SMU PHYSICS 3305: Modern Physics

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Contents

a Foundations of Modern Physics - Part I

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The Big Picture

- Newton's Mechanics - laws of motion, linking forces to changes in states of motion ("velocities")
  - Established via Isaac Newton's "Philosophiæ Naturalis Principia Mathematica," published in 1687.
- Conservation Laws - total energy (including internal forms of energy), total linear momentum, and total angular momentum in closed and isolated systems.
  - Established through careful chemical and physical work up through the 1700s, and continued in work on heat energy in the 1800s
- Newton's Law of Gravitation - the law relating distance between material bodies and the force between them that does not require physical contact
  - Also established in Newton's "Principia."
- Electromagnetism - laws of electricity and magnetism as forces that can induce changes in states of motion without physical contact between material bodies
  - Established in the 1700s-1800s, finally codified formally in "Maxwell's Equations" in 1862.

Vectors

Vectors are numbers that are essential to describing any multi-dimensional quantity. They have a defined algebra which you have exercised in previous physics (and likely math or engineering) courses:

- Component notation: $\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} + \ldots$
- Vector Length or Magnitude: $a \equiv |\vec{a}| = \sqrt{a_x^2 + a_y^2 + \ldots}$
- Unit vectors: a vector of length 1. The unit vectors denoted $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are special and point only along the x-, y-, and z-axes, respectively, of a Cartesian Coordinate System. (that means the angle between any pair of these is $90°$)
- Addition: $\vec{a} + \vec{b} = (a_x + b_x)\hat{\imath} + (a_y + b_y)\hat{\jmath} + \ldots$
- Dot Product: $c = \vec{a} \cdot \vec{b} = ab\cos\theta = (a_x b_x) + (a_y b_y) + \ldots \longrightarrow$ yields a pure scalar
- Cross Product: $\vec{c} = \vec{a} \times \vec{b} = (a_x b_y)(\hat{\imath} \times \hat{\jmath}) + (a_y b_x)(\hat{\jmath} \times \hat{\imath}) + \ldots \longrightarrow$ yields a pure vector with length $c = ab\sin\theta$.
  - The cross-products of coordinate axis unit vectors obey the following rules: $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0$; $\hat{\imath} \times \hat{\jmath} = \hat{k}$, $\hat{k} \times \hat{\imath} = \hat{\jmath}$, $\hat{\jmath} \times \hat{k} = \hat{\imath}$; swap the order the left side of these equations and the right side changes by a minus sign.

Newton's Laws of Motion

These are summarized in the famous "3 Laws" of motion:

- The state of motion of an object remains constant unless the object is acted upon by an external force.

$$\sum_{i=1}^{N} \vec{F}_i = 0 \longrightarrow \vec{a} \equiv \frac{d^2\vec{r}}{dt^2} = 0$$

- Force generates a change in the state of motion of an object and that change is proportional to the object's mass.

$$\vec{F} = m\vec{a} = m\frac{d^2\vec{r}}{dt^2}$$

- In every interaction of two material objects, A and B, two forces act; the direction of the force exerted by object A on object B is opposite the force of object B on object A, but they are otherwise equal in magnitude.

Energy and Momentum

Motion has certain quantities associated with it that vary in some proportion to the degree of motion:

- Kinetic Energy: proportional to mass and to the square of velocity of an object, $K = \frac{1}{2}mv^2$
- Linear Momentum: proportional to mass and to the velocity: $\vec{p} = m\vec{v} = m\frac{d\vec{r}}{dt}$
- Angular Momentum: proportional to moment of inertia and to the rotational velocity: $\vec{L} = I\vec{\omega} = m\frac{d\vec{\theta}}{dt}$, $I = \int r^2\, dm$

If an external *conservative force* acts, one where the work done by the force in moving an object from point a to point b is the negative of the work moving from point b to point a (by any path), then there is an associated *potential energy*, U.

For external *non-conservative forces*, there is no associated potential energy but other forms of energy, such as heat (the motion of atoms in the material object), can result (e.g. friction and drag can both cause heating).

Conservation of Energy and Momentum

For a system that is acted upon only by conservative forces and is otherwise closed to and isolated from all other kinds of forces, *mechanical energy* is completely conserved:

$$K_i + U_i = K_f + U_f$$

For a non-closed and non-isolated system where non-conservative forces can act, *total energy* will be conserved but not mechanical energy:

$$E_{total,i} = K_i + U_i + E_{internal,i} = E_{total,f} = K_f + U_f + E_{internal,f}$$

where $E_{internal}$ are forms of energy like heat or chemical energy.

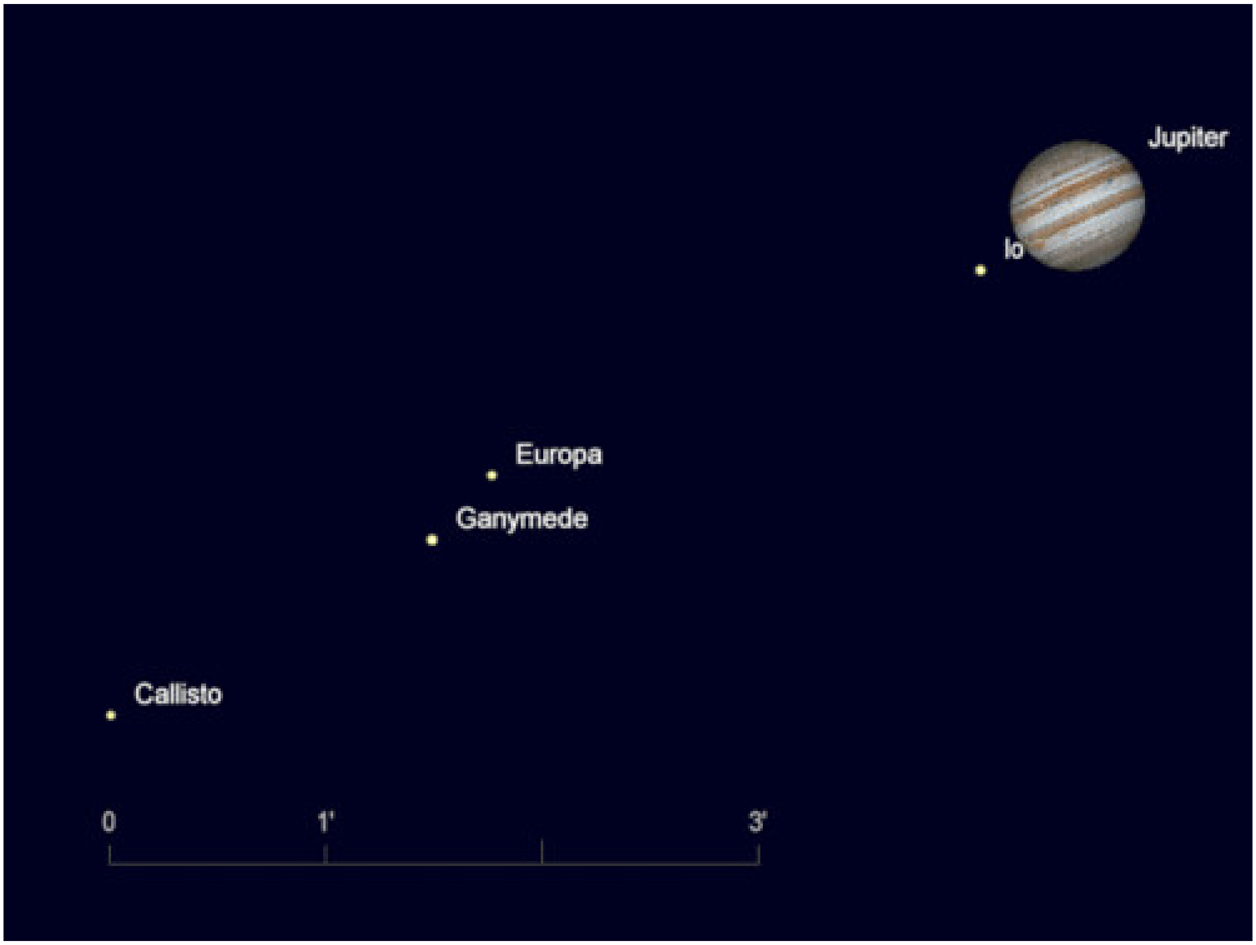
For a closed and isolated system of objects, total momentum is conserved:

$$\vec{p}_{1,i} + \vec{p}_{2,i} \ldots \vec{p}_{n,i} = \vec{p}_{1,f} + \vec{p}_{2,f} \ldots \vec{p}_{n,f}$$

$$\vec{L}_{1,i} + \vec{L}_{2,i} \ldots \vec{L}_{n,i} = \vec{L}_{1,f} + \vec{L}_{2,f} \ldots \vec{L}_{n,f}$$

If only elastic collisions of the objects are possible (the number and mass of objects never change), then total momentum and kinetic energy are conserved; if inelastic collisions are possible, only momentum is conserved.

The Law of Gravitation



The “Galilean Moons” of Jupiter. Image credit: "Jupsat220617" by blobrana2 is licensed under CC PDM 1.0

Of the many forces you explore in introductory physics, gravity is one of the strangest; it is a force between two objects that acts without physical contact, even across empty space. The gravitational force that object A exerts on object B is proportional to the masses of both objects and inversely to the square of the distance between them:

$$\vec{F}_{A,B} = G\frac{m_A m_B}{r_{A,B}^2}\hat{r}_{A,B}$$

This is the *force on A that is exerted by B*. Here, G is a universal constant that must be determined by experiment, and is known to be $G = 6.67430(15) \times 10^{-11}\mathrm{N} \cdot \mathrm{m}^2/\mathrm{kg}^2$. Gravity is *described* by this law but not *explained* by it.

Laws of Electricity and Magnetism

The electric and magnetic forces have something in common with gravitation: the can act without physical contact across stretches of empty space. However, they part ways from gravity there. Their strength is proportional to a different physical property: electric charge. Like gravity, the strength varies inversely with the square-distance between charges or flows of charges (currents). A density of electric charge (ρ) is the source of the electric *field of force*, $\vec{E}$; an electric current density, $\vec{J}$, is the source of a magnetic *field of force*, $\vec{B}$. Let $\vec{\nabla} \equiv (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$. The laws governing these fields are four in number:

- Gauss's Law for Electric Fields (from which Coulomb's Law is derived)

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

- Gauss's Law for Magnetic Fields

$$\vec{\nabla} \cdot \vec{B} = 0$$

- The Faraday-Maxwell Law

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

- The Ampere-Maxwell Law:

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

Electricity, Magnetism, and Light

Maxwell's Equations finally clarified the nature of light. If one solves for $\vec{E}$ and $\vec{B}$ in empty space, where no charges or current densities (no matter of any kind) are present, nonetheless a non-trivial solution is found:

$$\begin{array}{rcl} \vec{\nabla} \cdot \vec{E} = 0 & ; & \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & ; & \vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t} \end{array}$$

The non-trivial solutions are vector functions of space and time:

$$\begin{aligned} \vec{E}(\vec{x}, t) &= \vec{E}_0 \cos\Big(\hat{k} \cdot \vec{x} - c_0 t\Big) \\ \vec{B}(\vec{x}, t) &= \frac{1}{c_0} (\hat{k} \times \vec{E}) \cos\Big(\hat{k} \cdot \vec{x} - c_0 t\Big) \end{aligned}$$

describing an oscillatory phenomenon, where $\hat{k}$ is a unit vector in the direction of travel, and c_0 is the speed of the phenomenon in empty space and is given by $c_0 = 1/\sqrt{\mu_0 \varepsilon_0} = 2.998 \times 10^8 \mathrm{m/s}$ ... the speed of light. Light is an electromagnetic wave, and like a mechanical wave it was originally assumed that it must have a medium in which it must travel, and that empty space is not really empty.

Relativity

In introductory physics you get some exposure to relative motion and the idea of *relativity of observations*. For example, a person standing still on a train that is moving at constant velocity ($\vec{v}$) tosses a ball straight up into the air. From their perspective it goes up, slowing to a stop, then accelerates downward back to their hand... all along a straight vertical line. A person on the ground watching this sees the ball follow a parabolic trajectory.

To relate the observations in space and time, you assume that time passes the same for all observers and use the *Galilean Transformation* to relate spatial coordinates (x,y,z) and velocities (u_x, u_y, u_z) in the rest frame to those in the moving frame ((x', y', z') and (u'_x, u'_y, u'_z)):

$$
\begin{aligned}
x &= x' + \vec{v}_x t' \ \ldots \text{etc.} && (1) \\
t &= t' && (2) \\
u_x &= u'_x + \vec{v}_x \ \ldots \text{etc.} && (3)
\end{aligned}
$$

From Classical to Modern Physics

Classical physics often feels very intuitive, but you have to be careful. Your intuition is largely based on experience with events that involve:

- speeds very close to zero, as compared to the fastest known phenomenon in the universe: light.
- sizes that are very large by comparison to the known building blocks of the material universe: atoms and the stuff inside atoms.

As a result, as we begin to think about the very fast (objects moving close to the speed of light) and the very small (objects more at the atomic and sub-atomic scale), you will find classical physics needs to be modified to describe the universe more completely. You will also find that your intuition is wrong, but that only means that you are finally experiencing the breadth of the universe, rather than the limited scale of phenomena closer to human experience.

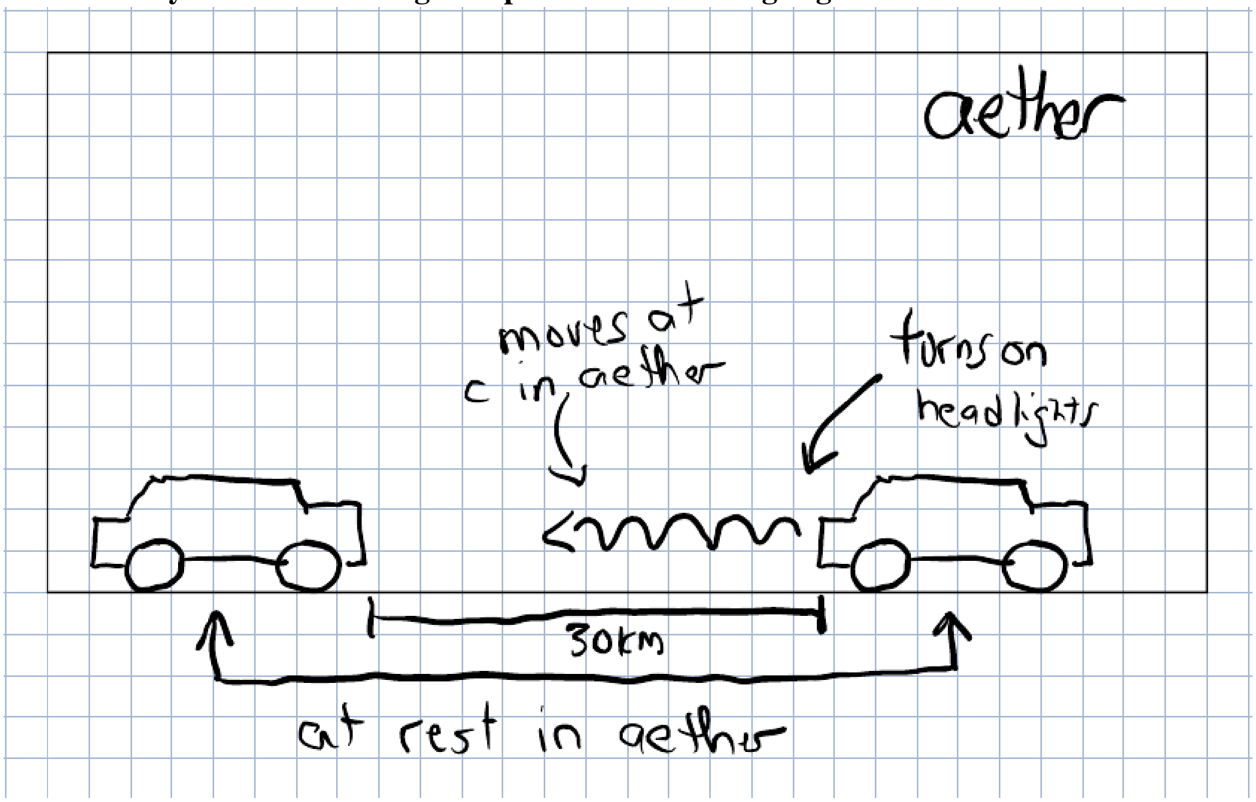
Classical Physics and Predictions

The tenets of classical physics, encoded in Newton's Laws and Maxwell's Equations, would apply to all phenomena in the natural world if they are the complete set of all laws of nature. What would the framework of classical physics then insist be true about light?

- Light is an oscillatory phenomenon; like other waves, it must be "mechanical" in nature, representing the distortion of a medium. This medium was named the "aether" and was believed to be the thing that actually fills empty space.
- The speed of light "in empty space" would really be the speed of light as measured relative to an observer at rest with respect to the aether. The aether, then, would be the "universal rest frame" in the view of Galilean Relativity and Newton's Mechanics.
- Maxwell's Equations were silent on the topic of the aether, and so were assumed to be incomplete as they failed to include this medium.

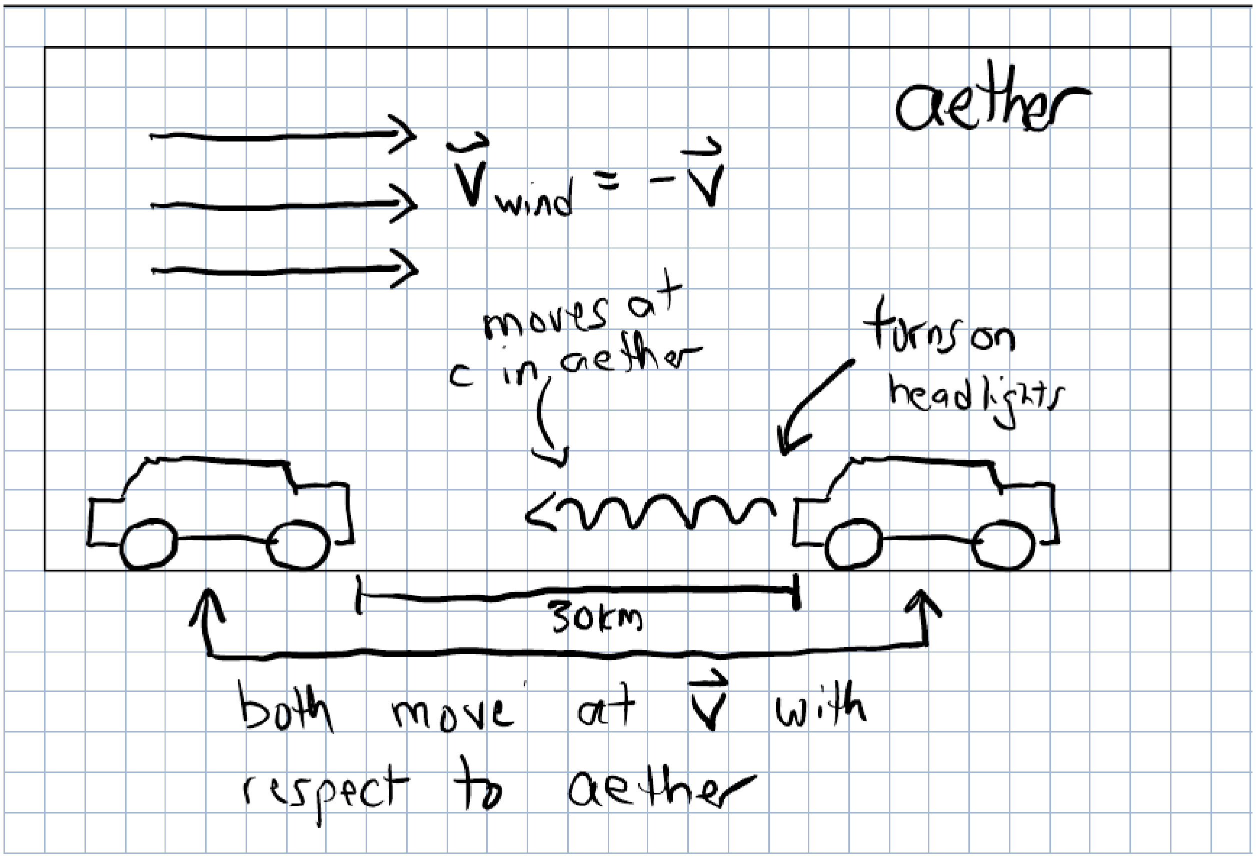
If we then apply this thinking to a problem involving light and travel and time, what would we predict?

Classical Physics and a "Thought Experiment" involving Light



- Imagine two cars at rest relative to the aether. One is 30 kilometers away from the other. One car switches on its head lights. How long does it take for an observer in the second car to see the light reach them?
  - Easy, right? You know the distance — 30km — and you assume that Maxwell's Equations, though they don't incorporate a medium, tell us that light travels at $2.998 \times 10^8 \mathrm{m/s}$ in aether. So the answer is $\Delta t = 1.0 \times 10^{-4}$s, or 0.10ms.

Classical Physics and a "Thought Experiment" involving Light



- Now imagine that both cars travel with the same velocity with magnitude (speed) half that of light relative to the aether, so $v = -1.5 \times 10^8 \mathrm{m/s}$. However, at all times they maintain a fixed distance of 30km. Once car switches on its headlights. Now how long does it take the light to reach the observer in the other car?

  - Light travels at c in the aether, but now the aether "rushes past" the cars like a wind, with velocity $\vec{v}_{wind} = -\vec{v}$. Galilean relativity and Newtonian Mechanics demand that $c_{obs} = c - v$. So you would answer: $\Delta t = \Delta x/(c - v) = 2.0 \times 10^{-4}$s, or about 0.20ms.

That's a prediction. It comports with all prior experience in the pre-20^{th} century world, about velocities adding in this way. But…is this what is observed?

A review of the ideas

- The foundations of modern physics are Newton's Mechanics, Energy and Momentum, the Law of Gravitation, and Electromagnetism.
- These are built to describe phenomena that comport with typical human experiences: phenomena at our scale (or slightly larger or smaller, essentially within our ability to see the world around us including with a microscope or telescope). The exception is Maxwell's Equations, which were developed by studying electric charges, which are very small and beyond the scale of everyday experience except in their macroscopic effects (electric and magnetic forces, electric currents, etc.)

- By the end of the 1800s, chemists and physicists were beginning to directly interact with scales beyond human experience. The electron is discovered in 1897, the first “subatomic particle,” and invisible radiation like x-rays are discovered in 1895. These and other phenomena at the same scale (e.g. atoms and general forms of light) are beyond human experience.
- Not only were such new phenomena small, they turned out to be capable of moving extremely fast, with speeds near or at the scale of the speed of light. Such speeds are also beyond human day-to-day experience.
- Will classical physics lead us to correct inferences about these phenomena?

Modern Physics is a pathway to many insights some consider to be . . . unintuitive.

b Introduction to the Course

Introduction to the Course

b.1 What are the Goals (Learning Outcomes) of this Course?

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Goals of this Course

As described in the SMU Undergraduate Course Catalog:

> *For science and engineering majors. Covers vector kinematics, Newtonian mechanics, gravitation, rotational motion, special relativity, and structure of matter.*

Goals of this Course

The specific learning goals of this course are as follows. Upon successful completion of this course, students will be able to:

1. Explain why relativity and quantum mechanics and are needed to understand natural phenomena that are central to the modern world;
2. Apply their understanding of relativity and quantum mechanics to a range of problems that occur in areas as diverse as medicine, communication and computation;
3. Demonstrate the basic understanding of relativity, quantum mechanics and statistical mechanics required to pursue more advanced topics in each of these related areas.

b.2 What is the Structure of this Course?

What is the Structure of this Course?

General Overview of Course Structure

- In-Class Periods (Attendance and Participation)
- Assigned Reading and Video Lectures
- Reading/Video Lecture-based Quizzes
- Assigned Homework
- The Grand Challenge Problem
- Exams (3 total, including the final exam)
- Office Hours

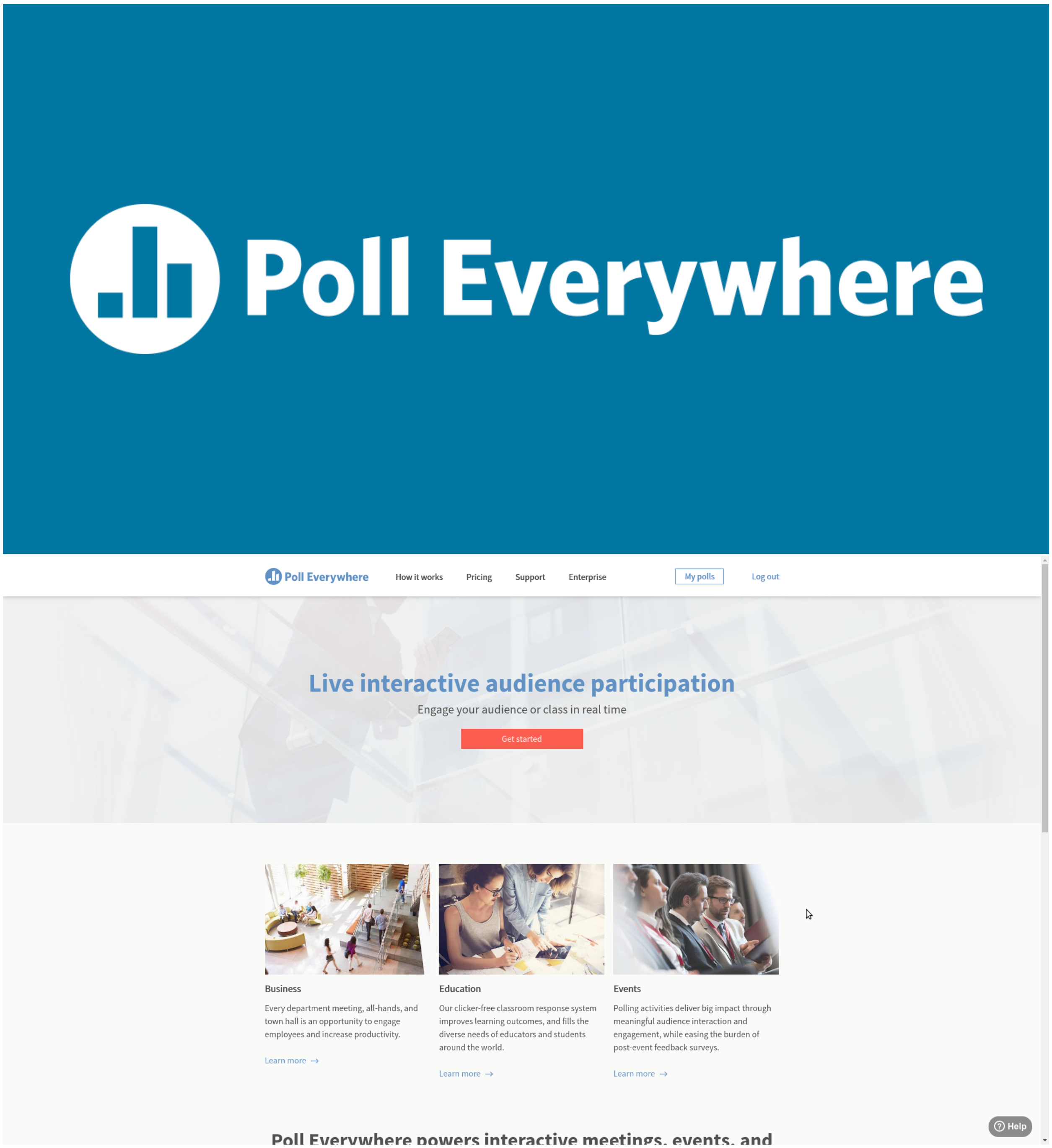
In-Class Periods

- Regular
  - Tuesdays and Thursdays
  - 12:30-1:50pm, FOSC 158
- Classroom Activities
  - Lecturettes: Problem-solving demonstrations and how to think about physics concepts in problem solving.
  - Demonstrations of physical phenomena (some stuff is more fun in person)
  - Problem Solving exercises: first attempts at solving problems individually and together.
  - Discussion and interaction will naturally occur as a result of the above.
- Worth 10% of your final grade. You are allotted 2 unexcused absences per semester (e.g. sick days). Otherwise, in-class participation is counted.

Student Response System

- What is a "student response system"?
  - Well, to be glib, it's any system by which I can collect your responses to questions (paper would do just fine!)
  - But these days, it usually refers to an *electronic, real-time system for collecting student responses*
- For this class, we will use **PollEverywhere**

- Go to PollEverywhere.com to register (free). **Use your SMU Email Address to register!** Then go to your settings on the site and register with me (use my SMU email address, written on the whiteboard and in Canvas)
- To participate in active polls, go to my poll site: PollEv.com/drsekula. Feel free to use a web browser or the iOS or Android apps.
- Active polls during class help us to further identify learning challenges in the class.



Pre-Class: Reading and Lecture Videos

- Pace: about 1 chapter every 2 weeks
  - I expect you to spend 3-4 hours outside of class periods reading and watching video lectures, taking notes as you do those activities, and reading/reviewing/condensing your notes.
- Assessment: Reading and Video Lecture Quizzes
  - Worth 5% of your final grade
  - Check your pace and comprehension of reading and lecture video material. You should be taking notes during reading and lecture video, and review your notes before class.
  - I will try to provide "guiding questions" to help you identify the core themes and ideas around which new material may be centered. There were examples of these to help guide preparation for today's class.
  - For the 24 hours before class, a reading quiz is posted on Canvas. When you are ready, start the quiz. It is timed. Once started, you have 5 minutes to complete. Your notes should be a primary reference to aid you.
  - Two lowest reading quiz grades automatically dropped.

Homework

- Homework
  - worth 15% of your final grade
  - I expect you to spend 4-6 hours outside of class working on homework
  - Assigned weekly — a mix of textbook and custom problems I write. Due Thursdays by 12:30pm.
  - You must make good quality written solutions to each problem. You will ***submit those by the due date electronically via Canvas***[bh]. A part of the grade on your homework will come from the quality and structure of your written presentation.

Exams

- 2 Monthly Exams ("In-Class Exams")
  - In-class, each covering a specific subset of topics; the first one is themed on "faster" (relativity) and the second "smaller" (quantum physics)
  - Each worth 15% of your final grade
- Final exam

[bh] A scan or photo of the written document can be uploaded to Canvas. Or, you can type up solutions fully electronically and submit a PDF file.

  - Cumulative, including material learned after the second in-class exam
  - Worth 20% of your grade
  - Goal is to assess your comprehensive learning, including on the subject material of your grand challenge problem solutions

- General Exam Policies
  - You bring a writing implement and a calculator.
  - You prepare and bring an equation sheet: one piece of paper (both sides allowed)
  - You demonstrate that you have achieved a level of proficiency with the concepts and methods of this course; no external help.

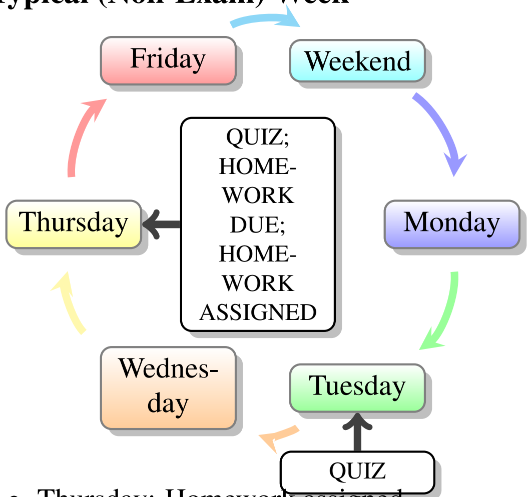
The Grand Challenge Problem

- The "Grand Challenge Problem" is a non-textbook, open-ended physics question with no obvious or simple textbook answer.
- You will begin working on this starting next week, by first being organized randomly into groups of about 3 people.
- You will be judged both on the quality of your individual work and the coherence of your overall group coordination; on application of physics principles, use of concepts beyond what we learn in class (built on classroom learning), and accuracy of calculations and conclusions; and on presentation quality, including both speaking and visualization.
- Your work product will be a series of short presentations, delivered by each member of your team, at the end of the semester (the last 2-3 class periods)
- Your presentations will be assessed both at the peer level (other students in this class) and using external referees (e.g. faculty and PhD students from Physics).

Don't Get Stuck...Get Help!

- Instructor office hours:
  - We will establish these today
  - Interacting with me outside of class is encouraged
  - Office hours will be subscribable using Canvas - you must sign up during a prescribed block of time in order to meet with me (no drop-ins).
- Peer Mentoring
  - I strongly urge you to form small study/work groups to help each other out on homework and learning
  - Peer mentoring, guided by an instructor, is one of the most effective and efficient ways to transmit information
  - There are also Canvas discussion forums for digitally asking for help, sharing information and ideas, etc.

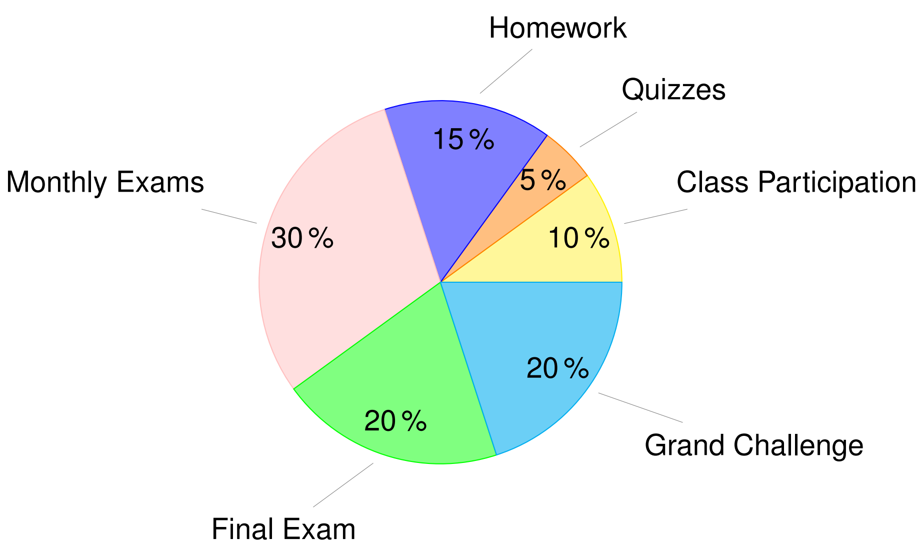
A Typical (Non-Exam) Week



- Thursday: Homework assigned
- Friday: Look through each and every problem assigned see which ones might cause you trouble. Try to complete 25%.
- Weekend: Really work the homework - try to complete 40-50%.
- Monday: You should have tried to submit answers to the online system for your first attempts by the end of Monday. Assemble questions for office hours.
- Tuesday and Wednesday: Try to complete 75% to 100% of homework.
- Homework due and new homework assigned

I expect you to put in 6-9 hours of your own time outside of class, roughly 4 on absorbing new ideas, principles, and material (reading, lecture videos) and 4 on beginning to exercise ideas in problem solving (homework).

Grade Composition

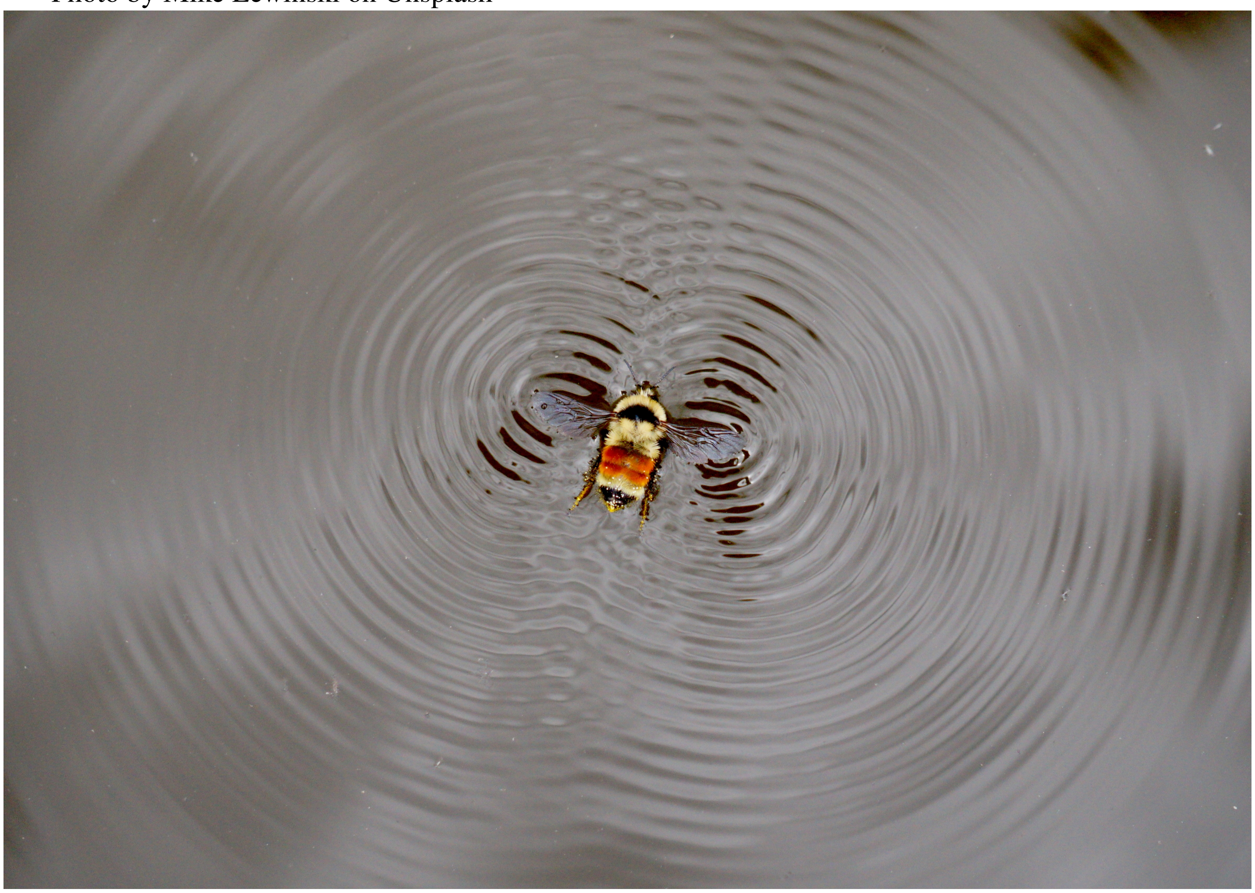


b.3 Problem Solving in the Foundations

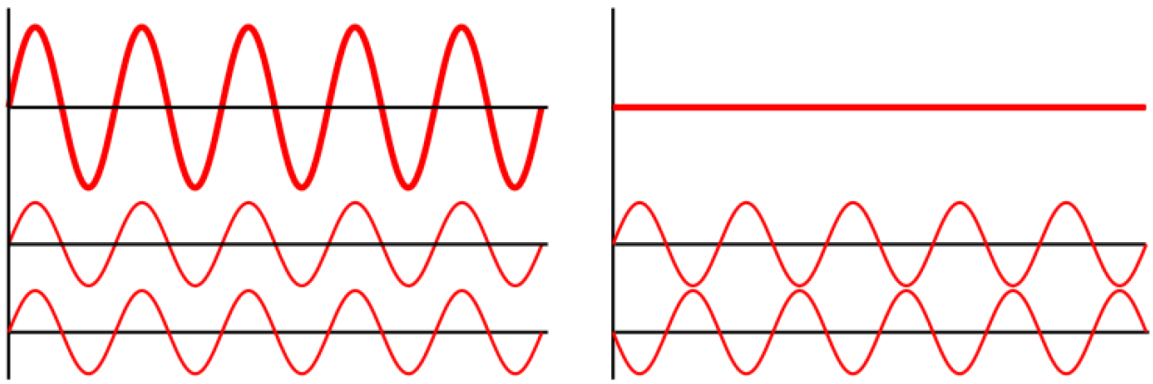
Problem Solving in the Foundations

Mechanical Waves

Photo by Mike Lewinski on Unsplash

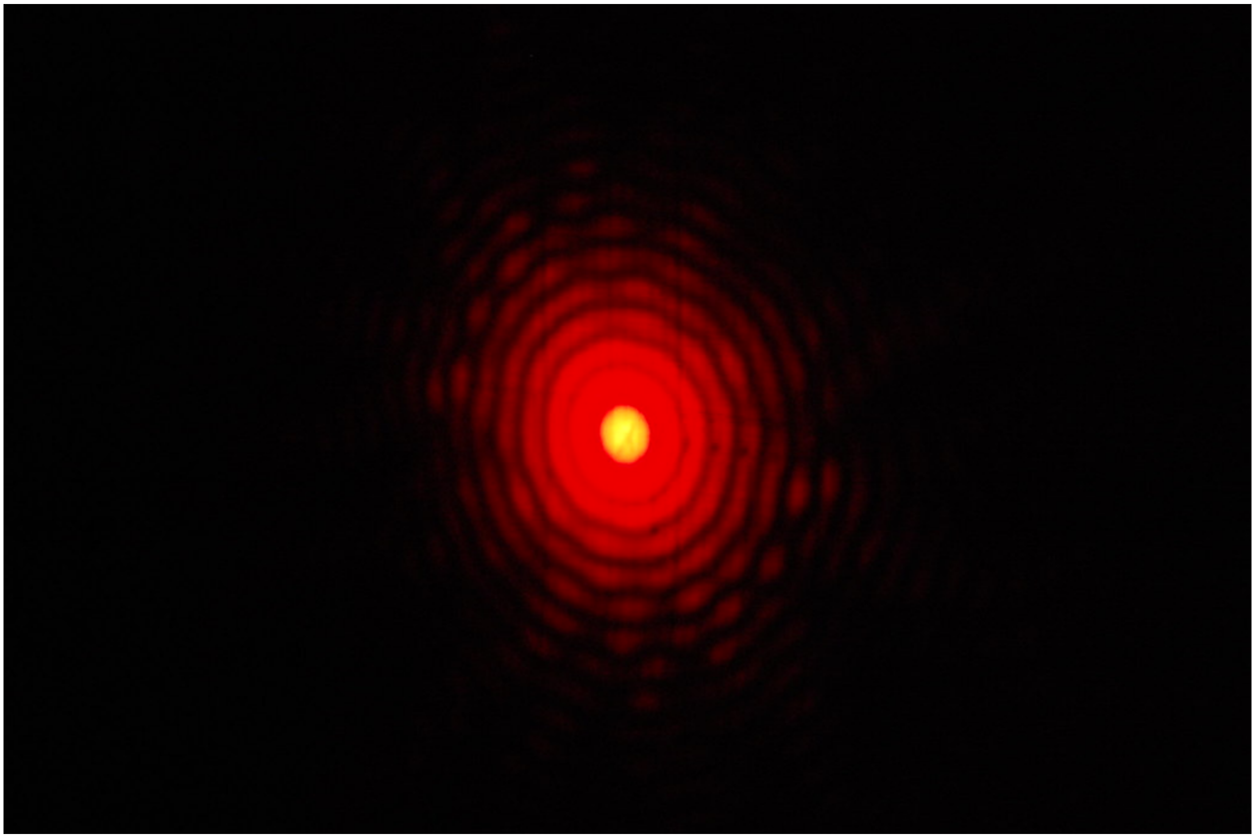


Wave Interference

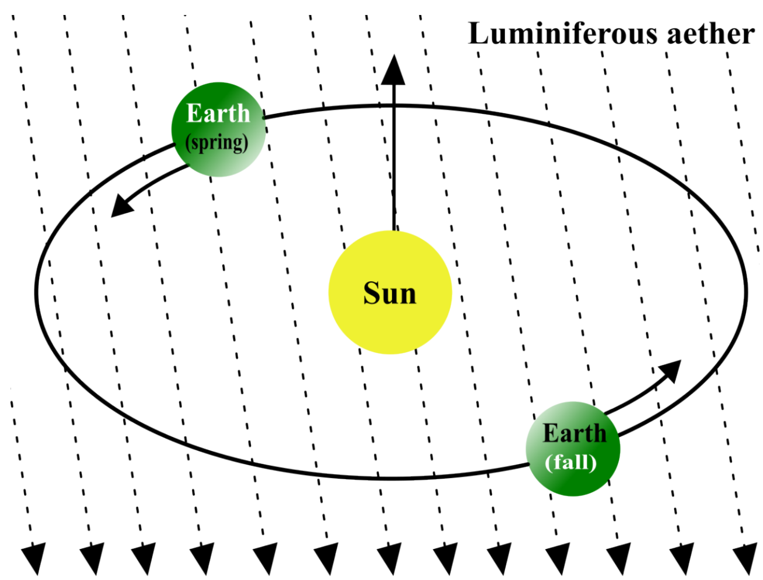


The above pictures demonstrate the extrema of wave interference: (left) completely constructive interference and (right) complete destructive interference.

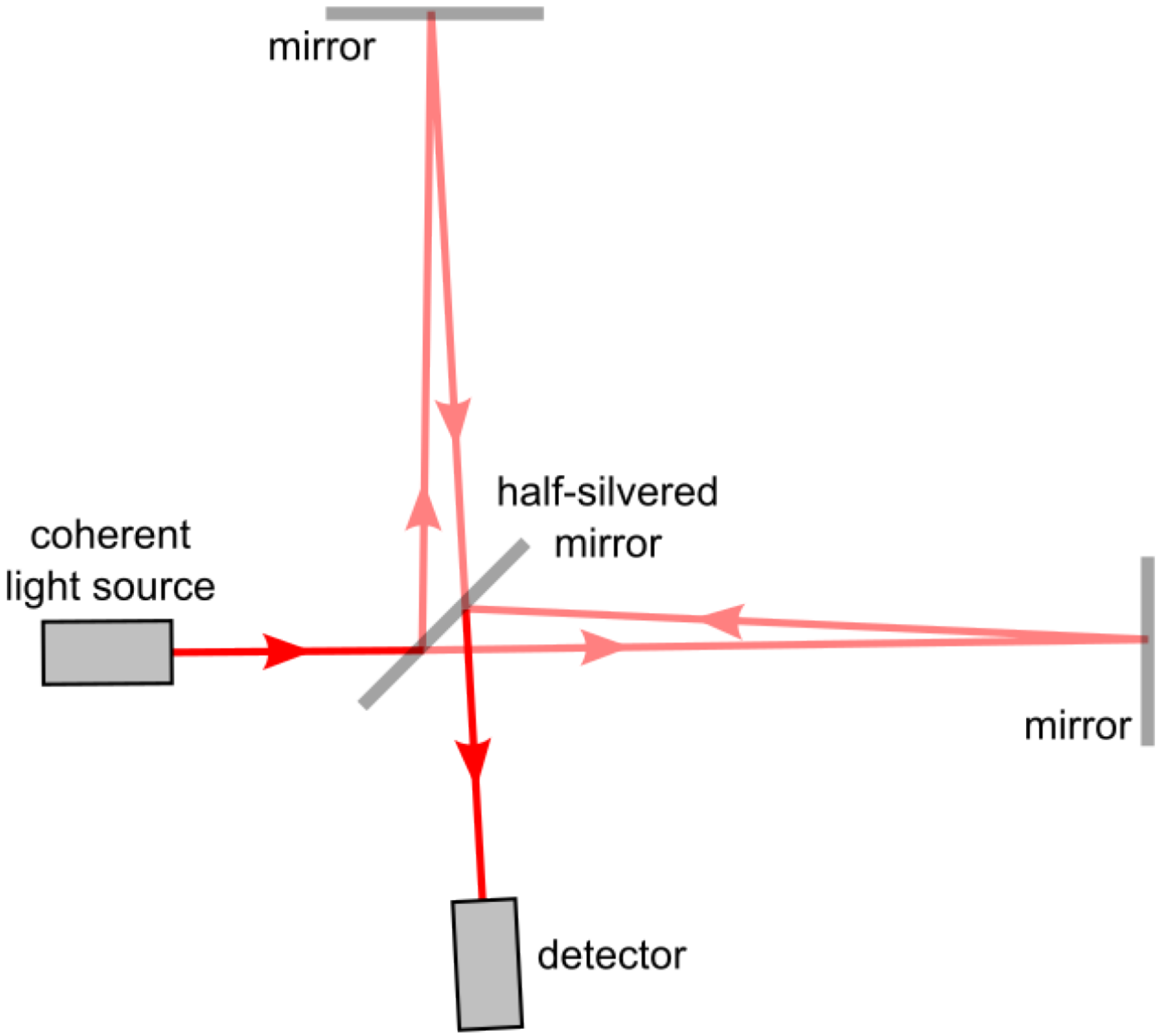
Light and Interference



Measuring the Aether

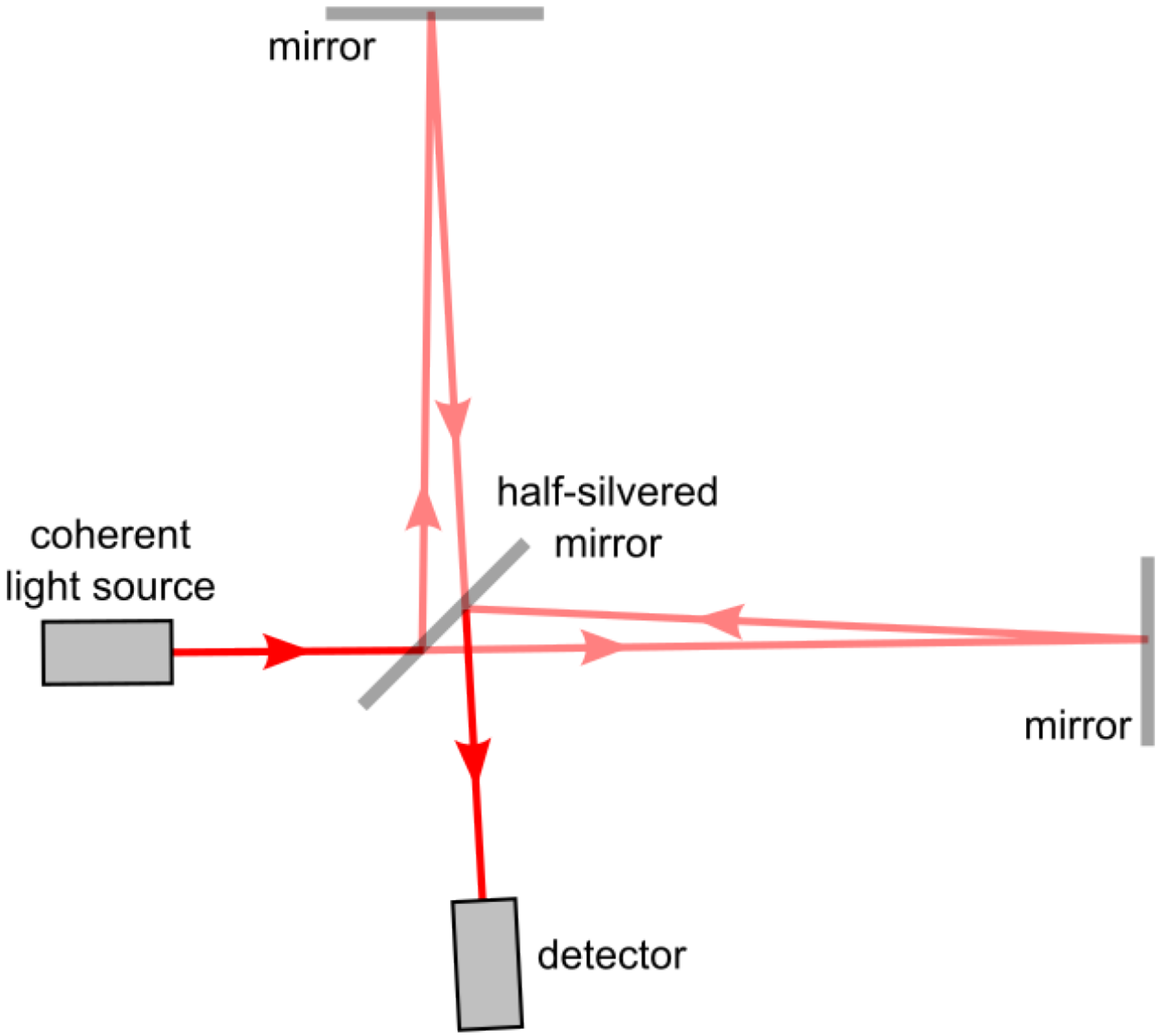


The Michelson-Morley Experiment



- First Albert Michelson, then he and Edward Morley, conducted a series of experiments to demonstrate the existence of the aether. This work culminated in 1887 in the most serious and precise sequence of experiments.
- One "arm" of the interferometer, ideally, is aimed into the "aether wind" experienced by the motion of the Earth around the Sun and through the aether. (Path 1)
- The other "arm" is perpendicular to the velocity of the aether wind. (Path 2)
- Light travels at $c = 2.998 \times 10^8\ \mathrm{m/s}$ in the rest frame of the aether.

The Michelson-Morley Experiment



- Light travels at $c = 2.998 \times 10^8\,\mathrm{m/\,s}$ in the rest frame of the aether.
- The length of Path 1 is the same as Path 2: $L = 11\,\mathrm{m}$.
- Assume that Galilean Relativity holds.
- Calculate the difference in travel time, $T_2 - T_1$, of a beam of light that travels from the half-silvered mirror along Path 1 and an identically prepared beam that travels instead along Path 2.
- What fraction of the characteristic length of green light (wavelength $\lambda = 500\,\mathrm{nm}$, about mid-spectrum for visible light) does the resulting path-length difference represent?

c The Special Theory of Relativity - Basic Ideas

The Special Theory of Relativity - Basic Ideas

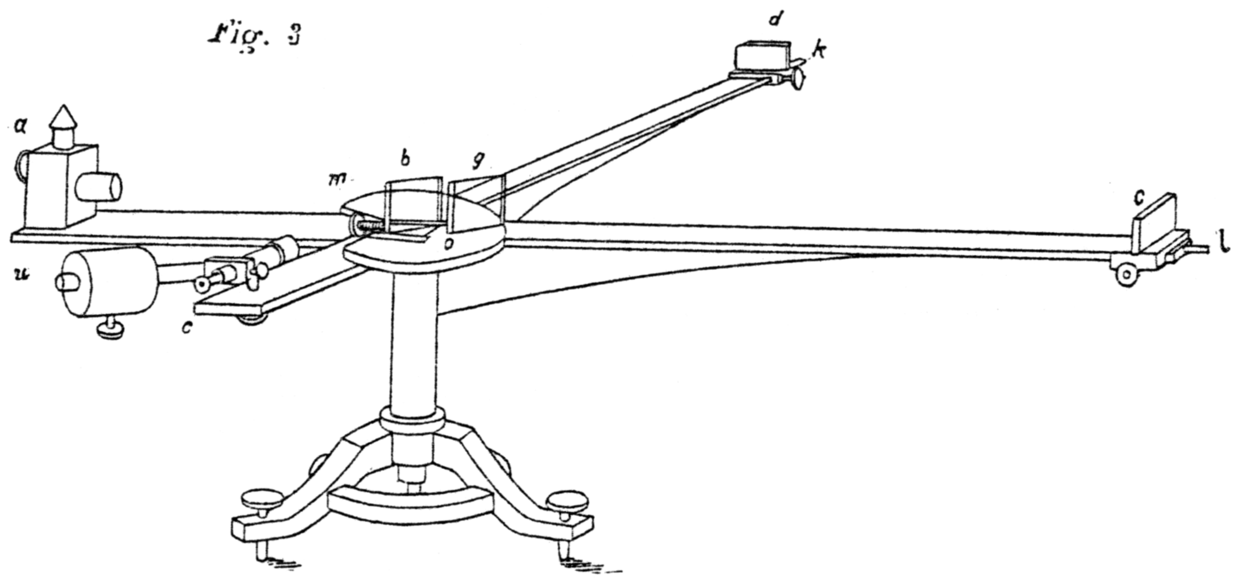
Overview

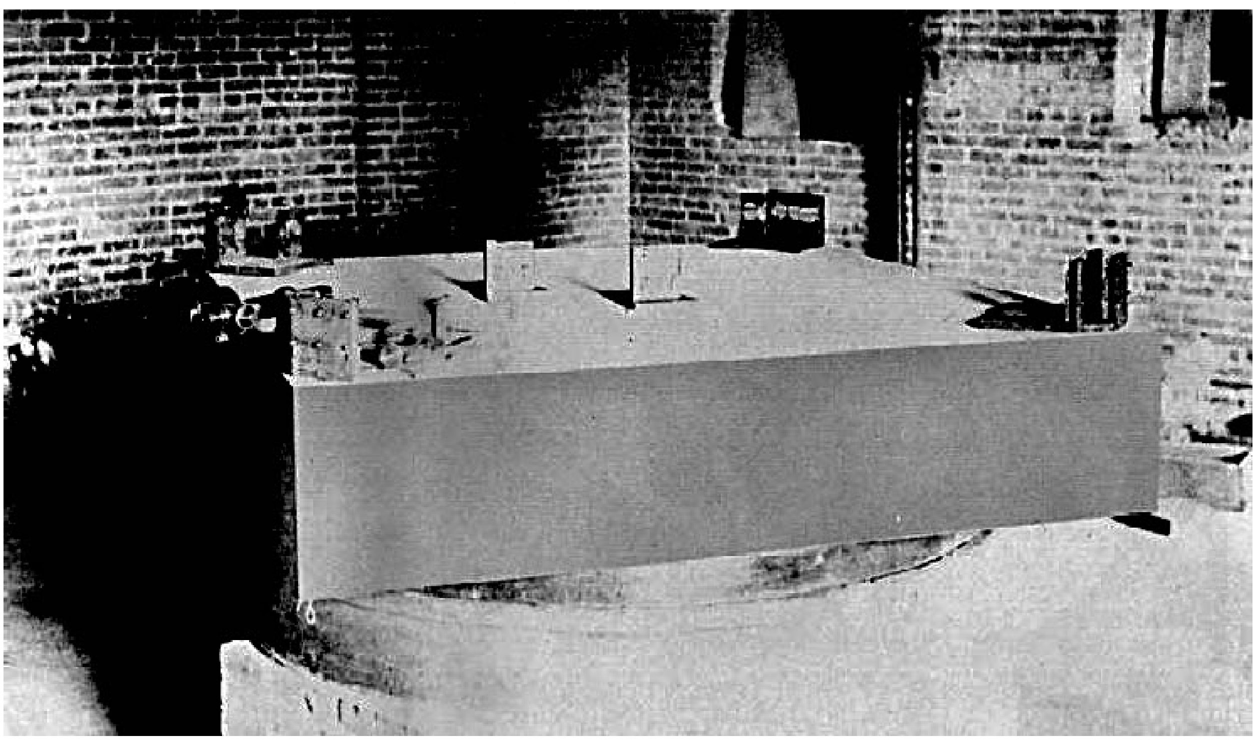
In this lecture, we will learn…

- The transition in thinking that led from Galilean Relativity to the Special Theory of Relativity in 1905.
- The postulates of special relativity, which are the basis of the mathematics of the framework.
- The consequences of the postulates.

Albert Einstein, 1904 (1879—1955)

The Lessons of the Michelson-Morley Experiment





The Michelson Interferometer (1881, top) and the definitive Michelson-Morley Experiment (1887, bottom)

- Light travels at a fixed and constant speed in any medium, regardless of the relative velocity of the light-source and the light-observer $\longrightarrow$ this is unlike any other phenomenon described in mechanics, and implies that Newton's Mechanics is incomplete.
- No medium is required for light to propagate; unlike a mechanical oscillatory phenomenon (wave), to exist light requires no medium to be distorted $\longrightarrow$ this implies Maxwell's Equations are complete.

- These lessons would not be absorbed fully until 1905, when Albert Einstein published the definitive papers explaining how to reconcile mechanics, electricity and magnetism, and the Michelson-Morley experiment

Hendrik Lorentz and "Compression of Bodies in the Aether"

- The mathematics that would later become the replacement for the Galilean Relativity equations would be laid down for a different purpose by Hendrik Lorentz.
- In considering the effects of the aether on bodies in motion through it[1, 2], bodies held together by chemical bonds (which are just electromagnetism in action), he arrived at a few hypotheses:
  - Mechanical bodies would compress along the direction of motion in the aether, with a precise mathematical description for the process;
  - In transforming observations from the aether frame to other frames of reference, he would conceive of an alteration of time that also had a mathematical description.
- Lorentz conceived of this during a period when the aether was still believed to exist - the results of the Michelson-Morley experiment were not fully digested. The aether's existence would be disproven in the decades that follows, but the mathematics would still prove useful.

Portrait of Hendrik A. Lorentz by Jan Veth

Albert Einstein and the "Miracle Year"

- It would be Albert Einstein in 1905, laboring on the side during regular work in the Swiss

Patent Office in Bern (because he was unable to secure a faculty job after completing his PhD [bh]), to change the thinking about the supremacy of assumptions in Newton's Mechanics vs. Electromagnetism (Maxwell's Equations).

- That year he published work resulting from his PhD research in a series of 4 papers[3, 4, 5, 6] - his "miracle year" - and in doing so reframed assumptions about space, time, and what is invariant to all observers in all frames of reference.

[bh]In part because Einstein could get no recommendations from his professors because he had so irritated them with his behavior in graduate school, including skipping classes and challenging his professors.


Albert Einstein, 1904 (1879—1955)

Albert Einstein and the “Postulates of Relativity”

- In short, he accepted the conclusion of the Michelson-Morley experiments that light has a fixed speed regardless of the motion of the source relative to the observer, which implied there was no aether as well; using a simple “thought experiment,” he explained why time is not absolute even in Newton’s Mechanics and abandoned it as the “constant” in transformations from frame to frame, choosing instead to preserve the laws of physics and the speed of light.
- The postulates that allowed him to define his mathematics:
  1. **The forms of the laws of physics are the same for all observers, regardless of their state of relative motion ("frame of reference")**
  2. **The speed of light is the same for all observers, regardless of their frame of reference.**


Albert Einstein, 1904 (1879—1955)

Breaking Down the Concepts

To dig into this, we need to break down these postulates into the underlying concepts so that we can build back up to a more complete understanding of the math that will rule relativity going forward.

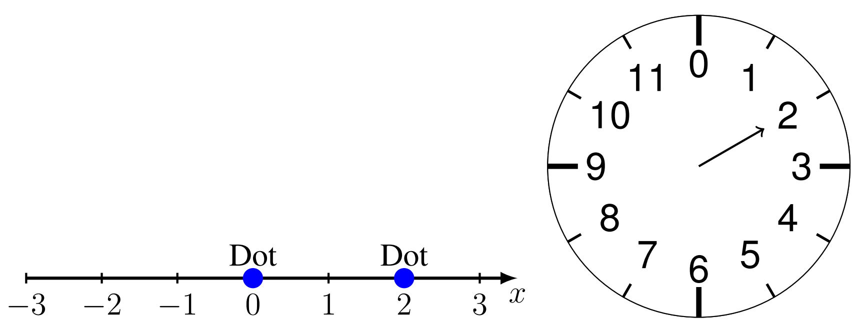
- "Event": you might think you know what this is, but in physics it is given a precise definition so that we can always try to define the concept mathematically.
- "Frame of Reference": we need to define this phrase, as it comes up a lot in physics discussions. Because descriptions of events can depend on the frame of reference in which the observation is made.
- "Simultaneity": we haven't used this word yet, but it is a subset of the discussion of events and it will play an important conceptual role in future discussions, so we need to define it.
- "Speed of Light": we should understand the number that is behind this phrase; it is a ridiculously large number, but is impressive only on the scale of things the size of planets or smaller.

Let's get started on the next few slides defining each of these.

Event

An "Event" is anything with a location in space and time.

Let's practice this concept. I will show you an event, and you try to describe it with words and numbers. This is good practice for defining events in any new situation.



A one-dimensional axis (x) in units of meters and a timer with units of seconds.

Describe the event depicted above.

Example description: "The dot is at position x=0m at time t=0s."

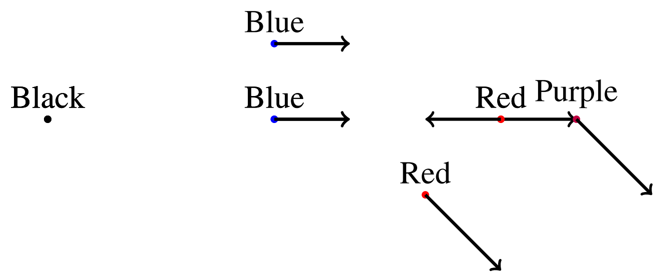
Describe the event depicted above.

Example description: "The dot is at position x=2m at time t=2s."

Frame of Reference

A "Frame of Reference" is any object or system all of whose parts move at the same velocity with respect to an agreed-upon reference point in space.

Consider the three objects shown below. One of them (black dot) is agreed upon by the others (e.g. red and blue dots) as the common reference point for all measurements.



As depicted, red and blue are in the same frame of reference because they have the same velocities.

Do the red dot and blue dot share the same or different frames of reference?

Answer: no - although their speeds are the same, the direction of the motion has changed; this means they have different velocities, different states of motion, and thus are in different frames of reference.

Practice more: how many unique frames of reference can you identify in the above picture?

Simultaneity

Two events (or more) are said to be simultaneous (that is, to possess of simultaneity), if they are observed to occur at the same moment in time.

This seemingly straight-forward definition of the concept should not fool you; you have to think really hard about whether events are simultaneous, and for whom (which observers in which frames of reference) they are simultaneous.

Let's look at a real physical example of two observers who disagree on whether two events happen at exactly the same time.

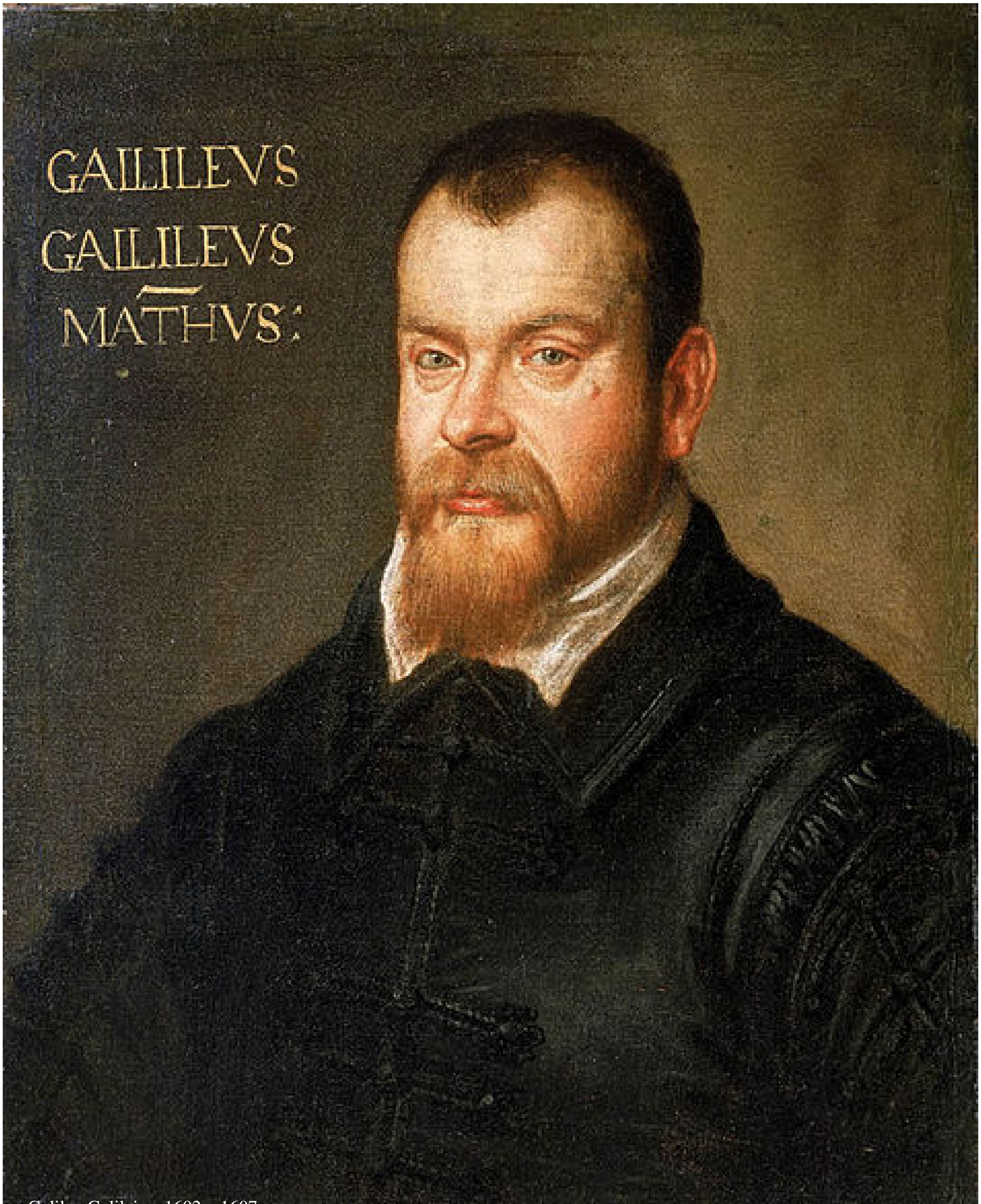
The Speed of Light

Let's be clear about the speed of light: it is the number of meters light can travel, once emitted by a source, in a certain amount of time.

- Galileo Galilei famously attempted to measure this by uncovering a lantern, having an assistant on a distant hill who uncovers their lantern upon seeing his, and upon seeing the assistant's lantern light he records the time for the round trip, taking into account human reaction time. Light moves too fast for this to work with 17th-century technology.

- Ole Roemer would use the period of Jupiter's moon, Io (discovered by Galileo using the telescope), and its cycle of eclipses by Jupiter, to make the first reasonable determination that light travels in finite time (ca 1676); it doesn't travel instantaneously from place to place.

- By the time of Einstein's publications, the speed of light had been established by multiple experimental methods to be within 50km/s of the precision of today's methods. That is remarkable for such a large number, representing such a large speed.



Galileo Galilei ca 1602—1607

Ole Roemer, painted by Jacob Coning ca 1700

Modern Speed of Light

The speed of light, based on modern definitions of the meter and the second, is defined to be exactly $299,792,458\mathrm{m/s}$. Light travels roughly one foot in one billionth of a second (1ft/ns).

The consequences of the posulates of “Special Relativity”

For his 1905 work, Einstein focused on *inertial frames of reference* - those in which there are no net, unbalanced forces (e.g. $\sum \vec{F}_{external} = 0 \rightarrow \vec{a} = 0$). Under this *special condition*, an object in motion will appear to all observers in all frames to have constant velocity, even if observers in different frames disagree on the magnitude and direction of the vector. Let’s recall his postulates again:

1. **The forms of the laws of physics are the same for all observers, regardless of their state of relative motion ("frame of reference")**
2. **The speed of light is the same for all observers, regardless of their frame of reference.**

No Absolute States of Motion — All Motion is Relative

1. **The forms of the laws of physics are the same for all observers, regardless of their state of relative motion ("frame of reference")**
2. The speed of light is the same for all observers, regardless of their frame of reference.

The consequences of the first postulate are both straight-forward and surprising:

1. All physical laws (e.g. Newton’s Laws or Maxwell’s Equations) all have the same observed form in all inertial reference frames. This is “helpful” in that the basic laws of physics are not dependent on your state of motion.
2. But as a consequence of this, *it is impossible to tell from the laws of physics in your frame whether you are in motion or not.*

As a result of this postulate, there is no such thing as an absolute state of rest or motion - all motion is relative.

Light’s Speed is a Fundamental Invariant of Nature

1. The forms of the laws of physics are the same for all observers, regardless of their state of relative motion ("frame of reference")
2. **The speed of light is the same for all observers, regardless of their frame of reference.**

The consequences of the second postulate are typically more surprising, and well outside of the comfort zone of typical human experience:

1. All observers agree that light moves at a fixed speed — this is the singular invariant independent of states of relative motion;
2. But as a consequence of this, *the belief that time or space or both are experienced in the same way by observers in different states of motion must be abandoned.*

As a result of this postulate, there is no such thing as an absolute measures of time or space; measurements in one frame of reference need not agree with those in another, but all observers will agree that light signals travel at a fixed speed.

The Relative Nature of Time: Einstein's *Gedanken* Experiment

- It was the abandonment of Newton's old idea of "absolute time" — time that passes the same way for all observers regardless of their state of motion — that was Einstein's moment of insight, the moment that ultimately freed him from Newton to re-think space, time, and invariants of motion.
- While riding on a streetcar in Bern, Switzerland (where he worked as a patent clerk), Einstein thought about what it meant to "know the time by observing the clocktower".
- Time is the measure of distance between events that occur, for example, at the same spatial coordinates. Imagine a blinking light as a means to measure time. The "gap" between two blinks could be used as a measure of 1 unit of time.
- Einstein realized that since light has to catch up to a moving object, even in Newton's view of the universe time measurements cannot be absolute.

The Relative Nature of Time: Einstein's *Gedanken* Experiment

- Imagine two observers using a blinking light to measure time. One is at rest on the ground with respect to the source of the light. The other is on a super-train racing away from the light source at half the speed of light.
- The two observers agree to count how many blinks occur while the super-train makes a journey of 2 million miles (this is about how far light can travel in 10s)
- On the ground, the observer at rest counts 10 blinks during the journey.
- For the observer in motion, not all those 10 blinks will have had time to reach the super-train by the time it arrives at its destination; it marks fewer blinks, and thus claims less time was required than 10 blinks to make the journey. Two observers disagree on how much time has passed using a common reference.

Even in Newton's view of space and time, the notion of absolute time measurements is not correct; while this thought experiment is essentially based on an optical effect, it nonetheless disproves that there is such a thing as a notion of absolute time and frees one to abandon this concept as a necessary tenet of reality.

Distances in space and time are not observed to be the same in different reference frames

1. The forms of the laws of physics are the same for all observers, regardless of their state of relative motion ("frame of reference")
2. **The speed of light is the same for all observers, regardless of their frame of reference.**

Since time and space displacements are not experienced the same in frames with different relative states of motion. . .

1. Observers at rest will observe the physical dimensions of objects in motion, relative to them, to be *contracted along the direction of their motion* $\rightarrow$ **length contraction**
2. Observers in motion relative to other observers will experience a slower passage of time $\rightarrow$ **time dilation**

These will be easier to appreciate as we explore the postulates of relativity in class and in the next section of the course on the *Lorentz Transformation*, the correct way to relate observations between frames of reference.

Review

In this lecture, we have learned about. . .

- The transition in thinking that led from Galilean Relativity to the Special Theory of Relativity in 1905.
- The postulates of special relativity, which are the basis of the mathematics of the framework.

- The consequences of the postulates.

Albert Einstein, 1904 (1879—1955)

d The Special Theory of Relativity - Basic Ideas

d.1 Problem Solving in the Postulates of Relativity

Problem Solving in the Postulates of Relativity

A Light-Year

The distance travelled by light in one year is called a "Light-Year", denoted "ly". What is this distance in...

1. kilometers?
2. miles?

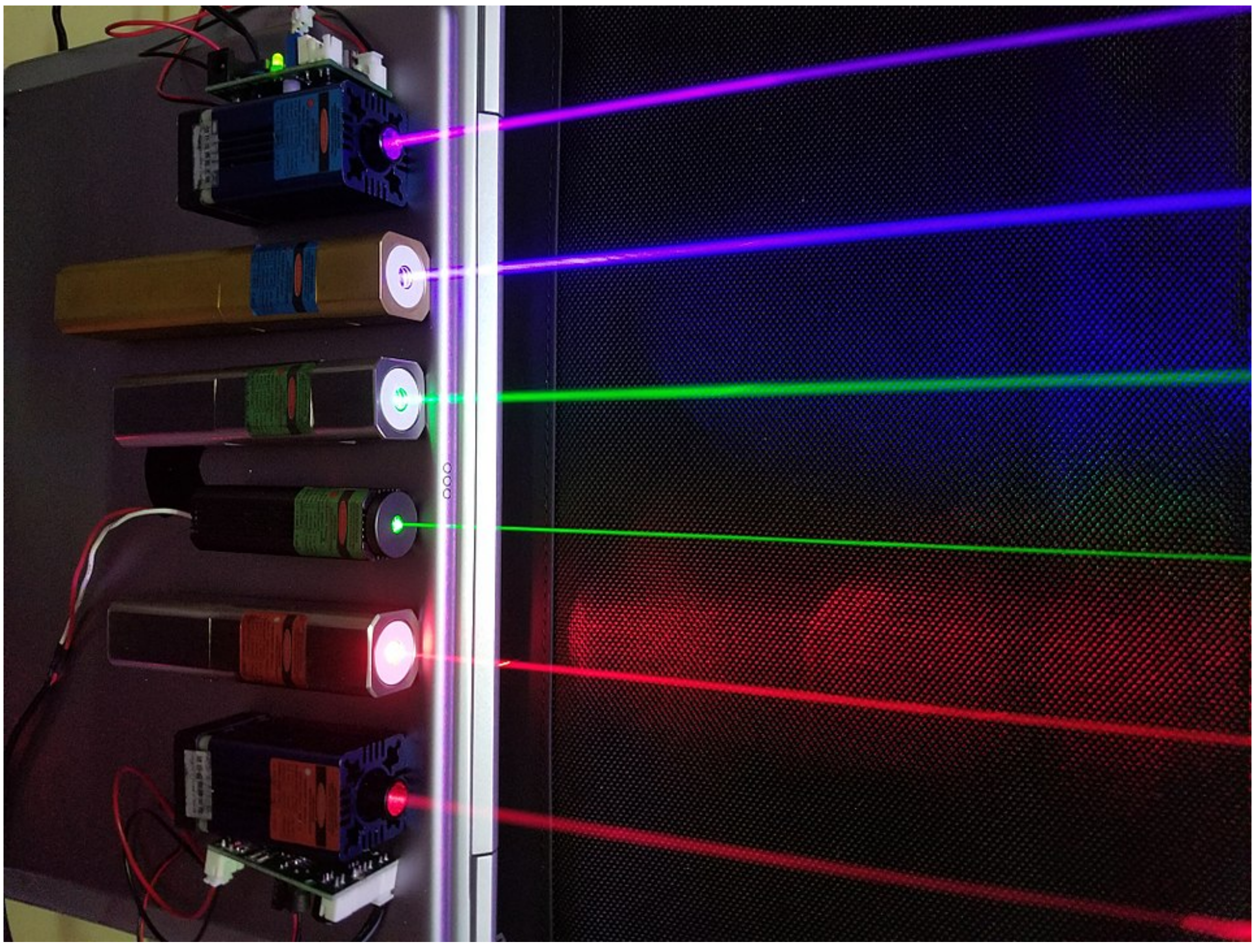


Image from Wikipedia

Student Problem: From the Earth to the Moon

How far (center-to-center, or thereabouts) is it from the Earth to the Moon in...

1. Light-Years?
2. Light-Seconds?

Photo by Gregory H. Revera

Student Problem: Into the Stellar Neighborhood

The Voyager 1 probe was launched 42 years, 4 months, and 17 days ago. It is presently 22.2 billion km from Earth.

1. What is its average speed, relative to Earth?

2. At this speed, how many years would it take Voyager 1 to reach the closest star to our sun, Proxima Centauri, which is 4.244ly away?

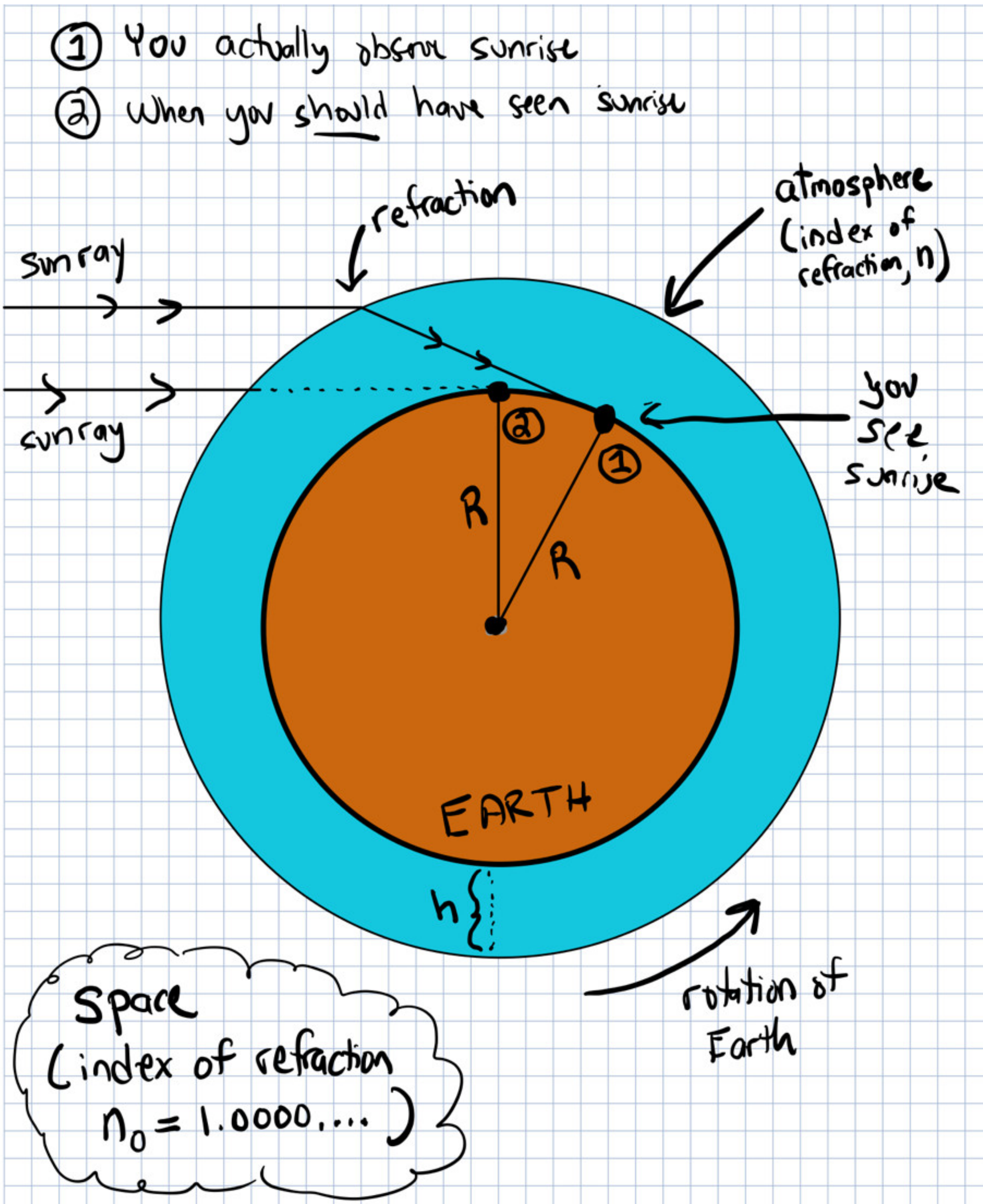
Image from ESA/Hubble & NASA

Challenge Problem: The Illusion of Sunrise (or Sunset)

Due to a trick of light - refraction - sunrise, in reality, does not occur when we observe it. Yes, we see the sun rise… but this turns out to happen before we should actually see light rays directly from the Sun. Why? Light enters the Earth's atmosphere from empty space, passes through the boundary between space and air, refracts, and as a result can reach us on Earth even though those rays should have missed our eyes.

Using the following information, calculate the time between when we see the sun rise and when we should have seen it rise (e.g. if the Earth's atmosphere had not been there).

- You are observing the sunrise at the equator, and the equatorial radius is $R = 6378.1370\text{km}$.
- Treat the atmosphere as being dominated by the Troposphere, which is 10km thick (high); use an index of refraction of $n_{atm} = 1.00030$
- The Earth requires 86,400 seconds to make 1 rotation.



A Discussion About the Postulates of Special Relativity

- Postulate 1: the laws of physics do not depend on the frame of reference in which they are determined.
  - Discussion Points
    - Why would this be useful?
    - Did it have to be this way?
    - How would we test this postulate?
    - What is a consequence of this postulate?
- Postulate 2: the speed of light is the same in all frames of reference.
  - Discussion Points
    - How would we test this postulate (beyond the Michelson-Morley Experiment)?
    - What is a consequence of this postulate?

