Mechanics

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December 4, 2022 7:48pm

Contents

1 Forces. Newton's laws.

Relevant book section: University Physics Volume 1, chapter 5.

1.1 Forces

We've learned how to describe and analyze motion, but we haven't talked about the causes of motion yet. Objects can modify each other's motion by pulling or pushing on each other. For example, in a free fall, the Earth pulls the falling object down. That's what makes it fall. If the Earth was not pulling down on the object, and the object had no initial velocity, then it would just stay in place.

When an object A pushes or pulls on an object B, we say that object A exerts a force on object B. This force has a direction: A could be pulling B in, pushing B away, dragging it sideways, etc. The force also has a magnitude (a *strength*): it can be a hard push/pull/drag or a soft one. In short, forces are vectors.

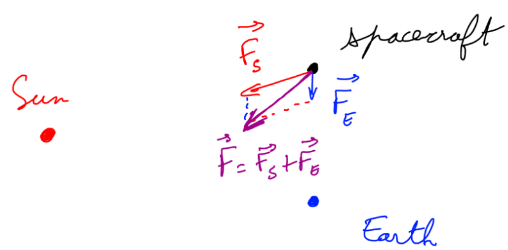
In a free fall, the only force is the weight of the object. It is exerted by the Earth, onto the falling object. Its direction is "down", toward the ground. Its magnitude depends on the mass of the object.

The comet of kinematics problem 28 also experiences a kind of free fall. This time the force of gravity is exerted by the Sun onto the comet. Its direction is from the comet to the Sun. Its magnitude depends on the mass of the Sun, the mass of the comet, and the distance between the Sun and the comet.

1.2 Net force

If there are multiple forces exerted on the same object, then their effect of the object's motion is equivalent to that of a single force that would equal to the sum of all the forces exerted on the object. That single force is called the *net force*.

Example:



$\vec F_E$ = force exerted by the Earth on the spacecraft

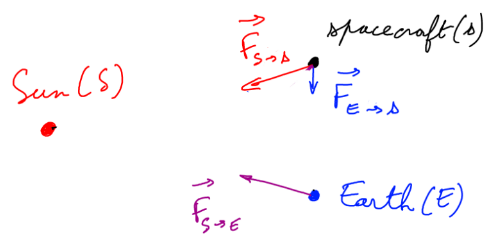
$\vec F_S$ = force exerted by the Sun on the spacecraft

$\vec F = \vec F_E + \vec F_S$ = net force on the spacecraft.

Notes:

- If there is only one force, then the net force is just that force.
- Since forces are vectors, we need to add them like vectors. In particular, we cannot just add their magnitudes.
- The net force is calculated for a specific object. We say that it is the net force *exerted on that object*, or the net force *felt by that object*. A different object will feel a different net force. Before computing a net force, always ask yourself "the net force on what object?". If you can't identify the object, then there's no point computing the net force.
- The net force on an object is the sum of the forces *exerted on that object*. Forces exerted by the object, or forces exerted by other objects on each other, should be ignored. For example, the force exerted by the Sun on the comet is not part of the net force exerted on the Sun.

To help keep track of forces, I will often write the force exerted by an object A onto an object B as $\vec{F}_{A\to B}$. When using that notation, I will often use $\vec{F}_A$ for the net force exerted on A and $\vec{F}_B$ for the net force exerted on B. For example:



The net force exerted on (or felt by) the spacecraft is $\vec{F}_s = \vec{F}_{S\to s} + \vec{F}_{E\to s}$. Only forces exerted on the spacecraft (subscript ending with $\to s$) matter as far as $\vec{F}_s$ is concerned. The force exerted by the Sun on the Earth, $\vec{F}_{S\to E}$, is not part of it. It is however part of the net force exerted on the Earth. There are other forces I did not show on the sketch, like the force exerted by the spacecraft on the Sun, $\vec{F}_{s\to S}$

Free-body diagrams

Identifying the object whose net force we want to compute and identifying the individual forces that contribute to that net force is so critical that there's a special sketch whose sole purpose is to do it correctly. It's called the object's *free-body diagram*. Here is how you construct it:

- Identify the object whose motion you want to study. Start referring to it as "the system".
- Draw "the system". Sometimes just as a dot. Don't draw any other object from the problem. Just the system
- Draw the forces exerted on the system. Don't draw the net force. Don't draw any other forces. Don't draw forces exerted Only the individual forces.

Problem 1: Free body diagram.

Draw the free-body diagram of the spacecraft from the earlier example.

1.3 Newton's second law

This is the central result of mechanics: The acceleration of an object is equal to the net force exerted on that object, divided by the mass of the object. It is usually written as

$$\vec{F} = m\vec{a}$$

although conceptually it makes more sense to write it as

$$\vec{a} = \frac{\vec{F}}{m}$$

because forces cause the object to have an acceleration, not the other way around.

As we discussed in kinematics, the acceleration controls the future motion of the object. In a way, $\vec{a} = \vec{F}/m$ is how the universe decides where everything is going to be next. The forces objects exert on each other depend on their current positions and velocities. As we'll see, there are formulas for computing those forces in terms of those positions and velocities. In principle, if you knew where every object in the universe is and how fast each object is going, you could:

- Compute every force in the universe.
- Use those forces to compute the net force on every object in the universe.
- Use those net forces and $\vec{a} = \vec{F}/m$ to compute the acceleration of every object in the universe.
- Use the current velocity of each object to predict their next position.
- Use the acceleration you just computed to predict the next velocity of each object, which will be used in the next round to predict its "next next" position.

Rinse and repeat at infinitely small time intervals and you'll have predicted the future of the universe. Of course, this is not doable in practice, however:

- There are mathematical techniques as well as computer-based techniques to perform a simplified version of the steps above. That *is* used daily to predict all sorts of things, like how a plane will behave in flight or how a drug molecule may interact with a target protein.
- In a way, the universe is performing those calculations every instant in order to know what to do next.

Problem 2: First consequences of Newton's second law.

1. [Newton's first law] What can you say about the motion of an object that feels no force at all?
2. What can you say about the net force felt by a car in each of the following situations? Draw a sketch for each.
   (a) Not moving at all.
   (b) Speeding up.
   (c) Slowing down by braking.
   (d) Slowing down by taking the foot off the gas pedal.
   (e) Turning at constant speed.
   (f) Driving straight at constant speed.

Problem 3: Free fall on Earth.

Based on Newton's second law and what we learned about free fall on Earth in the Kinematics module, what is the force exerted by the Earth on a falling object of mass m.

Problem 4: Object sitting on the ground.

1. What can you say about the net force on an object sitting on the ground?
2. An object sitting on the ground still feels the force of gravity, which is computed the same way we discovered in problem 3. Why must there be another force at play? What direction and magnitude must this force have?
3. What might be exerting this force?

Problem 5: Object sliding on a tilted plane.

An object is sitting on tilted ground (velocity equals zero). It then starts to slide down the tilted plane.

1. Draw a sketch of the tilted plane and the object.
2. On the sketch, draw the object's velocity once it is sliding.
3. What was the direction of the acceleration while the object was going from not moving to sliding down the plane?
4. Using Newton's second law, what was the direction of the net force during that time?
5. Draw a free-body diagram of the object before it starts sliding (no acceleration).
6. Draw a free-body diagram of the object while it's accelerating downslope. You will probably need to draw the net force to get to the result; just draw it in a different color.
7. Draw a free body diagram of the object once it's sliding at constant speed.

Unit of force

As seen in the exam, Newton's second law implies that the dimension of a force is $M.L.T^{-2}$. It follows that the SI unit of force is $\mathrm{kg\,m\,s^{-2}}$. For convenience, it's been given a shorter name: the *Newton*, symbol N. In other words, $1\,\mathrm{N} \equiv 1\,\mathrm{kg\,m\,s^{-2}}$.

Problem 6: Unit of force.

1 N is the net force needed to accelerate a 1 kg object from zero to 1 m/s in 1 s. Prove it by studying the motion of the object when its initial position is $\vec{0}$, initial velocity $\vec{0}$, and the net force on it $\vec{F} = \begin{bmatrix} 1\,\mathrm{N} \\ 0\,\mathrm{N} \end{bmatrix}$.

Inertia

$\vec{a} = \vec{F}/m$ implies that if you exert the same pushing force on two objects, the heavier one will move less. More precisely, for a given force, the acceleration is inversely proportional to the mass of the object.

Problem 7: Inertia.

Redo problem 6 with the same force but a 10 kg object. How do the speeds of the 1 kg object and the 10 kg object compare after 1 s?

Reading

Read sections 5.1, 5.2, and 5.3 in University Physics Volume 1.

Problem 8: Newton's second law examples from the book.

Work on examples 5.2 to 5.7 and "check your understanding" 5.5 from University Physics Volume 1. In other words, every example in section 5.3, plus the "check your understanding" at the end of the section.

1.4 Weight

The weight of an object is the gravitational force exerted by the Earth (or whatever celestial body the object is on the surface of, e.g., the Moon). It is often called $\vec{w}$. As discussed in problem 3, it is equal to

$$\vec{w} = m\vec{g}$$

where $\vec{g}$ is the acceleration of gravity on the Earth (or the relevant celestial body).

Depending on context, "weight" may also refer to the magnitude of the weight vector, $w = \|\vec{w}\| = mg$.

Note that the weight is not the same thing as the mass. This can be confusing at first because in every day speech they are pretty much interchangeable. The mass of an object is an intrinsic property of that object, meaning that it only depends on the properties of that object. In contrast, the weigth of an object depends both on the properties of the object (specifically, its mass) and the properties of the Earth (specifically, the acceleration of gravity on its surface, $\vec{g}$). For example, your mass (say, 70 kg) would be the same on the Moon as it is on the Earth, however your weight (measured in Newtons, like any other force) would be different because g is different ($g_{\text{Moon}} = 1.6\,\text{m}\,\text{s}^{-2}$ vs $g_{\text{Earth}} = 9.8\,\text{m}\,\text{s}^{-2}$).

Note: you can quickly tell that the weight is a force by comparing $F = m.a$ and $w = m.g$. The acceleration of gravity g has the dimension of an acceleration, therefore w has the same dimension as F, that of a force.

Problem 9: Weight.

Read University Physics Volume 1 section 5.4. Work on example 5.8 and do check-your-understanding problem 5.6.

1.5 Normal force and friction force

The normal force is the force exerted by a solid surface to prevent objects from passing through it. We've already discussed it a little bit in problems 4 and 5.

It is not limited to objects resting on the ground. Every time solid objects are in contact, specifically every time a situation arises in which the two objects are about to occupy the same space, a normal force appears at the surface of contact to prevent it. As the name *normal force* suggests, the force is always normal (perpendicular) to the surface of contact.

In addition to the normal force, two solids in contact with each other can exert solid friction force on each other. While normal forces are always perpendicular to the surface of contact, friction forces are always parallel to the surface of contact. In addition friction forces always oppose motion, i.e., they always point opposite the velocity vector. In the case of static friction, when the velocity vector is has no direction because it is $\vec{0}$, the friction force

Problem 10: Weight on an incline.

1. Solve for the acceleration vector in example 5.12 from University Physics Volume 1 section 5.6, but keep m, g, f, N, θ as literal.
2. Plug in the numerical values and compare with the book's results.

