings $\varepsilon_l\colon \mathcal{E}^l \subset J^{k+l}(\pi)$ for all $l \geq 0$. Since, in general, the sets $\mathcal{E}^l$ may not be smooth manifolds, we define a *function* on $\mathcal{E}^l$ as the restriction $f\,|_{\mathcal{E}^l}$ of a smooth function $f \in \mathcal{F}_{k+l}(\pi)$. The set $\mathcal{F}_l(\mathcal{E})$ of all functions on $\mathcal{E}^l$ forms an $\mathbb{R}$-algebra in a natural way and $\varepsilon_l^*\colon \mathcal{F}_{k+l}(\pi) \to \mathcal{F}_l(\mathcal{E})$ is a homomorphism of algebras. In the case of formally integrable equations, the algebra $\mathcal{F}_l(\mathcal{E})$ coincides with $C^\infty(\mathcal{E}^l)$. Let $I_l = \ker \varepsilon_l^*$. Evidently, $I_l(\mathcal{E}) \subset I_{l+1}(\mathcal{E})$. Then $I(\mathcal{E}) = \bigcup_{l\geq 0} I_l(\mathcal{E})$ is an ideal in $\mathcal{F}(\pi)$ which is called the *ideal of the equation* $\mathcal{E}$. The *function algebra* on $\mathcal{E}^\infty$ is the quotient algebra $\mathcal{F}(\mathcal{E}) = \mathcal{F}(\pi)/I(\mathcal{E})$ and coincides with $\operatorname{inj\,lim}_{l\to\infty} \mathcal{F}_l(\mathcal{E})$ with respect to the system of homomorphisms $\pi^*_{k+l+1,k+l}$. For all $l \geq 0$, we have the homomorphisms $\varepsilon_l^*\colon \mathcal{F}_l(\mathcal{E}) \to \mathcal{F}(\mathcal{E})$. When $\mathcal{E}$ is formally integrable, they are monomorphic, but in any case the algebra $\mathcal{F}(\mathcal{E})$ is filtered by the images of ε_l^*.

To construct differential calculus on $\mathcal{E}^\infty$, one needs the general algebraic scheme exposed in Section 1 and applied to the filtered algebra $\mathcal{F}(\mathcal{E})$. However, in the case of formally integrable equations, due to the fact that all $\mathcal{E}^l$ are smooth manifolds, this scheme may be simplified and combined with a purely geometrical approach (cf. with similar constructions of Subsection 3.3).

In special coordinates the infinite prolongation of the equation $\mathcal{E}$ is determined by the system similar to (3.28) on the preceding page with the only difference that $|\sigma|$ is unlimited now. Thus, the ideal $I(\mathcal{E})$ is generated by the functions $D_\sigma F^j$, $|\sigma| \geq 0$, $j = 1, \dots, m$. From these remarks we obtain the following important fact.

Remark 3.4. Let $\mathcal{E}$ be a formally integrable equations. Then from the above said it follows that the ideal $I(\mathcal{E})$ is stable with respect to the action of the total derivatives D_i, $i = 1, \dots, n$. Consequently, the restrictions $D_i^{\mathcal{E}} = D_i\,|_{\mathcal{E}} : \mathcal{F}(\mathcal{E}) \to \mathcal{F}(\mathcal{E})$ are well defined and $D_i^{\mathcal{E}}$ are filtered derivations. In other words, we obtain that the vector fields D_i are tangent to any infinite prolongation and thus determine vector fields on $\mathcal{E}^\infty$. We shall often skip the superscript $\mathcal{E}$ in the notation of the above defined restrictions.

Example 3.9. Consider a system of evolution equations of the form

$$u_t^j = f^j(x, t, \dots, u^\alpha, \dots, u_x^\alpha, \dots), \quad j, \alpha = 1, \dots, m.$$

Then the set of functions $x_1, \dots, x_n, t, \dots, u^j_{i_1,\dots,i_r,0}$ with $1 \leq i_k \leq n$, $j = 1, \dots, m$, where $t = x_{n+1}$, may be taken for *internal coordinates* on $\mathcal{E}^\infty$.

The total derivatives restricted to $\mathcal{E}^\infty$ are expressed as

$$
\begin{aligned}
D_i =& \frac{\partial}{\partial x_i} + \sum_{j=1}^{n} \sum_{|\sigma|\geq 0} u^j_{\sigma i} \frac{\partial}{\partial u^j_\sigma}, \ i = 1, \dots, n, \\
D_t =& \frac{\partial}{\partial t} + \sum_{j=1}^{n} \sum_{|\sigma|\geq 0} D_\sigma(f^j) \frac{\partial}{\partial u^j_\sigma}
\end{aligned}
\tag{3.29}
$$

in these coordinates, while the Cartan forms restricted to $\mathcal{E}^\infty$ are written down as

$$
\omega^j_\sigma = du^j_\sigma - \sum_{i=1}^{n} u^j_{\sigma i}\, dx_i - D_\sigma(f^j)\, dt. \tag{3.30}
$$

Let $\pi\colon E \to M$ be a vector bundle and $\mathcal{E} \subset J^k(\pi)$ be a formally integrable equation.

Definition 3.22. Let $\theta \in J^\infty(\pi)$. Then

(1) The *Cartan plane* $\mathcal{C}_\theta = \mathcal{C}_\theta(\pi) \subset T_\theta J^\infty(\pi)$ at θ is the linear envelope of tangent planes to all manifolds $j_\infty(\varphi)(M)$, $\varphi \in \Gamma(\pi)$, passing through θ.
(2) If $\theta \in \mathcal{E}^\infty$, then the intersection $\mathcal{C}_\theta(\mathcal{E}) = \mathcal{C}_\theta(\pi) \cap T_\theta\mathcal{E}^\infty$ is called *Cartan plane* of $\mathcal{E}^\infty$ at θ.

The correspondence $\theta \mapsto \mathcal{C}_\theta(\pi)$, $\theta \in J^\infty(\pi)$ (respectively, $\theta \mapsto \mathcal{C}_\theta(\mathcal{E}^\infty)$, $\theta \in \mathcal{E}^\infty$) is called the *Cartan distribution* on $J^\infty(\pi)$ (respectively, on $\mathcal{E}^\infty$).

Proposition 3.15. *For any vector bundle $\pi\colon E \to M$ and a formally integrable equation $\mathcal{E} \subset J^k(\pi)$ one has:*

(1) *The Cartan plane $\mathcal{C}_\theta(\pi)$ is n-dimensional at any point $\theta \in J^\infty(\pi)$.*
(2) *Any point $\theta \in \mathcal{E}^\infty$ is generic, i.e., $\mathcal{C}_\theta(\pi) \subset T_\theta\mathcal{E}^\infty$ and thus one has $\mathcal{C}_\theta(\mathcal{E}^\infty) = \mathcal{C}_\theta(\pi)$.*
(3) *Both distributions, $\mathcal{C}(\pi)$ and $\mathcal{C}(\mathcal{E}^\infty)$, are integrable.*

Proof. Let $\theta \in J^\infty(\pi)$ and $\pi_\infty(\theta) = x \in M$. Then the point θ completely determines all partial derivatives of any section $\varphi \in \Gamma_{\mathrm{loc}}(\pi)$ such that its graph passes through θ. Consequently, all such graphs have a common tangent plane at this point, which coincides with $\mathcal{C}_\theta(\pi)$. This proves the first statement.

To prove the second one, recall Remark 3.4 on the preceding page: locally, any vector field D_i is tangent to $\mathcal{E}^\infty$. But as it follows from (3.20) on page 46, $\mathrm{i}_{D_i}\omega^j_\sigma = 0$ for any D_i and for any Cartan form ω^j_σ. Hence, linear independent vector fields $D_1, \dots, D_n$ locally lie both in $\mathcal{C}(\pi)$ and in $\mathcal{C}(\mathcal{E}^\infty)$ which gives the result.

Finally, as it follows from the above said, the module

$$\mathcal{C}\mathrm{D}(\pi) = \{\, X \in \mathrm{D}(\pi) \mid X \text{ lies in } \mathcal{C}(\pi) \,\} \tag{3.31}$$

is locally generated by the fields $D_1, \dots, D_n$. But it is easily seen that $[D_\alpha, D_\beta] = 0$ for all $\alpha, \beta = 1, \dots, n$ and consequently $[\mathcal{C}\mathrm{D}(\pi), \mathcal{C}\mathrm{D}(\pi)] \subset \mathcal{C}\mathrm{D}(\pi)$. The same reasoning is valid for

$$\mathcal{C}\mathrm{D}(\mathcal{E}) = \{\, X \in \mathrm{D}(\mathcal{E}^\infty) \mid X \text{ lies in } \mathcal{C}(\mathcal{E}^\infty) \,\}. \tag{3.32}$$

This completes the proof of the proposition. □

Proposition 3.16. *Maximal integral manifolds of the Cartan distribution $\mathcal{C}(\pi)$ are graph of infinite jets of sections $j_\infty(\varphi)$, $\varphi \in \Gamma_{\mathrm{loc}}(\pi)$.*

Proof. Note first that graphs of infinite jets are integral manifolds of the Cartan distribution of maximal dimension (equaling to n) and that any integral manifold projects to $J^k(\pi)$ and M without singularities.

Let now $N \subset J^\infty(\pi)$ be an integral manifold and $N^k = \pi_{\infty,k} N \subset J^k(\pi)$, $N' = \pi_\infty N \subset M$. Hence, there exists a diffeomorphism $\varphi' \colon N' \to N^0$ such that $\pi \circ \varphi' = \mathrm{id}_{N'}$. Then by the Whitney theorem on extension for smooth functions (see [38]), there exists a local section $\varphi \colon M \to E$ satisfying $\varphi\,|_{N'} = \varphi'$ and $j_k(\varphi)(M) \supset N^k$ for all $k > 0$. Consequently, $j_\infty(\varphi)(M) \supset N$. □

Corollary 3.17. *Maximal integral manifolds of the Cartan distribution on $\mathcal{E}^\infty$ coincide locally with graphs of infinite jets of solutions.*

Consider a point $\theta \in J^\infty(\pi)$ and let $x = \pi_\infty(\theta) \in M$ be its projection to M. Let v be a tangent vector to M at the point x. Then, since the Cartan plane $\mathcal{C}_\theta$ isomorphically projects to $T_x M$, there exists a unique tangent vector $\mathcal{C}v \in T_\theta J^\infty(\pi)$ such that $(\pi_\infty)_*(\mathcal{C}v) = v$. Hence, for any vector field $X \in \mathrm{D}(M)$ we can define a vector field $\mathcal{C}X \in \mathrm{D}(\pi)$ by setting $(\mathcal{C}X)_\theta = \mathcal{C}(X_{\pi_\infty(\theta)})$. Then, by construction, the field $\mathcal{C}X$ is projected by $(\pi_\infty)_*$ to X while the correspondence $\mathcal{C} \colon \mathrm{D}(M) \to \mathrm{D}(\pi)$ is a $C^\infty(M)$-linear one. In other words, this correspondence is a linear connection in the bundle $\pi_\infty \colon J^\infty(\pi) \to M$.

Definition 3.23. The connection $\mathcal{C} \colon \mathrm{D}(M) \to \mathrm{D}(\pi)$ defined above is called the *Cartan connection* in $J^\infty(\pi)$.

For any formally integrable equation, the Cartan connection is obviously restricted to the bundle $\pi_\infty \colon \mathcal{E}^\infty \to M$ and we preserve the same notation $\mathcal{C}$ for this restriction.

Let $(x_1, \dots, x_n, \dots, u_\sigma^j, \dots)$ be a special coordinate system in $J^\infty(\pi)$ and $X = X_1 \partial/\partial x_1 + \dots + X_n \partial/\partial x_n$ be a vector field on M represented in this coordinate system. Then the field $\mathcal{C}X$ is to be of the form $\mathcal{C}X = X + X^v$, where $X^v = \sum_{j,\sigma} X^j_\sigma \partial/\partial u^j_\sigma$ is a π_∞-vertical field. The defining conditions $\mathrm{i}_{\mathcal{C}X}\omega^j_\sigma = 0$, where ω^j_σ are the Cartan forms on $J^\infty(\pi)$, imply

$$\mathcal{C}X = \sum_{i=1}^{n} X_i \left(\frac{\partial}{\partial x_i} + \sum_{j,\sigma} u^j_{\sigma i} \frac{\partial}{\partial u^j_\sigma} \right) = \sum_{i=1}^{n} X_i D_i. \tag{3.33}$$

In particular, $\mathcal{C}(\partial/\partial x_i) = D_i$, i.e., total derivatives are the liftings to $J^\infty(\pi)$ of the corresponding partial derivatives by the Cartan connection.

Let now V be a vector field on $\mathcal{E}^\infty$ and $\theta \in \mathcal{E}^\infty$ be a point. Then the vector V_θ can be projected parallel to the Cartan plane $\mathcal{C}_\theta$ to the fiber of the projection $\pi_\infty \colon \mathcal{E}^\infty \to M$ passing through θ. Thus we get a vertical vector field V^v. Hence, for any $f \in \mathcal{F}(\mathcal{E})$ a differential one-form $U_{\mathcal{C}}(f) \in \Lambda^1(\mathcal{E})$ is defined by

$$\mathrm{i}_V(U_{\mathcal{C}}(f)) = V^v(f), \quad V \in \mathrm{D}(\mathcal{E}). \tag{3.34}$$

The correspondence $f \mapsto U_{\mathcal{C}}(f)$ is a derivation of the algebra $\mathcal{F}(\mathcal{E})$ with the values in the $\mathcal{F}(\mathcal{E})$-module $\Lambda^1(\mathcal{E})$, i.e., $U_{\mathcal{C}}(fg) = fU_{\mathcal{C}}(g) + gU_{\mathcal{C}}(f)$ for all $f, g \in \mathcal{F}(\mathcal{E})$.

Definition 3.24. The derivation $U_{\mathcal{C}} \colon \mathcal{F}(\mathcal{E}) \to \Lambda^1(\mathcal{E})$ is called the *structural element* of the equation $\mathcal{E}$.

In the case $\mathcal{E}^\infty = J^\infty(\pi)$ the structural element $U_{\mathcal{C}}$ is locally represented in the form

$$U_{\mathcal{C}} = \sum_{j,\sigma} \omega^j_\sigma \otimes \frac{\partial}{\partial u^j_\sigma}, \tag{3.35}$$

where ω^j_σ are the Cartan forms on $J^\infty(\pi)$. To obtain the expression in the general case, one needs to rewrite (3.35) in local coordinates.

The following result is a consequence of definitions:

Proposition 3.18. *For any vector field $X \in \mathrm{D}(M)$ the equality*

$$j_\infty(\varphi)^*(\mathcal{C}X(f)) = X(j_\infty(\varphi)^*(f)), \quad f \in \mathcal{F}(\pi), \quad \varphi \in \Gamma_{\mathrm{loc}}(\pi), \tag{3.36}$$

takes place. Equality (3.36) *uniquely determines the Cartan connection in $J^\infty(\pi)$.*

Corollary 3.19. *The Cartan connection in $\mathcal{E}^\infty$ is flat, i.e., $\mathcal{C}[X,Y] = [\mathcal{C}X, \mathcal{C}Y]$ for any $X, Y \in \mathrm{D}(M)$.*

Proof. Consider the case $\mathcal{E}^\infty = J^\infty(\pi)$. Then from Proposition 3.18 we have

$$\begin{aligned} j_\infty(\varphi)^*(\mathcal{C}[X,Y](f)) &= [X,Y](j_\infty(\varphi)^*(f)) \\ &= X(Y(j_\infty(\varphi)^*(f))) - Y(X(j_\infty(\varphi)^*(f))) \end{aligned}$$

for any $\varphi \in \Gamma_{\mathrm{loc}}(\pi)$, $f \in \mathcal{F}(\pi)$. On the other hand,

$$\begin{aligned} j_\infty(\varphi)^*([\mathcal{C}X, \mathcal{C}Y](f)) &= j_\infty(\varphi)^*(\mathcal{C}X(\mathcal{C}Y(f)) - \mathcal{C}Y(\mathcal{C}X(f))) \\ &= X(j_\infty(\varphi)^*(Y(f))) - Y(j_\infty(\varphi)^*(\mathcal{C}X(f))) \\ &= X(Y(j_\infty(\varphi)^*(f))) - Y(X(j_\infty(\varphi)^*(f))). \end{aligned}$$

To prove the statement for an arbitrary formally integrable equation $\mathcal{E}$, it suffices to note that the Cartan connection in $\mathcal{E}^\infty$ is obtained by restricting the fields $\mathcal{C}X$ to infinite prolongation of $\mathcal{E}$. $\square$

The construction of Proposition 3.18 on the preceding page can be generalized. Let $\pi\colon E \to M$ be a vector bundle and $\xi_1\colon E_1 \to M$, $\xi_2\colon E_2 \to M$ be another two vector bundles.

Definition 3.25. Let $\Delta\colon \Gamma(\xi_1) \to \Gamma(\xi_2)$ be a linear differential operator. The *lifting* $\mathcal{C}\Delta\colon \mathcal{F}(\pi, \xi_1) \to \mathcal{F}(\pi, \xi_2)$ of the operator Δ is defined by

$$j_\infty(\varphi)^*(\mathcal{C}\Delta(f)) = \Delta(j_\infty(\varphi)^*(f)), \tag{3.37}$$

where $\varphi \in \Gamma_{\mathrm{loc}}(\pi)$, $f \in \mathcal{F}(\pi, \xi_1)$ are arbitrary sections.

Proposition 3.20. *Let $\pi\colon E \to M$, $\xi_i\colon E_i \to M$, $i = 1, 2, 3$, be vector bundles. Then*

(1) *For any $C^\infty(M)$-linear differential operator $\Delta\colon \Gamma(\xi_1) \to \Gamma(\xi_2)$, the operator $\mathcal{C}\Delta$ is an $\mathcal{F}(\pi)$-linear differential operator of the same order.*

(2) *For any $\Delta, \square\colon \Gamma(\xi_1) \to \Gamma(\xi_2)$ and $f, g \in \mathcal{F}(\pi)$, one has*

$$\mathcal{C}(f\Delta + g\square) = f\mathcal{C}\Delta + g\mathcal{C}\square.$$

(3) *For $\Delta_1\colon \Gamma(\xi_1) \to \Gamma(\xi_2)$ and $\Delta_2\colon \Gamma(\xi_2) \to \Gamma(\xi_3)$, one has*

$$\mathcal{C}(\Delta_2 \circ \Delta_1) = \mathcal{C}\Delta_2 \circ \mathcal{C}\Delta_1.$$

From this proposition and from Proposition 3.18 on the facing page it follows that if Δ is a scalar differential operator $C^\infty(M) \to C^\infty(M)$ locally represented as $\Delta = \sum_\sigma a_\sigma \partial^{|\sigma|}/\partial x_\sigma$, $a_\sigma \in C^\infty(M)$, then $\mathcal{C}\Delta = \sum_\sigma a_\sigma D_\sigma$ in the corresponding special coordinates. If $\Delta = \|\Delta_{ij}\|$ is a matrix operator, then $\mathcal{C}\Delta = \|\mathcal{C}\Delta_{ij}\|$. Obviously, $\mathcal{C}\Delta$ may be understood as a constant differential operator acting from sections of the bundle π to linear differential operators from $\Gamma(\xi_1)$ to $\Gamma(\xi_2)$. This observation is generalized as follows.

Definition 3.26. An $\mathcal{F}(\pi)$-linear differential operator Δ acting from the module $\mathcal{F}(\pi, \xi_1)$ to $\mathcal{F}(\pi, \xi_2)$ is called a *$\mathcal{C}$-differential operator*, if it admits restriction to graphs of infinite jets, i.e., if for any section $\varphi \in \Gamma(\pi)$ there exists an operator $\Delta_\varphi\colon \Gamma(\xi_1) \to \Gamma(\xi_2)$ such that

$$j_\infty(\varphi)^*(\Delta(f)) = \Delta_\varphi(j_\infty(\varphi)^*(f)) \tag{3.38}$$

for all $f \in \mathcal{F}(\pi, \xi_1)$.

Thus, $\mathcal{C}$-differential operators are nonlinear differential operators taking their values in $C^\infty(M)$-modules of linear differential operators.

Exercise 3.2. Consider a $\mathcal{C}$-differential operator $\Delta\colon \mathcal{F}(\pi,\xi_1) \to \mathcal{F}(\pi,\xi_2)$. Prove that if $\Delta(\pi^*(f)) = 0$ for all $f \in \Gamma(\xi_1)$, then $\Delta = 0$.

Proposition 3.21. *Let π, ξ_1, and ξ_2 be vector bundles over M. Then any $\mathcal{C}$-differential operator $\Delta\colon \mathcal{F}(\pi,\xi_1) \to \mathcal{F}(\pi,\xi_2)$ can be presented in the form $\Delta = \sum_\alpha a_\alpha \mathcal{C}\Delta_\alpha$, $a_\alpha \in \mathcal{F}(\pi)$, where Δ_α are linear differential operators acting from $\Gamma(\xi_1)$ to $\Gamma(\xi_2)$.*

Proof. Recall that we consider the filtered theory; in particular, there exists an integer l such that $\Delta(\mathcal{F}_k(\pi,\xi_1)) \subset \mathcal{F}_{k+l}(\pi,\xi_2)$ for all k. Consequently, since $\Gamma(\xi_1)$ is embedded into $\mathcal{F}_0(\pi,\xi_1)$, we have $\Delta(\Gamma(\xi_1)) \subset \mathcal{F}_l(\pi,\xi_2)$ and the restriction $\bar{\Delta} = \Delta\big|_{\Gamma(\xi_1)}$ is a $C^\infty(M)$-differential operator taking its values in $\mathcal{F}_l(\pi,\xi_2)$.

On the other hand, the operator $\bar{\Delta}$ is represented in the form $\bar{\Delta} = \sum_\alpha a_\alpha \Delta_\alpha$, $a_\alpha \in \mathcal{F}_l(\pi)$, with $\Delta_\alpha\colon \Gamma(\xi_1) \to \Gamma(\xi_2)$ being $C^\infty(M)$-linear differential operators. Define $\mathcal{C}\bar{\Delta} = \sum_\alpha a_\alpha \mathcal{C}\Delta_\alpha$. Then the difference $\Delta - \mathcal{C}\bar{\Delta}$ is a $\mathcal{C}$-differential operator such that its restriction to $\Gamma(\xi_1)$ vanishes. Therefore, by Exercise 3.2 $\Delta = \mathcal{C}\bar{\Delta}$. ☐

Corollary 3.22. *$\mathcal{C}$-differential operators admit restrictions to infinite prolongations: if $\Delta\colon \mathcal{F}(\pi,\xi_1) \to \mathcal{F}(\pi,\xi_2)$ is a $\mathcal{C}$-differential operator and $\mathcal{E} \subset J^k(\pi)$ is a k-th order equation, then there exists a linear differential operator $\Delta_\mathcal{E}\colon \mathcal{F}(\mathcal{E},\xi_1) \to \mathcal{F}(\mathcal{E},\xi_2)$ such that $\varepsilon^* \circ \Delta = \Delta_\mathcal{E} \circ \varepsilon^*$, where $\varepsilon\colon \mathcal{E}^\infty \hookrightarrow J^\infty(\pi)$ is the natural embedding.*

Proof. The result immediately follows from Remark 3.4 on page 55 and from Proposition 3.21. ☐

Example 3.10. Let $\xi_1 = \tau_i^*$, $\xi_2 = \tau_{i+1}^*$, where $\tau_p^*\colon \bigwedge^p T^*M \to M$ (see Example 3.2 on page 38), and $\Delta = d\colon \Lambda^i(M) \to \Lambda^{i+1}(M)$ be the de Rham differential. Then we obtain the first-order operator $\bar{d} = \mathcal{C}d\colon \bar{\Lambda}^i(\pi) \to \bar{\Lambda}^{i+1}(\pi)$, where $\bar{\Lambda}^p(\pi)$ denotes the module $\mathcal{F}(\pi,\tau_p^*)$. Due Corollary 3.22 the operators $\bar{d}\colon \bar{\Lambda}^i(\mathcal{E}) \to \bar{\Lambda}^{i+1}(\mathcal{E})$ are also defined, where $\bar{\Lambda}^p(\mathcal{E}) = \mathcal{F}(\mathcal{E},\tau_p^*)$.

Definition 3.27. Let $\mathcal{E} \subset J^k(\pi)$ be an equation.

(1) Elements of the module $\bar{\Lambda}^i(\mathcal{E})$ are called *horizontal i-forms* on the infinite prolongation $\mathcal{E}^\infty$.
(2) The operator $\bar{d}\colon \bar{\Lambda}^i(\mathcal{E}) \to \bar{\Lambda}^{i+1}(\mathcal{E})$ is called the *horizontal de Rham differential* on $\mathcal{E}^\infty$.

From Proposition 3.20 (3) on the preceding page it follows that $\bar{d} \circ \bar{d} = 0$. The sequence

$$0 \to \mathcal{F}(\mathcal{E}) \xrightarrow{\bar{d}} \bar{\Lambda}^1(\mathcal{E}) \to \cdots \to \bar{\Lambda}^i(\mathcal{E}) \xrightarrow{\bar{d}} \bar{\Lambda}^{i+1}(\mathcal{E}) \to \cdots$$

is called the *horizontal de Rham complex* of the equation $\mathcal{E}$. Its cohomology is called the *horizontal de Rham cohomology* of $\mathcal{E}$ and is denoted by $\bar{H}^*(\mathcal{E}) = \bigoplus_{i\geq 0} \bar{H}^i(\mathcal{E})$.

In local coordinates, horizontal forms of degree p on $\mathcal{E}^\infty$ are represented as $\omega = \sum_{i_1<\dots<i_p} a_{i_1\dots i_p} dx_{i_1} \wedge \dots \wedge dx_{i_p}$, where $a_{i_1\dots i_p} \in \mathcal{F}(\mathcal{E})$, while the horizontal de Rham differential acts as

$$\bar{d}(\omega) = \sum_{i=1}^{n} \sum_{i_1<\dots<i_p} D_i(a_{i_1\dots i_p})\, dx_i \wedge dx_{i_1} \wedge \dots \wedge dx_{i_p}. \tag{3.39}$$

In particular, we see that both $\bar{\Lambda}^i(\mathcal{E})$ and $\bar{H}^i(\mathcal{E})$ vanish for $i > \dim M$.

Consider the algebra $\Lambda^*(\mathcal{E})$ of all differential forms on $\mathcal{E}^\infty$ and note that one has the embedding $\bar{\Lambda}^*(\mathcal{E}) \hookrightarrow \Lambda^*(\mathcal{E})$. Let us extend the horizontal de Rham differential to this algebra as follows:

$$\bar{d}(d\omega) = -d(\bar{d}(\omega)), \quad \bar{d}(\omega\wedge\theta) = \bar{d}(\omega)\wedge\theta + (-1)^p\omega\wedge\bar{d}(\theta) \quad \omega \in \Lambda^p(\mathcal{E}).$$

Obviously, these conditions define the differential $\bar{d}\colon \Lambda^i(\mathcal{E}) \to \Lambda^{i+1}(\mathcal{E})$ and its restriction to $\bar{\Lambda}^*(\mathcal{E})$ coincides with the horizontal de Rham differential.

Let us also set $d_{\mathcal{C}} = d - \bar{d}\colon \Lambda^*(\mathcal{E}) \to \Lambda^*(\mathcal{E})$ and call $d_{\mathcal{C}}$ the *Cartan* (or *vertical*) *differential* on $\mathcal{E}^\infty$. Then from definitions we obtain

$$d = \bar{d} + d_{\mathcal{C}}, \quad \bar{d}\circ\bar{d} = d_{\mathcal{C}}\circ d_{\mathcal{C}} = 0, \quad d_{\mathcal{C}}\circ\bar{d} + \bar{d}\circ d_{\mathcal{C}} = 0,$$

i.e., the pair $(\bar{d}, d_{\mathcal{C}})$ forms a bicomplex in $\Lambda^*(\mathcal{E})$ with the total differential d. It is called the *variational bicomplex* and will be discussed in more details in Section 7.

Denote by $\mathcal{C}\Lambda^1(\mathcal{E})$ the *Cartan submodule* in $\Lambda^1(\mathcal{E})$, i.e., the module of 1-forms vanishing on the Cartan distribution on $\mathcal{E}^\infty$ (cf. with Definition 3.12 on page 46). Then the splitting $d = \bar{d} + d_{\mathcal{C}}$ implies the direct sum decomposition

$$\Lambda^1(\mathcal{E}) = \bar{\Lambda}^1(\mathcal{E}) \oplus \mathcal{C}\Lambda^1(\mathcal{E}),$$

which gives

$$\Lambda^i(\mathcal{E}) = \bigoplus_{p+q=i} \bar{\Lambda}^q(\mathcal{E}) \otimes_{\mathcal{F}(\mathcal{E})} \mathcal{C}^p\Lambda(\mathcal{E}), \tag{3.40}$$

where $\mathcal{C}^p\Lambda(\mathcal{E}) = \underbrace{\mathcal{C}\Lambda^1(\mathcal{E}) \wedge \dots \wedge \mathcal{C}\Lambda^1(\mathcal{E})}_{p \text{ times}}$.

To conclude this section, we shall write down the coordinate representation for the Cartan differential $d_{\mathcal{C}}$ and the extended differential $\bar{d}$. First note that by definition and due to representation (3.39), one has

$$d_{\mathcal{C}}(f) = \sum_{j,\sigma} \frac{\partial f}{\partial u^j_\sigma}\omega^j_\sigma, \quad f \in \mathcal{F}(\pi). \tag{3.41}$$

In particular, $d_{\mathcal{C}}$ takes coordinate functions u^j_σ to the corresponding Cartan forms. This is reason why we called $d_{\mathcal{C}}$ the Cartan differential on $\mathcal{E}^\infty$. It is easily seen that $d_{\mathcal{C}}\big|_{\mathcal{F}(\mathcal{E})} = U_{\mathcal{C}}(\mathcal{E})$ (see Definition 3.24 on page 58). To finish computations, it suffices to compute $\bar{d}(\omega^j_\sigma)$:

$$\bar{d}(\omega^j_\sigma) = \bar{d}d_{\mathcal{C}}(u^j_\sigma) = -d_{\mathcal{C}}\bar{d}(u^j_\sigma)$$

and thus

$$\bar{d}(\omega^j_\sigma) = -\sum_{i=1}^{n} \omega^j_{\sigma i} \wedge dx_i. \tag{3.42}$$

Note that from the results obtained it follows that

$$\begin{aligned}\bar{d}(\bar{\Lambda}^q(\mathcal{E}) \otimes \mathcal{C}^p\Lambda(\mathcal{E})) &\subset \bar{\Lambda}^{q+1}(\mathcal{E}) \otimes \mathcal{C}^p\Lambda(\mathcal{E}),\\ d_{\mathcal{C}}(\bar{\Lambda}^q(\mathcal{E}) \otimes \mathcal{C}^p\Lambda(\mathcal{E})) &\subset \bar{\Lambda}^q(\mathcal{E}) \otimes \mathcal{C}^{p+1}\Lambda(\mathcal{E}).\end{aligned}$$

Now let us define the *module of horizontal jets*. Let ξ be a vector bundle over M. Say that two elements of $P = \mathcal{F}(\mathcal{E}, \xi)$ are horizontally equivalent up to order $k \leq \infty$ at point $\theta \in \mathcal{E}^\infty$, if their total derivatives up to order k coincide at θ. The horizontal jet space $\bar{J}^k_\theta(P)$ is P modulo this relation, and the collection $\bar{J}^k(P) = \bigcup_{\theta \in \mathcal{E}^\infty} \bar{J}^k_\theta(P)$ constitutes the *horizontal jet bundle* $\bar{J}^k(P) \to \mathcal{E}^\infty$. We denote the module of sections of horizontal jet bundle by $\bar{\mathcal{J}}^k(P)$.

As with the usual jet bundles, there exist the natural $\mathcal{C}$-differential operators

$$\bar{\jmath}_k \colon P \to \bar{\mathcal{J}}^k(P),$$

and the natural projections $\nu_{k,l} \colon \bar{\mathcal{J}}^k(P) \to \bar{\mathcal{J}}^l(P)$ such that $\nu_{k,l} \circ \bar{\jmath}_k = \bar{\jmath}_l$. The operators $\bar{\jmath}_k$ and $\nu_{k,l}$ are restrictions of the operators $\mathcal{C}j_k$ and $\mathcal{C}\pi^*_{k,l}$ to $\mathcal{E}^\infty$.

$\mathcal{C}$-differential operators, horizontal forms and jets constitute a "subtheory" in the differential calculus on an infinitely prolonged equation. It is, roughly speaking, "the total derivatives calculus" and is called *$\mathcal{C}$-differential calculus*. It is easily shown that all components of usual calculus and the Lagrangian formalism discussed above have their counterparts in the framework of $\mathcal{C}$-differential calculus. All constructions of Sections 1 and 2 are carried over into $\mathcal{C}$-differential calculus word for word as long as the operators, jets, and forms in them are assumed respectively $\mathcal{C}$-differential and horizontal.

3.9. **Higher symmetries.** Let $\pi \colon E \to M$ be a vector bundle and $\mathcal{E} \subset J^k(\pi)$ be a differential equation. We shall still assume $\mathcal{E}$ to be formally integrable, though it not restrictive in this context.

Consider a symmetry $F : J^k(\pi) \to J^k(\pi)$ of the equation $\mathcal{E}$ and let θ_{k+1} be a point of the first prolongation $\mathcal{E}^1$ such that $\pi_{k+1,k}(\theta_{k+1}) = \theta_k \in$

$\mathcal{E}$. Then the R-plane $L_{\theta_{k+1}}$ is taken to the R-plane $F_*(L_{\theta_{k+1}})$, since F is a Lie transformation, and $F_*(L_{\theta_{k+1}}) \subset T_{F(\theta_k)}$, since F is a symmetry. Consequently, the lifting $F^{(1)}\colon J^{k+1}(\pi) \to J^{k+1}(\pi)$ is a symmetry of $\mathcal{E}^1$. By the same reasons, $F^{(l)}$ is a symmetry of the l-th prolongation of $\mathcal{E}$. From here it also follows that for any infinitesimal symmetry X of the equation $\mathcal{E}$, its l-th lifting is is a symmetry of $\mathcal{E}^l$ as well.

Proposition 3.23. *Symmetries of a formally integrable equation $\mathcal{E} \subset J^k(\pi)$ coincide with symmetries of any prolongation of this equation. The same is valid for infinitesimal symmetries.*

Proof. We showed already that to any (infinitesimal) symmetry of $\mathcal{E}$ there corresponds an (infinitesimal) symmetry of $\mathcal{E}^l$. Consider an (infinitesimal) symmetry of $\mathcal{E}^l$. By Theorems 3.12 on page 50 and 3.13 on page 52, it is $\pi_{k+l,k}$-fiberwise and therefore generates an (infinitesimal) symmetry of the equation $\mathcal{E}$. □

The result proved means that a symmetry of $\mathcal{E}$ generates a symmetry of $\mathcal{E}^\infty$ which preserves every prolongation of finite order. A natural step to generalize the concept of symmetry is to consider "all symmetries" of $\mathcal{E}^\infty$. Recall the notation

$$\mathcal{C}\mathrm{D}(\pi) = \{\, X \in \mathrm{D}(\pi) \mid X \text{ lies in } \mathcal{C}(\pi) \,\}$$

(cf. with (3.23) on page 51).

Definition 3.28. Let π be a vector bundle. A vector field $X \in \mathrm{D}(\pi)$ is called a *symmetry* of the Cartan distribution $\mathcal{C}(\pi)$ on $J^\infty(\pi)$, if $[X, \mathcal{C}\mathrm{D}(\pi)] \subset \mathcal{C}\mathrm{D}(\pi)$.

Thus, the set of symmetries coincides with $\mathrm{D}_{\mathcal{C}}(\pi)$ (see (3.24) on page 51) and forms a Lie algebra over $\mathbb{R}$ and a module over $\mathcal{F}(\pi)$. Note that since the Cartan distribution on $J^\infty(\pi)$ is integrable, one has $\mathcal{C}\mathrm{D}(\pi) \subset \mathrm{D}_{\mathcal{C}}(\pi)$ and, moreover, $\mathcal{C}\mathrm{D}(\pi)$ is an ideal in the Lie algebra $\mathrm{D}_{\mathcal{C}}(\pi)$.

Note also that symmetries belonging to $\mathcal{C}\mathrm{D}(\pi)$ are tangent to any integral manifold of the Cartan distribution. By this reason, we call such symmetries *trivial*. Respectively, the elements of the quotient Lie algebra

$$\mathrm{sym}(\pi) = \mathrm{D}_{\mathcal{C}}(\pi)/\mathcal{C}\mathrm{D}(\pi)$$

are called *nontrivial symmetries* of the Cartan distribution on $J^\infty(\pi)$.

Let now $\mathcal{E}^\infty$ be the infinite prolongation of an equation $\mathcal{E} \subset J^k(\pi)$. Then, since $\mathcal{C}\mathrm{D}(\pi)$ is spanned by the fields of the form $\mathcal{C}Y$, where $Y \in \mathrm{D}(M)$ (see Remark 3.4 on page 55), any vector field from $\mathcal{C}\mathrm{D}(\pi)$ is tangent to $\mathcal{E}^\infty$. Consequently, either all elements of the coset $[X] = X \bmod \mathcal{C}\mathrm{D}(\pi)$, $X \in \mathrm{D}(\pi)$, are tangent to $\mathcal{E}^\infty$ or neither of them do. In the first case we say that the coset $[X]$ is *tangent* to $\mathcal{E}^\infty$.

Definition 3.29. An element $[X] = X \bmod \mathcal{C}\mathrm{D}(\pi)$, $X \in \mathrm{D}(\pi)$, is called a *higher symmetry* of $\mathcal{E}$, if it is tangent to $\mathcal{E}^\infty$.

The set of all higher symmetries forms a Lie algebra over $\mathbb{R}$ and is denoted by $\mathrm{sym}(\mathcal{E})$. We shall usually omit the adjective *higher* in the sequel.

Consider a vector field $X \in \mathrm{D}(\pi)$. Then, substituting X into the structural element $U_\mathcal{C}$ (see (3.35) on page 58), we obtain a field $X^v \in \mathrm{D}(\pi)$. The correspondence $U_\mathcal{C} \colon X \mapsto X^v = \mathrm{i}_X U_\mathcal{C}$ possesses the following properties:

(1) The field X^v is vertical, i.e., $X^v(C^\infty(M)) = 0$.
(2) $X^v = X$ for any vertical field.
(3) $X^v = 0$ if and only if the field X lies in $\mathcal{C}\mathrm{D}(\pi)$.

Therefore, we obtain the direct sum decomposition of $\mathcal{F}(\pi)$-modules

$$\mathrm{D}(\pi) = \mathrm{D}^v(\pi) \oplus \mathcal{C}\mathrm{D}(\pi),$$

where $\mathrm{D}^v(\pi)$ denotes the Lie algebra of vertical fields. A direct corollary of these properties is the following result.

Proposition 3.24. *For any coset $[X] \in \mathrm{sym}(\mathcal{E})$ there exists a unique vertical representative and thus*

$$\mathrm{sym}(\mathcal{E}) = \{\, X \in \mathrm{D}^v(\mathcal{E}) \mid [X, \mathcal{C}\mathrm{D}(\mathcal{E})] \subset \mathcal{C}\mathrm{D}(\mathcal{E}) \,\}, \tag{3.43}$$

where $\mathcal{C}\mathrm{D}(\mathcal{E})$ is spanned by the fields $\mathcal{C}Y$, $Y \in \mathrm{D}(M)$.

Lemma 3.25. *Let $X \in \mathrm{sym}(\pi)$ be a vertical vector field. Then it is completely determined by its restriction to $\mathcal{F}_0(\pi) \subset \mathcal{F}(\pi)$.*

Proof. Let X satisfy the conditions of the lemma and $Y \in \mathrm{D}(M)$. Then for any $f \in C^\infty(M)$ one has

$$[X, \mathcal{C}Y](f) = X(\mathcal{C}Y(f)) - \mathcal{C}Y(X(f)) = X(Y(f)) = 0$$

and hence the commutator $[X, \mathcal{C}Y]$ is a vertical vector field. On the other hand, $[X, \mathcal{C}Y] \in \mathcal{C}\mathrm{D}(\pi)$ because $\mathcal{C}\mathrm{D}(\pi)$ is a Lie algebra ideal. Consequently, $[X, \mathcal{C}Y] = 0$. Note now that in special coordinates we have $D_i(u^j_\sigma) = u^j_{\sigma i}$ for all σ and j. From the above said it follows that

$$X(u^j_{\sigma i}) = D_i(X(u^j_\sigma)). \tag{3.44}$$

But a vertical derivation is determined by its values at the coordinate functions u^j_σ. $\square$

Let now $X_0 \colon \mathcal{F}_0(\pi) \to \mathcal{F}(\pi)$ be a derivation. Then equalities (3.44) allow one to reconstruct locally a vertical derivation $X \in \mathrm{D}(\pi)$ satisfying $X\,\big|_{\mathcal{F}_0(\pi)} = X_0$. Obviously, the derivation X lies in $\mathrm{sym}(\pi)$ over the neighborhood under consideration. Consider two neighborhoods $\mathcal{U}_1, \mathcal{U}_2 \subset J^\infty(\pi)$ with the corresponding special coordinates in each of them and two symmetries $X^i \in \mathrm{sym}(\pi\,|_{\mathcal{U}_i})$, $i = 1, 2$, arising by the described procedure. But the restrictions of X^1 and X^2 to $\mathcal{F}_0(\pi\,|_{\mathcal{U}_1\cap\mathcal{U}_2})$ coincide. Hence, by Lemma 3.25 on the preceding page, the field X^1 coincides with X^2 over the intersection $\mathcal{U}_1\cap\mathcal{U}_2$. Hence, the reconstruction procedure $X_0 \mapsto X$ is a global one. So we have established a one-to-one correspondence between elements of $\mathrm{sym}(\pi)$ and derivations $\mathcal{F}_0(\pi) \to \mathcal{F}(\pi)$.

Note now that due to vector bundle structure in $\pi\colon E \to M$, derivations $\mathcal{F}_0(\pi) \to \mathcal{F}(\pi)$ are identified with sections of $\pi_\infty^*(\pi)$, or with elements of $\mathcal{F}(\pi,\pi)$.