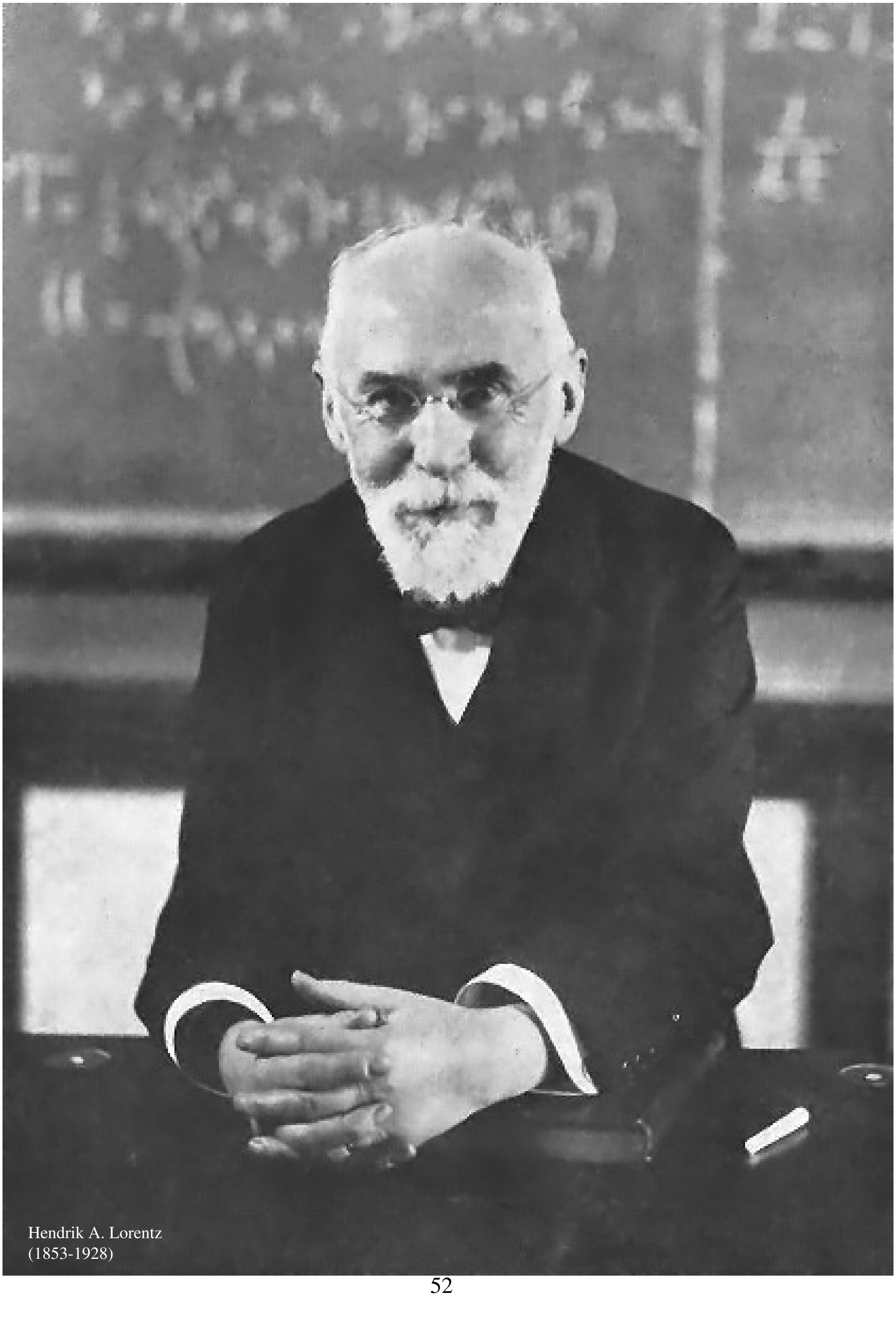
e The Special Theory of Relativity - The Lorentz Transformation

The Special Theory of Relativity - The Lorentz Transformation

Overview

In this lecture, we will learn…

- To appreciate the Galilean Transformation and its assumptions;
- A way to derive the form of the correct transformation between frames of reference;
- How to begin applying this transformation, consistent with the postulates of special relativity.



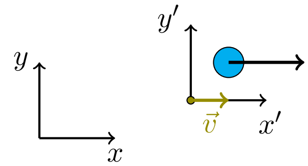
Hendrik A. Lorentz
(1853-1928)

The Galilean Transformation

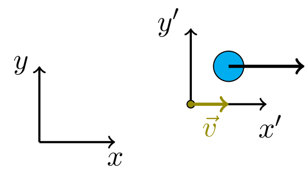
The *Galilean Transformation* was predicated on two assumptions, which may not have been made very clear when you originally learned about it. For observers in inertial frames of reference:

- Time passes in the same way for all observers, regardless of their state of motion.
- All observers agree that objects in each others' frames are in states of constant motion.

Define two frames: one, denoted S, taken to be at rest where observers measure positions (x, y, z) and object velocities $\vec{u}$; and another, denoted S', moving at velocity $\vec{v}$ relative to S in which observers measure coordinates (x', y', z') and object velocities $\vec{u}'$. Assume all of their coordinate axes are parallel between frames (eg. x is parallel to x'), and further simplify by assuming $\vec{v}$ is entirely along x and x'. In all frames, time is absolute so that $t = t'$.



The Galilean Transformation



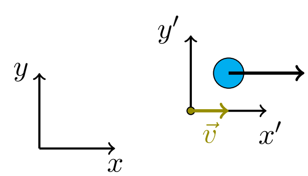
This picture, built from the postulates of the Galilean/Newtonian approach to space and time, then allows one to define the equations transforming observations in one frame to another (e.g. from S' to S):

$$\begin{aligned} x &= x' + \vec{v}t';\ y = y';\ z = z' \\ t &= t' \end{aligned}$$

Object velocities are related between frames by $\vec{u} = \vec{u}' + \vec{v}$.

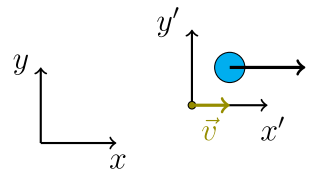
Toward the correct transformation

However, the Galilean transformation is immediately observed to violate the postulates of special relativity, which are based on observational evidence. For example, you can show that Maxwell's Equations are not invariant under a Galilean Transformation, violating the first postulate of special relativity. This implies that it's possible to know if you are in the absolute rest frame (e.g. the aether). It's a bit easier to see how the second postulate is violated using a simple example: a beam of light emitted in the aether frame, the absolute rest frame, can have very different speeds in other reference frames.



We can build up the transformation we need by starting from the exact same picture, since that is still valid; however, the new postulates will enable us to arrive at a mathematics consistent with observation.

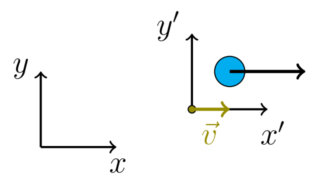
Building the mathematics



Our goal is to figure out what is the correct transformation. We will continue to work with frames of reference wherein object velocities are observed to be constant (inertial reference frames). It must be true that, in two inertial reference frames S and S':

$$\begin{aligned} x &= ut \\ x' &= u't' \end{aligned}$$

Building the mathematics



In order to further satisfy the first postulate, it must also be true that the transformation equations represent a *linear transformation between frames*. Otherwise, it cannot be true that all frames observe object velocities to be constant. Let me demonstrate why.

$$\begin{aligned} x &= A(x')^n + B(t')^m \\ t &= C(x')^n + D(t')^m \end{aligned}$$

Here, A, B, C, and D are constants.

Building the mathematics - why the transformation must be linear

$$\begin{aligned} x &= A(x')^n + B(t')^m \\ t &= C(x')^n + D(t')^m \end{aligned}$$

Recall that any velocity like u_x (or u'_x) is defined by dx/dt (or dx'/dt'). Transform the above equations into statements about the differentials rather than just the coordinates themselves:

$$\begin{aligned} dx &= An(x')^{n-1}dx' + Bm(t')^{m-1}dt' \\ dt &= Cn(x')^{n-1}dx' + Dm(t')^{m-1}dt' \end{aligned}$$

Building the mathematics - why the transformation must be linear

$$\begin{aligned} dx &= An(x')^{n-1}dx' + Bm(t')^{m-1}dt' \\ dt &= Cn(x')^{n-1}dx' + Dm(t')^{m-1}dt' \end{aligned}$$

To relate the *velocities* in each frame, take the ratio of the above two equations:

$$\begin{aligned} \frac{dx}{dt} \equiv u_x &= \frac{An(x')^{n-1}dx' + Bm(t')^{m-1}dt'}{Cn(x')^{n-1}dx' + Dm(t')^{m-1}dt'} \\ &= \frac{An(x')^{n-1}\frac{dx'}{dt'} + Bm(t')^{m-1}}{Cn(x')^{n-1}\frac{dx'}{dt'} + Dm(t')^{m-1}} \end{aligned}$$

Building the mathematics - why the transformation must be linear

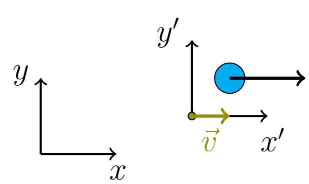
$$u_x = \frac{An(x')^{n-1}\frac{dx'}{dt'} + Bm(t')^{m-1}}{Cn(x')^{n-1}\frac{dx'}{dt'} + Dm(t')^{m-1}}$$

The above equation leads us to the final relationship:

$$u_x = \frac{An(x')^{n-1}u'_x + Bm(t')^{m-1}}{Cn(x')^{n-1}u'_x + Dm(t')^{m-1}}$$

See the problem with this? Unless $n = m = 1$, the above always has a space and time functional dependence on the right-hand side, which violates the first postulate of special relativity. To be compatible with that postulate, we are forced to choose a *linear transformation from frame-to-frame.*

Building the mathematics - how about those unknown constants?

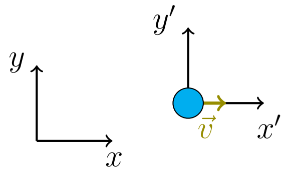


$$\begin{aligned} x &= Ax' + Bt' \\ t &= Cx' + Dt' \end{aligned}$$

Now that we have established the linear nature of the transform, let's nail down those constants - A, B, C, and D.

We need to think of some limiting cases of the picture above (some "special cases") where we can isolate the constants in small batches. This is a standard trick in algebra. We have two equations and four unknowns; we need four special cases to solve for all unknowns.

Building the mathematics - Case 1: object pinned at the origin of S'



$$
\begin{aligned}
x &= Ax' + Bt' \\
t &= Cx' + Dt' \\
x &= ut;\; x' = u't'
\end{aligned}
$$

Consider the case that the moving object is pinned to the origin of frame S', so that it moves with speed $u = v$ (the frame speed). In that case, $x' = 0$ and we can simplify the equations to:

$$
\begin{aligned}
x &= Bt' \\
t &= Dt' \\
x &= ut;\; 0 = u't'
\end{aligned}
$$

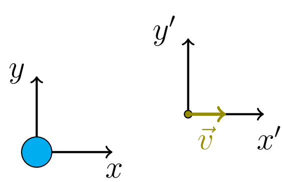
Divide the first two, substitute using third

$$
\begin{aligned}
\frac{x}{t} &= \frac{B}{D} \\
\frac{x}{t} &= u = v
\end{aligned}
$$

First Constraint

$$
v = \frac{B}{D}
$$

Building the mathematics - Case 2: object pinned at the origin of S



$$
\begin{aligned}
x &= Ax' + Bt' \\
t &= Cx' + Dt' \\
x &= ut;\; x' = u't'
\end{aligned}
$$

Consider the case that the moving object is pinned to the origin of frame S, so that it moves with speed $u' = -v$. In that case, $x = 0$ and we can simplify the equations to:

$$
\begin{aligned}
0 &= Ax' + Bt' \\
t &= Cx' + Dt' \\
0 &= ut;\; x' = -vt'
\end{aligned}
$$

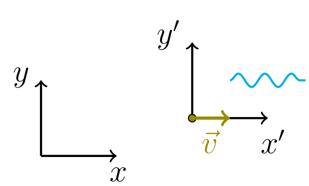
Use first and third equations...

$$
\begin{gathered}
-Ax' = Bt';\; x' = -vt' \\
Avt' = Bt' \rightarrow v = \frac{B}{A}
\end{gathered}
$$

Next Constraints

$$
\begin{aligned}
v &= \frac{B}{D};\; v = \frac{B}{A} \\
A &= D
\end{aligned}
$$

Building the mathematics - Case 3: object is a beam of light



$$
\begin{aligned}
x &= Ax' + Bt' \\
t &= Cx' + Dt' \\
x &= ut;\; x' = u't'
\end{aligned}
$$

Consider the case that the moving object is light. By the second postulate of special relativity, both frames must observe the velocity of the object to be exactly c, regardless of their relative motion. Thus:

$$
\begin{aligned}
x &= Ax' + Bt' \\
t &= Cx' + Dt' \\
x &= ct;\ x' = ct'
\end{aligned}
$$

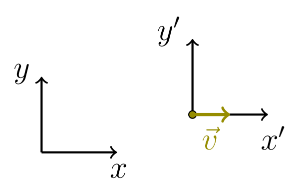
Combine first two, substitute using third...

$$
\begin{aligned}
\frac{x}{t} &= \frac{Ax' + Bt'}{Cx' + Dt'} \\
c &= \frac{Ac + B}{Cc + D} \\
C &= \frac{B}{c^2} = A\frac{v}{c^2}
\end{aligned}
$$

All Constraints

$$
\begin{aligned}
B &= Av;\ C = A\frac{v}{c^2} \\
D &= A
\end{aligned}
$$

Building the mathematics - Case 4: relative motion of the two frames



$$
\begin{aligned}
x &= Ax' + Avt' \\
t &= A\frac{v}{c^2}x' + At' \\
x &= ut;\ x' = u't'
\end{aligned}
$$

According to the guiding picture, the equations relating observations S and S' differ by changing $v \to -v$:

$$
\begin{aligned}
x &= Ax' + Avt' \\
t &= A\frac{v}{c^2}x' + At'
\end{aligned}
$$

Rearrange to solve for x' and t'

$$
\begin{aligned}
x' &= A^{-1}\left(1-\frac{v^2}{c^2}\right)^{-1}(x-vt) \\
t' &= A^{-1}\left(1-\frac{v^2}{c^2}\right)^{-1}\left(-\frac{v}{c^2}x+t\right)
\end{aligned}
$$

Solve for A

$$
\begin{aligned}
A &= A^{-1}\left(1-\frac{v^2}{c^2}\right)^{-1} \\
A &= \left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} \equiv \gamma
\end{aligned}
$$

The Lorentz Transformation

We have finally arrived at a mathematical transformation that obeys all of the postulates of special relativity:

$$
\begin{aligned}
x &= \gamma(x'+vt') \\
t &= \gamma\left(\frac{v}{c^2}x'+t'\right)
\end{aligned}
$$

Observations are made in S' and converted to S

$$
\begin{aligned}
x' &= \gamma(x-vt) \\
t' &= \gamma\left(-\frac{v}{c^2}x+t\right)
\end{aligned}
$$

Observations are made in S and converted to S'

The multiplicative factor, γ, depends on the relative velocity of the two frames and is a measure of the degree of the "relativistic effects" between the two frames, as we will see. It is given by:

$$
\gamma = \left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} = \frac{1}{\sqrt{1-v^2/c^2}}
$$

The Gamma Factor - Some Intuition-Building

The "gamma factor," γ, will appear *everywhere* in relativity calculations. It's largely unavoidable for all of the physics calculations you will do going forward. Let's build some intuition about this fascinating quantity.

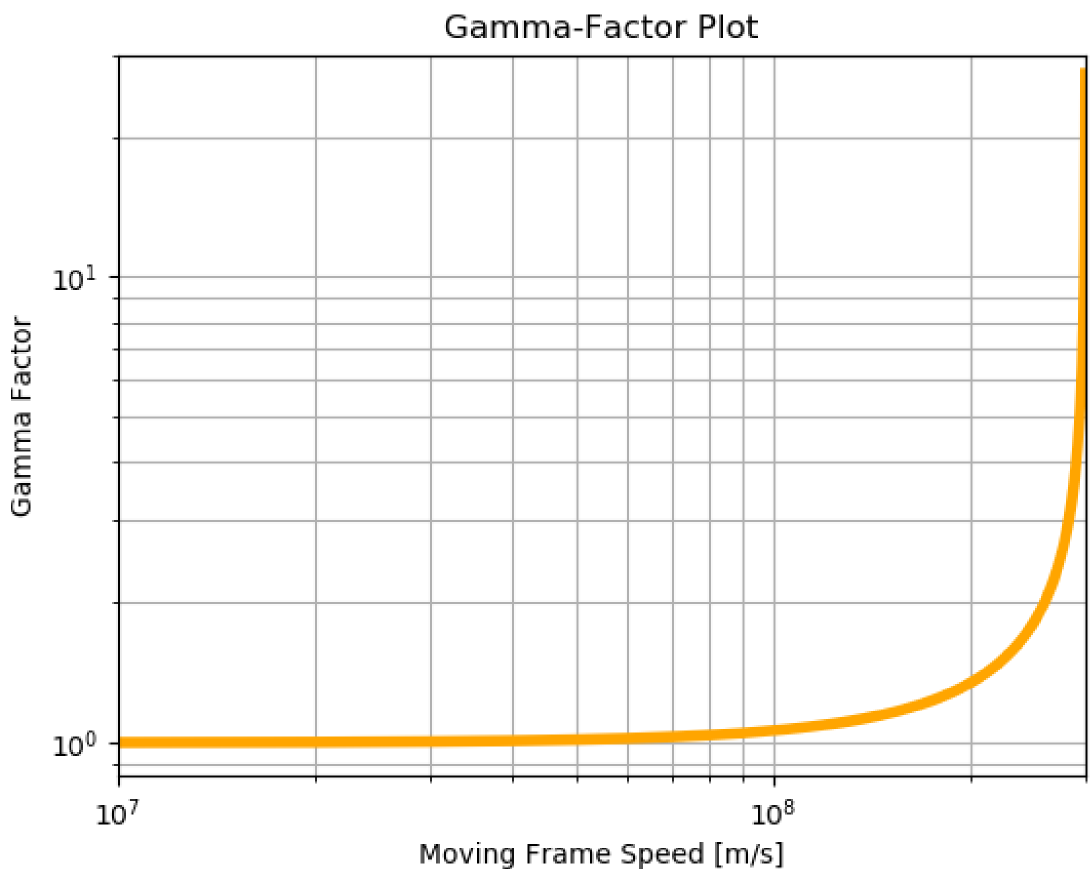
- What is γ for a frame S' at rest with respect to frame S?

- We would expect to find that the two frames are the same, since they are then in the same state of motion. Indeed, for $v = 0$ we observe that $\gamma(v = 0) = 1$.

- What is γ for a frame, S', that achieves a velocity of c relative to S?
  - This would be like riding a beam of light, moving at $v = c$. It's another special case, and we see that:

$$\gamma(v = c) = \frac{1}{\sqrt{1 - c^2/c^2}} = \frac{1}{0} = \infty$$

- So γ is a frame-velocity-dependent number whose range is $[1, \infty]$ (inclusive).



γ remains incredibly close to 1.0 until you reach frame speeds well in excess of 50% of c. It's no wonder relative time and space is so far beyond daily human experience!

Some expected effects due to special relativity - overview

- As teased in the previous lecture on the basics of special relativity (the postulates and their consequences), this theory of space and time has some consequences that can feel surprising to the average human being.

- For instance, objects in motion relative to Frame S will appear contracted along the direction of travel;

- To appreciate this effect, even from the Lorentz Transformation, requires you to really think about what it has ever meant to measure the length of something. That discussion is best saved for class time, as we can get very "hands on" with the concept before doing calculations.

- Frames in motion relative to Frame S will also observe a passage of time that, relative to S, seems slowed (time dilation);
- Finally, events that are simultaneous in one frame are not guaranteed at all to be simultaneous in another.

Let's explore time and simultaneity in this lecture.

Some expected effects due to special relativity - dilation of time with motion

Another intriguing effect has to do with the passage of time in different frames of reference. Consider a clock at rest in frame S' that provides regular information (e.g. pulses of light at t'_1 and t'_2) always at the same position x'. What is the time between pulses observed in the rest frame?

$$\begin{aligned} t_2 &= \gamma\left(\frac{v}{c^2}x'_2 + t'_2\right) \\ t_1 &= \gamma\left(\frac{v}{c^2}x'_1 + t'_1\right) \end{aligned}$$

Since the clock is always pulsing away at the same location in S', $x'_1 = x'_2 \ldots$

$$\begin{aligned} t_2 - t_1 &= \gamma\left(t'_2 - t'_1\right) \\ \Delta t &= \gamma \Delta t' \to \frac{\Delta t}{\Delta t'} = \gamma \geq 1 \end{aligned}$$

We see that in a frame moving relative to another, durations of time will always be observed to be shorter than in the rest frame; the degree of *dilation* of time depends on v/c. Time in the moving frame will appear to the rest frame to pass more slowly.

Some expected effects due to special relativity - simultaneity depends on frame

A third intriguing effect has to do with whether or not two events simultaneous in one frame will be simultaneous in another frame. Consider two events - e.g. pulses of light - which are observed to be *simultaneous* in frame S'; the events have coordinates (x'_1, t') and x'_2, t'. What is the time between the events observed in the rest frame?

$$\begin{aligned} t_2 &= \gamma\left(\frac{v}{c^2}x'_2 + t'_2\right) \\ t_1 &= \gamma\left(\frac{v}{c^2}x'_1 + t'_1\right) \end{aligned}$$

Since the events are simultaneous in frame S', $t'_1 = t'_2 \ldots$

$$
\begin{aligned}
t_2 - t_1 &= \gamma \frac{v}{c^2}\left(x_2' - x_1'\right) \\
\Delta t &= \gamma \frac{v}{c^2} \Delta x'
\end{aligned}
$$

We see that in the rest frame, the events cannot be observed to be simultaneous unless $x_2' = x_1'$ or $v = 0$, in which case the two frames become indistinguishable. Except under these conditions, simultaneity cannot be guaranteed in another frame.

This is a good example of how space and time are not inseparable in special relativity - they get "tangled up" in each other in moving from frame to frame. A spatial separation in S' becomes a temporal separation in frame S.

Recovering Classical Physics

Are the Galilean/Newton view of space and time and relative motion totally gone? Not really - after all, the Galilean Transformation worked in real computations for centuries before special relativity was needed. A good theory of nature describes all new phenomena while including the old ones that were successful in some more limited regime.

To recover the Galilean/Newtonian view, we need only slow nature down from speeds close to that of light. For example, in the special case that $v \ll c$, let's look at what happens to γ using the *Binomial Expansion*:

$$
\gamma = (1 - v^2/c^2)^{-1/2} \xrightarrow{\alpha^2 = v^2/c^2 \leq 1} = (1-\alpha^2)^{-1/2} \xrightarrow[Expansion]{Binomial} \approx 1 + \frac{1}{2}\alpha^2 + \ldots \text{higher order terms of } \mathcal{O}(\alpha^4) \ldots
$$

For the case where $\alpha \ll 1$ so that $\alpha \to 0$, we see that $\gamma \approx 1$. That makes sense - it's approaching the limiting case when $v = 0$. So what happens to the Lorentz Transformation Equations?

$$
\begin{aligned}
x &= \gamma(x' + vt') = \left(1 + \frac{1}{2}\alpha^2 + \mathcal{O}(\alpha^4)\right)(x' + vt') \xrightarrow{v \ll c} x' + vt' \\
t &= \gamma(\frac{v}{c^2}x' + t') = \left(1 + \frac{1}{2}\alpha^2 + \mathcal{O}(\alpha^4)\right)(\frac{\alpha}{c}x' + t') \xrightarrow{v \ll c} t'
\end{aligned}
$$

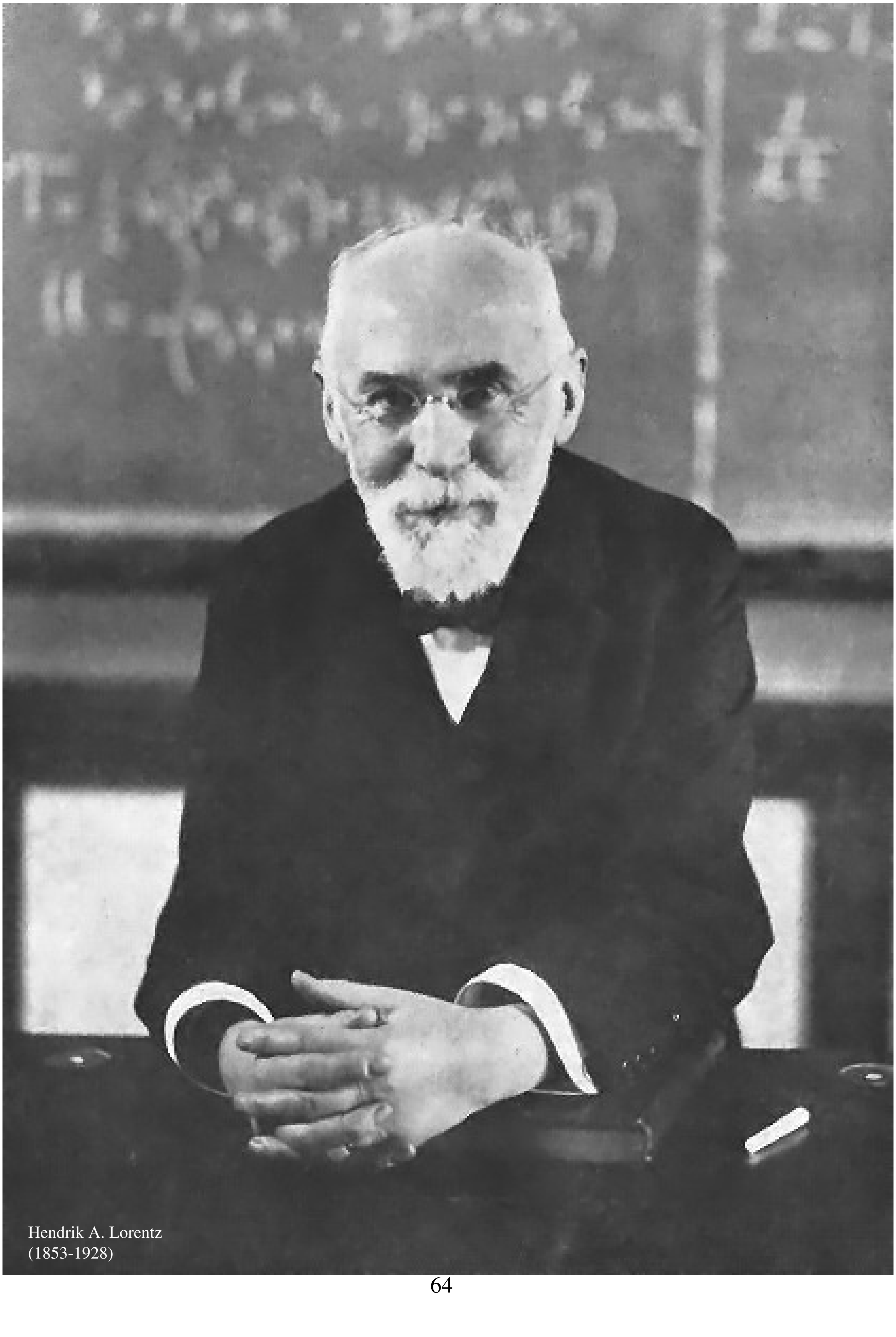
We have recovered the Galilean Transformation, and reconciled with classical physics in the limit of low velocities.

Review

In this lecture, we have learned...

- To appreciate the Galilean Transformation and its assumptions;
- A way to derive the form of the correct transformation between frames of reference;
- How to begin applying this transformation, consistent with the postulates of special relativity.

- We see that while all observers must agree that light moves at the same speed regardless of relative motion, observers in different frames of reference will disagree on lengths of objects, lengths of time that pass, and the simultaneity of events.
- We have also seen how to recover classical physics from special relativity, by allowing $v \ll c$ and observing how the Lorentz Transformation reduces in that limit to the Galilean Transformation.



Hendrik A. Lorentz
(1853-1928)

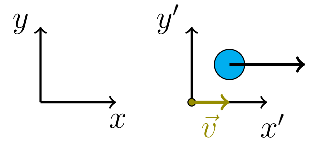
A Discussion About the Postulates of Special Relativity

- Postulate 1: the laws of physics do not depend on the frame of reference in which they are determined.
  - Discussion Points
    - Why would this be useful?
    - Did it have to be this way?
    - How would we test this postulate?
    - What is a consequence of this postulate?
- Postulate 2: the speed of light is the same in all frames of reference.
  - Discussion Points
    - How would we test this postulate (beyond the Michelson-Morley Experiment)?
    - What is a consequence of this postulate?

e.1 Problem Solving in the Lorentz Transformation

Problem Solving in the Lorentz Transformation

Refreshers



The Lorentz Transformation obeys all of the postulates of special relativity. A frame S is taken to be at rest, and a frame S' to be moving relative to it at velocity v:

$$
\begin{aligned}
x &= \gamma(x' + vt') \\
t &= \gamma(\frac{v}{c^2}x' + t')
\end{aligned}
$$

Observations are made in S' and converted to S

$$
\begin{aligned}
x' &= \gamma(x - vt) \\
t' &= \gamma(-\frac{v}{c^2}x + t)
\end{aligned}
$$

Observations are made in S and converted to S'

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Instructor Problem: The Gamma Factor of Commuting

The gamma factor, γ, is a kind of measure of how "relativistic" a situation is (that is, how seriously you need to take the application of pure special relativity, as opposed to using just classical physics). Calculate the gamma factor for an everyday situation:

- You are driving at 70 mph (113km/h) relative to the surface of the Earth

Photo by Charl Folscher on Unsplash

Student Problem: Practice with the Gamma Factor

The gamma factor, γ, is a kind of measure of how "relativistic" a situation is (that is, how seriously you need to take the application of pure special relativity, as opposed to using just classical physics). For each of the following, calculate the *difference of the gamma factor from 1.0*. You should not get 0.0 for any of your answers (if you are having trouble with this, let's talk!):

- A Global Positioning System (GPS) satellite makes two revolutions around the Earth each day, relative to a fixed reference point on the surface of the Earth, at an altitude of 20,200km above the surface of the Earth.
- The Sun orbits the center of the Milky Way Galaxy once every 226 million years, at a speed of 782,000km/h relative to the center.
- The "Crab Nebula" is the remnant of a star that was observed to explode in 1054 A.D., a cloud of debris now 10 ly across. It expands at a speed of 1,000 km/s relative to the stellar corpse at its center. For reference, the Crab Nebula is 6500ly from Earth.
- At its design collision energy, the Large Hadron Collider at the CERN Laboratory in Geneva, Switzerland, can accelerate a proton to a speed of 99.9999991% that of light, relative to the surface of the Earth.

The Crab Nebula Expansion over 13 years of observation

Movie from The Planetary Society.

Mini-Discussion: Commentary on the Consequences of the Lorentz Transformation

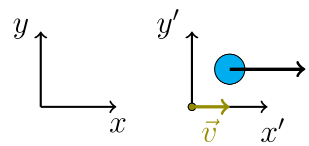
- Space and time get “tangled up” in each other: a spatial difference in one frame can become a time difference in another, etc.
  - It is more correct to refer to a single framework called “space-time” or “spacetime,” of which space and time are aspects and which motion can entangle in one another.
- Spatial displacements or distances in one frame are not necessarily the same in all other frames.

- Time ("temporal") displacements or distances in one frame are not necessarily the same in all other frames.
- Events that are simultaneous in one frame are not necessarily simultaneous in any other frame of reference.

e.2 Problem Solving in the Lorentz Transformation (Part 2)

Problem Solving in the Lorentz Transformation (Part 2)

Refreshers



The Lorentz Transformation obeys all of the postulates of special relativity. A frame S is taken to be at rest, and a frame S' to be moving relative to it at velocity v:

$$\begin{aligned} x &= \gamma(x' + vt') \\ t &= \gamma(\frac{v}{c^2}x' + t') \end{aligned}$$

Observations are made in S' and converted to S

$$\begin{aligned} x' &= \gamma(x - vt) \\ t' &= \gamma(-\frac{v}{c^2}x + t) \end{aligned}$$

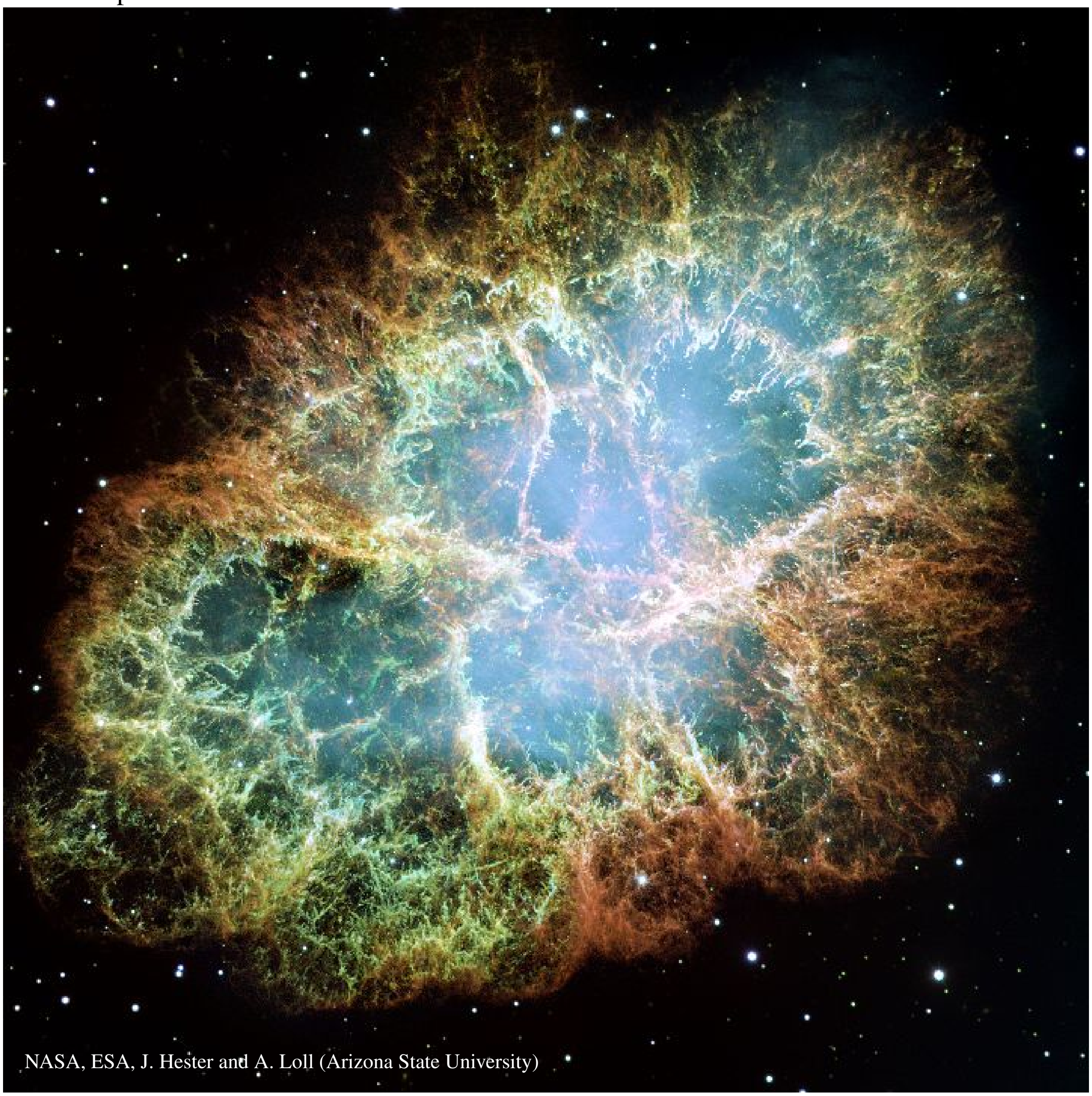
Observations are made in S and converted to S'

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Instructor Problem: Two Stars Explode

NASA sends out a probe at high velocity relative to Earth ($v = 0.87c$). Before leaving the solar system, the probe flies past Earth one last time. 25 years later, Earth observers record two simultaneous stellar explosions (supernovas); both lie along the flight path of the probe, one directly ahead and one directly behind. The original stars, long known to astronomers, were observed to have zero relative velocity to Earth. What does the probe observe to be the time difference between the two supernovas?



Student Problem: Measure the Length of a Moving Object

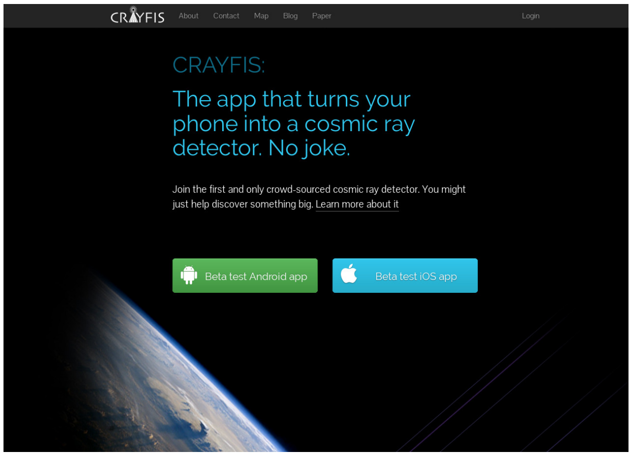
You are going to figure out how to measure the length of a moving object. You are never allowed to enter the frame of reference of the moving object. You must devise a strategy that can be done entirely in the rest frame of the room, without ever entering the moving frame.

Some inputs:

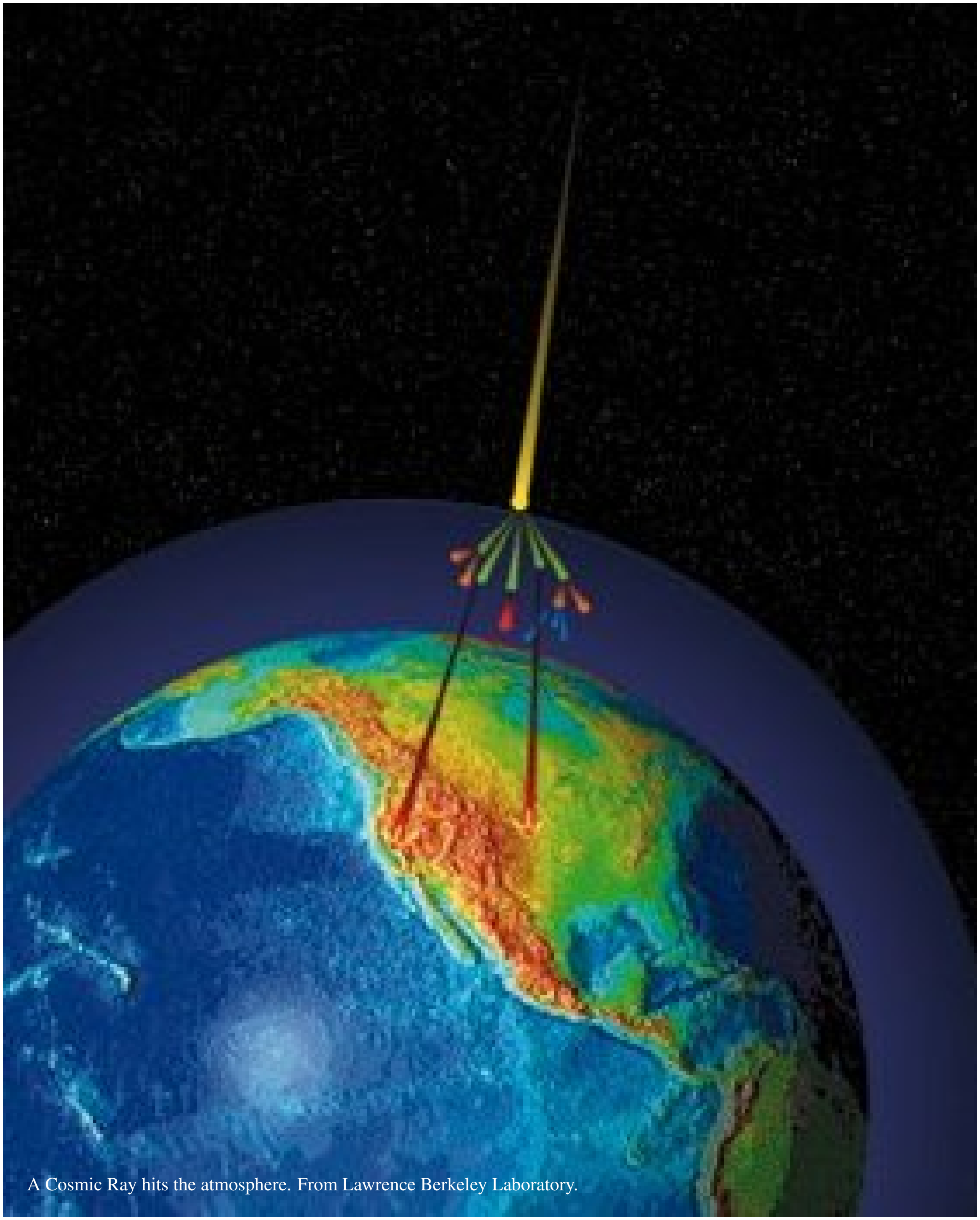
- The relative velocity of the object to the rest frame of the room is $v = 0.87c$.
- In the frame of the moving object, its length was determined to be $10.0\mathrm{cm}$.

Instructor Problem: Will the muon make it?

A muon is produced in a cosmic ray interaction with nitrogen 10km above the surface of the Earth. The muon has a decay lifetime, in its own rest frame, of $2.2\mu\mathrm{s}$. If the muon is headed straight toward the surface of the Earth at 0.87c, what is the probability it survives long enough to make it to the surface of the Earth?



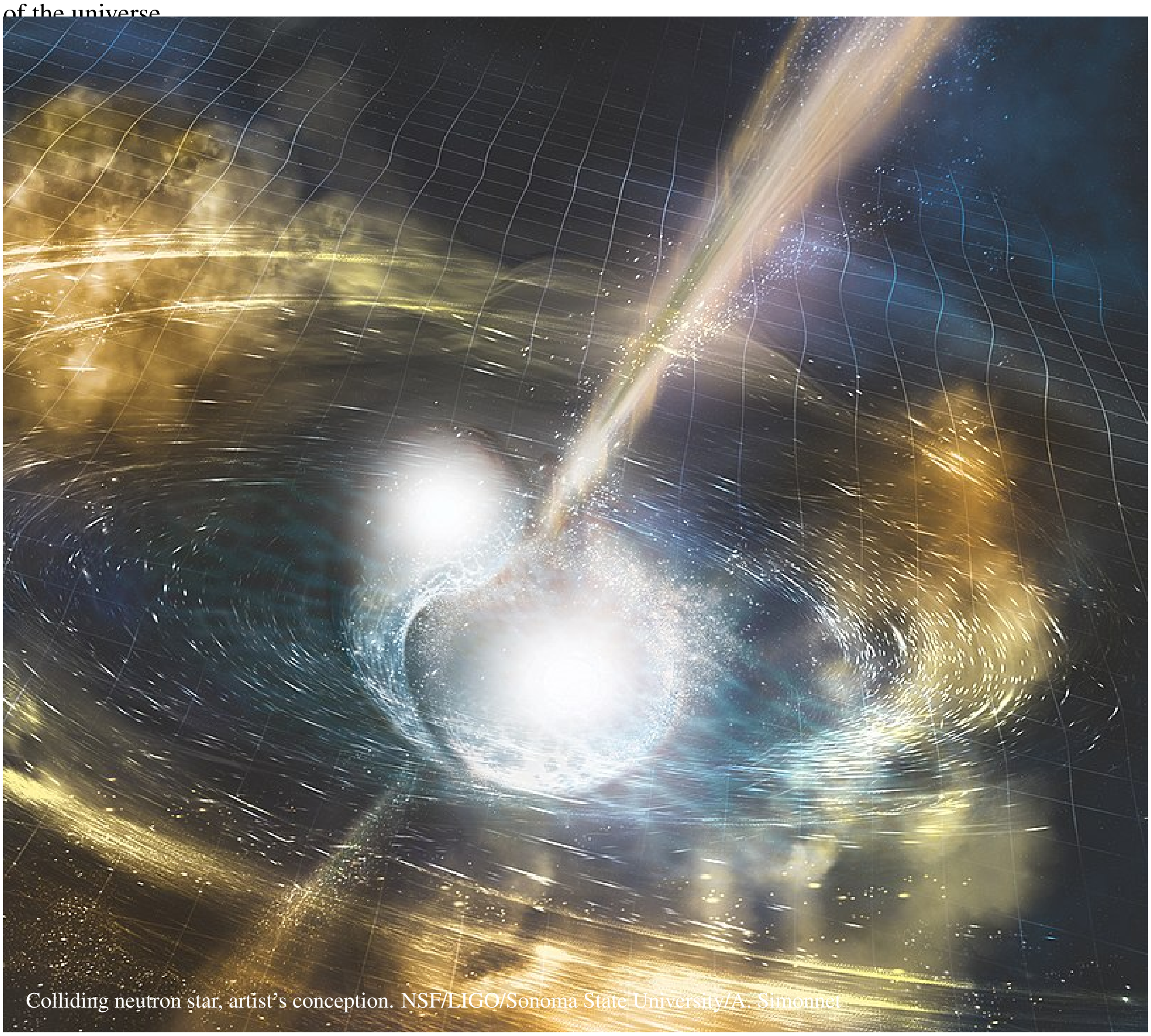
Crayfis.io cosmic ray muon app



A Cosmic Ray hits the atmosphere. From Lawrence Berkeley Laboratory.

Student Problem: Plutonium from the Collision of Binary Neutron Stars

Neutron Stars are one of the death-stages of massive stars; a star whose core is about 1.5 times that of our sun's whole mass will end its life by collapsing under its own weight, and the resulting explosion can leave behind a 10km-radius ball of mostly neutrons: a *neutron star*. In 2017 the collision of a pair of neutron stars was observed for the very first time. This collision resulted in a *kilonova*, a huge explosion that then produced many of the heavy elements we take for granted, including gold and platinum. Binary colliding neutron stars have likely been a key manufacturer of the heavy elements during the long lifetime of the universe.

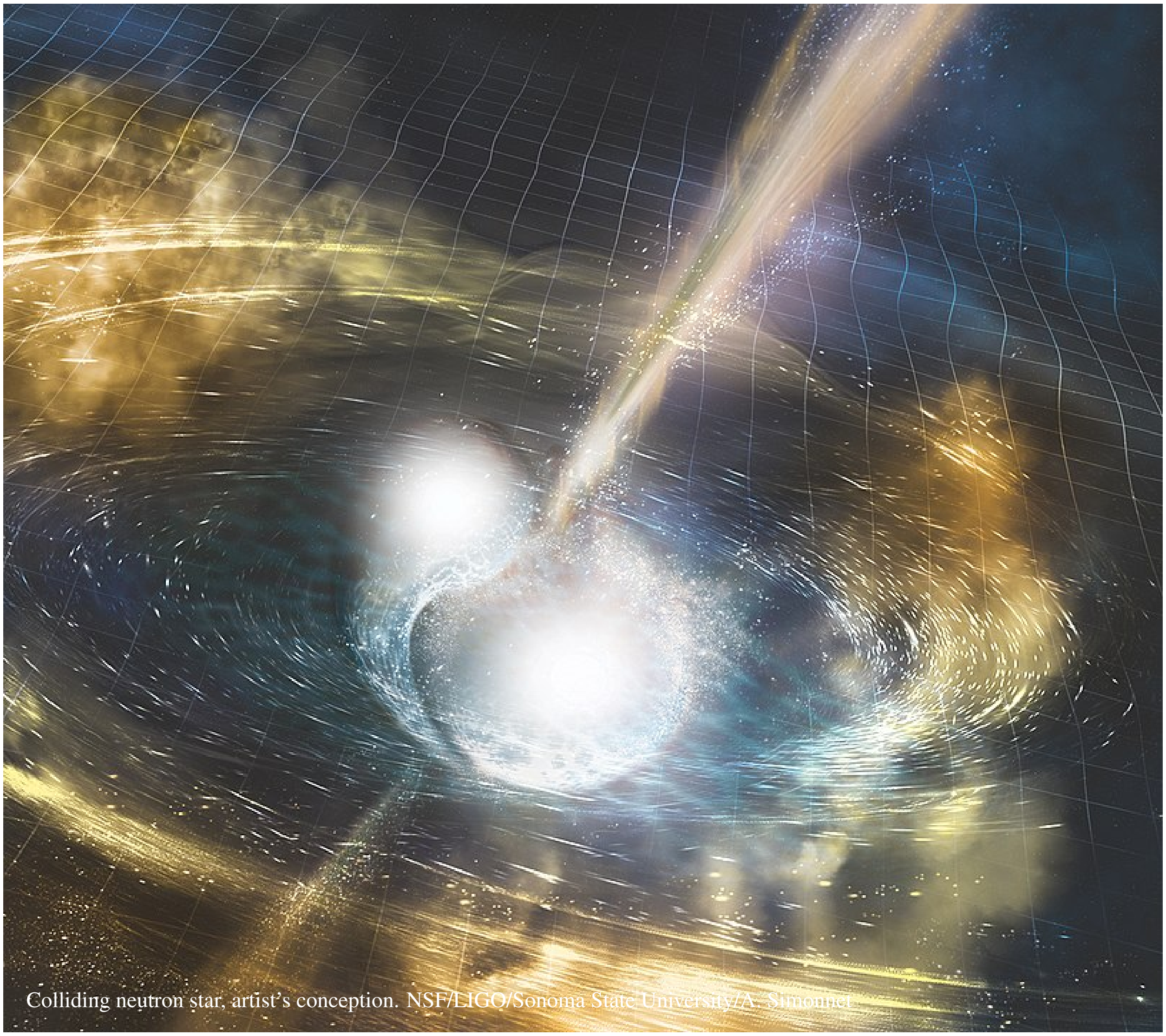


Colliding neutron star, artist's conception. NSF/LIGO/Sonoma State University/A. Simonnet

Student Problem: Plutonium from the Collision of Binary Neutron Stars

The collision observed in 2017 happened 130 million light-years from Earth. If the explosion produced the isotope of Plutonium, Pu-244 (half-life of 80.8 million years), and if the Plutonium was accelerated by the explosion to a speed of 0.97c…

- what is the probability that Plutonium (e.g. a single atom) made in this kilonova explosion could have reached Earth? Assume a straight-line flight path.
- what distance would an observer riding with the Plutonium at that speed determine separates the Earth from the kilonova explosion? Give the answer in light-years.


Colliding neutron star, artist's conception. NSF/LIGO/Sonoma State University/A. Simonnet

f The Special Theory of Relativity - Length Contraction

The Special Theory of Relativity - Length Contraction

Overview

In this lecture, we will learn…

- To appreciate what it truly *means* to measure the length of something;
- To apply a clearer understanding of the meaning of length measurement to objects in motion;
- That objects in motion relative to a Frame S will appear contracted along the direction of flight from S's perspective.

A blurry shot of a moving train from the platform in Leicester Square Station.

What does it mean to measure the length of an object?

You have probably been taking the act of measuring the length of something for granted your whole life. Why? Because for the most part, you have been spoiled by the fact that objects whose length needs to be determined have been in your same frame of reference, making it easy to perform the measurement. Think about how you measure a long object, like a wall in a room (e.g. for laying out furniture):

- You have a reference length, like a tape measure or a ruler;
- You lay down one end of the reference coincident with one end of the wall;
- You briefly enter a reference frame moving with respect to the wall by walking the reference length (e.g. the tape measure) to the other end of the wall. For this period, you and the wall are not in the same reference frame, but it doesn't affect what happens next.
- You stop where you want to make the measurement, re-entering the reference frame of the wall. You have now stretched the reference length to the end point of the measurement. You read off the coordinate on the reference length and consider the job done.

But imagine the wall had been moving the whole time, and you are not allowed to ever enter its reference frame. How then do you measure the length of the wall?

How do you measure the length of an object in motion?

Instead, imagine the object whose length you wish to measure is passing you at a velocity, $\vec{v}$, along your x-axis in your coordinate system. Let's call your frame the rest frame, Frame S. Let's call the frame of the moving object Frame S'. You no longer have the luxury of placing a length reference at one end, taking time to get it to the other end of the object, and then making the measurement. In fact, you are *never* allowed, in this scenario, to enter the reference frame of the moving object. How now do you make the measurement?

- You might have come up with the following idea. If you know the relative velocity of the object, you could wait for the front of the object to pass a reference point in your frame (e.g. the origin); then wait until you observe the back of the object to pass the same point some time later, marking the passage of time; then use the speed and the time to compute the length;
- Another idea uses *simultaneity*: you and a friend sit in Frame S as the object passes; one of you observes the front and the other the back of the moving object, and when each of you observes your assigned end pass you press a button to make one of two lights blink; when your blinks are observed to be simultaneous in Frame S, it means that you each located your assigned end at the same time; the physical distance between the two of you is the length of the moving object.

Let's take the second approach and apply the Lorentz Transformation to see what we observe to be the relationship between the length of the object observed in the moving frame (where a person can use the approach of walking around and taking as much time as they need locating the ends in their coordinate system) and in the rest frame.

Some expected effects due to special relativity - contraction of length with motion

Consider an object at rest in frame S' whose ends are determined to be located at position (start) x'_1 and (end) x'_2 in that frame. Its length is therefore $L' = x'_2 - x'_1$. What is its length as observed in the rest frame using the method of simultaneity in Frame S to co-locate the ends of the object?

The two observers in Frame S use button presses and lights to generate two events, (x_2, t_2) (locating the front) and (x_1, t_1) (locating the back). To measure the length of the object, we demand they reposition themselves until these events are *simultaneous*: $t_2 = t_1$. Let $\Delta t' = t'_2 - t'_1$, etc.

$$\begin{aligned} x_2 &= \gamma\left(x'_2 + vt'_2\right) \\ x_1 &= \gamma\left(x'_1 + vt'_1\right) \\ t_2 &= \gamma\left(\frac{v}{c^2}x'_2 + t'_2\right) \\ t_1 &= \gamma\left(\frac{v}{c^2}x'_1 + t'_1\right) \end{aligned}$$

Use simultaneity in Frame S...

$$\begin{aligned} L = x_2 - x_1 &= \gamma\left(L' + v\Delta t'\right) \\ t_2 - t_1 = 0 &= \gamma\left(\frac{v}{c^2}L' + \Delta t'\right) \to \Delta t' = -\frac{v}{c^2}L' \\ L &= \gamma\left(L' - \frac{v^2}{c^2}L'\right) = \gamma\frac{1}{\gamma^2}L' \\ L &= \gamma^{-1}L' \to \frac{L'}{L} = \gamma \geq 1 \end{aligned}$$

We see that the length of the object when it's observed at rest is longer than when it's observed in motion; this is the *length contraction* hinted at earlier.

What about the alternative method of measurement: a fixed location?

To show you that the answer is the same regardless of the method of measuring a moving object using observations in Frame S, let's quickly run through the "using the speed and time at a fixed location in Frame S" method. A single observer in Frame S fixes their location at a point and observes the times, t_1 and t_2, at which the front and then the back of the moving object, respectively, pass that point. One important subtlety: in this method, the front end (at x'_2 in frame S') passes first (at (x_1, t_1) in Frame S), followed by the back end (at x'_1 in frame S', but at (x_2, t_2) in Frame S). Thus:

$$\begin{aligned} x_2 - x_1 = 0 &= \gamma\left((x'_1 - x'_2) + v\Delta t'\right) = \gamma\left(-L' + v\Delta t'\right) \to v\Delta t' = L' \\ t_2 - t_1 = \Delta t &= \gamma\left(-\frac{v}{c^2}L' + \Delta t'\right) \xrightarrow[\text{sides by v}]{\text{multiply both}} v\Delta t = L = \gamma\left(-\frac{v^2}{c^2}L' + v\Delta t'\right) \\ L &= \gamma\left(1 - \frac{v^2}{c^2}\right)L' = \frac{1}{\gamma}L' \to \frac{L'}{L} = \gamma \end{aligned}$$

We again are forced to conclude that the consequence of special relativity is that the length of the object, observed in the rest frame, will be shorter than what observers see in the moving frame. Length contraction should be a thing, no matter how we choose to measure that length.

The Concept of "Proper Length"

We can now introduce a common concept in discussions of the implications of special relativity: *Proper Length*. The proper length, L_0, of an object will be that length observed in a frame where the object speed, relative to the frame, is zero (at rest with respect to the frame of observation.)

The proper length is the *longest length that can be measured/observed for the object in any frame of motion*. In all other frames in motion relative to that special frame where the object is observed to be at rest, observers will see a length that is shorter by L_0/γ.

Review

In this lecture, we have learned…

- To appreciate what it truly *means* to measure the length of something;
- To apply a clearer understanding of the meaning of length measurement to objects in motion;
- That objects in motion relative to a Frame S will appear contracted along the direction of flight from S's perspective.

A blurry shot of a moving train from the platform in Leicester Square Station.

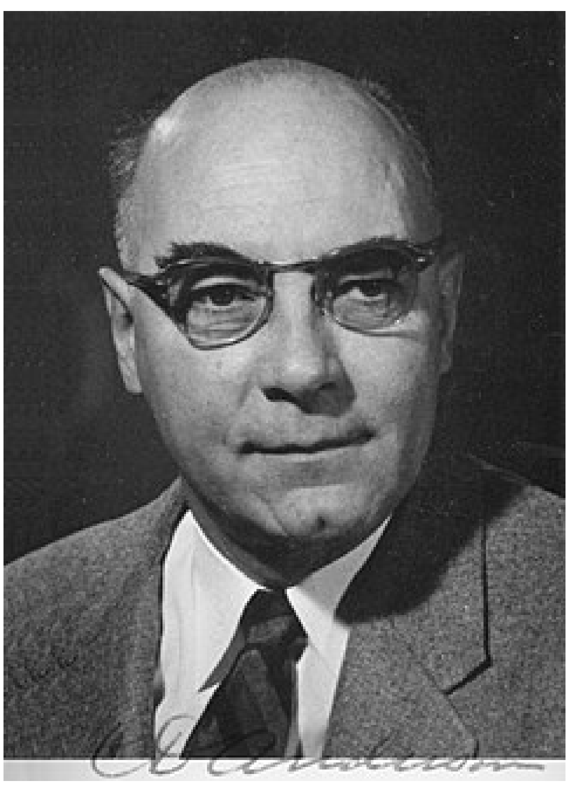
g The Special Theory of Relativity - The Relativity of Time

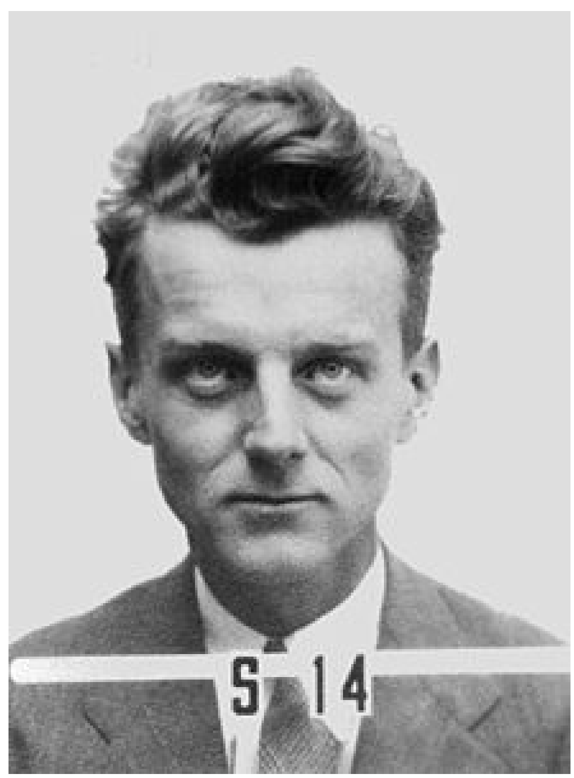
The Special Theory of Relativity - The Relativity of Time

Overview

In this lecture, we will learn…

- What is a "muon";
- How to use the muon as a laboratory for making predictions with the Lorentz Transformation;
- How the muon was the first direct test of the validity of Special Relativity.





Carl Anderson (left, 1905—1991) and Seth Neddermeyer (right, 1907—1988), who co-discovered the muon in 1936.

Reminder: The Lorentz Transformation

$$\begin{aligned} x &= \gamma(x' + vt') \\ t &= \gamma(\frac{v}{c^2}x' + t') \end{aligned}$$

Observations are made in moving frame S' and converted to rest frame S

$$\begin{aligned} x' &= \gamma(x - vt) \\ t' &= \gamma(-\frac{v}{c^2}x + t) \end{aligned}$$

Observations are made in rest frame S and converted to moving frame S'

$$\gamma \;=\; \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} = \frac{1}{\sqrt{1 - v^2/c^2}} \xrightarrow[\mathit{Expansion}]{\mathit{Binomial}} 1 + \frac{1}{2}\frac{v^2}{c^2} + \mathcal{O}(\frac{v^4}{c^4})$$

Some consequences of the Lorentz Transformation (Special Relativity):

- Length measurements depend on the frame of reference;
- Time measurements depend on the frame of reference;
- Events simultanous in one frame and not necessarily so in another.

Exploring the Relativity of Time

It's not easy to construct a vessel that can take humans up to speeds in excess of $0.5c$, where γ begins to take on values much in excess of 1.0. Instead, we need to identify a laboratory where such speeds can readily be achieved and where there is a natural "clock" of some kind that we can compare when at rest and when in motion.

Tiny particles like electrons ($m = 9.11 \times 10^{-31}$ kg) would be ideal for investigating fast-moving objects. For example, the ground-state energy of the Hydrogen atom is 13.6 eV $= 2.19 \times 10^{-18}$J, which equates to a kinetic energy for the electron in that state. The speed of such an electron, imagined as "orbiting" the proton at the center of the Hydrogen atom, would be

$$K = \frac{1}{2}mv^2 \rightarrow v = \sqrt{\frac{2K}{m}} = 2.2 \times 10^6 \mathrm{m/s}$$

That's already pretty fast on its own; J. J. Thompson discovered the electron by ripping them off parent atoms using the Coulomb force, accelerating the electrons to kinetic energies of tens of thousands of eV, which would equate to speeds of $\approx 10^8 \mathrm{m/s}$. So a particle like the electron would be easy to accelerate. But the electron is very stable - it doesn't do anything with regularity once isolated - so it lacks a useful "clock".

Exploring the Relativity of Time

Marie and Pierre Curie ca 1903

Radioactive decay of atomic nuclei, on the other hand, represents an excellent built-in clock provided by nature. For example, among her many discoveries two-time Nobel Prize winner Marie Curie isolated the element *Polonium*. It is highly unstable, and the natural isotope of Polonium, Po-210, transforms spontaneously into a stable Lead atom (Pb-206) after emitting energy in the form of radiation (specifically, ejecting two protons and two neutrons bonded into a Helium nucleus, known as an *alpha particle*). Po-210 has a “half life” of 138 days. What does that mean?

It means that if I have 100 atoms of Po-210 in a sealed container, and I wait 138 days and then look in the container, on average I will find that 50 atoms of Po-210 remain and the container now also is home to 50 Pb-206 atoms from spontaneous nuclear decay. If I further wait another 138 days and look again, I will on average have 25 Po-210 atoms left and 75 Pb-206 atoms in the container. Every half-life, I lose half of the remaining atoms due to spontaneous nuclear decay.

Unstable radioactive elements have a reliable built-in “clock” - a regular process that occurs at the same place (the nucleus) at regular time intervals. However, Polonium and other radioative elements were hard to come by in the days of the early 1900s and not easy to accelerate to near-light speeds (they’re thousands of times heavier) and particle accelerators of that quality were decades away.

Exploring the Relativity of Time

If only we had a tiny particle that combined the “lightness” and ease of acceleration of the electron with the regular instability of radioactive atoms.

Enter the “Mu Meson,” or “Muon.”

The mathematical description of nuclear behavior required new particles to transmit forces inside the nucleus. These intermediaries were dubbed the “mesons,” (from the Greek word *mesos*, meaning “intermediary”) and by the 1930s the hunt was on to find them.

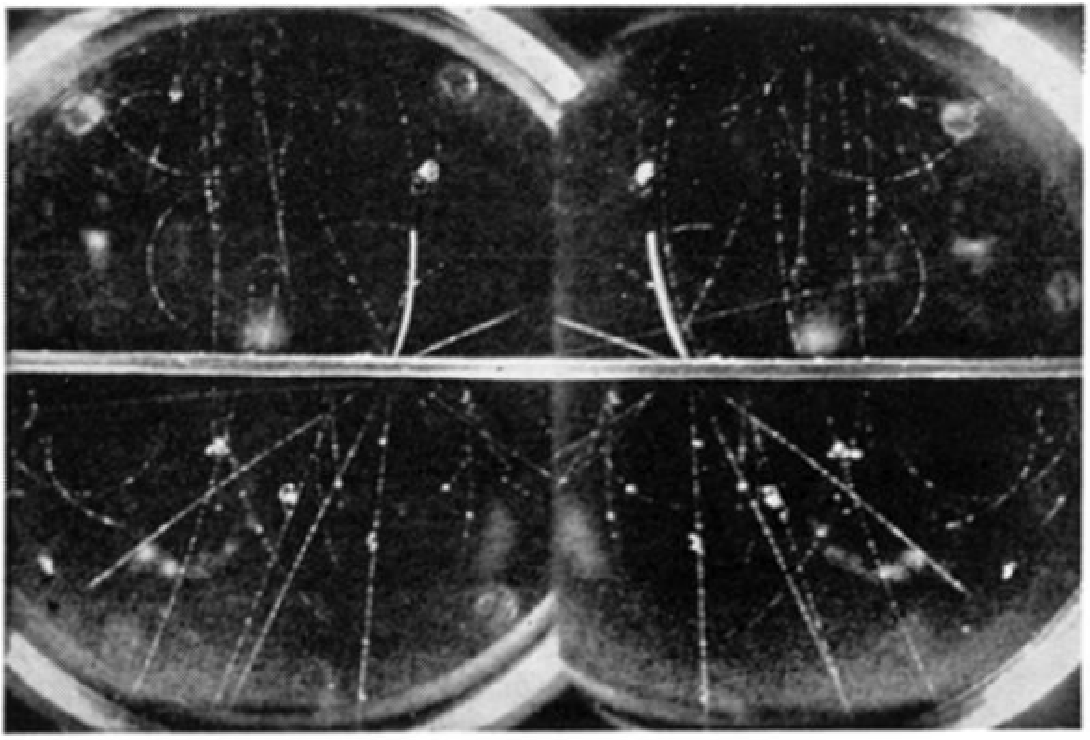


FIG. 12. Pike's Peak, 7900 gauss. A disintegration produced by a nonionizing ray occurs at a point in the 0.35 cm lead plate, from which six particles are ejected. One of the particles (strongly ionizing) ejected nearly vertically upward has the range of a 1.5 MEV proton. Its energy (given by its range) corresponds to an $H\rho = 1.7 \times 10^5$, or a radius of 20 cm, which is three times the observed value. If the observed curvature were produced entirely by magnetic deflection it would be necessary to conclude that this track represents a massive particle with an e/m much greater than that of a proton or any other known nucleus.

Shown left is an image taken by Anderson and Neddermeyer and published in 1936[7, 8]. A previously unobserved electrically charged particle punches through the lead target in the center of the picture, knocking apart nuclei in the process but barely losing any energy.

It would be dubbed the "mu meson" ("muon"), as physicists mistakenly thought it might be one of the sought-after nuclear force intermediaries. This turned out to be wrong, but would not be understood until the 1940s.

Its electric charge was determined from careful experimentation to be $q_\mu = -1.609 \times 10^{-19}$C, identical to the electron charge; its mass to be $m_\mu \approx 207 \times m_e$. Crucially, it is also unstable; trapped nearly at rest, on average it only lives $2.2\,\mu$s.

An Important Aside: Unstable Particles and their "Lifetime"

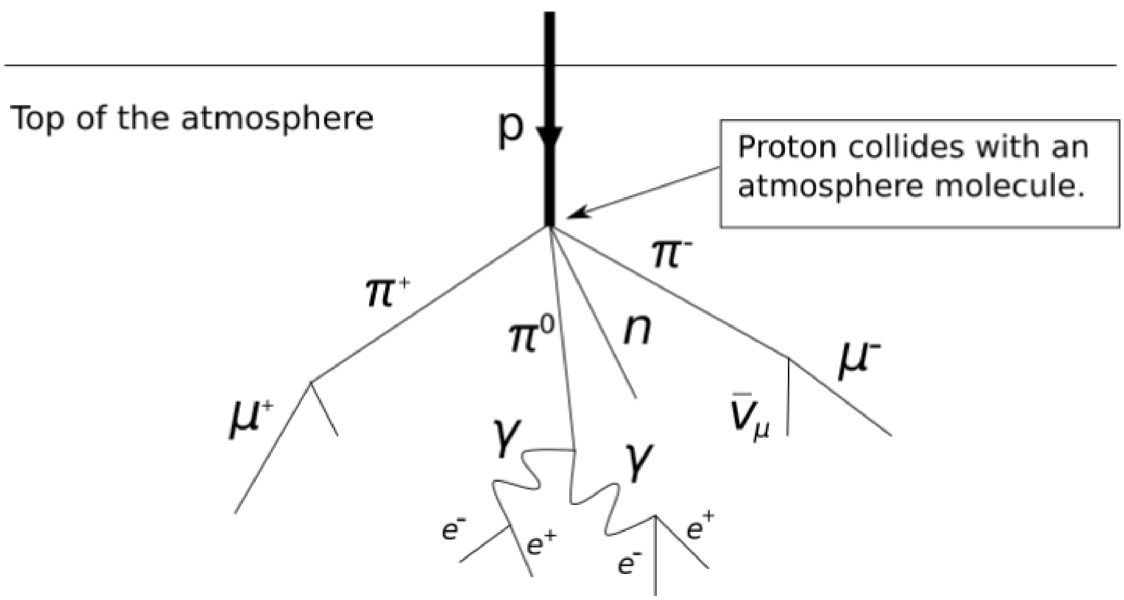
The mathematics of unstable particles was developed in response to the discovery of radioactive decay of atomic nuclei. It's a fairly straight-forward application of algebra and calculus, and it's instructive to run through it here.

Consider a system of N_0 unstable objects (e.g. nuclei or particles like the muon) at time t. Let us consider a time $t + dt$ at which time we discover that the number of objects has changed by $-dN$, where dt and dN are differential units of time and number, respectively. The number remaining, $N = N_0 - dN$, depends on how many we started with, N_0 - thus, there is a proportionality between the change of number, the change in time, and the original number of objects. We can express this as:

$$-dN = \lambda N\, dt \rightarrow \int_{N_0}^{N} -\frac{1}{N} dN = \int_0^t \lambda dt \rightarrow \ln N_0 - \ln N = \lambda(t - 0) \rightarrow \ln N = \ln N_0 - \lambda t \rightarrow N = N_0 e^{-\lambda t}$$

We find that the constant of proportionality, λ, has units of inverse-time; it's convenient to define $\lambda = 1/\tau$, where τ is known as the *time constant* of the phenomenon. When $t = \tau$, 63.2% of the original N_0 objects are gone. For unstable particles, this is what is known as the *lifetime of the particle*, and you can show mathematically that it's the average time an unstable particle exists. The *half-life* of an unstable particle, $t_{1/2}$, is related to τ by $t_{1/2} = \tau \ln(2)$. So when we say "The muon has a lifetime of $2.2\,\mu$s," we are referring to time at which there is a 63.2% chance a single muon has decayed.

The Muon and Cosmic Rays



The muon was discovered using "Cosmic Rays," particles from space that slam into the Earth's atmosphere, showering electrically charged and neutral particles toward the surface of the Earth (e.g. by hitting Nitrogen or Oxygen nuclei). Anderson and Neddermeyer found the muon in cosmic ray showers, using detectors located either on Pike's Peak ($4.3\,\mathrm{km}$ above sea level) or in Pasadena, CA (roughly at sea level). Most muons produced by cosmic rays are made about $15\,\mathrm{km}$ above the Earth's surface.

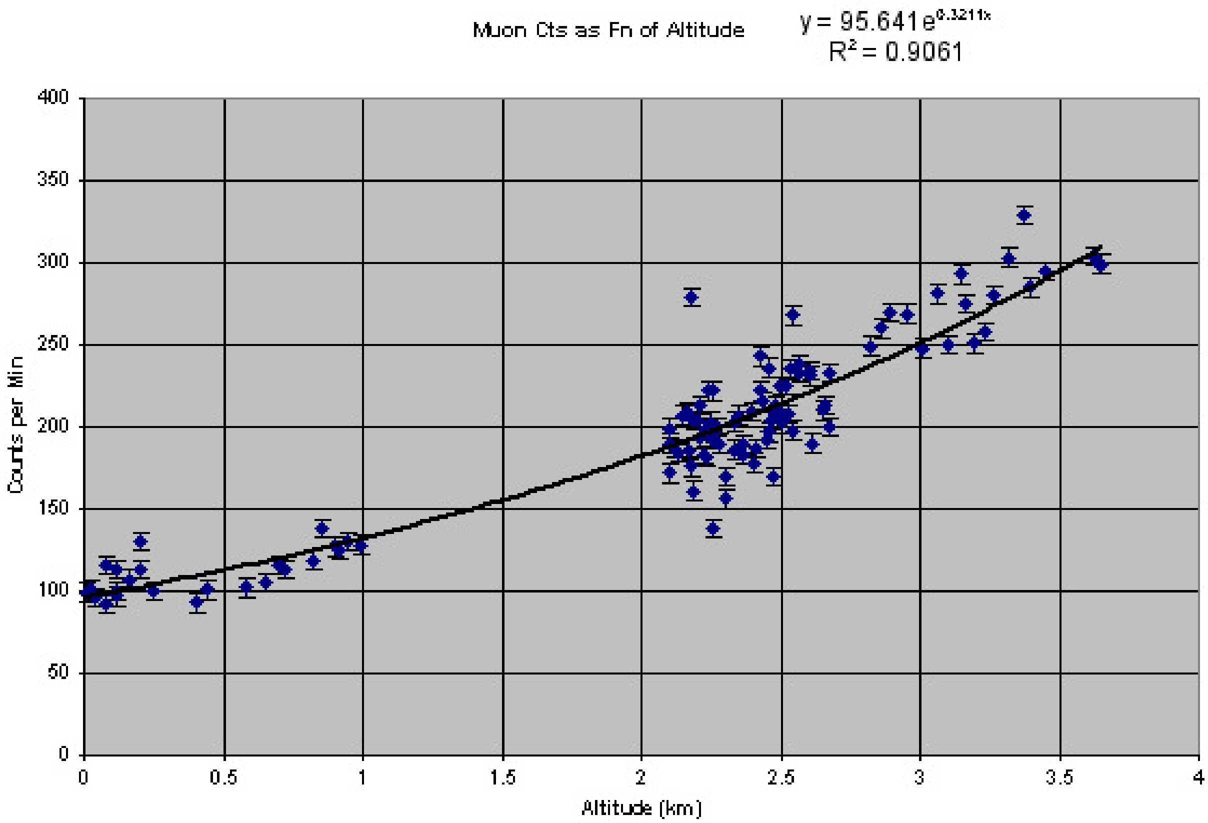
Based on the known instability of the muon, one might expect that if one counts a certain number of muons at high altitude, N_1, then by the math of unstable particle decay and using the known lifetime of the muon when it's nearly at rest, $\tau_\mu = 2.2\,\mu$s, one can accurately predict the number of muons you should expect at a lower altitude, N_2, where due to particle decay fewer total muons are expected to be found.

The muon's short lifetime should radically cut down its numbers as we go lower and lower into the atmosphere. Even traveling at $v = c$, in one lifetime a muon can travel just $0.66\,\mathrm{km}$, by which time it's already had a 63.2% chance of decaying; after two lifetimes, it could have traveled $1.3\,\mathrm{km}$ into the atmosphere, but with an 86.4% chance of having decayed. 10 lifetimes brings a muon only $6.6\,\mathrm{km}$ into the atmosphere (still $8\,\mathrm{km}$ above the surface of the Earth!) but by then it has had a 99.995% chance of having decayed.

Muons, Muon Decay, and Observations

So, what is actually observed? Shown right is real data on muon counting at different altitudes, in this case collected by teachers involved in a program called "QuarkNet" that engages K-12 teachers in real physics research environments[bh]

Based on the counts per minute at $3.5\,\mathrm{km}$ and assuming that time passes at the same rate whether the muon is moving or not (in violation of the conclusions of special relativity), we can predict how many counts per minute we expect at $0.5\,\mathrm{km}$. At $3.5\,\mathrm{km}$, $N_0 \approx 300\mathrm{counts/min}$. Assuming the best-case scenario of $v = c$ such that $t = y/c$ then yields $N = N_0 e^{-y/(c\tau)}$. After a height change of $y = 3.0\,\mathrm{km}$ we expect to find *at most* $N = 3.2\,\mathrm{counts/min}$.



Is that what is actually observed? **HECK NO.** In fact, the teachers observed $100\,\mathrm{counts/min}$. Why?

Muons, Time Dilation, and the Correctness of Special Relativity

[bh]Data available from `https://cosmic.lbl.gov/SnowMass/main.html`, "Quarknet Teachers take the Berkeley Detector to Colorado and Virginia."

Special Relativity, with its Lorentz Transformation valid for all speeds up to that of light, will help us to understand this. Let's relate what's going on in the muon's reference frame (S') and what is going on in the Earth's reference frame (S). In the reference frame of the muon, its lifetime is $2.2\,\mu$s - recall that this is the lifetime observed when the muon is nearly or exactly at rest...its *proper lifetime*. The Lorentz Transformation would predict that the time measured by an observer on the Earth will be different from a person who could ride along with the muon:

$$t_2 - t_1 \quad = \quad \gamma\left(\frac{v}{c^2}(x'_2 - x'_1) + (t'_2 - t'_1)\right)$$

All the events - being created in the upper atmosphere at time t'_1, decaying later at time t'_2 - *happen at the same place in the muon's reference frame*. That is, $x'_2 = x'_1$. Thus:

$$t_2 - t_1 \quad = \quad \gamma\left(t'_2 - t'_1\right)$$

The lifetime of the muon in its frame of reference is $\Delta t' = 2.2 \times 10^{-6}\,\mathrm{s}$. Special Relativity would predict that, from the perspective of an observer on the ground, the muon would appear to live longer than would be expected if it were at rest as well. This is completely in accordance with the data - more muons are observed to survive to a lower altitude than would be expected from classical physics and its assumption of absolute passage of time for all observers.

Tying Together the Data and Special Relativity

The data told us that of the $N_0 = 300$ muons per minute observed at $3.5\,\mathrm{km}$, about $N = 100$ per minute survive at $0.5\,\mathrm{km}$ above the Earth's surface. In the reference frame of the Earth, we can relate these numbers to the observed decay time of the muons in their rest frame (τ), the distance they travel (y), and their typical speed v):

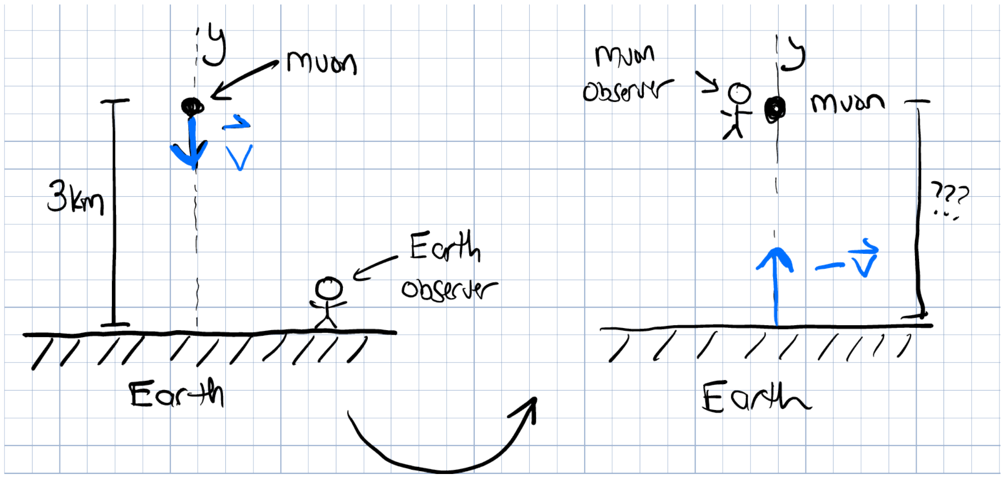
$$N = N_0 e^{-t/(\gamma\tau)} \rightarrow \ln(N/N_0) = -\frac{y}{\gamma v \tau} \rightarrow -\frac{y}{\tau \ln(N/N_0)} = \gamma v$$

We want either γ or v, and since they are related to each other[bh] we have some algebra to do. Try it yourself as an exercise. You should find that $\gamma \approx 4.3$ so that $v \approx 2.91 \times 10^8\mathrm{m/s}$; from the Earth observer's perspective, the journey takes $10.3\,\mu$s. It's no wonder muons make a fantastic early laboratory for tests of special relativity!

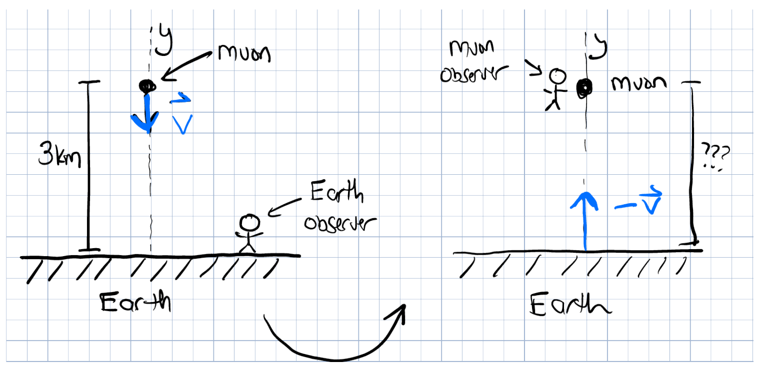
What is the Conclusion from the Muon's Perspective?

It is instructive to revisit the problem from the perspective of the muon. Imagine you are sitting with the muon in its reference frame. Suddenly you come into existence and below you the Earth is racing up toward you at a speed of v (whereas a person on Earth sees the muon heading downward at the same speed.).

[bh]Recall that $\gamma = (1 - v^2/c^2)^{-1/2}$, which means $v = c\sqrt{1 - 1/\gamma^2}$ and thus $\gamma v = c\sqrt{\gamma^2 - 1}$



What is the Conclusion from the Muon's Perspective?



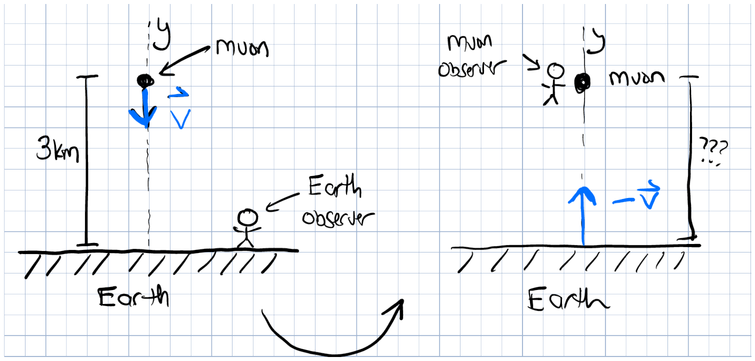
From the muon's perspective, it is standing still; all events (coming into existence and decaying) happen at the same location in its frame of reference. The muon sees the Earth below it when it comes into existence, rushing toward it at $-v$.

What is the Conclusion from the Muon's Perspective?

How far does the muon have to go to make it to its destination? The distance in the frame of the Earth is $h = 3\,\mathrm{km}$... the frame where the Earth and its atmosphere appear to be at rest. That makes this distance the *proper length*, h_0. The muon will see a *different length* that it is to cross, given by $h' = h_0/\gamma \approx 0.233h_0 = 0.699\,\mathrm{km}$.

We conclude that the muon observes the distance it will travel to be *contracted* compared to what observers on the Earth see, by a factor of $1/\gamma$. From the muon's perspective, the distance between the place in the atmosphere where it came into existence and where it ultimately decays is greatly shortened, requiring only a time $\Delta t' \approx 2.4\,\mu\mathrm{s}$, since the Earth-Atmosphere system is moving relative to it and is length-contracted.

What is the Conclusion from the Muon's Perspective?

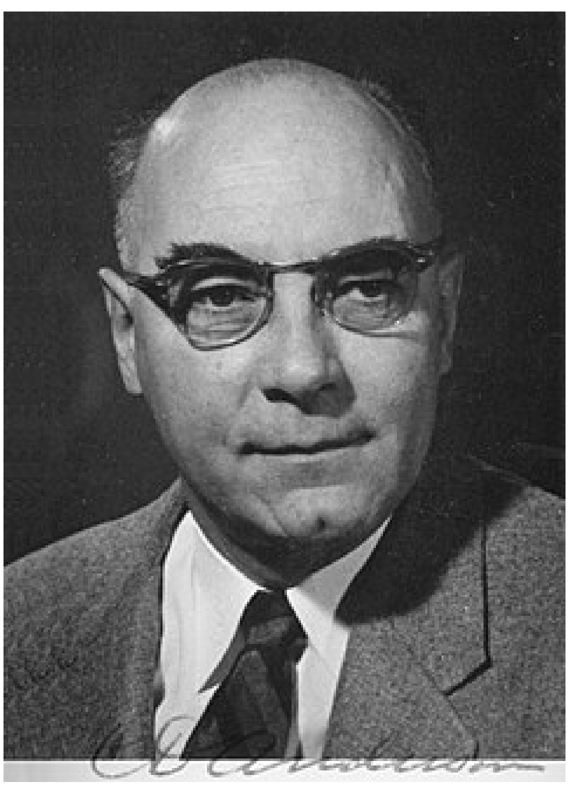


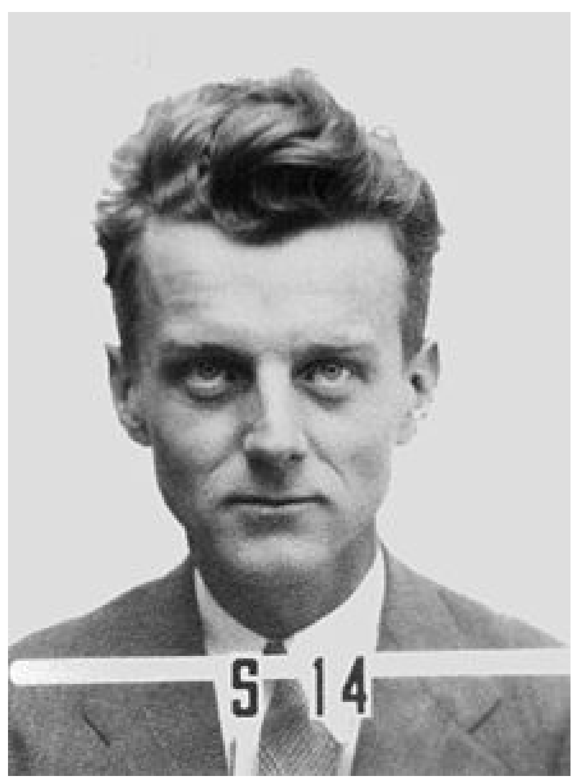
So while observers on Earth and an observer moving with the muon would *disagree on the reason for the muon reaching the lower measurement point, they agree that it's very likely to happen.* The Earth observer argues that it's because time is passing more slowly for the muon in motion, so it takes longer to decay; the muon observer argues it's because the distance is contracted, giving the muon a chance to get there in a time close to one lifetime ($2.2\,\mu$s). They are both right, and they can relate their perspectives using the Lorentz Transformation to explain each other's observations.