In the rest of the problem, we assume $f = 0$ (no friction).

3. Assume no friction ($f = 0$). Sketch a_x (x component of $\vec{a}$) as a function of θ. Explain why its values at $\theta = 0$ and $\theta = \pi/2$ make sense in light of past problems.
4. Let's call D the distance between the initial position of the object and the bottom of the slope. Compute the time it takes the object to reach the bottom if the initial velocity is zero. Write your answer in terms of m, g, f, N, θ. *Hint: Use a coordinate system with the same axes as before (parallel and perpendicular to the ground) and the initial condition as its origin. Write the position of the bottom of the slope in that coordinate system. Integrate the acceleration twice to get the velocity and position.*

1.6 Newton's third law (law of "action-reaction")

Whenever an object A exerts a force $\vec{F}_{A\to B}$ on an object B, object B simultaneously exerts a force $\vec{F}_{B\to A} = -\vec{F}_{A\to B}$ on object A.

Examples: See figures 5.16, 5.17, and 5.18 in University Physics Volume 1 section 5.5.

Note: It would be misleading to say that $\vec{F}_{B/A}$ and $\vec{F}_{A/B}$ "cancel each other" because they are not exerted on the same object. $\vec{F}_{A\to B}$ impacts the motion of object B, whereas $\vec{F}_{B\to A}$ impact the motion of object A.

Reading

Read University Physics Volume 1 section 5.5.

Problem 11: Multiple objects and choosing a system.

See the sketch in "check your understanding 5.7" from University Physics Volume 1 section 5.5. The force drawn on the sketch is exerted by someone the first block. The first block then pushes on the second block and the two blocks start moving together. The block stay together the whole time, and their only motion is horizontal (parallel to the surface the blocks are on).

1. Sketch all the forces acting on either block.
2. Write Newton's second law for each block. Call normal forces $\vec{N}_{A\to B}$ where A is the object exerting the force and B the object feeling it. For example, the force exerted by the ground on block 1 is $\vec{N}_{G\to 1}$ and its components are $N_{G\to 1x}$ and $N_{G\to 1y}$.
3. The two blocks stay against each other throughout. What mathematical constraint does that imply for their accelerations?
4. The blocks have no vertical motion. What constraint does that imply for their accelerations?
5. Use Newton's second law the two constraints to solve for the normal forces and the blocks' accelerations.
6. Start over, this time treating the two blocks as a single object.

Newton's third law and internal forces

In the problem above, you may have noticed that the normal forces exerted by each block on the other cancel out when we write the second law for the two blocks treated as a single system. This is sometimes presented as an additional rule: *When writing Newton's second law, ignore internal force, i.e., forces exerted by the system on itself.* That rule is not really needed though. It's really just a logical consequence of Newton's third law.

An internal force is a force exerted by a part of the system (call it *part A*) onto another part of the system (call it *part B*). Let's call that force $\vec{F}_{A\to B}$. According to Newton's third law, part B exerts a force back on part A, and that force is $-\vec{F}_{A\to B}$. Therefore, the sum of those two forces is $\vec{0}$, and they do not contribute to the net force on the system. We might as well ignore them.

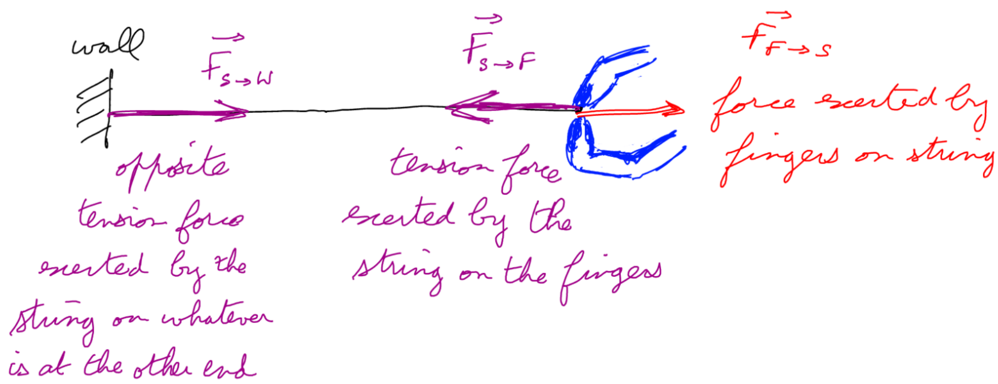
1.7 Tension force

See the **Tension** paragraph in University Physics Volume 1 section 5.6.

If you pull on the end of a fully extended string or rope, it pulls back. This is called the tension force. On a conceptual level, the tension force is similar to the normal force exerted by a solid object (like the ground). The normal force exerted by the ground prevents objects from sinking into the ground. It doesn't have a fixed value; rather, its value is whatever is takes to prevent objects from sinking into the ground. Similarly, the tension force exerted by a string exists to prevent the string from stretching past its fixed length. It doesn't have a fixed value; rather, its value is whatever it takes to prevent the string from stretching past its fixed length.

The important properties of the tension force are:

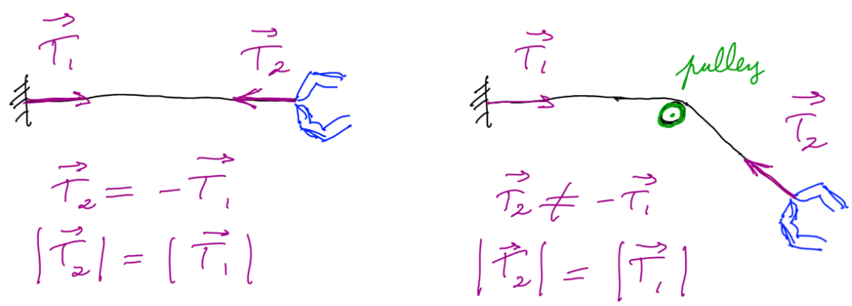
- Its value is whatever it takes to prevent the string from stretching past its length.
- It exists at both ends of the string.
- It is always directed along the string.
- It always pulls inward.
- It always has the same magnitude at both ends of the string.
- It exists everywhere in the string. You can pick any part of the string as your *system*; it will have a tension force at each end, pulling inward along the string, with the same magnitude at both ends.



If the string is straigth, then the properties above imply that the force exerted by the string on the outside world at one end is minus the force exerted by the string on the outside world at the other end. In the example sketched above, it means that the force exerted by the string on the wall is minus the force exerted by the string on the finger:

$$\vec{F}_{S\to W} = -\vec{F}_{S\to F}$$

However, if the string rests on something, e.g., a pulley, then the tension forces exerted by the string at one end is no longer equal to minus the one exerted at the other end. What remains true, though, is that the two forces are along the string (more specifically, along the direction of the string at the end at which the force is applied), that they both pull inward, and that they have the same magnitude:



In other words, a string (or a rope) and a pulley can redirect a force.

Lastly, unless a problem specifies otherwise, we always neglect the mass of strings and ropes. More specifically, we assume that, for a string or rope, the left-hand side of $m\vec{a} = \vec{F}_{\text{net}}$ can be neglected, leading to $\vec{F}_{\text{net}} = \vec{0}$.

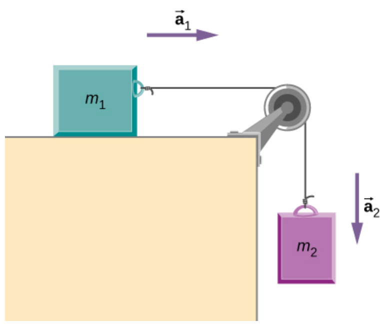
Problem 12: Force on a pulley.

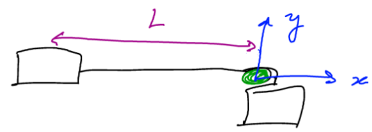
Redraw the sketch of the string and pulley above. Perform all computations graphically.

1. Draw a free-body diagram for the string. Construct the force exerted by the pulley on the string.
2. Draw a free-body diagram for the pulley. What must be in place for the pulley to remain where it is?

Problem 13: Coupled blocks.

See example 5.16 from University Physics Volume 1 section 5.7.





1. Compute the acceleration of each block.
2. Integrate the accelerations twice to get the position of each block as a function of time. Use the coordinate shown in the sketch on the left. The origin is the pulley. One block moves along the x axis while the other moves along the y axis. The initial velocities are zero. The initial positions are $\begin{bmatrix} -L \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Problem 14: Tension of a tightrope.

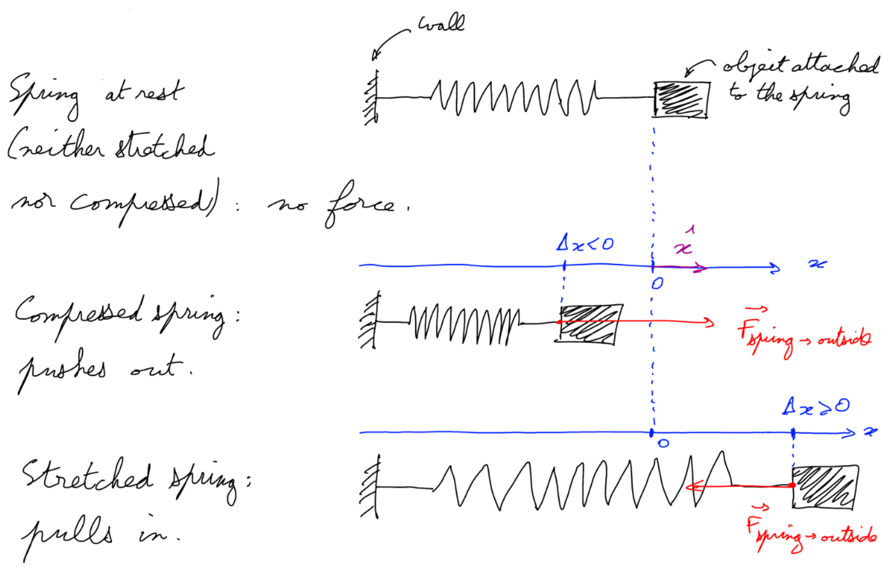
See example 5.13 from University Physics Volume 1 section 5.6
Note: While both are valid choices, I find it easier to pick the person + the bit of rope under them as the system under study, rather than just the bit of rope as suggested in the example.

Problem 15: Transverse push on a chain.

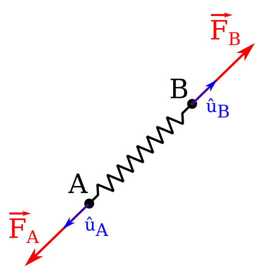
See "check your understanding 5.9" from University Physics Volume 1 section 5.6

1.8 Spring force

Springs have a preferred length, called their "rest length". When they are stretched (made longer than their preferred length), they pull inward. When they are compressed (made shorter than their preferred length), they push outward. Furthermore, the magnitude of the spring force is proportional to the length by which they've been stretched or compressed.



Hooke's law



The sketch shows a compressed spring, pushing outward at both ends.