Theorem 3.26. *Let $\pi\colon E \to M$ be a vector bundle. Then:*

(1) *The $\mathcal{F}(\pi)$-module* $\mathrm{sym}(\pi)$ *is in one-to-one correspondence with elements of the module $\mathcal{F}(\pi,\pi)$.*
(2) *In special coordinates the correspondence $\mathcal{F}(\pi,\pi) \to \mathrm{sym}(\pi)$ is expressed by the formula*[7]

$$\varphi \mapsto Э_\varphi = \sum_{j,\sigma} D_\sigma(\varphi^j)\frac{\partial}{\partial u_\sigma^j}, \tag{3.45}$$

where $\varphi = (\varphi^1,\dots,\varphi^m)$ is the component-wise representation of the section $\varphi \in \mathcal{F}(\pi,\pi)$.

Proof. The first part of the theorem has already been proved. To prove the second one, it suffices to use equality (3.44) on the preceding page. ☐

Definition 3.30. Let $\pi\colon E \to M$ be a vector bundle.

(1) The field $Э_\varphi$ of the form (3.45) is called an *evolutionary vector field* on $J^\infty(\pi)$.
(2) The section $\varphi \in \mathcal{F}(\pi,\pi)$ is called the *generating function* of the field $Э_\varphi$.

Remark 3.5. Let $\zeta\colon N \to M$ be an arbitrary smooth fiber bundle and $\xi\colon P \to M$ be a vector bundle. Then it easy to show that any ζ-vertical vector field X on N can be uniquely lifted up to an $\mathbb{R}$-linear map $X^\xi\colon \Gamma(\zeta^*(\xi)) \to \Gamma(\zeta^*(\xi))$ such that

$$X^\xi(f\psi) = X(f)\psi + fX^\xi(\psi), \quad f \in C^\infty(N), \quad \psi \in \Gamma(\zeta^*(\xi)). \tag{3.46}$$

In particular, taking π_∞ for ζ, for any evolutionary vector field $Э_\varphi$ we obtain the family of maps $Э_\varphi^\xi\colon \mathcal{F}(\pi,\xi) \to \mathcal{F}(\pi,\xi)$ satisfying (3.46).

Consider the map $Э_\varphi^\pi\colon \mathcal{F}(\pi,\pi) \to \mathcal{F}(\pi,\pi)$ and recall the element $\rho_0 \in \mathcal{F}_0(\pi,\pi) \subset \mathcal{F}(\pi,\pi)$ (see Example 3.1 on page 38). It can be easily seen that

$$Э_\varphi^\pi(\rho_0) = \varphi \tag{3.47}$$

[7]To denote evolutionary vector fields (see Definition 3.30), we use the Cyrillic letter Э, which is pronounced like "e" in "ten".

which can be taken for the definition of the generating section.

Let $\partial_\varphi, \partial_\psi$ be two evolutionary vector fields. Then, since $\mathrm{sym}(\pi)$ is a Lie algebra and by Theorem 3.26 on the page before, there exists a unique section $\{\varphi, \psi\}$ satisfying $[\partial_\varphi, \partial_\psi] = \partial_{\{\varphi,\psi\}}$.

Definition 3.31. The section $\{\varphi, \psi\} \in \mathcal{F}(\pi, \pi)$ is called the (*higher*) *Jacobi bracket* of the sections $\varphi, \psi \in \mathcal{F}(\pi)$.

Proposition 3.27. *Let $\varphi, \psi \in \mathcal{F}(\pi, \pi)$ be two sections. Then:*

(1) $\{\varphi, \psi\} = \partial^\pi_\varphi(\psi) - \partial^\pi_\psi(\varphi)$.

(2) *In special coordinates, the Jacobi bracket of φ and ψ is expressed by the formula*

$$\{\varphi, \psi\}^j = \sum_{\alpha,\sigma} \left(D_\sigma(\varphi^\alpha) \frac{\partial \psi^j}{\partial u^\alpha_\sigma} - D_\sigma(\psi^\alpha) \frac{\partial \varphi^j}{\partial u^\alpha_\sigma} \right), \tag{3.48}$$

where superscript j denotes the j-th component of the corresponding section.

Proof. To prove (1), let us use (3.47) on the preceding page:

$$\{\varphi, \psi\} = \partial^\pi_{\{\varphi,\psi\}}(\rho_0) = \partial^\pi_\varphi(\partial^\pi_\psi(\rho_0)) - \partial^\pi_\psi(\partial^\pi_\varphi(\rho_0)) = \partial^\pi_\varphi(\psi) - \partial^\pi_\psi(\varphi).$$

The second statement follows from the first one and from equality (3.45) on the page before. □

Consider now a nonlinear operator $\Delta\colon \Gamma(\pi) \to \Gamma(\xi)$ and let $\psi_\Delta \in \mathcal{F}(\pi, \xi)$ be the corresponding section. Then for any $\varphi \in \mathcal{F}(\pi, \pi)$ the section $\partial^\xi_\varphi(\psi_\Delta) \in \mathcal{F}(\pi, \xi)$ is defined and we can set

$$\ell_\Delta(\varphi) = \partial^\xi_\varphi(\psi_\Delta). \tag{3.49}$$

Definition 3.32. The operator $\ell_\Delta\colon \mathcal{F}(\pi, \pi) \to \mathcal{F}(\pi, \xi)$ defined by (3.49) is called the *universal linearization operator*[8] of the operator $\Delta\colon \Gamma(\pi) \to \Gamma(\xi)$.

From the definition and equality (3.45) on the page before we obtain that for a scalar differential operator

$$\Delta\colon \varphi \mapsto F(x_1, \dots, x_n, \dots, \frac{\partial^{|\sigma|}\varphi^j}{\partial x_\sigma}, \dots)$$

one has $\ell_\Delta = (\ell^1_\Delta, \dots, \ell^m_\Delta)$, $m = \dim \pi$, where

$$\ell^\alpha_\Delta = \sum_\sigma \frac{\partial F}{\partial u^\alpha_\sigma} D_\sigma. \tag{3.50}$$

If $\dim \xi = r > 1$ and $\Delta = (\Delta_1, \dots, \Delta_r)$, then

$$\ell_\Delta = \left\| \ell^\beta_{\Delta^\alpha} \right\|, \quad \alpha = 1, \dots, m, \quad \beta = 1, \dots, r. \tag{3.51}$$

[8] Cf. with the algebraic definition on page 34.

In particular, we see that the following statement is valid.

Proposition 3.28. *For any differential operator Δ, its universal linearization is a $\mathcal{C}$-differential operator.*

Now we can describe the algebra $\operatorname{sym}(\mathcal{E})$, $\mathcal{E} \subset J^k(\pi)$ being a formally integrable equation. Let $I(\mathcal{E}) \subset \mathcal{F}(\pi)$ be the ideal of the equation $\mathcal{E}$ (see page 55). Then, by definition, $\mathbf{\ni}_\varphi$ is a symmetry of $\mathcal{E}$ if and only if

$$\mathbf{\ni}_\varphi(I(\mathcal{E})) \subset I(\mathcal{E}). \tag{3.52}$$

Assume now that $\mathcal{E}$ is given by a differential operator $\Delta\colon \Gamma(\pi) \to \Gamma(\xi)$ and locally is described by the system of equations $F^1 = 0, \dots, F^r = 0$, $F^j \in \mathcal{F}(\pi)$. Then the functions $F^1, \dots, F^r$ are differential generators of the ideal $I(\mathcal{E})$ and condition (3.52) may be rewritten as

$$\mathbf{\ni}_\varphi(F^j) = \sum_{\alpha,\sigma} a^\alpha_{\sigma,j} D_\sigma(F^\alpha), \quad j = 1, \dots, m, \quad a^\alpha_\sigma \in \mathcal{F}(\pi). \tag{3.53}$$

Using of (3.49) on the preceding page, the last equation acquires the form[9]

$$\ell_{F^j}(\varphi) = \sum_{\alpha,\sigma} a^\alpha_{\sigma,j} D_\sigma(F^\alpha), \quad j = 1, \dots, m, \quad a^\alpha_\sigma \in \mathcal{F}(\pi). \tag{3.54}$$

But by Proposition 3.28, the universal linearization is a $\mathcal{C}$-differential operator and consequently can be restricted to $\mathcal{E}^\infty$ (see Corollary 3.22 on page 60). It means that we can rewrite (3.54) as

$$\ell_{F^j}\,|_{\mathcal{E}^\infty}\,(\varphi\,|_{\mathcal{E}^\infty}) = 0, \quad j = 1, \dots, m. \tag{3.55}$$

Combining these equations with (3.50) and (3.51) on the preceding page, we obtain the following fundamental result:

Theorem 3.29. *Let $\mathcal{E} \subset J^k(\pi)$ be a formally integrable equation and $\Delta = \Delta_{\mathcal{E}}\colon \Gamma(\pi) \to \Gamma(\xi)$ be the operator corresponding to $\mathcal{E}$. Then an evolutionary vector field $\mathbf{\ni}_\varphi$, $\varphi \in \mathcal{F}(\pi, \pi)$ is a symmetry of $\mathcal{E}$ if and only if*

$$\ell_{\mathcal{E}}(\bar{\varphi}) = 0, \tag{3.56}$$

where $\ell_{\mathcal{E}}$ and $\bar{\varphi}$ denote restrictions of ℓ_Δ and φ on $\mathcal{E}^\infty$ respectively. In other words, $\operatorname{sym}(\mathcal{E}) = \ker \ell_{\mathcal{E}}$.

Exercise 3.3. Show that classical symmetries (see Subsection 3.6) are embedded in $\operatorname{sym}\mathcal{E}$ as a Lie subalgebra. Describe their generating functions.

Remark 3.6. From the result obtained it follows that higher symmetries of $\mathcal{E}$ can be identified with elements of $\mathcal{F}(\mathcal{E}, \pi)$ satisfying equation (3.56). Below we shall say that a *section $\varphi \in \mathcal{F}(\mathcal{E}, \pi)$ is a symmetry* of $\mathcal{E}$ keeping

[9] We use the notation ℓ_F, $F \in \mathcal{F}(\pi, \xi)$, as a synonym for ℓ_Δ, where $\Delta\colon \Gamma(\pi) \to \Gamma(\xi)$ is the operator corresponding to the section F.

in mind this identification. Note that due to the fact that $\mathrm{sym}(\mathcal{E})$ is a Lie algebra, for any two symmetries $\varphi, \psi \in \mathcal{F}(\mathcal{E}, \pi)$ their Jacobi bracket $\{\varphi, \psi\}_{\mathcal{E}} \in \mathcal{F}(\mathcal{E}, \pi)$ is well defined and is a symmetry as well. If no confusion arises, we shall omit the subscript $\mathcal{E}$ in the notation of the Jacobi bracket.

Finally, we give a useful description of the modules $\mathrm{D}^v(\mathcal{E})$ and $\mathcal{C}^k\Lambda(\mathcal{E})$. Denote $\varkappa = \mathcal{F}(\mathcal{E}, \pi)$.

First consider the case $\mathcal{E}^\infty = J^\infty(\pi)$. From the coordinate expression (3.45) on page 65 for an evolutionary vector field it immediately follows that any vertical tangent vector at point $\theta \in J^\infty(\pi)$ can be realized in the form $\ni_\varphi|_\theta$ for some φ. This shows that the map $\varphi \mapsto \ni_\varphi$ yields the canonical isomorphism

$$\mathrm{D}^v(\pi) = \bar{\mathcal{J}}^\infty(\varkappa).$$

The dual isomorphism reads

$$\mathcal{C}^1\Lambda(\pi) = \mathcal{C}\mathrm{Diff}(\varkappa, \mathcal{F}).$$

In coordinates, this isomorphism takes the form ω^j_σ to the operator

$$(0, \dots, 0, D_\sigma, 0, \dots, 0),$$

with D_σ on j-th place.

It is clear that the Cartan k-forms can be identified with multilinear skew-symmetric $\mathcal{C}$-differential operators in k arguments:

$$\mathcal{C}^p\Lambda(\pi) = \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa, \mathcal{F}).$$

Now consider the general case. Suppose that the equation $\mathcal{E}$ is given by an operator $\Delta\colon \Gamma(\pi) \to \Gamma(\xi)$. Denote $P = \mathcal{F}(\mathcal{E}, \xi)$, so that $\ell_{\mathcal{E}}\colon \varkappa \to P$. From (3.55) on the preceding page we get

Proposition 3.30. (1) *The module $\mathrm{D}^v(\mathcal{E})$ is isomorphic to the kernel of the homomorphism $\psi^{\ell_{\mathcal{E}}}_\infty\colon \bar{\mathcal{J}}^\infty(\varkappa) \to \bar{\mathcal{J}}^\infty(P)$;*

(2) *the module $\mathcal{C}^p\Lambda(\mathcal{E})$ is isomorphic to $\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa, \mathcal{F})$ modulo the submodule consisting of the operators of the form $\nabla \circ \ell_{\mathcal{E}}$, where $\nabla \in \mathcal{C}\mathrm{Diff}(P, \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p-1)}(\varkappa, \mathcal{F}))$.*

4. Coverings and nonlocal symmetries

The facts exposed in this section constitute a formal base to introduce nonlocal variables to the differential setting, i.e., variables of the type $\int \varphi\, dx$, φ being a function on an infinitely prolonged equation. A detailed exposition of this material can be found in [33] and [34].

4.1. **Coverings.** We start with fixing up the setting. To do it, extend the universum of infinitely prolonged equations in the following way. Let $\mathcal{N}$ be a chain of smooth maps $\cdots \to N^{i+1} \xrightarrow{\tau_{i+1,i}} N^i \to \cdots$, where N^i are smooth finite-dimensional manifolds. Define the algebra $\mathcal{F}(\mathcal{N})$ of smooth functions on $\mathcal{N}$ as the direct limit of the homomorphisms $\cdots \to C^\infty(N^i) \xrightarrow{\tau^*_{i+1,i}} C^\infty(N^{i+1}) \to \cdots$. Then there exist natural homomorphisms $\tau^*_{\infty,i}\colon C^\infty(N^i) \to \mathcal{F}(\mathcal{N})$ and the algebra $\mathcal{F}(\mathcal{N})$ may be considered to be filtered by the images of these maps. Let us consider calculus (see Section 1) over $\mathcal{F}(\mathcal{N})$ agreed with this filtration. Define the category *Inf* as follows:

1. The *objects* of *Inf* are the above introduced chains $\mathcal{N}$ endowed with *integrable distributions* $\mathcal{D}_{\mathcal{N}} \subset \mathrm{D}(\mathcal{F}(\mathcal{N}))$, where the word "integrable" means that $[\mathcal{D}_{\mathcal{N}}, \mathcal{D}_{\mathcal{N}}] \subset \mathcal{D}_{\mathcal{N}}$.
2. If $\mathcal{N}_1 = \{N_1^i, \tau^1_{i+1,i}\}$, $\mathcal{N}_2 = \{N_2^i, \tau^2_{i+1,i}\}$ are two objects of *Inf*, then a *morphism* $\varphi\colon \mathcal{N}_1 \to \mathcal{N}_2$ is a system of smooth maps $\varphi_i\colon N_1^{i+\alpha} \to N_2^i$, where $\alpha \in \mathbb{Z}$ is independent of i, satisfying $\tau^2_{i+1,i} \circ \varphi_{i+1} = \varphi_i \circ \tau^1_{i+\alpha+1,i+\alpha}$ and such that $\varphi_{*,\theta}(\mathcal{D}_{\mathcal{N}_1,\theta}) \subset \mathcal{D}_{\mathcal{N}_2,\varphi(\theta)}$ for any point $\theta \in \mathcal{N}_1$.

Definition 4.1. A morphism $\varphi\colon \mathcal{N}_1 \to \mathcal{N}_2$ is called a *covering* in the category *Inf*, if $\varphi_{*,\theta}\big|_{\mathcal{D}_{\mathcal{N}_1,\theta}} : \mathcal{D}_{\mathcal{N}_1,\theta} \to \mathcal{D}_{\mathcal{N}_2,\varphi(\theta)}$ is an isomorphism for any point $\theta \in \mathcal{N}_1$.

In particular, manifolds $J^\infty(\pi)$ and $\mathcal{E}^\infty$ endowed with the corresponding Cartan distributions are objects of *Inf* and we can consider coverings over these objects.

Example 4.1. Let $\Delta\colon \Gamma(\pi) \to \Gamma(\pi')$ be a differential operator of order $\le k$. Then the system of maps $\Phi^{(l)}_\Delta\colon J^{l+l}(\pi) \to J^l(\pi')$ (see the proof of Lemma 3.3 on page 38) is a morphism of $J^\infty(\pi)$ to $J^\infty(\pi')$. Under unrestrictive conditions of regularity, its image is of the form $\mathcal{E}^\infty$ for some equation $\mathcal{E}$ in the bundle π' while the map $J^\infty(\pi) \to \mathcal{E}^\infty$ is a covering.

Definition 4.2. Let $\varphi'\colon \mathcal{N}' \to \mathcal{N}$ and $\varphi''\colon \mathcal{N}'' \to \mathcal{N}$ be two coverings.

1. A morphism $\psi\colon \mathcal{N}' \to \mathcal{N}''$ is said to be a *morphism of coverings*, if $\varphi' = \varphi'' \circ \psi$.
2. The coverings φ', φ'' are called *equivalent*, if there exists a morphism $\psi\colon \mathcal{N}' \to \mathcal{N}''$ which is a diffeomorphism.

Definition 4.3. A covering $\varphi : \mathcal{N}' \to \mathcal{N}$ is called *linear*, if

(1) φ is a linear bundle;
(2) any element $X \in \mathcal{D}(\mathcal{N}')$ preserves the submodule of fiber-wise linear (with respect to the projection φ) functions in $\mathcal{F}(\mathcal{N}')$.

Let $\mathcal{N}$ be an object of *Inf* and W be a smooth manifold. Consider the projection $\tau_W : \mathcal{N} \times W \to \mathcal{N}$ to the first factor. Then we can make a covering of τ_W by lifting the distribution $\mathcal{D}_{\mathcal{N}}$ to $\mathcal{N} \times W$ in a trivial way.

Definition 4.4. A covering $\tau : \mathcal{N}' \to \mathcal{N}$ is called *trivial*, if it is equivalent to the covering τ_W for some W.

Let again $\varphi' : \mathcal{N}' \to \mathcal{N}$, $\varphi'' : \mathcal{N}'' \to \mathcal{N}$ be two coverings. Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{N}' \times_{\mathcal{N}} \mathcal{N}'' & \xrightarrow{\varphi''^*(\varphi')} & \mathcal{N}'' \\ {\scriptstyle \varphi'^*(\varphi'')}\Big\downarrow & & \Big\downarrow{\scriptstyle \varphi''} \\ \mathcal{N}' & \xrightarrow{\varphi'} & \mathcal{N} \end{array}$$

where

$$\mathcal{N}' \times_{\mathcal{N}} \mathcal{N}'' = \{\, (\theta', \theta'') \in \mathcal{N}' \times \mathcal{N}'' \mid \varphi'(\theta') = \varphi''(\theta'') \,\}$$

while $\varphi'^*(\varphi'')$, $\varphi''^*(\varphi')$ are natural projections. The manifold $\mathcal{N}' \times_{\mathcal{N}} \mathcal{N}''$ is supplied with a natural structure of an object of *Inf* and the maps $(\varphi')^*(\varphi'')$, $(\varphi'')^*(\varphi')$ become coverings.

Definition 4.5. The composition

$$\varphi' \times_{\mathcal{N}} \varphi'' = \varphi' \circ \varphi'^*(\varphi'') = \varphi'' \circ \varphi''^*(\varphi') : \mathcal{N}' \times_{\mathcal{N}} \mathcal{N}'' \to \mathcal{N}$$

is called the *Whitney product* of the coverings φ' and φ''.

Definition 4.6. A covering is said to be *reducible*, if it is equivalent to a covering of the form $\varphi \times_{\mathcal{N}} \tau$, where τ is a trivial covering. Otherwise it is called *irreducible*.

From now on, all coverings under consideration will be assumed to be smooth fiber bundles. The fiber dimension is called the *dimension of the covering* φ under consideration and is denoted by $\dim \varphi$.

Proposition 4.1. *Let $\mathcal{E} \subset J^k(\pi)$ be an equation in the bundle $\pi : E \to M$ and $\varphi : \mathcal{N} \to \mathcal{E}^\infty$ be a smooth fiber bundle. Then the following statements are equivalent:*

(1) *The bundle φ is equipped with a structure of a covering.*
(2) *There exists a connection $\mathcal{C}^\varphi$ in the bundle $\pi_\infty \circ \varphi : \mathcal{N} \to M$, $\mathcal{C}^\varphi : X \mapsto X^\varphi$, $X \in \mathrm{D}(M)$, $X^\varphi \in \mathrm{D}(\mathcal{N})$, such that*
    (a) $[X^\varphi, Y^\varphi] = [X, Y]^\varphi$, *i.e., $\mathcal{C}^\varphi$ is flat, and*

(b) *any vector field X^φ is projectible to $\mathcal{E}^\infty$ under φ_* and $\varphi_*(X^\varphi) = \mathcal{C}X$, where $\mathcal{C}$ is the Cartan connection on $\mathcal{E}^\infty$.*

The proof reduces to the check of definitions.

Using this result, we shall now obtain coordinate description of coverings. Namely, let $x_1, \dots, x_n, u^1, \dots, u^m$ be local coordinates in $J^0(\pi)$ and assume that internal coordinates in $\mathcal{E}^\infty$ are chosen. Suppose also that over the neighborhood under consideration the bundle $\varphi\colon \mathcal{N} \to \mathcal{E}^\infty$ is trivial with the fiber W and $w^1, w^2, \dots, w^s, \dots$ are local coordinates in W. The functions w^j are called *nonlocal coordinates* in the covering φ. The connection $\mathcal{C}^\varphi$ puts into correspondence to any partial derivative $\partial/\partial x_i$ the vector field $\mathcal{C}^\varphi(\partial/\partial x_i) = \tilde{D}_i$. By Proposition 4.1 on the facing page, these vector fields are to be of the form

$$\tilde{D}_i = D_i + X_i^v = D_i + \sum_\alpha X_i^\alpha \frac{\partial}{\partial w^\alpha}, \quad i = 1, \dots, n, \tag{4.1}$$

where D_i are restrictions of total derivatives to $\mathcal{E}^\infty$, and satisfy the conditions

$$\begin{aligned}[\tilde{D}_i, \tilde{D}_i] &= [D_i, D_j] + [D_i, X_j^v] + [X_i^v, D_j] + [X_i^v, X_j^v] \\ &= [D_i, X_j^v] + [X_i^v, D_j] + [X_i^v, X_j^v] = 0\end{aligned} \tag{4.2}$$

for all $i, j = 1, \dots, n$.

We shall now prove a number of facts that simplify checking of triviality and equivalence of coverings.

Proposition 4.2. *Let $\varphi_1\colon \mathcal{N}_1 \to \mathcal{E}^\infty$ and $\varphi_2\colon \mathcal{N}_2 \to \mathcal{E}^\infty$ be two coverings of the same dimension $r < \infty$. They are equivalent if and only if there exists a submanifold $X \subset \mathcal{N}_1 \times_{\mathcal{E}^\infty} \mathcal{N}_2$ such that*

(1) $\operatorname{codim} X = r$;
(2) *The restrictions $\varphi_1^*(\varphi_2)\,|_X$ and $\varphi_2^*(\varphi_1)\,|_X$ are surjections.*
(3) *$(\mathcal{D}_{\mathcal{N}_1 \times_{\mathcal{E}^\infty} \mathcal{N}_2})_\theta \subset T_\theta X$ for any point $\theta \in X$.*

Proof. In fact, if $\psi\colon \mathcal{N}_1 \to \mathcal{N}_2$ is an equivalence, then its graph $G_\psi = \{\,(y, \psi(y)) \mid y \in \mathcal{N}_1\,\}$ is the needed manifold X. Conversely, if X is a manifold satisfying (1)–(3), then the correspondence

$$y \mapsto \varphi_1^*(\varphi_2)\big((\varphi_1^*(\varphi_2))^{-1}(y) \cap X\big)$$

is an equivalence. □

Submanifolds X satisfying assumption (3) of the previous proposition are called *invariant.*

Proposition 4.3. *Let $\varphi_1\colon \mathcal{N}_1 \to \mathcal{E}^\infty$ and $\varphi_2\colon \mathcal{N}_2 \to \mathcal{E}^\infty$ be two irreducible coverings of the same dimension $r < \infty$. Assume that the Whitney product of φ_1 and φ_2 is reducible and there exists an invariant submanifold X*

in $\mathcal{N}_1 \times_{\mathcal{E}^\infty} \mathcal{N}_2$ of codimension r. Then φ_1 and φ_2 are equivalent almost everywhere.

Proof. Since φ_1, φ_2 are irreducible, X is to be mapped surjectively almost everywhere by $\varphi_1^*(\varphi_2)$ and $\varphi_2^*(\varphi_1)$ to $\mathcal{N}_1$ and $\mathcal{N}_2$ respectively (otherwise, their images would be invariant submanifolds). Hence, the coverings are equivalent by Proposition 4.2 on the page before. □

Corollary 4.4. *If φ_1 and φ_2 are one-dimensional coverings over $\mathcal{E}^\infty$ and their Whitney product is reducible, then they are equivalent.*

Proposition 4.5. *Let $\varphi\colon \mathcal{N} \to \mathcal{E}^\infty$ be a covering and $\mathcal{U} \subset \mathcal{E}^\infty$ be a domain such that the the manifold $\tilde{\mathcal{U}} = \varphi^{-1}(\mathcal{U})$ is represented in the form $\mathcal{U} \times \mathbb{R}^r$, $r \leq \infty$, while $\varphi|_{\tilde{\mathcal{U}}}$ is the projection to the first factor. Then the covering φ is locally irreducible if the system*

$$D_1^\varphi(f) = 0, \dots, D_n^\varphi(f) = 0 \tag{4.3}$$

has only constant solutions.

Proof. Suppose that there exists a solution $f \neq \text{const}$ of (4.3). Then, since the only solutions of the system

$$D_1(f) = 0, \dots, D_n(f) = 0,$$

where D_i is the restriction of the i-th total derivative to $\mathcal{E}^\infty$, are constants, f depends on one nonlocal variable w^α at least. Without loss of generality we may assume that $\partial f/\partial w^1 \neq 0$ in a neighborhood $\mathcal{U}' \times V$, $\mathcal{U}' \subset \mathcal{U}$, $V \subset \mathbb{R}^r$. Define the diffeomorphism $\psi\colon \mathcal{U}' \subset \mathcal{U} \to \psi(\mathcal{U}' \subset \mathcal{U})$ by setting

$$\psi(\dots, x_i, \dots, p_\sigma^j, \dots, w^\alpha, \dots) = (\dots, x_i, \dots, p_\sigma^j, \dots, f, w^2, \dots, w^\alpha, \dots).$$

Then $\psi_*(D_i^\varphi) = D_i + \sum_{\alpha > 1} X_i^\alpha \partial/\partial w^\alpha$ and consequently φ is reducible.

Let now φ be a reducible covering, i.e., $\varphi = \varphi' \times_{\mathcal{E}^\infty} \tau$, where τ is trivial. Then, if f is a smooth function on the total space of the covering τ, the function $f^* = \big(\tau^*(\varphi')\big)^*(f)$ is a solution of (4.3). Obviously, there exists an f such that $f^* \neq \text{const}$. □

4.2. **Nonlocal symmetries and shadows.** Let $\mathcal{N}$ be an object of *Inf* with the integrable distribution $\mathcal{D} = \mathcal{D}_\mathcal{N}$. Define

$$\mathrm{D}_\mathcal{D}(\mathcal{N}) = \{\, X \in \mathrm{D}(\mathcal{N}) \mid [X, \mathcal{D}] \subset \mathcal{D} \,\}$$

and set $\operatorname{sym} \mathcal{N} = \mathrm{D}_\mathcal{D}(\mathcal{N})/\mathcal{D}_\mathcal{N}$. Obviously, $\mathrm{D}_\mathcal{D}(\mathcal{N})$ is a Lie $\mathbb{R}$-algebra and $\mathcal{D}$ is its ideal. Elements of the Lie algebra $\operatorname{sym} \mathcal{N}$ are called *symmetries* of the object $\mathcal{N}$.

Definition 4.7. Let $\varphi\colon \mathcal{N} \to \mathcal{E}^\infty$ be a covering. A *nonlocal φ-symmetry* of $\mathcal{E}$ is an element of $\operatorname{sym} \mathcal{N}$. The Lie algebra of such symmetries is denoted by $\operatorname{sym}_\varphi \mathcal{E}$.

A base for computation of nonlocal symmetries is the following two results.

Theorem 4.6. *Let $\varphi\colon \mathcal{N} \to \mathcal{E}^\infty$ be a covering. The algebra $\operatorname{sym}_\varphi \mathcal{E}$ is isomorphic to the Lie algebra of vector fields X on $\mathcal{N}$ such that*

(1) *X is vertical, i.e., $X(\varphi^*(f)) = 0$ for any function $f \in C^\infty(M) \subset \mathcal{F}(\mathcal{E})$;*
(2) *$[X, D_i^\varphi] = 0$, $i = 1, \dots, n$.*

Proof. Note that the first condition means that in coordinate representation the coefficients of the field X at all $\partial/\partial x_i$ vanish. Hence the intersection of the set of vertical fields with $\mathcal{D}$ vanish. On the other hand, in any coset $[X] \in \operatorname{sym}_\varphi \mathcal{E}$ there exists one and only one vertical element X^v. In fact, let X be an arbitrary element of $[X]$. Then $X^v = X - \sum_i a_i D_i^\varphi$, where a_i is the coefficient of X at $\partial/\partial x_i$. ☐

Theorem 4.7. *Let $\varphi\colon \mathcal{N} = \mathcal{E}^\infty \times \mathbb{R}^r \to \mathcal{E}^\infty$ be the covering locally determined by the fields*

$$D_i^\varphi = D_i + \sum_{\alpha=1}^{r} X_i^\alpha \frac{\partial}{\partial w^\alpha}, \quad i = 1, \dots, n, \quad X_i^\alpha \in \mathcal{F}(\mathcal{N}),$$

where $w^1, w^2, \dots$ are coordinates in $\mathbb{R}^r$ (nonlocal variables). Then any nonlocal φ-symmetry of the equation $\mathcal{E} = \{F = 0\}$ is of the form

$$\tilde{\mathfrak{I}}_{\psi,a} = \tilde{\mathfrak{I}}_\psi + \sum_{\alpha=1}^{r} a_\alpha \frac{\partial}{\partial w^\alpha}, \tag{4.4}$$

where $\psi = \psi^1, \dots, \psi^m$, $a = (a^1, \dots, a^r)$, ψ^i, $a^\alpha \in \mathcal{F}(\mathcal{N})$ are functions satisfying the conditions

$$\tilde{\ell}_F(\psi) = 0, \tag{4.5}$$

$$\tilde{D}_i(a^\alpha) = \tilde{\mathfrak{I}}_{\psi,a}(X_i^\alpha) \tag{4.6}$$

while

$$\tilde{\mathfrak{I}}_\psi = \sum_{j,\sigma} \tilde{D}_\sigma(\psi) \frac{\partial}{\partial u_\sigma^j} \tag{4.7}$$

and $\tilde{\ell}_F$ is obtained from ℓ_F by changing total derivatives D_i for D_i^φ.

Proof. Let $X \in \operatorname{sym}_\varphi \mathcal{E}$. Using Theorem 4.6, let us write down the field X in the form

$$X = {\sum_{\sigma,j}}' b_\sigma^j \frac{\partial}{\partial u_\sigma^j} + \sum_{\alpha=1}^{r} a^\alpha \frac{\partial}{\partial w^\alpha}, \tag{4.8}$$

where "prime" over the first sum means that the summation extends on internal coordinates in $\mathcal{E}^\infty$ only. Then, equaling to zero the coefficient at $\partial/\partial u_\sigma^j$ in the commutator $[X, D_i^\varphi]$, we obtain the following equations

$$D_i^\varphi(b_\sigma^j) = \begin{cases} b_{\sigma i}^j, & \text{if } b_{\sigma i}^j \text{ is an internal coordinate,} \\ X(u_{\sigma i}^j) & \text{otherwise.} \end{cases}$$

Solving these equations, we obtain that the first summand in (4.8) on the page before is of the form $\tilde{\mathfrak{I}}_\psi$, where ψ satisfies (4.5) on the preceding page. $\square$

Comparing the result obtained with the description on local symmetries (see Theorem 3.29 on page 67), we see that in the nonlocal setting an additional obstruction arises represented by equation (4.6) on the preceding page. Thus, in general, not every solution of (4.5) corresponds to a nonlocal φ-symmetry. We call vector fields $\tilde{\mathfrak{I}}_\psi$ of the form (4.7), where ψ satisfies equation (4.5), *φ-shadows*. In the next subsection it will be shown that for any φ-shadow $\tilde{\mathfrak{I}}_\psi$ there exists a covering $\varphi' \colon \mathcal{N}' \to \mathcal{N}$ and a nonlocal $\varphi \circ \varphi'$-symmetry S such that $\varphi'_*(S) = \tilde{\mathfrak{I}}_\psi$.

4.3. **Reconstruction theorems.** Let $\mathcal{E} \subset J^k(\pi)$ be a differential equation. Let us first establish relations between horizontal cohomology of $\mathcal{E}$ (see Definition 3.27 on page 60) and coverings over $\mathcal{E}^\infty$. All constructions below are realized in a local chart $\mathcal{U} \subset \mathcal{E}^\infty$.

Consider a horizontal 1-form $\omega = \sum_{i=1}^n X_i\, dx_i \in \bar{\Lambda}^1(\mathcal{E})$ and define on the space $\mathcal{E}^\infty \times \mathbb{R}$ the vector fields

$$D_i^\omega = D_i + X_i \partial/\partial w,\ X_i \in \mathcal{F}(\mathcal{E}), \tag{4.9}$$

where w is a coordinate along $\mathbb{R}$. By direct computations, one can easily see that the conditions $[D_i^\omega, D_j^\omega] = 0$ fulfill if and only if $\bar{d}\omega = 0$. Thus, (4.9) determines a covering structure in the bundle $\varphi : \mathcal{E}^\infty \times \mathbb{R} \to \mathcal{E}^\infty$ and this covering is denoted by φ^ω. It is also obvious that the covering φ^ω and $\varphi^{\omega'}$ are equivalent if and only if the forms ω and ω' are cohomologous, i.e., if $\omega - \omega' = \bar{d}f$ for some $f \in \mathcal{F}(\mathcal{E})$.

Let $[\omega_1], \dots, [\omega^\alpha], \dots$ be an $\mathbb{R}$-basis of the vector space $\bar{H}^1(\mathcal{E})$. Let us define the covering $\mathfrak{a}_{1,0} : \mathcal{A}^1(\mathcal{E}) \to \mathcal{E}^\infty$ as the Whitney product of all φ^{ω_α}. It can be shown that the equivalence class of $\mathfrak{a}_{1,0}$ does not depend on the basis choice. Now, literary in the same manner as it was done in Definition 3.27 on page 60 for $\mathcal{E}^\infty$, we can define horizontal cohomology for $\mathcal{A}^1(\mathcal{E})$ and construct the covering $\mathfrak{a}_{2,1} : \mathcal{A}^2(\mathcal{E}) \to \mathcal{A}^1(\mathcal{E})$, etc.

Definition 4.8. The inverse limit of the chain

$$\cdots \to \mathcal{A}^k(\mathcal{E}) \xrightarrow{\mathfrak{a}_{k,k-1}} \mathcal{A}^{k-1}(\mathcal{E}) \to \cdots \to \mathcal{A}^1(\mathcal{E}) \xrightarrow{\mathfrak{a}_{1,0}} \mathcal{E}^\infty \tag{4.10}$$

is called the *universal Abelian covering* of the equation $\mathcal{E}$ and is denoted by $\mathfrak{a}: \mathcal{A}(\mathcal{E}) \to \mathcal{E}^\infty$.

Obviously, $\bar{H}^1(\mathcal{A}(\mathcal{E})) = 0$.

Theorem 4.8 (see [21]). *Let $\mathfrak{a}: \mathcal{A}(\mathcal{E}) \to \mathcal{E}^\infty$ be the universal Abelian covering of the equation $\mathcal{E} = \{F = 0\}$. Then any $\mathfrak{a}$-shadow reconstructs up to a nonlocal $\mathfrak{a}$-symmetry, i.e., for any solution $\psi = (\psi^1, \dots, \psi^m)$, $\psi^j \in \mathcal{F}(\mathcal{A}(\mathcal{E}))$, of the equation $\tilde{\ell}_F(\psi) = 0$ there exists a set of functions $a = (a_{\alpha,i})$, where $a_{\alpha,i} \in \mathcal{F}(\mathcal{A}(\mathcal{E}))$ such that $\tilde{\mathcal{Э}}_{\psi,a}$ is a nonlocal $\mathfrak{a}$-symmetry.*

Proof. Let $w^{j,\alpha}$, $j \leq k$, be nonlocal variables in $\mathcal{A}^k(\mathcal{E})$ and assume that the covering structure in $\mathfrak{a}$ is determined by the vector fields $D_i^{\mathfrak{a}} = D_i + \sum_{j,\alpha} X_i^{j,\alpha}\partial/\partial w^{j,\alpha}$, where, by construction, $X_i^{j,\alpha} \in \mathcal{F}(\mathcal{A}^{j-1}(\mathcal{E}))$, i.e., the functions $X_i^{j,\alpha}$ do not depend on $w^{k,\alpha}$ for all $k \geq j$.

Our aim is to prove that the system

$$D_i^{\mathfrak{a}}(a_{j,\alpha}) = \tilde{\mathcal{Э}}_{\psi,a}(X_i^{j,\alpha}) \tag{4.11}$$

is solvable with respect to $a = (a_{j,\alpha})$ for any $\psi \in \ker \tilde{\ell}_F$. We do this by induction on j. Note that

$$[D_i^{\mathfrak{a}}, \tilde{\mathcal{Э}}_{\psi,a}] = \sum_{j,\alpha} \big(D_i^{\mathfrak{a}}(a_{j,\alpha}) - \tilde{\mathcal{Э}}_{\psi,a}(X_i^{j,\alpha})\big)\frac{\partial}{\partial w^{j,\alpha}}$$

for any set of functions $(a_{j,\alpha})$. Then for $j = 1$ one has $[D_i^{\mathfrak{a}}, \tilde{\mathcal{Э}}_{\psi,a}](X_k^{1,\alpha}) = 0$, or

$$D_i^{\mathfrak{a}}\big(\tilde{\mathcal{Э}}_{\psi,a}(X_k^{1,\alpha})\big) = \tilde{\mathcal{Э}}_{\psi,a}\big(D_i^{\mathfrak{a}}(X_k^{1,\alpha})\big),$$

since $X_k^{1,\alpha}$ are functions on $\mathcal{E}^\infty$.

But from the construction of the covering $\mathfrak{a}$ one has $D_i^{\mathfrak{a}}(X_k^{1,\alpha}) = D_k^{\mathfrak{a}}(X_i^{1,\alpha})$, and we finally obtain

$$D_i^{\mathfrak{a}}\big(\mathcal{Э}_\psi(X_k^{1,\alpha})\big) = D_k^{\mathfrak{a}}\big(\mathcal{Э}_\psi(X_i^{1,\alpha})\big).$$

Note now that the equality $\bar{H}^1(\mathcal{A}(\mathcal{E})) = 0$ implies existence of functions $a_{1,\alpha}$ satisfying

$$D_i^{\mathfrak{a}}(a_{1,\alpha}) = \mathcal{Э}_\psi(X_i^{1,\alpha}),$$

i.e., equation (4.11) is solvable for $j = 1$.

Assume now that solvability of (4.11) was proved for $j < s$ and the functions $(a_{1,\alpha}, \dots, a_{j-1,\alpha})$ are some solutions. Then, since $[D_i^{\mathfrak{a}}, \tilde{\mathcal{Э}}_{\psi,a}]\,\big|_{\mathcal{A}^{j-1}(\mathcal{E})} = 0$, we obtain the needed $a_{j,\alpha}$ literally repeating the proof for the case $j = 1$. □

Let now $\varphi\colon \mathcal{N} \to \mathcal{E}^\infty$ be an arbitrary covering. The next result shows that any φ-shadow is reconstructable.

Theorem 4.9 (see also [22])**.** *For any φ-shadow, i.e., for a solution $\psi = (\psi^1, \dots, \psi^m)$, $\psi^j \in \mathcal{F}(\mathcal{N})$, of the equation $\tilde{\ell}_F(\psi) = 0$, there exists a covering $\varphi_\psi \colon \mathcal{N}_\psi \to \mathcal{N} \xrightarrow{\varphi} \mathcal{E}^\infty$ and a φ_ψ-symmetry S_ψ, such that $S_\psi\,|_{\mathcal{E}^\infty} = \tilde{\mathfrak{Ə}}_\psi\,|_{\mathcal{E}^\infty}$.*

Proof. Let locally the covering φ be represented by the vector fields

$$D_i^\varphi = D_i + \sum_{\alpha=1}^{r} X_i^\alpha \frac{\partial}{\partial w^\alpha},$$

$r \le \infty$ being the dimension of φ. Consider the space $\mathbb{R}^\infty$ with the coordinates w_l^α, $\alpha = 1, \dots, r$, $l = 0, 1, 2, \dots$, $w_0^\alpha = w^\alpha$, and set $\mathcal{N}_\psi = \mathcal{N} \times \mathbb{R}^\infty$ with

$$D_i^{\varphi_\psi} = D_i + \sum_{l,\alpha} \left(\tilde{\mathfrak{Ə}}_\psi + S_w\right)^l (X_i^\alpha) \frac{\partial}{\partial w_l^\alpha}, \tag{4.12}$$

where

$$\tilde{\mathfrak{Ə}}_\psi = \sum_{\sigma,k}{}' D_\sigma^\varphi(\psi^k) \frac{\partial}{\partial u_\sigma^k}, \quad S_w = \sum_{\alpha,l} w_{l+1}^\alpha \frac{\partial}{\partial w_l^\alpha} \tag{4.13}$$

and "prime", as before, denotes summation over internal coordinates.