Review

In this lecture, we have learned...

- What is a "muon";
- How to use the muon as a laboratory for making predictions with the Lorentz Transformation;
- How the muon was the first direct test of the validity of Special Relativity.





Carl

Anderson (left, 1905—1991) and Seth Neddermeyer (right, 1907—1988), who co-discovered the muon in 1936.

h The Special Theory of Relativity - Light and the Doppler Effect

The Special Theory of Relativity - Light and the Doppler Effect

Overview

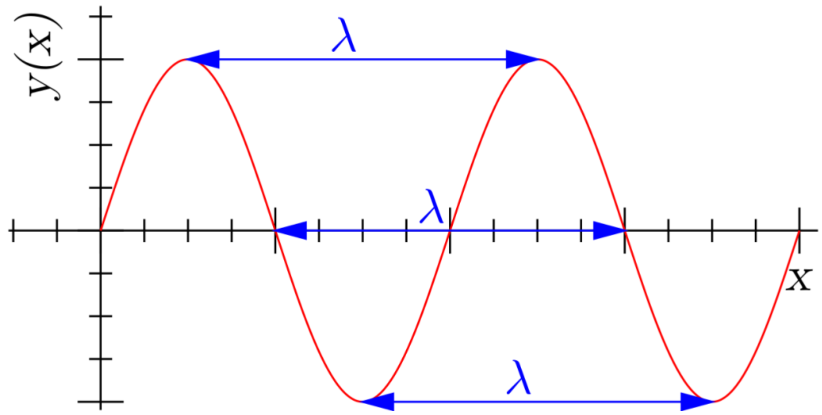
In this lecture, we will learn…

- What is the classical "Doppler Effect";
- The effect of motion of a light source on the characteristics of light other than speed;
- How to compute the special relativistic Doppler effect on light and interpret the effect on observations.

Christian Doppler (1803—1853)

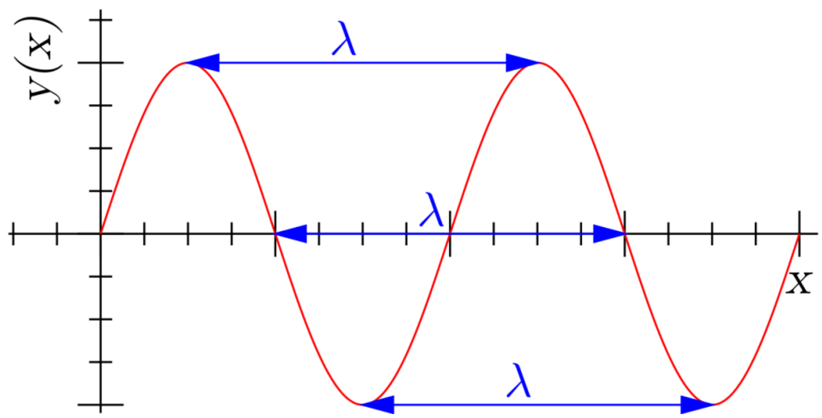
Properties of Waves

Recall *oscillatory phenomena* from introductory physics; specifically, recall simple harmonic motion. This kind of repetitious motion has a time and space structure that allows it to be described using sine or cosine functions of space and time:



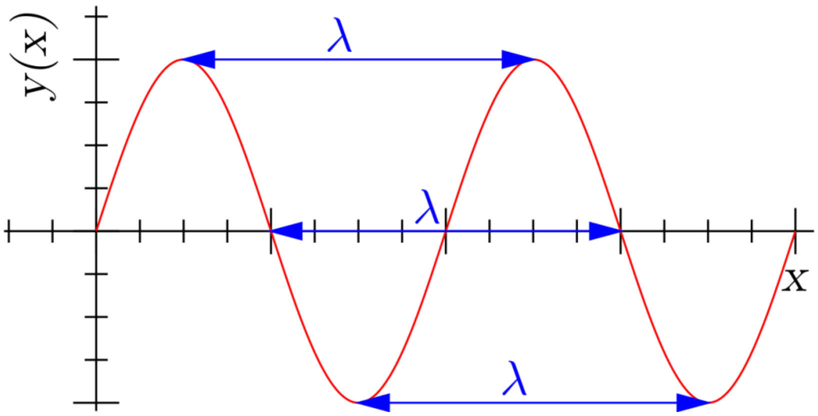
(Image from Wikipedia)

Properties of Waves



The distance between maxima (or minima) of the phenomenon is called the *wavelength*, denoted by λ. The time between maxima (or minima) passing the same spatial point is known as the period, T; the inverse of the period is the rate at which maxima (minima) pass that point and is known as the *frequency*, denoted $f = 1/T$ (or using the Greek letter "nu": $\nu = 1/T$).

Properties of Waves



The speed with which waves move in space during some unit of time is given by the product of frequency and wavelength:

$$v_{wave} = \lambda f \xrightarrow{\text{Light Wave}} c = \lambda f$$

Properties of Waves

We can think of waves of sound or waves of light as being represented by lines or planes; the location of a line (in 2-D) or a plane (in 3-D) indicates a location in space of a maximum of the traveling wave. This is a common way to quickly and simply sketch a wave. The distance between lines/planes is the wavelength. Such a line or plane would be referred to as a "wave front."

The frequency of such a phenomenon can be thought of as how many fronts per second are emitted by the source.

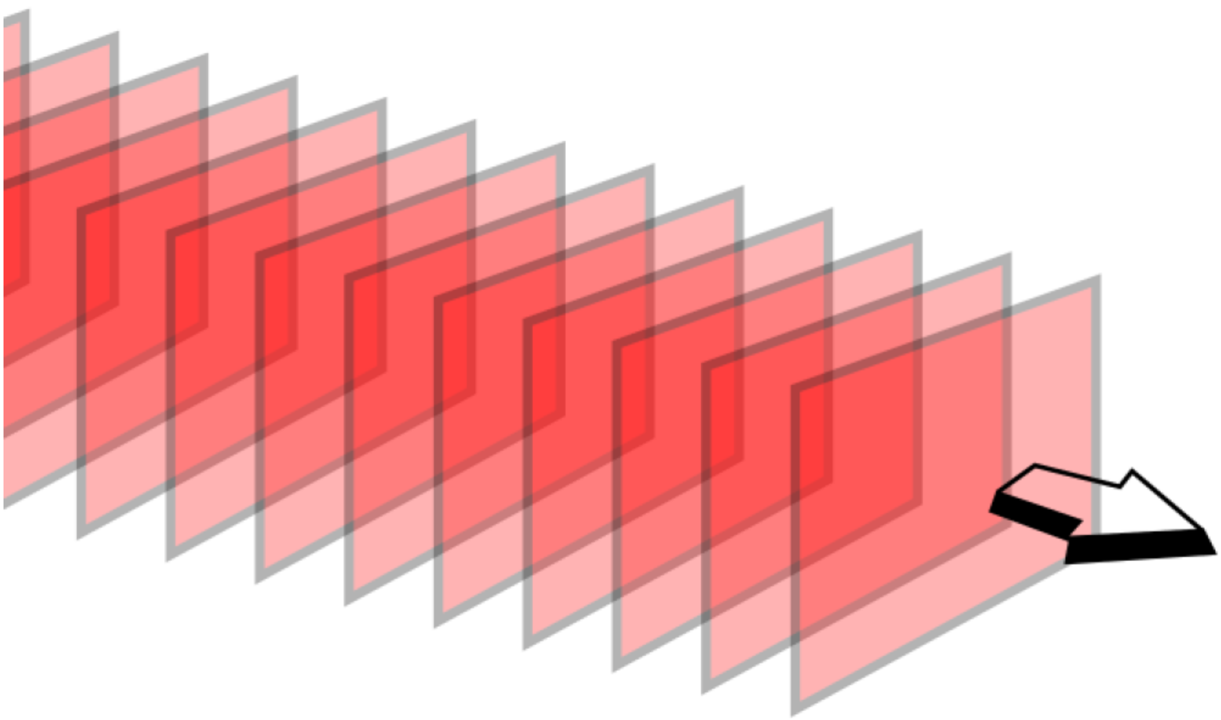
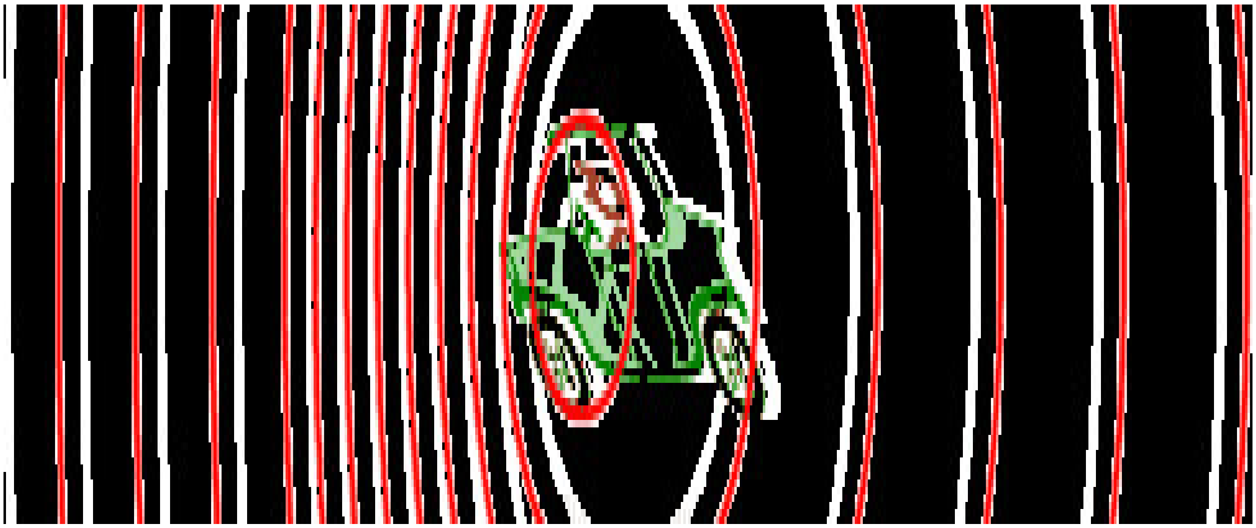


Image from Wikipedia

The Classical Doppler Effect: Sound Waves

The "Doppler Effect" occurs when the source emitting a wave is moving relative to an observer, or when the observer is moving relative to the source. This is illustrated in the image below.



Consider this example using sound waves. In the direction of motion, ahead of the source, the wave fronts are pressed together more densely, shrinking the wavelength and increasing the frequency of the waves. Against the direction of motion, behind the source, the wave fronts are more widely spread apart, increasing the wavelength and decreasing the frequency. To human ears, frequency determines pitch; high-pitched sounds are high-frequency sounds, and vice versa.

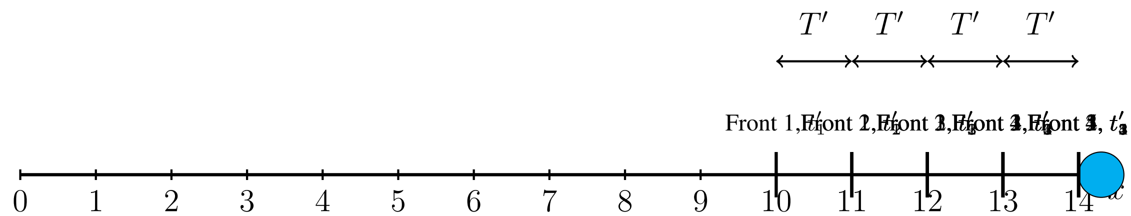
The Relativistic Doppler Effect on Light Waves

While all observers may agree on the speed of light, we know that special relativity leads to the conclusion that space and time measurements may differ. *Wavelength* is a space measurement and *frequency* is a time measurement. It should come as no surprise, then, that while observers in relative motion all agree that a wave of light travels at c, regardless of the frame of reference, observers in different frames will disagree on the wavelength and frequency of that light wave.

To measure *wavelength* is to be able to simultaneously locate maxima on a wave; we know that simultaneity is a frame-dependent statement, and in moving frames of reference lengths are contracted along the direction of motion; to measure *frequency* is to be able to measure the time displacement between two events at the same location in space, and in moving frames time passes more slowly.

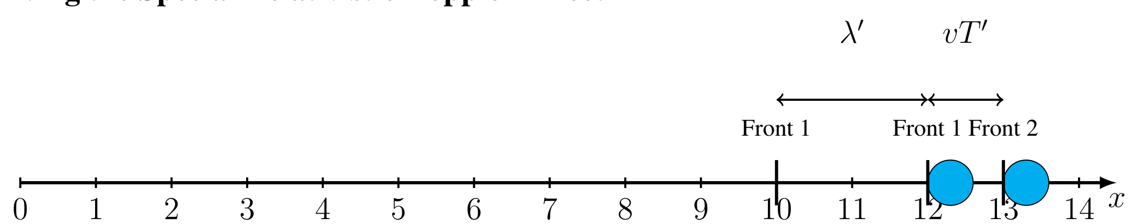
This is known as the "Special Relativistic Doppler Effect." We will derive it here using the Lorentz Transformation and discuss its implications for observing the universe.

Deriving the Special Relativistic Doppler Effect



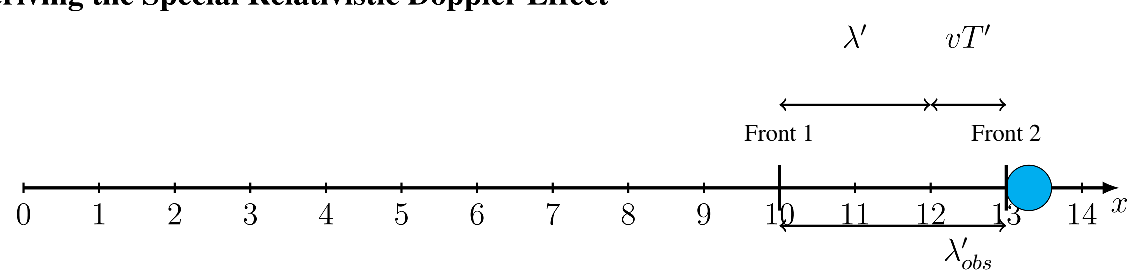
Imagine a light-emitting device that is sending out **wave fronts** at regular intervals in its rest frame; the time of the first wave front's emission is t'_1, the time of the second wave front's emission is t'_2, etc. In the source's frame of reference, the gap between the times is regular: $t'_2 - t'_1 = t'_3 - t'_2 = \ldots \equiv T'$ and defines the period of the light wave (and thus a regular *frequency* in the source frame, $f' = 1/T'$).

Deriving the Special Relativistic Doppler Effect



Now imagine that the light-emitting device that is sending out **wave fronts** at regular intervals in its own reference frame is *moving with respect to an observer* located at 0 in the above coorindate axis. Let the velocity of the source be $+v$ (moving away from the origin) and entirely along the x-axis. Treat the observer as being as rest (frame S) and the source as being in motion (frame S'). Let's think about what will be the distance between wave-fronts (arriving at the observer) from the perspective of the source. The source emits its first wave front, which moves at c, at time t'_1 in its frame. By the time it emits the second wave front at time t'_2, the first wave front is a distance $c(t'_2 - t'_1) = cT' = \lambda'$ from where it was emitted but the source is now farther from the observer by an amount $v(t'_2 - t'_1) = vT'$ when it emits the second wave front.

Deriving the Special Relativistic Doppler Effect



This is all illustrated above at time t'_2, and we can relate all the lengths:

$$\lambda' + vT' = \lambda'_{observed}$$

We can write this in terms of frequencies by remembering that $T = 1/f$ and $c = \lambda f$, so that $\lambda' = c/f'$ and $\lambda'_{obs} = c/f'_{obs}$. We find then that:

$$\frac{(1 + v/c)}{f'} = T'_{obs}$$

Deriving the Special Relativistic Doppler Effect

We are nearly there. Let's begin by defining a convenient symbol, $\beta = v/c$. Then:

$$T'_{obs} = \frac{(1+\beta)}{f'}$$

So far, all of this is in the S' frame - the original frequency of emission from the perspective of the source ($f' = f_{source}$), the relative speed of the source and observer (v), and the period (frequency) with which the person in the source frame would expect the observer to receive the wave fronts, T'_{obs} (f'_{obs}). However, if we now transform into the actual frame of reference of the observer, we know that there is an additional effect we must consider: the relativity of time. Time in the source frame, where all emissions happen at the same location, is proper time; the time dilation relation is given by $T_{obs} = \gamma T'_{obs}$. We then arrive at:

$$\begin{aligned} T_{obs} &= \gamma T'_{obs} = \gamma \frac{(1+\beta)}{f'} = \frac{1+\beta}{\sqrt{1-\beta^2}} \frac{1}{f'} = \sqrt{\frac{1+\beta}{1-\beta}} \frac{1}{f'} \\ f_{obs} &= \frac{1}{T_{obs}} = \sqrt{\frac{1-\beta}{1+\beta}} f' = \sqrt{\frac{1-\beta}{1+\beta}} f_{source} \end{aligned}$$

The Special Relativistic Doppler Effect

$$f_{obs} = \sqrt{\frac{1-\beta}{1+\beta}} f_{source} \text{ (Source moving away from Observer)}$$

The special relativistic Doppler effect, in this case for a source moving away from (receding from) an observer, is a combination of two effects: the classical doppler effect of a moving source that adds extra space between wave fronts and the dilation of time due to relative motion of the source and observer.

For a source that is moving toward an observer (approaching), the sign of the velocity is all that needs to be changed ($\beta \rightarrow -\beta$):

$$f_{obs} = \sqrt{\frac{1+\beta}{1-\beta}} f_{source} \text{ (Source moving toward an Observer)}$$

You can practice this calculation by checking for yourself that this second equation, for an approaching source, is correct.

Some Expectations from the Special Relativistic Doppler Shift

Let's look at some of the consequences we expect as a result of this Doppler shift. For example, if a light source is moving away from us or toward us, what do we expect to happen to the frequency of its light? For a source that is moving away from us at speed β along our line-of-sight, we expect to scale the source frequency by a quantity

$$\sqrt{\frac{1-\beta}{1+\beta}} \leq 1$$

The frequency we observe should always be *lower* than in the source's frame of reference, owing to the "stretching" of its wave fronts combined with the dilation of time. Because $\beta = [0, 1]$, we are taking the ratio of a number less than 1.0 and a number greater than 1.0.

If, instead, the source and observer are moving toward each other, then we scale the source frequency by a quantity:

$$\sqrt{\frac{1+\beta}{1-\beta}} \geq 1$$

This means the observed frequency is always greater than what is observed in the frame of the source, since we are dividing a number greater than (or equal to) 1.0 by a number less than (or equal to) 1.0.

The Special Relativistic Doppler Effect - Frequency and Wavelength

$$\begin{aligned} f_{obs} &= \sqrt{\frac{1-\beta}{1+\beta}} f_{source} \text{ (Source moving away from Observer)} \\ f_{obs} &= \sqrt{\frac{1+\beta}{1-\beta}} f_{source} \text{ (Source moving toward an Observer)} \end{aligned}$$

The above are for frequency. We can derive the equations for wavelength using $c = \lambda f$:

$$\begin{aligned} \lambda_{obs} &= \sqrt{\frac{1+\beta}{1-\beta}} \lambda_{source} \text{ (Source moving away from Observer)} \\ \lambda_{obs} &= \sqrt{\frac{1-\beta}{1+\beta}} \lambda_{source} \text{ (Source moving toward an Observer)} \end{aligned}$$

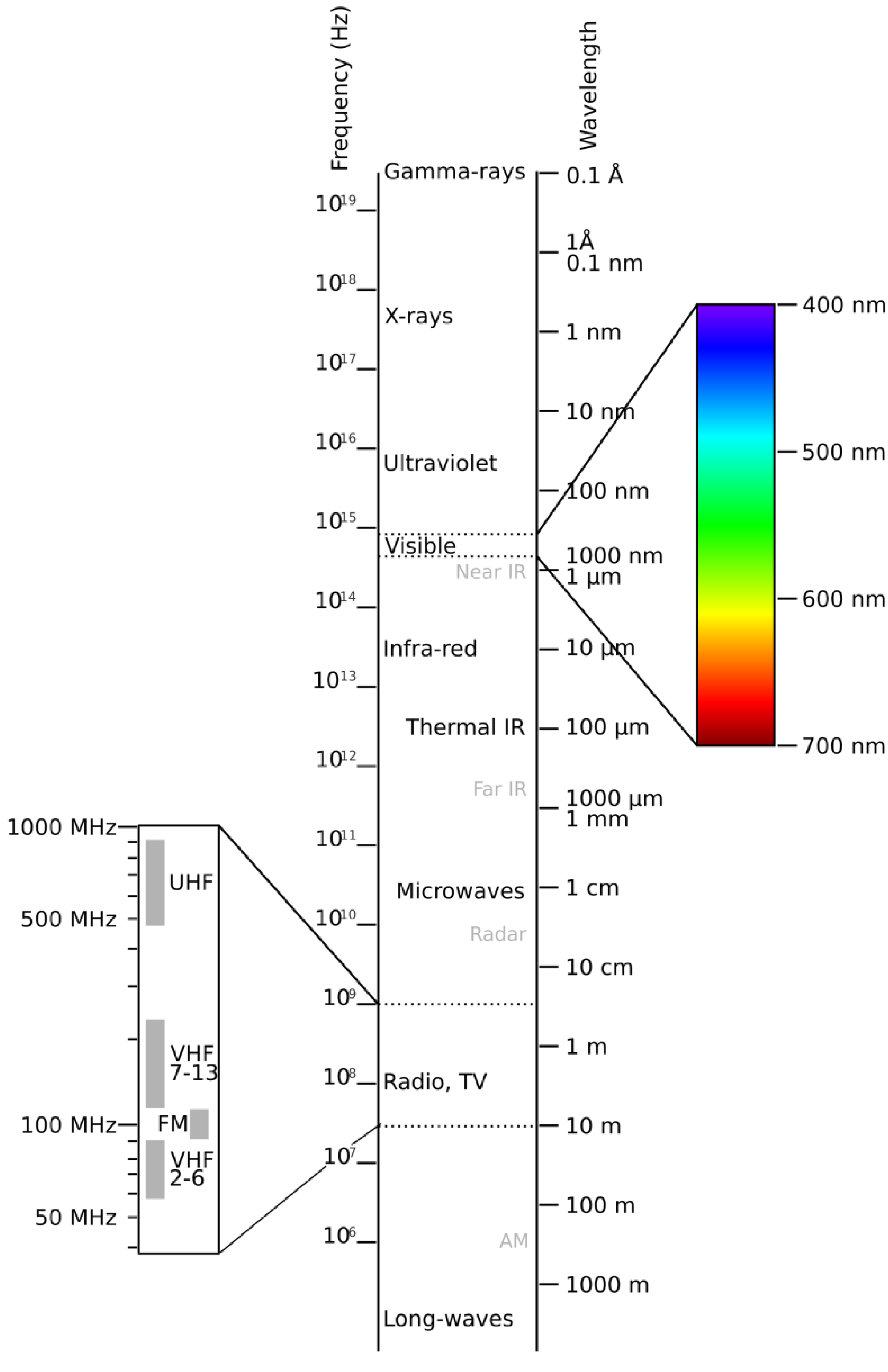
As expected, when a receding (approaching) source results in a lower (higher) frequency than in the frame of the source, this conversely results in a longer (shorter) wavelength.

Perceived Light Color Due to the Relativistic Doppler Shift

$$\begin{aligned} \lambda_{obs} &= \sqrt{\frac{1+\beta}{1-\beta}} \lambda_{source} \text{ (Source moving away from Observer)} \\ \lambda_{obs} &= \sqrt{\frac{1-\beta}{1+\beta}} \lambda_{source} \text{ (Source moving toward an Observer)} \end{aligned}$$

Receding sources of light are said to *Red Shift* compared to when they are at rest, since longer wavelengths are redder than shorter wavelengths of light; conversely, an approaching source is said to be *Blue Shifted* because shorter wavelengths are bluer than longer wavelengths.

This has implications for measuring our place in the cosmos. For instance, without making physical contact with distant stars and galaxies, it's possible to determine whether those objects are receding from Earth or approaching Earth based on the degree of the color shift of their atomic spectra.



Frequency (Hz)
Wavelength
Gamma-rays
X-rays
Ultraviolet
Visible
Near IR
Infra-red
Thermal IR
Far IR
Microwaves
Radar
Radio, TV
AM
Long-waves
10^{19}
10^{18}
10^{17}
10^{16}
10^{15}
10^{14}
10^{13}
10^{12}
10^{11}
10^{10}
10^{9}
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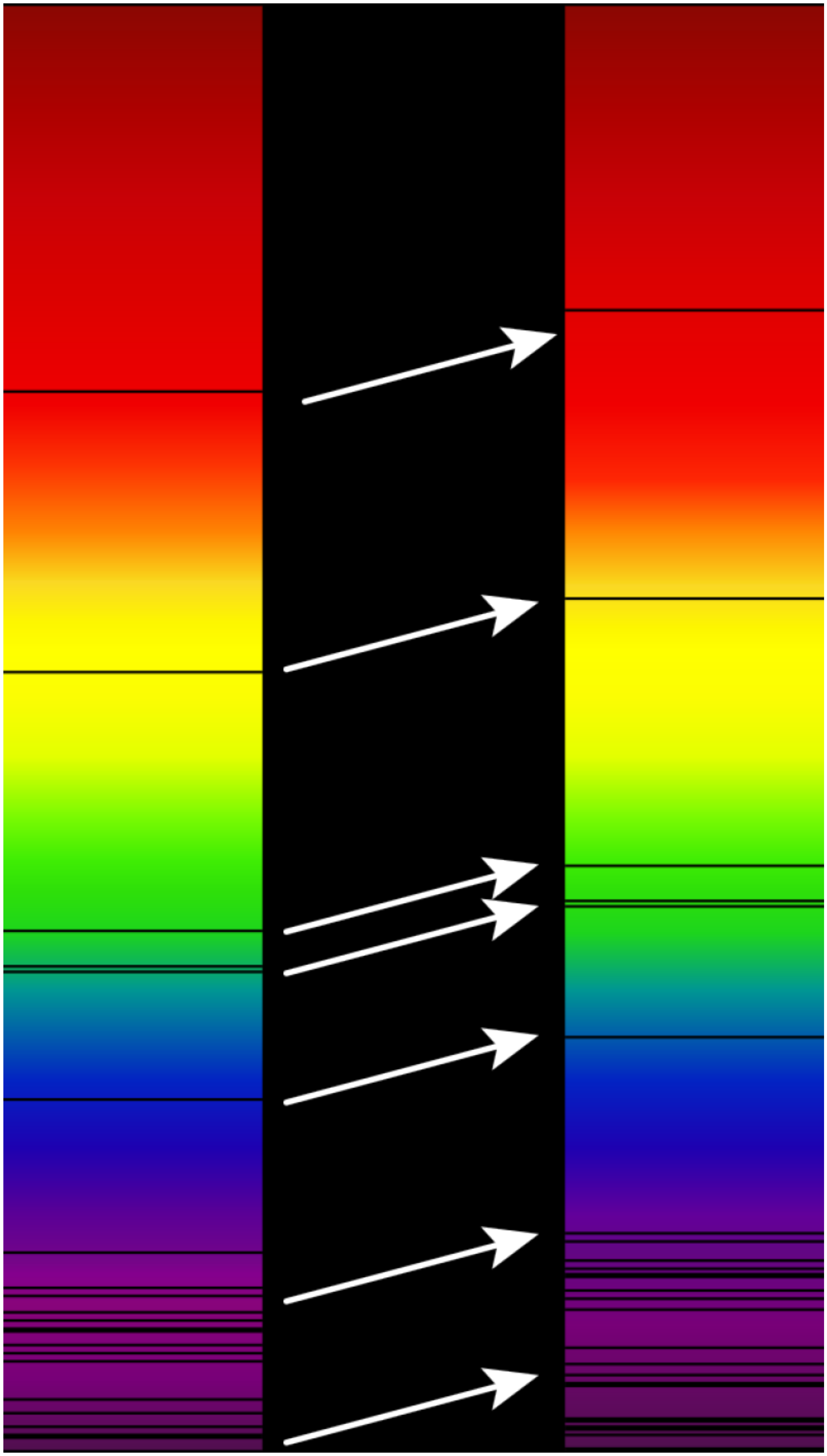
1000 nm
1 μm
10 μm
100 μm
1000 μm
1 mm
1 cm
10 cm
1 m
10 m
100 m
1000 m
400 nm
500 nm
600 nm
700 nm
1000 MHz
500 MHz
100 MHz
50 MHz
UHF
VHF
7-13
FM
VHF
2-6

The Red (or Blue) Shifting of Astrophysical Objects and Velocity Measurements

For example at the right is a pair of atomic spectra obtained for two objects: the Sun (left), which is not appreciably moving toward or away from us at any one time and the atomic spectrum of a distant supercluster of galaxies named BAS11 (right). Note the difference between the two? Both have the same pattern of missing colors (so-called "absorption lines"), but in the supercluster these are shifted toward the red.

The fact that those missing colors are ***red-shifted*** means the galaxy supercluster is moving away from us. You can use the difference between where the missing wavelengths are present in the Sun and where they are present in the galaxy supercluster to estimate the relative velocity ($\beta = v/c$) with which the supercluster is receding from us.

This kind of measurement is how we know the universe, as a whole, is expanding: all distant objects appear to be receding from Earth, implying the universe as a whole and on the largest distance scale is expanding with time.



Review

In this lecture, we have learned...

- What is the classical "Doppler Effect";
- The effect of motion of a light source on the characteristics of light other than speed;
- How to compute the special relativistic Doppler effect on light and interpret the effect on observations.


Christian Doppler (1803—1853)

i The Special Theory of Relativity - The Addition of Velocities

The Special Theory of Relativity - The Addition of Velocities

Overview

In this lecture, we will learn...

- How to think about object velocities in different frames of reference;
- How to properly add velocities of objects to frame velocities in special relativity;

In the Kelvin Timeline of Star Trek, the U.S.S. Vengeance fires on the U.S.S. Enterprise while both are engaged at high velocity.

Concrete Example: Space Wars!

Recently, in a globular cluster fairly nearby, two ships were engaged in a chase [bh] . The lead ship is moving away from the pursuing ship at a velocity $\vec{v}$. The pursuing ship fires a projectile straight at the lead ship, along the line of motion and at a velocity $\vec{u}$ relative to the firing ship. With what velocity does the lead ship observe the projectile to move?

In the Galilean/Newtonian view of space and time, the answer is rather straight-forward — $\vec{u}' = \vec{v} - \vec{u}$ — and also completely wrong when the velocities in the question approach that of light. For instance, if the projectile is a beam of light, we get all kinds of wrong answers here. So what is the *correct addition of velocities?*

Concrete Example: Space Wars!

We can begin by writing down the Lorentz Transformation equations, treating the pursing ship as the rest frame, the lead ship as the moving frame, and the projectile as an object to be located in either frame. The space-time coordinates in each frame are given by:

$$x' = \gamma(x - vt);\ t' = \gamma(-\frac{v}{c^2}x + t)$$

We can write differentials of space and time using calculus, to work toward obtaining equations with velocities ($u' = dx'/dt'$ and $u = dx/dt$) in them:

$$\begin{aligned} dx' = \gamma(dx - vdt);\ dt' &= \gamma(-\frac{v}{c^2}dx + dt) \\ \frac{dx'}{dt'} &\equiv u' = \frac{\gamma(dx - vdt)}{\gamma(-\frac{v}{c^2}dx + dt)} = \frac{dx - vdt}{-\frac{v}{c^2}dx + dt} = \frac{u - v}{1 - \frac{uv}{c^2}} \end{aligned}$$

Concrete Example: Space Wars!

[bh] Ship images are "Duhr's Nightfall" by FonsoSac and "RAK-R2_side" by lingonils, both licensed under CC BY-NC-SA 2.0.

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}}$$

We have arrived at a formula for combining the velocities of the moving frame with the projectile in the rest frame to compute the velocity in the moving frame. Let's try some numbers and see what we learn, Begin by picking a low-velocity situation where $v = 0.01c$ and $u = 0.03c$. From the above, we find $u' = 0.02c$... exactly what we would expect from the low-speed case which could have been answered by Newtonian/Galilean Relativity ($u' = u - v$). Let's instead pick $v = 0.5c$ and $u = 0.8c$. Plugging those numbers in, we find $u' = 0.5c$... definitely not what would have been predicted by the Newtonian/Galilean approach. What if the lead ship is flying *toward* the pursuer at $v = -0.5c$? In that case, $u' = 0.93c$... *not in excess of the speed of light, as the Newtonian/Galilean approach would have yielded!*

Adding Velocities in Special Relativity

You have the object velocity in the rest frame and want to determine it in the moving frame:

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}}$$

You have the object velocity in the moving frame and want to determine it in the rest frame:

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$$

These can always be derived directly from the Lorentz Transformation, or you can memorize them.

What if the object has velocity components u_y and/or u_z?

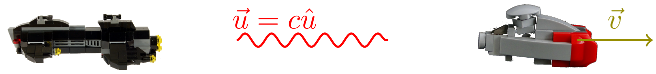
What if the object is moving in a direction that isn't just parallel, or antiparallel, to $\vec{v}$, the relative frame velocity (e.g. along x and x')? You might be tempted to assume that $u_y = u'_y$ and $u_z = u'_z$, since in the Lorentz Transformation the coordinates are transformed as $y = y'$ and $z = z'$ when all the frame motion is along x and x'. You'd be wrong! Why? Because object velocity necessarily involves the *time derivative* of a coordinate... and time does not transform as $t = t'$ between frames!

For example, consider the transformation of u'_y to u_y:

$$\begin{aligned} u_y &= \frac{dy}{dt} = \frac{dy'}{dt} = \frac{dy'}{\gamma\left(\frac{v}{c^2}dx' + dt'\right)} \\ u_y &= \frac{u'_y}{\gamma\left(1 + \frac{u'_x v}{c^2}\right)} \end{aligned}$$

and similarly if we have u'_y and want u_y, $u'_y = \frac{u_y}{1 - \frac{u_x v}{c^2}}$. You can obtain a similar equation for u_z and u'_z.

One Last Special Case: Shooting a Laser Beam



$$u' = \frac{u - v}{1 - \frac{uv}{c^2}}$$

What if the pursuing ship had fired a beam of light instead? We know that the second postulate of special relativity demands that all observers see light moving at c regardless of their state of motion. Does this velocity addition capture that postulate? Let's use $v = 0.5c$ and $u = c$. We find that

$$u' = \frac{c - v}{1 - \frac{cv}{c^2}} = \frac{c - v}{1 - \frac{v}{c}} = \frac{c - v}{\frac{1}{c}(c - v)} = c$$

In fact, we see that the value of v doesn't matter one bit. You could have $v = 0.5c$ or $v = -0.5c$... once $u = c$, v drops entirely out of the equation and we always recover $u' = c$!

Review

In this lecture, we have learned...

- How to think about object velocities in different frames of reference;
- How to properly add velocities of objects to frame velocities in special relativity;

In the Kelvin Timeline of Star Trek, the U.S.S. Vengeance fires on the U.S.S. Enterprise while both are engaged at high velocity.

i.1 Problem Solving in the Doppler Effect and Addition of Velocities

Problem Solving in the Doppler Effect and Addition of Velocities

Instructor Problem: The Cassini-Huygens Mission

The Cassini satellite recently ended its mission to Saturn with a fiery dive into the atmosphere of Saturn. In the early part of its mission, Cassini dropped a probe, called Huygens, onto the moon of Saturn called Titan. The original mission plan called for the Huygens probe to reach a velocity, relative to Cassini, that made Cassini approach Huygens at 5.6 km/s at the time it sent data back to Cassini. Huygens used a radio transmitter to relay that data to Cassini. At rest relative to Cassini, the radio transmitter emits 2.040 GHz electromagnetic waves.

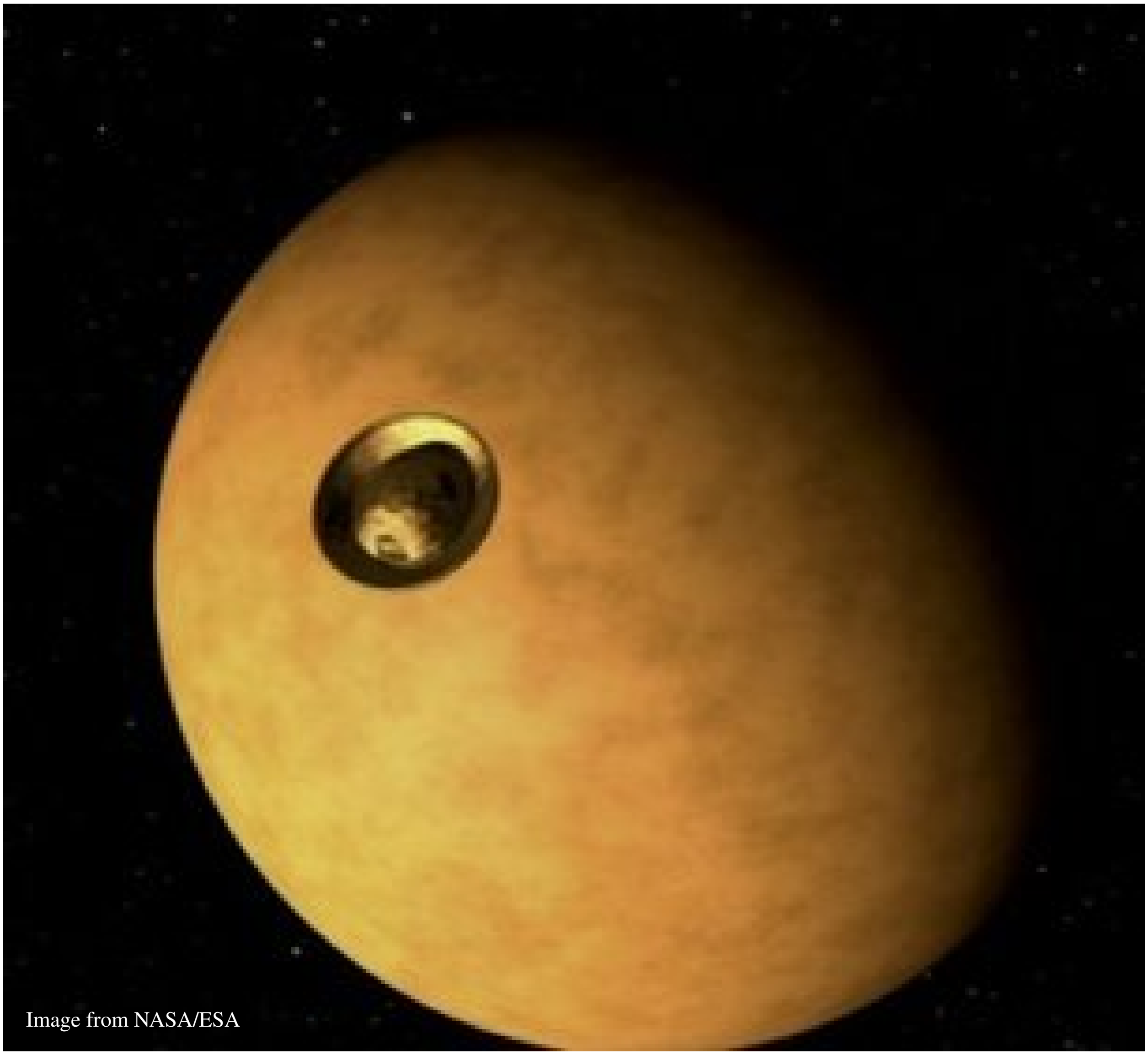


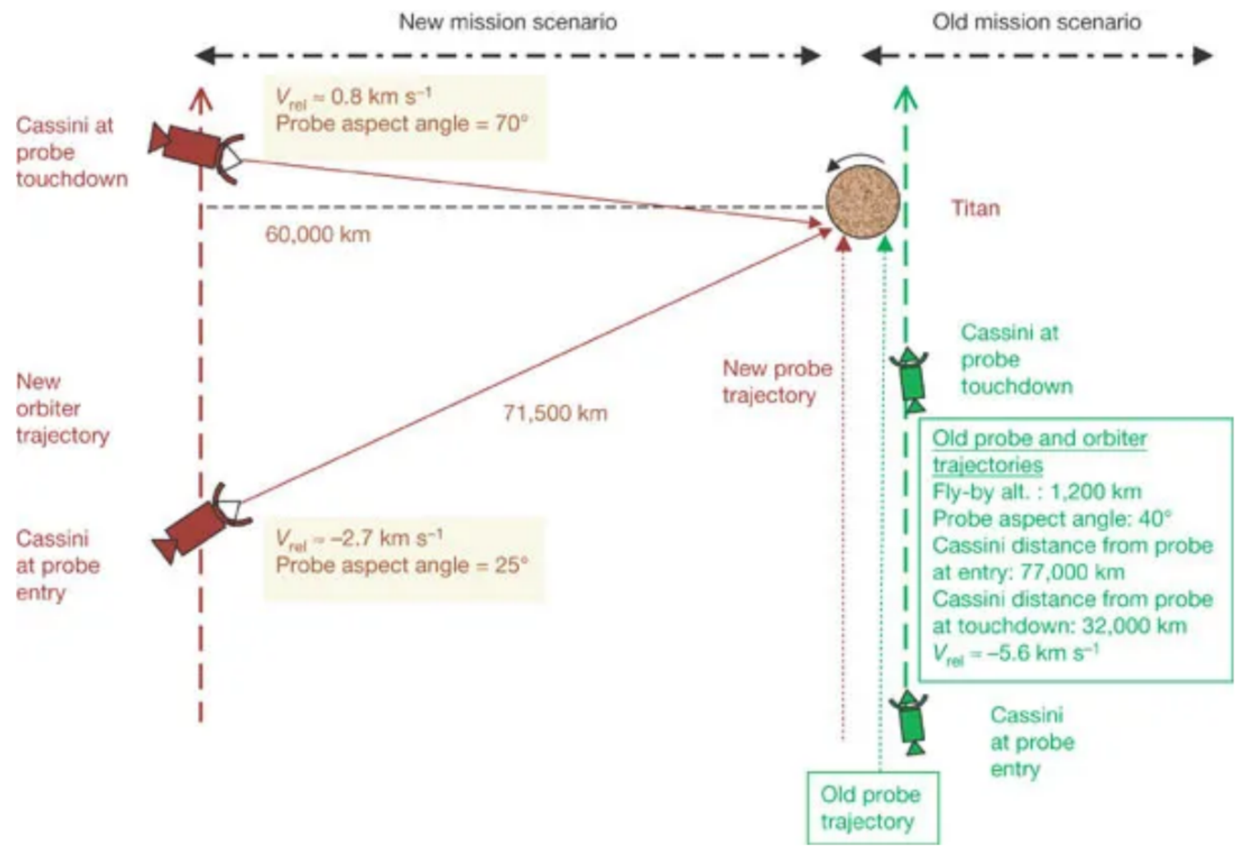
Image from NASA/ESA


Image from NAS

- What is the Doppler Shift of the radio waves due to the relative motion?
- The receiver on Cassini could tolerate, at most, a 15 parts-per-million deviation from the nominal frequency; would the mission, as planned, fail with these design parameters?

Cassini-Huygens: A Change of Plan

This problem with Cassini-Huygens really happened. The flaw in the receiver and the original mission plan was discovered after Cassini was launched and it was too late to fix it using hardware. Instead, flight engineers largely solved the problem by changing Cassini's flight trajectory during the Huygens descent and data transfer. Instead of a straight fly-by of Titan, Cassini's trajectory was curved around Titan to minimize motion along the line-of-sight during the communication phase.



Student Problem: A Moving Star

Specific atomic spectrum lines of Hydrogen are often used to determine the speed, along the line of sight, with which stars are moving relative to observers on Earth. A particular Hydrogen spectral line has a wavelength of 656.3nm (this is known as the $H - \alpha$ line of the Balmer Series of spectral lines, for reference). Compute the observed wavelength, and describe the "color" of the observed light, for the following scenarios.

1. The star is receding from us at 0.25c
2. The star is approaching us at 0.25c
3. The star is receding from us at 0.75c
4. The star is approaching us at 0.75c

The star approaching us at 0.75c explodes. A piece of the star's outer atmosphere is blown off at a speed, relative to the rest frame of the original unexploded star, at a speed of 0.90c. What is the speed of the projectile, viewed from Earth, if...

5. The velocity of the projectile is aimed at Earth and entirely along the line-of-motion of the star?

6. The velocity of the projectile is aimed at a $45°$ degree angle above the line-of-motion of the star?

j The Special Theory of Relativity - Energy and Momentum

The Special Theory of Relativity - Energy and Momentum

Overview

In this lecture, we will learn…

- How to define kinetic energy and momentum while incorporating special relativity;
- About the nature of mass, and the concept of *intrinsic mass*;
- About the relationship between energy, momentum, and mass.

Leo Szilard (1898—1964) Lecture on the process of Nuclear Fission. Photo from Argonne National Laboratory, CC BY-NC-SA

Newton's Second Law and Classical Momentum

In introductory physics, you are introduced to the concept of momentum roughly as follows:

- Historically, it was observed that there appeared to be a conserved, directional quantity associated with motion;
- This quantity, *momentum*, is well-defined in the classical domain of physics (low velocities and large scales) by the product of object mass (m) and object velocity ($\vec{u}$):

$$\vec{p} = m\vec{u}$$

- In a closed and isolated system, it is observed that this quantity is conserved:

$$\sum_i \vec{p}_i = \vec{P}_{total} = \text{constant}$$

- When a system is not closed and isolated - that is, subject to some net external force, $\vec{F}$, then Newton's Second Law is observed to be obeyed by the system:

$$\vec{F} = \frac{d\vec{P}_{total}}{dt}$$

From Classical Physics to Modern Physics

We can also see that if one subjects a classical momentum to consideration in another frame of reference, moving at constant relative velocity $\vec{v}$ to the closed and isolated frame where $\vec{u}$ is defined for an object, the conservation of momentum will hold:

$$\begin{aligned} \vec{p} &= m\vec{u} \xrightarrow[\text{moving frame}]{\text{viewed in}} \vec{p}' = m\vec{u}' = m\vec{u} - m\vec{v} \\ \frac{d\vec{p}'}{dt} &= m\frac{d\vec{u}}{dt} - m\frac{d\vec{v}}{dt} = \frac{d\vec{p}}{dt} \end{aligned}$$

This leaves the form of Newton's Second Law invariant under Galilean Transformations... at least, in classical physics domains like speeds low compared to that of light.

From Classical Physics to Modern Physics

However, the original definition of momentum was predicated on experiments and observations all in the low-velocity, large-scale regime of investigation (e.g. human-scale speeds and human-scale sizes). We know now that the Lorentz Transformation is the correct way to define relationships between frames, but if we apply that we see a problem:

$$\begin{aligned} \vec{p} &= m\vec{u} \xrightarrow[\text{moving frame}]{\text{viewed in}} \vec{p}' = m\vec{u}' = m\frac{u - v}{1 - \frac{uv}{c^2}} \\ d\vec{p}' &= \gamma^{-2}\left(1 - \frac{uv}{c^2}\right)^{-2} d\vec{p} \\ \frac{dp'}{dt'} &= \gamma^{-3}\left(1 - \frac{uv}{c^2}\right)^{-3} \frac{d\vec{p}}{dt} \end{aligned}$$

Why is this bad? It's bad because it violates the First Postulate of Special Relativity: the forms of the laws of physics must be invariant across all inertial frames of reference. Here we see that in one frame, $\vec{F} = d\vec{p}/dt$ but in the other frame, the law is velocity-dependent.

Relativistic Momentum

There are many, many alternative approaches to finding the correct definition of momentum (c.f. [9, 10]) but I prefer the one from my colleague Darin Acosta [11]. Let's assume the problem in the original definition was that of the time used in the time derivative of space. dt is not invariant from frame to frame. However, there is a time unit that all observers, regardless of their states of relative motion, can agree on: *proper time*, τ. If two events occur and are observed by all reference frames, *proper time* is the time defined in a frame where the two events happen at the same spatial location. It's always possible to find such a frame, and the time in any other frame t moving at velocity v with respect to the proper time frame is given by $t = \gamma\tau$.

Consider an object moving at velocity $\vec{u}$ with respect to the proper time frame. Let's trade the old time derivative in the definition of momentum, $\vec{p} = m(d\vec{r}/dt)$, for the derivative with respect to proper time:

$$\vec{p} \equiv m\frac{d\vec{r}}{d\tau} = m\frac{d\vec{r}}{\gamma_u^{-1}dt} = \gamma_u m\vec{u}$$

where $\gamma_u \equiv \left(1 - u^2/c^2\right)^{-1/2}$. This redefinition of momentum can be demonstrated, with a lot of algebraic pain, to leave Newton's Second Law invariant!

Relating Classical Momentum and the Correct Definition of Momentum

If Special Relativity is the more correct general framework for describing space and time, then in the appropriate limit (e.g. low velocity of the object) it should recover the classical definition of momentum. Let's give this a try! Begin by writing the gamma factor for the moving object as a binomial expansion, which we used in an earlier lecture:

$$\gamma_u = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} = 1 + \frac{1}{2}\frac{u^2}{c^2} + \mathcal{O}(u^4/c^4)$$

Now write relativistic momentum using this series expansion of the gamma factor:

$$\vec{p} = \left(1 + \frac{1}{2}\frac{u^2}{c^2} + \mathcal{O}(u^4)\right)m\vec{u} = \left(m\vec{u} + \frac{1}{2}m\vec{u}(u^2/c^2) + \mathcal{O}(\vec{u}\,u^4/c^4)\right)$$

As $u \ll c$ (essentially, as $u \to 0$), any terms that depend on u^3/c^2 (or higher) in the series expansion will vanish much faster than the leading term, leaving:

$$\vec{p} = \left(m\vec{u} + \frac{1}{2}m\vec{u}(\vec{u^2}/c^2) + \mathcal{O}(\vec{u}\,u^4/c^4)\right) \xrightarrow{u \ll c} m\vec{u}$$

We have recovered the classical definition of momentum.

Kinetic Energy in Special Relativity

We can proceed similarly to think about kinetic energy in special relativity. When an external force acts on an object over a displacement $\vec{s}$, this represents a unit of energy called *work*. Work done by an external force changes the kinetic energy of the object, and we know the relationship classically as the *Work-Kinetic Energy Theorem*: $\Delta K = W = \vec{F} \cdot \vec{s}$ (for a constant magnitude force).

Let's assume a constant force acts on an object from the perspective of an observer in frame S, given in special relativity by $\vec{F} = \frac{d(\gamma_u m\vec{u})}{dt}$ (using the correct definition of momentum now!). The force acts over a small displacement, $d\vec{s}$, which is related at any moment to the velocity of the object and the time over which the displacement occurs via $\vec{u} = \frac{d\vec{s}}{dt}$. We can write the differential of work represented by the applied force over the differential of displacement using the definition of work:

$$dW = \vec{F} \cdot d\vec{s} = \frac{d(\gamma_u m\vec{u})}{dt} \cdot d\vec{s} = \frac{d(\gamma_u m\vec{u})}{dt} \cdot \vec{u}dt$$

Assume the change in momentum is in the same direction as the force. To find the total work done by the force, which is to be related to the total change in kinetic energy, we integrate both sides:

$$\Delta K = W = \int \frac{d(\gamma_u m\vec{u})}{dt} \cdot \vec{u}dt = m \int u\, d(\gamma_u u)$$

Kinetic Energy in Special Relativity

$$\Delta K = W = m \int u\, d(\gamma_u u)$$

Let's integrate by parts to get a final form for the integral, using the trick that $\int udv = uv - \int vdu$. Identify $u = u$, $v = \gamma u$, so that $uv = \gamma_u u^2$ and $v\, du = \gamma_u u\, du$. The integral becomes:

$$\Delta K = m\gamma_u u^2 \Big|_{u_i}^{u_f} - m \int_{u_i}^{u_f} \gamma_u u\, du = \left(m\gamma_u u^2 + mc^2\sqrt{1 - u^2/c^2} \right) \Big|_{u_i}^{u_f}$$

Let's assume the initial speed of the object is zero, so that $K_i = 0$, and the final speed is u, where the kinetic energy is K. We obtain:

$$K = m\gamma_u u^2 + mc^2\gamma_u^{-1} - mc^2 = (\gamma_u - 1)mc^2$$

You can show that last step on your own. We find that relativistic kinetic energy is defined by $K = (\gamma_u - 1)mc^2$. You can use the binomial expansion trick once again to show that, in the limit $u \ll c$, this expression reduces to $K = (1/2)mu^2$.

Total Energy of an Object in Special Relativity

The total energy of a body in any system is composed of at least two parts: a kinetic part (describing the energy associated with its motion) and a potential part, describing any energy that is stored internally in the system (and could be released by some means). The total energy, then, is the sum of these two pieces:

$$E = K + U$$

We see that kinetic energy in special relativity is the difference of two pieces: $K = (\gamma_u - 1)mc^2 = \gamma_u mc^2 - mc^2$. If we rearrange the above total energy equation, and then plug in this expression for kinetic energy, we arrive at an interesting conclusion:

$$\begin{aligned} K &= E - U \\ K &= \gamma_u mc^2 - mc^2 \\ E &= \gamma_u mc^2;\ U = mc^2 \end{aligned}$$

The *total energy of an object in special relativity* is given by $\gamma_u mc^2$. In the limit that the object is at rest, we see the total energy then becomes mc^2... not zero. We note that this same quantity, mc^2, has been identified in the above exercise as a kind of energy stored in the object somewhere.

Mass as Energy and "Intrinsic Mass"

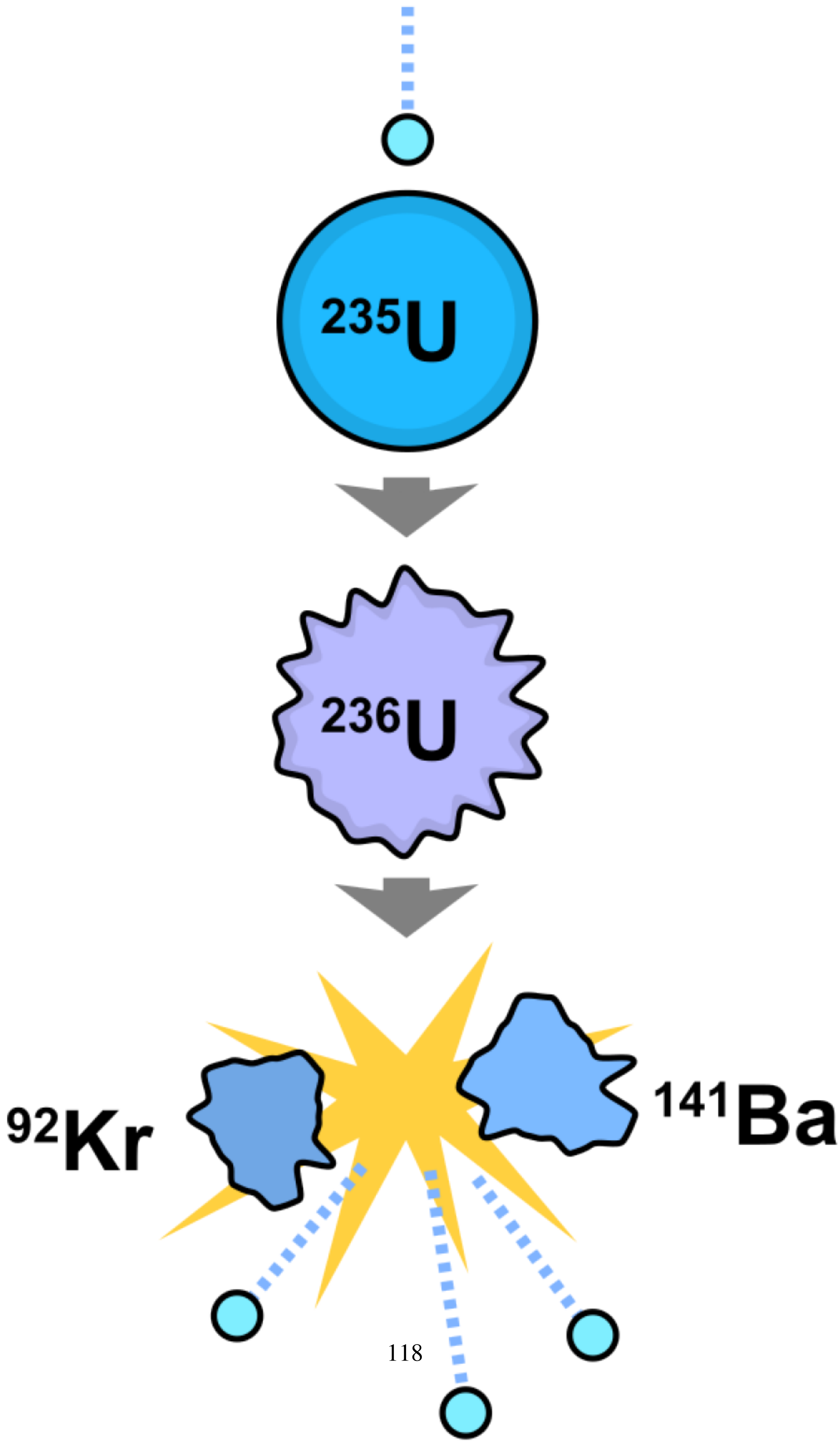
$$\begin{aligned} E &= \gamma_u mc^2 \\ K &= (\gamma_u - 1)mc^2 \end{aligned}$$

We arrive at one of the most profound conclusions drawn from special relativity: *mass is a form of stored energy, and even when a body is not moving its total energy is not zero, but rather decreases to a minimum given by $E = mc^2$.* The latter equation is one of the most famous in the history of science, and led to the development of nuclear weapons, nuclear power plants, the PET scan (a non-invasive medical invention), the particle collider, and many other technologies of the modern world. For an indivisible, fundamental particle (e.g. the electron is a good example), when it's at rest its energy is the result of its *intrinsic mass*, a fundamental property of matter (like electric charge).

How much energy is contained in mass?

A typical human being has a mass somewhere in the realm of $60\,\mathrm{kg}$. If, by some means, all of that could be converted to another form of energy (kinetic, chemical, radiation) the above equation tells us the energy, in Joules, that represents: $E = mc^2 = (60\,\mathrm{kg})\,(8.988 \times 10^{16}\mathrm{m}^2/\mathrm{s}^2) = 5.4 \times 10^{18}\mathrm{J}$. For comparison, the energy from the Sun that reaches Earth each second is about $10^{17}\mathrm{J}$.

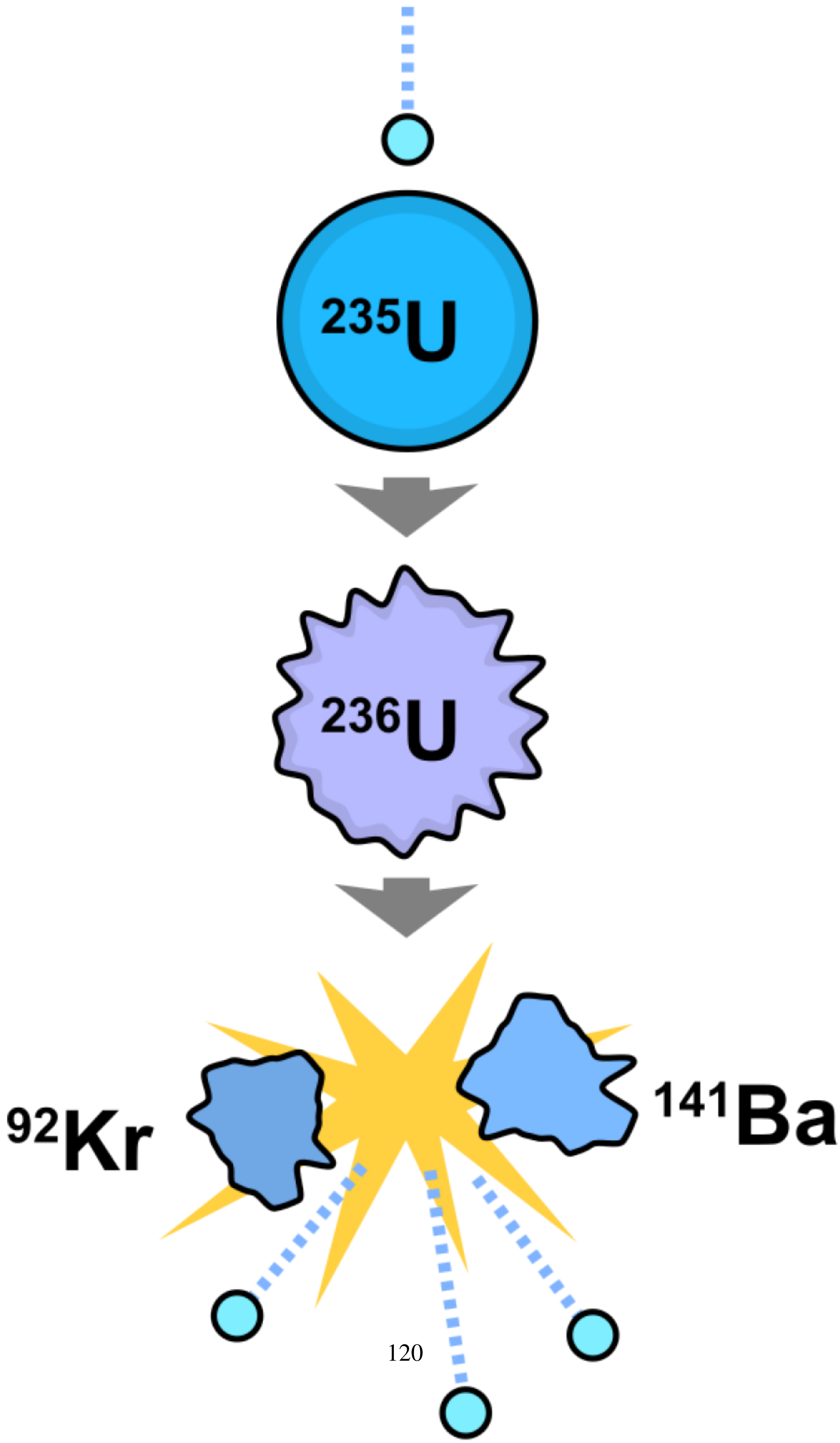
Example: The Mass Lost by a Uranium Nucleus during Fission



235U
236U
92Kr
141Ba

Once nuclear fission was discovered by Otto Hahn and Fritz Strassman in Germany in December, 1938, and understood shortly thereafter in January, 1939 by Lise Meitner and Otto Frisch, the physics community began to realize the vast power that lay in the hearts of unstable atoms. Consider the process shown left, whereby a neutron strikes a U-235 nucleus and splits it into nearly equal parts, a nucleus of Krypton (Kr-92) and Barium (Ba-141). The mass of the unsplit U-235 nucleus is $235.0439299\mathrm{u}$; the masses of the daughter nuclei, Kr-92 and Ba-141, are $83.798\mathrm{u}$ and $137.327\mathrm{u}$, respectively. $1\mathrm{u} = 1.6605402 \times 10^{-27}\,\mathrm{kg}$. If you check, the daughter masses do not add up to the parent mass; the mass lost in the fission process is $\Delta m = 13.919\mathrm{u}$.

Example: The Mass Lost by a Uranium Nucleus during Fission



235U
236U
92Kr
141Ba

If you check, the daughter masses do not add up to the parent mass; the mass lost in the fission process is $\Delta m = 13.919\mathrm{u}$. This represents a mass-energy of $E = 2.077 \times 10^{-9}$J. This doesn't sound like much, but since the fission process results in 3 neutrons also being produced, and those neutrons can hit other U-235 nuclei, a *chain reaction* can be initiated that exponentially multiplies the energy over each generation. The first split makes 3^1 neutrons, the second generation $3^2 = 9$, the third generation $3^3 = 27$, etc. A typical chain reaction in purified U-235 can go at least 40-50 generations before the device blows itself apart, a multiplication of $3^{45} \approx 3 \times 10^{21}$. If the energy of those neutrons is eventually converted to heat from collisions, you find that this level of multiplication is sufficient to explain the explosive yield of the first Uranium atomic weapon ("Little Boy"), which was equivalent to about 13-18 thousand tons of TNT, or 54-75 TeraJoules of energy. That weapon devastated the Japanese city of Hiroshima at the end of World War II.

Putting It All Together: Energy, Mass, and Momentum

In classical Newtonian/Galilean physics, there is a relationship between momentum and kinetic energy: $K = p^2/2m$. In the more correct description of space and time, the Special Theory of Relativity, we have kinetic energy, mass energy, and momentum. What is the correct relationship between these things?

Begin with the momentum equation:

$$p = \gamma_u m u = \gamma_u m \left(\frac{u}{c}\right) c$$

We know that the equation for total energy has c^2 and γ_u (which depends on u^2/c^2) in it. Try squaring the above:

$$p^2 = \gamma_u^2 m^2 c^2 \left(\frac{u^2}{c^2}\right) \xrightarrow{u^2/c^2 = 1 - \gamma_u^{-2}} m^2 c^2 (\gamma_u^2 - 1)$$

This has a piece that is awkwardly close the $E^2 = \gamma_u^2 m^2 c^4$ in it. . . multiply both sides by c^2:

$$p^2 c^2 = m^2 c^4 (\gamma_u^2 - 1) = \gamma_u^2 m^2 c^4 - m^2 c^4 = E^2 - m^2 c^4$$

Putting It All Together: Energy, Mass, and Momentum

We then arrive at the elegant relationship between and object's total energy, its momentum, and its mass-energy in special relativity:

$$E^2 = p^2 c^2 + m^2 c^4$$

Special Case: Massless Particles

We can use these relationships to look at a special case of objects: those devoid of mass-energy. Photons, the particles involved in light, have never been observed to have an intrinsic mass. In that case, $m = 0$ and

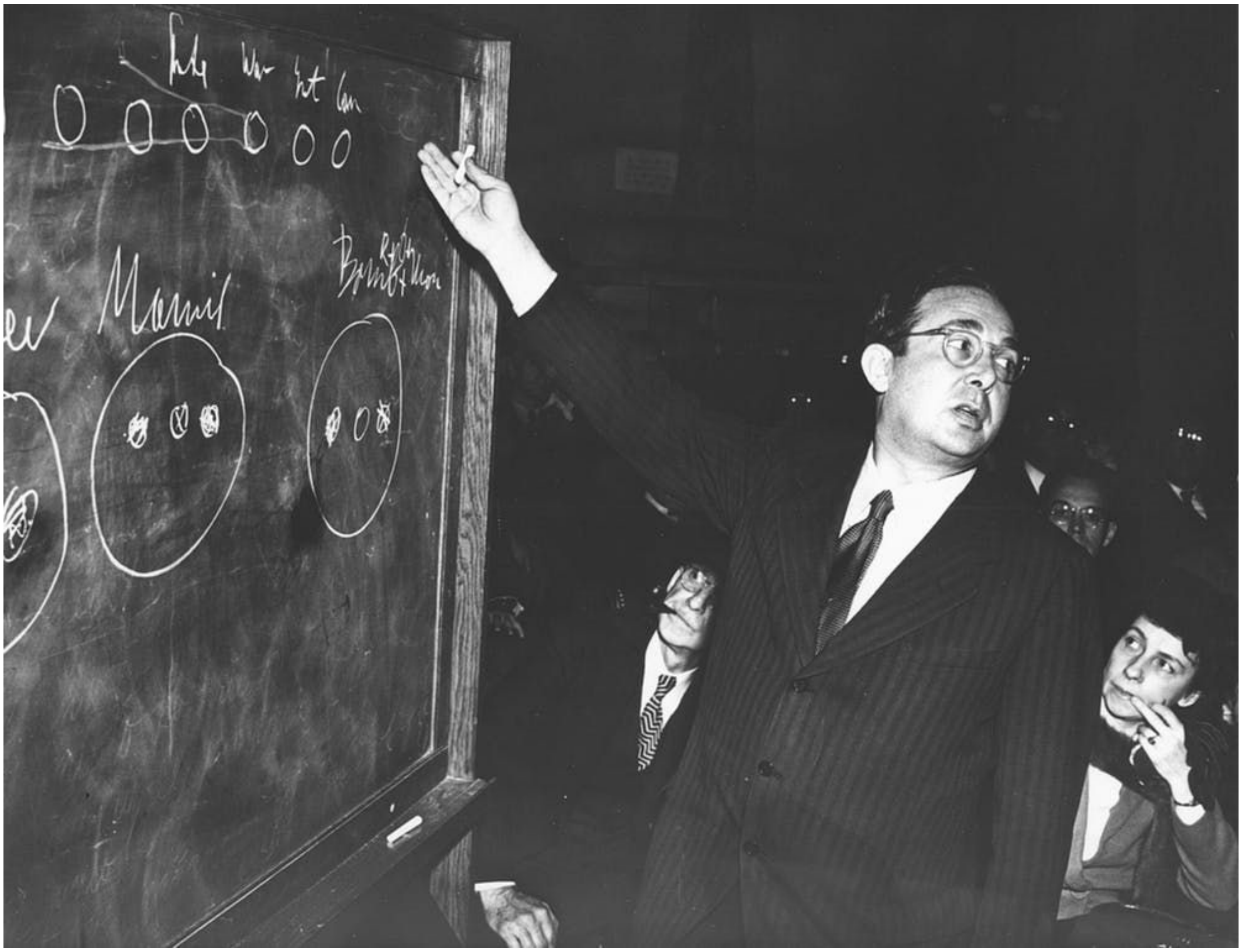
$$E^2 = p^2c^2 \rightarrow E = pc$$

Their total energy is entirely energy of motion; if such particles stopped moving, you might interpret it as meaning they cease to exist; this implies such particles can only be stopped when they are removed from the natural world. Note that $E = \gamma_u mc^2$ and $p = \gamma_u mu$ are indeterminate for such particles - they tell us nothing about how to determine either E or p! What determines energy and momentum, then, of a common particle like a photon? For that, we will have to wait and see.

Overview

In this lecture, we have learned...

- How to define kinetic energy and momentum while incorporating special relativity;
- About the nature of mass, and the concept of *intrinsic mass*;
- About the relationship between energy, momentum, and mass.



Leo Szilard (1898—1964) Lecture on the process of Nuclear Fission. Photo from Argonne National Laboratory, CC BY-NC-SA

j.1 Problem Solving in Energy, Momentum, and Mass

Problem Solving in Energy, Momentum, and Mass

Meet the Large Hadron Collider

Image from Wikimedia Commons

Meet the Large Hadron Collider

Image courtesy of CERN

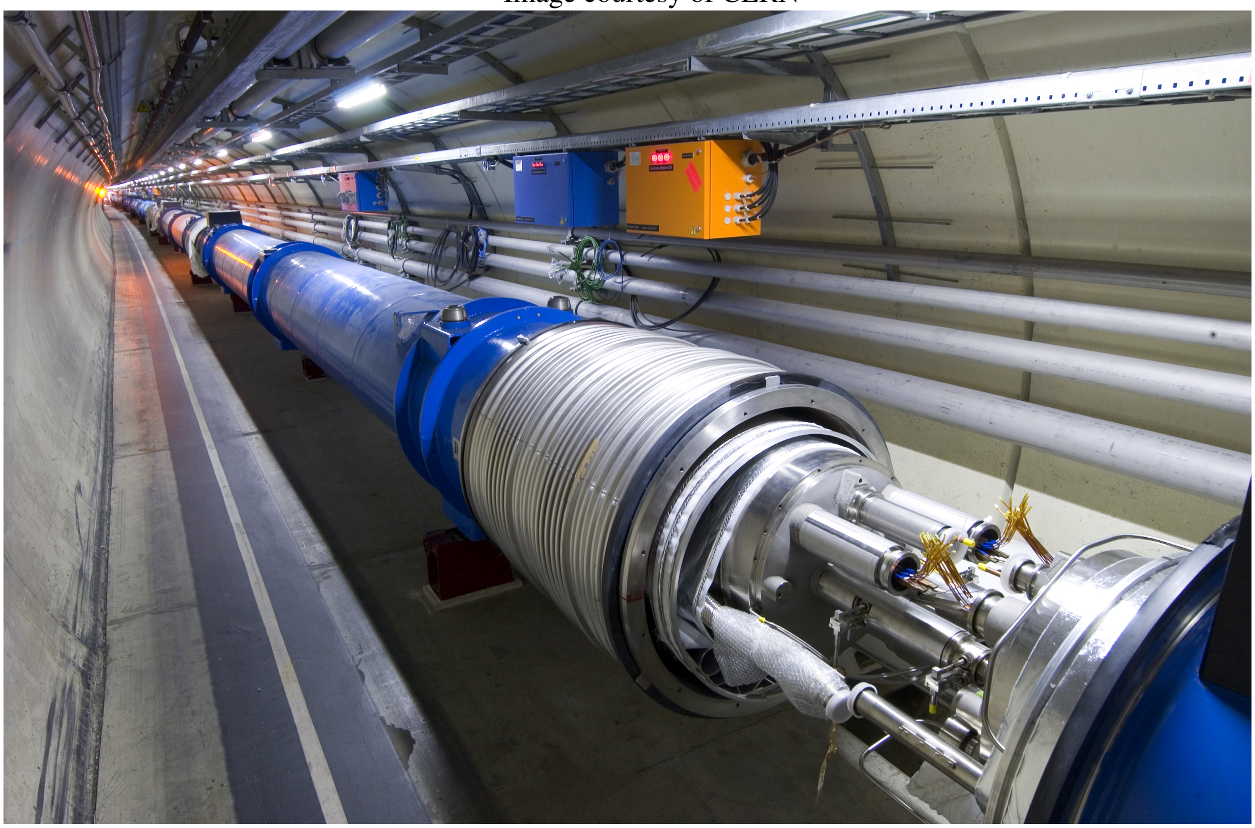


Image courtesy of CERN

Instructor Problem: The Large Hadron Collider

The Large Hadron Collider, or LHC, is a one-of-a-kind particle accelerator. Located in Geneva, Switzerland at the CERN Laboratory and straddling the border between France and Switzerland, this 17-mile (27-kilometer) circular particle accelerator was conceived of in the early 1980s and required 30 years to plan, design, build, and begin operating.

It presently accelerates protons, each with a mass of 1.67×10^{-27} kg, to a total kinetic energy of 6.5 TeV.

- What is this kinetic energy in Joules?
- What is the gamma factor for a proton relative to the laboratory rest frame?
- What is the velocity of this proton, as a fraction of the speed of light?
- What is the momentum of a proton?
- What is the intrinsic mass of a proton in "particle physicist units" of $\mathrm{MeV}/\mathrm{c}^2$?

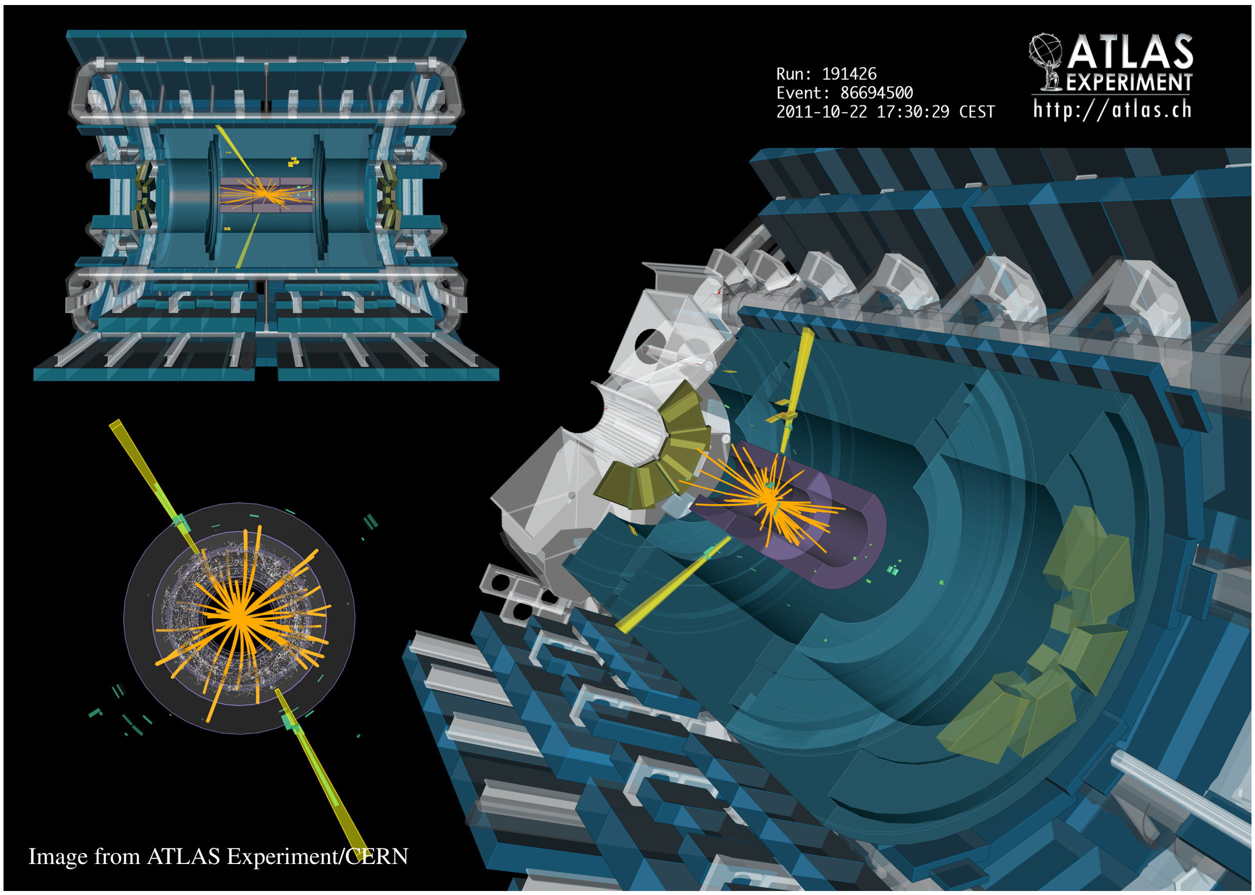


Image from ATLAS Experiment/CERN

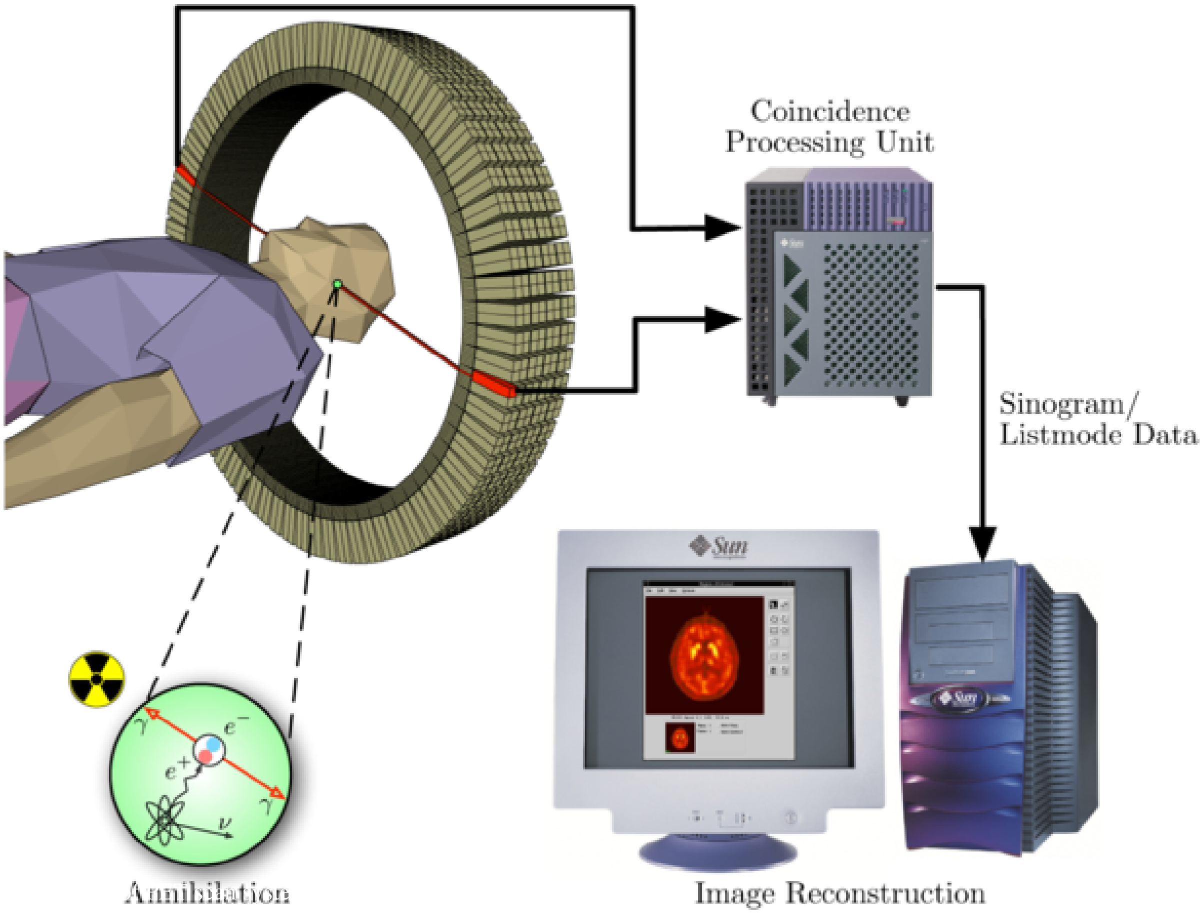
The Hitchhiker's Guide to Antimatter

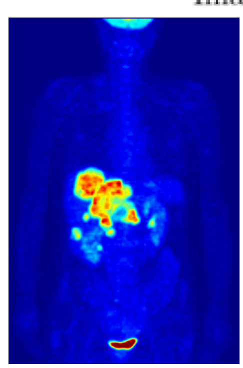
Antimatter was predicted in 1928 by brilliant theoretical physicist Paul Adrian Maurice Dirac, or P.A.M. Dirac. He was attempting to unify special relativity and quantum mechanics. He succeeded, but in doing so was forced by the math to predict the existence of particles in every way like the matter counterparts, but with opposite electric charge: antimatter. His theory predicted the existence of the antimatter electron, or *positron*, which was consequently discovered in 1932 by Carl Anderson, who in 1936 would go on to discover the muon (which was not predicted by a fundamental theory).

Paul Dirac
1902—1984
Photo from 1933

The Hitchhiker's Guide to Antimatter

Antimatter has many scientific and practical applications. One of these is the PET Scan, a fairly common medical imaging technique. A radioactive isotope (Fluorine-18) is attached as part of a glucose substitue (fluorodeoxyglucose) and is injected into a patient's bloodstream. The fluorodeoxyglucose is taken up like normal glucose in the body, and goes wherever glucose is needed. Cancers often need a lot of glucose.





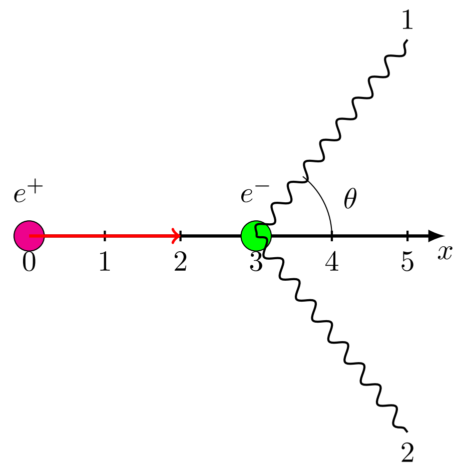
When the unstable isotope decays, the nuclear decay can produce a positron. The positron strikes an electron in a nearby atom, they annihilate into pure energy, and the result is a pair of light waves that have such short wavelengths they are gamma rays.

Student Problem: Matter Collides with Antimatter

The mass of the electron, denoted e^-, is 9.109×10^{-31} kg. The positron has an identical mass, but opposite electric charge (denoted e^+). Consider a positron is emitted along the x-axis with velocity $v_{e^+} =$

0.9512 c. The electron is at rest when the two collide.