The force exerted by the spring on whatever is attached to it at point B is

$$\vec{F}_B = -k\,(\ell - \ell_0)\,\hat{u}_B.$$

Definitions:

- k is called the spring's *spring constant*, also called the *stiffness*, measured in N/m. It is always positive.
- ℓ_0 is spring's preferred length, called the *rest length*.
- ℓ is the current length of the spring. If the spring is straight, it's just AB.
- $\ell - \ell_0$ is known as the spring's elongation. It measures how far the spring is from its preferred length. It is positive when the spring is stretched out and negative when it is compressed.
- $\hat{u}_B$ is the unit vector parallel to the spring at B, pointing outward. If the spring is straight, then $\hat{u}_B$ is just $\hat{AB}$.

Comments:

- k and ℓ_0 are intrinsic properties of the spring. Their values can vary from one spring to the next, but for a given spring they're constant, i.e., their values don't change mid-motion.
- The definitions above coincide with our intuition of what a spring does. If the spring is compressed ($\ell < \ell_0$), then $\ell - \ell_0 < 0$, therefore $-k(\ell - \ell_0) > 0$ and the force exerted by the spring at B, $\vec{F}_B = -k\,(\ell - \ell_0)\,\hat{AB}$, points in the same direction as $\hat{AB}$. In other words, the formula tells us that a compressed spring pushes outward. Or rather, experience shows that a compressed spring pushes outward, and the formula is designed to capture that.
- To get the formula for the force at A, just swap A and B in the formula and definitions above: the force exerted by the spring on whatever is attached to it at point A is $\vec{F}_A = -k\,(\ell - \ell_0)\,\hat{u}_A$ where k is the spring constant, ℓ is the spring's length, ℓ_0 is the spring's rest length, and $\hat{u}_A$ is the unit vector pointing away and tangent to the spring at A

I didn't provide a formula for the force exerted at the other end (A) because it follows from the formula for the force at B.

we can just swap don't need a new formula. If we want to The force exerted by the spring on whatever is attached to it at point A is

$$\vec{F}_A = -k(\ell - \ell_0)\,\hat{u}_A.$$

- Similar to strings, springs exerts forces at both ends, and both forces have the same magnitude. This is not an extra rule; it's a logical consequence of the formula above. Indeed, if we want to compute the force at A, we need to use the outward-pointing unit vector along the spring at A, which is not $\hat{AB}$ but $\overrightarrow{BA}$. The spring's length, on the other hand, remains the same; it doesn't matter whether we measure it from A to B or from B to A. Therefore, the force at A is $\vec{F}_A = -(\ell - \ell_0)\hat{BA} = -(\ell - \ell_0)(-\hat{AB}) = -\vec{F}_B$.
- What if nothing is attached to the spring at B? Can the spring exert a force on nothing? The answer to that paradox is that if nothing pulls or pushes on the spring, then the spring's length will relax to ℓ_0, at which point the force becomes zero.

Problem 16: Spring force.

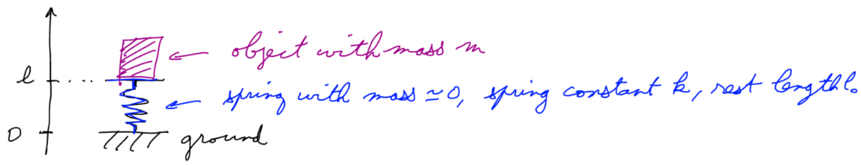
For each of the three spring configurations shown in the first sketch, identify sketch the quantities appearing in Hooke's law (ℓ, ℓ_0, $\hat{u}$, $\vec{F}$) and explain why Hooke's law's prediction for the force exerted by the spring on the object is in the expected direction.

Problem 17: Real string.

No string is truly inextensible. In reality, strings are quite like springs: when under tension, they stretch a little bit, and the amount they stretch is proportional to the amount of tension they're under. More accurately, real strings are one-side springs: they stretch according to Hooke's law when under tension, but they don't push back at all when compressed.

A string is attached to the ceiling. When an object of mass 10 kg is attached to it, it stretches by 1 mm. What is its spring constant?

Problem 18: Measuring weight with a spring.



Assume the object is static.

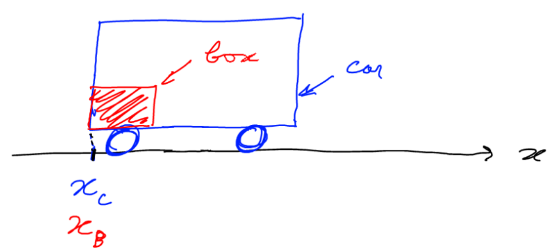
1. Write the second law for the mass. Compute the length of the spring as a function of ℓ_0, m, and k.
2. Explain how this can be used as a scale. *Note: This is a bit of a rabbit hole, but an interesting one. Generally speaking, building a sensor requires a calibration step. If you can, give precise instructions as to how you would construct then calibrate your scale.*
3. On the Moon, g is smaller. Would the scale show a different value? In other words, does this scale measure weight, or does it measure mass? Explain.

1.9 Non-inertial frames of reference

Problem 19: Box in a braking car.

A box is sitting in the back of a car. The car starts braking. The purpose of the problem is to use Newton's second law to understand what happens to the box.

As we will see, Newton's second law cannot be applied directly to the motion of the box relative to the car. We need to apply it to the motion of the box relative to the ground, then use relative motion formulas to find the actual relationship between the net force exerted on the box and the box's acceleration relative to the car.



To make things simpler, we treat this as a 1D problem along the x axis. In particular, we ignore the weight of the box and the normal force exerted by the floor of the car on the box, both of which are along the y axis. We also ignore any other force on the box, like friction between the box and the floor of the car.

x_C is the x coordinate of the left end of the car, measured relative to the ground. x_B is the x coordinate of the left end of the box, measured relative to the ground. The deceleration starts at $t = 0$. After that the acceleration is constant, equal to a_0. The initial conditions are $x_C(0) = 0$ (in other words, we choose the initial position of the back of the car as the origin of the x axis), $x_B(0) = x_C(0)$ (the box is initially all the way in the back of the car), $v_C(0) = v_0$ (that's the velocity prior to braking), and $v_{B/C}(0) = 0$ (before braking, the box is not moving within the car). The mass of the box is m_B.

Write all results in terms of a_0, v_0, m_B, and t.

Part 1:

1. Integrate the car's acceleration to get $v_C(t)$ and $x_C(t)$.

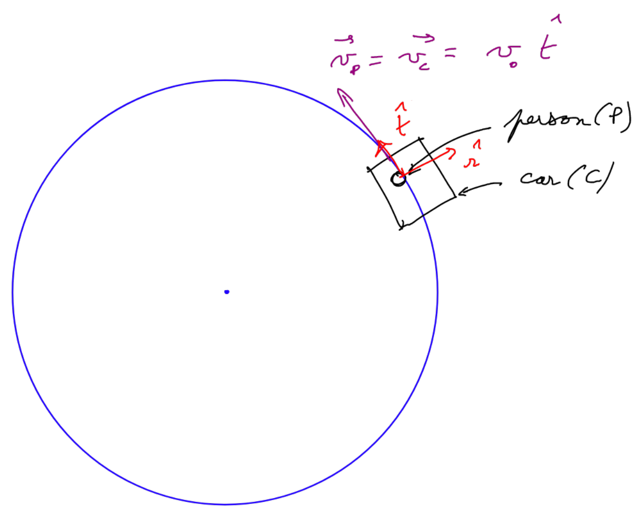
2. Write Newton's second law for the box "in the frame of reference of the ground". That means that the acceleration showing up in the second law (call it a_B) is measured relative to the ground. Integrate to get $v_B(t)$, then $x_B(t)$, both measured relative to the ground.
3. Compute the position of the box relative to the car, $x_{B/C}$. Use it to obtain the velocity and the acceleration of the box relative to the car, $v_{B/C}$ and $a_{B/C}$.
4. Show that the acceleration of the box relative to the car does not obey Newton's second law. What similar equation does it obey?

<u>Part 2</u>: This time, the box is anchored to the bottom of the car. That means that there's a new force, exerted by the car on the box, which prevents the box from moving relative to the car.

5. Write Newton's second law for the box in the frame of reference of the ground.
6. Compute the anchoring force as a function of the parameters of the problem.
7. Write the general relationship between x_B, x_C, and $x_{B/C}$. Use it to find (prove) the relationship between a_B, a_C, and $a_{B/C}$.
8. Use questions 5 and 7 to write the correct form of Newton's second law in the frame of reference of the car.
9. Reflecting on both parts of the problem, what is the general form of Newton's second law in an accelerated frame of reference? Explain why the additional term cause by the acceleration of the car is sometimes called a "fictitious force".

Problem 20: Centrifugal "force".

The sketch shows a car from above. The car drives in a circle of radius R at constant speed R. The person is attached; they don't move relative to the car.



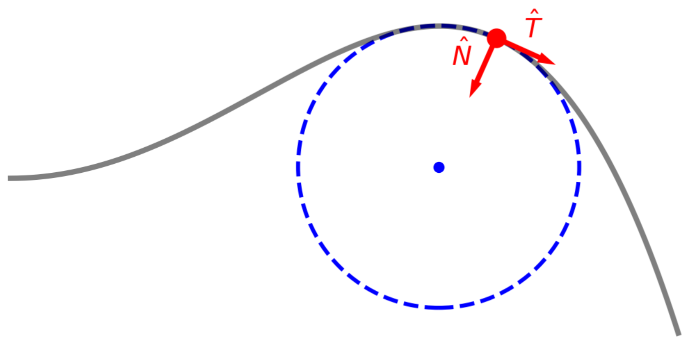
1. Write the acceleration of the car in the polar basis. What force must be exerted on the car for that acceleration to be possible? What is exerting it? What other two forces, discussed in earlier problems, are also exerted on the car? What exerts each force? Draw a different sketch that shows them. Show that they must cancel.
2. Same questions for the person.
3. What is the acceleration of the person relative to the car? Show that by writing Newton's second law for the person in the frame of reference of the ground, then subtracting the acceleration of the car from both sides of the equation, we get Newton's second law for the person in the frame of reference of the car. Discuss the forces, fake and real, that appear in the latter.

Centrifugal "force" and tangential/normal acceleration

As discussed in the kinematics chapter, the acceleration can always be written as

$$\vec{a} = \frac{dv}{dt}\hat{T} + \frac{v^2}{R}\hat{N}$$

Here $\hat{T}$ is the unit vector pointing in the direction of motion (known as the tangent vector, but it's really just another name for $\hat{v}$, the unit vector constructed from the velocity vector). $\hat{N}$ is called the normal vector; it's a unit vector perpendicular to $\hat{T}$ and pointing toward the inside of the curve. R is the radius of curvature, defined as the radius of the circle that best approximates the trajectory around the point of interest (dashed blue circle). $\hat{T}$ is also tangent to that circle, and $\hat{N}$ points toward the center of the circle.