Set $S_\psi = \tilde{\mathfrak{Ə}}_\psi + S_w$. Then

$$\begin{aligned}
[S_\psi, D_i^{\varphi_\psi}] &= \sum_{\sigma,k}{}' \tilde{\mathfrak{Ə}}_\psi(\bar{u}_{\sigma i}^k) \frac{\partial}{\partial u_\sigma^k} + \sum_{l,\alpha} \left(\tilde{\mathfrak{Ə}}_\psi + S_w\right)^{l+1} (X_i^\alpha) \frac{\partial}{\partial w_l^\alpha} \\
&\quad - \sum_{\sigma,k}{}' D_i^{\varphi_\psi}(D_\sigma^\varphi(\psi^k)) \frac{\partial}{\partial u_\sigma^k} - \sum_{l,\alpha} \left(\tilde{\mathfrak{Ə}}_\psi + S_w\right)^{l+1} (X_i^\alpha) \frac{\partial}{\partial w_l^\alpha} \\
&= \sum_{\sigma,k}{}' \left(\tilde{\mathfrak{Ə}}_\psi(\bar{u}_{\sigma i}^k) - D_{\sigma i}^\varphi(\psi^k)\right) \frac{\partial}{\partial u_\sigma^k} = 0.
\end{aligned}$$

Here, by definition, $\bar{u}_{\sigma i}^k = D_i^\varphi(u_\sigma^k)\,|_{\mathcal{N}}$.

Now, using the above proved equality, one has

$$\begin{aligned}
[D_i^{\varphi_\psi}, D_j^{\varphi_\psi}] &= \sum_{l,\alpha} \left(D_j^{\varphi_\psi} \big(\tilde{\mathfrak{Ə}}_\psi + S_w\big)^l (X_j^\alpha) - D_j^{\varphi_\psi} \big(\tilde{\mathfrak{Ə}}_\psi + S_w\big)^l (X_i^\alpha)\right) \frac{\partial}{\partial w_l^\alpha} \\
&= \sum_{l,\alpha} \big(\tilde{\mathfrak{Ə}}_\psi + S_w\big)^l \big(D_i^{\varphi_\psi}(X_j^\alpha) - D_j^{\varphi_\psi}(X_i^\alpha)\big) \frac{\partial}{\partial w_l^\alpha} = 0,
\end{aligned}$$

since $D_i^{\varphi_\psi}(X_j^\alpha) - D_j^{\varphi_\psi}(X_i^\alpha) = D_i^\varphi(X_j^\alpha) - D_j^\varphi(X_i^\alpha) = 0$. ☐

Let now $\varphi \colon \mathcal{N} \to \mathcal{E}^\infty$ be a covering and $\varphi' \colon \mathcal{N}' \to \mathcal{N} \xrightarrow{\varphi} \mathcal{E}^\infty$ be another one. Then, by obvious reasons, any φ-shadow ψ is a φ'-shadow as well.

Applying the construction of Theorem 4.9 to both φ and φ', we obtain two coverings, φ_ψ and φ'_ψ respectively.

Lemma 4.10. *The following commutative diagram of coverings*

$$\begin{array}{ccccc}
\mathcal{N}'_\psi & \longrightarrow & \mathcal{N}_\psi & & \\
\Big\downarrow & & \Big\downarrow & & \\
\mathcal{N}' & \longrightarrow & \mathcal{N} & \longrightarrow & \mathcal{E}^\infty
\end{array}$$

takes place. Moreover, if S_ψ and S'_ψ are nonlocal symmetries corresponding in $\mathcal{N}_\psi$ and $\mathcal{N}'_\psi$ constructed by Theorem 4.9 *on the preceding page, then $S'_\psi\big|_{\mathcal{F}(\mathcal{N}_\psi)} = S_\psi$.*

Proof. It suffices to compare expressions (4.12) and (4.13) on the facing page for the coverings $\mathcal{N}_\psi$ and $\mathcal{N}'_\psi$. □

As a corollary of Theorem 4.9 and of the previous lemma, we obtain the following result.

Theorem 4.11. *Let $\varphi\colon \mathcal{N} \to \mathcal{E}^\infty$, $\mathcal{E} = \{\, F = 0 \,\}$, be an arbitrary covering and $\psi_1, \dots, \psi_s \in \mathcal{F}(\mathcal{N})$, be solutions of the equation $\tilde{\ell}_F(\psi) = 0$. Then there exists a covering $\varphi_\Psi\colon \mathcal{N}_\Psi \to \mathcal{N} \xrightarrow{\varphi} \mathcal{E}^\infty$ and φ_Ψ-symmetries $S_{\psi_1}, \dots, S_{\psi_s}$, such that $S_{\psi_s}|_{\mathcal{E}^\infty} = \tilde{\Game}_{\psi_i}|_{\mathcal{E}^\infty}$, $i = 1, \dots, s$.*

Proof. Consider the section ψ_1 and the covering $\varphi_{\psi_1}\colon \mathcal{N}_{\psi_1} \xrightarrow{\bar{\varphi}_{\psi_1}} \mathcal{N} \xrightarrow{\varphi} \mathcal{E}^\infty$ together with the symmetry S_{ψ_1} constructed in Theorem 4.9 on the preceding page. Then ψ_2 is a φ_{ψ_1}-shadow and we can construct the covering $\varphi_{\psi_1,\psi_2}\colon \mathcal{N}_{\psi_1,\psi_2} \xrightarrow{\bar{\varphi}_{\psi_1,\psi_2}} \mathcal{N}_{\psi_1} \xrightarrow{\varphi_{\psi_1}} \mathcal{E}^\infty$ with the symmetry S_{ψ_2}. Applying this procedure step by step, we obtain the series of coverings

$$\mathcal{N}_{\psi_1,\dots,\psi_s} \xrightarrow{\bar{\varphi}_{\psi_1,\dots,\psi_s}} \mathcal{N}_{\psi_1,\dots,\psi_{s-1}} \xrightarrow{\bar{\varphi}_{\psi_1,\dots,\psi_{s-1}}} \cdots \xrightarrow{\bar{\varphi}_{\psi_1,\psi_2}} \mathcal{N}_{\psi_1} \xrightarrow{\bar{\varphi}_{\psi_1}} \mathcal{N} \xrightarrow{\varphi} \mathcal{E}^\infty.$$

with the symmetries $S_{\psi_1}, \dots, S_{\psi_s}$. But ψ_1 is a $\varphi_{\psi_1,\dots,\psi_s}$-shadow and we can construct the covering $\varphi_{\psi_1}\colon \mathcal{N}^{(1)}_{\psi_1} \to \mathcal{N}_{\psi_1,\dots,\psi_s} \to \mathcal{E}^\infty$ with the symmetry $S^{(1)}_{\psi_1}$ satisfying $S^{(1)}_{\psi_1}\Big|_{\mathcal{F}(\mathcal{N}_{\psi_1})} = S_{\psi_1}$ (see Lemma 4.10), etc. Passing to the inverse limit, we obtain the covering $\mathcal{N}_\Psi$ we need. □

5. Frölicher–Nijenhuis brackets and recursion operators

We return back to the general algebraic setting of Section 1 and extend standard constructions of calculus to form-valued derivations. It allows us to define *Frölicher–Nijenhuis brackets* and introduce a cohomology theory ($\mathcal{C}$-cohomologies) associated to commutative algebras with *flat connections*. Applying this theory to partial differential equations, we obtain an algebraic description of recursion operators for symmetries and describe efficient tools to compute these operators. For technical details, examples and generalizations and we refer the reader to the papers [24, 28, 27] and [29, 31, 30].

In Subsection 6.4, $\mathcal{C}$-cohomologies will be discussed again in the general framework of horizontal cohomologies with coefficients.

5.1. **Calculus in form-valued derivations.** Let $\Bbbk$ be a field of characteristic zero and A be a commutative unitary $\Bbbk$-algebra. Let us recall the basic notations from Section 1:

- $\mathrm{D}(P)$ is the module of P-valued derivations $A \to P$, where P is an A-module;
- $\mathrm{D}_i(P)$ is the module of P-valued skew-symmetric i-derivations. In particular, $\mathrm{D}_1(P) = \mathrm{D}(P)$;
- $\Lambda^i(A)$ is the module of differential i-forms of the algebra A;
- $d\colon \Lambda^i(A) \to \Lambda^{i+1}(A)$ is the de Rham differential.

Recall also that the modules $\Lambda^i(A)$ are representative objects for the functors $\mathrm{D}_i\colon P \Rightarrow \mathrm{D}_i(P)$, i.e., $\mathrm{D}_i(P) = \mathrm{Hom}_A(\Lambda^i(A), P)$. The isomorphism $\mathrm{D}(P) = \mathrm{Hom}_A(\Lambda^1(A), P)$ can be expressed in more exact terms: for any derivation $X\colon A \to P$, there exists a uniquely defined homomorphism $\varphi^X\colon \Lambda^1(A) \to P$ satisfying $X = \varphi^X \circ d$. Denote by $\langle Z, \omega\rangle \in P$ the value of the derivation $Z \in \mathrm{D}_i(P)$ at $\omega \in \Lambda^i(A)$.

Both $\Lambda^*(A) = \bigoplus_{i\geq 0} \Lambda^i(A)$ and $\mathrm{D}_*(A) = \bigoplus_{i\geq 0} \mathrm{D}_i(A)$ are endowed with the structures of superalgebras with respect to the wedge product operation $\wedge\colon \Lambda^i(A) \otimes \Lambda^j(A) \to \Lambda^{i+j}(A)$ and $\wedge : \mathrm{D}_i(A) \otimes \mathrm{D}_j(A) \to \mathrm{D}_{i+j}(A)$, the de Rham differential $d\colon \Lambda^*(A) \to \Lambda^*(A)$ becoming a derivation of $\Lambda^*(A)$. Note also that $\mathrm{D}_*(P) = \bigoplus_{i\geq 0} \mathrm{D}_i(P)$ is a $\mathrm{D}_*(A)$-module.

Using the paring $\langle\cdot,\cdot\rangle$ and the wedge product, we define the *inner product* (or *contraction*) $\mathrm{i}_X\omega \in \Lambda^{j-i}(A)$ of $X \in \mathrm{D}_i(A)$ and $\omega \in \Lambda^j(A)$, $i \leq j$, by setting

$$\langle Y, \mathrm{i}_X\omega\rangle = (-1)^{i(j-i)}\langle X \wedge Y, \omega\rangle, \tag{5.1}$$

where Y is an arbitrary element of $\mathrm{D}_{j-i}(P)$, P being an A-module. We formally set $\mathrm{i}_X\omega = 0$ for $i > j$. When $i = 1$, this definition coincides with the one given in Section 1. Recall that the following duality is valid:

$$\langle X, da \wedge \omega\rangle = \langle X(a), \omega\rangle, \tag{5.2}$$

where $\omega \in \Lambda^i(A)$, $X \in \mathrm{D}_{i+1}(P)$, and $a \in A$ (see Exercise 1.4 on page 14). Using the property (5.2), one can show that

$$\mathrm{i}_X(\omega \wedge \theta) = \mathrm{i}_X(\omega) \wedge \theta + (-1)^{X\omega}\omega \wedge \mathrm{i}_X(\omega)$$

for any $\omega, \theta \in \Lambda^*(A)$, where (as everywhere below) the symbol of a graded object used as the exponent of (-1) denotes the degree of that object.

We now define the *Lie derivative* of $\omega \in \Lambda^*(A)$ along $X \in \mathrm{D}_*(A)$ as

$$\mathrm{L}_X\omega = \big(\mathrm{i}_X \circ d - (-1)^X d \circ \mathrm{i}_X\big)\omega = [\mathrm{i}_X, d]\omega, \tag{5.3}$$

where $[\cdot,\cdot]$ denotes the supercommutator: if $\Delta, \Delta' : \Lambda^*(A) \to \Lambda^*(A)$ are graded derivations, then $[\Delta, \Delta'] = \Delta \circ \Delta' - (-1)^{\Delta\Delta'}\Delta' \circ \Delta$. For $X \in \mathrm{D}(A)$ this definition coincides with the one given by equality (1.9) on page 15.

Consider now the graded module $\mathrm{D}(\Lambda^*(A))$ of $\Lambda^*(A)$-valued derivations $A \to \Lambda^*(A)$ (corresponding to form-valued vector fields—or, which is the same—vector-valued differential forms on a smooth manifold). Note that the graded structure in $\mathrm{D}(\Lambda^*(A))$ is determined by the splitting $\mathrm{D}(\Lambda^*(A)) = \bigoplus_{i \geq 0} \mathrm{D}(\Lambda^i(A))$ and thus elements of grading i are derivations X such that $\operatorname{im} X \subset \Lambda^i(A)$. We shall need three algebraic structures associated to $\mathrm{D}(\Lambda^*(A))$. First note that $\mathrm{D}(\Lambda^*(A))$ is a graded $\Lambda^*(A)$-module: for any $X \in \mathrm{D}(\Lambda^*(A))$, $\omega \in \Lambda^*(A)$ and $a \in A$ we set $(\omega \wedge X)a = \omega \wedge X(a)$. Second, we can define the inner product $\mathrm{i}_X\omega \in \Lambda^{i+j-1}(A)$ of $X \in \mathrm{D}(\Lambda^i(A))$ and $\omega \in \Lambda^j(A)$ in the following way. If $j = 0$, we set $\mathrm{i}_X\omega = 0$. Then, by induction on j and using the fact that $\Lambda^*(A)$ as a graded A-algebra is generated by the elements of the form da, $a \in A$, we set

$$\mathrm{i}_X(da \wedge \omega) = X(a) \wedge \omega - (-1)^X da \wedge \mathrm{i}_X(\omega), \quad a \in A. \tag{5.4}$$

Finally, we can contract elements of $\mathrm{D}(\Lambda^*(A))$ with each other in the following way:

$$(\mathrm{i}_X Y)a = \mathrm{i}_X(Ya), \quad X, Y \in \mathrm{D}(\Lambda^*(A)), \quad a \in A. \tag{5.5}$$

Three properties of contractions are essential in the sequel.

Proposition 5.1. *Let $X, Y \in \mathrm{D}(\Lambda^*(A))$ and $\omega, \theta \in \Lambda^*(A)$. Then*

$$\mathrm{i}_X(\omega \wedge \theta) = \mathrm{i}_X(\omega) \wedge \theta + (-1)^{\omega(X-1)}\omega \wedge \mathrm{i}_X(\theta), \tag{5.6}$$

$$\mathrm{i}_X(\omega \wedge Y) = \mathrm{i}_X(\omega) \wedge Y + (-1)^{\omega(X-1)}\omega \wedge \mathrm{i}_X(Y), \tag{5.7}$$

$$[\mathrm{i}_X, \mathrm{i}_Y] = \mathrm{i}_{[\![X,Y]\!]^{\mathrm{rn}}}, \tag{5.8}$$

where

$$[\![X, Y]\!]^{\mathrm{rn}} = \mathrm{i}_X(Y) - (-1)^{(X-1)(Y-1)}\mathrm{i}_Y(X). \tag{5.9}$$

Proof. Equality (5.6) is a direct consequence of (5.4). To prove (5.7), it suffices to use the definition and expressions (5.5) and (5.6).

Let us prove (5.8) now. To do this, note first that due to (5.5) the equality is sufficient to be checked on elements $\omega \in \Lambda^j(A)$. Let us use induction on j. For $j = 0$ it holds in a trivial way. Let $a \in A$; then one has

$$\begin{aligned}[][\mathrm{i}_X, \mathrm{i}_Y](da \wedge \omega) &= \big(\mathrm{i}_X \circ \mathrm{i}_Y - (-1)^{(X-1)(Y-1)} \mathrm{i}_Y \circ \mathrm{i}_X\big)(da \wedge \omega) \\ &= \mathrm{i}_X(\mathrm{i}_Y(da \wedge \omega)) - (-1)^{(X-1)(Y-1)} \mathrm{i}_Y(\mathrm{i}_X(da \wedge \omega)).\end{aligned}$$

But

$$\begin{aligned}\mathrm{i}_X(\mathrm{i}_Y(da \wedge \omega)) &= \mathrm{i}_X(Y(a) \wedge \omega - (-1)^Y da \wedge \mathrm{i}_Y \omega) \\ &= \mathrm{i}_X(Y(a)) \wedge \omega + (-1)^{(X-1)Y} Y(a) \wedge \mathrm{i}_X \omega - (-1)^Y (X(a) \wedge \mathrm{i}_Y \omega \\ &\qquad - (-1)^X da \wedge \mathrm{i}_X(\mathrm{i}_Y \omega)),\end{aligned}$$

while

$$\begin{aligned}\mathrm{i}_Y(\mathrm{i}_X(da \wedge \omega) &= \mathrm{i}_Y(X(a) \wedge \omega - (-1)^X da \wedge \mathrm{i}_X \omega) \\ &= \mathrm{i}_Y(X(a)) \wedge \omega + (-1)^{X(Y-1)} X(a) \wedge \mathrm{i}_Y \omega - (-1)^X (Y(a) \wedge \omega \\ &\qquad - (-1)^Y da \wedge \mathrm{i}_Y(\mathrm{i}_X \omega)).\end{aligned}$$

Hence,

$$\begin{aligned}[][\mathrm{i}_X, \mathrm{i}_Y](da \wedge \omega) &= \big(\mathrm{i}_X(Y(a)) - (-1)^{(X-1)(Y-1)} \mathrm{i}_Y(X(a))\big) \wedge \omega \\ &\quad + (-1)^{X+Y} da \wedge \big(\mathrm{i}_X(\mathrm{i}_Y \omega) - (-1)^{(X-1)(Y-1)} \mathrm{i}_Y(\mathrm{i}_X \omega)\big).\end{aligned}$$

But, by definition,

$$\begin{aligned}&\mathrm{i}_X(Y(a)) - (-1)^{(X-1)(Y-1)} \mathrm{i}_Y(X(a)) \\ &\qquad = (\mathrm{i}_X Y - (-1)^{(X-1)(Y-1)} \mathrm{i}_Y X)(a) = [\![X, Y]\!]^{\mathrm{rn}}(a),\end{aligned}$$

whereas

$$\mathrm{i}_X(\mathrm{i}_Y \omega) - (-1)^{(X-1)(Y-1)} \mathrm{i}_Y(\mathrm{i}_X \omega) = \mathrm{i}_{[\![X,Y]\!]^{\mathrm{rn}}}(\omega)$$

by induction hypothesis. $\square$

Definition 5.1. The element $[\![X, Y]\!]^{\mathrm{rn}}$ defined by equality (5.9) is called the *Richardson–Nijenhuis bracket* of elements X and Y.

Directly from Proposition 5.1 we obtain the following

Proposition 5.2. *For any derivations* $X, Y, Z \in \mathrm{D}(\Lambda^*(A))$ *and a form* $\omega \in \Lambda^*(A)$ *one has*

$$[\![X, Y]\!]^{\mathrm{rn}} + (-1)^{(X+1)(Y+1)}[\![Y, X]\!]^{\mathrm{rn}} = 0, \tag{5.10}$$

$$\oint (-1)^{(Y+1)(X+Z)}[\![[\![X, Y]\!]^{\mathrm{rn}}, Z]\!]^{\mathrm{rn}} = 0, \tag{5.11}$$

$$[\![X, \omega \wedge Y]\!]^{\mathrm{rn}} = \mathrm{i}_X(\omega) \wedge Y + (-1)^{(X+1)\omega}\omega \wedge [\![X, Y]\!]^{\mathrm{rn}}. \tag{5.12}$$

Here and below the symbol $\oint$ denotes the sum of cyclic permutations.

Remark 5.1. Note that Proposition 5.2 means that $\mathrm{D}(\Lambda^*(A))^{\downarrow}$ is a Gerstenhaber algebra with respect to the Richardson–Nijenhuis bracket [23]. Here the superscript $\downarrow$ denotes the shift of grading by 1.

Similarly to (5.3) define the Lie derivative of $\omega \in \Lambda^*(A)$ along $X \in \mathrm{D}(\Lambda^*(A))$ by

$$\mathrm{L}_X\omega = (\mathrm{i}_X \circ d + (-1)^X d \circ \mathrm{i}_X)\omega = [\mathrm{i}_X, d]\omega \tag{5.13}$$

(the change of sign is due to the fact that $\deg(\mathrm{i}_X) = \deg(X) - 1$). From the properties of i_X and d we obtain

Proposition 5.3. *For any* $X \in \mathrm{D}(\Lambda^*(A))$ *and* $\omega, \theta \in \Lambda^*(A)$*, one has the following identities:*

$$\mathrm{L}_X(\omega \wedge \theta) = \mathrm{L}_X(\omega) \wedge \theta + (-1)^{X\omega}\omega \wedge \mathrm{L}_X(\theta), \tag{5.14}$$

$$\mathrm{L}_{\omega\wedge X} = \omega \wedge \mathrm{L}_X + (-1)^{\omega+X} d(\omega) \wedge \mathrm{i}_X, \tag{5.15}$$

$$[\mathrm{L}_X, d] = 0. \tag{5.16}$$

Our main concern now is to analyze the commutator $[\mathrm{L}_X, \mathrm{L}_Y]$ of two Lie derivatives. It may be done efficiently for smooth algebras (see Definition 1.9 on page 19).

Proposition 5.4. *Let* A *be a smooth algebra. Then for any derivations* $X, Y \in \mathrm{D}(\Lambda^*(A))$ *there exists a uniquely determined element* $[\![X, Y]\!]^{\mathrm{fn}} \in \mathrm{D}(\Lambda^*(A))$ *such that*

$$[L_X, L_Y] = L_{[\![X,Y]\!]^{\mathrm{fn}}}. \tag{5.17}$$

Proof. To prove existence, recall that for smooth algebras one has

$$\mathrm{D}_i(P) = \mathrm{Hom}_A(\Lambda^i(A), P) = P \otimes_A \mathrm{Hom}_A(\Lambda^i(A), A) = P \otimes_A \mathrm{D}_i(A)$$

for any A-module P and integer $i \geq 0$. Using this identification, represent elements $X, Y \in \mathrm{D}(\Lambda^*(A))$ in the form

$$X = \omega \otimes X' \text{ and } Y = \theta \otimes Y' \text{ for } \omega, \theta \in \Lambda^*(A), X', Y' \in \mathrm{D}(A).$$

Then it is easily checked that the element

$$\begin{aligned}Z &= \omega \wedge \theta \otimes [X', Y'] + \omega \wedge \mathrm{L}_{X'}\theta \otimes Y + (-1)^{\omega} d\omega \wedge \mathrm{i}_{X'}\theta \otimes Y' \\ &\quad - (-1)^{\omega\theta}\theta \wedge \mathrm{L}_{Y'}\omega \otimes X' - (-1)^{(\omega+1)\theta} d\theta \wedge \mathrm{i}_{Y'}\omega \otimes X' \\ &= \omega \wedge \theta \otimes [X', Y'] + \mathrm{L}_X\theta \otimes Y' - (-1)^{\omega\theta}\mathrm{L}_Y\omega \otimes X'\end{aligned} \tag{5.18}$$

satisfies (5.17).

Uniqueness follows from the fact that $\mathrm{L}_X(a) = X(a)$ for any $a \in A$. □

Definition 5.2. The element $[\![X, Y]\!]^{\mathrm{fn}} \in \mathrm{D}^{i+j}(\Lambda^*(A))$ defined by formula (5.17) is called the *Frölicher–Nijenhuis bracket* of elements $X \in \mathrm{D}^i(\Lambda^*(A))$ and $Y \in \mathrm{D}^j(\Lambda^*(A))$.

The basic properties of this bracket are summarized in the following

Proposition 5.5. *Let A be a smooth algebra, $X, Y, Z \in \mathrm{D}(\Lambda^*(A))$ be derivations and $\omega \in \Lambda^*(A)$ be a differential form. Then the following identities are valid:*

$$[\![X, Y]\!]^{\mathrm{fn}} + (-1)^{XY}[\![Y, X]\!]^{\mathrm{fn}} = 0, \tag{5.19}$$

$$\oint (-1)^{Y(X+Z)}[\![X, [\![Y, Z]\!]^{\mathrm{fn}}]\!]^{\mathrm{fn}} = 0, \tag{5.20}$$

$$\mathrm{i}_{[\![X,Y]\!]^{\mathrm{fn}}} = [\mathrm{L}_X, \mathrm{i}_Y] + (-1)^{X(Y+1)}\mathrm{L}_{\mathrm{i}_Y X}, \tag{5.21}$$

$$\begin{aligned}\mathrm{i}_Z[\![X, Y]\!]^{\mathrm{fn}} &= [\![\mathrm{i}_Z X, Y]\!]^{\mathrm{fn}} + (-1)^{X(Z+1)}[\![X, \mathrm{i}_Z Y]\!]^{\mathrm{fn}} \\ &\quad + (-1)^X \mathrm{i}_{[\![Z,X]\!]^{\mathrm{fn}}} Y - (-1)^{(X+1)Y}\mathrm{i}_{[\![Z,Y]\!]^{\mathrm{fn}}} X,\end{aligned} \tag{5.22}$$

$$\begin{aligned}[\![X, \omega \wedge Y]\!]^{\mathrm{fn}} &= \mathrm{L}_X\omega \wedge Y - (-1)^{(X+1)(Y+\omega)} d\omega \wedge \mathrm{i}_Y X \\ &\quad + (-1)^{X\omega}\omega \wedge [\![X, Y]\!]^{\mathrm{fn}}.\end{aligned} \tag{5.23}$$

Note that the first two equalities in the previous proposition mean that the module $\mathrm{D}(\Lambda^*(A))$ is a Lie superalgebra with respect to the Frölicher–Nijenhuis bracket.

Remark 5.2. The above exposed algebraic scheme has a geometrical realization, if one takes $A = C^\infty(M)$, M being a smooth finite-dimensional manifold. The algebra $A = C^\infty(M)$ is smooth in this case. However, in the geometrical theory of differential equations we have to work with infinite-dimensional manifolds[10] of the form $N = \operatorname{proj\,lim}_{\{\pi_{k+1,k}\}} N_k$, where all the maps $\pi_{k+1,k} \colon N_{k+1} \to N_k$ are surjections of finite-dimensional smooth manifolds. The corresponding algebraic object is a filtered algebra $A = \bigcup_{k\in\mathbb{Z}} A_k$, $A_k \subset A_{k+1}$, where all A_k are subalgebras in A. As it was already noted, self-contained differential calculus over A is constructed,

[10]Infinite jets, infinite prolongations of differential equations, total spaces of coverings, etc.

if one considers the category of all filtered A-modules with filtered homomorphisms for morphisms between them. Then all functors of differential calculus in this category become filtered, as well as their representative objects.

In particular, the A-modules $\Lambda^i(A)$ are filtered by A_k-modules $\Lambda^i(A_k)$. We say that the algebra A is *finitely smooth*, if $\Lambda^1(A_k)$ is a projective A_k-module of finite type for any $k \in \mathbb{Z}$. For finitely smooth algebras, elements of $\mathrm{D}(P)$ may be represented as formal infinite sums $\sum_k p_k \otimes X_k$, such that any finite sum $S_n = \sum_{k \le n} p_k \otimes X_k$ is a derivation $A_n \longrightarrow P_{n+s}$ for some fixed $s \in \mathbb{Z}$. Any derivation X is completely determined by the system $\{S_n\}$ and Proposition 5.5 obviously remains valid.

5.2. **Algebras with flat connections and cohomology.** We now introduce the second object of our interest. Let A be an $\Bbbk$-algebra, $\Bbbk$ being a field of zero characteristic, and B be an algebra over A. We shall assume that the corresponding homomorphism $\varphi\colon A \longrightarrow B$ is an embedding. Let P be a B-module; then it is an A-module as well and we can consider the B-module $\mathrm{D}(A, P)$ of P-valued derivations $A \longrightarrow P$.

Definition 5.3. Let $\nabla^\bullet\colon \mathrm{D}(A,\cdot) \Rightarrow \mathrm{D}(\cdot)$ be a natural transformations of functors $\mathrm{D}(A,\cdot)\colon A \Rightarrow \mathrm{D}(A,P)$ and $\mathrm{D}(\cdot)\colon P \Rightarrow \mathrm{D}(\cdot)$ in the category of B-modules, i.e., a system of homomorphisms $\nabla^P\colon \mathrm{D}(A,P) \longrightarrow \mathrm{D}(P)$ such that the diagram

$$\begin{array}{ccc} \mathrm{D}(A,P) & \xrightarrow{\nabla^P} & \mathrm{D}(P) \\ {\scriptstyle \mathrm{D}(A,f)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{D}(f)} \\ \mathrm{D}(A,Q) & \xrightarrow{\nabla^Q} & \mathrm{D}(Q) \end{array}$$

is commutative for any B-homomorphism $f\colon P \longrightarrow Q$. We say that $\nabla^\bullet$ is a *connection* in the triad (A, B, φ), if $\nabla^P(X)\,|_A = X$ for any $X \in \mathrm{D}(A, P)$.

Here and below we use the notation $Y\,|_A = Y \circ \varphi$ for any $Y \in \mathrm{D}(P)$.

Remark 5.3. When $A = C^\infty(M)$, $B = C^\infty(E)$, $\varphi = \pi^*$, where M and E are smooth manifolds and $\pi\colon E \longrightarrow M$ is a smooth fiber bundle, Definition 5.3 reduces to the ordinary definition of a connection in the bundle π. In fact, if we have a connection $\nabla^\bullet$ in the sense of Definition 5.3, then the correspondence

$$\mathrm{D}(A) \hookrightarrow \mathrm{D}(A,B) \xrightarrow{\nabla^B} \mathrm{D}(B)$$

allows one to lift any vector field on M up to a π-projectible field on E. Conversely, if ∇ is such a correspondence, then we can construct a natural transformation $\nabla^\bullet$ of the functors $\mathrm{D}(A,\cdot)$ and $\mathrm{D}(\cdot)$ due to the fact that for smooth finite-dimensional manifolds one has $\mathrm{D}(A,P) = P \otimes_A \mathrm{D}(A)$ and

$\mathrm{D}(P) = P \otimes_B \mathrm{D}(P)$ for an arbitrary B-module P. We use the notation $\nabla = \nabla^B$ in the sequel.

Definition 5.4. Let $\nabla^\bullet$ be a connection in (A, B, φ) and $X, Y \in \mathrm{D}(A, B)$ be two derivations. The *curvature form* of the connection $\nabla^\bullet$ on the pair X, Y is defined by

$$R_\nabla(X, Y) = [\nabla(X), \nabla(Y)] - \nabla(\nabla(X) \circ Y - \nabla(Y) \circ X). \tag{5.24}$$

Note that (5.24) makes sense, since $\nabla(X) \circ Y - \nabla(Y) \circ X$ is a B-valued derivation of A.

Consider now the de Rham differential $d = d_B \colon B \longrightarrow \Lambda^1(B)$. Then the composition $d_B \circ \varphi \colon A \longrightarrow B$ is a derivation. Consequently, we may consider the derivation $\nabla(d_B \circ \varphi) \in \mathrm{D}(\Lambda^1(B))$.

Definition 5.5. The element $U_\nabla \in \mathrm{D}(\Lambda^1(B))$ defined by

$$U_\nabla = \nabla(d_B \circ \varphi) - d_B \tag{5.25}$$

is called the *connection form* of ∇.

Directly from the definition we obtain the following

Lemma 5.6. *The equality*

$$\mathrm{i}_X(U_\nabla) = X - \nabla(X\,|_A) \tag{5.26}$$

holds for any $X \in \mathrm{D}(B)$.

Using this result, we now prove

Proposition 5.7. *If B is a smooth algebra, then*

$$\mathrm{i}_Y \mathrm{i}_X [\![U_\nabla, U_\nabla]\!]^{\mathrm{fn}} = 2R_\nabla(X\,|_A\,, Y\,|_A) \tag{5.27}$$

for any $X, Y \in \mathrm{D}(B)$.

Proof. First note that $\deg U_\nabla = 1$. Then using (5.22) and (5.19) we obtain

$$\begin{aligned}\mathrm{i}_X [\![U_\nabla, U_\nabla]\!]^{\mathrm{fn}} &= [\![\mathrm{i}_X U_\nabla, U_\nabla]\!]^{\mathrm{fn}} + [\![U_\nabla, \mathrm{i}_X U_\nabla]\!]^{\mathrm{fn}} - \mathrm{i}_{[\![X, U_\nabla]\!]^{\mathrm{fn}}} U_\nabla - \mathrm{i}_{[\![X, U_\nabla]\!]^{\mathrm{fn}}} U_\nabla \\ &= 2\big([\![\mathrm{i}_X U_\nabla, U_\nabla]\!]^{\mathrm{fn}} - \mathrm{i}_{[\![X, U_\nabla]\!]^{\mathrm{fn}}} U_\nabla\big).\end{aligned}$$

Applying i_Y to the last expression and using (5.20) and (5.22), we get now

$$\mathrm{i}_Y \mathrm{i}_X [\![U_\nabla, U_\nabla]\!]^{\mathrm{fn}} = 2\big([\![\mathrm{i}_X U_\nabla, \mathrm{i}_Y U_\nabla]\!]^{\mathrm{fn}} - \mathrm{i}_{[\![X, Y]\!]^{\mathrm{fn}}} U_\nabla\big).$$

But $[\![V, W]\!]^{\mathrm{fn}} = [V, W]$ for any $V, W \in \mathrm{D}(\Lambda^0(A)) = \mathrm{D}(A)$. Hence, by (5.26), we have

$$\mathrm{i}_Y \mathrm{i}_X [\![U_\nabla, U_\nabla]\!]^{\mathrm{fn}} = 2\big([X - \nabla(X\,|_A), Y - \nabla(Y\,|_A)] - ([X, Y] - \nabla([X, Y]\,|_A))\big).$$

It only remains to note now that $\nabla(X\,|_A)\,|_A = X\,|_A$ and $[X, Y]\,|_A = X \circ Y\,|_A - Y \circ X\,|_A$. ∎

Definition 5.6. A connection ∇ in (A, B, φ) is called *flat*, if $R_\nabla = 0$.

Thus for flat connections we have

$$[\![U_\nabla, U_\nabla]\!]^{\mathrm{fn}} = 0. \tag{5.28}$$

Let $U \in \mathrm{D}(\Lambda^1(B))$ be an element satisfying (5.28). Then from the graded Jacobi identity (5.20) we obtain $2[\![U, [\![U, X]\!]^{\mathrm{fn}}]\!]^{\mathrm{fn}} = [\![[\![U, U]\!]^{\mathrm{fn}}, X]\!]^{\mathrm{fn}} = 0$ for any $X \in \mathrm{D}(\Lambda^*(A))$. Consequently, the operator $\partial_U = [\![U, \cdot]\!]^{\mathrm{fn}} : \mathrm{D}(\Lambda^i(B)) \to \mathrm{D}(\Lambda^{i+1}(B))$ defined by the equality $\partial_U(X) = [\![U, X]\!]^{\mathrm{fn}}$ satisfies the identity $\partial_U \circ \partial_U = 0$.

Consider now the case $U = U_\nabla$, where ∇ is a flat connection.

Definition 5.7. An element $X \in \mathrm{D}(\Lambda^*(B))$ is called *vertical*, if $X(a) = 0$ for any $a \in A$. Denote the B-submodule of such elements by $\mathrm{D}^v(\Lambda^*(B))$.

Lemma 5.8. *Let ∇ be a connection in (A, B, φ). Then*

(1) *an element $X \in \mathrm{D}(\Lambda^*(B))$ is vertical if and only if $\mathrm{i}_X U_\nabla = X$;*
(2) *the connection form U_∇ is vertical, $U_\nabla \in \mathrm{D}^v(\Lambda^1(B))$;*
(3) *the map ∂_{U_∇} preserves verticality, $\partial_{U_\nabla}(\mathrm{D}^v(\Lambda^i(B))) \subset \mathrm{D}^v(\Lambda^{i+1}(B))$.*

Proof. To prove (1), use Lemma 5.6: from (5.26) it follows that $\mathrm{i}_X U_\nabla = X$ if and only if $\nabla(X\,|_A) = 0$. But $\nabla(X\,|_A)\,|_A = X\,|_A$. The second statements follows from the same lemma and from the first one:

$$\mathrm{i}_{U_\nabla} U_\nabla = U_\nabla - \nabla(U_\nabla\,|_A) = U_\nabla - \nabla\big((U_\nabla - \nabla(U_\nabla\,|_A)\,|_A\big) = U_\nabla.$$

Finally, (3) is a consequence of (5.22). $\square$

Definition 5.8. Denote the restriction $\partial_{U_\nabla}\big|_{\mathrm{D}^v(\Lambda^*(A))}$ by ∂_∇ and call the complex

$$0 \to \mathrm{D}^v(B) \xrightarrow{\partial_\nabla} \mathrm{D}^v(\Lambda^1(B)) \to \cdots \to \mathrm{D}^v(\Lambda^i(B)) \xrightarrow{\partial_\nabla} \mathrm{D}^v(\Lambda^{i+1}(B)) \to \cdots \tag{5.29}$$

the ∇-*complex* of the triple (A, B, φ). The corresponding cohomology is denoted by $H^*_\nabla(B; A, \varphi) = \bigoplus_{i \geq 0} H^i_\nabla(B; A, \varphi)$ and is called the ∇-*cohomology* of the triple (A, B, φ).

Introduce the notation

$$d^v_\nabla = \mathrm{L}_{U_\nabla} : \Lambda^i(B) \to \Lambda^{i+1}(B). \tag{5.30}$$

Proposition 5.9. *Let ∇ be a flat connection in the triple (A, B, φ) and B be a smooth (or finitely smooth) algebra. Then for any $X, Y \in \mathrm{D}^v(\Lambda^*(A))$*

and $\omega \in \Lambda^*(A)$ *one has*

$$\partial_\nabla [\![X, Y]\!]^{\mathrm{fn}} = [\![\partial_\nabla X, Y]\!]^{\mathrm{fn}} + (-1)^X [\![X, \partial_\nabla Y]\!]^{\mathrm{fn}}, \tag{5.31}$$

$$[\mathrm{i}_X, \partial_\nabla] = (-1)^X \mathrm{i}_{\partial_\nabla X}, \tag{5.32}$$

$$\partial_\nabla (\omega \wedge X) = (d^v_\nabla - d)(\omega) \wedge X + (-1)^\omega \omega \wedge \partial_\nabla X, \tag{5.33}$$

$$[d^v_\nabla, \mathrm{i}_X] = \mathrm{i}_{\partial_\nabla X} + (-1)^X \mathrm{L}_X. \tag{5.34}$$

Proof. Equality (5.31) is a direct consequence of (5.20). Equality (5.32) follows from (5.22). Equality (5.33) follows from (5.23) and (5.26). Finally, (5.34) is obtained from (5.21). □

Corollary 5.10. *The cohomology module* $H^*_\nabla(B; A, \varphi)$ *inherits the graded Lie algebra structure with respect to the Frölicher–Nijenhuis bracket* $[\![\cdot, \cdot]\!]^{\mathrm{fn}}$*, as well as to the contraction operation.*

Proof. Note that $\mathrm{D}^v(\Lambda^*(A))$ is closed with respect to the Frölicher–Nijenhuis bracket: to prove this fact, it suffices to apply (5.22). Then the first statement follows from (5.31). The second one is a consequence of (5.32). □

Remark 5.4. We preserve the same notations for the inherited structures. Note, in particular, that $H^0_\nabla(B; A, \varphi)$ is a Lie algebra with respect to the Frölicher–Nijenhuis bracket (which reduces to the ordinary Lie bracket in this case). Moreover, $H^1_\nabla(B; A, \varphi)$ is an associative algebra with respect to the inherited contraction, while the action

$$\mathcal{R}_\Omega \colon X \mapsto \mathrm{i}_X \Omega, \quad X \in H^0_\nabla(B; A, \varphi), \quad \Omega \in H^1_\nabla(B; A, \varphi)$$

is a representation of this algebra as endomorphisms of $H^0_\nabla(B; A, \varphi)$.

Consider now the map $d^v_\nabla \colon \Lambda^*(B) \longrightarrow \Lambda^*(B)$ defined by (5.30) and define $d^h_\nabla = d_B - d^v_\nabla$.

Proposition 5.11. *Let* B *be a (finitely) smooth algebra and* ∇ *be a smooth connection in the triple* $(B; A, \varphi)$*. Then*

(1) *The pair* (d^h_∇, d^v_∇) *forms a bicomplex, i.e.*

$$d^v_\nabla \circ d^v_\nabla = 0, \quad d^h_\nabla \circ d^h_\nabla = 0, \quad d^h_\nabla \circ d^v_\nabla + d^v_\nabla \circ d^h_\nabla = 0. \tag{5.35}$$

(2) *The differential* d^h_∇ *possesses the following properties*

$$[d^h_\nabla, \mathrm{i}_X] = -\mathrm{i}_{\partial_\nabla X}, \tag{5.36}$$

$$\partial_\nabla (\omega \wedge X) = -d^h_\nabla(\omega) \wedge X + (-1)^\omega \omega \wedge \partial_\nabla X, \tag{5.37}$$

where $\omega \in \Lambda^*(B)$, $X \in \mathrm{D}^v(\Lambda^*(B))$.

Proof. (1) Since $\deg d^v_\nabla = 1$, we have

$$2d^v_\nabla \circ d^v_\nabla = [d^v_\nabla, d^v_\nabla] = [\mathrm{L}_{U_\nabla}, \mathrm{L}_{U_\nabla}] = \mathrm{L}_{[\![U_\nabla, U_\nabla]\!]^{\mathrm{fn}}} = 0.$$

Since $d^v_\nabla = \mathrm{L}_{U_\nabla}$, the identity $[d_B, d^v_\nabla] = 0$ holds (see (5.16)), and it concludes the proof of the first part.

(2) To prove (5.36), note that

$$[d^h_\nabla, \mathrm{i}_X] = [d_B - d^h_\nabla, \mathrm{i}_X] = (-1)^X \mathrm{L}_X - [d^v_\nabla, \mathrm{i}_X],$$

and (5.36) holds due to (5.34). Finally, (5.37) is just the other form of (5.33). $\square$

Definition 5.9. Let ∇ be a connection in (A, B, φ).

(1) The bicomplex $(B, d^h_\nabla, d^v_\nabla)$ is called the *variational bicomplex* associated to the connection ∇.

(2) The corresponding spectral sequence is called the ∇*-spectral sequence* of the triple (A, B, φ).

Obviously, the ∇-spectral sequence converges to the de Rham cohomology of B.