- What is the gamma factor of the positron?
- What is the total energy of the positron and electron system before the collision? (this is the energy produced in their mutual annihilation). Get an answer in Joules and in MeV.
- Total energy, and momentum, must each be conserved in the annihilation, and so the two gamma rays will be constrained by these conservation laws. What is the momentum (including each separate component) of the gamma ray labelled "1"?
- What will be the angle, θ, indicated above, at which gamma ray 1 departs the system?



For discussion in class: *is this an example of an elastic, or an inelastic, collision?*

Toward the General Theory of Relativity

Overview

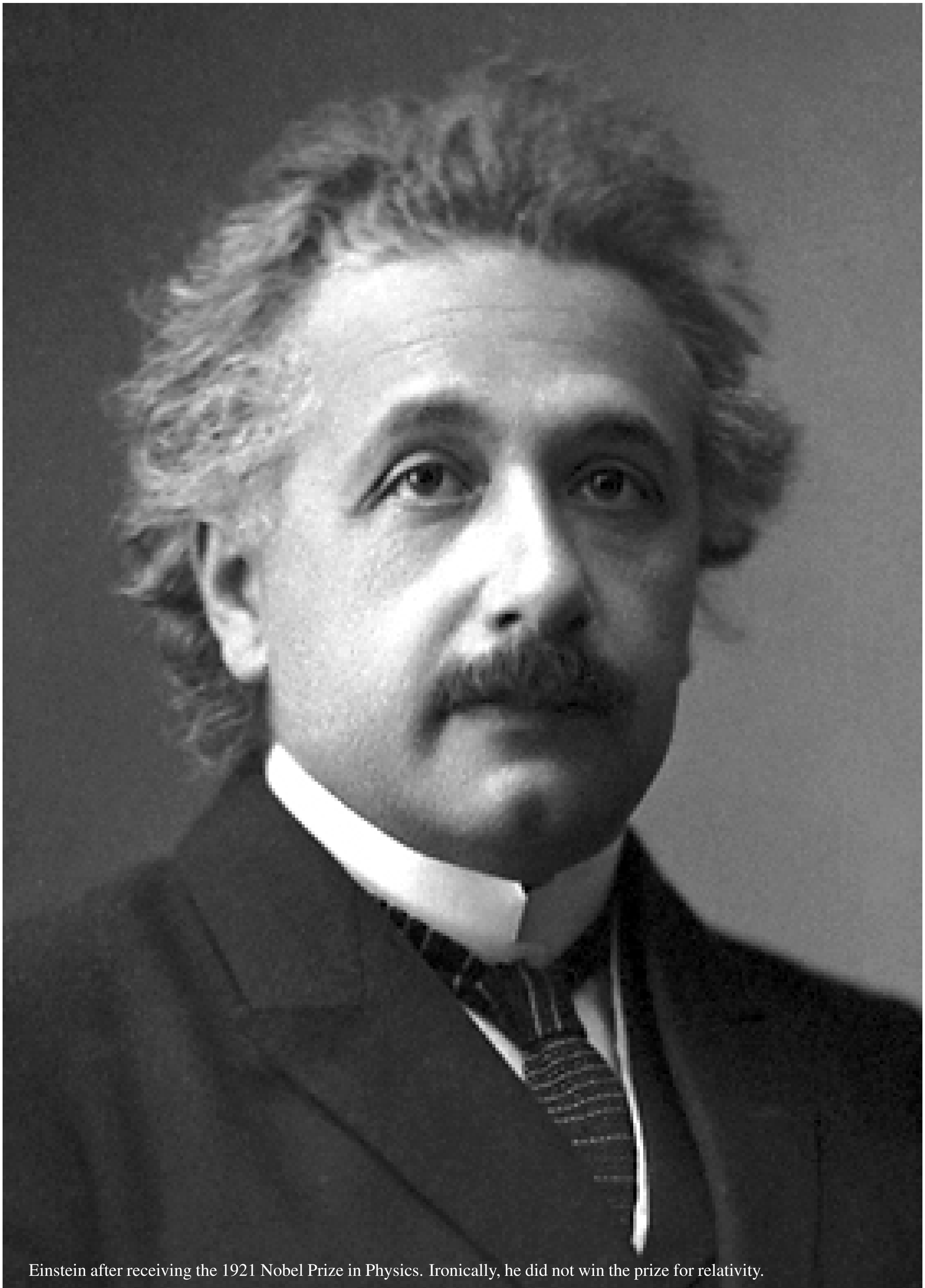
In this lecture, we will learn…

- The transition in thinking from the special to the general theory of relativity;
- Some implications of the general theory of relativity on physical phenomena;
- The large-scale implications for space and time.

Albert Einstein and Arthur Eddington ca. 1930 Photo by Winifred Eddington, from the Royal Greenwich Observatory

From Special to General Relativity

- Experiments on the speed of light in the first and second decades of the 1900s yielded no disconfirming evidence for the postulates of special relativity; Albert Einstein's physics work in 1904 and beyond earned him the faculty position he had sought after his PhD.
- He was sure that the special theory could be generalized to a complete theory of space and time, including an explanation of the nature of gravity — the prize that had eluded Isaac Newton
- It would take another decade, and require much of the advanced math that Einstein skipped out on in graduate school and had to re-learn to pursue his line of research. Ultimately, his labor was rewarded with the first complete General Theory of Relativity in 1915-1916[12, 13, 14, 15].
- We'll explore some of the basic ideas and tease the implications of the general theory in this lecture.



Einstein after receiving the 1921 Nobel Prize in Physics. Ironically, he did not win the prize for relativity.

A Tale of Two Masses

Mass appears in two distinct equations in introductory physics, two equations that don't necessarily have anything to do with one another:

$$\vec{F} \;= m\vec{a};\; \vec{F}_{gravity} = G\frac{m_1 m_2}{r^2}\hat{r}$$

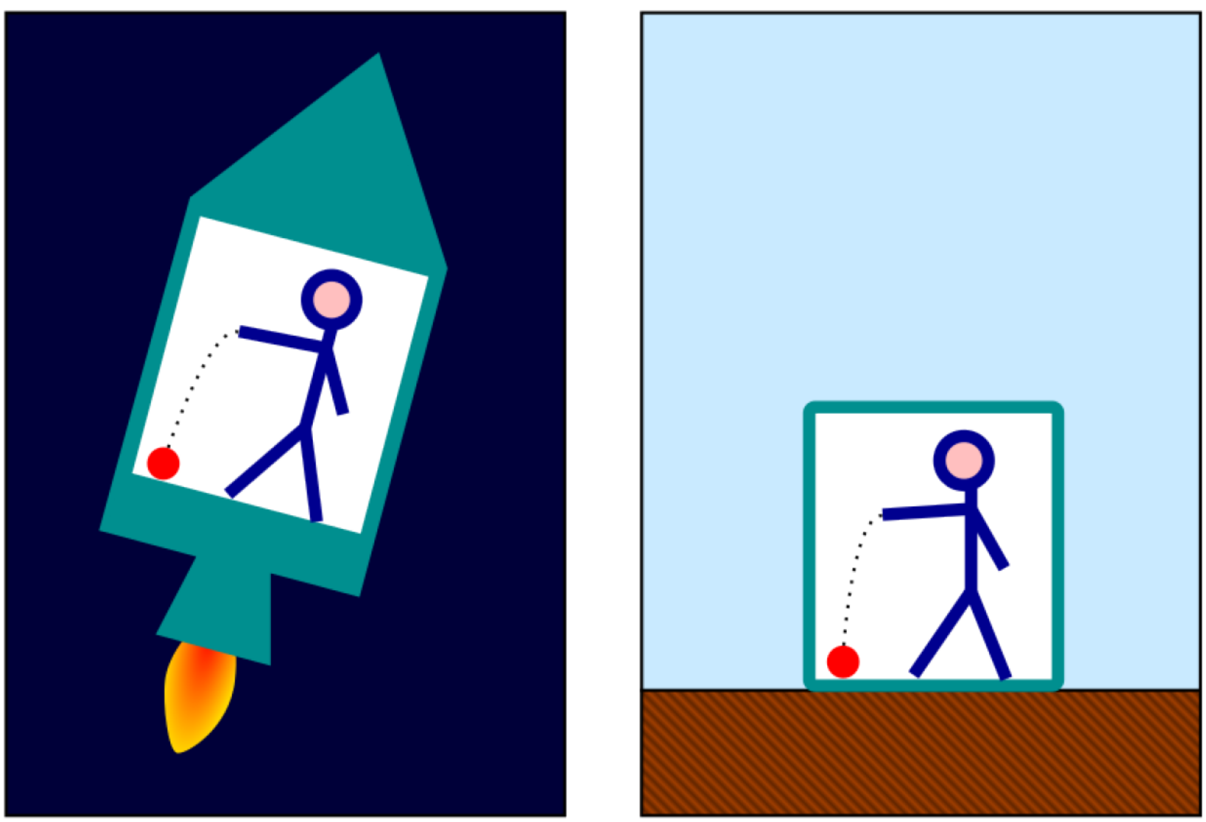
The m that appears in Newton's Second Law has to do with *inertia*, the tendency of a body to resist changes in the state of motion ("Inertial Mass"); the m that appears in Newton's Law of Gravitation has to do with the primary cause of the gravitational force ("Gravitational Mass"). There is nothing in these two laws that says these have to be the same mass; yet their equivalence has been tested to remarkable precision: inertial mass and gravitational mass appear to be one-in-the-same. A consequence of their equivalence is often taken for granted:

$$G\frac{M_{earth}m_{gravitational}}{R_{earth}^2} \;=\; m_{inertial}a_{gravity} \rightarrow a_{gravity} = \left(G\frac{M_{earth}}{R_{earth}^2}\right)\frac{m_{gravitational}}{m_{inertial}} \equiv g\frac{m_{gravitational}}{m_{inertial}}$$

The Postulate of General Relativity: The Equivalence Principle

The observational equivalence of gravitational and inertial mass leads to a larger consequence: *The Principle of Equivalence of Acceleration by a Constant Force or by a Constant Gravitational Field*. Einstein observed early on that, due to the equivalence of these two kinds of mass, there is no difference between being under the influence of a uniform and constant gravitational field or being placed in a *non-inertial reference frame* by the action of some other kind of force.

Einstein defined the concept of a *locally inertial frame* by imagining a system in free-fall in a gravitational field.

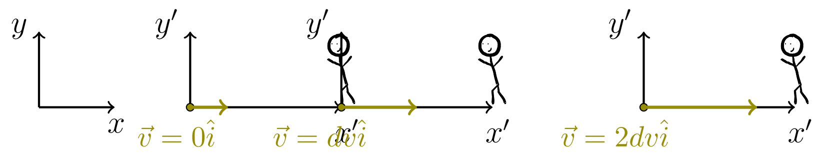


(Image from Wikipedia)

Since all parts of the system are accelerated at the same rate, they act as if no external forces are present on the system…as if they are in an inertial frame. Absent external information, no experiment can determine that this system is in free-fall in a gravitational field.

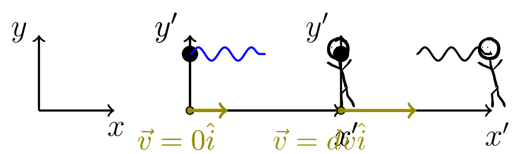
Gravity and Acceleration

Since there is no difference between gravitational acceleration and the act of changing a whole reference frame into a non-inertial reference frame, you can analyze phenomena in a situation where an inertial frame of reference is considered *instantaneously inertial*…although, it experiences overall accelerations, at any moment in time all elements of the frame have the same velocity.



Implications of General Relativity: Light in a Gravitational Field

For example, using this imagery of *instantaneously inertial frames of reference,* let's analyze an observation of light that was emitted during this period of slight accelerations of a frame of reference. Consider a light source that is fixed in the frame S'. Just as the frame begins to accelerate at $a = dv/dt$ (with an initial velocity of 0), the source emits a light wave at velocity $\vec{c} = c\hat{i}$. Once the light wave reaches the observer in Frame S', what will the observer see?



Light as Viewed in an Accelerating Frame of Reference

Let's take stock of the elements of this question:

1. The light was emitted from a source at rest with frequency f_{source} and wavelength λ_{source}.
2. The light source and observer remain a fixed distance apart because they accelerate together.
3. The light travels at c, no matter the frame of reference in which it is observed.
4. The light will take finite time to travel to the observer from the source.
5. By the time the light reaches the observer, they will have obtained a state of non-zero velocity $(0 + dv)$.

Therefore, the light wave will be *observed* in a frame of reference moving with respect to the frame of the source, which was a rest frame. This sounds a lot like a Doppler shift problem (light viewed in a frame moving with respect to the frame of emission). This will guide the math that we can do to calculate what the observer will see.

Light as Viewed in an Accelerating Frame of Reference

Let's do some basic calculations, assuming that we are in instantaneously inertial reference frames and that $v \ll c$ at any time, so that $\frac{v}{c} = \beta \ll 1$.

Let's assume the distance from the light source to the observer is some length, L. Because the light source travels in the same frame as the observer, L remains constant the whole time. The light will take a time $\Delta t = L/c$ to travel from the source to the observer. In that time, Δt, the frame of the observer will have accelerated by an amount a from rest up to velocity v. Thus, using equations of motion from introductory physics (still valid here), we find:

$$v = v_0 + a\Delta t \rightarrow v = 0 + a\left(\frac{L}{c}\right) \rightarrow v = \frac{aL}{c} \rightarrow \beta = \frac{aL}{c^2}$$

Light as Viewed in an Accelerating Frame of Reference

$$v = \frac{aL}{c} \rightarrow \beta = \frac{aL}{c^2}$$

Again, let's treat the case of small velocities relative to light. The Doppler Shift of the light wave by the time the observer sees it will be given by:

$$\begin{aligned} f_{observed} &= \sqrt{\frac{1-\beta}{1+\beta}} f_{source} \xrightarrow{\text{Binomial Expansions}} \left(1 - \frac{1}{2}\beta + \ldots\right)\left(1 - \frac{1}{2}\beta + \ldots\right) f_{source} \\ &\approx (1-\beta) f_{source} = \left(1 - \frac{aL}{c^2}\right) f_{source} \end{aligned}$$

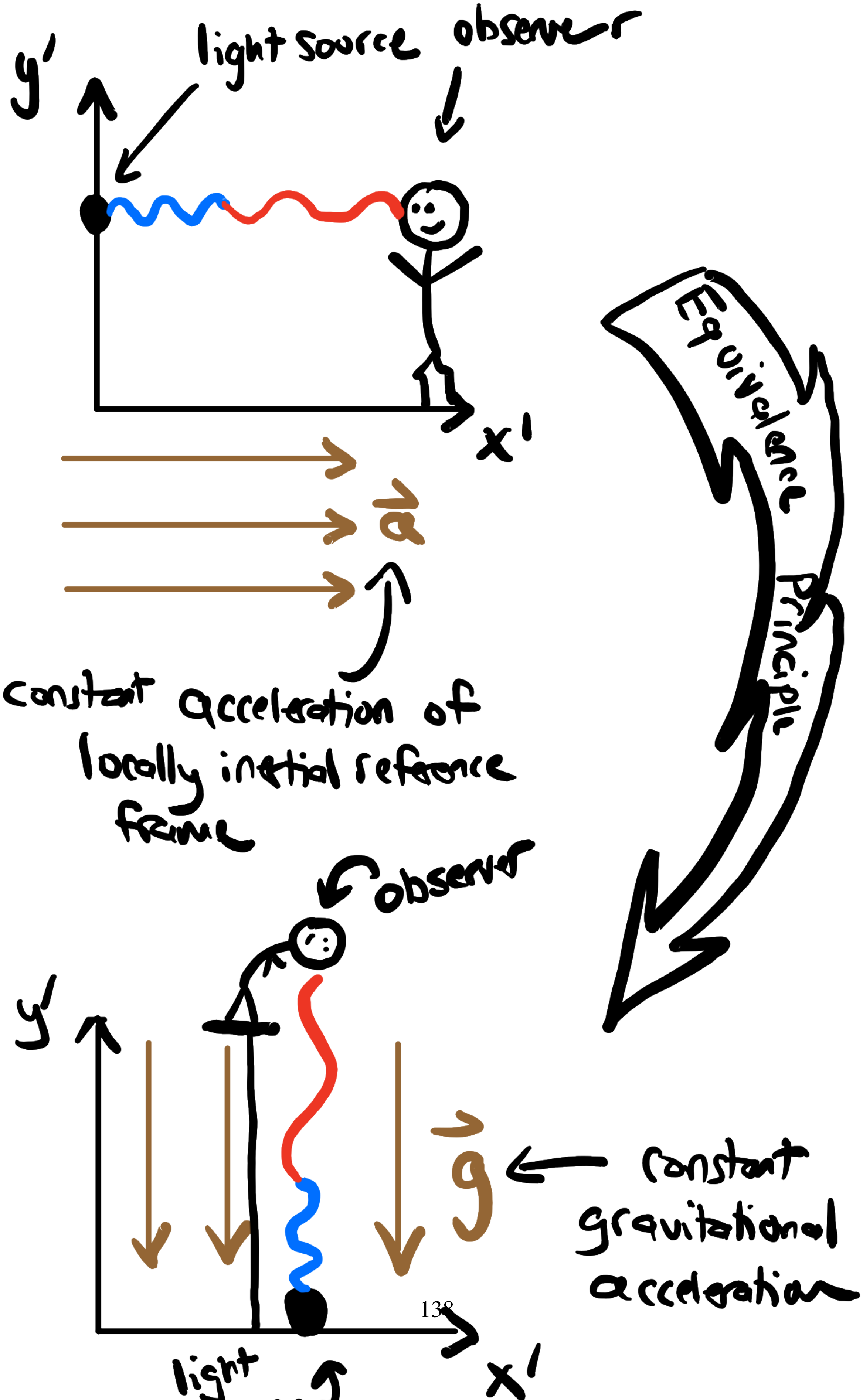
So we see that to the observer, who is now in a frame that is not that in which the light source was at rest when the emission occurred, the light will appear shifted from its source frequency. In this case, it's a *red-shift*. If the observer were in a frame accelerating in the opposite direction, $a \rightarrow -a$, the light would appear *blue-shifted*.

Light as Viewed in an Accelerating Frame of Reference

$$f_{observed} \approx \left(1 - \frac{aL}{c^2}\right) f_{source}$$

But according to the Equivalence Principle, the conclusion in this picture - that light is frequency-shifted when you accelerate a reference frame away from the rest frame of the light source - must apply to *a light source in a gravitational field*.

In other words, if the source of a is gravity, e.g. $a = GM_{earth}/R_{earth}^2 = g$, and instead we are viewing light from a source below us ("from upstream in the gravitational field") shining up toward us ("downstream in the field"), there must be a shift (in this case, a red-shift) in the frequency of the light. This phenomenon has been confirmed to be real and is known as *gravitational red-shift*.


y'
light source
observer
x'
a
constant acceleration of locally inertial reference frame
Equivalence Principle
observer
y'
g
constant gravitational acceleration
light
x'

Implications of General Relativity: Time in a Gravitational Field

By implication from the previous example - the Doppler-shifting of light by a gravitational field - one can also predict that time *itself* will pass at different rates in different locations in a uniform gravitational field. We saw that *frequency of light in a gravitational field* is altered depending on the degree of acceleration. Frequency is a measure of the rate at which events happen. Consider observing time at a height of 0 above the surface of the Earth (a "lower" observer) and at a height of h above that (a "higher" observer). We know that...

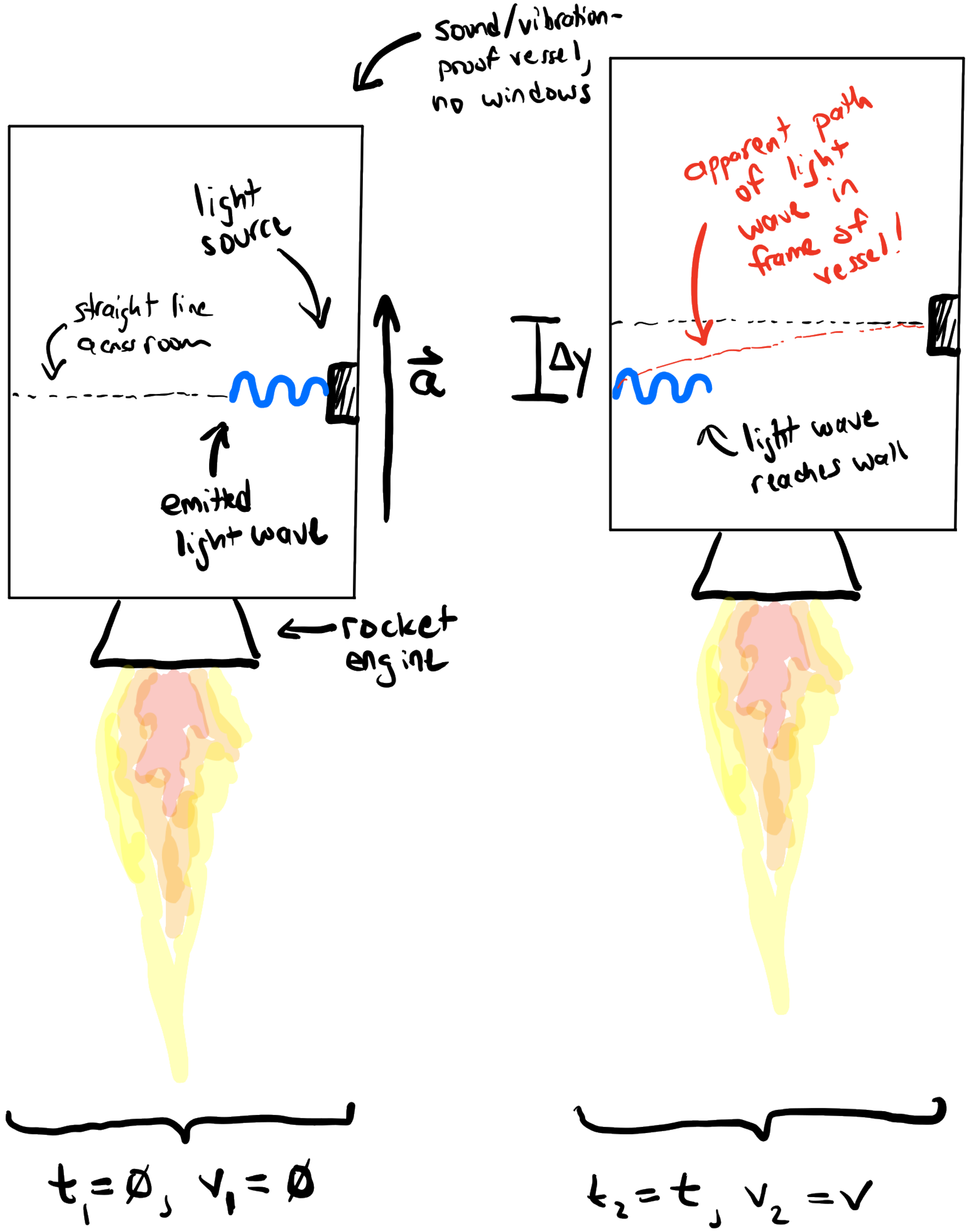
$$\begin{aligned} f &= 1/T \rightarrow T = t_2 - t_1 = \Delta t \rightarrow f = \frac{1}{\Delta t} \\ \Delta t_{higher} &= \frac{1}{f_{higher}} = \frac{\sqrt{1+\beta}}{\sqrt{1-\beta}} \frac{1}{f_{lower}} \approx (1+\beta)\,\Delta t_{lower} \approx \left(1 + \frac{gh}{c^2}\right) \Delta t_{lower} \end{aligned}$$

We would expect time to pass more slowly for observers who are lower down in a gravitational field! The time that passes higher up in the gravitational field is always multiplied by a number whose value is ≤ 1, meaning that less time passes lower in the field. This effect has also been confirmed, and plays a major role in the operation of the Global Positioning Satellite (GPS) system on which nearly all modern navigation relies.

Implications of General Relativity: Deflection of Light By Gravity

The deflection of light by a gravitational field was not a new idea by the time of Albert Einstein. In fact, it was realized soon after Isaac Newton established the laws of mechanics and gravitation that since all objects, regardless of their mass, fall at the same rate in a uniform gravitational field, light (though it appears to have no mass) should also fall in such a field. This idea was put on firmer footing, however, thanks to the Equivalence Principle. Consider the cartoon at the right...

The light wave is emitted in a frame that is instantaneously at rest, but by the time it reaches the far wall of the vessel the vessel itself has been accelerated at a and achieves a speed of v. The light wave strikes the wall at a lower height than it started, from the perspective of an observer in the vessel. With no way of hearing the rocket engine or looking for references outside, this is indistinguishable from being in a room on a planet whose gravitational field accelerates all things down at a rate a. Light *must fall in a gravitational field*. This is generalizable to any body with mass bending the path of light.



The Large-Scale Implications of the General Theory of Relativity

- Space and time are really part of a singular structure, Spacetime, providing a 4-dimensional framework in which matter and energy can move.

- General Relativity is really a theory of spacetime, and concludes that what we call the “force of gravity” is really due to the fact that mass and energy cause space and time to curve ("bend" or “warp”). Other bits of matter, or light, that travel in spacetime follow the curvature, and from our perspective in 3-dimensions appear to accelerate as a result.
- Space and time tell energy and matter how to move; energy and matter tell space and time how to bend. This summary of the General Theory is elegant, and comes from the brilliant theoretical physicist John Archibald Wheeler.
- The universe is expanding in all directions at once; spacetime is curved, even if space itself appears very flat and smooth on cosmic scales. The curvature of spacetime tells us about the origin and fate of the cosmos, which was born 13.78 billion years ago.

Review

In this lecture, we have learned…

- The transition in thinking from the special to the general theory of relativity;
- Some implications of the general theory of relativity on physical phenomena;
- The large-scale implications for space and time.

Albert Einstein and Arthur Eddington ca. 1930 Photo by Winifred Eddington, from the Royal Greenwich Observatory

k.1 Problem Solving in the First Steps in General Relativity

Problem Solving in the First Steps in General Relativity

Instructor Problem: Light is Falling

In General Relativity, light can be deflected by a gravitational field, which is just the curving of space-time. Newton's Mechanics also allowed for the possibility that light can fall in a gravitational field.

1. Perform a fully classical calculation of the deflection of light that starts out horizontally, near the surface of the earth, and is accelerated down while crossing a room of width 10m. Estimate the angle of deflection from horizontal.

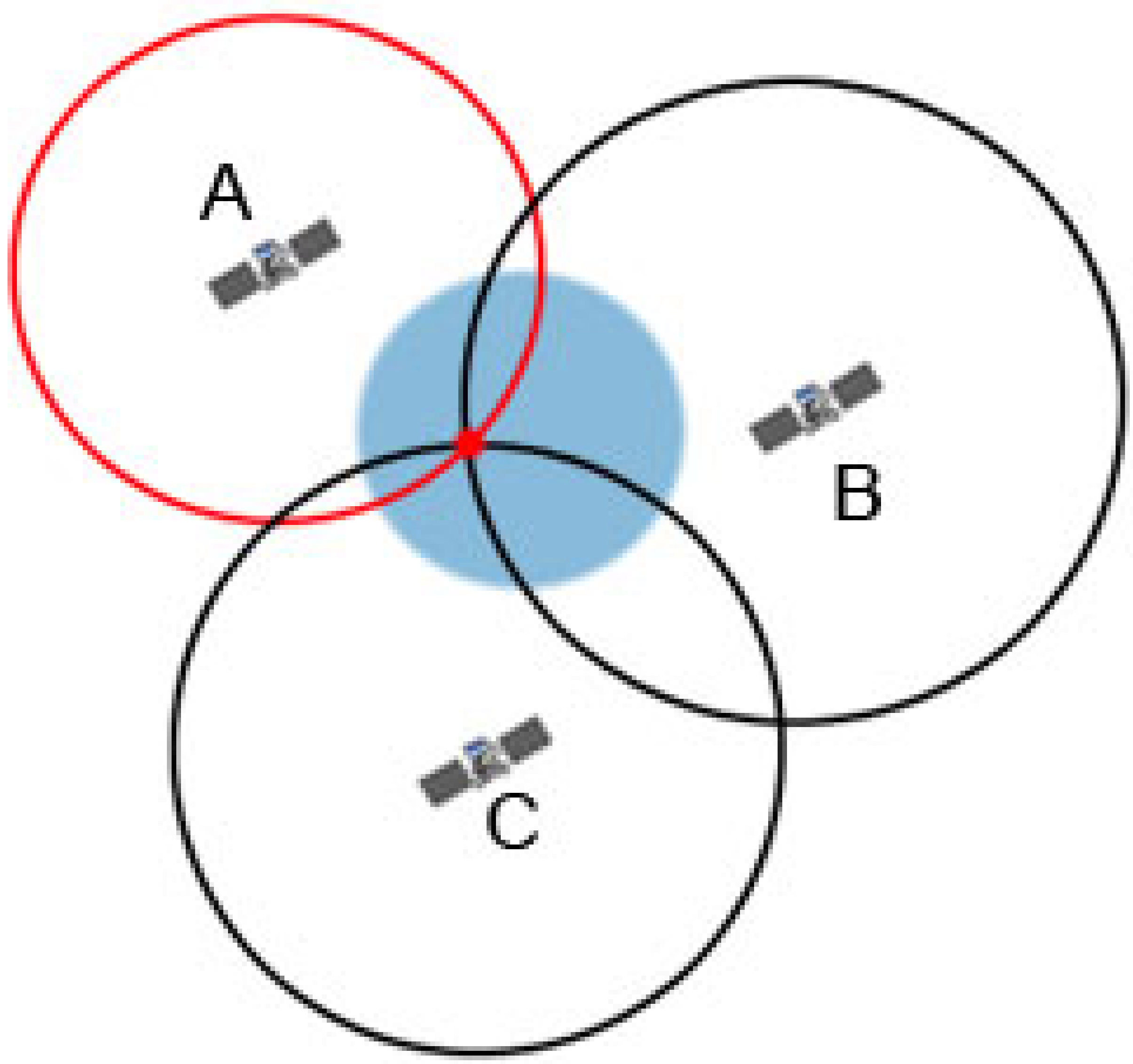
2. Using the special relativistic treatment of light in a constant gravitational field, calculate by how much a green laser pointer ($\lambda = 520\mathrm{nm}$) would be wavelength-shifted if aimed from the ceiling down toward the floor, 4m below.

Note that in full General Relativity, the deflection of light around massive bodies, like stars, is TWICE what would have ever been predicted classically. This is just to illustrate the challege of the problem, and why we would not have noticed the effect until we really wanted to understand space and time at the level required by General Relativity.

Student Problem: Relativity and the Global Positioning System

The Global Positioning System relies on the identical atomic clocks on satellites orbiting the Earth being synchronized to atomic clocks on the surface of the Earth. The signals sent by the satellites are time-stamped using their on-board atomic clocks, and the time at which the signal is received on Earth (known using the Earth-based clock) is used to calculate the time-of-flight of the light signal, and thus how far you are from the satellite.

However, satellites orbit the Earth twice per day, relative to a fixed point on Earth, at an altitude of 20,000km. They are moving very fast and are in a weaker gravitational field than their counterparts at rest on Earth.



Student Problem: Relativity and the Global Positioning System

GPS Satellites orbit the Earth twice per day, relative to a fixed point on Earth, at an altitude of 20,000km. They are moving very fast and are in a weaker gravitational field than their counterparts at rest on Earth.

1. Calculate the difference, due to motion, in time between two clocks, one on Earth and one on a GPS satellite, after exactly 1 day (86,400s) has passed on Earth. Pay close attention to the sign and magnitude of the difference.

2. Calculate the difference due to the fact that the satellite is in a weaker part of Earth's gravitational field. Note that since g has a very different value at the altitude of the satellite you should employ:

$$\Delta t_{Earth} = \Delta t_{GPS} \left(1 - \frac{1}{c^2} \left(\frac{GM_{Earth}}{r_{Earth}} - \frac{GM_{Earth}}{r_{GPS}}\right)\right)$$

3. Which effect is bigger: that due to motion, or that due to gravity?

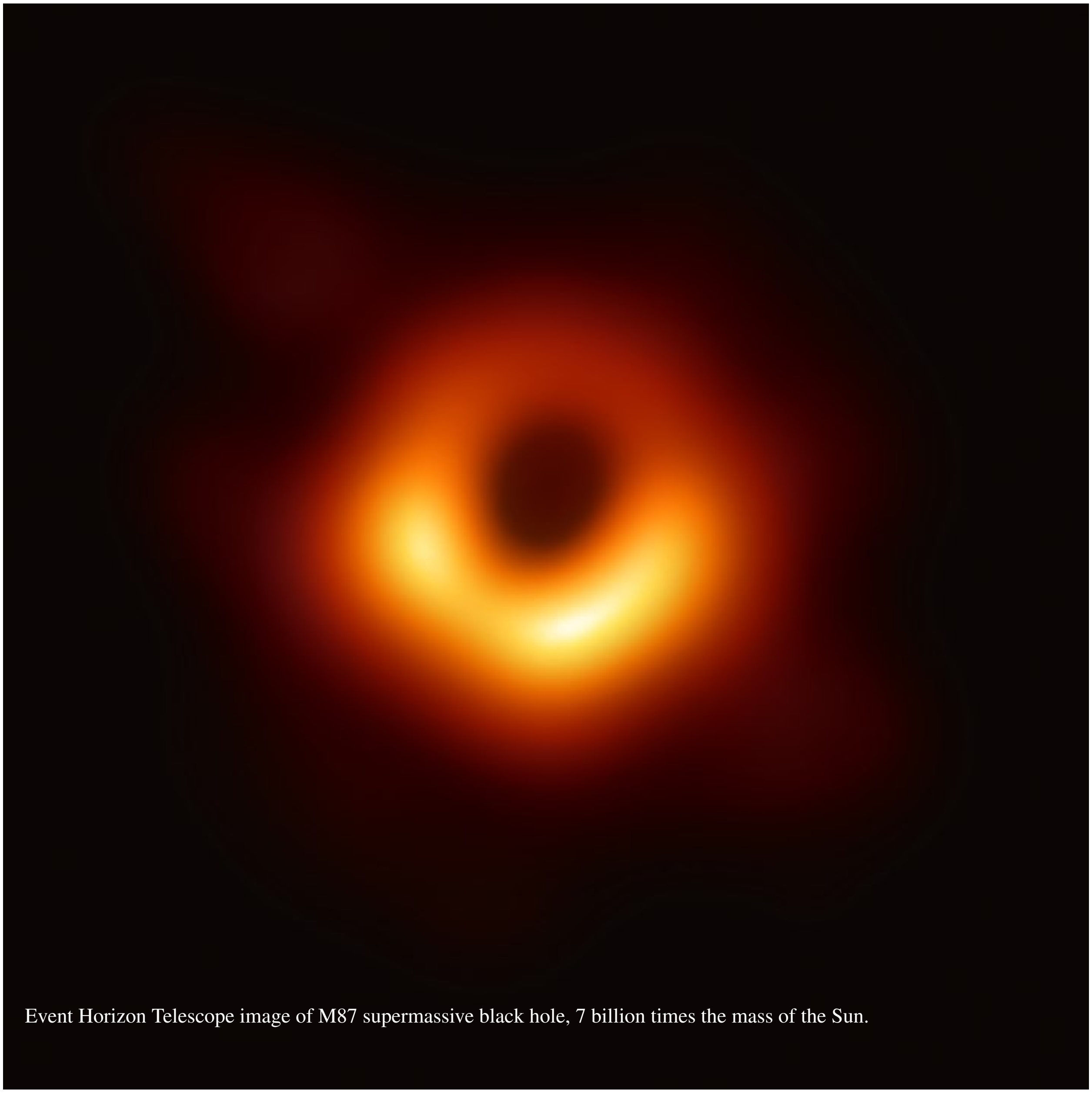
4. What is the total correction to 1 day, combining the two effects?

5. Atomic clocks are sensitive to drifts in time at better than the level of 10^{-16}s, and for GPS to work they need to be accurate at better than 10^{-9} s. Do you think special and general relativity effects are included in the design of GPS? Discuss in class.

Student Problem: Life at the Event Horizon

Imagine that you could be the first astronaut to journey close to the event horizon of a black hole, a dead star so massive and compact that there is a radius within which, if light approaches, it cannot escape. This radius is known as the "Event Horizon" of the black hole, though the dead star inside is compressed to something smaller than the size of an atom and is itself far from the event horizon.

Imagine you could be the first person to fly around and close to a black hole, just a little bit outside the event horizon. The event horizon of a dead collapsed star is given by $R_{eh} = \frac{2GM}{c^2}$. The mass of this particular dead star is 10×10^{30} kg. If, from your perspective, you spend 30 days orbiting the black hole, how long does the journey take from the perspective of someone who is very far away from the black hole (e.g. infinitely far away, so that its gravity has no influence on the outside observer)?



Event Horizon Telescope image of M87 supermassive black hole, 7 billion times the mass of the Sun.

Discussion: did we attack this question with the right framework? If so, why? If not, why not?

As a bonus exercise in this problem, show that the event horizon of a dead collapsed star of mass M can be estimated classically by calculating the radius of a star at whose surface the escape velocity (the velocity a second mass m would need to achieve in order to just reach an infinite distance from the star and come to rest) is the speed of light.

l Foundations of Modern Physics - Part II

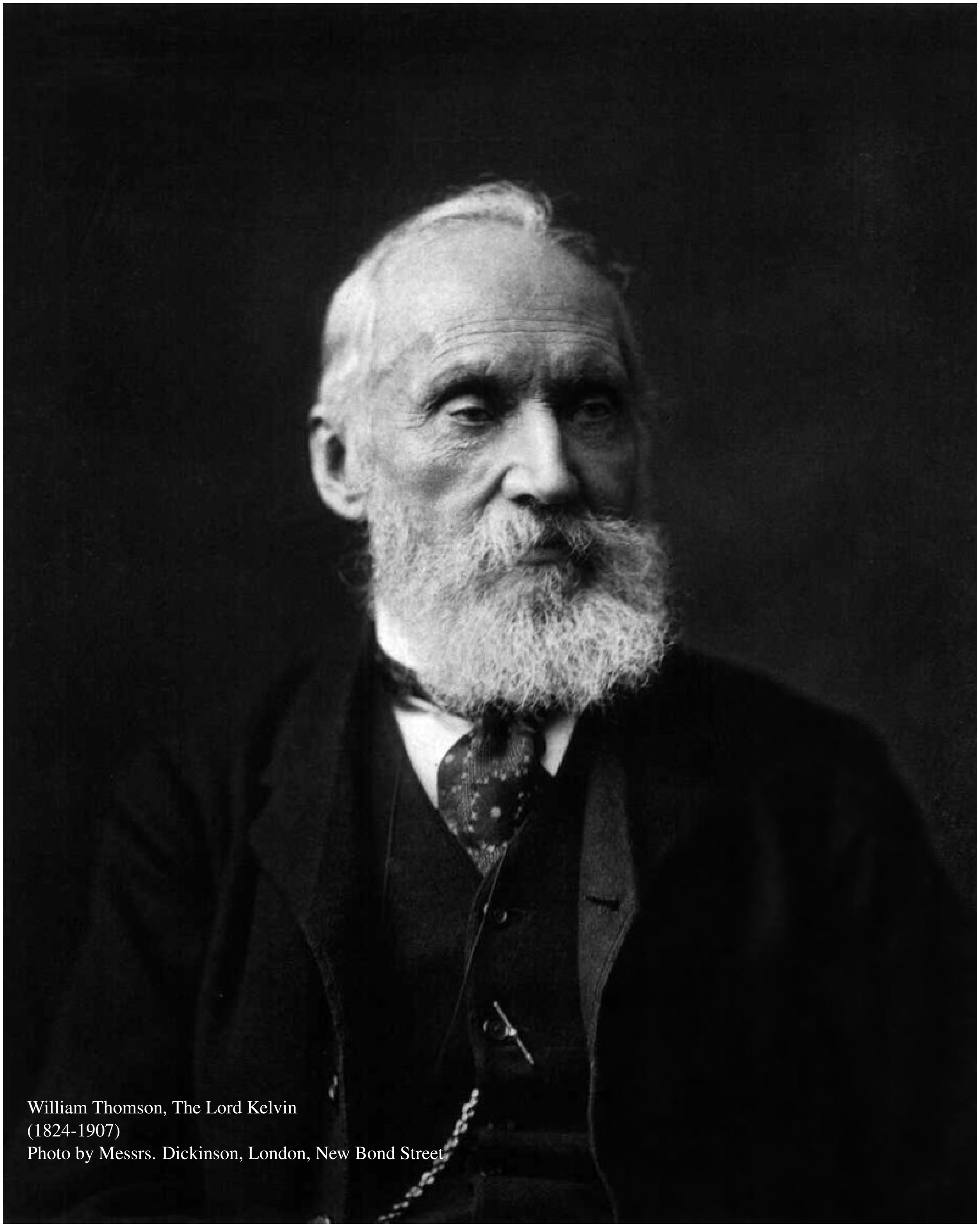
l.1 Temperature and Heat

Foundations of Modern Physics - Part II

Overview

In this lecture, we will learn…

- About the concept of "temperature" of a material body
- How to establish a scale and measure of temperature
- About the response of material bodies to changes in temperature
- About heat energy as the underlying agent connected to changes in temperature



William Thomson, The Lord Kelvin
(1824-1907)
Photo by Messrs. Dickinson, London, New Bond Street

Things Left Unsaid

- Introductory physics covers a prescribed range of topics: motion, force, the laws of motion, energy, momentum, conservation of energy and momentum, non-conservative forces, oscillatory motion, and rotational motion (first course); electric charge, electric force, electric fields, electric potential, electric currents, circuits, magnetic field and force, and the basic behaviors of light such as geometric optics or interference and diffraction (second course).

- In introductory physics, there is essentially no time to discuss the laws of heat energy: *Thermodynamics*.
- Nonetheless, Thermodynamics is an essential foundation of modern physics. It ultimately helped lead the way to quantum mechanics, the next subject of this course.
- In this part of the course, we will establish the second half of the foundations of modern physics: the concept of temperature, heat energy, and some of the behaviors of heat energy.

The concept of cold

Photo by Miguel Mansilla on Unsplash

The concept of hot

Photo by NeONBRAND on Unsplash

The concept of measuring temperature

Photo by Dan LeFebvre on Unsplash

Some critical issues and a plan

- We have a conception of hot and cold and a measure of these things.
- We need to establish the basis for a quantitative description of these concepts.
- Let us begin by establishing a scale on which we can quantify ideas like “hot” and “cold”
- Let us then look at the origins of hot and cold and how the underlying concept is really “heat energy”

Establishing a measure of hot and cold

- There are some phenomena in nature that occur at what appear to be very specific thermal conditions…that is to say, reproducible actions on a material body result in the same outcome under the same conditions.
- For example, the freezing or boiling of water - the only substance on Earth that can exist in solid, liquid, and gaseous states under Earth conditions - seems an attractive set of such phenomena.
- Materials other than water also change in response to temperature. Different metals expand and contract by differing amounts, even exposed to the same change in hot or cold. The stretching and squashing can be used to make “thermometers” - devices whose physical changes are a proxy for temprature changes.
- How do we “calibrate” their changes in response to temperature?

Establishing a measure of hot and cold

- We might imagine making water freeze, and at the moment that we observe this process to begin in a volume of water we use an external reference and call that temperature "0".
- We might then boil the water, and at that moment the process begins in the volume of water we take the same external reference and mark that moment off as temperature "100".
- We have then used two reproducible phenomena to establish two points on a scale. We could then divide the scale into 100 equal-sized units. Things can be colder than freezing and hotter than boiling, and we just keep marking off equal-sized units above and below the reference points (0 and 100).

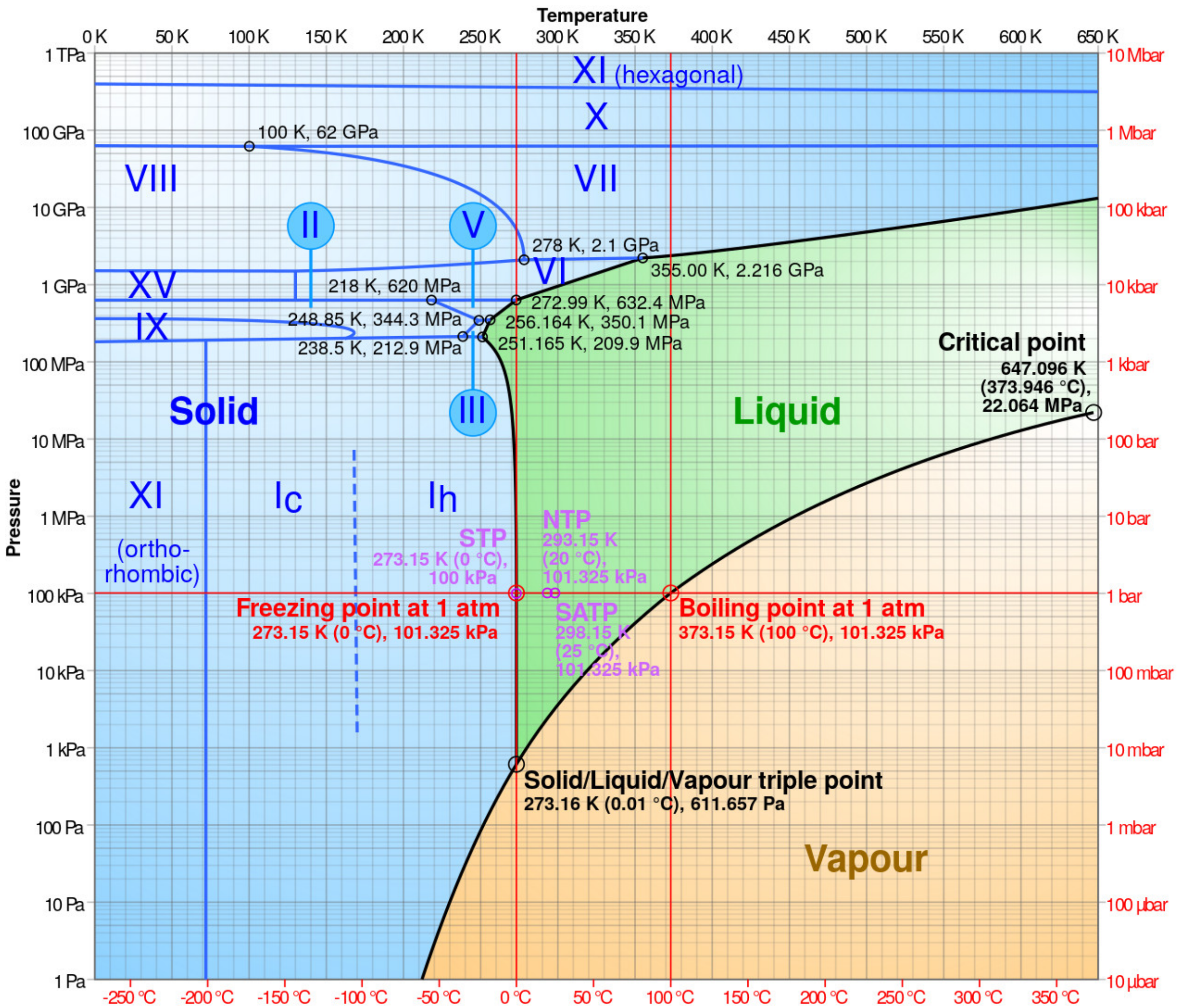
- What are the problems with this way of establishing a temperature scale?

Kelvin Scale and the Triple-Point of Water

- One problem with this idea is that the bottom of the scale doesn't have a physical meaning. The other problem is that the boiling and freezing of water depends strongly on other conditions, like the air pressure where the heating/cooling is conducted.

- It is the convention of the Systeme Internationale (SI) units of measure to employ the *Kelvin Temperature Scale*, whose smallest value (0K) has a physical meaning: a system at 0K has no heat energy content at all.

- The “Triple-Point” of water is then used to establish a second point on the scale. This is the temperature at which ice, liquid water, and water vapor all coexist, and it can happen only under one specific set of temperature and pressure conditions.
- For water, this only occurs when the exterior pressure (force per unit area exerted by air) is 611.657 Pa ("Pascals"), where 1 Pa = 1 $\mathrm{N/m^2}$. On the Kelvin scale, this is defined to occur at 273.16K (corresponding to roughly $0°$C at this air pressure).



Expansions and Contraction of Material Dimensions with Temperature

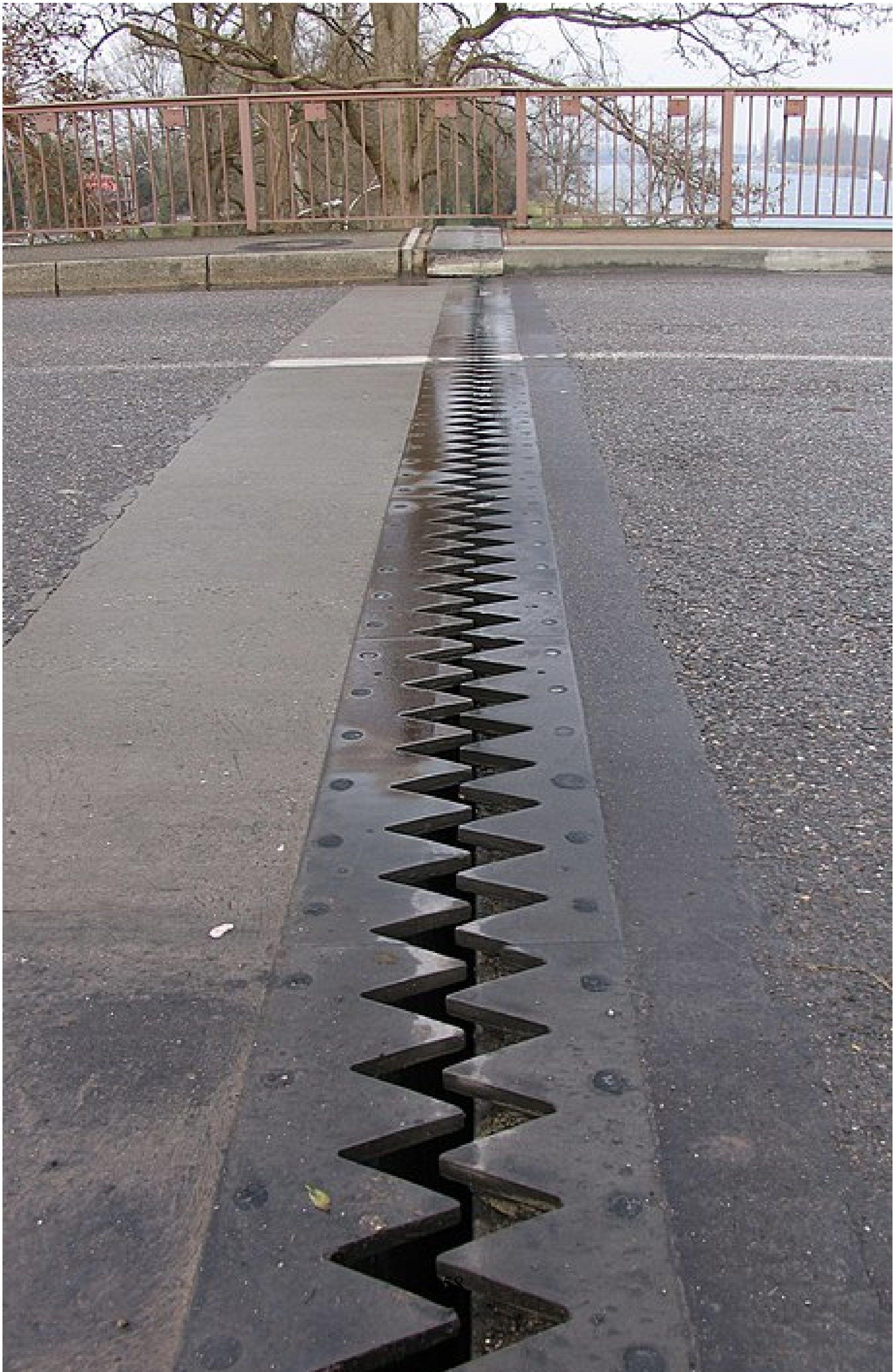
- I mentioned before that materials change physical dimension when exposed to changes in temperature.
- This is used to make many kinds of thermometers. For example, alcohol’s volume changes with temperature, so if we confine it to a vessel its level inside the vessel will be seen to rise or fall. Metals similarly change length and volume, and this can be used to make a mechanical thermometer (e.g. by having a metal spring that coils or uncoils with temperature changes, moving a marker)

- The length of a material of original length L is observed to change by an amount ΔL, in response to a measured change in temperature ΔT, according to:

$$\Delta L = \alpha L \Delta T$$

  where α is the *coefficient of linear expansion* of a material and must be determined by experiment.

- Similarly, whole volumes of material change by $\Delta V = \beta V \Delta T$, where $\beta = 3\alpha$ is the coefficient of volume expansion.



Temperature and Heat

- We can define an object embedded in a larger space as a *system*.
- The enveloping space is then defined as the *environment*.
- We observe systems at temperature T_S placed in an environment at temperature $T_E$and note the following:
  - If $T_S < T_E$, then over time T_S will *increase* until $T_S = T_E$, at which point T_S will cease to change.
  - Alternative, if $T_S > T_E$, then over time T_S will *decrease* until $T_S = T_E$, at which point T_S will cease to change.
  - When $T_S = T_E$, this is referred to as "thermal equilibrium."
- There is energy transfered between the system and the environment in the above examples. This energy is referred to as *heat energy*, and a quantity of this energy is denoted Q

Photo by Matt Hoffman on Unsplash

Matter Absorbing (or Releasing) Heat Energy

- Different material objects, composed of different substances (e.g. elements, molecules, alloys, etc.) react to the same amount of heat energy, Q, in different ways.

- The reaction can be measured by observing how the temperature changes with the absorption (or release) of heat energy from (to) the environment.

- For example, the same number of Joules of energy provided to a metal or to water will lead to very different changes in temperature. Metals readily take up (conduct) heat energy and result in a rapid change in temperature; water is not as conductive as metals, and the same amount of heat energy results in a more modest change in temperature.

Matter Absorbing (or Releasing) Heat Energy

- This is described mathematically by

$$Q = C\Delta T = C(T_f - T_i)$$

  where C is the *heat capacity* of the material in question. A large heat capacity corresponds to a smaller change in temperature, and vice versa.

- There is not a limit to the absorption of heat energy. So long as a temperature difference between a system and the environment can be maintained, heat energy can be transferred to a system. Phase changes can occur, but this does not stop the transfer of heat energy unless $T_S = T_E$.

Matter Absorbing (or Releasing) Heat Energy

$$Q = C\Delta T = C(T_f - T_i)$$

- The more mass of a material is supplied, the more heat energy it can absorb before achieving a given temperature change. It is convenient to define the specific heat of a material, c - the amount of heat energy per unit mass needed to achieve a given temperature change.
- The above equation then becomes

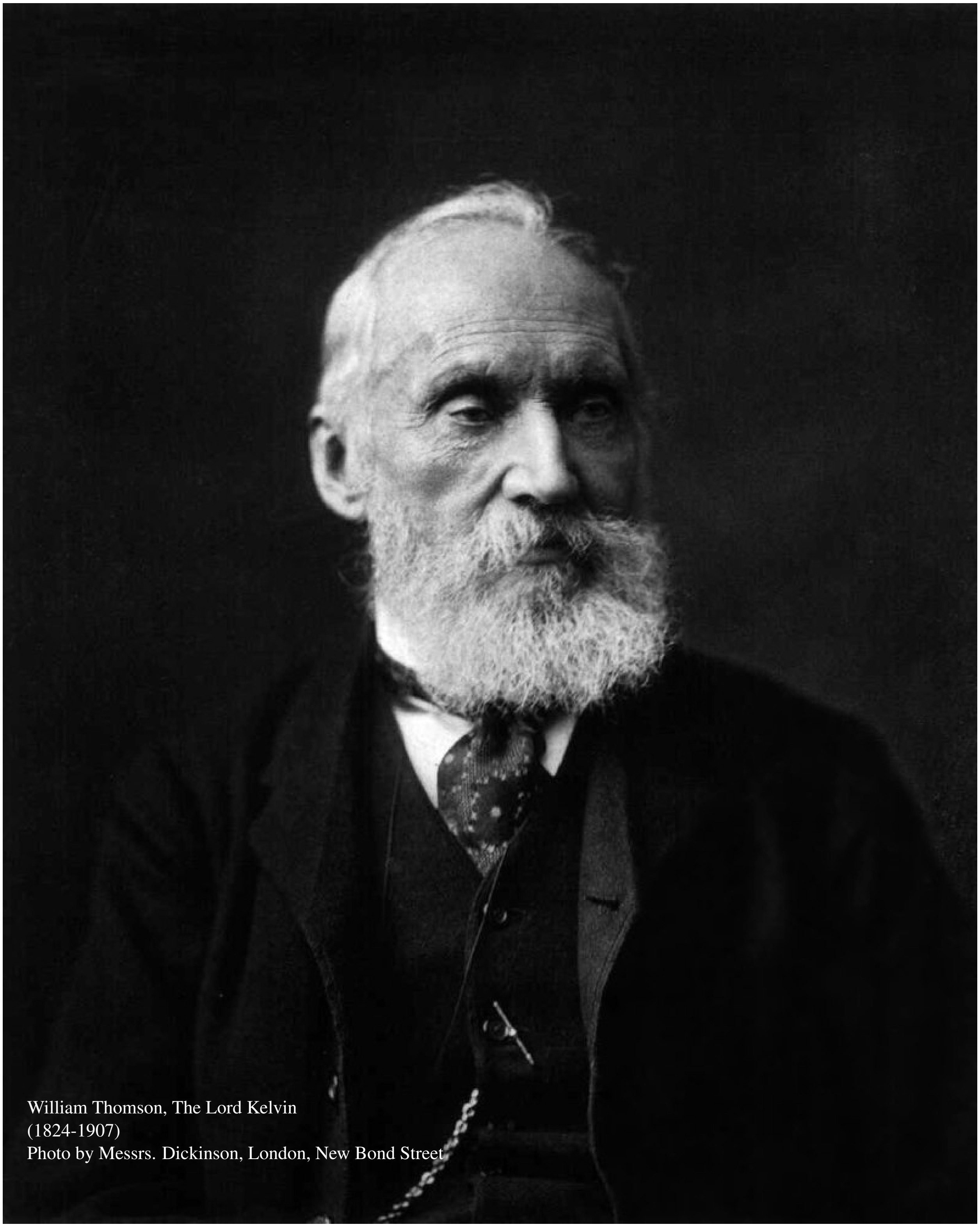
$$Q = mc\Delta T = mc(T_f - T_i)$$


Photo by Buzz Andersen on Unsplash

Review

In this lecture, we have learned…

- About the concept of "temperature" of a material body
- How to establish a scale and measure of temperature
- About the response of material bodies to changes in temperature
- About heat energy as the underlying agent connected to changes in temperature



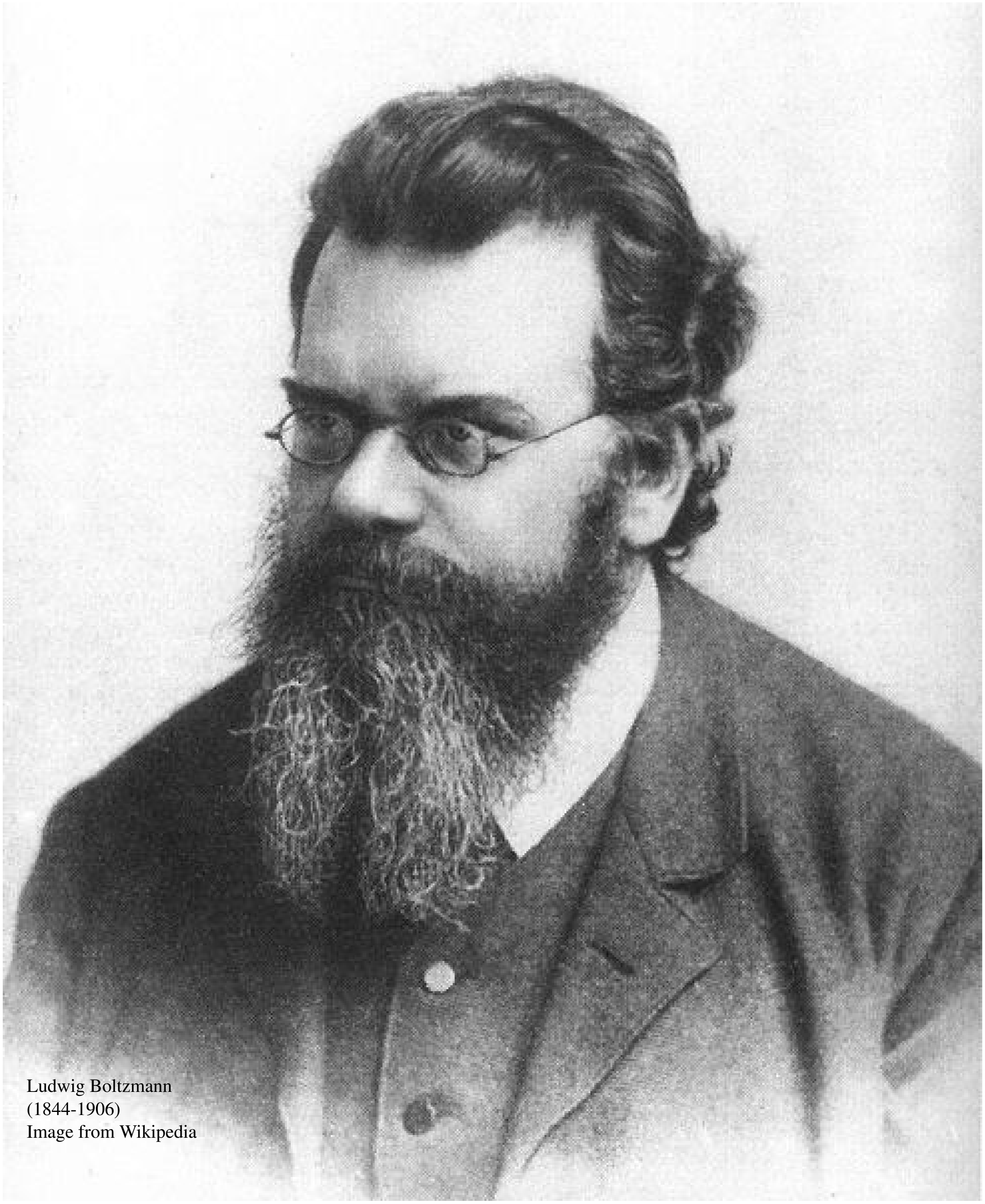
William Thomson, The Lord Kelvin
(1824-1907)
Photo by Messrs. Dickinson, London, New Bond Street

l.2 Heat, Matter, and Radiation

Overview

In this lecture, we will learn…

- About the connection between temperature and the constituents of a material body
- About the precise nature and cause of heat energy
- About radiation of energy from a material body



Ludwig Boltzmann
(1844-1906)
Image from Wikipedia

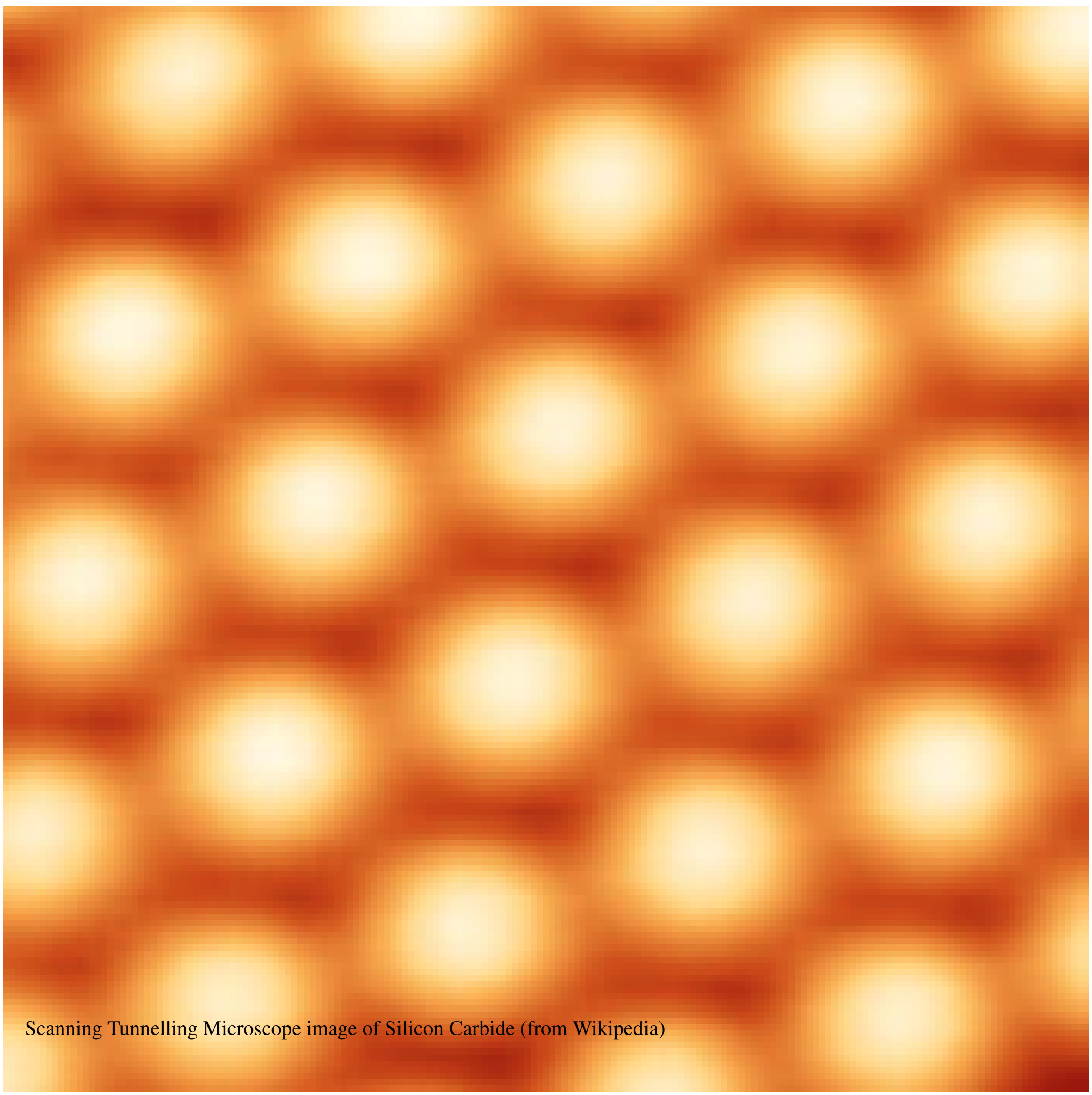
Matter is Made from Building Blocks

- Matter is ultimately made from building blocks; for example, a liquid may be made of a large number atoms or molecules.

- "1 mole" is the number of atoms in a 12g sample of C-12. Experimentally, 1 mole's worth of things is given by Avogadro's Number: $N_A = 6.02 \times 10^{23} \mathrm{mol}^{-1}$

- Heat energy must have a connection to the behavior of the building blocks of matter. For example, depositing a unit of heat energy ΔQ into a material must affect the properties of its constituents. This helps answer the questions "What is heat energy?" and "where does heat energy go (or come from)?"

- Let's look at an ideal gas as a laboratory for connection macroscopic concepts, such as the volume of a material, the temperature of a material, and the pressure exerted by a material on its environment, to microscopic concepts like the position and velocity of an atom or molecule.



Scanning Tunnelling Microscope image of Silicon Carbide (from Wikipedia)

An Ideal Gas

An ideal gas is a diffuse system of matter whose constituents experience interactions that are perfectly elastic collisions. These interactions include collisions with the wall(s) of the container holding the gas, and collisions between gas constituents.

There are many gases that are nearly ideal in nature: all the Noble gases (e.g. Helium, Argon, etc.), and even many other substances under a range of conditions.

There is an empirical law that relates the macroscopic properties of a gas: the number of moles of gas constituents, n; the volume of the gas, V; the temperature of the gas, T; and the pressure exerted by the gas on its container, P. This is *the ideal gas law*:

$$PV = nRT$$

where $R = 8.314 \mathrm{J}/(\mathrm{K} \cdot \mathrm{mol})$ is the *ideal gas constant*, named in honor of French chemist Henri Victor Regnault. But since a gas is made from small constituents, can we connect the microscopic properties to this macroscopic statement?

Connecting the Microscopic to the Macroscopic using Classical Physics

- Mass:
  - The molar mass of a gas, M (mass per unit mole), is given by adding up Avogadro's Number of individual constituent masses, m. The relationship is $M = N_A m$.
- Volume:
  - Consider a cubical space containing an ideal gas. The lengths of all sides are L. The area of any side is $A = L^2$. The volume is $V = L^3$.
- Pressure:
  - Pressure is the sum total of the force F_{total} per unit area exerted by all gas constituents on the walls of the container at any moment in time. How might we describe this using concepts of motion, Newton's Laws, and conservation laws from classical physics?

Connecting the Microscopic to the Macroscopic using Classical Physics

- Exploring Pressure using Classical Physics:
  - Each constituent has a velocity $\vec{v} = (v_x, v_y, v_z)$. Let's consider an elastic collision between a constituent of mass m and the wall of the container, along the x-axis. The wall doesn't move in the process, so its velocity before and after the collision is zero. Conserving kinetic energy and momentum yields that $p_i = mv_x$ and $p_f = -mv_x$. Thus the impulse is $\Delta p = p_f - p_i = -2mv_x$.

– If we knew the time over which the impulse occurs, we can compute the force exerted by the constituent on the wall using $F = \Delta p/\Delta t$. The time between collisons in 1-dimension is just the time it takes for the constituent to cross the width of the volume, L, collide, and cross back to where it started. Thus $\Delta t = 2L/v_x$. The force of the gas constituent on the wall is equal in magnitude and opposite in direction to the force of the wall on the constituent, so is then

$$F_{wall} = -F_{constituent} = 2mv_x/(2L/v_x) = mv_x^2/L$$

Connecting the Microscopic to the Macroscopic using Classical Physics

- The pressure is then the sum of all such forces across all constituents, divided by the area:

$$\begin{aligned} P &= F_{total}/(L^2) = mv_{1x}^2/L^3 + mv_{2x}^2/L^3 + \dots \\ &= (m/L^3)\sum v_{ix}^2 = (m/L^3)N(v_x^2)_{avg} = (m/L^3)nN_A(v_x^2)_{avg} \end{aligned}$$

- On average, the x-component of a constituent's squared-speed will be 1/3 of its total squared-speed, so:

$$\begin{aligned} P &= \frac{mnN_A(v_x^2)_{avg}}{L^3} = \frac{m}{V}nN_A\frac{1}{3}(v^2)_{avg} \\ PV &= \frac{mnN_Av_{avg}^2}{3} = \frac{Mnv_{avg}^2}{3} = nRT \\ v_{avg} &= \sqrt{\frac{3RT}{M}} \end{aligned}$$

The Kinetic Energy of Constituents of an Ideal Gas

The average kinetic energy of any single constituent is $K_{avg} = \frac{1}{2}mv_{avg}^2$. From the ideal gas relationship between average speed, temperature, molar mass, and the gas constant, we learn that:

$$\begin{aligned} K_{avg} &= \frac{1}{2}mv_{avg}^2 = \frac{1}{2}m\frac{3RT}{M} = \frac{1}{2}m\frac{3RT}{mN_A} \\ &= \frac{3}{2}\frac{R}{N_A}T = \frac{3}{2}k_BT \end{aligned}$$

where k_B is known as the Boltzmann Constant, $1.381 \times 10^{-23}\,\mathrm{J/K}$. When we measure the temperature of an ideal gas, we are really measuring the average kinetic energy of its individual constituents. *Heat Energy is determined to be related to the average kinetic energy of constituents of a material body.* As one adds heat energy to a system, this raises the average kinetic energy of the constituents. To remove heat energy from a system is to reduce the average kinetic energy of the system. A system with no kinetic energy of its constituents (no motion at all) is then identified as being the lowest temperature state, a state of zero heat energy. . . 0K on the Kelvin scale.

Transferring Heat Energy To/From a System

There are three established mechanisms for transferring heat energy from a system. Let's consider cooling (heating is just the reverse of these):

- Conduction: you place a second system, at a lower temperature, in contact with the first system. Collisions between the constituents at the contact interface of the systems will transfer kinetic energy (higher kinetic energy constituents will lose some kinetic energy to the slower moving constituents in the other system). This decreases the temperature of the hot system and increases the temperature of the cold system until such time $T_1 = T_2$ and equilibrium is reached.
- Convection: You pass a fluid (e.g. gas or liquid) across or around a system. Collisions at the boundary between constituents of system 1 and constituents of the fluid will transfer kinetic energy, on average, to the fluid. This cools the system.
- Radiation: the constituents lose kinetic energy by giving it up in the form of radiation of light (e.g. infrared). Radiation requires no physical contact between the system and the environment; radiation carries kinetic energy away from the system to the environment, even without physical contact.

Radiation from a Heated Material Body

There is a mathematical relationship between the energy emitted (or absorbed) by a heated material body and the temperature of the body. This was determined empirically by Josef Stefan to be:

$$\frac{\Delta Q}{\Delta t} = \sigma \varepsilon A T^4$$

where $\sigma = 5.670 \times 10^{-8} \mathrm{W}/(\mathrm{m}^2\mathrm{K}^4)$ is a constant of nature, the Stefan-Boltzmann Constant; ε is the *emissivity* of the surface of the material, ranging between 0 (no emission) and 1 (perfect emission); and A is the surface area of the material body. Note that all material bodies above 0K radiate energy in the form of electromagnetic radiation! A perfect emitter, $\varepsilon = 1$, is known as a "blackbody" and is a system that absorbs all incident radiation and can subsequently re-emits its own radiation.

Concept: Power per Unit Wavelength $\rightarrow$Spectral Radiance

In a situation where an amount of energy, ΔQ, is radiated by a body in some period of time, Δt, it is typical to ask the following question:

> If I consider a range of the radiation with minimum wavelength λ and maximum wavelength $\lambda + \Delta\lambda$, how much energy per unit time is radiated by wavelengths in that range?

This is known as "spectral radiance," $B(\lambda)$ - energy radiated, per unit time, per unit wavelength (or, alternatively, frequency). If you want to know the power radiated around a specific wavelength, you need to pick a small range ($\Delta\lambda$) around that wavelength and compute the product:

$$B(\lambda) \times \Delta\lambda = P$$

If you have a continuous function representing $B(\lambda)$, you can integrate in a range to get the answer:

$$\int_{\lambda_1}^{\lambda_2} B(\lambda)d\lambda = P$$

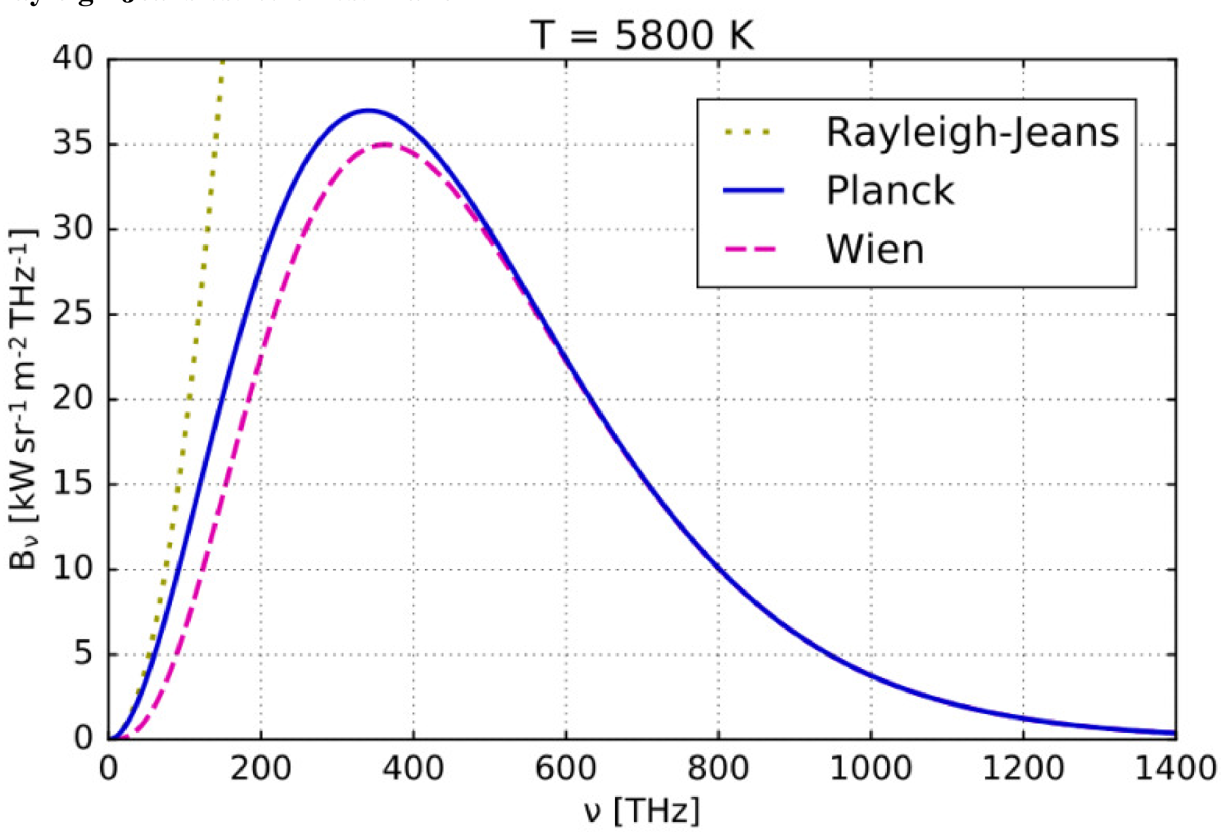
The Rayleigh-Jeans Law and Radiation from a Heated Object

Using classical physics, the amount of energy emitted per unit time (P) about a given wavelength, λ, was worked out. This is the Rayleigh-Jeans Law (from 1905):

$$P = \frac{8\pi Ack_BT}{\lambda^4}\Delta\lambda$$

This equation tells you that for a spherical heated body of surface area A at temperature T, the shorter the wavelength of the radiation you consider being emitted from the body, the more and more power is radiated around that wavelength. If true, this would be a catastrophic feature of nature: for example, a small sphere of surface area $1\,\mathrm{m}^2$ and emissivity $\varepsilon = 1$, when heated to 6000K (a small propane torch can reach 3000K) would emit about 10^{16} Watts alone in dangerous ultraviolet radiation…easily lethal to a living organism. But this is just not what is observed in reality!

Rayleigh-Jeans vs. Wien vs. Planck

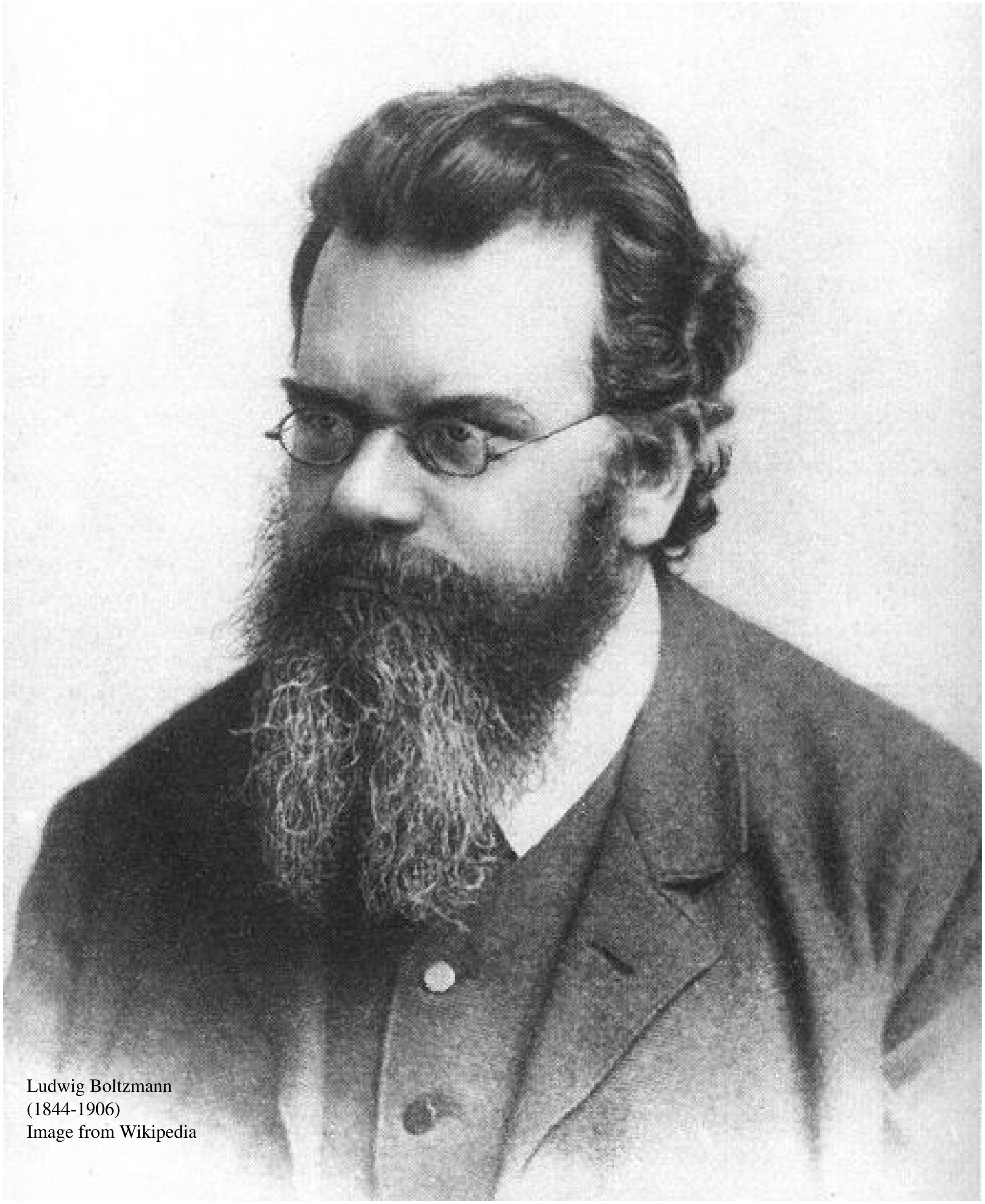


In reality, the radiation at a specific temperature looks like the blue curve in the graph above (derived in 1900). The pink curve was a separate attempt at the derivation by Wilhelm Wien in 1896. The Rayleigh-Jeans curve is the yellow one with the huge slope.

Review

In this lecture, we have learned…

- About the connection between temperature and the constituents of a material body
- About the precise nature and cause of heat energy
- About radiation of energy from a material body



Ludwig Boltzmann
(1844-1906)
Image from Wikipedia

Problem Solving in Temperature and Heat

Instructor Problem: Heating or Cooling a Metal

You have a sphere of brass at room temperature. You heat it from room temperature ($T_{room} = 295\mathrm{K}$) up to $T_{final} = 672\mathrm{K}$. The sphere had an original radius of $R = 1.0\mathrm{cm}$. The coefficient of length expansion of brass is $\alpha = 5.7 \times 10^{-5}$ /K. The density of brass is $\rho_{brass} = 8.5 \times 10^3\ \mathrm{kg/m^3}$. The specific heat capacity of brass is $c_{brass} = 380\mathrm{J/(kg \cdot K)}$

1. How much heat energy is required to achieve this temperature change?
2. By how much will the radius of the brass increase when heated by this degree?

A demonstration of this problem will be done in class

Student Problem: Expansion of a Metal Rod vs. Special Relativity

A copper metal rod of mass 2.0kg and length *exactly* $L = 1\mathrm{m}$ at room temperature is then heated from room temperature $T_{room} = 295\mathrm{K}$ to $T_{final} = 1200\mathrm{K}$.

1. What is the new length of the copper rod once it reaches its target temperature? (*HINT: go lookup the information needed to work this part of the problem*)
2. How fast would you have to move, parallel to the length of the rod, to observe it to contract back to its original length?
3. Copper has a resistivity of $\rho_{copper} = 16.78 \times 10^{-6}\Omega\cdot\mathrm{m}$. If you wanted to achieve this heating in a time of 30s by using an electric current, how much electric current and electric potential difference (voltage) is needed? (*HINT: assume the density and resistance are negligibly influenced by the change in temperature*)

Student Problem: Heating the Atmosphere

Since the industrial revolution, the combustion of hydrocarbons has led to the addition of carbon dioxide to the atmosphere. The added carbon dioxide prevents infrared radiation - heat - from escaping back into space. The trapping of additional heat energy raises the temperature of the atmosphere.

Since the industrial revolution, the atmosphere has heated by an average of $0.8°\mathrm{C}$ (which corresponds to an average temperature increase of 0.8K). If the average specific heat capacity of the atmosphere is about $1000\mathrm{J}/(\mathrm{kg} \cdot \mathrm{K})$, the average density of the atmosphere is about $0.8\mathrm{kg}/\mathrm{m}^3$, and we consider just the Troposphere (which extends from sea level to a height of 11km above the surface of the Earth)...