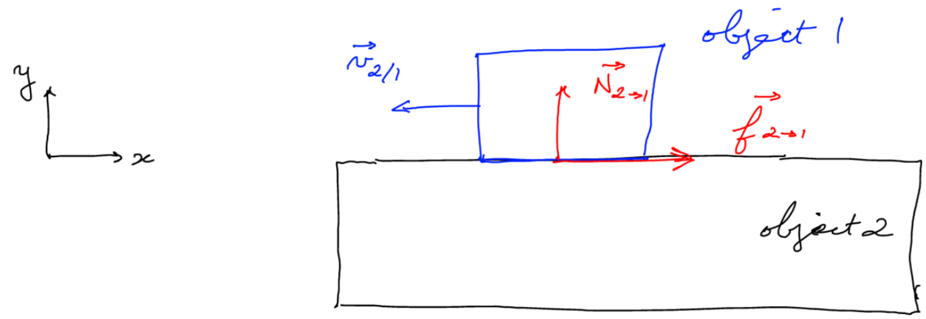


In the context of non-inertial frames of reference, what that means is that the centrifugal force can always be written as $\vec{F} = m\frac{v^2}{R}\hat{N}$ where v is the speed of the frame of reference, R is the radius of curvature of the trajectory of the frame of reference, $\hat{N}$ is the normal unit vector of the trajectory of the frame of reference, and m is the mass of the object feeling the centrifugal force.

1.10 Friction & Drag

Solid friction – Moving case

Solid friction forces are the ones that appear at the contact surface between two solids:



$\vec{f}_{2\to1}$ is the friction force exerted by object 2 on object 1. It is parallel to the contact surface. It opposes the motion, i.e., it points in the direction opposite to the velocity of object 2 relative to object 1 ($\vec{v}_{2/1}$). Its magnitude is proportional to the normal force exerted by object 2 on object 1: $f_{2\to1} = \mu N_{2\to1}$ where μ is the friction coefficient. μ depends on the nature of the two surfaces in contact (what materials they're made of, how smooth they are, how dry they are, etc), but it is constant through the motion; in particular, it doesn't depend on the velocity.

Problem 21: Weight on an incline with solid friction.

Recompute the acceleration of the skier from problem 10 with a friction coefficient μ between the skis and the snow.

Viscous drag

An object moving slowly through a viscous fluid experiences a drag force proportional to its velocity vector $\vec{v}$: $\vec{F} = -\alpha\vec{v}$ where α is a positive constant called the viscous drag coefficient. Like μ above, α is constant in the sense that it doesn't change through the motion, but it does depend on the shape and size of the object and the nature of the fluid (air, water, etc).

Situations in which this type of drag force applies include a marble falling through syrup and a single-cell organism moving through water.

If the object happens to be spherical, then $\alpha = 6\pi\eta R$ where η is the dynamic viscosity of the fluid (a positive number that you can look up online; for example, search "water dynamic viscosity").

Problem 22: Free fall with viscous drag.

A sphere with density ρ (density $= \frac{\text{mass}}{\text{volume}}$) falls vertically (along the y axis) through a viscous fluid with dynamic viscosity η. We neglect buoyancy, so the only two forces are the sphere's weight and the viscous drag force.

1. The sphere's initial velocity is $\vec{v} = \vec{0}$. What is the acceleration vector? Show that the sphere gains speed in the expected direction.
2. Eventually, the velocity reaches a steady value and no longer changes. What is $\vec{a}$ then? What does that imply for $\vec{v}$? Show that $\vec{v}$ has the expected direction. $|\vec{v}|$ is called the terminal speed. What happens to it if we double the sphere's size?

Turbulent drag

When the fluid is less viscous, or the motion is faster, e.g., a car moving through air, the magnitude of the fluid drag force is proportional to v^2 rather than v: $\vec{F} = -\alpha v\vec{v} = -\alpha v^2\hat{v}$ where α is a different friction coefficient that depends on the shape and size of the object and the density of the fluid.

Similar to the viscous drag, Newton's second law is difficult to integrate when this force is present, however the terminal velocity is easy to compute by writing that the net force (propulsive + drag) must be $\vec{0}$ once the velocity has reached its final value.

2 Energy

2.1 Kinetic energy

Every object has a kinetic energy defined as

$$K = \frac{1}{2}mv^2$$

where m is the object's mass and v is the object's speed.

The formula is sometimes written $K = \frac{1}{2}m\vec{v}^2$, which is the same thing because $\vec{v}^2$ means $\vec{v}\cdot\vec{v}$ (dot product), the magnitude of $\vec{v}$ is v, and the angle between $\vec{v}$ and itself is 0, therefore $\vec{v}^2 = \vec{v}\cdot\vec{v} = v.v.\cos(0) = v^2$.

The SI unit of energy is the Joule, symbol J.

Problem 23: Joule.

Show that a Joule is the same thing as a $(\mathrm{kg\,m^2/s^2})$.

Reading

Read section 7.2 from University Physics Volume 1. Make sure you understand the examples in it. If you don't, bring it up.

Problem 24: Horizontal kinetic energy.

Example 7.8 in University Physics Volume 1, section 7.2 introduces the horizontal kinetic energy, obtained by using v_x instead of v in the regular kinetic energy formula. It's not very common (I had never heard of it before seeing it in this book), but it raises a question. If you add the horizontal kinetic energy and the vertical kinetic energy, do you recover the full kinetic energy? Prove your answer.

2.2 Work-energy theorem

Rate of change of the kinetic energy

When an object experiences a net force $\vec{F}$, its velocity changes, which in turn can change it kinetic energy. To be more precise, the rate of change of K is

$$\frac{dK}{dt} = \vec{F}\cdot\vec{v}$$

Connection with the tangential/normal decomposition

To understand why this is the right kind of formula to write, you have to think about the tangential/normal decomposition. $K = \frac{1}{2}mv^2$ does not depend on the direction of $\vec{v}$, only on its magnitude v. As a result, $\frac{dK}{dt}$ does not care about the rate at which $\vec{v}$ changes direction, only about the rate at which the speed changes. Changes of direction are controlled by the normal part of the acceleration, which is controlled by the normal part of the net force through Newton's second law. Changes of speed are controlled by the tangential part of the acceleration, which is controlled by the tangential part of the force through Newton's second law. Therefore, $\frac{dK}{dt}$ does not depend on the normal part of the net force, only on its tangential part.

The dot product $\vec{F}\cdot\vec{v}$ achieves exactly that. By definition, the tangential direction is along $\vec{v}$. The dot product is equal to the projection of $\vec{F}$ onto $\vec{v}$ times the magnitude of $\vec{v}$. Projecting $\vec{F}$ onto $\vec{v}$ discards the component of $\vec{F}$ that is perpendicular to $\vec{v}$, which is the normal component, while keeping the component parallel to $\vec{v}$, which is the tangential component.

In other words, the dot product discards the part of $\vec{F}$ that causes direction changes (the normal one) while keeping the one that causes changes of speed (the tangential one), and that's exactly what we need because K depends on the speed but not on the direction.

Work-energy theorem

Integrating the rate of change between two times t_1 and t_2 gives us the net change of K between those two times:

$$K(t_2) - K(t_1) = \int_{t_1}^{t_2} \vec{F}(t) \cdot \vec{v}(t)\, dt.$$

The right-hand side of this equation is called the work of the net force, and the formula as a whole is known as *work-energy theorem.*

Note that this is not useful yet. The formula tells us that in order to know how much $\frac{1}{2}mv^2$ is going to change between t_1 and t_2, we can integrate $\vec{F} \cdot \vec{v}(t)$ between t_1 and t_2. In order to compute that integral, we need to know $\vec{F}$ and $\vec{v}$ at any time between t_1 and t_2. But if we know $\vec{v}$ at any time between t_1 and t_2, we don't need to compute the integral, we can just compute $\frac{1}{2}mv(t_2)^2$ and $\frac{1}{2}mv(t_1)^2$ directly. So what's the point? The answer is: this is really just a foundation on which we're going to build something useful. But first, we have to make sense of the work-energy theorem and practice working with it.

Case of a constant force

If the force is constant (for example, the weight), then the work-energy theorem takes a simpler form. We can take $\vec{F}$ out of the integral and integrate $\vec{v}$ explicitly:

$$K(t_2) - K(t_1) = \vec{F} \cdot \int_{t_1}^{t_2} \vec{v}(t)\, dt = \vec{F} \cdot \left[\vec{r}(t_2) - \vec{r}(t_1)\right].$$

This is often written in the more compact form $\Delta K = \vec{F} \cdot \Delta\vec{r}$ where ΔK is the change in the object kinetic energy K, $\vec{F}$ is the net force exerted on the object, and $\Delta\vec{r}$ is the displacement vector of the object. The idea here is that if the object moved by $\Delta\vec{r}$ while experiencing a net force $\vec{F}$, then we know that its kinetic energy must have changed by ΔK. Since K is directly related to the object's speed, we can use that to predict how much that speed has changed.

Problem 25: Braking distance.

A car of mass m, initially moving at $\vec{v} = v_0\hat{x}$, experiences a constant braking force $\vec{F} = -F\hat{x}$.

1. Compute the initial kinetic energy.
2. Compute the change of kinetic energy after traveling a distance d along $\hat{x}$.
3. What is the car's kinetic energy when it reaches a full stop? Use that to compute the braking distance.
4. Show that the result is consistent with that of the braking distnace problem from the kinematics module (kinematics problem 10).

Problem 26: Projectile motion.

Note: This problem sets up the next section about potential energy.

1. An object of mass m is dropped from a height h with no initial velocity. Compute its initial kinetic energy. Compute the work of its weight between $y = h$ and $y = 0$ (the ground). What is its kinetic energy right before it hits the ground? What about its speed?
   Hint: To compute the work, use the simplification we discussed above in "Case of a constant force".
2. Answer the same questions with an unspecified initial velocity $\vec{v}_0 = \begin{bmatrix} v_{0x} \\ v_{0y} \end{bmatrix}$.
3. How does ΔK compare between questions 1 and 2? Try to articulate why this is the case.

2.3 Potential energy and conservation of energy in a free fall

When the only force is the weight $\vec{w} = m\vec{g}$, we can rewrite the work-energy theorem in a much more useful form. First, we simplify the work using the fact that $\vec{w}$ is constant:

$$K(t_2) - K(t_1) = \int_{t_1}^{t_2} \vec{w} \cdot \vec{v}(t)dt = \vec{w} \cdot \int_{t_1}^{t_2} \vec{v}(t)dt = \vec{w} \cdot \left[\vec{r}(t_2) - \vec{r}(t_1)\right] = \vec{w} \cdot \vec{r}(t_2) - \vec{w} \cdot \vec{r}(t_1)$$

Let's assume the usual coordinate system with the y axis pointing up vertically, then

$$K(t_2) - K(t_1) = \begin{bmatrix} 0 \\ -mg \end{bmatrix} \cdot \begin{bmatrix} x(t_2) \\ y(t_2) \end{bmatrix} - \begin{bmatrix} 0 \\ -mg \end{bmatrix} \cdot \begin{bmatrix} x(t_1) \\ y(t_1) \end{bmatrix} = -mgy(t_2) + mgy(t_1)$$

At this point, the right-hand side still represents the work of the weight between t_1 and t_2. What's remarkable is that that work does not depend at all on what the object does between t_1 and t_2, only where it is at t_1 and where it is at t_2.

The next step is to put the terms that depend on t_1 on one side and the terms that depend on t_2 on the other:

$$K(t_2) + mgy(t_2) = K(t_1) + mgy(t_1)$$

We now define the *gravitational potential energy*

$$U = mgy$$

and rewrite the work-energy theorem as

$$K(t_2) + U(t_2) = K(t_1) + U(t_1)$$

What this means is that $K + U$ does not change between t_1 and t_2. Since we didn't assume anything about t_1 or t_2, this is true for any values of t_1 and t_2 (as long as $\vec{w}$ is the only force on the object between t_1 and t_2), therefore $K + U$ remains constant throughout the free fall.