To finish this section, note the following. Since the module $\Lambda^1(B)$ is generated by the image of the operator $d_B\colon B \longrightarrow \Lambda^1(B)$ while the graded algebra $\Lambda^*(B)$ is generated by $\Lambda^1(B)$, we have the direct sum decomposition

$$\Lambda^*(B) = \bigoplus_{i\geq 0} \bigoplus_{p+q=i} \Lambda^p_v(B) \otimes \Lambda^q_h(B),$$

where

$$\Lambda^p_v(B) = \underbrace{\Lambda^1_v(B) \wedge \cdots \wedge \Lambda^1_v(B)}_{p\ \text{times}}, \quad \Lambda^q_h(B) = \underbrace{\Lambda^1_h(B) \wedge \cdots \wedge \Lambda^1_h(B)}_{q\ \text{times}},$$

while the submodules $\Lambda^1_v(B) \subset \Lambda^1(B)$, $\Lambda^1_h(B) \subset \Lambda^1(B)$ are spanned in $\Lambda^1(B)$ by the images of the differentials d^v_∇ and d^h_∇ respectively. Obviously, we have the following embeddings:

$$d^h_\nabla\left(\Lambda^p_v(B) \otimes \Lambda^q_h(B)\right) \subset \Lambda^p_v(B) \otimes \Lambda^{q+1}_h(B),$$
$$d^v_\nabla\left(\Lambda^p_v(B) \otimes \Lambda^q_h(B)\right) \subset \Lambda^{p+1}_v(B) \otimes \Lambda^q_h(B).$$

Denote by $\mathrm{D}^{p,q}(B)$ the module $\mathrm{D}^v(\Lambda^p_v(B) \otimes \Lambda^q_h(B))$. Then, obviously, $\mathrm{D}^v(B) = \bigoplus_{i\geq 0} \bigoplus_{p+q=i} \mathrm{D}^{p,q}(B)$, while from equalities (5.36) and (5.37) we obtain

$$\partial_\nabla\big(\mathrm{D}^{p,q}(B)\big) \subset \mathrm{D}^{p,q+1}(B).$$

Consequently, the module $H^*_\nabla(B; A, \varphi)$ is split as

$$H^*_\nabla(B; A, \varphi) = \bigoplus_{i\geq 0} \bigoplus_{p+q=i} H^{p,q}_\nabla(B; A, \varphi) \tag{5.38}$$

with the obvious meaning of the notation $H^{p,q}_\nabla(B; A, \varphi)$.

5.3. **Applications to differential equations: recursion operators.** Now we apply the above exposed algebraic results to the case of infinitely prolonged differential equations. Let us start with establishing a correspondence between geometric constructions of Section 3 and algebraic ones presented in the previous two subsections.

Let $\mathcal{E} \subset J^k(\pi)$ be a formally integrable equation (see Definition 3.20 on page 54) and $\mathcal{E}^\infty \subset J^\infty(\pi)$ be its infinite prolongations. Then the bundle $\pi_\infty : \mathcal{E}^\infty \longrightarrow M$ is endowed with the Cartan connection $\mathcal{C}$ (Definition 3.23 on page 57) and this connection is flat (Corollary 3.19 on page 58). Thus the triple

$$\big(A = C^\infty(M),\, B = \mathcal{F}(\mathcal{E}),\, \varphi = \pi_\infty^*\big)$$

with $\nabla = \mathcal{C}$ is an algebra with a flat connection, A being a smooth and B being a finitely smooth algebra. The corresponding connection form is exactly the structural element $U_{\mathcal{C}}$ of the equation $\mathcal{E}$ (see Definition 3.24 on page 58).

Thus, to any formally integrable equation $\mathcal{E} \subset J^k(\pi)$ we can associate the complex

$$0 \longrightarrow \mathrm{D}^v(\mathcal{E}) \xrightarrow{\partial_{\mathcal{C}}} \mathrm{D}^v(\Lambda^1(\mathcal{E})) \longrightarrow \cdots \longrightarrow \mathrm{D}^v(\Lambda^i(\mathcal{E})) \xrightarrow{\partial_{\mathcal{C}}} \mathrm{D}^v(\Lambda^{i+1}(\mathcal{E})) \longrightarrow \cdots \tag{5.39}$$

and the cohomology theory determined by the Cartan connection. We denote the corresponding cohomology modules by $H^*_{\mathcal{C}}(\mathcal{E}) = \bigoplus_{i\geq 0} H^i_{\mathcal{C}}(\mathcal{E})$. In the case of the "empty" equation, we use the notation $H^*_{\mathcal{C}}(\pi) = \bigoplus_{i\geq 0} H^i_{\mathcal{C}}(\pi)$.

Definition 5.10. Let $\mathcal{E} \subset J^k(\pi)$ be a formally integrable equation and $\mathcal{C}$ be the Cartan connection in the bundle $\pi_\infty \colon \mathcal{E}^\infty \longrightarrow M$. Then the module $H^*_{\mathcal{C}}(\mathcal{E})$ is called the *$\mathcal{C}$-cohomology* of $\mathcal{E}$.

Remark 5.5. Let us also note that the above introduced modules $\Lambda^q_h(B)$ are identical to the modules $\bar{\Lambda}^q(\mathcal{E})$ of horizontal q-forms on $\mathcal{E}^\infty$, the modules $\Lambda^p_v(B)$ coincide with the modules of Cartan forms $\mathcal{C}^p\Lambda(\mathcal{E})$, the differential d^h_∇ is the extended horizontal de Rham differential $\bar{d}$, while d^v_∇ is the Cartan differential $d_{\mathcal{C}}$ (cf. with constructions on pp. 60–62). Thus we again obtain a complete coincidence between algebraic and geometric approaches. In particular, the ∇-spectral sequence (Definition 5.9 on the page before (2)) is the Vinogradov $\mathcal{C}$-spectral sequence (see the Section 7).

The following result contains an interpretation of the first two of $\mathcal{C}$-cohomology groups.

Theorem 5.12. *For any formally integrable equation $\mathcal{E} \subset J^k(\pi)$, one has the following identities:*

(1) *The module* $H^0_{\mathcal{C}}(\mathcal{E})$ *as a Lie algebra is isomorphic to the Lie algebra* $\operatorname{sym}\mathcal{E}$ *of higher symmetries*[11] *of the equation* $\mathcal{E}$*.*
(2) *The module* $H^1_{\mathcal{C}}(\mathcal{E})$ *is the set of the equivalence classes of nontrivial vertical deformations of the equation structure* (*i.e., of the structural element*) *on* $\mathcal{E}$*.*

Proof. To prove (1), take a vertical vector field $Y \in \mathrm{D}^v(\mathcal{E})$ and an arbitrary field $Z \in \mathrm{D}(\mathcal{E})$. Then, due to (5.22) on page 82, one has

$$\begin{aligned} \mathrm{i}_Z\partial_{\mathcal{C}}Y = \mathrm{i}_Z[\![U_{\mathcal{C}}, Y]\!]^{\mathrm{fn}} &= [\mathrm{i}_Z U_{\mathcal{C}}, Y] - \mathrm{i}_{[Z,Y]}U_{\mathcal{C}} \\ &= [Z^v, Y] - [Z, Y]^v = [Z^v - Z, Y]^v, \end{aligned}$$

where $Z^v = \mathrm{i}_Z U_{\mathcal{C}}$. Hence, $\partial_{\mathcal{C}}Y = 0$ if and only if $[Z - Z^v, Y]^v = 0$ for any $Z \in \mathrm{D}(\mathcal{E})$. But the last equality holds if and only if $[\mathcal{C}X, Y] = 0$ for any $X \in \mathrm{D}(M)$ which means that

$$\ker\big(\partial_{\mathcal{C}} \colon \mathrm{D}^v(\mathcal{E}) \to \mathrm{D}^v(\Lambda^1(\mathcal{E}))\big) = \operatorname{sym}\mathcal{E}.$$

Consider the second statement now. Let $U(\varepsilon) \in \mathrm{D}^v(\Lambda^1(\mathcal{E}))$ be a deformation of the structural element satisfying the conditions $[\![U(\varepsilon), U(\varepsilon)]\!]^{\mathrm{fn}} = 0$ and $U(0) = U_{\mathcal{C}}$. Then $U(\varepsilon) = U_{\mathcal{C}} + U_1\varepsilon + O(\varepsilon^2)$. Consequently,

$$[\![U(\varepsilon), U(\varepsilon)]\!]^{\mathrm{fn}} = [\![U_{\mathcal{C}}, U_{\mathcal{C}}]\!]^{\mathrm{fn}} + 2[\![U_{\mathcal{C}}, U_1]\!]^{\mathrm{fn}}\varepsilon + O(\varepsilon^2) = 0,$$

from which it follows that $[\![U_{\mathcal{C}}, U_1]\!]^{\mathrm{fn}} = \partial_{\mathcal{C}}U_1 = 0$. Hence the linear part of the deformation $U(\varepsilon)$ determines an element of $H^1_{\mathcal{C}}(\mathcal{E})$ and vice versa. On the other hand, let $A \colon \mathcal{E}^\infty \to \mathcal{E}^\infty$ be a diffeomorphism[12] of $\mathcal{E}^\infty$. Define the action A^* of A on the elements $\Omega \in \mathrm{D}(\Lambda^*(\mathcal{E}))$ in such a way that the diagram

$$\begin{tikzcd} \Lambda^*(\mathcal{E}) \arrow[r, "\mathrm{L}_\Omega"] \arrow[d, "A^*"'] & \Lambda^*(\mathcal{E}) \arrow[d, "A^*"] \\ \Lambda^*(\mathcal{E}) \arrow[r, "\mathrm{L}_\Omega"] & \Lambda^*(\mathcal{E}) \end{tikzcd}$$

is commutative. Then, if A_t is a one-parameter group of diffeomorphisms, we have, obviously,

$$\left.\frac{d}{dt}\right|_{t=0} A_{t,*}(\mathrm{L}_\Omega) = \left.\frac{d}{dt}\right|_{t=0} A_t^* \circ \mathrm{L}_\Omega \circ (A_t^*)^{-1} = [\mathrm{L}_X, \mathrm{L}_\Omega] = \mathrm{L}_{[\![X,\Omega]\!]^{\mathrm{fn}}}.$$

Hence, the infinitesimal action is given by the Frölicher–Nijenhuis bracket. Taking $\Omega = U_{\mathcal{C}}$ and $X \in \mathrm{D}^v(\mathcal{E})$, we see that $\operatorname{im}\partial_{\mathcal{C}}$ consists of infinitesimal

[11]See Definition 3.29 on page 64.

[12]Since $\mathcal{E}^\infty$ is, in general, infinite-dimensional, vector fields on $\mathcal{E}^\infty$ do not usually possess one-parameter groups of diffeomorphisms. Thus the arguments below are of a heuristic nature.

deformations arising due to infinitesimal action of diffeomorphisms on the structural element. Such deformations are naturally called trivial. □

Remark 5.6. From the general theory [14], we obtain that the module $H^2_{\mathcal{C}}(\mathcal{E})$ consists of obstructions to prolongation of infinitesimal deformations to formal ones. In the case under consideration, elements $H^2_{\mathcal{C}}(\mathcal{E})$ have another nice interpretation discussed later (see Remark 5.8 on page 95).

We shall now compute the modules $H^p_{\mathcal{C}}(\pi)$, $p \geq 0$. To do this, recall the splitting $\Lambda^i(\mathcal{E}) = \bigoplus_{p+q=i} \mathcal{C}^p\Lambda(\mathcal{E}) \otimes \bar{\Lambda}^q(\mathcal{E})$ (see Subsection 5.1).

Theorem 5.13. *For any $p \geq 0$, one has*

$$H^p_{\mathcal{C}}(\pi) = \mathcal{F}(\pi,\pi) \otimes_{\mathcal{F}(\pi)} \mathcal{C}^p\Lambda(\pi).$$

Proof. Define a filtration in $\mathrm{D}^v(\Lambda^*(\pi))$ by setting

$$F^l\mathrm{D}^v(\Lambda^p(\pi)) = \{X \in \mathrm{D}^v(\Lambda^p(\pi)) \mid X\big|_{\mathcal{F}_{l-p-1}} = 0\}.$$

Evidently,

$$F^l\mathrm{D}^v(\Lambda^p(\pi)) \subset F^{l+1}\mathrm{D}^v(\Lambda^p(\pi)), \quad \partial_{\mathcal{C}}\big(F^l\mathrm{D}^v(\Lambda^p(\pi))\big) \subset F^l\mathrm{D}^v(\Lambda^{p+1}(\pi)).$$

Thus we obtain the spectral sequence associated to this filtration. To compute the term E_0, choose local coordinates $x_1, \dots, x_n, u^1, \dots, u^m$ in the bundle π and consider the corresponding special coordinates u^j_σ in $J^\infty(\pi)$. In these coordinates, the structural element is represented as

$$U_{\mathcal{C}} = \sum_{|\sigma|\geq 0} \sum_{j=1}^m \left(du^j_\sigma - \sum_{i=1}^n u^j_{\sigma i}\, dx_i \right) \otimes \frac{\partial}{\partial u^j_\sigma}, \tag{5.40}$$

while for $X = \sum_{\sigma,j} \theta^j_\sigma \otimes \partial/\partial u^j_\sigma$, $\theta \in \Lambda^*(\pi)$, one has

$$\partial_{\mathcal{C}}(X) = \sum_{|\sigma|\geq 0} \sum_{j=1}^m \sum_{i=1}^n dx_i \wedge \big(\theta^j_{\sigma i} - D_i(\theta^j_\sigma)\big) \otimes \frac{\partial}{\partial u^j_\sigma}. \tag{5.41}$$

Obviously, the term

$$E_0^{p,-q} = F^p\mathrm{D}^v(\Lambda^{p-q}(\pi))/F^{p-1}\mathrm{D}^v(\Lambda^{p-q}(\pi)), \quad p \geq 0, \quad 0 \leq q \leq p,$$

is identified with the tensor product $\Lambda^{p-q}(\pi) \otimes_{\mathcal{F}(\pi)} \Gamma(\pi^*_{\infty,q-1}(\pi_{q,q-1}))$. These modules can be locally represented as $\mathcal{F}(\pi,\pi) \otimes \Lambda^{p-q}(\pi)$-valued homogeneous polynomials of order q, while the differential $\partial_0^{p,-q}\colon E_0^{p,-q} \longrightarrow E_0^{p,-q+1}$ acts as the δ-Spencer differential (or, which is the same, as the Koszul differential; see Exercise 1.7 on page 20). Hence, all homology groups are trivial except for those at the terms $E_0^{p,0}$ and one has

$$\operatorname{coker} \partial_0^{p,0} = \mathcal{F}(\pi,\pi) \otimes_{\mathcal{F}(\pi)} \mathcal{C}^p\Lambda(\pi).$$

Consequently, only the 0-th line survives in the term E_1 and this line is of the form

$$\mathcal{F}(\pi,\pi) \xrightarrow{\partial_1^{0,0}} \mathcal{F}(\pi,\pi) \otimes_{\mathcal{F}(\pi)} \mathcal{C}^1\Lambda(\pi) \longrightarrow \cdots$$
$$\cdots \longrightarrow \mathcal{F}(\pi,\pi) \otimes_{\mathcal{F}(\pi)} \mathcal{C}^p\Lambda(\pi) \xrightarrow{\partial_1^{p,0}} \mathcal{F}(\pi,\pi) \otimes_{\mathcal{F}(\pi)} \mathcal{C}^{p+1}\Lambda(\pi) \longrightarrow \cdots$$

But the image of $\partial_{\mathcal{C}}$ contains at least one horizontal component (see equality (5.33) on page 86, where, by definition, $d_\nabla^v - d = d_{\mathcal{C}} - d = -\bar{d}$). Therefore, all differentials $\partial_1^{p,0}$ vanish. $\square$

Let us now establish the correspondence between the last result (describing $\mathcal{C}$-cohomology in terms of $\mathcal{C}^*\Lambda(\pi)$) and representation of $H^*_{\mathcal{C}}(\pi)$ as classes of derivations $\mathcal{F}(\pi) \longrightarrow \Lambda^*(\pi)$. To do this, for any $\omega = (\omega^1, \dots, \omega^m) \in \mathcal{F}(\pi,\pi) \otimes_{\mathcal{F}(\pi)} \mathcal{C}^*\Lambda(\pi)$ set

$$\ni_\omega = \sum_{\sigma,j} D_\sigma(\omega^j) \otimes \frac{\partial}{\partial u_\sigma^j}, \tag{5.42}$$

where $D_\sigma = D_1^{\sigma_1} \circ \cdots \circ D_n^{\sigma_n}$ for $\sigma = (\sigma_1, \dots \sigma_n)$.

Definition 5.11. The element $\ni_\omega \in \mathrm{D}^v(\Lambda^*(\pi))$ defined by (5.42) is called the *evolutionary superderivation* with the *generating section* $\omega \in \mathcal{C}^*\Lambda(\pi)$.

Proposition 5.14. *The definition of $\ni_\omega$ is independent of coordinate choice.*

Proof. It is easily checked that

$$\ni_\omega\bigl(\mathcal{F}(\pi)\bigr) \subset \Lambda_v^*(\pi), \quad \ni_\omega \in \ker \partial_{\mathcal{C}}.$$

But derivations possessing these properties are uniquely determined by their restriction to $\mathcal{F}_0(\pi)$ which coincides with the action of the derivation $\omega : \mathcal{F}_0(\pi) \longrightarrow \mathcal{C}^*\Lambda(\pi)$. Let us prove this fact.

Set $X = \ni_\omega$ and recall that the derivation X is uniquely determined by the corresponding Lie derivative $\mathrm{L}_X : \Lambda^*(\pi) \longrightarrow \Lambda^*(\pi)$. Further, since $\mathrm{L}_X d\theta = (-1)^X d(\mathrm{L}_X\theta)$ (see (5.16) on page 81) for any $\theta \in \Lambda^*(\pi)$, the derivation L_X is determined by its restriction to $\Lambda^0(\pi) = \mathcal{F}(\pi)$.

Now, from the identity $\partial_{\mathcal{C}} X = 0$ it follows that

$$0 = [\![U_{\mathcal{C}}, X]\!]^{\mathrm{fn}}(f) = \mathrm{L}_{U_{\mathcal{C}}}(\mathrm{L}_X(f)) - (-1)^X \mathrm{L}_X(\mathrm{L}_{U_{\mathcal{C}}}(f)), \quad f \in \mathcal{F}(\mathcal{E}). \tag{5.43}$$

Let now X be such that $\mathrm{L}_X\big|_{\mathcal{F}_0(\pi)} = 0$ and assume that $\mathrm{L}_X\big|_{\mathcal{F}_r(\pi)} = 0$ for some $r > 0$. Then taking $f = u_\sigma^j$, $|\sigma| = r$, and using (5.43) we obtain

$$\mathrm{L}_X\left(du_\sigma^j - \sum_{i=1}^n u_{\sigma i}^j dx_i\right) = \mathrm{L}_X d_{\mathcal{C}} u_\sigma^j = (-1)^X d_{\mathcal{C}}(L_X(u_\sigma^j)) = 0.$$

In other words,

$$\mathrm{L}_X\left(\sum_{i=1}^{n} u^j_{\sigma i}dx_i\right) = \sum_{i=1}^{n}\mathrm{L}_X(u^j_{\sigma i}dx_i)) = \sum_{i=1}^{n}\mathrm{L}_X(du^j_\sigma)$$
$$= (-1)^X\sum_{i=1}^{n} d(\mathrm{L}_X u^j_\sigma) = 0.$$

Hence, $\mathrm{L}_X(u^j_\sigma) = 0$ and thus $\mathrm{L}_X\left.\right|_{\mathcal{F}_{r+1}(\pi)} = 0$. $\square$

From this result and from Corollary 5.10 on page 86, it follows that if two evolutionary superderivations $\ni_\omega, \ni_\theta$ are given, the elements

(i) $[\![\ni_\omega, \ni_\theta]\!]^{\mathrm{fn}}$,
(ii) $\mathrm{i}_{\ni_\omega}(\ni_\theta)$

are evolutionary superderivations as well.

In the first case, the corresponding generating section is called the *Jacobi superbracket* of elements $\omega = (\omega^1, \dots, \omega^m)$ and $\theta = (\theta^1, \dots, \theta^m)$ and is denoted by $\{\omega, \theta\}$. The components of this bracket are expressed by the formula

$$\{\omega, \theta\}^j = \mathrm{L}_{\ni_\omega}(\theta^j) - (-1)^{\omega\theta}\mathrm{L}_{\ni_\theta}(\omega^j),\ j = 1, \dots, m. \tag{5.44}$$

Obviously, the module $\mathcal{F}(\pi, \pi) \otimes_{\mathcal{F}(\pi)} \mathcal{C}^*\Lambda(\pi)$ is a graded Lie algebra with respect to the Jacobi superbracket. The restriction of $\{\cdot, \cdot\}$ to $\mathcal{F}(\pi, \pi) \otimes \mathcal{C}^0\Lambda(\pi) = \mathcal{F}(\pi, \pi)$ coincides with the higher Jacobi bracket (see Definition 3.31 on page 66).

In the case (ii), the generating section is $\mathrm{i}_{\ni_\omega}(\theta)$. Note now that any element $\rho \in \mathcal{C}^1\Lambda(\pi)$ is of the form $\rho = \sum_{\sigma,\alpha} a_{\sigma,\alpha}\omega^\alpha_\sigma$, where, as before, $\omega^\alpha_\sigma = d_{\mathcal{C}}u^\alpha_\sigma = du^\alpha_\sigma - \sum_{i=1}^n u^\alpha_{\sigma i}dx_i$ are the Cartan forms on $J^\infty(\pi)$. Hence, if $\theta \in \mathcal{F}(\pi, \pi) \otimes_{\mathcal{F}(\pi)} \mathcal{C}^1\Lambda(\pi)$ and $\theta^j = \sum_{\sigma,\alpha} a^j_{\sigma,\alpha}\omega^\alpha_\sigma$, then

$$\left(\mathrm{i}_{\ni_\omega}(\theta)\right)^j = \sum_{\sigma,\alpha} a^j_{\sigma,\alpha} D_\sigma(\omega^\alpha). \tag{5.45}$$

In particular, we see that (5.45) establishes an isomorphism between the modules $\mathcal{F}(\pi, \pi) \otimes_{\mathcal{F}(\pi)} \mathcal{C}^*\Lambda(\pi)$ and $\mathcal{C}\mathrm{Diff}(\pi, \pi)$ and defines the action of $\mathcal{C}$-differential operators on elements of $\mathcal{C}^*\Lambda(\pi)$. This is a really well-defined action because of the fact that $\mathrm{i}_{\mathcal{C}X}\omega = 0$ for any $X \in \mathrm{D}(M)$ and $\omega \in \mathcal{C}^*\Lambda(\pi)$.

Consider now a formally integrable differential equation $\mathcal{E} \subset J^k(\pi)$ and assume that it is determined by a differential operator $\Delta \in \mathcal{F}(\pi, \xi)$. Denote, as in Section 3, by $\ell_{\mathcal{E}}$ the restriction of the operator of universal linearization ℓ_Δ to $\mathcal{E}^\infty$. Let $\ell^{[p]}_{\mathcal{E}}$ be the extension of $\ell_{\mathcal{E}}$ to $\mathcal{F}(\pi, \pi) \otimes_{\mathcal{F}(\pi)} \mathcal{C}^p\Lambda(\mathcal{E})$ which is well defined due to what has been said above. Then the module $H^{p,0}_{\mathcal{C}}(\mathcal{E})$ is identified with the set of evolution superderivations $\Game_\omega$ whose generating sections $\omega \in \mathcal{F}(\pi,\pi) \otimes_{\mathcal{F}(\pi)} \mathcal{C}^p\Lambda(\mathcal{E})$ satisfy the equation

$$\ell_{\mathcal{E}}^{[p]}(\omega) = 0 \tag{5.46}$$

If, in addition, $\mathcal{E}$ satisfies the assumptions of the two-line theorem, then $H_{\mathcal{C}}^{p,1}(\mathcal{E})$ is identified with the cokernel of $\ell_{\mathcal{E}}^{[p-1]}$ and thus

$$H_{\mathcal{C}}^i(\mathcal{E}) = \ker \ell_{\mathcal{E}}^{[i]} \oplus \operatorname{coker} \ell_{\mathcal{E}}^{[i-1]}$$

in this case. These two statements will be proved in Subsection 6.4.

As it was noted in Remark 5.4 on page 86, $H_{\mathcal{C}}^1(\mathcal{E})$ is an associative algebra with respect to contraction and is represented in the algebra of endomorphisms of $H_{\mathcal{C}}^0(\mathcal{E})$. It is easily seen that the action of the $H_{\mathcal{C}}^{0,1}(\mathcal{E})$ is trivial while $H_{\mathcal{C}}^{1,0}(\mathcal{E})$ acts on $H_{\mathcal{C}}^0(\mathcal{E}) = \operatorname{sym}\mathcal{E}$ as $\mathcal{C}$-differential operators (see above).

Definition 5.12. Elements of the module $H_{\mathcal{C}}^{1,0}(\mathcal{E})$ are called *recursion operators* for symmetries of the equation $\mathcal{E}$.

We use the notation $\mathcal{R}(\mathcal{E})$ for the algebra of recursion operators.

Remark 5.7. The algebra $\mathcal{R}(\mathcal{E})$ is always nonempty, since it contains the structural element $U_{\mathcal{E}}$ which is the unit of this algebra. “Usually” this is the only solution of (5.46) for $p = 1$ (see Example 5.1 below). This fact apparently contradicts practical experience (cf. with well-known recursion operators for the KdV and other integrable systems [43]). The reason is that these operators contain nonlocal terms like D^{-1} or of a more complicated form. An adequate framework to deal with such constructions will be described in the next subsection.

Example 5.1. Let

$$u_t = uu_x + u_{xx} \tag{5.47}$$

be the Burgers equation. For internal coordinates on $\mathcal{E}^\infty$ we choose the functions $x, t, u = u_0, \dots, u_k, \dots$, where u_k corresponds to the partial derivative $\partial^k u/\partial x^k$.

We shall prove here that the only solution of the equation $\ell_{\mathcal{E}}^{[1]}(\omega) = 0$ for (5.47) is $\omega = \alpha\omega_0$, $\alpha = \text{const}$, where

$$\omega_k = d_{\mathcal{C}} u_k = du_k - u_{k+1}dx - D_x^k(uu_1 + u_2)dt.$$

Let $\omega = \phi^0\omega_0 + \dots + \phi^r\omega_r$. Then the equation (5.46) on the page before for $p = 1$ transforms to

$$
\begin{aligned}
&u_0 D_x(\phi^0) + D_x^2(\phi^0) = D_t(\phi^0) + \sum_{j=1}^{r} u_{j+1}\phi^j,\\
&u_0 D_x(\phi^1) + D_x^2(\phi^1) + 2D_x(\phi^0) = D_t(\phi^1) + \sum_{j=2}^{r}(j+1)u_j\phi^j,\\
&\dots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5.48)\\
&u_0 D_x(\phi^i) + D_x^2(\phi^i) + 2D_x(\phi^0) = D_t(\phi^i) + \sum_{j=i+1}^{r}\binom{j+1}{i}u_{j-i+1}\phi^j,\\
&\dots\\
&u_0 D_x(\phi^r) + D_x^2(\phi^r) + 2D_x(\phi^{r-1}) = D_t(\phi^r) + r u_1\phi^r,\\
&D_x(\phi^r) = 0.
\end{aligned}
$$

To prove the result, we apply the scheme used in [64] to describe the symmetry algebra of the Burgers equation.

Denote by $\mathcal{K}_r$ the set of solutions of (5.48). Then a direct computation shows that

$$\mathcal{K}^1 = \{\alpha\omega_0 \mid \alpha \in \mathbb{R}\} \tag{5.49}$$

and that any element $\omega \in \mathcal{K}_r$, $r > 1$, is of the form

$$\omega = \alpha_r + \left(\frac{r}{2}u_0\alpha_r + \frac{1}{2}x\alpha_r^{(1)} + \alpha_{r-1}\right) + \Omega[r-2], \tag{5.50}$$

where $\alpha_r = \alpha_r(t)$, $\alpha_{r-1} = \alpha_{r-1}(t)$, $\alpha^{(i)}$ denotes the derivative $d^i\alpha/dt^i$, and $\Omega[s]$ is an arbitrary linear combination of $\omega_0, \dots, \omega_s$ with coefficients in $\mathcal{F}(\mathcal{E})$.

Note now that for any evolution equation the embedding

$$[\![\operatorname{sym}\mathcal{E}, \ker \ell_{\mathcal{E}}^{[1]}]\!]^{\mathrm{fn}} \subset \ker \ell_{\mathcal{E}}^{[1]}$$

is valid. Consequently, if $\psi \in \operatorname{sym}\mathcal{E}$ and $\omega \in \ker \ell_{\mathcal{E}}^{[1]}$, then $\{\psi, \omega\} \in \ker \ell_{\mathcal{E}}^{[1]}$.

Since the function u_1 is a symmetry of the Burgers equation (translation along x), one has

$$\{u_1, \omega\} = \left(\sum_k u_{k+1}\frac{\partial}{\partial u_k}\right)\omega - D_x\omega = -\frac{\partial}{\partial x}\omega.$$

Hence, if $\omega \in \mathcal{K}_r$, then from (5.50) we obtain that

$$\operatorname{ad}_{u_1}^{(r-1)}(\omega) = \alpha_r^{(r-1)}\omega_1 + \Omega[0] \in \mathcal{K}_1,$$

where $\mathrm{ad}_\psi = \{\psi, \cdot\}$. Taking into account equation (5.49), we get that $\alpha_r^{r-1} = 0$, or

$$\alpha_r = a_0 + a_1 t + \cdots + a_{r-2} t^{r-2}, \quad a_i \in \mathbb{R}. \tag{5.51}$$

We shall use now the fact that the element $\Phi = t^2 u_2 + (t^2 u_0 + tx) u_1 + t u_0 + 1$ is a symmetry of the Burgers equation (see [64]). Then, since the action of symmetries is permutable with the Cartan differential $d_{\mathcal{C}}$, we have

$$\{\Phi, \phi^s \omega_s\} = \ni_\Phi(\phi^s \omega_s) - \ni_{\phi^s \omega_s}(\Phi) = \ni_\Phi(\phi^s)\omega_s + \phi^s \ni_\Phi(\omega_s) - \ni_{\phi^s \omega_s}(\Phi).$$

But

$$\begin{aligned} \ni_\Phi(\omega_s) &= \ni_\Phi d_{\mathcal{C}}(u_s) = d_{\mathcal{C}} \ni_\Phi(u_s) = d_{\mathcal{C}} D_x^s(\Phi) \\ &= d_{\mathcal{C}}\big(t^2 u_{s+2} + (t^2 u_0 + tx) u_{s+1} + (s+1)(t^2 u_1 + t) u_s\big) + \Omega[s-1]. \end{aligned}$$

On the other hand,

$$\begin{aligned} \ni_{\phi^s \omega_s}(\Phi) = t^2 \phi^s \omega_{s+2} &+ \big(2t^2 D_x^2(\phi^s) + (t^2 u_0 + tx)\phi^s\big)\omega_{s+1} \\ &+ \big(t^2 D_x^2(\phi^s) + (t^2 u_0 + tx) D_x(\phi^s) + (t^2 u_1 + t)\phi^s\big)\omega_s. \end{aligned}$$

Thus, we finally obtain

$$\begin{aligned} \{\Phi, \phi^s \omega_s\} = \{\Phi, \phi^s\}\omega_s &+ (s+1)(t^2 u_1 + t)\omega_s \\ &- 2t^2 D_x(\phi^s)\omega_{s+1} + \Omega[s-1]. \end{aligned} \tag{5.52}$$

Applying (5.52) to (5.50), we get

$$\mathrm{ad}_\Phi(\omega) = (rt\alpha_r - t^2 \alpha_r^{(1)})\omega_r + \Omega[r-1]. \tag{5.53}$$

Let now $\omega \in \mathcal{K}_r$ and assume that ω has a nontrivial coefficient α_r of the form (5.51), and a_i be the first nontrivial coefficient in α_r. Then, by representation (5.53), we have

$$\mathrm{ad}_\Phi^{r-i}(\omega) = \alpha_r' \omega_r + \Omega[r-1] \in \mathcal{K}_r,$$

where α_r' is a polynomial of degree $r-1$. This contradicts to (5.51) and thus concludes the proof.

Remark 5.8. Let $\varphi \in \operatorname{sym}\mathcal{E}$ be a symmetry and $R \in \mathcal{R}(\mathcal{E})$ be a recursion operator. Then we obtain a sequence of symmetries $\varphi_0 = \varphi$, $\varphi_1 = R(\varphi)$, $\dots, \varphi_n = R^n(\varphi), \dots$. Using identity (5.22) on page 82, one can compute the commutators $[\varphi_m, \varphi_n]$ in terms of $[\![\varphi, R]\!]^{\mathrm{fn}} \in H_{\mathcal{C}}^{1,0}(\mathcal{E})$ and $[\![R, R]\!]^{\mathrm{fn}} \in H_{\mathcal{C}}^{2,0}(\mathcal{E})$. In particular, it can be shown that when both $[\![\varphi, R]\!]^{\mathrm{fn}}$ and $[\![R, R]\!]^{\mathrm{fn}}$ vanish, all symmetries φ_n mutually commute (see [27]).

For example, if $\mathcal{E}$ is an evolution equation, $H_{\mathcal{C}}^{p,0}(\mathcal{E}) = 0$ for all $p \geq 2$ (see Theorem 6.8 on page 112). Hence, if φ is a symmetry and R is a φ-*invariant* recursion operator (i.e., such that $[\![\varphi, R]\!]^{\mathrm{fn}} = 0$), then R generates a commutative sequence of symmetries. This is exactly the case for the KdV and other integrable evolution equations.

5.4. **Passing to nonlocalities.** Let us now introduce nonlocal variables into the above described picture. Namely, let $\mathcal{E}$ be an equation and $\varphi : \mathcal{N} \to \mathcal{E}^\infty$ be a covering over its infinite prolongation. Then, due to Proposition 4.1 on page 70, the triad $\big(\mathcal{F}(\mathcal{N}), C^\infty(M), (\pi_\infty \circ \varphi)^*\big)$ is an algebra with the flat connection $\mathcal{C}^\varphi$. Hence, we can apply the whole machinery of Subsections 5.1–5.3 to this situation. To stress the fact that we are working over the covering φ, we shall add the symbol φ to all notations introduced in these subsections. Denote by $U_\mathcal{C}^\varphi$ the connection form of the connection $\mathcal{C}^\varphi$ (the structural element of the covering φ).

In particular, on $\mathcal{N}$ we have the $\mathcal{C}^\varphi$-differential $\partial_\mathcal{C}^\varphi = [\![U_\mathcal{C}^\varphi, \cdot]\!]^{\mathrm{fn}} : \mathrm{D}^v(\Lambda^i(\mathcal{N})) \to \mathrm{D}^v(\Lambda^{i+1}(\mathcal{N}))$, whose 0-cohomology $H_\mathcal{C}^0(\mathcal{E}, \varphi)$ coincides with the Lie algebra $\mathrm{sym}_\varphi \mathcal{E}$ of nonlocal φ-symmetries, while the module $H_\mathcal{C}^{1,0}(\mathcal{E}, \varphi)$ identifies with recursion operators acting on these symmetries and is denoted by $\mathcal{R}(\mathcal{E}, \varphi)$. We also have the horizontal and the Cartan differential $\bar{d}^\varphi$ and $d_\mathcal{C}^\varphi$ on $\mathcal{N}$ and the splitting $\Lambda^i(\mathcal{N}) = \bigoplus_{p+q=i} \mathcal{C}^p\Lambda^p(\mathcal{N}) \otimes \bar{\Lambda}^q(\mathcal{N})$.

Choose a trivialization of the bundle $\varphi : \mathcal{N} \to \mathcal{E}^\infty$ and nonlocal coordinates $w^1, w^2, \dots$ in the fiber. Then any derivation $X \in \mathrm{D}^v(\Lambda^i(\mathcal{N}))$ splits to the sum $X = X_\mathcal{E} + X^v$, where $X_\mathcal{E}(w^j) = 0$ and X^v is a φ-vertical derivation.

Lemma 5.15. *Let $\varphi : \mathcal{E}^\infty \times \mathbb{R}^N \to \mathcal{E}^\infty$, $N \le \infty$, be a covering. Then $H_\mathcal{C}^{p,0}(\mathcal{E}, \varphi) = \ker \partial_\mathcal{C}^\varphi \big|_{\mathcal{C}^p\Lambda(\mathcal{N})}$. Thus $H_\mathcal{C}^{p,0}(\mathcal{E}, \varphi)$ consists of derivations $\Omega : \mathcal{F}(\mathcal{N}) \to \mathcal{C}^p\Lambda(\mathcal{N})$ such that*

$$[\![U_\mathcal{C}^\varphi, \Omega]\!]_\mathcal{E}^{\mathrm{fn}} = 0, \quad \big([\![U_\mathcal{C}^\varphi, \Omega]\!]^{\mathrm{fn}}\big)^v = 0. \tag{5.54}$$

Proof. In fact, due to equality (5.33) on page 86, any element lying in the image of $\partial_\mathcal{C}^\varphi$ contains at least one horizontal component, i.e.,

$$\partial_\mathcal{C}^\varphi\big(\mathrm{D}^v(\mathcal{C}^p\Lambda(\mathcal{N}))\big) \subset \mathrm{D}^v(\mathcal{C}^p\Lambda(\mathcal{N}) \otimes \bar{\Lambda}^1(\mathcal{N})).$$

Thus, equations (5.54) should hold. □

We call the first equation in (5.54) the *shadow equation* while the second one is called the *relation equation*. This is explained by the following result (cf. with Theorem 4.7 on page 73).

Proposition 5.16. *Let $\mathcal{E}$ be an evolution equation of the form*

$$u_t = f(x, t, u, \dots, \frac{\partial^k u}{\partial u^k})$$

and $\varphi : \mathcal{N} = \mathcal{E}^\infty \times \mathbb{R}^N \longrightarrow \mathcal{E}^\infty$ *be a covering given by the vector fields*[13]

$$\tilde{D}_x = D_x + X, \quad \tilde{D}_t = D_t + T,$$

where $[\tilde{D}_x, \tilde{D}_t] = 0$ *and*

$$X = \sum_s X^s \frac{\partial}{\partial w^s}, \quad T = \sum_s T^s \frac{\partial}{\partial w^s},$$

$w^1, \dots, w^s, \dots$ *being nonlocal variables in* φ*. Then the group* $H_{\mathcal{C}}^{p,0}(\mathcal{E}, \varphi)$ *consists of elements*

$$\Psi = \sum_i \Psi_i \otimes \frac{\partial}{\partial u_i} + \sum_s \psi^s \frac{\partial}{\partial w^s} \in \mathrm{D}^v(\mathcal{C}^p\Lambda(\mathcal{N}))$$

such that $\Psi_i = \tilde{D}_x^i \Psi_0$ *and*

$$\tilde{\ell}_{\mathcal{E}}^{[p]}(\Psi_0) = 0, \tag{5.55}$$

$$\sum_\alpha \frac{\partial X^s}{\partial u_\alpha} \tilde{D}_x^\alpha(\Psi_0) + \sum_\beta \frac{\partial X^s}{\partial w^\beta} \psi^\beta = \tilde{D}_x(\psi^s), \tag{5.56}$$

$$\sum_\alpha \frac{\partial T^s}{\partial u_\alpha} \tilde{D}_x^\alpha(\Psi_0) + \sum_\beta \frac{\partial T^s}{\partial w^\beta} \psi^\beta = \tilde{D}_t(\psi^s), \tag{5.57}$$

$s = 1, 2, \dots,$ *where* $\tilde{\ell}_{\mathcal{E}}^{[p]}$ *is the natural extension of the operator* $\ell_{\mathcal{E}}^{[p]}$ *to* $\mathcal{N}$*.*

Proof. Consider the Cartan forms

$$\omega_i = du_i - u_{i+1}dx - D_x^i(f)\, dt, \quad \theta^s = dw^s - X^s dx - T^s dt$$

on $\mathcal{N}$. Then the derivation

$$U_{\mathcal{C}}^{\varphi} = \sum_i \omega_i \otimes \frac{\partial}{\partial u_i} + \sum_s \theta^s \otimes \frac{\partial}{\partial w^s}$$

[13]To simplify the notations of Section 4, we denote the lifting of a $\mathcal{C}$-differential operator Δ to $\mathcal{N}$ by $\tilde{\Delta}$.

is the structural element of the covering φ. Then, using representation (5.18) on page 82, we obtain

$$
\begin{aligned}
\partial_{\mathcal{C}}^{\varphi}\Psi = dx \wedge \sum_i \big(\Psi_{i+1} - \tilde{D}_x(\Psi_i)\big) \otimes \frac{\partial}{\partial u_i} \\
+ dt \wedge \sum_i \Big(\sum_\alpha \frac{\partial(D_x^i f)}{\partial u_\alpha}\Psi_\alpha - \tilde{D}_t\Psi_i\Big) \otimes \frac{\partial}{\partial u_i} \\
+ dx \wedge \sum_s \Big(\sum_\alpha \frac{\partial X^s}{\partial u_\alpha}\Psi_\alpha + \sum_\beta \frac{\partial X^s}{\partial w^\beta}\psi^\beta - \tilde{D}_x(\psi^s)\Big) \otimes \frac{\partial}{\partial w^s} \\
+ dt \wedge \sum_s \Big(\sum_\alpha \frac{\partial T^s}{\partial u_\alpha}\Psi_\alpha + \sum_\beta \frac{\partial T^s}{\partial w^\beta}\psi^\beta - \tilde{D}_t(\psi^s)\Big) \otimes \frac{\partial}{\partial w^s},
\end{aligned}
$$

which gives the needed result. □

Note that relations $\Psi_i = \tilde{D}_x^i(\Psi_0)$ together with equation (5.55) are equivalent to the shadow equations. In the case $p = 1$, we call the solutions of equation (5.55) the *shadows of recursion operators* in the covering φ. Equations (5.56) and (5.57) on the page before are exactly the relation equations on the case under consideration.

Exercise 5.1. Generalize the above result to general equations using the proof similar to that of Theorem 4.7 on page 73.

Thus, any element of the group $H_{\mathcal{C}}^{1,0}(\mathcal{E},\varphi)$ is of the form

$$
\Psi = \sum_i \tilde{D}_x^i(\psi) \otimes \frac{\partial}{\partial u_i} + \sum_s \psi^s \otimes \frac{\partial}{\partial w^s}, \tag{5.58}
$$

where the forms $\psi = \Psi_0$, $\psi^s \in \mathcal{C}^1\Lambda(\mathcal{N})$ satisfy the system of equations (5.55)–(5.57).