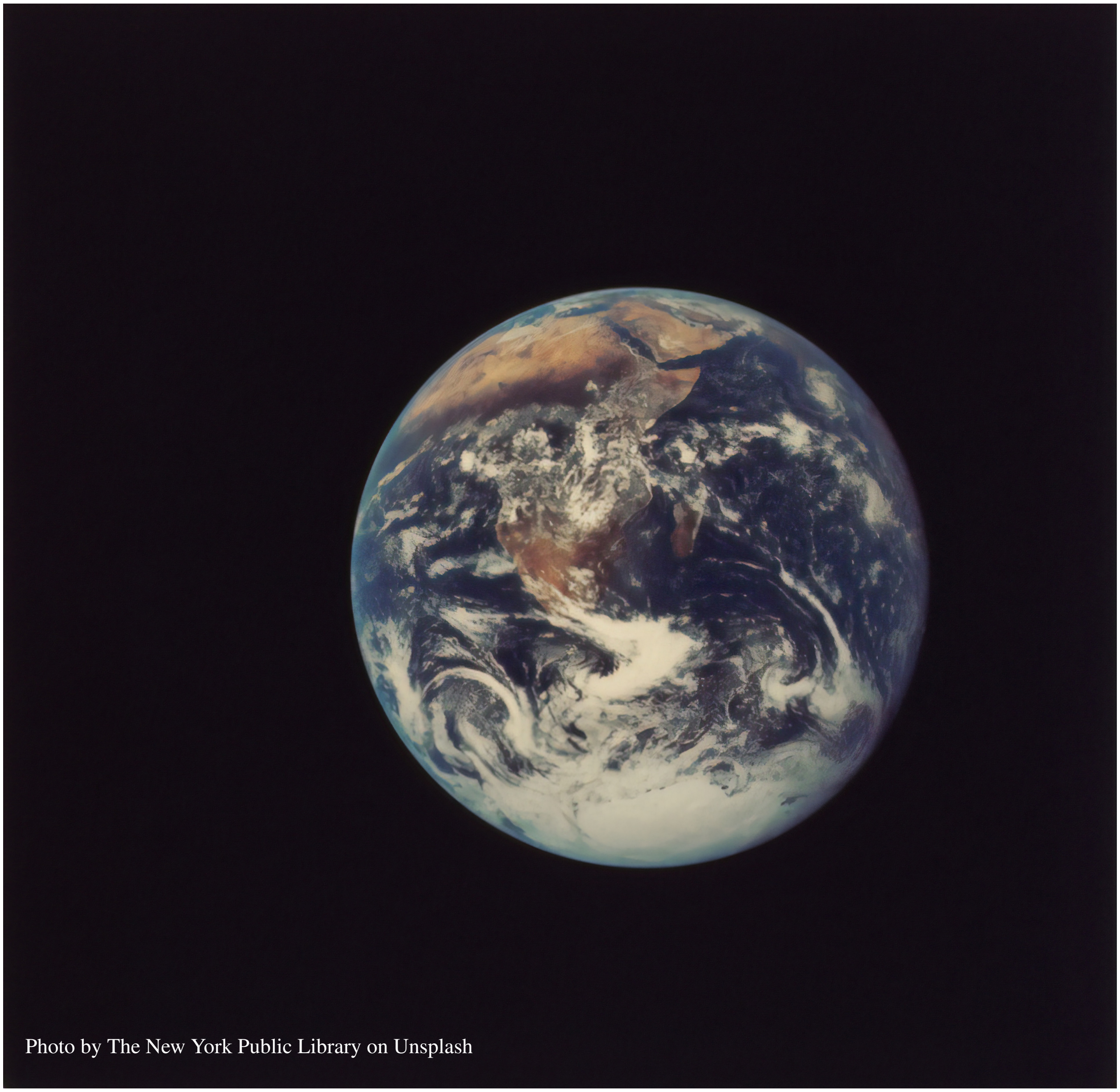


Photo by The New York Public Library on Unsplash

1. How much additional heat energy has been trapped by the atmosphere since the industrial revolution?

2. Compare that to the detonation of a hydrogen bomb, which releases 63,000 TeraJoules of energy. This was the explosive energy of the Castle Bravo test on the Bikini Atoll in 1954, the most powerful weapon ever tested by the United States. How many such hydrogen bombs would have to be detonated in the Earth's Troposphere to raise the temperature by the amount accomplished by a small increase in atmospheric carbon dioxide since the industrial revolution?

l.4 Problem Solving in Heat, Motion, and Radiation

Problem Solving in Heat, Motion, and Radiation

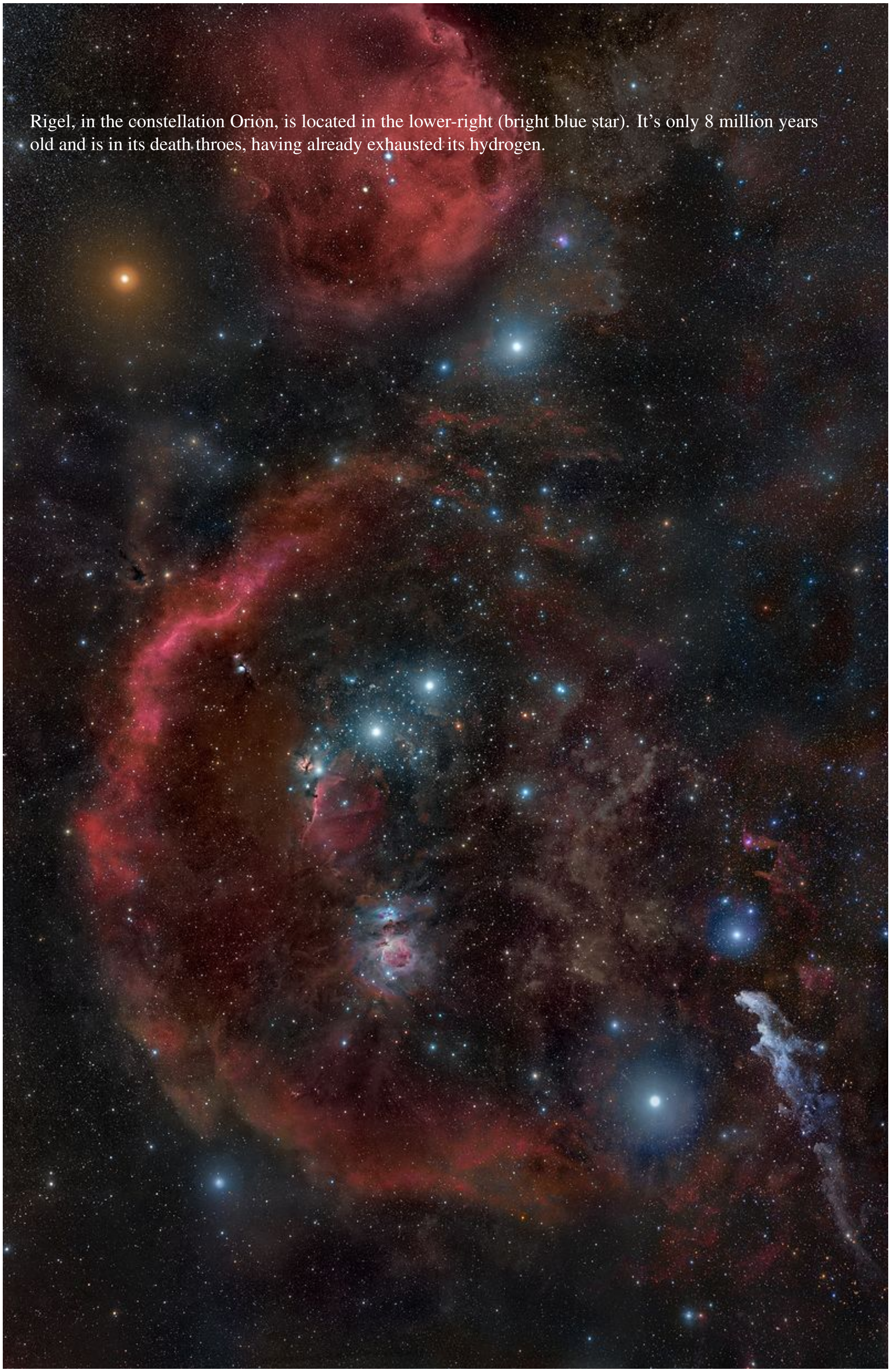
Instructor Problem: The Star Rigel

The star named Rigel is located in the constellation of "Orion". It is a blue-colored star (a blue super-giant), with a peak wavelength of emitted electromagnetic radiation of 252nm. Using information about heat energy, motion of constituents in a heated body, and Wien's Displacement Law,

$$\lambda_{peak} = \frac{b}{T},$$

where $b = 2.898 \times 10^{-3}\mathrm{m} \cdot \mathrm{K}$ is Wien's Displacement Constant, answer the following questions:

1. What is the temperature of Rigel's photosphere?
2. What is Rigel's current emitted power per unit area (also known in astronomy as *flux density*) assuming the photosphere is in thermal equilibrium and behaves like a blackbody?
3. If Rigel presently has a radius that is 74 times that of the Sun, what is its total emitted power and how does that compare to the Sun, if $P_{sun} = 3.83 \times 10^{26}\mathrm{W}$?
4. Treat the outer surface of Rigel like an ideal gas made from Hydrogen and estimate the root-mean-squared speed of a gas constituent on the surface.
5. For discussion: which star is hotter - Rigel or Betelgeuse (upper left star)?

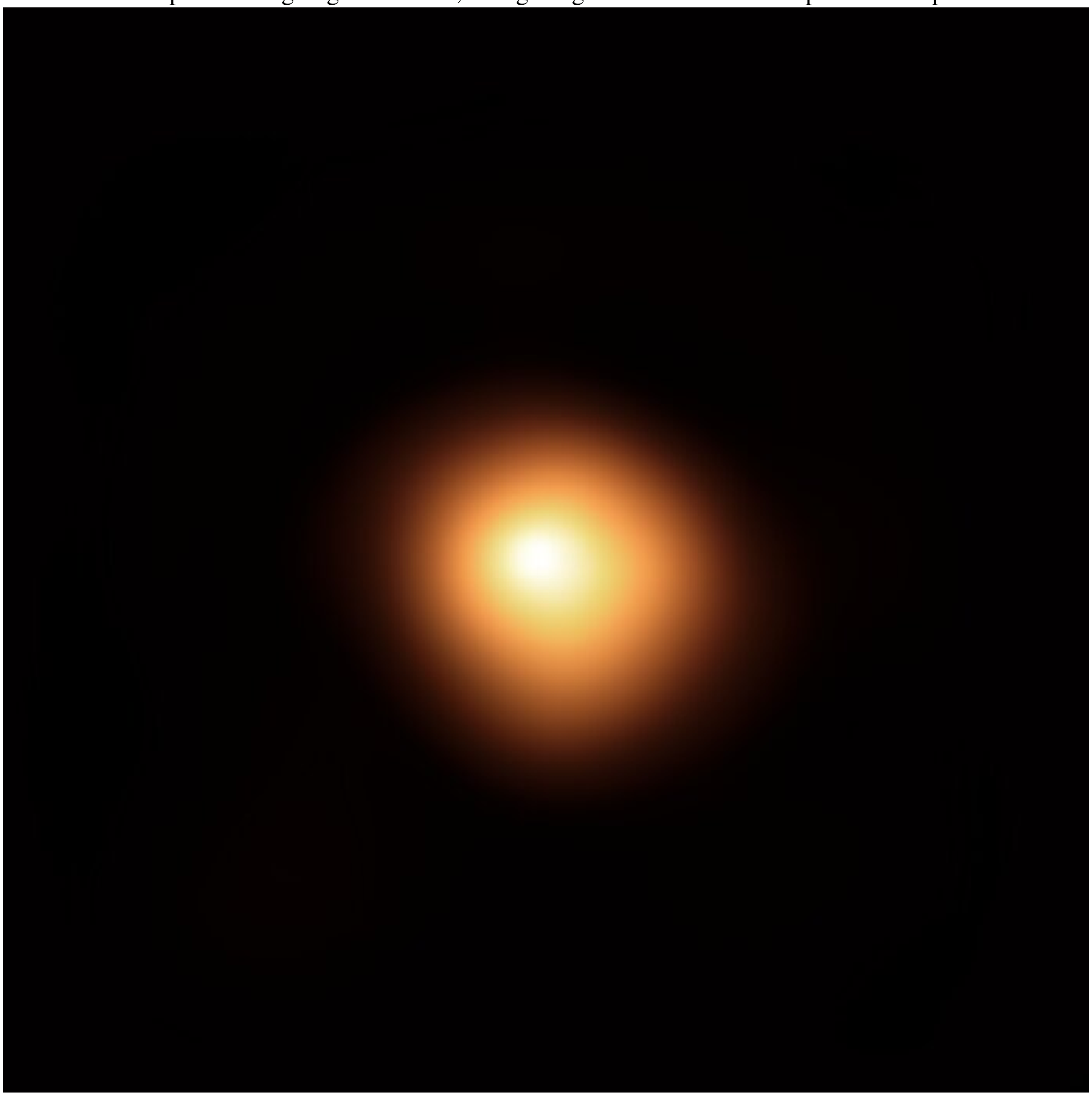


Rigel, in the constellation Orion, is located in the lower-right (bright blue star). It's only 8 million years old and is in its death throes, having already exhausted its hydrogen.

Aside: What's Going on with Betelgeuse?

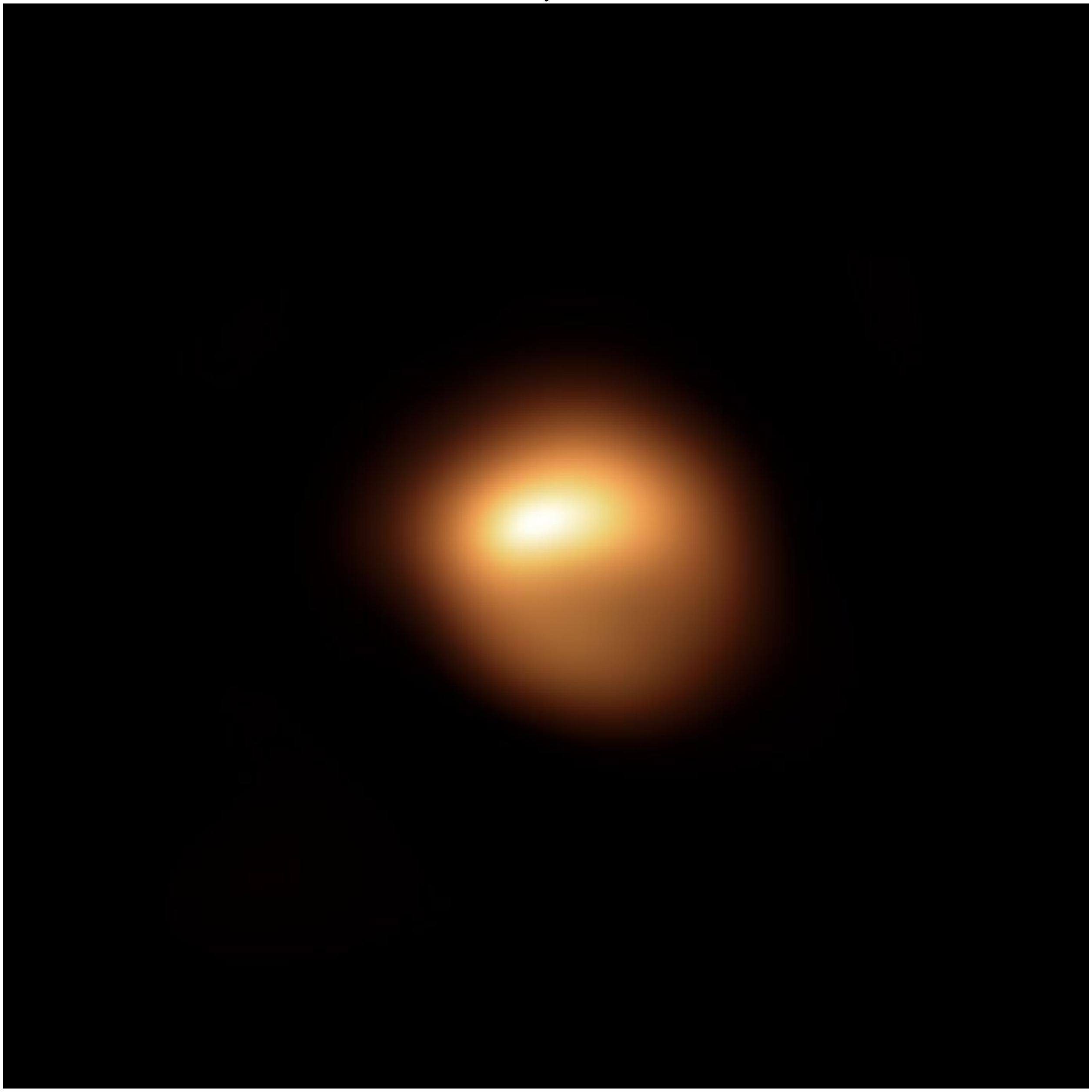
The star, Betelgeuse, in the upper-left part of Orion has been behaving rather strangely over the last year. We've known for some time that it is "off the Main Sequence," the astronomer's way of saying "in its death throes." Betelgeuse has used up its main Hydrogen fuel and is burning Helium. It has expanded to a vast size during the Helium-burning phase, a sneak preview of what our own Sun will do in 5 billion years. It is so large that, despite being 700 light-years away, we can image its surface using radio telescopes[bh]

Over the last six months, Betelgeuse's luminosity has decreased 36%. A recent radio telescope image shows the constrast of the star with an image taken 1 year ago. There is serious scientific debate about what, exactly, is going on: it might be a normal cycle not seen before; it might be sunspots; it might be a new dust plume sloughing off the star; it might signal its imminent collapse into a supernova.



[bh]It is one of only a handful of stars whose surface we have imaged; the Sun is the easiest to image, of course.

January 2019



December 2019

c.f. “ESO Telescope Sees Surface of Dim Betelgeuse” from the European Southern Observatory

Student Problem: The Vast Reaches of Outer Space

Imagine being in a vast pocket of "empty space," far from all stars, planets, etc. The temperature of empty space is just 2.7K (it’s not zero!).

- There is stray hydrogen, even in empty space. At this temperature, what is the root-mean-square speed of a random hydrogen molecule? (HINT: what is the molar mass of Hydrogen molecular gas?)
- The temperature of empty space is due to the light left over from the initiation of the universe,

an event known as the "big bang". The universe was once extremely small, extremely dense, and at very high temperature. After 380,000 years, it cooled enough for hydrogen to form and light has been free-streaming through the universe ever since then. Treat the entire universe like a blackbody and calculate the peak wavelength of light left over from the big bang.

- To what band of the electromagnetic spectrum does the light left over from the big bang correspond? With what modern technology might you detect it? (HINT: what are the frequency of the light waves left over from the big bang?)

Student Problem: A Beam of Hydrogen Molecules

A beam of hydrogen molecules[bh] (H_2) is directed at a target area. The beam makes an angle of $55°$ with respect to a normal line to the target surface area. Each molecule has a speed of $1.0\mathrm{km/s}$ and a mass of $3.3 \times 10^{-27}\mathrm{kg}$. If the target area is $2.0\mathrm{cm}^2$, and the beam delivers hydrogen at a rate of 10^{23} molecules per second...

1. What is the pressure exerted by the beam on the target?
2. What is the equivalent temperature of this beam of hydrogen molecules?

[bh]Molecular beams are useful not only for trying to answer basic questions about the properties of matter, but also for manufacturing. Molecular beam epitaxy is used in electronics manufacturing to deposit thin films on single crystals. This allows for production of devices like MOSFETs and diodes, as well as semiconductor lasers.

m Radiation and Matter - Part I

m.1 The Blackbody Spectrum and the Photoelectric Effect

The Blackbody Spectrum and the Photoelectric Effect

Overview

In this lecture, we will learn…

- How the blackbody radiation spectrum was understood
- About the possibility energy may come in discrete units
- About the photoelectric effect
- How Einstein resolved the photoelectric effect

Max Planck
(1858-1947)
Portrait of Max Planck in 1878

The "Ultraviolet Catastophe"

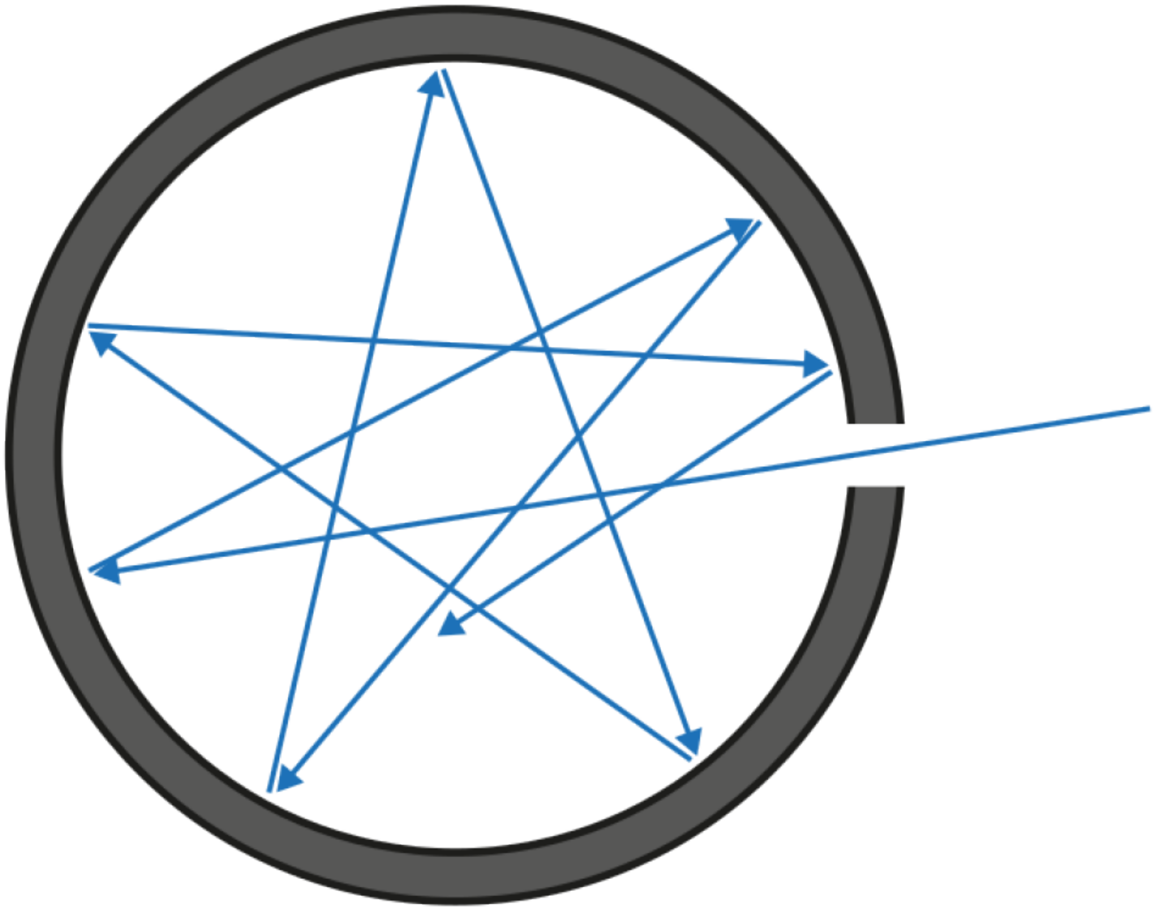
- In the last lecture, we saw how the Rayleigh-Jeans power spectrum prediction utterly failed to model nature correctly. Matter heated to a temperature, T, simply does not radiate according to that prediction. If it did, the effects would be catastrophic, as the shorter the wavelength (and more damaging the electromagnetic radiation), the more of it would be emitted from such a body.

- This mismatch between reality and the prediction of classical physics has been called the "ultraviolet catastophe," though historically this problem was not so threatening or important that anyone really panicked. It's a lovely and exciting name, though.[bh]

- We saw, however, that a physicist named Max Planck seems to have gotten the right answer. What did he do?

A Mathematical Model of a Perfect Blackbody - a cavity with a hole

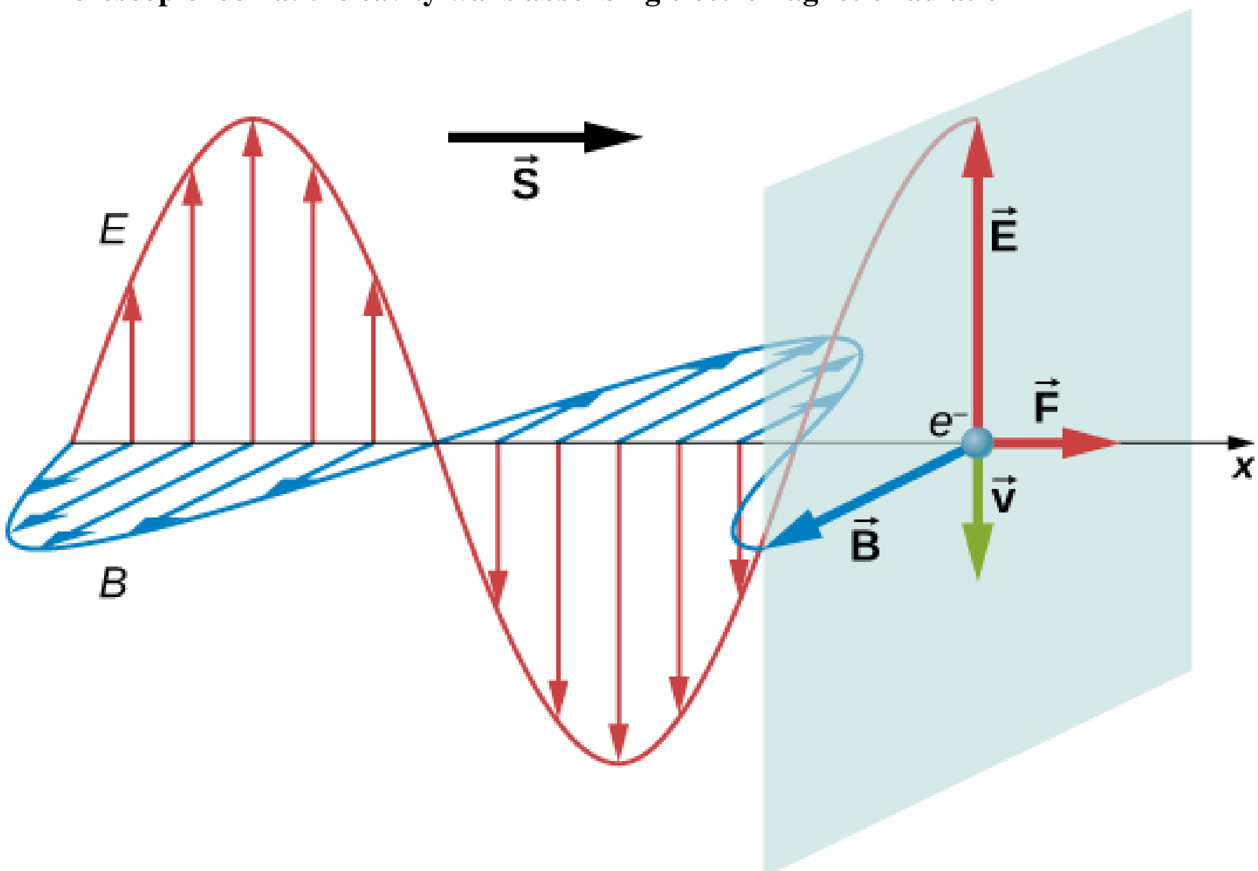
[bh]The term "ultraviolet catastrophe" does not date to the exact period when the Rayleigh-Jeans prediction was established, but appears to date to about 1911 and seems to have been coined by physicist Paul Ehrenfest.



- An ideal blackbody is one that absorbs all incident radiation and then emits its own radiation with perfect emissivity.
- Once one hypothesizes such a system, one then has to apply the laws of physics to predict or describe that emitted radiation *spectrum* - the amount of energy emitted, per unit solid angle, per unit time, per unit wavelength (or frequency)
- A cavity with a single, small hole in it is a good model for a perfect blackbody. 100% of radiation incident on the opening enters it and is "lost" in the cavity (*absorbed*), whereupon the radiation scatters inside and heats the walls of the cavity.
- The heated walls re-radiate energy. Some of it escapes through the hole. What will its spectrum look like?

Image from AG Caesar on Wikipedia

A microscopic look at the cavity walls absorbing electromagnetic radiation



- The "blackbody problem" can be boiled down to a very simple collection of phenomena.
- Radiation enters the cavity with any number of possible frequencies or wavelengths composing the incident radiation.
- The electric charges that make up the matter in the walls of the cavity will scatter or absorb the electromagnetic radiation.
- Absorbing an electromagnetic wave causes the charges to oscillate, and they can then re-radiate their own electromagnetic radiation.
- What will that re-radiated energy look like when it escapes the cavity? How much of each frequency is found in its power spectrum?

Image from AG Caesar on Wikipedia

The Rayleigh-Jeans Spectrum vs. Planck's Spectrum

The Rayleigh-Jeans Model:

$$P = \frac{8\pi A c k_B T}{\lambda^4}\Delta\lambda$$

Key Assumption: all frequencies are possible for oscillating charges
The Planck Model:

$$P = \frac{8\pi hAc^2}{\lambda^5} \times \frac{1}{e^{\frac{hc}{\lambda k_B T}} - 1}\Delta\lambda$$

Key Assumption: not all frequencies are possible for oscillating charges

Planck's effort resulted in the need to define a new constant, h, which came to be known as *Planck's Constant*. It is related to the degree of discretization of oscillation - the *quantization of the oscillatory motion of charges in the cavity walls*. The reason the spectrum "cuts off" at short wavelengths (high frequency) is that electromagnetic radiation requires a specific amount energy to make a specific wavelength; if you don't have that energy, you cannot make that wavelength. This implies the energy of radiation is also *quantized* (comes in units or packets).

Planck's Constant

The new constant, h, had to be determined from experiment. It has units of $\mathrm{J} \cdot \mathrm{s}$, which correspond to units of angular momentum. Its value was originally determined by finding the value that best described the shape of blackbody spectral energy densities, but there are many ways to measure it that have been developed over a century.

The currently accepted value of Planck's Constant is $h = 6.62607015 \times 10^{-34} \mathrm{J} \cdot \mathrm{s}$. This constant is crucially important in the modern world; its value is now the basis of the Systeme International definition of the *kilogram*; that change went into effect in 2018! h plays a fundamental and key role in all electronics devices; such devices rely on the exact properties of semiconducting materials, and semiconductors can be precisely engineered thanks to the quantization of radiation and matter, which stems from Planck's Constant.

A consequence of Planck's work, he realized in his paper on the subject, was that the energy of the electromagnetic radiation would be related to the frequency of that radiation:

$$E = hf$$

Planck and the context of his work

Planck concluded his work in 1900, but he did not accept the implication of h and the blackbody spectrum: that matter and energy are *quantized* into discrete units, and units in between are forbidden. He assumed this was all a convenient math trick and that someone would figure out the real answer.

> Moreover, it is necessary to interpret [the total energy of a blackbody radiator] not as a continuous, infinitely divisible quantity, but as a discrete quantity composed of an integral number of finite equal parts... (from his 1900 paper, "Ueber das Gesetz der Energieverteilung im Normalspectrum" published in *Annalen der Physik. Folge 4: v.4(1901).*)

> ...the whole procedure was an act of despair because a theoretical interpretation had to be found at any price, no matter how high that might be. (M. Planck to R. W. Wood, in a letter from 1931)

> ...I was ready to sacrifice any of my previous convictions about physics. (source unknown)

The quotes above were motivated, in part, by the fact that Planck had to use statistics to develop his ideas; this he had learned from Ludwig Boltzmann, but he detested the statistical interpretation of nature (that one can only know the probabilities of outcomes), as opposed to the more commonly embraced deterministic view of nature (that if you know all the initial conditions of a problem you can exactly and always reliably determine the outcome).

The Photoelectric Effect


Heinrich Hertz
(1857—1894)
Photo from no later than 1894 and in the public domain

- Heinrich Hertz was the first person to definitively demonstrate the existence of electromagnetic waves, a phenomenon that had been predicted using Maxwell's Equations. He showed that an oscillating charge at one place in a room could induce an oscillating charge elsewhere in the room, without physical contact.
- He was also the first person to demonstrate an intriguing physical phenomenon: *the photoelectric effect*

- Light, an electromagnetic wave, shone on a metal can liberate electrons from the metal, inducing an electric current.

- Knowing from Maxwell's Equations that the intensity of light (an EM wave) is proportional to the squared-strength (E_0^2) of the electric field of the wave, attempts were made to describe and predict the overall behavior of the photoelectric effect.

... this is not what is actually observed.

- The intensity of the light has no effect on initiating the photoelectric effect. Shine light on a metal. Observe it causes no electric current. Intensify the light. Still nothing will be observed, even if you wait a long time (allowing amplitude to "build up").

- Instead, change the wavelength (frequency) of the light. If a particular wavelength of light doesn't cause the photoelectric effect, switching to a longer wavelength (lower frequency) light will also cause no effect. But *shortening* the wavelength (high frequency) will be observed, at some point, to induce the effect.

- Lowering the intensity of the light, once it begins inducing a photoelectric response in the metal, may reduce the current, but it never switches off until you entirely switch off the light. At that threshold wavelength and frequency, it simply begins.

- Raising/lowering the intensity of light seemed to have no effect on the maximum kinetic energy of an ejected electron, despite the fact that intensity scales as E_0^2 for a EM wave... and shouldn't more E_0 produce more acceleration?

- This set of observational facts defied explanation using notions of classical waves and the laws of electromagnetism.

Albert Einstein: The Photoelectric Effect Explained

- It was Albert Einstein who cracked the photoelectric effect in one of his 1905 "Miracle Year" publications.

- To explain the phenomenon, Einstein reached back to Planck's 1900 paper on the blackbody spectrum, wherein a consequence of Planck's solution was that light has an energy given, not by the intensity of the electric field of the wave, but rather by the frequency (or wavelength) of the light:

$$E = hf = \frac{hc}{\lambda}$$

- A single unit of light is hypothesized to carry (or cost) hf for light of frequency, f. If even one unit of light of sufficient energy strikes an electron, liberation of the electron is immediately possible, independent of intensity of the light (number of units per second striking the metal).

Equations Decribing the Photoelectric Effect

In Einstein's solution to the photoelectric effect, which immediately resolved all the puzzles related to this phenomenon, it takes a certain minimum amount of energy to remove an electron from, say, a metal. This *minimum amount of energy* is called the *Work Function*,

$$\text{Work Function} \equiv \phi$$

If a quantum of light with energy $E_{light} > \phi$ strikes the electron, then it can scatter the electron or even be fully absorbed by the electron. The *maximum amount of energy that can be transferred to the electron* by such an interaction of matter and light is:

$$\begin{aligned} E_{max} &= E_{light} - \phi \\ &= hf - \phi \end{aligned}$$

The electron will gain kinetic energy as a result of this interaction, and so finally we arrive at:

$$KE_{max} = hf - \phi$$

The Birth of the "Photon": Light as Having Particle-like and Wave-like Aspects

Classical description of light (Maxwell's Equations):

$$\begin{aligned} \vec{E}(\vec{x}, t) &= \vec{E}_0 \cos\Big(\hat{k} \cdot \vec{x} - c_0 t\Big) \\ \vec{B}(\vec{x}, t) &= \frac{1}{c_0}(\hat{k} \times \vec{E}) \cos\Big(\hat{k} \cdot \vec{x} - c_0 t\Big) \end{aligned}$$

Energy (E) per unit area (A) of an EM wave in empty space:

$$E/A = \frac{1}{c\mu_0}\frac{E_0^2}{2}$$

Einstein special relativistic description of massless phenomena:

$$E = pc$$

Planck/Einstein description of light interacting with matter:

$$E = hf = \frac{hc}{\lambda}$$

The wave-like aspects of light had been well-confirmed prior to the 1900s (interference and diffraction; oscillating charges making EM waves, EM waves oscillating charges). The blackbody problem and the photoelectric effect hinted at *particle-like* aspects of light: light energy in units, energy defined by frequency, and light as a massless phenomenon. Einstein referred to these packets of light energy as "light quanta" (from the Latin, *quantum*, meaning "how much"); in a letter in 1926, physical chemist Gilbert Lewis coined the more common term "photon," which we still use today.

Trust but Verify: Experimenting on the Photoelectric Effect

Robert Millikan
1868—1953
Portrait from 1891

- Einstein's explanation was not readily accepted, of course, and met with serious scientific skepticism

- The physicist Robert Millikan, famous for the “oil drop experiment” that established the fundamental unit of electric charge, took the claims of Einstein’s explanation about the maximum kinetic energy of an ejected electron and used an experiment to test the claim.

- We’ll do a reproduction of the experiment in class, but the bottom line was this: Millikan confirmed Einstein’s hypothesis in 1914, and in the end the photoelectric effect paper of 1905 won the day. For this work, Einstein received the 1921 Nobel Prize in Physics.

- Einstein, extending Planck’s work to an entirely separate phenomenon, set the stage for another entirely new perspective on nature: the quantum view of matter and radiation.

Review

In this lecture, we have learned…

- How the blackbody radiation spectrum was understood

- About the possibility energy may come in discrete units

- About the photoelectric effect

- How Einstein resolved the photoelectric effect

Max Planck
(1858-1947)
Portrait of Max Planck in 1878

Student Activity Period: Exploring the Photoelectric Effect

I will demonstrate a mechanical example of the photoelectric effect in class. We will then begin working on a computational laboratory exercise. The software models the photoelectric effect. Each group will have a Canvas quiz to guide them through the activity.

Additional questions (use the Planck Blackbody Spectrum to solve):

1. For Platinum, to what temperature would you have to heat it such that the average kinetic energy of the constituents is sufficient to rip the electrons from the atoms via collisions alone?
2. What is the most prominent light emission from Platinum at that temperature?
3. How much energy would be emitted by such a heated body, per unit time, per unit area, per unit wavelength at that wavelength and at that temperature?

n Radiation and Matter - Part I

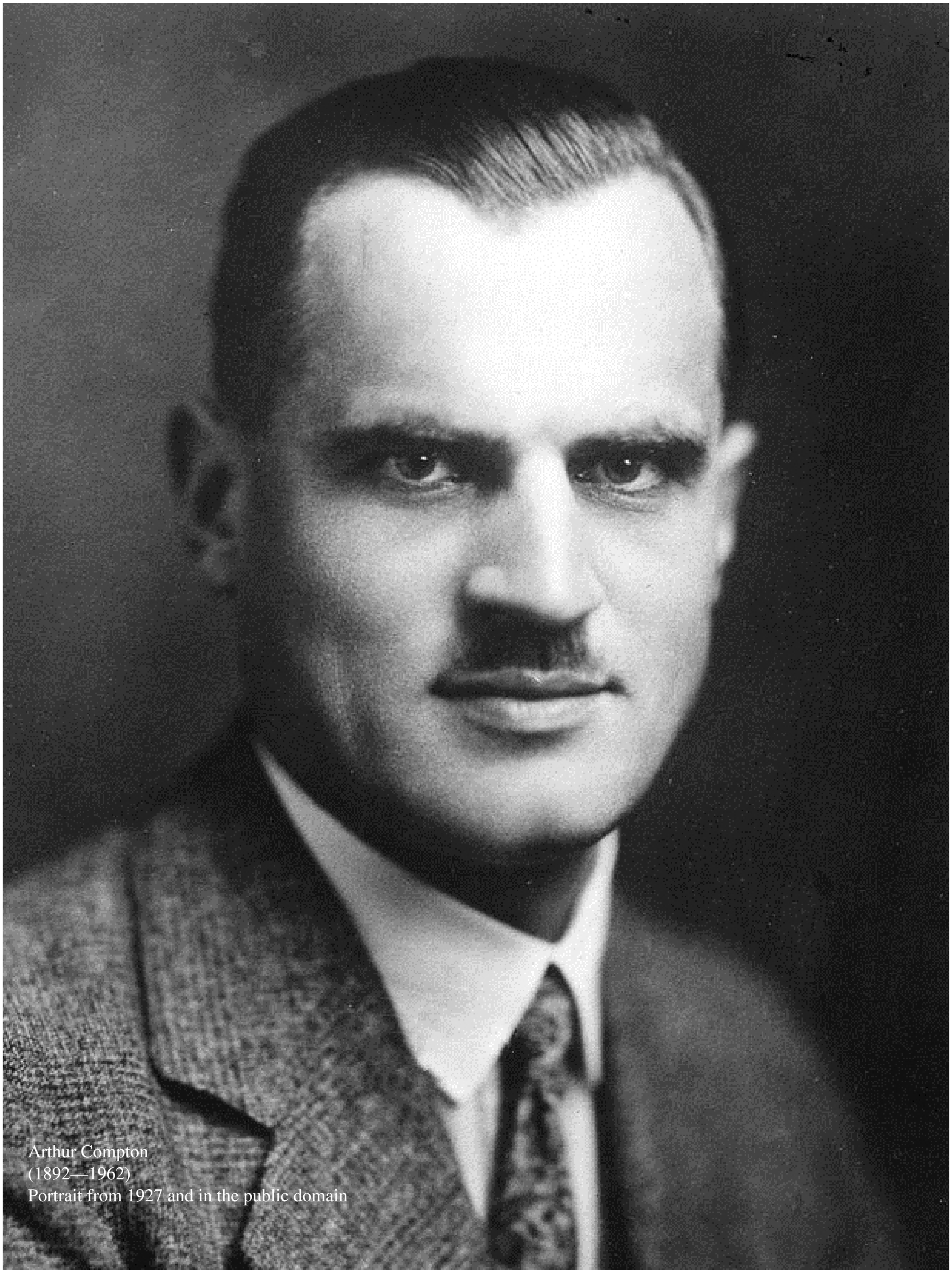
n.1 X-Rays and Compton Scattering

X-Rays and Compton Scattering

Overview

In this lecture, we will learn…

- About the nature of "x-rays"
- About the production of x-rays
- About the scattering of x-rays by matter, and the implications for the nature of electromagnetic radiation



Arthur Compton
(1892—1962)
Portrait from 1927 and in the public domain

Wilhem Roentgen and the Discovery of X-Rays

Roentgen discovers x-rays serendipitously in 1895 while experimenting with "cathode rays" (electrons boiled off a metal using a strong electric field)


Hand mit Ringen
igenthum von Prof. Zehnder
Freiburg i/B
PHYSIK. INSTITUT
DER UNIVERSITÄT

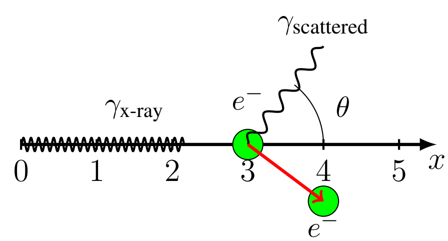
First medical x-ray from 1895 showing the hand of Anna Ludwig (Roentgen's spouse)

- We now know that x-rays are a kind of electromagnetic radiation, a very short-wavelength light
- They have wavelengths between 0.01nm and 10nm.
- As Roentgen discovered, they easily penetrate common low-density materials (e.g. cardboard, skin, muscle) and can be stopped by more dense materials (e.g. lead, bone)
- They would become an object of study, and would lead the way toward understanding more the particle-like aspects of light's behaviors.

The X-Ray Scattering Experiments of Arthur Holly Compton

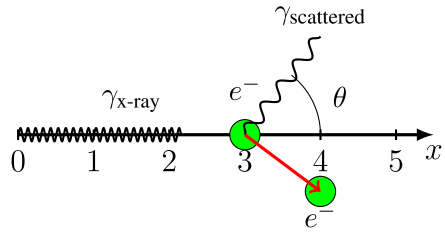
- There were not many notable physicists from the United States in the late 1800s or early 1900s (at least, compared to the European powerhouse of education and research across the Atlantic).
- One of these was Ohio-born Arthur Holly Compton. His PhD thesis was, in part, on the reflection of x-rays. He received National Research Council support to travel and research abroad, and selected to conduct work at the famous Cavendish Laboratory in Cambridge, England, in 1919. There he would experiment with very short-wavelength light, laying the groundwork for his eventual discovery what is now known as the "Compton Effect"
- As faculty at Washington University in St. Louis in 1922, he observed that x-ray quanta scattered by free electrons experienced a lengthening of their wavelength after the scatter. How to explain this?

Compton Scattering: A Model



We can hypothesize, as Compton did based on Einstein's 1905 photoelectric effect work, that the x-ray incident (i) on the electron, before scattering, carries total momentum given by $E_i = p_i c = hf_i = hc/\lambda_i \longrightarrow p_i = h/\lambda_i$. The final ($f$) scattered light quantum carries different momentum, $p_f = h/\lambda_f$. The initial electron state can be taken as being at rest, $u_i = 0$; the final state as the electron having total speed u_f, and thus total momentum $\gamma_{u_f} m_e u_f$. To analyze this, we need only conserve total energy, and momentum in x and y.

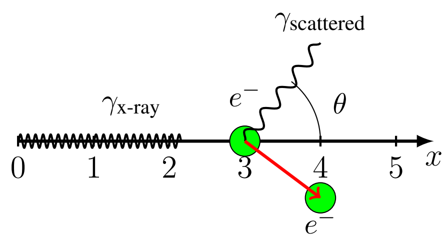
Compton Scattering: Conserve Momentum in x



$$
\begin{aligned}
P_{initial,x} &= P_{final,x} \longrightarrow \frac{hc}{\lambda_i} + 0 = \frac{hc}{\lambda_f}\cos(\theta) + \gamma_u m_e u \cos(\phi) \\
\frac{hc}{\lambda_i} &= \frac{hc}{\lambda_f}\cos(\theta) + \gamma_u m_e u \cos(\phi)
\end{aligned}
$$

That's about as far as we can go without knowing things like ϕ. We need more equations. Turn to conserving momentum in y.

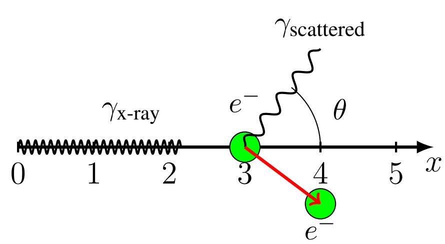
Compton Scattering: Conserve Momentum in y



$$
\begin{aligned}
P_{initial,y} &= P_{final,y} \longrightarrow 0 + 0 = \frac{hc}{\lambda_f}\sin(\theta) - \gamma_u m_e u \sin(\phi) \\
0 &= \frac{hc}{\lambda_f}\sin(\theta) - \gamma_u m_e u \sin(\phi)
\end{aligned}
$$

We could solve for ϕ, although that's a bit nasty. Let's see if the conservation of energy holds any comfort.

Compton Scattering: Conserve Total Energy



$$\begin{aligned}E_{initial} &= E_{final} \longrightarrow \frac{hc}{\lambda_i} + m_e c^2 = \frac{hc}{\lambda_f} + \gamma_u m_e c^2 \\ \frac{hc}{\lambda_i} + m_e c^2 &= \frac{hc}{\lambda_f} + \gamma_u m_e c^2\end{aligned}$$

Experimentally, Compton would have known three things: the incident x-ray wavelength λ_i, the scattered light wavelength λ_f, and the angle at which the light is scattered, θ. Can we relate these using this hypothesis and make a prediction for the relationship between them?

The Compton Effect

We have:

$$\begin{aligned}\frac{hc}{\lambda_i} &= \frac{hc}{\lambda_f}\cos(\theta) + \gamma_u m_e u \cos(\phi) \\ 0 &= \frac{hc}{\lambda_f}\sin(\theta) - \gamma_u m_e u \sin(\phi) \\ \frac{hc}{\lambda_i} + m_e c^2 &= \frac{hc}{\lambda_f} + \gamma_u m_e c^2\end{aligned}$$

Some lengthy algebra leads to the following:

$$\lambda_f - \lambda_i = \frac{h}{m_e c}\left(1 - \cos(\theta)\right)$$

In other words, Compton's analysis suggested that the *difference in wavelengths* after and before the scatter *depends only on the scattering angle* of the light and some constants of nature. Compton confirmed this with his experiments.

Implications of the Compton Effect

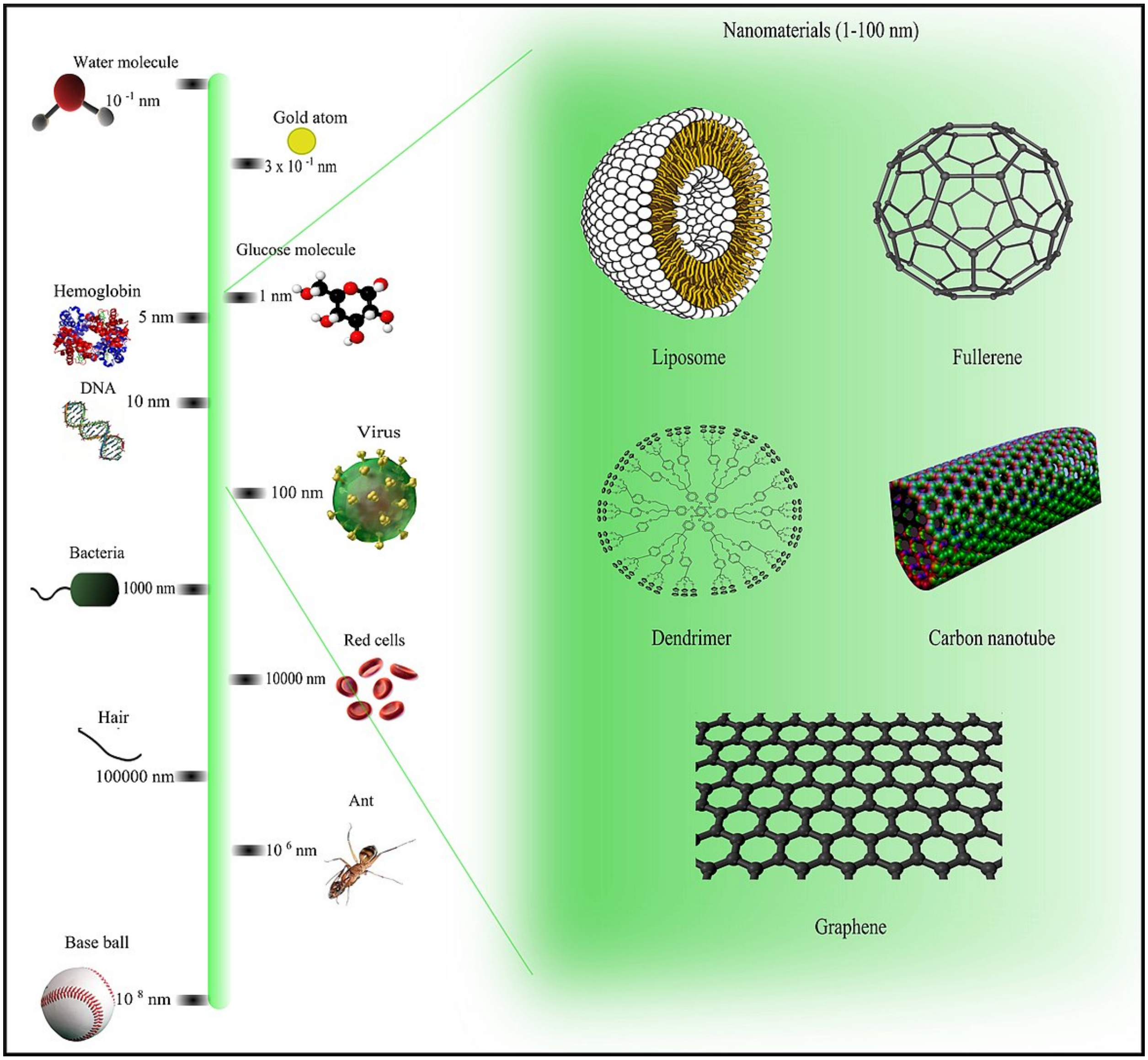
$$\lambda_f - \lambda_i = \frac{h}{m_e c}\left(1 - \cos(\theta)\right)$$

- An undeflected x-ray (θ=0) will experience no shift in wavelength;
- A completely deflected x-ray ($\theta = \pi$) - a *back-scatter* - will experience the maximal possible shift in wavelength, corresponding to the largest achievable kinetic energy for the electron.
- Compton did observe scattered light at angles other than those expected from scattering off electrons, because some x-rays were scattering off entire atoms, and not just electrons ($m_e \to m_N$ in the above equation. . . a smaller wavelength shift results at the same angle, θ)
- For light with wavelengths (frequencies) at those of x-rays, scattering of the light behaves more like scattering particles off of other particles rather than waves off of particles; the description of light as a large collection of *quanta works better than the wave hypothesis*

Deeper Implications of the Compton Effect

- Taken all together, the blackbody problem, the photoelectric effect, and Compton scattering all point toward a complex set of aspects of light behavior:
  - Under some conditions, light behaves according to classical Maxwell Equation theory - waves scattering off of or otherwise interacting with matter;
  - Under different conditions, light behaves according to a particle description, and not a wave description; then it is better described as a collection of *quanta* - photons - that can be thought of as particles interacting with the particles that compose matter.
- Ultimately, it came to be understood that particle-like aspects of light's behavior corresponded more to when the wavelength of the light was short (high-frequency), and wave-like aspects of light's behavior correponded more to when the wavelength was long (low-frequency). Somewhere in that space of wavelengths (frequencies), there is a transition in behavior.
- But what defines "short" and "long"? It turns out the answer is the dimension (D), or size, or scale, of the system with which the light interacts. If $\lambda \gg D$, then wave-like behavior rules; if $\lambda \ll D$, particle-like behavior rules; in the middle, as $\lambda \approx D$, you have to be careful and have the right theory to make predictions.

Sizes of Things

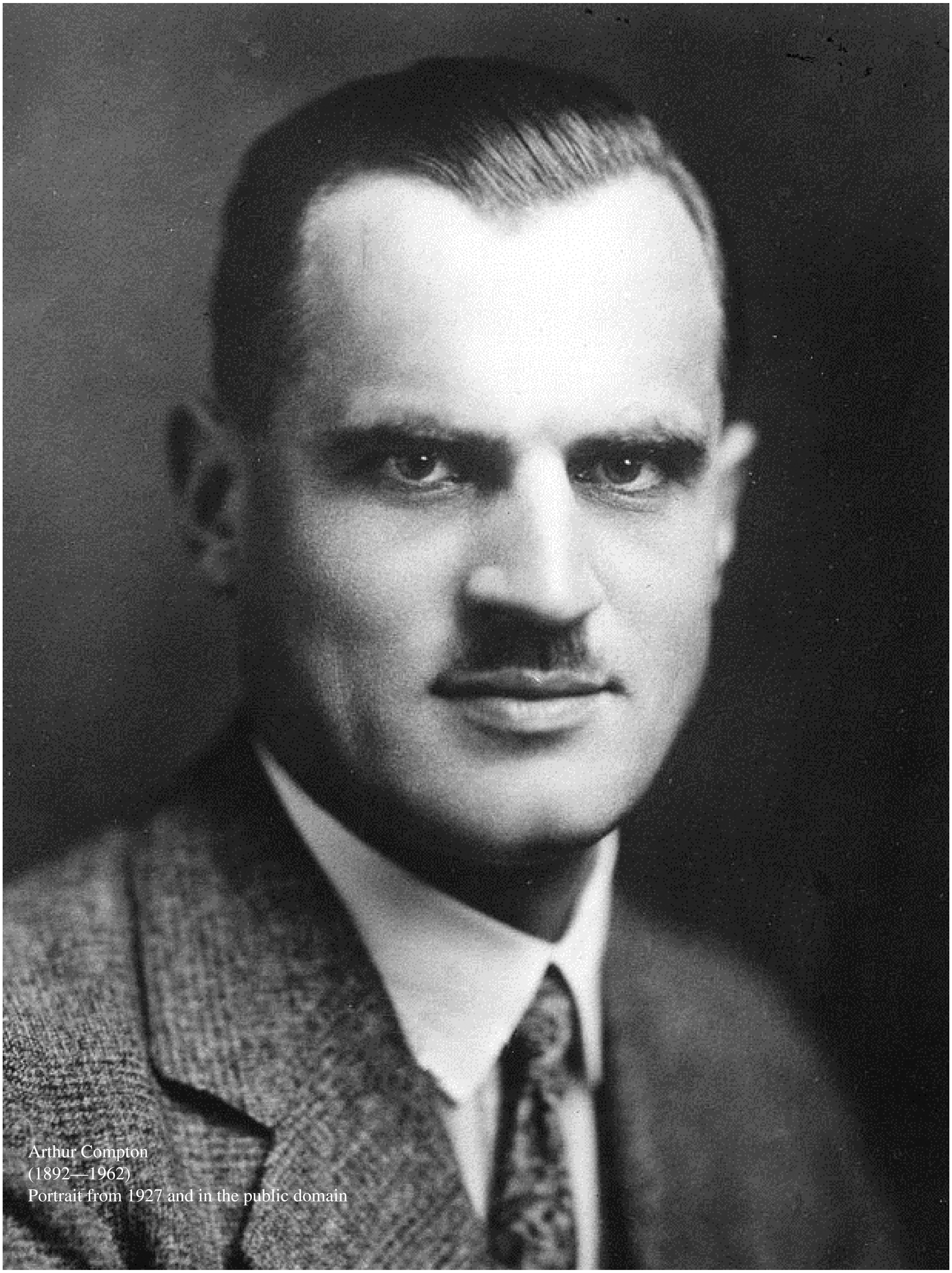


- To probe the scales of things with sizes larger than a virus (100nm), it is sufficient to use visible light. This is why microscopes work down to scales just around that of living cells and bacteria. Below that size, it's impossible to "resolve" features with visible light.
- To probe viruses, DNA, hemoglobin, or macromolecules, you need x-rays (10nm down to 1nm).
- To probe atoms and molecules, you need very short x-rays (pushing down to 0.01nm). The scale of atoms is about $10^{-10}\mathrm{m}$. X-rays can be comparable to, and smaller than, the size of atom, so the particle-like aspects of light emerge naturally at this scale.
- To probe the nucleus of the atom, at sizes around $10^{-15}\mathrm{m}$ (femtometers), you need gamma rays or smaller.

Review

In this lecture, we have learned...

- About the nature of "x-rays"
- About the production of x-rays
- About the scattering of x-rays by matter, and the implications for the nature of electromagnetic radiation



Arthur Compton
(1892—1962)
Portrait from 1927 and in the public domain

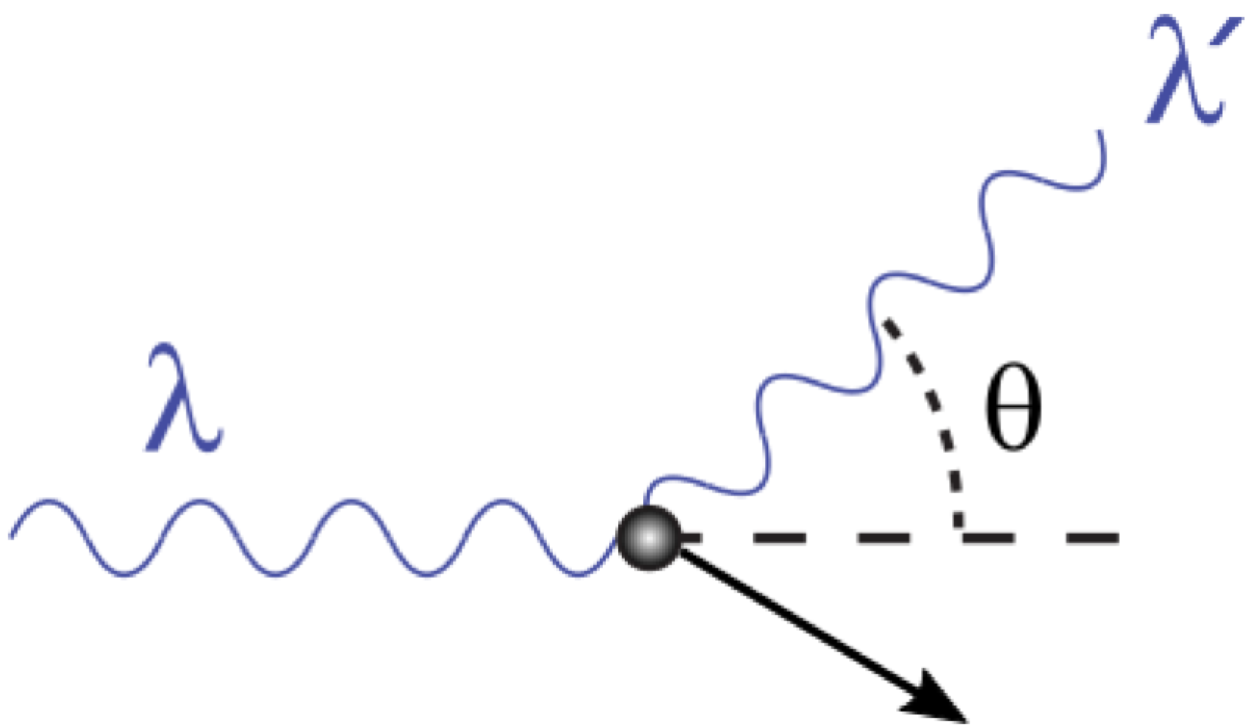
n.2 Problem Solving in Compton Scattering

Problem Solving in Compton Scattering

Instructor Problem: Tuning the X-ray Detector

You are conducting an experiment in which you scatter x-rays off a silicon semi-conducting wafer. The beam enters the system entirely along the x-axis, and is designed to have an energy of 19keV. Answer the following questions.

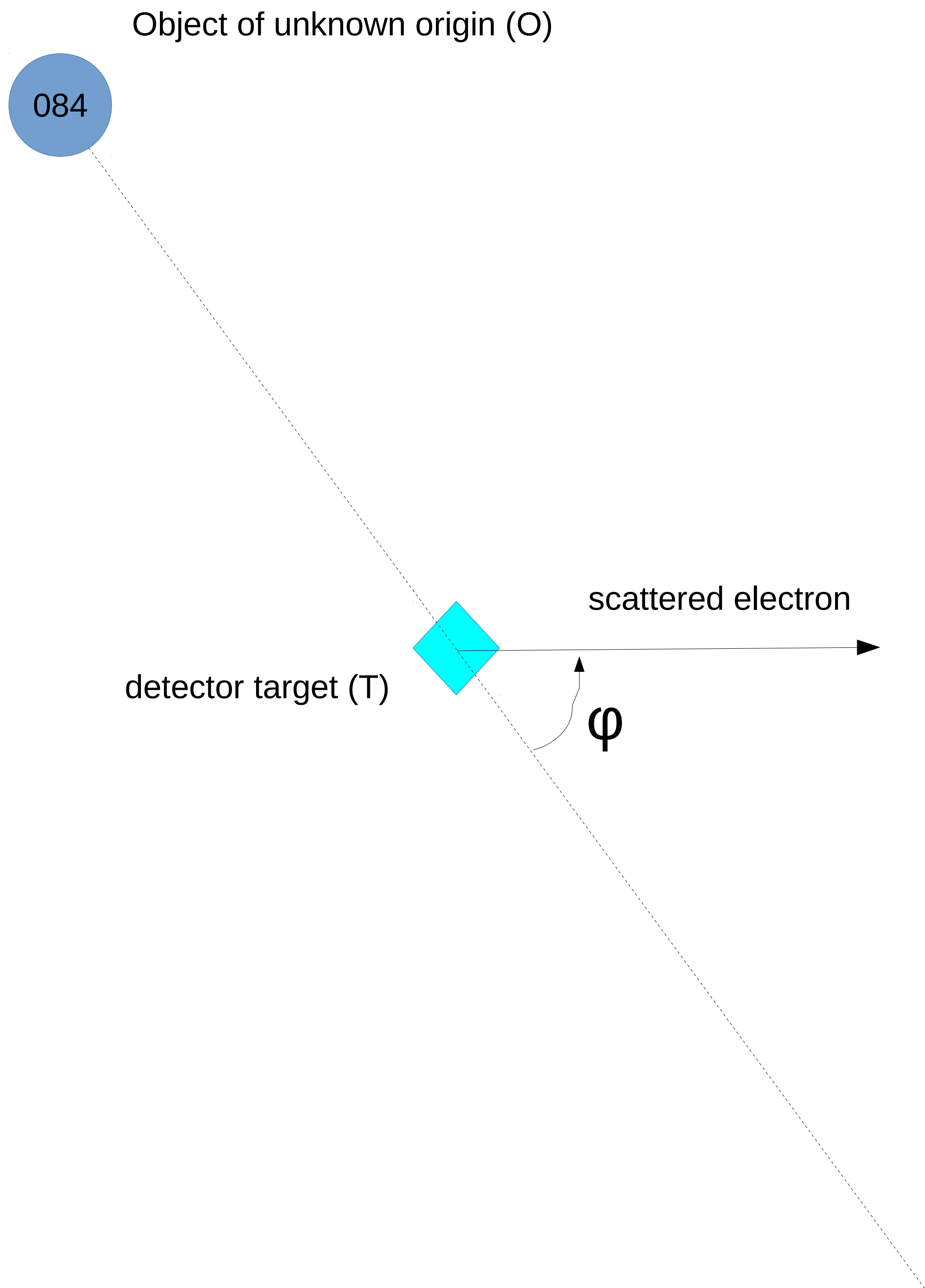
1. What is the wavelength of the x-rays in the incoming beam?
2. If your detector system is optimal at observing outgoing x-rays, scattered by electrons in the silicon, with wavelengths of 0.067nm, at what angle to the incident beam must you place your detector?
3. What is the kinetic energy of the scattered electron?



Student Problem: Emitter of Unknown Origin

Your S.H.I.E.L.D. science team has been sent in to investigate an object of unknown origin (designated "084"). Early and quick observations revealed that it is emitting a great deal of x-ray radiation. Your team is to place an electron detector in the vicinity of the object to assess the specific nature of the radiation. Your detector presents a target to the object and observes that electrons from the target are scattered at a speed of $4.5 \times 10^7 \mathrm{m/s}$ at an angle of $\varphi = 60°$ with respect to the line between the object and the target (designated "OT").

1. What is the momentum of the electron parallel to the line OT?
2. What is the momentum of the electron perpendicular to OT?
3. What is the kinetic energy of the ejected electron?
4. Since you don't know the angle at which the x-ray scatters, can you use momentum conservation to eliminate this unknown?
5. Since you don't know the wavelength of the scattered x-ray, λ_f, can you use momentum and energy conservation together to eliminate this unknown?
6. Having found ways to eliminate these unknowns, what is the wavelength, λ_i, of the x-ray radiation emitted by the source?


Object of unknown origin (O)
084
scattered electron
detector target (T)
φ

o Radiation and Matter - Part II

o.1 The Wave Nature of Matter

The Wave Nature of Matter

Overview

In this lecture, we will learn…

- About the structure of the atom as it was known in the late 1800s and very early 1900s
- About how matter itself can have wave aspects to its behavior
- About Louis de Broglie's experimentally verified conjectures about the wave properties of matter
- About how to conduct experiments to reveal the wave aspects of matter's behavior

Louie de Broglie
(1892—1987)
Photo from 1929 and available from Wikipedia

Wave and Particle Aspects of Electromagnetic Radiation - a Review

Wave description of light (Maxwell's Equations): spatially and temporally distributed phenomenon.

$$\begin{aligned}\left(c_0^2\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\vec{E} &= 0\\ \left(c_0^2\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\vec{B} &= 0\end{aligned}$$

Solution in vacuum (empty space):

$$\begin{aligned}\vec{E}(\vec{x},t) &= \vec{E}_0\cos\Big(\hat{k}\cdot\vec{x} - c_0 t\Big)\\ \vec{B}(\vec{x},t) &= \frac{1}{c_0}(\hat{k}\times\vec{E})\cos\Big(\hat{k}\cdot\vec{x} - c_0 t\Big)\end{aligned}$$

Energy (E) per unit area (A) of an EM wave in empty space:

$$E/A = \frac{1}{c\mu_0}\frac{E_0^2}{2}$$

Particle description of light: definite energy and momentum, localization in space and time.
Einstein special relativistic description of massless phenomena:

$$E = pc$$

Planck/Einstein description of light interacting with matter:

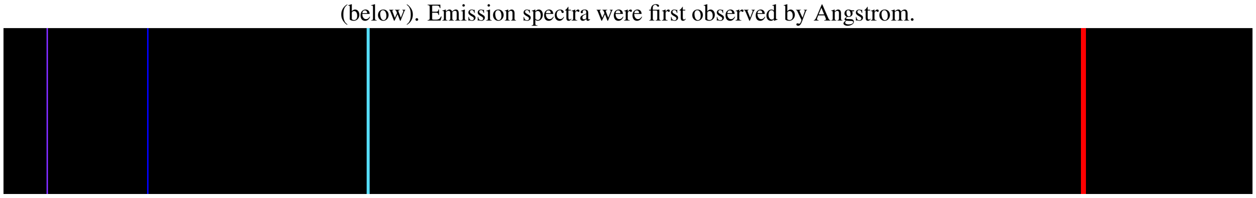
$$E = hf = \frac{hc}{\lambda} \longrightarrow p = \frac{h}{\lambda}$$

Particle vs. Wave - a Cartoon

But...what if?

But, what if matter also had wave properties? Nobody had directly observed them in the early 1900s, though there were hints. The fact that light emitted from heated atoms - the *atomic spectrum* - only contained certain colors and not others, depending on the element, was already a source of curiosity and investigation.

The emission spectrum of Hydrogen gas, revealed by scattering off an optical disk (above) and shown more clearly all in a plane (below). Emission spectra were first observed by Angstrom.



Anders Ångström
(1814—1874)
Photo available from Wikipedia

The Discovery of the Atom

In early 1800s John Dalton discerns that elements have weights, in proportion to Hydrogen, unique to each element and that elements can only combine in certain proportions into final products, with proportions in between not being allowed.

Thomson discovers the electron in 1897 when he estimates the mass of “Cathode Rays” to be 1000 times less than that of Hydrogen. He observes that they possess of electric charge. In 1905 he proposes a model of the atom as a central positive charge embedded in a mass of negative charges, the electrons.

John Dalton
(1766—1844)
Portrait from 1834 available from Wikipedia

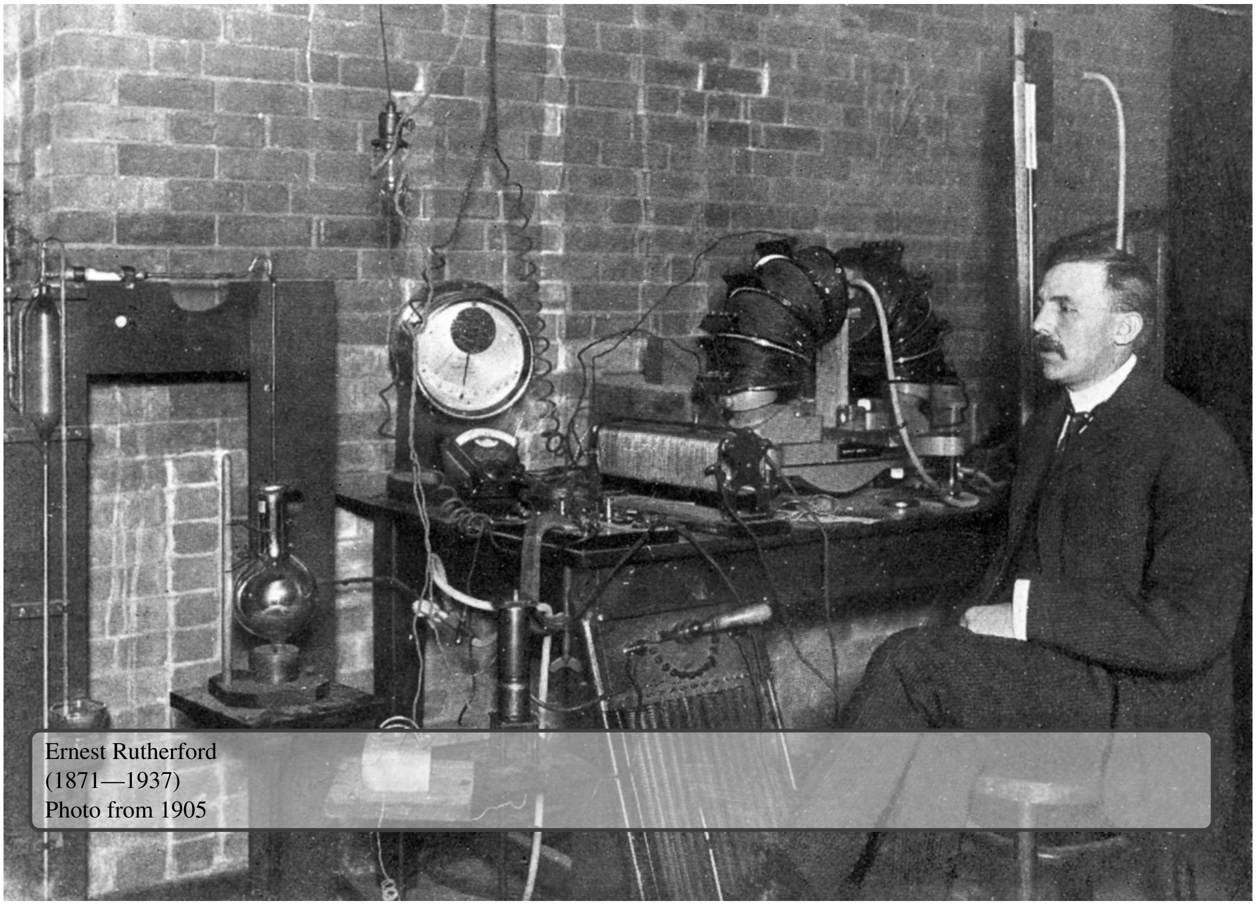
Joseph John (J.J.) Thomson
(1856—1940)
Photo available from Wikipedia

The Discovery of the Atom

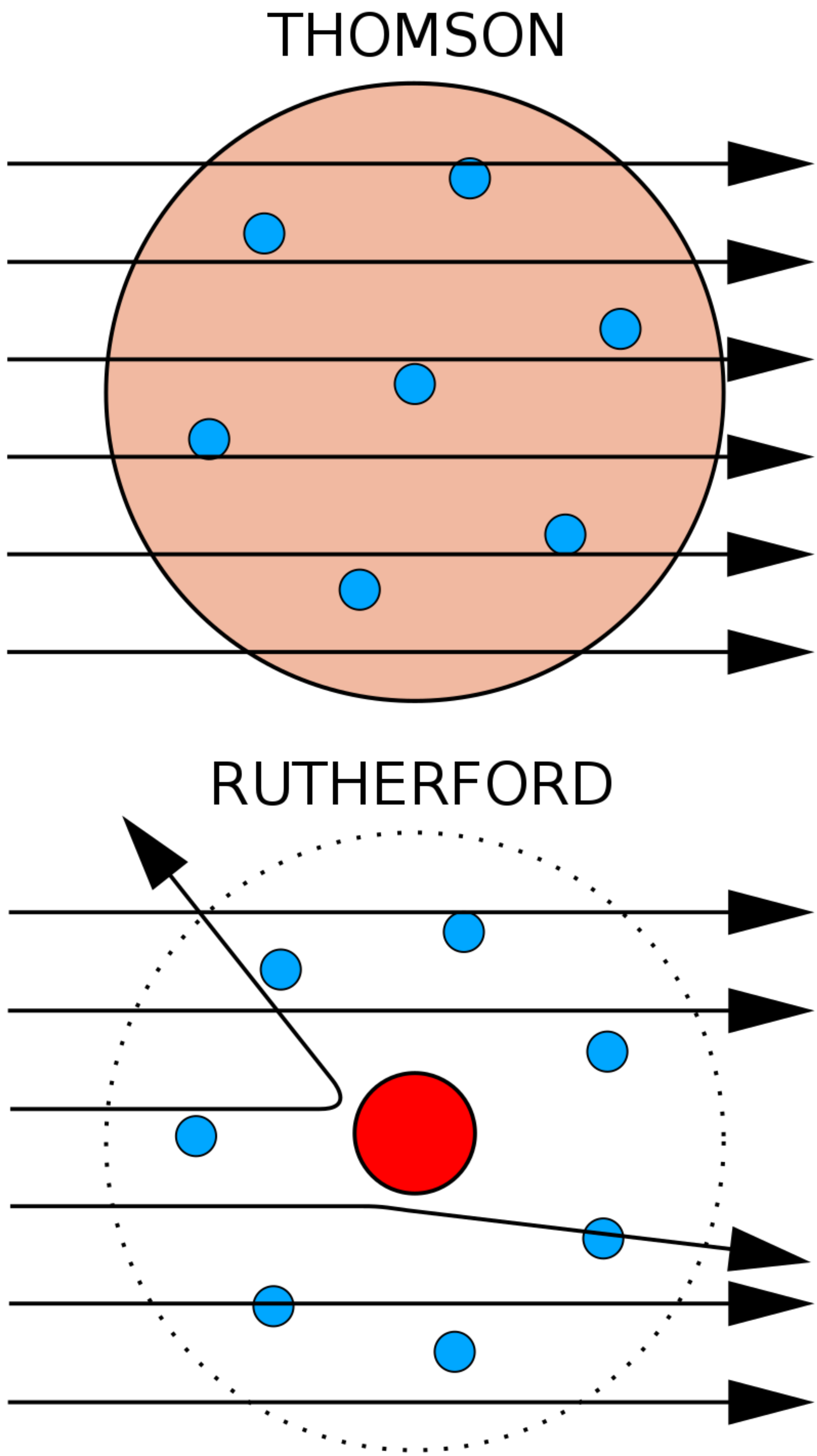
Marie and Pierre Curie, among others, come to understand that radioactive decay of elements results in new kinds of radiation. These would be classified into *alpha, beta, and gamma* radiation. Alpha radiation would be revealed to be ejected Helium nuclei consisting of two protons bound to two neutrons — heavy and able to penetrate into materials.

Ernest Rutherford, Hans Geiger, and Ernest Marsden scatter alpha radiation off metallic targets and reveals the existence of a tightly packed nucleus, wherein all the positive charge resides, deep inside each atom. This results in Rutherford reformulating the model of the atom in 1911 to what is now called *The Rutherford Model*.

Marie and Pierre Curie
(1766—1844)
Photo from 1903



Ernest Rutherford
(1871—1937)
Photo from 1905

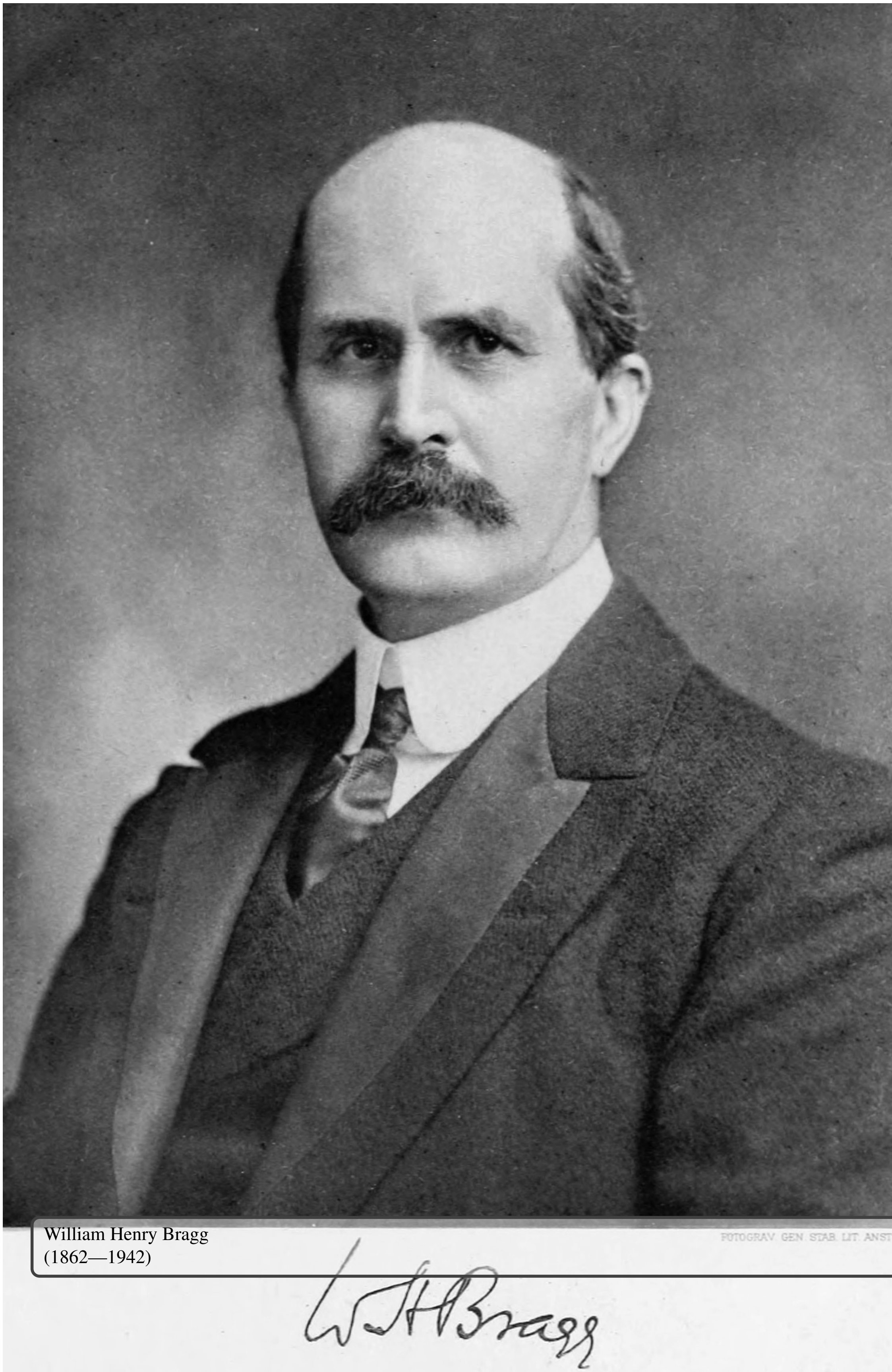

THOMSON
RUTHERFORD

The Discovery of the Atom

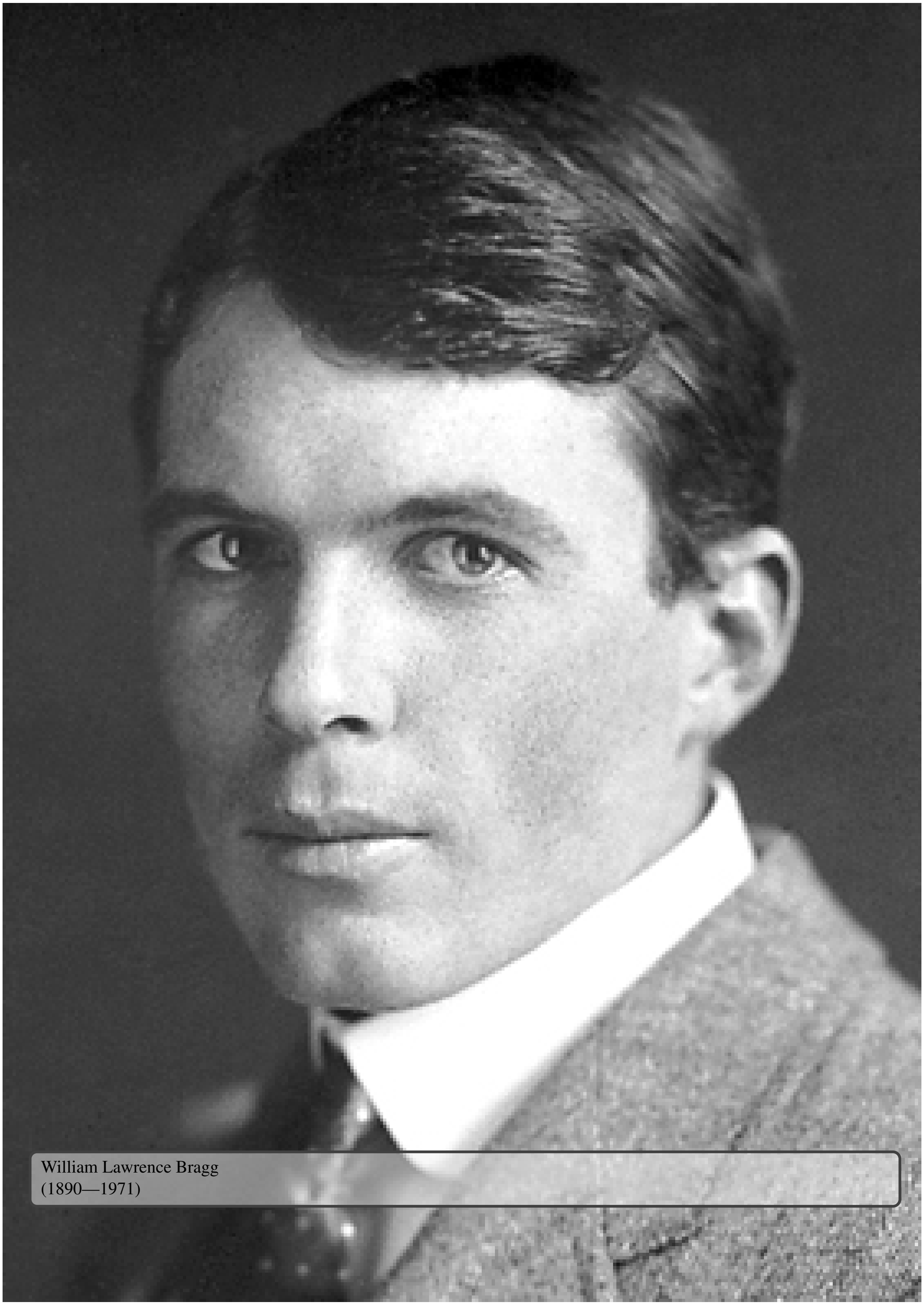
Scattering x-rays ($\lambda = 0.1\mathrm{nm}$) off crystalline solids (e.g. table salt, NaCl) it is observed that specific patterns appear in the scattered x-rays. Father and son team William Henry Bragg and Lawrence Bragg explain the origin of this from the perspective of crystals being regular arrangements of atoms with separation, d, between plans of atoms.