$E \equiv K + U$ is called the *mechanical energy* of the free-falling object, and we say that it is *conserved* (meaning it doesn't change).

Problem 27: Free fall and energy conservation.

An object of mass m is dropped from a height $y = h$ with no initial velocity.

1. Compute its speed when it reaches $y = 0$ by writing that $K + U$ doesn't change between the moment the object is dropped and the moment it reaches the ground.
2. Can we use the constantness of $K + U$ to predict v after the object hits the ground? Why or why not?

2.4 Multiple forces

Earlier we introduced the work of the *net* force between times t_1 and t_2:

$$W = \int_{t_1}^{t_2} \vec{F}_{\text{net}}(t) \cdot \vec{v}(t)\,dt.$$

This definition is actually more general than that. We can use it to compute the work of any force. If an object experiences multiples forces, say $\vec{F}_1$ and $\vec{F}_2$, then we can compute the work of each force individually:

$$W_1 = \int_{t_1}^{t_2} \vec{F}_1(t) \cdot \vec{v}(t)\,dt \text{ is the work of } \vec{F}_1 \text{ between } t_1 \text{ and } t_2.$$
$$W_2 = \int_{t_1}^{t_2} \vec{F}_2(t) \cdot \vec{v}(t)\,dt \text{ is the work of } \vec{F}_2 \text{ between } t_1 \text{ and } t_2.$$

We can also compute the work of the sum of the two forces: $W_{1+2} = \int_{t_1}^{t_2} \left[\vec{F}_1(t) + \vec{F}_2(t)\right] \cdot \vec{v}(t)\,dt$. If we distribute the dot product over the addition, then use the linearity of the integral, we see that the work of the sum of two forces is the sum of the works of the individual forces:

$$W_{1+2} = \int_{t_1}^{t_2} \left[\vec{F}_1(t) \cdot \vec{v}(t) + \vec{F}_2(t) \cdot \vec{v}(t)\right] dt = \left[\int_{t_1}^{t_2} \vec{F}_1(t) \cdot \vec{v}(t)\,dt\right] + \left[\int_{t_1}^{t_2} \vec{F}_2(t) \cdot \vec{v}(t)\,dt\right] = W_1 + W_2$$

This is not limited to two forces; it works for any number of forces. Since the net force exerted on an object is the sum of the individual forces exerted on that object, it follows that **the work of the net force** (the one that shows up in the work-energy theorem) **is equal to the sum of the works of all the individual forces**.

Notes:

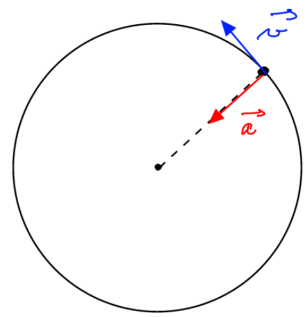
- Like the average velocity or the average acceleration in the kinematics module, the work is always computed between two times. It is not an instantaneous quantity. There is no such thing as the work of a force "at time t".
- University Physics Volume 1 introduces the work a little differently. Instead of framing it as the work between two times t_1 and t_2, they frame it as the work between two points A and B, or along the path from A to B. The points in question are the positions of the object at t_1 and t_2, and the path is the trajectory of the object between those two points. Mathematically, they write the work as an integral with respect to the position vector $\vec{r}$: $W = \int_A^B \vec{F}(\vec{r}) \cdot d\vec{r}$. This is completely equivalent to our definition, it just uses a fancier type of integral called line integral or curve integral. The equivalence between the two makes sense if you think of $\vec{v} = \dfrac{d\vec{r}}{dt}$ as a regular division and write $\displaystyle\int \vec{F} \cdot \vec{v}\,dt = \int \vec{F} \cdot \frac{d\vec{r}}{dt}\,dt = \int \vec{F} \cdot d\vec{r}$. In this course, though, I try to stick to the simpler time integral definition.

Forces that do not work

In some situations you can tell that the work of a specific force or group of forces is zero without computing the integral:

- The net force in any motion in which the speed is constant. There are two ways to see why the work has to be zero in this case:
  - Using the work-energy theorem: If the speed is constant, then the kinetic energy is constant, therefore the change of kinetic energy between any two times is zero, therefore the work of the net force during those two times, which is equal to the change of kinetic energy, has to be zero as well.
  - Using the tangential/normal decomposition: Constant speed means the acceleration has no tangential component, i.e., it is perpendicular to the motion: $\vec{a} \perp \vec{v}$. By Newton's second law, so is the net force: $\vec{F}_{\text{net}} \perp \vec{v}$. It follows that $\vec{F}_{\text{net}} \cdot \vec{v} = 0 \implies W = 0$.

Example: net force in a uniform circular motion. Note that it's possible for individual forces do have a non-zero work in a uniform circular motion (as long as they're not perpendicular to $\vec{v}$, i.e., radial). However the net force can never work, because in order to do work it would need a tangential component, at which point the motion would no longer be uniform.

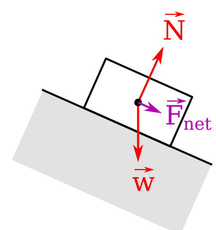


- Any force that remains perpendicular to the motion:

$$\vec{F} \perp \vec{v} \implies \vec{F} \cdot \vec{v} = 0 \implies W = \int \vec{F} \cdot \vec{v}\, dt = 0$$

  Note that there is some overlap with the previous case. For example, the net force in the uniform circular motion above is remains perpendicular to the motion throughout the motion. The difference is that we're now extending this to any force that is perpendicular to $\vec{v}$.

  For an example, consider the normal force on an object sliding down an incline without friction (similar to problem 10). This time the net force points in the direction of motion (downslope), i.e., it has a tangential component, so the work is non-zero. The result of that work is that the object speeds up down the slope, which means its kinetic energy is increasing. The normal force, however, is perpendicular to the motion the whole time, so its work is 0:



$$\vec{N} \perp \vec{v} \implies W_{\vec{N}} \equiv \int \vec{N} \cdot \vec{v}\, dt = 0.$$

- Any force applied on an object that doesn't move. Imagine you're holding an object still in your hand. In order to keep the object still, your hand must exert a force on the object that opposes and cancels the object's own weight. However, the work of the force exerted by your arm is zero because the object is not moving:

$$\vec{v} = \vec{0} \implies \vec{F} \cdot \vec{v} = 0 \implies W = 0.$$

  This is a bit counter-intuitive because if you hold an object steady for a while, you're definitely going to feel your arm doing some work. What's happening here that there is some work being done, but it is not done on the object. It is done by microscopic constituents of your arm muscles onto other microscopic constituents of those same arm muscles. It's a consequence of the design of muscles that it costs energy to maintain contraction even if there is no motion.

Problem 28: Sign of the work.

For each of the forces below, analyze the sign of $\vec{F} \cdot \vec{v}$, then discuss the sign of the work.

1. The weight of a free-falling object that was dropped without initial velocity.
2. The weight of a free-falling object that was thrown up, while it's still moving up.
3. The weight of a free-falling object that was thrown up, between a time when it's moving up and a time when it's moving down.
4. The fluid drag force on an object moving through a fluid.
5. The tension exerted by the string on the rock at the end of a sling before it's released (assuming the motion is circular).

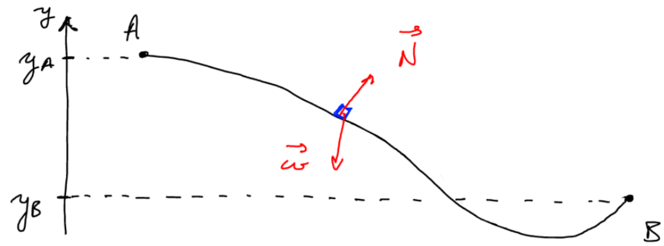
Problem 29: When the normal force does work.

There are cases in which the work of the normal force is nonzero. For example, when an object falls on the floor. During what part of the motion is that work done? What is different about this situation compared to the sliding object discussed above, which received no work from the normal force?

2.5 Potential energy and conservation of energy.

Problem 30: Frictionless slide 1.

Here is another situation in which the conservation of energy goes a long way. An object slides down a solid surface. There is no friction, only gravity and the normal force.



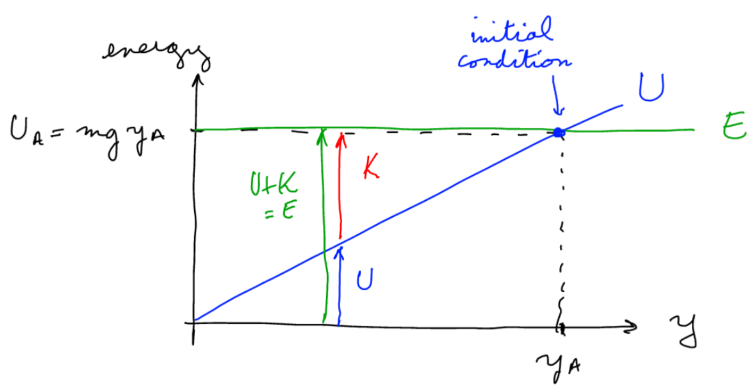
1. Write the work-energy theorem in its most general form. Split the work into the work of $\vec{w}$ and the work of $\vec{N}$.
2. Show that the work of $\vec{N}$ is zero.
3. Show that $K + mgy$ is constant.
4. The object starts at A with no initial speed. Compute its speed later on as a function of g and y. Where is it the fastest?

2.6 Potential energy diagrams

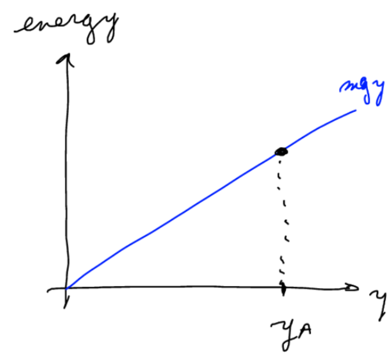
Vertical (1D) free fall

We just showed that the mechanical energy $E = \frac{1}{2}mv^2 + mgy$ is constant.

When we specify an initial condition, i.e., an initial v and an initial y, that sets E. As the object moves up or down, y changes, which changes U. Whatever changes U undergoes, K must undergo the exact opposite change to keep $E = U + K$ constant and equal to its initial value. We can visualize that on a $U(y)$ graph:



Problem 31: Vertical free fall.



Use the potential energy diagram to explain the evolution of y and v_y (when they increase, decrease, change sign, etc) in each of the following cases:

1. $y(0) = y_A$ and $v_y(0) = 0$.
2. $y(0) = y_A$ and $v_y(0) < 0$.
3. $y(0) = y_A$ and $v_y(0) > 0$.

Forbidden zone

Any part of a potential energy diagram where $U > E$ is inaccessible to the object because $K = E - U$ would have to be negative, but $K = \frac{1}{2}mv^2$ can only ever be positive.