As a direct consequence of the above said, we obtain the following

Corollary 5.17. *Let Ψ be a derivation of the form* (5.58) *with $\psi, \psi^s \in \mathcal{C}^p\Lambda(\mathcal{N})$. Then ψ is a solution of equation* (5.55) *on the preceding page in the covering φ if and only if $\partial_{\mathcal{C}}^{\varphi}(\Psi)$ is a φ-vertical derivation.*

We can now formulate the main result of this subsection.

Theorem 5.18. *Let $\varphi : \mathcal{N} \to \mathcal{E}^\infty$ be a covering, $S \in \mathrm{sym}_\varphi \mathcal{E}$ be a φ-symmetry, and $\psi \in \mathcal{C}^1\Lambda(\mathcal{N})$ be a shadow of a recursion operator in the covering φ. Then $\psi' = \mathrm{i}_S\psi$ is a shadow of a symmetry in φ, i.e., $\tilde{\ell}_{\mathcal{E}}(\psi') = 0$.*

Proof. In fact, let Ψ be a derivation of the form (5.58). Then, due to identity (5.32) on page 86, one has

$$
\partial_{\mathcal{C}}^{\varphi}(\mathrm{i}_S\Psi) = \mathrm{i}_{\partial_{\mathcal{C}}^{\varphi}S} - \mathrm{i}_S(\partial_{\mathcal{C}}^{\varphi}\Psi) = -\mathrm{i}_S(\partial_{\mathcal{C}}^{\varphi}\Psi),
$$

since S is a symmetry. But, by Corollary 5.17 on the facing page, $\partial_{\mathcal{C}}^{\varphi}\Psi$ is a φ-vertical derivation and consequently $\partial_{\mathcal{C}}^{\varphi}(\mathrm{i}_S\Psi) = -\mathrm{i}_S(\partial_{\mathcal{C}}^{\varphi}\Psi)$ is φ-vertical as well. Hence, $\mathrm{i}_S\Psi$ is a φ-shadow by the same corollary. □

Using the last result together with Theorem 4.11 on page 77, we can describe the process of generating a series of symmetries by shadows of recursion operators. Namely, let ψ be a symmetry and $\omega \in \mathcal{C}^1\Lambda(\mathcal{N})$ be a shadow of a recursion operator in a covering $\varphi : \mathcal{N} \longrightarrow \mathcal{E}^\infty$. In particular, ψ is a φ-shadow. Then, by Theorem 4.9 on page 76, there exists a covering $\varphi_\psi : \mathcal{N}_\psi \longrightarrow \mathcal{N} \xrightarrow{\varphi} \mathcal{E}^\infty$ where $\ni_\psi$ can be lifted to as a φ_ψ-symmetry. Obviously, ω still remains a shadow in this new covering. Therefore, we can act by ω on ψ and obtain a shadow ψ_1 of a new symmetry on $\mathcal{N}_\psi$. By Theorem 4.11 on page 77, there exists a covering, where both ψ and ψ_1 are realized as nonlocal symmetries. Thus we can continue the procedure applying ω to ψ_1 and eventually arrive to a covering in which the whole series $\{\psi_k\}$ is realized.

Example 5.2. Let $u_t = uu_x + u_{xx}$ be the Burgers equation. Consider the one-dimensional covering $\varphi : \mathcal{E}^\infty \times \mathbb{R} \longrightarrow \mathcal{E}^\infty$ with the nonlocal variable w and defined by the vector fields

$$D_x^\varphi = D_x + u_0\frac{\partial}{\partial w}, \quad D_t^\varphi = D_t + \left(\frac{u_0^2}{2} + u_1\right)\frac{\partial}{\partial w}.$$

Then it easily checked that the form

$$\omega = \omega_1 + \frac{1}{2}\omega_0 + \frac{1}{2}\theta,$$

where ω_0 and ω_1 are the Cartan forms $d_{\mathcal{C}}u_0$ and $d_{\mathcal{C}}u_1$ respectively and $\theta = dw - u_0dx - (u_0^2/2 + u_1)dt$, is a solution of the equation $\tilde{\ell}_{\mathcal{E}}^{[1]}\omega = 0$. If $\ni_\psi$ is a symmetry of the Burgers equation, the corresponding action of ω on ψ is

$$D_x\psi + \frac{1}{2}\psi + \frac{1}{2}D_x^{-1}\psi$$

and thus coincides with the well-known recursion operator for this equation, see [43].

Exercise 5.2. Let $u_t = uu_x + u_{xxx}$ be the KdV equation. Consider the one-dimensional covering $\varphi : \mathcal{E}^\infty \times \mathbb{R} \longrightarrow \mathcal{E}^\infty$ with the nonlocal variable w and defined by the vector fields

$$D_x^\varphi = D_x + u_0\frac{\partial}{\partial w}, \quad D_t^\varphi = D_t + \left(\frac{u_0^2}{2} + u_2\right)\frac{\partial}{\partial w}.$$

Solve the equation $\tilde{\ell}_{\mathcal{E}}^{[1]}\omega = 0$ in this covering and find the corresponding recursion operator.

Remark 5.9. Recursion operators can be understood as supersymmetries (cf. Subsection 7.9 on page 132) of a certain superequation naturally related to the initial one. To such symmetries and equations one can apply nonlocal theory of Section 4 and prove the corresponding reconstruction theorems, see [28, 30].

6. Horizontal cohomology

In this section we discuss the horizontal cohomology of differential equations, i.e., the cohomology of the horizontal de Rham complex (see Definition 3.27 on page 60). This cohomology has many physically relevant applications. To demonstrate this, let us start with the notion of a conserved current. Consider a differential equation $\mathcal{E}$. A *conserved current* is a vector-function $J = (J_1, \dots, J_n)$, where $J_k \in \mathcal{F}(\mathcal{E})$, which is conserved modulo the equation, i.e., that satisfies the equation

$$\sum_{k=1}^{n} D_k(J_k) = 0, \tag{6.1}$$

where D_k are restrictions of total derivatives to $\mathcal{E}^\infty$. For example, take the nonlinear Schrödinger equation[14]

$$i\psi_t = \Delta\psi + |\psi|^2\psi, \qquad \Delta = \sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j^2}. \tag{6.2}$$

Then it is straightforwardly verified, that the vector-function

$$J = (|\psi|^2, i(\bar\psi\psi_{x_1} - \psi\bar\psi_{x_1}), \dots, i(\bar\psi\psi_{x_{n-1}} - \psi\bar\psi_{x_{n-1}}))$$

is a conserved current, i.e., that

$$D_t(|\psi|^2) + \sum_{k=1}^{n-1} D_k(i(\bar\psi\psi_{x_k} - \psi\bar\psi_{x_k}))$$

vanishes by virtue of equation (6.2).

A conserved current is called *trivial*, if it has the form

$$J_k = \sum_{l=1}^{n} D_l(\mathcal{L}_{kl}) \tag{6.3}$$

for some skew-symmetric matrix, $\|\mathcal{L}_{kl}\|$, $\mathcal{L}_{kl} = -\mathcal{L}_{lk}$, $\mathcal{L}_{kl} \in \mathcal{F}(\mathcal{E})$. The name "trivial currents" means that they are trivially conserved regardless to the equation under consideration. Two conserved currents are said to be *equivalent* if they differ by a trivial one. *Conservation laws* are defined to be the equivalent classes of conserved currents.

Let us assign the horizontal $(n-1)$-form $\omega_J = \sum_{k=1}^{n}(-1)^{k-1} J_k\, dx_1 \wedge \dots \wedge \widehat{dx_k} \wedge \dots \wedge dx_n$ to each conserved current $J = (J_1, \dots, J_n)$. Then equations (6.1) and (6.3) can be rewritten as $\bar d\omega_J = 0$ and $\omega_J = \bar d\eta$ respectively, where $\eta = \sum_{k>l}(-1)^{k+l}\mathcal{L}_{kl}\, dx_1 \wedge \dots \wedge \widehat{dx_l} \wedge \dots \wedge \widehat{dx_k} \wedge \dots \wedge dx_n$. Thus, we

[14] Here ψ is a complex function and (6.2) is to be understood as a system of two equations.

see that the horizontal cohomology group in degree $n-1$ of the equation $\mathcal{E}$ consists of conservation laws of $\mathcal{E}$.

In physical applications one also encounters the horizontal cohomology in degree less than $n-1$. For instance, the Maxwell equations read

$$\bar{d}(*F) = 0,$$

where F is the electromagnetic field strength tensor and $*$ is the Hodge star operator. Clearly $*F$ is not exact. Another reason to consider the low-dimensional horizontal cohomology is that it appears as an auxiliary cohomology in calculation of the BRST cohomology [5]. Recently, by means of horizontal cohomology the problem of consistent deformations and of candidate anomalies has been completely solved in cases of Yang-Mills gauge theories and of gravity [6, 4].

The horizontal cohomology plays a central role in the Lagrangian formalism as well. Really, it is easy to see that the horizontal cohomology group in degree n is exactly the space of actions of variational problems constrained by equation $\mathcal{E}$.

For computing the horizontal cohomology there is a general method based on the Vinogradov $\mathcal{C}$-spectral sequence. It can be outlined as follows. The horizontal cohomology is the term $E_1^{0,\bullet}$ of the Vinogradov $\mathcal{C}$-spectral sequence and thereby related to the terms $E_1^{p,\bullet}$, $p > 0$. For each p, such a term is also a horizontal cohomology but with some nontrivial coefficients. The crucial observation is that the corresponding modules of coefficients are supplied with filtrations such that the differentials of the associated graded complexes are linear over the functions. Hence, the cohomology can be computed algebraically. A detailed description of these techniques is our main concern in this and the next sections.

6.1. **$\mathcal{C}$-modules on differential equations.** Let us begin with the definition of $\mathcal{C}$-modules, which are left differential modules (see Definition 1.7 on page 16) in $\mathcal{C}$-differential calculus and serve as the modules of coefficients for horizontal de Rham complexes.

Proposition 6.1. *The following three definitions of a $\mathcal{C}$-module are equivalent*:

(1) *An $\mathcal{F}$-module Q is called a* $\mathcal{C}$-module, *if Q is endowed with a left module structure over the ring $\mathcal{C}\mathrm{Diff}(\mathcal{F},\mathcal{F})$, i.e., for any scalar $\mathcal{C}$-differential operator $\Delta \in \mathcal{C}\mathrm{Diff}_k(\mathcal{F},\mathcal{F})$ there exists an operator $\Delta_Q \in \mathcal{C}\mathrm{Diff}_k(Q,Q)$, with*
   (1) $(\sum_i f_i\Delta_i)_Q = \sum_i f_i(\Delta_i)_Q, \quad f_i \in \mathcal{F},$
   (2) $(\mathrm{id}_{\mathcal{F}})_Q = \mathrm{id}_Q,$
   (3) $(\Delta_1 \circ \Delta_2)_Q = (\Delta_1)_Q \circ (\Delta_2)_Q.$

(2) *A $\mathcal{C}$-module is a module equipped with a* flat horizontal connection, *i.e., with an action on Q of the module $\mathcal{C}\mathrm{D} = \mathcal{C}\mathrm{D}(\mathcal{E})$, $X \mapsto \nabla_X$, which is $\mathcal{F}$-linear*:

$$\nabla_{fX+gY} = f\nabla_X + g\nabla_Y, \quad f, g \in \mathcal{F}, \quad X, Y \in \mathcal{C}\mathrm{D},$$

satisfies the Leibniz rule:

$$\nabla_X(fq) = X(f)q + f\nabla_X(q), \quad q \in Q, \quad X \in \mathcal{C}\mathrm{D}, \quad f \in \mathcal{F},$$

and is a Lie algebra homomorphism:

$$[\nabla_X, \nabla_Y] = \nabla_{[X,Y]}.$$

(3) *A $\mathcal{C}$-module is the module of sections of a linear covering, i.e., Q is the module of sections of a vector bundle $\tau\colon W \to \mathcal{E}^\infty$, $Q = \Gamma(\tau)$, equipped with a completely integrable n-dimensional linear distribution* (*see Definition* 4.3 *on page* 70) *on W which is projected onto the Cartan distribution on $\mathcal{E}^\infty$.*

The proof is elementary.

Exercise 6.1. Show that

(1) in coordinates, the operator $(D_i)_Q = \|\Delta_j^k\|$ is a matrix operator of the form

$$\Delta_j^k = D_i\delta_j^k + \Gamma_{ij}^k, \quad \Gamma_{ij}^k \in \mathcal{F},$$

where δ_j^k is the Kronecker symbol;

(2) the coordinate description of the corresponding flat horizontal connection looks as

$$\nabla_{D_i}(s_j) = \sum_k \Gamma_{ij}^k s_k$$

where s_j are basis elements of Q;

(3) the corresponding linear covering has the form

$$\tilde{D}_i = D_i + \sum_{j,k} \Gamma_{ij}^k w^j \frac{\partial}{\partial w^k},$$

where w^i are fiber coordinates on W.

Here are basic examples of $\mathcal{C}$-modules.

Example 6.1. The simplest example of a $\mathcal{C}$-module is $Q = \mathcal{F}$ with the usual action of $\mathcal{C}$-differential operators.

Example 6.2. The module of vertical vector fields $Q = \mathrm{D}^v = \mathrm{D}^v(\mathcal{E})$ with the connection

$$\nabla_X(Y) = [X, Y]^v, \quad X \in \mathcal{C}\mathrm{D}, \quad Y \in \mathrm{D}^v,$$

where $Z^v = U_{\mathcal{C}}(Z)$, is a $\mathcal{C}$-module.

Example 6.3. Next example is the modules of Cartan forms $Q = \mathcal{C}^k\Lambda = \mathcal{C}^k\Lambda(\mathcal{E})$. A vector field $X \in \mathcal{C}\mathrm{D}$ acts on $\mathcal{C}^k\Lambda$ as the Lie derivative L_X. It is easily seen that in coordinates we have

$$(D_i)_{\mathcal{C}^k\Lambda}(\omega^j_\sigma) = \omega^j_{\sigma i}.$$

Example 6.4. The infinite jet module $Q = \bar{\mathcal{J}}^\infty(P)$ of an $\mathcal{F}$-module P is a $\mathcal{C}$-module via

$$\Delta_{\bar{\mathcal{J}}^\infty(P)}(f\bar{\jmath}_\infty(p)) = \Delta(f)\bar{\jmath}_\infty(p),$$

where $\Delta \in \mathcal{C}\mathrm{Diff}(\mathcal{F},\mathcal{F})$, $f \in \mathcal{F}$, $p \in P$.

Example 6.5. Let us dualize the previous example. It is clear that for any $\mathcal{F}$-module P the module $Q = \mathcal{C}\mathrm{Diff}(P,\mathcal{F})$ is a $\mathcal{C}$-module. The action of horizontal operators is the composition:

$$\Delta_Q(\nabla) = \Delta \circ \nabla,$$

where $\Delta \in \mathcal{C}\mathrm{Diff}(\mathcal{F},\mathcal{F})$, $\nabla \in Q = \mathcal{C}\mathrm{Diff}(P,\mathcal{F})$.

Example 6.6. More generally, let $\Delta\colon P \to P_1$ be a $\mathcal{C}$-differential operator and $\psi^\Delta_\infty\colon \bar{\mathcal{J}}^\infty(P) \to \bar{\mathcal{J}}^\infty(P_1)$ be the corresponding prolongation of Δ. Obviously, ψ^Δ_∞ is a morphism of $\mathcal{C}$-modules, i.e., a homomorphism over the ring $\mathcal{C}\mathrm{Diff}(\mathcal{F},\mathcal{F})$, so that $\ker\psi^\Delta_\infty$ and $\operatorname{coker}\psi^\Delta_\infty$ are $\mathcal{C}$-modules.

On the other hand, the operator Δ gives rise to the morphism of $\mathcal{C}$-modules $\mathcal{C}\mathrm{Diff}(P_1,\mathcal{F}) \to \mathcal{C}\mathrm{Diff}(P,\mathcal{F})$, $\nabla \mapsto \nabla\circ\Delta$. Thus the kernel and cokernel of this map are $\mathcal{C}$-modules as well.

Example 6.7. Given two $\mathcal{C}$-modules Q_1 and Q_2, we can define $\mathcal{C}$-module structures on $Q_1 \otimes_{\mathcal{F}} Q_2$ and $\mathrm{Hom}_{\mathcal{F}}(Q_1,Q_2)$ by

$$\nabla_X(q_1 \otimes q_2) = \nabla_X(q_1) \otimes q_2 + q_1 \otimes \nabla_X(q_2),$$
$$\nabla_X(f)(q_1) = \nabla_X(f(q_1)) - f(\nabla_X(q_1)),$$

where $X \in \mathcal{C}\mathrm{D}$, $q_1 \in Q_1$, $q_2 \in Q_2$, $f \in \mathrm{Hom}_{\mathcal{F}}(Q_1,Q_2)$.

For instance, one has $\mathcal{C}$-module structures on $Q = \bar{\mathcal{J}}^\infty(P) \otimes_{\mathcal{F}} \mathcal{C}^k\Lambda$ and $Q = \mathcal{C}\mathrm{Diff}(P,\mathcal{C}^k\Lambda)$ for any $\mathcal{F}$-module P.

Example 6.8. Let $\mathfrak{g}$ be a Lie algebra and $\rho\colon \mathfrak{g} \to \mathfrak{gl}(W)$ a linear representation of $\mathfrak{g}$. Each $\mathfrak{g}$-valued horizontal form $\omega \in \bar{\Lambda}^1(\mathcal{E})\otimes_{\mathbb{R}}\mathfrak{g}$ that satisfies the horizontal Maurer–Cartan condition $\bar{d}\omega + \frac{1}{2}[\omega,\omega] = 0$ defines on the module Q of sections of the trivial vector bundle $\mathcal{E}^\infty \times W \to \mathcal{E}^\infty$ the following $\mathcal{C}$-module structure:

$$\nabla_X(q)_a = X(q)_a + \rho(\omega(X))(q_a),$$

where $X \in \mathcal{C}\mathrm{D}$, $q \in Q$, $a \in \mathcal{E}^\infty$, and $X(q)$ means the component-wise action.

Exercise 6.2. Check that Q is indeed a $\mathcal{C}$-module.

Such $\mathcal{C}$-modules are called *zero-curvature representations* over $\mathcal{E}^\infty$. Take the example of the KdV equation (in the form $u_t = uu_x + u_{xxx}$) and $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$. Then there exists a one-parameter family of Maurer–Cartan forms $\omega(\lambda) = A_1(\lambda)\,\bar{d}x + A_2(\lambda)\,\bar{d}t$, λ being a parameter:

$$A_1(\lambda) = \begin{pmatrix} 0 & -(\lambda+u) \\ \frac{1}{6} & 0 \end{pmatrix}$$

and

$$A_2(\lambda) = \begin{pmatrix} -\frac{1}{6}u_x & -u_{xx} - \frac{1}{3}u^2 + \frac{1}{3}\lambda u + \frac{2}{3}\lambda^2 \\ \frac{1}{18}u - \frac{1}{9}\lambda & \frac{1}{6}u_x \end{pmatrix}.$$

This is the zero-curvature representation used in the inverse scattering method.

Remark 6.1. In parallel with left $\mathcal{C}$-modules one can consider *right* $\mathcal{C}$-modules, i.e., right modules over the ring $\mathcal{C}\mathrm{Diff}(\mathcal{F},\mathcal{F})$. There is a natural way to pass from left $\mathcal{C}$-modules to right ones and back. Namely, for any left module Q set

$$\mathrm{B}(Q) = Q \otimes_{\mathcal{F}} \bar{\Lambda}^n(\mathcal{E}),$$

with the right action of $\mathcal{C}\mathrm{Diff}(\mathcal{F},\mathcal{F})$ on $\mathrm{B}(Q)$ given by

$$\begin{aligned} (q\otimes\omega)f &= fq\otimes\omega = q\otimes f\omega, \quad f\in\mathcal{F}, \\ (q\otimes\omega)X &= -\nabla_X(q)\otimes\omega - q\otimes L_X\omega, \quad X\in\mathcal{C}\mathrm{D}. \end{aligned}$$

One can easily verify that B determines an equivalence between the categories of left $\mathcal{C}$-modules and right $\mathcal{C}$-modules.

By definition of a $\mathcal{C}$-module, for a scalar $\mathcal{C}$-differential operator $\Delta\colon \mathcal{F}\to\mathcal{F}$ there exists the extension $\Delta_Q\colon Q\to Q$ of Δ to the $\mathcal{C}$-module Q. Similarly to Lemma 1.16 on page 16 one has more: for any $\mathcal{C}$-differential operator $\Delta\colon P\to S$ there exists the extension $\Delta_Q\colon P\otimes_{\mathcal{F}} Q\to S\otimes_{\mathcal{F}} Q$.

Proposition 6.2. *Let P, S be $\mathcal{F}$-modules. Then there exists a unique mapping*

$$\mathcal{C}\mathrm{Diff}_k(P,S)\to\mathcal{C}\mathrm{Diff}_k(P\otimes_{\mathcal{F}} Q, S\otimes_{\mathcal{F}} Q), \qquad \Delta\mapsto\Delta_Q,$$

such that the following conditions hold:

1. *if $P = S = \mathcal{F}$ then the mapping is given by the $\mathcal{C}$-module structure on Q,*
2. $(\sum_i f_i\Delta_i)_Q = \sum_i f_i(\Delta_i)_Q$, $\quad f_i\in\mathcal{F}$,
3. *if $\Delta\in\mathcal{C}\mathrm{Diff}_0(P,S) = \mathrm{Hom}_{\mathcal{F}}(P,S)$ then $\Delta_Q = \Delta\otimes_{\mathcal{F}}\mathrm{id}_Q$,*
4. *if R is another $\mathcal{F}$-module and $\Delta_1 : P\to S, \Delta_2 : S\to R$ are $\mathcal{C}$-differential operators, then $(\Delta_2\circ\Delta_1)_Q = (\Delta_2)_Q\circ(\Delta_1)_Q$.*

Proof. The uniqueness is obvious. To prove the existence consider the family of operators $\Delta(p, s^*)\colon \mathcal{F} \to \mathcal{F}$, $p \in P$, $s^* \in S^* = \mathrm{Hom}_{\mathcal{F}}(S, \mathcal{F})$, $\Delta(p, s^*)(f) = s^*(\Delta(fp))$, $f \in \mathcal{F}$. Clearly, the operator Δ is defined by the family $\Delta(p, s^*)$. The following statement is also obvious.

Exercise 6.3. For the family of operators $\Delta[p, s^*] \in \mathcal{C}\mathrm{Diff}_k(\mathcal{F}, \mathcal{F})$, $p \in P$, $s^* \in S^*$, we can find an operator $\Delta \in \mathcal{C}\mathrm{Diff}_k(P, S)$ such that $\Delta[p, s^*] = \Delta(p, s^*)$, if and only if

$$\begin{aligned} \Delta[p, \sum_i f_i s_i^*] &= \sum_i f_i \Delta[p, s_i^*], \\ \Delta[\sum_i f_i p_i, s^*] &= \sum_i \Delta[p_i, s^*] f_i. \end{aligned}$$

In view of this exercise, the family of operators

$$\Delta_Q[p \otimes q, s^* \otimes q^*](f) = q^*(\Delta(p, s^*)_Q(fq))$$

uniquely determines the operator Δ_Q. □

6.2. **The horizontal de Rham complex.** Consider a complex of $\mathcal{C}$-differential operators $\cdots \longrightarrow P_{i-1} \xrightarrow{\Delta_i} P_i \xrightarrow{\Delta_{i+1}} P_{i+1} \longrightarrow \cdots$. Multiplying it by a $\mathcal{C}$-module Q and taking into account Proposition 6.2 on the preceding page, we obtain the complex

$$\cdots \longrightarrow P_{i-1} \otimes Q \xrightarrow{(\Delta_i)_Q} P_i \otimes Q \xrightarrow{(\Delta_{i+1})_Q} P_{i+1} \otimes Q \longrightarrow \cdots.$$

Applying this construction to the horizontal de Rham complex, we get *horizontal de Rham complex with coefficients in* Q:

$$0 \longrightarrow Q \xrightarrow{\bar{d}_Q} \bar{\Lambda}^1 \otimes_{\mathcal{F}} Q \xrightarrow{\bar{d}_Q} \cdots \xrightarrow{\bar{d}_Q} \bar{\Lambda}^n \otimes_{\mathcal{F}} Q \longrightarrow 0,$$

where $\bar{\Lambda}^i = \bar{\Lambda}^i(\mathcal{E})$.

The cohomology of the horizontal de Rham complex with coefficients in Q is said to be *horizontal cohomology* and is denoted by $\bar{H}^i(Q)$.

Exercise 6.4. Proof that the differential $\bar{d} = \bar{d}_Q$ can also be defined by

$$\begin{aligned} &(\bar{d}q)(X) = \nabla_X(q), \quad q \in Q, \\ &\bar{d}(\omega \otimes q) = \bar{d}\omega \otimes q + (-1)^p \omega \wedge \bar{d}q, \quad \omega \in \bar{\Lambda}^p. \end{aligned}$$

One easily sees that a morphism $f\colon Q_1 \to Q_2$ of $\mathcal{C}$-modules gives rise to a cochain mapping of the de Rham complexes:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & Q_1 & \xrightarrow{\bar{d}} & \bar{\Lambda}^1 \otimes_{\mathcal{F}} Q_1 & \xrightarrow{\bar{d}} & \cdots & \xrightarrow{\bar{d}} & \bar{\Lambda}^n \otimes_{\mathcal{F}} Q_1 & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & & & \big\downarrow & & \\ 0 & \longrightarrow & Q_2 & \xrightarrow{\bar{d}} & \bar{\Lambda}^1 \otimes_{\mathcal{F}} Q_2 & \xrightarrow{\bar{d}} & \cdots & \xrightarrow{\bar{d}} & \bar{\Lambda}^n \otimes_{\mathcal{F}} Q_2 & \longrightarrow & 0. \end{array}$$

Let us discuss some examples of horizontal de Rham complexes.

Example 6.9. The horizontal de Rham complex with coefficients in the module $\bar{\mathcal{J}}^\infty(P)$

$$0 \to \bar{\mathcal{J}}^\infty(P) \xrightarrow{\bar{d}} \bar{\Lambda}^1 \otimes \bar{\mathcal{J}}^\infty(P) \xrightarrow{\bar{d}} \bar{\Lambda}^2 \otimes \bar{\mathcal{J}}^\infty(P) \xrightarrow{\bar{d}} \cdots \xrightarrow{\bar{d}} \bar{\Lambda}^n \otimes \bar{\mathcal{J}}^\infty(P) \to 0$$

is the project limit of the *horizontal Spencer complexes*

$$0 \to \bar{\mathcal{J}}^k(P) \xrightarrow{\bar{S}} \bar{\Lambda}^1 \otimes \bar{\mathcal{J}}^{k-1}(P) \xrightarrow{\bar{S}} \bar{\Lambda}^2 \otimes \bar{\mathcal{J}}^{k-2}(P) \xrightarrow{\bar{S}} \cdots, \tag{6.4}$$

where $\bar{S}(\omega \otimes \bar{\jmath}_l(p)) = \bar{d}\omega \otimes \bar{\jmath}_{l-1}(p)$. As usual Spencer complexes, they are exact in positive degrees and

$$H^0(\bar{\Lambda}^\bullet \otimes \bar{\mathcal{J}}^{k-\bullet}(P)) = P.$$

Recall that one proves this fact by considering the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \bar{\mathrm{S}}^k \otimes P \arrow[r] \arrow[d, "\bar{\delta}"] & \bar{\mathcal{J}}^k(P) \arrow[r] \arrow[d, "\bar{S}"] & \bar{\mathcal{J}}^{k-1}(P) \arrow[r] \arrow[d, "\bar{S}"] & 0 \\
0 \arrow[r] & \bar{\Lambda}^1 \otimes \bar{\mathrm{S}}^{k-1} \otimes P \arrow[r] \arrow[d, "\bar{\delta}"] & \bar{\Lambda}^1 \otimes \bar{\mathcal{J}}^{k-1}(P) \arrow[r] \arrow[d, "\bar{S}"] & \bar{\Lambda}^1 \otimes \bar{\mathcal{J}}^{k-2}(P) \arrow[r] \arrow[d, "\bar{S}"] & 0 \\
0 \arrow[r] & \bar{\Lambda}^2 \otimes \bar{\mathrm{S}}^{k-2} \otimes P \arrow[r] \arrow[d, "\bar{\delta}"] & \bar{\Lambda}^2 \otimes \bar{\mathcal{J}}^{k-2}(P) \arrow[r] \arrow[d, "\bar{S}"] & \bar{\Lambda}^2 \otimes \bar{\mathcal{J}}^{k-3}(P) \arrow[r] \arrow[d, "\bar{S}"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

(see page 20).

Exercise 6.5. Multiply this diagram by a $\mathcal{C}$-module Q (possibly of infinite rank) and prove that the complex

$$0 \to \bar{\mathcal{J}}^\infty(P) \mathbin{\widehat{\otimes}} Q \xrightarrow{\bar{d}} \bar{\Lambda}^1 \otimes \bar{\mathcal{J}}^\infty(P) \mathbin{\widehat{\otimes}} Q \xrightarrow{\bar{d}} \cdots \xrightarrow{\bar{d}} \bar{\Lambda}^n \otimes \bar{\mathcal{J}}^\infty(P) \mathbin{\widehat{\otimes}} Q \to 0$$

is exact in positive degrees and

$$H^0(\bar{\Lambda}^\bullet \otimes \bar{\mathcal{J}}^\infty(P) \mathbin{\widehat{\otimes}} Q) = P \otimes Q.$$

Here

$$\bar{\mathcal{J}}^\infty(P) \mathbin{\widehat{\otimes}} Q = \operatorname{proj\,lim} \bar{\mathcal{J}}^k(P) \otimes Q.$$

Example 6.10. The dualization of the previous example is as follows. The coefficient module is $\mathcal{C}\mathrm{Diff}(P, \mathcal{F})$. The corresponding horizontal de Rham complex multiplied by a $\mathcal{C}$-module Q has the form

$$0 \longrightarrow \mathcal{C}\mathrm{Diff}(P, \mathcal{F}) \otimes Q \xrightarrow{\bar{d}} \mathcal{C}\mathrm{Diff}(P, \bar{\Lambda}^1) \otimes Q \xrightarrow{\bar{d}} \cdots$$
$$\cdots \xrightarrow{\bar{d}} \mathcal{C}\mathrm{Diff}(P, \bar{\Lambda}^n) \otimes Q \longrightarrow 0.$$

As in the previous example, it is easily shown that

$$H^i(\mathcal{C}\mathrm{Diff}(P, \bar{\Lambda}^\bullet) \otimes Q) = 0 \quad \text{for } i < n,$$
$$H^n(\mathcal{C}\mathrm{Diff}(P, \bar{\Lambda}^\bullet) \otimes Q) = \hat{P} \otimes Q,$$

where $\hat{P} = \mathrm{Hom}_{\mathcal{F}}(P, \bar{\Lambda}^n)$.

One can use this fact to define the notion of adjoint $\mathcal{C}$-differential operator similarly to Definition 2.1 on page 27. The analog of Proposition 2.1 on page 27 remains valid for $\mathcal{C}$-differential operators.

Example 6.11. Take the $\mathcal{C}$-module

$$Q = \bigoplus_p \mathrm{D}^v(\mathcal{C}^p\Lambda) = \bigoplus_p \mathrm{Hom}_{\mathcal{F}}(\mathcal{C}^1\Lambda, \mathcal{C}^p\Lambda).$$

The horizontal de Rham complex with coefficients in Q can be written as

$$0 \longrightarrow \mathrm{D}^v \longrightarrow \mathrm{D}^v(\Lambda^1) \longrightarrow \mathrm{D}^v(\Lambda^2) \longrightarrow \cdots$$

Proposition 6.3. *The differential $d_{\mathrm{D}^v(\mathcal{C}^p\Lambda)}$ of this complex is equal to $-\partial_{\mathcal{C}}$ (see page 88), so that the complex coincides up to sign with the complex (5.39) on page 88.*

Proof. Take a vertical vector field $Y \in \mathrm{D}^v$ and an arbitrary vector field Z. By (5.22) on page 82 we obtain (cf. the proof of Theorem 5.12 on page 88) $\mathrm{i}_Z\partial_{\mathcal{C}}Y = [Z^v - Z, Y]^v$. Hence, $\partial_{\mathcal{C}}(\mathrm{D}^v) \subset \mathrm{D}^v \otimes \bar{\Lambda}^1$ and $\partial_{\mathcal{C}}|_{\mathrm{D}^v} = -d_{\mathrm{D}^v}$. This together with formula (5.37) on page 86 and Remark 5.5 on page 88 yields $\partial_{\mathcal{C}}(\mathrm{D}^v(\mathcal{C}^p\Lambda) \otimes \bar{\Lambda}^q) \subset \mathrm{D}^v(\mathcal{C}^p\Lambda) \otimes \bar{\Lambda}^{q+1}$ and $\partial_{\mathcal{C}}|_{\mathrm{D}^v(\mathcal{C}^p\Lambda)\otimes\bar{\Lambda}^q} = -d_{\mathrm{D}^v(\mathcal{C}^p\Lambda)}$. ☐

6.3. **Horizontal compatibility complex.** Consider a $\mathcal{C}$-differential operator $\Delta\colon P_0 \to P_1$. It is clear that by repeating word by word the construction of Subsection 1.4 on page 13 one obtains the *horizontal compatibility complex*

$$P_0 \xrightarrow{\Delta} P_1 \xrightarrow{\Delta_1} P_2 \xrightarrow{\Delta_2} P_3 \xrightarrow{\Delta_3} \cdots, \tag{6.5}$$

which is formally exact (see the end of Subsection 1.7 on page 25).

Consider the $\mathcal{C}$-module $\mathcal{R}_\Delta = \ker \psi_\infty^\Delta$ (cf. Example 6.6 on page 104). Then by Theorem 1.20 on page 21 the cohomology of complex (6.5) is isomorphic to the horizontal cohomology with coefficients in $\mathcal{R}_\Delta$:

Theorem 6.4.

$$\bar{H}^i(\mathcal{R}_\Delta) = H^i(P_\bullet).$$

Recall that this theorem follows from the spectral sequence arguments applied to the commutative diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \bar{\Lambda}^2 \otimes \bar{\mathcal{J}}^\infty(P_0) & \longrightarrow & \bar{\Lambda}^2 \otimes \bar{\mathcal{J}}^\infty(P_1) & \longrightarrow & \bar{\Lambda}^2 \otimes \bar{\mathcal{J}}^\infty(P_2) & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle \bar{d}} & & \uparrow{\scriptstyle \bar{d}} & & \uparrow{\scriptstyle \bar{d}} & & \\
0 & \longrightarrow & \bar{\Lambda}^1 \otimes \bar{\mathcal{J}}^\infty(P_0) & \longrightarrow & \bar{\Lambda}^1 \otimes \bar{\mathcal{J}}^\infty(P_1) & \longrightarrow & \bar{\Lambda}^1 \otimes \bar{\mathcal{J}}^\infty(P_2) & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle \bar{d}} & & \uparrow{\scriptstyle \bar{d}} & & \uparrow{\scriptstyle \bar{d}} & & \\
0 & \longrightarrow & \bar{\mathcal{J}}^\infty(P_0) & \longrightarrow & \bar{\mathcal{J}}^\infty(P_1) & \longrightarrow & \bar{\mathcal{J}}^\infty(P_2) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Let us multiply this diagram by a $\mathcal{C}$-module Q. This yields

$$\bar{H}^i(\mathcal{R}_\Delta \mathbin{\widehat{\otimes}} Q) = H^i(P_\bullet \otimes Q), \tag{6.6}$$

where $\mathcal{R}_\Delta \mathbin{\widehat{\otimes}} Q = \operatorname{proj\,lim} \mathcal{R}^l_\Delta \otimes Q$, with $\mathcal{R}^l_\Delta = \ker \psi^\Delta_{k+l}$, $\operatorname{ord} \Delta \le k$.

We can dualize our discussion. Namely, consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & \mathcal{C}\mathrm{Diff}(P_0, \bar{\Lambda}^{n-2}) & \longleftarrow & \mathcal{C}\mathrm{Diff}(P_1, \bar{\Lambda}^{n-2}) & \longleftarrow & \mathcal{C}\mathrm{Diff}(P_2, \bar{\Lambda}^{n-2}) & \longleftarrow & \cdots \\
 & & \downarrow{\scriptstyle \bar{d}} & & \downarrow{\scriptstyle \bar{d}} & & \downarrow{\scriptstyle \bar{d}} & & \\
0 & \longleftarrow & \mathcal{C}\mathrm{Diff}(P_0, \bar{\Lambda}^{n-1}) & \longleftarrow & \mathcal{C}\mathrm{Diff}(P_1, \bar{\Lambda}^{n-1}) & \longleftarrow & \mathcal{C}\mathrm{Diff}(P_2, \bar{\Lambda}^{n-1}) & \longleftarrow & \cdots \\
 & & \downarrow{\scriptstyle \bar{d}} & & \downarrow{\scriptstyle \bar{d}} & & \downarrow{\scriptstyle \bar{d}} & & \\
0 & \longleftarrow & \mathcal{C}\mathrm{Diff}(P_0, \bar{\Lambda}^{n}) & \longleftarrow & \mathcal{C}\mathrm{Diff}(P_1, \bar{\Lambda}^{n}) & \longleftarrow & \mathcal{C}\mathrm{Diff}(P_2, \bar{\Lambda}^{n}) & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

As above, we readily obtain

$$\bar{H}^i(\mathcal{R}^*_\Delta) = H_{n-i}(\hat{P}_\bullet)$$

and, more generally,

$$\bar{H}^i(\mathcal{R}_\Delta^* \otimes Q) = H_{n-i}(\hat{P}_\bullet \otimes Q), \tag{6.7}$$

where $\mathcal{R}_\Delta^* = \mathrm{Hom}(\mathcal{R}_\Delta, \mathcal{F})$. The homology in the right-hand side of these formulae is the homology of the complex

$$\hat{P}_0 \xleftarrow{\Delta^*} \hat{P}_1 \xleftarrow{\Delta_1^*} \hat{P}_2 \xleftarrow{\Delta_2^*} \hat{P}_3 \xleftarrow{\Delta_3^*} \cdots,$$

dual to the complex (6.5).

6.4. **Applications to computing the $\mathcal{C}$-cohomology groups.** Let $\mathcal{E}$ be an equation,

$$P_0 = \varkappa \xrightarrow{\ell_{\mathcal{E}}} P_1 \xrightarrow{\Delta_1} P_2 \xrightarrow{\Delta_2} P_3 \xrightarrow{\Delta_3} P_4 \xrightarrow{\Delta_4} \cdots$$

the compatibility complex for the operator of universal linearization, $\varkappa = \mathcal{F}(\mathcal{E}, \pi)$. Take a $\mathcal{C}$-module Q.

Theorem 6.5. $\bar{H}^i(\mathrm{D}^v(Q)) = H^i(P_\bullet \otimes Q)$.

Proof. The statement follows immediately from (6.6) on the page before and Proposition 3.30 on page 68. □

Let $Q = \mathcal{C}^p\Lambda$. The previous theorem gives a method for computing of the cohomology groups $\bar{H}^i(\mathrm{D}^v(\mathcal{C}^p\Lambda))$, which are the $\mathcal{C}$-cohomology groups (see Example 6.11 on page 108):

Corollary 6.6. $\bar{H}^i(\mathrm{D}^v(\mathcal{C}^p\Lambda)) = H^i(P_\bullet \otimes \mathcal{C}^p\Lambda)$.

Let us describe the isomorphisms given by this corollary in an explicit form.

Consider an element $\sum_{i\in I} \omega_i^q \otimes \bar{\jmath}_\infty(s_i) \in \bar{\Lambda}^q \otimes \mathrm{D}^v(\mathcal{C}^p\Lambda)$, where $\omega_i^q \in \bar{\Lambda}^q \otimes \mathcal{C}^p\Lambda$, $s_i \in \varkappa$, which is a horizontal cocycle. This means that

$$\sum_{i\in I} \omega_i^q \otimes \bar{\jmath}_\infty(\ell_{\mathcal{E}}(s_i)) = 0 \text{ and } \sum_{i\in I} \bar{d}\omega_i^q \otimes \bar{\jmath}_\infty(s_i) = 0.$$

From the second equality it easily follows that there exists an element $\sum_{i\in I_1} \omega_i^{q-1} \otimes \bar{\jmath}_\infty(s_i') \in \bar{\Lambda}^{q-1} \otimes \mathcal{C}^p\Lambda \otimes \bar{\mathcal{J}}^\infty(\varkappa)$, such that $\sum_{i\in I_1} \bar{d}\omega_i^{q-1} \otimes \bar{\jmath}_\infty(s_i') = \sum_{i\in I} \omega_i^q \otimes \bar{\jmath}_\infty(s_i)$. Denote $s_i^1 = \ell_{\mathcal{E}}(s_i')$. The element $\sum_{i\in I_1} \omega_i^{q-1} \otimes \bar{\jmath}_\infty(s_i^1) \in \bar{\Lambda}^{q-1} \otimes \mathcal{C}^p\Lambda \otimes \bar{\mathcal{J}}^\infty(P_1)$ satisfies

$$\sum_{i\in I_1} \omega_i^{q-1} \otimes \bar{\jmath}_\infty(\Delta_1(s_i^1)) = 0 \text{ and } \sum_{i\in I_1} \bar{d}\omega_i^{q-1} \otimes \bar{\jmath}_\infty(s_i^1) = 0.$$

Continuing this process, we obtain elements $\sum_{i\in I_l} \omega_i^{q-l} \otimes \bar{\jmath}_\infty(s_i^l) \in \bar{\Lambda}^{q-l} \otimes \mathcal{C}^p\Lambda \otimes \bar{\mathcal{J}}^\infty(P_l)$ such that

$$\sum_{i\in I_l} \omega_i^{q-l} \otimes \bar{\jmath}_\infty(\Delta_l(s_i^l)) = 0 \text{ and } \sum_{i\in I_l} \bar{d}\omega_i^{q-l} \otimes \bar{\jmath}_\infty(s_i^l) = 0.$$

For $l = q$ these formulae mean that the element $\sum_{i \in I_q} \omega_i^0 \otimes \bar{\jmath}_\infty(s_i^q)$ represents an element of the module $P_q \otimes \mathcal{C}^p\Lambda$ that lies in the kernel of the operator Δ_{q+1}. This is the element that gives rise to the cohomology class in the group $H^q(P_\bullet \otimes \mathcal{C}^p\Lambda)$ corresponding to the chosen element of $\bar{\Lambda}^q \otimes \mathrm{D}^v(\mathcal{C}^p\Lambda)$.