Photos of Braggs from 1915, available on Wikipedia

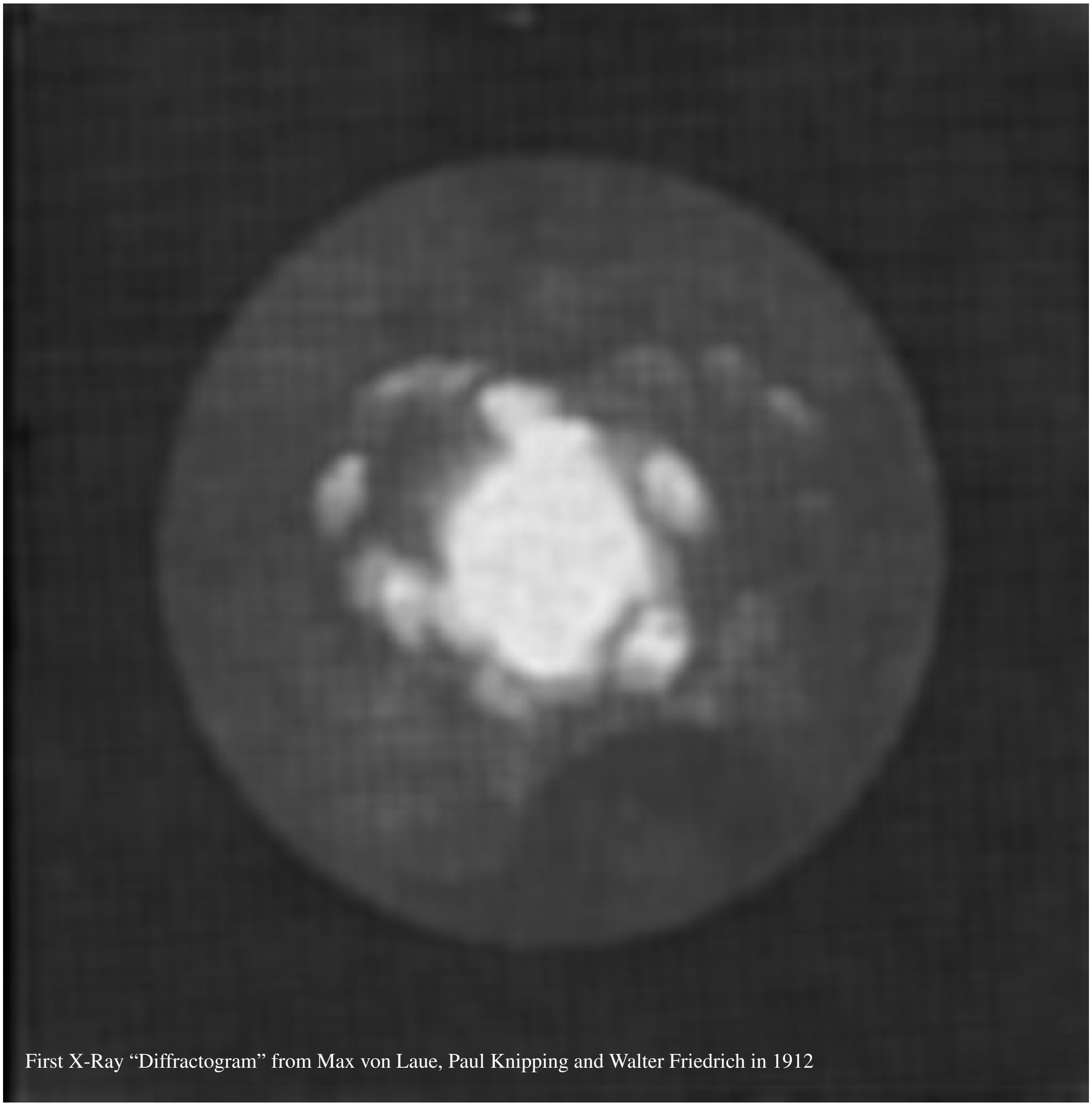
This led to the ability to determine the sizes of atoms, using x-ray diffraction patterns.



William Henry Bragg
(1862—1942)

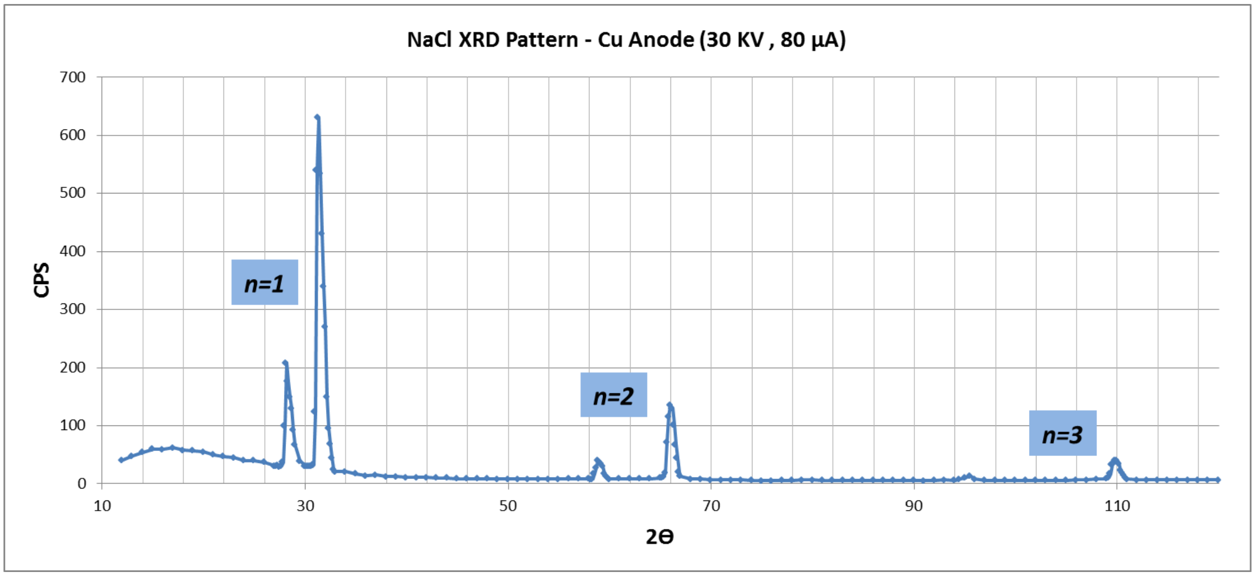


William Lawrence Bragg
(1890—1971)



First X-Ray "Diffractogram" from Max von Laue, Paul Knipping and Walter Friedrich in 1912

NaCl x-ray diffraction pattern

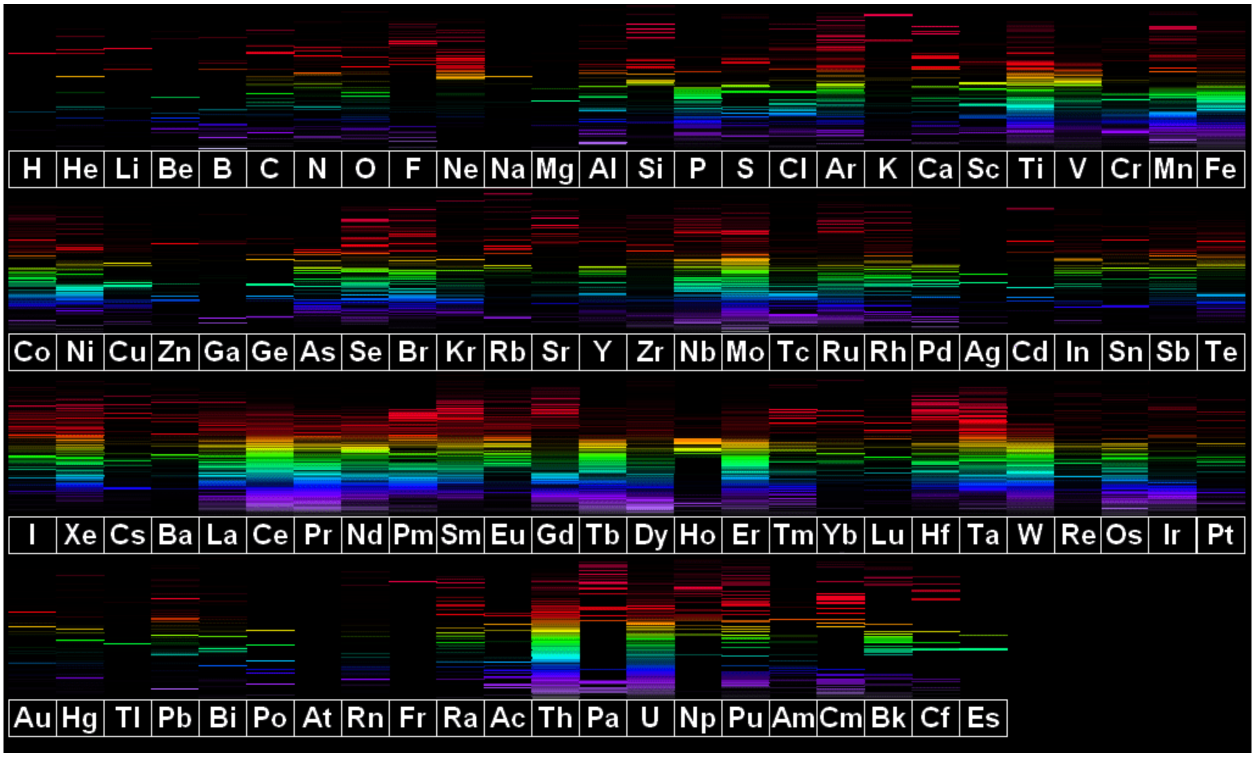


The copper x-ray emitter generates two x-ray wavelengths: Cu $K\alpha$, with $\lambda = 0.15406\text{nm}$, and $K\beta$, with $\lambda = 0.13923\text{nm}$. Using the above data, we can estimate $d = 0.28\text{nm}\ldots$ the spacing in NaCl is about the same scale as the x-ray wavelengths.

What can we learn about atoms and atomic spectra?

Atomic emission spectra are discrete. It's as if only certain energies are permitted to the electrons in an atom. Electrons in atoms behave like guitar strings: only certain frequencies are permissible from a string once you set its tension. Only certain energies seem possible from a specific atom.

We might conclude that when electrons are confined to spaces of the volume of an atom, $r \approx 0.1\text{nm}$, wave-like properties of the electron emerge. This could explain atomic spectra, and would be consistent with the observation that oscillating charges in a blackbody cavity have only certain allowed oscillation frequencies.



If Matter can also be Wave-like, What Determines the Wave Properties of Matter?

In his 1924 Ph.D. thesis, French physicist Louis de Broglie postulated that matter also has wave properties. Drawing from Planck's relationship between energy and frequency for light, and the relationship between momentum and wavelength that results from special relativity, de Broglie asserted the hypothesis that the same would be true for matter:

$$E = hf_{matter}, \; p = \frac{h}{\lambda_{matter}}$$

How would one prove this? Bragg scattering offers the possibility to test this hypothesis. We could compute the matter wave properties of electrons and then find a system off of which we might scatter them. Just as x-rays scattered from crystals allows the wave nature of x-rays to reveal the structure of crystals, once we know the structure of crystals we can look at electron scattering to see if it reveals the wave properties of electrons.

Estimating the wavelength and frequency of a matter wave

Consider the electron with mass $m_e = 9.11 \times 10^{-31}$ kg. Imagine accelerating it to a momentum $p_e = \gamma_e m_e u_e$. By de Broglie's postulates, $p_e = h/\lambda_e$. What momentum would we need to accelerate an electron to to probe the scale of a crystal, with spacing at the level of 0.1nm? We want to get the wavelength down to the same level, $\lambda \approx 0.1$nm.

That will require $p_e \approx 7 \times 10^{-24}$kg $\cdot$ m/s. What voltage would be needed to achieve that for an electron? [bh] This corresponds to a gamma factor of $\gamma_e = 1.0003$ and a kinetic energy of about 2×10^{-17}J.

[bh] Use special relativity to answer these questions: $E = \gamma_e m_e c^2 = \sqrt{p_e^2 c^2 + m_e^2 c^4}$, and $K_e = (\gamma_e - 1) m_e c^2$

A voltage of $K = |eV| \longrightarrow |V| = K/|e| = 150\text{V}$. No problem!

Lester Germer (right) with Clinton Davisson performed exactly this scattering experiment in 1927. They used a Nickel target.

Do the Wave Properties of Electrons Manifest in Scattering from Nickel?

Just as in x-ray scattering, if you scan over the scattering angle of the electrons from the crystal, and if wave properties manifest, then constructive and destructive wave interference should occur at different angles, for a fixed wavelength (and thus a fixed momentum):

$$n\lambda_e = 2d\sin\theta$$

The wavelength is determined by the momentum of the electron, $\lambda_e = h/p_e$. Thus, instead of scanning over scattering angle, you can observe the intensity of scattered electrons at a fixed angle and vary the momentum by varying the accelerating electric potential difference (the voltage, V). As you scan over voltage, sometimes you will make the electrons have the right wavelength to interfere constructively at the observing angle, sometimes destructively. This is what Davisson and Germer did, and here is what they saw:

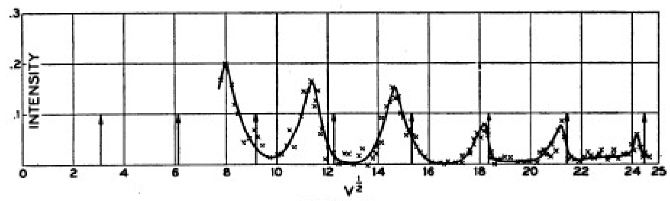
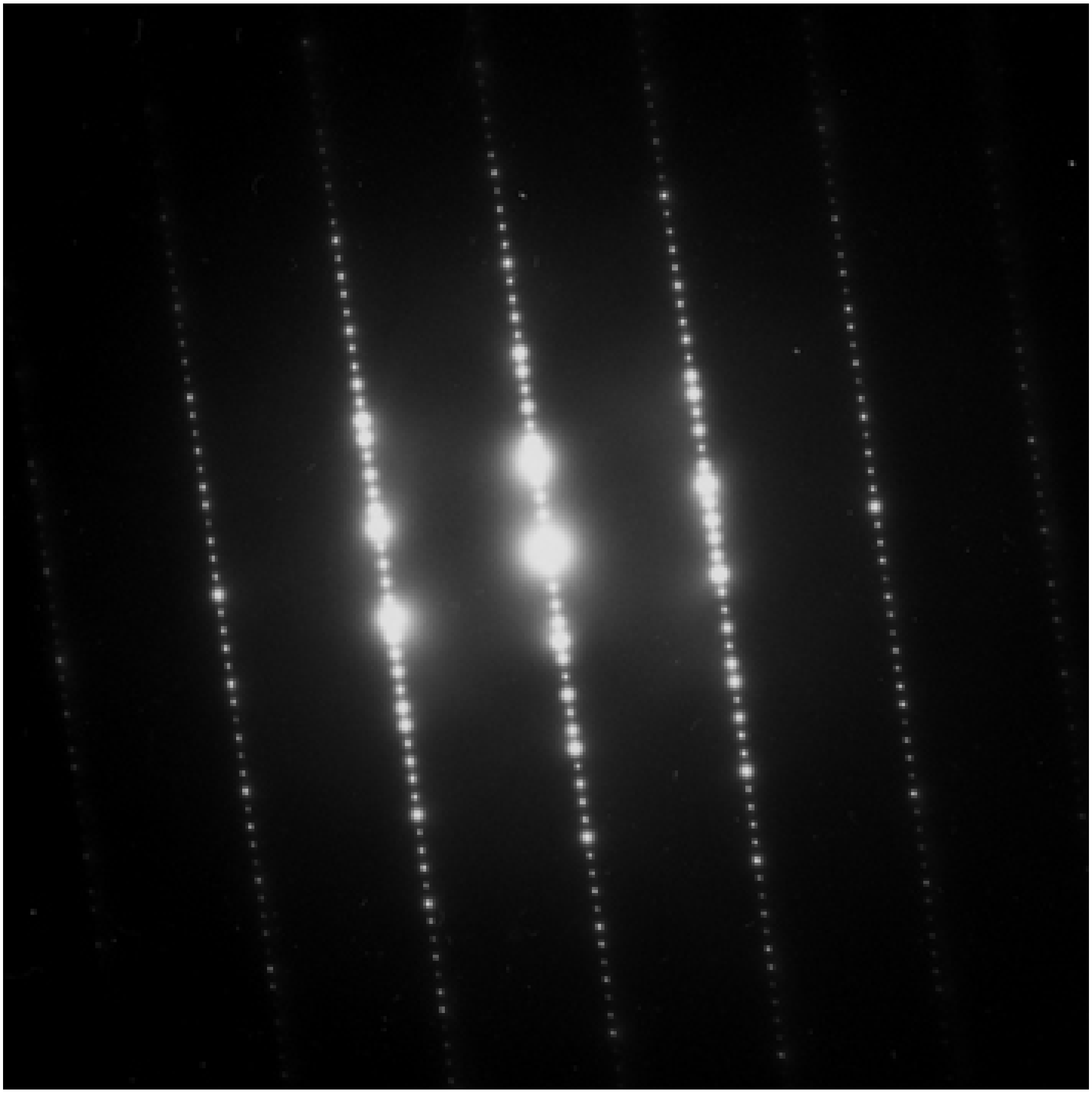


FIGURE 3
Variation of the intensity of the regularly reflected electron beam with bombarding potential, for 10° incidence—Intensity vs. $V^{1/2}$.

An Electron Diffraction Pattern

In two-dimensions we can see what an electron scattering diffraction pattern looks like. The bright spots show you where the electrons constructively interfered. Only waves can interference with each other like this. The only reason we get this behavior, however, is because crystalline solids have structures of size comparable to the *matter wavelengths* of modestly accelerated electrons. 

What Scattering and Interference Tell us About the True Nature of Matter and Radiation

Electromagnetic Radiation already has a wave equation that describes its wave nature: Maxwell's Equations. *What is the wave equation that describes the wave nature of matter?* Electric and Magnetic fields are the solutions to Maxwell's Equations. *What will the solutions to the matter wave equation look like?*

These are excellent questions. Physicists in the 1920s struggled with this issue. We have some hints for ourselves already: the solutions to the matter wave equation will be *probabilistic in nature*. The intensity of a scattering pattern seems to have everything to do with the probability of finding a particle at a certain location in space after the scattering process has occurred.

Review

In this lecture, we have learned…

- About the structure of the atom as it was known in the late 1800s and very early 1900s
- About how matter itself can have wave aspects to its behavior
- About Louis de Broglie's experimentally verified conjectures about the wave properties of matter
- About how to conduct experiments to reveal the wave aspects of matter's behavior

Louie de Broglie
(1892—1987)
Photo from 1929 and available from Wikipedia

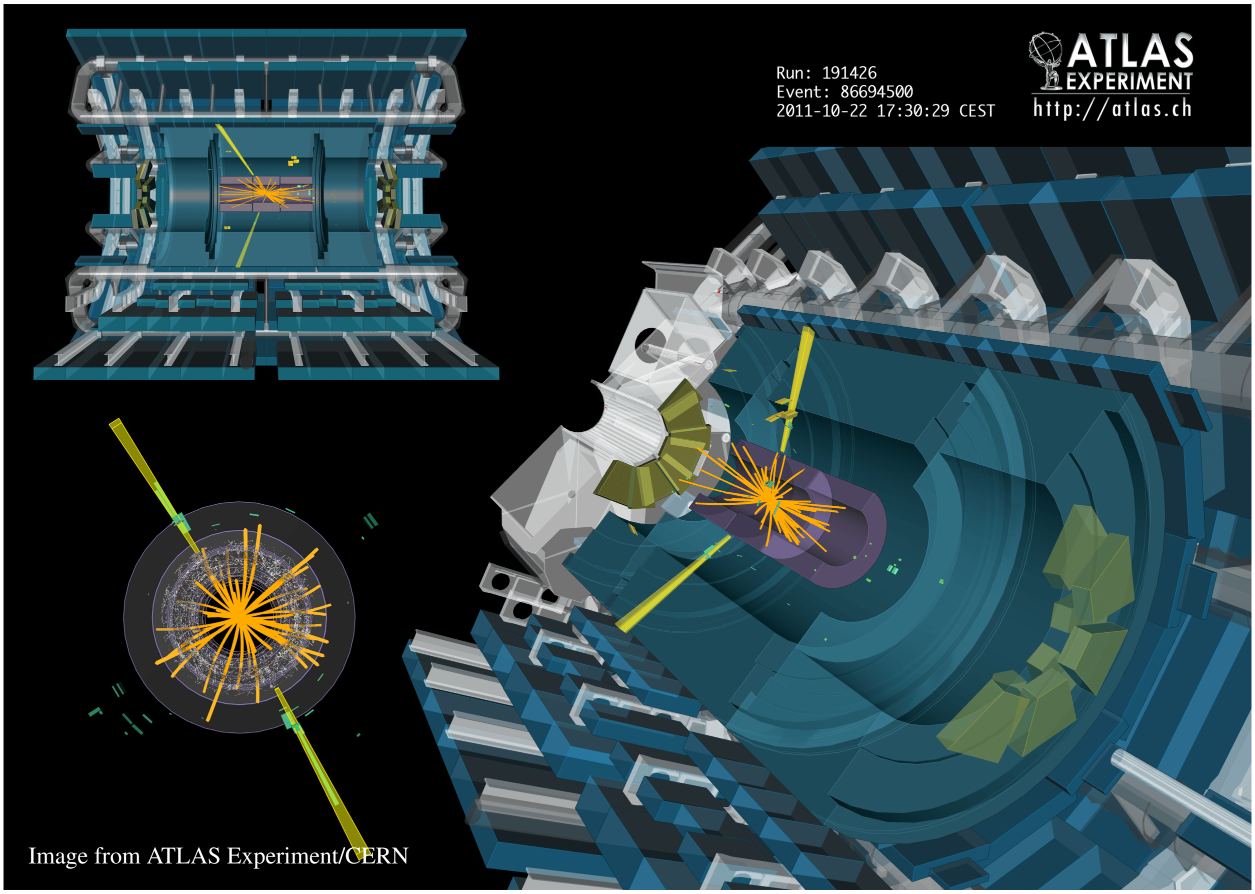
o.2 Problem Solving in Matter Waves

Problem Solving in Matter Waves

Instructor Problem: Protons in the Large Hadron Collider

Protons in the Large Hadron Collider are accelerated to kinetic energies of 6.5 TeV. Protons have rest masses of $1.67 \times 10^{-27}\mathrm{kg}$. Answer the following questions about the protons, taking into account their wave aspects:

1. What is the momentum of a single proton?
2. What is the matter wave wavelength of a single proton?
3. What is the total energy of a single proton?
4. What is the matter wave frequency of a single proton?
5. Treating matter waves like light waves, which can only "resolve" structures no smaller than their wavelength, can a proton with these properties "resolve" the nucleus of an atom ($\approx 10^{-15}\mathrm{m}$ in size)? How about an electron (whose "charge radius" is currently known to be no bigger than $10^{-18}\mathrm{m}$)?



Student Problem: Your Matter Wavelength and Frequency

1. Using your mass, estimate your matter wavelength and frequency, treating your entire body as a single particle (as opposed to a collection of many particles) and assuming you are walking with a speed of 3.0 mph.
2. Wave diffraction occurs when a wave passes through an opening whose size is comparable to the wavelength. You walk through a door of width 81cm. What is the maximum speed you could walk and still experience diffraction?
3. How long would it take you to walk through the door at the maximum speed you just calculated if the thickness of the door frame is 12cm?
4. Do you need to worry, in general, about what happens to lasers going through small slits (see next slide) also happening to you walking through doors?


Photo by Arek Adeoye on Unsplash

Student Problem: Further Investigation of the Sample of Unknown Origin

Your S.H.I.E.L.D. science team has brought back to the laboratory the object of unknown origin (designated "084") discussed in a previous in-class problem. It is determined to be crystalline in nature, meaning it appears to be made from atoms organized in structured layers. You scatter electrons off the sample, using a beam of electrons with kinetic energy of exactly $8\ \mathrm{keV}$. Your team observes that, relative to the surface of the crystal, the electrons form a brightest scattering intensity at 0.606 degrees and a second-brightest scattering intensity at 1.21 degrees.

1. What is the wavelength of an electron in the beam?

2. What is the frequency of an electron in the beam?
3. What is the spacing between atoms in the crystal?
4. To what crystal might this sample correspond (see table)?
5. How would you confirm your answer to question 3 using the information about the scattered electrons?

Table 1. Lattice Constants and Crystal Structures of some Semiconductors and Other Materials

Element or Compound	Type	Name	Crystal Structure	Lattice Constant at 300 K (Å)
C	Element	Carbon (Diamond)	Diamond	3.56683
Ge	Element	Germanium	Diamond	5.64613
Si	Element	Silicon	Diamond	5.43095
Sn	Element	Grey Tin	Diamond	6.48920
SiC	IV-IV	Silicon carbide	Wurtzite	a=3.086; c=15.117
AlAs	III-V	Aluminum arsenide	Zincblende	5.6605
AlP	III-V	Aluminum phosphide	Zincblende	5.4510
AlSb	III-V	Aluminum antimonide	Zincblende	6.1355
BN	III-V	Boron nitride	Zincblende	3.6150
BP	III-V	Boron phosphide	Zincblende	4.5380
GaAs	III-V	Gallium arsenide	Zincblende	5.6533
GaN	III-V	Gallium nitride	Wurtzite	a=3.189; c=5.185
GaP	III-V	Gallium phosphide	Zincblende	5.4512
GaSb	III-V	Gallium antimonide	Zincblende	6.0959
InAs	III-V	Indium arsenide	Zincblende	6.0584
InP	III-V	Indium phosphide	Zincblende	5.8686
InSb	III-V	Indium antimonide	Zincblende	6.4794
CdS	II-VI	Cadmium sulfide	Zincblende	5.8320
CdS	II-VI	Cadmium sulfide	Wurtzite	a=4.160; c=6.756
CdSe	II-VI	Cadmium selenide	Zincblende	6.050
CdTe	II-VI	Cadmium telluride	Zincblende	6.482
ZnO	II-VI	Zinc oxide	Rock Salt	4.580
ZnS	II-VI	Zinc sulfide	Zincblende	5.420
ZnS	II-VI	Zinc sulfide	Wurtzite	a=3.82; c=6.26
PbS	IV-VI	Lead sulfide	Rock Salt	5.9362
PbTe	IV-VI	Lead telluride	Rock Salt	6.4620

Table from eesemi.com

The Schrödinger Wave Equation

p.1 Wave Equations

Overview

In this lecture, we will learn…

- About mechanical and electromagnetic wave equations.
- How to infer the nature of the wave equation for matter.
- The meaning of the waves described by the matter wave equation, the Schrödinger Wave Equation.
- The limits of absolute knowledge imposed by the wave nature of matter.


Erwin Schrödinger
(1887—1961)
Photo from 1914 and available from Wikipedia

Waves in Classical Mechanics

A wave is a kind of *oscillatory phenomenon* that can be described by time and space-dependent functions. In introductory physics, we learn that a time-varying oscillation long one dimension (e.g. mass on the end of a spring) that is *simple-harmonic* in nature can be described by:

$$x(t) = x_0 \cos(\omega t - \phi)$$

where ω is the *angular frequency*, given in terms of the period of oscillation, T, by $\omega = 2\pi/T$, and in terms of the frequency of oscillation by $\omega = 2\pi f$. Angular frequency is the rate of angular displacement. For all considerations here, let's set the phase angle, ϕ, to zero. If you extend the phenomenon to two dimensions, so that it is a distortion in y that travels along x in time, that solution looks like this:

$$y(x,t) = y_0 \cos(kx - \omega t)$$

where k is known as the *wave number*, defined by $k = 2\pi/\lambda$, and describes the number of cycles per unit distance in the phenomenon.

Waves in Classical Mechanics

But if these functions are the answer to a question, what question are they answering? They are solutions to a *wave equation*: an equation that describes how changes in space relate to changes in time. The one-dimensional mechanical wave equation is:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Try applying this equation to the solution on the previous slide:

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2}{\partial t^2}\left(y_0 \cos(kx - \omega t)\right) = \frac{\partial}{\partial t}\left(\omega y_0 \sin(kx - \omega t)\right) = -\omega^2 y(x,t)$$

$$c^2 \frac{\partial^2 y}{\partial x^2} = c^2 \frac{\partial^2}{\partial x^2}\left(y_0 \cos(kx - \omega t)\right) = c^2 \frac{\partial}{\partial x}\left(-k y_0 \sin(kx - \omega t)\right) = -c^2 k^2 y(x,t)$$

This yields $\omega^2 = c^2 k^2 \longrightarrow c = \omega/k$. The speed of a mechanical wave is given by the ratio of the angular frequency and wave number. A dedicated waves course would spend more time on this.

Energy in a Mechanical Wave

Let's stick with a mechanical wave and think about the energy contained in that wave. A mechanical wave is a distortion of a physical medium. For instance, think about a string of total mass, M, and length, L, with a uniform distribution of mass, $\mu = M/L$ (linear mass density). We vibrate the string, such that a given part of it at time, t, and location, x, will have a small mass m with vertical velocity v_y, transverse to its length. That tiny chunk of the string will have length dx and mass $m = \mu dx$. The kinetic energy of that chunk will be defined by its mass and its velocity at a given moment in time, t:

$$dK = \frac{1}{2}(\mu dx) v_y^2 = \frac{1}{2}(\mu dx)\left(\frac{\partial y(x,t)}{\partial t}\right)^2 = \frac{1}{2}(\mu dx)\left(y_0 \omega \sin(kx - \omega t)\right)^2$$

The potential energy stored in that same chunk of mass will depend on the elasticity of the string. If we think of this as a tiny mass, m, connected to its neighbor masses by little springs with spring constants κ, then as in introductory mechanics $\omega^2 = \kappa/m \rightarrow \kappa = m\omega^2$. The little bit of potential energy stored at x is:

$$dU = \frac{1}{2}\kappa y(x,t)^2 = \frac{1}{2}(m\omega^2) y(x,t)^2 = \frac{1}{2}(\mu dx \omega^2) y(x,t)^2 = \frac{1}{2}(\mu dx \omega^2)\left(y_0 \cos(kx - \omega t)\right)^2$$

Energy in a Mechanical Wave

The total energy stored in mass m at location x and at time t is then:

$$\begin{aligned} dE &= dK + dU = \frac{1}{2}(\mu dx)\left(y_0 \omega \sin(kx - \omega t)\right)^2 + \frac{1}{2}(\mu dx \omega^2)\left(y_0 \cos(kx - \omega t)\right)^2 \\ &= \frac{1}{2} y_0^2 \omega^2 (\mu dx) \end{aligned}$$

This is just the energy stored in this little piece of mass, m, at (x, t) along the string in time. Note that the total energy depends on the *square of the angular frequency*. The presence of the ω^2 multiplier tells us something about the number of time derivatives, or the product of a number of time derivatives, that had been present in the original equation for the total energy.

Key Take-Aways from Mechanical Waves

- The mechanical wave equation relates the *second-derivative* with respect to space and the *second-derivative* with respect to time.
- Solutions to the wave equation, when acted upon by the derivatives in the wave equation, yield *squares* of the angular frequency and the wave number.
- The energy equation for the wave, or a part of the wave, is sensitive to the number of time (or space) derivatives in the underlying equation; these manifest as multipliers like ω^2.
- One might think about the presence of the squares of these quantities as *indicative of the underlying wave equation to have been solved*.

Waves in Electromagnetism

As in mechanical phenomena, Maxwell's Equations lead to another wave equation - this one for electric and magnetic fields, describing how they propagate in empty space. The wave equation is [bh]:

$$\left(c_0^2 \nabla^2 - \frac{\partial^2}{\partial t^2}\right) \vec{E} = 0$$

The solution is:

$$\vec{E}(\vec{x}, t) = \vec{E}_0 \cos\left(\hat{k} \cdot \vec{x} - c_0 t\right)$$

Applying the wave equation to the solution similarly yields quadratic multipliers of k^2 and $c_0^2 = \omega^2 / k^2$. This latter relationship turns out to be a direct consequence of the massless nature of light, allowing its speed to be related to its frequency and wavelength.

[bh] Yes, there is an identical equation for magnetic fields but it's redundant as the magnetic field travels perpendicular to the electric field and its strength is related to the electric field.

Electromagnetic Waves and their Energy and Momentum

The relationship between frequency and wavelength for a wave can be directly related to the energy present in the radiation quantum, the photon. For example, from above (and from our previous look at EM waves) we know that $c_0 = \lambda f = (2\pi)\lambda f(2\pi)^{-1} = \omega/k$. The energy of the quantum is $E = hf = (2\pi)hf(2\pi)^{-1} = \hbar\omega$. Here, $\hbar = h/(2\pi)$ - the *reduced Planck's constant*, which is convenient for all these angular wave concepts. Momentum of the quantum is given by $p = h/\lambda = \hbar k$. Thus $c_0 = (E/\hbar)(\hbar/p) \rightarrow E = pc_0$. We recover the Einstein energy relation for light, a massless phenomenon.

Overview of Wave Equations in Classical Mechanics and Electromagnetism

- They involve second-derivatives of both space and time.
- We can infer that from the results of the wave equations, as their application results in squares of time and spatial frequencies.
- Energy equations tell us the proportionality of frequency to wavelength, etc.
- This leaves you wondering: what is the equation that describes *matter waves*? Since its presence was not inferred directly from previous measurements, can we infer its form from what we know about particles and waves, as revealed by atomic spectra, the blackbody radiation spectrum, and the photoelectric effect?

Hints of the Matter Wave Equation

One can glimpse hints of the matter wave equation, like seeing the shadow of a greater thing cast on the wall, by considering the conservation of energy for a particle acted upon by an external force. Such a particle would have kinetic energy and potential energy. Conservation of energy would then require that:

$$E = K + U \xrightarrow{\text{classical physics}} \frac{1}{2}mv^2 + U = \frac{1}{2}\frac{m}{m}mv^2 + U = \frac{p^2}{2m} + U$$

Let us now inject de Broglie's postulates into this equation: $E = hf = \hbar\omega$ and $p = h/\lambda = \hbar k$. We now obtain the shadow of the matter wave equation:

$$\hbar\omega = \frac{1}{2m}\hbar^2 k^2 + U$$

ω is an indicator that a single time-derivative has acted on a solution to the equation, while k^2 indicates a second-derivative with respect to space has acted on the same solution to the equation.

Hints of the Matter Wave Equation

$$\hbar\omega = \frac{1}{2m}\hbar^2 k^2 + U$$

ω is an indicator that a single time-derivative has acted on a solution to the equation, while k^2 indicates a second-derivative with respect to space has acted on the same solution to the equation.

Let's try inserting this hypothesis into the above energy conservation equation with its matter-wave components:

$$\hbar\frac{\partial\Psi(x,t)}{\partial t} = \frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + U$$

This looks promising! It has all the hallmarks of a wave equation, but it's different from Maxwell's Equations or a Mechanical Wave in one key way: *it has a second-derivative in space, but only single-derivative in time.* This has implications for the kinds of functions that can solve such an equations. These solutions are denoted $\Psi(x,t)$.

Exploring Solutions to this Equation...and a Hint of Missing Piece

Let's begin by guessing the form of the solutions to our equation. To simplify matters, let's consider for now *free particles*, with no external forces acting on them. This means $U = 0$. To solve the equation, we need a kind of function that, when acted on by a derivative transmutes into another version of itself. For traditional waves, we used sines or cosines. Let's guess $\Psi(x,t) = A\cos(kx - \omega t)$.

$$\begin{aligned}
\hbar\frac{\partial\Psi(x,t)}{\partial t} &= \frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} \\
\hbar\frac{\partial}{\partial t}\left(A\cos(kx-\omega t)\right) &= \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left(A\cos(kx-\omega t)\right) \\
\hbar\omega A\sin(kx-\omega t) &= -\frac{\hbar^2 k^2}{2m}A\cos(kx-\omega t)
\end{aligned}$$

No good. It doesn't work to solve the equation. The left side does not match the right (one is a sine, one a negative cosine). Trying just a sine function will similarly fail. What about a *superposition* of sine and cosine?

Exploring Solutions to this Equation...and a Hint of Missing Piece

Try $\Psi(x,t) = A\cos(kx - \omega t) + A\sin(kx - \omega t)$:

$$
\begin{aligned}
\hbar\frac{\partial\Psi(x,t)}{\partial t} &= \frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} \\
\hbar\frac{\partial}{\partial t}\left(A\cos(kx-\omega t)+A\sin(kx-\omega t)\right) &= \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left(A\cos(kx-\omega t)+A\sin(kx-\omega t)\right) \\
\hbar\omega\left(A\sin(kx-\omega t)-A\cos(kx-\omega t)\right) &= \frac{\hbar^2}{2m}k^2\left(-A\cos(kx-\omega t)-A\sin(kx-\omega t)\right) \\
\hbar\omega\left(-A\cos(kx-\omega t)+A\sin(kx-\omega t)\right) &= \frac{\hbar^2}{2m}k^2\left(-A\cos(kx-\omega t)-A\sin(kx-\omega t)\right)
\end{aligned}
$$

Close, but still no good. The problem is that we keep generating a minus sign from the single-derivative of only the cosine, but the second-derivative on the right-hand side generates minus signs for both terms in the solution. We need to get rid of a minus sign, and when you get stray minus signs like this it helps to remember a special number: the imaginary number, i.

An Aside: The Imaginary Number, i

Let's recall the imaginary number, i. It's a special kind of 1. It's defined as:

$$i = \sqrt{-1}$$

This has consequences. For example:

$$i \cdot i = i^2 = \sqrt{-1}\sqrt{-1} = -1 \longrightarrow i^2 = -1.$$

The presence of extraneous minus signs when trying to solve equations using functions, as we are doing with the matter wave equation, can be an indicator: you are trying *real solutions*, using only real numbers, but maybe you need to solve the problem using *complex numbers*, those containing both real and imaginary numbers, e.g. $z = x + iy$.

Exploring Complex Solutions to this Equation

Try $\Psi(x,t) = A\cos(kx-\omega t) + Ai\sin(kx-\omega t)$:

$$
\begin{aligned}
\hbar\frac{\partial\Psi(x,t)}{\partial t} &= \frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} \\
\hbar\frac{\partial}{\partial t}\left(A\cos(kx-\omega t)+Ai\sin(kx-\omega t)\right) &= \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left(A\cos(kx-\omega t)+Ai\sin(kx-\omega t)\right) \\
\hbar\omega\left(A\sin(kx-\omega t)-Ai\cos(kx-\omega t)\right) &= \frac{\hbar^2}{2m}k^2\left(-A\cos(kx-\omega t)-Ai\sin(kx-\omega t)\right) \\
\hbar\omega\left(-Ai\cos(kx-\omega t)+A\sin(kx-\omega t)\right) &= \frac{\hbar^2}{2m}k^2\left(-A\cos(kx-\omega t)-Ai\sin(kx-\omega t)\right)
\end{aligned}
$$

Ridiculously close to working, but something is still missing. Let's try to move some minus signs around to see if we can find a clue to resolve this puzzle.

Exploring Complex Solutions to this Equation

$$\begin{aligned}
\hbar\omega\left(-Ai\cos(kx-\omega t)+A\sin(kx-\omega t)\right) &= \frac{\hbar^2}{2m}k^2\left(-A\cos(kx-\omega t)-Ai\sin(kx-\omega t)\right)\\
-\hbar\omega\left(-Ai\cos(kx-\omega t)+A\sin(kx-\omega t)\right) &= \frac{\hbar^2}{2m}k^2\left(A\cos(kx-\omega t)+Ai\sin(kx-\omega t)\right)\\
\hbar\omega\left(Ai\cos(kx-\omega t)-A\sin(kx-\omega t)\right) &= \frac{\hbar^2}{2m}k^2\left(A\cos(kx-\omega t)+Ai\sin(kx-\omega t)\right)
\end{aligned}$$

There is the clue. The difference between the function on the left-hand side (the thing in parentheses) and the function on the right-hand side is a missing factor of $-i$ on the left. If we we traded our original left-hand-side of the wave equation as follows:

$$\hbar\frac{\partial}{\partial t} \quad\rightarrow\quad -i\hbar\frac{\partial}{\partial t}$$

Then we'd solve our problem. To wit...

Exploring Complex Solutions to this Equation

Try $\Psi(x,t)=A\cos(kx-\omega t)+Ai\sin(kx-\omega t)$ in our modified wave equation:

$$\begin{aligned}
-i\hbar\frac{\partial\Psi(x,t)}{\partial t} &= \frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2}\\
-i\hbar\frac{\partial}{\partial t}\left(A\cos(kx-\omega t)+Ai\sin(kx-\omega t)\right) &= \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left(A\cos(kx-\omega t)+Ai\sin(kx-\omega t)\right)\\
-i\hbar\omega\left(A\sin(kx-\omega t)-Ai\cos(kx-\omega t)\right) &= \frac{\hbar^2}{2m}k^2\left(-A\cos(kx-\omega t)-Ai\sin(kx-\omega t)\right)\\
-i\hbar\omega\left(-Ai\cos(kx-\omega t)+A\sin(kx-\omega t)\right) &= \frac{\hbar^2}{2m}k^2\left(-A\cos(kx-\omega t)-Ai\sin(kx-\omega t)\right)\\
-i\hbar\omega\left(Ai\cos(kx-\omega t)-A\sin(kx-\omega t)\right) &= \frac{\hbar^2}{2m}k^2\left(A\cos(kx-\omega t)+Ai\sin(kx-\omega t)\right)\\
\hbar\omega\left(A\cos(kx-\omega t)+Ai\sin(kx-\omega t)\right) &= \frac{\hbar^2}{2m}k^2\left(A\cos(kx-\omega t)+Ai\sin(kx-\omega t)\right)
\end{aligned}$$

AHEM. *BAZINGA.*

Lessons from the Free-Particle Matter-Wave Equation

Starting from the total energy of a free particle, $\hbar\omega=\hbar^2k^2/2m$, we constructed a wave equation that had the right time and space derivatives in it to return these factors of ω and k^2. We played with oscillatory solutions and learned that only *complex functions* will satisfy an equation like this. From this, we inferred a missing imaginary number from our original guess at the equation. This yields:

$$-i\hbar\frac{\partial\Psi(x,t)}{\partial t} = \frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2}$$

The solutions to this free-particle equation are of the form:

$$\Psi(x,t) = A\cos(kx-\omega t) + Ai\sin(kx-\omega t)$$

Plugging them into the wave equation yields:

$$\hbar\omega = \frac{\hbar^2 k^2}{2m}$$

which is *precisely* the energy conservation equation from which we started. It's self-consistent!

The (One-Dimensional) Schrödinger Wave Equation

$$-i\hbar\frac{\partial\Psi(x,t)}{\partial t} = \frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x,t)\Psi(x,t)$$

This is one of the most important equations in history (adding back the potential energy term to complete the equation for a particle): *The (One-Dimensional) Schrödinger Wave Equation*. It is named after Erwin Schrödinger, the first person (1926) to fully determine the form of the matter wave equation.

Don't be deceived. The equation itself may seem a little daunting because of its need for complex numbers and its combination of space and time derivatives. But the real challenge of this equation is not in the equation itself, but rather in the *finding of solutions, $\Psi(x,t)$, to this equation*. We have effectively solved the *free-particle case*, and we'll explore the solutions to that (and their implications) in a moment. However, if a potential-energy term is present such that $V(x,t)\Psi(x,t) \to U$, we have to rework the problem to find the correct solution, and that always depends on the functional form that describes the potential, $V(x,t)$.

The Free-Particle Solutions

We can already learn a lot from this most basic scenario - a particle moving at constant speed free from external forces.

- This is a *wave equation,* which means the solutions will not have definite localization - the solutions will describe a phenomenon not specifiable to any one location in space, etc.
- The solutions describe something that is *oscillating*. What is it? Mechanical waves represent variations in a medium; electromagnetic waves are variations in the strength of electric and magnetic fields. What is waving in a matter wave?
- To better understand these solutions, we need to confront the mathematics of these complex functions a bit more closely.
- Don't be daunted by the presence of complex numbers. They are a representation of information that is necessary when a problem has too much information to be described only by one kind of number. It's a way of representing a thing with two components, to double the available information.

The Free-Particle Solutions: Complex Functions

Our working solution to the free-particle wave equation is:

$$\Psi(x,t) = A\cos(kx-\omega t) + Ai\sin(kx-\omega t)$$

This is a function of the form $z = x + iy$, a complex structure with a real part (x) and an imaginary part (y). Observations of the natural world are conventionally described by real numbers, not imaginary ones. We are used to dealing with vectors, e.g. $z = x\hat{i} + y\hat{j}$ and from those we are comfortable summarizing the information content of a vector using the concept of *length* or *magnitude*, a single real number. How does one get a single real number out of a complex one?

You might try z^2, but this yields $x^2+2ixy-y^2$. This is still a complex number, so it won't work. Instead, to get a real number you need to do this: $zz^* = (x+iy)(x-iy) = x^2+y^2$. That yields something more consistent with, say, the Pythagorean Theorem about the length of a vector with both real and imaginary components. The notation z^* denotes the "complex conjugate" of the number z. The complex conjugate is obtained by trading $i \to -i$. A short-hand notation for zz^* to indicate that it's the "square of the real length" is $|z|^2 \equiv zz^*$.

The Free-Particle Solutions: Sines, Cosines, and Exponential Functions

Writing out the sines and cosines in our complex function representing the free-particle solutions to the Schrödinger Wave Equation is clunky. Mathematics offers us a more compact representation of the same information: the exponential function. For example, consider a Taylor Expansion of the sine function $\sin(x)$:

$$\sin(x) \xrightarrow{\text{Taylor Expansion}} x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

and similarly

$$\cos(x) \xrightarrow{\text{Taylor Expansion}} 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

and recall the Taylor Expansion of the exponential function, e^x:

$$e^x \xrightarrow{\text{Taylor Expansion}} 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

The Free-Particle Solutions: Sines, Cosines, and Exponential Functions

$$e^x \xrightarrow{\text{Taylor Expansion}} 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

Note that this expansion involves the *sum* of a bunch of powers of x; the sine and cosine expansions have alternating sums and *subtractions*. If we think creatively for a moment, recalling that $i^2 = -1$, we can re-write terms like $-x^2$ as $i^2x^2 = (ix)^2$, while terms like $-x^3$ can be re-written as i^2x^3, such that if we have $-ix^3 = i^3x^3 = (ix)^3$. With those things in mind, recall our free-particle solutions are of the form $A(\cos(x) + i\sin(x))$:

$$
\begin{aligned}
A\left(1-\frac{x^2}{2!}\right)+iA\left(x-\frac{x^3}{3!}\right) &= A\left(1+ix+\frac{(ix)^2}{2!}+\frac{(ix)^3}{3!}+\ldots\right) \\
&= Ae^{ix}
\end{aligned}
$$

Thus the free-particle solutions can be re-written in a more compact form:

$$\Psi(x,t) = A\cos(kx-\omega t) + Ai\sin(kx-\omega t) = Ae^{i(kx-\omega t)}$$

The Free-Particle Solutions: Magnitude

What is the magnitude of our free-particle solution? Keep in mind that we don't know if the constant out in front of the function, A, is real or complex. Let's try it:

$$
\begin{aligned}
|\Psi(x,t)|^2 &= \Psi(x,t)\Psi^*(x,t) \\
&= \left(Ae^{i(kx-\omega t)}\right)\left(A^*e^{-i(kx-\omega t)}\right) \\
&= AA^*e^{i(kx-\omega t)-i(kx-\omega t)} = AA^* = |A|^2
\end{aligned}
$$

The measure of this function is a real number, $|A|^2$. But what is it that we have just evaluated? What is this function that solves the wave equation, and what is the meaning of its length?

The Free-Particle Solutions: Probability Density

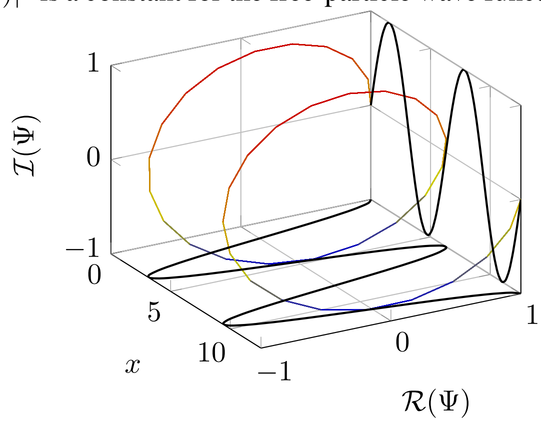
One has to interpret these functions and their meaning. In the history of physics, it is likely this step that has caused the most hand-rubbing and consternation, the most bitter disagreements and strong opinions…all over a function whose direct value has no physical meaning (because it is based, in part, on imaginary numbers). The most practical interpretation, which has also met with the most experimental success since Erwin Schrödinger first published his wave equation, is that of a *probabilistic meaning* to the square of the wave function.

The amplitude-squared, $|\Psi(x,t)|^2$, of the wave function is interpreted as representing a *probability, per unit distance, per unit time* in one-dimension (in two dimensions its *per unit area*, and in three it's *per unit volume*). To obtain raw probabilities, one has to specify the exact conditions under which the free particle is prepared (e.g. its exact position) and then answer questions such as, "What is the probability of finding this particle between 1cm and 2cm from the point of origin?" or "What is the probability of finding the particle a distance of 3cm from the point of origin, 1 second after it starts its journey?" We'll learn later how to get answers to questions like this.

Answering questions about free particles

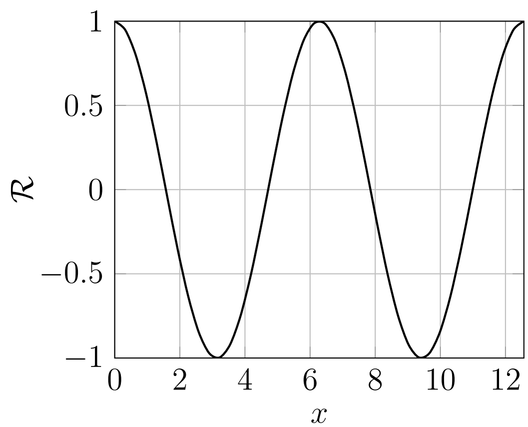
Let's visualize the solution by defining a real number axis; an imaginary number axis; and an x-axis that represents the direction of travel of the free particle.

This lets us visualize how the wave function, which is itself not physical (only its amplitude is physical), can be thought about. Remember that at all points $|\Psi(x,t)|^2$ is a constant for the free-particle wave function,



and is the only thing that is "physical" about the wave.

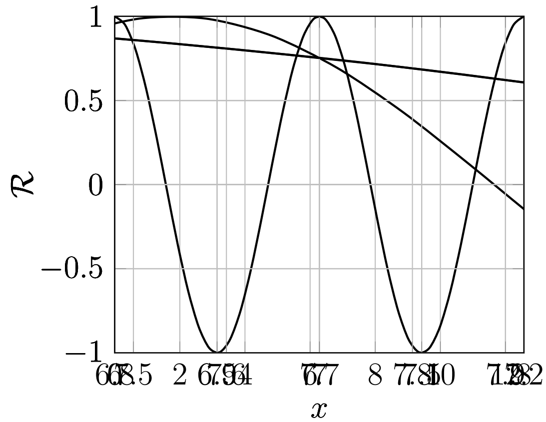
Thinking about what it means to try to measure both position and momentum



Measuring the *position* of a free particle boils down to determining where it is along the x-axis. Measuring the *momentum* of the same particle boils down to determining the curvature of the wave at a given point, e.g. $\partial^2/\partial x^2$ (which, when applied to the wave function, returns k^2, the square of the wave number, which provides a measure of the momentum).

As one locates the position of a particle more accurately (e.g. narrowing the location of a particle to a range $[x, x + \delta x]$, making $\delta x \to 0$), it becomes harder and harder to establish the curvature of the wave… and, thus, the momentum of the wave.

Thinking about what it means to try to measure both position and momentum



Measuring the *position* of a free particle boils down to determining where it is along the x-axis. Measuring the *momentum* of the same particle boils down to determining the curvature of the wave at a given point, e.g. $\partial^2/\partial x^2$ (which, when applied to the wave function, returns k^2, the square of the wave number, which provides a measure of the momentum).

As one locates the position of a particle more accurately (e.g. narrowing the location of a particle to a range $[x, x + \delta x]$, making $\delta x \to 0$), it becomes harder and harder to establish the curvature of the wave… and, thus, the momentum of the wave.

I actually changed the wavelength of the free particle on that last graph. You were staring at the wave itself and probably assumed the curvature was still the same, but it wasn't. Knowing the position too well comes at the cost of knowing the momentum.

The Heisenberg Uncertainty Principle

- Knowing the position very well comes at the cost of knowing the momentum with any precision.
- Knowing the momentum very well comes at the cost of knowing the position with any precision.

Werner Heisenberg worked out the mathematics of this in 1927, and it is codified in what is known as the *Heisenberg Uncertainty Principle*:

$$\delta p \delta x \geq \hbar/2$$

You cannot know the position and the momentum at the same time with infinite precision. If you know that $\delta x = 0$, the only way to satisfy the inequality is for $\delta p = \infty$ (and vice versa). This is a limit imposed by the wave nature of matter: you cannot know this pair of variables with simultaneous perfect precision.

Review

In this lecture, we have learned…

- About mechanical and electromagnetic wave equations.

- How to infer the nature of the wave equation for matter.
- The meaning of the waves described by the matter wave equation, the Schrödinger Wave Equation.
- The limits of absolute knowledge imposed by the wave nature of matter.

Erwin Schrödinger
(1887—1961)
Photo from 1914 and available from Wikipedia

p.2 Problem Solving in Complex Numbers

Problem Solving in Complex Numbers

Classroom Discussion

$$-i\hbar\frac{\partial\Psi(x,t)}{\partial t} \quad = \quad \frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x,t)\Psi(x,t)$$

A classroom discussion of the Schroedinger Wave Equation is useful at this stage. I invite your questions, concerns, confusions, etc. Let's air them out now before we begin a many-week deep dive into this equation.

- How does one begin to solve the Schroedinger Wave Equation?
- Why is this wave equation complex but others are not?
- What is the physical interpretation of the wave, if it involves complex functions/numbers?

Student Problem: Exercise in Complex Numbers

Find the real and imaginary parts of the following complex numbers:

1. $11 - 7i$
2. $21i$
3. 5

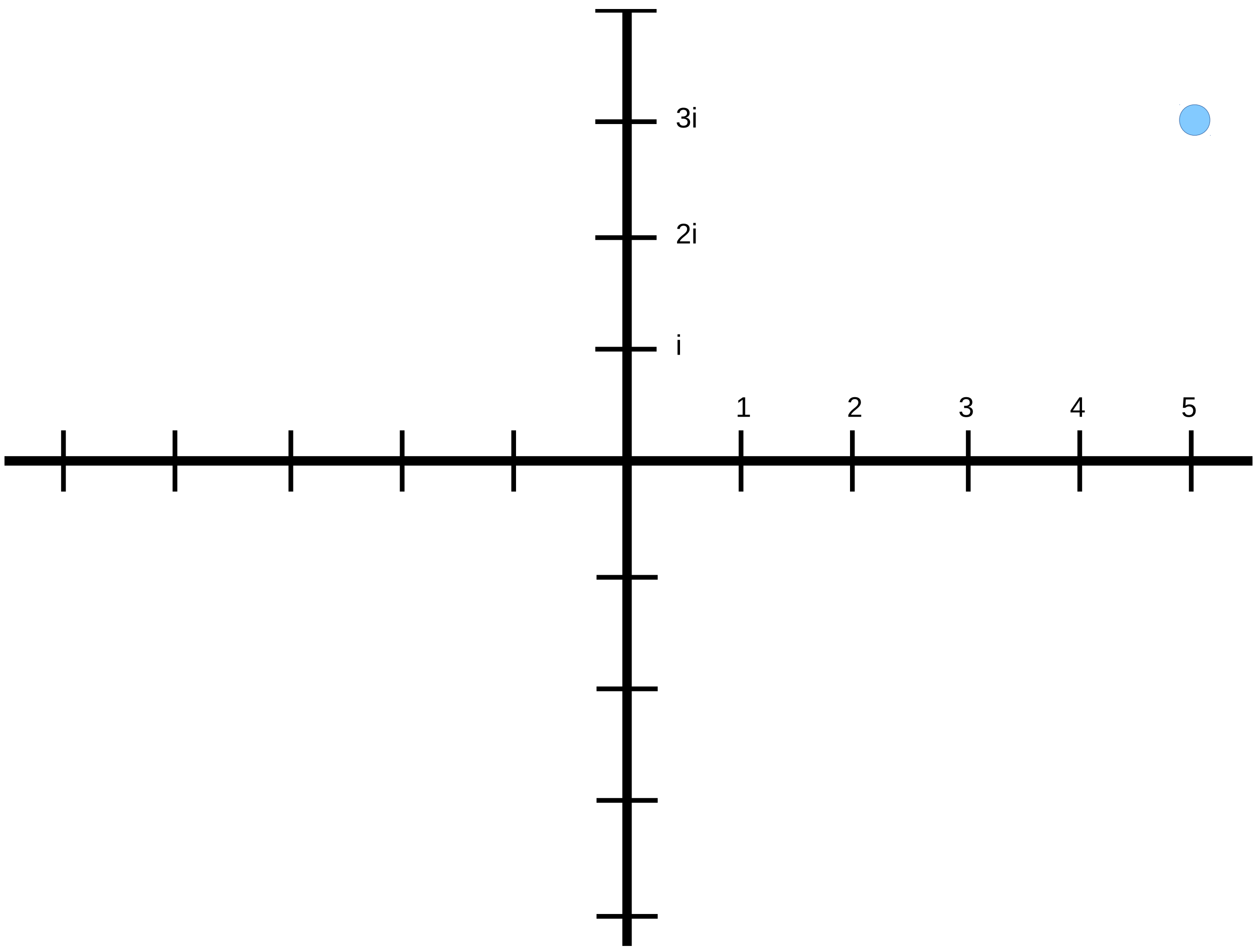
Find as simplified an expression as possible, eliminating all fractions wherever they appear, for the following:

1. $(11 - 7i)(11 - 7i)$
2. $(6 + 3i)(6 - 3i)$
3. $-\frac{1}{i}$

Student Problem: Thinking in the Complex Plane

A complex number, $a+bi$, can be represented in a kind of "real-imaginary" coordinate system in which the real number axis is the horizontal axis; and the imaginary number axis is the vertical axis. Consider the number $5+3i$ represented in such a way (shown right).

1. Redraw this coordinate system on your own paper and represent the following numbers on this system. Ask the instructor to evaluate your work.
   - $(3+4i)$
   - $(-4+3i)$
2. Using pure real-number geometry, determine by what angle you need to rotate $(3+4i)$ to get $(-4+3i)$.
3. Show that this same (counter-clockwise) rotation could have been accomplished merely by multiplying $(3+4i)$ by a single number. What is that number?

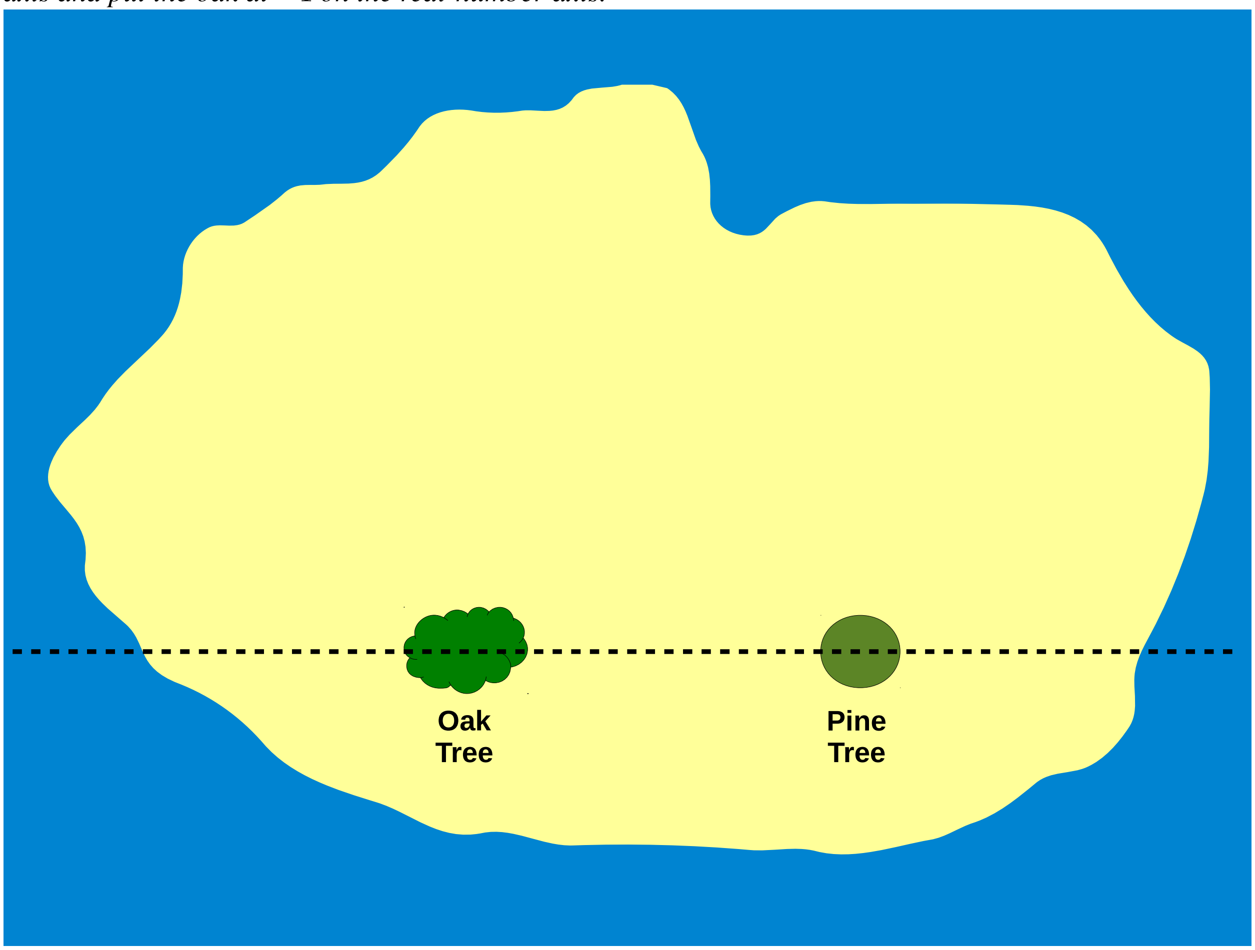


Student Problem: Find the Treasure!

You find an old family heirloom letter that describes a fabulous buried treasure! The note gives the coordinates of an island and says:

> When you arrive at the island you will find a solitary oak tree and a solitary pine tree. You will also see an old wooden house where caretaker lived. Start at the house. Walk to the oak, counting your steps. At the oak, turn *right* by a right angle and take the same number of steps. Place at that spot a stake. Return to the house. Walk to the pine, counting your steps. At the pine, turn *left* by a right angle and take the same number of steps. Place at that second spot a stake. The treasure is buried halfway between the two stakes along the line connecting them.

You arrive at the island, but the house is long gone, eroded by wind and sand. The oak and pine trees still stand, 100m apart. Use complex numbers to find the treasure. *HINT: put the pine at* $+1$ *on the real-number axis and put the oak at* -1 *on the real-number axis.*



Announcements

- Assigned
  - Read Harris, Ch 4.6; watch a lecture video associated with this material.
  - Homework 6: Exercise in matter behaving like waves and in complex numbers [Due Thursday]
- Second exam is the Thursday after Spring Break! More details on Thursday.
- **Always check Canvas for reading, homework, and lecture video assignments!**

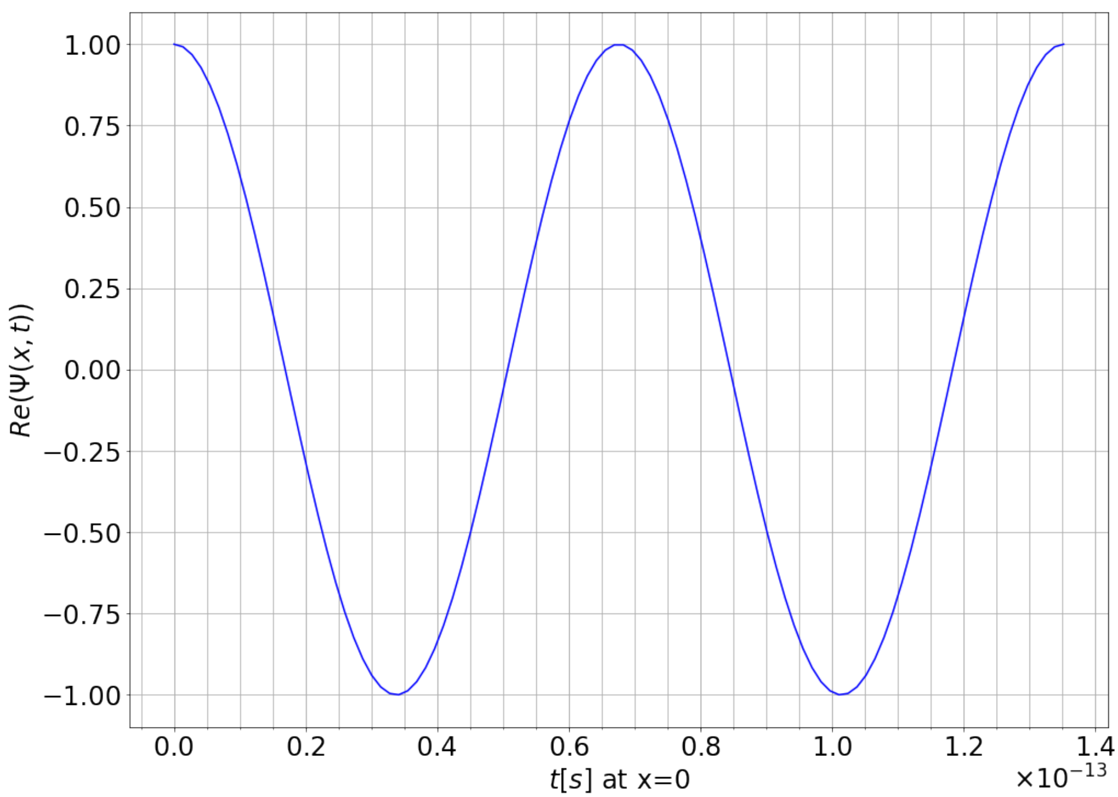
p.3 Problem Solving in Free Particle Matter Waves and the Heisenberg Uncertainty Principle

Problem Solving in Free Particle Matter Waves and the Heisenberg Uncertainty Principle

Instuctor Problem: Electron Matter Wave and Uncertainty

Consider the graph (right) of $Re(\Psi(x,t))$ vs. time (where the spatial coordinate, x, has been fixed to $x = 0$). This represents a part of the free-particle wave function of an electron.

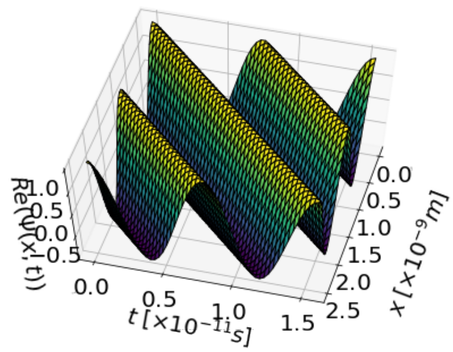
1. What is the frequency of this free-particle wave?
2. What is the angular frequency of this wave?
3. What is the total energy of this wave?
4. If you have localized this electron to less than an atomic diameter, $\delta x < 10^{-10}\mathrm{m}$, what is the minimum speed it can have?

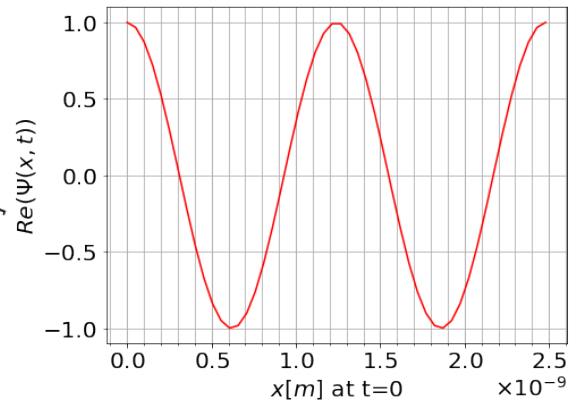


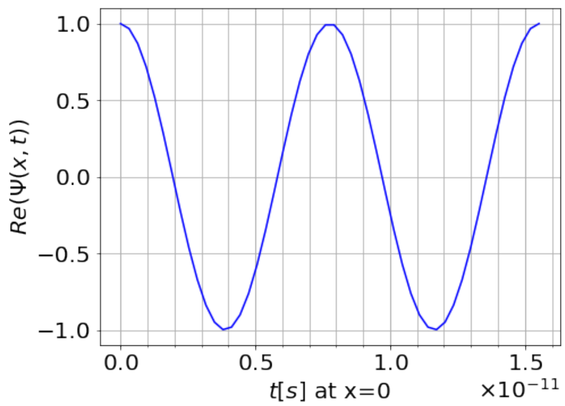
Student Problem: Exploring the Wave Function

Consider the wave function depicted right.

1. What is the angular frequency of this particle?
2. What is the wave number of this particle?
3. What is the momentum of this particle?
4. What is the energy of this particle?
5. What is the mass of this particle? (…and what particle is this?)







Student Problem: So Certain Are You

Consider a person sitting as still as they possible can.

1. What uncertainty would you place on their position? (*HINT: hold your hand out above the surface of the table; everybody's hand shakes a little. Based on the degree of shaking of your hand, what uncertainty would you place on the physical location in space of your whole body?*)

2. Based on your answer to part 1, what *minimum possible uncertainty* would be present on your momentum? (how certain can you ever be about the fact that you are "sitting still"?)

Consider an atom of Lithium ($m_{Li} = 6.94\text{u}$) in a gas of Lithium atoms that has been cooled to 2.7K.

3. Estimate your certainty about its position in space (*HINT: what is its matter wavelength at this temperature?*)

4. Based on your estimate in part 3, what *minimum possible uncertainty* would be present on its momentum?

5. Based on your answer to part 4, with what minimum speed can the Lithium atom be moving?

6. How does your answer to part 5 compare to the RMS speed of a Lithium atom in this gas? Are these answers consistent or not (in other words, do they contradict each other)?

Reminders

- Assigned
  - Read Harris, Ch 4.6; watch a lecture video associated with this material.
  - Homework 6: Exercise in matter behaving like waves and in complex numbers [Due Thursday]
- Second exam is the Thursday after Spring Break! More details on Thursday.
- **Always check Canvas for reading, homework, and lecture video assignments!**

q The Bohr Model of the Atom

The Bohr Model of the Atom

q.1 Wave Equations

Overview

In this lecture, we will learn…

- About a classical model of the atom.
- About how to impose the matter wave hypothesis on the atom.
- About making predictions about atoms using the Bohr Model of the Atom

Niels Bohr
(1885—1962)
Photo from 1922 and available from Wikipedia

The Hydrogen Atom Revisited

Atoms, excited by an ionizing potential, emit not a continuous rainbow of colors but a discrete set of colors - the atomic emission spectrum. Each atom has a characteristic spectrum. We are ready to confront this last mystery of the 1800s.

The Hydrogen Spectrum

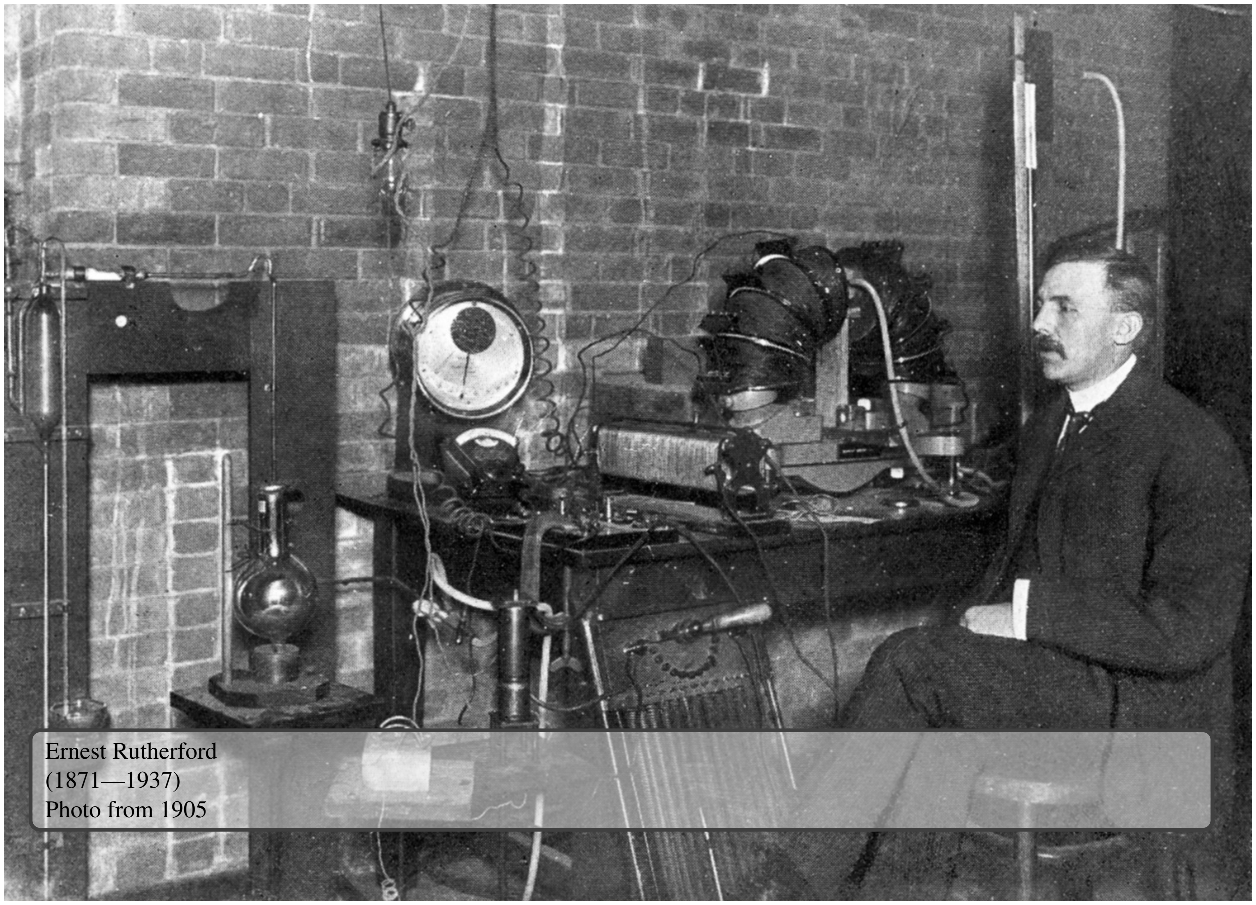
- Red Line: 656nm
- Cyan Line: 486nm
- Blue Line: 434nm

- Violet Line: 410nm

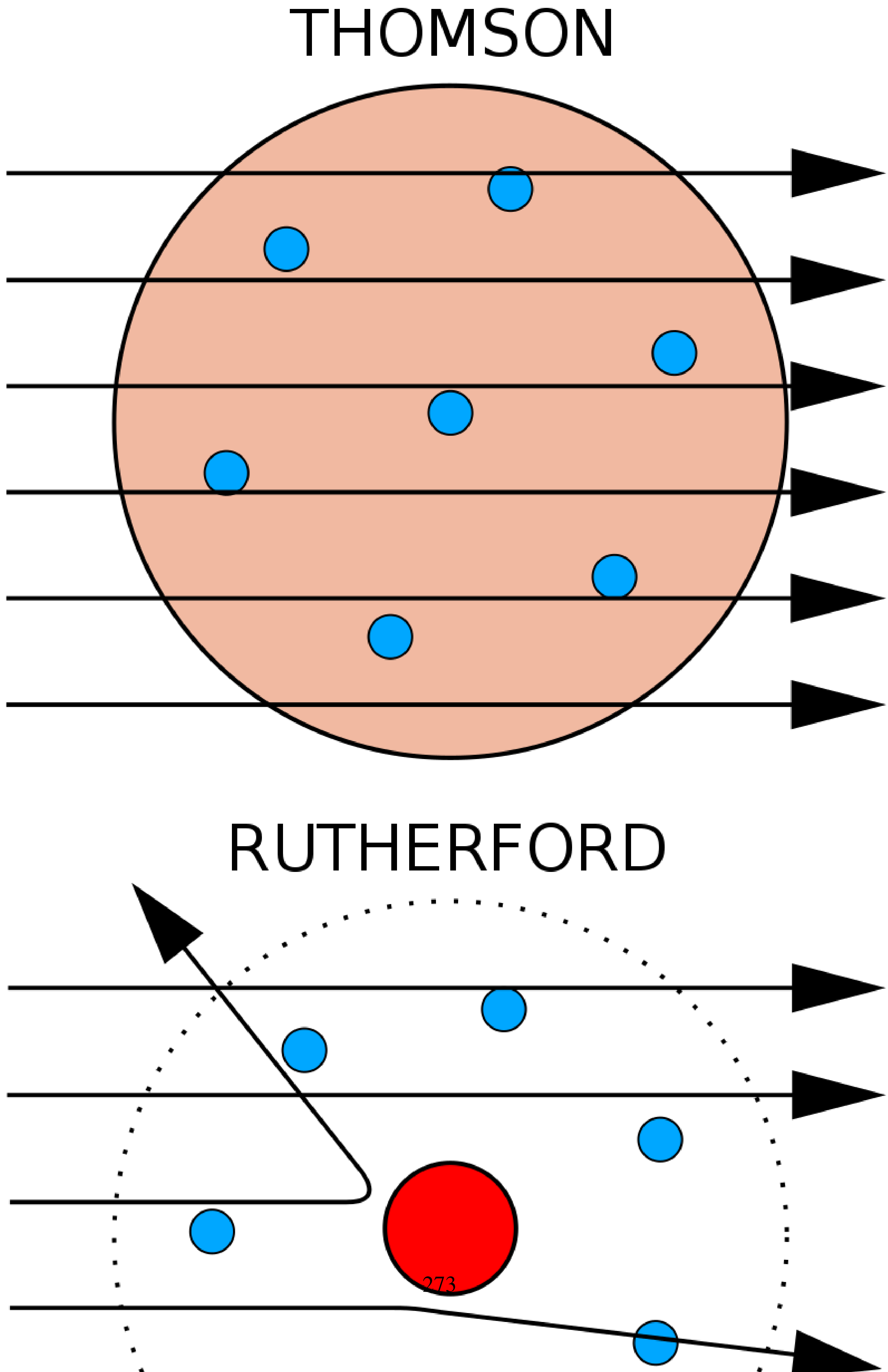
Johann Balmer worked out a mathematical relationship between the lines in 1885, but could not explain why this relationship existed. He observed that $\lambda = B(n^2/(n^2 - 2^2))$, where $B = 364.5\,\mathrm{nm}$ and $n = 3, 4, 5, 6, \ldots$

Early Models of the Atom

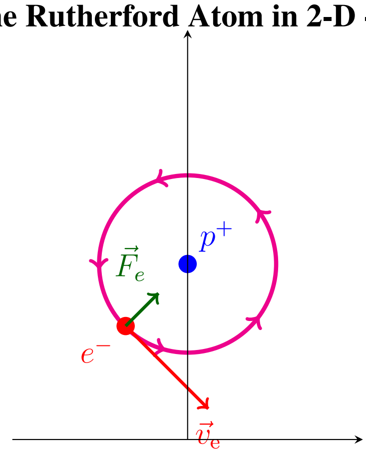
Joseph John (J.J.) Thomson
(1856—1940)
Photo available from Wikipedia



Ernest Rutherford
(1871—1937)
Photo from 1905


THOMSON
RUTHERFORD

The Rutherford Atom in 2-D - Force on the Electron



The electron is bound to the proton by the Coulomb Force:

$$\vec{F}_e = \frac{1}{4\pi\varepsilon_0}\frac{(-e)(e)}{r^2}\hat{r}$$

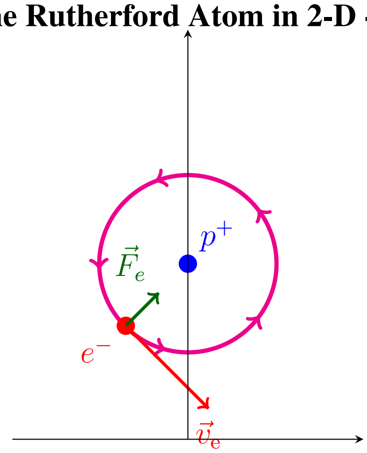
where $\hat{r}$ is a unit vector pointing from the source of the force (proton, p^+) to the recipient (electron, e^-). According to Newton's Second Law:

$$\sum \vec{F} = m\vec{a}$$

This is a centripetal force, leading to a centripetal acceleration:

$$\sum \vec{F} = m\frac{v_e^2}{r}(-\hat{r})$$

The Rutherford Atom in 2-D - Force on the Electron



Combining all of these, we learn that:

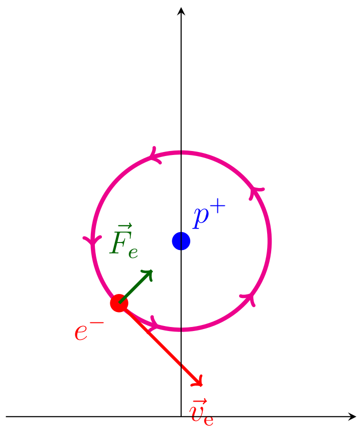
$$\begin{aligned}\frac{1}{4\pi\varepsilon_0}\frac{(-e)(e)}{r^2}\hat{r} &= m\frac{v_e^2}{r}(-\hat{r})\\ \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r} &= mv_e^2\end{aligned}$$

I leave it in this form because this force allows us to immediately write down the *classical* kinetic energy of this electron, $(1/2)mv_e^2$:

$$K_e = \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}$$

I will leave the equation unsimplified to ease the next step: computing the total energy of this electron.

The Rutherford Atom in 2-D - Energy of the Electron



$$K_e = \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}$$

The total energy is the sum of potential and kinetic energies for the electron at orbital radius, r:

$$E = K_e + U_e$$

The electric potential energy of this electron is:

$$U_e = -eV_p = \frac{1}{4\pi\varepsilon_0}\frac{-e^2}{r}$$

Thus the total energy is:

$$E = \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r} - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r} = -\frac{1}{8\pi\varepsilon_0}\frac{e^2}{r}$$

The Rutherford Atom Meets the Electron Wave

In the Rutherford model, any orbital radius is allowed. Therefore, any total energy is allowed for the electron. This cannot explain the discrete energy spectrum of electrons in a hydrogen atom. We have a sense already that *quantization* of some sort must be present in the atom - the atomic emission spectrum bears the fingerprint of a constrained system of allowed energies.

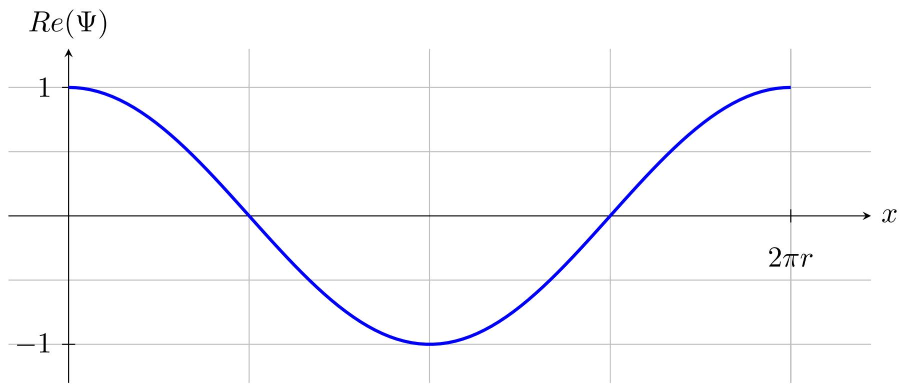
The de Broglie postulates enter to provide the crucial clue, the key step that leads to the quantization. For example:

$$p_e = \frac{h}{\lambda_e} = mv_e \text{ (classically)}$$

But is every λ_e possible for our electron? Consider our class discussion on the shapes of the matter waves associated with free particles. Let us apply that to the electron in orbit around the proton and see what conclusions we might draw.