Potential energy and force

The gravitational potential energy is defied from the work of the gravitational force, which is a type of integral of the force. Conversely, the force can be recovered from the potential energy by taking derivatives:

$$-\frac{dU}{dx} = -\frac{d}{dx}(mgy) = 0 \qquad \text{is the } x \text{ component of } \vec{w}$$
$$-\frac{dU}{dy} = -\frac{d}{dy}(mgy) = -mg \qquad \text{is the } y \text{ component of } \vec{w}$$

Here $\frac{dU}{dx}$ means "derive U with respect to x as if y was a constant", and $\frac{dU}{dy}$ means "derive U with respect to y as if x was a constant". Those are called partial derivatives. The proper notation is $\frac{\partial U}{\partial x}$ and $\frac{\partial U}{\partial y}$, but I don't mind if you use d instead of ∂ in this course. Here is a quick example to illustrate this concept. Let $f(x,y) = 3x - 2y$, then $\frac{\partial f}{\partial x} = 3$ and $\frac{\partial f}{\partial y} = -2$. When computing $\frac{\partial f}{\partial x}$, we treated y as a constant whose derivative is 0. Conversely, when computing $\frac{\partial f}{\partial y}$ we treated x as a constant whose derivative is 0.

Graphically, if you're plotting U against x, then the slope is $\frac{\partial U}{\partial x}$, which tells you about the x component of the force. If you're plotting U against y, then the slope is $\frac{\partial U}{\partial y}$, which tells you about the y component of the force. If the curve goes up, then the slope is positive. Since the component of the force is minus the slope, the force is negative. Conversely, if the curve goes down, then the slope is negative, therefore the force component is positive. Either way, the force points in the direction in which the potential energy decreases. If the slope is zero (horizontal tangent), then the component of the force at that location is 0.

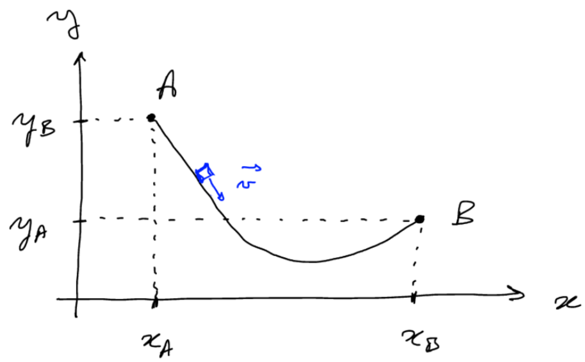
Turning points

When an object reaches the edge of the forbidden zone (an intersection of E and U on the potential diagram), it cannot go any further. This is called a turning point. We can use the relationship between energy and force we just discussed to understand what happens.

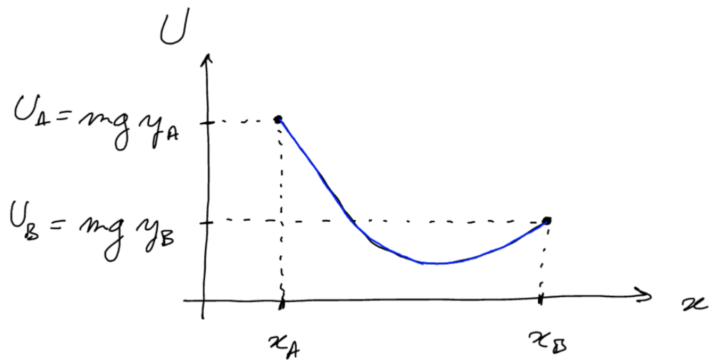
Example: Question 3 from problem 31 above.

Frictionless slide

Profile of the slide:

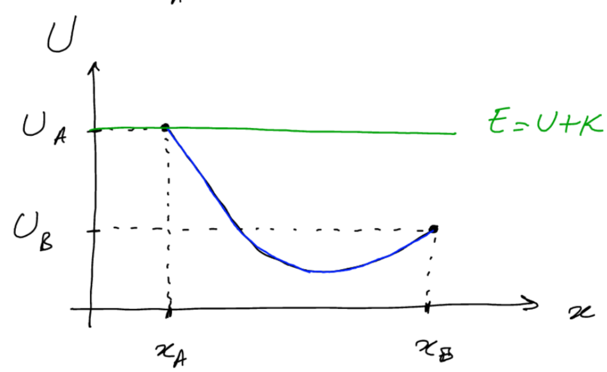


Potential energy:



Conservation of energy:

No initial velocity $\Longrightarrow K_A = 0 \Longrightarrow E = U_A$.
At any point later: $U + K = E \Longrightarrow K = E - U$.



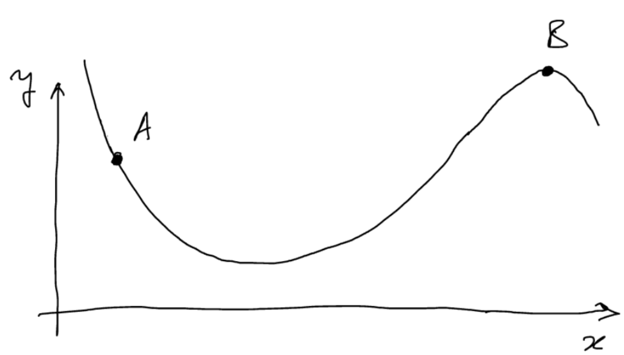
Problem 32: Frictionless slide 2.

Using the potential energy diagram above:

1. Compute the speed at B.
2. Where is the speed the largest?

Problem 33: Frictionless slide 3.

1. The object starts at A with no initial speed. Predict as much as you can of its motion.
2. What is the minimum amount of initial speed v_A needed to go over the hump at B?
3. What happens if the initial speed is above the threshold from question 2, but backward (to the left)?



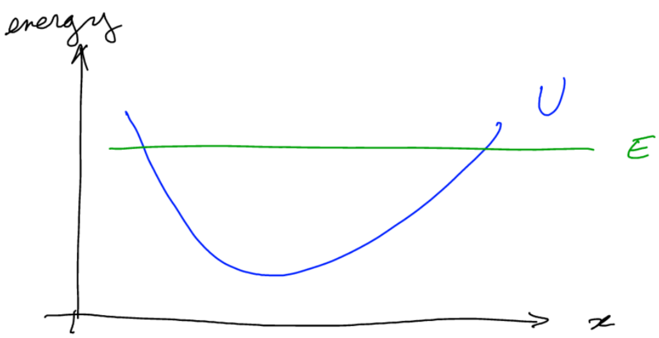
2.7 Non-conservative forces

Problem 34: Slide with friction.

The set-up is the same as the frictionless slide above except there is now a drag force $\vec{F} = -\alpha v \vec{v}$ where α is a positive constant (turbulent drag coefficient).

1. Write the work of $\vec{F}$ between two arbitrary times t_1 and $t_2 > t_1$. Do not attempt to compute the integral.
2. What is the sign of the work of $\vec{F}$?
3. Write the work energy theorem. Leave the integral of $\vec{F}$ as-is. Treat the rest like we did in the frictionless case. Based on the sign of the work of $\vec{F}$, what happens to $E = K + U$ between t_1 and t_2?

Potential energy diagram with friction



Work of a constant tangential force

The work of friction forces is often difficult to compute from the general work formula. One exception is when the friction force is constant: $\vec{F} = -F_0\hat{v}$ where F_0 is a positive constant ($-\hat{v}$ is a unit vector pointing opposite $\vec{v}$, so this is just a way to write that $\vec{F}$ has constant magnitude F_0 and points opposite the velocity $\vec{v}$ at all times). Then the work is:

$$W = \int_{t_1}^{t_2} (-F_0\hat{v}) \cdot \vec{v}(t)\,dt = -F_0 \int_{t_1}^{t_2} v(t)\,dt$$

where $v = |\vec{v}|$ is the speed. Since the speed is the amount of distance traveled per unit time, integrating it with respect to time between t_1 and t_2 yields the distance d traveled between t_1 and t_2:

$$W = -F_0 d$$

Since distances are always positive, the work of $\vec{F}$ is always negative. It had to be that way if you think about the work-energy theorem: a negative work means the friction force works to decrease the kinetic energy, i.e., the speed. You'll never see a friction force speed an object up. They only ever slow things down.

The same simplifications apply to propulsive forces that are tangential with constant magnitude: if $\vec{F} = F_0\hat{v}$ (note there is no minus in front of F_0 this time, so this force is in the direction of $\vec{v}$, i.e., in the direction of motion, i.e., it's a propulsive force), then $W = F_0 d$ where d is the distance traveled.

Problem 35: Car consumption.

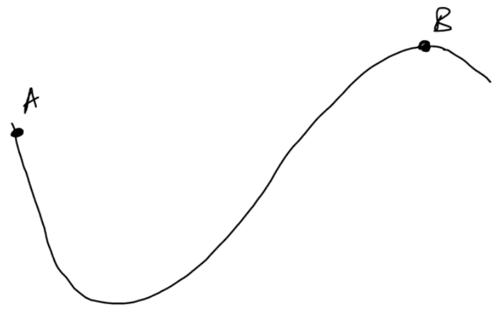
A car goes 25 miles on 1 gallon of gas.

1. Look up the energy available in a gallon of gas, in Joules. A ballpark value is fine; the exact value depends on the type of gas.
2. Assuming 25% of that energy is transfered to the car in the form of a constant propulsive force, what is that force (in N)?
3. Assuming the speed is constant, what is the net friction force (in N)?

Problem 36: Propelled frictionless slide.

An object of mass m is dropped without initial speed at A. An engine provides a propulsive force.

1. What minimum amount of work must the engine provide to reach B?
2. What is the speed at B if the engine provides exactly that minimum amount?

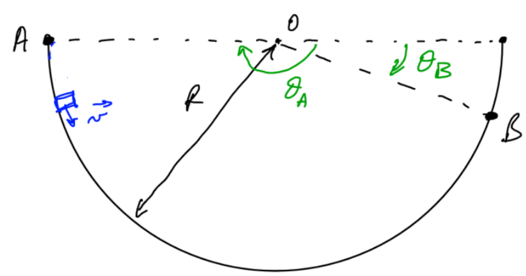


Problem 37: Height loss in a slide with friction.

An object of mass m is dropped without initial speed at A. A constant friction force F_0 causes it to turn around at point B located a little lower than A. The slide is circular with radius R. Its center O is level with A.

What is the height difference between A and B?

Hint: Write y_A and y_B in terms of θ_A and θ_B. At the end, assume that $\sin\theta_B \approx \theta_B$ (true if $\theta_B \ll 1$, i.e., if F_0 is small).



2.8 Power

The power of a force is the rate of change:

$$P = \frac{dW}{dt} = \vec{F} \cdot \vec{v}$$

The SI unit of power the Watt, symbol W. 1 W = 1 J/s.

As always with rates of change, the definition above is that of the instantaneous power, but we can also define the average power between times t_1 and t_2:

$$\overline{P} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} P(t)dt = \frac{W}{\Delta t}$$

where W is the work done between t_1 and t_2 and $\Delta t = t_2 - t_1$ is the duration over which that work was done.

2.9 Conservative forces

Here are some of the things we've learned about the work of the weight force:

(1) The work of the weight force between t_1 and t_2 can be written as $U(t_1) - U(t_2)$ where U is the gravitational potential energy ($U = mgy$).

(2) If the weight is the only force (free fall), then $K + U$ is conserved.

(3) The components of $\vec{w}$ are derivatives of $-U$: $w_x = -\dfrac{dU}{dx}$ and $w_y = -\dfrac{dU}{dy}$.