It follows from our results that if there is an integer k such that $P_k = P_{k+1} = P_{k+2} = \cdots = 0$, i.e., the compatibility complex has the form

$$P_0 = \varkappa \xrightarrow{\ell_\mathcal{E}} P_1 \xrightarrow{\Delta_1} P_2 \xrightarrow{\Delta_2} P_3 \xrightarrow{\Delta_3} \cdots \xrightarrow{\Delta_{k-2}} P_{k-1} \longrightarrow 0,$$

then

$$H^i(\mathrm{D}^v(\mathcal{C}^p\Lambda)) = 0 \quad \text{for } i \geq k.$$

This result is known as the *k-line theorem* for the $\mathcal{C}$-cohomology.

What are the values of the integer k for differential equations encountered in mathematical physics? The existence of a compatibility operator Δ_1 is usually due to the existence of dependencies between the equations under consideration: $\Delta_1(F) = 0$, $\mathcal{E} = \{F = 0\}$. The majority of systems that occur in practice consist of independent equations and for them $k = 2$. Such systems of differential equations are said to be *ℓ-normal*. In the case of ℓ-normal equations the *two-line theorem* for the $\mathcal{C}$-cohomology holds:

Theorem 6.7 (the two-line theorem)**.** *Let a differential equation $\mathcal{E}$ be ℓ-normal. Then:*

(1) $H^i(\mathrm{D}^v(\mathcal{C}^p\Lambda)) = 0$ *for $i \geq 2$,*
(2) $H^0(\mathrm{D}^v(\mathcal{C}^p\Lambda)) = \ker(\ell_\mathcal{E})_{\mathcal{C}^p\Lambda}$,
(3) $H^1(\mathrm{D}^v(\mathcal{C}^p\Lambda)) = \operatorname{coker}(\ell_\mathcal{E})_{\mathcal{C}^p\Lambda}$.

Further, we meet with the case $k > 2$ in gauge theories, when the dependencies $\Delta_1(F) = 0$ are given by the second Noether theorem (see page 128). For usual irreducible gauge theories, like electromagnetism, Yang-Mills models, and Einstein's gravity, the Noether identities are independent, so that the operator Δ_2 is trivial and, thus, $k = 3$. Finally, for an L-th stage reducible gauge theory, one has $k = L + 3$.

Remark 6.2. For the "empty" equation $J^\infty(\pi)$ Corollary 6.6 on the facing page yields Theorem 5.13 on page 90 (the *one-line theorem*).

6.5. **Example: Evolution equations.** Consider an evolution equation $\mathcal{E} = \{F = u_t - f(x, t, u_i) = 0\}$, with independent variables x, t and dependent variable u; u_i denotes the set of variables corresponding to derivatives of u with respect to x.

Natural coordinates for $\mathcal{E}^\infty$ are (x, t, u_i). The total derivatives operators D_x and D_t on $\mathcal{E}^\infty$ have the form

$$D_x = \frac{\partial}{\partial x} + \sum_i u_{i+1} \frac{\partial}{\partial u_i}, \quad D_t = \frac{\partial}{\partial t} + \sum_i D_x^i(f) \frac{\partial}{\partial u_i}.$$

The operator of universal linearization is given by

$$\ell_{\mathcal{E}} = D_t - \ell_f = D_t - \sum_i \frac{\partial f}{\partial u_i} D_x^i.$$

Clearly, for an evolution equation the two-line theorem holds, hence the $\mathcal{C}$-cohomology $\bar{H}^q(\mathrm{D}^v(\mathcal{C}^p\Lambda))$ is trivial for $q \geq 2$. Now, assume that the order of the equation $\mathcal{E}$ is greater than or equal to 2, i.e., $\operatorname{ord} \ell_f \geq 2$. Then one has more:

Theorem 6.8. *For any evolution equation of order ≥ 2, one has*

$$\bar{H}^0(\mathrm{D}^v(\mathcal{C}^p\Lambda)) = 0 \quad \textit{for } p \geq 2,$$

Proof. It follows from Theorem 6.7 on the preceding page that $\bar{H}^0(\mathrm{D}^v(\mathcal{C}^p\Lambda)) = \ker(\ell_{\mathcal{E}})_{\mathcal{C}^p\Lambda}$. Hence to prove the theorem it suffices to check that the equation

$$(D_t - \ell_f)(\omega) = 0, \tag{6.8}$$

with $\omega \in \mathcal{C}^p\Lambda$, has no nontrivial solutions for $p \geq 2$.

To this end consider the symbol of (6.8). Denote $\operatorname{smbl}(D_x) = \theta$. The symbol of ℓ_f has the form $\operatorname{smbl}(\ell_f) = g\theta^k$, $k = \operatorname{ord} \ell_f \geq 2$, where $g = \dfrac{\partial f}{\partial u_k}$. An element $\omega \in \mathcal{C}^p\Lambda$ can be identified with a multilinear $\mathcal{C}$-differential operator, so the symbol of ω is a homogeneous polynomial in p variables $\operatorname{smbl}(\omega) = \delta(\theta_1, \dots, \theta_p)$. Equation (6.8) yields

$$[g(\theta_1^k + \dots + \theta_p^k) - g(\theta_1 + \dots + \theta_p)^k] \cdot \delta(\theta_1, \dots, \theta_p) = 0.$$

The conditions $k \geq 2$ and $p \geq 2$ obviously imply that $\delta(\theta_1, \dots, \theta_p) = 0$. This completes the proof. $\square$

Remark 6.3. This proof can be generalized for determined systems of evolution equations with arbitrary number of independent variables (see [16]).

7. Vinogradov's $\mathcal{C}$-spectral sequence

7.1. **Definition of the Vinogradov $\mathcal{C}$-spectral sequence.** Suppose $\mathcal{E} \subset J^k(\pi)$ is a formally integrable differential equation. Consider the ideal $\mathcal{C}\Lambda^* = \mathcal{C}\Lambda^*(\mathcal{E})$ of the exterior algebra $\Lambda^*(\mathcal{E})$ of differential forms on $\mathcal{E}^\infty$ generated by the Cartan submodule $\mathcal{C}^1\Lambda(\mathcal{E})$ (see page 61): $\mathcal{C}\Lambda^* = \mathcal{C}^1\Lambda(\mathcal{E}) \wedge \Lambda^*(\mathcal{E})$. Clearly, this ideal and all its powers $(\mathcal{C}\Lambda^*)^{\wedge s} = \mathcal{C}^s\Lambda \wedge \Lambda^*$, where $\mathcal{C}^s\Lambda = \underbrace{\mathcal{C}^1\Lambda \wedge \cdots \wedge \mathcal{C}^1\Lambda}_{s \text{ times}}$, is stable with respect to the operator d, i.e.,

$$d((\mathcal{C}\Lambda^*)^{\wedge s}) \subset (\mathcal{C}\Lambda^*)^{\wedge s}.$$

Thus, in the de Rham complex on $\mathcal{E}^\infty$ we have the filtration

$$\Lambda^* \supset \mathcal{C}\Lambda^* \supset (\mathcal{C}\Lambda^*)^{\wedge 2} \supset \cdots \supset (\mathcal{C}\Lambda^*)^{\wedge s} \supset \cdots .$$

The spectral sequence $(E_r^{p,q}, d_r^{p,q})$ determined by this filtration is said to be *the Vinogradov $\mathcal{C}$-spectral sequence* of equation $\mathcal{E}$. As usual p is the filtration degree and $p+q$ is the total degree.

It follows from the direct sum decomposition (3.40) on page 61 that $E_0^{p,q}$ can be identified with $\mathcal{C}^p\Lambda \otimes \bar{\Lambda}^q$.

Exercise 7.1. Prove that under this identification the operator $d_0^{p,q}$ coincides with the horizontal de Rham differential $\bar{d}_{\mathcal{C}^p\Lambda}$ with coefficients in $\mathcal{C}^p\Lambda$ (cf. Example 6.3 on page 104).

Thus, the Vinogradov $\mathcal{C}$-spectral sequence is one of two spectral sequences associated with the variational bicomplex $(\mathcal{C}^p\Lambda \otimes \bar{\Lambda}^q), \bar{d}, d_{\mathcal{C}})$ constructed in Subsection 3.8 on page 61.

Remark 7.1. The second spectral sequences associated with the variational bicomplex can be naturally identified with the Leray–Serre spectral sequence of the de Rham cohomology of the bundle $\mathcal{E}^\infty \to M$.

Remark 7.2. The definition of the Vinogradov $\mathcal{C}$-spectral sequence given above remains valid for any object the category *Inf* (see page 69), whereas the variational bicomplex exists only for an infinite prolonged equation.

Exercise 7.2. Prove that any morphism $F\colon \mathcal{N}_1 \to \mathcal{N}_2$ in *Inf* gives rise to the homomorphism of the Vinogradov $\mathcal{C}$-spectral sequence for $\mathcal{N}_2$ into the Vinogradov $\mathcal{C}$-spectral sequence for $\mathcal{N}_1$.

7.2. **The term E_1 for $J^\infty(\pi)$.** Let us consider the term E_1 of the Vinogradov $\mathcal{C}$-spectral sequence for the "empty" equation $\mathcal{E}^\infty = J^\infty(\pi)$.

By definition the first term E_1 of a spectral sequence is the cohomology of its zero term E_0. Thus, to describe the terms $E_1^{p,q}(\pi)$ we must compute the cohomologies of complexes

$$0 \to \mathcal{C}^p\Lambda(\pi) \xrightarrow{\bar{d}} \mathcal{C}^p\Lambda(\pi) \otimes \bar{\Lambda}^1(\pi) \xrightarrow{\bar{d}} \cdots \xrightarrow{\bar{d}} \mathcal{C}^p\Lambda(\pi) \otimes \bar{\Lambda}^n(\pi) \to 0.$$

Using Proposition 3.30 on page 68, this complex can be rewritten in the form

$$0 \longrightarrow \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa(\pi), \mathcal{F}(\pi)) \xrightarrow{w} \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa(\pi), \bar{\Lambda}^1(\pi)) \xrightarrow{w} \cdots$$
$$\xrightarrow{w} \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa(\pi), \bar{\Lambda}^n(\pi)) \longrightarrow 0,$$

where $w(\Delta) = (-1)^p \bar{d} \circ \Delta$.

Now from Theorem 2.8 on page 32 we obtain the following description of the term E_1 for $J^\infty(\pi)$:

Theorem 7.1. *Let π be a smooth vector bundle over a manifold M,* $\dim M = n$. *Then*:

(1) $E_1^{0,q}(\pi) = \bar{H}^q(\pi)$ *for all* $q \geq 0$;
(2) $E_1^{p,q}(\pi) = 0$ *for* $p > 0$, $q \neq n$;
(3) $E_1^{p,n}(\pi) = L_p^{\mathrm{alt}}(\varkappa(\pi))$, $p > 0$,

where $L_p^{\mathrm{alt}}(\varkappa(\pi))$ was defined in Theorem 2.8 *on page* 32.

Since the the Vinogradov $\mathcal{C}$-spectral sequence converges to the de Rham cohomology of the manifold $J^\infty(\pi)$, this theorem has the following

Corollary 7.2. *For any smooth vector bundle π over an n-dimensional smooth manifold M one has*:

(1) $E_r^{p,q}(\pi) = 0$, $1 \leq r \leq \infty$, *if* $p > 0$, $q \neq n$ *or* $p = 0$, $q > n$;
(2) $E_1^{0,q}(\pi) = E_\infty^{0,q}(\pi) = H^q(J^\infty(\pi)) = H^q(J^0(\pi))$, $q < n$;
(3) $E_2^{p,n}(\pi) = E_\infty^{p,n}(\pi) = H^{p+n}(J^\infty(\pi)) = H^{p+n}(J^0(\pi))$, $p \geq 0$.

Exercise 7.3. Prove that $H^q(J^\infty(\pi)) = H^q(J^0(\pi))$.

We now turn our attention to the differentials $d_1^{p,n}$. They are induced by the Cartan differential $d_{\mathcal{C}}$. For $p = 0$, we have $d_{\mathcal{C}}(\omega) = \ell_\omega$, $\omega \in \bar{\Lambda}^n$. (Note that the expression ℓ_ω is correct, because ω is a horizontal form, i.e., a nonlinear operator from $\Gamma(\pi)$ to $\Lambda^n(M)$.) Therefore the operator

$$E_1^{0,n}(\pi) = \bar{H}^n(\pi) \xrightarrow{d_1^{0,n}} E_1^{1,n}(\pi) = \hat{\varkappa}(\pi)$$

is given by the formula $d_1^{0,n}([\omega]) = \mu(\ell_\omega) = \ell_\omega^*(1)$, where $\omega \in \bar{\Lambda}^n(\pi)$, $[\omega]$ is the horizontal cohomology class of ω.

Exercise 7.4. Write down the coordinate expression for the operator $d_1^{0,n}$ and show that it coincides with the standard Euler operator, i.e., with the operator that takes a Lagrangian to the corresponding Euler–Lagrange equation.

Let us compute the operators $d_1^{p,n}$, $p > 0$.

Consider an element $\nabla \in L_p^{\mathrm{alt}}(\varkappa(\pi))$ and define the operator $\square \in \mathcal{C}\mathrm{Diff}_{(p+1)}(\varkappa(\pi), \bar{\Lambda}^n(\pi))$ via

$$\begin{aligned}\square(\chi_1, \dots, \chi_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i+1} \ni_{\chi_i}(\nabla(\chi_1, \dots, \hat{\chi}_i, \dots, \chi_{p+1})) \\ &+ \sum_{1 \le i < j \le p+1} (-1)^{i+j} \nabla(\{\chi_i, \chi_j\}, \chi_1, \dots, \hat{\chi}_i, \dots, \hat{\chi}_j, \dots, \chi_{p+1}). \quad (7.1)\end{aligned}$$

Exercise 7.5. Prove that $d_1^{p,n}(\nabla) = \mu_{(p+1)}(\square)$ (see page 33 for the definition of $\mu_{(p+1)}$).

Remark 7.3. Needless to say that this fact follows from the standard formula for exterior differential. It needs however to be proved that one may use this formula even though ∇ as an element of $\mathcal{C}\mathrm{Diff}_{(p)}(\varkappa, \bar{\Lambda}^n)$ is not skew-symmetric.

From (7.1) we get

$$\begin{aligned}\square(\chi_1, \dots, \chi_{p+1}) &= \sum_{i=1}^{p} (-1)^{i+1} \ni_{\chi_i}(\nabla(\chi_1, \dots, \hat{\chi}_i, \dots, \chi_p))(\chi_{p+1}) \\ &+ \sum_{i=1}^{p} (-1)^{i+1} \nabla(\chi_1, \dots, \hat{\chi}_i, \dots, \chi_p, \ni_{\chi_i}(\chi_{p+1})) \\ &+ (-1)^p \ni_{\chi_{p+1}}(\nabla(\chi_1, \dots, \chi_p)) \\ &+ \sum_{1 \le i < j \le p} (-1)^{i+j} \nabla(\{\chi_i, \chi_j\}, \chi_1, \dots, \hat{\chi}_i, \dots, \hat{\chi}_j, \dots, \chi_{p+1}) \\ &+ \sum_{i=1}^{p} (-1)^{i+p+1} \nabla(\{\chi_i, \chi_{p+1}\}, \chi_1, \dots, \hat{\chi}_i, \dots, \chi_p) \\ &= \sum_{i=1}^{p} (-1)^{i+1} \ni_{\chi_i}(\nabla(\chi_1, \dots, \hat{\chi}_i, \dots, \chi_p))(\chi_{p+1}) \\ &+ \sum_{1 \le i < j \le p} (-1)^{i+j} \nabla(\{\chi_i, \chi_j\}, \chi_1, \dots, \hat{\chi}_i, \dots, \hat{\chi}_j, \dots, \chi_{p+1}) \\ &+ \sum_{i=1}^{p} (-1)^{i+1} \nabla(\chi_1, \dots, \hat{\chi}_i, \dots, \chi_p, \ell_{\chi_i}(\chi_{p+1})) + (-1)^p \ell_{\nabla(\chi_1, \dots, \chi_p)}(\chi_{p+1}).\end{aligned}$$

Therefore

$$\begin{aligned}
d_1^{p,n}(\nabla)(\chi_1,\dots,\chi_p) &= \mu_{(p+1)}(\Box)(\chi_1,\dots,\chi_p)\\
&= \sum_{i=1}^{p}(-1)^{i+1}\ni_{\chi_i}(\nabla(\chi_1,\dots,\hat{\chi}_i,\dots,\chi_p))\\
&\quad+\sum_{i<j}(-1)^{i+j}\nabla(\{\chi_i,\chi_j\},\chi_1,\dots,\hat{\chi}_i,\dots,\hat{\chi}_j,\dots,\chi_p)\\
&\quad+\sum_{i=1}^{p}(-1)^{i+1}\ell^*_{\chi_i}(\nabla(\chi_1,\dots,\hat{\chi}_i,\dots,\chi_p))+(-1)^p\ell^*_{\nabla(\chi_1,\dots,\chi_p)}(1). \qquad (7.2)
\end{aligned}$$

Exercise 7.6. Prove that

$$\ell^*_{\psi(\varphi)}(1)=\ell^*_\psi(\varphi)+\ell^*_\varphi(\psi), \qquad \varphi\in\varkappa(\pi), \quad \psi\in\hat{\varkappa}(\pi).$$

Using this formula, let us rewrite the last term of (7.2) in the following way:

$$\begin{aligned}
(-1)^p\ell^*_{(\nabla(\chi_1,\dots,\chi_p))}(1) &= \frac{1}{p}\sum_{i=1}^{p}(-1)^i(\ell^*_{(\nabla(\chi_1,\dots,\hat{\chi}_i,\dots,\chi_p,\chi_i))}(1)\\
&= \frac{1}{p}\sum_{i=1}^{p}(-1)^i(\ell^*_{\nabla(\chi_1,\dots,\hat{\chi}_i,\dots,\chi_p)}(\chi_i)+\ell^*_{\chi_i}(\nabla(\chi_1,\dots,\hat{\chi}_i,\dots,\chi_p))).
\end{aligned}$$

Finally we obtain

$$\begin{aligned}
\left(d_1^{p,n}(\nabla)\right)(\chi_1,\dots,\chi_p) &= \sum_{i=1}^{p}(-1)^{i+1}\ni_{\chi_i}(\nabla(\chi_1,\dots,\hat{\chi}_i,\dots,\chi_p))\\
&\quad+\sum_{i<j}(-1)^{i+j}\nabla(\{\chi_i,\chi_j\},\chi_1,\dots,\hat{\chi}_i,\dots,\hat{\chi}_j,\dots,\chi_p)\\
&\quad+\frac{1}{p}\sum_{i=1}^{p}(-1)^{i+1}((p-1)\ell^*_{\chi_i}(\nabla(\chi_1,\dots,\hat{\chi}_i,\dots,\chi_p))-\ell^*_{\nabla(\chi_1,\dots,\hat{\chi}_i,\dots,\chi_p)}(\chi_i)).
\end{aligned}$$

In particular, for $p=1$ we have $d_1^{1,n}(\psi)(\varphi)=\ni_\varphi(\psi)-\ell^*_\psi(\varphi)=\ell_\psi(\varphi)-\ell^*_\psi(\varphi)$, $\psi\in\hat{\varkappa}(\pi)$, $\varphi\in\varkappa(\pi)$, that is

$$d_1^{1,n}(\psi)=\ell_\psi-\ell^*_\psi.$$

Consider the following complex, which is said to be the (global) *variational complex*,

$$0\longrightarrow\mathcal{F}(\pi)\xrightarrow{\bar{d}}\bar{\Lambda}^1(\pi)\xrightarrow{\bar{d}}\cdots\xrightarrow{\bar{d}}\bar{\Lambda}^n(\pi)\xrightarrow{\mathbf{E}}E_1^{1,n}(\pi)\xrightarrow{d_1^{1,n}}E_1^{2,n}(\pi)\xrightarrow{d_1^{2,n}}\cdots,$$

where operator $\mathbf{E}$ is equal to the composition of the natural projection $\bar{\Lambda}^n(\pi) \to \bar{H}^n(\pi)$ and the operator $\bar{H}^n(\pi) \xrightarrow{d_1^{0,n}} E_1^{1,n}(\pi)$ [15].

In view of Corollary 7.2 on page 114, the cohomology of this complex coincides with $H^*(J^0(\pi))$.

The operator $\mathbf{E}$ is the Euler operator (see Exercise 7.4 on page 114). It takes each Lagrangian density $\omega \in \Lambda_0^n(\pi)$ to the left-hand part of the corresponding Euler–Lagrange equation $\mathbf{E}(\omega) = 0$. Thus the action functional

$$s \mapsto \int_M j_\infty(s)^*(\omega), \qquad s \in \Gamma(\pi),$$

is stationary on the section s if and only if $j_\infty(s)^*\left(\mathbf{E}(\omega)\right) = 0$.

If the cohomology of the space $J^0(\pi)$ is trivial, then the variational complex is exact. This immediately implies a number of consequences. The three most important are:

(1) $\ker \mathbf{E} = \operatorname{im} \bar{d}$ ("a Lagrangian with zero variational derivative is a total divergence");
(2) $\bar{d}\omega = 0$ if and only if ω is of the form $\omega = \bar{d}\eta$, $\omega \in \bar{\Lambda}^{n-1}(\pi)$ ("all zero total divergence are total curls");
(3) $\ell_\psi = \ell_\psi^*$ if and only if ψ is of the form $\psi = \mathbf{E}(\omega)$, $\psi \in \hat{\varkappa}(\pi)$ (this is the solution of the inverse problem to the calculus of variations).

Now suppose that we are given $\psi \in \hat{\varkappa}(\pi)$ such that $\ell_\psi = \ell_\psi^*$. How one can find a Lagrangian ω such that $\psi = \mathbf{E}(\omega)$? To this end take a one-parameter family of fiberwise transformations $G_t \colon E \to E$, $0 \le t \le 1$, of the space of the bundle $\pi \colon E \to M$, with $G_0 = 0$ and $G_1 = \mathrm{id}_E$. Consider the corresponding family of evolutionary vector field $\mathfrak{I}_{\varphi_t}$, i.e.,

$$\frac{d}{dt} G_t^{(\infty)*} = \mathfrak{I}_{\varphi_t} \circ G_t^{(\infty)*}$$

for $t > 0$. Let us compute the correspondent Lie derivative $\mathfrak{I}_{\varphi_t}(\psi)$ (which is different from the usual "component-wise" derivative). Take $\Omega \in \bar{\Lambda}^n \otimes \mathcal{C}^1\Lambda$, $d\Omega = 0$, that represents ψ. Then $\mathfrak{I}_{\varphi_t}(\Omega) = d_{\mathcal{C}}(\Omega(\mathfrak{I}_{\varphi_t}))$. Therefore $\mathfrak{I}_{\varphi_t}(\psi) = \mathbf{E}(\psi(\varphi_t))$. Hence

$$\frac{d}{dt} G_t^{(\infty)*}(\psi) = \mathbf{E}(G_t^{(\infty)*}(\psi(\varphi_t))),$$

and integrating this with respect to t, we obtain the following *homotopy* (or *inverse*) formula

$$\psi = \mathbf{E}\left(\int_0^1 G_t^{(\infty)*}(\psi(\varphi_t))\, dt \right).$$

[15]Below we use the notation $\mathbf{E}$ for the operator $d_1^{0,n} \colon \bar{H}^n(\pi) \to E_1^{1,n}(\pi)$ as well.

Take, for instance, $G_t(e_x) = te_x$, $e_x \in E_x = \pi^{-1}(x)$. Then $\varphi_t^i = \dfrac{u^i}{t}$ and we have

$$\psi = \mathbf{E}\left(\int_0^1 \sum_i u^i \psi^i(x, tu_\sigma^j)\,dt\right).$$

Exercise 7.7. Let $\Delta \in \mathcal{C}\mathrm{Diff}(P, \bar{\Lambda}^n(\pi))$. Using the Green formula and Exercise 7.6 on page 116, prove that for any $p \in P$ one has

$$\mathbf{E}(\Delta(p)) = \ell_p^*(\Delta^*(1)) + \ell_{\Delta^*(1)}^*(p).$$

Deduce from this formula that for any $\varphi \in \varkappa(\pi)$ and $\omega \in \bar{\Lambda}^n(\pi)$ the following equality holds

$$\mathbf{E}(\ni_\varphi(\omega)) = \ni_\varphi(\mathbf{E}(\omega)) + \ell_\varphi^*(\mathbf{E}(\omega)).$$

Exercise 7.8. Let $J = (J_0, J_1, \dots, J_n)$ be a conserved current for an evolution equation $\mathcal{E} = \{u_t = f(t, x, u, u_x, u_{xx}, \dots)\}$. Using the previous exercise, prove that the vector-function $\psi = \mathbf{E}(J_0)$, where J_0 is the t-component of the conserved current that is regarded as a function of $(t, x, u, u_x, u_{xx}, \dots)$, satisfies the equation

$$D_t(\psi) + \ell_f^*(\psi) = 0$$

(cf. Theorem 7.11 on page 124).

7.3. **The term E_1 for an equation.** Let $\mathcal{E}$ be an equation,

$$P_0 = \varkappa \xrightarrow{\ell_{\mathcal{E}}} P_1 \xrightarrow{\Delta_1} P_2 \xrightarrow{\Delta_2} P_3 \xrightarrow{\Delta_3} P_4 \xrightarrow{\Delta_4} \cdots$$

be the compatibility complex for the universal linearization operator, and

$$\hat{P}_0 = \hat{\varkappa} \xleftarrow{\ell_E^*} \hat{P}_1 \xleftarrow{\Delta_1^*} \hat{P}_2 \xleftarrow{\Delta_2^*} \hat{P}_3 \xleftarrow{\Delta_3^*} \hat{P}_4 \xleftarrow{\Delta_4^*} \cdots$$

be the dual complex. Take a $\mathcal{C}$-module Q.

Theorem 7.3. *For any equation $\mathcal{E}$ and a $\mathcal{C}$-module Q one has*

$$\bar{H}^{n-i}(\mathcal{C}^1\Lambda \otimes Q) = H_i(\hat{P}_\bullet \otimes Q).$$

Proof. The statement follows immediately from (6.7) on page 110 and Proposition 3.30 on page 68. □

Let $Q = \mathcal{C}^p\Lambda$. The theorem gives a method for computing the Vinogradov $\mathcal{C}$-spectral sequence. Namely, since the term $E_1^{p,q} = \bar{H}^q(\mathcal{C}^p\Lambda)$ of the Vinogradov $\mathcal{C}$-spectral sequence is a direct summand in the cohomology group $\bar{H}^q(\mathcal{C}^1\Lambda \otimes \mathcal{C}^{p-1}\Lambda)$, we have a description for the first term of the Vinogradov $\mathcal{C}$-spectral sequence. Thus:

Corollary 7.4. *The term $E_1^{p,q}$ of the Vinogradov $\mathcal{C}$-spectral sequence is the skew-symmetric part of the group $H_{n-q}(\hat{P}_\bullet \otimes \mathcal{C}^{p-1}\Lambda)$.*

It is useful to describe the isomorphisms given by this corollary in an explicit form.

Consider an operator $\nabla \in \mathcal{C}\mathrm{Diff}(\varkappa, \bar{\Lambda}^q \otimes \mathcal{C}^{p-1}\Lambda)$ that represents an element of $E_1^{p,q}$. This means that

$$\bar{d} \circ \nabla = \nabla_1 \circ \ell_{\mathcal{E}}$$

for an operator $\nabla_1 \in \mathcal{C}\mathrm{Diff}(P_1, \bar{\Lambda}^{q+1} \otimes \mathcal{C}^{p-1}\Lambda)$. Applying the operator $\bar{d}$ to both sides of this formula and using Exercise 1.3 on page 13, we get

$$\bar{d} \circ \nabla_1 = \nabla_2 \circ \Delta_1$$

for some operator $\nabla_2 \in \mathcal{C}\mathrm{Diff}(P_2, \bar{\Lambda}^{q+2} \otimes \mathcal{C}^{p-1}\Lambda)$. Continuing this process, we obtain operators $\nabla_i \in \mathcal{C}\mathrm{Diff}(P_i, \bar{\Lambda}^{q+i} \otimes \mathcal{C}^{p-1}\Lambda)$, $i = 1, 2, \dots, n-q$, such that

$$\bar{d} \circ \nabla_{i-1} = \nabla_i \circ \Delta_{i-1}.$$

For $i = n-q$, this formula means that the operator $\nabla_{n-q} \in \mathcal{C}\mathrm{Diff}(P_{n-q}, \bar{\Lambda}^n \otimes \mathcal{C}^{p-1}\Lambda)$ represents an element of the module $\hat{P}_{n-q} \otimes \mathcal{C}^{p-1}\Lambda$ that lies in the kernel of the operator Δ^*_{n-q-1}. This is the element that gives rise to the homology class in $H_{n-q}(\hat{P}_\bullet \otimes \mathcal{C}^{p-1}\Lambda)$ corresponding to the chosen element of $E_1^{p,q}$.

If the compatibility complex has the length k,

$$P_0 = \varkappa \xrightarrow{\ell_E} P_1 \xrightarrow{\Delta_1} P_2 \xrightarrow{\Delta_2} P_3 \xrightarrow{\Delta_3} \cdots \xrightarrow{\Delta_{k-2}} P_{k-1} \longrightarrow 0,$$

then $E_1^{p,q} = 0$ for $p > 0$ and $q \le n-k$. This is the *k-line theorem* for the Vinogradov $\mathcal{C}$-spectral sequence.

In the case $k = 2$, i.e., for ℓ-normal equations, the *two-line theorem* holds:

Theorem 7.5 (the two-line theorem)**.** *Let $\mathcal{E}$ be an ℓ-normal differential equation. Then:*

(1) $E_1^{p,q} = 0$ *for $p > 0$ and $q \le n-2$,*
(2) $E_1^{p,n-1} \subset \ker(\ell^*_E)_{\mathcal{C}^{p-1}\Lambda}$ *for $p > 0$,*
(3) $E_1^{p,n} \subset \operatorname{coker}(\ell^*_E)_{\mathcal{C}^{p-1}\Lambda}$ *for $p > 0$.*

This theorem has the following elementary

Corollary 7.6. *The terms $E_r^{p,q}(\mathcal{E})$ of the Vinogradov $\mathcal{C}$-spectral sequence satisfy the following:*

(1) $E_r^{p,q}(\mathcal{E}) = 0$ *if $p \ge 1$, $q \ne n-1, n$, $1 \le r \le \infty$;*
(2) $E_3^{p,q}(\mathcal{E}) = E_\infty^{p,q}(\mathcal{E})$;
(3) $E_1^{0,q}(\mathcal{E}) = E_\infty^{0,q}(\mathcal{E}) = H^q(\mathcal{E}^\infty)$, *$q \le n-2$;*
(4) $E_2^{0,n-1}(\mathcal{E}) = E_\infty^{0,n-1}(\mathcal{E}) = H^{n-1}(\mathcal{E}^\infty)$;
(5) $E_2^{1,n-1}(\mathcal{E}) = E_\infty^{1,n-1}(\mathcal{E})$.

Example 7.1. For an evolution equation $\mathcal{E} = \{F = u_t - f(x, t, u, u_x, u_{xx} \dots) = 0\}$ the two-line theorem implies that the Vinogradov $\mathcal{C}$-spectral sequence is trivial for $q \neq 1, 2$, $p > 0$, and exactly as in Example 6.5 on page 111 one proves that $E_1^{p,1} = 0$ for $p \geq 3$.

7.4. **Example: Abelian p-form theories.** Let M be a (pseudo-)Riemannian manifold and $\pi \colon E \to M$ the p-th exterior power of the cotangent bundle over M, so that a section of π is a p-form on M. Evidently, on the jet space $J^\infty(\pi)$ there exists a unique horizontal form $A \in \bar{\Lambda}^p(J^\infty(\pi))$ such that $j_\infty^*(\omega)(A) = \omega$ for all $\omega \in \Lambda^p(M)$. Consider the equation $\mathcal{E} = \{F = 0\}$, with $F = \bar{d}{*}\bar{d}A$, where $*$ is the Hodge star operator. Our aim is to calculate the terms of the Vinogradov $\mathcal{C}$-spectral sequence $E_1^{i,q}(\mathcal{E})$ for $q \leq n - 2$. We shall assume that $1 \leq p < n - 1$ and that the manifold M is topologically trivial.

Obviously, we have $P_0 = \varkappa = \bar{\Lambda}^p$, $P_1 = \bar{\Lambda}^{n-p}$, and $\ell_\mathcal{E} = \bar{d}{*}\bar{d} \colon \bar{\Lambda}^p \to \bar{\Lambda}^{n-p}$. Taking into account Example 1.2 on page 24, we see that the compatibility complex for $\ell_\mathcal{E}$ has the form

$$\begin{array}{ccccccccc} \bar{\Lambda}^p & \xrightarrow{\ell_\mathcal{E}} & \bar{\Lambda}^{n-p} & \xrightarrow{\bar{d}} & \bar{\Lambda}^{n-p+1} & \xrightarrow{\bar{d}} \cdots \xrightarrow{\bar{d}} & \bar{\Lambda}^n & \longrightarrow & 0 \\ \| & & \| & & \| & & \| & & \\ P_0 & & P_1 & & P_2 & & P_{k-1} & & \end{array} \tag{7.3}$$

Thus $k = p + 2$ and the k-line theorem yields $E_1^{i,q} = 0$ for $i > 0$ and $q < n - p - 1$. Since the Vinogradov $\mathcal{C}$-spectral sequence converges to the de Rham cohomology of $\mathcal{E}^\infty$, which is trivial, we also get $E_1^{0,q} = 0$ for $0 < q < n - p - 1$, and $\dim E_1^{0,0} = 1$, i.e., $\bar{H}^1 = \bar{H}^2 = \cdots = \bar{H}^{n-p-2} = 0$ and $\dim \bar{H}^0 = 1$. Next, consider the terms $E_1^{i,q}$ for $n - p - 1 \leq q < 2(n - p - 1)$ and $i > 0$. In view of Corollary 7.4 on page 118 one has

$$E_1^{i,q} \subset \bar{H}^{q-(n-p-1)}(\mathcal{C}^{i-1}\Lambda) = E_1^{i-1, q-(n-p-1)},$$

because the complex dual to the compatibility complex (7.3) has the form

$$\begin{array}{ccccccccc} \bar{\Lambda}^{n-p} & \xleftarrow{\ell_\mathcal{E}^*} & \bar{\Lambda}^p & \xleftarrow{\bar{d}} & \bar{\Lambda}^{p-1} & \xleftarrow{\bar{d}} \cdots \xleftarrow{\bar{d}} & \mathcal{F} & \longleftarrow & 0. \\ \| & & \| & & \| & & \| & & \\ \hat{P}_0 & & \hat{P}_1 & & \hat{P}_2 & & \hat{P}_{p+1} & & \end{array}$$

(Throughout, it is assumed that $q \leq n - 2$.) Thus we obtain $E_1^{i,q} = 0$ for $n - p - 1 < q < 2(n - p - 1)$, $i > 0$ and $\dim E_1^{1,n-p-1} = 1$. Again, taking into account that the spectral sequence converges to the trivial cohomology, we get $E_1^{0,q} = 0$ for $n - p - 1 < q < 2(n - p - 1)$ and $\dim E_1^{0,n-p-1} = 1$. In addition, the map $d_1^{0,n-p-1} \colon E_1^{0,n-p-1} \to E_1^{1,n-p-1}$ is an isomorphism. Explicitly, one readily obtains that the one-dimensional space $E_1^{0,n-p-1}$ is

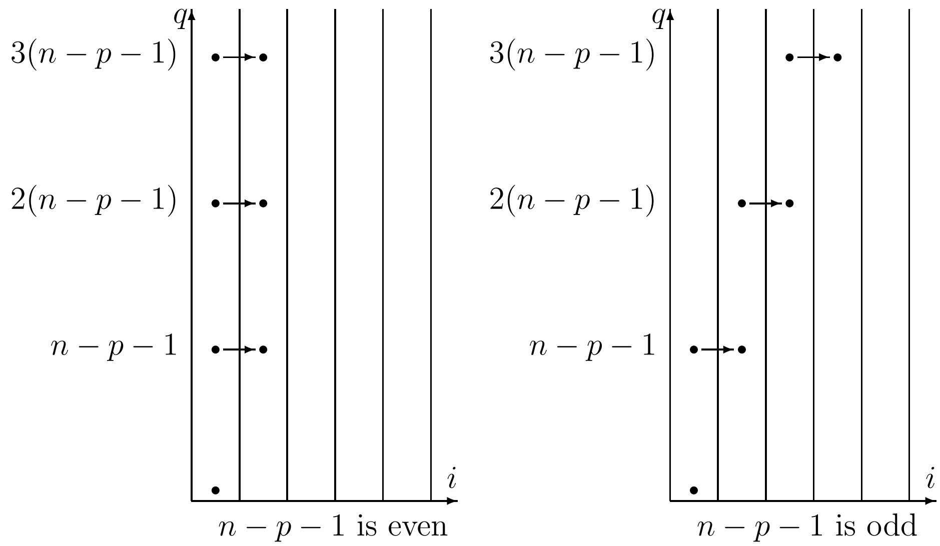


Diagram 7.1

generated by the element $*\bar{d}A \in \bar{\Lambda}^{n-p-1}$ and the map $d_1^{0,n-p-1}$ takes this element to the operator $*\bar{d}\colon \varkappa = \bar{\Lambda}^p \to \bar{\Lambda}^{n-p-1}$, which generates the space $E_1^{1,n-p-1}$.

Further, let us consider the terms $E_1^{i,q}$ for $2(n-p-1) \le q < 3(n-p-1)$. Arguing as before, we see that all these terms vanish unless $q = 2(n-p-1)$ and $i = 0, 1, 2$, with $\dim E_1^{1,2(n-p-1)} = 1$ and $\dim E_1^{i,2(n-p-1)} \le 1$, $i = 0, 2$. To compute the terms $E_1^{i,2(n-p-1)}$ for $i = 0$ and $i = 2$, we have to consider two cases: $n-p-1$ is even and $n-p-1$ is odd (see Diagram 7.1).

In the first case, the map $d_1^{1,2(n-p-1)}\colon E_1^{1,2(n-p-1)} \to E_1^{2,2(n-p-1)}$ is trivial. Indeed, the operator $(*\bar{d}A) \wedge *\bar{d}\colon \varkappa = \bar{\Lambda}^p \to \bar{\Lambda}^{2(n-p-1)}$, which generates the space $E_1^{1,2(n-p-1)}$, under the mapping $d_1^{1,2(n-p-1)}$ is the antisymmetrization of the operator $(\omega_1, \omega_2) \mapsto (*\bar{d}\omega_1) \wedge (*\bar{d}\omega_2)$, $\omega_i \in \varkappa = \bar{\Lambda}^p$. But this operator is symmetric, so that $d_1^{1,2(n-p-1)} = 0$. Consequently, $E_1^{2,2(n-p-1)} = 0$ and $\dim E_1^{0,2(n-p-1)} = 1$. This settles the case when $n-p-1$ is even.

In the case when $n-p-1$ is odd, the operator $(\omega_1, \omega_2) \mapsto (*\bar{d}\omega_1) \wedge (*\bar{d}\omega_2)$ is skew-symmetric, hence the map $d_1^{1,2(n-p-1)}$ is an isomorphism. Thus, $\dim E_1^{2,2(n-p-1)} = 1$ and $E_1^{0,2(n-p-1)} = 0$.

Continuing this line of reasoning, we obtain the following result.

Theorem 7.7. *For $i = q = 0$ one has $\dim E_1^{0,0} = 1$. If either or both i and q are positive, there are two cases:*

(1) *if $n-p-1$ is even then*

$$\dim E_1^{i,q} = \begin{cases} 1 & \text{for } i = l(n-p-1) \text{ and } q = 0,\ 1, \\ 0 & \text{otherwise}; \end{cases}$$

(2) *if $n-p-1$ is odd then*

$$\dim E_1^{i,q} = \begin{cases} 1 & \text{for } i = l(n-p-1) \text{ and } q = l-1,\ l, \\ 0 & \text{otherwise}. \end{cases}$$

Here $1 \le l < \dfrac{n-1}{n-p-1}$.

In other words, let $\bar{\mathcal{A}}$ be the exterior algebra generated by two forms: $\omega_1 = *\bar{d}A \in \bar{\Lambda}^{n-p-1}$ and $\omega_2 = \bar{d}_1(\omega_1) = *\bar{d} \in \bar{\Lambda}^{n-p-1} \otimes \mathcal{C}^1\Lambda$; then we see that the space $\bigoplus_{i,q\le n-2} E_1^{i,q}$ is isomorphic to the subspace of $\bar{\mathcal{A}}$ containing no forms of degree $q > n-2$.

7.5. **Conservation laws and generating functions.** We start by describing the differentials $d_1^{0,n-1}$ and $d_1^{1,n-1}$ for an ℓ-normal equation since they directly relate to the theory of conservation laws.

Suppose that an ℓ-normal equation $\mathcal{E} \subset J^k(\pi)$ is given by a section $F \in \mathcal{F}(\pi, \xi) = P$.

Proposition 7.8. *The operator*

$$d_1^{0,n-1} \colon E_1^{0,n-1}(\mathcal{E}) = \bar{H}^{n-1}(\mathcal{E}) \to E_1^{1,n-1}(\mathcal{E}) = \ker{(\ell_{\mathcal{E}})^*} \subset \hat{P}$$

has the form

$$d_1^{0,n-1}(h) = \square^*(1),$$

where $h = [\omega] \in \bar{H}^{n-1}(\mathcal{E})$, $\omega \in \bar{\Lambda}^{n-1}(\mathcal{E})$ *and* $\square \in \mathcal{C}\mathrm{Diff}(P, \bar{\Lambda}^n(\mathcal{E}))$ *is an operator satisfying* $\bar{d}\omega = \square(F)$.