Considering the electron matter wave and the orbital circumference

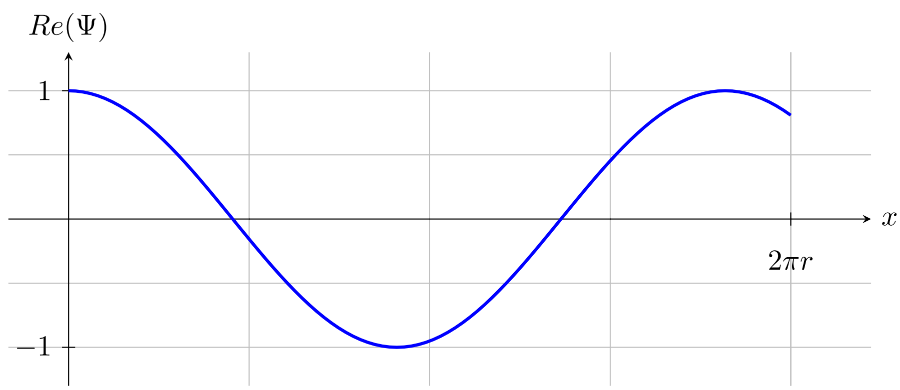
A circular orbit is one that, after one period, repeats again. Imagine the real part of the electron matter wave that might describe the electron traveling along such a circumference of an orbit of radius, r, at a specific time, $t = 0$. It might look like this:



This $Re(\Psi(x,t))$ nicely repeats itself when reaching $2\pi r$ - its behavior is smooth and continuous at the boundary where the orbit then begins to repeat ($2\pi \to 0$).

Considering the electron matter wave and the orbital circumference

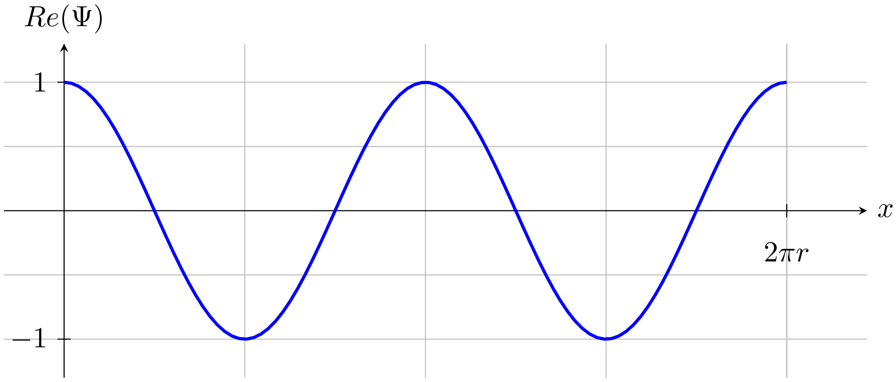
What about this matter wave for the same electron at the same radius, r. Is this a good wave function?



No. It has a jump discontinuity at 2π and does not cycle back to where it started. The jump discontinuity results in a first derivative $\partial/\partial x$ that is infinite, which corresponds to a place of *infinite momentum* in the wave function. This is physically forbidden.

Considering the electron matter wave and the orbital circumference

What about this matter wave for the same electron at the same radius, r. Is this a good wave function?



Yes. It contains double the wavelengths of the first good wave function we looked at and has no jump discontinuity. It differs from the first wave function by $\lambda_2 = 2\lambda_1$. In fact, all waves satisfying $\lambda_n = n\lambda_1$, where $n = 2, 3, 4, \ldots$, will work. None in between will work.

The Atom, Matter Waves, and Conditions on the Wave Function

Whatever the wave function that describes the electron in orbit around the proton, it must satisfy this condition to be physical:

$$n\lambda = 2\pi r$$

If we utilize the de Broglie postulate relating momentum and wavelength:

$$n\frac{h}{p} = 2\pi r \xrightarrow{classically} \frac{nh}{mv_e} = 2\pi r$$

If we substitute with the *reduced Planck's Constant*, $\hbar = h/(2\pi)$ we arrive at this equation for the speed of the electron and its relation to the radius of the orbit:

$$mv_e r = n\hbar \rightarrow v_e = \frac{n\hbar}{mr} \rightarrow v_e^2 = \frac{n^2\hbar^2}{m^2r^2}$$

The Bohr Postulate

It is worth noting at this point that the way Niels Bohr attacked this problem was to postulate that it was the *angular momentum*, L, of the electron that was quantized in its orbit around the proton. This was in 1913, prior to de Broglie's work (1924). Bohr's assertion was that, since h (and $\hbar$) have units of angular momentum (J $\cdot$ s), it might be in an atom that

$$L = n\hbar$$

where the angular momentum of the electron can only come in multiples of $\hbar$. For a circular orbit, $L = pr = mvr$ (classically), which leads to:

$$mvr = n\hbar$$

Later, de Broglie explained the reason why this works: based on what we saw in the previous slide, if $n\lambda_e = 2\pi r$ for an atom, the same condition results. This was a bold assertion by Bohr, long before the matter wave idea rose to the fore.

The Bohr Radius

We can now take the kinetic energy equation and eliminate the speed of the electron in favor of the quantization condition from the matter wave hypothesis:

$$K_e = \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r} = \frac{1}{2}mv_e^2$$

Substitute using $v_e^2 = \frac{n^2\hbar^2}{m^2r^2}$:

$$\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r} = \frac{1}{2}m\frac{n^2\hbar^2}{m^2r^2}$$

Some algebra leads you finally to:

$$r = n^2\left(\frac{4\pi\varepsilon_0\hbar^2}{me^2}\right)$$

Notice that, apart from the integer, n, the radial orbit(s) allowed by the quantization condition are given only by a multiplication of fundamental constants of nature: $\varepsilon_0 = 8.85\times 10^{-12}\mathrm{F/m}$, $\hbar = 1.05\times 10^{-34}\mathrm{J\cdot s}$, $e = 1.602\times 10^{-19}\mathrm{C}$, and $m = 9.11\times 10^{-31}$ kg.

The Bohr Radius

For $n = 1$, we arrive at the *Bohr Radius*:

$$a_0 = \frac{4\pi\varepsilon_0\hbar^2}{me^2} = 5.3\times 10^{-11}\,\mathrm{m}$$

For this model of the Hydrogen atom, based on a classical definition of kinetic energy and momentum but with matter wave quantization imposed, we suddenly find that only a fundamental orbit, and multiples of that orbit, are allowed. This begins to look much more like the atom that gives rise to a quantized atomic spectrum. But can we see that spectrum arise from this model?

The Electron Orbital Energy

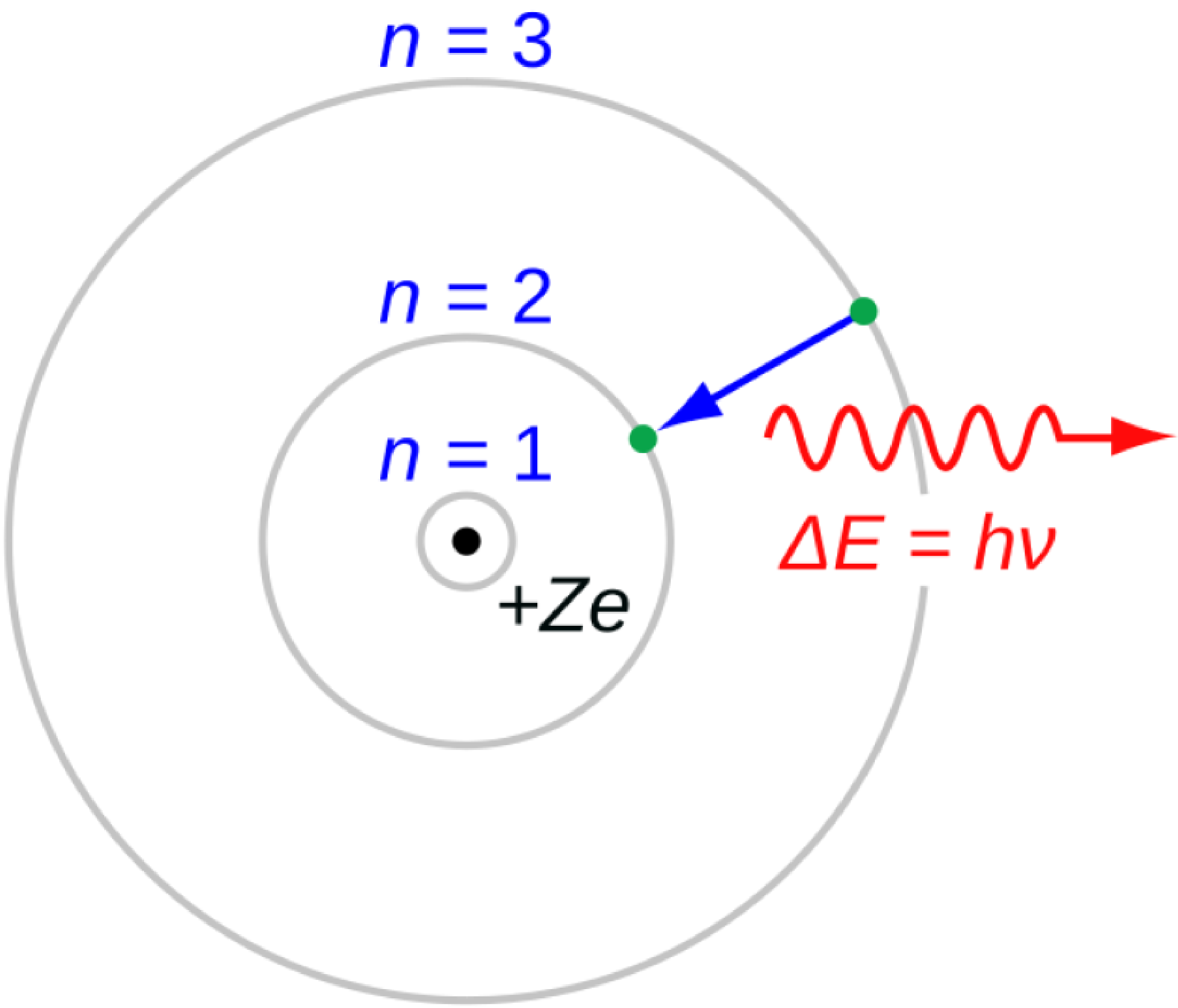
To answer that question, let's again consider the total classical energy of an electron orbiting a proton at radius r, but again impose the condition that $n\lambda_e = 2\pi r$ that led to $r = n^2a_0$.

$$E_{total} = -\frac{1}{8\pi\varepsilon_0}\frac{e^2}{r} = -\frac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2}\frac{1}{n^2}$$

The factor in front of $1/n^2$ is $E_0 = -2.19 \times 10^{-18}$J, or the famous -13.6eV - which turns out to be the energy required to fully ionize an electron out of its parent Hydrogen atom. Thus the energy of an allowed orbit of integer n, corresponding to radius $r = n^2 a_0$, is given by:

$$E = (-13.6\text{eV})\frac{1}{n^2}$$

Predicting Emitted Photon Energies



The atom conserves energy. For an electron to go to a wider orbit, it must absorb energy; a photon of the right frequency and wavelength can do that. To drop to a lower orbit, it must release energy; emitting a photon of a specific wavelength and frequency will do that, too.

Let's consider a transition that releases a photon, from an orbit marked by integer $n > m$ to one marked by integer m. The change in energy is:

$$\Delta E = E_m - E_n = (-13.6\text{eV})\left(\frac{1}{m^2} - \frac{1}{n^2}\right)$$

For example, for the transition $2 \to 1$ ($n = 2,\ m = 1$), we find $\Delta E = E_1 - E_2 = 10.2\text{eV}$. This lost energy from the electron would go into the energy of the photon emitted during the transition, $\hbar\omega$.

Photon Wavelengths from Electron Transitions in Hydrogen

Energy conservation combined with $E = hf = hc/\lambda$ will allow us to calculate the expected wavelengths of photons emitted from an ionized Bohr atom when electrons descend from higher orbits to lower orbits. Recall that the Balmer Series involved an empirical relationship between wavelengths of emitted light from Hydrogen as $\lambda = B(n^2/(n^2 - 2^2))$, where $B = 364.5\,\mathrm{nm}$ and $n = 3, 4, 5, 6, \ldots$. Let's tabulate transitions from $n = 3, 4, 5, \ldots$ to $m = 2$:

Transition	Emitted Photon Wavelength (nm)
$3 \to 2$	656
$4 \to 2$	484
$5 \to 2$	432
$6 \to 2$	409

These are, to very good accuracy, the Balmer Series lines. The pattern is thus explained by the quantization of orbits in the atom due to the matter-wave nature of the electron (and the resulting quantization of angular momentum, ala Bohr).

The Bohr Model vs. Real Atoms

It is wise to revisit our model and compare that to what we expect from real atoms. We've only made a very good approximation to what we would expect real atoms to need to be more accurately modeled.

- Bohr-Rutherford Model of the Atom
  - Two-dimensional
  - Electron free to move, but proton (or whole nucleus with Z protons) "pinned" and unmoving at the center
  - Obtained by combining classical picture of a "planetary" atom with matter wave idea
- More realistic model of the Atom
  - Fully three-dimensional
  - Motion of both electron and proton allowed (co-orbiting common center-of-mass)
  - Determined by exactly solving Schroedinger's Wave Equation using $V(x, y, z) = \frac{1}{4\pi\varepsilon_0}\frac{(Ze)}{|\vec{r}|}$

We are simply not ready at this stage to commit to more realism in describing the atom, but we will build up a toolkit to do so.

Review

In this lecture, we have learned...

- About a classical model of the atom.
- About how to impose the matter wave hypothesis on the atom.
- About making predictions about atoms using the Bohr Model of the Atom

Niels Bohr
(1885—1962)
Photo from 1922 and available from Wikipedia

Announcements

- Assigned
  - Read Harris, Ch 5.1–5.5; watch associated lecture video.
  - Study for Exam 2 (covers general relativity; temperature, heat, and classical microscopic heat energy; radiation behaving like particles; and particles behaing like waves - Homeworks 4–6 and associated in-class, video lecture, and reading material)
  - Homework 7: Exercise in the wave function, free-particle solutions, the uncertainty principle, the Bohr Model of the Atom, Infinite Square Well [Due in 3 weeks]
- Due Today
  - Homework 6 (submitted on Canvas)
- **Always check Canvas for reading, homework, and lecture video assignments!**

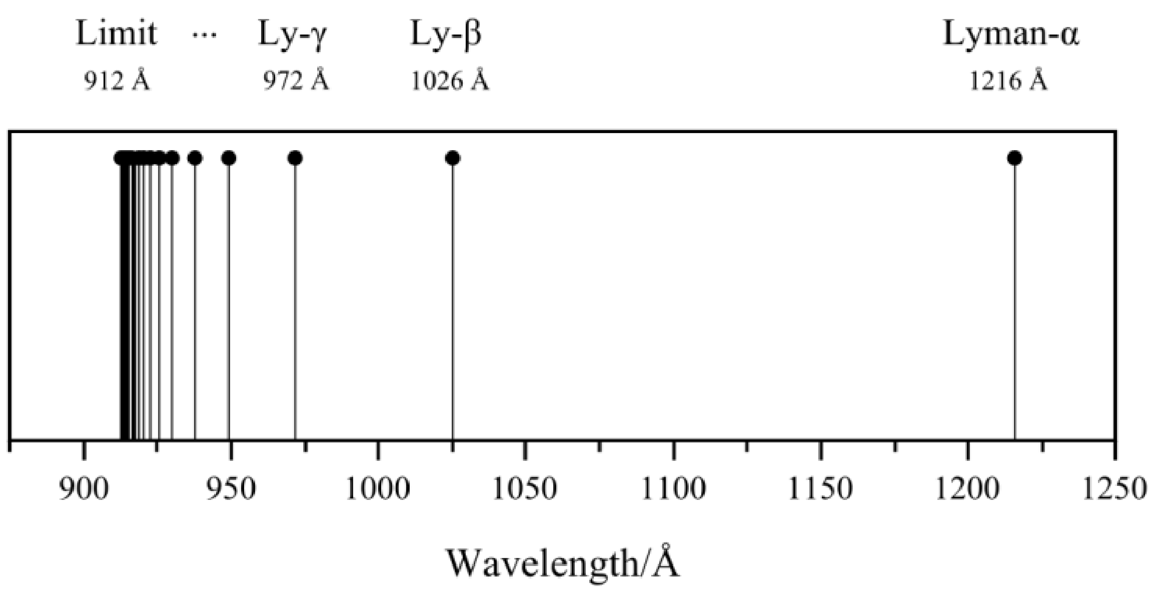
q.2 Problem Solving in the Bohr Model of the Atom

Problem Solving in the Bohr Model of the Atom

Instuctor Problem: The Lyman Hydrogen Spectral Lines

The Balmer Spectrum consists of the lines that fall into the visible part of the Hydrogen emission spectrum. There are other "lines" that lie outside the visible portion of the spectrum. The Lyman spectrum (known as the "Lyman-α Series") is another such spectrum. It consists of the photons emitted by electrons that transition from $n > 1$ to the $n = 1$ orbit.

1. Show that the Bohr Model of the Hydrogen Atom, considering the $2 \to 1$ and $3 \to 1$ transitions, generally reproduces the longest wavelengths present in the Lyman spectrum: $\lambda = 121.57\mathrm{nm},\ 102.56\mathrm{nm}$.
2. Where, in any direction around the Hydrogen nucleus, are you most likely to find the electron when it's in the $n = 2$ orbit?



Student Problem: The Paschen Hydrogen Spectral Lines

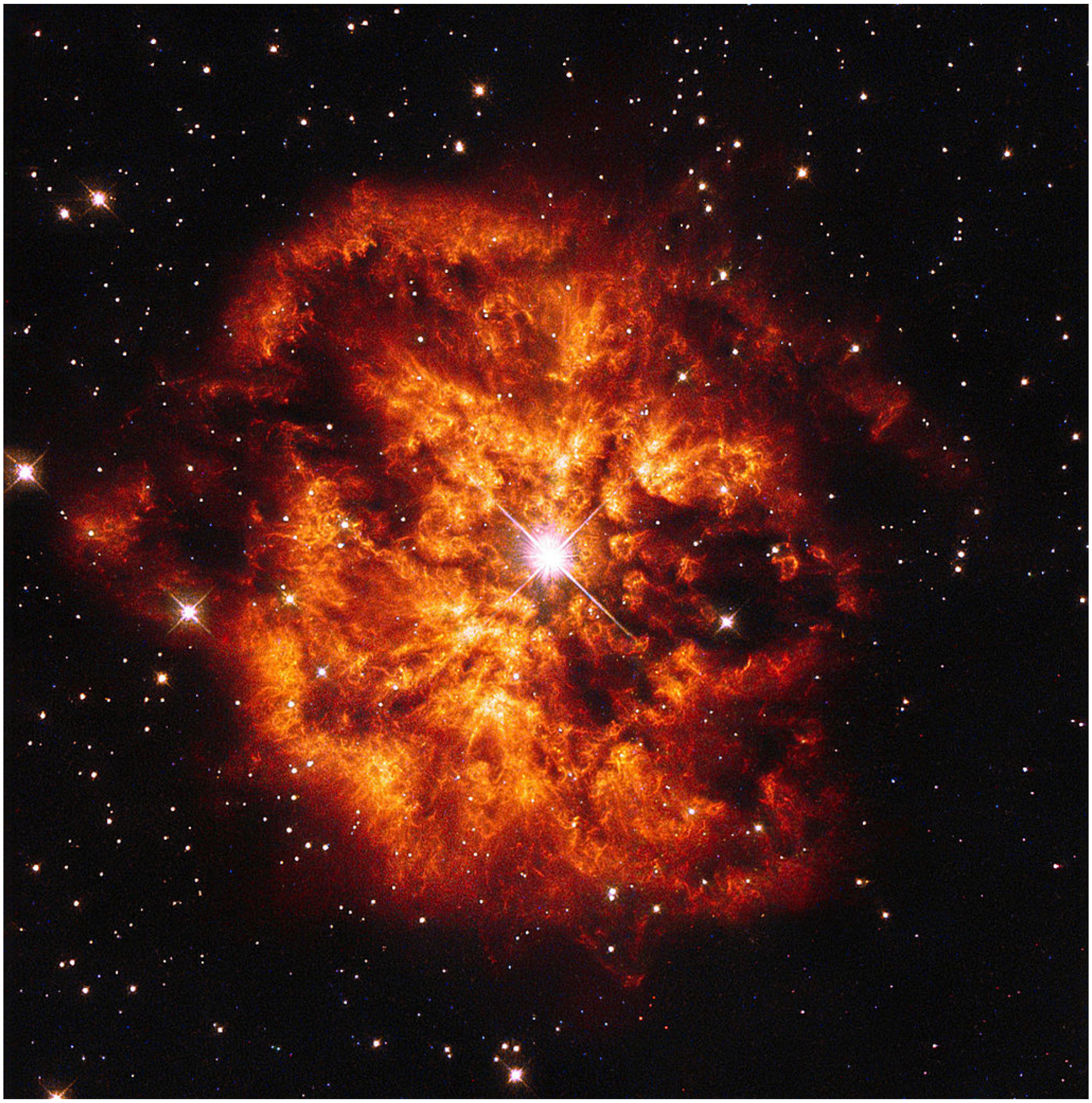
The Paschen spectrum (known as the "Paschen-α Series") is yet another part of the Hydrogen spectrum. It consists of the photons emitted by electrons that transition from $n > 3$ to the $n = 3$ orbit.

1. Show that the Bohr Model of the Hydrogen Atom, considering the $4 \to 3$ and $5 \to 3$ transitions, generally reproduces the longest wavelengths present in the Paschen spectrum: $\lambda = 1875\mathrm{nm},\ 1282\mathrm{nm}$.
2. If an electron was captured from $r = \infty$ into the $n = 3$ orbit, what photon wavelength would this capture emit?
3. Where, in any direction around the Hydrogen nucleus, are you most likely to find the electron when it's in the $n = 5$ orbit?

Student Problem: Singly Ionized Helium

Consider, not a Hydrogen-like atom with 1 proton, but a Helium-like atom with 2 protons and 2 neutrons, but only 1 electron. The protons each carry $+e$ electric charge while the neutrons are electrically neutral. This is *singly ionized Helium*, first discovered in stellar spectrum observations by E. C. Pickering in 1896 but not understood until over a decade later (it was originally mistaken for Hydrogen with additional spectral lines). The Pickering spectral lines (in nm) are: 1012.3, 656.0, 541.1, 485.9, 454.1. Another line at 468.6nm is visible in hot stars, but technically not part of the Pickering lines.

1. What is the smallest orbital radius you would predict for singly-ionized Helium?
2. What are the predicted wavelengths of the visible spectral lines in this atom, and how do they compared to the Pickering lines and the 486.6nm lines (what is the mean difference between prediction and observation)?



A Wolf-Rayet star, imaged by the Hubble Space Telescope. These stars exhibit the singly-ionized Helium spectrum lines.

Reminders

- Assigned
  - Read Harris, Ch 5.1–5.5; watch associated lecture video.
  - Study for Exam 2 (covers general relativity; temperature, heat, and classical microscopic heat energy; radiation behaving like particles; and particles behaing like waves - Homeworks 4–6 and associated in-class, video lecture, and reading material)

- Homework 7: Exercise in the wave function, free-particle solutions, the uncertainty principle, the Bohr Model of the Atom, Infinite Square Well [Due in 3 weeks]

- Due Today
  - Homework 6 (submitted on Canvas)

- **Always check Canvas for reading, homework, and lecture video assignments!**

Solving the Schrödinger Wave Equation

Overview

In this lecture, we will learn…

- About the postulates of quantum mechanics.
- About some guidelines for wave functions that can solve the Schrödinger Wave Equation (SWE).
- About classical analogs of quantum systems we might want to model.
- About solving the "particle in a box" model using the SWE.

Max Born
(1882—1970)
Photo ca 1930-1940 and available from Wikipedia

r.1 The Postulates of Quantum Mechanics

The (One-Dimensional) Schrödinger Wave Equation (SWE)

$$-i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x,t)\Psi(x,t)$$

The above is the one-dimensional SWE and allows generally for solutions that vary in space and time and forces (potentials) that vary in space and time. This is very complex. To utilize this equation, we will need to do the following:

- Represent physical situations with a model (e.g. a potential, V) that describes how the system constrains the particle(s) described by the wave function. This may involve simplifying assumptions.
- Define the basic rules of quantum mechanics — what are the inviolable tenets of problem-solving in quantum mechanics that, if untrue, mean the fundamental dissolution of quantum mechanics?
- Define some guidelines for how to write down wave functions that will work to solve the SWE. These may be violable depending on how you *approximate* physical situations, but don't represent a fundamental failure of quantum mechanics if violated (poor assumptions on the part of the problem solver are not to be held against the fundamental framework of quantum mechanics!)

The Postulates of Quantum Mechanics

The inviolable tenets of quantum mechanics are known as the *postulates of quantum mechanics*:

1. At each specific time the state of a system (e.g. a particle or collection of particles) can be entirely represented by a space of functions that are related to the wave function, $\Psi(x,t)$. For our purposes, we will concentrate on the wave function; a more advanced course will concentrate on that space of functions, called a "Hilbert Space" (after physicist David Hilbert)
2. Every observable quantity of a system (e.g. a measurement of momentum, energy, etc.) will be represented by the action of a mathematical *operator* on the state of the system. I'll elaborate more on this later, but think of how we derived the SWE - the total energy is "measured" in the equation by a time derivative acting on Ψ, etc.
3. The only possible results of a measurement of an observable are related to characteristic numbers, "eigenvalues," of the operators. The details behind this demand higher-level mathematics than can be assumed in this course, and so this one will be less emphasized here.

A dedicated course in quantum mechanics develops these ideas using fundamental mathematical frameworks, like linear algebra (Hermitian Matrices) or function spaces (Hilbert Spaces); I will gloss over those aspects and instead try to equip you to solve the SWE while using the implications of these postulates to get useful information from the SWE.

Implications of the Postulates

From the 1st postulate of quantum mechanics, there are some consequences:

- There exists a product of states that yields a measure of distance between states, in the same way that the components of a vector form the "basis" over which the full vector is composed; the distance measure of the vector is then given by combining the lengths of the components using the Pythagorean theorem. In our case, since the wave function represents the probability density of the system (probability per unit length in one dimension), we need:

$$\Psi(x,t)\Psi^*(x,t) \quad = \quad \text{Probability Per Unit Length of Outcome}$$

  The above interpretation of the link between the wave function and the probability density was introduced by physicist Max Born, and allows for interpretation of the wave function. To get the probability, rather than probability density, we need:

$$\int \Psi(x,t)\Psi^*(x,t)dx \quad = \quad \text{Probability of Outcome in Region of Space}$$

Implications of the Postulates

From the 2nd postulate of quantum mechanics, there are some consequences:

- Measuring the energy or momentum of a state will be represented by application of a mathematical operator on the state of the system. For example, from the SWE we might realize already that measuring energy, E, is accomplished by applying an operator, $\hat{E}$, to the wave function:

$$\hat{E}\Psi(x,t) = \left(i\hbar\frac{\partial}{\partial t}\right)\Psi(x,t) = E\Psi(x,t)$$

- Let's try the simple case of a free particle, $\Psi(x,t) = A(\cos(kx-\omega t) + i\sin(kx-\omega t))$:

$$\left(i\hbar\frac{\partial}{\partial t}\right)\Psi(x,t) \quad = \quad i\hbar\left(A(-\omega)(-\sin + i\cos)\right) = \hbar\omega\Psi(x,t)$$

  We recognize, from the de Broglie postulates, that the energy of a matter wave is indeed $\hbar\omega$! Momentum is similarly measured using a different operator:

$$\hat{p} \equiv -i\hbar\frac{\partial}{\partial x}$$

  Check for yourself that this works on the free-particle wave function to "return" the momentum, $\hbar k$.

r.2 Guidelines on Wave Functions

Guidelines on the Wave Functions

The following are guidelines for writing down well-behaved (e.g. physical) solutions to the SWE. They can be violated. I will list examples of how with each guideline. Violation says more about the mathematical model you have chosen to implement in quantum mechanics than about quantum mechanics itself.

1. The probability of finding the particle or particle(s) *anywhere at all in space* is 100%. This is reflected by the following *normalization condition*:

$$\int_{\text{all space}} \Psi(x,t)\Psi^*(x,t)dx = 1$$

   When is this violated? If a particle is unstable and can decay into other particles (think about a radioactive nucleus) then the probability of finding the original nucleus at a later time is not 100%. This can be accommodated in Ψ, but is for a more advanced class.

2. The wave function must be continuous both in the function itself and the first derivative. Jump discontinuities in space, for example, represent places of infinite momentum. When is this violated? If the force constraining the particle is represented by an insurmountable barrier (e.g. it would take infinite energy to overcome the force), then the first derivative need not be continuous. But can a *physical* force really be infinite?

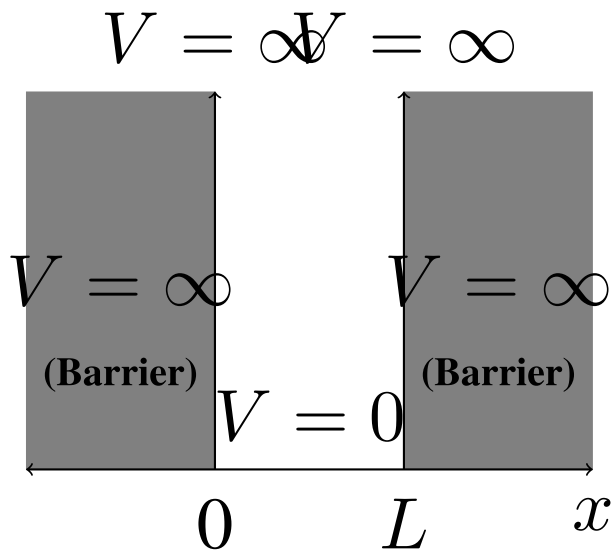
r.3 Models of Physical Situations

Models of Physical Situations

- Let's begin by considering a purely classical situaton - a particle free to move on a track that is bent in a U-shape. It can roll down to the bottom of the track (the lowest point) where it achieves the highest kinetic energy and the lowest gravitational potential energy. It can roll up the sides of the U until it trades all its kinetic energy for potential. But it is trapped on the track. It can never escape without help from an external force.
- A quantum analog of this is the trapping of an electron in a system from which it (ostensibly) is unable to escape. The confinement of an electron within such a system leads to strong emergence of its wave behavior. This is taken advantage of in modern electronic via the manufacturing of *quantum dots*. By tuning the size of the confinement cell, the properties of the dot (e.g. what light it will emit when excited by radiation) can be tightly controlled.
- The above lend themselves to an overly simplistic model - the "particle in a box" or "infinite square well" model - one in which a particle is confined within a completely insurmountable barrier. We'll look at this model in one dimension as a means to learn to solve the SWE. The classical analog of this would be a cart bouncing back and forth on a nearly frictionless air track, forever trapped between the ends of the track.

r.4 The Infinite Square Well, or "Particle in a Box"

The Infinite Square Well Model ("Particle in a Box")



Setting up the SWE and Preparing to Solve

By convention, the minus sign present in the time-derivative part of the SWE is moved to the space-derivative part of the SWE, and absorbed into the definition of the potential (e.g. $-V(x,t) \rightarrow V(x,t)$) so that

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x,t)\Psi(x,t)$$

Let's begin by making the following assumptions: the force is time-independent ($V(x,t) = V(x)$), which is true for our case; and the wave function can be separated into the product of two pieces: a time-dependent piece and a time-independent piece. This is known as *separation of variables* - it may not work in all problems, but let us see the implications if it does.

$$\Psi(x,t) \quad = \quad \psi(x)\phi(t)$$

Then:

$$\begin{aligned} i\hbar\psi(x)\frac{\partial}{\partial t}\phi(t) &= -\frac{\hbar^2}{2m}\phi(t)\frac{\partial^2}{\partial x^2}\psi(x) + V(x)\psi(x)\phi(t) \\ i\hbar\frac{1}{\phi(t)}\frac{\partial\phi(t)}{\partial t} &= -\frac{\hbar^2}{2m}\frac{1}{\psi(x)}\frac{\partial^2\psi(x)}{\partial x^2} + V(x) \end{aligned}$$

Setting up the SWE and Preparing to Solve

$$i\hbar\frac{1}{\phi(t)}\frac{\partial\phi(t)}{\partial t} \quad = \quad -\frac{\hbar^2}{2m}\frac{1}{\psi(x)}\frac{\partial^2\psi(x)}{\partial x^2} + V(x)$$

Note that if we evaluate the left-hand side at a space-time coordinate (x_1, t_1) and then again at a different space-time coordinate (x_2, t_1), its value remains constant; this implies that the right-hand side also remains constant, even though the spatial coordinate has changed from $x_1 \rightarrow x_2$. Thus the following must be true:

$$i\hbar\frac{1}{\phi(t)}\frac{\partial\phi(t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{1}{\psi(x)}\frac{\partial^2\psi(x)}{\partial x^2} + V(x) = C$$

C is known as the *separation constant*. Let's see what we can learn about the separation constant from the time-dependent part of the wave function $\phi(t)$.

The Separation Constant and the Time-Dependent Solution

If $\Psi(x,t) = \psi(x)\phi(t)$, we have learned the following must be true:

$$i\hbar\frac{1}{\phi(t)}\frac{d\phi(t)}{dt} = C$$

We can guess at the functional form of the solution to this time-dependent portion of the SWE. It must be a function that (a) when acted upon by a first derivative returns a copy of itself; (b) the argument of the function must contain time, but also the constants i, $\hbar$, and C; (c) such that when the results of the derivative are divided by $\phi(t)$ the function is completely gone, leaving only C behind. This sounds like a job for the exponential function!

$$\phi(t) = e^{-(iC/\hbar)t} \quad \xrightarrow{\text{Use in SWE}} \quad i\hbar\frac{1}{e^{-(iC/\hbar)t}}\frac{d}{dt}e^{-(iC/\hbar)t} = C$$

What is this constant? Well, the argument of the exponential function must be *dimensionless*. $t/\hbar$ has units of time divided by angular momentum, or $s/(J\cdot s)$, leaving $1/J$. Thus C has units of *energy* $\rightarrow$the constant is the total energy of the system, $E = \hbar\omega$!

The Space-Dependent Solution

From the exercise with the time-dependent solution, we have learned that:

$$i\hbar\frac{1}{\phi(t)}\frac{\partial\phi(t)}{\partial t} = E = -\frac{\hbar^2}{2m}\frac{1}{\psi(x)}\frac{\partial^2\psi(x)}{\partial x^2} + V(x)$$

Let us also consider what will be the probability density of these separable solutions:

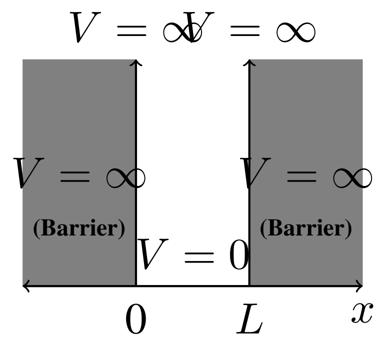
$$\Psi(x,t)\Psi^*(x,t) = \psi(x)e^{-iEt/\hbar}\psi^*(x)e^{iEt/\hbar} = \psi(x)\psi^*(x)$$

The probability density is indepenent of time, too, only having dependence on what happens spatially to the wave. Thus for situations where the space- and time-dependent portions of a wave function can be separated, we need only worry about solving the *time-independent SWE*:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + V(x)\psi(x) = E\psi(x)$$

r.5 The Infinite Square Well, or "Particle in a Box"

The Infinite Square Well Model ("Particle in a Box")



Consider a particle of mass m trapped inside the infinite square well. When the particle is located between $x = 0$ and $x = L$, we have that $V(x) = 0$. Its energy, E, will be entirely in the form of kinetic energy, $p^2/(2m)$, in this region. Because the "walls" of the potential are represented by $V(0) = V(L) = \infty$, the wave function must vanish at $x = 0$ and $x = L$ in order to avoid having infinite total energy. It is forbidden to find the particle in any other region of the problem.

Articulating a Strategy for Solving Problems using the SWE

1. Write down the SWE that includes the potential used to model the situations.
2. Separate the problem into any regions that can be treated distinctly from other regions.
3. Identify trial functions that might solve the SWE in each region.
4. Determine the values of unknown parameters in the functions using constraints: employ the guidelines of good wave function behavior (conservation of probability, smoothness); match the wave functions at boundaries between regions and see if that constrains unknowns.

Solving the Problem: Test Solution

We need only worry about the space-dependent portion of the Schroedinger Wave Equation, since if the wave function can be separated into space and time dependent portions then we already know the time-dependent part of $\Psi(x,t) = \psi(x)\phi(t)$:

$$i\hbar \frac{1}{\phi(t)} \frac{d\phi(t)}{dt} = E \xrightarrow{\text{Solve for } \phi\text{(t)}} \phi(t) = e^{-iEt/\hbar}$$

We need only concern ourselves with solving:

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x)}{\partial x^2} + V(x)\psi(x) = E\psi(x)$$

This demands a function that, after two space derivatives, returns to a version of itself. Any sine or cosine function would do. Let's begin by guessing:

$$\psi(x) = \begin{cases} A\sin(\kappa x) & \text{for } (0 \leq \text{x} \leq \text{L}) \\ 0 \text{ everywhere else} \end{cases}$$

where κ is an unknown coefficient modifying x in the sine function argument and A is an overall (and potentially complex) multiplicative constant, also unknown.

Solving the Problem: What is κ?

Let's plug our test function into the time-independent SWE inside the infinite square well (between the walls, such that $V(x) = 0$) and see what happens:

$$\begin{aligned}
-\frac{\hbar^2}{2m}\frac{\partial^2(\psi(x))}{\partial x^2} + 0 &= E\psi(x) \\
-\frac{\hbar^2}{2m}\frac{\partial^2(A\sin(\kappa x))}{\partial x^2} &= E(A\sin(\kappa x)) \\
\frac{\partial^2(A\sin(\kappa x))}{\partial x^2} &= \frac{-2mE}{\hbar^2}(A\sin(\kappa x)) \\
A(-\kappa^2)\sin(\kappa x) &= \frac{-2mE}{\hbar^2}(A\sin(\kappa x)) \\
-\kappa^2 &= \frac{-2mE}{\hbar^2} \\
\kappa &= \frac{\sqrt{2mE}}{\hbar}
\end{aligned}$$

But what is E? What are the allowed energies of this system?

Solving the Problem: What are the allowed energies, E?

To answer this, let's consider the guidelines on good wave function solutions: they should be smooth, either by being continuous in the function itself and possibly also in the first derivative. In our case, this means that at the left boundary $x = 0$ we need this to be true:

$$A\sin(\kappa \cdot 0) = 0$$

That one is *automatically satisfied* by our test solution. It must also be true at the right boundary, $x = L$, that:

$$A\sin(\kappa L) = 0$$

This can be satisfied if $\kappa L = \pi n$, for $n = 1, 2, 3, \ldots$ (any integer multiple of π). This is our *quantization condition*. Thus:

$$\frac{\sqrt{2mE}}{\hbar}L = \pi n \longrightarrow E_n = \frac{\pi^2 n^2 \hbar^2}{2mL^2}$$

Solving the Problem: What is A?

To answer this question, let's take advantage of the guideline that a good physical solution satisfies "conservation of probabilty":

$$\int_{\text{all space}} \Psi(x,t)\Psi^*(x,t)dx = 1$$

Plugging in our solution (and noting that $\psi(x) = 0$ if $x < 0$ or $x > L$):

$$\int_0^L \left(A\sin(\kappa x)e^{-iEt/\hbar}\right)\left(A^*\sin(\kappa x)e^{iEt/\hbar}\right)dx = 1$$
$$\int_0^L (AA^*)\sin^2(\kappa x)dx = 1$$

You can use a standard technique to solve the integral, look it up in a reference, etc.:

$$\int_0^L (AA^*)\sin^2(\kappa x)dx = AA^*\left(\frac{x}{2} - \frac{1}{4\kappa}\sin(2\kappa x)\right)\Bigg|_0^L = 1$$

Solving the Problem: What is A?

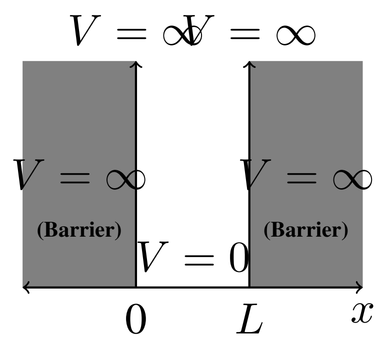
$$AA^*\left(\frac{x}{2} - \frac{1}{4\kappa}\sin(2\kappa x)\right)\Bigg|_0^L = 1$$
$$AA^*\left(\frac{L}{2} - \frac{1}{4\kappa}\sin(2\kappa L)\right) = 1$$

Since at $x = L$ or any integer multiple of L the function, by construction, goes to zero:

$$AA^*\left(\frac{L}{2} - \frac{1}{4\kappa}\sin(2\kappa L)\right) = 1$$
$$AA^*\left(\frac{L}{2}\right) = 1$$
$$AA^* = \frac{2}{L}$$

For sure, $|A|^2 = 2/L \rightarrow |A| = \sqrt{2/L}$. This implies $A = \sqrt{2/L}$ or $A = i\sqrt{2/L}$. There is no physical consequence to picking a real or purely imaginary coefficient, so it's convention to select $A = \sqrt{2/L}$.

The Full Solution to the Infinite Square Well



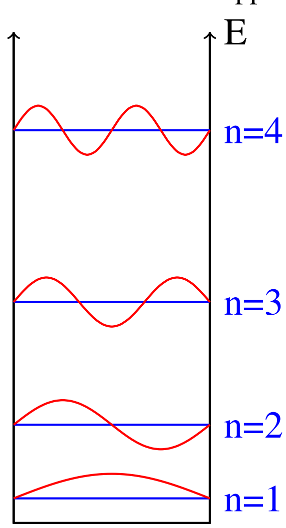
The wave functions that solve the problem are as follows:

$$\psi(x) = \begin{cases} \sqrt{\frac{2}{L}}\sin\left(\frac{\pi n}{L}x\right) & \text{for } (0 \le \text{x} \le \text{L}) \\ 0 \text{ everywhere else} \end{cases}$$

The allowed energies permitted to the particle in this confined system are given by

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

If you can construct a system that approximates an infinite well in one or more dimensions, you are effectively creating a "customized atom" with allowed energies that are tuneable using the width of the well and the mass of the trapped particle.



Reminder: a Strategy for Solving Problems using the SWE

1. Write down the SWE that includes the potential used to model the situations
2. Separate the problem into any regions that can be treated distinctly from other regions
3. Identify trial functions that might solve the SWE in each region.
4. Determine the values of unknown parameters in the functions using constraints: employ the guidelines of good wave function behavior (conservation of probability, smoothness); match the wave functions at boundaries between regions and see if that constrains unknowns.

Review

In this lecture, we have learned...

- About the postulates of quantum mechanics.
- About some guidelines for wave functions that can solve the Schrödinger Wave Equation (SWE).
- About classical analogs of quantum systems we might want to model.
- About solving the "particle in a box" model using the SWE.

Max Born
(1882—1970)
Photo ca 1930-1940 and available from Wikipedia

Announcements

- Assigned
  - Study for Exam 2 (covers general relativity; temperature, heat, and classical microscopic heat energy; radiation behaving like particles; and particles behaing like waves - Homeworks 4–6 and associated in-class, video lecture, and reading material)
  - Homework 7: Exercise in the wave function, free-particle solutions, the uncertainty principle, the Bohr Model of the Atom, Infinite Square Well [Due next week]
- **Always check Canvas for reading, homework, and lecture video assignments!**

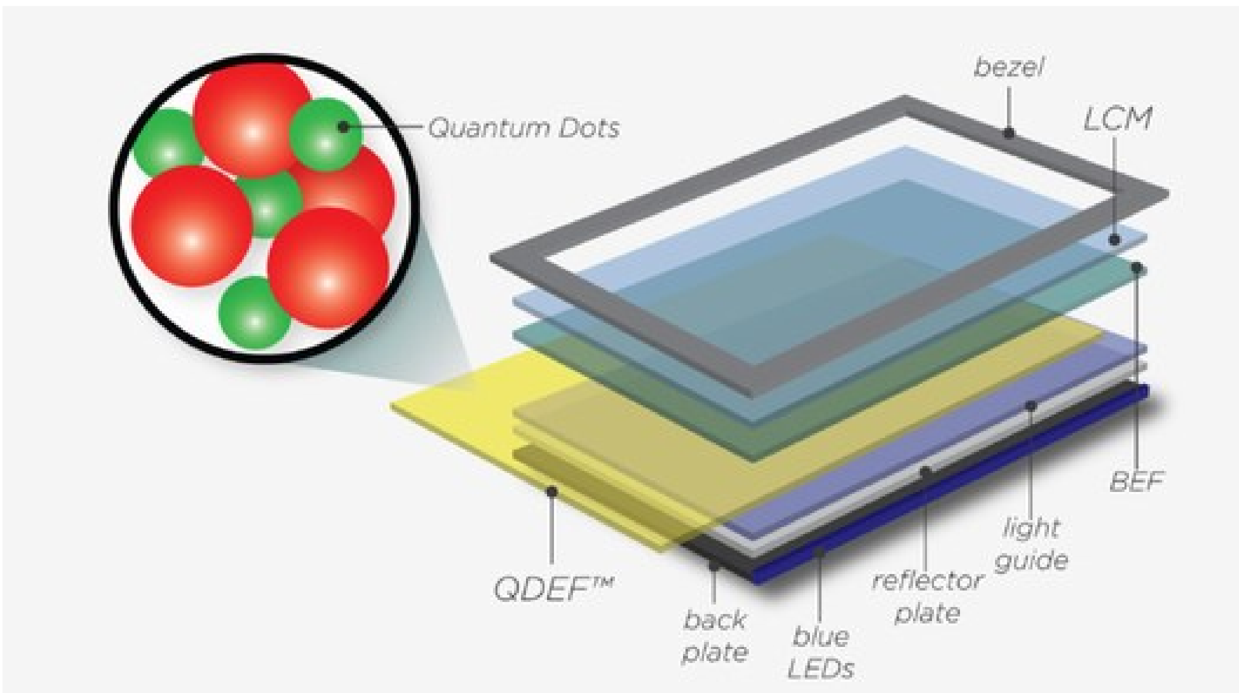
r.6 Problem Solving in the Infinite Square Well

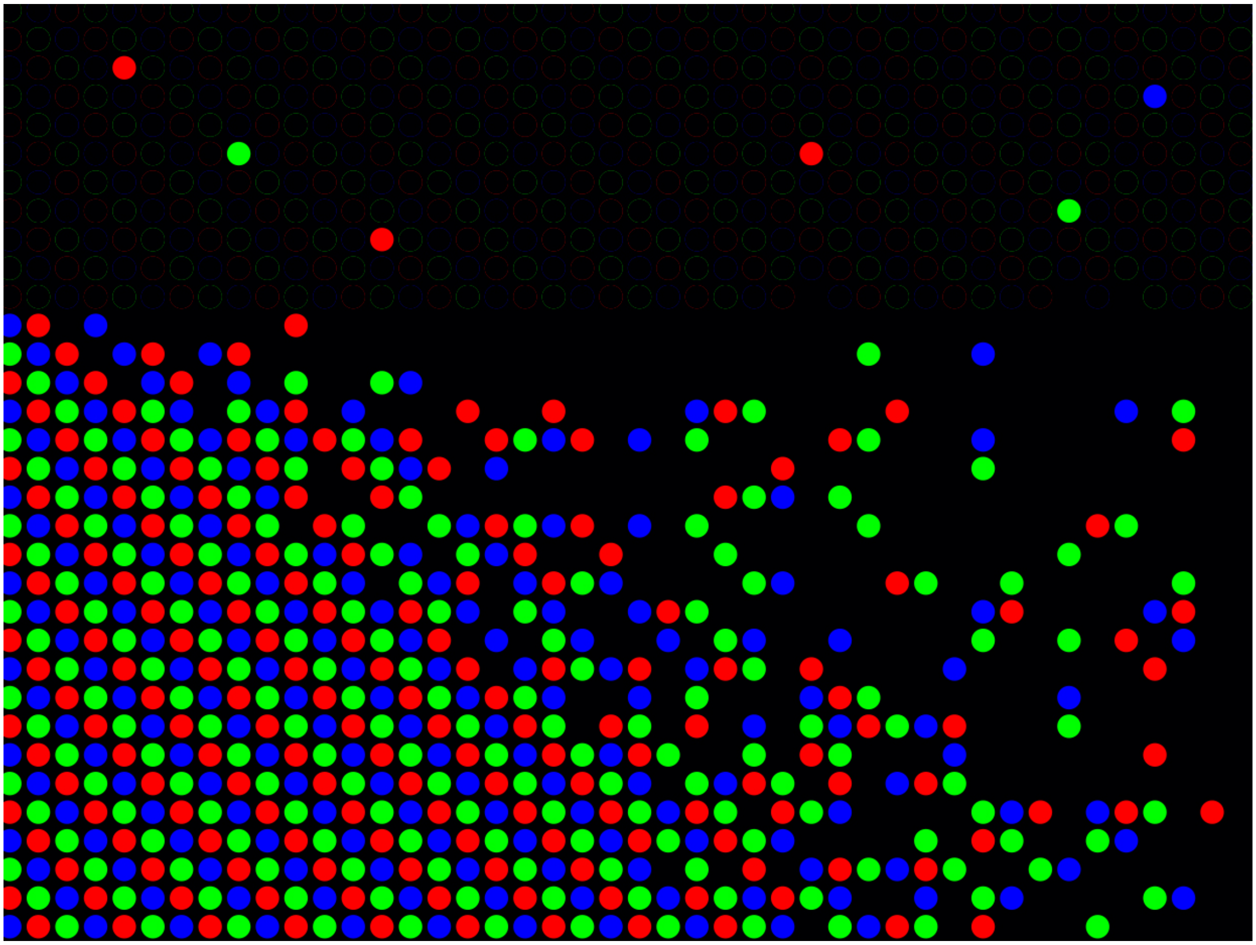
Problem Solving in the Infinite Square Well

Instructor Problem: Quantum Dot Video Screen

Quantum Dots allow for brilliant video screens to be produced. A blue light-emitting diode is used to provide the blue color elements in the screen. Quantum dots are used to provide the red and the green elements. The blue LED light is used to excite electrons in the red and green quantum dots, and then the electrons will emit red or green light when they "fall" back to the ground state of the dot. Model the dots at 1-D infinite square wells. If a red element needs to emit light of wavelength of 638nm. The blue LED emits light of wavelength 450nm.

- What size dots do you need such that the $n = 2$ to $n = 1$ (ground state) transition of an electron in the well emits red light of the desired wavelength?
- Write the full wave function (space and time) of the electron when it's in the $n = 2$ state.
- What is the probability of finding the electron in the range $[0, L/3]$ of the $n = 2$ state?





Student Problem: Practice Solving the SWE using the Infinite Square Well

Without referring to the textbook or notes from before class, solve the SWE for the case of the Infinite Square Well: $V(x) = 0$ for $0 < x < L$, but $V(x) = \infty$ for all other x values. Use the trick of treating the wave function, $\Psi(x,t)$, as separable in space and time, $\Psi(x,t) = \psi(x)\phi(t)$, so that you only have to solve the time-independent SWE:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

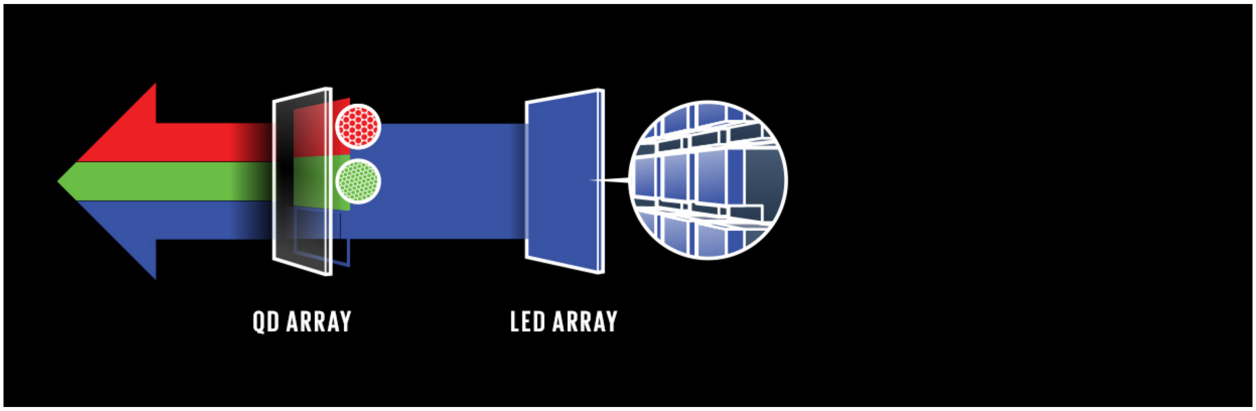
Use the guidelines for solving the SWE:

1. Write down the SWE that includes the potential used to model the situations.
2. Separate the problem into any regions that can be treated distinctly from other regions.
3. Identify trial functions that might solve the SWE in each region.
4. Determine the values of unknown parameters in the functions using constraints: employ the guidelines of good wave function behavior (conservation of probability, smoothness); match the wave functions at boundaries between regions and see if that constrains unknowns.

Student Problem: Moar Quantum Dot Video Screen

If a green element needs to emit light of wavelength of 527nm...

1. What size dots do you need such that the $n = 2$ to $n = 1$ (ground state) transition of an electron in the well emits green light of the desired wavelength?



2. Write the full wave function (space and time) of the electron when it's in the $n = 3$ state.

3. What is the probability of finding the electron in the range $[L/3, 2L/3]$ of the $n = 3$ state?

4. What energy photon is required to excite the electron to the $n = 3$ state?

5. For the TV design described earlier (using blue light of wavelength 450nm to stimulate the electrons to higher energy levels so that they emit red or green photons when they return to the ground state), is there any danger of getting light from the quantum dot that you don't want? To answer this, consider:

   - Is there any danger in this system of exciting electrons to the $n = 3$ state (and thus getting undesirable colors)?
   - Is there any danger of a photon, scattered in the process of exciting an electron from $n = 1$ to $n = 2$, of being visible to the human eye? What *do* you expect to get radiated from the screen of the TV?

Reminders

- Assigned
  - Study for Exam 2 (covers general relativity; temperature, heat, and classical microscopic heat energy; radiation behaving like particles; and particles behaing like waves - Homeworks 4–6 and associated in-class, video lecture, and reading material)
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r.7 Problem Solving in the Square Well

Problem Solving in the Square Well

Discussion: Well-behaved wave functions in finite potentials

- Today will be conducted as a workshop in the Schroedinger Wave Equation and solving it for the case where the potential energy is not infinite or zero, but rather finite.
- This makes the problem more realistic, of course - but it also makes solving it more mathematically challenging.
- In fact, as you will see from a discussion later in this class period and in the notes accompanying this class period, the finite square well ... the simplest model of a realistic potential energy situation ... is NOT EXACTLY SOLVABLE but rather must be solved by approximate or numerical methods.
- Let's look a little bit at thought processes and strategies for attacking problems with finite potential energy changes.

Student Problem: Practice Setting up the SWE solution for the Finite Square Well

Without referring to the textbook or notes from before class, try setting up the SWE for the case of the Finite Square Well: $V(x) = 0$ for $0 < x < L$, but $V(x) = U_0$ (constant and finite) for all other x values. Use the trick of treating the wave function, $\Psi(x,t)$, as separable in space and time, $\Psi(x,t) = \psi(x)\phi(t)$, so that you only have to solve the time-independent SWE:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

Use the guidelines for solving the SWE:

1. Write down the SWE that includes the potential used to model the situations.
2. Separate the problem into any regions that can be treated distinctly from other regions.
3. Identify trial functions that might solve the SWE in each region.
4. Determine the values of unknown parameters in the functions using constraints: employ the guidelines of good wave function behavior (conservation of probability, smoothness); match the wave functions at boundaries between regions and see if that constrains unknowns.

Student Problem: The Quantum Dot, Revisited

Consider a quantum dot that is a finite square well of width L, with a potential height of $U_0 = 5.0\text{eV}$ for the region of the dot where the electron is free, which we will denote $x < 0$ and $x > L$. An electron in the well has total energy $E = 1.7\text{eV}$.

- What is the penetration depth of an electron trapped in the well into the forbidden region when it's in the ground state?
- Can a TV made with quantum dots like this be left out in direct sunlight? Do a calculation to show whether or not this is a good idea. (HINT: what are the shortest electromagnetic radiation wavelengths from the sun?)

Discussion: How then to Solve Such Problems?

The energy quantization condition is a transcendental equation:

$$2\cot\left(\frac{\sqrt{2mE}}{\hbar}L\right) = \sqrt{\frac{E}{U_0 - E}} - \sqrt{\frac{U_0 - E}{E}}$$

Recall that the wave number of the wave function in the free particle region is $k = \sqrt{2mE}/\hbar$; the decay constant in the forbidden region is $\alpha = \sqrt{2m(U_0 - E)}/\hbar$. Observe that there is a "Pythagorean Theorem" that constrains k and α:

$$k^2 + \alpha^2 = \frac{2mU_0}{\hbar^2}$$

Define dimensionless variables u and v:

$$u = \alpha L/2 \longrightarrow \alpha = 2u/L \text{ and } v = kL/2 \longrightarrow k = 2v/L$$

Leads to

$$2\cot(v) = \frac{v}{u} - \frac{u}{v} \text{ and } u^2 + v^2 = \frac{mLU_0}{2\hbar^2}$$

The left equation holds without regard to the particulars of the problem (u and v are dimensionless numbers); the right equation ties the allowed values of u and v to the specifics of the problem (m, L, U_0).

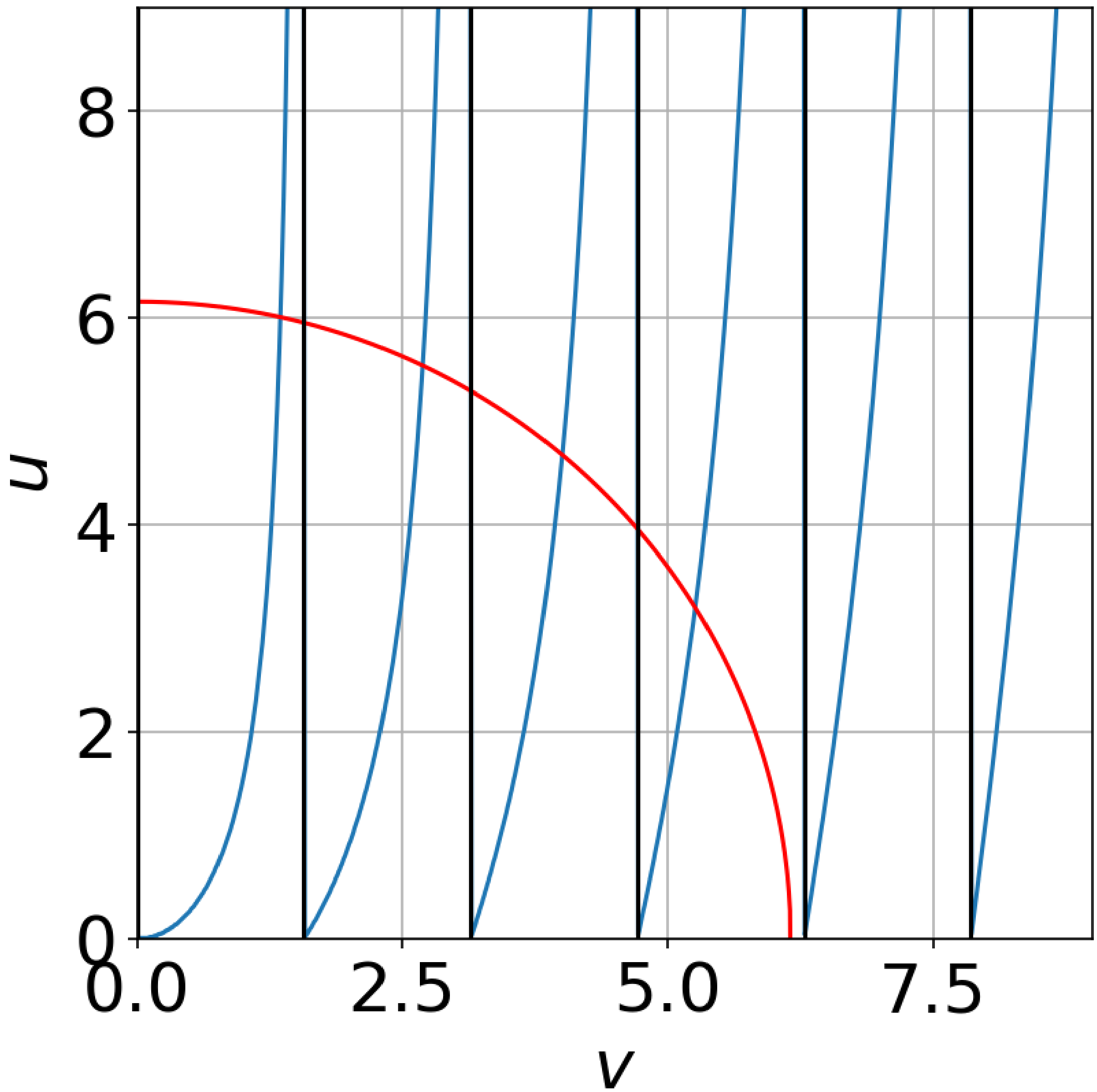
Discussion: How then to Solve Such Problems?

$$2\cot(v) = \frac{v}{u} - \frac{u}{v}$$
$$u^2 + v^2 = \frac{mLU_0}{2\hbar^2}$$

Numerically solve the problem: scan over $u = 0 \to \infty$; calculate the value of v that results from the energy quantization relationship (blue lines); independently, for your situation (m, U_0, L) scan over

$u = 0 \to \infty$ and then compute the allowed values of v from the second equation (red line). *The intersection of the two sets of curves correspond to the allowed energies for electrons in your potential well of wall height U_0 and width L!*

If this feels outside your comfort zone (even more so than wave functions and the SWE), that's OK ... it's intended to be. Solving even the most basic quantum mechanics problems is HARD.



r.8 Problem Solving in the Harmonic Oscillator

Problem Solving in the Harmonic Oscillator

Instructor Problem: Nitrogen Gas

Nitrogen gas is a diatomic molecule at room temperature on Earth. Chemically it is denoted N_2. The mass of a Nitrogen atom is 14.007 atomic mass units, or 2.3259×10^{-26} kg.

1. Two masses attached by a spring and free to vibrate can be mathematically modeled as if one of them is fixed in space while the other oscillates with a "reduced mass". What is the reduced mass of N_2?
2. What is the angular frequency of N_2 if the Coulomb Force holding it together has an effective spring constant (the "force constant" of the chemical bond) of 2300 N/m?
3. What is the energy of the ground-state of vibration of N_2?
4. What wavelength photon will excite a transition from the ground state to the first vibrational excited state of N_2?

Student Problem: Carbon Monoxide

Carbon monoxide is a diatomic molecule of Carbon (mass m_C = 12.011 amu) and Oxygen (mass m_O = 15.999 amu). The force constant of the molecule is k_{CO} = 1860 N/m.

1. What is the reduced mass of CO?
2. What is the angular frequency of CO?
3. What is the energy of the vibrational ground state of CO?
4. What wavelength photon is needed to exactly excite the first vibrational excited state of CO?
5. Could a collision with a water molecule from a hot cup of tea excite this vibrational mode? (*HINT: think about the steam rising from the surface of the tea and the motion of water molecules in that steam*)

Student Problem: Carbon Monoxide Wave Function

Let's stick with Carbon Monoxide again for this problem. This time, imagine a photon with wavelength 2331 nm is fully absorbed by the molecule and is excited to a new vibrational energy state from the ground state.

1. What is numerical value of the normalization constant of the wave function that describes this new state?
2. At what $|x|$ distance from equilibrium is the classical limit, x_{max}, of this energy level located?
3. What is the probability of finding the system at a separation between 0 to $(1/3)x_{max}$ from equilibrium?

r.9 Problem Solving in Expectation Values

Problem Solving in Expectation Values

A Basic Discussion of Statistical Quantities

Not all outcomes are absolutely certain. Statistics and probability are highly valuable skills in the real world. There is a lot of misunderstanding of statistics, but grasping the basic concepts and understanding the tools to use them will protect you somewhat from that misunderstanding.

A good basic example is the rolling of a 6-sided die. Let's assume that every roll of the die is uncorrelated with the previous or the next one, and that each roll on its own is *unbiased* - there is no process that favors one side of the die over any other.

- What is the probability of rolling the number "6"? Since all rolls are uncorrelated and each roll is unbiased, there is a one-in-six chance on any roll of getting the side "6" to face up. Therefore, $p_6 = \frac{1}{6}$. But what does this mean, practically and experimentally? It means that if I could roll the die an infinite number of times, the fraction of rolls that yield a "6" will be exactly 1/6 of the total. For finite statistics, the fraction of the population will tend toward 1/6 as one increases the number of rolls.

- What is the mean value I expect to roll? This is a very different question. This is asking, on average, if I roll the die an infinite number of time and write down all the numbers I get from my rolls, what is the average of all my rolls? Let x represent the outcome of any single roll.

$$\overline{x} = \frac{\sum_{i=1}^{6} x_i p_i}{\sum_{j=1}^{6} p_j} = \frac{1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \ldots 6 \cdot \frac{1}{6}}{\frac{1}{6} + \ldots + \frac{1}{6}} = \frac{\frac{1}{6}(1+2+3+4+5+6)}{1} = \frac{21}{6} = 3.5$$

A Basic Discussion of Statistical Quantities

What about a 20-sided die? That presents many more sides, and thus many more outcomes each with its own probability. But so long as the rolls of this die are uncorrelated and each roll is unbiased, $p_i = 1/20$ and the calculation proceeds as on the previous slide. But what if the die has been weighted in some way so as to make rolling a 20 twice as likely as rolling any other side? It's a little harder to figure out the probability now, so let's switch to *probability density*. Let's set the probability density of any roll of any side other than $i = 20$ to $P_i = 1$, and set $P_{20} = 2$. Reuse our framework, but use the probability densities in the computation:

$$\overline{x} = \frac{\sum_{i=1}^{20} x_i P_i}{\sum_{j=1}^{20} P_j} = \frac{1 \cdot 1 + 2 \cdot 1 + \ldots 20 \cdot 2}{1 + 1 + \ldots + 2} = \frac{230}{21} = 10.952$$

If you could do enough rolls of this die, you could detect the bias in the outcome if you computed the average precisely enough. More rolls = more precision = more chance of detecting bias. We can use this approach to compute the average of anything we like - for instance, the square of the number we roll:

$$\overline{x^2} = \frac{\sum_{i=1}^{20} x_i^2 P_i}{\sum_{j=1}^{20} P_j} = \frac{1 \cdot 1 + 4 \cdot 1 + \ldots 400 \cdot 2}{1 + 1 + \ldots + 2} = \frac{3270}{21} = 155.714$$

A Basic Discussion of Statistical Quantities

Another important statistical quantity is the *standard deviation* - how much you expect any roll to "typically" deviate from the mean value of all possible outcomes. This is related to the *variance* of the distribution of outcomes, which is defined as:

$$\begin{aligned} \mathrm{Var(x)} &\equiv \overline{x^2} - (\overline{x})^2 \\ \mathrm{StdDev(x)} &\equiv \sigma = \sqrt{\mathrm{Var(x)}} \end{aligned}$$

For the 20-sided die example,

$$\begin{aligned} \overline{x^2} - (\overline{x})^2 &= 155.714 - (10.952)^2 = 35.759 \\ \sigma &= 5.980 \end{aligned}$$

Because the outcomes are generally so spread out for a die roll, even a slightly biased one, (because the probability densities are so similar for each outcome) we expect the typical deviation from the mean to be large — it is!

A Basic Discussion of Statistical Quantities

It's impractical to do all this by-hand summing when P(x) is a continuous analytic function, so instead we can generalize this idea using calculus:

$$\overline{x} = \frac{\int_{\text{all space}} x P(x)\, dx}{\int_{\text{all space}} P(x)\, dx}$$

If the probability density function, $P(x)$, is *normalized* such that $\int P(x) dx = 1$, then the denominator is known automatically. You don't have to normalize $P(x)$, so long as you stick to the above rigorous formula for the mean - which is the formula for the *expectation value*.

The definitions of the variance and standard deviation are unchanged, as they merely utilize how you compute the mean (expectation value):

$$\begin{aligned}\mathrm{Var(x)} &\equiv \overline{x^2} - (\overline{x})^2 \\ \mathrm{StdDev(x)} &\equiv \sigma = \sqrt{\mathrm{Var(x)}}\end{aligned}$$

The *standard deviation* is used in statistics as a measure of *uncertainty* in the outcome of a series of experiments or measurements or observations.

Statistics and Quantum Mechanics

In quantum mechanics, we have to modify our language a little bit. Whereas in statistics, x would be taken to be a random variable whose distribution is governed by $P(x)$, in quantum mechanics the position of a particle, or the momentum of a particle, is determined by *operators*. An operator is a mathematical function whose action on the wave function returns a number - a measurement. The probability density is defined by the square of the *wave function*. The operator definitions we will use are given in what is called *position space* and are defined as follows:

- Position Operator ($\hat{x}$): the operator for position is just $\hat{x} = x$.
- Momentum Operator ($\hat{p}$): the operator for momentum is $\hat{p} = -i\hbar\frac{d}{dx}$ (in one dimension)

In quantum mechanics, we are computing not the expectation value of a random variable but of the outcome of an *operator acting on a wave function*. So when we "make a measurement of position of a particle" whose wave function is $\Psi(x,t)$, what we mean *mathematically* is:

$$\hat{x}\Psi(x,t) = \tilde{x}\Psi(x,t)$$

where $\tilde{x}$ is a real number - the outcome of the measurement.

Statistics and Quantum Mechanics

What if we want to know the *average outcome we expect from a series of measurements*? (expectation value) Let's work it out:

$$\begin{aligned}\hat{x}\Psi(x,t) &= \tilde{x}\Psi(x,t) \\ \Psi^*(x,t)\hat{x}\Psi(x,t) &= \Psi^*(x,t)\tilde{x}\Psi(x,t) = \tilde{x}\Psi^*(x,t)\Psi(x,t) \\ \int_{\text{all space}} \Psi^*(x,t)\hat{x}\Psi(x,t)dx &= \int_{\text{all space}} \tilde{x}\Psi^*(x,t)\Psi(x,t)dx = \tilde{x}\int_{\text{all space}} \Psi^*(x,t)\Psi(x,t)dx = \tilde{x} \\ \int_{\text{all space}} \Psi^*(x,t)\hat{x}\Psi(x,t)dx &\equiv \overline{x}\end{aligned}$$

Similarly for momentum:

$$\begin{aligned}\int_{\text{all space}} \Psi^*(x,t)\hat{p}\Psi(x,t)dx &\equiv \overline{p} \\ \int_{\text{all space}} \Psi^*(x,t)\left(-i\hbar\frac{d}{dx}\right)\Psi(x,t)dx &\equiv \overline{p}\end{aligned}$$

In general, the order in which you act with operators matter (e.g. operators can be derivatives, which you must first evaluate before moving terms around in the integral).

Instructor Problem: Expected Outcomes in Position Space for a Quantum Dot

Let's revisit the quantum dot from a few class periods ago. It had a width of 0.76nm and we modeled it as an infinite square well. Consider an electron in the first excited state (n=2) of this well, having been raised to that level by blue LED radiation of 450nm in wavelength.

1. What is the expectation value of the position of the electron in this state?
2. What is the expectation value of the position-squared of the electron in this state?
3. What is the uncertainty in its position?

Student Problem: Expectation Values in Momentum for a Quantum Dot

Go further with this same quantum dot of $L = 0.76\text{nm}$ in state $n = 2$.

1. What is the expectation value of the momentum of the electron in the n=2 state?
2. What is the expectation value of the momentum-squared of the electron in the n=2 state?

Student Problem: The Uncertainty Principle and the Quantum Dot

Let's delve deeper into this same quantum dot of $L = 0.76\text{nm}$ in state $n = 2$.

1. What is the uncertainty on the momentum of the electron in the n=2 state?
2. How does the product of Δx and Δp compare to the Heisenberg Limit?
3. What is the uncertainty on the energy of the electron in the n=2 state?

r.10 Problem Solving in the Step Potential

Problem Solving in the Step Potential

Instructor Problem: Accelerator Design Mishap!

While designing a particle accelerator for Carbon Ions (Z=6, A=12), an error is made in fabricating a part. This results in a sudden change in the potential to which accelerated particles are subjected. You are to diagnose the severity of the mistake.

The ions are injected with a kinetic energy of 100 keV into a region of the accelerator where they are supposed to experience no changes in potential energy; however, at $x = 3.0\mathrm{m}$ into this stage of the accelerator, they encounter a slight change in potential of 10V.

1. What is the corresponding change in potential energy?
2. What is the probability of transmitting across the change in potential?
3. What is the probability of reflecting off the change in potential?
4. If 3.0×10^9 carbon ions per second pass through this region, what is the rate of reflection?

Student Problem: Speed Bump

You are driving along a road in University Park at 30mph when you encounter a speed bump with a height of 4.0in. Your car has a mass of 2000kg. Treat your initial potential energy as 0J, and treat your car as a single particle of the given mass.

1. What is the kinetic energy of the car before encountering the speed bump?
2. What is the height of the "potential energy step" encountered by the car at the bump?
3. What is the matter wavelength of the car before it encounters the speed bump?

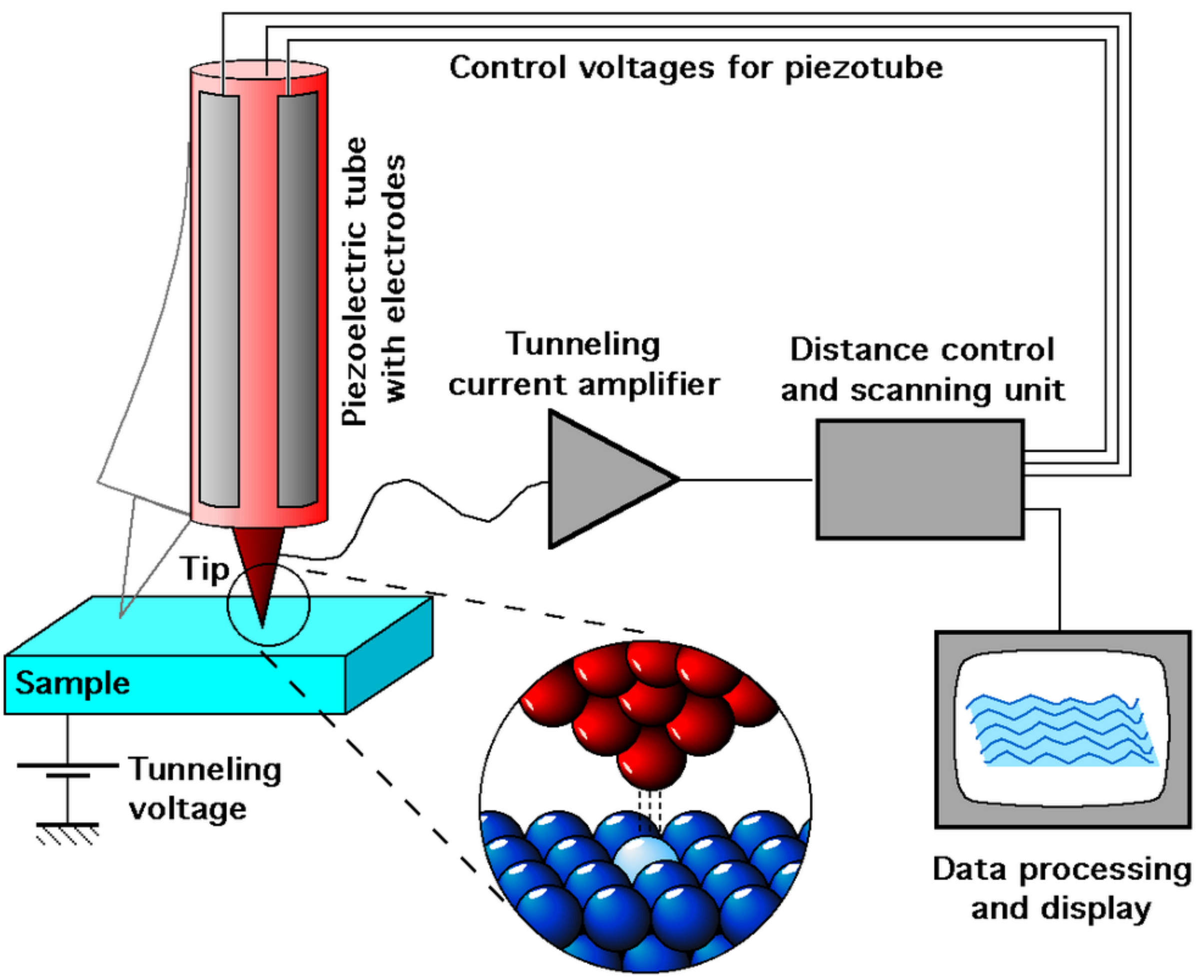
Photo by Makarios Tang on Unsplash

4. What is the change in the matter wavelength of the car before and after it encounters the bump? $(\Delta\lambda = \lambda_f - \lambda_i)$
5. What is the transmission probability of the car?
6. What is the reflection probability of the car?
7. Solve the following "Fermi Problem": *Estimate the number of times every day, in the United States, that people experience bumps in the road while driving.* Learn more about Fermi Problems.
8. Why is the modeling of this problem using a quantum step potential correct or incorrect? (In other words, can you explain why it is we don't live in fear of reflecting off speed bumps in residential neighborhoods or potholes on highways?)

Problem Solving in Quantum Tunneling

PhET Simulator of Quantum Tunneling

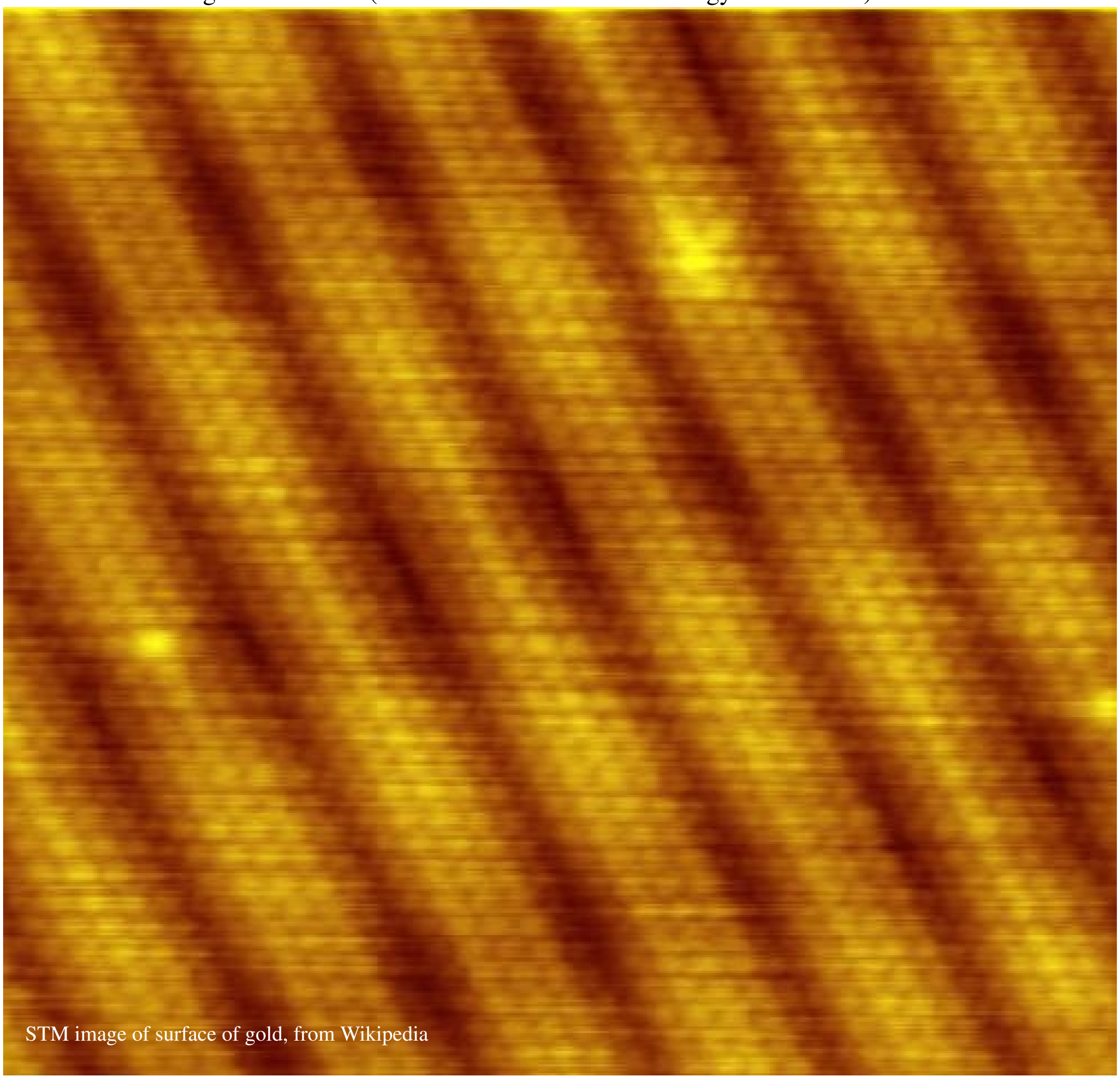
(Requires JAVA)



Instructor Problem: Scanning Tunneling Microscope

A Scanning Tunneling Microscope (STM) moves a very, very fine-tipped (e.g. approximately 1 atom at the tip) probe over the surface of a material, making no physical contact with the surface. The probe is held at a higher electric potential than the surface and there is a physical gap of 3nm between the two. Electrons in the atoms in the surface can tunnel into the probe across this gap and "overcome" this potential difference.

If the probe voltage is -0.1V (with the sample grounded), and the material is gold, the easiest-to-remove electrons have energies of 5.53 eV (this is known as the Fermi Energy of the metal)



STM image of surface of gold, from Wikipedia

1. What is the probability than an electron in the surface will tunnel into the probe tip?
2. Use the energy of the electron to determine the average speed of the electron.
3. If a gold atom has a radius of 144pm, and we think of the electron as bouncing back and forth across the parent atom, how many times per second does the electron reach the gap (and thus make itself "available" to tunnel)?

4. Based on this, estimate the "tunneling current" into the tip of the probe.

Student Problem: Fusion in the Sun

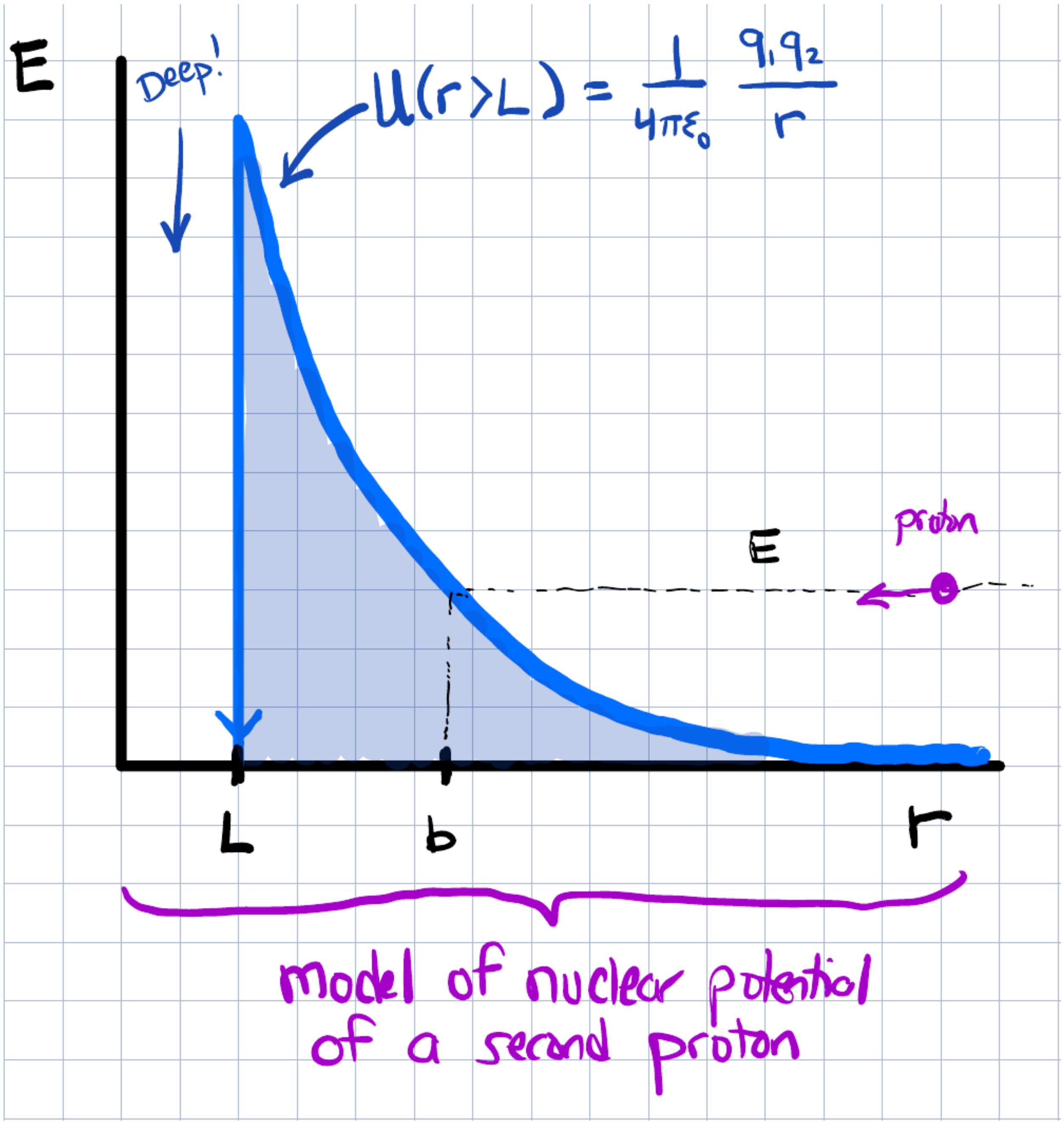
Without quantum tunneling, the fusion that powers the sun would not be possible. Let's explore this.