As it turns out, there are other forces that obey the same properties, except with a different potential energy formula. Those forces are said to be *conservative* (because of the energy conservation property). A force $\vec{F}$ is conservative if

(1) There exists a function $U(\vec{r})$ of the position $\vec{r}$ such that the work of $\vec{F}$ between t_1 and t_2 is equal to $U(\vec{r}(t_1)) - U(\vec{r}(t_2))$.

(2) If $\vec{F}$ is the only force acting on an object, then $K + U$ is conserved.

(3) The components of $\vec{F}$ are derivatives of $-U$: $F_x = -\dfrac{dU}{dx}$ and $F_y = -\dfrac{dU}{dy}$.

If any of those three properties is satisfied, then the other two are satisfied as well. U is called the potential energy associated with $\vec{F}$. We also say that $\vec{F}$ derives from U.

Forces that do not obey those properties are said to be *non-conservative*.

Notes:

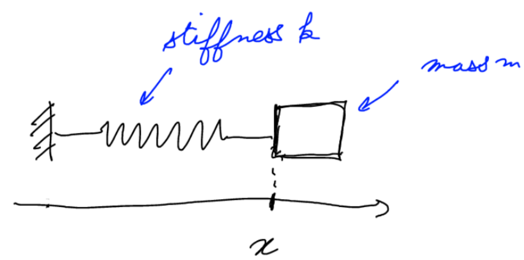
- U is a function of the object's position coordinates x and y. It can also depend on constants like m, g, etc. If the components of the force are derivatives of U with respect to x and y, then they themselves are function of x, y, and constants like m, g, etc. Therefore, a minimal (necessary but not sufficient) condition for a force to be conservative is that it can only depend on the position and constants like m, g, etc. In particular, it cannot depend on the velocity, like drag forces.
- The normal force is not conservative. It may appear to be because its work is often 0, which satisfies condition (1) above with $U = 0$. However, that is not always true. A force is only conservative if (1) is satisfied in every possible scenario, which is not the case for $\vec{N}$.

2.9.1 Elastic potential energy

The spring force is conservative. The corresponding potential energy is $U = \dfrac{1}{2}k(\ell - \ell_0)^2$.

Problem 38: Elastic potential energy.

1. Sketch the elastic potential energy $U(x)$.
2. The spring starts at rest ($\ell = \ell_0$). It's given a kick to the left that sets its initial speed to v_0. Add the initial condition and the mechanical energy on the sketch of question 1.
3. Use the potential energy diagram you just drew to predict and describe the motion.
4. What are the shortest and longest lengths reached by the spring during its motion?
5. Where is the spring the fastest? Where is it the slowest?



2.9.2 Work-energy theorem with multiple conservative forces

If there are multiple conservative forces $\vec{F}_a$, $\vec{F}_b$, ... with potential energies U_a, U_b, ..., then

$$K_2 - K_1 = \int_{t_1}^{t_2} \vec{F}_a \cdot \vec{v}\,dt + \int_{t_1}^{t_2} \vec{F}_b \cdot \vec{v}\,dt + ... = (U_{a1} - U_{a2}) + (U_{b1} - U_{b2}) + ..$$
$$\implies K_2 + U_{a2} + U_{b2} + ... = K_1 + U_{a1} + U_{b1} + ...$$

where the ellipses stand for the work of (or the potential energy associated with) optional additional conservative forces. K_1 is a short-hand for $K(t_1)$, the object's kinetic energy at time t_1, and U_{a1} means $U_a(\vec{r}(t_1))$, the potential energy associated with the force $\vec{F}_a$, evaluated at the object's position at time t_1. Similar definitions apply for K_2, U_{a2}, U_{b1}, U_{b2}, ... If every force is conservative, then this is proof that mechanical energy is conserved as long as we include the potential energy of every conservative force in our definition of the mechanical energy

$$E_2 = E_1 \quad \text{where} \quad E = K + U_a + U_b + ...$$

If there are both conservative and non-conservative forces, then their work must be evaluated separately and we get

$$\Delta E \equiv \Delta\left(K + U_a + U_b + ...\right) = \int_{t_1}^{t_2} \vec{F}_{nc}(t) \cdot \vec{v}(t)\,dt$$

where the ellipsis stands (again) for the potential energies associated with hypothetical additional conservative forces, and $\vec{F}_{nc}$ is the sum of all the non-conservative forces. What we've done here is leave the work of the non-conservative forces on the right-hand side of the equation where the original work-energy theorem had them, but move the work of the conservative forces to the left after recasting it a a change of potential energy.

The mechanical energy is still defined as $E = K + U_a + U_b + ...$, however it is no longer conserved.

Problem 39: Spring gun.

See problem 43 from University Physics Volume 1 chapter 8 (link). To simplify the calculation, assume the 5m height is measured from the initial position of the ball (rather than from end of the spring in its rest position, as shown in the book).

Problem 40: Looping slide.

See problem problem 42 from University Physics Volume 1 chapter 8 (link).
Additional questions:

3. What is the normal force $\vec{N}$ exerted by the track at the top of the loop?
4. Assuming the block rests on the track but is not held by it, then the normal force can only ever push the block away from the track. If you find that $\vec{N}$ points toward the track, i.e., the track pulls the block back, what it really means is that the assumption we made to compute the normal acceleration (that the object follows the track) is no longer valide, i.e., the object has left the track.

   Imagine the initial height is h instead of $4R$. What is the minimal height h required for the object to not fall off the track at the top of the loop?

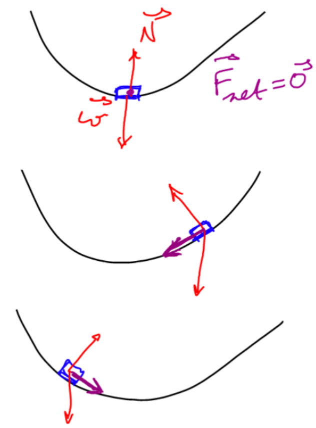
2.10 Equilibrium and stability

Definitions

An object is said to be in equilibrium (also static equilibrium, or mechanical equilibrium) if it is not moving $\vec{v} = \vec{0}$ and feels no net force ($\vec{F}_{\text{net}} = \vec{0}$). The idea is that if $\vec{F}_{\text{net}} = \vec{0}$, then $\dfrac{d\vec{v}}{dt} = \vec{0}$, therefore $\vec{v} = \vec{0}$ remains constant, i.e., the object remains static.

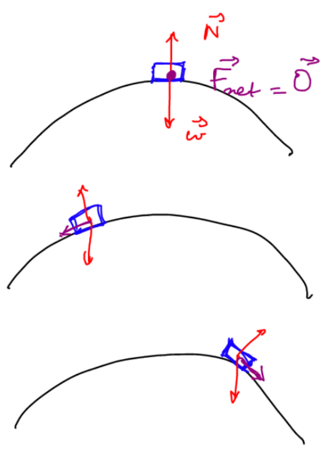
An equilibrium point is a location where the object, if it has no motion ($\vec{v} = \vec{0}$), will experience no net force ($\vec{F}_{\text{net}} = \vec{0}$). In other words, a location where the object could be in equilibrium, hypothetically.

In practice things never stay exactly in place. There is always some kind of small unpredictable motion cause by, e.g., air draft or vibrations. In order to know whether an object is truly going to remain at an equilibrium point once it has stopped there, we need to think about what happens to the object when it's moved ever so slightly from that equilibrium point. There are three possible scenarios, which we illustrate below with a slide:



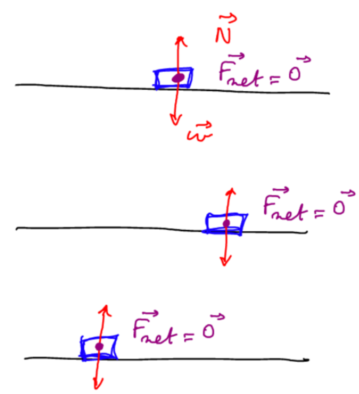
- Stable equilibrium:

  The net force at the bottom of the slide is $\vec{0}$, therefore that's an equilibrium point. When the object moves away from the equilibrium point, the force pushes it back toward the equilibrium point (the bottom of the slide). We say that the equilibrium point is stable.



- Unstable equilibrium:

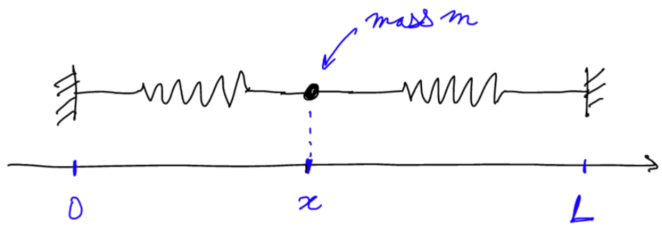
  The net force at the top of the slide is also $\vec{0}$, therefore that's also an equilibrium point. When the object moves away from the equilibrium point, the force pushes it further away from the equilibrium point (the top of the slide). We say that the equilibrium point is unstable.



- Neutral equilibrium:

  The slide is flat. The net force is zero everywhere. When the object moves away from the equilibrium point, the net force remains zero; it does not push the object either toward or away from the original equilibrium point. We say that the equilibrium point is neutral.

Problem 41: Double spring, part 1.



Both springs have the same rest length ℓ_0 and stiffness k. It's a 1D problem.

1. Compute the net force F on the mass as a function of x, L, ℓ_0, and k.
2. Where is the equilibrium point? Let's call it x_0.
3. What is the sign of F when $x > x_0$? What is the sign of F when $x < x_0$? What kind of equilibrium is this?
4. The directions of the forces are different when $L < 2\ell_0$ compared to the $L > 2\ell_0$ case. Explain what changes in plain English. Does it have an impact on the stability of the equilibrium?

Problem 42: Double spring, part 2.

The system is the same as in part 1, but this time we focus on energies rather than forces.

1. Compute the potential energy U of the system as a function of x, L, ℓ_0, and k.
2. Show that $-\dfrac{dU}{dx}$ is the net force from part 1. What happens to $U(x)$ at equilibrium points?
3. Graph $U(x)$ with $k = 1\,\mathrm{N/m}$, $L = 2\,\mathrm{m}$, and $\ell_0 = 0.7\,\mathrm{m}$ using desmos (`https://www.desmos.com/calculator`) or a graphing calculator. Analyze the slope around the equilibrium point to show what kind of equilibrium this is.
4. What is the general rule for determining the stability of an equilibrium point from the potential energy diagram.

Equilibrium and potential energy

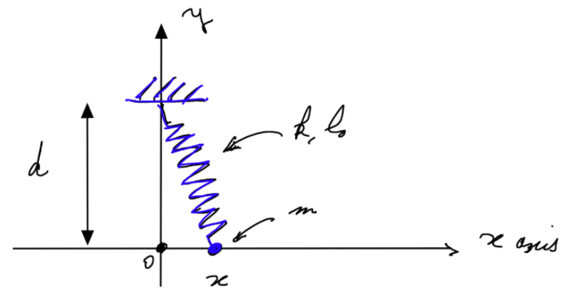
See the solution for question 4 of problem 42. In summary:

- Since the force is equal to minus the slope of the potential energy curve, equilibrium points are found where the slope of the potential energy is zero, i.e., at extrema of the potential energy curve.
- Stable equilibrium points are found at minima of the potential energy. That's because the force always points downslope of the potential energy, and near a minimum of the potential energy, downslope pushes you back towards the minimum.
- Unstable equilibrium points are found at maxima of the potential energy. That's because near a maximum, a force in the downslope direction pushes you away from the maximum.