Proof. We have $\bar{d} \circ \ell_\omega = \square \circ \ell_{\mathcal{E}}$. Thus $\square$ is an operator that represents the element $d_1^{0,n-1}(h) \in E_1^{1,n-1}(\mathcal{E})$. Hence $d_1^{0,n-1}(h) = \square^*(1)$. ☐

Proposition 7.9. *The term $E_1^{2,n-1}(\mathcal{E})$ can be described as the quotient*

$$\{\, \nabla \in \mathcal{C}\mathrm{Diff}(\varkappa, \hat{P}) \mid \ell_{\mathcal{E}}^* \circ \nabla = \nabla^* \circ \ell_{\mathcal{E}} \,\}/\theta,$$

where $\theta = \{\, \square \circ \ell_{\mathcal{E}} \mid \square \in \mathcal{C}\mathrm{Diff}(P, \hat{P}), \square^* = \square \,\}$.

Proof. Take a horizontal $(n-1)$-cocycle with coefficients in $\mathcal{C}^1\Lambda \otimes \mathcal{C}^1\Lambda$. Let an operator $\Delta \in \mathcal{C}\mathrm{Diff}(\varkappa, \hat{P})$ corresponds to this cocycle by Theorem 7.3 on page 118. Then there exists an operator $A \in \mathcal{C}\mathrm{Diff}(P, \hat{P})$ such that $\ell_{\mathcal{E}}^* \circ \Delta = A \circ \ell_{\mathcal{E}}$. By the Green formula we have

$$\langle \ell_{\mathcal{E}}^*(\Delta(\chi_1)), \chi_2 \rangle - \langle \Delta(\chi_1), \ell_{\mathcal{E}}(\chi_2) \rangle = \bar{d}(\Delta_1(\chi_1, \chi_2)),$$

where $\chi_1, \chi_2 \in \varkappa$, and $\Delta_1 \in \mathcal{C}\mathrm{Diff}_{(2)}(\varkappa, \Lambda^{n-1})$. The cocycle under consideration belongs to $E_1^{2,n-1}$, if the operator Δ_1 is skew-symmetric:

$$\Delta_1(\chi_1, \chi_2) = -\Delta_1(\chi_2, \chi_1) \bmod K,$$

where $K \subset \mathcal{C}\mathrm{Diff}_{(2)}(\varkappa, \bar{\Lambda}^{n-1})$ is the submodule consisting of the operators of the form $\gamma(\chi_1, \chi_2) = \gamma_1(\ell_{\mathcal{E}}(\chi_1), \chi_2) + \gamma_2(\ell_{\mathcal{E}}(\chi_2), \chi_1)$ for some operators $\gamma_1, \gamma_2 \in \mathcal{C}\mathrm{Diff}(P, \mathcal{C}\mathrm{Diff}(\varkappa, \bar{\Lambda}^{n-1}))$. In this case

$$\begin{aligned}\langle \ell_{\mathcal{E}}^*(\Delta(\chi_1)), \chi_2\rangle - \langle \Delta(\chi_1), \ell_{\mathcal{E}}(\chi_2)\rangle &= -\langle \ell_{\mathcal{E}}^*(\Delta(\chi_2)), \chi_1\rangle + \langle \Delta(\chi_2), \ell_{\mathcal{E}}(\chi_1)\rangle \\ &= -\langle \ell_{\mathcal{E}}^*(\Delta(\chi_2)), \chi_1\rangle + \langle \chi_2, \Delta^*(\ell_{\mathcal{E}}(\chi_1))\rangle \\ &= -\langle A(\ell_{\mathcal{E}}(\chi_2)), \chi_1\rangle + \langle \chi_2, \ell_{\mathcal{E}}^*(A^*(\chi_1))\rangle\end{aligned}$$

modulo $\bar{d}K$. This implies $\Delta = A^* + B \circ \ell_{\mathcal{E}}$ for an operator $B \in \mathcal{C}\mathrm{Diff}(P, \hat{P})$. One has $\ell_{\mathcal{E}}^* \circ B \circ \ell_{\mathcal{E}} = \ell_{\mathcal{E}}^* \circ \Delta - \ell_{\mathcal{E}}^* \circ A^* = \ell_{\mathcal{E}}^* \circ \Delta - \Delta^* \circ \ell_{\mathcal{E}}$, hence $B^* = -B$. Now we see that the operator $\nabla = \Delta - \frac{1}{2} B \circ \ell_{\mathcal{E}}$ satisfies $\ell_{\mathcal{E}}^* \circ \nabla = \nabla^* \circ \ell_{\mathcal{E}}$. The operator ∇ is defined modulo the operators of the form $\square \circ \ell_{\mathcal{E}}$. We have $\ell_{\mathcal{E}}^* \circ \square \circ \ell_{\mathcal{E}} = \ell_{\mathcal{E}}^* \circ \square^* \circ \ell_{\mathcal{E}}$, so that $\square^* = \square$. $\square$

Proposition 7.10. *The operator* $d_1^{1,n-1}\colon E_1^{1,n-1}(\mathcal{E}) = \ker \ell_{\mathcal{E}}^* \to E_1^{2,n-1}(\mathcal{E})$ *is given by*

$$d_1^{1,n-1}(\psi) = (\ell_\psi + \Delta^*) \bmod \theta,$$

where $\Delta \in \mathcal{C}\mathrm{Diff}(P, \hat{\varkappa})$ *is an operator satisfying* $\ell_F^*(\psi) = \Delta(F)$.

Proof. By Green's formula on $J^\infty(\pi)$ we have

$$\langle \psi, \ell_F(\chi)\rangle - \langle \ell_F^*(\psi), \chi\rangle = \bar{d}(\square(\chi)),$$

where $\chi \in \varkappa(\pi)$, $\square \in \mathcal{C}\mathrm{Diff}(\varkappa(\pi), \bar{\Lambda}^{n-1}(\pi)) = \mathcal{C}^1\Lambda(\pi) \otimes \bar{\Lambda}^{n-1}(\pi)$. Let us compute $\bar{d} \circ d_{\mathcal{C}}(\square) \in \mathcal{C}^2\Lambda(\pi) \otimes \bar{\Lambda}^n(\pi)$:

$$\begin{aligned}\bar{d}(d_{\mathcal{C}}(\square)(\chi_1, \chi_2)) &= \mathrm{Э}_{\chi_1}(\bar{d}(\square(\chi_2))) - \mathrm{Э}_{\chi_2}(\bar{d}(\square(\chi_1))) - \bar{d}(\square(\{\chi_1, \chi_2\})) \\ &= \mathrm{Э}_{\chi_1}(\langle \psi, \ell_F(\chi_2)\rangle) - \mathrm{Э}_{\chi_2}(\langle \psi, \ell_F(\chi_1)\rangle) - \langle \psi, \ell_F(\{\chi_1, \chi_2\})\rangle \\ &\quad - \mathrm{Э}_{\chi_1}(\langle \ell_F^*(\psi), \chi_2\rangle) + \mathrm{Э}_{\chi_2}(\langle \ell_F^*(\psi), \chi_1\rangle) + \langle \ell_F^*(\psi), \{\chi_1, \chi_2\}\rangle \\ &= \langle \ell_\psi(\chi_1), \ell_F(\chi_2)\rangle - \langle \ell_\psi(\chi_2), \ell_F(\chi_1)\rangle - \langle \ell_{\Delta(F)}(\chi_1), \chi_2\rangle + \langle \ell_{\Delta(F)}(\chi_2), \chi_1\rangle.\end{aligned}$$

Therefore, the restriction of $\bar{d} \circ d_{\mathcal{C}}(\square)$ to $\mathcal{E}^\infty$ equals to

$$\begin{aligned}&\bar{d} \circ d_{\mathcal{C}}(\square)\big|_{\mathcal{E}^\infty}(\chi_1, \chi_2) \\ &= \langle \ell_\psi(\chi_1), \ell_{\mathcal{E}}(\chi_2)\rangle - \langle \ell_\psi(\chi_2), \ell_{\mathcal{E}}(\chi_1)\rangle - \langle \Delta(\ell_{\mathcal{E}}(\chi_1)), \chi_2\rangle + \langle \Delta(\ell_{\mathcal{E}}(\chi_2)), \chi_1\rangle \\ &\quad = \langle (\ell_\psi + \Delta^*)(\chi_1), \ell_{\mathcal{E}}(\chi_2)\rangle - \langle (\ell_\psi + \Delta^*)(\chi_2), \ell_{\mathcal{E}}(\chi_1)\rangle + \bar{d}\gamma(\chi_1, \chi_2),\end{aligned}$$

where $\gamma \in K$. This completes the proof. $\square$

Now we apply these results to the problem of computing conservation laws of an ℓ-normal differential equation $\mathcal{E}$.

First, note that for a formally integrable equation $\mathcal{E}$ the projections $\mathcal{E}^{(k+1)} \to \mathcal{E}^{(k)}$ are affine bundles, therefore $\mathcal{E}^{(k+1)}$ and $\mathcal{E}^{(k)}$ are of the same homotopy type. Hence, $H^*(\mathcal{E}^\infty) = H^*(\mathcal{E})$.

Further, it follows from the two-line theorem that there exists the following exact sequence:

$$0 \longrightarrow H^{n-1}(\mathcal{E}) \longrightarrow \bar{H}^{n-1}(\mathcal{E}) \xrightarrow{d_1^{0,n-1}} \ker\left(\ell_{\mathcal{E}}\right)^* .$$

Recall that the group $\bar{H}^{n-1}(\mathcal{E})$ was interpreted as the group of conservation laws of the equation $\mathcal{E}$ (see the beginning of Section 6 on page 101). Conservation laws $\omega \in H^{n-1}(\mathcal{E}) \subset \bar{H}^{n-1}(\mathcal{E})$ are called *topological* (or *rigid*), since they are determined only by the topology of the equation $\mathcal{E}$. In particular, the corresponding conserved quantities do not change under deformations of solutions of the equation $\mathcal{E}$. Therefore topological conservation laws are not very interesting for us and we consider the quotient group $\mathrm{cl}(\mathcal{E}) = \bar{H}^{n-1}(\mathcal{E})/H^{n-1}(\mathcal{E})$, called the group of *proper* conservation laws of the equation $\mathcal{E}$. The two-line theorem implies immediately the following.

Theorem 7.11. *If $\mathcal{E}$ is an ℓ-normal equation, then*

$$\mathrm{cl}(\mathcal{E}) \subset \ker \ell_{\mathcal{E}}^*.$$

If, moreover, $H^n(\mathcal{E}) \subset \bar{H}^n(\mathcal{E})$ (in particular, $H^n(\mathcal{E}) = 0$), we have

$$\mathrm{cl}(\mathcal{E}) = \ker d_1^{1,n-1}.$$

Element $\psi \in \ker \ell_{\mathcal{E}}^*$ that corresponds to a conservation law $[\omega] \in \mathrm{cl}(\mathcal{E})$ is called its *generating function*.

Theorem 7.11 gives an effective method for computing conservation laws.

Remark 7.4. In view of Proposition 7.9 on page 122, elements of $E_1^{2,n-1}$ can be interpreted as mappings from $\ker \ell_{\mathcal{E}}^*$ to $\ker \ell_{\mathcal{E}}$, i.e., from generating functions of conservation laws to symmetries of $\mathcal{E}$.

Proposition 7.12. *Let $\mathcal{E} = \{u_t = f(t, x, u, u_x, u_{xx}, \dots)\}$ be an evolution equation and $J = (J_0, J_1, \dots, J_n)$ a conserved current for $\mathcal{E}$. Then the generating function of J is equal to $\psi = \mathbf{E}(J_0)$, where J_0 is the t-component of the conserved current that is regarded as a function of $(t, x, u, u_x, u_{xx}, \dots)$.*

Proof. The restriction of the total derivative D_t to the equation $\mathcal{E}^\infty$ has the form: $D_t = \dfrac{\partial}{\partial t} + \ni_f$. Hence $\dfrac{\partial J_0}{\partial t} + \ni_f(J_0) + \sum_{i=1}^n D_i(J_i) = 0$. On the other hand, $D_t = \dfrac{\partial}{\partial t} + \ni_{u_t}$, therefore $D_t(J_0) + \sum_{i=1}^n D_i(J_i) = \dfrac{\partial J_0}{\partial t} + \ni_{u_t}(J_0) - \dfrac{\partial J_0}{\partial t} - \ni_f(J_0) = \ni_{u_t - f}(J_0) = \ell_{J_0}(u_t - f)$. Thus $\psi = \ell_{J_0}^*(1) = \mathbf{E}(J_0)$. $\square$

Let $\varphi \in \ker \ell_{\mathcal{E}}$ be a symmetry and $[\omega] \in \bar{H}^{n-1}(\mathcal{E})$ a conservation law of the equation $\mathcal{E}$. Then $[\Im_\varphi(\omega)]$ is a conservation law of $\mathcal{E}$ as well.

Proposition 7.13. *If $\psi \in \ker \ell_{\mathcal{E}}^*$ is the generating function of a conservation law $[\omega]$ of an ℓ-normal equation $\mathcal{E} = \{F = 0\}$, then the generating function of the conservation law $[\Im_\varphi(\omega)]$ has the form $\Im_\varphi(\psi) + \Delta^*(\psi)$, where the operator $\Delta \in \mathcal{C}\mathrm{Diff}(P, P)$ is defined by $\Im_\varphi(F) = \Delta(F)$.*

Proof. First, we have

$$\langle \psi, \ell_{\mathcal{E}}(\chi) \rangle = \bar{d}\ell_\omega(\chi) + \bar{d}\gamma(\ell_{\mathcal{E}}(\chi)), \qquad \chi \in \varkappa,$$

where $\gamma \in \mathcal{C}\mathrm{Diff}(P, \bar{\Lambda}^{n-1})$. Using the obvious formula

$$\ell_{\Im_{\chi_1}(\eta)}(\chi_2) = \Im_{\chi_1}(\ell_\eta(\chi_2)) - \ell_\eta(\{\chi_1, \chi_2\}), \qquad \chi_1, \chi_1 \in \varkappa, \quad \eta \in \bar{\Lambda}^n,$$

where $\{\cdot, \cdot\}$ is the Jacobi bracket (see Definition 3.31 on page 66, we obtain

$$\begin{aligned}
\bar{d}\ell_{\Im_\varphi(\omega)}(\chi) &= \bar{d}(\Im_\varphi(\ell_\omega(\chi))) - \bar{d}(\ell_\omega(\{\varphi, \chi\})) = \Im_\varphi(\bar{d}(\ell_\omega(\chi))) - \bar{d}(\ell_\omega(\{\varphi, \chi\})) \\
&= \Im_\varphi(\langle \psi, \ell_{\mathcal{E}}(\chi) \rangle) - \langle \psi, \ell_{\mathcal{E}}(\{\varphi, \chi\}) \rangle - \Im_\varphi(\bar{d}\gamma(\ell_{\mathcal{E}}(\chi))) + \bar{d}\gamma(\ell_{\mathcal{E}}(\{\varphi, \chi\})) \\
&= \langle \Im_\varphi(\psi), \ell_{\mathcal{E}}(\chi) \rangle + \langle \psi, (\Im_\varphi(\ell_{\mathcal{E}}(\chi)) - \ell_{\mathcal{E}}(\{\varphi, \chi\})) \rangle + \bar{d}\gamma'(\ell_{\mathcal{E}}(\chi)) \\
&= \langle \Im_\varphi(\psi), \ell_{\mathcal{E}}(\chi) \rangle + \langle \psi, \ell_{\Im_\varphi(F)}\big|_{\mathcal{E}^\infty} \rangle + \bar{d}\gamma'(\ell_{\mathcal{E}}(\chi)) \\
&= \langle \Im_\varphi(\psi), \ell_{\mathcal{E}}(\chi) \rangle + \langle \psi, \Delta(\ell_{\mathcal{E}}(\chi)) \rangle + \bar{d}\gamma'(\ell_{\mathcal{E}}(\chi)) \\
&= \langle (\Im_\varphi + \Delta^*)(\psi), \ell_{\mathcal{E}}(\chi) \rangle + \bar{d}\gamma''(\ell_{\mathcal{E}}(\chi)),
\end{aligned}$$

where $\gamma', \gamma'' \in \mathcal{C}\mathrm{Diff}(P, \bar{\Lambda}^{n-1})$. This completes the proof. □

7.6. **Generating functions from the antifield-BRST standpoint.** A differential equation $\mathcal{E} = \{F = 0\}$, $F \in P$, is called *normal*, if any $\mathcal{C}$-differential operator Δ, such that $\Delta(F) = 0$, vanishes on $\mathcal{E}^\infty$. A normal equation is obviously ℓ-normal.

Consider a normal equation $\mathcal{E}$ and the complex on $J^\infty(\pi)$

$$0 \leftarrow \mathcal{F} \xleftarrow{\delta} \mathcal{C}\mathrm{Diff}(P, \mathcal{F}) \xleftarrow{\delta} \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(2)}(P, \mathcal{F}) \xleftarrow{\delta} \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(3)}(P, \mathcal{F}) \xleftarrow{\delta} \cdots,$$

$\delta(\Delta)(p_1, \dots, p_k) = \Delta(F, p_1, \dots, p_k)$, $p_i \in P$. This complex is exact in all terms except for the term $\mathcal{F}$. At points $\theta \in \mathcal{E}^\infty$, the exactness follows immediately from the normality condition. At points $\theta \notin \mathcal{E}^\infty$, this is a well known fact from linear algebra (see Example 8.5 on page 138). The homology in the term $\mathcal{F}$ is clearly equal to $\mathcal{F}(\mathcal{E})$.

In physics, this complex is said to be the *Koszul–Tate resolution*, and elements of $\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(k)}(P, \mathcal{F})$ are called *antifields*.

Consider the commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & \bar\Lambda^n & \xleftarrow{\delta} & \mathcal{C}\mathrm{Diff}(P,\bar\Lambda^n) & \xleftarrow{\delta} & \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(2)}(P,\bar\Lambda^n) & \xleftarrow{\delta} & \cdots \\
 & & \uparrow{\scriptstyle \bar d} & & \uparrow{\scriptstyle \bar d} & & \uparrow{\scriptstyle \bar d} & & \\
0 & \longleftarrow & \bar\Lambda^{n-1} & \xleftarrow{\delta} & \mathcal{C}\mathrm{Diff}(P,\bar\Lambda^{n-1}) & \xleftarrow{\delta} & \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(2)}(P,\bar\Lambda^{n-1}) & \xleftarrow{\delta} & \cdots \\
 & & \uparrow{\scriptstyle \bar d} & & \uparrow{\scriptstyle \bar d} & & \uparrow{\scriptstyle \bar d} & & \\
0 & \longleftarrow & \bar\Lambda^{n-2} & \xleftarrow{\delta} & \mathcal{C}\mathrm{Diff}(P,\bar\Lambda^{n-2}) & \xleftarrow{\delta} & \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(2)}(P,\bar\Lambda^{n-2}) & \xleftarrow{\delta} & \cdots \\
 & & \uparrow{\scriptstyle \bar d} & & \uparrow{\scriptstyle \bar d} & & \uparrow{\scriptstyle \bar d} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

From the standard spectral sequence arguments (see the Appendix) and Theorem 2.8 on page 32 it follows that $\bar H^q(\mathcal{E}) = H_{n-q}(L^{\mathrm{alt}}_\bullet(P),\delta)$. Since the complex $(L^{\mathrm{alt}}_\bullet(P),\delta)$ is a direct summand in the complex $(\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(\bullet)}(P,\hat P),\delta)$, it is exact in all degrees except for 0 and 1. This yields the *two-line theorem* for normal equations. We also get

$$\bar H^{n-1}(\mathcal{E}) = H_1(L^{\mathrm{alt}}_\bullet(P),\delta) = \{\, \psi \in \hat P \bmod T \mid \langle \psi, F\rangle \in \bar d\bar\Lambda^{n-1} \,\},$$

where $T = \{\, \psi \in \hat P \mid \psi = \square(F),\ \square \in \mathcal{C}\mathrm{Diff}(P,\hat P),\ \square^* = -\square \,\}$. The condition $\langle\psi, F\rangle \in \bar d\bar\Lambda^{n-1}$ is equivalent to $0 = \mathbf{E}\langle\psi, F\rangle = \ell^*_F(\psi) + \ell^*_\psi(F)$. So we again obtain the correspondence between conservation laws and generating functions together with the equation $\ell^*_{\mathcal{E}}(\psi) = 0$.

7.7. **Euler–Lagrange equations.** Consider the Euler–Lagrange equation $\mathcal{E} = \{\mathbf{E}(\mathcal{L}) = 0\}$ corresponding to a Lagrangian $\mathcal{L} = [\omega] \in \bar H^n(\pi)$. Let $\varphi \in \varkappa(\pi)$ be a *Noether symmetry* of $\mathcal{L}$, i.e., $\Game_\varphi(\mathcal{L}) = 0$ on $J^\infty(\pi)$.

Exercise 7.9. Using Exercise 7.7 on page 118, check that a Noether symmetry of $\mathcal{L}$ is a symmetry of the corresponding equation $\mathcal{E}$ as well, i.e., $\mathrm{sym}(\mathcal{L}) \subset \mathrm{sym}(\mathcal{E})$.

Exercise 7.10. Show that if $E_2^{0,n}(\mathcal{E}) = 0$, then finding of Noether symmetries of the Lagrangian $\mathcal{L} = [\omega]$ amounts to solution of the equation $\mathbf{E}(\ell_\omega(\varphi)) = \ell_{\mathbf{E}(\mathcal{L})}(\varphi) + \ell^*_\varphi(\mathbf{E}(\mathcal{L})) = 0$. (Thus, to calculate the Noether symmetries of an Euler–Lagrange equation one has no need to know the Lagrangian.)

Let $\Im_\varphi(\omega) = \bar{d}\nu$, where $\nu \in \bar{\Lambda}^{n-1}(\pi)$. By the Green formula we have

$$\Im_\varphi(\omega) - \bar{d}\nu = \ell_\omega(\varphi) - \bar{d}\nu = \ell_\omega^*(1)(\varphi) + \bar{d}\gamma(\varphi) - \bar{d}\nu \\ = \mathbf{E}(\mathcal{L})(\varphi) + \bar{d}(\gamma(\varphi) - \nu) = 0.$$

Set

$$\eta = (\nu - \gamma(\varphi))|_{\mathcal{E}^\infty} \in \bar{\Lambda}^{n-1}(\mathcal{E}).$$

Thus, $\bar{d}\,\eta|_{\mathcal{E}^\infty} = 0$, i.e., $[\eta] \in \bar{H}^{n-1}(\mathcal{E})$ is a conservation law of the equation $\mathcal{E}$. The map

$$\operatorname{sym}(\mathcal{L}) \to \bar{H}^{n-1}(\mathcal{E}), \qquad \varphi \mapsto [\eta],$$

is said to be the *Noether map*.

An arbitrariness in the choice of ω and ν leads to the multivaluedness of the Noether map.

Exercise 7.11. Check that the Noether map is well defined up to the image of the natural homomorphism $\bar{H}^{n-1}(\pi) \to \bar{H}^{n-1}(\mathcal{E})$.

Proposition 7.14. *If the Euler–Lagrange equation $\mathcal{E}$ corresponding to a Lagrangian $\mathcal{L}$ is ℓ-normal, then the Noether map considered on the set of Noether symmetries of $\mathcal{L}$ is inverse to the differential $d_1^{0,n-1}$.*

Proof. On $J^\infty(\pi)$ we have

$$\bar{d}\ell_\eta(\chi) = \ell_{\langle \mathbf{E}(\mathcal{L}),\varphi\rangle}(\chi) = \langle \ell_{\mathbf{E}(\mathcal{L})}(\chi), \varphi\rangle + \langle \mathbf{E}(\mathcal{L}), \ell_\varphi(\chi)\rangle.$$

Therefore on $\mathcal{E}^\infty$ we obtain $\bar{d}\ell_\eta(\chi) = \langle \ell_{\mathcal{E}}(\chi), \varphi\rangle$, i.e., $d_1^{0,n-1}([\eta]) = \varphi$. ☐

Remark 7.5. The Noether map can be understood as a procedure for finding a conserved current corresponding to a given generating function.

Thus, we see that if $\Im_\varphi$ is a Noether symmetry of a Lagrangian, then φ is the generating function of a conservation law for the corresponding Euler–Lagrange equation. This is the (first) *Noether theorem.* Note that since for Euler–Lagrange equations one has $\ell_{\mathcal{E}}^* = \ell_{\mathcal{E}}$, the inverse Noether theorem is obvious: if φ is the generating function of a conservation law for an Euler–Lagrange equation, then φ is a symmetry for this equation.

Let us discuss the Noether theorem from the antifield-BRST point of view. Consider a 1-cycle $\varphi \in \varkappa$ of the complex $L_\bullet^{\mathrm{alt}}(\hat{\varkappa})$. We have $\langle \varphi, \mathbf{E}(\omega)\rangle \in \bar{d}\bar{\Lambda}^{n-1}$, where ω is a density of the Lagrangian $\mathcal{L} = [\omega]$. Hence $\Im_\varphi(\omega) \in \bar{d}\bar{\Lambda}^{n-1}$ and, therefore, $\Im_\varphi(\mathcal{L}) = 0$, i.e., φ is a Noether symmetry. Thus, the Koszul–Tate resolution gives a homological interpretation of the Noether theorem.

Now, suppose that the Lagrangian has a *gauge symmetry*, i.e., there exist an $\mathcal{F}$-module $\mathfrak{a}$ and a $\mathcal{C}$-differential operator $R\colon \mathfrak{a} \to \varkappa$ such that $R(\alpha)$ is a Noether symmetry for any $\alpha \in \mathfrak{a}$. This means that $\Im_{R(\alpha)}(\mathcal{L}) = 0$ or

$\ell_{\mathcal{L}} \circ R = 0$. Hence $R^* \circ \ell^*_{\mathcal{L}} = 0$ and, finally, $R^*(\ell^*_{\mathcal{L}}(1)) = R^*(\mathbf{E}(\mathcal{L})) = 0$. Thus, if the Lagrangian is invariant under a gauge symmetry, then the*Noether identities* $R^*(\mathbf{E}(\mathcal{L})) = 0$ between the Euler–Lagrange equations hold (the second Noether theorem).

7.8. **The Hamiltonian formalism on $J^\infty(\pi)$.** Let $A \in \mathcal{C}\mathrm{Diff}(\hat{\varkappa}(\pi), \varkappa(\pi))$ be a $\mathcal{C}$-differential operator. Define the *Poisson bracket* on $\bar{H}^n(\pi)$ corresponding to the operator A by the formula

$$\{\omega_1, \omega_2\}_A = \langle A(\mathbf{E}(\omega_1)), \mathbf{E}(\omega_2)\rangle,$$

where $\langle\, ,\, \rangle$ denotes the natural pairing $\varkappa(\pi) \times \hat{\varkappa}(\pi) \to \bar{H}^n(\pi)$.

The lemma below shows that the operator A is uniquely determined by the corresponding Poisson bracket.

Lemma 7.15. *Let $\pi \colon E \to M$ be a vector bundle.*

(1) *Consider an operator $A \in \mathcal{C}\mathrm{Diff}_{(l)}(\hat{\varkappa}(\pi), P)$, where P is an $\mathcal{F}(\pi)$-module. If for all $\omega_1, \dots, \omega_l \in \bar{H}^n(\pi)$ one has*

$$A(\mathbf{E}(\omega_1), \dots, \mathbf{E}(\omega_l)) = 0,$$

then $A = 0$.

(2) *Consider an operator $A \in \mathcal{C}\mathrm{Diff}_{(l)}(\hat{\varkappa}(\pi), \bar{\Lambda}^n(\pi))$. If for all cohomology classes $\omega_1, \dots, \omega_l \in \bar{H}^n(\pi)$ the element $A(\mathbf{E}(\omega_1), \dots, \mathbf{E}(\omega_l))$ belongs to the image of $\bar{d}$, then $\operatorname{im} A \subset \operatorname{im} \bar{d}$, i.e., $\mu_{(l-1)}(A) = 0$ (see Subsection 2.4).*

(3) *Consider an operator $A \in \mathcal{C}\mathrm{Diff}_{(l-1)}(\hat{\varkappa}(\pi), \varkappa(\pi))$. If for all elements $\omega_1, \dots, \omega_l \in \bar{H}^n(\pi)$ one has $\langle A(\mathbf{E}(\omega_1), \dots, \mathbf{E}(\omega_{l-1})), \mathbf{E}(\omega_l)\rangle = 0$, then $A = 0$.*

Proof. (1) It suffices to consider the case $l = 1$. Obviously, on $J^\infty(\pi)$ every element of $\hat{\varkappa}(\pi) = \hat{\mathcal{F}}(\pi, \pi)$ of the form $\pi^*(f)$, with $f \in \hat{\Gamma}(\pi)$, (in other words, every element of $\hat{\varkappa}(\pi)$ depending on base coordinates x only) can locally be presented in the form $\pi^*(f) = \mathbf{E}(\omega)$ for some $\omega \in \bar{\Lambda}^n(\pi)$. Thus $A(\pi^*(f)) = 0$ for all f. Since A is a $\mathcal{C}$-differential operator, this implies $A = 0$.

(2) It is also sufficient to consider the case $l = 1$. We have $\mathbf{E}(A(\mathbf{E}(\omega))) = 0$. Using Exercise 7.7 on page 118, we get

$$0 = \mathbf{E}(A(\mathbf{E}(\omega))) = \ell^*_{\mathbf{E}(\omega)}(A^*(1)) + \ell^*_{A^*(1)}(\mathbf{E}(\omega))$$

for all $\omega \in \bar{\Lambda}^n(\pi)$. As above, we see that for any $f \in \hat{\Gamma}(\pi)$ there exists $\omega \in \bar{\Lambda}^n(\pi)$ such that $\pi^*(f) = \mathbf{E}(\omega)$. Since $\ell_{\pi^*(f)} = 0$, we obtain $\ell^*_{A^*(1)}(\pi^*(f)) = 0$. Hence $\ell^*_{A^*(1)} = 0$, so that $0 = \mathbf{E}(A(\mathbf{E}(\omega))) = \ell^*_{\mathbf{E}(\omega)}(A^*(1))$.

Exercise 7.12. Prove that locally there exists a form $\omega \in \bar{\Lambda}^n(\pi)$ such that $\ell_{\mathbf{E}(\omega)}$ is the identity operator.

Using this exercise, we get $0 = A^*(1) = \mu(A)$, which is our claim.

(3) The assertion follows immediately from (1) and (2) above. $\square$

Definition 7.1. An operator $A \in \mathcal{C}\mathrm{Diff}(\hat{\varkappa}(\pi), \varkappa(\pi))$ is called *Hamiltonian*, if its Poisson bracket defines a Lie algebra structure on $\bar{H}^n(\pi)$, i.e., if

$$\{\omega_1, \omega_2\}_A = -\{\omega_2, \omega_1\}_A, \tag{7.4}$$

$$\{\{\omega_1, \omega_2\}_A, \omega_3\}_A + \{\{\omega_2, \omega_3\}_A, \omega_1\}_A + \{\{\omega_3, \omega_1\}_A, \omega_2\}_A = 0. \tag{7.5}$$

The bracket $\{\,,\}_A$ is said to be a *Hamiltonian structure*.

Proposition 7.16. *The Poisson bracket* $\{\,,\}_A$ *is skew-symmetric, i.e., condition* (7.4) *holds, if and only if the operator* A *is skew-adjoint, i.e.,* $A = -A^*$.

Proof. Since

$$\{\omega_1, \omega_2\}_A + \{\omega_2, \omega_1\}_A = \langle (A + A^*)(\mathbf{E}(\omega_1)), \mathbf{E}(\omega_2)\rangle,$$

the claim follows immediately from the previous lemma. $\square$

Now we shall prove criteria for checking an arbitrary skew-adjoint operator $A \in \mathcal{C}\mathrm{Diff}(\hat{\varkappa}(\pi), \varkappa(\pi))$ to be Hamiltonian. For this, we need the following

Lemma 7.17. *Consider an operator* $A \in \mathcal{C}\mathrm{Diff}(\hat{\varkappa}(\pi), \varkappa(\pi))$ *and an element* $\psi \in \hat{\varkappa}(\pi)$. *Define the operator* $\ell_{A,\psi} \in \mathcal{C}\mathrm{Diff}(\varkappa(\pi), \varkappa(\pi))$ *by*

$$\ell_{A,\psi}(\varphi) = (\ell_A(\varphi))(\psi) \qquad \varphi \in \varkappa(\pi).$$

Then

$$\ell^*_{A,\psi_1}(\psi_2) = \ell^*_{A^*,\psi_2}(\psi_1). \tag{7.6}$$

Proof. By the Green formula,

$$\langle A(\psi_1), \psi_2\rangle = \langle \psi_1, A^*(\psi_2)\rangle.$$

Applying $\mathrm{Э}_\varphi$ to both sides, we get

$$\langle \mathrm{Э}_\varphi(A)(\psi_1), \psi_2\rangle = \langle \psi_1, \mathrm{Э}_\varphi(A^*)(\psi_2)\rangle,$$

and so

$$\langle \ell_{A,\psi_1}(\varphi), \psi_2\rangle = \langle \psi_1, \ell_{A^*,\psi_2}(\varphi)\rangle.$$

Again the Green formula yields

$$\langle \varphi, \ell^*_{A,\psi_1}(\psi_2)\rangle = \langle \ell^*_{A^*,\psi_2}(\psi_1), \varphi\rangle,$$

and the lemma is proved. $\square$

Theorem 7.18. *Let* $A \in \mathcal{C}\mathrm{Diff}(\hat{\varkappa}(\pi), \varkappa(\pi))$ *be a skew-adjoint operator; then the following conditions are equivalent:*

(1) *A is a Hamiltonian operator;*

(2) $\langle \ell_A(A(\psi_1))(\psi_2), \psi_3\rangle + \langle \ell_A(A(\psi_2))(\psi_3), \psi_1\rangle + \langle \ell_A(A(\psi_3))(\psi_1), \psi_2\rangle = 0$ *for all* $\psi_1, \psi_2, \psi_3 \in \hat{\varkappa}(\pi)$;
(3) $\ell_{A,\psi_1}(A(\psi_2)) - \ell_{A,\psi_2}(A(\psi_1)) = A(\ell^*_{A,\psi_2}(\psi_1))$ *for all* $\psi_1, \psi_2 \in \hat{\varkappa}(\pi)$;
(4) *the expression* $\ell_{A,\psi_1}(A(\psi_2)) + \frac{1}{2}A(\ell^*_{A,\psi_1}(\psi_2))$ *is symmetric with respect to* $\psi_1, \psi_2 \in \hat{\varkappa}(\pi)$;
(5) $[\ni_{A(\psi)}, A] = \ell_{A(\psi)} \circ A + A \circ \ell^*_{A(\psi)}$ *for all* $\psi \in \operatorname{im}\mathbf{E} \subset \hat{\varkappa}(\pi)$.

Moreover, it is sufficient to verify conditions (2)–(4) *for elements* $\psi_i \in \operatorname{im}\mathbf{E}$ *only.*

Proof. Let $\omega_1, \omega_2, \omega_3 \in \bar{H}^n(\pi)$ and $\psi_i = \mathbf{E}(\omega_i)$. The Jacobi identity (7.5) on the page before yields

$$\begin{aligned}
\oint \{\{\omega_1, \omega_2\}_A, \omega_3\}_A &= \oint -\ni_{A(\psi_3)}\langle A(\psi_1), \psi_2\rangle \\
= \oint -\langle \ni_{A(\psi_3)}(A)(\psi_1), \psi_2\rangle &- \langle A(\ell_{\psi_1}(A(\psi_3))), \psi_2\rangle - \langle A(\psi_1), \ell_{\psi_2}(A(\psi_3))\rangle \\
= \oint -\langle \ell_A(A(\psi_3))(\psi_1), \psi_2\rangle &+ \langle A(\psi_2), \ell_{\psi_1}(A(\psi_3))\rangle - \langle A(\psi_1), \ell_{\psi_2}(A(\psi_3))\rangle \\
&= \oint -\langle \ell_A(A(\psi_3))(\psi_1), \psi_2\rangle = 0,
\end{aligned}$$

where as above the symbol $\oint$ denotes the sum of cyclic permutations. It follows from Lemma 7.15 on page 128 that this formula holds for all $\psi_i \in \hat{\varkappa}(\pi)$. Criterion (2) is proved.

Rewrite the Jacobi identity in the form

$$\langle \ell_{A,\psi_1}(A(\psi_2)), \psi_3\rangle + \langle A(\psi_1), \ell^*_{A,\psi_3}(\psi_2)\rangle - \langle A(\ell^*_{A,\psi_2}(\psi_1)), \psi_3\rangle = 0.$$

Using (7.6) on the page before, we obtain

$$\langle \ell_{A,\psi_1}(A(\psi_2)), \psi_3\rangle - \langle \ell_{A,\psi_2}(A(\psi_1)), \psi_3\rangle - \langle A(\ell^*_{A,\psi_2}(\psi_1)), \psi_3\rangle = 0,$$

which implies criterion (3).

The equivalence of criteria (3) and (4) follows from (7.6) on the preceding page.

Finally, criterion (5) is equivalent to criterion (3) by virtue of the following obvious equalities:

$$[\ni_{A(\psi_2)}, A](\psi_1) = \ell_{A,\psi_1}(A(\psi_2)),$$
$$\ell_{A,\psi} \circ A = \ell_{A(\psi)} \circ A - A \circ \ell_\psi \circ A.$$

This concludes the proof. ☐

Example 7.2. Consider a skew-symmetric differential operator $\Delta\colon \hat{\Gamma}(\pi) \to \Gamma(\pi)$. Then its lifting (see Definition 3.25 on page 59) $\mathcal{C}\Delta\colon \hat{\varkappa}(\pi) \to \varkappa(\pi)$ is obviously a Hamiltonian operator.

Exercise 7.13. Check that in the case $n = \dim M = 1$ and $m = \dim \pi = 1$ operators of the form $A = D_x^3 + (\alpha + \beta u)D_x + \dfrac{\beta}{2}u_x$ are Hamiltonian.

Let $A\colon \hat{\varkappa}(\pi) \to \varkappa(\pi)$ be a Hamiltonian operator. For any $\omega \in \bar{H}^n(\pi)$ the evolutionary vector field $X_\omega = \mathsf{Э}_{A(\mathbf{E}(\omega))}$ is called *Hamiltonian* vector field corresponding to the Hamiltonian ω. Obviously,

$$X_{\omega_1}(\omega_2) = \langle A\mathbf{E}(\omega_1), \mathbf{E}(\omega_2)\rangle = \{\omega_1, \omega_2\}_A.$$

This yields

$$\begin{aligned} X_{\{\omega_1,\omega_2\}_A}(\omega) &= \{\{\omega_1, \omega_2\}_A, \omega\}_A = \{\omega_1, \{\omega_2, \omega\}_A\}_A - \{\omega_2, \{\omega_1, \omega\}_A\}_A \\ &= (X_{\omega_1} \circ X_{\omega_2} - X_{\omega_2} \circ X_{\omega_1})(\omega) = [X_{\omega_1}, X_{\omega_2}](\omega) \end{aligned}$$

for all $\omega \in \bar{H}^n(\pi)$. Thus

$$X_{\{\omega_1,\omega_2\}_A} = [X_{\omega_1}, X_{\omega_2}]. \tag{7.7}$$

As with the finite dimensional Hamiltonian formalism, 7.7 implies a result similar to the Noether theorem.

For each $\mathcal{H} \in \bar{H}^n(\pi)$, the evolution equation

$$u_t = A(\mathbf{E}(\mathcal{H})), \tag{7.8}$$

corresponding to the Hamiltonian $\mathcal{H}$ is called *Hamiltonian* evolution equation.

Example 7.3. The KdV equation $u_t = uu_x + u_{xxx}$ admits two Hamiltonian structures:

$$u_t = D_x\left(\mathbf{E}\left(\frac{u^3}{6} - \frac{u_x^2}{2}\right)\right)$$

and

$$u_t = \left(D_x^3 + \frac{2}{3}uD_x + \frac{1}{3}u_x\right)\left(\mathbf{E}\left(\frac{u^2}{2}\right)\right).$$

Theorem 7.19. *Hamiltonian operators take the generating function of a conservation law of equation* (7.8) *to the symmetry of this equation.*

Proof. Let A be a Hamiltonian operator and

$$\tilde{\omega}_0(t) + \tilde{\omega}_1(t) \wedge dt \in \bar{\Lambda}^n(\pi) \oplus \bar{\Lambda}^{n-1}(\pi) \wedge dt$$

be a conserved current of equation (7.8). This means that $D_t(\omega_0(t)) = 0$, where $\omega_0(t) \in \bar{H}^n(\pi)$ is the horizontal cohomology class corresponding to the form $\tilde{\omega}_0(t)$, and D_t is the restriction of the total derivative in t to the equation. Further,

$$D_t(\omega_0) = \frac{\partial \omega_0}{\partial t} + \mathsf{Э}_{A(\mathbf{E}(\mathcal{H}))}(\omega_0) = \frac{\partial \omega_0}{\partial t} + \{\mathcal{H}, \omega_0\}.$$

This yields

$$\frac{\partial}{\partial t} X_{\omega_0} + [X_{\mathcal{H}}, X_{\omega_0}] = 0.$$

Hence $X_{\omega_0} = \ni_{A(\mathbf{E}(\omega_0))}$ is a symmetry of (7.8) on the page before. It remains to recall that $\mathbf{E}(\omega_0)$ is the generating function of the conservation law under consideration (see Proposition 7.12 on page 124). ☐

Remark 7.6. Thus Hamiltonian operators are in a sense dual to elements of $E_1^{2,n-1}$ (cf. Remark 7.4 on page 124).

7.9. **On superequations.** The theory of this and preceding sections is based on the pure algebraic considerations in Sections 1 and 2. Therefore all results remain valid for the case of differential superequations, provided one inserts the minus sign where appropriate (detailed geometric definitions of superjets, super Cartan distribution, and so on the reader can find, for example, in [44, 45]). So we discuss here only a couple of somewhat less obvious points and the coordinates formula.

Let M be a supermanifold, $\dim M = n|m$, and π be a superbundle over M, $\dim \pi = s|t$. The following theorem is the superanalog of theorem 2.2 on page 28.

Theorem 7.20. (1) $\hat{A}_s = 0$ *for* $s \neq n$.