1. In order to fuse, two protons have to get to a separation of L=2fm. What is the height of the Coulomb potential energy between these two protons at that separation?

Once two protons get to within the fusion distance, the strong nuclear force takes over and overwhelms the Coulomb repulsion. So the potential energy drops off STEEPLY at L. The nuclear potential then can be approximated as $U(r < L) = 0$, while $U(r > L) = ke^2/r$.

2. Treating the protons in the Sun's core as an ideal gas, what temperature would the core have to be at in order for protons to get this close?
3. The temperature at the core of the Sun is actually only 10^7 K. How close, b, can protons get under these conditions?
4. The nuclear potential is not a square well… but we can try to approximate it as one. To do this, determine the height of the square potential whose base is $b - L$ and whose area is the same as that enclosed under the nuclear potential between L and b.
5. What is the tunneling probability?
6. Is it reasonable to apply the "wide barrier" approximation to this problem? Use a calculation to evaluate this.
7. How much does the exact tunneling probability differ from the wide barrier approximation?

Sketch of the Nuclear Potential Model (between two protons)

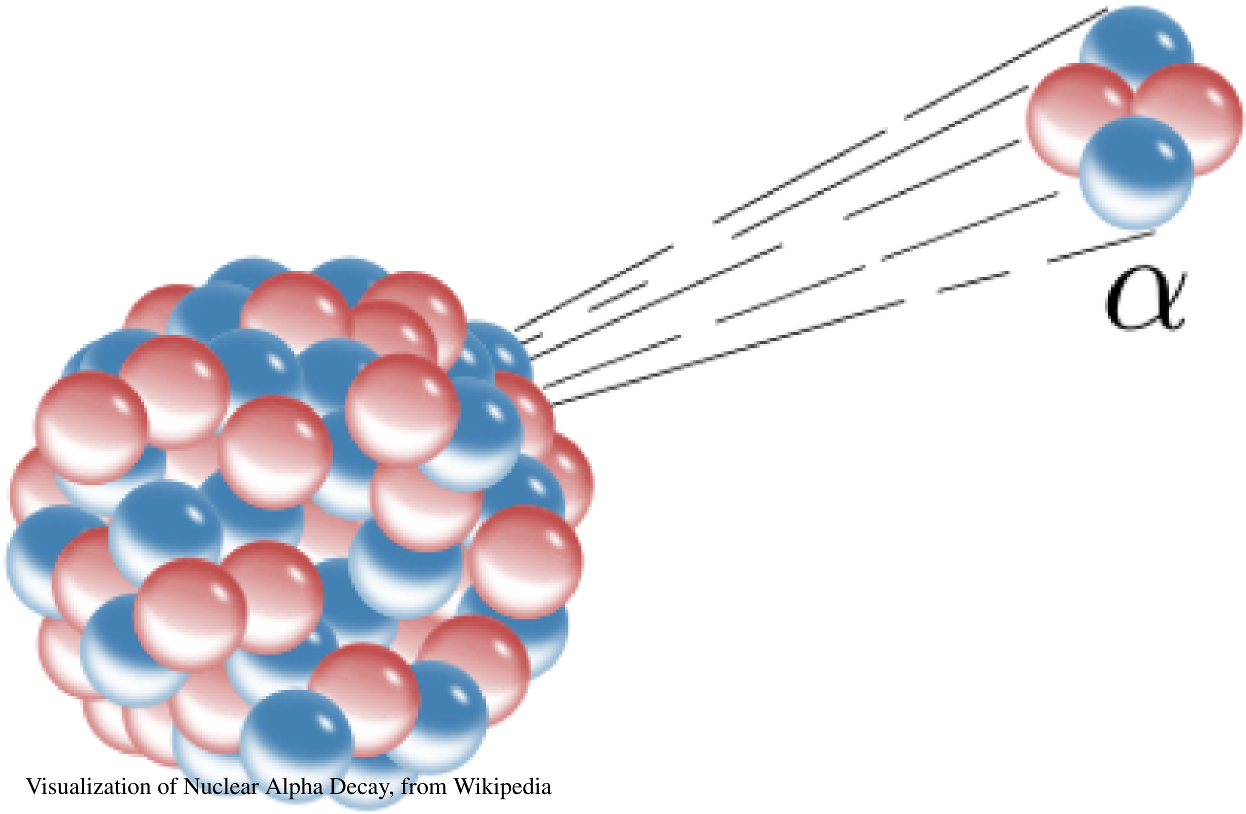

E
Deep!
U(r>L) = 1/4πε₀ q₁q₂/r
E
proton
L
b
r
model of nuclear potential of a second proton

r.12 Workshop in Quantum Tunneling Application

Workshop in Quantum Tunneling Application

Workshop Problem: The Half-Life of Radon-222

The book presents alpha decay in a rather hand-waving way, but provides a basic recipe for determining how long it will take for a nucleus to emit a helium nucleus (an alpha particle). We will break this down into steps and compute the half-life of Radon-222 (Rn-222). Rn-222 is a source of serious noise for sensitive experiments that demand a very low radioactive contamination level (e.g. equipment that hunts for dark matter or ultra-rare nuclear processes involving the production of particles called "neutrinos").



Visualization of Nuclear Alpha Decay, from Wikipedia

First, basic facts about Rn-222:

- Its mass number, A, is 222 (the total number of protons and neutrons)
- Its atomic number, Z, is 86 (the total number of protons)
- An alpha particle has a A=4 and Z=2
- When Radon emits an alpha particle, the alpha particle always has energy $E = 5.5904\ \mathrm{MeV}$.

When Radon decays in this way, the reaction equation is:

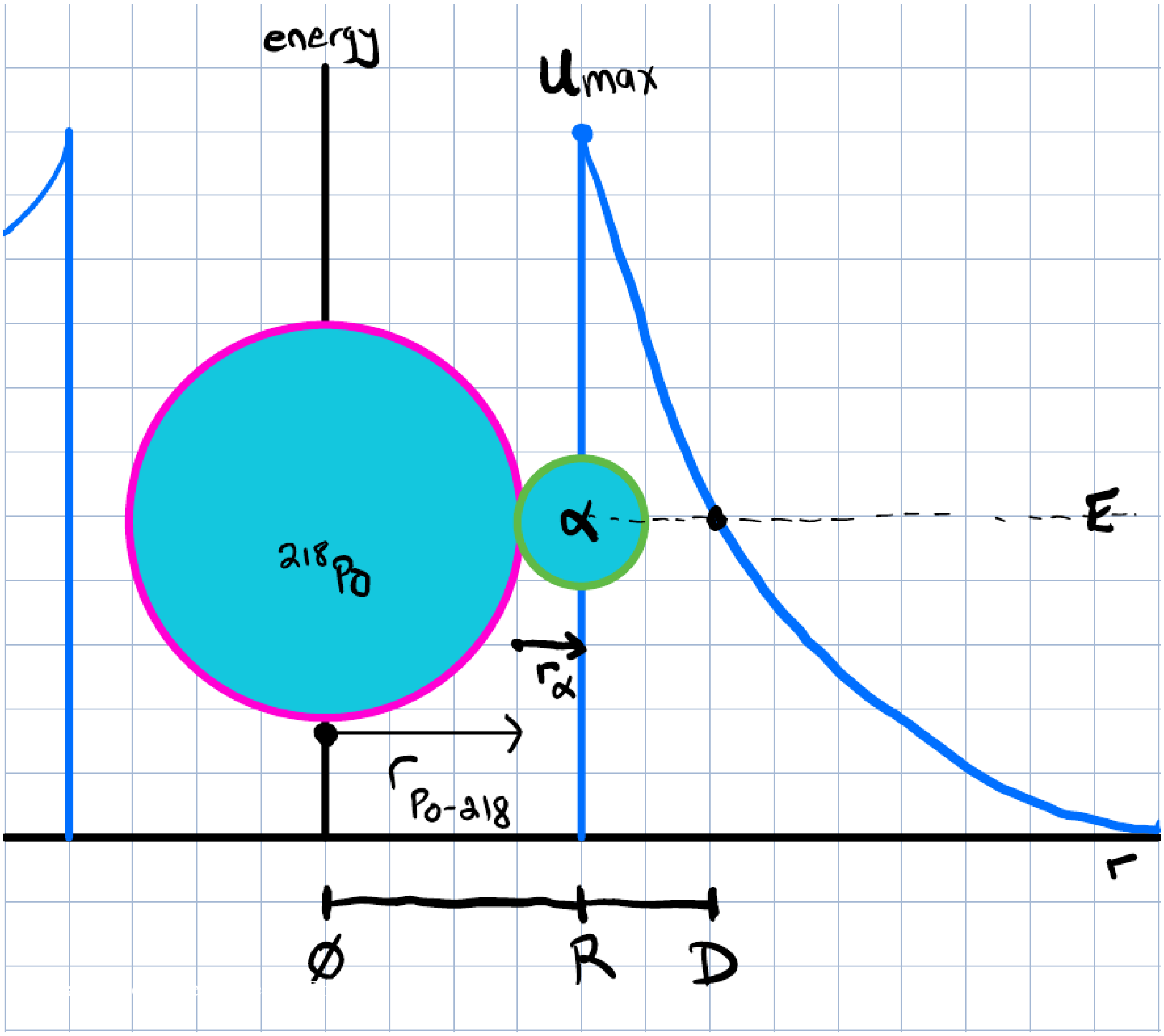
$$^{222}\mathrm{Rn} \rightarrow {}^{218}\mathrm{Po} + \alpha$$

Phase 1: The Nuclear Potential

The sketch of the nuclear potential of a heavy element is shown right, with key features labeled. The radius of a nucleus is given by $r = r_0 A^{1/3}$ where $r_0 = 1.2\mathrm{fm}$. Determine the following parameters:

- r_α - the nuclear radius of the α

- r_{Po} - the nuclear radius of the Polonium nucleus, ^{218}Po
- R - the maximum separation of the centers of the alpha particle and the Polonium nucleus before encountering the wall of the nuclear well
- U_{max} - the maximum height of the Coulomb barrier at the top of the nuclear well (*HINT: this is the Coloumb potential energy of the alpha particle in the electric field of the Polonium nucleus)*

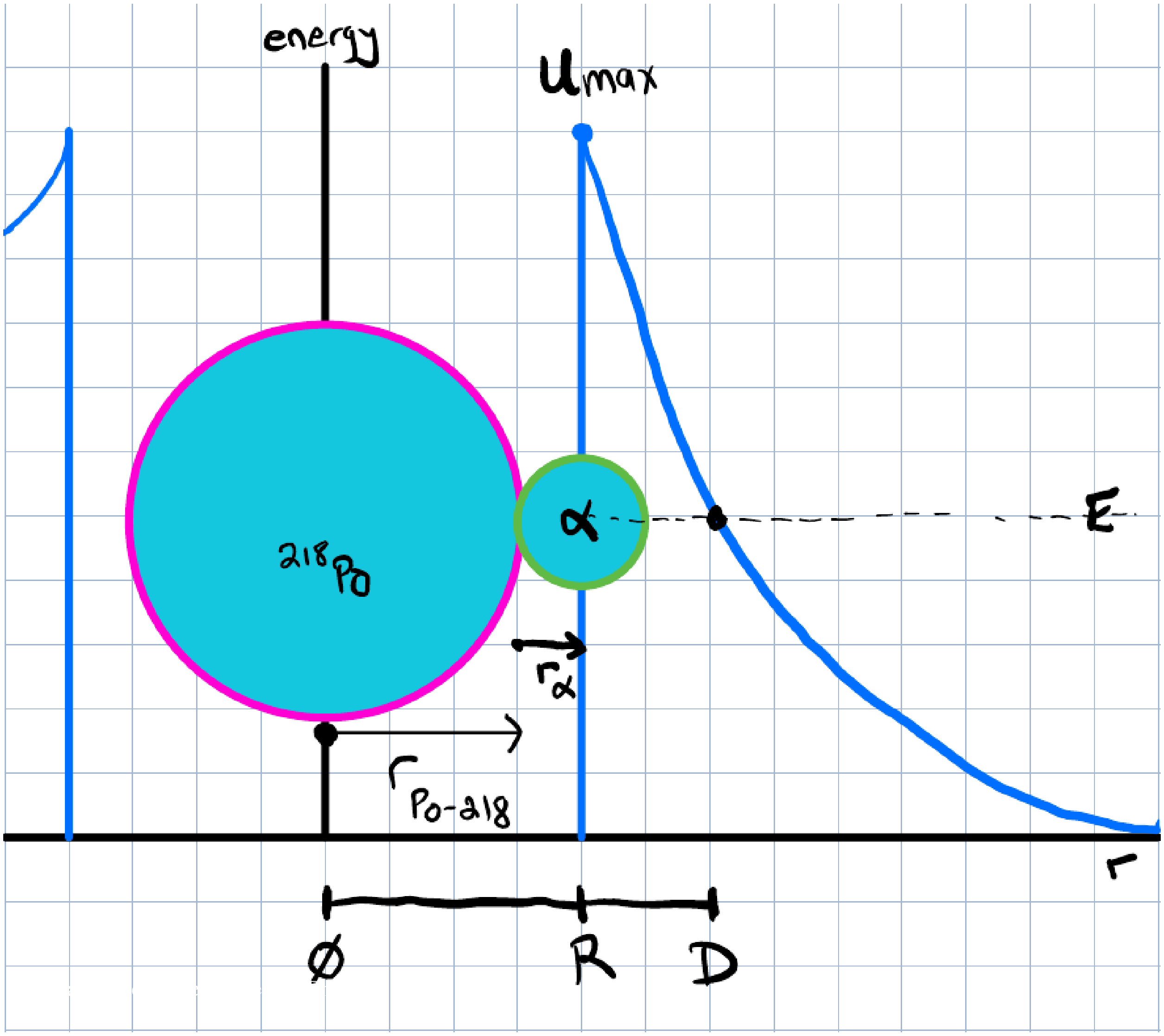


Phase 2: The Moving Alpha Particle

The α particle can be thought of as having a velocity v given its energy E. It moves around inside the nuclear well, banging into the wall of the nuclear potential on the left, recoiling, and striking the wall on the right (then recoiling again and repeating the whole motion).

Determine the following things:

- v - the speed of the alpha particle inside the nuclear potential
- f - the frequency with which the alpha particle encounters one wall or the other of the nuclear well

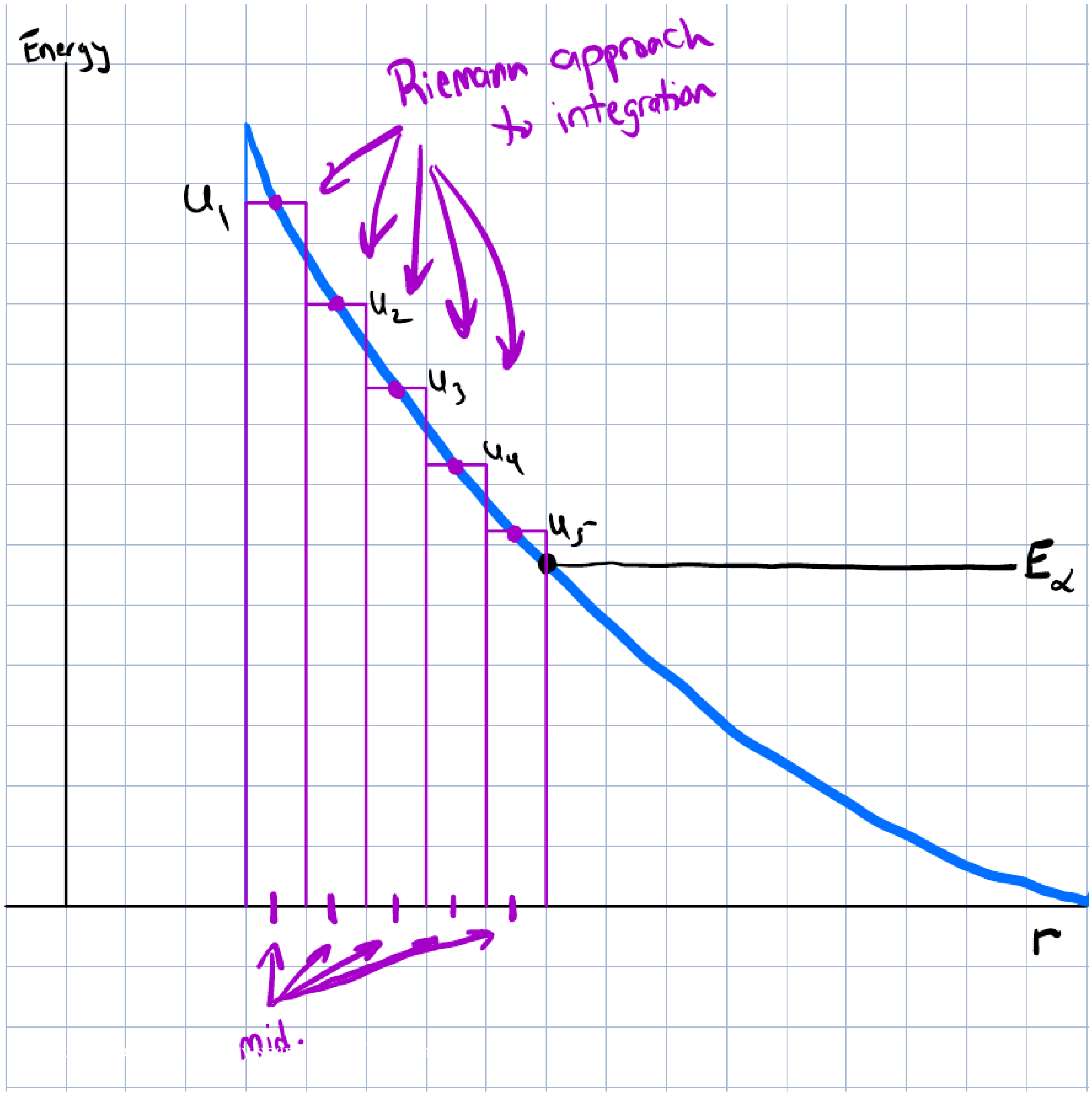


Phase 3: Approximating the Barrier Using Riemann Sums

To tunnel through the wall of the nuclear potential, the alpha particle with energy E must overcome a barrier of *maximum* height, U_{max}, but whose height *declines* the farther through the barrier the wave function penetrates. This is not easy to handle, and is poorly approximated by a single rectangular barrier.

Instead, let's use the method of Riemann Sums to break the barrier into 5 equal-width rectangles.

- w - the width of the barrier at the energy, E, of the alpha particle. (*HINT: if the alpha particle instead entered from outside the barrier and tried to get into the nuclear well, what is the distance of closest approach, D, that is possible classically? The width is* $w = D - R$)
- U_i - dividing the barrier into 5 equal-width pieces of thickness $w/5$, compute the height of the Coulomb potential at the mid-point of each segment. This is U_i (for $i = 1, 2, \ldots, 5$)



Phase 4: Compute the Tunneling Probability

The wave function in each segment of the barrier, i, is given by:

$$\psi_i(x) = A_i e^{-\alpha_i x}$$

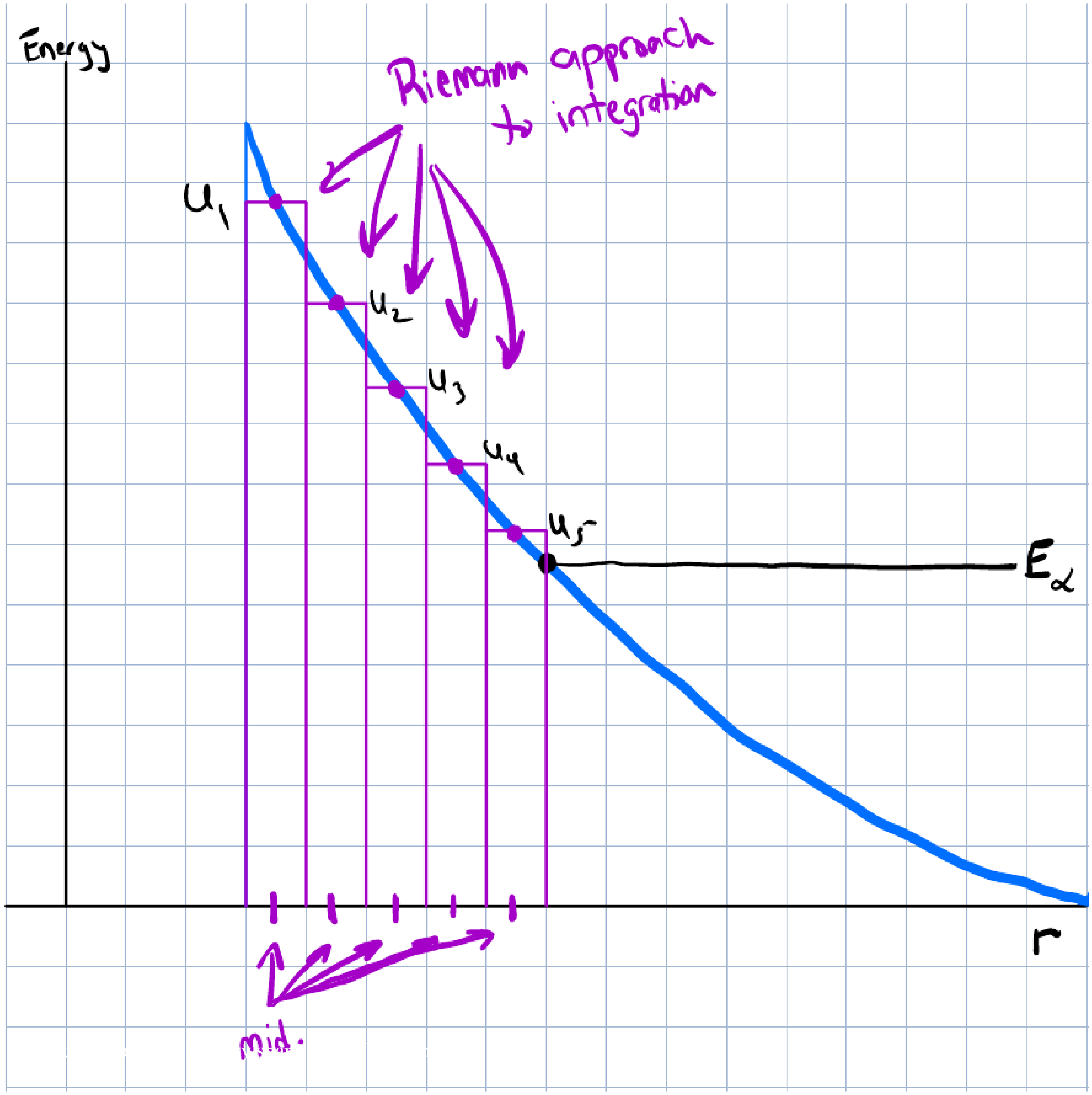
The probability to tunnel through each section of width w is given by:

$$P_i = \frac{|A_i|^2 e^{-2\alpha_i w}}{|A_i|^2 e^{-2\alpha_i 0}} = e^{-2\alpha_i w}$$

The total probability of tunneling through ALL 5 sections is $P_{total} = \prod_{i=1}^{5} P_i$.

- P_i - calculate the 5 tunneling probabilities, using U_i and w and E.

- P_{total} - combine the 5 probabilities to compute the total probability



Phase 5: The Half Life

- τ - Use the barrier encounter frequency ("encounters per second") and the tunneling probability ("probability to tunnel per encounter") to compute the lifetime of the nucleus (mean time until the alpha particle is emitted)

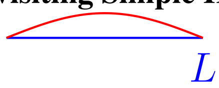
- $t_{1/2}$ - the Half-Life is given by $t_{1/2} = \tau \ln(2)$. Calculate this. Compare to the measured half-life, 3.8215 days. Discuss!

Graphic from the game “Half Life”

r.13 Workshop in Multi-Atom Systems

Workshop in Multi-Atom Systems

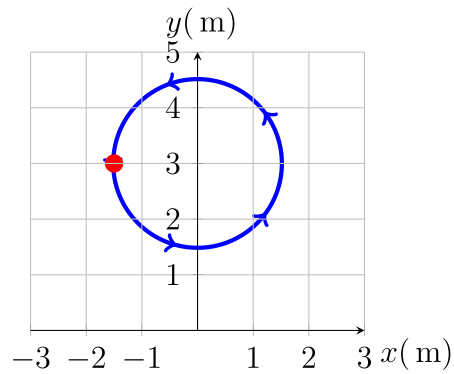
Revisiting Simple Harmonic Motion



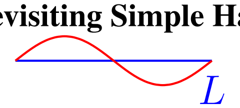
Consider the wave at the left. How many cycles of the wave are we seeing?

The half-cycle on the left traces out the sine function, which you can think of as representing the y-coordinate of an object in circular motion about a center

A half-cycle of the wave on the left represents a journey of π radians on the right. The wave number k is thus π/L, or "a half-cycle per unit length".



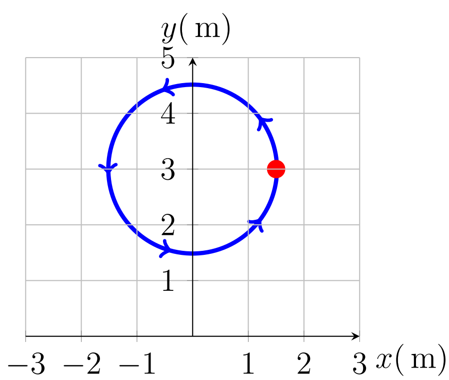
Revisiting Simple Harmonic Motion



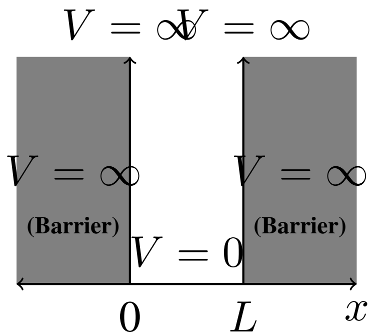
Consider the wave at the left. How many cycles of the wave are we seeing?

The full cycle on the left again can be related to circular motion of a point around a center.

A full cycle of the wave on the left represents a journey of 2π radians on the right. This has a wave number $k = 2\pi/L$, or "a full cycle per unit length".

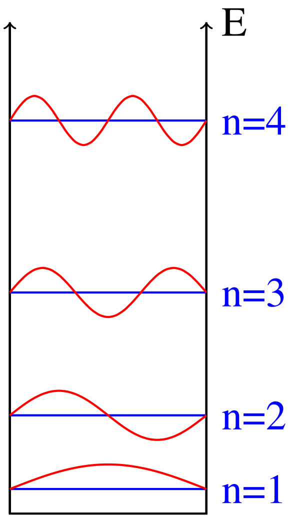


Revisiting the Infinite Square Well



Allowed solutions have energies:

$$
\begin{aligned}
E &= \frac{n^2\pi^2\hbar^2}{2mL} = \frac{p^2}{2m} \\
p &= \hbar k \rightarrow k = n\pi/L
\end{aligned}
$$



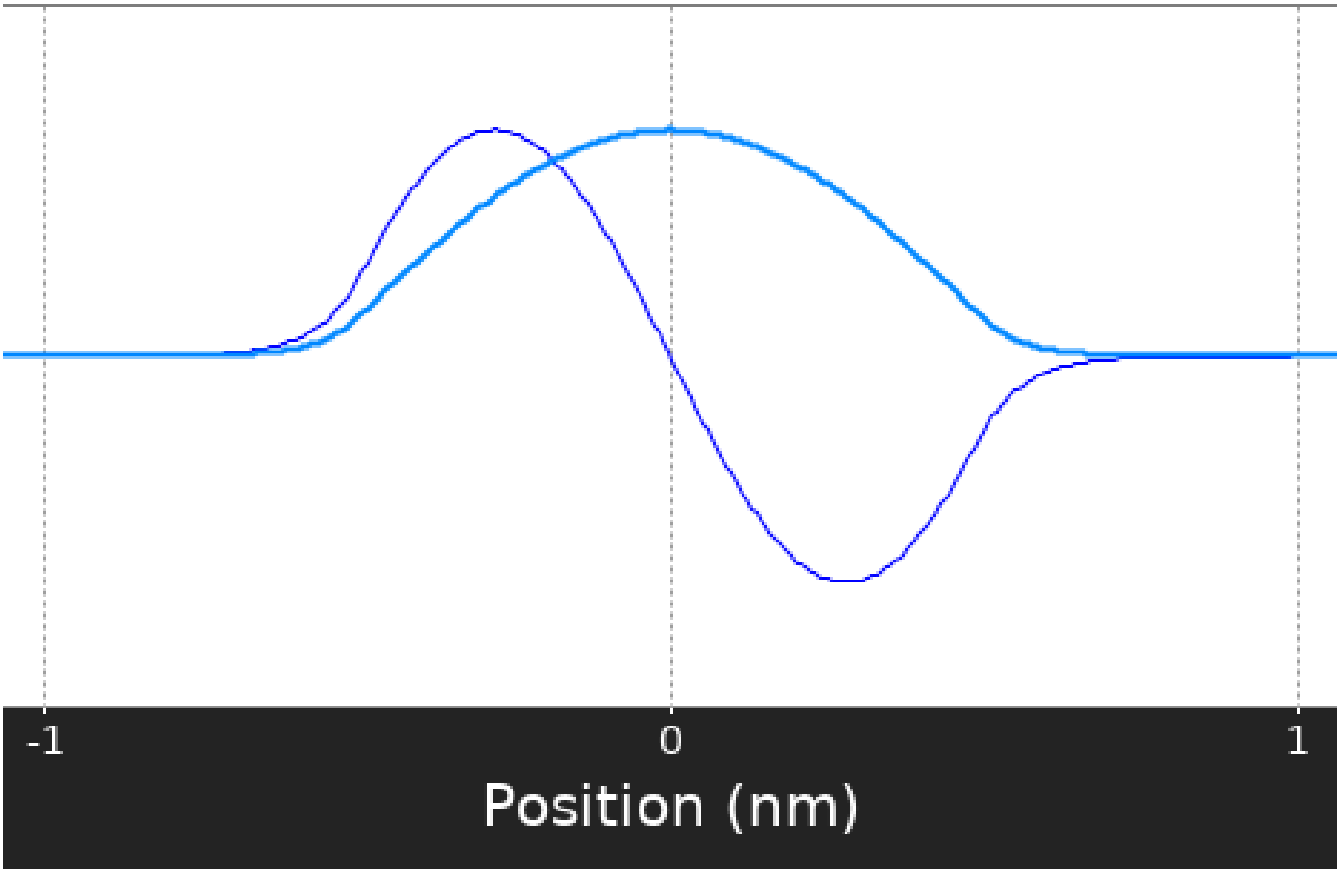
Note the relationship between momentum, wave number, n, and the number of "zero-crossings" the wave function makes.

n	zero crossings
1	0
2	1
3	2
4	3

Zero-crossings z are given by $n-1$, so $k = (z+1)\pi/L \rightarrow$ momentum is a function of zero-crossings. $n = 1 \rightarrow k = \pi/L$, or "half-cycle per unit length", whereas $n = 2 \rightarrow k = 2\pi/L$, or "full cycle per length," etc.

Class Discussion: Reading Wavefunctions and Potentials

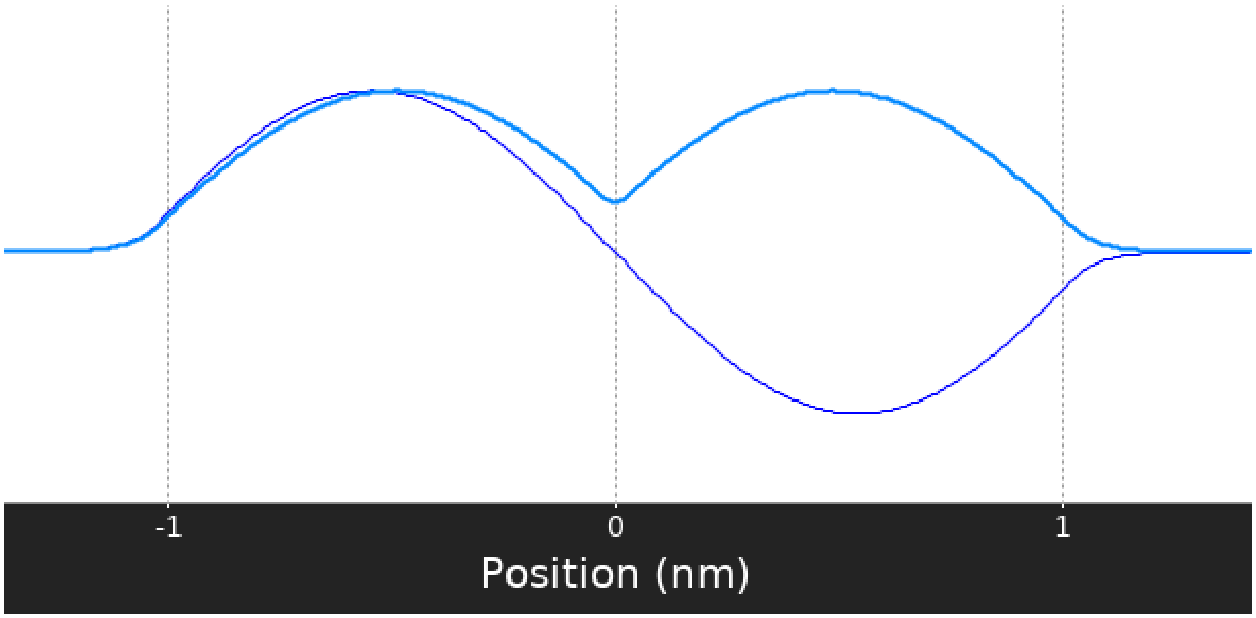
The following wave functions are solutions to the finite square well. Which of them has more total energy?



ANSWER: the purple one. Rule of thumb: the more zero-crossings, the higher the wave number; the higher the wave number, the highe the momentum; the higher the momentum, the higher the energy.

Class Discussion: Reading Wavefunctions and Potentials

The following wave functions are solutions to the finite square well, but now with two adjacent wells separated by a small potential wall. Which of them has more total energy?



ANSWER: the purple wave function $\rightarrow$ *more zero crossings!*

Bound State Simulator

PhET Simulator on Quantum Bound States

Worksheet distributed to class.

s Special Topics: Nuclear Medicine

Nuclear Medicine

s.1 Nuclear Medicine

What is Nuclear Medicine?

- The employment of radioactive materials in the diagnosis and treatment of disease $\rightarrow$ *from the inside to the outside*
- Examples

  Brachytherapy The introduction of small "seeds" containing radioisotopes into undesirable tissue (e.g. cancerous growths) with the intent to use emitted radiation to damage the tissue in the immediate region of the seed.

  PET Scan The injection of a radioisotope (usually a sugar substitute) into the bloodstream that is then taken up by blood-hungry tissues (e.g. cancers); where the radioisotope collects, radiation is emitted and an image can be formed from this radiation.

- In this part of the lecture, I will go into the details of the PET scan; the physics involved here neatly applies to brachytherapy, as the fundamental element is an unstable nuclear isotope, radioactive decay, and interaction of emitted radiation with the body.

s.2 A Brief History of Radioactivity

A Brief History of Radioactivity

Discovery of Radiation

Roentgen discovers x-rays serendipitously in 1895 while experimenting with “cathode rays” (electrons boiled off a metal using a strong electric field)


Hand mit Ringen
igenthum von Prof. Zehnder
Freiburg i/B
PHYSIK. INSTITUT
DER UNIVERSITÄT

First medical x-ray from 1895 showing the hand of Anna Ludwig (Roentgen's spouse)
The x-rays were sourced from the cathode ray tube; accelerated electrons crashed into material at the end of the tube and the scattering process caused the release of high-energy photons…gamma rays. Henri Bequeral would then discover similar radiation being emitted by Uranium salts…naturally occurring gamma radiation requiring no external input of energy.

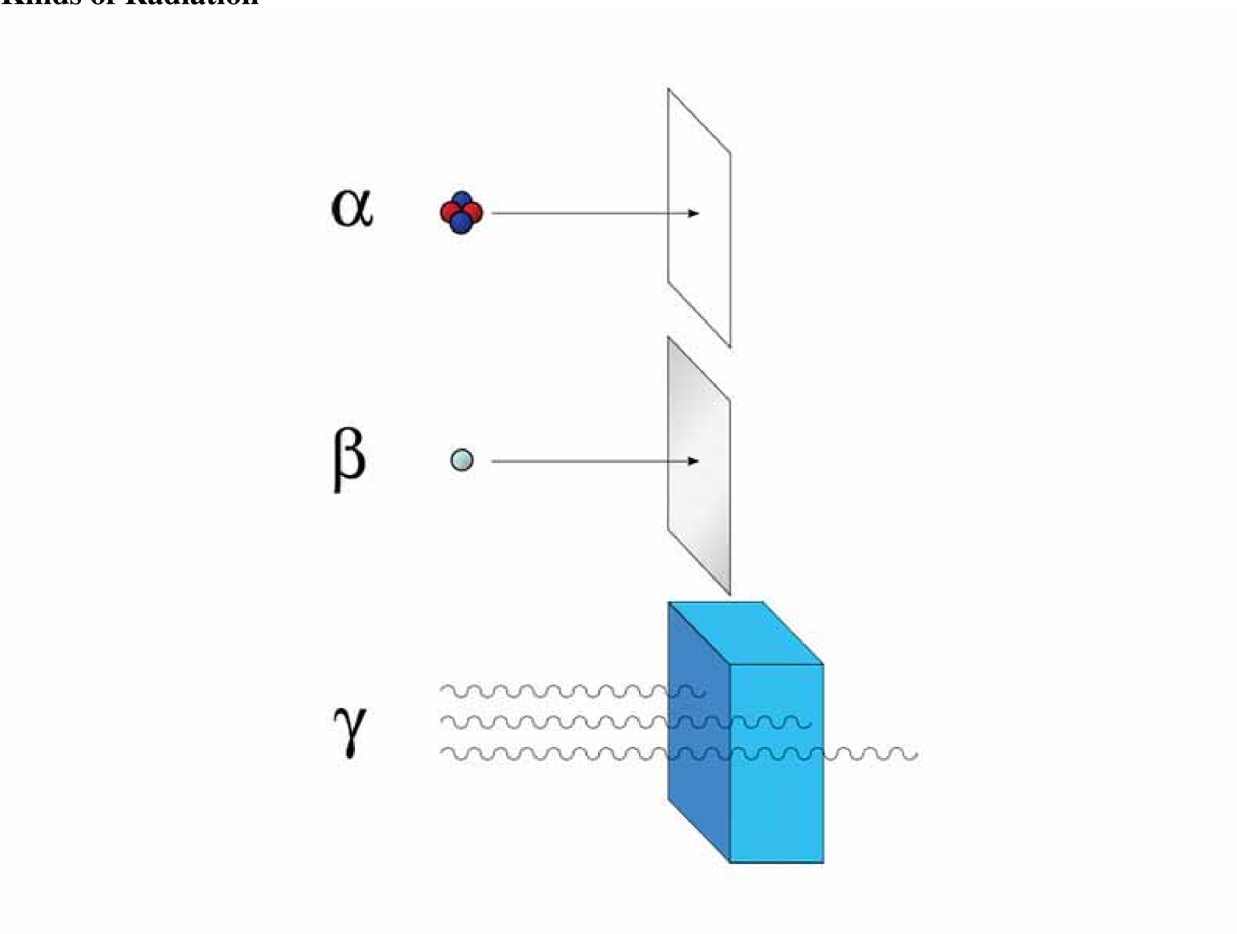
The Discovery of Radioactivity

Marie and Pierre Curie
(1766-1844)
Photo from 1903

Marie Curie performed detailed studies of the degree of radiation emitted by natural sources, like pitchblende and tobernite, and found that these were more active than Uranium alone. Pierre dropped his work on crystals to collaborate with Marie. They would discover Radium and Polonium and she would develop the first theory of radiation.

Marie Curie would go on to be the first female scientist to ever win a Nobel Prize, the first to win a second Nobel Prize, and is still the only person ever to have won prizes in two different categories (Physics in 1903 and Chemistry in 1911).

Kinds of Radiation



α Later understood to be ejected Helium nuclei (^{4}He), they are highly ionizing ($q = 2e$) but short-ranged, and can be stopped by soft tissue, paper, etc.

β Later understood to be high-speed electrons ejected from nuclei, they result from nuclear interactions induced by the "weak nuclear force", e.g. $n \to p^+ e^- \nu_e$.

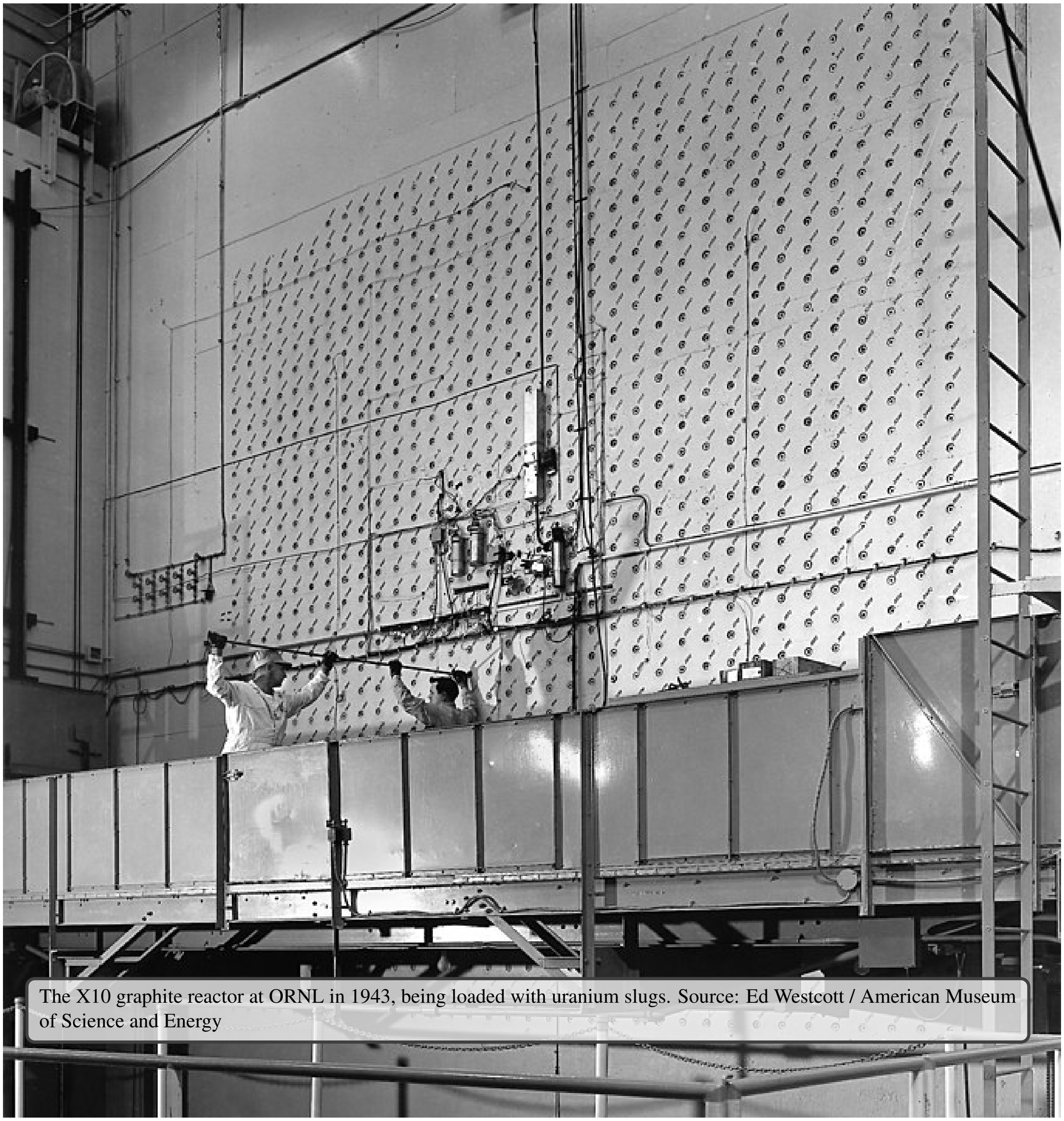
γ High-energy photons, these are emitted whenever a nucleon (n,p) drops from an excited state to a lower allowed nuclear bound state, and have typical energies at the MeV-scale.

Advent of the Nuclear Age and Nuclear Medicine

The race to build the first atomic weapons, in a bid to bring World War II to a rapid close, led to massive industrial nuclear capability, beginning with the ability to create artificial isotopes (e.g. by bombarding nuclei with neutrons) or enrich naturally occurring isotopes.

Example: Factory production of plutonium is shown at the right. Oak Ridge National Laboratory, created to support the war effort, constructed the X10 "breeder reactor" convert uranium into plutonium. Uranium slugs are fed in one side. The proximity of the slugs to one another promotes a high rate of fission through emission of neutrons. The fission rate is moderated by graphite. By pushing the uranium slugs through the reactor over a period of time, when they pop out the other end they will be filled with Plutonium, which can be chemically separated from Uranium and purified into bomb-making material.

Post-war, this production capability was bent toward peacetime applications, including the production of isotope for medicine at ORNL.



The X10 graphite reactor at ORNL in 1943, being loaded with uranium slugs. Source: Ed Westcott / American Museum of Science and Energy

The Particle Accelerator and Nuclear Medicine

One common medical isotope, utilized in the “PET Scan,” is ^{18}F. It is produced, not in a reactor, but by using a particle accelerator [16, 17, 18]. A linear accelerator (LINAC) or a circular accelerator ("cyclotron") brings a beam of protons up to a design kinetic energy. The protons are then fired into a target of ^{18}O-enriched water.

- ^{18}O has 18 nucleons, 8 of which are protons (determining its chemistry). This is a stable isotope of ^{16}O, occuring in 0.2% natural abundance.
- ^{18}F has 18 nucleons, 9 of which are protons.

- The protons from the beam ($E = 18\text{MeV}$) strike nuclei of ^{18}O, coming close enough to tunnel through the Coulomb barrier and become part of the nucleus; in the process, a neutron and gamma radiation are ejected, leaving ^{18}F

$$^{18}O + p^{+} \rightarrow ^{18}F + n^{0} + \gamma$$

FIGURE 5-5 Photograph of a negative-ion biomedical cyclotron. *Left,* Cyclotron within concrete shield. *Right,* The cyclotron itself. (*Courtesy Siemens Molecular Imaging Inc., Knoxville, TN.*)

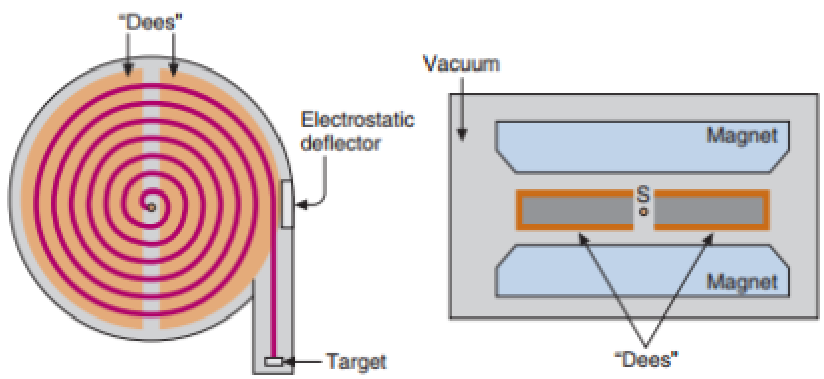


FIGURE 5-3 Schematic representation of a positive ion cyclotron: top (*left*) and side (*right*) views. The accelerating voltage is applied by a high-frequency oscillator to the two "dees." S is a source of positive ions.

Images from Ref. [19]

The Building Blocks of Matter, the Forces of Nature

We have focused a lot on the atom this semester, made from electrons in orbit around a tightly packed nucleus composed of protons and neutrons. Electrons are held in the atom by the electromagnetic interaction.

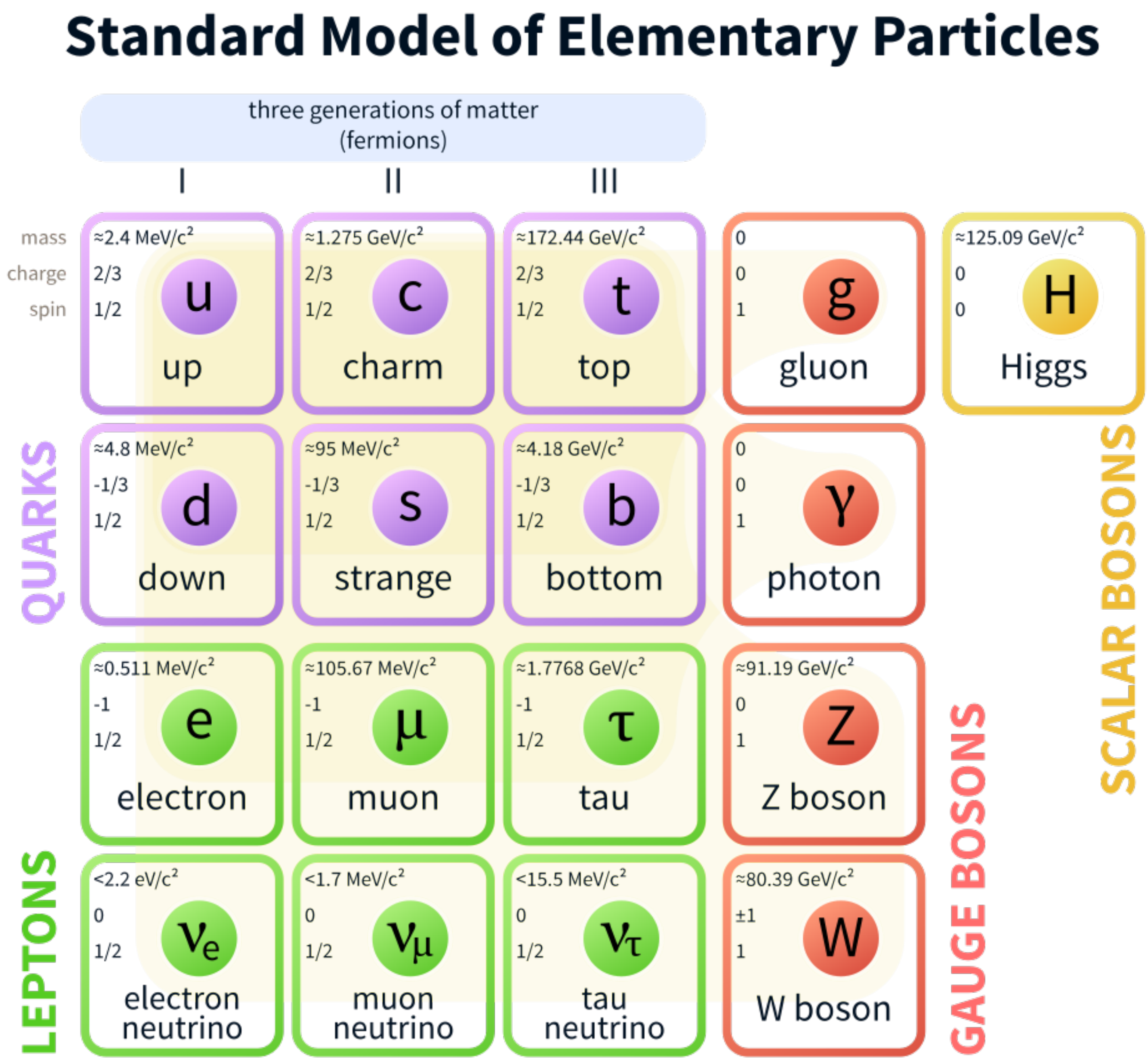
Matter

It is important to know that neutrons and protons are not fundamental - they are made from other particles.

To the best of our knowledge, the table below represents the fundamental building blocks of all matter - quarks and leptons. Up and down quarks make up protons (uud) and neutrons (udd), and the electron is one of six known "leptons."

Forces

To the best of our knowledge, there are four fundamental forces in nature: gravity, electromagnetism, and the weak and strong nuclear forces. The latter two are transmitted by the weak bosons ($W^{\pm}$ and Z^0) and the gluon (g), respectively, analogs of the photon.



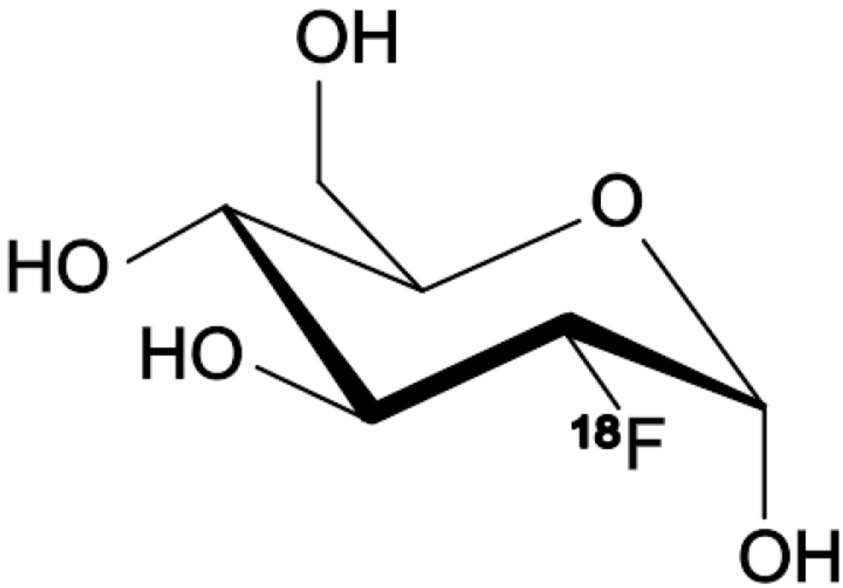
Gravitation is described by the General Theory of Relativity (not depicted), while all other forces are described by Quantum Field Theory ("The Standard Model")

s.3 PET Scan

The PET Scan

^{18}F, Biochemistry, and Radiology

The radioisotope ^{18}F has a half-life of just 110 minutes; it must be produced close to where it will be utilized (within a several-hour drive). It is used in the production of fluorodeoxyglucose (FDG), an analog of glucose that is readily taken up by cells that need glucose. **A typical medical dose represents 7.6 mSv (milli-Sieverts) of radiation exposure.**



By Anypodetos - Own work, Public Domain. Chemical structure of fluorodeoxyglucose (18F) [18F-FDG]

A "Sievert" is a unit of radiation exposure representing the biological impact of depositing a Joule of radiation into a kilogram of human tissue.

Some numbers

50nSv Being next to a person for 8 hours

90nSv Living within 50 miles of a nuclear power plant for 1 year

98nSv The "Banana Equivalent Dose", a measure of radiation from a typical banana

300nSv Living within 50 miles of a coal-fired power plant for 1 year

5-10μSv One set of dental x-rays

1.5-1.7mSv Annual dose for flight attendants.

10-30mSv Single full-body CT scan

80mSv 6-month stay on the International Space Station

250mSv 6-month trip to Mars

500mSv Maximum annual shallow-depth skin dose allowed by occupational health standards in the US

1Sv Maximum lifetime dose allowed for NASA Astronauts. At this dose level, a person has a 5.5% chance of developing a cancer in their lifetime.

4-5Sv Dose will kill a human with 50% risk in 30 days.

Decay of ^{18}F

When ^{18}F decays, it returns to ^{18}O and primarily emits β radiation in the process. Let's think about what it means for ^{18}F to be able to spontaneously decay to ^{18}O while emitting β radiation.

- ^{18}F has 18 nucleons, 9 of which are protons.
- ^{18}O has 18 nucleons, 8 of which are protons.
- ^{18}F must "lose" a proton in the process of becoming ^{18}O, while at the same time gaining a neutron to maintain $A = 18$ (mass number).
- The reaction equation for a proton in the nucleus of ^{18}F must look something like

  $$p^+ \rightarrow n^0 + \beta + X$$

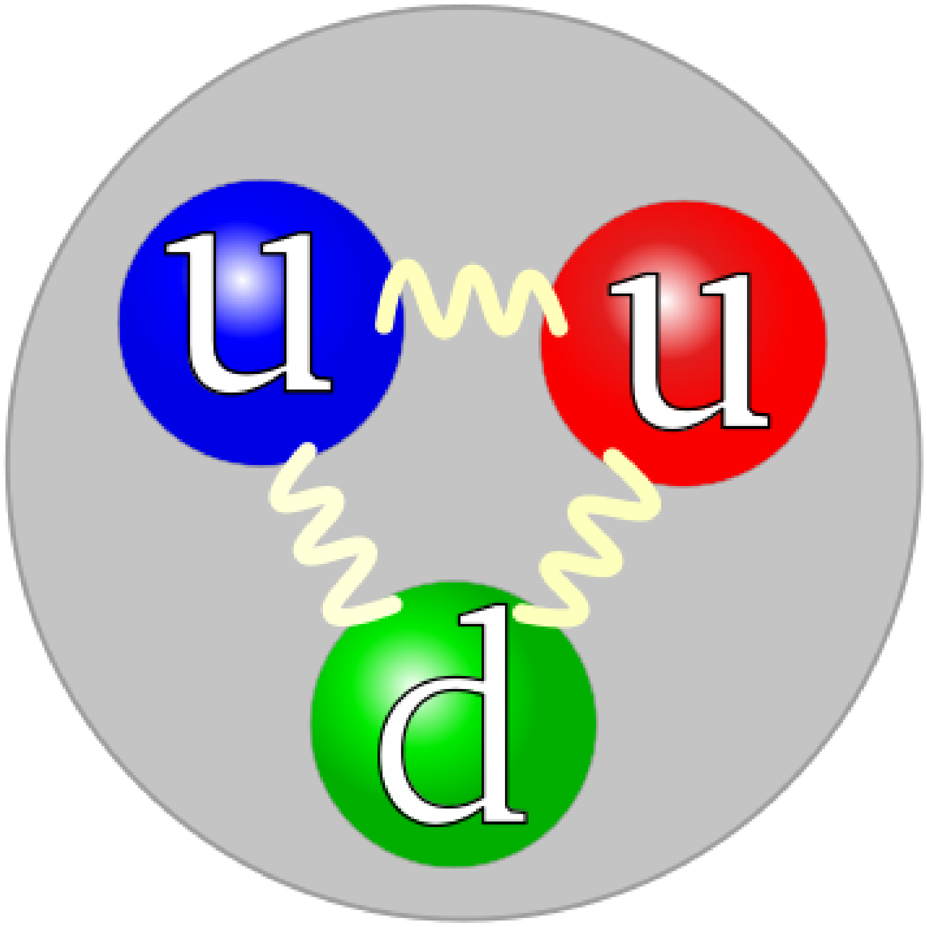
  where X stands for any other emission products that may be required to conserve charge, momentum, total energy, etc. (could be nothing at all)
- Issue: free protons do not spontaneously decay (the measured lifetime of free protons is well in excess of the age of the Universe). Is that a problem here? **DISCUSS**
- Issue: what electric charge must the β radiation have? **DISCUSS**

The Decay of ^{18}F: a quark-level view

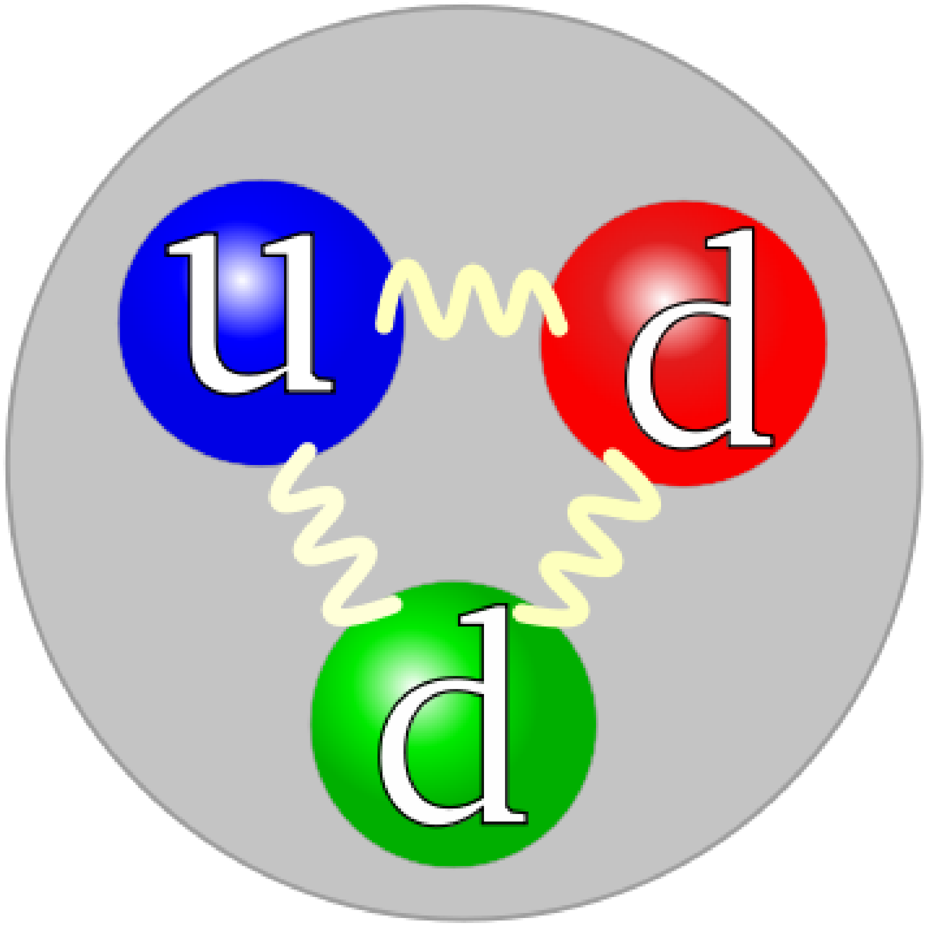
- Protons are not really "free" in the nucleus. They are bound up with other nucleons in close proximity to one another ($< 1\mathrm{fm}$ separations).
- Not only that, protons are not fundamental or point-line: they are built from quarks and gluons.
- The weak nuclear interaction is capable of changing one type of quark to another (e.g. $u \rightarrow d$) - it's fundamental transmutation of matter, induced by a force.
- Viewed as a collection of quarks, it's not really a surprise that the proton (uud) could, through a weak nuclear interaction, transmute into a neutron (udd)
- The reaction is:

  $$(ud)u \rightarrow (ud)d + W^+ \rightarrow n^0 + e^+ + \nu_e$$

  where ν_e is an electron-type neutrino, one of the leptons, and e^+ is an anti-matter electron - a *positron*.



$\Downarrow$

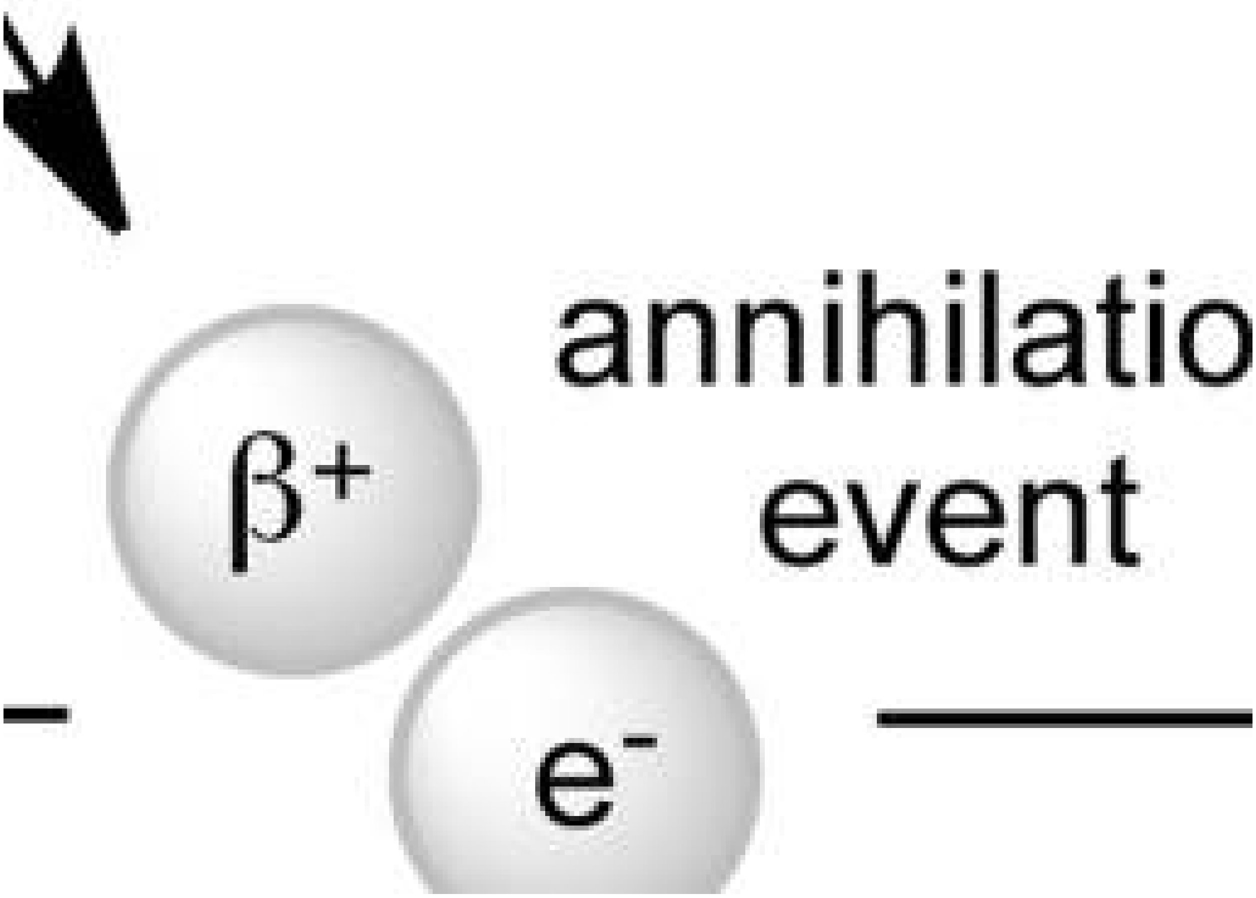


The Annihilation of the Positron

Having decayed and produced a positive-charged anti-matter electron ("positron"), ^{18}F becomes stable ^{18}O and is radiologically inert. The positron is the next step in making a map of where the FDG has collected in the body (and is being metabolized).

- Positrons with energies typical of this nuclear decay can travel only about 2mm in tissue.
- Within that typical range, they interact with an atomic electron and the two particles mutually *annihilate* one another into pure energy: a pair of gamma ray photons.
- We looked at this annihilation process in the class period on special relativity, energy, momentum, and mass.

- The pair of photons travel along straight lines until they interact with matter. The patient is surrounded by a ring of dense material that general electrical signals upon absorbing a gamma ray. By projecting along lines pointing from the face of each detector back through the patient, a 3-D image of the location of FDG can be formed.



The Next Steps - Academic Physics

This class has bridged from the classical physics concepts of space, time, energy, and matter to the modern concepts: special relativity and the wave nature of matter. What are the next steps in physics?

- *Enhance your mathematical methodology for physics* → PHYS 4321
- *Revisit classical mechanics in an energy context* → PHYS 3344

- *Explore the principles of the structure, origin, and fate of the universe* → PHYS 3368
- *Explore the stastical nature of microscopic behavior and the origin of thermodynamics* → PHYS 3374
- *Go deeper into electricity and magnetism and classical field theory* → PHYS 4392
- *Go deeper into quantum mechanics* → PHYS 5382
- *Go deeper into the connection between the subatomic and cosmic* → PHYS 5380
- *Go deeper into the nature, structure, and properties of solids* → PHYS 5337

A Physics Minor requires just 2 classes beyond this one, and a research course is permitted for one of these. A Physics Major (B.A or B.S.) requires more courses and is described on our Undergraduate Program Webpage.

The Next Steps - Physics Research

SMU's Physics Department is involved in multiple frontier areas of the field, with concentrations in Particle Physics and Astrophysics/Cosmology.

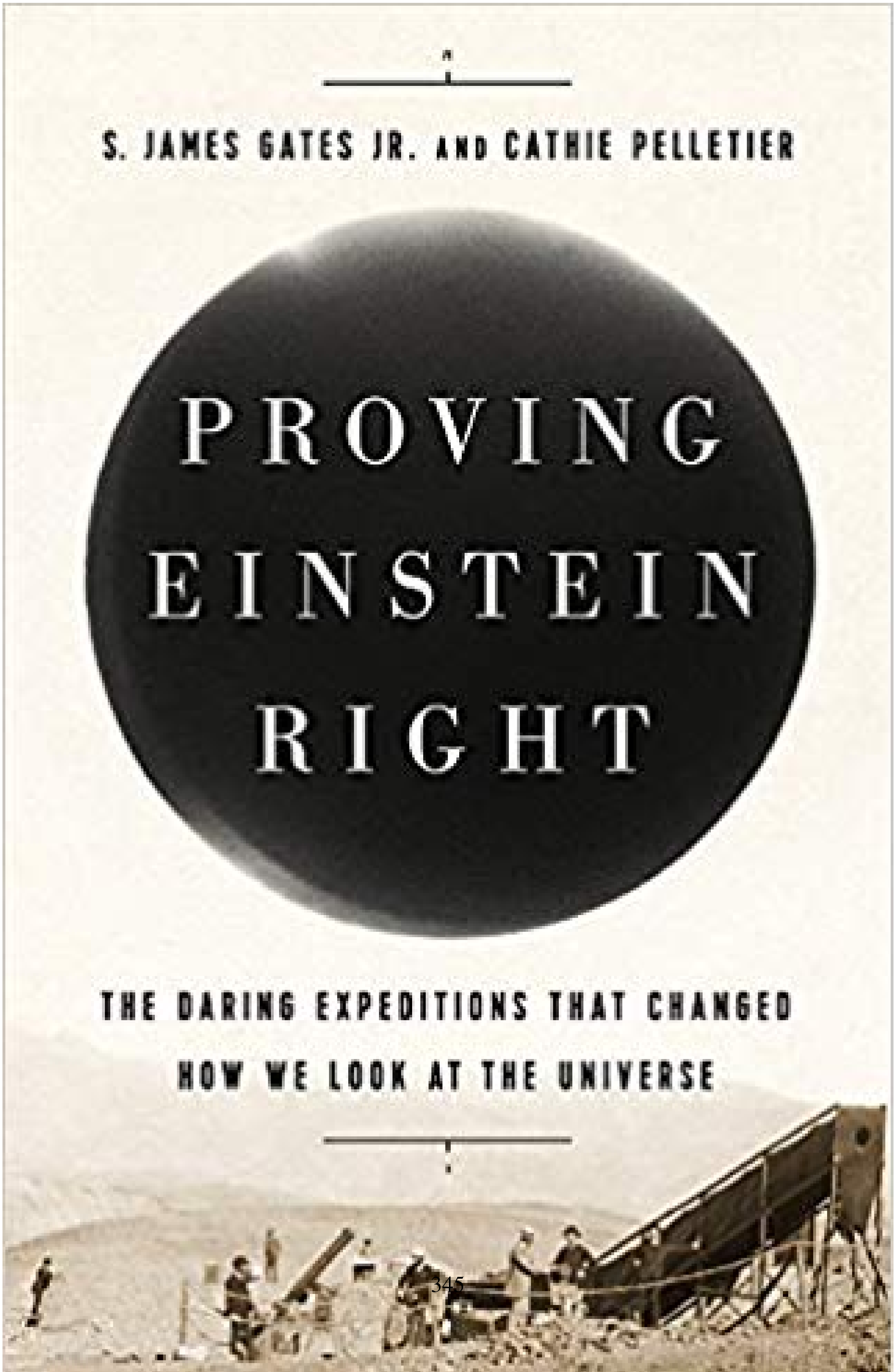
- The Large Hadron Collider - ATLAS Experiment [Deiana*, Sekula*, Stroynowski, Ye*]
- Particle Theory [Nadolsky*, Olness, Vega]
- NOνA and DUNE Experiments - experimental neutrino physics [Coan]
- Observational Astronomy and Astrophysics [Kehoe*, Smith]
- Experimental Dark Matter Searches [Cooley*]
- Theoretical Astrophysics and Cosmology [Meyers]

[*] Indicates faculty who frequently or recently take-on undergraduate researchers. An underline indicates junior faculty who may not yet have had a chance to work with undergraduate researchers but are open to the opportunity.

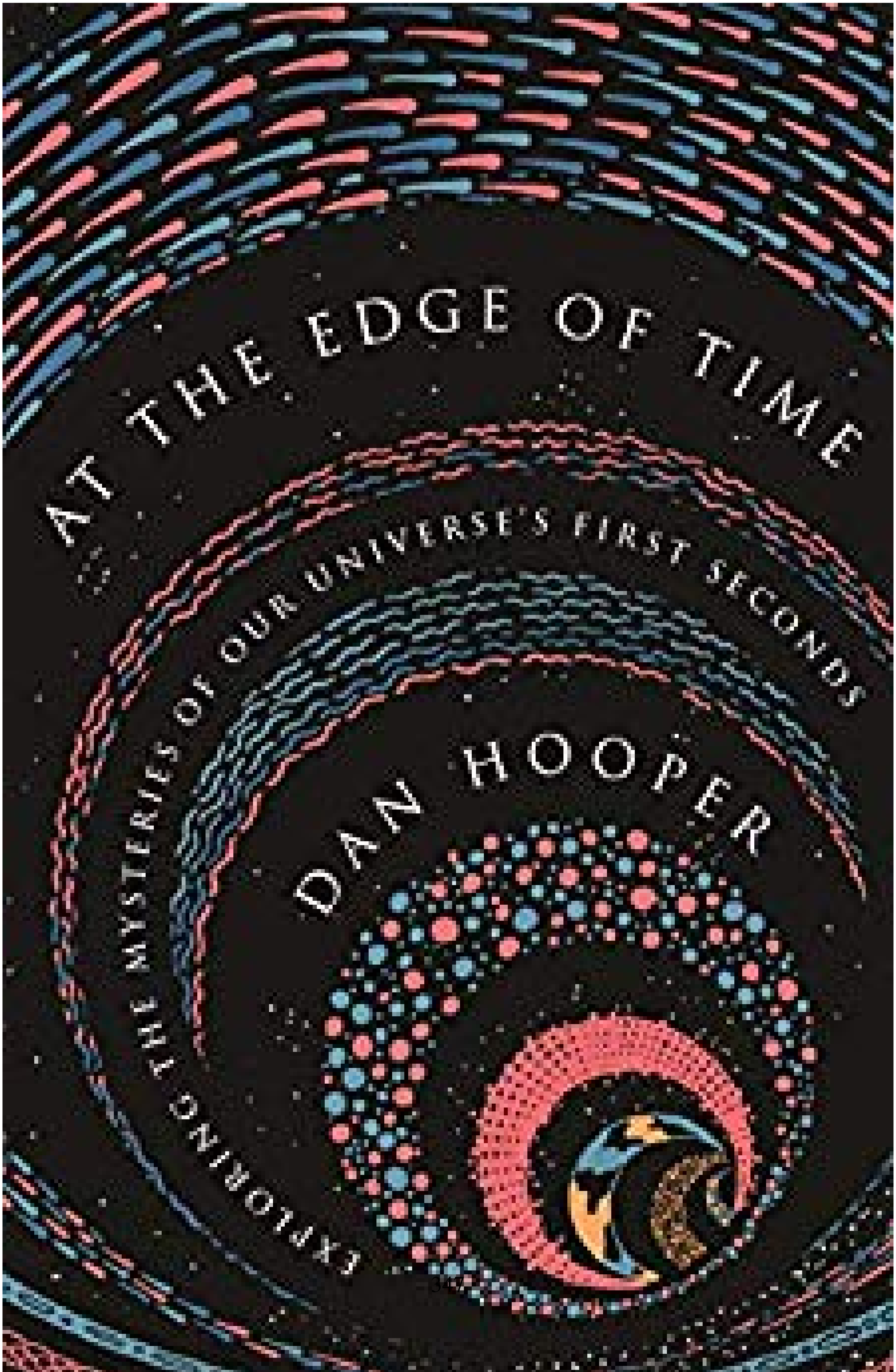
Our department Research Website has lots of information!

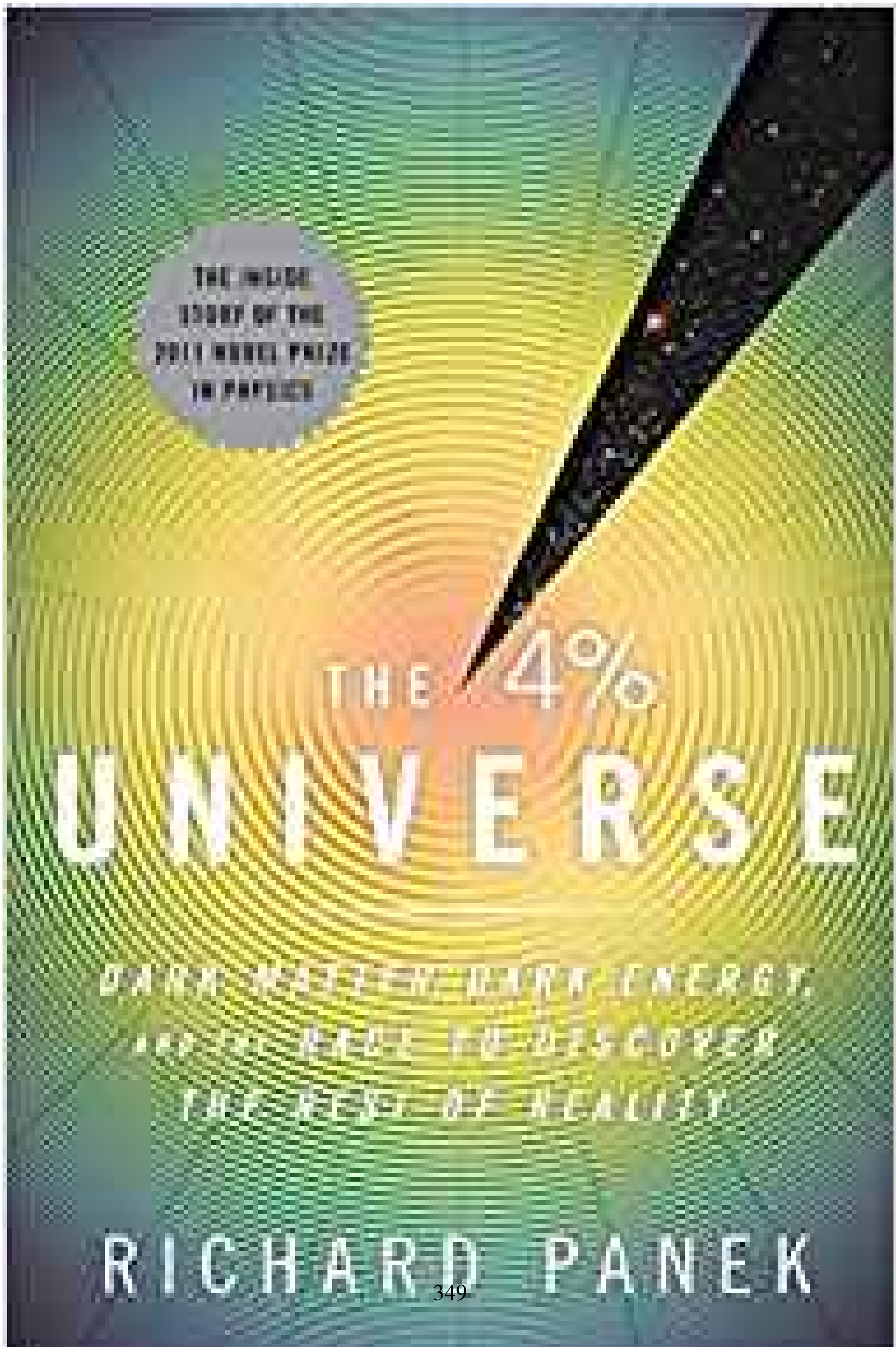
The Next Steps - Go Big

Holiday book recommendations! Broaden your perspective on physics and the universe:


S. JAMES GATES JR. AND CATHIE PELLETIER
PROVING EINSTEIN RIGHT
THE DARING EXPEDITIONS THAT CHANGED HOW WE LOOK AT THE UNIVERSE

Buy on Amazon.com


AT THE EDGE OF TIME
EXPLORING THE MYSTERIES OF OUR UNIVERSE'S FIRST SECONDS
DAN HOOPER


THE 4%
UNIVERSE
RICHARD PANEK

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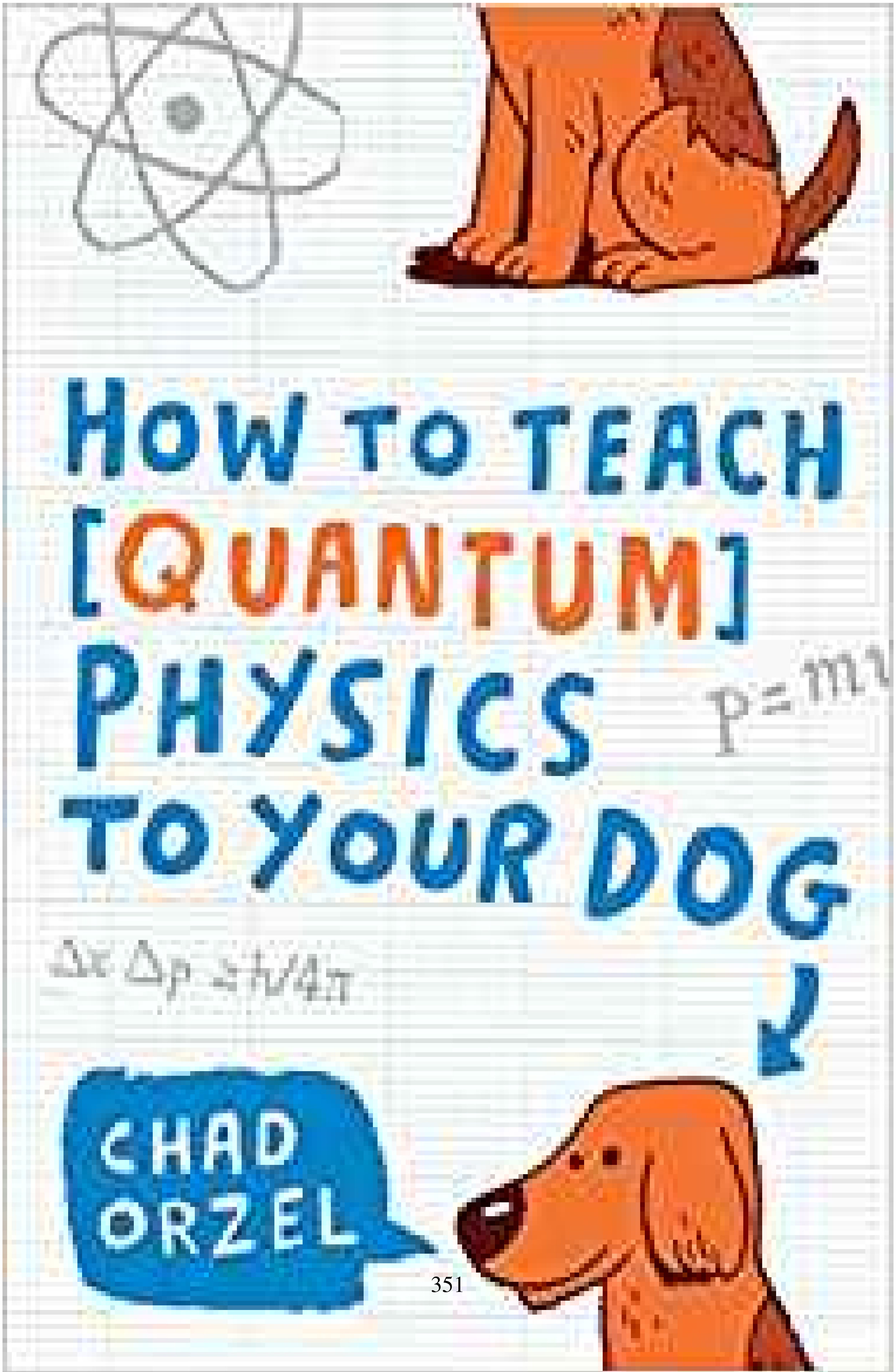

HOW TO TEACH [QUANTUM] PHYSICS TO YOUR DOG
p=mv
$\Delta x \Delta p \geq h/4\pi$
CHAD ORZEL

Buy on Amazon.com

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