Note: It may sound like we're just repeating the frictionless slide examples we used to define stable and unstable. In reality, by thinking about the potential energy curve rather than the physical shape of the slide, we've made the point much more general.

Problem 43: Constrained spring.

An object of mass m (blue dot) moves along a horizontal frictionless rail (the x axis). It is attached to a spring with stiffness k and rest length ℓ_0. The other end of the spring is attached to a point located at a height d above the origin.

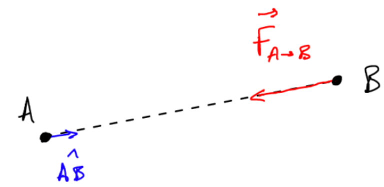


1. Sketch the three forces. Compute the x component of the net force (the only one that matters since the motion is restricted to the x axis).
2. Explain why the mechanical energy is conserved. Write the potential energy in terms of the problem's data.
3. Analyze $U(x)$ to locate equilibrium positions (values of x at which the dot is in mechanical equilibrium). How does this relate to question 1? What changes when $\ell_0 > d$ vs $\ell_0 < d$?
4. Determine the stability of each equilibrium position. Sketch $U(x)$ when $\ell_0 > d$ and when $\ell_0 < d$.
5. Most of those results could be predicted using intuition rather than math by thinking about the direction of the spring force and the spring rest configuration(s). Explain.

3 Law of gravitation

3.1 Gravitational force

The weight of an object is gravitational force exerted by the Earth on that object. On the surface of the Earth, it is $m\vec{g}$, and $\vec{g}$ is constant. This, however, is only an approximation. In reality, the gravitational force exerted by an object on another depends on the distance between their centers:



$$\underbrace{\vec{F}_{A\to B}}_{\text{gravitational force exerted by A on B}} = -G\,\frac{\overbrace{m_A}^{\text{mass of A}}\;\overbrace{m_B}^{\text{mass of B}}}{\underbrace{AB^2}_{AB=|\vec{AB}|=\text{distance between A and B}}}\;\underbrace{\hat{AB}}_{\text{unit vector from A to B}}$$

$$\hat{AB} = \frac{\vec{AB}}{AB}$$

universal gravitational constant

$$G = 6.67\times 10^{-11}\ \mathrm{N.m^2.kg^{-2}}$$

G, m_A, m_B, and AB are all positive numbers, therefore $\vec{F}_{A\to B}$ is along $-\hat{AB}$, i.e., along $-\vec{AB} = \vec{BA}$, i.e., it points from B to A. In other words, it's an attractive force.

The formula also implies that the gravitational force obeys Newton's third law. To get the force exerted by B on A, we swap A and B:

$$\vec{F}_{B\to A} = -\frac{G\,\overbrace{m_B m_A}^{=m_A m_B}}{\underbrace{BA^2}_{=|\vec{BA}|=|\vec{AB}|}}\;\underbrace{\hat{BA}}_{=\frac{\vec{BA}}{BA}=\frac{-\vec{AB}}{BA}} = -\frac{G\,m_A m_B}{AB^2}\left(-\hat{AB}\right)$$

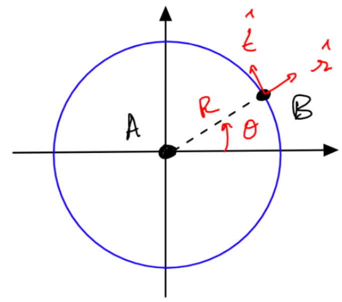
$$\Longrightarrow\quad \vec{F}_{B\to A} = -\vec{F}_{A\to B}$$

Reading

The weight formula we've been using, $\vec{w} = m\vec{g}$, is an approximation of the formula above valid near the surface of the Earth. Read about the relation between the two in University Physics Volume 1, Section 13.2.

Problem 44: Circular orbit.

A represents the Earth. B represents a satellite following a circular orbit around A with radius R at constant speed v. We work in the polar basis.



1. Write the acceleration vector of B in the polar basis.
2. Write $\vec{F}_{A\to B}$ in the polar basis.
3. Write Newton's second law. Use it to write v as a function of the other parameters. Does an object on a larger orbit go faster or slower? Explain why that makes sense (in plain English, but grounding your explanation in the result you just obtained).
4. What parameter does v somewhat remarkably not depend on? What is that reminiscent of?
5. Write v as a function of R and the orbiting period T, then write T^2 as a function of R, G, and $m_{A/B}$.
   Note: The result is a special case of Kepler's third law of planetary motion.
6. Geostationary satellites are satellites whose orbits always keeps them over the same spot on the surface of the Earth. They do that by occupying orbits whose period matches the period of revolution of the Earth. What altitude must they have to achieve that? *Hint: You'll need the radius of the Earth. Look it up online.*

3.2 Gravitational potential energy

The gravitational force $\vec{F}_{A\to B} = -\frac{Gm_A m_B \hat{AB}}{AB^2}$ is conservative. The corresponding potential energy is

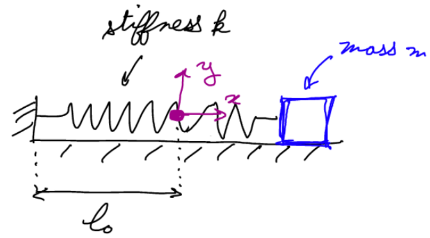
$$U = -\frac{Gm_A m_B}{AB}.$$

Problem 45: Escape energy.

Consider a rocket with mass m and distance to the center of the Earth r. Let's call m_E and R_E the mass and radius of the Earth.

1. Write the gravitational potential energy of the rocket. Sketch it as a function of r.
2. If the rocket has enough mechanical energy to reach any distance r from the Earth, no matter how far, then we say that it has escaped Earth's gravity. What is the minimal mechanical energy needed to achieve that?
3. The rocket starts off on the surface of the Earth without any kinetic energy. Based on question 2, assuming perfect efficiency (all energy received from the engine goes toward raising the mechanical energy), how much energy must the rocket's engine provide to allow the rocket to escape Earth's gravity?
4. For the purpose of launching into space, the velocity that matters is the velocity measured relative to the Earth's center rather than relative to the ground. Therefore, a rocket that launches from the equator actually starts with some kinetic energy because it travels $2\pi R_E$ per 24 hours. Compute the corresponding kinetic energy, then answer question 4 again accounting for this initial kinetic energy. How much difference does it make (how many percents less energy energy needed from the engine)?

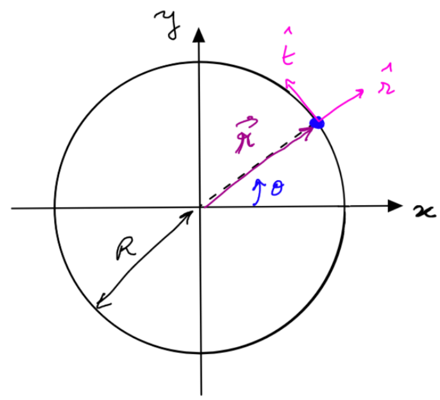
4 Harmonic motion



Problem 46: Horizontal spring.

The blue object has mass m. It moves frictionlessly on the horizontal surface underneath. The origin of the coordinate system at the rest position of the right end of the spring.

1. Use Newton's second law to compute the normal force $\vec{N}$ exerted by the horizontal ground on the object and show that $\frac{d^2x}{dt^2} = -\frac{k}{m}x$.
2. Show that $x(t) = A\cos(\omega t + \phi)$ where $\omega = \sqrt{\frac{k}{m}}$ is a solution of $\frac{d^2x}{dt^2} = -\frac{k}{m}x$ for any values of A and ϕ as long as neither depends on time (i.e., both remain constant throughout the motion). What is the period of this motion?
3. Sketch $x(t)$, $v(t)$, and $a(t)$ one above the other with their t axes lining up when $\phi = 0$ and $A > 0$. When is $v(t)$ largest, smallest, and zero? Show that we could have predicted as much using a potential energy diagram.
4. A and ϕ are determined by the initial position and velocity. First, write $x(0)$ and $v(0)$ as functions of A and ϕ.
5. Assume we release the spring at $x = x_0$ with no initial speed. Compute A and ϕ.
6. Another way to obtain A (but not ϕ) from the initial condition is to use the conservation of energy. Assume the spring's initial position and velocity are x_0 and v_0, respectively. What is v when $x = A$? Explain why we can used the conservation of energy here. Use the conservation of energy between the initial time and the time when $x = A$ to compute A as a function of x_0, v_0, m, and k.

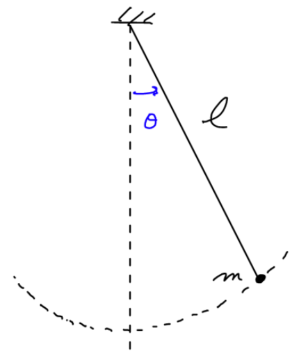


Problem 47: Non-uniform circular motion.

When studying uniform circular motion, we saw that the general form of the position vector of an object moving on a circle of radius R centered on the origin is $\vec{r}(t) = \begin{bmatrix} R\cos\theta(t) \\ R\sin\theta(t) \end{bmatrix}$. R is constant, but θ is not. In a uniform circular motion, $\theta(t)$ is a linear function of time. Here, we no longer assume linearity – $\theta(t)$ can be any function of t.

1. Show that $\vec{v} = R\dfrac{d\theta}{dt}\hat{t}$.
2. Show that the speed $v \equiv |\vec{v}|$ is equal to $\left|\dfrac{d(R\theta)}{dt}\right|$. What does $R\theta$ represent graphically? Explain why this makes sense if you think of v is the distance traveled per unit time.
3. Show that $\vec{a} = -R\left(\dfrac{d\theta}{dt}\right)^2\hat{r} + R\dfrac{d^2\theta}{dt^2}\hat{t}$. Show that it's consistent with the tangential/normal decomposition of $\vec{a}$.

Problem 48: Pendulum.



A mass m is attached at the end of a string of length ℓ hanging from a rigid support. θ is the angle between the string and the vertical axis.

1. How must we choose the x and y axes for ℓ and θ to be the polar coordinates of the mass?
2. Write the acceleration $\vec{a}$ of the mass in the polar basis in terms of ℓ and θ.
3. Draw the forces acting on the mass. Compute their coordinates in the polar basis.
4. Use Newton's second law to show that $T_r = -mg\cos\theta - m\ell\left(\frac{d\theta}{dt}\right)^2$ and $\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\sin\theta$.
5. When θ is small (small movements around the vertical position), we can assume $\sin\theta \approx \theta$. Show that this is now a type of harmonic motion. What is its period?
   Numerical example: Compute the period of a 1 ft-long pendulum. Does your answer seem reasonable?
6. The pendulum is released with an initial angle θ_0 and no initial speed $\vec{v} = \vec{0}$. What does $\vec{v} = \vec{0}$ mean for $\frac{d\theta}{dt}(0)$? Compute A and ϕ.