(2) $\hat{A}_n$ *is the module of sections for the bundle* $\mathrm{Ber}(M)$*, the latter being defined as follows: locally, sections of* $\mathrm{Ber}(M)$ *are written in the form* $f(x)\mathbf{D}(x)$*, where* $f \in C^\infty(\mathcal{U})$ *and* $\mathbf{D}$ *is a basis local section that is multiplied by the Berezin determinant of the Jacobi matrix under the change of coordinates. The Berezin determinant of an even matrix* $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *is equal to* $\det(A - BD^{-1}C)(\det D)^{-1}$.

Proof. The assertion is local, so we can consider the domain $\mathcal{U}$ with local coordinates $x = (y_i, \xi_j)$, $i = 1, \dots, n$, $j = 1, \dots, m$, and split the complex (2.1) on page 27 $\mathrm{Diff}^+(\Lambda^*)$ in the tensor product of complexes $\mathrm{Diff}^+(\Lambda^*)_{\mathrm{even}} \otimes \mathrm{Diff}^+(\Lambda^*)_{\mathrm{odd}}$, where $\mathrm{Diff}^+(\Lambda^*)_{\mathrm{even}}$ is complex (2.1) on the underlying even domain of $\mathcal{U}$ and $\mathrm{Diff}^+(\Lambda^*)_{\mathrm{odd}}$ is the same complex for the Grassmann algebra in variables $\xi_1, \dots, \xi_m$.

We have $H^i(\mathrm{Diff}^+(\Lambda^*)_{\mathrm{even}}) = 0$ for $i \neq n$ and $H^i(\mathrm{Diff}^+(\Lambda^*)_{\mathrm{even}}) = \Lambda^n_{\mathcal{U}}$, where $\Lambda^n_{\mathcal{U}}$ is the module of n-form on the underlying even domain of $\mathcal{U}$. To compute the cohomology of $\mathrm{Diff}^+(\Lambda^*)_{\mathrm{odd}}$ consider the quotient complexes

$$0 \longrightarrow \mathrm{Smbl}_k(A)_{\mathrm{odd}} \longrightarrow \mathrm{Smbl}_{k+1}(\Lambda^1)_{\mathrm{odd}} \longrightarrow \cdots,$$

where $\mathrm{Smbl}_k(P)_{\mathrm{odd}} = \mathrm{Diff}^+_k(P)_{\mathrm{odd}}/\mathrm{Diff}^+_{k-1}(P)_{\mathrm{odd}}$. Then an easy calculation shows that these complexes are the Koszul complexes, hence $H^i(\mathrm{Diff}^+(\Lambda^*))_{\mathrm{odd}} = 0$ for $i > 0$ and $H^0(\mathrm{Diff}^+(\Lambda^*))$ is a module of rank

1. Therefore $\hat{A}_i = H^i(\mathrm{Diff}^+(\Lambda^*)) = 0$ for $i \neq n$ and the only operators that represent non-trivial cocycles have the form $dy_1 \wedge \cdots \wedge dy_n \dfrac{\partial^m}{\partial\xi_1 \cdots \partial\xi_m} f(y,\xi)$.

To complete the proof it remains to check that $\hat{A}_n$ is precisely $\mathrm{Ber}(M)$, i.e., that changing coordinates we obtain:

$$dy_1 \wedge \cdots \wedge dy_n \frac{\partial^m}{\partial\xi_1 \ldots \partial\xi_m} f \\ = dv_1 \wedge \cdots \wedge dv_n \frac{\partial^m}{\partial\eta_1 \ldots \partial\eta_m} f \mathrm{Ber}\left(J\left(\frac{x}{z}\right)\right) + T,$$

where $z = (v_i, \eta_j)$ is a new coordinate system on $\mathcal{U}$, Ber denotes the Berezin determinant, $J\left(\dfrac{x}{z}\right)$ is the Jacobi matrix, T is cohomologous to zero. This is an immediate consequence of the following well known formula for the Berezin determinant: $\mathrm{Ber}\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) = \det A \cdot \det \widetilde{D}$, where $\widetilde{D}$ is defined by $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{smallmatrix}\right)$. □

The coordinate expression for the adjoint operator is as follows. Let $\Delta \in \mathrm{Diff}(A, \mathcal{B})$ be a scalar operator $\Delta = \sum_\sigma \mathbf{D} a_\sigma \dfrac{\partial^{|\sigma|}}{\partial x_\sigma}$. Then

$$\Delta^* = \sum_\sigma (-1)^{|\sigma| + a_\sigma x_\sigma} \mathbf{D} \frac{\partial^{|\sigma|}}{\partial x_\sigma} \circ a_\sigma.$$

Here the symbol of an object used in exponent denotes the parity of the object.

Now, consider a matrix operator $\Delta \colon P \to Q$, $\Delta = \|\Delta^i_j\|$, where the matrix elements are defined by the equalities $\Delta(\sum_\alpha e_\alpha f^\alpha) = \sum_{\alpha,\beta} e'_\alpha \Delta^\alpha_\beta(f^\beta)$, $\{e_i\}$ is a basis in P, $\{e'_i\}$ is a basis in Q. If $\mathbf{D}$ is even, then Δ^* has the form

$$\mathbf{D}(\Delta^*)^i_j = (-1)^{(e_i + e'_j)(\Delta + e_i)} (\mathbf{D}\Delta^j_i)^*.$$

If $\mathbf{D}$ is odd, then

$$\mathbf{D}((\Delta^*)^\Pi)^i_j = (-1)^{(e_i + \Delta)(e'_j + 1) + \Delta e_i} (\mathbf{D}\Delta^j_i)^*,$$

where $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)^\Pi = \left(\begin{smallmatrix} D & C \\ B & A \end{smallmatrix}\right)$is the Π-transposition.

Remark 7.7. One has $(\Delta^{**})^i_j = (-1)^{e_i + e'_j} \Delta^i_j$.

Remark 7.8. There is one point where we need to improve the algebraic theory of differential operators to extend it to the supercase. This is the definition of geometrical modules that should read:

Definition 7.2. A module P over $C^\infty(M)$ is called *geometrical*, if

$$\bigcap_{\substack{x \in M_{\mathrm{rd}} \\ k \geq 1}} \mu_x^k P = 0,$$

where M_{rd} is the underlying even manifold of M and μ_x is the ideal in $C^\infty(M)$ consisting of functions vanishing at point $x \in M_{\mathrm{rd}}$.

Appendix: Homological algebra

In this appendix we sketch the basics of homological algebra. For an extended discussion see, e.g., [37, 20, 7, 41, 8].

8.1. **Complexes.** A sequence of vector spaces over a field $\Bbbk$ and linear mappings

$$\cdots \longrightarrow K^{i-1} \xrightarrow{d^{i-1}} K^i \xrightarrow{d^i} K^{i+1} \xrightarrow{d^{i+1}} \cdots$$

is said to be a *complex* if the composition of any two neighboring arrows is the zero map: $d^i \circ d^{i-1} = 0$.

The maps d^i are called *differentials*. The index i is often omitted, so that the definition of a complex reads: $d^2 = 0$.

By definition, $\operatorname{im} d^{i-1} \subset \ker d^i$. The complex $(K^\bullet, d^\bullet)$ is called *exact* (or *acyclic*) in degree i, if $\operatorname{im} d^{i-1} = \ker d^i$. A complex exact in all degrees is called *acyclic* (or *exact*, or an *exact sequence*).

Example 8.1. The sequence $0 \longrightarrow L \xrightarrow{f} K$ is always a complex. It is acyclic if and only if f is injection. The sequence $K \xrightarrow{g} M \longrightarrow 0$ is always a complex, as well. It is acyclic if and only if g is surjection.

The sequence

$$0 \longrightarrow L \xrightarrow{f} K \xrightarrow{g} M \longrightarrow 0 \tag{8.1}$$

is a complex, if $g \circ f = 0$. It is exact, if and only if f is injection, g is surjection, and $\operatorname{im} f = \ker g$. In this case we can identify L with a subspace of K and M with the quotient space K/L. Exact sequence (8.1) is called a *short exact sequence* (or an *exact triple*).

Example 8.2. The *de Rham complex* is the complex of differential forms on a smooth manifold M with respect to the exterior derivation:

$$\cdots \longrightarrow \Lambda^{i-1} \xrightarrow{d} \Lambda^i \xrightarrow{d} \Lambda^{i+1} \xrightarrow{d} \cdots .$$

The *cohomology* of a complex $(K^\bullet, d^\bullet)$ is the family of the spaces

$$H^i(K^\bullet, d^\bullet) = \ker d^i / \operatorname{im} d^{i-1}.$$

Thus, the equality $H^i(K^\bullet, d^\bullet) = 0$ means that the complex $(K^\bullet, d^\bullet)$ is acyclic in degree i. Note that for the sake of brevity the cohomology is often denoted by $H^i(K^\bullet)$ or $H^i(d^\bullet)$. Elements of $\ker d^i \subset K^i$ are called i-dimensional *cocycles*, elements of $\operatorname{im} d^{i-1} \subset K^i$ are called i-dimensional *coboundaries*. Thus, the cohomology is the quotient space of the space of all cocycles by the subspace of all coboundaries. Two cocycles k_1 and k_2 from common cohomology coset, i.e., such that $k_1 - k_2 \in \operatorname{im} d^{i-1}$, are called *cohomologous*.

Remark 8.1. In the case of the complex of differential forms on a manifold cocycles are called *closed forms*, and coboundaries are called *exact forms*.

Remark 8.2. It is clear that the definition of a complex can be immediately generalized to modules over a ring instead of vector spaces.

Exercise 8.1. Prove that if

$$\cdots \longrightarrow Q^{i-1} \xrightarrow{d^{i-1}} Q^i \xrightarrow{d^i} Q^{i+1} \xrightarrow{d^{i+1}} \cdots$$

is a complex of modules (and d^i are homomorphisms) and P is a projective module, then $H^i(Q^\bullet \otimes P) = H^i(Q^\bullet) \otimes P$.

Complexes defined above are called *cochain* to stress that the differentials raise the dimension by 1. Inversion of arrows gives *chain* complexes

$$\cdots \xleftarrow{d_{i-1}} K_{i-1} \xleftarrow{d_i} K_i \xleftarrow{d_{i+1}} K_{i+1} \leftarrow \cdots,$$

homology, *cycles*, *boundaries*, etc. The difference between these types of complex is pure terminological, so we shall mainly restrict our considerations to cochain complexes.

A *morphism* (or a *cochain map*) of complexes $f\colon K^\bullet \to L^\bullet$ is the family of linear mappings $f^i\colon K^i \to L^i$ that commute with differentials, i.e., that make the following diagram commutative:

$$\begin{tikzcd}
\cdots \arrow[r] & K^{i-1} \arrow[r, "d_K^{i-1}"] \arrow[d, "f^{i-1}"] & K^i \arrow[r, "d_K^i"] \arrow[d, "f^i"] & K^{i+1} \arrow[r, "d_K^{i+1}"] \arrow[d, "f^{i+1}"] & \cdots \\
\cdots \arrow[r] & L^{i-1} \arrow[r, "d_L^{i-1}"] & L^i \arrow[r, "d_L^i"] & L^{i+1} \arrow[r, "d_L^{i+1}"] & \cdots .
\end{tikzcd}$$

Such a morphism induces the map $H^i(f)\colon H^i(K^\bullet) \to H^i(L^\bullet)$, $[k] \mapsto [f(k)]$, where k is a cocycle and $[\ \cdot\ ]$ denotes the cohomology coset. Clearly, $H^i(f \circ g) = H^i(f) \circ H^i(g)$ (so that H^i is a functor from the category of complexes to the category of vector spaces). A morphism of complexes is called *quasiisomorphism* (or *homologism*) if it induces an isomorphism of cohomologies.

Example 8.3. A smooth map of manifolds $F\colon M_1 \to M_2$ gives rise to the map of differential forms $F^*\colon \Lambda^\bullet(M_2) \to \Lambda^\bullet(M_1)$, such that $d(F^*(\omega)) = F^*(d(\omega))$. Thus F^* is a cochain map and induces the map of the de Rham cohomologies $F^*\colon H^\bullet(M_2) \to H^\bullet(M_1)$. In particular, if M_1 and M_2 are diffeomorphic, then their de Rham cohomologies are isomorphic.

Exercise 8.2. Check that the wedge product on differential forms on M induces a well-defined multiplication on the de Rham cohomology $H^*(M) = \bigoplus_i H^i(M)$, which makes the de Rham cohomology a (*super*)*algebra*, and not just a vector space. Show that for diffeomorphic manifolds these algebras are isomorphic.

Two morphisms of complexes $f^\bullet, g^\bullet \colon K^\bullet \to L^\bullet$ are called *homotopic* if there exist mappings $s^i \colon K^i \to L^{i-1}$, such that

$$f^i - g^i = s^{i+1}d^i + d^{i-1}s^i.$$

The mappings s^i are called (*cochain*) *homotopy.*

Proposition 8.1. *If morphisms $f^\bullet$ and $g^\bullet$ are homotopic, then $H^i(f^\bullet) = H^i(g^\bullet)$ for all i.*

Proof. Consider a cocycle $z \in K^i$, $dz = 0$. Then

$$f(z) - g(z) = (sd + ds)(z) = d(s(z)).$$

Thus, $f(z)$ and $g(z)$ are cohomologous, and so $H^i(f^\bullet) = H^i(g^\bullet)$. □

Two complexes $K^\bullet$ and $L^\bullet$ are said to be *cochain equivalent* if there exist morphisms $f^\bullet \colon K^\bullet \to L^\bullet$ and $g^\bullet \colon L^\bullet \to K^\bullet$ such that $g \circ f$ is homotopic to $\mathrm{id}_{K^\bullet}$ and $f \circ g$ is homotopic to $\mathrm{id}_{L^\bullet}$. Obviously, cochain equivalent complexes have isomorphic cohomologies.

Example 8.4. Consider two maps of smooth manifolds $F_0, F_1 \colon M_1 \to M_2$ and assume that they are homotopic (in the topological sense). Let us show that the corresponding morphisms of the de Rham complexes $F_0^*, F_1^* \colon \Lambda^\bullet(M_2) \to \Lambda^\bullet(M_1)$ are homotopic (in the above algebraic sense).

Let $F \colon M_1 \times [0,1] \to M_2$ be the homotopy between F_0 and F_1, $F_0(x) = F(x,0)$, $F_1(x) = F(x,1)$. Take a form $\omega \in \Lambda^i(M_2)$. Then

$$F^*(\omega) = \omega_1(t) + dt \wedge \omega_2(t),$$

where $\omega_1(t) \in \Lambda^i(M_1)$, $\omega_2(t) \in \Lambda^{i-1}(M_1)$ for each $t \in [0,1]$. In particular, $F_0^*(\omega) = \omega_1(0)$ and $F_1^*(\omega) = \omega_1(1)$. Set $s(\omega) = \int_0^1 \omega_2(t)\,dt$. We have $F^*(d\omega) = d(F^*(\omega)) = d\omega_1(t) + dt \wedge \omega_1'(t) - dt \wedge d\omega_2(t)$, where $'$ denotes the derivative in t. Hence, $s(d(\omega)) = \int_0^1 (\omega_1'(t) - d\omega_2(t))\,dt = \omega_1(1) - \omega_1(0) - d\int_0^1 \omega_2(t)\,dt = F_1^*(\omega) - F_0^*(\omega) - d(s(\omega))$, so s is a homotopy between F_0^* and F_1^*.

Exercise 8.3. Prove that if two manifolds M_1 and M_2 are homotopic (i.e., there exist maps $f \colon M_1 \to M_2$ and $g \colon M_2 \to M_1$ such that the maps $f \circ g$ and $g \circ f$ are homotopic to the identity maps), then their cohomology are isomorphic.

Corollary 8.2 (Poincaré lemma)**.** *Locally, every closed form $\omega \in \Lambda^i(M)$, $d\omega = 0$, $i \geq 1$, is exact: $\omega = d\eta$.*

A complex $K^\bullet$ is said to be *homotopic to zero* if the identity morphism $\mathrm{id}_{K^\bullet}$ homotopic to the zero morphism, i.e., if there exist maps $s^i \colon K^i \to K^{i-1}$ such that $\mathrm{id}_{K^\bullet} = sd + ds$. Obviously, a complex homotopic to zero has the trivial cohomology.

Example 8.5. Let V be a vector space. Take a nontrivial linear functional $u\colon V \to \Bbbk$ and consider the complex

$$0 \longleftarrow \Bbbk \xleftarrow{d} V \xleftarrow{d} \Lambda^2(V) \xleftarrow{d} \cdots \xleftarrow{d} \Lambda^{n-1}(V) \xleftarrow{d} \Lambda^n(V) \xleftarrow{d} \cdots,$$

where d is the inner product with u:

$$d(v_1 \wedge \cdots \wedge v_k) = \sum_{i=1}^{k} (-1)^{i+1} u(v_i) v_1 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \wedge v_k.$$

Take also a nontrivial element $v \in V$ and consider the complex

$$0 \longrightarrow \Bbbk \xrightarrow{s} V \xrightarrow{s} \Lambda^2(V) \xrightarrow{s} \cdots \xrightarrow{s} \Lambda^{n-1}(V) \xrightarrow{s} \Lambda^n(V) \xrightarrow{s} \cdots,$$

where s is the exterior product with v:

$$s(v_1 \wedge \cdots \wedge v_k) = v \wedge v_1 \wedge \cdots \wedge v_k.$$

Since d is a derivation of the exterior algebra $\Lambda^*(V)$, we have $(ds+sd)(w) = d(v\wedge w) + v \wedge dw = dv \wedge w = u(v)w$. This means that both complexes under consideration are homotopic to zero and, therefore, acyclic.

Example 8.6. Consider two complexes

$$0 \longleftarrow S^n(V) \xleftarrow{d} S^{n-1}(V) \otimes V \xleftarrow{d} S^{n-2}(V) \otimes \Lambda^2(V) \xleftarrow{d} \cdots, \tag{8.2}$$

$$0 \longrightarrow S^n(V) \xrightarrow{s} S^{n-1}(V) \otimes V \xrightarrow{s} S^{n-2}(V) \otimes \Lambda^2(V) \xrightarrow{s} \cdots, \tag{8.3}$$

where

$$d(w \otimes v_1 \wedge \cdots \wedge v_q) = \sum_{i=1}^{q} (-1)^{i+1} v_i w \otimes v_1 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \wedge v_q,$$

$$s(w_1 \cdots w_p \otimes v) = \sum_{i=1}^{p} w_1 \cdots w_{i-1} w_{i+1} \cdots w_p \otimes w_i \wedge v.$$

Both maps d and s are derivations of the algebra $S^*(V) \otimes \Lambda^*(V)$, equipped with the grading induced from $\Lambda^*(V)$, therefore their commutator is also a derivation. Noting that on elements of $S^1(V) \otimes \Lambda^1(V)$ the commutator is identical, we get the formula

$$(ds + sd)(x) = (p+q)x, \qquad x \in S^p(V) \otimes \Lambda^q(V).$$

Thus again both complexes under consideration are homotopic to zero (for $n > 0$). Complex (8.2) is called the *Koszul complex*. Complex (8.3) is the polynomial de Rham complex.

A complex $L^\bullet$ is called a *subcomplex* of a complex $K^\bullet$, if the spaces L^i are subspaces of K^i, and the differentials of $L^\bullet$ are restrictions of differentials of $K^\bullet$, i.e., $d_K(L^{i-1}) \subset L^i$. In this situation, differentials of $K^\bullet$ induce differentials on quotient spaces $M^i = K^i/L^i$ and we obtain the complex $M^\bullet$ called the *quotient complex* and denoted by $M^\bullet = K^\bullet/L^\bullet$.

The cohomologies of complexes $K^\bullet$, $L^\bullet$, and $M^\bullet = K^\bullet/L^\bullet$ are related to one another by the following important mappings. First, the inclusion $\varphi\colon L^\bullet \to K^\bullet$ and the natural projection $\psi\colon K^\bullet \to M^\bullet$ induce the cohomology mappings $H^i(\varphi)\colon H^i(L^\bullet) \to H^i(K^\bullet)$ and $H^i(\psi)\colon H^i(K^\bullet) \to H^i(M^\bullet)$. There exists one more somewhat less obvious mapping

$$\partial^i\colon H^i(M^\bullet) \to H^{i+1}(L^\bullet)$$

called the *boundary* (or *connecting*) mapping.

The map ∂^i is defined as follows. Consider a cohomology class $x \in H^i(M^\bullet)$ represented by an element $y \in M^i$. Take an element $z \in K^i$ such that $\psi(z) = y$. We have $\psi(dz) = d\psi(z) = dy = 0$, hence there exists an element $w \in L^{i+1}$ such that $\varphi(w) = dz$. Since $\varphi(dw) = d\varphi(w) = ddz = 0$, we get $dw = 0$, i.e., w is a cocycle. It can easily be checked that its cohomology class is independent of the choice of y and z. This class is the class $\partial^i(x)$.

Thus, given a short exact sequence of complexes

$$0 \to L^\bullet \xrightarrow{\varphi} K^\bullet \xrightarrow{\psi} M^\bullet \to 0 \tag{8.4}$$

(this means that φ and ψ are morphisms of complexes and for each i the sequences $0 \to L^i \xrightarrow{\varphi^i} K^i \xrightarrow{\psi^i} M^i \to 0$ are exact), one has the following infinite sequence:

$$\begin{aligned}\cdots \xrightarrow{H^{i-1}(\psi)} H^{i-1}(M^\bullet) \xrightarrow{\partial^{i-1}} H^i(L^\bullet) \xrightarrow{H^i(\varphi)} H^i(K^\bullet) \xrightarrow{H^i(\psi)} H^i(M^\bullet)\\ \xrightarrow{\partial^i} H^{i+1}(L^\bullet) \xrightarrow{H^{i+1}(\varphi)} \cdots\end{aligned} \tag{8.5}$$

The main property of this sequence is the following.

Theorem 8.3. *Sequence* (8.5) *is exact.*

Proof. The proof is straightforward and is left to the reader. □

Sequence (8.5) is called the *long exact sequence* corresponding to short exact sequence of complexes (8.4).

Exercise 8.4. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & 0. \end{array}$$

Prove using Theorem 8.3 that if f and h are isomorphisms, then g is also an isomorphism.

8.2. **Spectral sequences.** Given a complex $K^\bullet$ and a subcomplex $L^\bullet \subset K^\bullet$, the exact sequence (8.5) on the page before can tell something about the cohomology of $K^\bullet$, if the cohomology of $L^\bullet$ and $K^\bullet/L^\bullet$ are known. Now, suppose that we are given a filtration of $K^\bullet$, that is a decreasing sequence of subcomplexes

$$K^\bullet \supset K_1^\bullet \supset K_2^\bullet \supset K_3^\bullet \supset \cdots .$$

Then we obtain for each $p = 0, 1, 2, \ldots$ complexes

$$\cdots \longrightarrow E_0^{p,q-1} \longrightarrow E_0^{p,q} \longrightarrow E_0^{p,q+1} \longrightarrow \cdots ,$$

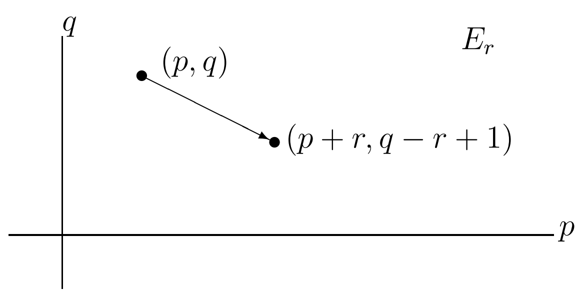
where $E_0^{p,q} = K_p^{p+q}/K_{p+1}^{p+q}$. The cohomologies $E_1^{p,q} = H^{p+q}(E_0^{p,\bullet})$ of these complexes can be considered as the first approximation to the cohomology of $K^\bullet$. The apparatus of spectral sequences enables one to construct all successive approximations E_r, $r \geq 1$.

Definition 8.1. A *spectral sequence* is a sequence of vector spaces $E_r^{p,q}$, $r \geq 0$, and linear mappings $d_r^{p,q}\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$, such that $d_r^2 = 0$ (more precisely, $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$) and the cohomology $H^{p,q}(E_r^{\bullet,\bullet}, d_r^{\bullet,\bullet})$ with respect to the differential d_r is isomorphic to $E_{r+1}^{p,q}$.

Thus E_r and d_r determine E_{r+1}, but do not determine d_{r+1}.

Usually, $p+q$, p, and q are called respectively the degree, the filtration degree, and the complementary degree.

It is convenient for each r to picture the spaces $E_r^{p,q}$ as integer points on the (p,q)-plane. The action of the differential d_r is shown as follows:



Take an element $\alpha \in E_r^{p,q}$. If $d_r(\alpha) = 0$ then α can be considered as an element of $E_{r+1}^{p,q}$. If again $d_{r+1}(\alpha) = 0$ then α can be considered as an element of $E_{r+2}^{p,q}$ and so on. This allows us to define the following two vector spaces:

$$C_\infty^{p,q} = \{\, \alpha \in E_0^{p,q} \mid d_0(\alpha) = 0,\ d_1(\alpha) = 0,\ \ldots,\ d_r(\alpha) = 0,\ \ldots \,\}, \tag{8.6}$$
$$B_\infty^{p,q} = \{\, \alpha \in C_\infty^{p,q} \mid \text{there exists an element } \beta \in E_r^{p,q} \text{ such that } \alpha = d_r(\beta) \,\}.$$

Set $E_\infty^{p,q} = C_\infty^{p,q}/B_\infty^{p,q}$. A spectral sequence is called *regular* if for any p and q there exists r_0, such that $d_r^{p,q} = 0$ for $r \geq r_0$. In this case there are natural projections

$$E_r^{p,q} \longrightarrow E_{r+1}^{p,q} \longrightarrow \cdots \longrightarrow E_\infty^{p,q}, \qquad r \geq r_0,$$

and $E_\infty^{p,q} = \operatorname{inj\,lim} E_r^{p,q}$.

Let E and $'E$ be two spectral sequences. A *morphism* $f\colon E \to {'E}$ is a family of mappings $f_r^{p,q}\colon E_r^{p,q} \to {'E}_r^{p,q}$, such that $d_r \circ f_r = f_r \circ d_r$ and $f_{r+1} = H(f_r)$. Obviously, a morphism $f\colon E \to {'E}$ induces the maps $f_\infty^{p,q}\colon E_\infty^{p,q} \to {'E}_\infty^{p,q}$. Further, it is clear that if f_r is an isomorphism, then f_s are isomorphisms for all $s \geq r$. Moreover, if the spectral sequences E and $'E$ are regular, then f_∞ is an isomorphism as well.

Exercise 8.5. Assume that $E_r^{p,q} \neq 0$ for $p \geq p_0$, $q \geq q_0$ only. Prove that in this case there exists r_0 such that $E_r^{p,q} = E_{r+1}^{p,q} = \cdots = E_\infty^{p,q}$ for $r \geq r_0$.

Consider a graded vector space $G = \bigoplus_{i\in\mathbb{Z}} G^i$ endowed with a decreasing filtration $\cdots \supset G_p \supset G_{p+1} \supset \cdots$, such that $\bigcap_p G_p = 0$ and $\bigcup_p G_p = G$. The filtration is called *regular*, if for each i there exists p, such that $G_p^i = 0$.

It is said that a spectral sequence E *converges* to G, if the spectral sequence and the filtration of G are regular and $E_\infty^{p,q}$ is isomorphic to G_p^{p+q}/G_{p+1}^{p+q}.

Exercise 8.6. Consider two spectral sequences E and $'E$ that converge to G and G' respectively. Let $f\colon E \to {'E}$ be a morphism of spectral sequences and $g\colon G \to G'$ be a map such that $f_\infty^{p,q}\colon E_\infty^{p,q} \to {'E}_\infty^{p,q}$ coincides with the map induced by g. Prove that if the map $f_r^{p,q}\colon E_r^{p,q} \to {'E}_r^{p,q}$ for some r is an isomorphism, then g is an isomorphism too.

Now we describe an important method for constructing spectral sequences.

Definition 8.2. An *exact couple* is a pair of vector spaces (D, E) together with mappings i, j, k, such that the diagram

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

is exact in each vertex.

Set $d = jk\colon E \to E$. Clearly, $d^2 = 0$, so that we can define cohomology $H(E, d)$ with respect to d. Given an exact couple, one defines the *derived* couple

$$\begin{tikzcd} D' \arrow[rr, "i'"] & & D' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$

as follows: $D' = \operatorname{im} i$, $E' = H(E, d)$, i' is the restriction of i to D', $j'(i(x))$ for $x \in D$ is the cohomology class of $j(x)$ in $H(E)$, the map k' takes a cohomology class $[y]$, $y \in E$, to the element $k(y) \in D'$.

Exercise 8.7. Check that mappings i', j', and k' are well defined and that the derived couple is an exact couple.

Thus, starting from an exact couple $C_1 = (D, E, i, j, k)$ we obtain the sequence of exact couples $C_r = (D_r, E_r, i_r, j_r, k_r)$ such that C_{r+1} is the derived couple for C_r.

A direct description of C_r in terms of C_1 is as follows.

Proposition 8.4. *The following isomorphisms hold for all* r:

$$\begin{aligned} D_r &= \operatorname{im} i^{r-1}, \\ E_r &= k^{-1}(\operatorname{im} i^{r-1})/j(\ker i^{r-1}). \end{aligned}$$

The map i_r *is the restriction of* i *to* D_r, $j_r(i^{r-1}(x)) = [j(x)]$, *and* $k_r([y]) = k(y)$, *where* $[\,\cdot\,]$ *denotes equivalence class modulo* $j(\ker i^{r-1})$.

Proof. The proof is by induction on r and is left to the reader. $\square$

Now suppose that the exact couple C_1 is bigraded, i.e., $D = \bigoplus_{p,q} D^{p,q}$, $E = \bigoplus_{p,q} E^{p,q}$, and the maps i, j, and k have bidegrees $(-1, 1)$, $(0, 0)$, $(1, 0)$ respectively. In other words, one has:

$$\begin{aligned} i^{p,q} &\colon D^{p,q} \to D^{p-1,q+1}, \\ j^{p,q} &\colon D^{p,q} \to E^{p,q}, \\ k^{p,q} &\colon E^{p,q} \to D^{p+1,q}. \end{aligned}$$

It is clear that the derived couples C_r are bigraded as well, and the mappings i_r, j_r, and k_r have bidegrees $(-1, 1)$, $(r-1, 1-r)$, $(1, 0)$ respectively. Therefore the differential d_r is a differential in E_r and has bidegree $(r, 1-r)$. Thus, $(E_r^{p,q}, d_r^{p,q})$ is a spectral sequence.

Now, suppose we are given a complex $K^\bullet$ with a decreasing filtration $K_p^\bullet$. Each short exact sequence

$$0 \to K_{p+1}^\bullet \to K_p^\bullet \to K_p^\bullet/K_{p+1}^\bullet \to 0$$

induces the corresponding long exact sequence:

$$\begin{aligned} \cdots \xrightarrow{k} H^{p+q}(K_{p+1}^\bullet) \xrightarrow{i} H^{p+q}(K_p^\bullet) \xrightarrow{j} H^{p+q}(K_p^\bullet/K_{p+1}^\bullet) \\ \xrightarrow{k} H^{p+q+1}(K_{p+1}^\bullet) \xrightarrow{i} \cdots . \end{aligned}$$

Hence, setting $D_1^{p,q} = H^{p+q}(K_p^\bullet)$ and $E_1^{p,q} = H^{p+q}(K_p^\bullet/K_{p+1}^\bullet)$ we obtain a bigraded exact couple, with mappings having bidegrees as above. Thus we assign a spectral sequence to a complex with a filtration.

Let us compute the spaces $E_r^{p,q}$ in an explicit form. Consider the upper term $k^{-1}(\operatorname{im} i^{r-1})$ from the expression for $E_r^{p,q}$ (see Proposition 8.4 on the facing page). An element of $E_1^{p,q}$ is a class $[x] \in H^{p+q}(K_p^\bullet/K_{p+1}^\bullet)$, $x \in K_p^{p+q}$, $dx \in K_{p+1}^{p+q}$. The class $[x]$lies in $k^{-1}(\operatorname{im} i^{r-1})$, if $k([x]) \in H^{p+q+1}(K_{p+r}^\bullet) \subset H^{p+q+1}(K_{p+1}^\bullet)$. This is equivalent to $dx = y + dz$, with $y \in K_{p+r}^{p+q}$, $z \in K_{p+1}^{p+q}$. Thus, we see that $x = (x - z) + z$, with $d(x - z) \in K_{p+r}^{p+q}$. Denoting

$$Z_r^{p,q} = \{\, w \in K_p^{p+q} \mid dw \in K_{p+r}^{p+q} \,\},$$

we obtain $k^{-1}(\operatorname{im} i^{r-1}) = Z_r^{p,q} + K_{p+1}^{p+q}$.

Further, consider the lower term $j(\ker i^{r-1})$ from the expression for $E_r^{p,q}$. The kernel of the map $i^{r-1}\colon H^{p+q}(K_p^\bullet) \to H^{p+q}(K_{p-r+1}^\bullet)$ consists of cocycles $x \in K_p^{p+q}$ such that $x = dy$ for $y \in K_{p-r+1}^{p+q-1}$. So $y \in Z_{r-1}^{p-r+1,q+r-2}$ and $\ker i^{r-1} = dZ_{r-1}^{p-r+1,q+r-2}$. Then $j(\ker i^{r-1}) = dZ_{r-1}^{p-r+1,q+r-2} + K_{p+1}^{p+q}$.

Thus, we get

$$E_r^{p,q} = \frac{Z_r^{p,q} + K_{p+1}^{p+q}}{dZ_{r-1}^{p-r+1,q+r-2} + K_{p+1}^{p+q}} = \frac{Z_r^{p,q}}{dZ_{r-1}^{p-r+1,q+r-2} + Z_{r-1}^{p+1,q-1}}.$$

Remark 8.3. The last equality follows from the well known Noether modular isomorphism

$$\frac{M+N}{M_1+N} = \frac{M}{M_1 + (M \cap N)}, \qquad M_1 \subset M.$$

Theorem 8.5. *If the filtration of the complex $K^\bullet$ is regular, then the spectral sequence of this complex converges to $H^\bullet(K^\bullet)$ endowed with the filtration $H_p^k(K^\bullet) = \operatorname{im} H^k(i_p)$, where $i_p\colon K_p^\bullet \to K^\bullet$ is the natural inclusion.*

Proof. Note first, that if the filtration of the complex $K^\bullet$ is regular, then the spectral sequence of this complex is regular too. Further, the spaces $C_\infty^{p,q}$ and $B_\infty^{p,q}$ (see (8.6) on page 140) can easily be described by

$$C_\infty^{p,q} = \frac{Z_\infty^{p,q}}{Z_\infty^{p+1,q-1}}, \qquad B_\infty^{p,q} = \frac{(K_p^{p+q} \cap d(K^{p+q-1})) + Z_\infty^{p+1,q-1}}{Z_\infty^{p+1,q-1}},$$

where $Z_\infty^{p,q} = \{\, w \in K_p^{p+q} \mid dw = 0 \,\}$, whence

$$E_\infty^{p,q} = \frac{Z_\infty^{p,q}}{(K_p^{p+q} \cap d(K^{p+q-1})) + Z_\infty^{p+1,q-1}}.$$

Since $H_p^{p+q}(K^\bullet) = \dfrac{Z_\infty^{p,q} + d(K^{p+q-1})}{d(K^{p+q-1})}$, we have

$$\frac{H_p^{p+q}(K^\bullet)}{H_{p+1}^{p+q}(K^\bullet)} = \frac{Z_\infty^{p,q} + d(K^{p+q-1})}{Z_\infty^{p+1,q-1} + d(K^{p+q-1})}$$

$$= \frac{Z_\infty^{p,q}}{Z_\infty^{p+1,q-1} + (K_p^{p+q} \cap d(K^{p+q-1}))} = E_\infty^{p,q}.$$

This concludes the proof. $\square$

Definition 8.3. A *bicomplex* is a family of vector spaces $K^{\bullet,\bullet}$ and linear mappings $d'\colon K^{p,q} \to K^{p+1,q}$, $d''\colon K^{p,q} \to K^{p,q+1}$, such that $(d')^2 = 0$, $(d'')^2 = 0$, and $d'd'' + d''d' = 0$.

Let $K^\bullet$ be the *total* (or *diagonal*) complex of a bicomplex $K^{\bullet,\bullet}$, i.e., by definition, $K^i = \bigoplus_{i=p+q} K^{p,q}$ and $d_K = d' + d''$. There are two obvious filtration of $K^\bullet$:

$$\text{filtration I:} \qquad {}'K_p^i = \bigoplus_{\substack{j+q=i\\ j\geq p}} K^{j,q},$$

$$\text{filtration II:} \qquad {}''K_q^i = \bigoplus_{\substack{p+j=i\\ j\geq q}} K^{p,j}.$$

These two filtrations yield two spectral sequences, denoted respectively by ${}'E_r^{p,q}$ and ${}''E_r^{p,q}$.

It is easy to check that ${}'E_1^{p,q} = {}''H^q(K^{p,\bullet})$ and ${}''E_1^{p,q} = {}'H^q(K^{\bullet,p})$, where ${}'H$ (resp., ${}''H$) denotes the cohomology with respect to d' (resp., d''), with the differential d_1 being induced respectively by d' and d''. Thus, we have:

Proposition 8.6. ${}'E_2^{p,q} = {}'H^p({}''H^q(K^{\bullet,\bullet}))$ *and* ${}''E_2^{p,q} = {}''H^p({}'H^q(K^{\bullet,\bullet}))$.

Now assume that both filtrations are regular.

Exercise 8.8. Prove that

(1) if $K^{p,q} = 0$ for $q < q_0$ (resp., $p < p_0$), then the first (resp., second) filtration is regular;
(2) if $K^{p,q} = 0$ for $q < q_0$ and $q > q_1$, then both filtration are regular.

In this case both spectral sequences converge to the common limit $H^\bullet(K^\bullet)$.

Remark 8.4. This fact does not mean that both spectral sequences have a common infinite term, because the two filtrations of $H^\bullet(K^\bullet)$ are different.

Let us illustrate Proposition 8.6.

Example 8.7. Consider the commutative diagram

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & K^{2,0} \arrow[u] \arrow[r, "d_2"] & K^{2,1} \arrow[u] \arrow[r, "d_2"] & K^{2,2} \arrow[u] \arrow[r] & \cdots \\
0 \arrow[r] & K^{1,0} \arrow[u, "d_1"'] \arrow[r, "d_2"] & K^{1,1} \arrow[u, "d_1"'] \arrow[r, "d_2"] & K^{1,2} \arrow[u, "d_1"'] \arrow[r] & \cdots \\
0 \arrow[r] & K^{0,0} \arrow[u, "d_1"'] \arrow[r, "d_2"] & K^{0,1} \arrow[u, "d_1"'] \arrow[r, "d_2"] & K^{0,2} \arrow[u, "d_1"'] \arrow[r] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

and suppose that the differential d_1 is exact everywhere except for the terms $K^{0,q}$ in the bottom row, and the differential d_2 is exact everywhere except for the terms $K^{p,0}$ in the left column. Thus, we have two complexes $L_1^\bullet$ and $L_2^\bullet$, where $L_1^i = H^0(K^{i,\bullet}, d_2)$, $L_2^i = H^0(K^{\bullet,i}, d_1)$ and the differential of L_1 (resp., L_2) is induced by d_1 (resp., d_2). Consider the bicomplex $K^{\bullet,\bullet}$ with $(d')^{p,q} = d_1^{p,q}$, $(d'')^{p,q} = (-1)^q d_2^{p,q}$. We easily get

$$'E_2^{p,q} = {'E_3^{p,q}} = \cdots = {'E_\infty^{p,q}} = \begin{cases} 0 & \text{if } q \neq 0, \\ H^p(L_1^\bullet) & \text{if } q = 0, \end{cases}$$

$$''E_2^{p,q} = {''E_3^{p,q}} = \cdots = {''E_\infty^{p,q}} = \begin{cases} 0 & \text{if } p \neq 0, \\ H^q(L_2^\bullet) & \text{if } p = 0. \end{cases}$$

Since both spectral sequences converge to a common limit, we conclude that $H^i(L_1^\bullet) = H^i(L_2^\bullet)$.

Let us describe this isomorphism in an explicit form. Consider a cohomology class from $H^i(L_1^\bullet)$. Choose an element $k^{i,0} \in K^{i,0}$, $d_1(k^{i,0}) = 0$, $d_2(k^{i,0}) = 0$, that represents this cohomology class. Since $d_1(k^{i,0}) = 0$, there exists an element $x \in K^{i-1,0}$ such that $d_1(x) = k^{i,0}$. Set $k^{i-1,1} = -d_2(x) \in K^{i-1,1}$. We have $d_2(k^{i-1,1}) = 0$ and $d_1(k^{i-1,1}) = -d_1(d_2(x)) = -d_2(d_1(x)) = -d_2(k^{i,0}) = 0$. Further, the elements $k^{i,0}$ and $k^{i-1,1}$ are cohomologous in the total complex $K^\bullet$: $k^{i,0} - k^{i-1,1} = d_1 x + d_2 x = (d' + d'')(x)$. Continuing this process we obtain elements $k^{i-j,j} \in K^{i-j,j}$, $d_1(k^{i-j,j}) = 0$, $d_2(k^{i-j,j}) = 0$, that are cohomologous in the total complex $K^\bullet$. Thus, the above isomorphism takes the cohomology class of $k^{i,0}$ to that of $k^{0,i}$.

Exercise 8.9. Discuss an analog of Example 8.7 on the page before for the commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 & K^{2,0} \arrow[l] \arrow[d, "d_1"] & K^{2,1} \arrow[l, "d_2"'] \arrow[d, "d_1"] & K^{2,2} \arrow[l, "d_2"'] \arrow[d, "d_1"] & \cdots \arrow[l] \\
0 & K^{1,0} \arrow[l] \arrow[d, "d_1"] & K^{1,1} \arrow[l, "d_2"'] \arrow[d, "d_1"] & K^{1,2} \arrow[l, "d_2"'] \arrow[d, "d_1"] & \cdots \arrow[l] \\
0 & K^{0,0} \arrow[l] \arrow[d] & K^{0,1} \arrow[l, "d_2"'] \arrow[d] & K^{0,2} \arrow[l, "d_2"'] \arrow[d] & \cdots \arrow[l] \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

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